VOl-TEX MOTION IN THIN FILMS

by

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B.Sc. The University of Toronto, 1976

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES
THE DEPARTMENT OF PHYSICS

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

December, 1979

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The classical theory of rectilinear vortex motion has been generalized to include vortices in thin fluids of varying depth on curved surfaces. The equations of motion are examined to lowest order in a perturbation expansion in which the depth of fluid is considered small in comparison with the principal radii of curvature of the surface. Existence of a generalized vortex streamfunction is proved and used to generate conservation laws. A number of simple vortex systems are described. In particular, criteria for the stability of rings of vortices on surfaces of revolution are found. In contradistinction to the result of von Karman, double rings (vortex streets) in both staggered and symmetric configurations may be stable. The effects of finite core size are examined. Departures from radial symmetry in core vorticity distributions are shown to introduce small wobbles in the vortex motion. The case of an elliptical core is treated in detail. Applications of the theory to atmospheric cyclones and superfluid vortices are discussed.
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ACKNOWLEDGEMENT

I would like to thank Prof. F.A. Kaempffer for initiating this work and for his advice and criticism throughout.

The financial assistance of both the University of British Columbia (Macmillan Graduate Fellowship) and the Natural Sciences and Engineering Research Council of Canada (Postgraduate Fellowship) is gratefully acknowledged.
I. INTRODUCTION

In 1858 Helmholtz published a paper which marked the beginning of the study of vorticity in fluids. He showed that in a fluid in which the momentum production at each point is the gradient of a scalar potential, the vortex lines are advected. Helmholtz went on to show that if the fluid is incompressible, two-dimensional, and its vorticity is zero except at N isolated points, then the partial differential equations describing the motion of the fluid can be reduced to a 2Nth order system of ordinary differential equations describing the positions of the points of vorticity. Although the original equations are non-linear, each point of vorticity is associated with a velocity field which obeys the principle of superposition: that is, the total velocity field is the sum of the velocity fields associated with the points of vorticity. Each point of vorticity, along with its velocity field, is called a vortex. In practice, the vorticity of a vortex is not confined to a single point, but is distributed over some small region known as the core. The position and velocity (not to be confused with the velocity field) of a vortex refer to the position and rate of change of position of the core.

Lord Kelvin (1869) later reformulated Helmholtz' theorems and introduced the idea of circulation; the circulation around a contour C is: \( \Gamma_C = \oint_C \mathbf{v} \cdot d\mathbf{s} \), where \( \mathbf{v} \) is the fluid velocity. Kelvin showed that, under similar assumptions to those made by Helmholtz, the circulation of any advected
contour is constant in time. Moreover, if $C$ and $C'$ both enclose the same set of vortices then $\Gamma_C = \Gamma_{C'}$. The strength, $\gamma$, of a vortex is proportional to the circulation around any contour enclosing that vortex and no other.

Routh (1881) first introduced the vortex streamfunction (a Hamiltonian for the system of equations governing the vortex motion) although his work was much generalized by Kirchoff (1876), Lagally (1921), Masotti (1931) and Lin (1943). As in any Hamiltonian system, there are conservation laws associated with invariances of the Hamiltonian under infinitesimal coordinate transformations. For vortex systems invariance under translations leads to conservation of centre of circulation, under rotations to conservation of moment of circulation, and under time translations to conservation of the vortex streamfunction (Batchelor (1967), p.530). In addition, there is another conserved quantity known as the angular moment of circulation. It is discussed in detail in Section IV.6.

Since the publication of Lin (1943) there has been little work on the formal theory of vortex motion. However, much has been written on the application to real fluid systems; in particular, the prediction of Onsager (1949) that superfluid HeII would support vortices of quantized circulation sparked renewed interest in the subject which persists to this day. Other applications are to atmospheric weather systems and to flows past bluff bodies (e.g., the Karman vortex street).
Almost all the work on vortices to date has assumed that the vortices are rectilinear: that is infinitely long and parallel so that the problem is reduced to two-dimensional flow in the plane. Lamb (1916), in his well-known text, briefly outlines a method for determining the motion of vortices on a curved surface under the assumption that the fluid depth is uniform and small in comparison with the principal radii of curvature of the surface. It is the purpose of this thesis to examine such systems in more detail and to generalize further to fluids of non-uniform depth but in which the depth variation is small.

After defining coordinates which prove useful in later sections (Section II), the equations governing the motion of an ideal fluid (one in which the momentum production is the gradient of a scalar) are derived and subjected to a perturbation scheme in which the depth of the fluid and the vertical component of the velocity field are assumed small (Section III). Vortex solutions to the lowest order equations of motion are examined in Section IV under the additional assumption of incompressible flow. Systems of vortices are shown to be governed by a generalized vortex streamfunction which casts the equations of motion into symplectic form. A relation between the vortex streamfunction and the kinetic energy is shown and conservation laws corresponding to invariances under simple infinitesimal transformations are generated. In particular, the conservation of angular moment of circulation is shown to be related to invariance under
scale transformations. When the fluid depth is uniform, it is shown that the vortex streamfunction transforms simply under conformal transformations of coordinates.

In Section V, the effects of surface curvature and depth variation on some simple vortex systems are examined. The motion of a single vortex and of pairs of vortices past localized surface bumps and depressions in depth are analyzed. Also, the stabilities of some rigidly rotating configurations of vortices on surfaces of revolution are determined. The results are generalizations of those first derived by Lord Kelvin (1878), Thomson (1883) and von Karman (1912). In particular, it is found that, in contradistinction to the results of von Karman, double rings of symmetrically placed vortices (vortex streets) may be stable while similar staggered rings are unstable.

In Section VI, the effects of finite distributions of vorticity within the vortex cores are examined. It is argued that small departures from circular cores introduce wobbles into the motion of each vortex, but that, for times long enough that the vortex travels the order of the mean vortex separation, the systematic drift of the core closely approximates that of a circular core. The simplest case of elliptical cores is examined in detail.

The evolution of the core can be examined in a perturbation scheme having a vortex in the plane as its lowest order solution. It is shown that if a vortex is to remain circular for an appreciable length of time then its vorticity distribution must have a precise form.
The work of Sections II-VI was motivated largely by an attempt to provide a first order model for atmospheric cyclones in which the curvature of the earth is treated exactly, not just as a linear perturbation of a plane or a cone. Almost every analytic model of terrestrial cyclones is derived via the β-plane approximation in which effects of surface curvature are included only in the latitudinal variation of the Coriolis parameter. The β-plane approximation is not consistent (see, e.g., Veronis (1963a)) since variations of the Coriolis parameter are important only if the system is large enough that other effects of surface curvature are also important.

Our approach is from the other direction; Coriolis forces have been neglected entirely (clearly a drastic assumption if one really expects a good model of a terrestrial cyclone) in an attempt to highlight the effects of surface curvature alone. A good model of a large cyclone must affect some compromise. The problems inherent in a simple vortex model including both Coriolis and surface curvature effects are discussed in Section VII.1.

Another problem motivating this work is the following. A rotating bucket of liquid HeII is known to support superfluid vortices. Its surface is also known to be a paraboloid of revolution (Osborne et.al. (1963)). How does the variation in depth of liquid helium affect the motion and distribution of the vortices? In Section VII.2 the extent to which the theory of Sections II-VI is applicable to superfluid vortices is discussed.
II. COORDINATES

Throughout this work it will be found that a particular choice of coordinates is necessary to make the mathematics tractable. Here the properties of these coordinates are discussed at length.

II.1 Harmonic Coordinates

Let $M$ be a two-dimensional, oriented, Riemannian manifold with metric $g_{ij}$ . It is possible to choose coordinates $(x,y)$ such that the metric has the form:

$$g_{ij}(x,y) = \delta_{ij}h^2(x,y) \quad (\text{II.1.1})$$

where $h(x,y)$ is a real-valued differentiable non-negative function (see, for example, Eisenhart (1909)). $(x,y)$ are called harmonic coordinates. In general we shall only be concerned with some sub-domain $D$ of $M$ ($D$ is that part of the surface over which the fluid flows). For simplicity it will always be assumed that $D$ is parametrized completely and unambiguously by $(x,y)$ . This amounts to a restriction to those surfaces topologically equivalent to multiply connected sub-domains of the complex plane or sphere. More complex surfaces (e.g., a torus) will not be considered.

Polar harmonic coordinates are defined by:

$$x = r \cos \phi ; \quad y = r \sin \phi \quad (\text{II.1.2})$$

The line element in terms $(r,\phi)$ is:

$$ds^2 = h^*(r,\phi)(dr^2 + r^2 d\phi^2) \quad (\text{II.1.3})$$

with $h^*(r,\phi) = h(x,y)$ . In future the asterisk will be dropped if there is no chance of confusion between $h^*$ and $h$ .
II.2 Surfaces of Revolution

The line element of a surface of revolution can be put in the form:

\[ ds^2 = g_u^2(u)du^2 + g_\phi^2(u)d\phi^2 \]  

(II.2.1)

where \( \phi \) is an angular coordinate of period \( 2\pi \). Let \( r \equiv r(u) \) and require that \( (r,\phi) \) be harmonic polars. Then:

\[
\begin{align*}
\frac{r'(u)}{r(u)} &= \frac{g_u(u)}{g_\phi(u)} \\
\frac{r(u)}{h(r(u))} &= \frac{g_\phi(u)}{r(u)}
\end{align*}
\]

(II.2.3)

so that:

\[
\begin{align*}
\frac{r'(u)}{r(u)} &= \frac{g_u(u)}{g_\phi(u)} \\
\frac{r(u)}{g_\phi(u)} &= \frac{r(u)}{h(r(u))}
\end{align*}
\]

(II.2.4)

(II.2.5)

It will prove convenient to define:

\[
p(r) = \frac{h'(r)r}{h(r)} + 1
\]

(II.2.6)

Then:

\[
p(r(u)) = \frac{r^2(u)}{g_\phi(u)r'(u)} \frac{d}{du} \left( \frac{g_\phi(u)}{r(u)} \right) + 1
\]

\[
= \frac{g_\phi(u)}{g_u(u)}
\]

(II.2.7)
Also:

\[ rp'(r) = \frac{r(u)}{r'(u)} \frac{d}{du} \left( \frac{g_\phi(u)}{g_u(u)} \right) \]
\[ = \frac{g_\phi(u)}{g_u(u)} \frac{d}{du} \left( \frac{g_\phi(u)}{g_u(u)} \right) \quad (\text{II.2.8}) \]

The Gaussian curvature of a surface is defined by:

\[ K = -\frac{g^{ik}}{2} \left[ \frac{\partial g^n}{\partial x^k} - \frac{\partial g^k}{\partial x^n} + \Gamma_i^{n} \Gamma^m_{kn} - \Gamma_i^{m} \Gamma^m_{kn} \right] \quad (\text{II.2.9}) \]

with:

\[ \Gamma_i^{k} = \frac{g^{km}}{2} \left[ \frac{\partial g_{im}}{\partial x^n} + \frac{\partial g_{mn}}{\partial x^i} - \frac{\partial g_{in}}{\partial x^m} \right] \quad (\text{II.2.10}) \]

In harmonic coordinates:

\[ K = \frac{-1}{h^2} \frac{\partial}{\partial x^i} \left( \frac{1}{h} \frac{\partial h}{\partial x^i} \right) = \frac{-1}{h^2} \nabla^2 \ln h \quad (\text{II.2.11}) \]

For a surface of revolution \( h = h(r) \) and

\[ K = \frac{-1}{h^2 r} \frac{d}{dr} \left( \frac{r}{h} \frac{dh}{dr} \right) = \frac{p'(r)}{rh^2(r)} \quad (\text{II.2.12}) \]

The above definitions apply for any manifold such that \( h = h(r) \). For physical applications one is generally only interested in those manifolds which can be imbedded in \( \mathbb{R}^3 \). We therefore determine \( r, h(r) \) and \( p(r) \) for surfaces obtained by the revolution of a function about an axis.

Let \( \rho, \phi \) and be cylindrical coordinates in \( \mathbb{R}^3 \) and consider the surface of revolution defined by: \( \rho = f(z) \) (see Figure I). One can use \( z \) and \( \phi \) to parametrize the surface. The line element in these coordinates is:
Fig. I: The Surface of Revolution $\rho = f(z)$
\[ ds^2 = (1 + (f'(z))^2)dz^2 + f^2(z)d\phi^2 \] (II.2.13)

whence, using (II.2.4-6):

\[ r = \exp \int_{z_0}^{z} \frac{(1 + (f'(s))^2)^{\frac{1}{2}}ds}{f(s)} \] (II.2.14)

\[ rh(r) = f(z) \] (II.2.15)

\[ p(r) = \frac{f'(z)}{(1 + (f'(z))^2)^{\frac{1}{2}}} \] (II.2.16)

Notice that \(-1 \leq p(z) \leq 1\). If the slope of \( f(z) \) is \( \tan \theta \) then \( p(z) = \sin \theta \).

Similarly, one can define a surface of revolution by:

\[ z = b(\rho) \]. The line element is:

\[ ds^2 = (1 + (b'(\rho))^2)d\rho^2 + \rho^2d\phi^2 \] (II.2.17)

and

\[ r = \exp \int_{\rho_0}^{\rho} \frac{(1 + (b'(s))^2)^{\frac{1}{2}}ds}{s} \] (II.2.18)

\[ rh(r) = \rho \] (II.2.19)

\[ p(r) = \frac{1}{(1 + (b'(\rho))^2)^{\frac{1}{2}}} \] (II.2.20)

If the slope of \( b(\rho) \) is \( \tan \theta \) then \( p(r) = \cos \theta \).

The function \( \frac{d}{dr}(\ln h(r)) \) will also be of importance later. Notice that:

\[ \frac{d}{dr}(\ln h(r)) = \frac{(p(r) - 1)}{r} \leq 0 \] (II.2.21)

since: \( p(r) \leq 1 \).
Special Cases

a) Plane

For the plane $b(\rho) = \text{const.}$ whence:

$$r = \rho \quad ; \quad h(r) = 1 \quad ; \quad rp'(r) = 0 \quad ; \quad K = 0 \quad \text{(II.2.22)}$$

b) Cylinder

For the cylinder $f(z) = R = \text{const.}$ whence:

$$r = \exp(z/R) \quad ; \quad h(r) = R/r = R\exp(-z/R) \quad ;$$
$$p(r) = 0 \quad ; \quad rp'(r) = 0 \quad ; \quad K = 0 \quad \text{(II.2.23)}$$

c) Sphere

For the sphere $g_{\theta}(\theta) = R \quad ; \quad g_{\phi}(\phi) = R\sin\theta$ where $\theta$ is the colatitude. Then:

$$r = \tan^{1/2}\theta \quad ; \quad h(r) = \frac{2R}{(1+r^2)} = 2\cos^{1/2}\theta \quad ;$$
$$p(r) = \frac{(1-r^2)}{(1+r^2)} = \cos\theta \quad ; \quad rp'(r) = \frac{-4r}{(1+r^2)} = -\sin^2\theta$$
$$K = \frac{1}{R^2} \quad \text{(II.2.24)}$$
II.3 Coordinates for a Thin Film

Consider, now a fluid bounded by two surfaces, \( M_1 \) and \( M_2 \) having similar topological characteristics. The distance between \( M_1 \) and \( M_2 \) is much smaller than their radii of curvature: the fluid is "thin".

Choose coordinates \( x, y \) on \( M_1 \) such that \( x, y \) are harmonic. Choose a coordinate \( z \) so that \( M_1 \) and \( M_2 \) are surfaces of constant \( z \). For simplicity we choose them to be \( z=0, z=1 \). The \( x, y \) coordinates on each of the surfaces \( z=constant \) are defined by requiring that the lines of constant \( (x, y) \) are orthogonal to the surfaces of constant \( z \). The line element in these coordinates then has the form:

\[
\text{d}s^2 = g_{xx} \text{d}x^2 + 2g_{xy} \text{d}x \text{d}y + g_{yy} \text{d}y^2 + g_{zz} \text{d}z^2 \quad \text{II.3.1}
\]

Since the fluid is thin the \( z \) coordinate can be chosen so that \( g_{zz} \) is nearly independent of \( z \).

\[
\left| \frac{1}{g_{zz}} \frac{\partial g_{zz}}{\partial z} \right| \ll 1 \quad \text{II.3.2}
\]

The depth of the fluid, \( k^*(x, y) \) is then:

\[
k^*(x, y) = \int_0^1 \sqrt{g_{zz}} \text{d}z = \sqrt{g_{zz}} \quad \text{II.3.3}
\]

The thinness of the fluid requires that the depth is small in comparison with the distance over which \( g_{xx}, g_{xy}, g_{yy} \) vary appreciably.
\[
\left| \frac{g_{zz}}{g_{xx}} \right| \left| \text{grad} \ g_{xx} \right| \ll 1; \quad \left| \frac{g_{zz}}{g_{yy}} \right| \left| \text{grad} \ g_{yy} \right| \ll 1;
\]
\[
\left| \frac{g_{zz}}{g_{xy}} \right| \left| \text{grad} \ g_{xy} \right| \ll 1
\]

(II.3.3)

The z component of grad is:

\[
\text{grad} \ z = \frac{1}{\sqrt{g_{zz}}} \frac{\partial}{\partial z} \approx \frac{1}{k^*(x,y)} \frac{\partial}{\partial z}
\]

(II.3.4)

Therefore \( g_{xx}, g_{xy}, g_{zz} \) are approximately independent of z.

Moreover, since on \( M_1 \) the \((x,y)\) coordinates are harmonic:

\[
g_{xx} \approx h^2(x,y), \quad g_{yy} \approx h^2(x,y)
\]

and \( g_{xy} \) nearly vanishes.

The horizontal components of grad are then approximately:

\[
\text{grad}_{\text{hor}} \approx \frac{\hat{x}}{h(x,y)} \frac{\partial}{\partial x} + \frac{\hat{y}}{h(x,y)} \frac{\partial}{\partial y}
\]

(II.3.6)

and the requirements that the fluid is thin become:

\[
\left| \frac{k^*}{h^2} \frac{\partial h}{\partial x} \right| \ll 1; \quad \left| \frac{k^*}{h^2} \frac{\partial h}{\partial y} \right| \ll 1
\]

(II.3.7)

It will also be assumed that the depth of the fluid varies appreciably only over distances much larger than the depth: i.e.,

\[
\left| \frac{1}{h} \frac{\partial k^*}{\partial x} \right| \ll 1; \quad \left| \frac{1}{h} \frac{\partial k^*}{\partial y} \right| \ll 1
\]

(II.3.8)

Equations II.3.2.8 suggest a perturbation scheme whereby:
where $\lambda$ is a small parameter which measures the ratio of the vertical and horizontal scales of the system.

The depth of the fluid, to lowest order, is

$$k^*(x,y) = \lambda k(x,y)$$

(II.3.9)
II.4 Vector Components: Notation

In any orthogonal coordinate system a vector can be represented in three ways: contravariant components, covariant components, and physical components. Contravariant components will always be denoted by upper case letters with superscripts (e.g., $V^i$), covariant components by upper case letters with subscripts (e.g., $V_i$), and physical components by lower case letters (e.g., $v_i$). For the most part physical components will be used.

If the line element is:

$$ds = h_1^2 dx^2 + h_2^2 dy^2 + h_3^2 dz^2$$  \hspace{1cm} (II.4.1)

then the vector components are related by:

$$v_x = h_1 V^x = \frac{V^x}{h_1}, \quad v_y = h_2 V^y = \frac{V^y}{h_2}, \quad v_z = h_3 V^z = \frac{V^z}{h_3}$$ \hspace{1cm} (II.4.2)

Covariant derivatives will be denoted by semi-colons and the symbol $\nabla$ will be reserved for the two-dimensional differential operator defined so that:

$$\nabla \cdot \mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y}$$ \hspace{1cm} (II.4.3)

for any vector $\mathbf{a}$. Also:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$ \hspace{1cm} (II.4.4)
III. IDEAL FLUIDS

III.1 Field Equations for an Ideal Fluid

An ideal fluid is defined to be one for which the stress can be derived as the gradient of a scalar function $\Pi$. The equation describing momentum conservation is then (in covariant form):

$$\frac{\partial V_i}{\partial t} + V^k V_{ik} = -\Pi ; i \quad (III.1.1)$$

where $V_i$ and $V^i$ are the covariant and contravariant velocity fields respectively and semi-colons denote covariant derivatives.

The equation of mass conservation is:

$$\frac{\partial \rho}{\partial t} + (\rho V^i) ; i = 0 \quad (III.1.2)$$

where $\rho$ is the mass density of the fluid.

In studies of vortices it is most convenient to replace (III.1.1) by the vorticity equation. The vorticity is defined by:

$$W^i \equiv \varepsilon^{ink} V_{k;i} g^{-\frac{1}{2}} \quad (III.1.3)$$

where $\varepsilon^{ink}$ is the antisymmetric tensor density having $\varepsilon^{123} = 1$, and $g \equiv \det g_{ij}$. An equation for $W^i$ is obtained by taking the covariant derivative of (III.1.1) contracting with $\varepsilon^{ink}$ and using (III.1.2) to eliminate $V^i ; i$:
\[
\frac{\partial}{\partial t} \left( \frac{W_i}{\rho} \right) + V^k \frac{\partial}{\partial x^k} \left( \frac{W_i}{\rho} \right) = \frac{W^k}{\rho} \frac{\partial V^i}{\partial x^k}
\]

(III.1.4)

(III.1.2), (III.1.3) and (III.1.4) are now regarded as the fundamental equations of motion.

Note that (III.1.4) may be written in terms of ordinary derivatives:

\[
\frac{\partial}{\partial t} \left( \frac{W_i}{\rho} \right) + V^k \frac{\partial}{\partial x^k} \left( \frac{W_i}{\rho} \right) = \frac{W^k}{\rho} \frac{\partial V^i}{\partial x^k}
\]

(III.1.5)

since the terms in the connections $\Gamma^i_{jk}$ cancel.
III.2 The Kelvin Circulation Theorem

In 1869, Lord Kelvin reformulated much of Helmholtz' earlier work. His most important contribution was the definition of the circulation and his proof of the Kelvin Circulation Theorem.

Let $C$ be a closed contour in the fluid:

$$C(t) = \{x^k(s,t) : 0 \leq s \leq 1\} \quad \text{(III.2.1)}$$

where $x^k(s,t)$, $k = 1,2,3$ are smooth real-valued functions of $s$ and $t$ such that: $x^k(0,t) = x^k(1,t)$, $k = 1,2,3$ for all $t$. The circulation around $C$ is defined by:

$$\Gamma_C = \int_0^1 V_k(x^i(s,t),t) \frac{\partial x^k}{\partial s}(s,t) ds \quad \text{(III.2.2)}$$

If $C$ is advected then it can be parametrized such that:

$$\frac{\partial x^k}{\partial t}(s,t) = V_k(x^i(x,t),t) \quad \text{(III.2.3)}$$

Therefore:

$$\frac{d\Gamma_C}{dt} = \frac{d}{dt} \int_0^1 V_k(x^i(s,t)) \frac{\partial x^k}{\partial s}(s,t) ds$$

$$= \int_0^1 \left[ \frac{\partial V_k}{\partial t}(x^i,t) + V_{k;n}(x^i,t) \frac{\partial x^n}{\partial t}(s,t) \frac{\partial x^k}{\partial s}(s,t) \right] ds$$

$$= \int_0^1 \left[ -V^n(x^i,t)V_{k;n}(x^i,t) + \Pi_k(x^i,t) \right.$$ 

$$+ V_{k;n}(x^i,t)V^n(x^i,t) \frac{\partial x^k}{\partial s}(s,t)$$

$$+ V_k(x^i,t) \frac{\partial V^k}{\partial s}(x^i,t) \left. \right] ds$$
\[ \pi(x^{i}(1,t),t) - \pi(x^{i}(0,t),t) + (V^{k}V_{k})(x^{i}(1,t),t) \]
\[ - (V^{k}V_{k})(x^{i}(0,t),t) = 0 \quad (III.2.4) \]

Thus, the circulation around any advected contour is conserved. This is the Kelvin Circulation Theorem.
III.3 The Thin Film Approximation

In this section a number of approximations are introduced allowing the expansion of the field equations in powers of the small parameter $\lambda$ of Sec.II.3.

