A NEW VARIATIONAL PRINCIPLE IN FLUID MECHANICS

by

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ABSTRACT

A new variational principle in fluid mechanics is presented, based on a generalized version of the conservation of particle label constraint. The variational principle represents an extension of the work of Clebsch (1859) and C.C. Lin (1959) and for the one-component case it describes a perfect fluid with a finite density of vortices; for the two-component fluid it yields the Khalatnikov equations for rapidly rotating superfluid ^{4}He. In the latter case two particle label constraints are needed, which express the possibility of labelling both an element of normal fluid and a superfluid vortex, averaged over many vortices. In addition a negative result for a variational formulation of viscous fluids based on a generalized particle label constraint is given.

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INTRODUCTION

The modern form of variational calculus is attributed to Euler (1701 - 1783) and Lagrange (1736-1783) although the very first results date back to Hero of Alexandria circa 140 A.D. One of the problems of variational calculus is to find a functional $L[\psi(x,t),x,t]$, called the Lagrangian density, such that the Euler-Lagrange equations $\delta L/\delta\psi = 0$ are equivalent to the equations of motion. It is straightforward to prove that the vanishing of the variational derivative of L is equivalent to the requirement that the action $A \equiv \iint L\, dxdt$ be an extremum for all variations of $\psi(x,t)$ with fixed boundary values. The basic mathematical framework was completed by Noether (1918) who showed explicitly how symmetry transformations of the Euler-Lagrange equations are connected with conservation laws.

The chief difficulty in variational calculus is that a Lagrangian may not exist when the equations of motion are expressed in terms of a given set of variables; in such a case the equations of motion must be rewritten in a transformed set of variables. For the electromagnetic field the decomposition of $\vec{E}$ and $\vec{B}$ in terms of the potentials $\vec{A}$ and ϕ is well known; however in general there is no way of knowing in advance which transformation, if any, will bring the equations of motion into the form of the Euler-Lagrange equations.

The advantages in constructing a Lagrangian formalism are three-fold: (1) Noether's theorem provides a convenient connection between symmetries and conservation laws; (2) symmetry arguments applied to the Lagrangian give a systematic way of extending the equations of motion; and (3) the variational equations may be easier to solve directly than the usual form of the equations of motion. In fluid mechanics (3) is especially true. For instance the variational equation associated with the mass density ρ is the most general form of the Bernoulli equation, which can be used to discuss properties of fluid flow without finding exact solutions. In summary, a variational formulation of hydrodynamic systems is extremely useful.

More than one hundred years elapsed between the simultaneous development of variational calculus and fluid mechanics and the discovery by Clebsch (1859) of a Lagrangian for isentropic, incompressible fluids. Clebsch's representation of the velocity field in terms of the Monge potentials, introduced by Monge (1787), succeeded in overcoming the two difficulties in formulating a variational principle for fluids, namely the occurrence in the equations of motion of non-linearities and first order derivatives of the velocity field. Because of the symmetry properties of the variational derivative, it is impossible to obtain odd order derivatives of the velocity field $\vec{V}(x,t)$ as a result of variations with respect to $\vec{V}(x,t)$. Clebsch's incorporation of both non-linear and first order derivative terms in a variational principle is absolutely unique in classical field theory.

Bateman (1929) and Lamb (1932) extended the Lagrangian to include compressible, isentropic flows. The adiabatic case was solved by C.C. Lin (1959) who recognized that the conservation of particle label constraint (Lin's constraint), an expression of the possibility of labelling an element of fluid, must be explicitly incorporated in the variational principle. The physical consequence of including Lin's constraint is the appearance of non-zero vorticity in the absence of entropy gradients.

Because of the difficulties with first order derivatives and non-linearities mentioned previously, extensions of Clebsch's variational principle are extremely difficult to find. The purpose of this thesis is to present such an extension based on a generalized version of Lin's constraint. The physical interpretation of the resulting theory is that of a fluid with a large number of vortices present, where all the hydrodynamic variables have been averaged over regions containing many vortices. In its two-fluid version the variational principle yields the Khalatnikov equations for rapidly rotating superfluid ^{4}He.

After reviewing the adiabatic Lagrangian in Chapter 1, the consequences of relaxing Lin's constraint for a classical one-component fluid without changing the conservation of mass equation are examined in Chapter 2. It is found that such theories represent a macroscopic (compared to the mean vortex separation) description of a fluid with a large number of vortices present.

As necessary background material Chapter 3 reviews Herivel's variational principle for the Landau two-fluid equations. Chapter 4 presents a new variational principle for the Khalatnikov equations of rapidly rotating superfluid ^{4}He. It is found necessary to use two constraint equations, the usual Lin's constraint associated with the normal velocity field and the other constraint expressing the possibility of labelling a superfluid vortex, averaged over many vortices. Chapter 5 concludes with an extension of the variational principle to higher order derivatives and with a negative result for viscous fluids, namely that a generalized Lin's constraint by itself is not sufficient to generate the additional viscous terms which occur in the Navier-Stokes momentum equations.

To summarize, the new variational principles presented in this thesis are given by Eqs. (2-6), (2-40) and (5-1) which describe one-component fluids with a density of vortices, and by Eq. (4-7) which yields the Khalatnikov equations for rotating superfluid ^{4}He.

CHAPTER 1

A REVIEW OF THE VARIATIONAL PRINCIPLE FOR A PERFECT FLUID IN ADIABATIC FLOW

1-1 An Historical Outline

The equations of motion of a perfect fluid were first developed by Euler[1] (1751) and Lagrange[2] (1781). More than one hundred years later Clebsch[3] (1859), using a representation for the velocity field introduced by Monge[4] (1787), succeeded in finding a Lagrangian for the incompressible, isentropic (constant entropy) flow of a perfect fluid. Clebsch proved that the isentropic, incompressible equations

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}P \quad , \quad \vec{\nabla}\cdot\vec{V} = 0 \tag{1-1}$$

where $\vec{V}$ is the fluid velocity, $d/dt = \partial/\partial t + \vec{V}\cdot\vec{\nabla}$, $P(x,t)$ is the pressure and ρ is the mass density, can be solved in terms of three scalar functions $\phi(x,t)$, $m(\mathbf{x},t)$ and $\psi(x,t)$ (the Monge potentials) which satisfy

$$\vec{V} = \vec{\nabla}\phi + m\vec{\nabla}\psi \ , \ \frac{P}{\rho} + \frac{\partial\phi}{\partial t} + m\,\frac{\partial\psi}{\partial t} + \tfrac{1}{2}\,V^2 = 0 \ , \ \frac{dm}{dt} = \frac{d\psi}{dt} = 0 \tag{1-2}$$

Furthermore Clebsch showed that the equations $\vec{\nabla}\cdot\vec{V} = 0$ and $dm/dt = d\psi/dt = 0$ are the variational equations of the Lagrangian density

$$L = \frac{\partial\phi}{\partial t} + m\,\frac{\partial\psi}{\partial t} + \tfrac{1}{2}\,(\vec{\nabla}\phi + m\vec{\nabla}\psi)^2 \tag{1-3}$$

which arise from variations in ϕ, ψ and m respectively. This result was extended by Bateman[5] (1929) and Lamb[6] (1932) to compressible, isentropic flows. Taub[7] (1949) and Herivel[8] (1955) attempted with only partial success to generalize the variational principle to the adiabatic case $ds/dt = 0$, where s is the entropy density.

It was C.C. Lin[9] (1959) who pointed out that Herivel's variational principle yielded only a subset of the solutions of the Euler equations, those for which $\vec{\nabla}\times\vec{V} = 0$ when s = constant, and who supplied the necessary additional constraint. Lin observed that even if the Lagrangian coordinates $\vec{z}(x,t)$ do not appear in the Euler equations, only those velocity fields for which the Lagrangian coordinates could be found should be used in the variational principle. Lin incorporated this constraint into the variational principle in the form of the conservation of identity of particles equation $d\vec{z}(x,t)/dt = 0$, where $\vec{z}(x,t)$ is the initial position of a fluid particle located at $\vec{x}$ at time t. By using Weber's transformation[10] it follows that Herivel's variational principle supplemented with Lin's constraint for the identity of particles includes all solutions of the Euler equations.

Following Lin's crucial step a number of papers appeared extending the variational principle. These include Serrin[10] (1959) and Eckart[11] (1960) on adiabatic flow, a special relativistic formulation of adiabatic flow by Penfield[12] (1966), Seliger and Whitham[13] (1967) on variational principles in continuum mechanics, general relativistic treatments of adiabatic flow by Schutz[14] (1970) and Schutz and Sorkin[15] (1977) and a variety of rigorous mathematical results by Rund[16] (1976). Other generalizations include variational principles for magnetohydrodynamics by Calkin[17] (1961), Katz[18] (1961) and Penfield and Haus[19] (1966) and for a number of two-fluid systems (see Chapter 3 for details and references). Worth noting are several negative results for variational principles yielding the Navier-Stokes equations : Millikan[20] (1929), Bateman[21] (1931), Gerber[22] (1950) and Bailyn[23] (1980).

1-2 The Equations of Motion

The equations of motion of a perfect fluid in adiabatic flow are well known[10] and are given in terms of the Eulerian variables $(\vec{x},t)$ (the x^i are just the spatial coordinates, t is the time) by

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \qquad (1\text{-}4)$$

$$\frac{ds}{dt} = 0 \qquad (1\text{-}5)$$

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\vec{\nabla}P \qquad (1\text{-}6)$$

which represent the conservation of mass, entropy and momentum of the fluid respectively. The variables $\vec{V}$, ρ and s and the material derivative d/dt have been defined previously while the pressure $P(\rho,s)$ and the temperature $T(\rho,s)$ are defined in terms of the internal energy density $e(\rho,s)$ by the Gibbs relation

$$de = Tds + (P/\rho^2)d\rho \qquad (1\text{-}7)$$

In the Eulerian variables $(\vec{x},t)$ the velocity $\vec{V}(x,t)$ is simply regarded as a vector field which obeys Eqs. (1-4)-(1-6). In the Lagrangian variables $(\vec{z},t)$ the fluid flow is described in terms of particle paths $\vec{x} = \vec{x}(z,t)$. If $\vec{z}$ is fixed while t varies then $\vec{x}(z,t)$ maps out the path of a fluid particle initially at $\vec{z}$. For fixed t, $\vec{x}(z,t)$ gives a mapping of the region initially occupied by the fluid into its position at time t.

Assuming that initially distinct points remain distinct implies that $\vec{x}(z,t)$ possesses an inverse $\vec{z} = \vec{z}(x,t)$ which is the initial position of a fluid particle with position $\vec{x}$ at time t ($\vec{x}$ and $\vec{z}$ denote the values of the functions $\vec{x}(z,t)$ and $\vec{z}(x,t)$ respectively). This implies that $\vec{x} \equiv \vec{x}(z(x,t),t)$ and $\vec{z} \equiv \vec{z}(x(z,t),t)$ and hence use of the chain rule yields the identities

$$\frac{\partial x^i}{\partial z^j}\frac{\partial z^j}{\partial x^k} = \frac{\partial x^j}{\partial z^i}\frac{\partial z^k}{\partial x^j} = \delta^{ik} \tag{1-8}$$

where $\vec{x}(z,t)$ and $\vec{z}(x,t)$ are assumed to possess continuous derivatives up to third order in all derivatives; i, j,... = 1,2,3; repeated indices are summed and unnecessary indices are omitted.

In the Lagrangian picture of fluid flow the velocity of a fluid particle $\vec{V}(z,t)$ is defined as

$$V^i = \frac{\partial x^i(z,t)}{\partial t} = \frac{dx^i}{dt} \tag{1-9}$$

Multiplying Eq. (1-9) by $\partial z^j/\partial x^i$ and summing i yields the equivalent form

$$\frac{dz^j(x,t)}{dt} = 0 \tag{1-10}$$

which just states that the identity of the fluid particles is conserved during the motion. Note that Eq. (1-9) and use of the chain rule imply

$$\frac{df(x,t)}{dt} = \frac{\partial f(x(z,t),t)}{\partial t} \tag{1-11}$$

and hence d/dt and $\partial/\partial\vec{z}$ commute. Eqs. (1-4)-(1-6) and the Lin's constraint Eq. (1-10) will henceforth be referred to as the hydrodynamic equations which will be shown to be equivalent to a variational principle in the following section.

1-3 The Lagrangian

The following discussion is due to Serrin[10] in the Handbuch article. The Lagrangian density for the hydrodynamic equations is given by

$$L = \tfrac{1}{2}\,\rho V^2 - \rho e(\rho,s) - \alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\} + \rho\beta\,\frac{ds}{dt} + \rho\gamma^j\,\frac{dz^j}{dt} \tag{1-12}$$

where the dependent variables are $\rho, s, \vec{V}, \vec{z}$ and the Monge potentials $\alpha, \beta, \vec{\gamma}$ while the independent variables are $(\vec{x}, t)$. The variational equations are obtained by setting the variational derivatives $\delta L/\delta\psi^\alpha = \partial L/\partial\psi^\alpha - \vec{\nabla}\cdot(\partial L/\partial(\vec{\nabla}\psi^\alpha)) - \partial(\partial L/\partial(\partial\psi^\alpha/\partial t))/\partial t = 0$ where the ψ^α are the dependent variables; for a review of variational principles in mathematical physics see Hill[24] (1951). Variations of the Monge potentials $\alpha, \beta, \vec{\gamma}$ just give Eqs. (1-4),(1-5) and (1-10) respectively while the other variations give

$$\delta\vec{V}: \quad \vec{V} = -\,\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j \tag{1-13}$$

$$\delta\rho: \quad \frac{d\alpha}{dt} + \tfrac{1}{2}\,V^2 - e - \frac{P}{\rho} = 0 \tag{1-14}$$

$$\delta s: \quad \frac{d\beta}{dt} = -\,T \tag{1-15}$$

$$\delta\vec{z}: \quad \frac{d\vec{\gamma}}{dt} = 0 \tag{1-16}$$

Eqs. (1-4),(1-5),(1-10),(1-13)-(1-16) will be collectively referred to as the variational equations of L.

When use is made of Clebsch's lemma

$$\frac{d\vec{V}}{dt} = - \vec{\nabla}(\frac{d\alpha}{dt} + \tfrac{1}{2} V^2) - \frac{d\beta}{dt} \vec{\nabla}s - \beta\vec{\nabla} \frac{ds}{dt} - \frac{d\gamma^j}{dt} \vec{\nabla}z^j - \gamma^j\vec{\nabla} \frac{dz^j}{dt} \qquad (1\text{-}17)$$

which follows as an identity from the definitions $d/dt = \partial/\partial t + \vec{V}\cdot\vec{\nabla}$ and $\vec{V} = - \vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j$ (see Appendix A for a proof) then substitution of Eqs. (1-14)-(1-16),(1-5),(1-10) into Eq. (1-17) gives

$$\frac{d\vec{V}}{dt} = - \vec{\nabla}(e + \frac{P}{\rho}) + T\vec{\nabla}s = - \frac{1}{\rho}\nabla P \qquad (1\text{-}18)$$

which is just Eq. (1-6). Hence all solutions of the variational equations are also solutions of the hydrodynamic equations.

The converse statement can be proven using Weber's transformation. Eq. (1-9) or equivalently Eq. (1-10) implies the identity

$$\frac{d}{dt}(V^j \frac{\partial x^j}{\partial z^i}) = \frac{dV^j}{dt}\frac{\partial x^j}{\partial z^i} + V^j \frac{d}{dt}\frac{\partial x^j}{\partial z^i} = \frac{dV^j}{dt}\frac{\partial x^j}{\partial z^i} + \frac{\partial}{\partial z^i}(\tfrac{1}{2} V^2) \qquad (1\text{-}19)$$

Substituting Eq. (1-6) into Eq. (1-19) gives

$$\frac{d}{dt}(V^j \frac{\partial x^j}{\partial z^i}) = \frac{\partial}{\partial z^i}(\tfrac{1}{2} V^2 - e - \frac{P}{\rho}) + T\frac{\partial s}{\partial z^i} \qquad (1\text{-}20)$$

By defining $\alpha = \int_0^t [e + \frac{P}{\rho} - \tfrac{1}{2} V^2]dt$ and $\beta = \int_0^t Tdt$ (the integration is carrried out by constant $\vec{z}$) then it immediately follows that Eq.(1-20) can be written as

$$\frac{d}{dt}(\frac{\partial x^j}{\partial z^i}[V^j + \nabla^j\alpha + \beta\nabla^j s]) = 0 \qquad (1\text{-}21)$$

Since $\alpha(z,o) = 0$, $\beta(z,o) = 0$ and $\vec{x}(z,o) = \vec{z}$ Eq. (1-21) can be intergrated as

$$\vec{V} = -\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j \tag{1-22}$$

where $\gamma^j = -V^j(z,o)$. Eq. (1-22) is just Eq. (1-13) and it is easy to verify that α,β and $\vec{\gamma}$ as defined above satisfy Eqs. (1-14)-(1-16) respectively. Hence the variational equations of L are equivalent to the hydrodynamic equations.

1-4 Symmetry Transformations and Conservation Laws

Before considering the transformation properties of the specific Lagrangian given by Eq. (1-12) a more general treatment is needed. The following discussion can be found in greater detail in Hill[24]. If the variational equations of a Lagrangian $L[\psi] \equiv L(\psi,\nabla\psi,\partial\psi/\partial t,x,t)$ maintain the same functional form under the infinitesimal transformations $L[\psi] \to L'[\psi'] = L[\psi] + \delta L[\psi]$, $\psi^\alpha \to \psi'^\alpha = \psi^\alpha + \delta\psi^\alpha$, $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{x}$, $t \to t' = t + \delta t$ and $L'[\psi']d^3x'dt' \equiv L[\psi]d^3xdt$ (the latter condition just maintains the numerical invariance of the action) then they are said to be form invariant. This implies

$$\frac{\delta L'[\psi']}{\delta\psi'^\alpha} = \frac{\delta L[\psi']}{\delta\psi'^\alpha} \tag{1-23}$$

Hence if ψ^α is a solution of the equations of motion then so is ψ'^α and the transformation is said to be a symmetry transformation. A necessary and sufficient condition that Eq. (1-23) hold for arbitrary $\psi^\alpha(x,t)$ or equivalently for ψ^α and their derivatives considered as independent variables (not just for those ψ^α which satisfy the equations of motion) is that the old and new Lagrangians be related by a total divergence

$$L'[\psi'] = L[\psi'] + \vec{\nabla}\cdot\delta\vec{\Omega} + \partial\,\delta\Omega_o/\partial t \tag{1-24}$$

Eq. (1-24) just says that L is invariant under the transformation, if $\delta\vec{\Omega} = \delta\Omega_o = 0$ then L is said to be form invariant.

If Eq. (1-24) holds for ψ^α and their derivatives considered as independent variables (hence the equations of motion may not be used in verifying Eq. (1-24)) then Noether's theorem gives a conservation law in the form

$$\frac{\partial\sigma}{\partial t} + \vec{\nabla}\cdot\vec{s} = 0 \tag{1-25}$$

where the infinitesimal forms of σ and $\vec{s}$ are given by

$$\delta\sigma = \Big(L - \frac{\partial L}{\partial(\frac{\partial\psi^\alpha}{\partial t})}\frac{\partial\psi^\alpha}{\partial t}\Big)\delta t - \frac{\partial L}{\partial(\frac{\partial\psi^\alpha}{\partial t})}\,\delta\vec{x}\cdot\vec{\nabla}\psi^\alpha + \frac{\partial L}{\partial(\frac{\partial\psi^\alpha}{\partial t})}\,\delta\psi^\alpha + \delta\Omega_o \tag{1-26}$$

$$\delta\vec{s} = -\frac{\partial L}{\partial\vec{\nabla}\psi^\alpha}\frac{\partial\psi^\alpha}{\partial t}\,\delta t + \Big(L\delta\vec{x} - \frac{\partial L}{\partial\vec{\nabla}\psi^\alpha}\,\delta\vec{x}\cdot\vec{\nabla}\psi^\alpha\Big) + \frac{\partial L}{\partial\vec{\nabla}\psi^\alpha}\,\delta\psi^\alpha + \delta\vec{\Omega} \tag{1-27}$$

Hence to test for a possible symmetry transformation which generates a conservation law via Noether's theorem either Eqs. (1-23) or (1-24) may be used, with ψ^α and their derivatives considered as independent variables.

