MULTIPOLE ANALYSIS OF SINGLE GRAVITON STATE FUNCTIONS

by

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The statefunctions of massless particles described by a tensor field are classified by expressing them in terms of the eigenstates of the operator of angular momentum. The general tensor statefunction can also be separated into functions of different parity. By identifying the graviton as a special case obtained by imposing certain auxiliary conditions, familiar from the classical theory of gravitation, one arrives at a multipole analysis of single graviton statefunctions.

Employing standard composition methods one can use these results to arrive at selection rules governing the decay of objects into two or more gravitons.
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I. INTRODUCTION AND SUMMARY

1. Introduction

Multipole analysis is a powerful tool in electromagnetic theory as it uses, in general method, language fashioned after classical precedent. It is particularly useful in obtaining selection rules for the decay of atomic states into two or more photons.

The photon is considered as a special case of the solution of the wave equation for a vector field plus an auxiliary condition (Lorentz condition) which can be interpreted as a transversality condition. For well known reasons one considers in quantum mechanics generalised photon states, including longitudinal and timelike polarisations which do not occur as free particles, and eliminates these after the complete classification of all four-polarisation states has been affected.

This raises the question whether one cannot proceed in a similar fashion to treat the case of the gravitational field. Here again, the classical theory invokes only transverse gravitons, corresponding to particles of spin two, by the imposition of auxiliary conditions on the solution of the wave equation for a tensor field.

The analogy with quantum electrodynamics suggests to first of all treat the general solutions of the wave equation for a tensor field, then to make the corresponding multipole
analysis, and then to impose the auxiliary conditions only after this has been accomplished.

2. **Summary**

In Chapter II the suggestion is made that, in seeking a gravitational theory, the best approach is probably one involving the use of a rank two tensor to describe the field. This leads to the consideration of a quantum mechanical tensor field as a possible basis for a gravitational theory. Using the linear approximation to the field equations for a vacuum in the Einstein theory expressions identical to those describing a particle of zero mass and spin two are obtained.

Chapter III is devoted to obtaining the eigenstates of the angular momentum operator for a tensor field by combining spin one states using the Clebsch-Gordan formulae, and then using the same formulae to combine the spin and orbital angular momentum eigenstates. The functions obtained are called tensor spherical harmonics by an analogy with the spin one case.

In Chapter IV, certain auxiliary conditions are imposed on the general state function for a spin s, and in the case of spin two, the functions obtained when the spin is parallel to the motion of the particle are in accord with those of Zhirnov and Shirokov (Zhirnov and Shirokov 1957). The analysis is carried through for all cases, and the parity of
each of the functions obtained is evaluated.

Using the work of Zhirnov and Shirokov (Zhirnov and Shirokov 1957), in Chapter V a brief discussion is given of selection rules governing the decay of positronium into two gravitons with, for comparison, the selection rules for decay into two photons.
II. MOTIVATION FOR CONSIDERING A TENSOR FIELD

When considering the field seen by a freely falling observer under the gravitational attraction of a mass point (or spherically uniform mass distribution), the quadrupole nature of gravitation becomes evident. We can examine the stresses in a body $A$, with centre of mass $O$, falling freely under the influence of a massive body $E$, and find that the forces $F$ are in the directions shown.

![Diagram showing forces and field lines](image)

Thus we may construct field lines for the force as seen by an observer at $O$ as shown, and these are seen to be those corresponding to a quadrupole field. To describe this field, a three-dimensional tensor of second rank is required, so we conclude that in seeking a theory of gravitation the most fruitful approach is likely to be one using a tensor theory.

If one thinks in terms of Quantum Mechanics, one can now ask the obvious question: does a quantum theory of particles described by a tensor field give a tenable gravitational theory? The answer to this question is yes, if we place some
restrictions on the particles allowed. Pauli and Fierz (Pauli and Fierz 1939) showed that the equations for the description of massless particles of spin two, with the auxiliary condition of Lorentz invariance, are identical to those for the linear approximation to Einstein's theory for a vacuum. These particles are called gravitons. Thus we are led to considering the statefunctions of massless particles described by a tensor field in the work that follows.

For an empty space, the field equations of general relativity may be written as

$$ R_{ij} = 0 \quad i,j \in \{1,2,3,4\} $$  \hspace{1cm} (2.1)

where \( R_{ij} \) is the Ricci tensor,

$$ R_{ij} = \Gamma^\kappa_{i\kappa,j} - \Gamma^\kappa_{ij,\kappa} + \Gamma^\kappa_{i\kappa} \Gamma^\lambda_{j\lambda} - \Gamma^\lambda_{ij} \Gamma^\kappa_{\lambda\kappa} $$  \hspace{1cm} (2.2)

A comma here denotes partial differentiation with respect to the appropriate co-ordinate component, and the \( \Gamma^\kappa_{ij} \) are the Christoffel symbols of the second kind; namely

$$ \Gamma^\kappa_{ij} = \frac{1}{2} g^{\kappa\ell} (g_{\ell i,j} + g_{\ell j,i} - g_{ij,\ell}) $$

where \( g_{ij} \) is the metric tensor. In the linear approximation to the theory, the metric tensor is assumed to depart little from that of Minkowski space

$$ g_{ij} = \eta_{ij} + \chi_{ij} \quad , \quad g^{ij} = \eta^{ij} + \chi^{ij} $$  \hspace{1cm} (2.3)

where \( \eta_{ij} = \eta^{ij} = \text{diag}(1,1,1,-1) \) and \( \chi_{ij} , \chi^{ij} \) are assumed to be small quantities with continuous second partial derivatives. Then,

$$ g_{ij,\kappa} = \chi_{ij,\kappa} \quad , \quad g_{ij,\kappa\ell} = \chi_{ij,\kappa\ell} $$  \hspace{1cm} (2.4)
Using these, (2.1) becomes
\[ \frac{1}{2} \left[ g^{kh}(\delta_{ih,k} + \delta_{kh,i} - \delta_{ik,h}) \right]_{ij} = \frac{1}{2} \left[ g^{kh}(\delta_{ih,j} + \delta_{jh,i} - \delta_{ij,h}) \right]_{kh} \]

\[ + \frac{1}{4} g^{kh} g^{lj} [(\delta_{jkh} + \delta_{hjk} - \delta_{kjh})(\delta_{ij,k} + \delta_{kj,i} - \delta_{ij,k}) - (\delta_{kh,i} + \delta_{ik,j} - \delta_{ih,j})(\delta_{ij,j} + \delta_{jj,i} - \delta_{ij,j})] = 0 \]  