The approximations to be imposed are as follows:

a) The fluid is "thin" so that the approximation of equations (II.3.8) may be used.

b) The density of the fluid is nearly constant:

$$\rho(x,y,z,t) = \rho_0 + \rho^{(1)}(x,y,z,t) + \ldots$$  \hspace{1cm} (III.3.1)

where $\rho_0$ is constant.

c) We define:

$$v_x \equiv V^x h, \quad v_y \equiv V^y h, \quad v_z \equiv V^z k^*$$  \hspace{1cm} (III.3.2)

where $V^x, V^y, V^z$ are the components of the contravariant velocity field. To lowest order in $\lambda$, the metric is diagonal so that, to this order, $v_x, v_y, v_z$ are the components of the physical velocity field.

It is assumed that the vertical component of the velocity is small in comparison with the horizontal velocity and varies little with height.

$$v_x(x,y,z,t) = v_x^{(0)}(x,y,t) + \lambda v_x^{(1)}(x,y,t) + \ldots$$  \hspace{1cm} (III.3.3a)

$$v_y(x,y,z,t) = v_y^{(0)}(x,y,t) + \lambda v_y^{(1)}(x,y,t) + \ldots$$  \hspace{1cm} (III.3.3b)

$$v_z(x,y,z,t) = \lambda v_z^{(1)}(x,y,t) + \lambda^2 v_z^{(2)}(x,y,z,t) + \ldots$$  \hspace{1cm} (III.3.3c)

Similarly, vorticity components are defined:
\[ w_x = W^x h ; \quad w_y = W^y h ; \quad w_z = W^z k^* \quad (\text{III.3.4}) \]

(\text{III.1.3}) and (\text{III.3.3}) then imply:

\[ w_x = -\frac{1}{k} \frac{\partial v_y}{\partial z} + \lambda w_x(x,y,z,t) + \ldots \quad (\text{III.3.5a}) \]

\[ w_y = \frac{1}{k} \frac{\partial v_x}{\partial z} + \lambda w_y(x,y,z,t) + \ldots \quad (\text{III.3.5b}) \]

\[ w_z = \frac{1}{h^2} \left[ \frac{\partial (hv_y^0)}{\partial x} - \frac{\partial (hv_x^0)}{\partial x} \right] + \lambda w_z(x,y,z,t) + \ldots \quad (\text{III.3.5c}) \]

Written explicitly in terms of \( w_x, w_y, w_z \), (III.1.5) becomes:

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{w_x}{\rho h} \right) + \frac{v_x}{h} \frac{\partial}{\partial x} \left( \frac{w_x}{\rho h} \right) + \frac{v_y}{h} \frac{\partial}{\partial y} \left( \frac{w_x}{\rho h} \right) + \frac{v_z}{k} \frac{\partial}{\partial z} \left( \frac{w_x}{\rho h} \right) \\
= \frac{w_x}{\rho h} \frac{\partial}{\partial x} \left( \frac{v_x}{h} \right) + \frac{w_y}{\rho h} \frac{\partial}{\partial y} \left( \frac{v_x}{h} \right) + \frac{w_z}{\rho k} \frac{\partial}{\partial z} \left( \frac{v_x}{h} \right) \quad (\text{III.3.6a})
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{w_y}{\rho h} \right) + \frac{v_x}{h} \frac{\partial}{\partial x} \left( \frac{w_y}{\rho h} \right) + \frac{v_y}{h} \frac{\partial}{\partial y} \left( \frac{w_y}{\rho h} \right) + \frac{v_z}{k} \frac{\partial}{\partial z} \left( \frac{w_y}{\rho h} \right) \\
= \frac{w_x}{\rho h} \frac{\partial}{\partial x} \left( \frac{v_z}{h} \right) + \frac{w_y}{\rho h} \frac{\partial}{\partial y} \left( \frac{v_z}{h} \right) + \frac{w_z}{\rho k} \frac{\partial}{\partial z} \left( \frac{v_z}{h} \right) \quad (\text{III.3.6b})
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{w_z}{\rho k} \right) + \frac{v_x}{h} \frac{\partial}{\partial x} \left( \frac{w_z}{\rho h} \right) + \frac{v_y}{h} \frac{\partial}{\partial y} \left( \frac{w_z}{\rho h} \right) + \frac{v_z}{h} \frac{\partial}{\partial z} \left( \frac{w_z}{\rho h} \right) \\
= \frac{w_x}{\rho h} \frac{\partial}{\partial x} \left( \frac{v_z}{k} \right) + \frac{w_y}{\rho h} \frac{\partial}{\partial y} \left( \frac{v_z}{k} \right) + \frac{w_z}{\rho h} \frac{\partial}{\partial z} \left( \frac{v_z}{k} \right) \quad (\text{III.3.6c})
\end{align*}
\]

which to the lowest order in \( \lambda \) becomes:
\[
\frac{\partial}{\partial t}\left(\frac{w_z}{k}\right) + \frac{v_x}{h} \frac{\partial}{\partial x}\left(\frac{w_z}{k}\right) + \frac{v_y}{h} \frac{\partial}{\partial z}\left(\frac{w_z}{k}\right) = 0 \quad \text{(III.3.7)}
\]

To lowest order the equation of mass conservation (III.1.2) is:

\[
\frac{\partial (h k v_x(0))}{\partial x} + \frac{\partial (h k v_y(0))}{\partial y} = 0 \quad \text{(III.3.8)}
\]

The equation expressing the production of \(w_x^{(0)}\) and \(w_y^{(0)}\) are also of this order. However, according to equations (III.3.5a,b), to this order of approximation, \(v_x, v_y\) and \(w_z\) are completely independent of \(w_x\) and \(w_y\): that is, changes in \(w_x, w_y\) of order \(\lambda^0\) induce changes in \(v_x, v_y\) and \(w_z\) of order \(\lambda\). The equations for \(w_x\) and \(w_y\) may therefore be disregarded to this order of approximation.

Henceforth, the superscripts will be omitted. All physical quantities will be assumed to be taken to lowest order.

In order that (III.3.3a,b,c) remain valid at the boundaries, it is necessary that all boundaries be of the form: \(f(x,y) + O(\lambda) = \text{const.}\) for some function \(f\) which for simplicity is assumed analytic. (Since any function may be approximated to arbitrary accuracy by an analytic function, departures from analyticity may be absorbed in the term of order \(\lambda\).) The assumed boundary conditions are that there is no flux of fluid through any boundary. However, slip at the boundaries; no slip conditions require that \(v_x^{(0)} = v_y^{(0)} = 0\) at \(z = 0\) and \(z = 1\), but \(\frac{\partial v_x^{(0)}}{\partial z} = \frac{\partial v_y^{(0)}}{\partial z} = 0\) so that the only solution compatible with no slip boundary conditions is the trivial solution \(v^{(0)} = 0\). Notice that
since \( v_z^{(0)} = 0 \), and the upper and lower boundaries are surfaces \( z = \) const., the lowest order solution is consistent with zero flux through these surfaces.

The problem of finding the fluid motion has now been reduced to a two-dimensional problem since all fields and boundaries (other than \( w_x \) and \( w_y \)) are, to lowest order, independent of \( z \).
IV. VORTICES IN THIN FILMS

In this section the main problem of the thesis is addressed: what are the equations of motion of a system of N vortices in a thin film of ideal fluid described by the metric function \( h(x,y) \) and the depth function \( k(x,y) \)?

IV.1 The Vortex Velocity Field

The equation of mass conservation (III.3.8) is:

\[
\frac{\partial (hk \psi_x)}{\partial x} + \frac{\partial (hk \psi_y)}{\partial y} = 0 \tag{IV.1.1}
\]

which has general solution:

\[
v_x = \frac{1}{hk} \psi_y(x,y); \quad v_y = -\frac{1}{hk} \psi_x(x,y) \tag{IV.1.2}
\]

for some real valued function \( \psi(x,y) \) having continuous mixed partial derivatives of second order. \( \psi(x,y) \) is the streamfunction for the flow. The velocity field of a vortex at \((x',y')\) is defined to be the incompressible velocity field having \( w_z = 0 \) everywhere throughout the region of flow, \( D \), except at \((x',y')\): that is, satisfying (IV.1.1) and:

\[
w_z = \frac{1}{h^2} \left[ \frac{\partial (hk \psi_y)}{\partial x} - \frac{\partial (hk \psi_x)}{\partial y} \right] = \frac{2\pi \gamma}{h^2} \delta(x-x')\delta(y-y') \tag{IV.1.3}
\]

Notice that if \( C \) is a contour with interior \( G \) such that \((x',y') \in G\), then:

\[
2\pi \gamma = \int_G 2\pi \gamma \delta(x-x')\delta(y-y')dxdy
\]

\[
= \int_G \left[ \frac{\partial (hk \psi_y)}{\partial x} - \frac{\partial (hk \psi_x)}{\partial y} \right]dxdy
\]
Thus, by the Kelvin Circulation Theorem \( \gamma \) is a constant. \( \gamma \) is called the vortex strength. Substituting (IV.1.2) into (IV.1.3) one has:

\[
\nabla \cdot \left( \frac{1}{k} \nabla \psi(x, y; x', y') \right) = -2\pi \delta(x-x') \delta(y-y') \quad (IV.1.5)
\]

where \( \psi(x, y; x', y') \) is the streamfunction for a vortex of unit strength at \( (x', y') \). \( \psi(x, y; x', y') \) is a Green's function of a self-adjoint elliptic differential operator.

The region of flow is \( \{(x,y) \in D\} \) where the boundary of \( D \) is:

\[
\partial D = \bigcup_{i=0}^{M} \partial_{i} D, \quad \partial_{i} D \cap \partial_{k} D = \phi, \quad i \neq k. \quad (IV.1.6)
\]

The boundary conditions for \( \psi(x, y; x_{i}, y_{i}) \) are:

\[
\psi = (\text{unspecified}) \text{ const. on } \partial_{i} D, i = 0, \ldots, M ; \quad \Gamma_{\partial_{i} D} \text{ given, } i = 1, \ldots, M. \quad (IV.1.7)
\]

It is often assumed, although this is not necessary, that \( \Gamma_{\partial_{i} D} = 0 \) for \( i = 1, \ldots, M \). Notice that the Kelvin Circulation Theorem implies that the boundary conditions are constant in time.

The existence and uniqueness of \( \psi \) satisfying (IV.1.5) and the boundary conditions:

\[
\psi(x, y; x', y') = \psi_{i} = \text{const. on } \partial_{i} D, i = 0, \ldots, M \quad (IV.1.8)
\]

is well known (Courant and Hilbert (1962)). We use this to demonstrate the existence and uniqueness up to an additive
constant of $\Psi$ satisfying the boundary conditions (IV.1.7).

Set $\Psi_0 = 0$ and consider the vector space $A$ of $M$-tuples, $(\Psi_1, \ldots, \Psi_M)$ corresponding to the possible values of the boundary condition (IV.1.8). Let $B$ be the vector space of $M$-tuples, $(\Gamma_{\partial_i D_1}, \ldots, \Gamma_{\partial_i D_M})$ corresponding to the possible values of the boundary condition (IV.1.7). There is a natural map $f: A \rightarrow B$, defined by finding the unique $\Psi(x, y; x', y')$ having boundary conditions $\Psi = \Psi_i$ on $\partial_i D, i = 1, \ldots, M$, $\Psi = 0$ on $\partial_0 D$, and setting $\Gamma_{\partial_i D}$ equal to the circulation around $\partial_i D$ for this $\Psi$. Clearly $f$ is linear.

To prove that $\Psi$ exists and is unique up to an additive constant under the boundary conditions (IV.1.7) it is only necessary to prove that $f$ has an inverse (the freedom of the additive constant arises when one does not fix $\Psi_0$). Thus it is only necessary to show that $(\Gamma_{\partial_1 D}, \ldots, \Gamma_{\partial_M D}) = (0, \ldots, 0)$ has a unique pre-image.

Suppose $\Psi(x, y; x', y')$ and $\Psi^*(x, y; x', y')$ satisfy (IV.1.5) and (IV.1.7) with $\Gamma_{\partial_i D} = 0, i = 1, \ldots, M$. Then:

$$\int_{\partial D} \frac{1}{k} \nabla \cdot \nabla (\Psi - \Psi^*) \, ds = \int_D \nabla \left( \frac{1}{k} \nabla (\Psi - \Psi^*) \right) \, dx \, dy$$

$$+ \int_D \frac{1}{k} \nabla (\Psi - \Psi^*) \cdot \nabla (\Psi - \Psi^*) \, dx \, dy$$

(IV.1.9)

where $\frac{\partial}{\partial \eta}$ denotes a derivative normal to the boundary. The left side vanishes, since from (IV.1.2) and (IV.1.7):

$$\int_{\partial_i D} \frac{1}{k} \frac{\partial}{\partial \eta} (\Psi - \Psi^*) \, ds = \text{const.} \times \Gamma_{\partial_i D} = 0, \ i = 1, \ldots, M$$

(IV.1.10)
and when \( i = 0, \psi - \psi* = 0 \) on \( \partial_0 D \). The first term on the right of (IV.1.9) vanishes by (IV.1.15). Hence:

\[
\int_D \frac{1}{k} \nabla(\psi-\psi*) \cdot \nabla(\psi-\psi*) \, dx \, dy = 0 \tag{IV.1.11}
\]

whence \( \nabla(\psi-\psi*) = 0 \) and, since \( \psi = \psi* \) on \( \partial_0 D \), \( \psi = \psi* \).

Thus, \((\Gamma_{\partial_1 D}, \ldots, \Gamma_{\partial_0 D}) = (0, \ldots, 0)\) has a unique pre-image and therefore there exists a \( \Psi(x,y;x',y') \) satisfying (IV.1.5) and (IV.1.7) which is unique up to an additive constant.

\( \Psi(x,y;x',y') \) also has the reciprocity property:

\[
\Psi(x,y;x',y') = \Psi(x',y';x,y) \tag{IV.1.12}
\]

and can be written in the form:

\[
\Psi(x,y;x',y') = -A(x,y;x',y') \ln r + B(x,y;x',y') \tag{IV.1.13}
\]

with \( r \equiv [(x-x')^2 + (y-y')^2]^{1/2} \) and with \( A \) and \( B \) analytic in \( D \) if \( k \) is analytic (see, for example, Sommerfeld (1949)). The circulation around the small contour \( r = \varepsilon \) is:

\[
2\pi = -\int_0^{2\pi} \frac{1}{k} \frac{\partial \Psi}{\partial r} \, rd\theta + \frac{2\pi A(x',y';x',y')}{k(x',y')} \text{ as } \varepsilon \to 0
\]

whence:

\[
A(x',y';x',y') = k(x',y') \tag{IV.1.14}
\]

Substituting (IV.1.13) into (IV.1.5) one finds that as \( r \to 0 \):

\[
\frac{\partial}{\partial r} \left( \frac{A}{k} \right) + \frac{1}{k} \frac{\partial A}{\partial r} \to 0 \tag{IV.1.15}
\]
whence:

$$\nabla A(x', y'; x', y') = \frac{1}{2} \nabla k(x', y') \tag{IV.1.16}$$

The total streamfunction for a system of $N$ vortices with positions $(x_n, y_n)$ and respective strengths $\gamma_n, n=1, \ldots, N$ is:

$$\psi(x, y) = \sum_{n=1}^{N} \gamma_n \psi(x, y; x_n, y_n) + \psi^*(x, y) \tag{IV.1.17}$$

where $\psi^*(x, y)$ is the streamfunction due to other imposed flows (e.g., a uniform stream). $\psi^*(x, y)$ must satisfy:

$$\nabla^2 \left( \frac{1}{k} \nabla \psi^* \right) = 0 \tag{IV.1.18}$$

So far it has been tacitly assumed that $D$ is bounded. The extension of $D$ to infinity presents no real problems (Courant and Hilbert (1962)) and will not be considered further. The special case of vortices on a closed unbounded surface (e.g., a sphere) is, however, of particular interest. Let $C$ be a closed contour on such a surface, such that no vortex lies on $C$. Denote its interior by $C_{\text{int}}$ and its exterior by $C_{\text{ext}}$ (the choice of what is the interior and what is the exterior is, of course, arbitrary). Then:

$$\oint_C \left( \psi \frac{\partial}{\partial x} + \psi^* \frac{\partial}{\partial y} \right) dx dy = \oint_{C_{\text{int}}} \frac{1}{k} \left[ \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial y} \right] dx dy = \sum_{z_n \in C_{\text{int}}} \gamma_n \tag{IV.1.19}$$
Integrating around $C$ in the other direction one finds:

$$\oint_C h(v_x \, dx + v_y \, dy) = \sum_{n=1}^{N} \gamma_n$$

whence:

$$\sum_{n=1}^{N} \gamma_n = 0$$

Thus, there cannot be a single vortex on such a surface.

The velocity field due to the streamfunction $\psi$ satisfying (IV.1.5) must be interpreted as that due to a vortex of unit strength at $(x',y')$ and one of negative unit strength at infinity. (Note that on a closed surface the point infinity is just like any other point: e.g., on the sphere it is the south pole.) Upon the superposition of $N$ such streamfunctions satisfying (IV.1.21) the vortex at infinity disappears.

If the depth of the fluid is constant, $k$ is constant (we put $k=1$ for simplicity) and $\psi(x,y;x',y')$ becomes the Green's function of the Laplacian. $\psi$ can then be written:

$$\psi(x,y;x',y') = -\ln r + A(x,y;x',y')$$

(see, e.g., Sommerfeld (1949)), whence, upon comparison with (IV.1.13)

$$A(x,y;x',y') = 1$$

Moreover, since: $\nabla^2 \ln r = 2\pi \delta(x-x')\delta(y-y')$, if there are no boundaries, then:

$$B(x,y;x',y') = 0 ; \psi(x,y;x',y') = -\frac{1}{2} \ln [(x-x')^2 + (y-y')^2]$$

When the fluid depth is constant one can therefore regard the flow induced by $B$ to be due to the boundaries. If $k$ is not constant there is no such simple decomposition of the flow.
IV.2 The Velocity of a Vortex in a Fluid of Uniform Depth

The motion of the vortices is governed by equation (III.3.7):

\[ \frac{\partial}{\partial t} \left( \frac{W_z}{k} \right) + \frac{V_x}{h} \frac{\partial}{\partial x} \left( \frac{W_z}{k} \right) + \frac{V_z}{h} \frac{\partial}{\partial y} \left( \frac{W_z}{k} \right) = 0 \]  

(IV.2.1)

which implies that \( \left( \frac{W_z}{k} \right) \) is advected: that is, each fluid element maintains its value of \( \left( \frac{W_z}{k} \right) \) as it moves around in the fluid. Thus, the vorticity concentrated at the points \((x^n, y^n)\) remains concentrated and the flow retains the characteristics of a vortex system: i.e., the streamfunction always satisfies:

\[ \nabla \cdot \left( \frac{1}{k} \nabla \psi \right) = -2\pi \sum_{n=1}^{N} \gamma_n \delta(x-x_n)\delta(y-y_n) \]  

(IV.2.2)

but the \( x_n \) and \( y_n \) are time-dependent, moving as if carried along by the flow.

In order to determine the velocity of each vortex we expand the velocity field near its singularity. The case \( k=1 \) (uniform depth fluid) is treated first.

It will prove convenient to introduce the complex variable \( z = x + iy \) (there should be no confusion with the variable \( z \) of Section II.3). The independent variables are then \( z \) and \( \bar{z} = x - iy \). For simplicity the present symbols are retained for all functions despite their change in arguments. Equation (IV.1.2) is then:

\[ v_x - iv_y = \frac{2i}{h(z, \bar{z})} \frac{\partial \psi}{\partial z} (z, \bar{z}) \]  

(IV.2.3)
Which, upon substitution of (IV.1.17) and (IV.1.22) becomes:

\[ v_x - iv_y = \frac{1}{h(z,\bar{z})} \left\{ \sum_{k=1}^{N} \left[ -i\gamma_k \frac{\partial B(z,\bar{z};z_k,\bar{z}_k)}{\partial z} + 2i\gamma_k \frac{\partial B(z,\bar{z};z_k,\bar{z}_k)}{\partial \bar{z}} \right] + 2i\frac{\partial \psi^*(z,\bar{z})}{\partial z} \right\} (IV.2.4) \]

If \(|z - z_n|\) is small:

\[ v_x - iv_y = \frac{i}{h_n} \left\{ \gamma_n \frac{\partial h^{-1}_n}{\partial z} \left( \frac{\bar{z} - \bar{z}_n}{(z - z_n)} \right) - \gamma_n \frac{\partial h^{-1}_n}{\partial \bar{z}} \right\} + 2i\frac{\partial \psi^*(z,\bar{z})}{\partial z} + \gamma_n B(z,\bar{z};z_n,\bar{z}_n) + \sum_{k \neq n} \gamma_k \psi(z,\bar{z};z_k,\bar{z}_k) \right\} + O(|z - z_n|) (IV.2.5) \]

where \( h_n = h(z_n,\bar{z}_n) \).

The velocity field: \( v_x - iv_y = \frac{\gamma_n}{h(z_n,\bar{z}_n)(z - z_n)} \) is concentric about \( z_n \). Preferring no direction it cannot impart a velocity to the vortex.

The velocity field \( v_z - iv_y = \frac{i\gamma_n}{h(z_n,\bar{z}_n)} \frac{\partial h^{-1}_n}{\partial z} \left( \frac{\bar{z} - \bar{z}_n}{(z - z_n)} \right) \) flows radially from \( z_n \) and hence cannot impart a velocity to the vortex either. It might seem paradoxical that a vortex velocity field should contain radial terms such as this. They arise because \( \nabla \cdot \mathbf{v} \) does not satisfy \( \nabla \cdot \mathbf{v} = 0 \) but \( \nabla \cdot \mathbf{v} = -\nabla \times \mathbf{h} \). By projecting the flow onto the plane "fictitious" source terms are introduced giving rise to the radial terms.

The velocity field:
\[ v_x - i v_y = \frac{1}{\hbar_n} \left\{ -i \gamma_n \frac{\partial h^{-1}}{\partial z} + 2i \frac{\partial}{\partial z} \left[ \psi^*(z, \bar{z}) + \gamma_n B(z, \bar{z}; z_n, \bar{z}_n) \right] \right\} \]

is uniform and therefore carries the vortex at \( z_n \) along with it.

All other terms vanish as \( z \to z_n \) so that they cannot induce any motion in the vortex at \( z_n \). The velocity of the vortex is therefore:

\[ u_x - i u_y = \frac{i}{\hbar_n} \left\{ -\gamma_n \frac{\partial h^{-1}}{\partial z} + \frac{\partial}{\partial z} \left[ \psi^*(z, \bar{z}) + \gamma_n B(z, \bar{z}; z_n, \bar{z}_n) \right] \right\} \]

But:

\[ u_x = \hbar_n \dot{x}_n, \quad u_y = \hbar_n \dot{y}_n \]

Therefore:

\[ \dot{z}_n = \frac{2i}{\hbar_n^2} \frac{\partial}{\partial z} \left[ \frac{\gamma_n}{2} \kappa h(z, \bar{z}) + \psi^*(z, \bar{z}) + \gamma_n B(z, \bar{z}; z_n, \bar{z}_n) \right] \]

\[ + \sum_{k \neq n} \gamma_k \psi(z, \bar{z}; z_k, \bar{z}_k) \]

or, reverting to \((x, y)\) coordinates:

\[ \dot{x}_n = \frac{1}{\hbar^2(x_n, y_n)} \left. \frac{\partial \Omega_n}{\partial y} \right|_{x=x_n, y=y_n}; \quad \dot{y}_n = \frac{-1}{\hbar^2(x_n, y_n)} \left. \frac{\partial \Omega_n}{\partial x} \right|_{x=x_n, y=y_n} \]
with:

\[ \Omega_n = \frac{\gamma_n}{2} \lambda \kappa h(x,y) + \psi^*(x,y) + \gamma B(x,y;x_n,y_n) \]

+ \sum_{k \neq n} \gamma_k \psi(x,y;x_k,y_k) \quad (IV.2.10) \]

These are the equations of motion for the vortex system.