1-5 The Infinitesimal "Gauge" Transformations

The discussion in Sec. 1-3 shows that the essential step in finding a variational principle for the hydrodynamic equations is the representation of the velocity field given by Eq. (1-13). However a definite value of $\vec{V}$ does not uniquely determine the values of the variables which appear on the R.H.S. of Eq. (1-13). In fact $\alpha,\beta,\vec{\gamma}$ and $\vec{z}$ may be subjected to "gauge" transformations which do not change the value of $\vec{V}$ and which keep L invariant and hence the variational equations of L form invariant (transformations of s are not allowed since this destroys the form invariance of Eqs. (1-14) and (1-15)). The form invariance of Eq. (1-13) implies that the infinitesimal gauge transformations $\alpha \to \alpha' = \alpha + \delta\alpha$, $\beta \to \beta' = \beta + \delta\beta$, $\vec{\gamma} \to \vec{\gamma}' = \vec{\gamma} + \delta\vec{\gamma}$, $\vec{z} \to \vec{z}' = \vec{z} + \delta\vec{z}$ and $\vec{V} \to \vec{V}' = \vec{V}$ satisfy

$$\vec{V}' = -\vec{\nabla}\alpha' - \beta'\vec{\nabla}s' - \gamma'^j\vec{\nabla}z'^j = \vec{V} = -\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j \tag{1-28}$$

or equivalently

$$\vec{\nabla}(\delta\alpha + \gamma^j\delta z^j) + \delta\beta\vec{\nabla}s + \delta\gamma^j\vec{\nabla}z^j - \delta z^j\vec{\nabla}\gamma^j = 0 \ . \tag{1-29}$$

The form invariance of the other variational equations implies that

$$\frac{d}{dt}(\delta\alpha) = \frac{d}{dt}(\delta\beta) = \frac{d}{dt}(\delta\vec{\gamma}) = \frac{d}{dt}(\delta\vec{z}) = 0 \tag{1-30}$$

which have the solutions

$$\begin{aligned} \delta\alpha &= \delta\alpha(s,\gamma,z), \qquad & \delta\beta &= \delta\beta(s,\gamma,z), \\ \delta\vec{\gamma} &= \delta\vec{\gamma}(s,\gamma,z), \qquad & \delta z &= \delta z(s,\gamma,z). \end{aligned} \tag{1-31}$$

Substitution of these results into Eq. (1-29) yields

$$(\varepsilon\frac{\partial G}{\partial s} + \delta\beta)\vec{\nabla}s + (\varepsilon\frac{\partial G}{\partial z^j} + \delta\gamma^j)\vec{\nabla}z^j + (\varepsilon\frac{\partial G}{\partial\gamma^j} - \delta z^j)\nabla\gamma^j = 0 \tag{1-32}$$

where for convenience $\delta\alpha + \gamma^j\delta z^j \equiv \varepsilon G(s,\gamma,z)$ and ε is an infinitesimal constant.

Since s, $\vec{\gamma}$ and $\vec{z}$ are independent variables Eq. (1-32) implies that the infinitesimal gauge transformations are given by

$$\delta\alpha = \varepsilon G - \varepsilon\gamma^j \frac{\partial G}{\partial \gamma^j}\ , \ \delta\beta = -\varepsilon \frac{\partial G}{\partial s}\ , \ \delta\gamma^j = -\varepsilon \frac{\partial G}{\partial z^j}\ , \ \delta z^j = \varepsilon \frac{\partial G}{\partial \gamma^j} \quad (1\text{-}33)$$

The conservation law associated with the gauge transformations is

$$\frac{\partial}{\partial t} (\rho G(s,\gamma,z)) + \vec{\nabla}\cdot(\rho\vec{V}G(s,\gamma,z)) = 0 \quad (1\text{-}34)$$

which is easily verified from Eq. (1-25). Note that if $\delta\vec{z} \equiv 0$ then the gauge transformations take the form

$$G = G(s,z) \quad (1\text{-}35)$$

and the conservation law becomes

$$\frac{\partial}{\partial t} (\rho G(s,z)) + \vec{\nabla}\cdot(\rho\vec{V}G(s,z)) = 0 \quad (1\text{-}36)$$

1-6 The Galilean Transformations

In this thesis all equations are invariant under the Galilean transformations: spatial translation, time translation, rotation of coordinates and Galilean boosts. The transformation properties of $\vec{V}$,ρ,s and $\vec{z}$ are well known but the transformation properties of the Monge potentials α,β and $\vec{\gamma}$ are not known a priori and must be deduced by requiring that the Lagrangian be invariant under the Galilean group (i) Under the infinitesimal spatial displacement $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{x}$ where $\delta\vec{x}$ is an infinitesimal constant, $\vec{z} \to \vec{z}' = \vec{z} + \delta\vec{x}$. The results of Sec. 1-5 show that the Monge potentials α, β and $\vec{\gamma}$ remain fixed apart from a gauge transformation of the type G = G(s,z). From Eq. (1-25) it is easy to verify that the conservation law associated with this symmetry transformation is the conservation of momentum equation

$$\frac{\partial \rho v^i}{\partial t} + \nabla^j(\rho V^i V^j + P\delta^{ij}) = 0 \tag{1-37}$$

(ii) Under the time translation $t \to t' = t + \delta t$ where δt is an infinitesimal constant, $\vec{z} \to \vec{z}' = \vec{z} - \vec{V}(z,o)\delta t$ (since $\vec{z}$ was defined as the initial position of a fluid particle, a shift in the time origin shifts $\vec{z}$ as well). The form invariance of Eq. (1-13) gives $\vec{\gamma} \to \vec{\gamma}' = \vec{\gamma} - \gamma^j(\partial V^j(z,o)/\partial z)\delta t$ apart from a gauge tranformation of the type $G = G(s,z)$. From Eq. (1-25) follows the conservation of energy equation

$$\frac{\partial}{\partial t}(\rho e + \tfrac{1}{2}\rho V^2) + \nabla^j([\rho e + \tfrac{1}{2}\rho V^2 + P]V^j) = 0 \tag{1-38}$$

(iii) Under the rotation of axes $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{\theta} \times \vec{x}$, where $\delta\vec{\theta}$ is an infinitesimal constant vector in the direction of the axis of rotation with a magnitude equal to the angle of rotation, $\vec{V} \to \vec{V}' = \vec{V} + \delta\vec{\theta} \times \vec{V}$ and $\vec{z} \to \vec{z}' = \vec{z} + \delta\vec{\theta} \times \vec{z}$. The form invariance of Eq. (1-13) gives $\vec{\gamma} \to \vec{\gamma}' = \vec{\gamma} + \delta\vec{\theta} \times \vec{\gamma}$, apart from a gauge transformation of the type $G = G(s,z)$. From Eq. (1-25) the conservation of angular momentum equation is

$$\frac{\partial}{\partial t}(\rho\vec{V} \times \vec{x}) + \nabla^j(\rho V^j\vec{V} \times \vec{x}) - \vec{x} \times \vec{\nabla}P = 0 \tag{1-39}$$

(iv) Under the Galilean boost $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{V}_o t$, where $\delta\vec{V}_o$ is an infinitesimal constant vector, $\vec{V} \to \vec{V}' = \vec{V} + \delta\vec{V}_o$. The form invariance of Eq. (1-13) gives $\alpha \to \alpha' = \alpha - \vec{x}\cdot\delta\vec{V}_o$ apart from a gauge transformation $G = G(s,z)$. From Eq. (1-25) the conservation of center of mass equation is

$$\frac{\partial}{\partial t}(\rho V^i t - \rho x^i) + \nabla^j(\rho V^i V^j t - \rho x^i V^j + Pt\delta^{ij}) = 0 \tag{1-40}$$

1-7 An Alternative Lagrangian

The variational principle given by Eq. (1-12) requires that the Lagrangian picture of fluid flow be adjoined to the hydrodynamic equations in the form $d\vec{z}/dt = 0$. Seliger and Whitham[13] have shown that this is not necessary. Consider the Lagrangian

$$L = \tfrac{1}{2}\rho V^2 - \rho e(\rho,s) - \alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\} + \rho\beta\frac{ds}{dt} + \rho\gamma\frac{dz}{dt} + \rho H(\gamma,z,t) \qquad (1\text{-}41)$$

where α,β,γ and z are to be interpreted as Monge potentials and H is an arbitrary function of γ, z and t. Variations of α and β give Eqs. (1-4) and (1-5). The other variations give

$$\delta\vec{V}: \quad \vec{V} = -\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma\vec{\nabla}z \qquad (1\text{-}42)$$

$$\delta\rho: \quad \frac{d\alpha}{dt} + \tfrac{1}{2}V^2 - e - \frac{P}{\rho} + \gamma\frac{dz}{dt} + H = 0 \qquad (1\text{-}43)$$

$$\delta s: \quad \frac{d\beta}{dt} = -T \qquad (1\text{-}44)$$

$$\delta\gamma: \quad \frac{d\gamma}{dt} = \frac{\partial H}{\partial z} \qquad (1\text{-}45)$$

$$\delta z: \quad \frac{dz}{dt} = -\frac{\partial H}{\partial\gamma} \qquad (1\text{-}46)$$

The Clebsch lemma gives

$$\frac{dV}{dt} = -\vec{\nabla}(\frac{d\alpha}{dt} + \tfrac{1}{2}V^2) - \frac{d\beta}{dt}\vec{\nabla}s - \beta\vec{\nabla}\frac{ds}{dt} - \frac{d\gamma}{dt}\vec{\nabla}z - \gamma\vec{\nabla}\frac{dz}{dt} \qquad (1\text{-}47)$$

while substituting Eqs. (1-43)-(1-46), (1-5) into Eq. (1-47) gives

$$\frac{dV}{dt} = -\vec{\nabla}(e + \frac{P}{\rho} - H + \gamma\frac{\partial H}{\partial\gamma}) + T\vec{\nabla}s - \frac{\partial H}{\partial z}\vec{\nabla}z + \gamma\vec{\nabla}\frac{\partial H}{\partial\gamma} = -\frac{1}{\rho}\vec{\nabla}P \qquad (1\text{-}48)$$

which is just the conservation of momentum equation (1-6). Hence all solutions of the variational equations of L are also solutions of

Eqs. (1-4)-(1-6).

The converse statement can be proven using Pfaff's theorem which states that an arbitrary 3-vector $\vec{V} + \beta\vec{\nabla}s$ can be written in the form

$$\vec{V} + \beta\vec{\nabla}s = -\vec{\nabla}\alpha - \gamma\vec{\nabla}z \ . \tag{1-49}$$

Since β is arbitrary it can be chosen such that $d\beta/dt = -T$. Clebsch's lemma and Eq. (1-6) then give

$$-\frac{1}{\rho}\vec{\nabla}P = \frac{d\vec{V}}{dt} = -\vec{\nabla}\left(\frac{d\alpha}{dt} + \tfrac{1}{2}V^2\right) - \frac{d\beta}{dt}\vec{\nabla}s - \beta\vec{\nabla}\frac{ds}{dt} - \frac{d\gamma}{dt}\vec{\nabla}z - \gamma\vec{\nabla}\frac{dz}{dt} \tag{1-50}$$

which can be rewritten using Eq. (1-5) and $d\beta/dt = -T$ as

$$\vec{\nabla}\left(\frac{d\alpha}{dt} + \tfrac{1}{2}V^2 - e - \frac{P}{\rho} + \frac{dz}{dt}\right) + \frac{d\gamma}{dt}\vec{\nabla}z - \frac{dz}{dt}\vec{\nabla}\gamma = 0 \tag{1-51}$$

Self-consistency conditions imply that α, γ and z must satisfy Eqs. (1-43), (1-45) and (1-46) respectively. Hence the variational principle given by Eq. (1-41) is completely equivalent to Eqs. (1-4)-(1-6) and no reference to the Lagrangian picture of fluid flow is needed. Note that the addition of the term $H(\gamma,z,t)$ means that the Lagrangian is invariant under the smaller group of gauge transformations $G = G(s)$. This completes the review of the variational principle for adiabatic flow.

CHAPTER 2

CONSEQUENCES OF RELAXING THE CONSERVATION OF PARTICLE LABEL

2-1 The "Gauge" Invariance Problem

As shown in Chapter 1 (see Eq. (1-12)) the Lagrangian for the perfect fluid in adiabatic flow is just the kinetic energy minus the internal energy plus a sum of Lagrange multipliers times the constraints $\partial\rho/\partial t+\vec{\nabla}\cdot(\rho\vec{V})=0$, $ds/dt=0$ and $d\vec{z}/dt=0$. Variation of L with respect to $\vec{V}$ gave the representation for $\vec{V}$, then use of Clebsch's lemma and the variational equations for $\rho, s, \vec{z}, \alpha, \beta$ and $\vec{\gamma}$ yielded the conservation of momentum equation. In fact the precise form of the constraint equations for ρ, s and $\vec{z}$ given above is crucial to the success of the variational principle.

For instance consider the Navier-Stokes entropy production equation $\rho ds/dt = kT^{-1}\nabla^2 T + T^{-1}\nabla^i V^j T^{ij}$ where $T^{ij} = \lambda\vec{\nabla}\cdot\vec{V}\delta^{ij} + \mu(\nabla^i V^j + \nabla^j V^i)$, k, λ and μ=constant. If this constraint is incorporated in the variational principle simply by using a Lagrange multiplier β then the Lagrangian is

$$L' = \tfrac{1}{2}\rho V^2 - \rho e(\rho,s) - \alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\} + \beta\{\rho\frac{ds}{dt} - kT^{-1}\nabla^2 T - T^{-1}\nabla^i V^j T^{ij}\} + \rho\gamma^j\frac{dz^j}{dt} \qquad (2\text{-}1)$$

Variations of $\alpha, \vec{\gamma}$ and $\vec{z}$ are unchanged while variations of ρ, s, β and $\vec{V}$ give

$$\delta\rho: \quad \frac{d\alpha}{dt} + \tfrac{1}{2}V^2 - e - \frac{P}{\rho} + \beta\,(1+\frac{\rho}{T}\frac{\partial T}{\partial\rho})\frac{ds}{dt} - k\frac{\partial T}{\partial\rho}\nabla^2(\frac{\beta}{T}) = 0 \qquad (2\text{-}2)$$

$$\delta s: \quad -\rho\frac{d\beta}{dt} - \rho T + \frac{\rho}{T}\frac{\partial T}{\partial s}\beta\frac{ds}{dt} - k\frac{\partial T}{\partial s}\nabla^2(\frac{\beta}{T}) = 0 \qquad (2\text{-}3)$$

$$\delta\beta: \quad \rho\frac{ds}{dt} = kT^{-1}\nabla^2 T + T^{-1}\nabla^i V^j T^{ij} \qquad (2\text{-}4)$$

$$\delta V^i: \quad V^i = -\nabla^i\alpha - \beta\nabla^i s - \gamma^j\nabla^i z^j - \frac{1}{\rho}\nabla^j(2\beta T^{-1}T^{ij}) \tag{2-5}$$

A straightforward calculation yields an equation for V^i

$$\frac{dV^i}{dt} + \frac{1}{\rho}\nabla^i P + \frac{1}{\rho}\nabla^j T^{ij} = \nabla^i\left(\frac{\rho\beta}{T}\frac{\partial T}{\partial\rho}\frac{ds}{dt} - k\frac{\partial T}{\partial\rho}\nabla^2\left(\frac{\beta}{T}\right)\right) + \left(-\frac{\beta}{T}\frac{\partial T}{\partial s}\frac{ds}{dt} + \frac{k}{\rho}\frac{\partial T}{\partial s}\nabla^2\left(\frac{\beta}{T}\right)\right)\nabla^i s - \frac{ds}{dt}\nabla^i\beta - \frac{\vec{\nabla}\cdot\vec{V}}{\rho}\nabla^j\left(\frac{2\beta}{T}T^{ij}\right) - \frac{1}{\rho}\nabla^l\left(\frac{2\beta}{T}T^{kl}\right)\nabla^i V^k + \frac{1}{\rho}\nabla^l\left(\frac{2\beta}{T}T^{ij}\right)\nabla^j V^l - \frac{1}{\rho}\nabla^j\left(2\beta\frac{d}{dt}\left(\frac{T^{ij}}{T}\right)\right) - \frac{1}{\rho}\nabla^j\left(\left\{\frac{2\beta}{T^2}\frac{\partial T}{\partial s}\frac{ds}{dt} - \frac{2k}{\rho T}\frac{\partial T}{\partial s}\nabla^2\left(\frac{\beta}{T}\right)\right\}T^{ij}\right) - \frac{1}{\rho}\nabla^j T^{ij} \tag{2-6}$$

which is not the Navier-Stokes momentum equation unless the RHS of Eq. (2-6) vanishes. This constraint would have to be added to the Lagrangian with no guarantee of a solution to the resulting closure problem.

In addition the Monge potential β which was introduced as a Lagrange multiplier and has no unique physical interpretation, cannot be eliminated from the R.H.S. of Eq. (2-6) by using the variational equations of L'. The conclusion is that the form of the constraint equations for ρ, s and $\vec{z}$ determines whether the Monge potentials α,β and $\vec{\gamma}$ (which have no unique physical interpretation) can be eliminated from the equation for $d\vec{V}/dt$. The latter case will be summarized by saying that the equation for $d\vec{V}/dt$ is "gauge" invariant i.e. the Monge potentials α,β and $\vec{\gamma}$ can be eliminated in terms of $\vec{V},\rho,s,\vec{z}$ and their derivatives.

Sec. 2-2 will explore an extension of the variational principle discovered by the author in which the equation for $d\vec{V}/dt$ is "gauge" invariant while the conservation of particle label and entropy equations are modified and the conservation of mass equation remains unchanged. In Sec. 2-3 the equations of motion are interpreted as describing a fluid with a finite density of vortices where the hydrodynamic variables $\vec{V},\rho,s$ and $\vec{z}$ have been averaged over a region containing many vortices. A variational principle for which $d\vec{V}/dt$ is "gauge" invariant and only the conservation of particle label equation is changed is given in Sec. 2-4, it is found that because of the requirements of Galilean invariance such a theory must be non-linear in the gradient of the Monge potential $\vec{\gamma}$.

2-2 A Theory of Hydrodynamics with $d\vec{z}/dt \neq 0$, $ds/dt \neq 0$ and $\partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{V}) = 0$.

Consider the Lagrangian

$$L = \tfrac{1}{2}\rho V^2 - \rho e(\rho,s) - \rho g(\omega) - \alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\} + \rho\beta\,\frac{ds}{dt} + \rho\gamma^j\,\frac{dz^j}{dt} \tag{2-7}$$

where the dependent variables are $\vec{V},\rho,s,\vec{z},\alpha,\beta$ and $\vec{\gamma}$, $\vec{\omega} \equiv -\vec{\nabla}\beta\times\vec{\nabla}s - \vec{\nabla}\gamma^j\times\vec{\nabla}z^j$ and $g(\omega)$ is an arbitrary function of $\omega = |\vec{\omega}|$. After the variation of $\vec{V}$ is carried out $\vec{\omega} = \vec{\nabla}\times\vec{V}$ and the $-\rho g(\omega)$ term in L may be interpreted as adding a vorticity dependent contribution to the internal energy $e(\rho,s)$ of the fluid. This is similar to an assumption made by Khalatnikov and Bekarevitch[25] (KB) in deriving the equations of superfluid helium with a finite density of superfluid vortices where the internal energy of the fluid is allowed to depend on the superfluid vorticity.

The following identities will prove helpful in finding the variational equations of L

$$\frac{\delta}{\delta\beta}\,(-\rho g(\omega)) = \vec{\nabla}s\cdot\vec{\nabla}\times(\rho\,\frac{\partial g}{\partial\omega}\,\frac{\vec{\omega}}{\omega})\ ,\ \frac{\delta}{\delta s}\,(-\rho g) = -\vec{\nabla}\beta\cdot\vec{\nabla}\times(\rho\,\frac{\partial g}{\partial\omega}\,\frac{\vec{\omega}}{\omega}) \tag{2-8}$$

with analogous expressions for $\vec{\gamma}$ and $\vec{z}$. By using Eqs. (2-8) the variational equations are easily shown to be

$$\delta\vec{V}: \quad \vec{V} = -\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j \tag{2-9}$$

$$\delta\alpha: \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \tag{2-10}$$

$$\delta\rho: \quad \frac{d\alpha}{dt} + \beta\frac{ds}{dt} + \gamma^j\frac{dz^j}{dt} + \tfrac{1}{2}V^2 - e - \frac{P}{\rho} - g(\omega) = 0 \tag{2-11}$$

$$\delta\beta: \quad \rho\frac{ds}{dt} = -\vec{\nabla}s\cdot\vec{\nabla}\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega}) \tag{2-12}$$

$$\delta s: \quad \rho\frac{d\beta}{dt} = -\vec{\nabla}\beta\cdot\nabla\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega}) - \rho T \tag{2-13}$$

$$\delta\gamma^j: \quad \rho\frac{dz^j}{dt} = -\vec{\nabla}z^j\cdot\vec{\nabla}\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega}) \tag{2-14}$$

$$\delta z^j: \quad \rho\frac{d\gamma^j}{dt} = -\vec{\nabla}\gamma^j\cdot\vec{\nabla}\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega}) \tag{2-15}$$

Since the representation for $\vec{V}$ remains unchanged Clebsch's lemma Eq. (1-17) is unchanged. Substitution of Eq. (2-11) into Eq. (1-17) yields

$$\rho\frac{dV^i}{dt} + \nabla^i P = -\rho\nabla^i g - \rho(\frac{d\beta}{dt} + T)\nabla^i s + \rho\frac{ds}{dt}\nabla^i\beta - \rho\frac{d\gamma^j}{dt}\nabla^i z^j$$

$$+ \rho\frac{dz^j}{dt}\nabla^i\gamma^j \tag{2-16}$$

By using Eqs. (2-12)-(2-15), Eq. (2-16) can be rewritten as

$$\rho\frac{dV^i}{dt} + \nabla^i P = -\rho\nabla^i g + \vec{\nabla}\beta\cdot\vec{\nabla}\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega})\nabla^i s - \vec{\nabla}s\cdot\vec{\nabla}\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega})\nabla^i\beta$$

$$+ \vec{\nabla}\gamma^j\cdot\nabla\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega})\nabla^i z^j - \vec{\nabla}z^j\cdot\vec{\nabla}\times(\rho\frac{\partial g}{\partial\omega}\frac{\vec{\omega}}{\omega})\nabla^i\gamma^j \tag{2-17}$$

The vector identity $(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} = \vec{A}\times(\vec{B}\times\vec{C})$ and the representation for $\vec{V}$ mean that the Monge potentials β and $\vec{\gamma}$ can be eliminated from Eq. (2-17) in the form

$$\rho \frac{d\vec{V}}{dt} + \vec{\nabla}P = - \rho\vec{\nabla}g + \vec{\nabla}\times(\rho \frac{\partial g}{\partial\omega} \frac{\vec{\omega}}{\omega})\times \vec{\omega} \tag{2-18}$$

or equivalently

$$\rho \frac{dV^i}{dt} + \nabla^j(P\delta^{ij} + \frac{\rho}{\omega}\frac{\partial g}{\partial\omega} [\omega^2\delta^{ij} - \omega^i\omega^j]) = 0 \tag{2-19}$$

Hence the effect of the $- \rho g(\omega)$ term in the Lagrangian is to add a symmetric contribution to the stress tensor $T^{(ij)} = - \frac{\rho}{\omega}\frac{\partial g}{\partial\omega} [\omega^2\delta^{ij} - \omega^i\omega^j]$. As in Chapter 1, Eqs. (2-10),(2-12), (2-14) and (2-18) will be collectively referred to as the hydrodynamic equations. Note that the representation for $\vec{V}$ implies that $\vec{\omega} = \vec{\nabla}\times\vec{V}$ and hence the hydrodynamic equations are "gauge" invariant.