(2.5)

Selecting only those terms to first order of \( \gamma \) or its derivatives gives the equation
\[ \eta^{kh} (\delta_{ih,k} + \delta_{kh,i} - \delta_{ik,h} - \delta_{ij,k} + \delta_{ij,h}) = 0 \]  

(2.6)

By the assumed continuity of \( \delta_{ih,jk} \)
\[ \delta_{ih,jk} = \delta_{ih,kj} \]  

(2.7)

so
\[ \eta^{kh} (\delta_{khij} + \delta_{ij,kh} - \delta_{ik,hj} - \delta_{ij,ik}) = 0 \]  

(2.8)

and so
\[ \eta^{ij} \eta^{kh} (\delta_{khij} + \delta_{ij,kh} - \delta_{ik,hj} - \delta_{ij,ik}) = 0 \]  

(2.9)

This becomes
\[ \eta^{ij} \eta^{kh} (\delta_{khij} + \delta_{ij,kh} - \delta_{ik,hj} - \delta_{ij,ik}) = 0 \]  

(2.10)

These equations may now be written in Minkowskian co-ordinates. (2.8) becomes
\[ \delta_{el,j} + \delta_{ij,el} - \delta_{il,j} - \delta_{ij,el} = 0 \]  

(2.11)

and (2.10) becomes
\[ \delta_{el,jj} - \delta_{jj,el} = 0 \]  

(2.12)

or, more fully
\[ \square \delta_{ij} + \frac{\partial^2 \delta_{ij}}{\partial x^i \partial x^j} - \frac{\partial^2 \delta_{ij}}{\partial x^i \partial x^e} - \frac{\partial^2 \delta_{ij}}{\partial x^e \partial x^i} = 0 \]

and
\[ \square \gamma - \frac{\partial^2 \gamma}{\partial x^i \partial x^e} = 0 \]  

(2.13)

where
\[ \delta_{el} = \gamma \]
The equations (2.11), and hence (2.12), are also satisfied by
\[ \phi_{ij} = \gamma_{ij} + \Lambda_{ij} + \Lambda_{ji}, \]
where \( \Lambda_i \) is an arbitrary vector.

To see this, substitute \( \phi_{ij} \) for \( \gamma_{ij} \) in the left hand side of equation (2.11). Then
\[ \begin{align*}
\phi_{ll,ij} + \phi_{ij,le} - \phi_{ie,ei} - \phi_{ji,il} &= \gamma_{ll,ij} + 2\Lambda_{l,ij} + \Lambda_{ij,le} + \Lambda_{ji,ll} + \Lambda_{ij,eli} - \Lambda_{ii,ij} - \Lambda_{ij,ll} - \Lambda_{ji,il} - \Lambda_{ji,ei} - \Lambda_{ji,il} \\
&= 0
\end{align*} \]
by the assumed continuity of \( \Lambda_{i,jl} \).

If the requirement that
\[ \phi_{ii} = 0 \]
is imposed, the gauge is restricted only by the following equation
\[ \Lambda_{ii} = 0 \]
as \( \phi_{ii} = \phi'_{ii} + 2\Lambda_{ii} \) and \( \phi'_{ii} = 0 \).

The additional requirement that
\[ \phi_{ik,k} = 0 \]
implies
\[ \phi_{ik,k} = \phi'_{ik,k} + \Lambda_{i,kk} + \Lambda_{k,ik} = 0 \]
so
\[ \Lambda_{i,kk} = 0 \]
Then (2.11) becomes, by the symmetry of \( \phi_{ij} \), and the two conditions (2.16) and (2.18)
\[ \phi_{ik,il} = 0 \]
Thus $\phi_{ik}$ must satisfy the wave equation $\Box \phi_{ik} = 0$, and the supplementary conditions that $\frac{\partial \phi_{ik}}{\partial x^k} = 0$, and that $\phi_{ik}$ should be symmetric and have vanishing trace. The gauge is then restricted by the conditions $\frac{\partial \Lambda_i}{\partial x^i} = 0$ and $\Box \Lambda_i = 0$. The solutions $\phi_{ik}$ then correspond to the transverse-transverse graviton states obtained in Chapter IV.
III. TENSOR SPHERICAL HARMONICS

The component operators of angular momentum, denoted by $J_1, J_2, J_3$ satisfy the commutation relations

$$J_\kappa J_\ell - J_\ell J_\kappa = i \epsilon_{\kappa\ell\mu} J_\mu$$

(3.1)

where each index has the range 1 to 3, and repeated indices are summed over in this range. $\epsilon_{\kappa\ell\mu}$ is the fully antisymmetric unit tensor of rank three in three dimensions.

There exist (e.g. Kaempffer 1965) simultaneous eigenstates of $J^2 = J_\kappa J_\kappa$ and $J_3$, denoted by $|j, m\rangle$, satisfying

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle \quad j \in \{0, \frac{1}{2}, 1, \ldots\}$$

$$J_3 |j, m\rangle = m |j, m\rangle \quad m \in \{-j, -j+1, \ldots, j-1, j\}$$

(3.2)

Considering first the spin one case, i.e. the case where $j = 1$, the states $|1, m\rangle$ and the operators can be represented in the following way

$$|1, 1\rangle = \begin{pmatrix} 1 \cr 0 \cr 0 \cr 0 \end{pmatrix} \quad |1, 0\rangle = \begin{pmatrix} 0 \cr 1 \cr 0 \cr 0 \end{pmatrix} \quad |1, -1\rangle = \begin{pmatrix} 0 \cr 0 \cr 1 \cr 0 \end{pmatrix}$$

with

$$J_1 = \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad J_2 = \frac{i}{2} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \cr 0 & 0 & 0 \cr 0 & 0 & -1 \end{pmatrix}$$

and

$$J^2 = 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(3.3)

These can now be transformed to a new representation, under the unitary transformation

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$S_i = U J_i U^+ \quad \xi(m) = U |1, m\rangle$$

(3.4)

Then the eigenstates of $S^2, S_3$, denoted here by $\xi(m)$, satisfy

$$S^2 \xi(m) = 2 \xi(m), \quad S_3 \xi(m) = m \xi(m)$$

(3.5)
In matrix form, these quantities are

\[
\begin{align*}
\xi^{(1)} &= -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\xi^{(0)} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\xi^{(-1)} &= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
S^2 &= 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{align*}
\]

\[
S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \\
S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\] (3.6)

Then \(\xi^{(1)}\) corresponds to right hand, \(\xi^{(-1)}\) to left hand circular polarisation, and \(\xi^{(0)}\) to longitudinal polarisation (Akhiezer and Berestetskii 1962).