If there are no boundaries then from (IV.2.8) and (IV.1.24)

\[ \frac{\hat{z}}{n} = \frac{i}{h_n} \sum_{k \neq n} \left[ \frac{-i \gamma_k}{h_n} \frac{\partial h_n}{\partial z} + \frac{i \gamma_n}{h_n} \frac{\partial h_n}{\partial \bar{z}} \right] \quad (IV.2.11) \]

Example: The Velocity of the Vortex on a Sphere with No Boundaries and Uniform Depth

In order to check that (IV.2.9-10) do indeed give the correct vortex velocity, we determine the velocity of a vortex on the sphere. This can also be determined by an alternative method which is in agreement with the first.

Consider a vortex of strength \( \gamma \) at \((x',y')\) on the sphere and its counterpart of strength \(-\gamma\) at infinity. (From (II.2.4), infinity in harmonic coordinates corresponds to \( \theta = \frac{\pi}{2} \), i.e., the south pole). Since there are no external flows, no boundaries and uniform depth, \( \psi^* = 0 \), \( B = 0 \), and \( k = 1 \). The velocity of the vortex is therefore:

\[ u_x - i u_y = \frac{i}{h(z',\bar{z}')} \frac{\partial \lambda \kappa h(z',\bar{z}')} {\partial \bar{z}} \quad (IV.2.12) \]

or, in polar harmonic coordinates:

\[ u_r = 0 \quad ; \quad u_p = \frac{-\gamma h'(r')}{2h(r')} = \frac{-\gamma(p(r')-1)}{2r'} \quad (IV.2.13) \]
Using (II.2.4):

\[ u_r = u_\theta = 0 \quad ; \quad u_\phi = \frac{\chi \tan \frac{1}{2} \phi}{R} \quad \text{(IV.2.14)} \]

This can be derived alternatively, for this case only, as follows:

Since the sum of all vortex strengths is zero, consider the velocity field of a single vortex of strength \( \gamma \) to be that such that it is incompressible and the vorticity is everywhere constant and equal to \( \alpha \gamma \). Then upon superposing the velocity fields one gets zero vorticity. The velocity field of each vortex is completely symmetric about its core owing to the symmetry of the sphere; hence, there is no self-induced vortex motion. The velocity field of the vortex at the south pole then satisfies (using ordinary polar coordinates):

\[ v_\theta = 0 \quad ; \quad w = \frac{1}{R \sin \theta} \left[ \frac{\partial (v \sin \theta)}{\partial \theta} - \frac{\partial v_\theta}{\partial \phi} \right] = -\alpha \gamma \quad \text{(IV.2.15)} \]

which has solution:

\[ v_\theta = 0 \quad ; \quad v_\phi = \frac{-\alpha \gamma R}{\sin \theta} \left[ \beta - \cos \theta \right] , \beta \text{ const.} \quad \text{(IV.2.16)} \]

If \( v_\phi \) is to be bounded at \( \theta = 0 \), \( \beta = 1 \), \( \alpha \) is evaluated by requiring that:

\[ -2\pi \gamma = \lim_{\theta \to \pi} \int_0^{2\pi} v_\phi R \sin \theta d\phi = \lim_{\theta \to \pi} -2\pi \alpha \gamma R^2 (1-\cos \theta) \]

\[ = -4\pi \alpha \gamma R^2 \quad \text{(IV.2.17)} \]
Therefore:

\[ \alpha = \frac{1}{2R^2} \quad \text{(IV.2.18)} \]

and

\[ v_\phi = \frac{\gamma(1-\cos \theta)}{2R \sin \theta} = \frac{\gamma \tan \frac{1}{2} \theta}{2R} ; \quad v_\phi = 0 \quad \text{(IV.2.19)} \]

The velocity of the vortex at \((\theta', \phi')\) is therefore given by (IV.2.14) since it is carried along in the flow of the vortex at the south pole. The two methods are in agreement. (Of course, the vortex at the south pole is also moving so that (IV.2.19) is only correct at the instant this vortex is at the pole.)
IV.3 The Velocity of a Vortex in a Fluid of Varying Depth

The velocity of a vortex in a fluid of varying depth is more complicated than that in a fluid of uniform depth owing to the more complex nature of the singularity in $\Psi(x,y;x',y')$. The physical reason for this is that the vortex core is no longer straight but must curve slightly in order to meet the upper and lower bounding surfaces perpendicularly so that the boundaries are streaming surfaces. (See Figure II.) It was shown by Helmholtz (1858) that an infinitesimally small curved vortex core will propagate infinitely fast. Thus, one must expect at the outset that the velocity of a vortex in a fluid of varying depth must depend in some way on the structure of its core.

We proceed as in Section II.2, by examining the physical velocity field in the neighbourhood of $z_n$. From (IV.1.2), (IV.1.15), and (IV.1.17):

$$v_x - iv_y = \frac{2i}{h(z,z)k(z,z)} \frac{\partial \Psi}{\partial z} (z,z)$$

$$= \frac{2i}{(hk)(z_n,z_n)} \left( -\gamma_n \frac{A(z_n,z_n;z_n,z_n)}{(z-z_n)} \right)$$

$$+ \gamma_n \left( \frac{A(z_n,z_n;z_n,z_n)}{(hk)(z_n,z_n)} \frac{\partial (hk) (z_n,z_n)}{\partial z} \right)$$

$$- \gamma_n \frac{\partial A(z,z;z_n,z_n)}{\partial z} \bigg|_{z=z_n} \frac{z-z_n}{z=z_n}$$

$$- \gamma_n \frac{\partial A(z,z;z_n,z_n)}{\partial z} \bigg|_{z=z_n} \frac{x \in |z-z_n|}{z=z_n}$$
Fig. II: Vortex Core in a Fluid of Varying Depth
\[\gamma_n A(z_n, \overline{z}_n) + \frac{\partial (hk)}{\partial z}(z_n, \overline{z}_n) + \frac{\partial}{\partial z} \left[ \gamma_n B(z, \overline{z}; z_n, \overline{z}_n) \right] + \psi(z, \overline{z}) + \sum_{k \neq n} \gamma_k \psi(z, \overline{z}; z_k, \overline{z}_k) \bigg|_{z = z_n, \overline{z} = \overline{z}_n} + O(|z - z_n| |\ln| z - z_n|) \]  

(IV.3.1)

From (IV.1.14) and (IV.1.16) and writing:

\[h_n = h(z_n, \overline{z}_n); \quad k_n = k(z_n, \overline{z}_n)\]  

(IV.3.2)

one gets:

\[v_x - i v_y = \frac{2i}{h_n k_n} \left\{ -\gamma_n k_n + \gamma_n \left( \frac{k_n}{h_n} \frac{\partial h_n}{\partial z} + \frac{\partial k_n}{\partial z} \right) \right\} \left( z - z_n \right) \]

\[+ \frac{\partial}{\partial z} \left[ \gamma_n B(z, \overline{z}; z_n, \overline{z}_n) + \psi(z, \overline{z}) \right] + \sum_{k \neq n} \gamma_k \psi(z, \overline{z}; z_k, \overline{z}_k) \bigg|_{z = z_n, \overline{z} = \overline{z}_n} + O(|z - z_n| |\ln| z - z_n|) \]  

(IV.3.3)

As before, the terms in \(\frac{1}{z - z_n}\) and \(\frac{\overline{z} - \overline{z}_n}{z - z_n}\) can impart no velocity to the vortex. The remaining terms with the exception of the term in \(\ln|z - z_n|\) are uniform and therefore carry the vortex with them. The term in \(\ln|z - z_n|\) is the source of the difficulties outlined at the beginning of the section. It is divergent as \(z \to z_n\) but has a definite direction (along the curves of constant \(k\)). It therefore implies that the vortex moves with infinite velocity along
k = const. If, however, it is assumed that the vortex has a small but finite circular core of radius \( \varepsilon_n \), then at the surface of the core the velocity due to this term is uniform and equals:

\[
-i\gamma_n \frac{\partial k_n}{\partial z} \varepsilon_n. \]

The boundary is therefore carried along in the uniform flow:

\[
v_x - i v_y = \frac{2i}{h_n k_n} \left( -\frac{\partial k_n}{\partial z} \left( \varepsilon_n - 1 \right) + \frac{k_n}{h_n} \frac{\partial h_n}{\partial z} \right)
+ \frac{\partial}{\partial z} \left[ \gamma_n B(z, \bar{z}; z_n, \bar{z}_n) + \psi(z, \bar{z}) \right]
+ \sum_{k \neq n} \frac{\gamma_k \psi(z, \bar{z}; z_k, \bar{z}_k)}{z - z_n \bar{z} - \bar{z}_n} .
\]

Now, \( \omega \) is advected and, since the fluid is incompressible, the volume of each core must be constant. Hence:

\[
h_n^2 k_n \varepsilon_n^2 = \alpha_n + O \left( \frac{h_n \varepsilon_n}{k_n} \right). \tag{IV.3.5}
\]

where \( \lambda \) is the small perturbation parameter of Section II.3 and Section III.3. \( \lambda \sim O(k|\nabla \varphi_n|) \), and \( \alpha_n \) is a constant. As \( k \) increases the radius of the core decreases. However, the streamlines of the flow, while approximately circular, are not concentric (See Figure III). In shrinking, the centre of the vortex appears to move in the direction of \( -\mathbf{V} k \). The calculation of this velocity term will be deferred until Section VI when the effects of finite sized cores are treated in a more rigorous manner. Suffice it to say, for the present, that this term may be incorporated by "renormalizing" \( \alpha_n \):
Fig. III: Streamlines near the Core of a Vortex in a Fluid of Varying Depth
i.e., by replacing $\alpha_n$ by:

$$\alpha^*_n = \alpha_n e^{-\beta_n} \quad (IV.3.6)$$

where:

$$\beta_n = \frac{E_{cn}}{\pi \gamma_n^2 k_n^2 \rho_n} \quad (IV.3.7)$$

$E_{cn}$ is the kinetic energy within the $n$th core. Thus, the velocity of the vortex is, substituting (IV.3.5) into (IV.3.4) and renormalizing $\alpha_n$:

$$u_x - i u_y = \frac{2i}{\hbar^2 k_n^2} \left\{ \frac{\partial}{\partial z} \left[ \ln \left( \frac{\alpha^*_n}{h^2 k_n^2} \right) - 2 \right] + \frac{\gamma_n k_n^2}{\hbar_n} \frac{\partial h_n}{\partial z} + \right. \left. + \frac{3}{\partial z} \left[ \gamma_n B(z, \overline{z}; z_n, \overline{z}_n) + \psi^*(z, \overline{z}) \right] + \sum_{k \neq n} \gamma_k \psi(z, \overline{z}; z_k, \overline{z}_k) \right\} \quad (IV.3.8)$$

whence, using (IV.2.7) and simplifying the equations of motion are:

$$x_n = \frac{1}{h^2 k_n^2} \frac{\partial \Omega^*_n}{\partial y} \quad ; \quad y_n = -\frac{1}{h^2 k_n^2} \frac{\partial \Omega^*_n}{\partial x} \quad (IV.3.9)$$

with:

$$\Omega^*_n = \frac{\gamma_n}{4} k(x, y) \ln \left\{ \frac{h^2(x, y) k(x, y)}{\alpha^*_n} \right\} + \gamma_n B(x, y; x_n, y_n)$$

$$+ \psi^*(x, y) + \sum_{k \neq n} \gamma_n \psi(x, y; x_n, y_n) \quad (IV.3.10)$$

Notice that (IV.3.10) reduces to (IV.2.10) (within an additive constant) if one puts $k=1$. 
The velocity of the vortex has been derived assuming that the core is circular and remains circular. However, circular cores will in general be distorted by advection so that (IV.3.9-10) cease to be valid. Order of magnitude estimates for the periods over which (IV.3.9-10) are good approximations, are given in Section VI.3.
IV.4 The Vortex Streamfunction

It is now shown that one can derive a generalized vortex streamfunction.

From (IV.1.14) and (IV.1.16):

\[ A(x,y;x',y') = k(x',y') + \frac{(x-x')}{2} \frac{\partial k(x',y')}{\partial x} \]

\[ + \frac{(y-y')}{2} \frac{\partial k(x',y')}{\partial y} + O(r^2) \]

\[ = k^2(x,y)k^2(x',y') + O(r^2) \]

\[ r = [(x-x')^2 + (y-y')^2]^{\frac{1}{2}} \quad (IV.4.1) \]

whence:

\[ \frac{\partial A}{\partial x}(x,y;x',y') \bigg|_{x=x'} = \frac{\partial A}{\partial x}(x',y';x,y) \bigg|_{x=x'} \quad (IV.4.2) \]

and a similar equation in \( \frac{\partial}{\partial y} \). Thus, since \( \Psi \) and \( \lambda n r \)
both obey the reciprocity property:

\[ \frac{\partial B}{\partial x}(x,y;x',y') \bigg|_{x=x'} = \frac{\partial B}{\partial y}(x',y';x,y) \bigg|_{y=y'} \quad (IV.4.3) \]

(IV.3.9) can therefore be written in the form:

\[ \dot{x}_n = \frac{1}{\gamma_n h^2 k_n} \frac{\partial \Omega}{\partial y_n} ; \quad \dot{y}_n = \frac{-1}{\gamma_n h^2 k_n} \frac{\partial \Omega}{\partial x_n} \quad (IV.4.4) \]

where:

\[ \Omega = \frac{1}{2} \sum_{n=1}^N \sum_{k \neq n} \gamma_n \gamma_k \psi(x_n,y_n;x_k,y_k) + \frac{1}{2} \sum_{n=1}^N \left\{ \gamma_n^2 B(x_n,y_n;x_n,y_n) \right\} \]

\[ + \frac{\gamma_n^2 k(x_n,y_n)}{2} \lambda_n \left[ \frac{h^2(x_n,y_n)k(x_n,y_n)}{\alpha_n^*} \right] + 2\gamma_n \psi^2(x_n,y_n) \quad (IV.4.5) \]
\( \Omega \) is a generalization of the vortex streamfunction given by C.C. Lin (1943). As is shown in Appendix A, (IV.4.4-5) may be put in the symplectic form

\[
\dot{x} = \sigma^{-1} \nabla \Omega \tag{IV.4.6}
\]

where \( \dot{x} \) is a 2N-dimensional vector, \( \sigma \) is a symplectic 2-form, and \( \nabla \) is the exterior derivative.

The vortex streamfunction may be related to the kinetic energy of the fluid. The core \( G_n \) of the \( n \)th vortex is:

\[
G_n = \{ (x,y) : [(x-x_n)^2 + (y-y_n)^2]^{\frac{1}{2}} < \epsilon_n \} \tag{IV.4.7}
\]

Denote its boundary by \( \partial G_n \). The region of fluid outside the cores is:

\[
D^* = D \setminus \bigcup_{n=1}^{N} \partial G_n \tag{IV.4.8}
\]

It is assumed that the \( \epsilon_n \)'s are sufficiently small that the cores are disjoint.

The kinetic energy of the fluid in \( D^* \) is:

\[
E^* = \frac{\rho}{2} \int_{D^*} (v_x^2 + v_y^2) h^2 k \, dx \, dy
\]

\[
= \frac{\rho}{2} \int_{D^*} \frac{1}{k} \left[ \left( \frac{\partial \psi}{\partial x} \right)^2 + \left( \frac{\partial \psi}{\partial y} \right)^2 \right] \, dx \, dy
\]

\[
= \frac{\rho}{2} \int_{D^*} \left[ \frac{\partial}{\partial x} \left( k \frac{\psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\psi}{\partial y} \right) \right] \, dx \, dy \tag{IV.4.9}
\]

since:

\[
\frac{\partial}{\partial x} \left( \frac{1}{k} \frac{\partial \psi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{1}{k} \frac{\partial \psi}{\partial y} \right) = 0 \quad \text{in} \quad D^*
Applying Green's Theorem to (IV.4.9):

$$E^* = \frac{\rho}{2}\int_{\partial D^*} \frac{\partial \psi}{\partial n} ds$$  \hspace{1cm} (IV.4.10)

where $\frac{\partial}{\partial n}$ is the directional derivative normal to $\partial D^*$. 

Since:

$$\psi = \sum_{k=1}^{N} \gamma_k \psi_k(x,y;x_k,y_k) + \psi^*(x,y)$$  \hspace{1cm} (IV.4.11)

and since (using $\psi_n = \psi(x,y;x_n,y_n)$):

$$\int_{\partial D^*} \frac{\psi_k}{k} \frac{\partial \psi}{\partial n} ds = \int_{\partial D^*} \frac{\nabla \psi \cdot \nabla \psi}{k} ds = \int_{\partial D^*} \frac{\psi_n}{k} \frac{\partial \psi_n}{\partial n} ds$$  \hspace{1cm} (IV.4.12)

one has:

$$E^* = \frac{\rho}{2}\int_{\partial D^*} \frac{\psi_0}{k} \frac{\partial \psi}{\partial n} ds + \rho \sum_{n=1}^{N} \gamma_n \int_{\partial D^*} \frac{\psi_n}{k} \frac{\partial \psi_n}{\partial n} ds$$

$$+ \rho \sum_{n=1}^{N} \sum_{k=1}^{N} \frac{\gamma_n \gamma_k}{2} \int_{\partial D^*} \frac{\psi_n}{k} \frac{\partial \psi_n}{\partial n} ds$$  \hspace{1cm} (IV.4.13)

The first term represents the energy due to the flow induced by $\psi^*$ and will be denoted by $E_{\psi^*}$.

The boundary $\partial D^*$ may be decomposed.

$$\partial D^* = \bigcup_{i=1}^{M} D_i \cup \bigcup_{j=1}^{N} G_j$$  \hspace{1cm} (IV.4.14)

so that:

$$\int_{\partial D^*} \frac{\psi_k}{k} \frac{\partial \psi}{\partial n} ds = \sum_{i=0}^{M} \int_{\partial D_i} \frac{\psi_k}{k} \frac{\partial \psi}{\partial n} ds + \sum_{j=1}^{N} \int_{\partial G_j} \frac{\psi_k}{k} \frac{\partial \psi}{\partial n} ds$$  \hspace{1cm} (IV.4.15)

Since $\psi^*$ is constant on $\partial D_n$ and $\int_{\partial D_n} \frac{\partial \psi}{\partial n} ds = -\Gamma \gamma_{D_n} = \text{const.}$, the first term on the right is constant. Since $\epsilon_i$...
is very small, \( \frac{\psi^*}{k_n} = \frac{\psi^*}{k_n}(x_n, y_n) \) on \( \partial G_n \).

Therefore:

\[
\int_{\partial G_j} \frac{\psi^*}{k} \frac{\partial \psi^*}{\partial n} ds \approx 2\pi \psi^*(x_j, y_j) \tag{IV.4.16}
\]

Similarly:

\[
\int_{\partial D^*} \frac{\psi^*}{k} \frac{\partial \psi^*}{\partial n} ds = \sum_{i=1}^{M} \int_{\partial D_i} \frac{\psi^*}{k} \frac{\partial \psi^*}{\partial n} ds + \sum_{j=1}^{N} \int_{\partial G_j} \frac{\psi^*}{k} \frac{\partial \psi^*}{\partial n} ds \tag{IV.4.17}
\]

and the first term is constant as before. If \( n \neq j \), then

\[
\psi^* = \psi(x_j, y_j; x_n, y_n) \text{ on } \partial G_j \quad \text{and:}
\]

\[
\int_{\partial G_j} \frac{\psi^*}{k} \frac{\partial \psi^*}{\partial n} ds \approx 2\pi \psi(x_k, y_k; x_n, y_n) \tag{IV.4.18}
\]

If \( n = j \) but \( k \neq j \) one can proceed as in (IV.4.12) to transpose \( n \) and \( k \), whence (IV.4.18) holds unless \( n = k = j \). Since on \( \partial G_n \):

\[
\psi^* = -k_n \psi_n + B(x_n, y_n; x_n, y_n), \tag{IV.4.19}
\]

\[
\int_{\partial G_n} \frac{\psi^*}{k} \frac{\partial \psi^*}{\partial n} ds \approx (-k_n \psi_n + B(x_n, y_n; x_n, y_n)) \]

\[
\int_{e_n}^{2\pi - 1} \frac{1}{k_n} \frac{\partial}{\partial \phi} \left( -k_n \psi_n + B(x, y; x_n, y_n) \right) |_{r = e_n} \delta \phi \]

\[
= -2\pi (k_n \psi_n - B(x_n, y_n; x_n, y_n)) \tag{IV.4.20}
\]

Therefore, using (IV.4.13, 15, 17, 18, 20) one finds:

\[
E^* = E^* + \pi \rho \sum_{n=1}^{N} \left\{ 2\gamma_n \psi^*(x_n, y_n) + \gamma_n^2 B(x_n, y_n; x_n, y_n) \right. \\
- k_n \psi_n \left. \right\} + \pi \rho \sum_{k=1}^{N} \gamma_n \gamma_k \psi(x_n, y_n; x_k, y_k) + \text{const.} \tag{IV.4.21}
\]
The kinetic energy of all the fluid is therefore:

\[ E = E_\psi^* + \sum_{n=1}^{N} E_n + 2\pi \rho \left\{ \frac{1}{2} \sum_{k=1}^{N} \sum_{k \neq n}^{N} \gamma_n \gamma_k \psi(x_n, y_n; x_k, y_k) 
\right. \\
+ \frac{1}{4} \sum_{n=1}^{N} \left[ \gamma_n^2 B(x_n, y_n; x_n, y_n) + \gamma_n^2 \frac{k_n}{\alpha_n} \left( \frac{h_n^2}{\alpha_n} \right) \right] \\
\left. + 2\gamma_n \psi^*(x_n, y_n) \right\} + \text{const.} \]  

(IV.4.22)

which, upon substitution of (IV.3.7) and (IV.3.6) for \( E_{cn} \) becomes:

\[ E = E_\psi^* + 2\pi \rho \Omega + \text{const.} \]

Thus, \( \Omega \) is proportional to the energy of that part of the flow which is due to the vortices.
IV.5 Conformal Transformations

Since, when the fluid depth is constant, the stream-function of a single vortex is the Green's function of the Laplacian, it is natural to ask how the vortex motion changes under a conformal transformation of coordinates, $z \mapsto \tilde{z}$, since these leave the Laplacian invariant. This question was first considered by Routh (1881) and later, in more generality, by Lin (1943) who showed that under the conformal transformation the vortex streamfunction transforms as:

$$\tilde{\Omega} = \Omega - \frac{1}{\pi} \sum_{n=1}^{\infty} \gamma_n \left| \frac{d\tilde{z}_n}{dz_n} \right|$$  \hspace{1cm} (IV.5.1)

where tildes denote transformed quantities (Lin's $\kappa = 2\pi\gamma$ and $W = -2\pi\Omega$). We now show that (IV.5.1) remains valid on curved surfaces of constant depth provided the surface is invariant under the transformation.