By using a suitable generalization of Weber's transformation it can be shown that all solutions of the hydrodynamic equations are also solutions of the variational equations (2-9)-(2-15). If $\vec{z}(x,t)$ is assumed to be invertible and differentiable then Eqs. (1-8) are unchanged. Multiplication of Eq. (2-14) by $\partial x^i/\partial z^j$ gives the equivalent form

$$V^i = \frac{\partial x^i(z,t)}{\partial t} - \frac{1}{\rho} [\vec{\nabla}\times(\rho \frac{\partial g}{\partial\omega} \frac{\vec{\omega}}{\omega})]^i \tag{2-20}$$

Hence it is $\vec{V}' \equiv \vec{V} + \frac{1}{\rho} \vec{\nabla}\times(\rho \frac{\partial g}{\partial\omega} \frac{\vec{\omega}}{\omega}) = \frac{\partial x(z,t)}{\partial t}$ which is the tangent to the particle paths $\vec{x} = \vec{x}(z,t)$, not $\vec{V}$ as in Eq. (1-9). If $d'/dt \equiv \partial/\partial t + \vec{V}'\cdot\vec{\nabla}$ then use of the chain rule implies that the analogue of Eq. (1-11) is

$$\frac{d'}{dt} f(x,t) = \frac{\partial}{\partial t} f(x(z,t),t) \tag{2-21}$$

and hence d'/dt and $\partial/\partial\vec{z}$ commute, not d/dt and $\partial/\partial\vec{z}$ as in the Lagrangian picture of fluid flow.

From Eq. (2-21) follows the identity

$$\frac{d'}{dt} (V^j \frac{\partial x^j}{\partial z^i}) = \frac{d'V^j}{dt} \frac{\partial x^j}{\partial z^i} + V^j \frac{\partial x^j}{\partial z^i} = \frac{dV^j}{dt} \frac{\partial x^j}{\partial z^i}$$

$$+ [V^\ell \nabla^j V'^\ell + (V'^\ell - V^\ell) \nabla^\ell V^j] \frac{\partial x^j}{\partial z^i} \tag{2-22}$$

Substitution of Eq. (2-18) into Eq. (2-22) gives

$$\frac{d'}{dt} (V^j \frac{\partial x^j}{\partial z^i}) = \frac{\partial}{\partial z^i} (-e- \frac{P}{\rho} - g + \tfrac{1}{2} V^2 + \vec{V}\cdot(\vec{V}'-\vec{V})) + T\nabla^j s \frac{\partial x^i}{\partial z^i} \tag{2-23}$$

By defining $\beta = - \int_0^t T\, d't$ and $\alpha = - \int_0^t [-e- \frac{P}{\rho} - g + \frac{1}{2} V^2 + \vec{V}\cdot(\vec{V}'-\vec{V})] d't$ (the integration is carried out at constant $\vec{z}$) and since $d's/dt = 0$ from Eqs. (2-12), (2-21) then Eq. (2-23) can be rewritten as

$$\frac{d'}{dt} (\frac{\partial x^j}{\partial z^i} [V^j + \nabla^j\alpha + \beta\nabla^j s]) = 0 \tag{2-24}$$

The latter equation can be integrated by defining $\gamma^j = -V^j(z,o)$ as

$$V^j = - \nabla^j\alpha - \beta\nabla^j s - \gamma^\ell \nabla^j z^\ell \tag{2-25}$$

which is just Eq. (2-9). From the definition of α,β and $\vec{\gamma}$ given above

$$\frac{d'\alpha}{dt} - e - \frac{P}{\rho} - g + \tfrac{1}{2} V^2 + \vec{V}\cdot(\vec{V}'-\vec{V}) = 0 \tag{2-26}$$

$$\frac{d'\beta}{dt} = - T \tag{2-27}$$

$$\frac{d'\gamma^j}{dt} = 0 \tag{2-28}$$

By using the definition $\vec{V}' = \vec{V} + \frac{1}{\rho} \vec{\nabla}\times(\rho \frac{\partial g}{\partial \omega} \frac{\vec{\omega}}{\omega})$ and Eq. (2-25) it immediately follows that Eqs. (2-26)-(2-28) are identical with Eqs. (2-11), (2-13) and (2-14) respectively. Hence the hydrodynamic equations are completely equivalent to the variational equations.

Note that since the representation for $\vec{V}$ is unchanged, the "gauge" transformations of $\alpha,\beta,\vec{\gamma}$ and $\vec{z}$ have the same form as given by Eq.(1-35). Furthermore since $\vec{\omega} = \vec{\nabla}\times[- \vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j]$ is clearly invariant under these transformations the Lagrangian given by Eq. (2-7) is also invariant. As the reader may easily verify from Eq. (1-25) the conservation law which arises from the gauge invariance of L is

$$\frac{\partial}{\partial t} (\rho G(s,\gamma,z)) + \vec{\nabla}\cdot(\rho\vec{V}'G(x,\gamma,z)) = 0 \tag{2-30}$$

which follows from Eqs. (2-10),(2-12), (2-14) and (2-15). Since ω is also a Galilean invariant the Lagrangian given by Eq. (2-7) is invariant under the Galilean group. The conservation laws which arise from Galilean invariance can be derived from Eq. (1-25) as

$$\frac{\partial}{\partial t} (\rho V^i) + \nabla^j(\rho V^i V^j + P\delta^{ij} + \frac{\rho}{\omega}\frac{\partial g}{\partial \omega} [\omega^2\delta^{ij} - \omega^i\omega^j]) = 0 \tag{2-31}$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\,\rho V^2+\rho e+\rho g\right)+\vec{\nabla}\cdot\left(\left[\tfrac{1}{2}\,\rho V^2+\rho e+\rho g+P\right]\vec{V}+\frac{\rho}{\omega}\frac{\partial g}{\partial \omega}\,\vec{\omega}\times[T\vec{\nabla}s+\vec{V}'\times\vec{\omega}]\right) = 0 \qquad (2\text{-}32)$$

$$\frac{\partial}{\partial t}(\varepsilon^{ijk}\rho x^j V^k)+\nabla^\ell(\varepsilon^{ijk}\rho x^j V^k V^\ell+\varepsilon^{ijk}x^j\{\rho\delta^{k\ell}+\frac{\rho}{\omega}\frac{\partial g}{\partial\omega}[\omega^2\delta^{k\ell}-\omega^k\omega^\ell]\})=0 \qquad (2\text{-}33)$$

$$\frac{\partial}{\partial t}\,(\rho V^i t-\rho x^i)+\nabla^j(\rho V^i V^j t-\rho x^i V^j+Pt\delta^{ij}+\frac{\rho}{\omega}\frac{\partial g}{\partial\omega}\,[\omega^2\delta^{ij}-\omega^i\omega^j]t) = 0 \qquad (2\text{-}34)$$

which represent the conservation of momentum, energy, angular momentum and center-of-mass respectively (ε^{ijk} is the permutation symbol).

2-3 Interpretation of the Equations of Motion

The Lagrangian given by Eq. (2-7) differs from that of Eq. (1-12) by the addition of a vorticity dependent contribution to the internal energy. This is analogous to the theory of (KB) which describes the motion of superfluid helium with a finite density of superfluid vortices in which the hydrodynamic variables are averaged over a macroscopic region containing a large number of vortices. In fact, in Chapter 4 a modification of the Lagrangian given by Eq. (2-7) will be used to derive the equations of (KB). This suggests that a similar interpretation can be made for the hydrodynamic equations given in Sec. 2-2, that they describe a fluid with a finite density of vortices where the hydrodynamic variables $\vec{V},\rho,s$ and $\vec{z}$ have been averaged over many vortices. Once the form of the total energy of the fluid is specified, then the equations of motion follow from the standard technique in hydrodynamics. Conservation laws for mass, momentum, entropy and energy are assumed which give six equations in five variables. The resulting self-consistency conditions fix the hydrodynamic equations; see Appendix B for details.

Since the turbulent flow of a fluid is characterized by a distribution of vorticity it is worthwhile to examine the equations of motion for turbulent solutions. By adopting Kolmogoroff's assumption[26] that only the energy dissipation ε_0 (a constant with dimensions $\ell^2 t^{-3}$) be used in the inertial subrange of turbulent flow then by dimensional analysis $g(\omega) = K'\varepsilon_0\omega^{-1}$, K' = dimensionless constant. Provided a closure relation is assumed for the two-point vorticity correlation function of the form

$$\langle\vec{\omega}(x+r)\cdot\vec{\omega}(x)(\omega(x))^{-3}\rangle = K''\langle\vec{\omega}(x+r)\cdot\vec{\omega}(x)\rangle^{-\frac{1}{2}} \tag{2-35}$$

(K" = dimensionless constant) then the hydrodynamic equations of Sec. 2-2 provide a closed equation for the two-point velocity correlation function, which can be solved as

$$\langle(\vec{V}(x+r) - \vec{V}(x))^2\rangle = 9/2(10K'K''/9)^{2/3}\varepsilon_0^{2/3}r^{2/3} \tag{2-36}$$

Eq. (2-36) agrees with Kolmogoroff's prediction for the inertial subrange, provided Kolmogoroff's constant $K(K\simeq .5)$ is given by $K = 9/8(10K'K''/9)^{2/3}$. Hence it is possible to model the inertial subrange of turbulent flow with the hydrodynamic equations of Sec. 2-2 provided the form of $g(\omega)$ is given by dimensional analysis as $g = K'\varepsilon_0\omega^{-1}$ and the closure relation Eq. (2-35) is assumed. Since this subject is peripheral to the main topic of this thesis the details of the foregoing discussion are relegated to Appendix C.

The Beltrami diffusion equation

$$\frac{d}{dt}\left(\frac{\omega^i}{\rho}\right) = \frac{\omega^j}{\rho}\nabla^j V^i + \frac{1}{\rho}\left(\vec{\nabla} \times \frac{d\vec{V}}{dt}\right)^i \tag{2-37}$$

which follows as an identity from the definition of d/dt, when combined with Eq. (2-18) in the barotropic case $P = P(\rho)$ yields an equation for the vorticity

$$\frac{d'}{dt}\left(\frac{\omega^i}{\rho}\right) = \frac{\omega^j}{\rho}\nabla^j V'^i \tag{2-38}$$

Eq. (2-38) can be integrated using Eq. (2-21) as

$$\frac{\omega^i}{\rho} = \frac{\omega^j(z,o)}{\rho(z,o)}\frac{\partial x^i}{\partial z^j} \tag{2-39}$$

which states that $\frac{\omega^i}{\rho}$ is transported with velocity $\vec{V}'$ not $\vec{V}$ as in the Lagrangian picture of the fluid flow. Note that since $\vec{\nabla}\cdot(\rho\vec{V}') = \vec{\nabla}\cdot(\rho\vec{V})$ then $d'\rho/dt = -\rho\vec{\nabla}\cdot V'$ which can be integrated as $\rho/\rho(z,o) = J \equiv \det\frac{\partial z}{\partial x}$, just as in the Lagrangian picture of fluid flow.

2-4 A Theory of Hydrodynamics with $d\vec{z}/dt \neq 0$, $ds/dt = 0$ and $\partial\rho/\partial t + \vec{\nabla}\cdot(\rho\vec{V}) = 0$.

To simplify the search for a Lagrangian such that only the conservation of particle label constraint is altered, assume that the representation for $\vec{V}$ remains unchanged and that the new Lagrangian maintains the invariance under the gauge transformations given by Eq. (1-35). The only expressions which involve the Monge potentials α, β and $\vec{\gamma}$ and are invariant under the gauge transformations are $\partial\alpha/\partial t + \beta\partial s/\partial t + \gamma^j\partial z^j/\partial t$ and $-\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j$ (see Ref. 16 for a proof). To keep the conservation of mass and entropy equations unchanged, additional terms involving α and β cannot appear in the Lagrangian (since variation of these Monge potentials just yield the conservation of mass and entropy equations). This means that the Monge potentials can appear in any additional terms in the Lagrangian only in the form $\vec{\omega}\cdot\vec{\nabla}s \equiv -\vec{\nabla}s\cdot(\vec{\nabla}\gamma^j\times\vec{\nabla}z^j)$.

The representation for $\vec{V}$ is unchanged, thus $\vec{V}$ may not appear in any addition to the Lagrangian. Furthermore since $\partial/\partial t$ is not invariant under Galilean boosts (see Sec. 1-6) no time derivatives may appear (d/dt cannot be used since this would involve $\vec{V}$). Since $\vec{z} \to \vec{z}\,' = \vec{z} - \vec{V}(z,o)\delta t$ under time translation (see Sec. 1-6) the variable $\vec{z}$ can appear only in the term $\vec{\omega}\cdot\vec{\nabla}s$. It is easy to see that $\vec{\omega}\cdot\vec{\nabla}s$ is invariant under spatial translation, time translation, Galilean boosts, rotations and inversion of coordinates ($\vec{\omega} \to -\vec{\omega}$) however under time inversion $\vec{\omega} \to -\vec{\omega}$ and thus $\vec{\omega}\cdot\vec{\nabla}s \to -\vec{\omega}\cdot\vec{\nabla}s$. Hence to maintain the Galilean invariance of the Lagrangian, any additional terms in the Lagrangian must have the form $h(\rho,s,\vec{\nabla}\rho,\vec{\nabla}s,(\vec{\omega}\cdot\vec{\nabla}s)^2)$, where h is a differentiable scalar function of its arguments, and therefore are non-linear in $\nabla^i\gamma^j$.

In view of the preceeding discussion consider the Lagrangian

$$L'' = \tfrac{1}{2}\rho V^2 - \rho e(\rho,s) - h(\rho,s,(\vec{\omega}\cdot\vec{\nabla}s)^2) - \alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\} + \rho\beta\,\frac{ds}{dt} + \rho\gamma^j\,\frac{dz^j}{dt} \qquad (2\text{-}40)$$

where for simplicity $\vec{\nabla}\rho$ and $\vec{\nabla}s$ have been eliminated from h apart from $(\vec{\omega}\cdot\vec{\nabla}s)^2$ terms. The variational equations of L'' are

$$\delta\vec{V}: \quad \vec{V} = -\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j \qquad (2\text{-}41)$$

$$\delta\alpha: \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \tag{2-42}$$

$$\delta\rho: \quad \frac{d\alpha}{dt} + \beta\,\frac{ds}{dt} + \gamma^j\,\frac{dz^j}{dt} + \tfrac{1}{2}\,V^2 - e - \frac{P}{\rho} - \frac{\partial h}{\partial\rho} = 0 \tag{2-43}$$

$$\delta\beta: \quad \frac{ds}{dt} = 0 \tag{2-44}$$

$$\delta s: \quad \frac{d\beta}{dt} = -T - \frac{1}{\rho}\,\frac{\partial h}{\partial s} - \frac{1}{\rho}\,\vec{\nabla}\left[\frac{\partial h}{\partial(\vec{\omega}\cdot\vec{\nabla}s)}\right]\times\vec{\nabla}s\cdot\vec{\nabla}\beta + \frac{1}{\rho}\,\vec{\nabla}\left[\frac{\partial h}{\partial(\vec{\omega}\cdot\vec{\nabla}s)}\right]\cdot\vec{\omega} \tag{2-45}$$

$$\delta\gamma^j: \quad \frac{dz^j}{dt} = -\frac{1}{\rho}\,\left[\frac{\partial h}{\partial(\vec{\omega}\cdot\vec{\nabla}s)}\right]\times\vec{\nabla}s\cdot\vec{\nabla}z^j \tag{2-46}$$

$$\delta z^j: \quad \frac{d\gamma^j}{dt} = -\frac{1}{\rho}\,\vec{\nabla}\left[\frac{\partial h}{\partial(\vec{\omega}\cdot\vec{\nabla}s)}\right]\times\vec{\nabla}s\cdot\vec{\nabla}\gamma^j \tag{2-47}$$

Substitution of Eqs. (2-43)-(2-47) into Eq.(1-17), which is unchanged, and use of the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ yields the conservation of momentum equation

$$\frac{\partial}{\partial t}\,(\rho V^i) + \nabla^j(\rho V^i V^j + P\delta^{ij} - T^{ij}) = 0 \tag{2-48}$$

where the symmetric stress tensor T^{ij} is given by

$$T^{ij} = (h - \rho\,\frac{\partial h}{\partial\rho})\,\delta^{ij} + \frac{\partial h}{\partial(\vec{\omega}\cdot\vec{\nabla}s)}\left[(\vec{\omega}\cdot\vec{\nabla}s)\,\delta^{ij} - (\omega^i\nabla^j s + \omega^j\nabla^i s)\right] \tag{2-49}$$

Multiplying Eq. (2-46) by $\partial x^i/\partial z^j$ yields the equivalent form

$$\vec{V} = \frac{\partial\vec{x}(z,t)}{\partial t} - \frac{1}{\rho}\,\vec{\nabla}\left[\frac{\partial h}{\partial(\vec{\omega}\cdot\vec{\nabla}s)}\right]\times\vec{\nabla}s \tag{2-50}$$

which states that $\vec{V} + \rho^{-1}[\partial h/\partial(\vec{\omega}\cdot\vec{\nabla}s)]\times\vec{\nabla}s \equiv \vec{V}'$ is the tangent to the particle paths, not $\vec{V}$ as in the Lagrangian picture of fluid flow.

Using the techniques developed in Sec. 2-2 it is straightforward to show that all solutions of the hydrodynamic equations (2-42), (2-44), (2-46) and (2-48) are also solutions of the variational equations of L".

The conservation law associated with the gauge invariance of L" is unchanged from Eq. (2-30). The conservation laws for the energy, angular momentum and center-of-mass are given respectively by

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho V^2+\rho e+h\right)+\nabla^j\left(V^j\left[\tfrac{1}{2}\rho V^2+\rho e+h+P\right]+v'^j(\vec{\omega}\cdot\vec{\nabla}s)\frac{\partial h}{\partial(\vec{\omega}\cdot\vec{\nabla}s)}\right) = 0 \tag{2-51}$$

$$\frac{\partial}{\partial t}\left(\varepsilon^{ijk}\rho x^j V^k\right)+\nabla^\ell\left(\varepsilon^{ijk}\rho x^j V^k V^\ell+\varepsilon^{ijk}x^j\{P\delta^{k\ell}-T^{k\ell}\}\right) = 0 \tag{2-52}$$

$$\frac{\partial}{\partial t}\left(\rho V^i t-\rho x^i\right)+\nabla^j\left(\rho V^i V^j t-\rho x^i V^j+Pt\delta^{ij}-T^{ij}t\right) = 0 \tag{2-53}$$

which follow from Eq. (1-25).

Just as in Sec. 2-3, the hydrodynamic equations of L" may be interpreted as describing a fluid with a finite density of vortices (see Appendix C for details). Note that both Eqs. (2-48) and (2-18) have stress tensors which depend on the vorticity and thus describe non-Stokesian fluids. In conclusion, the relaxation of the conservation of particle label constraint in a one-component fluid is equivalent to vorticity dependent contributions to the stress tensor and the energy of the fluid. The role of the relaxation of the conservation of particle label constraint for superfluid helium will be considered in Chapter 4, as a prelude to this work Chapter 3 will review Zilsel's[27] variational principle for the Landau two-fluid equations.

CHAPTER 3

THE VARIATIONAL PRINCIPLE FOR THE LANDAU TWO-FLUID EQUATIONS

3-1 Introduction

The superfluidity of He^4 was first observed by Kapitza[28] in 1938 who found that liquid helium below $T_\lambda = 2.17°K$ could flow through thin capilliary tubes with zero resistance. On the other hand experiments with rotating liquid helium showed that the superfluid could not be interpreted as a classical one-component fluid with zero viscosity. These two observations led Landau[29] to develop the two-fluid model of superfluidity as consisting of the flow of two interpenetrating fluids, the entropy carrying normal fluid with velocity $\vec{V}_n$ and the zero-entropy superfluid with velocity $\vec{V}_s$.

The Landau two-fluid equations which Landau postulated to describe this model consist of conservation laws for the mass, entropy and total momentum of the fluid and an equation of motion for the superfluid velocity for a total of eight equations. The eight independent variables may be taken as $\rho, s, \vec{V}_n$ and $\vec{V}_s$ where ρ and s are the total mass and entropy of the fluid per unit volume. It has long been suggested that superfluidity is a quantum phenomenon which occurs when an appreciable fraction of the He^4 atoms enter the groundstate in a Bose condensation giving rise to long-range order in the phase of the wavefunction of the Bose condensate.[30]

In fact, once the independent variables $\rho, s, \vec{V}_n$ and $\vec{V}_s$ and their Galilean transformation properties are specified then the Landau two-fluid equations follow without further recourse to the quantum theory from Galilean invariance arguments and by requiring that the conservation of energy equation be redundant (otherwise when combined with the conservation laws for the mass, entropy and momentum and the equation for the superfluid this would yield nine equations in eight unknowns, see Putterman[31] for a detailed derivation).