The states resulting from the addition of two such spin one states are obtained in the normal way, these final states having angular momentum \(j = 2, 1\) or \(0\).

\[j = 2:\]

\[
\chi^{j=2} = \xi^{(1)} \otimes \xi^{(1)}
\]

These can be written more conveniently as 3×3 matrices \((a_{ij})\) such that

\[a_{ij} = \xi_i^{(1)} \xi_j^{(1)}\]

so

\[
\chi^{2}_{ij} = \xi_i^{(1)} \xi_j^{(1)}
\] (3.8)

Then

\[
\chi^{2} = -\frac{1}{2} [\xi^{(1)} \otimes \xi^{(0)} + \xi^{(0)} \otimes \xi^{(1)}]
\]

corresponds to
\[ \chi^2_{ij}(z) = -\frac{i}{2 \sqrt{2}} \left[ \xi_{i}(0) \xi_{j}(-1) + \xi_{i}(1) \xi_{j}(1) \right] \]  

(3.9)

\[ \chi^2(0) = \frac{1}{\sqrt{2}} \left[ \xi_{i}(0) \xi_{j}(-1) + 2 \xi_{i}(0) \xi_{j}(0) + \xi_{i}(-1) \xi_{j}(0) \right] \]

corresponds to

\[ \chi^2_{ij}(0) = \frac{1}{\sqrt{2}} \left[ \xi_{i}(0) \xi_{j}(-1) + 2 \xi_{i}(0) \xi_{j}(0) + \xi_{i}(-1) \xi_{j}(0) \right] \]  

(3.10)

\[ j=1: \quad \chi^{1}_{ij}(z) = \frac{i}{\sqrt{2}} \left[ \xi_{i}(0) \xi_{j}(0) - \xi_{i}(0) \xi_{j}(1) \right] \]

(3.11)

Note that the five states belonging to \( j=2 \) are symmetric with zero trace; the \( j=1 \) triplet of states is antisymmetric; and the \( j=0 \) state is symmetric. It is easily verified that

\[ \sum_{pq} \chi^{s*}_{p,q}(m_{s}) \chi^{s'}_{p,q}(m'_{s}) = \delta_{ss'} \delta_{m_{s}m'_{s}} \]

These new statefunctions are isomorphic to the eigenstates of the operator

\[ S_{3} = I \otimes s_{3} + s_{3} \otimes I = s_{12} \quad \text{etc.} \]

\[ S_{12} = \begin{pmatrix} S_{3} & -iI & 0 \\ iI & S_{3} & 0 \\ 0 & 0 & S_{3} \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

(3.15)

Let \( \langle p,q | S_{ij} | r,s \rangle \) be the element of \( S_{ij} \) in the \( p,q ; r,s \)th...
place, with the matrix $S_{ij}$ labelled as indicated above. Then the equation (suppressing the angular momentum)

$$S_{ij} \chi = m \chi$$

becomes

$$\sum_{rs} \sum_{i,j} <p,q | S_{ij} | r,s> \chi_{rs}(m) = m \chi_{pq}(m) \quad (3.16)$$

Using this notation, the orbital angular momentum can be written as

$$<p,q | L_{ij} | r,s> = -i \delta_{pr} \delta_{qs} \left( p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i} \right)$$

where the symbols have their usual meanings, and the total angular momentum operator $J_{ij} = L_{ij} + S_{ij}$.

The eigenstates of the orbital angular momentum operator are the spherical harmonics $Y_{l,m}(\alpha)$, as is well known, where $\alpha$ is a unit vector in the $\theta, \phi$ direction.

$$Y_{l,m}(\theta, \phi) = \frac{1}{2^l l!} \left( \frac{(2l+1)(l-m)!}{4\pi (l+m)!} \right)^{1/2} e^{im\phi} \left( \frac{1}{\sin \theta} \frac{d}{d\theta} \right)^{l-m} \sin^m \theta$$

(3.17)

At this point it becomes convenient to introduce the concept of "tensor spherical harmonics" in complete analogy with the "vector spherical harmonics" for the spin one case. They can be defined as the eigenstates of $J_{12}$ obtained by combining the eigenstates of $L_{12}$ and $S_{12}$ in the usual manner. These are

$$Y^S_{pq}(j, l, m; \alpha) = \sum_{m'_s = -s}^s C(j, l, s, m, m_s, m'_s) Y^{(m)}_{l, m-m'_s}(\alpha) \chi^{s(m'_s)}_{pq} \quad (3.18)$$

where $j$ is the total angular momentum number;

$l$ the orbital part;

$s \in \{0, 1, 2\}$ the spin part;

$m$ the projection of $j$ on the z axis;
and \( m_s \) the projection of \( S \) on the \( z \) axis.