Let $\tilde{z} = f(z)$ be a conformal transformation of complex coordinates. Notice that $(\tilde{x}, \tilde{y})$ such that $x + iy = z$ are harmonic coordinates since:

$$d\tilde{s}^2 = h^2(\tilde{z}, \overline{\tilde{z}})(dx^2 + dy^2) = h^2(z, \overline{z})dzd\overline{z}$$

$$= \frac{h^2(z, \overline{z})dzd\overline{z}}{f'(\tilde{z})f'(\overline{\tilde{z}})} = \frac{h^2(\tilde{z}, \overline{\tilde{z}})(d\tilde{x}^2 + d\tilde{y}^2)}{f'(\tilde{z})f'(\overline{\tilde{z}})}$$  \hspace{1cm} (IV.5.2)

Thus:

$$h(\tilde{z}, \overline{\tilde{z}}) = \left| \frac{dz}{d\tilde{z}} \right| h(z, \overline{z})$$  \hspace{1cm} (IV.5.3)

The streamfunction for a vortex in the transformed system is:

$$\tilde{\psi}(\tilde{z}, \overline{\tilde{z}}; \tilde{z}', \overline{\tilde{z}}') = \psi(z, \overline{z}; z', \overline{z}')$$  \hspace{1cm} (IV.5.4)
\[ \bar{\psi}^*(\bar{z}, \bar{z}) = \psi^*(\bar{z}, \bar{z}) \] (IV.5.5)

since the Laplacian is invariant. From (IV.1.22)

\[ -\ln|\bar{z} - \bar{z}'| + \bar{B}(\bar{z}, \bar{z}; \bar{z}', \bar{z}') = -\ln|z - z'| + B(z, z'; z', z') \] (IV.5.6)

or:

\[ \bar{B}(\bar{z}, \bar{z}; \bar{z}', \bar{z}') = B(z, z; z', z') + \ln \left| \frac{\bar{z} - \bar{z}'}{z - z'} \right| \] (IV.5.7)

Therefore:

\[ \bar{B}(\bar{z}', \bar{z}'; \bar{z}', \bar{z}') = \lim_{z \to \bar{z}'} \left[ B(z, \bar{z}; z', \bar{z}') + \ln \left| \frac{\bar{z} - \bar{z}'}{z - z'} \right| \right] \]

\[ = B(z', \bar{z}'; z', \bar{z}') + \ln \left| \frac{d\bar{z}'}{dz'} \right| \] (IV.5.8)

In the transformed system the motion is derived from the transformed streamfunction:

\[ \tilde{\Omega} = \frac{1}{\Delta_2} \sum_{n=1}^{N} \gamma_n \delta_n \tilde{\psi}(\tilde{z}_n, \tilde{z}_n; \tilde{z}_k, \tilde{z}_k) + \frac{N}{\Delta_2} \sum_{k=1}^{N} \gamma_n^2 \bar{B}(\tilde{z}_n, \tilde{z}_n; \tilde{z}_n, \tilde{z}_n) \]

\[ + \gamma_n^2 \Delta_2 h^2(\tilde{z}_n, \tilde{z}_n) + 2\gamma_n \bar{\psi}^*(\tilde{z}_n, \tilde{z}_n) \]

\[ = \Omega - \frac{1}{\Delta_2} \sum_{k=1}^{N} \Delta_2 \ln \left| \frac{d\bar{z}_n}{dz_n} \right| \] (IV.5.8)

verifying (IV.5.1).
IV.7 Constants of the Motion

As in any symplectic system, there are conservation laws associated with infinitesimal transformations which leave the vortex streamfunction invariant (see, e.g., J.M. Souriau (1969)). In particular, if the metric function $h$ and the depth function $k$ and all boundaries are invariant under an infinitesimal coordinate transformation then the vortex streamfunction must also be invariant. Symmetries of the fluid therefore induce conservation laws.

Let $G(x_1, y_1, \ldots, x_n, y_n)$ be a real valued function of the vortex positions. Then:

$$\frac{dG}{dt} = \sum_{n=1}^{N} \left[ \frac{\partial G}{\partial x_n} \dot{x}_n + \frac{\partial G}{\partial y_n} \dot{y}_n \right]$$

$$= \sum_{n=1}^{N} \frac{1}{\gamma_n h^2 k_n} \left[ \frac{\partial G}{\partial x_n} \frac{\partial \Omega}{\partial y_n} - \frac{\partial G}{\partial y_n} \frac{\partial \Omega}{\partial x_n} \right] = [G, \Omega] \quad (IV.7.1)$$

$G$ is a constant of the motion if and only if $[G, \Omega] = 0$.

In particular, $\Omega$ is itself conserved.

Consider the infinitesimal transformation:

$$\xi_n = x_n + \frac{\varepsilon}{\gamma_n h^2 k_n} \frac{\partial G}{\partial y_n}; \quad \eta_n = y_n - \frac{\varepsilon}{\gamma_n h^2 k_n} \frac{\partial G}{\partial x_n}$$

(IV.7.2)

for some small $\varepsilon$. $G$ is called the generator of the transformation. Then:

$$\Omega(\xi_1, \eta_1, \ldots, \xi_n, \eta_n) - \Omega(x_1, y_1, \ldots, x_n, y_n)$$
\[
= \sum_{n=1}^{N} \frac{1}{\gamma_n h_n k_n} \left[ \frac{\partial \Omega}{\partial x_n} \frac{\partial G}{\partial y_n} - \frac{\partial \Omega}{\partial y_n} \frac{\partial G}{\partial x_n} \right] + O(\epsilon^2)
\]

\[
= -[G,\Omega] + O(\epsilon^2)
\]  

(IV.7.3)

for some small \( \epsilon \).

Examples:

a) If \( h \equiv h(x) \), \( k \equiv k(x) \), and the only boundaries are curves \( x = \text{const.} \) then \( \Omega \) is invariant under: \( \xi_n = x_n, \eta_n = y_n - \epsilon \).

The generator of this transformation is:

\[
G = \sum_{n=1}^{N} \gamma_n \int h^2(x_n) k(x_n) dx_n
\]

(IV.7.4)

which is therefore conserved. When the surface of flow is a plane and \( k \) is constant this yields the conservation of centre of circulation.

b) If \( h \equiv h(r) \), \( k \equiv k(r) \), and all boundaries are curves \( r = \text{const.} \) then \( \Omega \) is invariant under

\[
\xi_n = x_n + \epsilon y_n \; ; \; \eta_n = y_n - \epsilon x_n
\]

Therefore:
\[
\frac{1}{\gamma_n h_n^* k_n} \frac{\partial G}{\partial y_n} = y_n, \quad \frac{1}{\gamma_n h_n^* k_n} \frac{\partial G}{\partial x_n} = x_n \quad (IV.7.5)
\]

which can be rewritten:

\[
\frac{\partial G}{\partial r_n} = 2\gamma_n h^2(r_n) k(r_n) r_n \quad ; \quad \frac{\partial G}{\partial \phi_n} = 0 \quad (IV.7.6)
\]

Thus:

\[
G = 2 \sum_{n=1}^{N} \gamma_n \int h^2(r_n) k(r_n) r_n dr_n \quad (IV.7.7)
\]

is conserved. For flow in the plane with \( k = \text{const} \), this yields the conservation of moment of circulation.

Vortex systems in the plane with \( k = \text{const} \) exhibit another conserved quantity known as the angular moment of circulation:

\[
N \sum_{n=1}^{N} \gamma_n (x_n \dot{y}_n - \dot{x}_n y_n) = \text{const.} \quad (IV.7.8)
\]

One can generalize to curved surfaces as follows:

Suppose \( h \) and \( k \) are homogeneous functions of order \( \nu \) : i.e., \( h(ax,ay) = a^\nu h(x,y) \), \( k(ax,ay) = a^\nu k(x,y) \). Physically this means that the fluid is invariant under scale transformations \( (x,y) \rightarrow (ax,ay) \). If the boundaries are also preserved under the transformation, then the vortex stream-function must transform as:

\[
\Omega(ax_1,ay_1,\ldots,ax_n,ay_n) = a^H \Omega(x_1,y_1,\ldots,x_n,y_n) + b(a)
\]

in order that the motion in the transformed system (with appropriately scaled time) is similar to the motion in the original system. Differentiating by \( a \) and then setting \( a=1 \):

\[
\frac{\partial \Omega}{\partial a} = \frac{\partial b}{\partial a}
\]
\[
\frac{N}{\Sigma} \left[ x_n \frac{\partial \Omega}{\partial x_n} + y_n \frac{\partial \Omega}{\partial y_n} \right] = \mu \Omega + \text{const.}
\]

Using (IV.4.4) and the fact that \( \Omega \) is itself conserved, one has:

\[
\frac{N}{\Sigma} \gamma_n h_n^2 k_n \left[ \dot{x}_n \dot{y}_n - \dot{x}_n \dot{y}_n \right] = \text{const.} \quad (IV.7.10)
\]

The conservation of angular moment of circulation is therefore related to the invariance of the fluid under scale transformations. That this is the case for rectilinear vortices has been pointed out by Chapman (1978).
V. SIMPLE VORTEX SYSTEMS

The behaviour of some simple vortex systems is now examined in order to gain some insight into the qualitative differences in the motion of vortices on a curved surface with varying depth and rectilinear vortices on a plane.

V.1 The Motion of a Single Vortex:
No External Velocity Field

Consider first the motion of a single vortex. Since $\Omega$ is conserved its path is given by:

$$\frac{2\Omega}{\gamma^2} = B(x,y;x,y) + k(x,y)\ln\left[\frac{h^2(x,y)k(x,y)}{\alpha^*}\right] = \text{const.}$$

(V.1.1)

The vortex path may close but may never cross itself. There are two special cases of interest.

a) Curved Surface, Uniform Depth

If the depth is uniform one can put $k=1$ and:

$$B(x,y;x,y) + \ln h^2(x,y) = \text{const.}$$

(V.1.2)

If one supposes further that there are no boundaries then $B=0$ and the vortex moves along a curve $h=\text{const}$. More explicitly, its velocity is:

$$v = \gamma^2 x \sqrt{h(x,y)}$$

(V.1.3)

The motion is in marked contrast to the motion of a vortex in the plane, which remains stationary. Notice that a knowledge of $h(x,y)$ in the vicinity of the core is sufficient to determine $v$. 
b) Plane Surface, Non-Uniform Depth

If the surface is planar one may put $h=1$ so that:

$$B(x,y;x,y) + \frac{k(x,y)}{2} \ln \left(\frac{k(x,y)}{\alpha}\right) = \text{constant} \quad (V.1.4)$$

Now, however, even in the absence of boundaries, $B$ does not necessarily vanish. One can only obtain $B$ in general by solving (IV.1.5). For non-constant $k$ very few analytic solutions are known.

It is possible, however, to treat the case in which the fluid has uniform depth but for a small depression near the origin (see Figure IV).

$$k = k_0 , \ r > \epsilon \ ; \ k = k(r) , \ r < \epsilon \quad (V.1.5)$$

The vortex is assumed to be at $(x,y) = (-a,0)$ with $a \gg \epsilon$. Let the streamfunction for the flow be:

$$\psi(x,y) = \psi_0(x,y) + \chi(x,y) \quad (V.1.6)$$

where $\psi_0$ is the streamfunction if $k=k_0$ everywhere. Then $\chi(x,y)$ satisfies:

$$\nabla \cdot \left( \frac{1}{k(r)} \nabla \chi \right) = -\nabla \cdot \left( \frac{1}{k(r)} \nabla \psi_0 \right) \quad (V.1.7)$$

If $\epsilon$ is sufficiently small $\nabla \psi_0$ is nearly constant across the region in which $k$ varies. Thus:

$$\nabla \cdot \left( \frac{1}{k(r)} \nabla \chi \right) \approx \frac{-k'(r)\gamma}{k^2(r)a} \hat{r} \cdot \hat{x} = \frac{-k_0k'(r)\gamma}{k^2(r)a} \cos \phi \quad (V.1.8)$$

since: $\psi_0(x,y) = -\gamma k_0 \ln [(x+a)^2 + y^2]^{\frac{1}{2}}$. $\chi$ therefore has the form:
Fig. IV: Fluid with Depression at the Origin
\[ \chi = \frac{\gamma}{a} f(r) \cos \phi \]

Since \( \nabla^2 \chi = 0 \) for \( r > \varepsilon \), \( f(r) \sim \frac{b \gamma}{ar} \) for \( r >> \varepsilon \) and \( b \) some constant depending only on \( k(r) \). At the vortex \( X = \chi \). Thus, if the vortex is at \((x,y)\) and the depression at \((x_0, y_0)\), then:

\[
B(x,y;x,y) \approx \frac{b \gamma}{(x-x_0)^2 + (y-y_0)^2} \quad (V.1.9)
\]

The velocity of the vortex induced by the depth variation varies as \( d^{-3} \) for large \( d \), where \( d \) is the distance of the vortex to the depression. The direction of the velocity is perpendicular to the line joining the vortex and the depression.

When the depth of the fluid is varying at the position of the vortex there is a component of velocity:

\[
v = -\frac{\gamma}{2k} \left[ \frac{k(x,y)}{a} \right] + 1 \right] \hat{e}_x \nabla k(x,y) \quad (V.1.10)
\]

directec along curves of constant \( k \). This velocity component is induced by the bending of the vortex core.

Notice that, unlike the previous case, a knowledge of \( k \) throughout the region of flow is necessary to determine \( v \). It is largely due to this property that vortex systems in fluids of varying depth are more difficult to analyze than those of constant depth.
Consider the motion of a single vortex in a fluid of constant thickness on a surface that is planar at infinity but is curved near the origin; for simplicity it is assumed radially symmetric (Figure V). One may then suppose the surface is defined by \( z = b(\rho) \) (see Section II.2) where \( b'(0) = 0 \) (so that the surface is smooth at the origin and \( b+0 \) as \( \rho+\infty \)).

The polar harmonic coordinate \( r \) is given by (II.2.18) and if one chooses \( \rho_0 \) so that:

\[
\frac{1}{\rho_0} \int_{\rho_0}^{\infty} \frac{1}{\sqrt{1 + (b'(s))^2}} \, ds = 1 \tag{V.2.1}
\]

then as \( \rho \to \infty \), \( r \to \rho \), \( k \to 1 \).

The external field is such that it is a uniform stream as \( \rho \to \infty \). If one chooses its direction to be the \( \hat{x} \)-direction then:

\[
\psi^*(r,\phi) = Uy = Ursin\phi \tag{V.2.2}
\]

The path of the vortex is therefore:

\[
xh(r) + \frac{U(y-y_0)}{\gamma} = 0
\]

where \( y = y_0 \) is the path of the vortex far upstream.

Clearly, \( y = y_0 \) is the path of the vortex far downstream too, and since \( y \to \rho\sin\phi \) at infinity, the paths at infinity do not depend on the curvature near the origin at all. In general, though, the time taken for the vortex to pass the origin will depend upon the details of the curvature.
Fig. V: Fluid with Surface Curvature Near the Origin
The equations of motion of the vortex are:

\[
\begin{align*}
\dot{x} &= \frac{\gamma y h'(r)}{2h^3(r)r} + \frac{U}{h^2(r)} ; \\
\dot{y} &= \frac{\gamma x h'(r)}{2h^3(r)r} \\
\end{align*}
\]  
(V.2.4)

Assuming \( \frac{U}{y} > 0 \) there is a stationary point at

\[
x = 0 , \ y = r_0
\]  
(V.2.5)

where:

\[
\frac{h'(r_0)}{h(r_0)} = -\frac{2U}{\gamma}
\]

(Note that, from (II.2.21), \( h'(r)/h(r) < 0 \)). The singular point is stable if \( \Omega \) has a maximum or minimum there: i.e., if:

\[
\begin{bmatrix}
\frac{\partial^2 \Omega(o, r_0)}{\partial x^2} & \frac{\partial^2 \Omega(o, r_0)}{\partial x \partial y} \\
\frac{\partial^2 \Omega(o, r_0)}{\partial y \partial x} & \frac{\partial^2 \Omega(o, r_0)}{\partial y^2}
\end{bmatrix}
\]  
\[
\begin{bmatrix}
\gamma^2 [p(r_0) - (p(r_0) - 1)] & 0 \\
0 & \frac{\gamma^2 [p(r_0) - (p(r_0) - 1)]}{2r_0^2}
\end{bmatrix}
\]

is positive or negative definite. Since \( p(r) < 1 \) this occurs when:

\[
r_0 p'(r_0) - p(r_0) + 1 < 0
\]  
(V.2.8)

Since the Gaussian curvature of the surface is: \( K = -\frac{p'(r)}{rh^2(r)} \),

(V.2.8) predicts stability only if \( K(r_0) > 0 \). If \( b(p) \) is decreasing this occurs only if the stationary point is sufficiently close to the origin.
V.3 The Motion of a Close Vortex Pair: $\gamma_1 = -\gamma_2$

Consider the motion of a pair of vortices with equal but opposite strengths separated by a distance much smaller than $|\text{curl} \nu h|^{-1}$ in a fluid of constant depth. The equations of motion are (using complex notation):

$$
\ddot{z}_1 = \frac{i\gamma}{h_1^2(z_1-z_2)} + \frac{i\gamma}{h_1^2} \frac{\partial h_1}{\partial z}, \quad \ddot{z}_2 = \frac{i\gamma}{h_2^2(z_1-z_2)} - \frac{i\gamma}{h_2^2} \frac{\partial h_2}{\partial z}
$$

(V.3.1)

Therefore, defining $Z = \frac{1}{2}(z_1 + z_2)$, $d = \frac{1}{2}(z_1 - z_2)$:

$$
\ddot{Z} = \frac{i\gamma}{4d} \left[ \frac{1}{h^2(Z+d)} + \frac{1}{h^2(Z-d)} \right] + \frac{i\gamma}{2} \left[ \frac{1}{h^3(Z+d)} \frac{\partial h(Z+d)}{\partial z} \right.
$$

$$
\left. - \frac{1}{h^3(Z-d)} \frac{\partial h(Z-d)}{\partial z} \right] \quad (V.3.2a)
$$

$$
\ddot{d} = \frac{i\gamma}{4d} \left[ \frac{1}{h^2(Z+d)} - \frac{1}{h^2(Z-d)} \right] + \frac{i\gamma}{2} \left[ \frac{1}{h^3(Z+d)} \frac{\partial h(Z+d)}{\partial z} \right.
$$

$$
\left. + \frac{1}{h^3(Z-d)} \frac{\partial h(Z-d)}{\partial z} \right] \quad (V.3.2b)
$$

(The argument $Z$ of $h$ has been dropped for simplicity)

Expanding and dropping terms of order $\frac{d^2}{h^2} \frac{\partial h}{\partial z}$:

$$
\ddot{Z} = \frac{i\gamma}{2dh^2(Z)}, \quad \ddot{d} = \frac{-i\gamma}{h^3(Z)} \frac{\partial h(Z)}{\partial z} \frac{\partial}{\partial d} \quad (V.3.3)
$$

Note that:

$$
\frac{\ddot{d}}{\partial d} = - \left( \frac{i\gamma}{2dh^2(Z)} \right) h(Z) \frac{\partial h(Z)}{\partial z} = \frac{-2Z}{h(Z)} \frac{\partial h(Z)}{\partial z} \quad (V.3.4)
$$

so that:
\[
\frac{d}{dt}(\ln d\tilde{d}) = \frac{d}{dt} + \frac{d}{dt} = -2\left[\frac{\tilde{Z}}{h(Z)} \frac{\partial h(Z)}{\partial Z} + \frac{\tilde{Z}}{h(Z)} \frac{\partial h(Z)}{\partial Z}\right]
\]

whence:

\[h^2(Z)d\tilde{d} = E = \text{const.} \quad (V.3.6)\]

This is the conservation of energy to this level of approximation.

From (V.3.3) and (V.3.6):

\[v^2 = h^2(Z)\frac{d^2}{dz^2} = \frac{v^2}{4E} = \text{const.} \quad (V.3.7)\]

The pair moves with constant speed.

One can eliminate \(d\) from the equations of motion by differentiating (V.3.3):

\[\ddot{Z} = \frac{i\gamma}{4E} \frac{\partial}{\partial Z} \frac{\partial h(Z)}{\partial Z} \frac{\dot{Z}}{Z} = -2Z^2 \frac{\partial \ln h(Z)}{\partial Z} \quad (V.3.8)\]

Suppose \(h = h(r)\) and put \(Z = Re^{i\phi}\). Then (V.3.8) becomes:

\[\dot{R} - R\dot{\phi}^2 = -(\dot{R}^2 + R^2\dot{\phi}^2) R_d \ln h(R) \quad (V.3.9a)\]

\[R\ddot{\phi} + 2R\dot{\phi} = -2R\dot{\phi} \frac{d}{dr} \ln h(R) \quad (V.3.9b)\]

\[\gamma^2/4E = h^2(R)(\dot{R}^2 + R^2\dot{\phi}^2) \quad (V.3.10)\]

There is another constant of the motion since:

\[\frac{\ddot{\phi}}{\dot{\phi}} + \frac{2\dot{R}}{R} = -2\frac{d}{dr} \ln h(R) \quad (V.3.11)\]

whence:
\[ R^2 \dot{R} h^2(R) = J = \text{const.} \quad (V.3.12) \]

Using (V.3.12) to eliminate \( \dot{\phi} \) in (V.3.10) gives:

\[ \dot{R}^2 + \frac{J^2}{R^2 h^4(r)} = \frac{\gamma^2}{4E h^2(r)} \quad (V.3.13) \]

If the surface is \( Z = b(\rho) \) (see Section III.2) then (V.3.11) and (V.3.12) may be rewritten:

\[ \rho^2 \dot{\phi} = J \quad (V.3.14) \]

\[ \dot{\rho}^2 = \left[ \frac{\gamma^2}{4E} - \frac{J^2}{\rho^2} \right] \left( \frac{1}{1 + \left[ b'(\rho) \right]^2} \right) \quad (V.3.15) \]

(V.3.14) and (V.3.15) are identical to the equations of motion of a particle moving under the influence of the central potential:

\[ V(\rho) = \left[ \frac{\gamma^2}{4E} - \frac{J^2}{\rho^2} \right] \left[ b'(\rho) \right]^2 \left( \frac{1}{1 + \left[ b'(\rho) \right]^2} \right) \quad (V.3.16) \]

For each \( J \) and \( E \) there is exactly one circular orbit with \( \rho = \frac{2J\sqrt{E}}{\gamma} \equiv a \). \( \rho \) can never be less than \( a \). The path of the vortex pair is given by:

\[ \frac{d\rho}{d\phi} = \left( \frac{\rho^2 \gamma^2}{4J^2 E} - 1 \right)^{1/2} \left( \frac{\rho}{1 + \left[ b'(\rho) \right]^2} \right)^{1/2} \quad (V.3.17) \]

from which one finds that the distance of closest approach to the origin is also \( \rho = a \). The circular orbit is therefore unstable and is the envelope of all the possible orbits for fixed \( E \) and \( J \).

If \( b \rightarrow \text{const.} \) as \( \rho \rightarrow \infty \) the path of the vortex pair is qualitatively similar to that shown in Figure VI. The
impact parameter is:

\[ l = \frac{J}{V} = \frac{2\sqrt{EJ}}{Y^2} = a \]  

(V.3.18)
Fig. VI: The Path of a Vortex Pair
V.4 The Stability of a Single Ring of Vortices on a Surface of Revolution

Historically the stability of configurations of vortices which move as rigid bodies has received a great deal of attention. Lord Kelvin (1878) first examined equal strength vortices placed at the vertices of regular polygons. His results were completed by J.J. Thomson (1883) who concluded that six or fewer such vortices are stable while seven or more are unstable. Thomson, however, erred in the heptagonal case. Morton (1933) showed that Thomson's method was in fact insufficient to determine the stability of a regular heptagon of vortices. It was only in 1977 that Mertz showed that this configuration is stable.

The stability of other rigid configurations has also been studied, notably, the vortex streets of von Karman (1912), vortices at polygonal vertices with a circular container (Havelock (1931), Chapman (1977)) and infinite vortex lattices (Tkachenko (1966)).

