Tachyons, Boundary Interactions, and the Genus Expansion in String Theory

by

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Abstract

This thesis examines the interaction of both bosonic- and superstrings with various backgrounds with a view to understanding the interplay between tachyon condensation and world-sheet conformal invariance, and to understanding the d-branes that overlap with closed string modes. We briefly review the development of both background independent string field theory and cubic string field theory, as these provide insight into the problem of tachyon condensation. We then develop the boundary state and show that in backgrounds of interest to tachyon condensation the conformal invariance of the string world-sheet is broken, which suggests a generalized boundary state obtained by integrating over the conformal group of the disk. We find that this prescription reproduces particle emission amplitudes calculated from the string sigma model for both on- and off-shell boundary interactions. The boundary state appears as a coherent superposition of closed string states, and using this a method for calculating amplitudes beyond tree level is developed. The interaction of closed strings with other backgrounds is also discussed. An extension of the boundary state to encode fields other than a gauge or tachyon field is described. A modification of the boundary state which encodes the time dependence of tachyon condensation is reviewed, and an examination of spherically symmetric tachyon condensation in the $1/D$ expansion is presented.

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Preface

This work investigates a number of aspects of the interplay between the conformal invariance in string theory and interaction terms confined to the boundaries of the string world-sheet. A brief synopsis of some of the theoretical basis for the work is presented in chapter 2, while chapters 3 and 4 have sections of extensive overlap with, respectively, [4, 69, 70] and [54], works on which the author collaborated.

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Chapter 1

Introduction

Max Born was attributed, in 1928, with the statement that 'Physics, as we know it, will be over in six months'. [59] This confidence was reportedly based on the recent discovery of the Dirac equation describing the electron, and the assumption that a similar equation could be found for the proton. Indeed, the spectacular success in the development of quantum theory to describe the emission spectrum of hydrogen, the work functions of metals, and the radiation of black bodies, as well as previous triumphs such as Maxwell's theory of electricity and magnetism can be seen as justifying that optimism. It may, without much exaggeration, be asserted that most of the progress in the discipline over the past century has been related to the quantum effects that govern exactly those particles of which Born was speaking, and that quantum mechanics forms the cornerstone of our current understanding of the physics of the small. Few physicists today would be willing to suggest that their discipline will be solved in short order, and many of the more pessimistic will suggest that the best we can ever hope to do is achieve some effective field theory description of the world. They would point to the difficulty quantizing gravity, and the success of the Standard Model in predicting and describing the results of most particle scattering experiments, and might perhaps suggest that the vein of fundamental discoveries accessible to us is played out, nearly exhausted. We take a more optimistic attitude, and so we ask the indulgence of the reader as we briefly touch on some of the major developments in the field of physics over the past hundred years and allude to the current state of knowledge. It is our hope that this will serve to put the work contained within this thesis into perspective, both as to its interest and its applicability to future developments within the field.

The early years of the century were marked by two developments that forced radical changes in the way the world was perceived. The first was the exposition of the theory of relativity which, for the first time, put the concepts of time and space on an equal footing and predicted apparently counterintuitive effects such as length contraction and time dilation for objects moving close to the speed of light. It was vindicated in many tests such as the precession of the perihelion of Mercury and the aberration of stars' light by the sun. The second was the discovery of the quantum nature of atoms, which facilitated an explanation of the spectra of the elements and compounds.

Coming close upon the heels of the initial understanding of the quantum nature of atoms was the discovery of the constituents of the nuclei, protons and neutrons, and the tantalizing hint of more particles through clues such as β-decay and the observation in cosmic ray experiments of particles intermediate in weight between the nucleons and the electron. The promise of more 'fundamental' particles was realized in a number of accelerator and reactor experiments in the early 1950s, with the discovery of strange particles and neutrinos, and the identification of the muon as a lepton with similar properties to the electron (see [53] for a review). As the number of particles known to physicists increased, so did the ability of physicists to make sense of their interactions. Between experiments which revealed the internal structure of the hadrons and mesons and others which sought to understand weak decay processes, a picture emerged of three families of particles, interacting with a spontaneously broken $SU(3) \times SU(2) \times U(1)$ gauge group. This picture which was greatly strengthened by the discovery in the early 1980s of the W and Z bosons [12–14], and the t quark in the 1990s [1].

Simultaneously, the understanding of the large scale structure of the universe has undergone a revolution in the past hundred years. It was originally observed in the 1920s that distant galaxies recede from us faster than the nearby galaxies. This observation admits the interpretation that we live in an expanding universe. The discovery of the cosmic microwave background in the 1960s was a window into an epoch when the universe was both hotter and denser than it is now. The powerful modern telescopes give a window into the past by allowing us to understand the formation of galaxies and the evolution of the universe. The history and evolution of this universe is also explored by calculations like big bang nucleosynthesis, which predicts the abundances of the light elements to great accuracy. In addition, recent precision measurements of the cosmic microwave background [23, 63] give insight into the small fluctuations in density that were the seeds for the large scale structure of the universe.

It appears that there exists a consistent and complete understanding of the world we live in. Many of the masses, couplings, and mixings of the Standard Model are known or measured, and the observed scattering processes are by and large calculated to better that 1% accuracy. We have a model of the early universe that makes use of our knowledge of nuclear processes, predicts the abundances of elements, and offers an explanation of the observed spectrum and describes the fluctuations in the cosmic microwave background. The dynamics of large objects are described very well by classical general relativity which has also been tested in an astrophysical setting by watching the decay of rotation time for binary pulsars. In short, a large variety of physical processes on many scales are well known and well described by current understanding.

However, there are, just as there were one hundred years ago, a number of gaps in our understanding that may well provide windows into new and exciting regimes and effects. One particle not yet observed to complete the description of the Standard Model is the Higgs boson, and its absence raises a question, is it truly a fundamental particle, distinguished as the only such scalar in nature, or does its mass-generating effect come from a more complicated mechanism such as technicolor? Recent observations have discovered masses and mixings between the species of neutrinos, which are not predicted in the Standard Model [2, 3, 42]. Recent cosmological observations have shown two facts that are very interesting, that the matter content of the universe accounts for roughly 30% of its observed energy density, with the other 70% coming from so-called vacuum energy [87], and further that the familiar particles from the Standard Model represent only a small fraction of the matter content of the universe, a fact previously suggested by data on galactic rotation curves [37]. In addition to this there are serious suggestions that gravity might be testably modified, both at the sub-millimeter level, and at length scales much greater than the size of our galaxy [35].

While this list is far from a comprehensive exposition of all the current areas of research, it suffices to give the impression that there are a number of very interesting and currently unresolved issues in the field. In a very real sense the discipline of Physics is currently at an exciting crossroads where it is possible to get precision experimental information about the parameters in a number of theories spanning orders of magnitude in size and energy. However, as many of the fundamental questions about the nature of the universe are laid open to inspection and resolution by diligent work, other questions arise to which the answers are not currently known.

A concrete question that is often asked is whether the current known particles exhaust the spectrum of the theory describing the world, and there are many currently popular suggestions. It may be that the world exhibits broken supersymmetry, in which case for each known particle there will exist a superpartner with identical charges and couplings, and many theorists expect that the lightest of these superpartners is a viable candidate for the dark matter that affects galaxy rotation curves [39]. Another possibility is that there may exist a larger gauge group which unifies the existing particles and couplings, but is broken at some higher scale. The simplest forms of such a grand unification which sees $SU(5)$ break to $SU(3) \times SU(2) \times U(1)$ have been experimentally ruled out, and the related supersymmetric models have been strongly constrained [8, 38], but larger groups have not. The breaking of a large enough group could result in matter in a 'hidden sector', which is to say light matter that is uncharged with respect to the matter we are made of, but which may couple at higher energies through interactions mediated by massive particles much like the leptoquarks in standard GUTs.

Other intriguing scenarios have been proposed as well [10, 18, 83]. A prominent recent theme being the existence of extra dimensions in addition to the three spatial and one time dimension so familiar from everyday experience. This idea has a number of interesting consequences, the first being that for extra dimensions with a very small spatial extent, wrapped up on themselves (compactified), there could very well be an infinite number of new particles which are massive partners of the known particles coming from the Fourier modes of the known particles around these compact dimensions. Equally well, larger, but still small, compact extra dimensions have been proposed [9] which give a natural way to interpret the relative weakness of gravitational interactions as compared with the other forces of nature. These and similar ideas have given rise to a number of scenarios in which large, or even non-compact extra dimensions are invoked with the assertion that our universe resides on some topological feature which describes a subspace of the extended space. In addition to all of these things, a large body of work describes the attempts to quantize gravity (for example, [11, 16, 30, 47, 82]), which is currently a classical theory.