A variational principle for the Landau two-fluid equations was first given by Zilsel[27] in 1950. Although all solutions of Zilsel's variational equations satisfy the Landau two-fluid equations the converse result does not hold. Zilsel's representation for the normal velocity $\vec{V}_n$ implies $\vec{\nabla} \times \vec{V}_n = 0$ for $\rho s/\rho_n$ = constant which is too restrictive (ρ_n is the mass density of the normal fluid). Schultz and Sorkin[15] have pointed out that this difficulty may be eliminated by postulating a Lin's constraint for $\vec{V}_n$ in analogy with the variational principle for the adiabatic flow of a classical one-component fluid (see Chapter 1). In addition Zilsel's variational principle has been criticized[32,33] on the grounds that $\chi = \rho_n/\rho$ is treated as an independent variable in contradiction with the Landau model. In spite of this it can be shown that Zilsel's variational principle supplemented with Lin's constraint for $\vec{V}_n$ is completely equivalent to the Landau two-fluid equations. The absence of a Lin's constraint for $\vec{V}_s$ ensures that the superfluid remains irrotational i.e. $\vec{\nabla} \times \vec{V}_s = 0$. For a review of these points see the article by Jackson.[34]

Sec. 3-2 reviews the Landau two-fluid equations; the notation will follow London.[35] The equivalence of Zilsel's variational principle supplemented with Lin's constraint for $\vec{V}_n$ and the Landau two-fluid equations is proven in Sec. 3-3 while the symmetries and conservation laws associated with Zilsel's Lagrangian are discussed in Sec. 3-4.

3-2 The Landau Two-Fluid Equations

From London[35] the Landau two-fluid equations for superfluid He^4 are given by

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho_n\vec{V}_n + \rho_s\vec{V}_s) = 0 \tag{3-1}$$

$$\frac{\partial(\rho s)}{\partial t} + \vec{\nabla}\cdot(\rho s\vec{V}_n) = 0 \tag{3-2}$$

$$\frac{d_s\vec{V}_s}{dt} + \vec{\nabla}\mu = 0 \quad , \quad \vec{\nabla}\times\vec{V}_s = 0 \tag{3-3}$$

$$\frac{\partial(\rho_n V_n^i + \rho_s V_s^i)}{\partial t} + \nabla^j(\rho_s V_s^i V_s^j + \rho_n V_n^i V_n^j + P\delta^{ij}) = 0 \tag{3-4}$$

where $\frac{d_\gamma}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}_\gamma\cdot\vec{\nabla}$. Eqs. (3-1), (3-2) and (3-4) represent the conservation of mass, entropy and total momentum of the fluid respectively while Eq. (3-3) gives an equation of motion for the superfluid.

The variables $\rho, s, \vec{V}_n$ and $\vec{V}_s$ have been defined previously while ρ_n and ρ_s are the densities of the normal and superfluid components respectively and the total mass density is given by $\rho = \rho_n + \rho_s$. The internal energy differential has the form

$$de = Tds + (P/\rho^2)d\rho + \tfrac{1}{2}(\vec{V}_n - \vec{V}_s)^2 d\chi \tag{3-5}$$

where $e(\rho,s,\chi)$ is the specific internal energy, T is the temperature, the pressure $P = \rho(-e + Ts + \frac{1}{2}(\vec{V}_n-\vec{V}_s)^2\chi + \mu)$ and μ is the chemical potential. Note that Eq. (3-5) implies $(\partial e/\partial\chi)\rho,s = \frac{1}{2}(\vec{V}_n-\vec{V}_s)^2$ and hence there exist functional relationships of the form $\rho_n = \rho_n(\rho, s,$

$(\vec{V}_n-\vec{V}_s)^2)$ and $\rho_s = \rho_s(\rho,s,(\vec{V}_n-\vec{V}_s)^2)$. The independent variables of the Landau two-fluid equations may therefore be taken as the eight variables $\rho, s, \vec{V}_n$ and $\vec{V}_s$.

Following Zilsel the factor Γ is defined as

$$\frac{\partial\rho_n}{\partial t} + \vec{\nabla}\cdot(\rho_n\vec{V}_n) = -\left[\frac{\partial\rho_s}{\partial t} + \vec{\nabla}\cdot(\rho_s V_s)\right] = \Gamma \tag{3-1)'$$

Using Eq. (3-1)' it follows that Eqs. (3-2) and (3-4) can be rewritten in the convenient equivalent forms

$$\frac{d_n(s/\chi)}{dt} = -\frac{s}{\rho\chi^2}\,\Gamma \tag{3-2)'$$

$$\frac{d_n\vec{V}_n}{dt} + \frac{1}{\rho}\vec{\nabla}P + \frac{\rho_s}{\rho}s\vec{\nabla}T + \frac{P_s}{2\rho}\vec{\nabla}(\vec{V}_n-\vec{V}_s)^2 + (\vec{V}_n-\vec{V}_s)\frac{\Gamma}{\rho\chi} = 0 \tag{3-4)'$$

Assume that there exists a Lin's constraint for the normal velocity field of the form

$$\frac{d_n z^j(x,t)}{dt} = 0 \tag{3-6}$$

Multiplication of Eq. (3-6) by $\partial x^i/\partial z^j$ yields the equivalent form

$$V_n^i = \frac{\partial x^i(z,t)}{\partial t} \tag{3-7}$$

where the function $\vec{x} = \vec{x}(z,t)$ is the inverse of $\vec{z} = \vec{z}(x,t)$ and the identities given by Eqs. (1-8) still hold. Eq. (3-7) just states that $\vec{V}_n$ is the tangent to the particle paths $\vec{x} = \vec{x}(z,t)$. For the classical one-component fluid the particle paths $\vec{x} = \vec{x}(z,t)$ were associated with the movement of small fluid elements. The Landau two-fluid model consists of two interpenetrating fluids and it is no longer clear what physical interpretation the particle paths have; see Jackson[34] for a discussion of this point. In the absence of any further progress on this matter, Eq. (3-6) should be viewed simply as an integrability condition on the normal velocity field.

Use of the chain rule and Eq. (3-7) yields

$$\frac{d_n}{dt} f(x,t) = \frac{\partial}{\partial t} f(x\ (z,t),t) \tag{3-8}$$

which implies that d_n/dt and $\partial/\partial\vec{z}$ commute. Eqs. (3-1), (3-2), (3-3), (3-4)' and (3-6) will be collectively referred to the hydrodynamic equations, which are shown in the following section to be equivalent to a variational principle.

3-3 Zilsel's Variational Principle

Zilsel's Lagrangian supplemented with a Lin's constraint for $\vec{V}_n$ is given by

$$L = \rho[\tfrac{1}{2}\ (1-\chi)V_s^2 + \tfrac{1}{2}\ \chi V_n^2] - \rho e(\rho,s,\chi) - \alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho(1-\chi)\vec{V}_s$$
$$+ \rho\chi\vec{V}_n)\} - \beta\{\frac{\partial(\rho s)}{\partial t} + \vec{\nabla}\cdot(\rho s V_n)\} + \rho\chi\gamma^j\{\frac{\partial z^j}{\partial t} + \vec{V}_n\cdot\vec{\nabla}z^j\} \tag{3-9}$$

where the dependent variables of L are $\rho, s, \vec{V}_n, \vec{V}_s, \chi, \vec{z}, \alpha, \beta$ and $\vec{\gamma}$ and the independent variables are $(\vec{x},t)$. The internal energy density $e(\rho,s,\chi)$ is defined by Eq. (3-5). Actually in Zilsel's procedure Eq. (3-5) is not assumed; instead the variation of χ gives the equation $(\partial e/\partial\chi)_{\rho,s} = \tfrac{1}{2}\ (\vec{V}_n-\vec{V}_s)^2$; when Eq. (3-5) is assumed initially then the variation of χ gives an identity.

The variational equations of L are given by

$$\delta\alpha: \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho(1-\chi)\vec{V}_s + \rho\chi\vec{V}_n) = 0 \tag{3-10}$$

$$\delta\beta: \quad \frac{\partial(\rho s)}{\partial t} + \vec{\nabla}\cdot(\rho s\vec{V}_n) = 0 \tag{3-11}$$

$$\delta\vec{V}_s: \quad \vec{V}_s = -\vec{\nabla}\alpha \tag{3-12}$$

$$\delta V_n: \quad \vec{V}_n = -\vec{\nabla}\alpha - \frac{s}{\chi}\vec{\nabla}\beta - \gamma^j\vec{\nabla}z^j \tag{3-13}$$

$$\delta\rho: \quad \tfrac{1}{2}(1-\chi)V_s^2 + \tfrac{1}{2}\chi V_n^2 - e - \frac{P}{\rho} + \frac{\partial\alpha}{\partial t} + ((1-\chi)\vec{V}_n + \chi\vec{V}_s)\cdot\vec{\nabla}\alpha + s\frac{d_n\beta}{dt} = 0 \tag{3-14}$$

$$\delta s: \quad \frac{d_n\beta}{dt} = T \tag{3-15}$$

$$\delta\chi: \quad -\tfrac{1}{2}\rho V_s^2 + \tfrac{1}{2}\rho V_n^2 + \tfrac{1}{2}\rho(\vec{V}_n - \vec{V}_s)^2 + \rho(\vec{V}_n - \vec{V}_s)\cdot(\vec{\nabla}\alpha) = 0 \tag{3-16}$$

$$\delta\vec{\gamma}: \quad \frac{d_n\vec{z}}{dt} = 0 \tag{3-17}$$

$$\delta\vec{z}: \quad \frac{d_n\vec{\gamma}}{dt} = -\frac{\Gamma}{\rho\chi}\vec{\gamma} \tag{3-18}$$

Eqs. (3-1), (3-2) and (3-6) are recovered as the variational equations of α, β and $\vec{\gamma}$ respectively. When Clebsch's lemma is used Eq. (3-12) implies

$$\frac{d_s\vec{V}_s}{dt} = -\vec{\nabla}\left(\frac{\partial\alpha}{\partial t} + \vec{V}_s\cdot\vec{\nabla}\alpha + \tfrac{1}{2}V_s^2\right) \tag{3-19}$$

Substitution of Eqs. (3-12), (3-15) into Eq. (3-14) yields

$$\frac{\partial\alpha}{\partial t} + \vec{V}_s\cdot\vec{\nabla}\alpha + \tfrac{1}{2}V_s^2 = e + \frac{P}{\rho} - T_s - \tfrac{1}{2}(\vec{V}_n - \vec{V}_s)^2\chi \equiv \mu \tag{3-20}$$

which when combined with Eq. (3-19) gives the equation of motion of the superfluid Eq. (3-3). The irrotational condition $\vec{\nabla}\times\vec{V}_s = 0$ follows from Eq. (3-12).

Clebsch's lemma applied to Eq. (3-13) yields

$$\frac{d_n\vec{V}_n}{dt} = -\nabla\left(\frac{d_n\alpha}{dt} + \tfrac{1}{2}V_n^2\right) - \frac{d_n}{dt}\left(\frac{s}{\chi}\right)\vec{\nabla}\beta - \frac{s}{\chi}\vec{\nabla}\frac{d_n\beta}{dt} - \frac{d_n\gamma^j}{dt}\vec{\nabla}z^j - \gamma^j\vec{\nabla}\frac{d_n z^j}{dt} \tag{3-21}$$

Eq. (3-14) can be rewritten using Eqs. (3-12), (3-15) as

$$\frac{d_n\alpha}{dt} + \tfrac{1}{2}V_n^2 = e + \frac{P}{\rho} - Ts + \tfrac{1}{2}(1-\chi)(\vec{V}_n-\vec{V}_s)^2 \tag{3-22}$$

Substitution of Eqs. (3-22), (3-15), (3-17), (3-18) and (3-2)' (which follows from Eqs. (3-10) and (3-11)) into Eq. (3-21) gives

$$\frac{d_n\vec{V}_n}{dt} = -\vec{\nabla}\left(e + \frac{P}{\rho} - Ts + \tfrac{1}{2}(1-\chi)(\vec{V}_n-\vec{V}_s)^2\right) - \frac{s}{\chi}\vec{\nabla}T$$
$$+ \left(\frac{s}{\chi}\vec{\nabla}\beta + \gamma^j\vec{\nabla}z^j\right)\frac{\Gamma}{\rho\chi} \tag{3-23}$$

Eqs. (3-12) and (3-13) and the identity

$$\frac{1}{\rho}\vec{\nabla}P = \vec{\nabla}\left(e + \frac{P}{\rho} - Ts\right) + s\vec{\nabla}T - \tfrac{1}{2}(\vec{V}_n-\vec{V}_s)\vec{\nabla}\chi \tag{3-24}$$

shows that Eq. (3-23) is just Eq. (3-4)'. Hence all solutions of the variational equations are also solutions of the hydrodynamic equations.

The irrotational condition $\vec{\nabla}\times\vec{V}_s = 0$ implies that $\vec{V}_s$ can be written as a gradient

$$\vec{V}_s = -\vec{\nabla}\alpha \tag{3-25}$$

Clebsch's lemma and Eq. (3-3) give

$$\frac{d_s\vec{V}_s}{dt} + \left(\frac{d_s\alpha}{dt} + \tfrac{1}{2} V_s^2\right) = \left(\frac{d_s\alpha}{dt} + \tfrac{1}{2} V_s^2 - \mu\right) = 0 \tag{3-26}$$

or equivalently

$$\frac{d_s\alpha}{dt} + \tfrac{1}{2} V_s^2 - \mu = 0$$

(a function of time can be absorbed into α) which is identical with Eq. (3-14) provided $\beta \equiv \int_0^t T dt$ (the integration is carried out at constant $\vec{z}$).

From Eq. (3-6) follows the identity

$$\frac{d_n}{dt}\left((V_n^j - V_s^j)\frac{\partial x^j}{\partial z^i}\right) = \left(\frac{d_n V_n^j}{dt} - \frac{d_s V_s^j}{dt}\right)\frac{\partial x^j}{\partial z^i} + (V_n^k - V_s^k)(\nabla^j V_n^k - \nabla^k V_s^j)\frac{\partial x^j}{\partial z^i} \tag{3-27}$$

Substitution of Eqs. (3-3) and (3-4)' into Eq. (3-27) gives

$$\frac{d_n}{dt}\left((V_n^j - V_s^j)\frac{\partial x^j}{\partial z^i}\right) = -\frac{s}{\chi}\nabla^j T \frac{\partial x^j}{\partial z^i} + (V_s^j - V_n^j)\frac{\Gamma}{\rho\chi}\frac{\partial x^j}{\partial z^i} \tag{3-28}$$

From the definition of β given above and using Eq. (3-2)' then Eq. (3-28) can be rewritten as

$$\frac{d_n}{dt}\left(\frac{\chi}{s}\left(V_n^j - V_s^j + \frac{s}{\chi}\nabla^j\beta\right)\frac{\partial x^j}{\partial z^i}\right) = 0 \tag{3-29}$$

which can be integrated as

$$\vec{V}_n - \vec{V}_s = -\frac{s}{\chi}\vec{\nabla}\beta - \gamma^j\vec{\nabla}z^j \tag{3-30}$$

where $\gamma^j \equiv -(s\chi(z,o)/s(z,o)\chi)(V_n^j(z,o) - V_s^j(z,o))$. Eqs. (3-25) and

(3.30) are identical with Eqs. (3-12) and (3-13) respectively, furthermore it is easy to show that α, β and $\vec{\gamma}$ defined above satisfy Eqs. (3-14), (3-15) and (3-18) respectively. Thus the variational equations are completely equivalent to the hydrodynamic equations.

3-4 Symmetries and Conservation Laws

As pointed out in Sec. 1-5 the Monge potentials α, β, $\vec{\gamma}$ and $\vec{z}$ may be subjected to "gauge" transformations which do not change the value of $\vec{V}$ and which keep the variational equations form invariant, leading to a conservation law via Noether's theorem. For the Landau two-fluid equations the requirement that $\vec{V}_s$ be unchanged and that Eq. (3-12) be form invariant gives $\vec{\nabla}\delta\alpha = 0$ or equivalently $\alpha \to \alpha' = \alpha + \delta\alpha(t)$. If $\vec{V}_n$ is unchanged and Eq. (3-13) is form invariant then the infinitesimal transformations $\beta \to \beta' = \beta + \delta\beta$, $\gamma^j \to \gamma'^j = \gamma^j + \delta\gamma^j$ and $z^j \to z'^j = z^j + \delta z^j$ must satisfy

$$\vec{\nabla}\left(\frac{s}{\chi}\,\delta\beta + \gamma^j\delta z^j\right) = \delta\beta\vec{\nabla}\,\frac{s}{\chi} - \delta\gamma^j\vec{\nabla}z^j + \delta z^j\vec{\nabla}\gamma^j = 0 \tag{3-31}$$

The form invariance of Eqs. (3-14)-(3-18) gives

$$\frac{\partial}{\partial t}\,(\delta\alpha) = \frac{d_n}{dt}\,(\delta\beta) = \frac{d_n}{dt}\,(\delta\vec{z}) = \frac{d_n}{dt}\,(\delta\vec{\gamma}) = 0 \tag{3-32}$$

which have the solutions

$$\delta\alpha = \varepsilon\alpha_o, \ \delta\beta = \delta\beta(s,\gamma,z), \ \delta\vec{\gamma} = \delta\vec{\gamma}(s,\gamma,z), \ \delta\vec{z} = \delta\vec{z}(s,\gamma,z) \tag{3-33}$$

Substitution of Eqs. (3-33) into Eqs. (3-31) implies that the infinitesimal gauge transformations have the form

$$\delta\beta = \varepsilon\,\frac{\partial G}{\partial(s/\chi)}, \quad \delta\gamma^j = -\,\varepsilon\,\frac{\partial G}{\partial z^i}, \quad \delta z^j = \varepsilon\,\frac{\partial G}{\partial\gamma^j} \tag{3-34}$$

where α_0 = constant, ε is an infinitesimal constant and G is a function of s/χ, $\vec{\gamma}$ and $\vec{z}$ homogeneous of degree one in s/χ and $\vec{\gamma}$ i.e.

$$G = \frac{s}{\chi} \frac{\partial G}{\partial (s/\chi)} + \gamma^j \frac{\partial G}{\partial \gamma^j} \tag{3-35}$$

From Eq. (1-25) the conservation law associated with this symmetry transformation is

$$\frac{\partial}{\partial t} (\rho\alpha_0 + \rho\chi G) + \vec{\nabla}\cdot(\rho\alpha_0\vec{V} + \rho\chi\vec{V}\, G) = 0 \tag{3-36}$$

By choosing G = 0 the conservation of mass equation is recovered while the choice $\alpha_0 = 0$, $G = s/\chi$ gives the conservation of entropy equation.

The Galilean transformation properties of ρ, s, $\vec{V}_n$, $\vec{V}_s$ and $\vec{z}$ are known;however the transformation properties of α, β and $\vec{\gamma}$ must be deduced by requiring that the Lagrangian be invariant.

(i) Under the translation of axes $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{x}$, $\vec{z} \to \vec{z}' = \vec{z} + \delta\vec{z}$ ($\rho, s, \chi, \vec{V}_n$ and $\vec{V}_s$ are unchanged). The Monge potentials α, β and $\vec{\gamma}$ transform as

$$\delta\alpha = \varepsilon\alpha_0 \ , \ \delta\beta = \varepsilon \frac{\partial G}{\partial (s/\chi)} \ , \ \delta\gamma^j = \varepsilon \frac{\partial G}{\partial z^j} \tag{3-37}$$

where $G = G(\frac{s}{\chi}, z)$ is homogeneous of degree one in s/χ. Eqs. (3-37) completely specify the Galilean tranformation properties of α, β and $\vec{\gamma}$. From Eq. (1-25) the conservation law associated with this symmetry is just the conservation of momentum equation (3-4).

