# Localization of a Particle due to Dissipation in 1 and 2 Dimensional Lattices

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# Abstract

We study two aspects of the problem of a particle moving on a lattice while subject to dissipation, often called the "Schmid model." First, a correspondence between the Schmid model and boundary sine-Gordon field theory is explored, and a new method is applied to the calculation of the partition function for the theory. Second, a traditional condensed matter formulation of the problem in one spatial dimension is extended to the case of an arbitrary two-dimensional Bravais lattice.

A well-known mathematical analogy between one-dimensional dissipative quantum mechanics and string theory provides an equivalence between the Schmid model at the critical point and boundary sine-Gordon theory, which describes a free bosonic field subject to periodic interaction on the boundaries. Using the tools of conformal field theory, the partition function is calculated as a function of the temperature and the renormalized coupling constants of the boundary interaction. The method pursues an established technique of introducing an auxiliary free boson, fermionizing the system, and constructing the boundary state in fermion variables. However, a different way of obtaining the fermionic boundary conditions from the bosonic theory leads to an alternative renormalization for the coupling constants that occurs at a more natural level than in the established approach.

Recent renormalization group analyses of the extension of the Schmid model to a two-dimensional periodic potential have yielded interesting new structure in the phase diagram for the mobility. We extend a classic one-dimensional, finite temperature calculation to the case of an arbitrary two-dimensional Bravais lattice. The duality between weak-potential and tight-binding lattice limits is reproduced in the two-dimensional case, and a perturbation expansion in the potential strength used to verify the change in the critical dependence of the mobility on the strength of the dissipation. With a triangular lattice the possibility of third order contributions arises, and we obtain some preliminary expressions for their contributions to the mobility.

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# 1. Introduction

Ever since Caldeira and Leggett proposed their model of dissipative quantum mechanics [1], attempts have been made to apply the formalism to simple situations and determine the effects of dissipative forces in the quantum regime. A particularly important and general problem is the question of the mobility of a particle in a periodic potential in the presence of friction.

The application of the Caldeira-Leggett theory to the particle mobility problem at zero temperature was first studied by Schmid [14]. His results suggested a sudden transition in the dimensionless mobility from 1 (diffusive hopping) to 0 (localization) as the magnitude of the friction passes through a critical value. Subsequently, Fisher and Zwerger [7] generalized Schmid's model to non-zero temperature, obtaining general expressions for the mobility at arbitrary temperature. These results apply to the case of a one-dimensional periodic potential, or to hyper-cubic lattices where the behaviour in the different dimensions is decoupled.

Following the discovery of a mathematical analogy between dissipative quantum mechanics and open string theory [4], the problem was opened up to attacks and extensions from a string theory perspective. The similarity arises in the context of open strings with end-points tied to some world-sheet boundary [3]. When the closed string modes in the bulk of the world-sheet are integrated away, the resulting effective action term for the open strings on the boundary is identical to the non-local interaction found in dissipative quantum mechanics upon elimination of the oscillator bath representing the environment.

More recently, Yi and Kane attacked the two-dimensional particle mobility problem for the additional cases of equilateral triangular and hexagonal lattices [15]. Their interpretation of

the renormalization group results of Schmid led them to propose an intermediate fixed point for the non-square geometries that was unstable in the triangular case but stable in the hexagonal case. They conclude that in a certain transition regime, the fixed point for the mobility in the hexagonal lattice varies continuously from 0 to 1.

In chapters 2 and 3 of this work, we look at the string theoretical formulation of the Schmid model at the critical point in terms of a free bosonic field theory with interactions on a boundary. We demonstrate a new calculation of the partition function using fermionization. The solution is given in a form such that the renormalized parameters may be more closely related to the bare parameters of a given condensed matter problem. While acknowledging certain weaknesses of the model's applicability to real systems, there is some hope that attempts at analyzing similar systems might benefit from the alternative approach outlined here.

In chapters 4 and 5, we attempt to clarify the origins and consequences of the behaviour Yi and Kane suggest. The analysis of Fisher and Zwerger is generalized to non-rectangular two-dimensional Bravais lattices, and traces of the behaviour predicted by Yi and Kane are investigated.

Each of the two parts of this thesis contains an introductory chapter that provides a basic explanation of the origins and machinery of the problem at hand. In the remainder of this chapter, we will introduce Caldeira-Leggett dissipative quantum mechanics and the Schmid model, and then outline the nature of the connection between the Schmid model and open string theory.

## 1.1   Overview of Problem

The models studied here consider a particle of mass $M$ moving in a periodic potential $V(\boldsymbol{x})$, and subject to some constant applied force $\boldsymbol{F}$. Dissipation is introduced by Caldeira and Leggett's prescription of coupling the particle to a bath of oscillators with a particular spectrum. The bath variables are eliminated, leaving an effective action for the particle.

In the absence of dissipation, the effect of the periodic potential is to create Bloch bands in the energy spectrum. The Hamiltonian eigenfunctions are also eigenfunctions of the crystal momentum, and these states are extended. A particle cannot be restricted to any small region; it will quickly disperse to fill the entire volume.

In the presence of dissipative forces, on the other hand, we might expect some restriction of the particle's mobility. Classically, "friction" impedes the free propagation of the particle. Quantum mechanically, long-distance correlations in the wavefunction may be destroyed, leading to the localization of the particle.

Classically, friction is added to a system by including a force that is proportional to the particle velocity. The constant of proportionality is the friction coefficient $\eta$, and the equation of motion for the system is then

$$M\ddot{\boldsymbol{q}} + \eta\dot{\boldsymbol{q}} + \boldsymbol{\nabla}V(\boldsymbol{q}) = \boldsymbol{F}. \tag{1.1}$$

The mobility $\mu$ indicates the ratio of the terminal velocity $v$ to the applied force. In one dimension we have

$$\mu \;\; = \;\; \frac{v}{F} \tag{1.2}$$

while in higher dimensions the more general mobility tensor $\mu^{ij}$ describes the particle's response in the $i$ direction to an applied force in the $j$ direction:

$$v^i \;\; = \;\; \mu^i_j F^j. \tag{1.3}$$

The tensor may in general depend on $\boldsymbol{F}$ and $\boldsymbol{v}$. For vanishingly small force we might only take the constant part of $\mu^i_j$; this is the linear mobility.

In the absence of the potential $V$, the classical mobility is found from the steady state

solution to (1.1):

$$\mu_0^{ij} = \frac{1}{\eta}\delta^{ij}. \tag{1.4}$$

In quantum mechanics, where a particle's position is not well defined, we approach the mobility in a way that generalizes to the classical result in the high temperature limit. Using the expectation value of position as a function of time in response to an applied force, the linear mobility is then

$$\mu^{ij} = \left. \frac{\delta}{\delta F_j} \lim_{t\to\infty} \frac{X^i(t) - X^i(0)}{t} \right|_{\boldsymbol{F}=0} \tag{1.5}$$

with the particle's mean position $X$ obtained from the reduced density matrix $\rho(\boldsymbol{q}, \boldsymbol{q}'; t)$, where $\boldsymbol{q}$ and $\boldsymbol{q}'$ represent spatial coordinates, which takes into account the effects of dissipation:

$$\boldsymbol{X}(t) = \langle \boldsymbol{q}(t) \rangle = \int d\boldsymbol{q} \; \rho(\boldsymbol{q}, \boldsymbol{q}; t) \; \boldsymbol{q}. \tag{1.6}$$

In the absence of an applied force, the mobility may be studied by analyzing the natural tendency of a localized state to spread out in space with time. Such behaviour is encoded in the two-point correlation function $\langle q^i(t)q^j(0) \rangle$. An alternative definition of mobility is then to take the (suitably normalized) coefficient of the logarithmic dependence of $\langle x^i(t)x^j(0) \rangle - \langle x^i(0)x^j(0) \rangle$. This quantity can be extracted from the two-point function's Fourier transform:

$$\mu^{ij} = \lim_{\omega\to 0} |\omega| \int dt \; e^{i\omega t} \left\langle x^i(t)x^j(0) \right\rangle. \tag{1.7}$$

As we will see, Schmid's original analysis was made using this form of the mobility, while Fisher and Zwerger took the "terminal velocity" approach and used the definition (1.5).

## 1.2 Caldeira-Leggett Dissipative Quantum Mechanics

In Caldeira and Leggett's model of dissipative quantum mechanics (DQM) [1, 2], a heat bath is represented as a set of oscillators, indexed by $\alpha$, with coordinates $x_\alpha$, mass $m_\alpha$, and frequencies $\omega_\alpha$. The oscillators are coupled linearly to the particle, with coupling constants $C_\alpha$.

The action for the entire system is then

$$S[q, \{x_\alpha\}] = S_0[q] + S_{bath}[\{x_\alpha\}] + S_{int}[q, \{x_\alpha\}] \tag{1.8}$$

where

$$S_0[q] \quad = \quad \int_0^t d\tau \left( \tfrac{1}{2} M \dot{\boldsymbol{q}}(\tau)^2 - V(\boldsymbol{q}) \right) \tag{1.9}$$

$$S_{bath}[\{x_\alpha\}] \quad = \quad \sum_\alpha \int_0^t d\tau \left( \tfrac{1}{2} m_\alpha \dot{\boldsymbol{x}}_\alpha^2 - \tfrac{1}{2} m_\alpha \omega_\alpha^2 \boldsymbol{x}_\alpha^2 \right) \tag{1.10}$$

$$S_{int}[q, \{x_\alpha\}] \quad = \quad \sum_\alpha \int_0^t d\tau C_\alpha \boldsymbol{x}_\alpha \cdot \boldsymbol{q} \tag{1.11}$$

The density matrix $\rho$ in position space, as a function of time $t$, is obtained by propagation of the initial configuration:

$$\rho(q, q'; \{x_\alpha\}, \{x'_\alpha\}; t) =$$
$$\int dQ \int dQ' \prod_\alpha \int dX_\alpha \int dX'_\alpha \, K(q, q', \{x_\alpha\}, \{x'_\alpha\}; Q, Q', \{X_\alpha\}, \{X'_\alpha\}; t)$$
$$\times \, \rho(Q, Q'; \{X_\alpha\}, \{X'_\alpha\}; 0) \tag{1.12}$$

with the propagator $K$ given by

$$K \quad = \quad \int_Q^q \mathcal{D}\tilde{q} \int_{Q'}^{q'} \mathcal{D}\tilde{q}' \int_{X_\alpha}^{x_\alpha} \mathcal{D}\tilde{x}_\alpha \int_{X'_\alpha}^{x'_\alpha} d\tilde{x}'_\alpha \exp \left( \tfrac{i}{\hbar} S[\tilde{q}, \{\tilde{x}_\alpha\}] - \tfrac{i}{\hbar} S[\tilde{q}', \{\tilde{x}'_\alpha\}] \right) \tag{1.13}$$

For our purposes it is reasonable to assume that the initial density matrix factors cleanly into some initial distribution for the particle and a bath distribution in thermal equilibrium at temperature $T = 1/k\beta$:

$$\rho(q, q', \{x_\alpha\}, \{x'_\alpha\}; 0) = \rho(q, q') \times \rho_\beta(\{x_\alpha\}, \{x'_\alpha\}). \tag{1.14}$$

Since we are only concerned with the particle's properties at time $t$, we can trace out the final state of the bath from the complete density matrix to leave the part relevant to particle expectation values:

$$\rho(q, q'; t) = \prod_\alpha \int dx_\alpha \, \rho(q, q'; \{x_\alpha\}, \{x_\alpha\}; t). \tag{1.15}$$

Caldeira and Leggett showed that these integrations can be done for a general $S_0$, and that this leaves a simplified expression for the particle density matrix that incorporates the bath through an influence phase $i\Phi$:

$$\rho(q, q'; t) = \int dQ \int dQ' \rho(Q, Q'; 0) \int_Q^q \mathcal{D}\tilde{q} \int_{Q'}^{q'} \mathcal{D}\tilde{q}' \, e^{\frac{i}{\hbar} S_0[\tilde{q}] - \frac{i}{\hbar} S_0[\tilde{q}'] + i\Phi[\tilde{q}, \tilde{q}']} \tag{1.16}$$

The influence phase $\Phi[q, q']$ couples the "forward" and "backward" paths $q$ and $q'$,

$$i\Phi[q, q'] = -\frac{2i}{\hbar} \int_0^t dt' \int_{t'}^t ds \, y(s)\alpha_I(s - t')x(t') - S_2[y], \tag{1.17}$$

with $x$ and $y$ the centre of mass and difference coordinates

$$x = \tfrac{1}{2}(q + q') \qquad y = q - q', \tag{1.18}$$

and $S_2$ a temperature-dependent term coupling difference paths only,

$$S_2[y] = \frac{1}{\hbar} \int_0^t dt' \int_0^{t'} ds \ y(s)\alpha_R(s-t')y(t'). \tag{1.19}$$

The functions $\alpha_I$ and $\alpha_R$ determined by the oscillator spectrum and the temperature:

$$\alpha_I(s) = \int_0^\infty d\omega J(\omega)\sin\omega s \tag{1.20}$$

$$\alpha_R(s) = \int_0^\infty d\omega J(\omega)\cos\omega s \coth\left(\tfrac{1}{2}\beta\hbar\omega\right) \tag{1.21}$$

$$J(\omega) = \sum_\alpha \frac{C_\alpha}{m_\alpha\omega_\alpha^2}\delta(\omega-\omega_\alpha). \tag{1.22}$$

The most interesting behaviour results from taking an "ohmic" spectrum for the oscillators, where $J(\omega) = \eta\omega$. In the high temperature limit, this choice of spectrum yields the classical frictional force $-\eta\dot{q}$. It is thus the spectrum most appropriate for systems where such friction is observed at high temperatures.

## 1.3  The Schmid Model

Schmid studied the general problem of a particle in one dimension moving in a potential

$$V(q) = -V_0\cos(2\pi q/a_0). \tag{1.23}$$

In order to extract correlation functions from the system, he worked with a generating functional in imaginary time:

$$Z[F] = \int \mathcal{D}q \ e^{-S_{eff}[q]-\int dt \ q(t)F(t)} \tag{1.24}$$

where $F(t)$ is a time-dependent source term and $S_{eff}[q]$ is obtained from (1.8) by tracing out the bath modes with ohmic dissipation (this is a simplified version of the approach described

in the last section):

$$S_{eff}[q] = \int_{-\infty}^{\infty} dt \left( \frac{M}{2} \dot{q}(t)^2 + V_0 \cos q(t) \right) + \frac{\eta}{4\pi} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \frac{(q(t) - q(t'))^2}{(t - t')^2}. \quad (1.25)$$

Both the mass term and the interesting non-local influence functional term can be Fourier transformed (with the convention $\widetilde{f}(\omega) = \int dt\ e^{i\omega t} f(t)$) to give

$$S_{eff}[q] = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left( M\omega^2 + \eta|\omega| \right) q(\omega)q(-\omega) + V_0 \int_{-\infty}^{\infty} dt\ \cos(q(t)). \quad (1.26)$$

With the friction term associated with a linear term in $\omega$, the mass term $M\omega^2$ serves only as an high energy cut-off. In renormalization group language, the terms in $\omega$ are marginal while the $\omega^2$ terms are irrelevant. We may then often treat $M$ as zero in what follows, while picking up a characteristic frequency

$$\gamma = \frac{\eta}{M} \quad (1.27)$$

that acts as an effective ultraviolet cut-off for the low-energy theory.

Dealing with the cosinusoidal potential in this framework is achieved by a "Coulomb gas expansion." This is essentially perturbative in $V_0$. First the exponential of $S_V[q] = V_0 \int dt\ \cos q(t)$ is expanded:

$$e^{-S_V[q]} = \sum_{n=0}^{\infty} \frac{(-V_0)^n}{n!} \left( \frac{1}{2} \sum_{\sigma=\pm 1} \int_{-\infty}^{\infty} dt\ e^{2\pi i \sigma q(t)/a_0} \right)^n. \quad (1.28)$$

Now the exponentials from the cosine interaction provide effective terms for the particle action:

$$e^{-S_V[q]} = \sum_{n=0}^{\infty} \left( \frac{-V_0}{2} \right)^n \sum_{\{\sigma_i = \pm 1\}} \int dt_1 \int dt_2 \ldots \int dt_n \exp\left( \frac{2\pi i}{a_0} \sum_{i=1}^{n} \sigma_i q(t_i) \right). \quad (1.29)$$

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The exponent is rewritten as an integral over a charge density $\rho(t')$,

$$\frac{2\pi i}{a_0} \sum_{i=1}^{n} \sigma_i q(t_i) \;=\; -\frac{i}{\hbar} \int dt' \; q(t')\rho(t') \tag{1.30}$$

where

$$\rho(t') \;=\; -\frac{2\pi\hbar}{a_0} \sum_{i=0}^{n} \sigma_i \delta(t_i - t'). \tag{1.31}$$

Using this expansion we may write the generating functional (1.24) for the correlation functions as

$$Z[F(t)] \;=\; \int \mathcal{D}q \; e^{-\frac{1}{2} \int \frac{d\omega}{2\pi} \widetilde{D}^{-1}(\omega)|q(\omega)|^2}$$
$$\times \sum_n \int dt_1 \ldots dt_{2n} \left(\frac{(-V_0/2)^n}{n!}\right)^2 e^{-\frac{i}{\hbar} \int dt' (iF(t') + \rho(t'))q(t')} \tag{1.32}$$

where (anticipating the imminent functional integration) the propagator is

$$\widetilde{D}(\omega) \;=\; \left(M\omega^2 + \eta|\omega|\right)^{-1} \tag{1.33}$$

and we have used that only neutral charge distributions (i.e. with $\sigma_i = (-1)^i$ for $i = 1, \ldots, 2n$) contribute finitely to the path integral. Note that for $\omega \ll \gamma$, the dominant term in $\widetilde{D}(\omega)$ is $1/\eta\omega$.