There are thus a large number of directions in which physical research can progress, adding more particles to the theory of the universe with interactions described by larger groups, adding extra dimensions to space and observing their effect, examining dynamics on a topological defect in this larger space, quantizing gravity. To try to do any one of these is a non-trivial task, and it would appear that to attempt to do many simultaneously would be much more difficult, but there has emerged over the past decades a physical theory that can apparently address all of these called 'String Theory'. String theory can naturally accommodate many of these directions, it can describe gravitons, it can have particles with complicated gauge interactions, it must describe a world with more dimensions than our familiar space and time. For all this, there is a challenge, that many of the descriptions of string theory have a tachyon, a particle that travels faster than light. We neither see nor expect such a particle, and to explain why it is not present is a challenge that many have undertaken. In this thesis, we discuss a possible mechanism that explains the absence of the tachyon, and also can explain why we observe less dimensions than the number one might expect from string theory. This mechanism is tachyon condensation, and the details presented later show how it can force particles to inhabit a small subspace within a higher dimensional volume.

We will now present a brief overview of string theory, both with a view to narrative exposition, and with a view to fixing some conventions (for the most part following [51, 52, 78, 79]) that will be used later in this work.

Field theories are naturally concerned with point-like quanta and so a natural generalization is to ask how to quantize extended objects. These would have a generalization of a world-line with more than one dimension. For a point-like object the action is the proper length of the world-line swept out by the propagation in space and time, and for a one dimensional extended object the natural action is the surface area swept out by its propagation in time. This area can be calculated as an integral over the world sheet of the positions of each point along the sheet in space and time, which will be

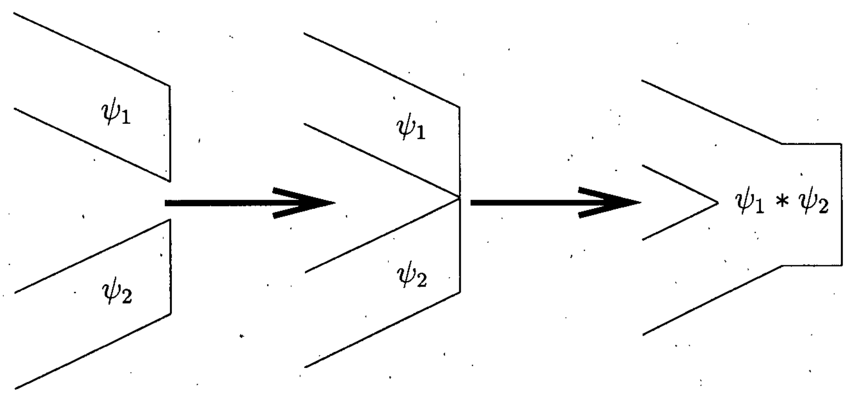

Figure 1.1: A representation of world-sheet interactions between open strings. On the left two open strings, ψ_1 and ψ_2, are propagating. (The previous positions of the two strings are indicated by the regions diagonally above and below ψ_1 and ψ_2 respectively.) In the center they interact by connecting at one end, and on the right they propagate as a single string, $\psi_1 * \psi_2$ which encodes the particle information in both of the original strings. This can also be thought of as a series of incomplete pictures of the string world sheet, the first showing only the portion with $t < t_0 - \epsilon$, the second showing the portion $t < t_0$, and the third $t < t_0 + \epsilon$, where t_0 is some measure of the time coordinate where they appear to merge, and ϵ is some small constant, and target space time increasing on the horizontal axis.

regarded as fields in what follows. The action is given as

$$S = \int d^2\sigma\sqrt{h}h^{\alpha\beta}(\sigma)g^{\mu\nu}(X_\gamma)\,\partial_\alpha X_\mu(\sigma)\partial_\beta X_\nu(\sigma) \tag{1.1}$$

where X_μ is the position of some point of the world-sheet in space and time, the string world-sheet has a metric $h^{\alpha\beta}$ and h is defined as the determinant of $h^{\alpha\beta}$. The pair of coordinates denoted σ parameterize the world-sheet, and $g_{\mu\nu}$ is the space-time metric, which is generically a function of the position. This action has both Weyl and reparameterization invariance and these can be used to eliminate the world-sheet metric from this equation [51]. It is also possible to add a term proportional to the two dimensional Ricci scalar R,

$$S = \int d^2\sigma\sqrt{h}R(h) \tag{1.2}$$

but this is purely a total derivative and, while not important in determining the spectrum of this theory, and it is possible to see that this term is responsible for the coupling constant that governs the string loop expansion because up to a constant this term evaluates nothing but the Euler number of the string world-sheet. As mentioned the reparameterization can eliminate the metric and this gives a sigma model action for the Xs, which is free when expanding around Minkowskian space. It can also be shown that not specializing to $g_{\mu\nu} \to \eta_{\mu\nu}$ will give the spacetime gravity action and stringy corrections that vanish in the limit of large string tension [51].

In the free case, which is of interest for perturbative calculations, it is possible to make a Laurent expansion of the modes of X_μ, observing that the right and left movers decouple in the bulk of the string world sheet. The conformal invariance of the string world sheet can be used to fix a flat metric and then it is possible to Wick rotate from a Minkowskian signature to a Euclidean signature through the transformation $\sigma^0 \to i\sigma^2$ [51]. The left and right movers can be expressed in terms of the holomorphic and antiholomorphic coordinates (z and $\bar{z}$) on the Euclideanized world sheet using the expansion [78]

$$X^\mu(z,\bar{z}) = x^\mu + p^\mu \ln|z^2| + \sum_{m\neq 0} \frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right). \tag{1.3}$$

In the following the terms holomorphic and antiholomorphic will be used interchangeably with left and right mover. When quantized the commutation relation between the Fourier modes of X is

$$[\alpha_a^\mu, \alpha_b^\mu] = a\eta^{\mu\nu}\delta_{a+b,0} \quad [\tilde{\alpha}_a^\mu, \tilde{\alpha}_b^\mu] = \eta^{\mu\nu}\delta_{a+b,0} \tag{1.4}$$

In the same way the Fourier coefficients of the two-dimensional energy momentum tensor can be written in terms of these αs, and for the holomorphic part we find

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} : \alpha_{m-n}\cdot\alpha_n : \tag{1.5}$$

where : { } : denotes the normal ordering of any expression within { }, which is moving the creation (negatively moded) operators to the right, and the dot represents contraction with respect to the Lorentz indices and α_0 is proportional to the momentum. An identical expression for the antiholomorphic $\tilde{L}$s can also be written. Since the energy momentum tensor is traceless it appears that all the L_ms should annihilate the physical states, however this strong condition would eliminate the spectrum of the theory, so the condition is relaxed to be that positively moded Ls will annihilate physical states. This coincides with the choice of positively moded αs as the annihilation operators, and also imposes that physical states are eigenstates of L_0 with eigenvalue a. Further, the Ls obey the Virasoro algebra

$$[L_m, L_n] = (m-n)\,L_{m+n} + \delta_{m+n,0}A(m) \tag{1.6}$$

where the $A(m)$ is the central charge which turns out to be proportional to the dimension of space-time, which is the number of world-sheet fields Xs.

Furthermore, it was briefly discussed above that the spacetime metric is expanded around the Minkowskian metric, which gives rise to a particular difficulty due to its negative signature, namely that there may be some excitations in the spectrum which have a negative norm. It is not difficult to show, demonstrated in [51, 78] that the condition for eliminating these negative norm states is equivalent to a condition on the number of Xs and on the value of a. It turns out that the number of dimensions for the bosonic string must be either 2 or 26, and the value of a in units of the string tension α' is fixed to -1 when the dimension is 26 [51]. The consequences of this are interesting to investigate. First the spectrum of this theory is built from a Fock space vacuum with the αs and $\tilde{\alpha}$s, and it must satisfy level matching conditions, as well as conditions on the polarization tensors for the various states which are obtained by requiring that the positively moded Ls do annihilate the state. The ground state of this theory is tachyonic, as it has negative mass squared, and the massless state consists of a symmetric traceless tensor, an antisymmetric tensor, and a trace term. These may be identified as a graviton, some gauge field, and a dilaton. This appears to be both good and bad, because while a particle exists with the appropriate quantum numbers for a graviton, the tachyon mode intimates an instability in the vacuum, which will be explored more later in this work.