(ii) Under the time translation $t \to t' = t + \delta t$ the initial position vector $\vec{z} \to \vec{z}' = \vec{z} - \vec{V}(z,o)\delta t$. Using the arguments developed in (i), the invariance of L implies $\vec{\gamma} \to \vec{\gamma}'^{j} = \vec{\gamma}^{j} - \gamma^{j}(\partial V^{j}(z,o)/\partial\vec{z})\delta t$ apart from a gauge transformation given by Eqs. (3-37). The conservation of energy equation associated with this symmetry follows from Eq. (1-25) as

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_n V_n^2 + \tfrac{1}{2}\rho_s V_s^2 + \rho e) + \vec{\nabla}\cdot(\tfrac{1}{2}\rho_n V_n^2 \vec{V}_n + \tfrac{1}{2}\rho_s V_s^2 \vec{V}_s + (\rho e + P)\vec{V}$$
$$+ \rho s T(\vec{V}_n - \vec{V}) + \tfrac{1}{2}\rho_n(\vec{V}_n - \vec{V}_s)^2(\vec{V}_n - \vec{V}))$$
$$= 0 \tag{3-38}$$

(iii) Under the rotation of axes $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{\theta} \times \vec{x}$, $\vec{V}_n \to \vec{V}_n' + \delta\vec{\theta} \times \vec{V}_n$, $V_s \to \vec{V}_s' = \vec{V}_s + \delta\theta \times \vec{V}_s$ and $\vec{z} \to \vec{z}' = \vec{z} + \delta\vec{\theta} \times \vec{z}$. The invariance of L gives $\vec{\gamma} \to \vec{\gamma}' = \vec{\gamma} + \delta\vec{\theta} \times \vec{\gamma}$ apart from a gauge transformation given by Eqs. (3-37). The conservation of angular momentum equation associated with this symmetry is

$$\frac{\partial}{\partial t}(\varepsilon^{ijk}x^{j}[\rho_n V_n^k + \rho_s V_s^k]) + \nabla^{\ell}(\varepsilon^{ijk}x^{j}[\rho_s V_s V_s^k + \rho_n V_n^{\ell} V_n^k$$
$$+ P\delta^{\ell k}]) = 0 \tag{3-39}$$

(iv) Under the Galilean boost $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{V}_o t$, $\vec{V}_n \to \vec{V}_n' + \delta\vec{V}_o$ and $\vec{V}_s \to \vec{V}_s' = \vec{V}_s + \delta\vec{V}_o$. The invariance of L gives $\alpha \to \alpha' = \alpha - \vec{x}\cdot\delta\vec{V}_o$ apart from a gauge transformation given by Eqs. (3-37). The conservation of center-of-mass associated with this symmetry is

$$\frac{\partial}{\partial t}(t[\rho_n V_n^i + \rho_s V_s^i] - \rho x^i) + \nabla^j(t[\rho_n V_n^i V_n^j + \rho_s V_s^i V_s^j] + Pt\delta^{ij}$$
$$- x^i[\rho_n V_n^j + \rho_s V_s^j]) = 0 \tag{3-40}$$

This completes the review of Herivel's variational principle for the Landau two-fluid equations. Chapter 4 will extend this variational principle to the two-fluid equations of rotating superfluid helium as formulated by Khalatnikov and Bekarevitch.

CHAPTER 4

A VARIATIONAL PRINCIPLE FOR SUPERFLUID HELIUM WITH VORTICITY

4-1 Introduction

It has been known for some time that superfluid vortices with circulation quantized in units of (h/m) can exist in superfluid helium.[36,37] The quantization of circulation is connected with the multiple-valuedness of the phase of the wavefunction of the Bose condensate.[38,39] The superfluid vorticity $\vec{\nabla} \times \vec{V}_s$ still vanishes everywhere on a microscopic scale except in the cores of vortices; however when $\vec{V}_s$ is averaged over a macroscopic region which contains a finite density of vortices then $\vec{\nabla} \times \vec{V}_s \neq 0$. If the averaging is done over a region large compared to the separation between vortices then the normal velocity $\vec{V}_n$ and the superfluid velocity $\vec{V}_s$ will be smoothly varying functions throughout the fluid.

Khalatnikov and Bekarevitch[25] (KB) have derived the equations of motion for the latter case with a phenomenological approach by allowing the internal energy of the fluid to depend on the absolute value of the superfluid vorticity. The hydrodynamic equations are then derived by the standard method[31] from Galilean invariance requirements and by manipulating the redundant conservation of energy equation. In this procedure a number of phenomenological coefficients appear which can be derived from a detailed vortex model. Hall[40] has examined the same problem by using a microscopic model of excitations interacting with vortices; the two-fluid equations he derives agree with KB. For a short review of this subject see the article by Chester.[41]

Lin[9] and more recently Lhuillier, Francois and Karatchentzeff[42] (LFK) have given variational principles incorporating a generalized Lin's constraint to describe the Landau two-fluid equations with microscopic superfluid vorticity, in distinction from the two-fluid equations of KB where only the macroscopic superfluid vorticity is non-vanishing. The purpose of Chapter 4 is to find an extension of Zilsel's variational principle which is equivalent to the two-fluid equations of KB with zero entropy production. The hydrodynamic equations are summarized in Sec. 4-2 and a Lagrangian for these equations is given in Sec. 4-3. It is found necessary to use two constraint equations, one constraint for $\vec{V}_n$ and as shown in Sec. 4-4, the other constraint giving the superfluid vortex equations of motion. A discussion of the symmetries and conservation laws is given in Sec. 4-5.

4-2 The Hydrodynamic Equations

Following KB the fundamental assumption is that the internal energy differential has the form

$$de = Tds + (P/\rho^2)d\rho + \tfrac{1}{2}(\vec{V}_n-\vec{V}_s)^2 d\chi + (\lambda/\rho)d\omega \tag{4-1}$$

where e is the specific internal energy, T is the temperature, s is the specific entropy, the pressure $P = \rho(-e + Ts + \tfrac{1}{2}(\vec{V}_n-\vec{V}_s)^2\chi + \mu)$, $\chi = (\rho_n/\rho)$ where ρ_n is the normal fluid density, μ is the chemical potential, λ is a phenomenological coefficent and $\omega = |\vec{\omega}|$ where $\vec{\omega} = \vec{\nabla} \times \vec{V}_s$. To make the notation agree with Zilsel the specific internal energy e differs from that of KB (denoted (ε/ρ)) by (ε/ρ)-e

$= \frac{1}{2} (\vec{V}_n - \vec{V}_s)\chi$. The meaning of P and μ is unchanged. From Eq. (4-1) and the definition of P follows the useful identity

$$\vec{\nabla}\mu = \frac{1}{\rho}\vec{\nabla}P - s\vec{\nabla}T - \frac{\chi}{2}\vec{\nabla}(\vec{V}_n - \vec{V}_s)^2 + \frac{\lambda}{\rho}\vec{\nabla}\omega \qquad (4\text{-}1)'$$

From KB the hydrodynamic equations with zero entropy production are

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho_s\vec{V}_s + \rho_n\vec{V}_n) = 0 \qquad (4\text{-}2)$$

$$\frac{\partial(\rho s)}{\partial t} + \vec{\nabla}\cdot(\rho s\vec{V}_n) = 0 \qquad (4\text{-}3)$$

$$\frac{d_s\vec{V}_s}{dt} + \vec{\nabla}\mu = (\beta' - \frac{1}{\rho_s})\vec{\omega} \times (\vec{\nabla}\times(\lambda\vec{\nu})) - \beta'\rho_s\vec{\omega} \times (\vec{V}_n - \vec{V}_s) \qquad (4\text{-}4)$$

$$\frac{\partial(\rho_s V_s^i + \rho_n V_n^i)}{\partial t} + \nabla^j(\rho_s V_s^i V_s^j + \rho_n V_n^i V_n^j + P\delta^{ij} + \lambda\omega\delta^{ij} - \lambda\omega^i\omega^j/\omega) = 0 \qquad (4\text{-}5)$$

where i,j, = 1,2,3 (sum repeated indices), $\vec{\nu} = \vec{\omega}/\omega$, $\frac{d_\gamma}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}_\gamma\cdot\vec{\nabla}$ and the superfluid density $\rho_s = \rho - \rho_n$. Following Zilsel the factor Γ is defined as

$$\frac{\partial\rho_n}{\partial t} + \vec{\nabla}\cdot(\rho_n\vec{V}_n) = -(\frac{\partial\rho_s}{\partial t} + \vec{\nabla}\cdot(\rho_s\vec{V}_s)) = \Gamma \qquad (4\text{-}2)'$$

Using Eq. (4-2)' if follows that Eqs. (4-3) and (4-5) can be rewritten respectively as

$$\frac{d_n(s/\chi)}{dt} = -\frac{s}{\rho\chi^2}\Gamma \qquad (4\text{-}3)'$$

$$\frac{d_n\vec{V}_n}{dt} + \frac{1}{\rho}\vec{\nabla}P + \frac{\rho_s}{\rho_n}S\vec{\nabla}T + \frac{\rho_s}{2\rho}\vec{\nabla}(\vec{V}_n-\vec{V}_s)^2 + (\vec{V}_n-\vec{V}_s)\frac{\Gamma}{\rho\chi} =$$

$$-\frac{\lambda}{\rho}\vec{\nabla}\omega + \frac{\beta'\rho_s}{\rho_n}\vec{\omega}\times(\vec{V}_n-\vec{V}_s-\frac{1}{\rho_s}\vec{\nabla}\times(\lambda\vec{\nu})) \qquad (4\text{-}5)'$$

The phenomenological coefficients λ,β' have been computed by KB from a vortex model as

$$\beta' = \frac{B'\rho_n}{2\rho\rho_s} \quad , \quad \lambda = \frac{h}{m}\rho_s \ln\left(\frac{R}{a}\right) \qquad (4\text{-}6)$$

where B' = constant, m is the mass of a He^4 atom and $\frac{R}{a}$ is the ratio of the distance between vortices to the effective radius of a vortex.

4-3 The Lagrangian

Consider the Lagrangian $L = L[\rho,s,\chi,\vec{V}_n,\vec{V}_s,\alpha,\beta,\gamma^j,z^j,\tilde{\gamma}^j,\tilde{z}^j]$ given by

$$L = \rho[\tfrac{1}{2}(1-\chi)V_s^2+\tfrac{1}{2}\chi V_n^2]-\rho e(\rho,s,\chi,\vec{\nabla}\tilde{\gamma}^j\times\vec{\nabla}\tilde{z}^j)-\alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho(1-\chi)\vec{V}_s+\rho\chi\vec{V}_n)$$

$$-\beta\{\frac{\partial(\rho s)}{\partial t}+\vec{\nabla}\cdot(\rho s\vec{V}_n) +\rho\chi\gamma^j\{\frac{\partial z^j}{\partial t}+\vec{V}_n\cdot\vec{\nabla}z^j\}$$

$$+\rho\tilde{\gamma}^j\{\frac{\partial z^j}{\partial t} + [(1-\chi)\vec{V}_s + \chi\vec{V}_n]\cdot\vec{\nabla}\tilde{z}^j \qquad (4\text{-}7)$$

where the representation for e is given by

$$de = Tds + (P/\rho^2)d\rho + \tfrac{1}{2}(\vec{V}_n-\vec{V}_s)^2 d\chi + (\lambda/\rho)d|\vec{\nabla}\tilde{\gamma}^j\times\vec{\nabla}\tilde{z}^j| \qquad (4\text{-}8)$$

From Eq. (4-8) it is easy to derive the following useful variational derivatives

$$\frac{\delta(\rho e)}{\delta \tilde{z}^j} = -[\vec{\nabla}\times(\lambda\vec{\nabla}\tilde{\gamma}^i\times\vec{\nabla}\tilde{z}^i/|\vec{\nabla}\tilde{\gamma}^k\times\vec{\nabla}\tilde{z}^k|)]\cdot\vec{\nabla}\tilde{\gamma}^j \tag{4-9}$$

$$\frac{\delta(\rho e)}{\delta \tilde{\gamma}^j} = [\vec{\nabla}\times(\lambda\vec{\nabla}\tilde{\gamma}^i\times\vec{\nabla}\tilde{z}^i/|\vec{\nabla}\tilde{\gamma}^k\times\vec{\nabla}\tilde{z}^k|)]\cdot\vec{\nabla}\tilde{z}^j \tag{4-10}$$

The variational equations of L are

$$\delta\alpha:\quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho(1-\chi)\vec{V}_s + \rho\chi\vec{V}_n) = 0 \tag{4-11}$$

$$\delta\beta:\quad \frac{\partial(\rho s)}{\partial t} + \vec{\nabla}\cdot(\rho s\vec{V}_n) = 0 \tag{4-12}$$

$$\delta\vec{V}_s:\quad \vec{V}_s = -\vec{\nabla}\alpha - \tilde{\gamma}^j\vec{\nabla}\tilde{z}^j \tag{4-13}$$

$$\delta\vec{V}_n:\quad \vec{V}_n = -\vec{\nabla}\alpha - \frac{s}{\chi}\vec{\nabla}\beta - \tilde{\gamma}^j\vec{\nabla}\tilde{z}^j - \gamma^j\vec{\nabla}z^j \tag{4-14}$$

$$\delta\rho:\quad \tfrac{1}{2}(1-\chi)V_s^2+\tfrac{1}{2}\chi V_n^2-e-\frac{P}{\rho}+\frac{\partial\alpha}{\partial t}+((1-\chi)\vec{V}_s+\chi\vec{V}_n)\cdot\vec{\nabla}\alpha+s\frac{d_n\beta}{dt}$$
$$+\tilde{\gamma}^j\{\frac{\partial\tilde{z}^j}{\partial t}+[(1-\chi)\vec{V}_s+\chi\vec{V}_n]\cdot\nabla\tilde{z}^j\} = 0 \tag{4-15}$$

$$\delta s:\quad \frac{d_n\beta}{dt} = T \tag{4-16}$$

$$\delta\chi:\quad -\tfrac{1}{2}\rho V_s^2+\tfrac{1}{2}\rho V_n^2-\tfrac{1}{2}\rho(\vec{V}_n-\vec{V}_s)^2+\rho(\vec{V}_n-\vec{V}_s)\cdot(\vec{\nabla}\alpha+\tilde{\gamma}^j\vec{\nabla}\tilde{z}^j) = 0 \tag{4-17}$$

$$\delta\gamma^j:\quad \frac{d_n z^j}{dt} = 0 \tag{4-18}$$

$$\delta z^j:\quad \frac{d_n\gamma^j}{dt} = -\frac{\Gamma}{\rho\chi}\gamma^j \tag{4-19}$$

$$\delta\tilde{\gamma}^j:\quad \frac{\partial\tilde{z}^j}{\partial t} + [(1-\chi)\vec{V}_s+\chi\vec{V}_n]\cdot\vec{\nabla}\tilde{z}^j = -\frac{1}{\rho}[\vec{\nabla}\times(\lambda\vec{v})]\cdot\vec{\nabla}\tilde{z}^j \tag{4-20}$$

$$\delta z^j: \frac{\partial \gamma^j}{\partial t} + [(1-\chi)\vec{V}_s + \chi\vec{V}_n]\cdot\vec{\nabla}\tilde{\gamma}^j = \frac{1}{\rho}[\vec{\nabla}\times(\lambda\vec{\nu})]\cdot\vec{\nabla}\tilde{\gamma}^j \tag{4-21}$$

Eqs. (4-2) and (4-3) are recovered as Eqs. (4-11) and (4-12) while Eqs. (4-13) and (4-8) give Eq. (4-1). When use is made of Clebsch's lemma

$$\frac{d\vec{w}}{dt} = -\vec{\nabla}\left(\frac{d\psi}{dt} + w^2/2\right) - \frac{d\eta^\alpha}{dt}\vec{\nabla}\xi^\alpha - \eta^\alpha\vec{\nabla}\frac{d\xi^\alpha}{dt} \tag{4-22}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{w}\cdot\vec{\nabla}$ and $\vec{w} = -\vec{\nabla}\psi - \eta^\alpha\vec{\nabla}\xi^\alpha$ ($\alpha=1,\ldots,m$) and the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ then Eqs. (4-11) - (4-21) imply after a lengthy but straightforward calculation

$$\frac{d_s\vec{V}_s}{dt} + \vec{\nabla}\mu = -\frac{1}{\rho}\vec{\omega}\times(\vec{\nabla}\times(\lambda\vec{\nu})) - \chi\vec{\omega}\times(\vec{V}_n-\vec{V}_s) \tag{4-23}$$

$$\frac{d_n\vec{V}_n}{dt} + \frac{1}{\rho}\vec{\nabla}P + \frac{\rho_s}{\rho_n}s\vec{\nabla}T + \frac{\rho_s}{2\rho}\vec{\nabla}(\vec{V}_n-\vec{V}_s)\frac{\Gamma}{\rho\chi} =$$

$$+\frac{\lambda}{\rho}\vec{\nabla}\omega - \frac{1}{\rho}\vec{\omega}\times(\vec{\nabla}\times(\lambda\vec{\nu})) + (1-\chi)\vec{\omega}\times(\vec{V}_n-\vec{V}_s) \tag{4-24}$$

which are Eqs. (4-4) and (4-5)' with B'=2. Note that due to the $(\lambda/\rho)d\omega$ terms in de the exact form of Eq. (4-20) is crucial to the gauge invariance of Eqs. (4-23) and (4-24) (i.e. that the potentials α and $\tilde{\gamma}^i$ can be eliminated from Eqs. (4-23) and (4-24)). Hence the variational principle is confined to the case B'=2.

The preceeding arguments have shown that all solutions of Eqs. (4-8) - (4-21) are solutions of Eqs. (4-1) - (4-6) with B'=2. The converse statement can be proven using a generalization of Weber's transformation of a classical one component fluid provided Eqs. (4-18) and (4-20) are adopted at the outset and it is assumed that $\tilde{z}^i = \tilde{z}^i(x,t)$ and $z^i - z^i(x,t)$ ($\tilde{z}^i$ and z^i denote the values of the functions $\tilde{z}^i(x,t)$ and $z^i(x,t)$ respectively) possess differentiable inverses $\tilde{x}^i(\tilde{z},t)$ and $x^i(z,t)$ respectively (indices are omitted unless necessary). This means that $x^i(z,(x,t),t) = x^i$, $\tilde{x}^i(\tilde{z}(x,t),t) = x^i$, $z^i(x(z,t),t) = z^i$ and $\tilde{z}^i(\tilde{x}(\tilde{z},t),t) = \tilde{z}^i$ (the x^i are the coordinates, $x^i(z,t)$ and $\tilde{x}^i(\tilde{z},t)$ are functions). The chain rule implies the relations

$$\frac{\partial x^i}{\partial z^j}\frac{\partial z^j}{\partial x^k} = \frac{\partial x^i}{\partial z^i}\frac{\partial z^k}{\partial x^j} = \frac{\partial \tilde{x}^j}{\partial \tilde{z}^i}\frac{\partial \tilde{z}^k}{\partial x^j} = \frac{\partial \tilde{x}^i}{\partial \tilde{z}^j}\frac{\partial \tilde{z}^j}{\partial x^k} = \delta^{ik} \tag{4-25}$$

where the partial derivatives have the meaning $\partial x^i/\partial z^j = \partial x^i(z,t)/\partial z^j$, $\partial z^j/\partial x^k = \partial z^j(x,t)/\partial x^k$, $\partial \tilde{x}^j/\partial \tilde{z}^i = \partial \tilde{x}^j(\tilde{z},t)/\partial \tilde{z}^i$, $\partial \tilde{z}^k/\partial x^j = \partial \tilde{z}^k(x,t)/\partial x^j$.

Define

$$\vec{V}_L = (1-\chi)\vec{V}_s + \chi\vec{V}_n + \frac{1}{\rho}\,\vec{\nabla}\times(\lambda\vec{\nu}) \tag{4-26}$$

then by using the chain rule the derivatives $\frac{d_L}{dt}$ and $\frac{d_n}{dt}$ have the meaning

$$\frac{d_L}{dt} f(x,t) = \frac{\partial}{\partial t} f(\tilde{x}(\tilde{z},t),t), \quad \frac{d_n}{dt} g(x,t) = \frac{\partial}{\partial t} g(x(z,t),t) \tag{4-27}$$

Eqs. (4-27) implies that $\frac{d_L}{dt}$ and $\frac{\partial}{\partial z^i}$ commute as do $\frac{d_n}{dt}$ and $\frac{\partial}{\partial z^i}$. By setting $f^i = x^i$ and $g^i = x^i$ Eqs. (4-27) imply

$$V_L^i = \frac{\partial \tilde{x}^i(\tilde{z},t)}{\partial t} = \frac{d_L x^i}{dt} \tag{4-28}$$

$$V_n^i = \frac{\partial x^i(z,t)}{\partial t} = \frac{d_n x^i}{dt} \tag{4-29}$$

which states that V_L^i and V_n^i are the tangents to the particle paths $\tilde{x}^i(\tilde{z},t)$ and $x^i(z,t)$ respectively.

Eq. (4-28) implies the identity

$$\frac{d_L}{dt}\,(V_s^j \frac{\partial \tilde{x}^j}{\partial \tilde{z}^j}) = (\frac{d_s V_s^j}{dt} + [\vec{\omega}\times(\vec{V}_L-\vec{V}_s)]^j)\,\frac{\partial \tilde{x}^j}{\partial \tilde{z}^j} + \frac{\partial}{\partial \tilde{z}^i}\,(V_s{}^2/2 + \vec{V}_s\cdot(\vec{V}_L-\vec{V}_s)) \tag{4-30}$$

Substituting Eq. (4-23) into Eq. (4-30) yields

$$\frac{d_L}{dt}\{\frac{\partial \tilde{x}^j}{\partial \tilde{z}^i}\,(V_s^j + \nabla^j[\int_0^t (\mu - V_s{}^2/2 - \vec{V}_s\cdot(\vec{V}_L-\vec{V}_s))\tilde{d}t])\} = 0 \tag{4-31}$$

where the integration is carried out at constant $\tilde{z}^i$. Eq. (4-31) can be integrated as

$$V_s^j = -\nabla^j[\int_0^t (\mu - V_s{}^2/2 - \vec{V}_s\cdot(\vec{V}_L-\vec{V}_s))\tilde{d}t] + V_s^i(\tilde{z},o)\nabla^j\tilde{z}^i = -\nabla^j\alpha - \tilde{\gamma}^i\nabla^j\tilde{z}^i \tag{4-32}$$

where $\alpha = \int_0^t (\mu - V_s{}^2/2 - \vec{V}_s\cdot(\vec{V}_L-\vec{V}_s))\tilde{d}t$ and $\tilde{\gamma}^i = -V_s^i(\tilde{z},o)$. It is easy to verify that $\vec{V}_s$, α and $\tilde{\gamma}^i$ satisfy Eqs. (4-13), (4-15) and (4-21) respectively.