Given \( j, m, s \), there are \( 2s+1 \) such tensor spherical harmonics, those belonging to

\[ l = j+s, j+s-1, \ldots, j-s. \]

The most general state of energy \( \omega \), spin \( S \) will therefore be a linear combination of \( 2s+1 \) tensor spherical harmonics, corresponding to the \( 2s+1 \) possible values of \( l \), and may therefore be written

\[
Y^S_{pq}(\omega, j, m; \eta) = \sum_{l, j, m} a^S_l(\omega) Y^S_{pq}(j, l, m; \eta)
\]  

(3.19)

For later use, the normalisation condition on the tensor spherical harmonics is written out explicitly

\[
\sum_{pq=1}^{3} \int Y^S_{pq}^*(j, l, m; \eta) Y^S_{pq}^*(j', l', m'; \eta) d\Omega
\]

\[
= \sum_{pq=1}^{3} \sum_{s=3}^{-s} \sum_{s'} \int C(j, l, s, m, m-m, m_s) C(j', l', s', m', m'-m', m'_s) Y^S_{pq}^*(m_s) Y^S_{pq}^*(m'_s) \chi^S_{pq}(m_s) \chi^S_{pq}(m'_s) d\Omega
\]

\[
= \sum_{s=3}^{-s} \sum_{m=3}^s C(j, l, s, m, m-m, m_s) C(j', l', s', m', m'-m', m'_s) \int Y^S_{pq}^*(m_s) Y^S_{pq}^*(m'_s) d\Omega \sum_{pq} \chi^S_{pq}(m_s) \chi^S_{pq}(m'_s)
\]

\[
= \sum_{m=3}^s \sum_{s=3}^{-s} C(j, l, s, m, m-m, m_s) C(j', l', s', m', m'-m', m'_s) \delta_{ls} \delta_{l'm'} \delta_{m'm} \delta_{ss'} \delta_{s's}
\]

\[
= \delta_{ll'} \delta_{mm'} \delta_{ss'} \delta_{jj'} \sum_{m=3}^s \sum_{s=3}^{-s} C(j, l, s, m, m-m, m_s) C(j', l', s', m', m'-m', m'_s)
\]

\[
= \delta_{ll'} \delta_{mm'} \delta_{ss'} \delta_{jj'}
\]

(3.20)
IV. ORTHOGONALITY CONDITIONS AND PARITY

In the case where the spin \( s = 2 \), the most general state is

\[
Y^2_{pq}(\omega, j, m; n) = \sum_{l=0}^{j-2} a^2_{l}(\omega) Y^2_{pq}(j, l, m; n) \tag{4.1}
\]

For a massless particle Lorentz invariance requires that the angular momentum and the motion be parallel (Wigner 1957) so the transversality condition

\[
\sum_p n_p Y^2_{pq}(\omega, j, m; n) = 0 \tag{4.2}
\]

is imposed.

The states satisfying this condition are called transverse-transverse states as the expectation value of the spin component in the \( \theta, \varphi \) direction is then \( \pm 2 \). The states giving \( \mp 1 \) for this component are called longitudinal-transverse states; if the value is zero, then the state is longitudinal-longitudinal (Appendix A).

The vector \( \vec{n} \) can be represented by its components \( n(\mu) \) defined by

\[
\vec{n} = \sum_{\mu=1}^{3} n(\mu) \vec{e}(\mu) \tag{4.3}
\]

Then, in the representation of the \( \vec{e} \) previously given, one finds for the \( n(\mu) \) in terms of the cartesian components

\[
\begin{align*}
n_1 &= \sin \theta \cos \varphi, & n_2 &= \sin \theta \sin \varphi, & n_3 &= \cos \theta \\
n(\uparrow) &= -\frac{1}{\sqrt{2}} (n_1 - i n_2) = \frac{1}{\sqrt{2}} \sin \theta e^{i \varphi} \\
n(\downarrow) &= n_3 = \cos \theta \\
n(\leftarrow) &= \frac{1}{\sqrt{2}} (n_1 + i n_2) = \frac{1}{\sqrt{2}} \sin \theta e^{-i \varphi} \tag{4.4}
\end{align*}
\]

Note also that
Thus the transversality condition written out in full is

\[
\sum_{\ell^2} a^2(x) \left[ C(j, l, 2, m, m-2, 2) Y_{\ell, m-2}^{(m)} \sum_p \xi_p^{(\ell)} \xi_p^{(\ell)} \left( n(1) \xi_p^{(1)} + n(0) \xi_p^{(0)} + n(-1) \xi_p^{(-1)} \right) 
- C(j, l, 2, m, m-1, 1) Y_{\ell, m-1}^{(m)} \sum_p \xi_p^{(\ell)} \xi_p^{(\ell)} \left( n(1) \xi_p^{(1)} + n(0) \xi_p^{(0)} + n(-1) \xi_p^{(-1)} \right) 
+ C(j, l, 2, m, m, 0) Y_{\ell, m}^{(m)} \sum_p \xi_p^{(\ell)} \xi_p^{(\ell)} \left( n(1) \xi_p^{(1)} + n(0) \xi_p^{(0)} + n(-1) \xi_p^{(-1)} \right) 
- C(j, l, 2, m, m+1, -1) Y_{\ell, m+1}^{(m)} \sum_p \xi_p^{(\ell)} \xi_p^{(\ell)} \left( n(1) \xi_p^{(1)} + n(0) \xi_p^{(0)} + n(-1) \xi_p^{(-1)} \right) 
+ C(j, l, 2, m, m+2, -2) Y_{\ell, m+2}^{(m)} \sum_p \xi_p^{(\ell)} \xi_p^{(\ell)} \left( n(1) \xi_p^{(1)} + n(0) \xi_p^{(0)} + n(-1) \xi_p^{(-1)} \right) \right] 
= \sum_{\ell^2} a^2(x) \left[ - \frac{1}{\sqrt{2}} C(j, l, 2, m, m-2, 2) Y_{\ell, m-2}^{(m)} \sin \theta e^{i\phi} \xi_q^{(1)} 
- \frac{1}{\sqrt{2}} C(j, l, 2, m, m-1, 1) Y_{\ell, m-1}^{(m)} \sin \theta e^{i\phi} \xi_q^{(0)} 
+ \frac{1}{\sqrt{3}} C(j, l, 2, m, m, 0) Y_{\ell, m}^{(m)} \sin \theta e^{i\phi} \xi_q^{(0)} 
- \frac{1}{\sqrt{3}} C(j, l, 2, m, m+1, -1) Y_{\ell, m+1}^{(m)} \cos \theta \xi_q^{(-1)} 
- \frac{1}{\sqrt{2}} C(j, l, 2, m, m+2, -2) Y_{\ell, m+2}^{(m)} \cos \theta \xi_q^{(-1)} \right] 
= 0
\]

(4.6)

\(\xi_q^{(1)}\), \(\xi_q^{(0)}\), \(\xi_q^{(-1)}\) are components of an orthogonal system of vectors so the coefficients of each must be zero.