Secondly, there are a number of extraneous degrees of freedom in the bosonic string, as witnessed by the restrictions on possible polarization tensors for the various excited states. One way to accommodate this is to work in so called light cone gauge where two directions are singled out as distinct and only oscillations transverse to those are permitted to propagate [49]. While very effective at reducing the number of degrees of freedom and enforcing the no-ghost conditions, this has the price of eliminating the manifest Lorentz invariance of the theory. There is a more elegant way to compensate for these extra degrees of freedom, and that is to introduce ghost fields in the manner of Fadeev and Popov to the string action, as exemplified in [80, 81]. These will be a pair of anticommuting fields, b and c with conformal weights 2 and -1 respectively whose action term

$$S_{b,c} = \int d^2\sigma b\partial c \tag{1.7}$$

These can also be broken into holomorphic and antiholomorphic degrees of freedom, and satisfy anticommutation relations

$$\begin{aligned} \{c_n, b_m\} &= \delta_{m+n,0} \\ \{b_m, b_n\} &= \{c_m, c_n\} = 0. \end{aligned} \tag{1.8}$$

Their fermionic nature gives a contribution to the determinant of the path integral which cancels the contributions of two of the X fields. Naively these ghosts appear to add to the number of possible Fock space excitations, but there is now an additional constraint, that the physical states must be annihilated by an operator composed of these ghosts, namely the BRST operator Q [22]

$$Q = \sum : \left(L^{\alpha}_{-m} + \frac{1}{2} L^{b,c}_{-m} - a\delta_m \right) c_m : \tag{1.9}$$

with

$$L^{b,c}_m = \sum_n (m-n) b_{m+n} c_{-n} \tag{1.10}$$

The constraint is then that $Q + \tilde{Q}$ must annihilate a physical state. This can be thought of as analogous to a gauge condition, that just as in the case of an Abelian gauge theory the transformation $A^\mu \to A^\mu + \partial^\mu \lambda$ for some scalar function λ leaves the field strength F invariant, the BRST transformation $|\phi\rangle \to |\phi\rangle + (Q + \tilde{Q})|\psi\rangle$ will result in a state that is still annihilated by the BRST operator even if $|\psi\rangle$ is not, because Q is nilpotent.

This is an attractive picture so far, but to mimic nature there is still a need for fermions charged under gauge groups and the corresponding gauge bosons in the spectrum. The simplest way to add fermions to the action is to generalize to a supersymmetric theory on the world-sheet [51]. The result of this is that the action changes

$$S \to S_{bosonic} + \int d^2\sigma i\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \tag{1.11}$$

where ρ is the world-sheet γ matrix, and ψ is a two dimensional Majorana spinor which can be decomposed into holomorphic and antiholomorphic parts which decouple. The convention which will be used for the Laurent expansion of the ψ field into modes (following [79]) is

$$\psi^\mu = \sum_n \frac{\psi_n^\mu}{z^{n+\frac{1}{2}}} \tag{1.12}$$

where n is either an integer or a rational number of the form $n = \frac{2m+1}{2}$ for integer m. This condition occurs the boundary conditions on the fermions impose that fermion bilinears are single valued. The anticommutation relations which arise are

$$\{\psi_n^\mu, \psi_m^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \tag{1.13}$$

The choice between integral and half integral modes for the fermions arises in the following way in the case of open strings. We require that the boundary variation vanishes which in turn implies the equality of certain holomorphic and antiholomorphic fermion bilinears, and this means that on the boundary the fermions are equal up to a sign, and the relative sign between the boundaries determines whether the fermions can admit a zero mode. In the case of closed strings the same considerations apply, only the choice of periodic and antiperiodic is independent for ψ and $\tilde{\psi}$, with the periodic sector known as the Ramond (R) sector and the antiperiodic as the Neveu-Schwarz (NS) sector [79]. These periodicities have the consequence of producing space-time fermions because the zero modes in the Ramond sector carry a representation of a Clifford algebra. This can be seen by examining equation 1.13 and noting that ψ_0^μ acting on any state in the Fock space will not change the eigenvalue of that state under action by L_0.

The addition of the new fermionic term to the string world sheet action has the following consequence, that there are additional parts in the two dimensional energy momentum tensor, coming from the fermions, and also a sort of superpartner for this, the Noether current for non-constant supersymmetry transformations over the world-sheet. In terms of modes these are [51]

$$L_m \rightarrow L_m^\alpha + \frac{1}{2}\sum_r \left(r + \frac{m}{2}\right) : \psi_{-r} \cdot \psi_{m+r} : \tag{1.14}$$

$$G_m = \sum_r \alpha_{-r} \cdot \psi_{m+r} \tag{1.15}$$

where the sum is implicitly over integers or half integers as appropriate for the sector of the theory. The Virasoro algebra remains unchanged, but there are additional terms that must be calculated

$$[L_m, G_n] = \left(\frac{m}{2} - n\right) G_{m+n} \tag{1.16}$$

$$\{G_a, G_b\} = 2L_{a+b} + B(a)\delta_{a+b,0} \tag{1.17}$$

where as in (1.6) $B(a)$ is a central charge. The physical states are annihilated by the positive modes of these currents (and in the R sector, the zero mode of G).

Again, it is possible to reduce constraints such as the imposition of the light cone gauge by the introduction of commuting ghost fields with conformal weights $\frac{3}{2}$ and $-\frac{1}{2}$, β and γ. These will be integrally or half integrally moded as appropriate from the fermionic sector, and contribute to the operators L, G, and Q in a well known way.

A similar exercise to that performed in the case of purely bosonic string reveals that the critical dimension is 10, and that depending upon whether the string is in the NS or R sector (for left and right movers) there is a different ground state energy. Between this and level matching it is possible to determine the spectrum in each of the sectors: for the case of both left and right moving NS sectors, there is a tachyon, massless modes with quantum numbers matching those of the graviton, Kalb-Ramond field, and the dilaton, in addition to the spectrum of massive modes [51]. When both left and right movers are in the R sector there is no tachyon but a massless field that transforms with two spinorial indices under Lorentz transformation. In the sectors where the left and right moving fermions obey different boundary conditions level matching makes the lowest state massless and it has both vector and spinor indices, making it a combination of spin $\frac{1}{2}$ and spin $\frac{3}{2}$. The number of fields described here is apparently too many to fill a supergravity multiplet, but more sophisticated analysis reveals that there is a condition which reduces the spectrum, the GSO projection [48], which gives supersymmetry by projecting out particles of a given chirality. At the level of interacting theories it is necessary to have a number of different combinations of left and right moving boundary conditions and GSO projections. This construction reveals essentially two types of spacetime supersymmetric theories, those that are chiral and non-chiral (respectively IIA and IIB theories).

Any orientation in string theory would be incomplete without mention of another class of theories, the heterotic string theories. The fact that the left and right movers decouple makes it possible to write different theories for the left and right movers. In the heterotic theories the left moving degrees of freedom are written as fermions which obey some internal symmetry [55, 56]. Consistency requires that there be 32 such fermions, and further that they either all obey the same boundary conditions (in that case there is an

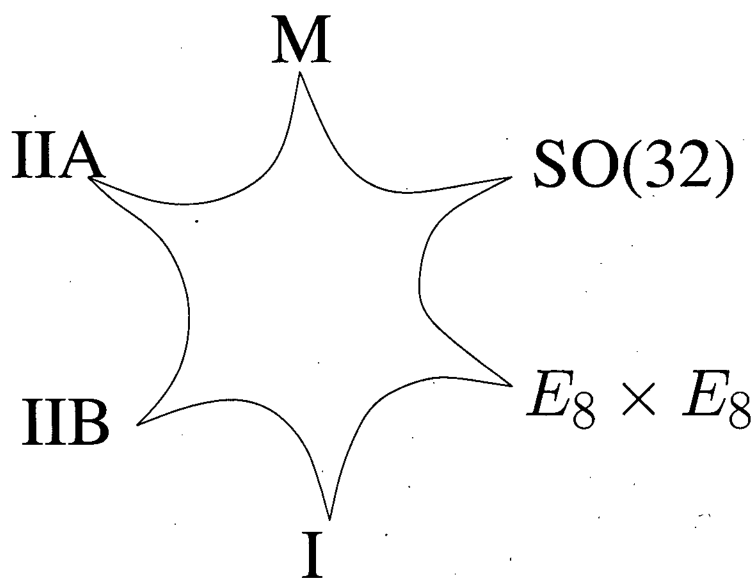

Figure 1.2: The modulus space of string theory, which consists of the five known perturbative string theories, and M theory. The edges on the figure represent duality transformations which will map one theory, usually at strong coupling to another at weak coupling, allowing perturbative calculations in one to probe non-perturbative effects in the other.

$SO(32)$ symmetry), or half obey one set of conditions and half obey the other, and when properly projected this theory has gauge group $E_8 \times E_8$. For these theories, the index from a left moving fermion is a gauge index, and the theories both have a symmetry group large enough to be broken to the Standard Model.