Eq. (4-29) implies the identity

$$\frac{d_n}{dt}((V_n^j - V_s^j)\,\frac{\partial x^j}{\partial z^i}) = (\frac{d_n V_n^j}{dt} - \frac{d_s V_s^j}{dt})\,\frac{\partial x^j}{\partial z^i} + (V_n^k - V_s^k)(\nabla^j V_n^k - \nabla^k V_s^j)\,\frac{\partial x^j}{\partial z^i} \tag{4-33}$$

Substituting Eq. (4-25) into Eq. (4-33) gives

$$\frac{d_n}{dt}\{\frac{\chi}{s}\,\frac{\partial x^j}{\partial z^i}\,(V_n^j - V_s^j + \frac{s}{\chi}\,\nabla^j\beta)\} = 0 \tag{4-34}$$

where $\beta = \int_0^t T dt$ (the integration is carried out at constant z^i). Eq. (4-34) can be integrated as

$$V_n^j - V_s^j = -\frac{s}{\chi}\nabla^j\beta + \frac{s}{\chi}\frac{\chi(z,o)}{s(z,o)}(V_n^i(z,o)-V_s^i(z,o))\nabla^j z^i = -\frac{s}{\chi}\nabla^j\beta - \gamma^i\nabla^j z^i \quad (4\text{-}35)$$

where $\gamma^i = -(s\chi(z,o)/\chi s(z,o))(V_n^i(z,o)-V_s^i(z,o))$. It is easy to verify that $\vec{V}_n$, β and $\vec{\gamma}$ satisfy Eqs. (4-14), (4-16) and (4-19) respectively.

This proves that Eqs. (4-1) - (4-6) with B'=2 and Eqs. (4-18) and (4-20) are completely equivalent to the variational equations of L. Since Eqs. (4-18) and (4-20) are added to the hydrodynamic equations suitable physical interpretations must be given these equations. Just as in classical one-component hydrodynamics Eq. (4-18) may be viewed as an integrability constraint on the normal velocity field. In Sec. 4-3 Eq. (4-20) is interpreted as the statement that the superfluid vortices, averaged over many vortices, move with velocity $\vec{V}_L$.

By following Lin or LFK and using only Eq. (4-20) as a constraint an interesting problem develops. The Lagrangian and variational equations for this case are obtained just by dropping the terms involving γ^j and z^j. The equation for $\vec{V}_s$ remains unchanged but the equation for $\vec{V}_n$ becomes

$$\vec{V}_n - \vec{V}_s = -\frac{s}{\chi}\vec{\nabla}\beta \quad (4\text{-}36)$$

Multiplying Eqs. (35) and (36) by $\frac{\chi}{s}$ and taking the curl gives respectively

$$[\vec{\nabla}\times\{\frac{\chi}{s}(\vec{V}_n-\vec{V}_s)\}]^i = \varepsilon^{ijk}\nabla^j z^m \nabla^k z^p \frac{\partial}{\partial z^p}\{\frac{\chi(z,o)}{s(z,o)}(V_n^m(z,o)-V_s^m(z,o))\} \quad (4\text{-}37)$$

$$[\vec{\nabla} \times \{\frac{\chi}{s} (\vec{V}_n - \vec{V}_s)\}]^i = 0 \qquad (4\text{-}38)$$

where ε^{ijk} is the Levi-Civita symbol. A necessary condition that Eq. (4-37) agree with Eq. (4-28) for all values of z^m is

$$\varepsilon^{\ell mp} \frac{\partial}{\partial z^p} \{\frac{\chi(z,o)}{s(z,o)} (V_n^m(z,o) - V_s^m(z,o))\} = 0 \qquad (4\text{-}39)$$

or $(\chi(z,o)/s(z,o))(V_n^m(z,o) - V_s^m(z,o)) = \partial\psi(z)/\partial z^m$. But the β which appears in Eq. (35) may be subjected to a gauge transformation $\beta \to \beta' = \beta + \psi(z)$ (which does not alter Eq. (4-16)) which just cancels the $\gamma^i \nabla^j z^i$ terms. Hence Eq. (4-39) is a necessary and sufficient condition that Eq. (4-35) reduce to Eq. (4-36). Note that the preceeding arguments did not depend on the presence of the λ terms. Thus a necessary and sufficient condition that the variational principle given by Eq. (4-7) be equivalent to the hydrodynamic equations is that either constraint Eqs. (4-18) and (4-20) are used with $\vec{\nabla} \times \{(\vec{V}_n - \vec{V}_s)\chi/s\}$ arbitrary or only Eq. (4-20) is used with the initial constraint $\vec{\nabla} \times \{(\vec{V}_n - \vec{V}_s)\chi/s\} = 0$. Note that this is the proof of a claim made by LFK for pure $\vec{V}_s$ vortices. Since KB do not assume $\vec{\nabla} \times \{(\vec{V}_n - \vec{V}_s)\chi/s\} = 0$ both Eqs. (4-18) and (4-20) are needed.

4-4 Interpretation of $\vec{z}(x,t)$, $\vec{V}_L$ and the restriction B' = 2

Taking the curl of Eq. (4-23) yields an equation for the superfluid vorticity

$$\frac{d_L}{dt} (\frac{\omega^i}{\rho}) = \frac{\omega^j}{\rho} \nabla^j V_L^i \qquad (4\text{-}40)$$

which can be integrated using Eq. (4-28) as

$$\frac{\omega^i}{\rho} = \frac{\omega^j(z,o)}{\rho(z,o)} \frac{\partial x^i}{\partial z^j} \tag{4-41}$$

Eq. (4-41) just states that $\vec{\omega}/\rho$ is fixed relative to a set of coordinate axes composed of the same particles $\vec{x}(z,t)$ for all t or equivalently that $\vec{\omega}/\rho$ is transported with velocity $\vec{V}_L$. Since the macroscopic vorticity $\vec{\omega}$ was assumed to arise as a result of averaging over many superfluid vortices in some small volume this means it is consistent to interpret $\vec{x}(z,t)$ as a mean vortex path. Hence $\vec{z}(x,t)$ would be the initial position of a mean vortex with position $\vec{x}$ at time t. Eq. (4-28) then states that the mean superfluid vortices move with velocity $\vec{V}_L$. Apart from the λ terms this corresponds to the pure $\vec{V}_s$ vortices of LFK.

Note that the variational principle given by Eq. (4-7) is confined to the case B' = 2. If B' is arbitrary then taking the curl of Eq. (4-4) implies that the velocity of the vortices is given by

$$\vec{V}_L{}' = \frac{1}{\rho} (\rho_s\vec{V}_s + \rho_n\vec{V}_n) + \frac{\rho_n}{2\rho} (B' - 2)\vec{V}_n - \frac{\rho_n}{2\rho} (B' - 2)\vec{V}_s + \left(\frac{1}{\rho_s} - \frac{B'\rho_n}{2\rho\rho_s}\right) \vec{\nabla} \times (\lambda\vec{\nu}) \tag{4-42}$$

Since the λ terms arise solely as a result of the averaging procedure the usual Landau two-fluid equations must be regained in the limit $\lambda \to 0$. Hence the condition B' = 2 is necessary and sufficient for the superfluid vortices to travel with the mass flux velocity $\vec{V} = \frac{1}{\rho} (\rho_s\vec{V}_s+\rho_s\vec{V}_s)$.

On the other hand the work of LFK shows that the latter condition is equivalent to the requirement that superfluid vortices be regarded as singularities in the superfluid velocity $\vec{V}_s$. Thus the physical interpretation of the mathematically necessary restriction B' = 2 is that the superfluid vortices (before the averaging is carried out) be regarded as singularities in the superfluid velocity $\vec{V}_s$. Note that this view of superfluid vortices is consistent with Eq. (B-6).

A direct measurement of B' has been made by Snyder[43] (1963) and Snyder and Linekin[44] (1966) by observing the mode splitting in a rotating second-sound resonator.They found that B' was approximately zero and slightly temperature dependent. Lin[9] (1963) has critized this experiment on the grounds that secondary motion along the axis of rotation may occur. From Eq. (4-42),B' = 0 implies that superfluid vortices travel with velocity $\vec{V}_s$ (apart from the λ terms). LFK show that this corresponds to superfluid vortices being regarded as combinations of singularities in $\vec{V}_s$ and $\vec{A} = (\chi/s)\ (\vec{V}_n - \vec{V}_s)$, which is not consistent with Eq. (B-6). Hence there appears to be a contradiction between the experimental results of Snyder and Linekin and Eq. (4-1) of the KB equations. For further discussion of the experimental results and the case B' = 0 see Appendix D.

4-5 Symmetries and Conservation Laws

Just as in Sec. 3-3 there are certain gauge transformations of the Monge potentials α, β, γ^i, z^i, γ^i and z^i which do not change the values of $\vec{V}_n$ or $\vec{V}_s$ and which keep the variational equations form invariant. The requirement that $\vec{V}_n$ and $\vec{V}_s$ be unchanged and that Eqs. (4-13) and (4-14) be form invariant under the infinitesimal gauge transformations gives

$$\vec{\nabla}(\delta\alpha + \tilde{\gamma}^j\delta\tilde{z}^j) = -\delta\tilde{\gamma}^j\vec{\nabla}\tilde{z}^j + \delta\tilde{z}^j\vec{\nabla}\tilde{\gamma}^j \tag{4-43}$$

$$\vec{\nabla}(\delta\beta\,\frac{s}{\chi} + \gamma^j\delta z^j) = \delta\beta\vec{\nabla}\,(\frac{s}{\chi}) - \delta\gamma^j\vec{\nabla}z^j + \delta z^j\vec{\nabla}\gamma^j \tag{4-44}$$

The form invariance of Eqs. (4-15)-(4-21) gives

$$\frac{d'}{dt}(\delta\alpha) = \frac{d'}{dt}(\delta\tilde{\gamma}^j) = \frac{d'}{dt}(\delta\tilde{z}^j) = 0 \tag{4-45}$$

$$\frac{d_n}{dt}(\delta\beta) = \frac{d_n}{dt}(\delta\vec{\gamma}) = \frac{d_n}{dt}(\delta\vec{z}) = 0 \tag{4-46}$$

which have the solutions

$$\delta\alpha = \delta\alpha(\tilde{\gamma},\tilde{z}), \quad \delta\tilde{\gamma}^j = \delta\tilde{\gamma}^j(\tilde{\gamma},\tilde{z}), \quad \delta\tilde{z}^j = \delta\tilde{z}^j(\tilde{\gamma},\tilde{z}) \tag{4-47}$$

$$\delta\beta = \delta\beta(s/\chi,\gamma z), \; \delta\vec{\gamma} = \delta\vec{\gamma}(s/\chi,\gamma,z), \; \delta\vec{z} = \delta\vec{z}(s/\chi,\gamma,z) \tag{4-48}$$

Substitution of Eqs. (4-47) and (4-48) into Eqs. (4-43) and (4-44) give

$$(\varepsilon \frac{\partial G}{\partial \tilde{z}^j} + \delta\tilde{\gamma}^j)\vec{\nabla}\tilde{z}^j + (\varepsilon \frac{\partial G}{\partial \tilde{\gamma}^j} - \delta\tilde{z}^j)\vec{\nabla}\tilde{\gamma}^j = 0$$

$$(\varepsilon \frac{\partial H}{\partial(s/\chi)} - \delta\beta)\vec{\nabla}s + (\varepsilon \frac{\partial H}{\partial z^j} + \delta\gamma^j)\vec{\nabla}z^j + (\varepsilon \frac{\partial H}{\partial \gamma^j} - \delta z^j)\vec{\nabla}\gamma^j = 0 \tag{4-49}$$

where $\varepsilon G(\tilde{\gamma},\tilde{z}) \equiv \delta\alpha + \tilde{\gamma}^j\delta\tilde{z}^j$ and $\varepsilon H(s/\chi,\gamma,z) \equiv (s/\chi)\delta\beta + \gamma^j\delta z^j$. Eqs. (4-49) imply that the infinitesimal gauge transformations have the form

$$\delta\alpha = \varepsilon G + \varepsilon\tilde{\gamma}^j \frac{\partial G}{\partial\tilde{\gamma}^j}, \quad \delta\tilde{\gamma}^j = -\varepsilon \frac{\partial G}{\partial \tilde{z}^i}, \quad \delta\tilde{z}^j = \varepsilon \frac{\partial G}{\partial\tilde{\gamma}^j} \tag{4-50}$$

$$\delta\beta = \varepsilon \frac{\partial H}{\partial(\frac{s}{\chi})}, \quad \delta\gamma^j = -\varepsilon \frac{\partial H}{\partial z^j}, \quad \delta z^j = \varepsilon \frac{\partial H}{\partial \gamma^j} \tag{4-51}$$

where $G = G(\tilde{\gamma},\tilde{z})$ is an arbitrary function of $\tilde{\gamma}^j$ and $\tilde{z}^j$ and $H = H(\frac{s}{\chi},\gamma,z)$ is homogeneous of degree one in $\frac{s}{\chi}$ and $\tilde{\gamma}^j$. From Eq. (1-25) the conservation laws which arise from these symmetries are

$$\frac{\partial}{\partial t}(\rho G) + \vec{\nabla}\cdot(\rho G\vec{V}_L) = 0 \tag{4-52}$$

$$\frac{\partial}{\partial t}(\rho\chi H) + \vec{\nabla}\cdot(\rho\chi H\,\vec{V}_n) = 0 \tag{4-53}$$

Under the Galilean transformations the transformation properties of ρ, s, χ, $\vec{V}_n$, $\vec{V}_s$, z^i and $\tilde{z}^i$ are known while the transformation properties of α, β, $\tilde{\gamma}^i$ and γ^i are deduced by requiring that the Lagrangian remain invariant.

(i) Under the translation of axes $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{x}$ and $z^i \to z'^i = z^i + \delta\chi^i$, $\tilde{z}^i \to \tilde{z}'^i = \tilde{z}^i + \delta\chi^i$. In this case the gauge transformation is given by $G = G(\tilde{z})$ and $H = H(\frac{s}{\chi}, z)$. The conservation of momentum equation (4-5) follows from Eq. (1-25).

(ii) Under the time translation $t \to t' = t + \delta t$ and $z^i \to z'^i = z^i - V_n^i(z,o)\delta t$, $\tilde{z}^i \to \tilde{z}'^i = \tilde{z}^i - V_L^i(\tilde{z},o)\Delta t$. The Monge potentials $\tilde{\gamma}^i$ and γ^i transform as

$$\tilde{\gamma}^i \to \tilde{\gamma}'^i = \tilde{\gamma}^i + \gamma^j(\partial V^j(z,o)/\partial z^i)\delta t$$

and

$$\gamma^i \to \gamma'^i = \gamma^i + \gamma^j(\partial V^j(z,o)/\partial z^i)\delta t$$

apart from a guage transformation of the type $G = G(\tilde{z})$ and $H = H(\frac{s}{\chi}, z)$. From Eq. (1-25) the conservation of energy equation is

$$\begin{aligned}\frac{\partial}{\partial t}(\tfrac{1}{2}\rho_n V_n + \tfrac{1}{2}\rho_s V_s^2 + \rho e) &+ \vec{\nabla}\cdot(\tfrac{1}{2}\rho_n V_n^2\vec{V}_n + \tfrac{1}{2}\rho_s V_s^2\vec{V}_s + (\rho e + P)\vec{V} \\ &+ \rho s T(\vec{V}_n - \vec{V}) + \tfrac{1}{2}\rho_n(\vec{V}_n - \vec{V}_s)^2(\vec{V}_n - \vec{V}) + \lambda\frac{\vec{\omega}}{\omega}\times(\vec{V}_L\times\vec{\omega})) \\ &= 0\end{aligned} \tag{4-54}$$

(iii) Under the rotation of axes $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{\theta}\times\vec{x}$ and $\vec{V}_n \to \vec{V}_n' = \vec{V}_n + \delta\vec{\theta}\times\vec{V}_n$, $\vec{V}_s \to \vec{V}_s' = \vec{V}_s + \delta\vec{\theta}\times\vec{V}_s$, $\vec{z} \to \vec{z}' = \vec{z} + \delta\vec{\theta}\times\vec{z}$,

$\vec{\tilde{z}} \to \tilde{z}' = \tilde{z} + \delta\vec{\theta} \times \vec{\tilde{z}}$. The Monge potentials transform as $\vec{\gamma} \to \vec{\gamma} = \vec{\gamma}' + \delta\vec{\theta} \times \vec{\gamma}$ and $\vec{\tilde{\gamma}} \to \vec{\tilde{\gamma}}' = \vec{\tilde{\gamma}} + \delta\vec{\theta} \times \vec{\tilde{\gamma}}$ apart from a gauge transformation of the type $G = G(\tilde{z})$ and $H = H(\frac{s}{\chi}, z)$. From Eq. (1-25) the conservation of angular momentum equation is

$$\frac{\partial}{\partial t} (\varepsilon^{ijk}\rho x^j V^k) + \nabla^\ell(\varepsilon^{ijk}x^j[\rho_n V_n^k V_n^\ell + \rho_s V_s^k V_s^\ell + P\delta^{k\ell} + \lambda\omega\delta^{k\ell} - \lambda\omega^k\omega^\ell/\omega]) = 0 \tag{4-55}$$

(iv) Under the Galilean boost $\vec{x} \to \vec{x}' = \vec{x} + \delta\vec{V}_o t$ and $\vec{V}_n \to \vec{V}_n' = \vec{V}_n + \delta\vec{V}_o$, $\vec{V}_s \to \vec{V}_s' = \vec{V}_s + \delta\vec{V}_o$. The Monge potential $\alpha \to \alpha' = \alpha - \vec{x}\cdot\delta\vec{V}_o$ apart from a gauge transformation of the type $G = G(\tilde{z})$ and $H = H(\frac{s}{\chi}, z)$. From (1-25) the conservation of center-of-mass equation is

$$\frac{\partial}{\partial t} (\rho V^i t - \rho x^i) + \nabla^j([\rho_n V_n^i V_n^j + \rho_s V_s^i V_s^j + P\delta^{ij} + \lambda\delta\sigma^{ij} - \lambda\omega^i\omega^j/\omega]t - \rho x^i V^j) = 0 \tag{4-56}$$

In conclusion, the relaxation of the conservation of particle label constraint in two-fluid hydrodynamics gives rise to superfluid vorticity dependent contributions to the internal energy. The hydrodynamic equations describe superfluid helium with a finite density of superfluid vortices where all hydrodynamic variables are averaged over many superfluid vortices.

CHAPTER 5

LIN'S CONSTRAINT GENERALIZED TO INCLUDE HIGHER ORDER DERIVATIVES

5-1 An Extension of the Results of Chapter 2

The variational principle given in Chapter 2, which depends on derivatives no higher than first order, can easily be extended to arbitrary order derivatives. The fundamental assumption is that the internal energy of the fluid contains an additional contribution $\rho g(\omega^i, \nabla^j\omega^i, \nabla^{jk\cdots}\omega^i)$ which depends on the vorticity and gradients of the vorticity. Just as in Chapter 2, this theory may be interpreted as describing a fluid with a finite density of vortices where the hydrodynamic variables $\vec{V}$, ρ, s, and $\vec{z}$ have been averaged over a region containing many vortices. The additional $\nabla^i\omega^j$ terms arise from considering interactions between neighboring vortices.

The Lagrangian for this case is given by

$$L = \tfrac{1}{2}\rho V^2 - \rho e(\rho,s) - \rho g(\omega^i, \nabla^j\omega^i, \nabla^{jk\cdots}\omega^i) - \alpha\{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\} + \rho\beta\frac{ds}{dt} + \rho\gamma^j\frac{dz^j}{dt} \tag{5-1}$$

The dependent variables are $\{\vec{V},\rho,s,\vec{z},\alpha,\beta,\vec{\gamma}\}$ and $\vec{\omega} \equiv -\vec{\nabla}\beta \times \vec{\nabla}s - \vec{\nabla}\gamma^j \times \vec{\nabla}z^j$ (only after the variation of $\vec{V}$ is carried out can the identification $\vec{\omega} = \vec{\nabla} \times \vec{V}$ be made). The variational derivatives of ρg with respect to β, s, $\vec{\gamma}$ and $\vec{z}$ are straightforward to compute:

$$\frac{\delta}{\delta\beta}(-\rho g) = \vec{\nabla}s\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}(\rho g)], \quad \frac{\delta}{\delta s}(-\rho g) = -\vec{\nabla}\beta\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}(\rho g)] \tag{5-2}$$

with similar expressions for $\vec{\gamma}$ and $\vec{z}$.