Experimental confirmation of the stability of some configurations within a circular boundary has recently been obtained for vortices in HeII, by a technique in which the positions of the vortices are actually photographed (Yarmchuk, et.al.(1979)).

We consider, now, an extension of the question considered originally by Lord Kelvin and Thomson: is a single ring of vortices on a surface of revolution stable?

Let \( N \) vortices of equal strength \( \gamma \) be placed initially at:
\[ r_n = r_0, \quad \phi_n = \frac{2\pi n}{N} \]  \hspace{1cm} (V.4.1)

on a surface of revolution described by \( h \equiv h(r) \) and in a fluid whose depth is constant (\( k=1 \)) and has no boundary (\( B=0 \)). Since \( \sum_{n=1}^{N} \gamma_n \neq 0 \) the surface cannot be closed.

The equations of motion are, from (IV.2.11):

\[ \frac{\dot{z}_n}{h^2(r_n)} = \frac{1}{2} \left[ \sum_{k \neq n} \gamma \frac{z_n - z_k}{z_n^2 z_k^2} + \frac{i \gamma z_n h'(r_n)}{2 r_n h(r_n)} \right] \]  \hspace{1cm} (V.4.2)

The symmetry of the initial configuration suggests a solution of the form:

\[ z_n = r(t) \exp\left[ \frac{2\pi n}{N} + \omega_1 t \right] \]  \hspace{1cm} (V.4.3)

Substituting into (V.4.2) one finds:

\[ r(t) = r_0; \omega_1 = \frac{\gamma}{r_0^2 h^2(r_0)} \left[ \sum_{k=1}^{N-1} \frac{1}{1 - \exp(2\pi k/N)} - \frac{r_0 h'(r_0)}{2 h(r_0)} \right] \]  \hspace{1cm} (V.4.4)

The sum is evaluated in Appendix B giving:

\[ \sum_{k=1}^{N} \frac{1}{1 - \exp(2\pi k/N)} = \frac{N-1}{2} \]  \hspace{1cm} (V.4.5)

so that:

\[ \omega_1 = \frac{\gamma}{2} [N - p(r_0)] \]  \hspace{1cm} (V.4.6)

where, for convenience, the unit of time has been taken to be: \( r_0^2 h^2(r_0)/\gamma \).

The ring of vortices rotates rigidly about the axis of revolution with angular velocity \( \omega_1 \). To examine the stability of the configuration consider small deviations from
the motion:

\[ z_n(t) = [r_0 \exp(2\pi i n/N) + \varepsilon_n(t)] e^{i\omega_1 t} \] (V.4.7)

Substituting into (V.4.2) and expanding to first order in the \( \varepsilon \)'s:

\[ \dot{\varepsilon}_n = \left[ \sum_{k \neq n} \frac{(\varepsilon_n - \varepsilon_k)}{(1 - \exp(2\pi i (k-n)/N))^2} + P(r_0, \omega_1) \varepsilon_n \right. \]

\[ \left. + Q(r_0, \omega_1) \bar{\varepsilon}_n \exp(4\pi i n/N) \right] i \exp(-4\pi i n/N) \] (V.4.8)

where:

\[ P(r, \omega) \equiv \frac{1}{4} r p'(r) + (p(r) - 1)(\omega - \frac{1}{2}) \] (V.4.9)

\[ Q(r, \omega) \equiv \frac{1}{4} r p'(r) + p(r) \omega \] (V.4.10)

The solutions to (V.4.8) are of the form:

\[ \varepsilon_n = a_M \exp[(2\pi i (1+M)n/N) + i\lambda_M t] \]

\[ + b_M \exp[2\pi i (1-M)n/N - i\lambda_M t] \] (V.4.11)

Substituting into (V.4.9) and equating coefficients of \( e^{i\lambda_M t} \) and \( e^{-i\lambda_M t} \) separately yields:

\[ (\lambda_M + Q(r_0, \omega_1))a_M + (S_{1-M} + P(r_0, \omega_1))\overline{b}_M = 0 \] (V.4.12a)

\[ (S_{1+M} + P(r_0, \omega_1))a_M + (-\lambda_M + Q(r_0, \omega_1))\overline{b}_M = 0 \] (V.4.12b)

with:

\[ S_L = \sum_{k=1}^{N-1} \frac{1 - \exp(2\pi i k L/N)}{(1 - \exp(2\pi i k/N))^2} = \frac{1}{2}(N-L)(2-L), \ L=1, \ldots, N \] (V.4.13)

(see Appendix B). There are non-trivial solutions of (V.4.12) if and only if:
\[ \lambda^2 M - Q^2(r_0, \omega_1) + (S_{1+M} + P(r_0, \omega_1))(S_{1-M} + P(r_0, \omega_1)) = 0 \]  
(V.4.14)

The Mth mode is stable if \( \lambda_M \) is real; i.e., if:

\[ Q^2(r_0, \omega_1) - (S_{1+M} + P(r_0, \omega_1))(S_{1-M} + P(r_0, \omega_1)) > 0 \]  
(V.4.15)

Using (V.4.6), (V.4.9), (V.4.10) and (V.4.13), (V.4.15) becomes:

\[ \frac{p'(r_0)}{2} r_0 + p(r_0)(N-p(r_0)) - \frac{M(N-M)}{2} > 0 \quad M=1, \ldots, N-1 \]  
(V.4.16)

In a linear stability analysis of any periodic solution of a symplectic system one always expects two zero-frequency solutions: one corresponding to small displacements of the system along its orbit, the other corresponding to small displacements of the system off the hyper-surface \( \Omega = \text{const.} \).

The first is a stable mode; the second is forbidden by the conservation of \( \Omega \). Here these modes are the \( M=N \) modes which have been neglected in (V.4.16). They cannot affect the stability of the configuration.

(V.4.16) has a minimum when \( M=\frac{1}{2}N \), \( N \) even or \( M=\frac{1}{2}(N+1) \), \( N \) odd. Hence, for stability of the ring of vortices:

\[ \frac{p'(r_0)}{2} r_0 + p(r_0)(N-p(r_0)) - \frac{N^2}{8} > 0 \quad N \text{ even} \]  
(V.4.17)  
\[ \times -1 \quad N \text{ odd} \]

Notice that, in contradistinction to the system in Section V.2, the stability is enhanced by negative curvature of the surface at \( r_0 \). By making the curvature at \( r_0 \) large and negative it is possible to accommodate arbitrarily large numbers of vortices.
in a stable ring.

Examples.

a) Plane

For the plane: \( p(r) = 1, p'(r) = 0 \) (Section II.2).

For stability:

\[
-N^2 + 8N - 8 > 0 \quad N \text{ even}
\]

\[
> -1 \quad N \text{ odd}
\]

\((V.4.18)\)

whence there is stability if \( N < 7 \). If \( N=7 \) the stability criterion is inconclusive and one must use higher order perturbation theory to determine the stability. This is the case on which Thomson erred.

b) Cylinder

A cylinder has: \( p(r) = 0, rp'(r) = 0 \). However, since the cylinder extends to infinity both as \( r \to 0 \) and as \( r \to \infty \), there is an arbitrariness in the choice of \( \psi(x, y; x', y') \) depending on the circulation around \( r=r' \) as \( r' \to \infty \) and \( r' \to 0 \). In other words:

\[
\psi(x, y; x', y') = -(1+\alpha)\ln((x-x')^2 + (y-y')^2)^{1/2} + \alpha\ln(x^2 + y^2)^{1/2}
\]

\((V.4.19)\)

is a perfectly valid streamfunction for a unit vortex. The circulation around \( r=r' \) as \( r' \to \infty \) is \( 2\pi(1+\alpha) \) while that around \( r=r' \) as \( r' \to 0 \) is \( -2\pi\alpha \). In writing the equations \((V.4.2)\) we have arbitrarily chosen \( \alpha = 0 \). The stability criterion is then:
\[ N^2 < 0 \quad , \quad N \text{ even} \quad , \quad N^2 < -1 \quad , \quad N \text{ odd} \]

whence all rings of vortices are unstable.

The velocity field induced by non-zero $\alpha$ in (V.4.19) is of the form: $v_r = 0$, $v_\phi = \text{const}$. The addition of such a velocity field cannot affect the stability of a ring of vortices: (V.4.20) is valid for all $\alpha$. 
V.5  The Stability of Vortex Streets on Surfaces of Revolution

If in addition to rotational symmetry the surface of flow is invariant under reflection in a plane perpendicular to the axis of rotation then double rings of vortices can also rotate rigidly about the axis of rotation. In view of the qualitative similarities to the configurations of von Karman (1912) these are called vortex streets.

Let the surface of revolution be described by:

\[ \rho = f(z) \]  (Section II.2) with \( f \) an even function.

Then, from (II.2.14)

\[
r = \exp \left[ z \int_{z_0}^{Z_0} \frac{1 + (f'(s))^2}{f(s)} \right]^{\frac{1}{2}} ds
\]  (V.5.1)

Choosing \( z_0 = 0 \) and since \( f(s) = f(-s) \):

\[
r(-z) = \frac{1}{r(z)}
\]  (V.5.2)

Moreover, (II.2.15) implies:

\[
 rh(r) = \frac{h(\frac{1}{r})}{r}
\]  (V.5.3)

(V.5.3) is the condition which implies that the surface is invariant upon reflection in the plane containing the curve \( r=1 \ (z=0) \).

We examine the stability of a ring of \( N \) vortices of strength \( \gamma \) at \( r_0(z_0) \) and \( N \) of strength \( -\gamma \) at \( \frac{1}{r_0}(-z_0) \). There are two distinct cases: staggered and symmetric vortex streets.
a) Staggered Vortex Streets

The vortices of a staggered vortex street are situated initially at:

\[ r_n = r_0 , \quad \phi_n = 2\pi n/N , \quad n=1,\ldots,N , \quad \text{strength } \gamma \quad \text{(V.5.4a)} \]

\[ r_m = \frac{1}{r_0} , \quad \phi_n = (2m+1)\pi i/N , \quad m=1,\ldots,N , \quad \text{strength } -\gamma \quad \text{(V.5.4b)} \]

In the absence of boundaries and in a fluid of uniform depth the equations of motion become:

\[ \frac{\dot{z}_n}{h^2(r_n)} = \frac{1}{N} \sum_{k=1, k\neq n}^{N} \frac{-i\gamma}{z_n - z_k} + \frac{\gamma}{N} \sum_{m=1}^{N} \frac{i\gamma z_n h'(r_n)}{z_m - z_n} + \frac{i\gamma z_n h'(r_n)}{2r_n h(r_n)} \quad \text{(V.5.5a)} \]

\[ \frac{\dot{z}_m}{h^2(r_m)} = \frac{1}{N} \sum_{k=1, k\neq m}^{N} \frac{i\gamma}{z_m - z_k} - \frac{\gamma}{N} \sum_{n=1}^{N} \frac{i\gamma z_m h'(r_n)}{z_m - z_n} - \frac{i\gamma z_m h'(r_n)}{2r_n h(r_n)} \quad \text{(V.5.5b)} \]

The symmetry of the initial configuration suggests solutions of the form:

\[ r_n = r(t) , \quad \phi_n = 2\pi n/N + \omega_2 t , \quad r_m = \frac{1}{r(t)} , \quad \phi_m = (2m+1)\pi i/N + \omega_2 t \quad \text{(V.5.6)} \]

Substituting into (V.5.5) and making use of (V.5.3) gives:

\[ r(t) = r_0 ; \quad \omega_2 = \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{1-\exp(2\pi ik/N)} + \frac{1}{2h(r_0)} \sum_{m=1}^{N} \frac{1}{1-r_0^{-2}\exp((2m+1)\pi i/N)} - \frac{r_0 h'(r_0)}{2h(r_0)} \quad \text{(V.5.7)} \]

The first sum was encountered in the last section (V.4.5). The second sum is (Appendix B):
Thus:

\[ \omega_2 = -\frac{1}{2} \left[ \frac{N(r_0^N - r_0^{-N})}{(r_0^N + r_0^{-N})} + p(r_0) \right] \]  

(V.5.9)

The unit of time has again been taken to be: \( r_0^2 \gamma \).

To determine the stability of the configuration set:

\[ z_n = [r_0 \exp(2\pi in/N) + \epsilon_n(t)] e^{i\omega_2 t} \]  

(V.5.10a)

\[ z_m = [r_0^{-1} \exp((2m+1)\pi i/N) + \delta_m(t)] e^{i\omega_2 t} \]  

(V.5.10b)

Substituting into the equations of motion and expanding to first order in the \( \epsilon \)'s and \( \delta \)'s:

\[
\frac{\cdot}{\cdot} \epsilon_n = \left[ \sum_{k=1}^{N} \frac{(\epsilon_n - \epsilon_k)}{(1 - \exp(2\pi i(k-n)/N))^2} - \sum_{m=1}^{N} \frac{(\epsilon_n - \delta_m)}{(1 - \exp(2\pi i(k-n)/N))^2} \right] \\
+ P(r_0, \omega_2) \epsilon_n + Q(r_0, \omega_2) \overline{\epsilon_n} \exp(4\pi in/N) \exp(-4\pi in/N) 
\]

(V.5.11a)

\[
\frac{\cdot}{\cdot} \delta_m = -\sum_{k=1}^{N} \frac{(\delta_m - \delta_k)}{(1 - \exp(2\pi i(k-n)/N))^2} + \sum_{n=1}^{N} \frac{(\delta_m - \epsilon_n)}{(1 - \exp(2\pi i(k-n)/N))^2} 
\]

(V.5.11b)

where \( P \) and \( Q \) are defined by (V.4.9,10). Using (V.5.3):

\[ p(r) = -p\left(\frac{1}{r}\right); \quad rp'(r) = \frac{1}{r}p'(\frac{1}{r}) \]  

(V.5.12)

so that:

\[ p\left(\frac{1}{r}, -\omega\right) = p(r) + 2\omega + p(r); \quad Q(r, \omega) = Q\left(\frac{1}{r}, -\omega\right) \]  

(V.5.13)
The solutions to (V.5.11) are of the form:

\[ \epsilon_n = a_M \exp[2\pi i (1+M)n/N + i\lambda_M t] \]

\[ + b_M \exp[2\pi i (1-M)n/N - i\lambda_M t] \]  

(V.5.14a)

\[ \delta_n = c_M \exp[-(2m+1)(1+M)\pi i/N + i\lambda_M t] \]

\[ + d_M \exp[-(2m+1)(1-M)\pi i/N - i\lambda_M t] \]  

(V.5.14b)

Substituting into (V.5.11) and equating coefficients of \( e^{i\lambda_M t} \) and \( e^{-i\lambda_M t} \) separately yields (using (V.5.9) and (V.5.13)):

\[ (\lambda_M + Q(r_0, \omega_2))a_M + A\bar{b}_M + r_0^N T_{1-M}(r_0^N) d_M = 0 \]  

(V.5.15a)

\[ Aa_M + (-\lambda_M + Q(r_0, \omega_2))\bar{b}_M + r_0^N T_{1-M}(r_0^N) \bar{c}_M = 0 \]  

(V.5.15b)

\[ T_{1-M}(r_0^N)\bar{b}_M + (-\lambda_M + Q(r_0, \omega_2)) \bar{c}_M + Ad_M = 0 \]  

(V.5.15c)

\[ T_{1-M}(r_0^N)a_M + A\bar{c}_M + (\lambda_M + Q(r_0, \omega_2))d_M = 0 \]  

(V.5.15d)

where

\[ A = S_{1+M} + P(r_0, \omega_2) + T_N(r_0^N) = Q(r_0, \omega_2) - \frac{1}{2}M(N-M) \]

\[ + \frac{N^2}{(r_0^N + r_0^{-N})^2} \]  

(V.5.16)

\[ T_L(x) = -x^{-2} T_{N-L+2}(x^{-1}) = \sum_{k=1}^{N} \frac{\exp((2k+1)\pi i/N)}{(1-x\exp((2k+1)\pi i/N))^2} \]

\[ = \frac{N^L((L-1)x^N - (N-L+1))}{(1+x^N)^2} , \quad L=1, \ldots, N \]  

(V.5.17)
(see Appendix B). \( S_L \) is defined in (IV.4.13). There are non-trivial solutions of (V.5.15) only if \( a = \pm r_0^2d, \) and \( b = \pm r_0^2c \).

Then:
\[
(\lambda_M + Q(r_0, \omega_2) \pm r_0^2 T_{1-M}(r_0^2)) a_M + A \delta_M = 0 \quad (V.5.18)
\]
\[
A a_M + (-\lambda_M + Q(r_0, \omega_2) \pm r_0^2 T_{1+M}(r_0^2)) b_M = 0 \quad (V.5.19)
\]

whence for non-trivial solutions:
\[
\lambda_M^2 \pm r_0^2 (T_{1+M}(r_0^2) - T_{1-M}(r_0^2)) \lambda_M + A^2
\]
\[- (Q(r_0, \omega_2) \pm r_0^2 T_{1-M}(r_0^2))(Q(r_0, \omega_2) \pm r_0^2 T_{1+M}(r_0^2)) = 0 \quad (V.5.20)
\]

The Mth modes are stable if \( \lambda_M \) is real, i.e., if:
\[
r_0^4 (T_{1+M}(r_0^2) - T_{1-M}(r_0^2))^2 \geq 4(A^2 - Q(r_0, \omega)
\]
\[
\pm r_0^2 T_{1+M}(r_0^2)(Q(r_0, \omega) \pm r_0^2 T_{1-M}(r_0^2)) \quad (V.5.21)
\]

which can be simplified to:
\[
(2C \pm D)(4q_2 - 2C \pm D) \geq 0 \quad (V.5.22)
\]

where:
\[
C \equiv (Q - A)/N^2 = \frac{1}{2}x(1 - x) - \frac{1}{4}\text{sech}^2(\frac{1}{2}y) \quad (V.5.23)
\]
\[
D \equiv r_0^2 (T_{1+M}(r_0^2) + T_{1-M}(r_0^2))/N^2
\]
\[
= \frac{xcosh((1-x)y) - (1-x)cosh(xy)}{2cosh^2(\frac{1}{2}y)} \quad (V.5.24)
\]
\[
q_2 \equiv Q(r_0, \omega_2)/N^2 \quad (V.5.25)
\]
\[
x \equiv M/N, \quad y \equiv N\pi r_0^2
\]

If \((2C \pm D)(4q_2 - 2C \pm D) = 0\) the solution is stable unless:
\[
T_{1+M}(r_0^2) = T_{1-M}(r_0^2) \quad (V.5.26)
\]
The stability criterion (V.5.22) is invariant under $r_0 + r_0^{-1}$ and $M \to N - M$. It is therefore sufficient to consider only $y \geq 0$ and $\frac{1}{2} \leq x \leq 1$.

The $N$th mode corresponds to perturbations of four types (Fig. VII; the $U$ denotes the upper signs in V.5.22, the $L$ the lower signs). The two $L$ modes are the expected zero frequency modes. The $U$ modes are stable if:

$$Q(r_0, \omega_2) = \frac{r_0 p'(r_0)}{4} - \frac{p(r_0)}{2} \left[ \frac{N(N - N)}{r_0 + r_0^{-1}} + p(r_0) \right] \leq 0$$

(V.5.27)

If $p(r_0) > 0$ and $K(r_0) > 0$ (Figure VIIIa) then this mode is stable. If $p(r_0) < 0$ and $K(r_0) < 0$ (Figure VIIIb) then it is unstable. Other cases must be examined separately.

If $N$ is even then the modes $M = \frac{1}{2} N$ are important as they tend to be the first to go unstable. The four possibilities are shown in Figure IX. They are all similar. This is reflected in the stability criterion which is the same for upper and lower signs:

$$\left(\frac{1}{4} - \frac{1}{2} \text{sech}^2 \frac{1}{2} y\right)(4q_2 - \frac{1}{4} + \frac{1}{2} \text{sech}^2 \frac{1}{2} y) > 0$$

(V.5.28)

If $q$ is small (as occurs when $N \to \infty$) then stability is restricted to a small region around:

$$y = 2 \arccosh \sqrt{2}$$

(V.5.29)

This criterion is analogous to the von Karman condition for the stability of infinite staggered vortex streets (in fact, if the surface of flow is a cylinder, our results must approach those of von Karman as $N \to \infty$). However, as will be seen for the case of the sphere, (V.5.28) is often incompatible with
Fig. VII: Staggered Vortex Street: The Modes M=N. Dots denote initial positions, crosses perturbed positions.
a) Staggered vortex streets stable. Symmetric vortex streets unstable.

\[ p(r_o) > 0, K(r_o) > 0 \]

b) Staggered vortex streets unstable. Symmetric vortex streets stable.

\[ p(r_o) < 0, K(r_o) < 0 \]

Fig. VIII: Surface for which the Mode \( M=N \) has Definite Stability
Fig. XI: Staggered Vortex Street: The Modes $M=\frac{1}{2}N$. The dots denote initial positions, the crosses perturbed positions.
(V.5.27) leading to overall instability.

Other modes are considerably more complicated and must be examined separately for specific surfaces of flow.

b) Symmetric Vortex Streets

The vortices of a symmetric vortex street are located initially at:

\[ r_n = r_0 ; \phi_n = \frac{2\pi n}{N} ; \quad n=1,\ldots,N, \quad \text{strength} \quad \gamma \]

\[(V.5.30)\]

\[ r_m = \frac{1}{r_0} ; \quad \phi_m = \frac{2\pi m}{N} ; \quad m=1,\ldots,N, \quad \text{strength} \quad -\gamma \]

\[(V.5.31)\]

Trying a solution to (V.5.5) of the form

\[ z_n = r(t)\exp\left(\frac{2\pi n}{N} + \omega_3 t\right) ; \quad z_m = r^{-1}(t)\exp\left(\frac{2\pi m}{N} + \omega_3 t\right) \]

\[(V.5.32)\]

one finds:

\[ r(t) = r_0 ; \quad \omega_3 = \omega_3 \left[ N\left(\frac{r_0^N - r_0^{-N}}{r_0 - r_0^{-N}}\right) + p(r_0) \right] \]

\[(V.5.33)\]

where use has been made of (V.4.5) and:

\[ \sum_{k=1}^{N} \frac{1}{1 - x \exp\left(\frac{2\pi ik}{N}\right)} = \frac{N}{1 - x^N} \]

(see Appendix B). The stability of the configuration is determined by putting:

\[ z_n = r_0 \exp\left(\frac{2\pi n}{N} + \epsilon_n(t)\right)e^{i\omega_3 t} \]

\[(V.5.35a)\]

\[ z_m = r_0^{-1} \exp\left(\frac{2\pi m}{N} + \delta_m(t)\right)e^{i\omega_3 t} \]

\[(V.5.35b)\]

substituting into (V.5.5), expanding to first order in the \( \epsilon \) 's and \( \delta \) 's and trying solutions of the form:

\[ \epsilon_n = a_n \exp\left[2\pi i(1+M)n/N + i\lambda M t\right] + b_n \exp\left[2\pi i(1-M)n/N - i\lambda M t\right] \]

\[(V.5.36a)\]
\[
\delta_m = c_m \exp\left[2\pi i (1+M)m/N + i\lambda_M t\right] + d_m \exp\left[2\pi i (1-M)m/N - i\lambda_M t\right]
\]
(E.5.36n)

The solutions satisfy:

\[
\begin{align*}
(\lambda_M + Q(r_0, \omega_3))a + B\bar{c}_M + r_0^4 R_{1-M}(r_0^2)d_M &= 0 \quad \text{(E.5.37a)} \\
B a_M + (-\lambda_M + Q(r_0, \omega_3))\bar{b}_M + r_0^4 R_{1+M}(r_0^2)\bar{c}_M &= 0 \quad \text{(E.5.37b)} \\
R_{1+M}(r_0^2)\bar{b}_M + (-\lambda_M + Q(r_0, \omega_3))\bar{c}_M + B d_M &= 0 \quad \text{(E.5.37c)} \\
R_{1-M}(r_0^2)a_M + B\bar{c}_M + (\lambda_M + Q(r_0, \omega_3))d_M &= 0 \quad \text{(E.5.37d)}
\end{align*}
\]

where:

\[
B = S_{1-M}R_N(r_0^2) + P(r_0, \omega_3) = Q(r_0, \omega_3) - \frac{1}{2}M(N-M)
\]
(E.5.38)

\[
R_L(x) = x^{-2}R_{N-L+2}(x^{-1}) \equiv \frac{N}{\sum_{k=1}^{N} \exp(2\pi i Lk/N)} \frac{\exp(2\pi i k/N)}{1-\exp(2\pi i k/N)}^2
\]
(E.5.39)

Equations (E.5.37) are exactly analogous to (E.5.15) so that the stability criterion is the equation analogous to (E.5.21): i.e.,

\[
\begin{align*}
& \quad r_0^4 (R_{1+M}(r_0^2) + R_{1-M}(r_0^2))^2 \geq 4(B^2 - (Q(r_0, \omega_3) \\
& \quad \pm r_0^2 R_{1+M}(r_0^2))(Q(r_0, \omega_3) \pm r_0^2 R_{1-M}(r_0^2))) \quad \text{(E.5.40)}
\end{align*}
\]

which can be simplified to:

\[
(2E \pm F)(4q_3 - 2E \pm F) \geq 0 , \quad M=1, \ldots, N \quad \text{(E.5.41)}
\]

with:

\[
E = \frac{1}{2}x(1-x) + \frac{1}{4}\text{csch}^2(\frac{1}{2}y) \quad \text{(E.5.42)}
\]
\[ F = \frac{(1-x)\cosh(xy) + xcosh((1-x)y)}{2\sinh^2(\frac{1}{2}y)} \quad (V.5.43) \]

\[ q_3 = Q(r_0, \omega_3)/N^2 \quad (V.5.44) \]

\[ x = M/N, \quad y = N\ln r_0^2 \quad (V.5.45) \]

Again one may suppose that \( y \geq 0, \frac{1}{2} \leq x \leq 1 \).