There remains one small difficulty: the world neither exhibits ten dimensions nor supersymmetry, nor these large gauge groups, and it is another matter to further constrain the theories to give a good simulation of the particles seen now. This is obtained in many ways, all essentially similar in that they force several dimensions to have another topology to that of the real axis, they are either periodically identified, or identified under reflection, or both, and this has the effect of breaking the large number of symmetries of the system. It is these techniques that allow the various string theories outlined to be related to one another. There is a famous 'web' of dualities that allow one theory compactified in a certain way at strong coupling to be related to another at weak coupling with a different compactification.

From this overarching framework, in this thesis we concentrate on the problem of tachyon condensation. In a number of the string theories described there is a tachyon in the spectrum: The state which is annihilated by all positively moded oscillators. Since tachyons are not observed in nature this indicates that the naive Fock space vacuum we have chosen to expand around is not the true ground state of the theory. The naive Fock space vacuum is the one, identified above, which is annihilated by all positively moded Virasoro generators L, and by all positively moded αs and ϕs. By contrast, the operational definition we choose for the 'true ground state' is one where all the excitations have positive semi-definite mass squared, eliminating tachyonic modes. We must therefore attempt to describe the ground state of the theory. This problem is commonly addressed in string field theory, of which there are two principal types. The first is the open string field theory [97, 98, 104, 106] which concerns itself solely with interactions at the string's boundaries. The second is the cubic, Chern-Simons like, string field theory [103] which consists solely of a kinetic term for the string field and a cubic interaction vertex. For each of these, the problem is the same, to describe what happens to the open string degrees of freedom as the tachyon condenses. From the world-sheet point of view it is possible to consider the following picture of what is happening: The volume in which open strings can end describes a d-brane. As the tachyon condenses the number of dimensions of this brane decreases, and the final stage of the condensation is a state in which the brane has been reduced to a point and the two ends of the open string coincide. This gives rise to closed strings, so at the endpoint of tachyon condensation only closed string degrees of freedom remain, and there are also predictions for the height of the tachyon potential [91].

As noted in [68] it is difficult to reproduce the properties of tachyon condensation in the cubic string theory because the calculations involve interactions of an infinite number of fields whose mass can be arbitrarily high. It is possible in some cases to investigate the structure of cubic string theory using techniques such as level truncation [77, 100]. This method yields results which tend to agree with those expected for some quantities, such as the vacuum energy of the condensate, but it remains unclear why the procedure works and whether level truncation is generally applicable, considering that there is no natural small parameter being expanded in.

In this thesis we study tachyon condensation within the framework of the open string field theory better known as 'boundary string field theory'. The idea is to consider backgrounds that interact with the boundary of the string, and analysis suggests that there is a set of coordinates on 'the space of all string fields' [68] that is better suited to the study of tachyon condensation. Further from the point of view of the world-sheet the tachyon condensation is described by a renormalization group flow from the UV, where the open strings end on d-branes, to the IR where only closed string degrees of freedom persist.

Here we formulate a boundary state in order to reproduce string sigma model amplitudes. This is constructed by modifying the definition of the boundary state [26] to include an integral over conformal reparameterizations. This boundary state then encodes the effect of these conformal reparameterizations, and is useful for many circumstances such as computing d-brane tensions, cylinder amplitudes, and looking for the gravity counterparts of d-branes [32, 34]. In the operator approach to string perturbation theory the boundary state contains the couplings of closed strings to a d-brane. This method gives an algebraic approach for calculations, and it suggests a method to generalize to higher loops, which reproduces the known results for the annulus.

The plan of the thesis is as follows. In Chapter 2 we introduce both boundary string field theory and cubic string theory, and we also describe in the language of boundary string field theory how to obtain actions for the fields which parameterize the boundary interactions. The discussion of cubic string field theory is intended to illustrate another approach to the problem of tachyon condensation which has offered some concrete evidence of the dynamics and end point of this process, and to stand in contrast with the methods used in subsequent chapters. In Chapter 3 we develop the 'boundary state' describing the boundary interactions that parameterize tachyon condensation. This state is a generalization of that discussed in [26], and it correctly reproduces the sigma model amplitudes for emission of closed string states. It is also a description of a state which is at neither fixed point of renormalization group flow and so interpolates between Neumann and Dirichlet boundary conditions. We also show that this boundary state can be used to reproduce known partition functions for boundary fields at the closed string tree level in the case of conformal invariance, compare with recent work on the definition of boundary string field theory at open string one loop level, and speculate on the applicability to more complicated surfaces. We also briefly develop boundary states for the world-sheet fermions of the superstring and the (super)conformal ghosts. In Chapter 4 we present some other calculations related to the question of tachyon condensation. We write a boundary state describing non-local boundary interactions [74]. We also summarize some recent work on time dependent tachyon condensation to show the general applicability of the boundary state method, and investigate the issue of spherically symmetric tachyon condensation. Chapter 5 we conclude and mention some ideas for future directions of research, and certain calculational details have been relegated to Appendices A and B.

Chapter 2

String Field Theory

A classic problem in string theory is to understand how the background space-time on which the string propagates arises in a self-consistent way. For open strings, there are two main approaches to this problem, discussed below, cubic string field theory [103] and background independent string field theory [97, 104].

The latter approach is defined as a problem in boundary conformal field theory, and the analysis begins with the partition function of open-string theory where the world-sheet is a disc. The strings in the bulk are considered to be on-shell and a boundary interaction with arbitrary operators is added. The configuration space of open string field theory is then taken to be the space of all possible boundary operators modulo gauge symmetries and the possibility of field redefinition. Renormalization fixed points, which correspond to conformal field theories, are solutions of classical equations of motion and should be viewed as the solutions of classical string field theory.

Despite many problems which are both technical and matters of principle, background independent string field theory has been useful for finding the classical tachyon potential energy functional and the leading derivative terms in the tachyon effective action [44, 68, 101]. Boundary field theories which can be used to study tachyons are the subject of the a large portion of the presented work.

The existence of a tachyon in the bosonic string theory indicates that the 26-dimensional Minkowski space background about which the string is quantized is unstable. An unstable state is likely to decay and the nature of

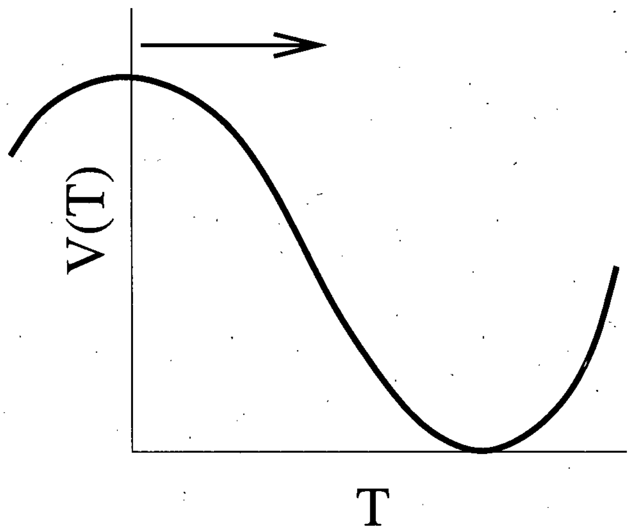

Figure 2.1: Tachyon Condensation: This schematic representation shows the idea behind tachyon condensation, that the tachyon is indicative of an instability in the perturbative vacuum. The perturbative (Fock space) vacuum is defined at the maximum of $V(T)$, and as the (open string) tachyon T rolls toward the minimum of its potential $V(T)$ it represents a decay of the space filling brane. At the minimum of the tachyon potential only closed string degrees of freedom survive.

both the decay process and the endpoint of the decay are interesting questions [77]. Recently, some understanding of this process has been achieved for the open bosonic string. The picture is that elaborated by Sen [91, 92], that the open bosonic string tachyon reflects the instability of the d-25 brane. This unstable d-brane should decay by condensation of the open string tachyon field. The energy per unit volume released in the decay should be the d-25 brane tension and the end-point of the decay is the closed string vacuum [36, 57, 68, 91]. There are also intermediate unstable states which are the d-branes of all dimensions between zero and 25.