\[
\sum_{\ell^2} a^2(x) \left[ - \frac{1}{\sqrt{2}} C(j, l, 2, m, m-2, 2) Y_{\ell, m-2}^{(m)} \sin \theta e^{i\phi} 
- \frac{1}{\sqrt{2}} C(j, l, 2, m, m-1, 1) Y_{\ell, m-1}^{(m)} \cos \theta 
+ \frac{1}{\sqrt{3}} C(j, l, 2, m, m, 0) Y_{\ell, m}^{(m)} \sin \theta e^{i\phi} \right] = 0
\]

(4.7)
We have the identities (Bethe and Salpeter 1957)

\[
\begin{align*}
\sum_{l,m_1} \frac{1}{2} C(j, l, 2, m, m_1, 1) Y_{l, m_1}^{(m)} &= \sin \theta e^{i \varphi} \\
+ \sqrt{3} C(j, l, 2, m, 0) Y_{l, m_1}^{(m)} \cos \theta \\
- \frac{1}{2} C(j, l, 2, m, m_1, -1) Y_{l, m_1}^{(m)} \sin \theta e^{-i \varphi} 
\end{align*}
\]

(4.8)

\[
\sum_{l,m_1} \frac{1}{2} C(j, l, 2, m, m_1, 0) Y_{l, m_1}^{(m)} \sin \theta e^{i \varphi} \\
+ \frac{1}{2} C(j, l, 2, m, m_1, -1) Y_{l, m_1}^{(m)} \cos \theta \\
+ \frac{1}{2} C(j, l, 2, m, m_2, -2) Y_{l, m_2}^{(m)} \sin \theta e^{-i \varphi} 
\]

(4.9)

We have the identities (Bethe and Salpeter 1957)

\[
\begin{align*}
\sum_{l,m_1} \frac{1}{2} C(j, l, 2, m, m_1, 1) Y_{l, m_1}^{(m)} &= \sin \theta e^{i \varphi} \\
+ \sqrt{3} C(j, l, 2, m, 0) Y_{l, m_1}^{(m)} \cos \theta \\
- \frac{1}{2} C(j, l, 2, m, m_1, -1) Y_{l, m_1}^{(m)} \sin \theta e^{-i \varphi} 
\end{align*}
\]

(4.10)

Substituting (4.10) in (4.7), one obtains

\[
\begin{align*}
\sum_{l,m_1} \frac{1}{2} C(j, l, 2, m, m_1, 1) Y_{l, m_1}^{(m)} &= \sin \theta e^{i \varphi} \\
+ \sqrt{3} C(j, l, 2, m, 0) Y_{l, m_1}^{(m)} \cos \theta \\
- \frac{1}{2} C(j, l, 2, m, m_1, -1) Y_{l, m_1}^{(m)} \sin \theta e^{-i \varphi} 
\end{align*}
\]

(4.11)

Using (4.8) and (4.9) similar and consistent expressions for the \(a_{j}^{2}\) may be obtained. The calculation outlined below is written out in Appendix B.
Substituting the values of the Clebsch-Gordan coefficients (Condon and Shortley 1935) in (4.11), and recalling that the $\gamma_{\lambda, m}^{(n)}$ are orthonormal functions so that the coefficients of each must be zero, we obtain for the $a^2_{\lambda} \gamma^{(n)}$ the following relations

$$a^2_j \left( \frac{j(2j-1)}{6(2j+1)} \right)^{\lambda^2} = a^2_{j+2} (j_2) \lambda^2$$

$$a^2_j \left( \frac{(j+1)(2j+3)}{6(2j+1)} \right)^{\lambda^2} = a^2_{j-2} (j_1) \lambda^2$$

$$a^2_{j+1} (j+2)^{\lambda^2} = a^2_{j-1} (j-1)^{\lambda^2}$$

Hence

$$\gamma^2_{pq} (\omega, j, m; n) = \frac{1}{N} \left[ (j-1)^{\lambda^2} \gamma^2_{pq} (j, j_1, m; n) + (j+2)^{\lambda^2} \gamma^2_{pq} (j, j-1, m; n) \right]$$

and

$$\gamma^2_{pq} (\omega, j, m; n) = \frac{1}{M} \left[ \left( \frac{j(2j+1)}{6(j+2)(2j+3)} \right)^{\lambda^2} \gamma^2_{pq} (j, j+2, m; n) + \gamma^2_{pq} (j, j, m; n) + \left( \frac{(j+1)(2j+3)}{6(j-1)(2j+1)} \right)^{\lambda^2} \gamma^2_{pq} (j, j-2, m; n) \right]$$

where $M$ and $N$ are normalising factors.

The $\gamma_{pq} (j, l, m; n)$ were shown in the previous section to be orthonormal functions so, using this fact, obtain

$$\gamma^2_{pq} (\omega, j, m; n) = \left( \frac{j+1}{2j+1} \right)^{\lambda^2} \gamma^2_{pq} (j, j_1, m; n) + \left( \frac{j_2}{2j+1} \right)^{\lambda^2} \gamma^2_{pq} (j, j-1, m; n)$$

and
for the states satisfying the condition (4.2), the transverse-transverse states.

Other conditions may be imposed to yield a complete set of orthonormal functions, the longitudinal-longitudinal and the longitudinal-transverse polarisations, corresponding respectively to the conditions $\epsilon_{pqr} n_q Y_{rt}^2 = 0$ and $\epsilon_{pqr} n_q n_t Y_{rt}^2 = 0$

From $\epsilon_{pqr} n_q n_t Y_{rt}^2 (w_j;m;\eta) = 0$, we obtain

$$Y_{pq}^2 (w_j;j;\eta;\eta)^2 = \left( \frac{j^2}{2j+1} \right)^2 Y_{pq}^2 (j;j;\eta;\eta) - \left( \frac{j^2}{2j+1} \right)^2 Y_{pq}^2 (j;j;\eta;\eta) (4.17)$$

and

$$Y_{pq}^2 (w_j;j;\eta;\eta)^4 = - \left( \frac{j^2}{2j+1} \right)^2 Y_{pq}^2 (j;j;\eta;\eta) - \left( \frac{j^2}{2j+1} \right)^2 Y_{pq}^2 (j;j;\eta;\eta) (4.18)$$

From $\epsilon_{pqr} n_q Y_{rt}^2 = 0$, there is only one possible solution

$$Y_{pq}^2 (w_j;j;\eta;\eta)^5 = \left( \frac{j^2}{2j+1} \right)^2 Y_{pq}^2 (j;j;\eta;\eta) - \left( \frac{j^2}{2j+1} \right)^2 Y_{pq}^2 (j;j;\eta;\eta) (4.19)$$

For the $s = 1$ case a similar analysis can be performed.