The four modes with \( M=N \) are shown in Figure X. (L denotes lower signs in V.5.41, U upper signs). As before, the L modes are the expected zero frequency modes. The upper signs give:

\[ Q(r_0, \omega_3) = \frac{r_0 p'(r_0)}{4} - p(r_0) \left[ N\frac{r_0^N + r_0^{-N}}{r_0^N - r_0^{-N}} + p(r_0) \right] \geq 0 \quad (V.5.46) \]

If \( p(r_0) < 0 \) and \( K(r_0) < 0 \) (see Figure VIIIb) then (V.5.46) is true, since:

\[\frac{N(r_0^N + r_0^{-N})}{r_0^N - r_0^{-N}} + p(r) > N - 1 > 0\]

If \( p(r_0) > 0 \) and \( K(r_0) > 0 \) then \( Q(r_0, \omega_3) < 0 \).

Other cases must be determined separately for each surface of interest.

We now show that \( 2E \pm F \geq 0 \) so that the stability criterion may be simplified further.

Let \( g(x, y) = 2x(1-x)\sinh^{2\frac{1}{2}}y + 1 - (1-x)\cosh(xy) - xcosh(1-x)y \)

Then: \( \frac{\partial g}{\partial y} = x(1-x)[\sinh y - \sinh(xy) - \sinh((1-x)y)] \)

Now: \( \frac{\partial}{\partial x} (\sinh(xy) + \sinh((1-x)y)) = y(\sinh(xy) - \sinh((1-x)y)) \geq 0 \)

for \( y \geq 0 \) and \( \frac{1}{2} \leq x \leq 1 \) since \( \sinhx \) is increasing.
Fig. X. Symmetric Vortex Street: The Modes $M=N$. Dots denote initial positions, crosses perturbed positions.
Therefore: \( \frac{1}{2} \sinh \frac{1}{2} y \leq \sinh(xy) + \sinh((1-x)y) \leq \sinh y \)

whence \( \frac{\partial g}{\partial y} > 0 \). Therefore:

\[ g(x, y) > g(x, 0) = 1-(1-x)-x = 0 \]

Therefore:

\[ 2E \pm F \geq 2E - F = \frac{g(x, y)}{2 \sinh \frac{1}{2} y} > 0 \quad \text{if } y > 0 \quad (V.5.47) \]

The stability criterion may therefore be simplified to:

\[ 4q_3 - 2E - F > 0 \quad , \quad M=1,\ldots,N-1 \quad (V:5.48) \]

The critical modes are in general the L modes with \( M = \frac{1}{2}N \) (N even; see Figure XI) for which the stability criterion is:

\[ q_3 > \frac{(1 + \cosh \frac{1}{2} y)^2}{16 \sinh^2 \frac{1}{2} y} = \frac{1}{16 \tanh^2 (\frac{1}{2} y)} \quad (V.5.49) \]

For stability \( q_3 > \frac{1}{16} \). As \( N \to \infty \), \( q_3 \to 0 \) so that for sufficiently large \( N \) a symmetric vortex street will always become unstable.

c) Example: The Cylinder

The stability of vortex streets on the cylinder is now examined.

For the cylinder \( p(r) = 0 \), \( p'(r) \) so that \( q_2 = q_3 = 0 \). (V.5.49) then immediately implies that all symmetric vortex streets are unstable.

Staggered vortex streets are stable if:

\[ 4C^2 - D^2 \leq 0 \quad (V.5.50) \]

For \( M = 1 \) all four modes now yield zero frequency modes. The L modes have already been explained. The conservation of moment of circulation and angular
Fig. XI: Symmetric Vortex Street: The Modes $M = \frac{1}{2} N$. The dots denote initial positions, the crosses perturbed positions.
moment of circulation for the cylinder are:

\[ \sum \gamma_n \phi_n = \text{const.} \]  \hspace{1cm} (V.5.51)

\[ \sum \gamma_n \rho_n = \text{const.} \]  \hspace{1cm} (V.5.52)

These conservation laws are violated by the \( M=N \), \( U \) modes. These modes must therefore be neglected and do not affect the stability of the configuration.

If \( N \) is even one may put \( M=\frac{1}{2}N(x=\frac{1}{2}) \). \( D \) is then zero, and for stability (from \( (V.5.22) \)),

\[ C^2 = \left[ \frac{1}{8} - \frac{1}{4} \sech^2(\frac{1}{2}y) \right]^2 \leq 0 \]  \hspace{1cm} (V.5.53)

i.e., the mode is unstable unless:

\[ y = 2\arccosh \sqrt{2} \]  \hspace{1cm} (V.5.54)

From \( (II.2.23) \) and \( (V.5.26) \), the separation of the rings of vortices is:

\[ d = \frac{R \arccosh \sqrt{2}}{N} \]  \hspace{1cm} (V.5.55)

while the separation of the vortices within each ring is:

\[ d_\phi = 2\pi R/N \]  \hspace{1cm} (V.5.56)

where \( R \) is the radius of the cylinder. Therefore:

\[ \frac{d}{d_\phi} = \frac{\arccosh \sqrt{2}}{\pi} \]  \hspace{1cm} (V.5.57)

in agreement with von Karman. When \( M=\frac{1}{2} \), \( T_{1+M}(x) = T_{1-M}(x) \) so that the stability of this mode is indeterminate. It can be determined only by higher order perturbation theory.

The stabilities of the other modes for \( N \) even and for \( N \) odd are now determined.

\[ \frac{\partial}{\partial y} [(D-2C)\cosh^2(\frac{1}{2}y)] = \frac{x(1-x)}{2} [\sinh((1-x)y) - \sinh(xy)] - 2cosh(\frac{1}{2}y)\sinh(\frac{1}{2}y) < 0 \]
since \( \sinh x \) is increasing if \( x > 0 \). But:
\[
(D-2C)\cosh^2 \frac{1}{2} y = x^2 > 0 \quad \text{when} \quad y = 0
\]
\[
\xrightarrow{y \to \pm \infty} (x \neq 1) \quad \text{as} \quad y \to \infty
\]
Thus, if \( x \neq 1 \), \( D-2C \) has exactly one zero for each \( x \).

Denote these zeros: \( y^- (x) \).

\[
\frac{\partial (D+2C)\cosh^2 \left( \frac{1}{2} y \right)}{\partial y} = \frac{x(1-x)}{2} \left[ \sinh((1-x)y) - \sinh(xy) + 2\cosh \left( \frac{1}{2} y \right) \sinh \left( \frac{1}{2} y \right) \right]
\]
\[
= \frac{(1-x^2)}{8} \left[ \sinh \left( \frac{1}{2} y \right) (1-e^{-2}) - \sinh \left( \frac{1}{2} y (1+e) \right) + 2\cosh \left( \frac{1}{2} y \right) \sinh \left( \frac{1}{2} y \right) \right]
\]
\[
= \frac{(1-x^2)}{8} \left[ -2\sinh \left( \frac{1}{2} y e \right) \cosh \left( \frac{1}{2} y \right) + 2\cosh \left( \frac{1}{2} y \right) \sinh \left( \frac{1}{2} y \right) \right] > 0
\]

where \( e = 2x-1 \). Moreover:

\[
D + 2C = -(1-x)^2 \quad \text{when} \quad y=0
\]
\[
\xrightarrow{y \to \infty} \quad \text{as} \quad y \to \infty
\]

Therefore \( D + 2C \) has exactly one zero for each \( x \ (x \neq 1) \).
These are denoted \( y^+(x) \).

When \( y = \arccosh \sqrt{2} \), \( C=0 \) and \( D > 0 \) ; therefore
\[
y^- (x) \leq 2\arccosh \sqrt{2} \leq y^+(x) \quad \text{and} \quad D + 2C > 0 \quad \text{when}:
\]
\[
y^- (x) \leq \cdots \quad y \quad \leq y^+(x) \quad \frac{1}{2} < x < 1 \quad (V.5.58)
\]

Define

\[
y^- = \max_x y^- (x) \quad , \quad y^+ = \min_x y^+(x) \quad (V.5.59)
\]

The vortex street is stable if:

\[
y^- < y < y^+ \quad (V.5.60)
\]

Figure XII is a graph of \( y^- (x) \) and \( y^+(x) \). Clearly \( y^+ \)
and \( y^- \) correspond to the values of \( x \) closest to \( \frac{1}{2} \). For \( N \) odd
Fig. XII: $y^+(x)$ and $y^-(x)$
this is: \( x = \frac{1}{N}(1 + \frac{1}{N}) \). For \( N \) even: \( x = \frac{1}{2} \).

Thus, if \( N \) is odd there is a small region of stability around \( y = \arccosh\sqrt{2} \). For the first few \( N \) these regions are given in Table 1.

Table 1: Regions of Stability of Staggered Vortex Streets on a Cylinder

<table>
<thead>
<tr>
<th>( N )</th>
<th>Lower Bound of ( y )</th>
<th>Upper Bound of ( y )</th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.5522</td>
<td>2.0794</td>
<td>0.0879</td>
</tr>
<tr>
<td>5</td>
<td>1.6278</td>
<td>1.9344</td>
<td>0.0307</td>
</tr>
<tr>
<td>7</td>
<td>1.6634</td>
<td>1.8806</td>
<td>0.0155</td>
</tr>
<tr>
<td>9</td>
<td>1.6842</td>
<td>1.8525</td>
<td>0.0094</td>
</tr>
</tbody>
</table>

\( \Delta \) is the actual width of these regions measured in cylinder radii:

\[
\Delta = \frac{y^+ - y^-}{2N}
\]

If \( N \) is even stability is only possible if \( y = \arccosh\sqrt{2} \), \( \Delta = 0 \) but one needs higher order perturbation theory to check the mode \( M = \frac{1}{2}N \).

d) Example: The Sphere

The analysis of the stability of vortex streets on the sphere is somewhat more complicated than on the cylinder.

From (II.2.24), (V.5.33) and (V.5.45):

\[
Q(r_0, \omega_3) = -\frac{1}{4} - \frac{1}{2} \tanh^2(y/2N) + \frac{1}{2}N\tanh(y/2N)\coth(\frac{1}{2}y) \tag{V.5.61}
\]

Since: \( \tanh(ax)\coth x \leq 1 \) if \( a < 1 \):

\[
Q(r_0, \omega_3) \leq -\frac{1}{4} + \frac{1}{2}N
\]
Moreover:

\[ E = \frac{1}{2}x(1-x) + \frac{1}{2}\text{cosech}^2(\frac{1}{2}y) > \frac{1}{2}x(1-x) ; \quad F \geq 0 \quad (V.5.63) \]

Therefore:

\[ 4q_3 - 2E - F < -\frac{1}{4N^2} + \frac{1}{2N} - x(1-x) \quad (V.5.64) \]

If \( N \) is even there is a mode \( x = \frac{1}{2} \), whence

\[ 4q_3 - 2E - F < -(N^2 - 8N + 4)/N^2 < 0 \quad \text{if } N > 7 \]

If \( N \) is odd there is a mode \( x = \frac{1}{2}(1 + \frac{1}{N}) \), whence

\[ 4q_3 - 2E - F < -(N^2 - 8N + 3)/N^2 < 0 \quad \text{if } N > 7 \]

Therefore all symmetric vortex streets with \( N > 7 \) are unstable.

The stability criterion (V.5.48) has been examined numerically for \( N = 2, \ldots, 6 \).

Table 2: Regions of Stability of Symmetric Vortex Streets on a Sphere

<table>
<thead>
<tr>
<th>( N )</th>
<th>Stability Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( y &gt; 4.2451 ), ( \theta &lt; 38.1727^0 )</td>
</tr>
<tr>
<td>3</td>
<td>( y &gt; 5.3020 ), ( \theta &lt; 44.9072^0 )</td>
</tr>
<tr>
<td>4</td>
<td>( y &gt; 7.5957 ), ( \theta &lt; 42.3078^0 )</td>
</tr>
<tr>
<td>5</td>
<td>( y &gt; 10.4306 ), ( \theta &lt; 38.8225^0 )</td>
</tr>
<tr>
<td>6</td>
<td>( y &gt; 17.7602 ), ( \theta &lt; 25.6478^0 )</td>
</tr>
<tr>
<td>( N \geq 7 )</td>
<td>Unstable</td>
</tr>
</tbody>
</table>

From: (II.2.24), (V.5.9) and (V.5.26):

\[ Q(r_0, \omega_2) = -\frac{1}{4} -\frac{1}{4}\text{tanh}^2(y/2N) + \frac{1}{2N}\text{tanh}(y/2N)\text{tanh}(\frac{1}{2}y) \quad (V.5.65) \]

Therefore:
\[
\frac{\partial Q}{\partial y} = \frac{\text{sech}^2(y/2N)}{4} \left[ \tanh\left(\frac{y}{2N}\right) - \frac{1}{N} \tanh(y/2N) \right] + \frac{N}{4} \tanh(y/2N) \text{sech}^2\left(\frac{y}{2N}\right) \geq 0 \text{ for } y > 0 \quad (V.5.66)
\]

Thus, for each \( N \) there is exactly one \( y \) such that \( Q = 0 \).

Denote it by \( y(N) \). The vortex street is unstable if:

\[ y > y(N) \quad \text{(see (V.5.27)).} \]

Let \( \alpha = y/2N \). Then \( N = y/2\alpha \) and:

\[
Q = -\frac{1}{4} \left[ 1 + \tanh^2 \alpha - \frac{y}{\alpha} \tanh \alpha \tanh\left(\frac{y}{2N}\right) \right]
= -\frac{\tanh \alpha}{4\alpha} \left[ \frac{(1+\tanh^2 \alpha)\alpha}{\tanh \alpha} - y \tanh\left(\frac{y}{2N}\right) \right] \quad (V.5.67)
\]

Now \( \frac{(1+\tanh^2 \alpha)\alpha}{\tanh \alpha} > \frac{\alpha}{\tanh \alpha} > 1 \). Therefore \( Q < 0 \) if

\[ y \tanh\left(\frac{y}{2N}\right) < 1 \quad \text{i.e., if } y < 1.543 \text{. Therefore:} \]

\[ y(N) > 1.543 \quad (V.5.68) \]

Let \( f(\alpha) = \alpha - (1+\alpha) \tanh \alpha \). Then:

\[
f'(\alpha) = (1-\tanh \alpha)(1-(1+\alpha)(1+\tanh \alpha)) < 0 \quad \text{if } \alpha > 0
\]

But \( f(0) = 0 \). Therefore: \( \alpha < (1+\alpha) \tanh \alpha \) and:

\[
\frac{\alpha(1+\tanh^2 \alpha)}{\tanh \alpha} = \frac{\alpha}{\tanh \alpha} + \alpha \tanh \alpha < 1+2\alpha \quad \text{Thus if:}
\]

\[ y \tanh\left(\frac{y}{2N}\right) > 1 + 2\alpha = 1 + \frac{y}{2N} \quad , \text{then } Q > 0 \,
\]

Let \( y^*(N) \) be the largest value of \( y \) such that

\[
y^*(N) \tanh\left(\frac{y^*(N)}{2N}\right) = 1 + \frac{y^*(N)}{2N} \quad (V.5.69)
\]

Then:

\[ 1.543 < y(N) < y^*(N) \quad (V.5.70) \]

Differentiating (V.5.69) by \( N \):
\[
\left\{\tanh\left(\frac{1}{2}y^*(N)\right) + \frac{1}{2}y^*(N)\sech^2\left(\frac{1}{2}y^*(N)\right) - \frac{1}{N}\right\}\frac{\partial y^*(N)}{\partial N} = -\frac{y^*(N)}{N}
\]

Since \( y^*(N) > 1.543 \):
\[
\tanh\left(\frac{1}{2}y^*(N)\right) + \frac{1}{2}y^*(N)\sech^2\left(\frac{1}{2}y^*(N)\right) > \tanh\left(\frac{1}{2} \times 1.543\right) = 0.648
\]
whence \( \frac{\partial y^*(N)}{\partial N} < 0 \).

One can check numerically that \( y^*(4) < 1.6 \). Thus \( y^*(N) < 1.6 \) for all \( N > 4 \), and:

\[
1.543 < y(N) < 1.6 \quad \text{for} \quad N > 4 \quad \text{(V.5.71)}
\]

A staggered vortex street on the sphere is therefore always unstable if \( N \geq 4 \) and \( y > 1.6 \).

It is now shown that if \( y < 1.6 \) and \( N \geq 6 \), then the modes \( M = \frac{1}{2}N \) or \( M = \frac{1}{2}(N+1) \) are unstable.

Let \( f(x,y) = \cosh(y(1-x)) + (1-\frac{1}{x})\cosh(yx) \). Then:

\[
\frac{\partial f}{\partial x} = \frac{1}{x^2}\cosh(xy)[1-xy\tanh(xy)] + y(\sinh(xy)) - \sinh((1-x)y)) > 0 \quad \text{if} \quad xy < 1.1997 \quad \text{(V.5.72)}
\]

With \( y < 1.6 \) (V.5.72) holds for \( x < 3/4 \). Thus:
\[
f(x,y) > f(\frac{1}{2},y) = 0 \quad \text{if} \quad \frac{1}{2} \leq x < 3/4 \quad \text{. Therefore:}
\]

\[
D = \frac{1}{2}xf(x,y)c\sinh^2(\frac{1}{2}y) > 0 \quad \text{if} \quad \frac{1}{2} \leq x < 3/4 , y < 1.6 \quad \text{(V.5.73)}
\]

and:

\[
\frac{\partial D}{\partial x} = \frac{1}{2}c\sinh^2(\frac{1}{2}x)\frac{\partial (xf)}{\partial x} > 0 \quad \text{if} \quad \frac{1}{2} \leq x < 3/4 , y < 1.6 \quad \text{(V.5.74)}
\]

Moreover, from (V.5.65) and (V.5.66):

\[
q_2 \geq -\frac{1}{4N^2} \quad \text{(V.5.75)}
\]
The modes $M = \frac{1}{2}N$ or $M = \frac{1}{2}(N+1)$ have: $x \leq \frac{1}{5}(1 + \frac{1}{N})$

whence:

$$C = \frac{1}{2}x(1-x) - \frac{1}{2}\text{sech}^2(\frac{1}{2}y) \leq \frac{1}{2} - \frac{1}{8N} - \frac{1}{2}\text{sech}^2(\frac{1}{2}y)$$

(V.5.76)

and, using (V.5.73, 75, 76):

$$4q_2 - 2C + D > \frac{-3}{4N^2} + \left[\frac{3}{16} - \frac{1}{2}\text{sech}^2(\frac{1}{2}X1.6)\right] = \frac{-3}{4N^2} + 0.0297$$

(V.5.77)

if $y < 1.6$. Therefore:

$$4q_2 - 2C + D > 0 \quad \text{if} \quad y < 1.6, \quad N \geq 6 \quad (V.5.78)$$

Moreover, one can check numerically that if $y = 1.6$ and $x = 0.58$ then: $2C + D < 0$. Since $\frac{3D}{8x} < 0$

for $y < 1.6$, $x < \frac{3}{4}$:

$$2C + D < 0, \quad y < 1.6, \quad \frac{3}{2} \leq x \leq 0.58$$

(V.5.79)

For $N \geq 6$ the modes $M = \frac{1}{2}N$ or $M = \frac{1}{2}(N+1)$ have $x < 0.58$.

Using (V.5.22), (V.5.78) and (V.5.79) these modes are therefore unstable if $y < 1.6$. Hence all staggered vortex streets on the sphere with $N \geq 6$ are unstable.

The stability of staggered vortex streets for $N \leq 5$ has been determined numerically (Table 3).

Table 3: Regions of Stability of Staggered Vortex Streets on a Sphere

<table>
<thead>
<tr>
<th>$N$</th>
<th>Stability Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Stable if $0 &lt; y &lt; 1.7627, 65.5302^0 &lt; \theta &lt; 70.9^0$</td>
</tr>
<tr>
<td>3</td>
<td>Stable if $1.5522 &lt; y &lt; 1.6306, 74.6171^0 &lt; \theta &lt; 75.3404^0$</td>
</tr>
<tr>
<td>$&gt; 3$</td>
<td>Unstable</td>
</tr>
</tbody>
</table>


These results differ qualitatively from those on the cylinder (or von Karman vortex streets) where the staggered vortex streets exhibit greater stability than symmetric vortex streets.
VI. VORTICES WITH FINITE CORES

Until now, it has been assumed that all vortices have infinitesimally small cores. In this section the effects of finite distributions of vorticity are considered. In particular, it is shown that if the surface of flow is curved and the core is not radially symmetric then a wobble is introduced into the motion of the vortex.

VI.1. The Position and Velocity of a Vortex

When the core extends over a finite region $G$ one can define the position of the vortex by:

$$X = \frac{1}{2\pi\gamma} \int_G w_z(x,y,t)xh^2(x,y)dx dy$$  \hspace{1cm} (VI.1.1a)

$$Y = \frac{1}{2\pi\gamma} \int_G w_z(x,y,t)yh^2(x,y)dx dy$$  \hspace{1cm} (VI.1.1b)

$$2\pi\gamma = \int_G w_z(x,y,t)h^2(x,y)dx dy$$  \hspace{1cm} (VI.1.2)

Since:

$$w_z = \frac{1}{h^2} \left[ \frac{\partial (hv_y)}{\partial x} - \frac{\partial (hv_x)}{\partial y} \right]$$  \hspace{1cm} (VI.1.3)

$$2\pi\gamma = \left[ \frac{\partial (hv_y)}{\partial x} - \frac{\partial (hv_x)}{\partial y} \right] dx dy = \oint_{\partial G} (v_x dx + v_y dy) = \Gamma_{\partial G}$$  \hspace{1cm} (VI.1.4)

so that $2\pi\gamma$ is still the circulation around a contour containing the core. By Kelvin's Circulation Theorem $\gamma$ does not depend on time.