These considerations are generally stated in the following way, that the tachyon field has a potential of the form

$$V(T) = Mf(T) \tag{2.1}$$

where $f(T)$ is a function of the tachyon field which is both universal and independent of the field theory describing the d-brane, and M is the mass of the d-brane for vanishing tachyon field. Further, the potential is defined with an additive constant such that at its minimum, T_0, it cancels the mass of the d-brane [92]. With these conventions the tachyon potential can be written

$$V(T) = M\left(1 + f(T)\right) \quad \text{with} \quad 1 + f(T_0) = 0. \tag{2.2}$$

The conjectures about the dynamics of tachyon condensation also contend that at the minimum of the tachyon potential the corresponding brane system is indistinguishable from that where there is no d-brane [90, 92]. This is to say that the open string degrees of freedom have condensed to leave only closed string modes.

2.1 Background Independent String Field Theory

In this section we review the basic formalism of background independent off-shell string theory because they will be of some use in the motivation of the subsequent work on the boundary state. This review follows very closely the work presented in [97, 104–106] and, for concreteness, focuses on the tachyon field, although the results are much more general. It was first demonstrated in [104] that in an attempt to define a Lagrangian in the 'space of all open-string world-sheet theories' we discover that the boundary interaction on the string world-sheet is constrained in a certain interesting way. In particular we find that the classical equations of motion which are derived from S, the Lagrangian on the space of possible interactions, are equivalent to BRST invariance of the theory on the string world-sheet. Further we find that if it is possible to decouple the matter and the ghosts in the world-sheet theory by a gauge condition on the boundary interaction, then the equations of motion in that particular gauge are equivalent to conformal invariance, and that the infinitesimal generators of a gauge transformation are the BRST operators. The result from all of this is that if matter and ghosts are decoupled then the on shell action S for the particular interaction is equal to the partition function of the world-sheet matter. [106]

2.1.1 Bosonic String Case

The starting point for this analysis is the string action with both bulk and boundary terms

$$S = S_0 + S_{bdy}. \tag{2.3}$$

In this equation S_0 is the standard action on a closed string world sheet

$$S_0 = \int_M d^2z(\partial X_\mu \bar{\partial} X^\mu + b^{ij}\partial_i c_j), \tag{2.4}$$

and we have already specialized to the case of flat metrics on both the world sheet and a Minkowskian (or Euclidean) metric on space-time. The terms b and c are the standard anticommuting ghosts from the quantization of bosonic string theory. Similarly S_{bdy} is a local boundary term made up of both the bosonic fields and ghost terms on the string world sheet and so can be written

$$S_{bdy} = \int_{\partial M} \mathcal{V}. \tag{2.5}$$

In particular the boundary operator $\mathcal{V}$ satisfies

$$\mathcal{V} = b_{-1}\mathcal{O} \tag{2.6}$$

where b is a ghost field and $\mathcal{O}$ is some combination of fields of ghost number one [104]. The reason for this choice stems particularly from the Batalin-Vilkovisky formalism [20, 21], and can be summarized in the following way. In this treatment we consider the string world-sheet as a super-manifold with a $U(1)$ symmetry, referred to as a ghost number symmetry in the literature [104]. For this kind of manifold the defining characteristic is a structure ω which is a non-degenerate fermionic two-form that is closed; ω can be thought of, and has been motivated in the literature as a fermionic symplectic form. With this symplectic form it is possible to define a Poisson bracket on the space

$$\{A, B\} = \frac{\partial A}{\partial u^K}\omega^{KL}\frac{\partial B}{\partial u^L}, \tag{2.7}$$

and in terms of this bracket the Master equation for the action is $\{S, S\} = 0$. In this definition of the Poisson bracket the us are local super-coordinates. It is also possible to define a vector field V (notice that it is distinct from $\mathcal{V}$ the boundary interaction term) which is a contraction on the symplectic form satisfying

$$V^K\omega_{KL} = \frac{\partial}{\partial u^L}S \tag{2.8}$$

and in this case S need not satisfy the Master equation. This condition is equivalent to (with indices suppressed)

$$V\omega = dS. \tag{2.9}$$

Now, under a diffeomorphism such as

$$u^L \to u^L + \epsilon V^L \tag{2.10}$$

the two-form ω transforms as

$$\omega_{KL} \to \omega_{KL} + \epsilon \left(V^M d_K \omega_{ML} + d_K V^M \omega_{ML} \right) \tag{2.11}$$

but since ω is closed we have $d\omega = 0$ and then V generates a symmetry of ω if $d(V^M \omega_{ML}) = 0$. This also implies that for a given vector V it is possible to construct an S that will give the required V according to the definition above. It is also possible to write the two-form as

$$\omega = \oint d\theta_1 d\theta_2 \langle \mathcal{O}(\theta_1) \mathcal{O}(\theta_2) \rangle \tag{2.12}$$

for some basis of operators $\mathcal{O}$ of appropriate ghost number, and with $\langle \ldots \rangle$ denoting an expectation value calculated through the usual path integral weighted by the string action. It is possible to decompose a particular vector in terms of these bases and we find that

$$dS = \oint d\theta_1 d\theta_2 \langle V \mathcal{O}(\theta_1) \mathcal{O}(\theta_2) \rangle. \tag{2.13}$$

Thus for the special case of choosing the vector V as the BRST operator

$$dS = \oint d\theta_1 d\theta_2 \langle d\mathcal{O}(\theta_1) \left\{ Q_{BRST}, \mathcal{O}(\theta_2) \right\} \rangle. \tag{2.14}$$

Now we can specialize to a particular theory that contains the germs of generality. In particular we note that the matter part of $\mathcal{V}$ can be Taylor expanded in the bosonic field X as

$$\mathcal{V} = T(X) + A_\mu(X) \partial_\theta X^\mu + B_{\mu\nu} \partial_\theta X^\mu \partial_\theta X^\nu + \ldots \tag{2.15}$$

and we now restrict our attention to the tachyon field term, $T(X)$. We also note in passing that later in this work the $A_\mu(X)\partial_\theta X^\mu$ term, which gives rise to a background gauge field, will also be important. A particularly simple solvable model is that of the quadratic tachyon

$$T(X) = \frac{a}{2\pi} + \sum_i \frac{u_i}{8\pi} X_i^2. \tag{2.16}$$

This model was originally considered [106] because it was a simple quadratic model, and therefore exactly solvable. The addition of non-zero u_is corresponds to a breaking of translational invariance, and because the term adds a potential energy to the zero mode of the string the strings oscillations are limited to a finite volume, and in the limit of a particular $u \to \infty$ the string end is fixed to a particular point in the space in which it is embedded, and we will argue later that this provides an interesting model to describe a d-brane. (This model can easily be generalized to the more general quadratic term $U_{\mu\nu}X^\mu X^\nu$ and this is often desirable if additional background fields are also being considered.) For this interaction term we note that the ghost fields decouple, and so the world sheet action can be written as

$$S = \int_M d^2z \partial X^\mu \bar{\partial} X_\mu + \int_{\partial M} d\theta \left(\frac{a}{2\pi} + \sum_i \frac{u_i}{8\pi} X_i^2 \right). \tag{2.17}$$

After some manipulation [106] (presented explicitly in section 3.2.4) we can find the partition function for this world sheet models

$$Z = e^{-a} \prod_i \sqrt{u_i}\Gamma(u_i). \tag{2.18}$$

Note that this differs from the result found in [66] by a factor relating to the normalization of the zero modes. It is also possible to note in passing that there are several good features of this function that are suggestive of it playing a role similar to the action for a space-time theory. First note that for any individual u the function goes as $\frac{1}{\sqrt{u}}$ for $u \to \epsilon > 0$. This arises because u plays the role of localizing the function near $X = 0$, in fact it is remarked elsewhere in the literature that u interpolates between Neumann and Dirichlet boundary conditions. (A convenient way to see this is in the bosonic component of the boundary state written in (3.10) and (3.11) go between the the boundary states for Neumann and Dirichlet boundary conditions [26, 34] as U goes from 0 to ∞.) This implies that the divergence near $u = 0$ can be interpreted as associated with the delocalization of the string over the volume of spacetime. Also, the expression for the world sheet partition function has divergences when $u < 0$, reflecting the fact that in that case the world-sheet action is not bounded below.