The path integral is gaussian and gives

$$Z \;=\; \frac{1}{\sqrt{\det D}} \sum_n \int dt_1 \ldots dt_{2n} \left(\frac{(-V_0/2)^n}{n!}\right)^2$$
$$\times \exp\left(-\frac{1}{2} \int ds \; ds' \; (iF(s) + \rho(s)) \, D(s - s') \, (iF(s') + \rho(s'))\right). \tag{1.34}$$

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with $D(t)$ the inverse Fourier transform of $\widetilde{D}(\omega)$,

$$D(t) \ = \ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \widetilde{D}(\omega) \ = \ \begin{cases} -\frac{1}{2\eta}\gamma|t| & , \ |t| \ll 1/\gamma \\ -\frac{1}{\pi\eta}\ln\gamma|t| & , \ |t| \gg 1/\gamma \end{cases} . \tag{1.35}$$

From this we may pull down the correlation function

$$\begin{aligned} \langle q(t)q(0) \rangle \ &= \ \frac{1}{Z}\frac{\delta}{\delta F(t)}\frac{\delta}{\delta F(0)}Z\Big|_{F=0} \\ &= \ D(t) - \int ds \int ds' \ D(t-s) \left\langle \rho(s)\rho(s') \right\rangle D(s') \end{aligned} \tag{1.36}$$

with charge density correlation function given by

$$\begin{aligned} \langle \rho(t')\rho(0) \rangle \ &= \ \frac{1}{Z[0]}\frac{1}{\sqrt{\det D}}\sum_n \int dt_1 \ldots dt_{2n} \left( \frac{(-V_0/2)^n}{n!} \right)^2 \rho(t')\rho(0) \\ &\qquad \times \exp\left( -\frac{1}{2} \int ds \int ds' \ \rho(s)D(s-s')\rho(s') \right). \end{aligned} \tag{1.37}$$

We see from this expression that the "charges" in $\rho$ interact via the essentially logarithmic potential $D(t')$. This analogy to electric charges moving in one dimension is the origin of the term *Coulomb gas expansion*. Using the expression (1.31) we can rewrite the exponential in (1.37) as

$$\begin{aligned} \exp\left( -\frac{1}{2} \int ds \int ds' \ \rho(s)D(s-s')\rho(s') \right) \ &= \ \exp\left( -\frac{1}{2}\sum_{j,k} \left( \frac{2\pi\hbar}{a_0} \right)^2 \sigma_j\sigma_k D(t_j - t_k) \right) \\ &= \ \exp\left( -\frac{1}{2}\frac{2\pi\hbar}{\alpha}\sum_{j,k} \sigma_j\sigma_k D(t_j - t_k) \right). \end{aligned} \tag{1.38}$$

where we have introduced the very important dimensionless dissipation parameter

$$\alpha \ = \ \frac{\eta a_0^2}{2\pi\hbar}. \tag{1.39}$$

From (1.36) the mobility (1.7) is then

$$\mu = \lim_{\omega \to 0} |\omega| \left( D(\omega) - D(\omega)S(\omega)D(\omega) \right) \tag{1.40}$$

$$= \frac{1}{\eta} \left( 1 - \lim_{\omega \to 0} D(\omega)S(\omega) \right) \tag{1.41}$$

where $S(\omega)$ is the Fourier transform of the charge density correlation function (1.37).

This perturbative expansion in powers of $V_0$ should be valid for weak $V_0$. Renormalization group arguments [7, 14] show that in fact $V_0$ flows to 0 provided that the dimensionless friction $\alpha$ given in (1.39) is less than one. We then have the dimensionless mobility

$$\mu/\mu_0 = 1 \quad \text{for} \quad \alpha < 1. \tag{1.42}$$

The region where the periodic potential does not flow to zero is not accessible to the perturbation theory. The approach to this side of the "phase diagram" has been to instead work in the strong potential limit. In this sort of tight-binding limit, the particle lives mostly in harmonic oscillator levels localized in the wells of the potential. The possibility of motion is provided by tunneling through the barrier between minima; these are instantons [6].

A single instanton tunnels from, for example $x = 0$ to $x = a_0$ through a barrier of height $V_0(1 - \cos(2\pi x/a_0)$. In an inverted potential formulation, we obtain the action of the classical path associated with this (this is the WKB phase):

$$s = \int_0^{a_0} dx \sqrt{2MV_0(1 - \cos(2\pi x/a_0))} = \frac{4a_0}{\pi} \sqrt{MV_0}. \tag{1.43}$$

The paths associated with these jumps have the form

$$f(t) = \frac{2a_0}{\pi} \tan^{-1} e^{\omega_0 t} \tag{1.44}$$

with $\omega_0 = \frac{2\pi}{a_0} \sqrt{2V_0/M}$ the effective harmonic oscillator frequency in the base of the wells. A

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good approximation to a multi-instanton path is constructed as a sum of jumps of the type (1.44). For $n$ jumps at times $t_1, \ldots, t_n$ in direction $e_j = \pm 1$ we have

$$q(t') = \sum_{j=1}^{n} e_j f(t - t_j), \tag{1.45}$$

which we may Fourier transform (using an integration by parts) to get

$$q(\omega) = \frac{i}{\omega} h(\omega) \sum_{j} e_j e^{i\omega t_j} \tag{1.46}$$

where $h(\omega)$ is the Fourier transform of $\frac{d}{dt} f(t)$. Note that since the path (1.44) is a smoothed-out step function, its derivative $h(t)$ is a smoothed-out delta function. When integrated alongside functions that vary on time scales longer than $1/\omega_0$, $h(t) \approx a_0 \delta(t)$.

The effective action is infinite unless the paths start and end at the same position, which allows us to impose charge neutrality $\sum_{j} e_j = 0$. For the $n$ instantons described by (1.45) we get effective action

$$
\begin{aligned}
S_{eff} &= ns + \frac{1}{2} \int \frac{d\omega}{2\pi} D^{-1}(\omega) q(\omega) q(-\omega) + i \int dt' F(t') q(t') \\
&= ns + \frac{1}{2} \sum_{j,k} e_j e_k \Delta(t_j - t_k) + i \int \frac{d\omega}{2\pi} \frac{\widetilde{F}(-\omega) h(\omega))}{\omega} \sum_{j} e_j e^{i\omega t_j}
\end{aligned} \tag{1.47}
$$

where $\Delta(t')$ is the inverse Fourier transform of

$$\widetilde{\Delta}(\omega) = \frac{\eta |h(\omega)|^2}{|\omega|}. \tag{1.48}$$

As mentioned above, on time scales shorter than $1/\omega_0$, the function $h(t)$ acts as a delta function $a_0 \delta(t)$. In this regime we then have

$$\widetilde{\Delta}(\omega) \approx \frac{\eta a_0^2}{|\omega|} = \frac{2\pi \hbar \alpha}{|\omega|} \tag{1.49}$$

and we see that the double sum term in the effective action (1.47) has the same form as that in (1.38) with the replacement $\alpha \to 1/\alpha$.

Calculating the generating functional using the effective action (1.47) by integrating over times $t_i$ we obtain an expression very similar to (1.34), but with no quadratic term in the source $F$. The lack of quadratic term yields a mobility (calculated from the two point correlation function as in (1.36) that lacks the leading classical behaviour of (1.41):

$$\mu_{TB} \quad = \quad \lim_{\omega \to 0} \widetilde{\Delta}(\omega) \Sigma(\omega). \tag{1.50}$$

Just as $\widetilde{\Delta}(\omega)$ and $\widetilde{D}(\omega)$ are very similar on long time scales, so are the factor $\Sigma(\omega)$ and $S(\omega)$. This leads to an approximate duality between the mobility $\mu$ in the weak potential limit given by (1.41) and the mobility $\mu_{TB}$ in the tight-binding limit described above:

$$\frac{\mu(\alpha)}{\mu_0} \quad = \quad 1 - \frac{\mu_{TB}(1/\alpha)}{\mu_0}, \tag{1.51}$$

with $\alpha$ defined in (1.39). The implication of this, given the renormalization argument that $\mu/\mu_0 = 0$ for $\alpha < 1$, is that

$$\mu/\mu_0 \quad = \quad 0 \quad \text{for} \quad \alpha \; > \; 1. \tag{1.52}$$

In (1.42) and (1.52) we see that there is a sudden transition from classical dissipative particle behaviour ($\mu = 1/\eta$) to localized tight-binding behaviour ($\mu = 0$) as the friction $\alpha$ passes through 1.

A finite-temperature approach to this problem was undertaken by Fisher and Zwerger [7]. Their approach involved the full evolution of the density matrix according to the Caldeira-Leggett prescription. An outline of their approach and a generalization of it to more complicated lattices is investigated in chapters 4 and 5.

## 1.4 The String Theory Connection

The connection between Caldeira-Leggett dissipative quantum mechanics and string theory was first pointed out by Callan and Thorlacius [4]. An essential technique in string theory for dealing with world-sheets of complicated topology is to cut up the world-sheet into various "fixtures" of simple topology (for example a disk or a cylinder). Calculations may be done on each fixture, and the results sewn together in a prescribed fashion.

When dealing with closed strings, which are closed loops, cutting the world-sheet into fixtures results in the severing of some of the strings. The severed closed strings become open strings with endpoints that live on the boundary of the fixture. The behaviour of these open strings is influenced by interactions in the bulk of the fixture, but these degrees of freedom can be integrated away to leave a *boundary state*. The boundary state is a functional of the boundary field degrees of freedom. Two adjacent fixtures are re-linked by taking the product of their boundary states; this is equivalent to a functional integral over their shared (boundary) degrees of freedom.

Without developing too much formalism (some of which will be covered in chapters 2 and 3), the boundary state for free bosons without interactions is

$$|B\rangle_{free} \;=\; \exp\left(-\sum_{m=1}^{\infty}\frac{1}{m}\alpha_{-m}\widetilde{\alpha}_{-m}\right)|0\rangle \tag{1.53}$$

where $\alpha_m$ and $\widetilde{\alpha}_m$ are mode operators for the string fields in the bulk of the world-sheet and $|0\rangle$ is the famous $SL(2,C)$ invariant vacuum, annihilated by the positive ($m>0$) modes. In the presence of the gauge field the boundary state is modified to [3]

$$|B\rangle \;=\; \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\alpha_{-m}\widetilde{\alpha}_{-m}\right)\int \mathcal{D}X(s)\exp\left(-S_{reg}-S_0-S_A-S_{ls}\right)|0\rangle \tag{1.54}$$

where the path integral is over the field configurations on the boundary, with $s \in [0,2\pi]$

parameterizing the boundary (which is assumed to be periodically identified). The various action terms $S$ are

$$S_0[X] \quad = \quad \frac{1}{8\pi\alpha'} \int_0^{2\pi} \frac{ds}{2\pi} \int_0^{2\pi} \frac{ds'}{2\pi} \frac{(X(s) - X(s'))}{\sin^2 \frac{1}{2}(s - s')} \tag{1.55a}$$

$$S_A[X] \quad = \quad \frac{i}{2} \int_0^{2\pi} \frac{ds}{2\pi} A_\mu(X(s)) \frac{dX^\mu(s)}{ds} \tag{1.55b}$$

$$S_{ls}[X] \quad = \quad \int_0^{2\pi} \frac{ds}{2\pi} \alpha(s) \cdot X, \quad \text{where} \quad \alpha^\mu(s) \quad = \quad \sum_{m=1}^{\infty} i(\widetilde{\alpha}^\mu_{-m} e^{-ims} + \alpha^\mu_{-m} e^{ims}) \tag{1.55c}$$

$$S_{reg}[X] \quad = \quad \int_0^{2\pi} \frac{ds}{2\pi} \tfrac{1}{2} M(\dot{X}(s))^2. \tag{1.55d}$$

The non-local term $S_0$ is very similar to the dissipative term in (1.25), and in fact can be obtained from this other by enforcing $2\pi$-periodicity of the paths $q(t')$ to rewrite the second integral. Alternatively, we may take the limit of the $S_0$ term as the boundary length goes to infinity, and recover the $(s - s')^2$ denominator seen in (1.25).

The term $S_A$ is the topological term arising from the gauge field in the bulk. We may freely take this to be zero now that we have used it to extract the other terms. (The inclusion of this term in the problem described in chapters 2 and 3 has been discussed in [10]).

The linear source term $S_{ls}$ we choose to ignore, since its function is to share the boundary fields back into the bulk modes, and may be treated as a c-number and shifted away. The mass term $S_{reg}$ is necessary to regulate the theory, which is the role of the identical mass term in the DQM model once friction has been introduced.

The conclusion we draw from this analogy is that calculations done with the DQM action (1.25) may be relevant to calculations in string theory for models of the type (1.54). Conversely, to find the boundary state in string theory (perhaps by some other means than the formula (1.54)) is also to perform the path integral in (1.54). A calculation based on the string theory side can thus provide insight into condensed matter problems.

To construct the Schmid model, we can consider the free bosonic field theory of the closed

strings in the bulk with a periodic interaction on the boundary. By finding the boundary state for the theory, we are evaluating the path integral expression (1.54) with the periodic interaction included. We end up working with the theory defined by action

$$S = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau \int_0^{\pi} d\sigma \left( (\partial_\tau X)^2 + (\partial_\sigma X)^2 \right) - \int_{-\infty}^{\infty} d\tau \left( \frac{g}{2} e^{iX(\tau,0)} + \frac{\bar{g}}{2} e^{-iX(\tau,0)} \right). \qquad (1.56)$$

This is the boundary sine-Gordon model, and is discussed in chapter 3.

Since string theory and thus the boundary state formalism is developed in the context of conformal field theory, any condensed matter problem being studied by these methods will inherit symmetries from the string model. In condensed matter systems, extreme reparametrization invariance tends to arise at the critical points of phase transitions (between localized and delocalized phases for example), so it is these critical theories that might be approached using the string theory analogy.

# 2. Free Bosonic Field Theory

Conformal field theories arise in a variety of contexts, most famously in string theory and in the statistical mechanics of condensed matter systems. The basic feature of conformal theories is a total insensitivity to the parametrization of the underlying space (and thus is often associated with scale invariance). This feature leads to tight constraints on correlation functions and limits the spectrum of possible theories (by dictating the number of space-time dimensions in string theory, for example).

The theory of a free boson in 2 dimensions is one of the simplest conformal field theories, and is the basis for many others. In this chapter we review the important features of free bosonic field theory that will be applied in order to find the partition function of the boundary sine-Gordon model. At the end of the chapter we also touch on the subject of free fermion field theory. The material here is mainly drawn from [12] and [8].

## 2.1   Action and Equations of Motion

Let us consider fields $X^\mu(\sigma, \tau)$ which live on the world-sheet parametrized by the spatial coordinate $\sigma$ and the euclidean time $\tau$. In string theory, the fields $X^\mu$ represent coordinates in space-time, so the field represents a time-dependent embedding of a one-dimensional entity in space-time, or *string*.

Our world sheet will live on a strip in the $\sigma, \tau$ plane:

$$\tau \in (-\infty, \infty), \qquad \sigma \in [0, \pi]. \tag{2.1}$$

Greek indices will be used to index the fields, and these will always contract with the Euclidean metric, $g_{\mu\nu} = \delta_{\mu\nu}$. Roman indices, as in $\partial_a$, refer to the coordinates $(\sigma, \tau)$ which also contract with a Euclidean metric. The action that interests us (for now) is the one that minimizes the area of the string:

$$S = \frac{1}{4\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_{0}^{\pi} d\sigma \, \partial_a X^\mu \partial^a X_\mu. \tag{2.2}$$

We may obtain the classical equations of motion by varying the fields and integrating by parts:

$$\delta S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left( -(2\partial_a \partial^a X_\mu)\delta X^\mu + \partial_a (\partial^a X_\mu \delta X^\mu) \right) \tag{2.3}$$

This gives an equation of motion

$$\partial_a \partial^a X^\mu = 0, \tag{2.4}$$

and we must impose boundary conditions to ensure that the total derivative term is zero as well. This is usually done by either forcing the derivative term normal to the boundaries to be zero (the Neumann boundary condition), or by preventing the variation of $X$ at the boundaries by fixing it to some specific coordinate (the Dirichlet boundary condition).

Since we have boundaries at $\sigma = 0$ and $\sigma = \pi$ the Neumann condition reduces to

$$\partial_\sigma X_\mu(0, \tau) = 0$$
$$\partial_\sigma X_\mu(\pi, \tau) = 0. \tag{2.5}$$

For the Dirichlet condition, fixing the endpoints of the string is equivalent to asking that the

derivative of the field *along* the boundary be zero. The Dirichlet condition is then

$$\partial_\tau X_\mu(0, \tau) = 0$$
$$\partial_\tau X_\mu(\pi, \tau) = 0. \qquad (2.6)$$

Note that these are the most common boundary conditions but not the only possibilities. When we later add boundary interactions to the action, our boundary conditions will become more complicated.

Expectation values are defined by a functional integral of the fields, weighted by the exponential of the action. The expectation value of the operator $\mathcal{F}$ is

$$\langle \mathcal{F} \rangle = \int \mathcal{D}X e^{S[X]} \mathcal{F}[X] \qquad (2.7)$$

where the functional integral is over all field configurations that satisfy the specified boundary conditions.

It will be advantageous to work in an alternative coordinate basis defined by

$$w = \tau + i\sigma \qquad \overline{w} = \tau - i\sigma \qquad (2.8)$$

(it is quite standard to use the complex variable $z$ here, but we wish to reserve that symbol for the radial quantization coordinate transformation). This is simply a linear coordinate transformation, and $w$ and $\overline{w}$ should be treated as independent variables. Since $\tau$ and $\sigma$ are real, however, it will always be true that $\overline{w} = w^*$. We will commonly abbreviate $w\overline{w} = |w|^2$.

Our fields $X^\mu(w, \overline{w})$ are functions of the two new independent variables. All of our previous

expressions can be rewritten by applying

$$\partial_w = \partial_\tau - i\partial_\sigma \tag{2.9}$$

$$\partial_{\overline{w}} = \partial_\tau + i\partial_\sigma, \tag{2.10}$$

which gives us an action

$$S = \frac{1}{2\pi} \int dw \, d\overline{w} \; \partial_w X^\mu \partial_{\overline{w}} X_\mu \tag{2.11}$$

and equations of motion

$$\partial_w \partial_{\overline{w}} X^\mu = 0. \tag{2.12}$$

The equation of motion tells us that the fields $X^\mu$ are harmonic functions

$$X^\mu(w, \overline{w}) = X_L^\mu(w) + X_R^\mu(\overline{w}) \tag{2.13}$$

with the "left-moving" field $X_L$ holomorphic ($\partial_{\overline{w}} X_L(w) = 0$), and the "right-moving" $X_R(\overline{w})$ antiholomorphic ($\partial_w X_R = 0$). Since a derivative with respect to $w$ (or $\overline{w}$) makes $X$ holomorphic (or, respectively, antiholomorphic), we can omit the dependence on $\overline{w}$ (or $w$), and write simply $\partial_w X(w)$ (or $\partial_{\overline{w}} X(\overline{w})$).

In quantum field theory, the equation of motion holds only as an *operator equation*, meaning that expectation values involving $\partial_w \partial_{\overline{w}} X$ are zero as long as there are no other field operators near $w$ or $\overline{w}$. In a path integral formulation, this is easily obtained using the properties of functional derivatives:

$$\partial_w \partial_{\overline{w}} \langle X^\mu(w,\overline{w}) \ldots \rangle = \int \mathcal{D}X \; e^{-S[X]} \partial_w \partial_{\overline{w}} X^\mu(w,\overline{w}) \ldots$$

$$= \int \mathcal{D}X \; \pi \frac{\delta e^{-S[X]}}{\delta X_\mu(w,\overline{w})} \ldots$$

$$= 0 \qquad\qquad (2.14)$$

The final line in this equation only follows if $\ldots$ contains no field operators at $w, \overline{w}$; in this case the functional derivative is a total derivative and its integral is 0.