The velocity of the core is:
\[ U^X = \dot{X} = \frac{1}{2\pi \gamma} \int_G \frac{\partial w_z(x,y,t)}{\partial t} x h^2(x,y) \, dx \, dy \]

\[- \frac{1}{2\pi \gamma} \int_G w_z(x,y,t) x h(x,y) \left[ (v_x - v_{Gx}) \, dy - (v_y - v_{Gy}) \, dx \right] \]  

(VI.1.5a)

\[ U^Y = \dot{Y} = \frac{1}{2\pi \gamma} \int_G \frac{\partial w_z(x,y,t)}{\partial t} y h^2(x,y) \, dx \, dy \]

\[- \frac{1}{2\pi \gamma} \int_G w_z(x,y,t) y h(x,y) \left[ (v_x - v_{Gx}) \, dy - (v_y - v_{Gy}) \, dx \right] \]  

(VI.1.5b)

where \( v_{G} \) is the velocity of the core boundary. Since \( w = 0 \) on \( \partial G \) the boundary terms vanish. Using (III.3.7) and (III.3.8):

\[ U^X = \frac{-1}{2\pi \gamma} \int_G \nabla \cdot \left( \frac{w_z}{k} \right) x h k \, dx \, dy \]

\[ = \frac{1}{2\pi \gamma} \int_G \frac{w_z}{k} \nabla \cdot (v x h k) \, dx \, dy \]

\[ = \frac{1}{2\pi \gamma} \int_G w_z v_x h k \, dx \, dy \]  

(VI.1.6a)

Similarly:

\[ U^Y = \frac{1}{2\pi \gamma} \int_G w_z v_y h k \, dx \, dy \]  

(VI.1.6b)

In terms of the streamfunction \( \psi \) defined by (IV.1.2):

\[ U^X = \frac{-1}{2\pi \gamma} \int_G \frac{1}{h^2 k} \nabla \cdot \left( \frac{1}{k} \nabla \psi \right) \frac{\partial \psi}{\partial y} \, dx \, dy \]  

(VI.1.7a)

\[ U^Y = \frac{1}{2\pi \gamma} \int_G \frac{1}{h^2 k} \nabla \cdot \left( \frac{1}{k} \nabla \psi \right) \frac{\partial \psi}{\partial x} \, dx \, dy \]  

(VI.1.7b)
VI.2 Circular Cores

A "circular" vortex is defined to be one which is derived by a streamfunction of the form:

\[
\psi(x,y) = -\gamma A(x,y;X_0, Y_0)f(r) + \gamma B(x,y;X_0, Y_0) + \psi^*(x,y)
\]

where \( r \equiv [(x-X)^2 + (y-Y)^2]^{\frac{1}{2}} \) and

\[
f(r) = \ln r, \quad r > \varepsilon
\]

f has continuous derivatives (so that the velocity field is continuous). The functions A and B are those defined by (IV.1.13) and \( \psi^* \) satisfies:

\[
\nabla \cdot \left( \frac{1}{k} \nabla \psi^* \right) = 0
\]

Thus, \( w = 0 \) for \( r > \varepsilon \) and the core boundary is: \( r = \varepsilon \).

Using (VI.2.1), (VI.1.2) and (VI.1.3) to evaluate (VI.1.1) one finds:

\[
X = X_0 + O(\nu) \quad ; \quad Y = Y_0 + O(\nu)
\]

where \( \nu \) is a small parameter of order \( \varepsilon |\nabla \nu| \), the ratio of the core radius and the distance over which \( h \) and \( k \) vary appreciably.

We wish to determine \( U^X \) and \( U^Y \) up to terms which vanish as \( \varepsilon \to 0 \) with \( \gamma \) constant.

\[
f(r) \sim O(\ln \varepsilon), \quad f'(r) \sim O\left(\frac{1}{\varepsilon}\right), \quad f''(r) \sim O\left(\frac{1}{\varepsilon^2}\right).
\]

Therefore: \( w \sim O\left(\frac{\gamma}{h^2 \varepsilon^2}\right) \), \( \nabla \sim \left(\frac{\gamma}{h \varepsilon}\right) \), and since \( \int \cdots dxdy \sim O(\varepsilon^2) \) one finds \( \bar{U} \sim O\left(\frac{\gamma}{h \varepsilon}\right) \), so that terms of
order $\varepsilon^{-1}$ must be retained in $\nabla \cdot \left( \frac{1}{k} \nabla \psi \right)$, terms of order $\varepsilon^0$ in $\nabla \psi$ and terms of order $\varepsilon$ in $h^{-2}$. Using (IV.4.1):

$$\nabla \cdot \left( \frac{1}{k} \nabla \psi \right) = -\frac{k_0^{1/2} \nabla^2 f(r)}{k^{1/2}(x,y)} + O(\varepsilon n \varepsilon) \quad (VI.2.5)$$

$$\frac{\partial \psi}{\partial y} = -\frac{y f(r)}{2} \frac{\partial k_0}{\partial y} - \frac{\gamma y f'(r)}{r} \left[ k_0 + \frac{x}{2} \frac{\partial k_0}{\partial x} + \frac{\gamma}{2} \frac{\partial k_0}{\partial y} \right] + \frac{\gamma \psi}{\partial y} + \frac{\partial \psi}{\partial y} + O(\varepsilon n \varepsilon), \quad r < \varepsilon \quad (VI.2.6)$$

$$\frac{1}{h^2 k^{1/2}} = \frac{1}{h_0^2 k_0^{1/2}} \left[ 1 - \frac{2x}{h_0} \frac{\partial h_0}{\partial x} - \frac{3x}{2k_0} \frac{\partial k_0}{\partial x} - \frac{2y}{h_0} \frac{\partial h_0}{\partial y} - \frac{3y}{2k_0} \frac{\partial k_0}{\partial y} \right] + O(\varepsilon^2) \quad (VI.2.7)$$

For convenience $(x, y)$ has been made the origin and subscripts zero denote evaluation at the origin. Substituting into (VI.1.7a):

$$U^X = \frac{1}{2 \pi} \int_0^E \int_0^{2\pi} \frac{1}{h_0 k_0^{1/2}} \left[ 1 - \frac{2x}{h_0} \frac{\partial h_0}{\partial x} - \frac{3x}{2k_0} \frac{\partial k_0}{\partial x} - \frac{2y}{h_0} \frac{\partial h_0}{\partial y} - \frac{3y}{2k_0} \frac{\partial k_0}{\partial y} \right]$$

$$\frac{\gamma k_0^{1/2} d(r \Phi(r))}{r \frac{d f(r)}{dr} \left[ -\frac{\gamma f(r) \frac{\partial k_0}{\partial y} - \frac{\gamma y f'(r)}{r} \frac{\partial k_0}{\partial y} + \frac{\partial \psi}{\partial y} \right] d\Phi(r) + O(\varepsilon n^2 \varepsilon)$$

with $x = r \cos \phi$, $y = r \sin \phi$. Therefore:

$$U^X = \frac{1}{k_0 h_0^{1/2}} \int_0^E d \left( r f(r) \right) \left[ -\frac{\gamma f(r) \frac{\partial k_0}{\partial y} + \frac{\gamma y f'(r)}{r} \frac{\partial k_0}{\partial y} + \frac{\partial \psi}{\partial y} \right] d\Phi(r) + O(\varepsilon n^2 \varepsilon)$$

$$+ \frac{\gamma k_0}{h_0 \frac{\partial h_0}{\partial y} + \frac{1}{2k_0} \frac{\partial k_0}{\partial y}} \left[ d \Phi(r) + O(\varepsilon n^2 \varepsilon) \right]$$
The kinetic energy of the fluid in the core is:

\[ E_c = \frac{1}{2} \left[ \int_0^E \int_0^{2\pi} r^2 v^2 h^2 k \rho d\phi dr \right] = \frac{1}{2} \left[ \int_0^E \int_0^{2\pi} \kappa \left( \frac{3}{4} \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{3}{4} \frac{\partial \psi}{\partial x} \right)^2 \right] d\phi dr \]

\[ = \frac{1}{2} \left[ \int_0^E \int_0^{2\pi} \kappa \gamma^2 \left( \frac{df}{dr} \right)^2 d\phi dr \right] = \pi \rho \gamma^2 k_0 \beta_0 \quad (VI.2.11) \]

As in Section IV.3, \( \alpha_0 = h_0^2 k_0 \varepsilon^2 \) is constant since the volume of the core is constant. Therefore, the velocity of the core is:

\[ U^x = \frac{1}{k_0 h_0^2} \frac{\partial}{\partial y} \left[ \gamma B(x,y;X,Y) + \psi^*(x,y) \right] + \frac{\gamma k(x,y) \ln \left( \frac{h^2(x,y) k(x,y)}{\alpha_0} \right)}{\varepsilon^2} + O(\varepsilon \ln^2 \varepsilon) \quad (VI.2.12a) \]

and similarly:
\[ U^Y = \frac{-1}{k_0 h_0} \frac{\partial}{\partial x} \left[ \gamma B(x,y;X,Y) + \psi^*(x,y) \right] + \frac{\gamma k(x,y)}{4} \left[ \ln \left( \frac{h^2(x,y) k(x,y)}{\alpha_0^*} \right) \right] + O(\epsilon \ln^2 \epsilon) \] (VI.2.12b)

in agreement with (IV.3.9) (the effects of other vortices have been included in \( \psi^* \) in (VI.2.12)).

A vortex with a core of finite size will therefore propagate as a vortex with a core of infinitesimal size provided its core is circular and remains circular. In general, the core will be distorted due to the advection of the vorticity within it. It is necessary, therefore, to have some idea of the effects of the distortion of the core. This is discussed in Section VI.3.

The velocity of the vortex induced by the surface curvature is of the order: \( \frac{\gamma |\nabla \lambda h|}{h} \sim v^{(0)} \epsilon |\nabla \lambda h| \). The velocities neglected in the approximation of Section II.3 are of the order: \( \lambda v^{(0)} \sim k \frac{\epsilon}{h} v^{(0)} |\nabla \lambda h| \). Thus, if the motion of the vortex induced by the surface curvature is to be non-negligible, it is necessary that \( \epsilon >> k \): the core radius must be considerably larger than the depth of the fluid. If \( \epsilon < k \) but the vortex motion induced by the surface curvature is not negligible, then \( v \) must be more nearly horizontal than suggested by the approximation (III.3.4a): i.e.,

\[ \frac{v^{(1)}}{x} = \frac{v^{(1)}}{y} = \frac{v^{(1)}}{z} = 0. \]
VI.3 The Validity of the Circular Approximation

Suppose that the vorticity distribution within the core of a vortex is nearly circular: that is:

$$w_z = w_z^{(0)}(r) + \nu w_z^{(1)}(r,\phi)$$  \hspace{1cm} (VI.3.1)

where $\nu$ is the small parameter of (VI.2.12). Within the core:

$$v = v^{(0)}(r) + \nu v^{(1)}(r,\phi)$$  \hspace{1cm} (VI.3.2)

$$h = h_0 + \nu h^{(1)}(r,\phi)$$  \hspace{1cm} (VI.3.3)

$$w_z^{(0)} = \frac{\partial v^{(0)}}{\partial x} - \frac{\partial v^{(0)}}{\partial y} + O(\nu)$$  \hspace{1cm} (VI.3.4)

The velocity of the vortex is:

$$U = \frac{1}{2\pi} \int_G w_z v h \, dx \, dy$$

$$= \frac{h_0}{2\pi} \int_G v^{(0)} w^{(0)} \, dx \, dy + \frac{\nu}{2\pi} \int_G \left( v^{(0)} w_z^{(0)} h_0^{(1)} + v^{(0)} w_z^{(1)} h_0^{(1)} \right) + v^{(1)} w_z^{(0)} h_0^{(1)} \, dx \, dy + O(\nu^2)$$

$$= \frac{\nu}{2\pi} \int_G \left( v^{(0)} w_z^{(1)} h_0^{(1)} + v^{(0)} w_z^{(1)} h_0^{(1)} + v^{(1)} w_z^{(0)} h_0^{(1)} \right) \, dx \, dy + O(\nu^2)$$  \hspace{1cm} (VI.3.5)

The first term in (VI.3.5) is that obtained for a circular core. The other terms are of comparable magnitude so that even small deviations from a circular core can produce relatively large changes in the velocity of the vortex.

However, consider a circular distribution of vorticity $w_z(r)$ and a small localized perturbation $w_z^{(1)}$. Since $\frac{w_z}{k}$ is advected $w_z^{(1)}$ is carried around the core on a nearly periodic orbit with period of order $\frac{2\pi}{w_z(r_0)}$, where $r_0$ is the
radius at which \( w^{(1)}_z \) is localized initially. As \( w^{(1)}_z \) orbits the core, the direction of the velocity induced by \( w^{(1)}_z \) also sweeps through an angle \( 2\pi \). The net displacement of the vortex due to this velocity in the time \( \frac{2\pi}{w^{(0)}_z} \) is almost zero (i.e., it is of order: \( \frac{\nu^2 v^{(1)}}{w^{(0)}_z} \approx \nu^2 \epsilon \)) so that the time averaged component of the velocity due to the departure from a circular core is only of order \( \nu^2 v^{(0)}_z \). The motion of the vortex is therefore a periodic wobble about the path of a circular vortex (to order \( \nu \)).

Similarly, one expects for more complicated \( w^{(1)}_z \) that the motion of the vortex is, to lowest order, a superposition of wobbles with frequencies of order \( \gamma / \epsilon^2 \) upon the motion of a circular vortex. The amplitude of the wobbles is of order: \( \nu v^{(0)}_z \sqrt{\frac{\epsilon}{\gamma}} = \nu \epsilon \) which is quite negligible.

The net systematic addition to the vortex velocity is at most of order: \( \nu U_C \) where \( U_C \) is the velocity of a circular vortex. For small \( \nu \) this is negligible.
VI.4 Elliptical Cores

The wobble in the motion of a vortex can be demonstrated explicitly if its core is elliptical with a uniform vorticity distribution within it. The depth of the fluid is constant. This is the generalization of Kirchhoff's elliptical vortex (see, e.g., Lamb (1916) p.226) to non-planar surfaces.

Suppose:

\[ w_z = w_0 = \text{const.} \quad (x, y) \in G \]
\[ = 0 \quad (x, y) \notin G \quad (VI.4.1) \]

with:

\[ G = \left\{ (x, y) ; \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 < 1 \right\} \quad (VI.4.2) \]

If there are no boundaries or external flows:

\[ \nabla^2 \psi(x, y) = -w_0 h^2(x, y) = -w_0 h_0 \left[ 1 + \alpha x + \beta y \right] \]
\[ + O(\nu^2), (x, y) \in G \]
\[ = 0, \quad (x, y) \notin G \quad (VI.4.3) \]

where:

\[ \alpha = \frac{2}{h_0} \frac{\partial h_0}{\partial x} \quad \beta = \frac{2}{h_0} \frac{\partial h_0}{\partial y} \quad (VI.4.4) \]

It is convenient to introduce complex coordinates so that one may use conformal transformations. Then:

\[ \frac{\partial^2 \psi}{\partial z \partial \bar{z}}(z, \bar{z}) = -i w_0 h_0^2 \left[ 1 + \frac{i}{2}(\alpha - i \beta)z + \frac{i}{2}(\alpha + i \beta)\bar{z} \right] \quad (z, \bar{z}) \in G \]
\[ = 0 \quad (z, \bar{z}) \notin G \quad (VI.4.5) \]
whence:

\[
\psi(z, \bar{z}) = -\frac{1}{4}\psi_0 h_0^2 \left[ z\bar{z} + \frac{1}{4}(\alpha - i\beta)z^2 \bar{z} + \frac{1}{4}(\alpha + i\beta)zz^2 \right] \quad (z, \bar{z}) \in G
\]

\[+ \phi_i(z) + \bar{\phi}_i(\bar{z})\]

\[= \phi_e(z) + \bar{\phi}_e(\bar{z}) \quad (z, \bar{z}) \notin G\]

(VI.4.6)

the subscripts \(e\) and \(i\) denoting complex potentials for the interior and exterior of the core.

Consider the mapping:

\[z = c \left( \zeta + \frac{d}{\zeta} \right), \quad c = \frac{a+b}{2}, \quad d = \frac{(a-b)}{(a+b)}\]

In terms of \(\zeta\) the core boundary is: \(|\zeta| = 1\).

Moreover: \(\frac{dz}{d\zeta} = c \left( 1 - \frac{d}{\zeta^2} \right) \neq 0\) if \(|\zeta| > 1\) since \(|d| < 1\).

Therefore the mapping is conformal outside the core. Thus, if the velocity field is to vanish at infinity:

\[\phi_e = p_0 n \zeta + \sum_{n=1}^{\infty} p_n \zeta^{-n} \quad (VI.4.7)\]

Moreover:

\[\phi_i = \sum_{n=1}^{\infty} q_n \zeta^n = \sum_{n=1}^{\infty} q_n c^n \left( \zeta + \frac{d}{\zeta} \right)^n \quad (VI.4.8)\]

The coefficients \(p_n\) and \(q_n\) are determined by requiring that the velocity field be continuous on the boundary. One finds:

\[p_n = q_n = 0 \quad , \quad n > 4 \quad (VI.4.9)\]

\[q_2 = \frac{w_0 h_0^2 d}{8} \quad (VI.4.11)\]

\[q_3 = \frac{w_0 h_0^2 d}{48} \quad (VI.4.10)\]
\[ q_1 = -\frac{w_0 h_0 c^2 (1+d^2)}{8} \left[ \alpha(1+d) - i\beta(1-d) \right] \quad (VI.4.12) \]

\[ p_0 = -\frac{w_0 h_0 c^2 (1-d^2)}{4} \quad (VI.4.13) \]

\[ p_1 = \frac{w_0 h_0 c^3 (1-d^2)}{16} \left[ \alpha(1+d)^2 + i\beta(1-d)^2 \right] \quad (VI.4.14) \]

\[ p_2 = -\frac{w_0 h_0 c^2 (1-d^2) d}{8} \quad (VI.4.15) \]

\[ p_3 = \frac{w_0 h_0 c^3 (1-d^2) d}{48} \left[ \alpha(1+d)^2 + i\beta(1-d)^2 \right] \quad (VI.4.16) \]

Since \( w_z \) is distributed uniformly over the core it remains uniformly distributed at all future times since it is advected. The structure of the core is therefore determined solely by the shape of the boundary.

The contravariant velocity field on the boundary is:

\[ V^X - i V^Y = \frac{2i}{h^2(z,\bar{z})} \frac{\partial \psi}{\partial \bar{z}} \bigg|_{|\zeta|=1} = -\frac{i w_0 c (1-d^2)}{2} \left[ 1 - (\alpha - i\beta) z - (\alpha + i\beta) \bar{z} \right] \]

\[ X \left[ \frac{1}{\zeta} + \frac{c}{4\zeta^2} (\alpha(1+d)^2 + i\beta(1-d)^2) \right] |_{|\zeta|=1} + O(\psi^2) \]

\[ = -\frac{i w_0 ab}{(a+b)} \left[ -a + i\beta + e^{-i\theta} \frac{e^{-2i\theta}}{4} \left[ \alpha a(d-3) - i\beta b(d+3) \right] \right] \quad (VI.4.17) \]

where \( \theta \) is defined by: \( \zeta = se^{i\theta} \)

The first two terms are constant implying a translational velocity:
Consider, now, the term: \( V^x - iV^y = -i \omega \theta \frac{-i \omega abe^{-i \theta}}{a+b} \)

Only the component perpendicular to the surface of the core can change its shape. This component is:

\[
V^\perp = \text{Re} \left[ \frac{(e^{i \theta} - de^{-i \theta})(-i \omega abe^{-i \theta})}{|e^{i \theta} - de^{-i \theta}|(a+b)} \right]
\]

\[
= \frac{2 \omega ab \sin 2 \theta}{(a+b)|e^{i \theta} - de^{-i \theta}|} \quad \text{(VI.4.19)}
\]

The velocity field of a uniform rotation with angular velocity \( \omega \) is:

\[
V^x - iV^y = -i \omega \tilde{z} = -i \omega c(e^{-i \theta} + de^{i \theta}) , \ |\xi|=1 \quad \text{(VI.4.20)}
\]

The component of this velocity field perpendicular to the core boundary is:

\[
V^\perp = \frac{2 \omega dc \sin 2 \theta}{|e^{i \theta} - de^{-i \theta}|} \quad \text{(VI.4.21)}
\]

which is identical to (VI.4.19) if:

\[
\omega = \frac{2 \omega ab}{(a+b)^2} \quad \text{(VI.4.22)}
\]

Thus, the third term causes the ellipse to rotate with angular velocity given by (VI.4.22).

The perpendicular component of the fourth term is:

\[
V^\perp = -\frac{w_0 ab}{4(a+b)} \left[ a(d-3)(\sin \theta - d \sin 3 \theta) + b(d+3)(\cos \theta - d \cos 3 \theta) \right] \quad \text{(VI.4.23)}
\]

The terms in \( \sin \theta \) and \( \cos \theta \) imply an additional contribution to
the translational velocity of the vortex:

$$U^x - iU^y = -\frac{w_0ab}{8} \left[ \beta(d+3) - i\alpha(d-3) \right]$$  \hspace{1cm} (VI.4.23)

Since the perpendicular component of a constant velocity field, $U^x - iU^y$, is:

$$V^+ = \frac{2}{a+b} \left[ U^x b\cos \theta + U^y a\sin \theta \right]$$  \hspace{1cm} (VI.4.24)

The contribution from the terms in $\sin 3\theta$ and $\cos 3\theta$ causes a distortion of the ellipse. However, in the time $T = \frac{(a+b)^2}{\pi w_0 ab}$ taken for the ellipse to rotate once the displacement of the boundary due to the above terms is of the order:

$$D \sim \varepsilon^2 a \sim \nu \varepsilon \ .$$

During this time, as seen from a frame in which the ellipse is at rest, the gradient of $h$ appears to rotate through $2\pi$ (or, in other words, the phase of $\alpha + i\beta$ changes by $2\pi$). The net displacement of the boundary therefore very nearly cancels: $D_{\text{net}} \sim \nu^2 \varepsilon$. Thus the core remains nearly elliptical for times of order $\frac{2\pi}{w_0}$.

The small distorting term will henceforth be disregarded.

The net velocity of translation of the core is, from (VI.4.18) and (VI.4.23):

$$U^x - iU^y = \frac{w_0 ab}{4(a+b)} \left[ \beta(b-2a) + i\alpha(a-2b) \right]$$  \hspace{1cm} (VI.4.25)

The circulation around the core is:

$$2\pi \gamma = \int_G w_0 h^2 dx dy = \pi ab w_0 h_0^2 + O(\nu^2)$$  \hspace{1cm} (VI.4.26)

Therefore:

$$\gamma = \frac{w_0 ab}{2}$$  \hspace{1cm} (VI.4.27)
Using the definitions of $\alpha$ and $\beta$ the velocity of the core is then:

$$U^x - iU^y = \frac{\gamma}{(a+b)} \left[ \frac{b-2a}{h} \frac{\partial h_0}{\partial y} + \frac{i(a-2b)}{h^3} \frac{\partial h_0}{\partial x} \right]$$  \hspace{1cm} (VI.4.28)

Notice that when $a=b$ this is in agreement with the result for circular cores.