The variational equations of L are given by

$$\delta\vec{V}: \quad \vec{V} = -\nabla\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j \tag{5-3}$$

$$\delta\alpha: \quad \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho V) = 0 \tag{5-4}$$

$$\delta\rho: \quad \frac{d\alpha}{dt} + \beta\,\frac{ds}{dt} + \gamma^j\,\frac{dz^j}{dt} + \tfrac{1}{2}\,V^2 - e - \frac{P}{\rho} - g = 0 \tag{5-5}$$

$$\delta\beta: \quad \rho\,\frac{ds}{dt} = -\vec{\nabla}s\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)] \tag{5-6}$$

$$\delta s: \quad \rho\,\frac{d\beta}{dt} = -\vec{\nabla}\beta\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)] - \rho T \tag{5-7}$$

$$\delta\gamma^j: \quad \rho\,\frac{dz^j}{dt} = -\vec{\nabla}z^j\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)] \tag{5-8}$$

$$\delta z^j: \quad \rho\,\frac{d\gamma^j}{dt} = -\vec{\nabla}\gamma^j\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)] \tag{5-9}$$

Clebsch's lemma yields the identity

$$\frac{d\vec{V}}{dt} = -\vec{\nabla}\,(\frac{d\alpha}{dt} + \tfrac{1}{2}\,V^2) - \frac{d\beta}{dt}\,\vec{\nabla}s - \beta\vec{\nabla}\,\frac{ds}{dt} - \frac{d\gamma^j}{dt}\,\nabla z^j - \gamma^j\nabla\,\frac{dz^j}{dt} \tag{5-10}$$

Substitution of Eqs. (5-3)-(5-9) into Eq. (5-10) gives

$$\begin{aligned}\rho\,\frac{dV^i}{dt} + \nabla^i P = {} & -\rho\nabla^i g + \vec{\nabla}\beta\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)]\nabla^i s \\ & - \vec{\nabla}s\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)]\nabla^i\beta + \vec{\nabla}\gamma\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)]\nabla^i z^j \\ & - \vec{\nabla}z^j\cdot\vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)]\nabla^i\gamma^j\end{aligned} \tag{5-11}$$

By using the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ Eq. (5-11) becomes

$$\rho\,\frac{dV}{dt} + \vec{\nabla}P = -\vec{\nabla}g + \vec{\nabla}\times[\frac{\delta}{\delta\vec{\omega}}\,(\rho g)]\times\vec{\omega} \tag{5-12}$$

or equivalently

$$\rho\,\frac{dV^i}{dt} + \nabla^j(P\delta^{ij} + T^{ij}) = 0 \tag{5-13}$$

The symmetric stress tensor T^{ij} is given by

$$T^{ij} = \delta^{ij}\omega^k \frac{\delta}{\delta\omega^k}(\rho g) - \omega^j \frac{\delta}{\delta\omega^i}(\rho g) + \nabla^i\omega^k \frac{\partial}{\partial\nabla^j\omega^k}(\rho g)$$

$$+ 2\nabla^{i\ell}\omega^k \frac{\partial}{\partial\nabla^{j\ell}\omega^k}(\rho g) - \nabla^\ell[\nabla^i\omega^k \frac{\partial}{\partial\nabla^{j\ell}\omega^k}(\rho g)] + \ldots \qquad (5\text{-}14)$$

Multiplication of Eq. (5-8) by $\partial z^j/\partial x^i$ yields the equivalent form

$$V = \frac{\partial\vec{x}(z,t)}{\partial t} - \frac{1}{\rho}\vec{\nabla} \times [\frac{\delta}{\delta\vec{\omega}}(\rho g)] \qquad (5\text{-}15)$$

which states that $\vec{V}' \equiv \vec{V} + \frac{1}{\rho}\vec{\nabla} \times [\frac{\delta}{\delta\vec{\omega}}(\rho g)]$ is the tangent to the particle paths $\vec{x} = \vec{x}(z,t)$. The equivalence of the hydrodynamic equations (5-4), (5-6) and (5-12) with the variational equations (5-3)-(5-9) follows from a straightforward extension of Weber's transformation as developed in Sec. 2-2. In addition the conservation law associated with the gauge invariance of L is unchanged from Eq. (2-30).

5-2 A Negative Result for Viscous Fluids

The variational principle given in Chapter 2 adds terms to the momentum equation which depend on second derivatives of the velocity. A natural question arises: Can the velocity terms $\rho^{-1}\vec{\nabla} \times \vec{\nabla} \times \vec{V}$ which occur in the momentum equation for incompressible, viscous fluids be derived solely from a generalized Lin's constraint of the type given in Chapter 2? Note that there is no chance of obtaining the compressible terms $\rho^{-1}\vec{\nabla}(\vec{\nabla}\cdot\vec{V})$ since these terms involve α which would alter the conservation of mass equation.

The variational principle of Chapter 2 uses the fact that the Lin's constraint may be generalized from $dz^j/dt = 0$ to $dz^j/dt = \rho^{-1}(\vec{\nabla}\times A)\cdot\vec{\nabla}z^j$

without affecting the conservation of mass equation and without destroying the gauge invariance of the momentum equation. Consider the Lagrangian

$$L = \tfrac{1}{2}\,\rho V^2 - \rho e(\rho,s) - \alpha\left\{\frac{\partial P}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V})\right\} + \rho\beta\left\{\frac{ds}{dt} + \frac{1}{\rho}\,\vec{\nabla}\times\vec{A}\cdot\vec{\nabla}s\right\}$$

$$+ \rho\gamma^j\left\{\frac{dz^j}{dt} + \frac{1}{\rho}\,\vec{\nabla}\times\vec{A}\cdot\vec{\nabla}z^j\right\} \qquad (5\text{-}16)$$

where $\vec{A} = \vec{A}(\vec{\nabla}z)$. The variational equations are

$$\delta\vec{V}:\quad V = -\vec{\nabla}\alpha - \beta\vec{\nabla}s - \gamma^j\vec{\nabla}z^j \qquad (5\text{-}17)$$

$$\delta\alpha:\quad \frac{\partial P}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \qquad (5\text{-}18)$$

$$\delta\rho:\quad \frac{d\alpha}{dt} + \beta\,\frac{ds}{dt} + \gamma^j\,\frac{dz^j}{dt} - e - \frac{P}{\rho} = 0 \qquad (5\text{-}19)$$

$$\delta\beta:\quad \rho\,\frac{ds}{dt} = -\vec{\nabla}s\cdot\vec{\nabla}\times\vec{A} \qquad (5\text{-}20)$$

$$\delta s:\quad \rho\,\frac{d\beta}{dt} = -\vec{\nabla}\beta\cdot\vec{\nabla}\times\vec{A} - \rho T \qquad (5\text{-}21)$$

$$\delta\gamma^j:\quad \rho\,\frac{dz^j}{dt} = -\vec{\nabla}z^j\cdot\vec{\nabla}\times\vec{A} \qquad (5\text{-}22)$$

$$\delta z^j:\quad \rho\,\frac{d\gamma^j}{dt} = -\vec{\nabla}\gamma^j\cdot\vec{\nabla}\times\vec{A} - \frac{\delta}{\delta z^j}\,(\vec{A}\cdot\vec{\nabla}\times\vec{V}) \qquad (5\text{-}23)$$

Clebsch's lemma combined with the vector identity $\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ yields the momentum equation

$$\frac{dV^i}{dt} + \frac{1}{\rho}\,\nabla^i P = \frac{1}{\rho}\,[(\vec{\nabla}\times\vec{A})\times(\vec{\nabla}\times\vec{V})]^i + \frac{1}{\rho}\,\frac{\partial z^j}{\partial x^i}\,\frac{\delta}{\delta z^j}\,(\vec{A}\cdot\vec{\nabla}\times\vec{V}) \qquad (5\text{-}24)$$

By expanding the variational derivative for z^j Eq. (5-24) can be written as

$$\frac{dV^i}{dt} + \frac{1}{\rho} \nabla^i P = - \frac{1}{\rho} \left[\varepsilon^{kmn} \frac{\partial A^k}{\partial \nabla^\ell z^j} \frac{\partial z^j}{\partial x^i} \right] \nabla^{m\ell} V^n$$

$$- \frac{1}{\rho} \nabla^\ell \left[\frac{\partial A^k}{\partial \nabla^\ell z^j} \right] \frac{\partial z^j}{\partial x^i} \nabla^m V^n + \frac{1}{\rho} [(\vec{\nabla}\times\vec{A}) \times (\nabla\times V)]^i \quad (5\text{-}25)$$

A necessary condition to obtain $(\mu/\rho)\vec{\nabla} \times \vec{\nabla} \times \vec{V}$ on the R.H.S. of Eq. (5-25) is ($[\mu] \simeq \frac{M}{LT}$)

$$\varepsilon^{kn(m} \frac{\partial A^k}{\partial \nabla^{\ell)} z^j} \frac{\partial z^j}{\partial x^i} = \mu \varepsilon^{iP(m}{}_{\varepsilon}{}^{\ell)Pn} \quad (5\text{-}26)$$

Multiplying Eq. (5-26) by $\partial x^i/\partial z^r$ and summing over m and ℓ gives

$$\varepsilon^{kmn} \frac{\partial A^k}{\partial \nabla^m z^j} = 2\mu \frac{\partial x^n}{\partial z^j} \quad (5\text{-}27)$$

Differentiating Eq. (5-27) by $\partial/\partial(\nabla^n z^j)$ gives

$$0 = 2\mu \frac{\partial(\partial x^n/\partial z^j)}{\partial(\partial z^j/\partial x^n)} \equiv -2\mu \frac{\partial x^n}{\partial z^j} \frac{\partial x^n}{\partial z^j} \neq 0 \quad (5\text{-}28)$$

where the derivatives of $\partial x^n/\partial z^j$ are computed from Eqs. (1-8). Eq. (5-28) shows that no solution for $\vec{A}$ exists, hence the following theorem: The velocity terms $\rho^{-1}\vec{\nabla}(\vec{\nabla}\cdot\vec{V})$ and $\rho^{-1}\vec{\nabla} \times \vec{\nabla} \times \vec{V}$ which occur in the momentum equation for viscous fluids cannot be derived solely from a generalized Lin's constraint of the form $dz^j/dt = \rho^{-1}(\vec{\nabla} \times \vec{A})\cdot\vec{\nabla}z^j$. The theorem can easily be extended to include $\vec{A} = \vec{A}(\rho,s,\nabla z)$. It is possible that a different modification of the Lin's constraint could generate the incompressible terms $\rho^{-1}\vec{\nabla} \times \vec{\nabla} \times \vec{V}$, but this would give no insight into the compressible terms $\rho^{-1}\vec{\nabla}(\vec{\nabla}\cdot\vec{V})$ or the entropy production equation. This suggests that a generalized Lin's constraint plays no role in describing viscous fluids,whose variational formulation remains an unsolved problem.

5-3 Conclusion

This thesis has presented a new extension of Clebsch's variational principle for perfect fluids, based on a generalized version of the conservation of particle label constraint. For the one-component case the variational principle gave a macroscopic description of a fluid with a finite density of vortices, for the two-fluid case it yielded the Khalatnikov equations for rapidly rotating superfluid ^{4}He. To the author's knowledge this variational principle has not been described in the literature and represents original research.

The considerable difficulties in finding a variational principle for fluids are connected with the presence of first order derivatives and non-linearities in the equations of motion. Clebsch's solution of this problem sharply restricts the form of the constraint equations for the mass density, the entropy and the particle label. The author has shown that the conservation of particle label constraint may be generalized to $\partial z^j/\partial t + (\vec{V} + \rho^{-1}\vec{\nabla} \times \vec{A})\cdot\vec{\nabla}z^j = 0$ without destroying the "gauge" invariance of the conservation of momentum equation.

A review of C.C. Lin's Lagrangian for the adiabatic case was given in Chapter 1. The consequences of relaxing Lin's constraint for a one-component fluid were examined in Chapter 2 and yielded vorticity dependent contributions to the internal energy and the stress tensor of the fluid. In Appendix B, the hydrodynamic equations were interpreted as describing a fluid with a finite density of vortices, where all hydrodynamic variables have been averaged over regions containing many vortices.

As background material, Chapter 3 reviewed Herivel's variational principle for the Landau two-fluid equations. In Chapter 4 a new variational principle for the Khalatnikov equations of rapidly rotating superfluid ^{4}He was presented. It was found necessary to use two Lin's constraints, one constraint for the normal velocity field and the other constraint expressing the possibility of labelling a superfluid vortex, averaged over many vortices.

Chapter 5 concluded with an extension of the variational principle to arbitrary orders of derivatives in the vorticity. In addition it was shown that a generalized Lin's constraint cannot be used to describe viscous fluids.

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APPENDIX A

PROOF OF CLEBSCH'S LEMMA

Define

$$\vec{V}(x,t) = \vec{\nabla}A(x,t) + B^{\alpha}(x,t)\vec{\nabla}C^{\alpha}(x,t) \tag{A-1}$$

where $\alpha = 1,\ldots,m$. Straightforward differentiation yields the results

$$\frac{\partial\vec{V}}{\partial t} = \vec{\nabla}\left(\frac{\partial A}{\partial t}\right) + \frac{\partial B^{\alpha}}{\partial t}\vec{\nabla}C^{\alpha} + B^{\alpha}\vec{\nabla}\left(\frac{\partial C^{\alpha}}{\partial t}\right) \tag{A-2}$$

while a lengthy algebraic manipulation gives the identity

$$(\vec{V}\cdot\vec{\nabla})\vec{V} + \vec{\nabla}(\tfrac{1}{2}\,V^2) = \vec{\nabla}(\vec{V}\cdot\vec{\nabla}A) + (\vec{V}\cdot\vec{\nabla}B^{\alpha})\vec{\nabla}C^{\alpha} + B^{\alpha}\vec{\nabla}(\vec{V}\cdot\vec{\nabla}C^{\alpha}) \tag{A-3}$$

Addition of Eqs. (A-2) and (A-3) gives Clebsch's lemma

$$\frac{d\vec{V}}{dt} = \vec{\nabla}\left(\frac{dA}{dt} - \tfrac{1}{2}\,V^2\right) + \frac{dB^{\alpha}}{dt}\vec{\nabla}C^{\alpha} + B^{\alpha}\vec{\nabla}\left(\frac{dC^{\alpha}}{dt}\right) \tag{A-4}$$

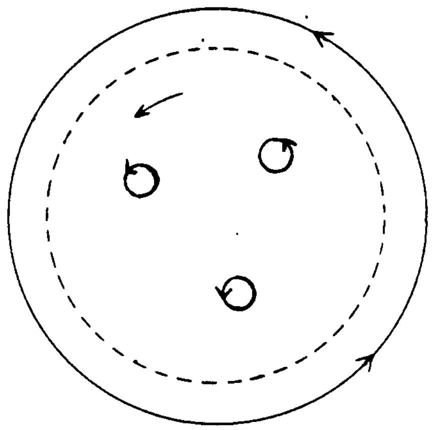
where $d/dt \equiv \partial/\partial t + (\vec{V}\cdot\vec{\nabla})$.

APPENDIX B

INTERPRETATION OF THE EQUATIONS OF MOTION

First consider the case of the rotating superfluid as discussed in Chapter 4. To maintain the irrotational condition $\vec{\nabla} \times \vec{V}_s = 0$ in a rotating superfluid, superfluid vortices are formed (possibly at the boundary) which rotate rigidly with the container, i.e.

Figure 1. Rotating Superfluid ^{4}He



Integrating around a closed path enclosing all the vortices gives a relation between the angular rotation ω and the total number of vortices N. If the vortices each have a strength $h/2\pi m$ then

$$2\omega A = \int \vec{\nabla} \times \vec{V}_s \cdot d\vec{A} = \oint \vec{V}_s \cdot d\vec{\ell} = Nh/m \qquad \text{(B-1)}$$

or $N/A = 2\omega m/h$. Hence for a rapidly rotating superfluid many superfluid vortices are formed to maintain the irrotational condition.

Consider an array of such vortices each with core size a and some mean separation b. Since the velocity field of each vortex is

$$\vec{V}_s = (h/2\pi mr)\vec{\theta} \tag{B-2}$$

then the energy per unit length of a vortex is

$$E' = \int_a^b (\tfrac{1}{2}\rho_s V_s^2)\, 2\pi r\, dr = (\rho_s h^2/4\pi m^2)\ln(b/a) \tag{B-3}$$

Now define an average velocity field $\langle\vec{V}_s\rangle$ by averaging over a region containing many superfluid vortices in such a way that the circulation is due to the enclosed vortices. If $\langle V_s\rangle$ does not vary appreciably over the area enclosed then

$$|\vec{\nabla} \times \langle\vec{V}_s\rangle| A = \int \vec{\nabla} \times \langle\vec{V}_s\rangle \cdot d\vec{A} = \int \langle\vec{V}_s\rangle \cdot d\vec{\ell} = Nh/m \tag{B-4}$$

Hence the number of vortices per unit area is

$$\frac{N}{A} = |\vec{\nabla} \times \vec{V}_s| m/h \tag{B-5}$$

(dropping the average symbol $\langle\ \rangle$) and the energy per unit volume due to the vortices is

$$\varepsilon = (\rho_s h/4\pi m)\ln(b/a)|\vec{\nabla} \times \vec{V}_s| \equiv \lambda|\vec{\nabla} \times \vec{V}_s| \tag{B-6}$$

The total internal energy is given by the generalized Gibbs relation

$$de(\rho,s,\chi,\omega) = T\,ds + (P/\rho^2)d\rho + \tfrac{1}{2}(\vec{V}_n - \vec{V}_s)^2 d\chi + (\lambda/\rho)d\omega \tag{B-7}$$

where $\chi = \rho_n/\rho$ and $\vec{\omega} \equiv \vec{\nabla} \times \vec{V}_s$. The Khalatnikov equations follow from Eq. (B-7) by the standard technique in hydrodynamics, namely manipulating the redundancy of the conservation of total energy equation.

The conservation of mass equation remains unchanged

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \tag{B-8}$$

where $\vec{J} = \rho_n\vec{V}_n + \rho_s\vec{V}_s$, $\rho = \rho_n + \rho_s$. The conservation of momentum and total energy equations become

$$\frac{\partial J^i}{\partial t} + \nabla^j(\Pi^{ij} + \pi^{ij}) = 0 \tag{B-9}$$

$$\frac{\partial E}{\partial t} + \nabla\cdot(\vec{Q} + \vec{q}) = 0 \tag{B-10}$$

where

$$\Pi^{ij} = \rho_n V_n^i V_n^j + \rho_s V_s^i V_s^j + P\delta^{ij} \quad ,$$

$$P = \rho(-e + T_s + \tfrac{1}{2}(\vec{V}_n - \vec{V}_s)^2\chi + \mu) \quad ,$$

$$E = \rho e + \tfrac{1}{2}\,\rho_n V_n^2 + \tfrac{1}{2}\,\rho_s V_s^2 \quad ,$$

$$\vec{Q} = \tfrac{1}{2}\,\rho_n V_n^2\vec{V}_n + \tfrac{1}{2}\,\rho_s V_s^2\vec{V}_s + (\rho e + P)\vec{J}/\rho + \rho T_s(\vec{V}_n - \vec{V}_s)$$
$$+ \tfrac{1}{2}\,\rho(\vec{V}_n - \vec{V}_s)^2(\vec{V}_n - \vec{J}/\rho)$$

and π^{ij}, $\vec{q}$ remain to be determined. The entropy equation remains unchanged.

$$\frac{\partial}{\partial t}(\rho s) + \vec{\nabla}\cdot(\rho s\vec{V}_n) = 0 \tag{B-11}$$

while the superfluid equations become

$$\frac{\partial\vec{V}_s}{\partial t} + (\vec{V}_s\cdot\vec{\nabla})\vec{V}_s + \vec{\nabla}\mu = \vec{f} \tag{B-12}$$

where f is to be determined.

Eqs. (B-8)-(B-12) give nine equations in eight unknowns $\vec{V}_n$, $\vec{V}_s$, s, ρ; the self-consistency conditions determine π^{ij}, $\vec{q}$ and $\vec{f}$. From the definition of E and using Eqs. (B-7), (B-9) and (B-12) it follows that

$$\frac{\partial E}{\partial t} = -\vec{\nabla}\cdot\vec{Q} - \nabla^i(\pi^{ij}V_n^j) + T[\frac{\partial}{\partial t}(\rho s) + \vec{\nabla}\cdot(\rho s\vec{V}_n)$$

$$+ \lambda\frac{\partial\omega}{\partial t} + \lambda V_n^i\nabla^i\omega + \pi^{ij}\nabla^j V_n^j + (\vec{J} - \rho\vec{V}_n)\cdot(\vec{f} + \vec{\omega}\times(\vec{V}_n - \vec{V}_s)) \tag{B-13}$$

Taking the curl of Eq. (B-12) gives

$$\lambda\frac{\partial\omega}{\partial t} = \lambda\vec{\nu}\cdot\nabla x[\vec{f} + \omega\times(V_n - V_s)] - \lambda\vec{\nu}\cdot\vec{\nabla}\times(\vec{\omega}\times\vec{V}_n) \tag{B-14}$$

where $\vec{\nu} \equiv \vec{\omega}/\omega$. Substitution of Eqs. (B-11) and (B-14) into Eq. (B-13) gives

$$\frac{\partial E}{\partial t} + \nabla^i[Q^i + \pi^{ij}V_n^j + \lambda\{\vec{\nu}\times(\vec{f} + \vec{\omega}\times(\vec{V}_n - \vec{V}_s)\}^i$$

$$= (\pi^{ij} - \lambda\omega\delta^{ij} + \lambda\omega^i\omega^j/\omega)\nabla^j V_n^i$$

$$+ [\vec{f} + \vec{\omega}\times(\vec{V}_n - \vec{V}_s)]\cdot[\vec{J} - \rho V_n + \vec{\nabla}\times(\lambda\vec{\nu})] \tag{B-15}$$

Comparison of Eqs. (B-10) and (B-15) shows that

$$q^i = \pi^{ij}V_n^j + \lambda\{\vec{\nu}\times(\vec{f} + \vec{\omega}\times(V_n - V_s)\}^i \tag{B-16}$$

$$\pi^{ij} = \lambda\omega\delta^{ij} - \lambda\omega^i\omega^j/\omega \tag{B-17}$$

$$[\vec{f} + \vec{\omega}\times(\vec{V}_n - \vec{V}_s)]\cdot[\vec{J} - \rho\vec{V}_n + \vec{\nabla}\times(\lambda\vec{\nu})] = 0 \tag{B-18}$$

The lowest order solution of Eq. (B-18) which holds for all values of ρ, $\vec{V}_n$ and $\vec{V}_s$ is

$$\vec{f} = -\vec{\omega}\times(\vec{V}_n - \vec{V}_s) + \alpha\vec{\omega}\times(\vec{J} - \rho\vec{V}_n + \vec{\nabla}\times(\lambda\vec{\nu})) \tag{B-19}$$

In order that the Landau equations be regained in the limit $\lambda \to 0$, the parameter $\alpha = \rho_s^{-1}$. Substitution of Eqs. (B-17) and (B-19) into Eqs. (B-9) and (B-12) give the Khalatnikov equations as presented in Chapter 4, justifying their interpretation as a macroscopic theory of superfluid helium. Requiring that there be many vortices per unit area and that the superfluid velocity be less than the critical velocity limits the applicability of the Khalatnikov equations to the region $(h/mL^2) \leq \omega \leq (h/mL^2)\ln(\frac{L}{a})$ where a is the core radius and L is some length characteristic of the flow.