The complications that arise when field operators coincide are dealt with in the next section.

## 2.2  Normal Ordering

We now consider the two-point function, or propagator, $\langle X^\mu(w,\overline{w}) X^\nu(0,0) \rangle$. Now allowing $w, \overline{w}$ to approach 0, we find the "naive" equation of motion is adjusted as follows:

$$\partial_w \partial_{\overline{w}} \langle X^\mu(w,\overline{w}) X^\nu(0,0) \rangle = \int \mathcal{D}X \; e^{-S[X]} \partial_w \partial_{\overline{w}} X^\mu(w,\overline{w}) X^\nu(0,0) \ldots$$

$$= \int \mathcal{D}X \; \pi \frac{\delta e^{-S[X]}}{\delta X_\mu(w,\overline{w})} X^\nu(0,0)$$

$$= \pi \int \mathcal{D}X \left( \frac{\delta(e^{-S[X]} X^\nu(0,0))}{\delta X_\mu(w,\overline{w})} - e^{-S[X]} \frac{\delta X^\nu(0,0)}{\delta X_\mu(w,\overline{w})} \right)$$

$$= -\pi \eta^{\mu\nu} \delta^{(2)}(w,\overline{w}) \qquad\qquad (2.15)$$

The propagator is thus the green function for the operator $\partial_w \partial_{\overline{w}}$ (which is just the two dimensional Laplacian if we return to $\sigma, \tau$ coordinates):

$$\langle X^\mu(w,\overline{w}) X^\nu(0,0) \rangle = -\tfrac{1}{2} \eta^{\mu\nu} \ln |w|^2 \qquad\qquad (2.16)$$

A normal ordered operator is one that, in each of its constituent fields, satisfies the equation

of motion. An example is the normal ordered two point function:

$$:X^\mu(w,\overline{w})X^\nu(0,0): \quad = \quad X^\mu(w,\overline{w})X^\nu(0,0) + \tfrac{1}{2}\eta^{\mu\nu}\ln|w|^2 \tag{2.17}$$

Then the expectation value of $:X^\mu(w,\overline{w})X^\nu(0,0):$ satisfies

$$\partial_w\partial_{\overline{w}}\langle :X^\mu(w,\overline{w})X^n u(0,0): \rangle \quad = \quad 0 \tag{2.18}$$

for all $w$, since the double derivative of the $\ln|w|^2$ term will exactly cancel the delta function from (2.15). The normal ordered operator is well-defined even if its constituent fields are coincident; $:X^\mu(w,\overline{w})X^\nu(w,\overline{w}):$ is a meaningful operator.

An unordered operator differs from its normal ordered form by the sum of all possible contractions of its constituent fields. For the two-point function, this means

$$:X^\mu(w,\overline{w})X^\nu(0,0): \quad = \quad X^\mu(w,\overline{w})X^\nu(0,0) - \langle X^\mu(w,\overline{w})X^\nu(0,0)\rangle$$

$$\tag{2.19}$$

Informally, the normal ordered operator is the sum of all possible ways of pairing fields in the unordered operator and replacing pair with the corresponding propagator. We may express this more formally: let $\mathcal{F}(\{w_i,\overline{w}_i\})$ be an unordered product of local field operators at $w_i, \overline{w}_i$, for $i = 1, \ldots, n$. Then let

$$
\begin{aligned}
:\mathcal{F}: \quad &= \quad \exp\left(-\int d^2w \int d^2w'\ \langle X^\mu(w,\overline{w})X^\nu(w',\overline{w}')\rangle \frac{\delta}{\delta X^\mu(w,\overline{w})}\frac{\delta}{\delta X^\nu(w',\overline{w}')}\right)\mathcal{F} \\
&= \quad \exp\left(\int d^2w \int d^2w'\ \ln|w-w'|^2 \frac{\delta}{\delta X^\mu(w,\overline{w})}\frac{\delta}{\delta X_\mu(w',\overline{w}')}\right)\mathcal{F}.
\end{aligned}
$$

$$\tag{2.20}$$

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(Here the integration measures are abbreviated, e.g. $d^2w = dw\ d\overline{w}$.) This will guarantee that $\partial_{w_i}\partial_{\overline{w}_i}\langle:\mathcal{F}:\rangle = 0$ for each of $\mathcal{F}$'s coordinates $w_i$.

## 2.3   Operator Product Expansion

We have already seen in (2.15) that the product of two local operators may become ill-defined as the coordinates approach each other on the world-sheet. This can be dealt with by using the operator product expansion (O.P.E.) which approximates the product of nearby operators as a sum of local operators. For a set of operators $\{\mathcal{F}\}$, for each pair $\mathcal{F}_i$ and $\mathcal{F}_j$ there is a neighbourhood of $w$ about $w_0$ such that

$$\mathcal{F}_i(w,\overline{w})\mathcal{F}_j(w_0,\overline{w}_0) \;\; = \;\; \sum_k c_{ij}^k(w - w_0, \overline{w} - \overline{w}_0)\mathcal{F}_k(w_0,\overline{w}_0) \tag{2.21}$$

where the coefficient functions $c_{ij}(w,\overline{w})$ are holomorphic in $w$ and antiholomorphic in $\overline{w}$ in this neighbourhood of 0 except possibly at 0.

O.P.E.s are used to move the singularity of the operator product into the singularity of the otherwise holomorphic/antiholomorphic coefficient functions. For that reason it is usually only the singular terms of the O.P.E. that are of interest. Normal ordering can be used to identify the singular terms (since the normal ordered form of the operator satisfies the naive equations of motion, only the contractions of the operator will be present as non-singular terms). For example, consider the product of two field operators at neighbouring points $(w,\overline{w})$ and $(0,0)$.

$$X^\mu(w,\overline{w})X^\nu(0,0) \;\; = \;\; :X^\mu(w,\overline{w})X^\nu(0,0): -\tfrac{1}{2}\eta^{\mu\nu}\ln|w|^2 \tag{2.22}$$

The normal-ordered product satisfies (2.16), and thus is well behaved at 0 and can be expanded

in a Taylor series.

$$
\begin{aligned}
X^\mu(w)X^\nu(0) &= -\tfrac{1}{2}\eta^{\mu\nu}\ln|w-w'|^2 + :X^\mu(0)X^\nu(0): \\
&\quad + \sum_{k=1}^\infty \tfrac{1}{k!}\left(:\partial_w^k X^\mu(0)X^\nu(0): w^k + :\partial_{\overline{w}}^k X^\mu(0)X^\nu(0): \overline{w}^k\right) \\
&\sim -\tfrac{1}{2}\eta^{\mu\nu}\ln|w-w'|^2 \quad\quad\quad\quad\quad\quad\quad\quad\quad (2.23)
\end{aligned}
$$

The first line above is the complete O.P.E., with all fields evaluated at 0. The second line introduces the equivalence $\sim$, which indicates that the expressions on either side are equal up to non-singular terms (so that we may drop all "well behaved" terms). Note that in the power series expansion the mixed derivative terms are zero by the equation of motion (which applies to the normal ordered product).

## 2.4    Conformal Transformations

Our action should be invariant under world-sheet reparametrization. Going from coordinates $w$ to $w' = f(w)$ for some analytic $f$ and requiring that the new fields satisfy

$$
X'^\mu(w',\overline{w}') = X^\mu(w,\overline{w}) \quad\quad\quad\quad\quad\quad (2.24)
$$

means that the action naively written in the new coordinates

$$
S' = \int dw'\, d\overline{w}'\, \partial_{w'}X'^\mu(w')\partial_{\overline{w}'}X'_\mu(\overline{w}') = \int dw\, d\overline{w}\, \partial_{w'}X^\mu(w)\partial_{\overline{w}}X_\mu(\overline{w}) = S \quad (2.25)
$$

is equal to the original action. Thus any change of world-sheet coordinates does not change the basic theory; this is conformal invariance. In boundary conformal field theory, the world-sheet boundaries will move with the transformation but the boundary operators appearing in the action and thus the derived boundary conditions must retain their form under conformal

transformations. The set of boundary operators that is conformally invariant is not the same as the set of operators that are conformally invariant in the bulk of the space.

The world-sheet energy-momentum tensor is a set of Noether currents that arise from variation of world-sheet coordinates. Operating locally, it generates infinitesimal coordinate changes. In the $w, \overline{w}$ coordinates it can be shown that the energy-momentum tensor has only two independent components. The first,

$$T(w) \quad = \quad - :\partial_w X^\mu \partial_w X^\mu:, \tag{2.26}$$

generates transformations in $w$, while

$$T(\overline{w}) \quad = \quad - :\partial_{\overline{w}} X^\mu \partial_{\overline{w}} X^\mu:, \tag{2.27}$$

generates transformations in $\overline{w}$. In the next section we discuss how these tensors may be used to determine the conformal transformation properties of field operators.

## 2.5 Primary Fields and Conformal Dimension

A primary field $\mathcal{A}(w, \overline{w})$ is a field that, under a conform transformation $w' = f(w)$ transforms according to

$$\mathcal{A}'(w', \overline{w}') = \left( \frac{\partial w'}{\partial w} \right)^{-h} \left( \frac{\partial \overline{w}'}{\partial \overline{w}} \right)^{-\widetilde{h}} \mathcal{A}(w, \overline{w}), \tag{2.28}$$

where $(h, \widetilde{h})$ are called the conformal weights of the primary field $\mathcal{A}$.

From (2.24), for example, we know that $X^\mu$ is a primary field with weights (0,0). The energy-momentum tensor is useful for identifying and determining the weights of primary fields. It can be shown that the O.P.E. of the energy-momentum tensor acting on a primary field satisfying

(2.28) is

$$T(w)\mathcal{A}(0,0) \quad \sim \quad \frac{h}{w^2}\mathcal{A}(0,0) + \frac{1}{w}\partial_w\mathcal{A}(0,0) \tag{2.29}$$

Using this we may confirm, for example, that $\partial_w X^\mu(w)$ is a primary field of weights $(1,0)$. We accomplish this by finding the O.P.E. of $T(w)\partial X^\mu(w_0)$, by expanding the normal ordering of $T(w)$ and then re-normal ordering the resulting operator products (this is equivalent to cross-contracting the fields in $T(w)$ with the $\partial X^\mu(w_0)$ but it is instructive to work it out very explicitly):

$$\begin{aligned}
T(w)\partial X^\mu(w_o) &= -:\partial X^\nu(w)\partial X_\nu(w):\partial X^\mu(w_o) \\
&= -\lim_{w' \to w} \partial_w\partial_{w'}\left(X^\nu(w',\overline{w}')X_\nu(w,\overline{w}) + \tfrac{1}{2}\eta^\nu{}_\nu \ln|w'-w|^2\right)\partial_{w_o}X^\mu(w_o) \\
&= -\lim_{w' \to w}\left(\tfrac{1}{2}\eta^\nu{}_\nu \frac{1}{(w-w')^2}\partial X^\mu(w_o)\right. \\
&\qquad + \partial_w\partial_{w'}\partial_{w_o}\left[:X^\nu(w')X_\nu(w)X^\mu(w_0): - \tfrac{1}{2}\eta^\nu{}_\nu \ln|w'-w|^2 X^\mu(w_o)\right. \\
&\qquad\qquad \left.\left. -\tfrac{1}{2}\eta^\mu{}_\nu \ln|w-w_o|^2 X^\nu(w') - \tfrac{1}{2}\eta^{\mu\nu}\ln|w_o-w'|^2 X_\nu(w)\right]\right) \\
&\sim \quad \frac{1}{(w-w_o)^2}\partial X^\mu(w) \tag{2.30}
\end{aligned}$$

We want local operators at $w_o$, so we Taylor expand $X^\mu(w) = X^\mu(w_o) + (w-w_o)\partial X^\mu(w_o) + \dots$ to get

$$T(w)\partial X^\mu(w_o) \quad \sim \quad \frac{1}{(w-w_o)^2}X^\mu(w_o) + \frac{1}{w-w_o}\partial X^\mu(w_o); \tag{2.31}$$

from which we conclude that $\partial X^\mu$ is a primary field of weights $(1,0)$ (since $\partial X^\mu$ has no $\overline{w}$ dependence, it trivially has weight $\widetilde{h} = 0$).

It is also important for our purposes to know the weights of an exponential of the fields:

$$
\begin{aligned}
T(w) :e^{k_\mu X^\mu(w_o)}: \;&=\; -:\partial X^\nu(w)\partial X_\nu(w):\!:e^{ik_\mu X^\mu(\overline{w}_o)}: \\
&=\; -:\partial X^\nu(w)\partial X_\nu(w)e^{ik_\mu X^\mu(w_o,\overline{w}_0)}: \\
&\quad -(-\tfrac{1}{2}ik_\mu\partial_w\ln|w-w_o|^2)(-\tfrac{1}{2}ik^\mu\partial_w\ln|w-w_o|):e^{ik_\mu X^\mu(w_o)}: \\
&\quad -2\partial X^\mu(w)(-\tfrac{1}{2}ik_\mu\partial_w\ln|w-w_o|):e^{ik_\mu X^\mu(w_o)}: \\
&\sim\; \left(\frac{|k|^2/4}{(w-w_o)^2}+\frac{1}{w-w_o}\partial_{w_o}\right):e^{ik_\mu X^\mu(w_o)}:\,. \qquad\qquad (2.32)
\end{aligned}
$$

The O.P.E. has terms from contracting both or just one of the fields in $T(w)$ with the exponential. Acting with $T(\overline{w})$ gives a similar result, and we find that $:e^{ik_\mu X^\mu}:$ has weights $(k_\mu k^\mu/4, k_\mu k^\mu/4)$.

## 2.6   Radial Quantization

It is often advantageous to perform the coordinate transformation

$$
w \to z = e^w \qquad \overline{w} \to \bar{z} = e^{\overline{w}}. \qquad\qquad (2.33)
$$

Referring to our original coordinates, circles about the origin in $z$ coordinates are lines of constant $\tau$; $\tau = -\infty$ maps to $z = 0$, $\tau = \infty$ maps to $z = \infty$.

In radial quantization it is very advantageous to make use of a connection between operators and states. Consider an operator very close to $z = \bar{z} = 0$. This position corresponds to $\tau = -\infty$, and thus in the original $\tau, \sigma$ coordinates it establishes an "ingoing" state for our subsequent calculations. A path integral on the $z$ world-sheet with the operator at the origin corresponds to a path integral on the strip ($w$ world-sheet) with some initial state defined at $\tau = -\infty$. This relationship will be explored further in the next chapter.

## 2.7  Mode Expansions

Since $\partial X^\mu(z)$ satisfies $\bar\partial(\partial X(z)) = 0$ everywhere except perhaps at $z = 0$, it has a Laurent expansion about 0. This is customarily expanded as

$$\partial X^\mu(z) = -i\sqrt{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \alpha_k z^{-k-1} \tag{2.34a}$$

where the $\alpha_k$ are operators. (Note that in the original $\sigma, \tau$ coordinates, this is just a Fourier decomposition; so it should not surprise us if the $\alpha_k$ turn out to act as harmonic raising and lowering operators.) Similarly,

$$\partial X^\mu(\bar z) = -i\sqrt{\frac{1}{2}} \sum_{k=-\infty}^{\infty} \widetilde\alpha_k \bar z^{-k-1}. \tag{2.34b}$$

These can be integrated to give the general mode expansion for $X^\mu(z,\bar z) = X_L^\mu(z) + X_R^\mu(\bar z)$:

$$X_L^\mu(z) = \frac{1}{\sqrt{2}} \left( x_L^\mu - ip_L \ln z + i\sum_{n=0} \frac{\alpha_n}{n} z^{-n} \right) \tag{2.35a}$$

$$X_R^\mu(\bar z) = \frac{1}{\sqrt{2}} \left( x_R^\mu - ip_R \ln \bar z + i\sum_{n=0} \frac{\widetilde\alpha_n}{n} \bar z^{-n} \right) \tag{2.35b}$$

The $\alpha_0$ and $\widetilde\alpha_0$ mode operators have been reinterpreted as momenta. It can be shown, for example by careful commutation of contour integrals that extract the various modes from the field $X^\mu$, that the operators $x_L, p_L, \alpha_L$ satisfy commutation relations

$$[x_L, p_L] = i \quad \text{and} \quad [\alpha_n, \alpha_{-n}] = n, \tag{2.36}$$

with all other commutators zero. The analogous relations hold for the right-moving modes:

$$[x_R, p_R] = i \quad \text{and} \quad [\widetilde\alpha_n, \widetilde\alpha_{-n}] = n, \tag{2.37}$$

and the left- and right-moving modes commute with each other.

The vacuum states consist of eigenstates $|k_L, k_R\rangle$ of the momentum operators

$$p_L \, |k_L, k_R\rangle \;=\; k_L \, |k_L, k_R\rangle \qquad\qquad p_R \, |k_L, k_R\rangle \;=\; k_R \, |k_L, k_R\rangle \qquad (2.38)$$

are annihilated by positive mode oscillators;

$$\alpha_n \, |k_L, k_R\rangle \;=\; \widetilde{\alpha}_n \, |k_L, k_R\rangle \;=\; 0 \quad \text{for} \;\; n > 0 \qquad (2.39)$$

## 2.8  Free Fermion Theory

The discussion of fermions will not be as deep as the discussion of bosons. Free fermionic field theory has many similarities with the bosonic development, and several more complicated aspects.

A free fermion $\psi(w, \overline{w}) = \psi_L(w) + \psi_R(\overline{w})$ has action

$$S = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \left( \psi_L^\dagger(w) \partial_w \psi_L(w) + \psi_R^\dagger(\overline{w}) \partial_{\overline{w}} \psi_R(\overline{w}) \right). \qquad (2.40)$$

where $w = \tau + i\sigma$ is on the Euclidean cylinder.

Since only fermion bilinears must be periodic under $\sigma \to \sigma + 2\pi$, there are two sectors of fermions, depending on whether they are periodic (Ramond) or antiperiodic (Neveu-Schwarz) in their spatial coordinate. Explicitly,

$$\sigma \to \sigma + 2\pi \quad \Longrightarrow \quad \begin{cases} (\psi_L, \psi_R) \to \; (-\psi_L, -\psi_R) & \text{``Neveu-Schwarz'' (NS)} \\ (\psi_L, \psi_R) \to \quad (\psi_L, \psi_R) & \text{``Ramond'' (R)} \end{cases} \qquad (2.41)$$

with all fields evaluated at $\tau, \sigma$ The resulting mode expansions, in radial quantization $z = e^w$,

$\bar{z} = e^{\overline{w}}$, are

$$\psi_L(z) = \sum_n \psi_n z^{-n} \qquad \psi_L^\dagger(z) = \sum_n \psi_n^\dagger z^{-n} \qquad (2.42)$$

$$\psi_R(\bar{z}) = \sum_n \widetilde{\psi}_n \bar{z}^{-n} \qquad \psi_R^\dagger(\bar{z}) = \sum_n \widetilde{\psi}_n^\dagger \bar{z}^{-n} \qquad (2.43)$$

where the index $n$ ranges over integers or half-odd integers depending on the sector:

$$\begin{array}{lll} \text{NS sector} & n - \frac{1}{2} & \in \mathbb{Z} \\[2mm] \text{R sector} & n & \in \mathbb{Z}. \end{array} \qquad (2.44)$$

We will refer to the modes $\psi_n, \psi_n^\dagger$ as left-moving, and the $\widetilde{\psi}_n, \widetilde{\psi}_n^\dagger$ modes as right-moving. We will frequently use the index $r$ to sum over half-odd integers in the NS sector.