Now, following the previous derivation in (2.14) and explicitly writing the dependence of the fields X on the boundary coordinate θ, we find that

$$dS = \oint d\theta_1 d\theta_2 \langle T(X)(\theta_1) \left(1 + \epsilon^i \partial_i\right) T(X)(\theta_2)\rangle. \tag{2.19}$$

Now, use the fact that (as in [104, 106]) the derivative with index i refers to the parameters within the tachyon field, and we have

$$\left(1 + \epsilon^i \partial_i\right) T(X) = \left(1 + 2\sum_i \frac{\partial^2}{\partial X_i^2}\right) T(X). \tag{2.20}$$

Including the ghost contribution, and using both the explicit form of the X two point function, and the relationships

$$\begin{aligned} \oint d\theta \langle X_i^2(\theta)\rangle &= -8\pi \frac{\partial Z}{\partial u_i} && (2.21) \\ \oint d\theta_1 d\theta_2 \langle X_i^2(\theta_1) X_j^2(\theta_2)\rangle &= (8\pi)^2 \frac{\partial^2 Z}{\partial u_i \partial u_j}, && (2.22) \end{aligned}$$

it is easy to obtain

$$dS = d\left(\sum_i u_i Z - \sum_j u_j \frac{\partial}{\partial u_j} Z + (1 + a) Z\right). \tag{2.23}$$

This is equivalent to obtaining the action for the boundary fields

$$S = \left(\beta_i \frac{\partial}{\partial \lambda^i} + 1\right) Z \tag{2.24}$$

where β_i is the β-function for the ith coupling λ^i, which is a parameter of the boundary interaction terms. This way of looking at the effect of terms on the string world sheet boundary will be particularly useful while considering the boundary state in subsequent sections.

2.1.2 Superstring Case

There have been several attempts to generalize the method given above to the superstring [67, 68, 76], and we give an account of one of them here [76], which has the consequence of proposing a modification to the boundary field action (2.24). The proposal for the action is

$$S = S_0 + S_\Gamma + \int_{\partial M} G_{-1/2}\mathcal{O} \tag{2.25}$$

where S_0 is the usual bulk action of the RNS superstring, including both the bosonic and fermionic ghosts,

$$S \quad = \quad \int d^2z \left(\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu + b\bar{\partial} c + \beta \bar{\partial} \gamma\right) \tag{2.26}$$

with b, c and β, γ anticommuting and commuting ghosts respectively. The second and third terms of (2.25) can be thought of as the perturbation due to the addition of the field on the boundary. Explicitly they are given as

$$S_\Gamma = \int_{\partial M} \int d\theta \Gamma D \Gamma, \tag{2.27}$$

with the defining relation for Γ,

$$\Gamma = \mu + \theta F \tag{2.28}$$

where μ is a fermionic component of the boundary action, and F is a bosonic component, and θ is a supercoordinate on the string world sheet. Finally, the derivative operator D is defined as

$$D = \partial_\theta + \theta\partial_{\|} \tag{2.29}$$

on the boundary of the string world sheet. The subscript $\|$ refers to the tangential orientation with respect to the boundary while ∂_θ is a Grassmanian derivative. $\mathcal{O}$ is also identified as the lowest component of a world-sheet superfield, Ψ satisfying

$$\Psi = \mathcal{O} + \theta G_{-1/2}\mathcal{O}. \tag{2.30}$$

The proposal for the string field action, in analogy with the development leading up to (2.14) is to write the two form ω as

$$\omega(\mathcal{O}_\infty, \mathcal{O}_\in) = \frac{1}{8}\oint \frac{d\tau d\tau'}{4\pi^2}\langle \mathcal{O}_\infty(\tau)\mathcal{O}_\in(\tau')\rangle \tag{2.31}$$

where the contribution from the conformal ghosts has been suppressed for clarity (as in (2.21) and (2.22)) as well as the factors appropriate to the inclusion of bosonized fermions $e^{-\phi}$ which are appropriate for the (-1) picture. Similarly we can write

$$dS = \frac{1}{8}\oint \frac{d\tau d\tau'}{4\pi^2}\langle d\mathcal{V}(\tau)\,\{Q_{BRST}, \mathcal{V}(\tau')\}\rangle \tag{2.32}$$

which completes the analogy with equation (2.14).

As discussed in [58, 67] the superfield which describes a tachyon profile is

$$\Psi = \Gamma T(X), \tag{2.33}$$

so following their argument we get

$$\mathcal{O} = \mu T(X) \tag{2.34}$$

so it is easy to show that

$$G_{-1/2}\mathcal{O} = FT(X) + \psi^a\mu\partial_a T(X). \tag{2.35}$$

Inserting these expressions and integrating we obtain

$$\begin{aligned} S_\Gamma + \int_{\partial M} G_{-1/2}\mathcal{O} &= \frac{1}{2\pi}\int d\tau d\theta \left(\mu + \theta F\right)\left(\partial_\theta + \theta\partial_{||}\right)\left(\mu + \theta F\right) \\ &\quad + \frac{1}{2\pi}\int d\tau \left(FT(X) + \psi^a \mu \partial_a T(X)\right) \\ &= \int \frac{d\tau}{2\pi}\left(F^2 + \mu\partial_{||}\mu + FT(X) + \psi^a\mu\partial_a T(X)\right). \end{aligned} \tag{2.36}$$

It is immediately apparent that integrating out the auxiliary field F will give a term like e^{-T^2} in the partition function. This is appropriate because the tachyon profile T used in the superstring case is analogous to the square root of that used in the previous section.

Two cases are of special note in the literature, the first is the case of the constant tachyon, in which, again up to ghost contributions we find that

$$\{Q_{BRST}, \mu T(X)\} \propto T(X) \tag{2.37}$$

and upon integration of the various modes we have

$$\begin{aligned} dS &= -\frac{1}{2} T dT e^{-\frac{1}{4}T^2} \\ &= d\left(e^{-\frac{1}{4}T^2}\right) \end{aligned} \tag{2.38}$$

and since $Z \propto e^{-\frac{1}{4}T^2}$ we find upon integration $S = Z$. For the case of a linear tachyon

$$T(X) = u_\mu X^\mu \tag{2.39}$$

the calculation is somewhat more involved, but the result is known [76], and can be summarized in the following way. The world sheet action (2.36) includes a term linear in μ now and with a field redefinition to account for this

$$\mu \to \mu + \frac{1}{2}\frac{1}{\partial_{||}}\psi^\nu \partial_\nu T \tag{2.40}$$

the integral becomes simpler in terms of the redefined field, but there is now a term of the form $\psi\frac{1}{\partial_{||}}\psi$ in the action. It is well known how to show that the expression for dS becomes [76]

$$dS = -\frac{1}{4}\langle X^2 + \psi\frac{1}{\partial_{||}}\psi\rangle Z(y)dy \tag{2.41}$$

where $y = u^2$. However, since

$$\frac{d\ln Z}{dy} = -\frac{1}{4}\langle X^2 + \psi\frac{1}{\partial_{||}}\psi\rangle \tag{2.42}$$

we have that the world-sheet partition function is equal to the action for the boundary field.

2.2 Cubic String Field Theory

In addition to the discussion of background independent string field theory above, another important motivation for the discussion of tachyons and other string fields comes from understanding cubic string field theory, and the recent conjectures of Sen [24, 91, 96] about the condensation of open string tachyons. This brief review draws heavily on lectures on cubic string field theory delivered at TASI-2001 by Taylor and Zwiebach [99, 100, 107]. This method is interesting in the context of this thesis for two reasons. The first reason is that it provides a distinct and independent check on the ideas describing the condensation of tachyons, and the final state of this decay. The second is that the coherent states that can be used to describe the product of string states resemble those which we will detail in constructing boundary states, and that similar manipulations can be performed on both.

Cubic string theory is an attempt to treat string theory as a field theory, with a kinetic term and a cubic, Chern-Simons like, interaction term. The string fields are constructed by operating creation operators on a vacuum, and since there are an infinite number of such possible interactions it is possible to think of string field theory as an interacting field theory with an infinite number of massive particles in its spectrum. The recent interest in cubic string field theory can, in large part, be traced to work [85] on the conjectures recently raised by Sen [89, 90], which address the question of the bosonic open string tachyon in the following way.

- In analogy with the Higgs mechanism familiar from the study of the Standard Model, the bosonic open string tachyon can be thought of as an instability of a space filling D-brane.

- There exists a locally stable minimum of the tachyon potential, and around that minimum there exist no open string excitations.

- That the height of this potential is given by $\frac{1}{2\pi^2 g^2} = \frac{\Delta E}{V}$, where g is the string coupling constant, and V is the volume of space time.

These conjectures appeal to our physical intuition and are under active investigation.