The most general state here, is

$$Y_{pq}^i (w_j;j;\eta;\eta) = \sum_{\ell=j-1}^{2j+1} a_{q,p}^i (w) Y_{pq}^i (j;\ell;\eta;\eta) (4.20)$$

as in the previous section.
Here the transverse states are characterised by the equation 
\[ \varepsilon_{pq} n_p \gamma_{qr}' = 0, \]
and the longitudinal by \( n_p \gamma_{pq}' = 0, \) as \( \gamma_{pq}'(\omega, j, m; n) \) is a skew-symmetric tensor. The transverse modes are
\[ \gamma_{pq}'(\omega, j, m; n) = \gamma_{pq}'(j, j, m; n) \tag{4.21} \]
and the longitudinal
\[ \gamma_{pq}'(\omega, j, m; n) = (\frac{j}{2j+1})^{\frac{1}{2}} \gamma_{pq}'(j, j, m; n) + (\frac{j+1}{2j+1})^{\frac{1}{2}} \gamma_{pq}'(j, j, m; n) \tag{4.22} \]
and the longitudinal
\[ \gamma_{pq}'(\omega, j, m; n) = (\frac{j}{2j+1})^{\frac{1}{2}} \gamma_{pq}'(j, j, m; n) - (\frac{j}{2j+1})^{\frac{1}{2}} \gamma_{pq}'(j, j, m; n) \tag{4.23} \]
In the case \( s = 0, \) the most general state is the only state
\[ \gamma_{pq}^0(\omega, j, m; n) = \gamma_{pq}^0(j, j, m; n) \tag{4.24} \]
The parity operator \( \Pi, \) when acting on a vector \( \xi \) gives
\[ \Pi \xi = -\xi \] and for a spherical harmonic
\[ \Pi \gamma_{\ell, m}^\ell = (-1)^\ell \gamma_{\ell, m}^\ell \tag{4.25} \]
So
\[ \Pi \chi^s(m_s) = \chi^s(m_s) \tag{4.26} \]
Using these, the parity of each of the states obtained above may be investigated, remembering that
\[ \gamma_{pq}^s(j, j, m; m) = \sum_{m_s} C(j, j, s, m, m, m, m, m_s, m_s) \gamma_{\ell, m}^\ell \chi^s(m_s) \]
Then
\[ \Pi \gamma_{pq}^2(\omega, j, m; n) = (-1)^{j+1} \gamma_{pq}^2(\omega, j, m; n) \]
\[ \Pi \gamma_{pq}^2(\omega, j, m; n) = (-1)^j \gamma_{pq}^2(\omega, j, m; n) \]
\[ \Pi \gamma_{pq}^2(\omega, j, m; n) = (-1)^{j+1} \gamma_{pq}^2(\omega, j, m; n) \]
\[ \Pi \gamma_{pq}^2(\omega, j, m; n) = (-1)^j \gamma_{pq}^2(\omega, j, m; n) \]
\[ \Pi \gamma_{pq}^2(\omega, j, m; n) = (-1)^{j+1} \gamma_{pq}^2(\omega, j, m; n) \]
Thus the statefunctions obtained for each transversality condition are separated into states of differing parity.

These results are summarised for ready reference in the tabular form below, where the \( a^S_j \) is the coefficient of the tensor spherical harmonic \( Y^S_{pq}(j, \ell, m; \pi) \) in the linear combination \( Y^S_{pq} = \sum_\ell a^S_\ell Y^S_{pq} \) as in (3.19), with the auxiliary condition shown.

<table>
<thead>
<tr>
<th>s spin</th>
<th>Auxiliary condition</th>
<th>( a^S_{j+2} )</th>
<th>( a^S_{j+1} )</th>
<th>( a^S_j )</th>
<th>( a^S_{j-1} )</th>
<th>( a^S_{j-2} )</th>
<th>( P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( n_p y^2_{pq} = 0 )</td>
<td>0</td>
<td>( \frac{(j-1)}{2(j+1)} )</td>
<td>0</td>
<td>( \frac{(j+2)}{2(j+1)} )</td>
<td>0</td>
<td>( (-1)^{j+1} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{j(j-1)}{2(2j+1)} )</td>
<td>0</td>
<td>( \frac{3(j-j+2)}{2(j-1)(j+3)} )</td>
<td>0</td>
<td>( \frac{(j+1)(j+2)}{2(j+2)(j+1)} )</td>
<td>0</td>
<td>( (-1)^j )</td>
</tr>
<tr>
<td></td>
<td>( \varepsilon_{pqr} n_q y^2_{rt} = 0 )</td>
<td>0</td>
<td>( \frac{j(j+2)}{2(j+1)} )</td>
<td>0</td>
<td>( \frac{(j-1)}{2(j+1)} )</td>
<td>0</td>
<td>( (-1)^{j+1} )</td>
</tr>
<tr>
<td></td>
<td>( \frac{3(j+2)(j+2)}{2(2j+1)(2j+3)} )</td>
<td>0</td>
<td>( \frac{3(j-1)}{2(j+1)(j+3)} )</td>
<td>0</td>
<td>( \frac{3(j+1)(j+1)}{2(j-1)(2j+1)} )</td>
<td>0</td>
<td>( (-1)^j )</td>
</tr>
<tr>
<td>1</td>
<td>( \varepsilon_{pqr} n_p y^I_{qr} = 0 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( (-1)^j )</td>
</tr>
<tr>
<td></td>
<td>( \frac{j}{2(j+1)} )</td>
<td>0</td>
<td>( \frac{j+1}{2(j+1)} )</td>
<td>0</td>
<td>( \frac{j}{2(j+1)} )</td>
<td>0</td>
<td>( (-1)^{j+1} )</td>
</tr>
<tr>
<td></td>
<td>( n_p y^I_{pq} = 0 )</td>
<td>0</td>
<td>( \frac{j+1}{2(j+1)} )</td>
<td>0</td>
<td>( \frac{j}{2(j+1)} )</td>
<td>0</td>
<td>( (-1)^{j+1} )</td>
</tr>
<tr>
<td>0</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( (-1)^j )</td>
</tr>
</tbody>
</table>
V. SELECTION RULES GOVERNING THE DECAY OF POSITRONIUM INTO TWO GRAVITONS

Since one graviton is described by a tensor of second rank, the two graviton state can be represented by a tensor of fourth rank

\[ \Psi_{klimn}(\mathbf{p}) \]

with \( \mathbf{p} \), referring to the relative momentum of the two particles, the only variable in the centre of mass system.