The modes have anticommutation relations

$$\left\{\psi_n, \psi_{-n}^\dagger\right\} = 1 \qquad\qquad \left\{\widetilde{\psi}_n, \widetilde{\psi}_{-n}^\dagger\right\} = 1, \qquad (2.45)$$

and all other inter-mode anticommutators are 0.

The vacuum state is different in the Ramond and Neveu-Schwarz sectors. In both cases, all positive mode $(n > 0)$ oscillators annihilate the vacuum and negative mode operators $(n < 0)$ create excited states. In the Ramond sector, the $n = 0$ modes complicate the notion of vacuum somewhat; we define our Ramond sector vacuum $|0\rangle_R$ such that

$$\widetilde{\psi}_0^\dagger |0\rangle_R = \psi_0 |0\rangle_R = 0 \qquad (2.46)$$

In the Neveu-Schwarz sector the Hamiltonian is

$$H_{NS} = \sum_{r=\frac{1}{2}}^{\infty} \left( r \psi_{-r} \psi_r + r \widetilde{\psi}_{-r} \widetilde{\psi}_r \right) - \frac{1}{12}. \qquad (2.47)$$

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The Hamiltonian for the Ramond sector is

$$H_R \;=\; \sum_{n=1}^{\infty} \left( n\psi_{-n}\psi_n + n\widetilde{\psi}_{-n}\widetilde{\psi}_n \right) - \frac{1}{6}. \tag{2.48}$$

We will ultimately be fermionizing a bosonic system, and most of our difficult work will involve the fermion operators.

# 3. Boundary Sine-Gordon Model

In this chapter we will calculate the partition function of the boundary sine-Gordon model at the self-dual radius. The theory is free in the bulk, while the periodic interaction at the boundaries modifies the boundary conditions. The results of this section, along with several extensions of the model, have been published in [10]. The approach was previously applied to the rolling tachyon problem [11]

We will first develop basic ideas about calculating partition functions using boundary states. In a path integral expression for the partition function, the field configurations are restricted to be periodic in the euclidean time; at the same time we have to impose our boundary conditions at spatial extremes of the world-sheet. To re-enter familiar string theory territory, we interchange the spatial and time coordinates so that the spatial component is periodic while the boundary conditions are recast as incoming and outgoing states.

The success of this approach comes from fermionizing the system. In free field theory, fermions are obtained as exponentials of bosonic fields, with the exponent containing a factor of $i/\sqrt{2}$ to ensure the correct conformal weights for the fermions. Since the bosonic boundary interactions are constrained to have the right conformal scaling dimensions, the exponentials appearing in the action appear without the factor $1/\sqrt{2}$. The disparity is remedied by introducing an additional boson field $Y$ with trivial boundary conditions, and forming new boson fields $(X \pm Y)/\sqrt{2}$. The new fields can be fermionized, and the boundary interaction is still marginal.

Upon fermionizing, the momenta of the bosons are related to fermion number. Since fermion

number is discrete, so must the boson momenta be. This leads to a requirement that the target space of the boson be compactified at a certain radius. Once this has been ensured, the partition functions may be calculated in the fermion system.

## 3.1   The Model

The boundary sine-Gordon model is the model of a free bosonic field $X$ on a 2d space, subject to a periodic interaction at the boundary of the space. As in section (2.1), we let world-sheet Euclidean time parameter $\tau$ run over all real values, while the spatial coordinate $\sigma$ is restricted to the interval $[0, \pi]$. The sinusoidal interaction is present at the $\sigma = 0$ boundary, and we impose either Neumann or Dirichlet conditions at the $\sigma = \pi$ boundary. The action is then

$$S = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau \int_{0}^{\pi} d\sigma \partial_a X \partial^a X - \int_{-\infty}^{\infty} d\tau \left( \frac{g}{2} e^{iX(\tau,0)} + \frac{\bar{g}}{2} e^{-iX(\tau,0)} \right). \tag{3.1}$$

One would typically take $g = \bar{g}$ to make the Hamiltonian hermitian. We need not insist on this for now, however. The equations of motion are as in (2.4). However, the derivative term in (2.3) that leads to the Neumann boundary condition equation is modified to

$$\partial_\sigma X(\tau, 0) + i \frac{g}{2} e^{iX(\tau,0)} - i \frac{\bar{g}}{2} e^{-iX(\tau,0)} = 0. \tag{3.2}$$

This is the new boundary condition at $\sigma = 0$.

## 3.2   Boundary States

The thermal partition function can be obtained in a path integral formalism from

$$Z \;\; = \;\; \int \mathcal{D} X e^{-\beta H} \tag{3.3}$$

where the path integral is over configurations of $X(\sigma, \tau)$ that are periodic in the $\tau$ direction (remember $\tau$ is already the Euclidean time) with period $\beta = 1/T$. In this problem we must also enforce the boundary conditions at $\sigma = 0$ and $\sigma = \pi$.

In string theory we are much more accustomed to $\sigma$ being the periodically identified coordinate, while $\tau$ is subject to notions of "in-coming" and "out-going" states. For this reason we redefine our coordinates, using conformal invariance. In particular we want to make the $\beta$-periodic $\tau$ into a $2\pi$-periodic $\sigma'$:

$$\sigma' = \frac{2\pi}{\beta}\tau \qquad\qquad \tau' = -\frac{2\pi}{\beta}\sigma$$

Now $\sigma'$ is $2\pi$-periodic and our boundaries are at $\tau' = 0$ and $-2\pi^2/\beta$. So $(\tau', \sigma')$ parametrize a finite length of Euclidean cylinder.

In the Boltzmann factor (3.3), the conformal transformation to $\sigma', \tau'$ will preserve the form of the double integral over Hamiltonian density $\mathcal{H}$, but the integration limits are adjusted:

$$\beta H = \int_0^\beta d\tau \int_0^\pi d\sigma \, \mathcal{H} = \int_0^{2\pi} d\sigma' \int_0^{2\pi^2/\beta} d\tau' \, \mathcal{H} = \frac{2\pi^2}{\beta}\int_0^{2\pi} d\sigma' \mathcal{H} = \frac{2\pi^2}{\beta} H'. \tag{3.4}$$

So our partition function (3.3) becomes

$$Z = \int \mathcal{D}'X \exp(-\alpha H) \tag{3.5}$$

where $\alpha = 2\pi^2/\beta$ and the prime on the integration measure indicating that the paths are restricted to satisfy the boundary conditions at $\tau' = 0$ and $\alpha$. We may let $X$ roam freely over all paths, provided that we enforce the boundary conditions some other way. Let $\Psi_1[X]$ be a functional of the path that enforces the boundary condition at $\tau' = 0$, and $\Psi_2[X]$ do the same

at $\tau' = \alpha$. Then

$$
\begin{aligned}
Z &= \int \mathcal{D}X \Psi_2[X] \exp(-\alpha H) \Psi_1[X]. \\
&= \langle B_1 | \exp(-\alpha H) | B_2 \rangle
\end{aligned}
\tag{3.6}
$$

where $|B_i\rangle$ are the *boundary states* corresponding to functionals $\Psi_i[X]$ that enforce the boundary conditions [12]. To do this they must satisfy

$$
\mathcal{B}_i |B_i\rangle = 0
\tag{3.7}
$$

where $\mathcal{B}_i$ are the operators obtained from the left hand side of the boundary condition equation (3.2).

We also need the form of the new Hamiltonian $H'$. It is obtained in the canonical fashion from the free action in the new coordinates. The action preserves its form (other than integration limits) under the conformal transformation, so the new Lagrangian is

$$
L' = \frac{1}{4\pi} \int_0^{2\pi} d\sigma' \partial_a X \partial^a X
\tag{3.8}
$$

and the new Hamiltonian is then

$$
\begin{aligned}
H' &= \frac{\partial \mathcal{L}}{\partial(\partial_{\tau'} X)} - L \\
&= \frac{1}{4\pi} \int_0^{2\pi} d\sigma' \left( (\partial_{\tau'} X)^2 - (\partial_{\sigma'} X)^2 \right) \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\sigma' \left( (z \partial_z X(z))^2 + (\bar{z} \partial_{\bar{z}} X(\bar{z}))^2 \right)
\end{aligned}
\tag{3.9}
$$

where now the derivatives are with respect to $z = e^{\tau' + i\sigma'}$, $\bar{z} = e^{\tau' - i\sigma'}$. The fields $\partial_z X(z)$ and

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$\partial_{\bar{z}} X(\bar{z})$ have mode expansions given in (2.34a) and (2.34b), and we may may integrate them very easily by realizing that $\sigma' \in [0, 2\pi]$ is a contour integral in the complex plane. Then with $d\sigma' = dz/iz = d\bar{z}/i\bar{z}$ we find

$$
\begin{aligned}
H' &= \oint \frac{dz}{2\pi i} z (\partial_z X(z))^2 + \oint \frac{d\bar{z}}{2\pi i} \bar{z} (\partial_{\bar{z}} X(\bar{z}))^2 \\
&= \tfrac{1}{2} p_L^2 + \tfrac{1}{2} p_R^2 + \sum_{n=1}^{\infty} (\alpha_{-n} \alpha_n + \widetilde{\alpha}_{-n} \widetilde{\alpha}_n) - \tfrac{1}{12}.
\end{aligned}
\tag{3.10}
$$

(Here the 1/12 comes from the commutation of $\alpha_n$ past $\alpha_{-n}$ which results in a $\sum_{n=1}^{\infty} n$. The requirement of modular invariance demonstrates that this sum is best interpreted as $\zeta(-1) = -1/12$, with $\zeta$ the Riemann zeta function.)

## 3.3  Auxiliary Boson

In order to facilitate fermionization, we must introduce an additional bosonic field $Y$ to the system [9, 13]. Our free action, on the cylinder in $\sigma', \tau'$ space is then

$$
S_o = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau' \int_0^{2\pi} d\sigma' \left( \partial_a X \partial^a X + \partial_a Y \partial^a Y \right).
\tag{3.11}
$$

where we may extend the $\tau'$ domain to $\pm\infty$ since our boundary conditions will be enforced by the boundary states. To make $Y$ easy to deal with, we give it Dirichlet boundary conditions at both boundaries.

Our real purpose is to change the scale of interaction term, however. We perform a unitary rotation on the fields $X, Y$ to get

$$
\phi_1 = \frac{X + Y}{\sqrt{2}} \qquad \phi_2 = \frac{X - Y}{\sqrt{2}}.
\tag{3.12}
$$

The free part of the action (3.11) becomes

$$S_o \;=\; \frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau' \int_0^{2\pi} d\sigma' \left( \partial_a \phi_1 \partial^a \phi_1 + \partial_a \phi_2 \partial^a \phi_2 \right). \qquad (3.13)$$

Our mode expansion for the new fields $\phi_i(z, \bar{z}) = \phi_{iL}(z) + \phi_{iR}(\bar{z})$ has the same form as (2.35), explicitly

$$\phi_{iL}(z, \bar{z}) \;=\; \frac{1}{\sqrt{2}} \left( \varphi_{iL} - i\pi_{iL} \ln z + \sum_{n \neq 0} \tfrac{1}{n} \beta_n z^{-n} \right) \qquad (3.14)$$

and similarly for $\phi_{iR}$.

We will also write the consequences that simple boundary conditions on $X$ and $Y$ have for $\phi_i$. For the case of the boundary state $|N, D\rangle$ where $X$ has Neumann boundary conditions and $Y$ has Dirichlet boundary conditions,

$$\left.\begin{aligned} (X_L - X_R)|N, D\rangle &= 0 \\ (Y_L + Y_R)|N, D\rangle &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} (\phi_{1L} - \phi_{2R})|N, D\rangle &= 0 \\ (\phi_{2L} - \phi_{1R})|N, D\rangle &= 0 \end{aligned}\right. \qquad (3.15)$$

with all fields evaluated at $\tau = 0$, arbitrary $\sigma$.

Similarly the boundary state $|D, D\rangle$, for the case of $X$ and $Y$ both having Dirichlet boundary conditions at $\tau = 0$, satisfies

$$\left.\begin{aligned} (X_L - X_R)|D, D\rangle &= 0 \\ (Y_L - Y_R)|D, D\rangle &= 0 \end{aligned}\right\} \Longrightarrow \left\{\begin{aligned} (\phi_{1L} - \phi_{1R})|D, D\rangle &= 0 \\ (\phi_{2L} - \phi_{2R})|D, D\rangle &= 0. \end{aligned}\right. \qquad (3.16)$$

## 3.4 Fermionization

Fermionization is a process by which the bosonic fields of a system are combined into operators that have fermionic commutation relations. If the inherited properties of the derived fermions can be shown to follow from some fermionic action, then expressions obtained in the fermionic

theory (in particular the form of the partition function) are valid in the bosonic context as well. To perform our computation in the fermion system, we will need the Hamiltonian, the boundary conditions, and thus the boundary state must be obtained in terms of the fermionic operators.

For the boson fields $\phi_i$, the fermionic field operators are [8, 11]

$$
\begin{aligned}
\psi_{1L}(z) &= \zeta_{1L} :e^{-\sqrt{2}i\phi_{1L}}: & \psi_{1L}^\dagger(z) &= :e^{\sqrt{2}i\phi_{1L}}: \zeta_{1L}^\dagger \\
\psi_{2L}(z) &= \zeta_{2L} :e^{\sqrt{2}i\phi_{2L}}: & \psi_{2L}^\dagger(z) &= :e^{-\sqrt{2}i\phi_{2L}}: \zeta_{2L}^\dagger \\
\psi_{1R}(\bar{z}) &= \zeta_{1R} :e^{\sqrt{2}i\phi_{1R}}: & \psi_{1R}^\dagger(\bar{z}) &= :e^{-\sqrt{2}i\phi_{1R}}: \zeta_{1R}^\dagger \\
\psi_{2R}(\bar{z}) &= \zeta_{2R} :e^{-\sqrt{2}i\phi_{2R}}: & \psi_{2R}^\dagger(\bar{z}) &= :e^{\sqrt{2}i\phi_{2R}}: \zeta_{2R}^\dagger.
\end{aligned}
\tag{3.17}
$$

The operators $\zeta_{iH}$ (where $H$ is the handedness, $L$ or $R$) are *cocycles*, and are necessary in the two boson case to ensure that the $\psi_1$ and $\psi_2$ fields anticommute (without it, the two kinds of fermions would commute instead). It is enough to use bosonic momentum operators in the cocycles [11]:

$$
\begin{aligned}
\zeta_{1L} &= \zeta_{1R} = \exp \tfrac{-i\pi}{2}(\pi_{1L} + \pi_{1R} + 2\pi_{2L} + 2\pi_{2R}) \\
\zeta_{2L} &= \zeta_{2R} = \exp \tfrac{-i\pi}{2}(\pi_{2L} + \pi_{2R})
\end{aligned}
\tag{3.18}
$$

In resolving the anticommutator of, for example, $\psi_{1L}$ and $\psi_{1R}$, factors of $e^{\pm i\pi/2}$ are obtained when commuting the cocycle from one operator through the exponential factor of the other.

The exponential in the boundary condition (3.2), acting on the boundary state $|B, D\rangle$ can

then rewritten in the following way:

$$
\begin{aligned}
\frac{g}{2}e^{iX}|B,D\rangle &= \frac{g}{2}e^{i(X+Y)}|B,D\rangle \\
&= \frac{g}{2}e^{i\sqrt{2}\phi_1}|B,D\rangle \\
&= \frac{g}{2}e^{i\sqrt{2}\phi_{1L}}e^{i\sqrt{2}\phi_{1R}}|B,D\rangle \\
&= \frac{g'}{2}:e^{i\sqrt{2}\phi_{1L}}::e^{i\sqrt{2}\phi_{1R}}:|B,D\rangle \\
&= \frac{g'}{2}\psi_{1L}^{\dagger}\psi_{1R}|B,D\rangle
\end{aligned}
\tag{3.19}
$$

with all fields evaluated at the boundary.

In the first line of this equation, the introduction of $Y$ is allowed since $X$ and $Y$ are independent and $e^{iY}|B,D\rangle = |B,D\rangle$ from $Y$'s Dirichlet boundary condition. In the third line, the left- and right-moving parts of $\phi_1$ are independent and the exponential can be factored.