We start with the proposal by Witten [103] for a cubic string field theory action, particularly

$$S = -\frac{1}{2}\int \psi * Q\psi - \frac{g}{3}\int \psi * \psi * \psi \tag{2.43}$$

where ψ is an open string field, g is the string coupling constant, $*$ is a product on the space of string fields, and Q is an operator which is roughly analogous to a derivative operator (in fact it will be the BRST operator). A number of properties that are important in the realization of the theory. First, with respect to ghost number, the $*$ product is additive, which is to say that if

$$\psi_1 * \psi_2 = \psi_3 \rightarrow G_{\psi_3} = G_{\psi_1} + G_{\psi_2} \tag{2.44}$$

where G_ψ is the ghost number of the string field ψ. In a similar way to this, the operator Q adds one to the ghost number of the field:

$$Q\psi = \psi' \rightarrow G_{\psi'} = 1 + G_\psi, \tag{2.45}$$

and the integration picks out the components of the integrand which have ghost number 3,

$$\int \psi = 0 \quad \forall \quad \psi : G_\psi \neq 3. \tag{2.46}$$

To avoid difficulty with boundary terms, it is desirable for the integration to vanish for total derivatives

$$\int Q\psi = 0 \quad \forall \quad \psi. \tag{2.47}$$

There are further properties motivated by thinking of Q as an exterior derivative. The first is that Q is nilpotent (satisfied for the BRST operator in the critical dimension), secondly a Leibnitz rule for the Q operator

$$Q\,(\psi_1 * \psi_2) \;=\; (Q\psi_1) * \psi_2 + (-1)^{G_{\psi_1}} \psi_1 * Q\psi_2, \tag{2.48}$$

and also a commutativity condition

$$\int \psi_1 * \psi_2 = (-1)^{G_{\psi_1}+G_{\psi_2}} \int \psi_2 * \psi_1. \tag{2.49}$$

It is also interesting to note that under the analog of a non-Abelian gauge transformation $\psi \rightarrow \psi + \delta\psi$ subject to

$$\delta\psi = Q\Lambda + g\,(\psi * \Lambda - \Lambda * \psi) \tag{2.50}$$

and $G_\Lambda = 0$, the cubic string field theory action is invariant.

The Fock space which the bosonic string fields inhabit is defined by the general state

$$\prod \alpha^\mu_{-m} \dots c_{-n} \dots b_{-k} \dots |0\rangle \tag{2.51}$$

where the ... refer to any arbitrary insertion of oscillators similar to the preceding α, c, or b. The α oscillators come from the quantizing of the bosonic X field and the b, c are the ghost fields. These have the property that on the Fock space vacuum [51]

$$\begin{aligned} b_n|0\rangle &= 0,\; n \geq -1 \\ c_n|0\rangle &= 0,\; n \geq 2 \\ \alpha_n|0\rangle &= 0,\; n \geq 1 \end{aligned} \tag{2.52}$$

and α_0 is the momentum operator. It is well known that it is possible to write both the BRST operator and the Virasoro generators in terms of the raising and lowering operators α, b, and c,

$$Q_B = \sum_n c_n L^m_{-n} + \sum_{mn} \frac{m-n}{2} : c_m c_n b_{-m-n} : -c_0 \tag{2.53}$$

where the matter part of the Virasoro generator is given by

$$L^m_k = \frac{1}{2} \sum_n : \alpha^\mu_{k-n} \alpha_{\mu n} : + a\delta_{k0} \tag{2.54}$$

with a a normal ordering constant associated with the mass-shell condition for the strings as in (1.5) and (1.9) Given this information it is relatively easy to explicitly construct Fock space states which have inner products with the string fields which satisfy all the enumerated requirements for the integral. For the 'kinetic' term of the string field theory action we need to construct a state which is the tensor product of two such Fock spaces because the string fields each carry a raising operator Fock space. In particular it can be shown that the state

$$\begin{aligned} \langle I_2| &= \int dp\,(\langle 0,p|_1 \otimes \langle 0,p|_2) \exp\Bigg[-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \alpha_n^{(1)} \alpha_n^{(2)} - \\ & \qquad \sum_{n=0}^{\infty} (-1)^n \left(c_n^{(1)} b_n^{(2)} + c_n^{(2)} b_n^{(1)} \right) \Bigg] \end{aligned} \tag{2.55}$$

is a representation of the integral. In the above, note that the operators numbered 1 and 2 operate on their respective ground states only. We note in passing that this is tantalizingly similar to the boundary states to be discussed in chapter 3, in that it is a coherent superposition of independent Fock spaces. In a similar manner it is possible to construct a state which is a tensor product of three Fock spaces to represent the integral for the interaction term, schematically given by

$$\langle I_3| = \int dp\,(\langle 0,p|_1 \otimes \langle 0,p|_2 \otimes \langle 0,p|_3) \exp\left[-\alpha^{(i)} N_{ij} \alpha^{(j)} - c^{(i)} X_{ij} b^{(j)}\right] \tag{2.56}$$

Level	# Fields	V/T_{25}
(0,0)	1	-0.685
(2,4)	3	-0.949
(4,8)	10	-0.986
(6,12)	31	-0.995
(8,16)	91	-0.998
(10,20)	252	-0.999

Table 2.1: Level truncation in string field theory. The term V/T_{25} would be -1 to confirm Sen's conjectures. [100]

where the oscillator indices have been suppressed, and the terms N_{ij} and X_{ij} are known [100] and of similar form to the terms in $\langle I_2|$. At this point, in principle it is now possible to calculate all terms and contributions to the cubic string field theory action. This approach has not been applied for arbitrary excitations because there are an infinite number of terms in the expansion of the string field and the number of terms at each level increase very quickly.

This problem has been approached with some success [77] using level truncation. The general idea of this method is to include only states up to some finite level in both oscillators and the sum of the level numbers for those oscillators. A complete exposition of this interesting field of study is well beyond the scope of this discussion, however we pause to note that this line of research has yielded some compelling 'experimental' evidence in favor of the famous conjectures about the minimum energy of the open string tachyon.

To illustrate the method of level truncation we perform explicitly the calculation at the lowest level. The potential can be written as

$$V = \sum d_{ij}\phi^i\phi^j + g\kappa \sum t_{ijk}\phi^i\phi^j\phi^k \tag{2.57}$$

where d_{ij} and t_{ijk} can be calculated, g is the string coupling constant of equation 2.43, and κ is chosen to be $\kappa = \sqrt{3^7}/2^6$ so that $t_{111} = 1$. [99] With this convention we find that for the ansatz

$$\phi = \frac{t}{g}|0\rangle \tag{2.58}$$

we obtain

$$V = -\frac{1}{2}\frac{t^2}{g^2} + \kappa g \frac{t^3}{g^3} \tag{2.59}$$

and inserting the value for κ we get the equation that must be satisfied for t to find an extremum of the string field theory action is

$$-t + \frac{3^{9/2}}{2^6}t^2 = 0 \tag{2.60}$$

and solving for t and substituting into (2.59) we obtain

$$V = -\frac{2^{11}}{3^{10}}\frac{1}{g^2} \tag{2.61}$$

and since the tension of the d-25 brane is $\frac{1}{2\pi^2 g^2}$ their ratio is -0.685 as mentioned in Table 2.2.

There is another way to examine the cubic string field theory, which is equivalent to that described above, but offers a more geometric picture of the construction. It is to represent the star product as a functional integral over the fields of the string world sheet. Explicitly, because of the decoupling between the matter and ghost sectors of the string, it can be written for an open string world sheet with width π as

$$\begin{aligned}(\psi_1 * \psi_2)[z(\sigma)] &= \int \prod_{0\le\tau\le\pi/2} dy(\tau)dx(\pi-\tau)\times \\ &\prod_{\pi/2\le\tilde{\tau}\le\pi} \delta\left[x(\tilde{\tau}) - y(\pi-\tilde{\tau})\right]\psi_1[x(\tau)]\,\psi_2[y(\tau)]\end{aligned} \tag{2.62}$$

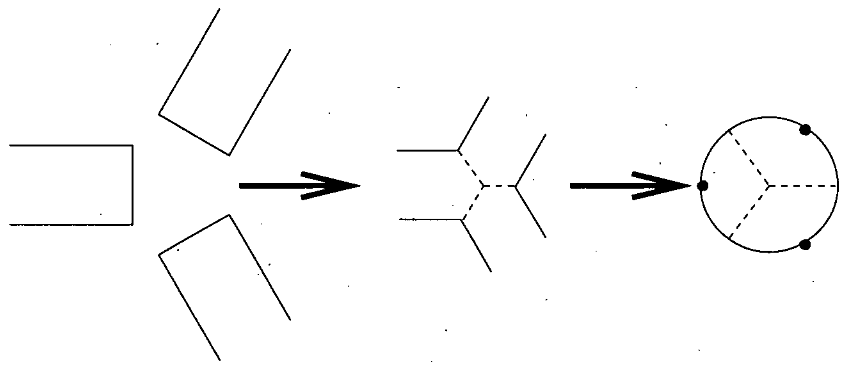

Figure 2.2: Cubic String Field theory integration: this schematic diagram outlines the three string field interaction. The string world-sheets are conformally mapped onto the unit disk, with the boundaries forming triangular wedges as indicated, with identification along the boundaries. The arrows indicate the two steps involved in the process, first identifying boundaries of the open string using the $*$ product, and then mapping the resulting world sheet to a disk using residual conformal invariance.