Angular momentum eigenfunctions for this case are obtained by the use of the Clebsch-Gordan formula with the \( \chi^2_{kl}(\mu_s) \) obtained in Chapter III. These are

\[ \Psi_{klimn}(J, L, M, S) = \sum_{\mu} \mathcal{C}(J, L, S, M, M - \mu, \mu) \chi^{(\mu)}_{LM} \chi^S_{mn} \]  \hspace{1cm} (5.1)

where

\[ \chi^S_{mn} = \sum_{\mu_2} \mathcal{C}(S, 2, 2, \mu_2, \mu_2, \mu_2) \chi^2_{kl}(\mu_2) \chi^2_{mn}(\mu_2) \]  \hspace{1cm} (5.2)

Here \( J \) is the total angular momentum of the two gravitons; \( L \) the orbital angular momentum; \( M \) the projection of \( J \) on the z axis; \( S \) the spin of the two-graviton system; and \( \mu \) the z component of \( S \).

The general state with angular momentum \( J \), z component \( M \), is then

\[ \Psi_{klimn}(J, M) = \sum_{L, S} \rho_{L, S} \Psi_{klimn}(J, L, M, S) \]  \hspace{1cm} (5.3)

with \( L = J, J \pm 1, J \pm 2 \) and \( S = 0, 1, 2, 3, 4 \).

Using the orthogonality requirement that

\[ n_{\kappa} \Psi_{klimn}(J, M) = 0 \]  \hspace{1cm} (5.4)
for a massless system, Zhirnov and Shirokov (Zhirnov and Shirokov 1957) determined the coefficients $\rho_{\nu,s}$, and arrived at the following table giving the number of possible states of the system for a given parity and given value of $J$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Even States</th>
<th>Odd States</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$2n \geq 4$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$2n+1 \geq 5$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

This may be compared with the similar table for the possible two-photon states (Kaempffer 1965).

<table>
<thead>
<tr>
<th>$J$</th>
<th>Even States</th>
<th>Odd States</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$2n \geq 4$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$2n+1 \geq 5$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Hence we may construct a combined table giving selection rules for the decay of positronium into two gravitons or two photons, under the conservation of angular momentum and of parity. Selection rules with respect to charge conjugality $C$ are given also, noting that $C = (-1)^n$ for $n$ photon decay, and $+1$ for $n$ gravitons. This is because under charge
conjugation the electric current density $j$ is transformed to $-j$, whereas mass remains invariant. Hence a single photon source changes sign under $C$, where the single graviton source $g_{\mu\nu}$ does not. The general result follows by induction.

<table>
<thead>
<tr>
<th>Positronium State</th>
<th>$J$</th>
<th>$P=(-1)^{2J}$</th>
<th>$C=(-1)^{2S}$</th>
<th>Decay into two photons by conservation of $P, J, J_3, C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1S_0$</td>
<td>0</td>
<td>$-1$</td>
<td>$+1$</td>
<td>Allowed</td>
</tr>
<tr>
<td>$^3P_0$</td>
<td>0</td>
<td>$+1$</td>
<td>$+1$</td>
<td>Allowed</td>
</tr>
<tr>
<td>$^1P_1$</td>
<td>1</td>
<td>$+1$</td>
<td>$-1$</td>
<td>Forbidden</td>
</tr>
<tr>
<td>$^3S_1, ^3D_1$</td>
<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
<td>Forbidden</td>
</tr>
<tr>
<td>$^3P_1$</td>
<td>1</td>
<td>$+1$</td>
<td>$+1$</td>
<td>Forbidden</td>
</tr>
<tr>
<td>$^1D_2$</td>
<td>2</td>
<td>$+1$</td>
<td>$+1$</td>
<td>Allowed</td>
</tr>
<tr>
<td>$^3P_2, ^3F_2$</td>
<td>2</td>
<td>$-1$</td>
<td>$+1$</td>
<td>Allowed</td>
</tr>
<tr>
<td>$^3D_2$</td>
<td>2</td>
<td>$+1$</td>
<td>$-1$</td>
<td>Allowed</td>
</tr>
<tr>
<td>$^1F_3$</td>
<td>3</td>
<td>$-1$</td>
<td>$-1$</td>
<td>Forbidden</td>
</tr>
<tr>
<td>$^3D_3, ^3G_3$</td>
<td>3</td>
<td>$+1$</td>
<td>$-1$</td>
<td>Forbidden</td>
</tr>
<tr>
<td>$^3F_3$</td>
<td>3</td>
<td>$-1$</td>
<td>$+1$</td>
<td>Forbidden</td>
</tr>
</tbody>
</table>

For $J \geq 4$ the selection rules are identical for decay into two photons or two gravitons.

From the table it is evident that the first significant difference between decay into two photons and two gravitons occurs for the $^3D_3$ and $^3G_3$ states of positronium. If angular momentum, parity, and charge conjugation are conserved, however, then any state which can decay into two gravitons can decay into two photons. A test of $C$ for other positronium
states requires decay into an odd number of gravitons or photons, as then the charge conjugation number differs in the two cases.

At present nobody knows how to make positronium decay into two photons from any but the ground state, so experimental verification remains beyond reach.
The equation of motion for a particle may be written as

\[
(P \cdot S) \phi = mE\phi
\]  \hspace{1cm} (A1)

where \(S\) is the spin operator; \(P\) the momentum of the particle; \(E\) its energy; and \(m\) the component of spin in the \(P\) direction.