The normal ordering in the fourth line, however, introduces an infinite constant that must be absorbed into the coupling constant. This is a standard coupling renormalization. We will continue using the symbols $g$ and $\bar{g}$ in the fermion theory, but we should remember that these are renormalized versions of the original couplings. In a similar fashion we obtain

$$
\frac{\bar{g}}{2}e^{-iX}|B,D\rangle = \frac{\bar{g}'}{2}\psi_{2L}^{\dagger}\psi_{2R}|B,D\rangle
\tag{3.20}
$$

The operator product in bosonic operators can be used to show that, for a bosonic holomorphic field $X(z)$, [8]

$$
:e^{i\alpha X(z)}e^{-i\alpha X(z)}: = i\alpha\partial_z X(z).
\tag{3.21}
$$

So our bosonic field derivatives may be expressed as fermion bilinears. In terms of the original

$X$ and $Y$ fields we then have

$$:\psi_{1L}^\dagger\psi_{1L}(w): \quad = \quad \sqrt{2}i\partial_w\phi_{1L}(w) \tag{3.22a}$$

$$:\psi_{2L}^\dagger\psi_{2L}(w): \quad = \quad -\sqrt{2}i\partial_w\phi_{2L}(w) \tag{3.22b}$$

$$:\psi_{1R}^\dagger\psi_{1R}(\overline{w}): \quad = \quad -\sqrt{2}i\partial_{\overline{w}}\phi_{1R}(\overline{w}) \tag{3.22c}$$

$$:\psi_{2R}^\dagger\psi_{2R}(\overline{w}): \quad = \quad \sqrt{2}i\partial_{\overline{w}}\phi_{2R}(\overline{w}) \tag{3.22d}$$

The derivative operator in the boundary condition (3.2) is expressed as a fermion bilinear using (3.22):

$$
\begin{aligned}
\partial_\tau X \quad &= \quad (\partial_w + \partial_{\overline{w}})\tfrac{1}{\sqrt{2}}(\phi_1 + \phi_2) \\
&= \quad \tfrac{1}{\sqrt{2}}\partial_w\left(\phi_{1L} + \phi_{2L}\right) + \tfrac{1}{\sqrt{2}}\partial_{\overline{w}}\left(\phi_{1R} + \phi_{2R}\right) \\
&= \quad \tfrac{1}{2i}\left(:\psi_{1L}^\dagger\psi_{1L}: - :\psi_{2L}^\dagger\psi_{2L}: - :\psi_{1R}^\dagger\psi_{1R}: + :\psi_{2R}^\dagger\psi_{2R}:\right) \tag{3.23}
\end{aligned}
$$

Introducing the vector notation

$$\psi_L = \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} \qquad \psi_L^\dagger = \begin{pmatrix} \psi_{1L}^\dagger & \psi_{2L}^\dagger \end{pmatrix} \tag{3.24}$$

and similarly for $\psi_R$ and $\psi_R^\dagger$, along with the Pauli matrices $\sigma^i$ we can write the complete boundary equation in fermion variables as

$$\left(:\psi_L^\dagger\sigma^3\psi_L: - :\psi_R^\dagger\sigma^3\psi_R: + \pi g\psi_L^\dagger(1+\sigma^3)\psi_R - \pi\bar{g}\psi_L^\dagger(1-\sigma^3)\psi_R\right)|B,D\rangle \quad = \quad 0. \tag{3.25}$$

In a similar fashion, the analogue of the Dirichlet boundary condition (2.6) for $Y$ is

$$\left(:\psi_L^\dagger \psi_L: - :\psi_R^\dagger \psi_R:\right)|B,D\rangle \;\; = \;\; 0 \tag{3.26}$$

## 3.5   Boson momenta and fermion numbers

The boson momenta $\pi_{iL}$ and $\pi_{iR}$ can be obtained from the spatial integral of the derivative of the boson fields $\phi_i$; this follows from the mode expansion (2.34a). In radial coordinates we have

$$
\begin{aligned}
\pi_{iL} \;\; &= \;\; \oint \frac{dz}{2\pi i}(i\sqrt{2}\partial_z \phi_{iL}) \\
&= \;\; -\oint \frac{dz}{2\pi i}(-1)^i :\psi_{iL}^\dagger \psi_{iL}: \\
&= \;\;
\begin{cases}
\sum_{r=1}\left(\psi_{i,-r}^\dagger \psi_{i,r} - \psi_{i,-r}\psi_{i,r}^\dagger\right) & \text{Neveu Schwarz} \\
\sum_{n=1}\left(\psi_{i,-n}^\dagger \psi_{i,n} - \psi_{i,-n}\psi_{i,n}^\dagger\right) + \psi_{i,0}^\dagger \psi_{i,0} - \frac{1}{2} & \text{Ramond}
\end{cases}
\end{aligned} \tag{3.27a}
$$

and similarly

$$
\pi_{iR} \;\; = \;\;
\begin{cases}
-\sum_{r=1}\left(\psi_{i,-r}^\dagger \widetilde{\psi}_{i,r} - \psi_{i,-r}\widetilde{\psi}_{i,r}^\dagger\right) & \text{Neveu Schwarz} \\
-\sum_{n=1}\left(\psi_{i,-n}^\dagger \widetilde{\psi}_{i,n} - \psi_{i,-n}\widetilde{\psi}_{i,n}^\dagger\right) - \widetilde{\psi}_{i,0}^\dagger \widetilde{\psi}_{i,0} + \frac{1}{2} & \text{Ramond.}
\end{cases} \tag{3.27b}
$$

Since the fermion numbers are integers (in the Neveu Schwarz sector) or half-odd-integers (in the Ramond sector), this limits the spectrum of the momenta in the bosonic theory. The way to handle this is to compactify the boson theory, which identifies target field values separated by $2\pi R$ where $R$ is the *compactification radius*.

From the bosonic mode expansions (2.35a) and (2.35b), the parts of $X(\sigma,\tau)$ linear in $\sigma$ and $\tau$ are

$$\tfrac{-i}{\sqrt{2}}\left((p_{X,L}+p_{X,R})\tau + (p_{X,L}-p_{X,R})i\sigma\right) \tag{3.28}$$

which leads to the definition of total momentum $p$ and wrapping number $w$:

$$p \ = \ p_L + p_R \qquad\qquad w \ = \ p_L - p_R \qquad\qquad (3.29)$$

Since taking $\sigma \rightarrow \sigma + 2\pi$ must map $X$ to an equivalent $X$ (in the sense that $X \sim X + 2\pi$), the wrapping number for $X$ must satisfy

$$\tfrac{1}{\sqrt{2}} w \ = \ mR \quad \implies \quad w = \sqrt{2}Rm \qquad\qquad (3.30)$$

with $m \in \mathbb{Z}$. The total momentum, is quantized according to

$$2\pi R \tfrac{1}{\sqrt{2}} p \ = \ 2\pi n \quad \implies \quad p = \frac{\sqrt{2}}{R} n \qquad\qquad (3.31)$$

with $n \in \mathbb{Z}$. At $R = 1$, $\sqrt{2}p_{X,L}$ and $\sqrt{2}p_{X,R}$ are integers and either both are even or both are odd. The same must apply to $Y$, and the implication for the $\phi$ bosons is that $2\phi_{1,L}, 2\phi_{1,R}, 2\phi_{2,L}$, and $2\phi_{2,R}$ all must be integers with the same parity. The significance of this is that our partition function expression (which originated with a trace over all possible states) must respect these relationships between momenta, and thus between fermion number as described in (3.27). This means that our partition function must be the sum of fermionic expressions for the Ramond and Neveu-Schwarz sectors:

$$Z \ = \ Z_{NS} + Z_R. \qquad\qquad (3.32)$$

By compactifying at $R = 1$, we are guaranteed that the half-integral spectrum of the fermion number operators spans the same values as the $\phi$ boson momenta; and by separating the partition function into the sum over the two fermion sectors we are enforcing the relationship between those $\phi$ boson momenta.

## 3.6   Gluing relations

Having established that the boson momenta are integers, we are set to establish another important set of relationships for the fermion fields acting on boundary states. Using the "gluing relations" (3.15) for the boson fields at the boundary, we obtain gluing relations for the fermion operators. For example, acting on the $|N, D\rangle$ state with the vector $\psi_L^\dagger(0, \sigma)$ gives

$$\begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} |N, D\rangle = \begin{pmatrix} \zeta_{1L} : e^{-\sqrt{2}i\phi_{1L}} : \\ \zeta_{2L} : e^{\sqrt{2}i\phi_{2L}} : \end{pmatrix} |N, D\rangle = \begin{pmatrix} \zeta_{1L} : e^{\sqrt{2}i\phi_{2R}} : \\ \zeta_{2L} : e^{-\sqrt{2}i\phi_{1R}} : \end{pmatrix} |N, D\rangle \qquad (3.33)$$

with all fields in this expression evaluated at $(\tau = 0, \sigma)$. Since the boson momenta have been restricted to integers, we have $\zeta_{1L}^\dagger \zeta_{2R} |B, D\rangle = |B, D\rangle$ and thus

$$\begin{aligned} \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} |N, D\rangle &= \begin{pmatrix} \zeta_{1L} : e^{\sqrt{2}i\phi_{2R}} : \zeta_{1L}^\dagger \zeta_{2R} \\ \zeta_{2L} : e^{-\sqrt{2}i\phi_{1R}} : \zeta_{2L}^\dagger \zeta_{1R} \end{pmatrix} |N, D\rangle = \begin{pmatrix} -i\zeta_{2R} : e^{\sqrt{2}i\phi_{2R}} : \\ -i\zeta_{1R} : e^{-\sqrt{2}i\phi_{1R}} : \end{pmatrix} |N, D\rangle \\ &= -i \begin{pmatrix} \psi_{2R} \\ \psi_{1R} \end{pmatrix} |N, D\rangle \qquad (3.34) \end{aligned}$$

A similar computation can be performed for $\psi_L^\dagger$ and $\psi_R^\dagger$. The result is the two gluing relations for the vectors (3.24):

$$(\psi_R^\dagger(0, \sigma) + \psi_L^\dagger(0, \sigma) \, i\sigma^1)|N, D\rangle = 0 \qquad (\psi_R(0, \sigma) + i\sigma^1 \psi_L(0, \sigma))|N, D\rangle = 0, \qquad (3.35)$$

where $\sigma^1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the Pauli matrix acting left on the row vector $\psi_L^\dagger$ or right on the column $\psi_L$.

If instead we impose Dirichlet boundary conditions on $X$, this leads to the much simpler

relations

$$(\psi_R^\dagger + \psi_L^\dagger)|D, D\rangle \;=\; 0 \qquad (\psi_R - \psi_L)|D, D\rangle \;=\; 0. \tag{3.36}$$

## 3.7  Fermion boundary state

We have found the expression for the partition function as a matrix element (3.6) of boundary states, and we have developed an equivalent fermionic theory with Hamiltonian (2.47) or (2.48) and boundary conditions (3.25). We will now establish the form of the boundary states in fermion variables. Then the computation of the partition function will be possible.

The action for the pair of fermions $\psi_1, \psi_2$ can be written exactly as (2.40), though now the field operators, such as $\psi_L$, represent the vectors defined in (3.24):

$$S = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \left( \psi_L^\dagger(w)\partial_w \psi_L(w) + \psi_R^\dagger(\overline{w})\partial_{\overline{w}} \psi_R(\overline{w}) \right). \tag{3.37}$$

This action is invariant under the unitary transformation

$$\psi_L \to U\psi_L \quad \text{and} \quad \psi_L^\dagger \to \psi_L^\dagger U^{-1} \tag{3.38}$$

and similarly (and independently) for $\psi_R$. These symmetries have corresponding currents [8]

$$J_L^a = \tfrac{1}{2} : \psi_L^\dagger \sigma^a \psi_L : \tag{3.39}$$

where $\sigma^a$ are Pauli matrices acting on the vectors $\psi_L$ and $\psi_L^\dagger$. In fact

$$[\psi_L, J_L^a] \;=\; \tfrac{1}{2}\sigma^a \psi_L \qquad [J_L^a, \psi_L^\dagger] \;=\; \tfrac{1}{2}\psi_L^\dagger \sigma^a. \tag{3.40}$$

For a vector $\theta_a$ of angles, the currents $J_L^a$ generate finite SU(2) transformations according

to

$$e^{-i\theta_a J_L^a} \psi_L e^{+i\theta_a J_L^a} = U\psi_L \tag{3.41a}$$

$$e^{-i\theta_a J_L^a} \psi_L^\dagger e^{+i\theta_a J_L^a} = \psi_L^\dagger U^{-1} \tag{3.41b}$$

where $U = e^{i\theta_a \sigma^a/2}$.

Note that the gluing relations (3.35) for the $|N, D\rangle$ and $|D, D\rangle$ states appear related by this sort of rotation, in particular it can be shown that

$$|N, D\rangle = e^{i\pi J_L^1}|D, D\rangle \tag{3.42}$$

We will attempt to construct the boundary state $|B, D\rangle$ that satisfies the boundary conditions (3.25) as an intermediate rotation of the $|N, D\rangle$ state in the $\sigma^1$ direction. Our ansatz is thus that

$$|B, D\rangle = \exp(-i\theta_a J_L^a)|N, D\rangle \tag{3.43}$$

for some vector of (possibly complex) angles $(\theta_a)$.

Using (3.41b) in (3.35) we obtain gluing relations for the boundary state:

$$\begin{aligned}
0 &= e^{-i\theta_a J_L^a} \left( \psi_R^\dagger(0, \sigma) + \psi_L^\dagger(0, \sigma)i\sigma^1 \right) e^{i\theta_a J_L^a}|B, D\rangle \\
&= \left( \psi_R^\dagger(0, \sigma) + \psi_L^\dagger(0, \sigma)U^{-1}i\sigma^1 \right) e^{i\theta_a J_L^a}|B, D\rangle \tag{3.44}
\end{aligned}$$

and similarly

$$0 = \left( \psi_R(0, \sigma) + i\sigma^1 U\psi_L(0, \sigma) \right)|B, D\rangle. \tag{3.45}$$

Applying these to the boundary state equation (3.25) gives an equation for $U$. Beginning

with the boundary state equation, we apply (3.41a) to change $\psi_R$ to $\psi_L$:

$$
\begin{aligned}
0 &= \Big(:\psi_L^\dagger \sigma^3 \psi_L: \; - \; :\psi_R^\dagger \sigma^3 \psi_R: \\
&\quad + \; \pi g \psi_L^\dagger (1 + \sigma^3)\psi_R \; - \; \pi \bar{g} \psi_L^\dagger (1 - \sigma^3)\psi_R \Big) |B, D\rangle \\
&= \Big(:\psi_L^\dagger \sigma^3 \psi_L: - :\psi_R^\dagger \sigma^3 (-i\sigma^1 U \psi_L): \\
&\quad + \; \pi g \psi_L^\dagger (1 + \sigma^3)(-i\sigma^1 U \psi_L) \; - \; \pi \bar{g} \psi_L^\dagger (1 - \sigma^3)(-i\sigma^1 U \psi_L) \Big) |B, D\rangle \quad (3.46)
\end{aligned}
$$

We will suppress normal ordering for a while. We can then separate the $\psi_L$ vector on the right, and anticommute the fermion operators left (note the introduction of indices $a$ to handle the vector product). The anticommutators of the $\psi_L, \psi_L^\dagger$ fields vanish because the matrix between them is traceless. The first two terms on the right hand side of (3.46), then, satisfy

$$
\begin{aligned}
\Big(\psi_L^\dagger \sigma^3 \psi_L \; - \; \psi_R^\dagger \sigma^3 (-i\sigma^1 U \psi_L)\Big) |B, D\rangle &= \; - (\sigma^2 U \psi_L)^a \Big(\psi_L^\dagger \sigma^3 U^{-1} \sigma^2 \; - \; \psi_R^\dagger\Big)_a |B, D\rangle \\
&= \; - (\sigma^2 U \psi_L)^a \Big(\psi_L^\dagger \sigma^3 U^{-1} \sigma^2 \; + \; i\psi_L^\dagger U^{-1} \sigma^1\Big)_a |B, D\rangle \\
&= \; \psi_L^\dagger \left(\sigma^3 - U^{-1} \sigma^3 U\right) \psi_L |B, D\rangle \quad (3.47)
\end{aligned}
$$

and so the complete boundary state equation is

$$
0 \; = \; \psi_L^\dagger \left[\sigma^3 - U^{-1} \sigma^3 U + \pi g (\sigma^2 - i\sigma^1)U + \pi \bar{g}(\sigma^2 + i\sigma^1)U\right] \psi_L |B, D\rangle. \quad (3.48)
$$

The matrix $U$ is then obtained by solving

$$
0 \; = \; \sigma^3 U^{-1} - U^{-1}\sigma^3 + \pi g(\sigma^2 - i\sigma^1) + \pi \bar{g}(\sigma^2 + i\sigma^1), \quad (3.49)
$$

to obtain

$$
U = \begin{bmatrix} \sqrt{1 - \pi^2 |g|^2} & -i\pi g \\ -i\pi \bar{g} & \sqrt{1 - \pi^2 |g|^2} \end{bmatrix} \tag{3.50}
$$

Here, the diagonal elements of $U$ are fixed by demanding that $U$ be unitary. This also forces $\bar{g} = g^*$.

The boundary state for the full boundary condition is a rotation of the Neumann and thus of the Dirichlet boundary states. In the two possible fermion sectors, the Dirichlet-Dirichlet boundary state $|D, D\rangle$, which satisfies (3.16), is

$$
|D, D\rangle_{NS} = 2^{-\frac{1}{2}} \prod_{r=\frac{1}{2}}^{\infty} \exp\left( \psi_{-r}^\dagger \widetilde{\psi}_{-r} + \widetilde{\psi}_{-r}^\dagger \psi_{-r} \right) |0\rangle_{NS} \tag{3.51}
$$

$$
|D, D\rangle_R = 2^{-\frac{1}{2}} \prod_{n=1}^{\infty} \exp\left( \psi_{-n}^\dagger \widetilde{\psi}_{-n} + \widetilde{\psi}_{-n}^\dagger \psi_{-n} \right) \exp\left( \psi_0^\dagger \widetilde{\psi}_0 \right) |0\rangle_R . \tag{3.52}
$$

The operators $\psi_r$, etc., are vectors containing the modes of the $L$ and $R$ fermion operator vectors (3.24), for example

$$
\psi_i(z) = \sum_n \psi_{i,n} z^{-n} \qquad \psi_n \equiv \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix} . \tag{3.53}
$$

We are interested in the action of the Hamiltonian on the boundary state. Since the currents $J_L^a$ commute with the Hamiltonian, we can move them left,

$$
e^{-\alpha H} |B, D\rangle = e^{-\alpha H} e^{-i\theta_a J_L^a} e^{i\pi J_L^1} |D, D\rangle = e^{-i\theta_a J_L^a} e^{i\pi J_L^1} e^{-\alpha H} |D, D\rangle .
$$

and work with the $|D, D\rangle$ state instead. The action of the Hamiltonian on the Dirichlet-Dirichlet state can be examined mode by mode. We will work it out carefully in the Neveu-Schwarz

sector. Combining (2.47) and (3.52) we have

$$e^{-\alpha H_{NS}}|D,D\rangle_{NS} \;=\; e^{\alpha/6}\frac{1}{\sqrt{2}}\prod_{r=\frac{1}{2}}^{\infty}\exp(-\alpha H_r^L - \alpha H_r^R)\exp(D_r)\,|0\rangle_{NS} \tag{3.54}$$

with

$$
\begin{aligned}
H_r^L &= r(\psi_{-r}^{\dagger}\psi_r + \psi_{-r}\psi_r^{\dagger}) \\
H_r^R &= r(\widetilde{\psi}_{-r}^{\dagger}\widetilde{\psi}_r + \widetilde{\psi}_{-r}\widetilde{\psi}_r^{\dagger}) \\
D_r &= (\psi_{-r}^{\dagger}\widetilde{\psi}_{-r} + \widetilde{\psi}_{-r}^{\dagger}\psi_{-r}).
\end{aligned} \tag{3.55}
$$

For different values of $r > 0$, the operators in (3.55) commute with each other. For equal $r$, we have

$$[H_r^L, D_r] \;=\; rD_r \qquad\qquad [H_r^R, D_r] \;=\; rD_r \tag{3.56}$$

and thus

$$e^{-\alpha(H_r^L + H_r^R)}D_r \;=\; D_r e^{-\alpha(H_r^L + r)}e^{-\alpha(H_r^R + r)} \;=\; D_r e^{-2\alpha r}e^{-\alpha(H_r^L + H_r^R)} \tag{3.57}$$

and finally

$$e^{-\alpha(H_r^L + H_r^R)}e^{D_r} \;=\; e^{e^{-2\alpha r}D_r}e^{-\alpha(H_r^L + H_r^R)}. \tag{3.58}$$