(VI.4.28) has been derived using coordinates whose axes are the principle axes of the ellipse. If, instead, one chooses the x-axis to be along $\nabla h$ so that the major axis of the ellipse now makes an angle $\theta$ with the x-axis (Figure XIII) the velocity of the vortex is:

$$U^x - iU^y = -\frac{i\gamma}{2h_0} \frac{\partial h_0}{\partial x} \left[ 1 + \frac{3(a-b)}{(a+b)} e^{2i\theta} \right]$$  \hspace{1cm} (VI.4.29)

Since the ellipse rotates with angular velocity $\omega$ given by (VI.4.22):

$$(U^x - iU^y)(t) = -\frac{i\gamma}{2h_0} \frac{\partial h_0}{\partial x} \left[ 1 + \frac{3(a-b)}{(a+b)} e^{2i\omega t} \right]$$  \hspace{1cm} (VI.4.30)

Over many periods $h$ changes very little so that it may be considered nearly constant. Integrating then gives:

$$X = \frac{\gamma}{2\omega h_0^3} \frac{\partial h_0}{\partial x} \left[ \omega t + \frac{3(a-b)}{2(a+b)} \sin 2\omega t \right]$$  \hspace{1cm} (VI.4.31a)

$$Y = \frac{3\gamma}{4\omega h_0^3} \frac{\partial h_0}{\partial x} \frac{(a-b)}{(a+b)} \cos 2\omega t$$  \hspace{1cm} (VI.3.31b)

This is the equation of a trochoid (Figure XIV). The amplitude of the oscillation in the motion is:

$$A = \frac{3\gamma}{4\omega h_0^3} \frac{\partial h_0}{\partial x} \frac{(a-b)}{(a+b)} = \frac{3(a^2-b^2)}{16h_0^3} \frac{\partial h_0}{\partial x} \sim O(\nu e)$$
Fig. XIII: A Vortex with an Elliptical Core
The Paths of a Vortex with an Elliptical Core.
The eccentricities of the ellipses are correct, but their sizes are much reduced.

Fig. XIV: The Paths of a Vortex with an Elliptical Core.
It is interesting to note that if one puts $d=0$ in (VI.4.23) the terms causing distortion disappear. A circular core with constant vorticity within it is therefore distorted at a much slower rate than other cores and will therefore obey the circular approximation for correspondingly greater lengths of time. In the next section another such circular core is found which has a continuous vorticity distribution.
VI.5 Perturbations of Planar Solutions

Several exact vortex solutions are known for planar uniform depth flows (see, for example, Batchelor (1967) p.534). By using the streamfunction of such a solution as a first approximation to the streamfunction for non-planar but uniform depth flows one can obtain solutions to this more general vortex problem as perturbation expansions in the small parameter $\nu$.

Let $\psi(x,t)$ be the streamfunction of the non-planar flow and let $\psi^{(0)}(x,t)$ be the known streamfunction of the planar flow. It is assumed that:

$$w^{(0)}_z = -\nabla^2 \psi^{(0)} + O$$

for $r > e$ so that $h$ and $k$ may be approximated by the first few terms of their Taylor expansions. One can expand $\psi$:

$$\psi(x,t) = \psi^{(0)}(x - Ut,t) + \nu \psi^{(1)}(x,t) + ... \quad (VI.5.1)$$

$\nu U$ is the velocity of translation of the zeroth order solution. $U$ can be expanded:

$$U = \nu U^{(1)} + \nu^2 U^{(2)} + ... \quad (VI.5.2)$$

$h$ is also expanded:

$$h = h_0 + \nu h^{(1)} + ... = h_0 + x \frac{\partial h_0}{\partial x} + y \frac{\partial h_0}{\partial y} + ... \quad (VI.5.3)$$

The equation governing $\psi$ is (from (III.3.7) and (IV.1.2)):

$$\frac{\partial}{\partial t} \left[ \frac{1}{h^2} \nabla^2 \psi \right] + \frac{1}{h^2} \left( \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) \left[ \frac{1}{h^2} \nabla^2 \psi \right] = 0 \quad (VI.5.4)$$

Substituting (VI.5.1) and expanding one has to lowest order:
\[
\frac{\partial}{\partial t} \nabla^2 \psi(0) + \frac{1}{h_0^2} \left[ \frac{\partial \psi(0)}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi(0)}{\partial x} \frac{\partial}{\partial y} \right] \nabla^2 \psi(0) = 0
\]

(VI.5.5)

which is the requirement that \( \psi(0) \) be a solution for planar, uniform depth flow.

To order \( \psi \):

\[
\frac{\partial}{\partial t} \left( \nabla^2 \psi(1) \right) + \frac{1}{h_0^2} \left[ \left( \frac{\partial \psi(1)}{\partial y} - u_y(1) \right) \frac{\partial}{\partial x} - \left( \frac{\partial \psi(1)}{\partial x} + u_x(1) \right) \frac{\partial}{\partial y} \right] \nabla^2 \psi(0)
\]

\[
+ \frac{1}{h_0^2} \left( \frac{\partial \psi(0)}{\partial y} - \frac{\partial \psi(0)}{\partial x} \right) \left[ \nabla^2 \psi(1) - \frac{2h(1)}{h_0} \nabla^2 \psi(0) \right] = 0
\]

(VI.5.6)

This is a linear equation to be solved subject to the boundary conditions:

\[
\nabla^2 \psi(1) \to 0 \text{ as } r \to \infty ; \psi(1) = 0 , t=0
\]

(VI.5.7)

In general the solution is complicated and can be found only numerically.

We wish to look for solutions to (VI.5.4) which are initially circular and retain their circular form for relatively long periods of time. Thus, we look for solutions of (VI.5.6) having:

\[
\psi^{(0)} = \psi^{(0)}(r) \text{ and } \psi^{(1)} = 0.
\]

(VI.5.6) becomes:

\[
\left[ \frac{1}{r} \frac{d}{dr} \left( \frac{rd\psi^{(0)}}{dr} \right) \frac{d\psi^{(0)}}{dr} \right] \frac{\partial h_0}{\partial r} \sin \phi - \psi^{(1)} \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( \frac{rd\psi^{(0)}}{dr} \right) \right] \sin \phi = 0
\]

\[
\left[ \frac{1}{r} \frac{d}{dr} \left( \frac{rd\psi^{(0)}}{dr} \right) \frac{d\psi^{(0)}}{dr} \right] \frac{\partial h_0}{\partial r} \cos \phi - \psi^{(1)} \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} \left( \frac{rd\psi^{(0)}}{dr} \right) \right] \cos \phi = 0
\]

(VI.5.8)
There is a solution if:

\[
\frac{1}{r} \left( \frac{d}{dr} \frac{rd\psi(0)}{dr} \right) \frac{d\psi(0)}{dr} + \alpha \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} \left( \frac{rd\psi(0)}{dr} \right) \right) = 0 \quad \text{(VI.5.9)}
\]

with:

\[
\alpha = u\gamma(1) \frac{\partial h_0}{\partial \alpha} -1 = -u\gamma(1) \frac{\partial h_0}{\partial y} -1 \quad \text{(VI.5.10)}
\]

Putting \( f(r) = \frac{r}{\alpha} \frac{d\psi(0)}{dr} \) and \( u = r^2 \), (VI.5.9) becomes:

\[
f \frac{df}{du} + 2u \frac{d^2f}{du^2} = 0 \quad \text{(VI.5.11)}
\]

The boundary conditions require that: \( f \to \text{const.} \) as \( r \to \infty \); \( f/r \to 0 \) as \( r \to 0 \). There is a one-parameter family of solutions:

\[
f = \frac{4u}{a^2 + u} = \frac{4r^2}{a^2 + r^2} \quad \text{(VI.5.12)}
\]

Therefore:

\[
\psi(0) = - \frac{d\psi(0)}{dr} = -\frac{4\alpha r}{(a^2 + r^2)} \quad \text{and} \quad \psi(0) = 2\alpha \gamma (a^2 + r^2) \quad \text{(VI.5.13)}
\]

The strength of the vortex is:

\[
\gamma = \lim_{r \to \infty} r\psi(0) = -4\alpha \quad \text{(VI.5.14)}
\]

so that:

\[
\psi(0) = \frac{\gamma r \hat{\varphi}}{a^2 + r^2} \quad , \quad \psi(1) = \frac{\gamma}{2h_0} \frac{\partial h_0}{\partial x} \quad , \quad \psi(1) = \frac{\gamma}{2h_0} \frac{\partial h_0}{\partial y} \quad \text{(VI.5.15)}
\]

The vorticity is, to lowest order:

\[
\omega_z = \frac{2\gamma a^2}{(a^2 + r^2)} \quad \text{(VI.5.16)}
\]

The radius of the core is of order \( a \). A vortex whose initial vorticity distribution has the form (VI.5.16) will
Fig. XV: The Velocity and Vorticity of a Quasi-Steady Vortex
therefore remain circular for long periods of time. Such a vortex might be called quasi-steady.

In general it is not possible to perform a similar analysis of the depth of fluid is varying since $\psi(0)$ need not be a good approximation to $\psi$ since outside the core they satisfy different equations: $\nabla^2 \psi(0) = 0$, $\nabla \cdot \left( \frac{1}{k} \nabla \psi \right) = 0$. 
VII. APPLICATIONS TO REAL FLUIDS

VII.1 Atmospheric Cyclones

As explained in the introduction, part of the motivation for this thesis was an attempt to provide a simple model for an atmospheric cyclone taking full account of the effects of the curvature of the earth. However, the observation that at mid-latitudes the wind is roughly geostrophic is evidence that the Coriolis force, which we have neglected entirely, plays a dominant role in the atmosphere. In this section, the possibility of including the Coriolis force is examined. It is also shown that the unrealistic assumption that the atmosphere has constant density may be relaxed somewhat.

The equations of motion for the atmosphere may be written:

\[ \frac{\partial \mathbf{V}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = - \frac{\nabla P}{\rho} - 2\omega \times \mathbf{v} - \mathbf{g} \]  

(VII.1.1)

where \( \mathbf{v} \) and \( \mathbf{V} \) are both three-dimensional vectors. \( P \) is the hydrostatic pressure and effects of viscosity and the centrifugal force are ignored. \( \mathbf{g} \) is the acceleration due to gravity and \( \omega \) is the angular velocity of the earth. Taking the curl of (VII.1.1) and using the equation of continuity:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]  

(VII.1.2)

one obtains the vorticity equation:

\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( \frac{\mathbf{W} + 2\omega}{\rho} \right) = \left( \frac{\mathbf{W} + 2\omega}{\rho} \right) \cdot \nabla \mathbf{v} + \frac{\nabla \rho}{\rho} \times \frac{\nabla P}{\rho^3} \]  

(VII.1.3)
We suppose as in Section III that the velocity is, to lowest order, tangential to the surface of the earth. We also suppose that the density is a function only of the height above the earth. While a very restrictive assumption this is certainly a much better approximation than that the density is uniform. Equation (VII.1.2) is then satisfied if: \( \nabla \cdot \mathbf{v} = 0 \). However:

\[ \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0 \tag{VII.1.4} \]
to lowest order. One can then show (see, for example, Veronis (1963b)) that:

\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \left( \frac{w + 2\omega}{\rho} \right) = 0 \tag{VII.1.5} \]

which in terms of the thin film coordinates of Section II.3 becomes to first order.

\[ \left( \frac{\partial}{\partial t} + \frac{v_x}{h} \frac{\partial}{\partial x} + \frac{v_y}{h} \frac{\partial}{\partial y} \right) \left( \frac{w_z + 2\omega_z}{k} \right) = 0 \tag{VII.1.6} \]

with equation of continuity:

\[ \frac{\partial (hk v_x)}{\partial x} + \frac{\partial (hk v_y)}{\partial y} = 0 \tag{VII.1.7} \]

These equations are (III.3.6) and (III.3.7) with \( w_z \) replaced by \( w_z + \omega_z \). Thus, allowing the density to vary with height produces no changes in the equations of motion, but including the Coriolis force does. \( 2\omega_z \) is known as the Coriolis parameter and on the sphere is \( 2\omega \cos \theta \) where \( \theta \) is the colatitude. The Coriolis parameter acts as a constant source of vorticity so that it is no longer possible to have flows in which the vorticity is isolated in vortex cores. The whole idea of a vortex as a region of isolated vorticity then breaks down. The
only satisfactory resolution seems to be the approximation that
$k$ and $\omega_z$ are both constant which amounts to neglecting all
effects of surface curvature and depth variation.

Thus one must conclude that the vortices discussed in
this thesis cannot be extended in a simple way to account for
Coriolis effects. A vortex on a curved rotating surface will
necessarily have vorticity throughout the flow and not just
at the core.

However, many planar flows with extended vorticity have
been modelled by large numbers of point vortices. In a similar
way, one might model an atmospheric vortex by a large number of
the vortices discussed in Section IV. It is possible to account
for the Coriolis force by requiring that the vortex strengths
vary according to:

$$\gamma_n = \frac{v(x_n, y_n)}{h(x_n, y_n)}, \gamma_n = \left[\frac{\omega_z(x_n, y_n)}{k(x_n, y_n)}\right]$$

**(VII.1.8)***

which may be restated more simply as:

$$\gamma_n = \gamma_{n0} + \frac{\gamma_{n1} \omega_z(x_n, y_n)}{k(x_n, y_n)}, \gamma_{n0} = \text{const.}, \gamma_{n1} = \text{const.}$$

**(VII.1.9)***

Unfortunately, such a system does not respect the required
constraint $\sum_n \gamma_n = 0$. Thus, there will appear to be a vortex
fixed at infinity (the south pole) whose strength waxes and
wanes as the flow progresses. If the number of vortices is
very large the effect of this single vortex will become
negligible and the system should still provide a satisfactory
model for the flow. A proper investigation of this system
(which would involve solving the equations of motion by
computer for large numbers of vortices, long times and varied initial conditions) is beyond the scope of this thesis.
IV.2 Superfluid Vortices

In recent years work on isolated vortices has been almost exclusively in relation to the behaviour of liquid HeII. While the complete set of partial differential equations which describe the flow of liquid HeII are still not completely understood (see Putterman(1974)) it is accepted that the superfluid component of the fluid velocity is irrotational and, outside vortex cores, very nearly incompressible. Vortices having circulation quantized in units of \( h/m = 10^{-3} \text{cm}^2/\text{sec} \) (\( h = \) Planck's constant, \( m = \) mass of helium atom) are observed. The cores of the vortices are characterized, not by regions of vorticity, but by a region in which the density decreases rapidly. The vorticity is truly singular but the density of the fluid is zero at the singularity. The radius of the core is constant and is the order of one or two Angstroms.

One expects, therefore, that equation (IV.3.4) should describe the motion of vortices but that now:

\[
h_n c_n = \delta_n = \text{const.} \quad (\text{VII.2.1})
\]

The equations of motions for a vortex system are then:

\[
\frac{\dot{z}_n}{\hbar} = \frac{2i}{h_n} \left( \gamma_n \left( \frac{\hbar}{2 \delta_n} \right) + 1 \right) + \frac{k_n}{h_n} \left( \frac{\hbar}{\delta_n} \right) + \frac{\partial}{\partial z} \left[ \gamma_n B(z, \bar{z}; z_n, \bar{z}_n) \right] + \psi(z, \bar{z}) + \sum_{k \neq n} \gamma_k \psi(z, \bar{z}; z_k, \bar{z}_k) \quad \text{(VII.2.2)}
\]

There is no vortex streamfunction and the kinetic energy of the fluid outside the core is not conserved. However, in
both the case of uniform depth flow and of flow on a plane
vortex streamfunctions exist. These are, respectively:

\[
\Omega = \frac{1}{2} \sum_{n=1}^{N} \sum_{k \neq n} \gamma_n \gamma_k \psi(x_n, y_n; x_k, y_k) + \frac{1}{2} \sum_{n=1}^{N} \left\{ \gamma_n^2 B(x_n, y_n; x_k, y_k) + \gamma_n^2 h(x_n, y_n) + 2\gamma_n \psi^*(x_n, y_n) \right\}, k=1 \quad (\text{VII.2.3})
\]

\[
\Omega = \frac{1}{2} \sum_{n=1}^{N} \sum_{k \neq n} \gamma_n \gamma_k \psi(x_n, y_n; x_k, y_k) + \frac{1}{2} \sum_{n=1}^{N} \left\{ \gamma_n^2 B(x_n, y_n; x_k, y_k) - \frac{\gamma_n}{2} k(x_n, y_n) (\ell n \delta_n - 1) + 2\gamma_n \psi^*(x_n, y_n) \right\}, h=1 \quad (\text{VII.2.4})
\]

The conservation of \( \Omega \) corresponds to the conservation of the kinetic energy of the fluid external to the core.

These equations can now be used to discuss vortex problems in more general geometries than have been used to date. A problem of particular interest is that of vortices in a cylindrical container with \( k = k_0 + a r^2 \) since this is the shape of the free surface of a rotating fluid. Unfortunately, it is difficult to solve for \( \psi(x, y; x', y') \) and numerical methods seem necessary.
VIII. CONCLUSION

In this thesis the conventional theory of rectilinear vortex motion has been generalized to include vortices in fluids of small but varying depth on curved surfaces.

To the author's knowledge, the only published work of a similar nature is by Lamb (1916) who suggested that, if the fluid is of constant depth, vortex velocity fields might be obtained by "orthomorphic projection onto the plane." Lamb assumes without justification that the fluid motion may be approximated by purely two-dimensional flow. The only example discussed is two vortices on a sphere.

The author has shown that, by a suitable choice of coordinates, it is possible to determine the vortex velocity fields in fluids of varying depth. The approximations used are shown to be consistent in a systematic perturbation expansion in a small parameter $\lambda$ representing the ratio of vertical to horizontal scales; only lowest order equations are examined.

He has also shown that there is a function $h(x,y)$ suited for describing the effects of the surface curvature on vortex motion. If the depth of fluid is uniform, there are no boundaries and the surface of flow is topologically similar to a plane or a sphere, the equations of motion may be written explicitly in terms of $h(x,y)$.

In the more general case with boundaries and varying depth, the equations of motion can be written in terms of the Green's functions of elliptic partial differential operators. As did Lin (1943) for rectilinear vortices, the author has shown that there is a vortex streamfunction for these equations.
of motion, that it is related to the kinetic energy of the fluid, and that, if the depth is uniform, it transforms simply under conformal transformations.

The behaviour of simple vortex systems has been examined in Section V, showing that surface curvature and depth variation can cause marked qualitative differences between vortex systems. Studies of the stability of rigidly rotating systems showed that these qualitative differences also extend to considerations of stability.

In Section VI the effects of non-infinitesimal vortex cores were discussed. It was shown that the velocity of radially symmetric cores depends only on their strength but that asymmetries introduces small wobbles into the motion of a vortex. If the core is approximately circular the amplitude of the wobble is much smaller than the core radius and may be neglected.

The theory developed in Sections III-VI is of theoretical interest as a generalization of the classical theory of vortex motion. It is natural, however, to ask how it may be applied to atmospheric motion. As shown in Section VII.1, it seems impossible to reconcile the effects of the Coriolis force and of surface curvature in simple vortices of this type. It is quite possible that in a similar, but non-rotating, atmosphere that these vortices would provide a simple first order model of a cyclone. Of course, if the Coriolis force plays an important role in cyclogenesis, then the incidence of cyclones in such an atmosphere would be greatly suppressed.
Vortices in superfluid helium can be described by the theory of Section IV if one makes a modification to require that the core radii remain constant. In general, there is no longer a vortex streamfunction for the flow, but for the special cases of uniform depth and of planar flow a vortex streamfunction does exist. For the most interesting problems of flow with varying depth there is still the practical problem of calculating the functions $\Psi$ and $B$ which describe the velocity field.
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APPENDIX A: MATHEMATICAL FORMALISM

Let $M$ be a two-dimensional smooth Riemannian manifold with metric $g$. One may define a two-form $\sigma^*$ by:

$$\sigma^* = \varepsilon (\det g)^{1/2} k$$  \hspace{1cm} (A1)

where $\varepsilon$ is the antisymmetric tensor density with $\varepsilon^{12} = 1$ and $k$ is a scalar function on $M$. There is a natural extension of $\sigma^*$ to a two-form on the $2N$-dimensional manifold $M^N$, namely, the unique two-form $\sigma$ satisfying:

$$\left( dx^1 x ... x dx^N, dy^1 x ... x dy^N \right) = \sum_{n=1}^{N} \gamma_n^{N} \sigma^*(dx^n, dy^n)$$  \hspace{1cm} (A2)

where the $\gamma_n, n=1, ..., N$ are constants. Note that $\ker \sigma = 0$ and $\sigma$ is differentiable everywhere. $\sigma$ therefore induces a symplectic structure on $M^N$. (Note that $\nabla \sigma^* = 0$ since non-trivial three-forms cannot exist on a two-dimensional manifold.)

Suppose that $\Omega$ is some scalar function on $M^N$. A natural flow is induced, the equations of motion of which are:

$$\frac{dx}{dt} = \sigma^{-1} \nabla \Omega = \text{grad} \Omega$$  \hspace{1cm} (A3)

where $x$ denotes position on $M^N$ and $\nabla$ is the exterior derivative. Notice that:

$$\frac{d\Omega}{dt} = \nabla \Omega \frac{dx}{dt} = \sigma(\Omega, \Omega) = 0$$

so that $\Omega$ is conserved. In harmonic coordinates $\sigma^* = h^2(x, y)k(x, y)\varepsilon$ and (A3) becomes (IV.4.4). For further reference on the mathematics of symplectic systems see Souriau (1969).
APPENDIX B: EVALUATION OF SUMS

All the special sums necessary for the calculations of Section V may be evaluated easily once:

\[ R_L(z) = \sum_{n=1}^{N} \frac{\exp(2\pi i n \ln/N)}{(1-z \exp(2\pi i n/N))^z}, \quad z \text{ complex, } L=1,\ldots,N \]  

(B1)

is known. Suppose first that \(|z| < 1\). Then:

\[ R_L(z) = \sum_{n=1}^{N} \exp(2\pi i n \ln/N) \sum_{k=1}^{\infty} kz^{k-1} \exp(2\pi i (k-1)/N) \]  

(B2)

The infinite series is absolutely convergent allowing the reordering of the sums:

\[ R_L(z) = \sum_{k=1}^{\infty} kz^{k-1} \sum_{n=1}^{N} \exp(2\pi i (L+k-1)n/N) \]  

(B3)

The second sum vanishes unless \( L+k-1 = rN, \quad r \text{ an integer}. \)

\[ R_L(z) = \sum_{r=1}^{\infty} N(rN-L+1)z^{rN-L} = \frac{N}{dz} \left( z^{N-L+1} \right)_{1-z^{N}} \]

(B4)

The right sides of both (B1) and (B4) are analytic in all regions of the complex plane excluding the \( N \)th roots of one: hence, by analytic continuation, (B4) is valid for all \( z \).

To evaluate \( T_L(x) \) put \( z = xe^{\pi i /n} \):

\[ T_L(x) = \sum_{n=1}^{N} \frac{\exp((2n+1)\pi i L/N)}{(1-\exp((2n+1)\pi i/N))^2} = e^{\pi i L/N} R_L(xe^{\pi i /n}) \]

(B5)

\[ S_L = \sum_{n=1}^{N-1} \frac{1-\exp(2\pi i Lk/N)}{(1-\exp(2\pi i k/N))^z} = \lim_{z \to 1} \left( R_N(z) - R_L(z) \right) \]
\[
\lim_{z \to 1} \frac{N(1+(N-1)z^{N-L}-(N-L+1)z^{N-L}-(L-1)z^{2N-L})}{1-z^N} \\
= \lim_{z \to 1} \frac{N(N(N-1)z^{N-1}-(N-L+1)(N-L)z^{N-L-1}-(L-1)(2N-L)z^{2N-L-1})}{-2NZ^{N-1}(1-z^N)} \\
= \lim_{z \to 1} \frac{(N(N-1)-(N-L+1)(N-L)z-L-(L-1)(2N-L)z^{N-L})}{2(1-z^N)} \\
= \frac{L(N-L+1)(N-L)-(N-L)(L-1)(2N-L)}{2N} \\
= \frac{1}{2}(N-L)(2-L) \tag{B6}
\]

L'Hopital's rule has been used twice.

\[
\sum_{n=1}^{N-1} \frac{1}{1-\exp(2\pi in/N)} = S_r = \frac{1}{2}(N-1) \tag{B7}
\]