If \(\phi\) is to be considered as a tensor, this may be rewritten as

\[
\frac{1}{2} \epsilon_{ij} \phi_{ij} = mE\phi
\]  \hspace{1cm} (A2)

using the notation of Chapter III. But

\[
\langle pq | S_{ij} | rs \rangle = i \left[ (\delta_{ip} \delta_{jr} - \delta_{ir} \delta_{jp}) \delta_{qs} + (\delta_{iq} \delta_{js} - \delta_{is} \delta_{jq}) \delta_{pr} \right]
\]  \hspace{1cm} (A3)

Hence (A2) becomes

\[
iP_i \epsilon_{ij} \phi_{qs} + iP_i \epsilon_{ij} \phi_{jr} = -mE\phi
\]  \hspace{1cm} (A4)

Iterating this equation yields

\[
2P_i P_r \phi_{ir} = 2[ \phi_{ir} + \phi_{qi} ] - P_r P_i [2\phi_{si} + \phi_{is}] - P_i P_r [2\phi_{jr} + \phi_{jr}]
\]  \hspace{1cm} (A5)

If the conditions \(P_i \phi_{ij} = 0, \phi_{ij} = 0\), are now imposed, corresponding to (4.2), (A5) reduces to

\[
4P_i P_r \phi_{ir} = m^2 E^2 \phi
\]  \hspace{1cm} (A6)

For a massless particle \(p^2 = E^2\), thus we have the requirement that \(|m| = 2\). Hence the component of spin in the direction of motion must be \(\pm 2\).
Appendix B.  RELATIONS BETWEEN THE COEFFICIENTS $a_i$ FOR THE TRANSVERSE-TRANSVERSE STATES

In accordance with the plan set forth on page 17, we write out the coefficient of $Y_{j+1, m-1}$ in (4.11). Then

$$a_j^2 \left\{ - C(j, j, 2, m, m-2, 2) \left( \frac{(j+m-1)(j+m)}{(2j+1)(2j+3)} \right)^{\frac{1}{2}} \right.$$  

$$- C(j, j, 2, m, m-1, 1) \left( \frac{(j+m)(j+m+2)}{(2j+1)(2j+3)} \right)^{\frac{1}{2}} - \frac{1}{\sqrt{6}} C(j, j, 2, m, m, 0) \left( \frac{(j-m+1)(j-m+2)}{(2j+1)(2j+3)} \right)^{\frac{1}{2}} \right.$$  

$$+ a_{j+2}^2 \left\{ C(j, j+2, 2, m, m-2, 2) \left( \frac{(j+m+2)(j-m+3)}{(2j+5)(2j+3)} \right)^{\frac{1}{2}} \right.$$  

$$- C(j, j+2, 2, m, m-1, 1) \left( \frac{(j+m+1)(j-m+3)}{(2j+5)(2j+3)} \right)^{\frac{1}{2}} + \frac{1}{\sqrt{6}} C(j, j+2, 2, m, m, 0) \left( \frac{(j+m+2)(j+m+4)}{(2j+5)(2j+3)} \right)^{\frac{1}{2}} \right\} = 0 \quad (B1)$$

Rearranging and substituting in the values of the Clebsch-Gordan coefficients

$$a_j^2 \left\{ \left( \frac{3}{2} \frac{(j+m-1)^2(j+m)^2(j-m+1)(j-m+3)}{(2j+1)(2j+3)^2} \right)^{\frac{1}{2}} + (1-2m) \left( \frac{3(j-m+1)(j-m+2)(j+m)^2}{2(2j+1)(2j+3)(2j+5)^2} \right)^{\frac{1}{2}} \right.$$  

$$+ \frac{1}{\sqrt{6}} \left(3m^2-j(j+1)\right) \left( \frac{(j-m+1)(j-m+2)}{(2j+1)(2j+3)^2} \right)^{\frac{1}{2}} \right\}$$  

$$= a_{j+2}^2 \left\{ \left( \frac{(j+m+2)(j-m+3)^2(j-m+4)^2}{2(j+1)(2j+3)^2} \right)^{\frac{1}{2}} + \left( \frac{(j-m+3)^2(j-m+2)(j-m+1)(j+m+1)}{(j+1)(2j+3)^2(j+2)(2j+5)^2} \right)^{\frac{1}{2}} \right.$$  

$$+ \frac{1}{\sqrt{6}} \left( \frac{3(j-m+2)(j-m+1)(j+m+2)^2(j+m+1)^2}{(2j+5)^2(2j+3)^2(j+2)(j+2)} \right)^{\frac{1}{2}} \right\} \quad (B2)$$

On cancelling common factors obtain

$$a_j^2 \left[ \frac{1}{6j(j+1)(2j+1)} \left[ 3(j+m-1)(j+m) + 3(1-2m)(j+m) + 3m^2 - j(j+1) \right] \right]$$  

$$= a_{j+2}^2 \frac{1}{2(2j+5)\sqrt{j+2}} \left[ (j-m+3)(j-m+4) + 2(j-m+3)^2(j+m+1) + (j+m+2)^2(j+m+1) \right] \quad (B3)$$
\[ a_j^2 \left( \frac{1}{6j(2j-1)(2j+1)} \right)^{\frac{1}{2}} \left( 2j^2 - j \right) \]

\[ = a_{j+2}^2 \frac{1}{2(2j+5)(j+2)^{\frac{1}{2}}} \left( (j-m+3)(2j+5) + (j+m+1)(2j+5) \right) \quad \text{(B4)} \]

\[ \therefore \quad a_j^2 \left( \frac{(j-1)j}{6(2j+1)} \right)^{\frac{1}{2}} = a_{j+2}^2 (j+2)^{\frac{1}{2}} \]

The other equations (4.12) may be obtained by considering the coefficients of \( Y_{j-1,m-1} \) and \( Y_{j, m} \) to give respectively

\[ a_j^2 \left( \frac{(j+1)(2j+3)}{6(2j+1)} \right)^{\frac{1}{2}} = a_{j-2}^2 (j-1)^{\frac{1}{2}} \]

\[ a_{j+1}^2 (j+2)^{\frac{1}{2}} = a_{j-1}^2 (j-1)^{\frac{1}{2}} \]
BIBLIOGRAPHY