Since $H_r^L|D,D\rangle = H_r^R|D,D\rangle = 0$, (3.54) becomes

$$e^{-\alpha H_{NS}}|D,D\rangle_{NS} \;=\; \frac{e^{\alpha/6}}{\sqrt{2}}\prod_{r=\frac{1}{2}}^{\infty}\exp\left(e^{-2\alpha r}(\psi_{-r}^{\dagger}\widetilde{\psi}_{-r} + \widetilde{\psi}_{-r}^{\dagger}\psi_{-r})\right)|0\rangle_{NS} \tag{3.59}$$

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Re-rotating the left-moving fields gives

$$
e^{-\alpha H_{NS}}|B,D\rangle_{NS} \;=\; \frac{e^{\alpha/6}}{\sqrt{2}} \prod_{r=\frac{1}{2}}^{\infty} \exp\left( e^{-2\alpha r}(i\psi^{\dagger}_{-r}U^{-1}\sigma^{1}\widetilde{\psi}_{-r} - i\widetilde{\psi}^{\dagger}_{-r}\sigma^{1}U\psi^{\dagger}_{-r}) \right)|0\rangle_{NS} \quad (3.60)
$$

Not surprisingly, the result in the Ramond sector is simply

$$
e^{-\alpha H_{R}}|B,D\rangle_{R} \;=\; \frac{e^{\alpha/3}}{\sqrt{2}} \prod_{n=1}^{\infty} \exp\left( e^{-2\alpha n}(i\psi^{\dagger}_{-n}U^{-1}\sigma^{1}\widetilde{\psi}_{-n} - i\widetilde{\psi}^{\dagger}_{-n}\sigma^{1}U\psi^{\dagger}_{-n}) \right)
$$
$$
\exp(i\psi^{\dagger}_{0}U^{-1}\sigma^{1}\widetilde{\psi}_{0})\,|0\rangle_{R} \quad (3.61)
$$

## 3.8 Evaluation of the Partition Function

Using the boundary states found in the last section, we can construct the fermion theory matrix elements associated with the boson partition function. Continuing from above, the NS sector partition function is

$$
\begin{aligned}
Z_{NS} \;&=\; \langle D,D|\,e^{-\alpha H_{NS}}|B,D\rangle \\
&=\; \frac{1}{\sqrt{2}}\,\langle 0|_{NS} \prod_{r'=\frac{1}{2}}^{\infty} \exp\left( \psi^{\dagger}_{r'}\widetilde{\psi}_{r'} + \widetilde{\psi}^{\dagger}_{r'}\psi_{r'} \right) \\
&\quad \times \frac{e^{\alpha/6}}{\sqrt{2}} \prod_{r=\frac{1}{2}}^{\infty} \exp\left( \gamma(\psi^{\dagger}_{-r}M^{-1}\widetilde{\psi}_{-r} + \widetilde{\psi}^{\dagger}_{-r}M\psi^{\dagger}_{-r}) \right)|0\rangle_{NS} \quad (3.62)
\end{aligned}
$$

where we have abbreviated

$$
M \;=\; i\sigma^{1}U \qquad M^{-1} \;=\; -iU^{-1}\sigma^{1}. \quad (3.63)
$$

and $\gamma = e^{-2\alpha r}$.

The exponentials can be broken up and grouped:

$$Z_{NS} = \frac{e^{\alpha/6}}{2} \langle 0|_{NS} \prod_{r=\frac{1}{2}}^{\infty} \left( e^{\psi_r^\dagger \widetilde{\psi}_r} e^{\gamma \psi_{-r}^\dagger M^{-1} \widetilde{\psi}_{-r}} \right) \left( e^{\widetilde{\psi}_r^\dagger \psi_r} e^{\gamma \widetilde{\psi}_{-r}^\dagger M \psi_{-r}^\dagger} \right) |0\rangle_{NS}. \qquad (3.64)$$

Then we can obtain

$$
\begin{aligned}
e^{\widetilde{\psi}_r^\dagger \psi_r} e^{\gamma \widetilde{\psi}_{-r}^\dagger M \psi_{-r}^\dagger} |0\rangle &= \left( 1 + \widetilde{\psi}_r^\dagger \psi_r + \widetilde{\psi}_{1,r}^\dagger \psi_{1,r} \widetilde{\psi}_{2,r}^\dagger \psi_{2,r} \right) \left( 1 + \gamma \widetilde{\psi}_{-r}^\dagger M \psi_{-r}^\dagger + \right. \\
&\quad + \gamma^2 M_{11} M_{22} \widetilde{\psi}_{1,-r}^\dagger \psi_{1,-r}^\dagger \widetilde{\psi}_{2,-r}^\dagger \psi_{2,-r}^\dagger \\
&\quad \left. + \gamma^2 M_{12} M_{21} \widetilde{\psi}_{1,-r}^\dagger \psi_{2,-r}^\dagger \widetilde{\psi}_{2,-r}^\dagger \psi_{1,-r}^\dagger \right) |0\rangle \\
&= (1 + \gamma \mathrm{tr} M + \gamma^2 \det M) |0\rangle \qquad (3.65)
\end{aligned}
$$

where in the first line, the first two terms in each set of brackets are still in our vector notation, while the remaining terms have been broken down into their components. Since $\det M = 1$, its eigenvalues are $\zeta$ and $\zeta^{-1}$ with

$$\zeta = \frac{\pi(g + \bar{g})}{2} + i\sqrt{1 - \frac{\pi(g + \bar{g})}{2}}. \qquad (3.66)$$

Then our factor from (3.65) is

$$
\begin{aligned}
(1 + \gamma \mathrm{tr} M + \gamma^2 \det M) &= 1 + e^{-2\alpha r}(\zeta + \zeta^{-1}) + e^{-4\alpha r} \\
&= (1 + \zeta e^{-2\alpha r})(1 + \zeta^{-1} e^{-2\alpha r}). \qquad (3.67)
\end{aligned}
$$

The other pair of exponentials in (3.64) works out the same way and the partition function is

$$Z_{NS} = \frac{e^{\alpha/6}}{2} \prod_{r=\frac{1}{2}}^{\infty} \left( 1 + \zeta e^{-2\alpha r} \right)^2 \left( 1 + \zeta^{-1} e^{-2\alpha r} \right)^2 \qquad (3.68)$$

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Using the Jacobi triple product identity,

$$\sum_{k=-\infty}^{\infty} z^k q^{n^2} = \prod_{n=0}^{\infty}(1 - q^{2n+2})(1 + zq^{2n+1})(1 + z^{-1}q^{2n+1}), \tag{3.69}$$

we have

$$Z_{NS} = \tfrac{1}{2}e^{\alpha/6}\left(\sum_n \zeta^n e^{-\alpha n^2} \prod_{k=1}^{\infty} \frac{1}{1 - e^{-2\alpha k}}\right). \tag{3.70}$$

By a similar procedure in the Ramond sector we obtain

$$Z_R = \tfrac{1}{2}e^{\alpha/6}\left(\sum_n \zeta^{(n+\frac{1}{2})} e^{-\alpha(n+\frac{1}{2})^2} \prod_{k=1}^{\infty} \frac{1}{(1 - e^{-2\alpha k})}\right)^2. \tag{3.71}$$

As explained leading up to (3.32), the partition function for the bosonic system is obtained as the sum of the Ramond and Neveu-Schwarz partition functions. In this sum we face the combination

$$\left[\sum_n \zeta^n e^{-\alpha n^2}\right]^2 + \left[\sum_n \zeta^{n+\frac{1}{2}} e^{-\alpha(n+\frac{1}{2})^2}\right]^2 = \sum_{n,m} \zeta^{n-m}\left(e^{-\alpha(n^2+m^2)} + e^{-\alpha((n+\frac{1}{2})^2+(n+\frac{1}{2})^2)}\right)$$

$$= \sum_{n,m} \zeta^{n-m} e^{-\frac{1}{2}\alpha(m-n)^2}\left(e^{-\frac{1}{2}\alpha(m+n)^2} + e^{-\frac{1}{2}\alpha(m+n+1)^2}\right)$$

$$= \sum_{n,n'} \zeta^n e^{-\frac{1}{2}\alpha n^2} e^{-\frac{1}{2}\alpha n'^2}. \tag{3.72}$$

Then our partition function reads

$$
\begin{aligned}
Z &= \tfrac{1}{2}e^{\alpha/6}\left(\prod_{k=1}^{\infty}\frac{1}{1-e^{-2\alpha k}}\right)^2 \sum_{m,n} e^{-\frac{1}{2}\alpha(m-n)^2}\zeta^{m-n}\left(e^{-\frac{1}{2}\alpha(m+n)^2}+e^{-\frac{1}{2}\alpha(m+n+1)^2}\right) \\
&= \left(\frac{1}{\sqrt{2}}e^{\alpha/12}\sum_n e^{-\frac{1}{2}\alpha n^2}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-2\alpha k})}\right) \\
&\quad \times \left(\frac{1}{\sqrt{2}}e^{\alpha/12}\sum_n \zeta^n e^{-\frac{1}{2}\alpha n^2}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-2\alpha k})}\right).
\end{aligned}
\tag{3.73}
$$

The first bracketed term is $\zeta$ independent, and is exactly the bosonic partition function of the free Dirichlet boson $Y$. We may remove it to get the desired partition function for $X$ only:

$$
Z_X = \frac{1}{\sqrt{2}}e^{\alpha/12}\sum_n \zeta^n e^{-\frac{1}{2}\alpha n^2}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-2\alpha k})}.
\tag{3.74}
$$

Recalling that $\beta = 2\pi^2/\alpha$ is our physical parameter, we use Poisson resummation to get

$$
\sum_n \zeta^n e^{-\frac{1}{2}\alpha n^2} = \sum_n e^{-\frac{1}{2}\alpha n^2 + n\ln\zeta} = \sqrt{\frac{2\pi}{\alpha}}\sum_k e^{-\frac{2\pi^2}{a}(k-i\ln\zeta/2\pi)} = \sqrt{\frac{\pi}{\beta}}\sum_k e^{-\beta(k+\delta)}
\tag{3.75}
$$

where from (3.66) we know

$$
\delta = \frac{-i\ln\zeta}{2\pi} = \frac{1}{2\pi}\cos^{-1}\left(\tfrac{1}{2}\pi(g+\bar{g})\right).
\tag{3.76}
$$

We can also write

$$
e^{\alpha/12}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-2\alpha k})} = \left[\eta\left(\frac{i\alpha}{\pi}\right)\right]^{-1} = \left[\sqrt{\frac{\pi}{\alpha}}\eta\left(-\frac{\pi}{i\alpha}\right)\right]^{-1} = \sqrt{\frac{2\pi}{\beta}}e^{\beta/24}\prod_{k=1}^{\infty}\frac{1}{(1-e^{-\beta k})}
\tag{3.77}
$$

where $\eta(\tau)$ is the Dedekind eta function (for which the identity $\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau)$ holds generally).

Combining these results, we have the partition function for the compact boson subject to a periodic potential on one boundary and a Dirichlet boundary condition on the other:

$$Z_{B,D} \;=\; e^{\beta/24} \sum_n e^{-\beta(n+\delta)} \prod_{k=1}^{\infty} \frac{1}{(1 - e^{-\beta k})}. \tag{3.78}$$

## 3.9   Conclusions

The expression for the partition function (3.78) is not new, nor is the idea of attacking the problem using fermionization [13]. However, our approach differs in the way that the renormalization of the coupling constants is performed. In the reference [13], the coupling is redefined inside an exponential of the boundary condition, leading to a relationship between their couplings $g', \bar{g}'$ and the ones obtained here:

$$\sin^2 \pi \sqrt{g'\bar{g}'} = \pi^2 g\bar{g} \qquad \text{and} \qquad \frac{g'}{\bar{g}'} = \frac{g}{\bar{g}}. \tag{3.79}$$

Our calculation, which involves a more direct renormalization of the couplings, in some sense justifies their renormalization arguments and provides an alternative approach for constructing boundary states using periodic operators.

Ultimately, the realization of a system described by the boundary sine-Gordon model at the self-dual radius is physically very difficult. At the self-dual radius, other interactions besides the periodic boundary potential become marginal; in particular the *bulk* operator $e^{i\cdot X:}$ is *relevant* and destroys any hope of maintaining the critical boundary theory.

An interesting extension, discussed in [10], involves compactification at rational radii $R = n/m$. The boundary state and partition function are then obtained by fermionizing and projecting out combinations of fermion number that coincide with the boson momenta at the rational radius. Perhaps these rational CFTs might be more easily realized in real systems.

Future work using this approach should be possible, in particular it should be possible to

apply it to the dissipative Hofstadter model [5], where there are two bosonic degrees of freedom, a periodic potential for each, and a magnetic field term that couples the two directions.

# 4. Mobility at finite temperature

In this chapter we return to the condensed matter context and consider two other important results relating to the Schmid model. The first is a finite temperature analysis of the one-dimensional problem, and the second is a renormalization group analysis of the zero temperature problem on a two-dimensional lattice.

## 4.1  Non-zero temperature approach

Fisher and Zwerger [7] attacked the Schmid model at arbitrary temperature, obtaining interesting results for the temperature dependence of the mobility. Their approach was also very explicit in its development of the duality between the weak potential and tight-binding limits of the problem. In anticipation of generalizing the results to two dimensions, we present some details of their calculations.

Working at finite temperature requires the full density matrix approach to Caldeira-Leggett dissipative quantum mechanics, presented in section 1.2. We will use much of the notation from that section. We work towards an expression for the mobility as defined in (1.5). We are now working in real time $t$.

Our applied force $F$ is now a constant that we include in the potential:

$$V(q) = -V_0 \cos(2\pi q/a_0) - Fq \tag{4.1}$$

Putting this into the expression for the reduced density matrix (1.16) involves the difference in

particle actions

$$
\begin{aligned}
S_0[q] - S_0[q'] &= S_0[x + \tfrac{1}{2}y] - S_0[x - \tfrac{1}{2}y] \\
&= M \int_0^t dt'\, \dot{x}(t')\dot{y}(t') + \int_0^t dt' \int_0^t dt'\, F y(t') \\
&\quad + \int_0^t dt' \left[ \cos \tfrac{2\pi}{a_0}(x + \tfrac{1}{2}y) - \cos \tfrac{2\pi}{a_0}(x - \tfrac{1}{2}y) \right]
\end{aligned}
\tag{4.2}
$$

where as before, $x = \tfrac{1}{2}(q+q')$ and $y = q - q'$. Each cosine is then expanded in a Coulomb gas as in section 1.3. The first cosine is represented as a sum over $n$ charges $\sigma_i = \pm 1$ and the second is indexed by $n'$ charges $e_j = \pm 1$. For each $n, n'$ and times $t_i, t_j$ we have charge distributions

$$
\rho(t) = \frac{2\pi\hbar}{a_0} \sum_{i=1}^{n} \sigma_i \delta(t - t_i) \qquad \rho'(t) = \frac{2\pi\hbar}{a_0} \sum_{j=1}^{n} e_j \delta(t - t_j).
\tag{4.3}
$$

We are interested in the classical probability distribution $P(X;t)$ for the position of the particle as a function of time, which is obtained as the diagonal component of the reduced density matrix $\hat{\rho}(q, q'; t)$ described in (1.16)

$$
\begin{aligned}
P(X;t) &\equiv \hat{\rho}(X, X; t) \\
&= \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \left(\tfrac{iV}{2\hbar}\right)^n \left(\tfrac{-iV}{2\hbar}\right)^{n'} \sum_{\sigma_i, e_j = \pm 1} \int dt_1 \ldots \int dt_n \int dt'_1 \ldots \int dt'_{n'} \\
&\quad \int dx_0 \int dy_0\, \hat{\rho}(x_0 + \tfrac{1}{2}y_0, x_0 - \tfrac{1}{2}y_0; 0) \times G(\rho, \rho'; x_0, y_0)
\end{aligned}
$$

$$
\tag{4.4}
$$
$$
\tag{4.5}
$$

where the propagator corresponding to charge distributions $\rho$ and $\rho'$ is

$$
\begin{aligned}
G(\rho, \rho'; x_0, y_0) &= \int_{x_0}^{X} \mathcal{D}x \int_{y_0}^{0} \mathcal{D}y \exp\left[ \tfrac{i}{\hbar} \int_0^t dt' \left( M\dot{x}\dot{y} + \eta x\dot{y} + Fy + x(\rho - \rho') - \tfrac{1}{2}y(\rho + \rho') \right) \right. \\
&\qquad\qquad \left. - S_2[y] \right].
\end{aligned}
\tag{4.6}
$$

The path integrals may be performed to yield

$$G(\rho, \rho'; x_0, y_0) = \frac{M}{2\pi\hbar d(t)} \exp\left( \frac{i}{\hbar} MX\dot{y}_{cl}\Big|_0^t + \frac{i}{\hbar} \int_0^t dt' \left[ F - \tfrac{1}{2}(\rho - \rho') \right] y_{cl}(t') - S_2[y_{cl}] \right) \quad (4.7)$$

where

$$d(t) = \tfrac{1}{\gamma}(1 - e^{\gamma t}), \quad (4.8)$$

$$\gamma = \eta/M, \quad (4.9)$$

and the classical path $y_{cl}(t')$ solves

$$\ddot{y}_{cl} - \gamma\dot{y}_{cl} = (\rho' - \rho)/M \quad (4.10)$$

subject to $y_{cl}(0) = y_0$ and $y_{cl}(t) = 0$.

Note that if the charge distribution is not neutral, then the integral in $S_2$ generates terms linear in $t$ which effectively kill that configuration's contribution to the probability distribution. Thus we may restrict our attention to the case of "neutral" total charge distributions, which means $n + n'$ is even and $\sum \sigma^i - \sum e^j = 0$.

To go further, we take the spatial Fourier transform of $P(X)$,

$$\tilde{P}(\lambda, t) \equiv \int dX e^{i\lambda X} P(X, t), \quad (4.11)$$

and use it as a generating function for the expectation value of the position at time $t$:

$$\bar{X}(t) \equiv \langle X(t) \rangle = \int dX \, X P(X) = -i \frac{d}{d\lambda} \tilde{P}(\lambda, t)\Big|_{\lambda=0} \quad (4.12)$$

In the generating function (4.11), the integral over $X$ produces a delta function in $y_0$, and

the resulting $x_0$ integral involving the initial particle density matrix has no dependence on $\lambda$ (which comes from the initial state having zero total momentum; the details are given the appendix of [7]). The result is that the expectation value of $X(t)$ is

$$\langle X(t) \rangle = \frac{F}{\eta}t - \frac{1}{\eta}\left\langle \tfrac{1}{2}\int_0^t dt'(\rho + \rho')\right\rangle_0 \tag{4.13}$$

where the average on the right hand side is a weighted average over the set of configurations of the charges:

$$\langle A \rangle_0 \equiv \sum_{\substack{n,n'=0 \\ n+n' \text{ even}}}^{\infty} \left(\tfrac{iV}{2\hbar}\right)^n \left(\tfrac{-iV}{2\hbar}\right)^{n'} \sum_{\substack{\sigma_i,e_j=\pm 1 \\ \text{neutral}}} \int dt_1 \ldots dt'_{n'} A \exp(\Omega[y_p]). \tag{4.14}$$

The influence phase is

$$\Omega[y_p] = \tfrac{i}{\hbar}\int_0^t dt'\left[F - \tfrac{1}{2}(\rho + \rho')\right]y_p(t') - S_2[y_p]. \tag{4.15}$$

with $y_p(t')$ the particular solution to (4.10) for the given charge distributions,

$$y_p(t') = \frac{a_0}{\alpha}\left[\sum_{i=1}^{n} e_i h(t' - t_i) - \sum_{j=1}^{n'} \sigma_j h(t' - t_j)\right]. \tag{4.16}$$

The dimensionless friction $\alpha$ is defined in (1.39), and

$$h(t') = \theta(t') + \theta(-t')e^{\gamma t'} \tag{4.17}$$

is the Green function for the operator $\frac{1}{\gamma}\frac{d^2}{dt^2} - \frac{d}{dt}$.