A similar interpretation can be given the one-component fluids discussed in Chapter 2. Consider a perfect fluid with an array of vortices present, each vortex separated by a mean distance b, with a velocity field

$$\vec{V} = (\gamma/r)\vec{\theta} \tag{B-20}$$

and a core size a. The energy of each vortex per unit length is

$$E' = \int_a^b (\tfrac{1}{2}\,\rho V^2)2\pi r\;dr = \pi\rho\gamma^2\;\ln(b/a) \tag{B-21}$$

Now define an average velocity field $\langle\vec{V}\rangle$ by requiring that the circulation of $\langle\vec{V}\rangle$ around some closed path be equal to the circulation due to the enclosed vortices. If $\langle\vec{V}\rangle$ does not vary appreciably over the enclosed area then

$$|\vec{\nabla} \times \langle\vec{V}\rangle|A = \int \vec{\nabla} \times \langle\vec{V}\rangle\cdot d\vec{A} = \oint \langle\vec{V}\rangle\cdot d\ell = 2\pi N\gamma \tag{B-22}$$

Thus the number of vortices/unit area is

$$\frac{N}{A} = \frac{\omega}{2\pi\gamma} \tag{B-23}$$

where $\vec{\omega} \equiv \vec{\nabla} \times \vec{V}$ (the average symbol is omitted). The energy/unit volume due to the vortices is

$$\varepsilon = \tfrac{1}{2}\,\rho\gamma\;\ln(b/a)\omega \tag{B-24}$$

If the vortex strength γ is allowed to depend on the number of vortices/unit area then Eq. (B-23) defines an implicit function for $N/A = g'(\omega)$ and hence Eq. (B-24) can be written as

$$\varepsilon = \rho g(\omega) \tag{B-25}$$

The total internal energy of the fluid is just $\rho e(\rho,s,\omega) = \rho e(\rho,s) + \rho g(\omega)$ where $e(\rho,s)$ is the usual expression for the internal energy i.e.,

$$de(\rho,s) = T\,ds + (P/\rho^2)d\rho \tag{B-26}$$

The hydrodynamic equations follow from the redundancy of the total energy equation. The conservation of mass equation remains unchanged.

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0 \tag{B-27}$$

while the conservation of momentum, energy and entropy equations take the form

$$\frac{\partial}{\partial t}(\rho V^i) + \nabla^j(\Pi^{ij} + \pi^{ij}) = 0 \tag{B-28}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(\vec{Q} + \vec{q}) = 0 \tag{B-29}$$

$$\frac{\partial}{\partial t}(\rho s) + \vec{\nabla}\cdot(\rho s\vec{V}) + R = 0 \tag{B-30}$$

where $\Pi^{ij} = \rho V^i V^j + P\delta^{ij}$, $E = \frac{1}{2}\rho V^2 + \rho e + \rho g$, $\vec{Q} = [\frac{1}{2}\rho V^2 + \rho e + \rho g]\vec{V}$ and π^{ij}, $\vec{q}$ and R remain to be determined.

From the definition of E and Eqs. (B-27), (B-29) it follows that

$$\frac{\partial E}{\partial t} + \nabla^i(Q^i) = T\left(\frac{\partial}{\partial t}(\rho s) + \vec{\nabla}\cdot(\rho s\vec{V})\right) - V^j\nabla^i\pi^{ij} + \rho\,\frac{\partial g}{\partial\omega}\left(\frac{\partial\omega}{\partial t} + \vec{V}\cdot\vec{\nabla}\omega\right) \tag{B-31}$$

Taking the curl of Eq. (B-28) gives

$$\frac{\partial \omega}{\partial t} = \varepsilon^{ijk} \nu^i \nabla^j [-(\vec{\omega} \times \vec{V})^k + T\nabla^k s - \rho^{-1} \nabla^\ell \pi^{k\ell}] \tag{B-32}$$

where $\vec{\nu} \equiv \vec{\omega}/\omega$. Eqs. (B-31), (B-32) yield

$$\frac{\partial E}{\partial t} + \nabla^i (Q^i + \rho \frac{\partial g}{\partial \omega} \varepsilon^{ijk} \nu^j [-(\vec{\omega} \times \vec{V})^k + T\nabla^k s - \rho^{-1} \nabla^\ell \pi^{k\ell}])$$

$$= T \frac{\partial}{\partial t} (\rho s) + \vec{\nabla} \cdot (\rho s \vec{V}) + \vec{\nabla} s \cdot \vec{\nabla} \times (\rho \frac{\partial g}{\partial \omega} \vec{\nu}))$$

$$+ [-\rho^{-1} \nabla^j \pi^{ij} - (\vec{\omega} \times \vec{V})^i][\rho V^i + \vec{\nabla} \times (\rho \frac{\partial}{\partial \omega} \vec{\nu})^i] \tag{B-33}$$

Comparison of Eqs. (B-29), (B-30) and (B-33) shows that

$$R = \vec{\nabla} s \cdot \vec{\nabla} \times (\rho \frac{\partial g}{\partial \omega} \vec{\nu}) \tag{B-34}$$

$$[-\rho^{-1} \nabla^j \pi^{ij} - (\vec{\omega} \times \vec{V})^i][\rho V^i + \vec{\nabla} \times (\rho \frac{\partial}{\partial \omega} \vec{\nu})^i] \equiv 0 \tag{B-35}$$

The lowest order solution of Eq. (B-35) is

$$\nabla^i \pi^{ij} = \alpha\, \varepsilon^{ijk} w^j [V^k + \rho^{-1} \vec{\nabla} \times (\rho \frac{\partial g}{\partial \omega} \vec{\nu})^k] - \rho^{-1} (\vec{\omega} \times \vec{V})^i \tag{B-36}$$

The requirement that the usual equations be regained in the limit $g \to 0$ fixes $\alpha = p^{-1}$. Eq. (B-36) becomes

$$\nabla^i \pi^{ij} = \nabla^i (\rho \frac{\partial}{\partial \omega} [\omega^2 \delta^{ij} - \omega^i \omega^j]) \tag{B-37}$$

and hence Eqs. (B-27)-(B-30) reduce to the hydrodynamic equations of Sec. 2-2, justifying their interpretation as a macroscopic description of a perfect fluid with a density of vortices present.

APPENDIX C

TURBULENT SOLUTIONS OF THE EQUATIONS OF MOTION

The irregular and disordered flow of a fluid known as turbulence is characterized by vorticity in three dimensions and by energy transfer from the large scales of motion to the small scales, ending in dissipation. It is usually assumed that turbulent fluid flow is described by the Navier-Stokes equations and experimentally it is found that turbulence occurs when the Reynolds number $R \gtrsim 20{,}000$. Since the hydrodynamic equations of Sec. 2-2 describe a fluid with a distribution of vorticity, it is worthwhile to check whether they can provide a model for turbulence.

A useful concept in the statistical description of turbulence is the two-point velocity correlation function defined as the time average

$$\langle V^i(x,t)V^j(x',t\;)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T V^i(x,t+t')V^j(x',t+t')dt' \qquad \text{(C-1)}$$

which relates adjacent fluctuations in $V^i(x,t)$ and $V^j(x,t)$. Turbulence is usually assumed to be an incompressible flow, where the statistical properties are homogeneous, isotropic and time-independent. Since space and time derivatives are assumed to commute with the averaging process, homogeneity implies the relations

$$\left\langle\frac{\partial V^i(x)}{\partial x^j} V^k(x')\right\rangle = \frac{\partial}{\partial x^j}\langle V^i(x)V^k(x')\rangle = \frac{\partial}{\partial r^j}\langle V^i(x)V^k(x+r)\rangle \qquad \text{(C-2)}$$

where $\vec{x}' = \vec{x}+\vec{r}$. Isotropy implies that

$$\langle V^i(x+r)P(x)\rangle \equiv 0 \qquad \text{(C-3)}$$

for any scalar function $P(x)$.

The following relations involving two-point correlations of the vorticity will prove useful

$$<\omega^i(x)\omega^j(x')> = -\varepsilon^{ik\ell}\varepsilon^{jmn} \frac{\partial^2}{\partial r^k \partial r^m} <V^\ell(x)V^n(x + r)> \tag{C-4}$$

Contraction over i and j gives

$$<\vec{\omega}(x)\cdot\vec{\omega}(x + r)> = - \frac{\partial^2}{\partial r^i \partial r^i} <\vec{V}(x)\cdot\vec{V}(x + r)>$$

$$= \tfrac{1}{2} \frac{\partial^2}{\partial r^i \partial r^i} <[\vec{V}(x) - \vec{V}(x + r)]^2> \tag{C-5}$$

In Kolmogoroff's theory of turbulence the fluid flow is pictured as a superposition of eddies of various sizes with energy being transferred from larger to smaller eddies at a constant rate $\varepsilon_o[L^2T^{-3}]$. The energy is ultimately dissipated by viscous effects in the smallest eddies of length scales $(\varepsilon_o \nu^{-3})^{-1/4}$ (ν is the kinematic viscosity). It is assumed that in the time-independent regime the statistics are completely determined by ε_o and ν. Furthermore for those eddies smaller than the largest scales but larger than viscous scales, the inertial subrange, the statistics are completely determined by ε_o.

By dimensional analysis Kolmogoroff's theory then predicts the form of the two-point correlation function in the inertial subrange as

$$<[\vec{V}(x) - \vec{V}(x + r)]^2> \simeq 4K\, \varepsilon_o^{2/3}\, r^{2/3} \tag{C-6}$$

where $K \simeq .5$ is Kolmogoroff's constant. Eq. (C-6) has been well-verified by experiment and any successful model of turbulence must reproduce this result.

Adopting Kolmogoroff's assumption that only ε_o be used in the inertial subrange implies by dimensional analysis that $g(\omega) = -K'\varepsilon_o\omega^{-1}$ where K' is a dimensionless constant. In the incompressible, isentropic case the hydrodynamic equations of Sec. 2-2 reduce to

$$\vec{\nabla}\cdot\vec{V} = 0 \tag{C-7}$$

$$\frac{\partial\vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} = \vec{\nabla}\left(-\frac{P}{\rho} - g\right) - \vec{\omega}\times\vec{\nabla}\times(K'\varepsilon_o\vec{\omega}\omega^{-3}) \tag{C-8}$$

$$\vec{V} = \frac{\partial x(z,t)}{\partial t} - \vec{\nabla}\times(K'\varepsilon_o\vec{\omega}\omega^{-3}) \tag{C-9}$$

Now assume that $\partial\vec{x}/\partial t$ is identified with the mean flow $\langle\vec{V}\rangle$ and that $-\vec{\nabla}\times(K'\varepsilon_o\vec{\omega}\omega^{-3})$ is the turbulent part of the velocity field. If $\langle\vec{V}\rangle = 0$ then Eq. (C-7) reduces to an identity and Eqs. (C-8) and (C-9) become

$$\frac{\partial\vec{V}}{\partial t} = \vec{\nabla}\left(-\frac{P}{\rho} - g\right) \tag{C-10}$$

$$\vec{V} = -\vec{\nabla}\times(K'\varepsilon_o\vec{\omega}\omega^{-3}) \tag{C-11}$$

Multiplying Eq. (C-10) by $V^i(x + r)$ and averaging gives the time independent condition

$$\frac{\partial}{\partial t}\langle V^i(x)V^i(x + r)\rangle = -\frac{\partial}{\partial r^i}\langle V^i(x + r)\left(\frac{P}{\rho} + g\right)\rangle \equiv 0 \tag{C-12}$$

Taking the curl of Eq. (C-11) gives

$$\omega^i = -\nabla^i(K'\varepsilon_o\vec{\nabla}\cdot(\vec{\omega}\omega^{-3})) + \nabla^2(K'\varepsilon_o\omega^i\omega^{-3}) \tag{C-13}$$

Multiplying Eq. (C-13) by $w^i(x + r)$ and averaging gives

$$\langle\vec{\omega}(x)\cdot\vec{\omega}(x+r)\rangle = K'\varepsilon_o\frac{\partial^2}{\partial r^i\partial r^i}\langle\vec{\omega}(x+r)\cdot\vec{\omega}(x)\omega(x)^{-3}\rangle \tag{C-14}$$

To close Eq. (C-14) it is sufficient to assume the closure relation

$$<\vec{\omega}(x+r)\cdot\omega(x)\,\omega(x)^{-3}> \;=\; K''<\vec{\omega}(x+r)\cdot\vec{\omega}(x)>^{-\frac{1}{2}} \tag{C-15}$$

Note that Eq. (C-15) does not follow from the Lagrangian. Eq. (C-14) gives

$$<\vec{\omega}(x)\cdot\vec{\omega}(x+r)> \;=\; K'K''\varepsilon_o \frac{\partial^2}{\partial r^i \partial r^i} <\vec{\omega}(x+r)\cdot\vec{\omega}(x)>^{-\frac{1}{2}} \tag{C-16}$$

which has the solution

$$<\vec{\omega}(x)\cdot\vec{\omega}(x+r)> \;=\; (10\ K'K''\varepsilon_o/9)^{2/3}\ r^{-4/3} \tag{C-17}$$

The identity Eq. (C-5) implies that the two-point velocity correlation function is given by

$$<[\vec{V}(x) - V(x+r)]^2> \;=\; 9/2(10\ K'K''\varepsilon_o/9)^{2/3}\ r^{2/3} \tag{C-18}$$

which agrees with Eq. (C-6) provided Kolmogoroff's constant is identified as

$$K = 9/8(10\ K'K''/9)^{2/3} \tag{C-19}$$

Hence the hydrodynamic equations of Sec. 2-2 provide a model of turbulence which is consistent with Kolmogoroff's theory in the inertial subrange. From Eqs. (B-23)-(B-25) it follows that this model of turbulence consists of a fluid with a density of vortices, whose individual vortex strengths depend inversely on the vortex density to the two thirds power, i.e.

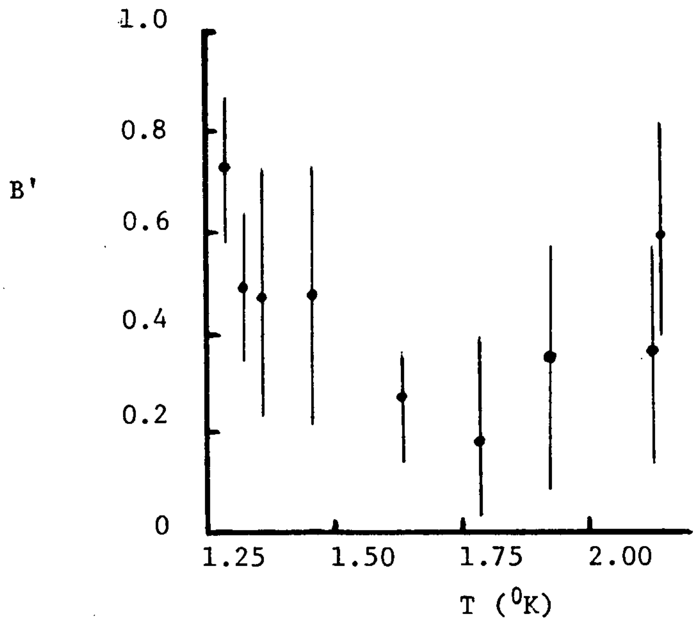
$$\gamma\ \alpha(N/A)^{-2/3} \tag{C-20}$$

APPENDIX D

MEASUREMENT OF B' FROM SECOND SOUND IN ROTATING ^4He

The parameter B' has been measured by Snyder[43], Snyder and Linekin[44] and Lucas[45]. The experiment of Snyder and Linekin is described briefly as follows: a standing wave of second sound with frequency σ was excited in a second sound resonator, rotating with angular velocity $\Omega\hat{z}$; the Coriolis force and the B' term removed the degeneracy of the two lowest second sound normal modes; the frequency of these normal modes was determined by observing the resonant frequencies of the temperature fluctuation on the resonator wall. Using a cylindrical resonator Lucas found B' = .08 ± .08 at 1.603 K and B' = .2 ± .25 at 1.426 K, the results of Snyder and Linekin using a square resonator are given in Figure 2.

Figure 2. B' versus T



Both these experiments suggest that B' is much less than 2 and temperature dependent.

Donnelly[46] has given a detailed derivation of the frequency splitting of the degenerate second sound modes; a brief summary of this derivation is given as follows. Linearization of Eqs. (4-2)-(4-5) yields

$$\dot{\vec{V}}_n = -\frac{1}{\rho}\vec{\nabla}P - \frac{\rho_s}{\rho_n}s\vec{\nabla}T + \frac{B'\rho_s}{2\rho}\vec{\omega}\times\vec{q} \tag{D-1}$$

$$\dot{\vec{V}}_s = -\frac{1}{\rho}\vec{\nabla}P + s\vec{\nabla}T - \frac{B'\rho_n}{2\rho}\vec{\omega}\times\vec{q} \tag{D-2}$$

$$\dot{\rho} + \rho_s\vec{\nabla}\cdot\vec{V}_s + \rho_n\vec{\nabla}\cdot\vec{V}_n = 0 \tag{D-3}$$

$$\rho\dot{s} + s\dot{\rho} + \rho s\vec{\nabla}\cdot\vec{V}_n = 0 \tag{D-4}$$

where $\vec{q} = \vec{V}_n - \vec{V}_s$ and it is assumed that $\vec{\omega} = 2\Omega\hat{z}$. Subtracting Eq. (D-2) from Eq. (D-1) and transforming to the rotating frame gives

$$\dot{\vec{q}} + (2-B')\vec{\Omega}\times\vec{q} = -\frac{\rho s}{\rho_n}\vec{\nabla}T \tag{D-5}$$

Using $\dot{s} = C\dot{T}$ and combining Eqs. (D-3) and (D-4) yields

$$\dot{T} = -\frac{\rho_s s}{C\rho}\vec{\nabla}\cdot\vec{q} \tag{D-6}$$

where C is the specific heat. Differentiation of Eq. (D-5) combined with Eq. (D-6) gives a wave equation for $\vec{q}$

$$\ddot{\vec{q}} + (2-B')\vec{\Omega}\times\dot{\vec{q}} = u_2^2\vec{\nabla}(\vec{\nabla}\cdot\vec{q}) \tag{D-7}$$

where $u_2 = \frac{\rho_s s^2}{\rho_n C}$ is the speed of second sound.

Comparison with the case of first sound propagation in a classical rotating fluid then gives the second sound normal mode frequencies as

$$\sigma_m^{\pm} = \sigma_{om} \pm \frac{4(2-B')\Omega}{m^2\pi^2} \tag{D-8}$$

where m is an odd interger and σ_{om} is the degenerate normal mode frequency in the non-rotating case corresponding to the intergers $\pm m$. Hence the Coriolis force and the B' term cause a split in the normal mode frequencies $\Delta\sigma_m = \frac{8(2-B')\Omega}{m^2\pi^2}$. Note that gradients in ρ_n and ρ_s would contribute a term $\frac{\Gamma}{\rho_n}\dot{\vec{q}}$ to the LHS of Eq. (D-7), which causes attenuation but does not alter the splitting of the normal modes. If B' = 2 then no splitting occurs, in contradiction with the results of Snyder and Linekin. Lin[9] has suggested that some sort of macroscopic secondary motion may occur, supported by the rotation, in contradiction to the assumptions leading to Eq. (D-8).

If B' = 0 then Eq. (4-42) states that the superfluid vortices travel with velocity $\vec{V}_s$ (apart from the λ terms). According to LFK this corresponds to the superfluid vortices being regarded as combined $\vec{A}$ and $\vec{V}_s$ singularities. This suggests that normal fluid vortices may be formed in the experiment of Snyder and Linekin. If this is the case then both $\vec{\nabla}\times\vec{V}_s$ and $\vec{\nabla}\times\vec{V}_n$ must be treated as thermodynamic variables and Eq. (4-1) will have an extra contribution due to $\vec{\nabla}\times\vec{V}_n$. This poses an interesting question for future consideration: Can the contributions of $\vec{\nabla}\times\vec{V}_s$ and $\vec{\nabla}\times\vec{V}_n$ to Eq. (4-1) be appropriately adjusted such that a variational principle can be found and such that the temperature dependence B' agrees with the results of Snyder and Linekin ? This remains an open question at present. Note that for the experiment of Snyder and Linekin the Reynolds number of the normal fluid at 1.8 °K is given by $R = r^2\Omega\rho_n/\eta_n \approx (2\text{ cm})^2(2\pi\text{ rad/sec})(.05\text{ g/cm}^3)(10^{-5}\text{ g cm/sec})^{-1}$ $\approx 13{,}000$ which suggests that normal fluid vortices may have an important effect on the measurement of B'.