Using (4.13) in the expression for the mobility (1.5) we get

$$\frac{\mu}{\mu_0} = 1 - \lim_{t\to\infty}\frac{1}{Ft}\left\langle \tfrac{1}{2}\int_0^t dt'(\rho + \rho')\right\rangle_0. \tag{4.18}$$

As in Schmid's original calculation, we can show a duality between this expression and an expression originating in a tight-binding approximation. The approach of Fisher and Zwerger is to reorganize and relabel the terms in (4.18) so that the smoothed-out paths become series of tight-binding hops, and the smoothness is absorbed into the form of the bath spectrum $J(\omega)$.

We begin by defining paths $q_s$ and $q'_s$ on a tight-binding lattice with lattice constant $\tilde{a}_0 = a_0/\alpha$ according to

$$q_s(t) \;=\; \tilde{a}_0 \sum_{i=1}^{n} e_i \theta(t - t_i) \;=\; \tfrac{1}{\eta} \int_0^t dt' \rho(t') \tag{4.19a}$$

$$q'_s(t) \;=\; \tilde{a}_0 \sum_{i=1}^{n} \sigma_i \theta(t - t'_i) \;=\; \tfrac{1}{\eta} \int_0^t dt' \rho'(t'). \tag{4.19b}$$

Now by creating sum and difference paths $x_s(t) = \tfrac{1}{2}(q_s + q'_s)$ and $y_s(t) = q_s - q'_s$ we can rewrite (4.13) and (4.15) as functionals of these sharp tight-binding trajectories instead of $y_p$. The result is that (4.15) becomes

$$\Omega = \tfrac{iF}{\hbar} \int_0^t dt' y_s(t') + i\Phi^\gamma[x_s, y_s] \tag{4.20}$$

where $i\Phi$ is defined in (1.17), and the superscript $\gamma$ indicates that instead of an ohmic spectrum given by (1.22), the weighted density of states is to be taken as

$$J(\omega) = \frac{\eta\omega}{1 + (\omega/\gamma)^2}. \tag{4.21}$$

The expectation value term in the expression for the mobility (4.18) can then be written

$$\lim_{t\to\infty} \frac{1}{Ft} \left\langle \tfrac{1}{2} \int_0^t dt' (\rho + \rho') \right\rangle_0 \;=\; \eta \lim_{t\to\infty} \frac{1}{Ft} \langle x_s \rangle_\gamma \tag{4.22}$$

$$= \; \frac{\mu_{tb}}{\mu_0} \tag{4.23}$$

where the average $\langle \cdot \rangle_\gamma$ indicates the use of weight (4.20) with spectral function (4.21). The mobility $\mu$ in the weak potential limit with lattice spacing $a_0$ is thus related to the mobility $\mu_{TB}$ of the same particle on a tight-binding lattice with spacing $\tilde{a}_0 = a_0/\alpha$. This disparity in lattice spacing $a_0 \leftrightarrow a_0/\alpha$ is equivalent, given the definition of $\alpha$, to $\alpha \leftrightarrow 1/\alpha$. The other parameter of the mobility is the force $F$; this is regrouped into energy change due to a single hop $\epsilon = Fq_0$. Then we may write the duality equation for the mobility

$$\frac{\mu(\alpha, \epsilon)}{\mu_0} = 1 - \frac{\mu_{TB}(1/\alpha, \epsilon/\alpha)}{\mu_0}. \tag{4.24}$$

Working in the tight-binding framework, Fisher and Zwerger calculate $\mu_{TB}$ as a function of temperature to order $V^2$, by looking at all possible "one-blip" paths. This involves summing over all paths with $n = n' = 1$, $\sigma_1 = e_1$. The four contributing paths are written

$$y_b = \xi\tilde{a}_0(\theta(t' - t_1) - \theta(t' - t_1')) \tag{4.25}$$

$$x_b = \zeta\tilde{a}_0(\theta(t' - t_1) + \zeta(t' - t_1')) \tag{4.26}$$

where $\xi$ and $\zeta$ each take on the values $\pm 1$. For these simple paths the integrals can be simplified to leave

$$\frac{\mu}{\mu_0} = 1 - \frac{2\pi V_0^2}{\epsilon\hbar} \int_0^\infty dt \ \sin(\epsilon t/\alpha\hbar) \sin[(2/\alpha)\widetilde{Q}_1(t)] \exp[-(2/\alpha)\widetilde{Q}_2(t)] \tag{4.27}$$

where

$$\widetilde{Q}_1(t) = \int_0^\infty d\omega \frac{\sin \omega t}{\omega(1 + (\omega/\gamma)^2)} \tag{4.28}$$

$$\widetilde{Q}_2(t) = \int_0^\infty d\omega \frac{1 - \cos \omega t}{\omega(1 + (\omega/\gamma)^2)} \coth(\tfrac{1}{2}\beta\hbar\omega). \tag{4.29}$$

This is enough to reproduce the zero-temperature result of Schmid. In taking the $\beta \to \infty$

limit it is necessary to replace the cut-off function $(1 + (\omega/\gamma)^2)^{-1}$ with an exponential cut-off $e^{-\gamma t}$; then the integrals can be performed to give

$$\frac{\mu(\epsilon)}{\mu_0} \;=\; 1 - \frac{\pi^2}{\alpha\Gamma(\frac{2}{\alpha})} \left(\frac{V_0}{\hbar\gamma}\right)^2 \left(\frac{\epsilon}{\alpha\hbar\gamma}\right)^{2(1/\alpha-1)} \exp(-\epsilon/\alpha\hbar\gamma), \qquad (4.30)$$

where $\epsilon = Fa_0$ is the potential energy difference between adjacent minima. The mobility then clearly has a critical dependence on $\alpha$, and it can be seen that as the applied force $F \to 0$ (the linear mobility limit), the coefficient of $V_0^2$ goes to zero as long as $\alpha < 1$. The perturbation expansion breaks down for $\alpha > 1$, but the duality argument implies an abrupt transition in the dimensionless mobility to 0 as $\alpha$ passes through 1.

The finite temperature expression is also used by Fisher and Zwerger to divine the behaviour of the mobility with temperature. In particular they note in the weak-potential limit where $\mu = mu_0$ at zero temperature, there is a drop in the mobility as the temperature rises above zero before it starts to approach its classical value of $1/\eta$.

## 4.2   Yi-Kane Generalization to 2d

Yi and Kane [15] looked at the renormalization behaviour of a generalized two-dimensional Schmid model. From the lattice vectors $\{\boldsymbol{R}\}$ they construct an arbitrary periodic potential that contains Fourier components corresponding to each reciprocal lattice vector $\boldsymbol{G}$. They eliminate the friction as a parameter in favour of the lattice spacing, and show using Schmid's renormalization approach that the perturbative stability of the mobility depends (in this parametrization) only on the lengths of the shortest lattice vector and the shortest reciprocal lattice vector. Having stated the problem in this way, they apply their simple rules to more general cases, such as the triangular and hexagonal lattice.

Beginning from (1.26), we rescale the particle coordinate and define imaginary time path $r(\tau) = q(\tau)\sqrt{\eta/2\pi} = (q/a_0)\sqrt{\alpha}$, with $\alpha$ the dimensionless friction defined in (1.39). Now the

free parameter of the system is the lattice spacing, corresponding to $\sqrt{\alpha}$. We may also go to two dimensions, in which case the particle coordinate is the vector $\boldsymbol{r}(\tau)$ and the action, adapted from (1.26), is

$$S[\boldsymbol{r}] = \tfrac{1}{2} \int d\omega |\omega| e^{|\omega|\tau_c} |\boldsymbol{r}(\omega)|^2 - \int \frac{d\tau}{\tau_c} \sum_{\boldsymbol{G}} v_{\boldsymbol{G}} e^{2\pi i \boldsymbol{G}\cdot\boldsymbol{r}(\tau)}. \tag{4.31}$$

In lieu of the mass term, we now have an exponential which enforces a short time cut-off $\tau_c$. The cosine potential has been generalized to include Fourier components $v_{\boldsymbol{G}} = v^*_{-\boldsymbol{G}}$ at all reciprocal lattice vectors $\boldsymbol{G}$ (here the reciprocal lattice is defined as all vectors $\boldsymbol{g}$ such that for any lattice vector $\boldsymbol{R}$, $\boldsymbol{g} \cdot \boldsymbol{R}$ is an integer).

The duality between weak potential and tight binding limits is established very explicitly by considering the dual action of the tight-binding model; for a momentum space path $\boldsymbol{k}(\tau)$ the tunnelling amplitudes $t_{\boldsymbol{R}}$ between sites separated by $\boldsymbol{R}$ leads to the dual action

$$S[\boldsymbol{k}] = \tfrac{1}{2} \int d\omega |\omega| e^{|\omega|\tau_c} |\boldsymbol{k}(\omega)|^2 - \int \frac{d\tau}{\tau_c} \sum_{\boldsymbol{R}} t_{\boldsymbol{R}} e^{2\pi i \boldsymbol{R}\cdot\boldsymbol{k}(\tau)}. \tag{4.32}$$

From Schmid's analysis in 1.3 we know that in the absence of any potential (all $v_{\boldsymbol{G}} = 0$) we have dimensionless mobility $\mu/\mu_0 = 1$. In the tight-binding case, in the absence of tunnelling ($t_{\boldsymbol{R}} = 0$) we have localization $\mu/\mu_0 = 0$. The stability of these couplings is obtained by a standard renormalization. For the weak potential limit, renormalization of the couplings $v_{\boldsymbol{G}}$ gives flow equations

$$dv_{\boldsymbol{G}} = (1 - |\boldsymbol{G}|^2) v_{\boldsymbol{G}} \tag{4.33}$$

and thus if all reciprocal lattice vectors are sufficiently small then the couplings $v_{\boldsymbol{G}}$ flow to zero;

i.e.

$$|\boldsymbol{G}_{min}| > 1 \implies \frac{\mu}{\mu_0} = 1 \tag{4.34}$$

Analogously, the flow equations in the tight-binding limit send all tunnelling couplings $t_{\boldsymbol{R}}$ to zero provided that the lattice spacing is sufficiently large:

$$|\boldsymbol{R}_{min}| > 1 \implies \frac{\mu}{\mu_0} = 0 \tag{4.35}$$

For a square lattice we will once again have the transition from free particle to tight binding behaviour as $\alpha$ passes through 1. This is only the case for certain geometries however, where $|\boldsymbol{R}_{min}| = |\boldsymbol{G}_{min}|^{-1}$. For a general Bravais lattice, there is some structure factor $\Sigma$ which describes the relationship between the shortest scales of the lattice and its reciprocal:

$$|\boldsymbol{R}_{min}| = \sqrt{\Sigma}|\boldsymbol{G}_{min}|^{-1}. \tag{4.36}$$

For lattice spacing $|\boldsymbol{R}_{min}| = \sqrt{\alpha}$, the reciprocal lattice spacing is $|\boldsymbol{G}_{min}| = \sqrt{\Sigma/\alpha}$. The perturbative stability of the $\mu/\mu_0 = 0$ regime is still guaranteed for $\alpha > 1$, but the $\mu/\mu_0 = 1$ limit is now stable for $\alpha < \Sigma$.

For a general Bravais lattice in two dimensions, we may take a basis $\{\boldsymbol{R}_1, \boldsymbol{R}_2\}$ such that $\boldsymbol{R}_1$ is the shortest non-trivial lattice vector and $\boldsymbol{R}_2$ is the shortest lattice vector that is not parallel to $\boldsymbol{R}_1$. The relative length of the two basis vectors is $\gamma \equiv |\boldsymbol{R}_2|/|\boldsymbol{R}_1| > 1$, and $\phi$ the angle between them. (It can be shown that in order to satisfy the other restrictions, $\gamma \cos\phi \leq \frac{1}{2}$, with equality in the case of a rhombic lattice). The reciprocal lattice has the same structure, but oriented differently and with structure factor $\Sigma = 1/\gamma^2 \sin^2\phi$. For the square lattice, this recovers $S = 1$, while for an equilateral triangular lattice we have $\Sigma_{tri} = 4/3$.

For the equilateral triangular lattice, in the region where $1 < \alpha < 4/3$, both the tight-

binding and zero-potential limits are stable. (This implies that there is then some intermediate unstable fixed point as well.)

Very interesting consequences result from Yi and Kane's extension of their argument to the case of non-Bravais lattices. They note in particular that changing the sign of the equilateral triangle potential yields its dual lattice, which is hexagonal. Since only the sign of the potential has changed, the structure of the reciprocal lattice is the same. However, the lattice constant of the triangular lattice is a factor of $\sqrt{3}$ larger than the nearest neighbour separation on the hexagonal lattice, and thus the structure factor is $\Sigma_{hex} = 4/9$. Now, for $4/9 < \alpha < 1$, *neither* of the two perturbative limits is stable. The result is that for these values of $\alpha$ there must exist stable intermediate fixed points $0 < \mu/\mu_0 < 1$.

The behaviour predicted by Yi and Kane for non-square lattices is an interesting and potentially observable phenomenon. In the next chapter we investigate the two-dimensional problem from the perspective of Fisher and Zwerger, and attempt to generalize their approach to a general two dimensional Bravais lattice.

# 5. Particle Mobility on a Bravais Lattice

Here we apply the methods of Fisher and Zwerger to the problem of the mobility of a particle moving on a two dimensional Bravais lattice subject to dissipation. The lattice is modelled using a potential that is a sum of plain waves oriented along a small number of reciprocal lattice vectors. The generalization of Fisher and Zwerger's expressions from the 1d case is straightforward, though cumbersome.

We then proceed to expose the duality between the weak potential and tight-binding mobilities, commenting on subtleties that arise in two dimensions. For the case of a triangular lattice, there is the possibility of terms arising at third order in the perturbation expansion in the potential strength. A useful parameterization and expressions for these terms is developed.

## 5.1   Reformulation in two dimensions

We will consider a general two-dimensional Bravais lattice, with an associated potential that is a sum of cosine plane waves along certain reciprocal lattice vectors. We will denote these reciprocal lattice vectors as $\boldsymbol{g}_b$, and generally use the index $b$ to sum over the corresponding potential components. Note that in the simplest rectangular lattice potential, the $\boldsymbol{g}_b$ would consist of two elements, each parallel to a rectangular axis of the lattice. For an equilateral triangular lattice, however, it is necessary to take three cosine plane waves (corresponding to the presence of six equivalent nearest neighbours in the triangle's reciprocal lattice).

We write our potential, as a function of the particle position $\boldsymbol{q}$, as

$$V(\boldsymbol{q}) \;\; = \;\; \sum_b V_b \cos\left(2\pi \boldsymbol{q} \cdot \boldsymbol{g}_b\right). \tag{5.1}$$

We anticipate that this will need to be expanded in a Coulomb gas,

$$
\exp\left(\frac{i}{\hbar} \int_0^t V[\boldsymbol{q}]\right) \;\; = \;\; \prod_b \exp\left(\frac{-iV_b}{2\hbar} \int_0^{t'} dt' (e^{2\pi i \boldsymbol{g}_b \cdot \boldsymbol{q}} + e^{-2\pi i \boldsymbol{g}_b \cdot \boldsymbol{q}})\right)
$$

$$
\;\; = \;\; \prod_b \left(\sum_{n_b=0}^{\infty} \left(\frac{-iV_b}{2\hbar}\right)^{n_b} \sum_{\sigma_i^b = \pm 1} \int_0^t dt' \ldots dt'_{n_b} \exp\left(-\frac{i}{\hbar}\int_0^t ds\; \boldsymbol{q}(s) \cdot \boldsymbol{\rho}_b(s)\right)\right), \quad \tag{5.2}
$$

and as usual we will need a corresponding construction for the forward-going potential $V[\boldsymbol{q}']$. We now have component charge densities that are vectors parallel to their associated $\boldsymbol{g}_b$,

$$
\begin{aligned}
\boldsymbol{\rho}_b(s) &\;\; = \;\; 2\pi\hbar\boldsymbol{g}_b \sum_i \sigma_i^b \delta(s - t_i^b) \\
\boldsymbol{\rho}_b'(s) &\;\; = \;\; 2\pi\hbar\boldsymbol{g}_b \sum_j e_j^b \delta(s - t_i^b),
\end{aligned}
\tag{5.3}
$$

and the total charge density is just the sum of these:

$$
\boldsymbol{\rho}(s) \;\; = \;\; \sum_b \boldsymbol{\rho}_b \qquad \boldsymbol{\rho}'(s) \;\; = \;\; \sum_b \boldsymbol{\rho}_b'. \tag{5.4}
$$

The early steps of the problem factor exactly, and the initial expressions from section 4.1 may be immediately adapted by replacing the paths $q, q', x, y$ with their vector equivalents, with dot products forming the necessary linear combinations. This is true of the particle position probability distribution (4.5), although the propagator (4.7) picks up another prefactor of $M/2\pi\hbar d(t)$. Ultimately the behaviours in the two dimensions are not independent because the two components of the charge distributions $\boldsymbol{\rho}$ are correlated (assuming a non-rectangular lattice).

We once again encounter the issue of charge neutrality; unless the distributions $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ have the same net charge the $S_2[\boldsymbol{y}]$ will kill the contribution to the probability distribution $P(\boldsymbol{X})$. As a result, the neutrality condition

$$\int_0^t dt' \ (\boldsymbol{\rho}'(t') - \boldsymbol{\rho}(t')) \ = \ 0 \tag{5.5}$$

is imposed on the charge distributions.

Without any new complications, we come to the expression for the linear mobility analogous to (4.18):

$$\frac{\mu^{ij}}{\mu_0} \ = \ \delta^{ij} - \lim_{t \to \infty} \frac{\delta}{\delta F_j} \frac{1}{t} \left\langle \tfrac{1}{2} \int_0^t dt' (\rho + \rho')^i \right\rangle_0 \bigg|_{F=0} . \tag{5.6}$$

The average $\langle \cdot \rangle_0$ involves all the $\boldsymbol{g}_b$ contributing to the potential

$$\langle A^i \rangle_0 \ = \ \prod_b \left( \sum_{n_b, n'_b} \left( \frac{iV_b}{2\hbar} \right)^{n_b} \left( \frac{-iV_b}{2\hbar} \right)^{-n'_b} \sum_{\sigma_b, e_b} \int_0^t dt_1 \dots dt'_{n_b} \right) A^i \exp(\Omega[\boldsymbol{y}_p]) \tag{5.7}$$

with the weight $\Omega$ the obvious generalization of (4.15), and the particular solution $\boldsymbol{y}_p(s)$ to the vector version of (4.10) given concisely as

$$\boldsymbol{y}_p(s) \ = \ \frac{1}{M\gamma} \int ds' \ h(s - s')(\boldsymbol{\rho}'(s') - \boldsymbol{\rho}(s')). \tag{5.8}$$

The function $h(s)$ is defined in equation (4.17).

## 5.2 Tight-binding duality

We can rewrite this as a tight-binding expression and expose the duality as in section 4.1. The only complications are in dealing with the structure factor $\Sigma$ (defined in section 4.2) for non-square lattices, and the relative orientation of the tight-binding lattice to the original lattice.

We give our original (real-space, not tight-binding) lattice a basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ whose members satisfy the relations imposed on $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ in section 4.2, namely that they open at acute angle $\phi$, $|\boldsymbol{a}_2|/|\boldsymbol{a}_1| = \gamma \geq 1$, and $\gamma \cos \phi \leq \frac{1}{2}$. The structure factor is $\Sigma = 1/\gamma^2 \sin \phi^2$.

We may then take the reciprocal lattice basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ to satisfy

$$\boldsymbol{g}_i \cdot \boldsymbol{a}_j = \epsilon_{ij} \tag{5.9}$$

where $\epsilon_{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, which implies

$$|\boldsymbol{g}_1| = \frac{1}{|\boldsymbol{a}_2| \sin \phi} \qquad\qquad |\boldsymbol{g}_2| = \frac{1}{|\boldsymbol{a}_1| \sin \phi}. \tag{5.10}$$

Just as in the one-dimensional case, the particular solution $\boldsymbol{y}_p(s)$ is the basis for our tight-binding expansion. We then take basis $\{\widetilde{\boldsymbol{a}}_1, \widetilde{\boldsymbol{a}}_2\}$ for our tight-binding lattice as

$$\widetilde{\boldsymbol{a}}_i = \frac{|\boldsymbol{a}_1 \times \boldsymbol{a}_2|}{\alpha} \boldsymbol{g}_i = \frac{2\pi\hbar}{\eta} \boldsymbol{g}_i. \tag{5.11}$$

where in two dimensions the dimensionless friction coefficient involves the area of the unit cell:

$$\alpha \equiv \frac{\eta |\boldsymbol{a}_1 \times \boldsymbol{a}_2|}{2\pi\hbar} = \frac{\eta |\boldsymbol{a}_1||\boldsymbol{a}_2| \sin \phi}{2\pi\hbar}. \tag{5.12}$$

Now we may produce tight-binding paths

$$\boldsymbol{q}_s(t') = \sum_b \widetilde{\boldsymbol{a}}_b \sum_i \sigma_i^b \theta(t - t_i^b) \qquad\qquad \boldsymbol{q}_s'(t') = \sum_b \widetilde{\boldsymbol{a}}_b \sum_i e_i^b \theta(t - t_i^b) \tag{5.13}$$

and note that this definition leads to

$$\frac{1}{\eta} \int_0^{t'} ds\, \boldsymbol{\rho}(s) = \boldsymbol{q}(t') \tag{5.14}$$

as desired. We can then rewrite (5.6) as

$$\frac{\mu^{ij}}{\mu_0} \;=\; \delta^{ij} - \frac{\mu_{TB}^{ij}}{\mu_0} \tag{5.15}$$

with

$$\mu_{TB}^{ij} \;=\; \frac{1}{\eta} \lim_{t\to\infty} \frac{\delta}{\delta F_j} \frac{1}{t} \left. \langle \boldsymbol{x}_s(t) \rangle_\gamma \right|_{F=0} \tag{5.16}$$

the mobility on the tight binding lattice defined in (5.11) and the average (5.7) is taken with respect to the revised spectrum (4.21).

Since the reciprocal lattice, and thus the tight-binding lattice, is rotated relative to the original lattice, directional information should be extracted with care.

The relation between the dimensionless friction $\alpha$ and the corresponding quantity $\widetilde{\alpha}$ of the tight binding lattice is

$$\widetilde{\alpha} \;=\; \frac{\eta |\widetilde{\boldsymbol{a}}_1 \times \widetilde{\boldsymbol{a}}_2|}{2\pi\hbar} \;=\; \frac{\sin\phi}{\alpha}. \tag{5.17}$$

## 5.3  Second order contributions

In the original one-dimensional analysis, the zero-temperature stability of the weak-potential limit was shown to second order by showing that the coefficient of $V_0^2$ in (4.27) is zero as long as $\alpha' < 1$. For $\alpha' > 1$ the perturbation expansion breaks down. (We use $\alpha'$ to refer to the dimensionless dissipation parameter in the 1d problem.)

The dependence on $\alpha'$ in the 1-d calculation comes from the integrals in the influence phase (1.16), which involves products of the paths $x$ and $y$. These paths, on the tight-binding lattice, each contribute a factor $\tilde{q}_0 = q_0/\alpha'$ which combine with the other factors to leave a residual $1/\alpha'$. In two dimensions, the paths in the order $V_b^2$ term each contribute $|\widetilde{\boldsymbol{a}}_b| = |\boldsymbol{a}_b|/\alpha$, and

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combine with the surrounding factor $\eta/2\pi\hbar$ to produce, in the case of $V_1$,

$$\frac{\eta}{2\pi\hbar}\left(\frac{|\boldsymbol{a}_1|}{\alpha}\right)^2 = \frac{1}{\alpha}\left(\frac{|\boldsymbol{a}_1|}{|\boldsymbol{a}_2|\sin\phi}\right) = \frac{\sqrt{\Sigma}}{\alpha} \tag{5.18}$$

From this we conclude that at zero-temperature, $V_1 = 0$ for $\alpha < \sqrt{\Sigma}$. For the case of $V_2$, the exponential factors combine to give

$$\frac{1}{\alpha\sin^2\phi\sqrt{\Sigma}} \tag{5.19}$$

and thus at $T = 0$, $V_2 = 0$ for $\alpha < \sin^2\phi\sqrt{\Sigma}$.

When $|\boldsymbol{a}_1| \neq |\boldsymbol{a}_2|$, the limits on $\alpha$ are different, and this suggests that the behaviour renormalizes differently along the two directions, or that it is simply determined by the smaller bound. When $|\boldsymbol{a}_1| = |\boldsymbol{a}_2|$, such as for the equilateral triangular lattice, $\sqrt{\Sigma} = 1/\sin\phi$ and these limits on $\alpha$ are the same. (In the triangular case there would also be a $V_3$ term which would have the same features).

Upon accounting for the different definition of $\alpha$ used in this section and 4.2, we are not surprised to find that the limit $\alpha < 1/\sin\phi$ agrees with Yi and Kane's renormalization group argument for the stability of the $\mu/\mu_0 = 1$ fixed point in the equilateral triangular lattice.

## 5.4  Third order contributions

A potentially interesting aspect of the triangular lattice is the possibility for paths of odd order. For a reasonably constructed triangular lattice, the potential will consist of 3 terms where any two of the associated reciprocal lattice vectors $\boldsymbol{g}_b$ form a basis for the reciprocal lattice, and the third can be formed from some combination $\boldsymbol{g}_3 = \pm\boldsymbol{g}_1 \pm \boldsymbol{g}_2$. Then there are many paths of length 2 and length 1 that end at the same point. Here we will work out some of the details of the third order contributions.

Our potential in this case consists of components in three directions, $\boldsymbol{g}_i$ for $i = 1, 2, 3$, with

$\boldsymbol{g}_1 = \boldsymbol{g}_2 + \boldsymbol{g}_3$. One contribution to the average position at time $t$ results from the tight-binding paths

$$\boldsymbol{q}_s(t') = \boldsymbol{a}_1\theta(t' - t_1) \tag{5.20}$$

$$\boldsymbol{q}'_s(t') = \boldsymbol{a}_2\theta(t' - t_2) + \boldsymbol{a}_3\theta(t' - t_3). \tag{5.21}$$

This contribution is weighted by the potential strengths

$$\left(\frac{iV_1}{2}\right)\left(\frac{-iV_2}{2}\right)\left(\frac{-iV_3}{2}\right) = \frac{-iV_1V_2V_3}{8}. \tag{5.22}$$

The integrals in the influence phase contribute terms depending on the relative positions of the times $t_i$. Assuming that $t_1 < t_2 < t_3$ we have corresponding centre and difference vectors $\boldsymbol{x}_s(t')$ and $\boldsymbol{y}_s(t')$ satisfying (here we suppress the subscript $s$)

$$\boldsymbol{x}(t') = \begin{cases} 0 & , & t' < t_1 \\ \frac{1}{2}\boldsymbol{a}_1 & , & t_1 < t' < t_2 \\ \frac{1}{2}(\boldsymbol{a}_1 + \boldsymbol{a}_2) & , & t_2 < t' < t_3 \\ \boldsymbol{a}_1 & , & t_3 < t' \end{cases} \tag{5.23a}$$

$$\boldsymbol{y}(t') = \begin{cases} \boldsymbol{a}_1 & , & t_1 < t' < t_2 \\ \boldsymbol{a}_1 - \boldsymbol{a}_2 & , & t_2 < t' < t_3 \\ 0 & , & \text{otherwise} \end{cases} \tag{5.23b}$$

Defining $\tau_1 = t_2 - t_1$, $\tau_2 = t_3 - t_2$ our weighting factor for this path is $e^\Omega$ with (here we combine equations (4.20), (1.17) and (1.19) and extend them to two dimensions)

$$\Omega = i\Omega_1 + i\Omega_2 + \Omega_3 \tag{5.24}$$

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with

$$i\Omega_1 = \frac{i}{\hbar} \int_0^t dt' \, \boldsymbol{F} \cdot \boldsymbol{y}_s(t') \tag{5.25}$$

$$i\Omega_2 = -\frac{2i}{\hbar} \int_0^t dt' \int_{t'}^t ds \, \boldsymbol{y}(s) \cdot \boldsymbol{x}(t')\alpha_I^\gamma(s - t') \tag{5.26}$$

$$\Omega_3 = -\frac{1}{\hbar} \int_0^t dt' \int_0^{t'} ds \, \boldsymbol{y}(s) \cdot \boldsymbol{y}(t')\alpha_R^\gamma(s - t'). \tag{5.27}$$

Working these out one by one for the particular paths 5.23, for all $t_1 < t_2 < t_3 < t$ we have

$$\begin{aligned}
i\Omega_1 &= \frac{i}{\hbar} \left( \int_0^{\tau_1} dt' \boldsymbol{F} \cdot \boldsymbol{a}_1 + \int_0^{\tau_2} dt' \boldsymbol{F} \cdot (\boldsymbol{a}_1 + \boldsymbol{a}_2) \right) \\
&= \frac{i}{\hbar} \boldsymbol{F} \cdot (\boldsymbol{a}_1 \tau_1 + (\boldsymbol{a}_1 + \boldsymbol{a}_2)\tau_2);
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
i\Omega_2 &= -\frac{i}{\hbar} \left[ \int_{t_1}^{t_2} dt' \int_{t'}^{t_2} ds + \int_{t_1}^{t_2} dt' \int_{t_2}^{t_3} ds + \int_{t_2}^{t_3} dt' \int_{t'}^{t_3} ds \right] \boldsymbol{y}(s) \cdot \boldsymbol{x}(t')\alpha_I^\gamma(s - t') \\
&= -\frac{i}{\hbar} \left( (\boldsymbol{a}_1 \cdot \boldsymbol{a}_1) \int_0^{\tau_1} dt' \int_{t'}^{\tau_1} ds \, \alpha_I(s - t') + \boldsymbol{a}_1 \cdot (\boldsymbol{a}_1 - \boldsymbol{a}_2) \int_0^{\tau_1} dt' \int_0^{\tau_2} ds \, \alpha_I(s - t' - \tau_2) \right. \\
&\qquad \left. + (\boldsymbol{a}_1 + \boldsymbol{a}_2) \cdot (\boldsymbol{a}_1 - \boldsymbol{a}_2) \int_0^{\tau_2} dt' \int_{t'}^{\tau_2} ds \, \alpha_I(s - t') \right) \\
&= -\frac{i}{\hbar} \left( (\boldsymbol{a}_1 \cdot \boldsymbol{a}_1)A_0(\tau_1) + \boldsymbol{a}_1 \cdot (\boldsymbol{a}_1 - \boldsymbol{a}_2)A_1(\tau_1; \tau_2) + (\boldsymbol{a}_1 + \boldsymbol{a}_2) \cdot (\boldsymbol{a}_1 - \boldsymbol{a}_2)A_0(\tau_2) \right)
\end{aligned}$$

$$\tag{5.29}$$

where we have defined the required double integrals of $\alpha_I^\gamma$ as

$$A_0(\tau) = \int_0^\tau dt' \int_{t'}^\tau ds \, \alpha_I^\gamma(s - t') \tag{5.30}$$

$$A_1(\tau_1; \tau_2) = \int_0^{\tau_1} dt' \int_0^{\tau_2} ds \, \alpha_I^\gamma(s - t' - \tau_1). \tag{5.31}$$

Similarly, we have

$$
\begin{aligned}
\Omega_3 &= -\frac{1}{\hbar} \int_0^t dt' \int_0^{t'} ds\; \boldsymbol{y}(s) \cdot \boldsymbol{y}(t') \alpha_R^\gamma(s-t') \\
&= -\frac{1}{\hbar} \left( (\boldsymbol{a}_1)^2 B_0(\tau_1) + \boldsymbol{a}_1 \cdot (\boldsymbol{a}_1 - \boldsymbol{a}_2) B_1(\tau_1, \tau_2) \right. \\
&\qquad \left. + (\boldsymbol{a}_1 - \boldsymbol{a}_2)^2 B_0(\tau_2) \right).
\end{aligned}
\tag{5.32}
$$

where

$$
B_0(\tau) = \int_0^\tau dt' \int_{t'}^\tau ds\; \alpha_R^\gamma(s-t')
\tag{5.33}
$$

$$
B_1(\tau_1; \tau_2) = \int_0^{\tau_1} dt' \int_0^{\tau_2} ds\; \alpha_R^\gamma(s-t'-\tau_1).
\tag{5.34}
$$

These expressions for $\Omega_i$ represent only one of 72 similar paths contributing at order $V^3$. The others are obtained by the following 4 independent operations:

1. **Re-ordering the times**; $t_1 < t_2 < t_3$ is only one of 6 possibilities. Reordering of the times makes a significant difference to the phases $\Omega_2$ and $\Omega_3$, particularly in the cross terms associated with $A_1$ and $B_1$.

2. **Taking an alternative single vector** into $\boldsymbol{\rho}$; we chose $\boldsymbol{a}_1$ but there are 3 distinct options. This has no effect on factors $\Omega_2$ and $\Omega_3$ if the lattice is equilateral, since $\boldsymbol{x}$ and $\boldsymbol{y}$ are rotated in the same way and their dot products feel nothing. The force term $\Omega_1$ does change, as does the final coordinate $\boldsymbol{x}(t) = \boldsymbol{a}_i$ that $e^\Omega$ is weighting.

3. **Changing the overall parity**; we get a new configuration by taking the negative of all three vectors. There are 2 ways to do this; doing it flips the sign of $\Omega_1$ as well as the sign of $\boldsymbol{x}(t)$ that $e^\Omega$ is weighting.

4. **Interchanging the forward and backward paths $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$**; this removes the asymmetry in putting only one vector in $\boldsymbol{\rho}$ and two in $\boldsymbol{\rho}'$. There are 2 choices for this assignment,

which changes the sign of $\Omega_1$ and $\Omega_2$. It also changes the sign of the contribution in the average, since $(iV_1)(-iV_2)(-iV_3) = -i\prod_b V_b \to (iV_2)(iV_3)(-iV_1) = i\prod_b V_b$.

Using index $j = 1, \ldots, 6$ for the timing arrangements and $k = 1, 2, 3$ for the single vector choice $\boldsymbol{\rho}(t') = \boldsymbol{a}_k \theta(t' - t_1)$, we then have weights $\Omega_i^{jk}(\tau_1, \tau_2)$ associated with these six possibilities. Letting $\xi = \pm 1$ represent the parity and $\zeta = \pm 1$ the forward or backward path assignment, we can write the order $V^3$ contribution to the expectation value of $\boldsymbol{x}_s(t)$:

$$
\langle \boldsymbol{x}_s(t) \rangle^{(3)} = \frac{-iV_1V_2V_3}{8\hbar^3} \sum_{j,k} \sum_{\substack{\xi=\pm 1 \\ \zeta=\pm 1}} \xi\zeta \, \boldsymbol{a}_k \int_0^t \!\! dt_1 \int_0^t \!\! d\tau_1 \int_{\tau_1}^t \!\! d\tau_2 \, \exp(i\xi\zeta\Omega_1^{jk} + i\zeta\Omega_2^{jk} + \Omega_3^{jk}) \tag{5.35}
$$

$$
= \frac{-V_1V_2V_3}{2\hbar^3} \sum_{j,k} \boldsymbol{a}_k \int_0^t \!\! dt_1 \int_0^t \!\! d\tau_1 \int_{\tau_1}^t \!\! d\tau_2 \, \sin(\Omega_1^{jk})\cos(\Omega_2^{jk})\exp(\Omega_3^{jk}) \tag{5.36}
$$

$$
\tag{5.37}
$$

Having parametrized the terms and reduced the problem to a sum over 18 similar pieces, the integrals remain difficult to simplify because of the cross-coupling of the integration times $\tau_1$ and $\tau_2$ in the $A_1$ and $B_1$ terms. It may be interesting to pursue this calculation further, to verify that the critical dependence on $\alpha$ is not altered by these third order terms.

## 5.5   Closing remarks

In chapters 4 and 5 we have presented the core results of Fisher and Zwerger's analysis of a particle in a periodic potential subject to dissipation, and extended the analysis into two dimensions. This has included adapting the second order perturbative expressions for the mobility at finite temperature to the case of an arbitrary Bravais lattice, as well as making investigations into the form of the third order contributions (which appear in the case of a triangular lattice).

The generalization is consistent with the renormalization group arguments of Yi and Kane for the zero temperature problem, but is formulated at finite-temperature and permits perturbative calculation of the non-linear mobility.

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