subject to the identification that

$$\begin{aligned} x(\sigma) &= z(\sigma),\ 0 \leq \sigma \leq \frac{\pi}{2} \\ y(\sigma) &= z(\sigma),\ \frac{\pi}{2} \leq \sigma \leq \pi. \end{aligned} \tag{2.63}$$

This can be thought of as joining the right half of one string to the left half of another to make a single string. A similar expression to (2.62) can be found for the integral over string fields, and again because of the decoupling the matter integral is

$$\int \psi = \int \prod_{0\leq\sigma\leq\pi} dx(\sigma) \prod_{0\leq\tilde{\sigma}\leq\pi/2} \delta\left[x(\tilde{\sigma}) - x(\pi - \tilde{\sigma})\right] \psi\left[x(\tilde{\sigma})\right]. \tag{2.64}$$

Similar to (2.62) being thought of as gluing the left and right halves of their respective strings together the integral (2.64) can be thought of as gluing the left and right halves of the same string together with this δ function interaction. This is illustrated for the cubic interaction term in Figure 2.2, and the residual conformal invariance of the string world sheet is used to map the semi-infinite string world sheets to a disk, with vertex operator insertions containing the asymptotic description of the open strings.

Chapter 3

Boundary States

In this chapter we use the boundary state formalism for both the bosonic string and the superstring to calculate the emission amplitude for closed string states from particular d-branes and show that the amplitudes are exactly those obtained from world-sheet sigma model calculations. We find that the construction of the boundary state automatically enforces the requirement for integrated vertex operators, even in the case of an off-shell boundary state. Using the boundary state and a similar expansion for the cross-cap, we produce higher order terms in the string loop expansion for the partition function of the backgrounds considered.

3.1 Introduction

The study of off-shell string theory has been addressed many times in the literature within the context of background independent string field theory [97, 98, 104–106] which has been the subject of a considerable amount of interest in that it can provide useful information about the properties of unstable d-branes [44, 46, 68]. Despite this there are several subtleties that have been examined, and in particular a great deal of effort has been expended in determining an action for a tachyon field coupled to a bosonic string [5–7, 15, 28, 29, 44–46, 66, 68, 84, 102], and while great progress had been made the understanding of higher loop effects is incomplete at best.

The boundary state for the superstring was first examined in [26] and the overarching idea of the system is to produce a state that vanishes when the boundary conditions, acting as operators, act on it. This state is then supposed to reproduce the overlap with closed string states, and if the surface upon which the strings end and have their boundary conditions is regarded as a dynamical object with fields upon it, the amplitudes for the emission of various closed string states determine the stringy self-interactions, the brane coupling to bulk fields, and the brane-brane interaction by string exchange. A source for a great deal of the formalism is [32, 34] and this idea has been generalized to non-quadratic interactions in, among other places, [86, 93–95]. The principal focus of this chapter is to first develop the boundary state for the case of a background tachyon field and a background constant Abelian gauge field strength, then examine the effect of world sheet coordinate reparameterization invariance upon these states, and finally to examine and explore a way in which the boundary state could be used to generate amplitudes more complicated than simply tree level closed string exchange. The general form of the boundary state is particularly simple for the case of quadratic boundary interactions, precisely because they are an exactly solvable model, and while more general interactions are discussed in Chapter 4 and in [85, 95], in this section the emphasis is on conformal properties, and for the moment we restrict attention to the quadratic case.

Since the bosonic and fermionic world-sheet oscillators, as well as the conformal ghosts, do not have non-trivial (anti)commutation relationships it is possible to decompose the boundary state into the direct product of boundary states $|B\rangle$ for the bosons and fermions respectively,

$$|B\rangle = \mathcal{N}|B_X\rangle|B_\psi\rangle|B_{bc}\rangle|B_{\beta\gamma}\rangle \tag{3.1}$$

where $\mathcal{N}$ is a normalization constant which generically depends upon the various background fields which appear as coupling constants in the string σ model. In addition to this property, the (b, c) and (β, γ) ghosts do not interact with any of the fields on the world-sheet boundary and so in a sense these contributions are trivial, and do not contribute to the amplitudes other than as a multiplicative factor which reproduces the known, conformally invariant, free case. There are a number of constraints that the boundary state must satisfy in order to encode physical degrees of freedom [26, 34], specifically

$$\left(Q_{BRST} + \tilde{Q}_{BRST}\right)|B\rangle = 0 \tag{3.2}$$

which is to say that it is BRST invariant. The strategy espoused here to determine the boundary state will be to examine in detail, in the next few sections, the bosonic string in particular backgrounds, and then look at the fermions and ghosts in a similar manner.

3.2 The Bosonic Boundary State

A tractable problem within this genre is the study of the off-shell theory in the background of a quadratic tachyon profile, a problem that is similar in spirit and detail to the examination of string theory in the background of a constant electromagnetic field [40]. In the following we combine these naturally compatible studies using the boundary state formalism [4, 19, 25, 31–33, 41, 70, 72, 73]. It allows us to calculate the probability for a topological defect which supports these quadratic fields to emit any number of closed string states into its bulk space-time. The loss of conformal invariance introduced by the background tachyon field is naturally accommodated by a conformal transformation which induces a calculable change in the boundary state. This new boundary state can be shown to reproduce the sigma model expectation values for the insertion of a vertex operator at an arbitrary point on the string world-sheet.

Using the correspondence between the sigma model calculation and that in the operator formalism the question of higher genus surfaces with some number of boundaries interacting with the background fields is considered.

The insertion of both loops and boundaries is included naturally in this method, and the results obtained are compared with known results.

Throughout this work the bosonic action under consideration is

$$\begin{aligned} S\left(g,F,T_0,U\right) &= \frac{1}{4\pi\alpha'}\int_{\Sigma} d\rho d\phi\; g_{\mu\nu}\partial^a X^\mu \partial_a X_\mu \\ &+ \int_{\partial\Sigma} d\phi \left(\frac{1}{2}F_{\mu\nu}X^\nu\partial_\phi X^\mu + \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu\right), \end{aligned} \tag{3.3}$$

where α' is the inverse string tension, Σ is the string world-sheet, $\partial\Sigma$ is the boundary of the string world-sheet, $d\rho d\phi$ is the integration measure of the string bulk, $d\phi$ is the integration measure of the string world-sheet boundary, and ∂_ϕ is the derivative tangential to that boundary. This action is motivated in [66, 106]. The field content in this are a constant $U(1)$ gauge field strength $F_{\mu\nu}$ and the tachyon profile,

$$T(X) = \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu, \tag{3.4}$$

is characterized by a constant, T_0, and a constant symmetric matrix $U_{\mu\nu}$. This provides a simple generalization for the discussion given in [28, 106] and (2.16), and similarly to avoid divergences we impose that it is positive semi-definite.

The virtue of the boundary state as a tool in the analysis of the action above is that it allows calculations that previously took careful integration to be reduced to algebraic manipulations. We wish to carefully construct the boundary state and to show that it reproduces with ease the particle emission amplitudes that would be obtained from the string sigma model. The starting point for this analysis is the action (3.3). By varying it, we obtain the equation

$$\left(\frac{1}{2\pi\alpha'}g_{\mu\nu}\partial_\rho + F_{\mu\nu}\partial_\phi + \frac{1}{4\pi}U_{\mu\nu}\right)X^\nu = 0 \tag{3.5}$$

as the boundary condition for the string world-sheet. Recalling the conventions from the action, ∂_σ is the derivative normal to the boundary and ∂_ϕ is the derivative tangential to the boundary. We now create a state $|B\rangle$ that obeys the above condition as an operator equation. To do this we use reparameterize the string world sheet in terms of holomorphic and antiholomorphic variables $z = \rho e^{i\phi}$ and $\bar{z} = \rho e^{-i\phi}$ and use the standard mode expansion for X as a function of z 1.3

$$X^\mu(z,\bar{z}) = x^\mu + p^\mu \ln |z^2| + \sum_{m\neq 0} \frac{1}{m}\left(\frac{\alpha^\mu_m}{z^m} + \frac{\tilde{\alpha}^\mu_m}{\bar{z}^m}\right). \tag{3.6}$$

we find that in terms of the mode operators the boundary conditions read

$$\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \alpha^\mu_n + \left(g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \tilde{\alpha}^\mu_{-n} = 0. \tag{3.7}$$

The condition for the boundary state to obey (3.5) can then be restated in terms of (3.7) to be

$$\left[\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \alpha^\mu_n + \left(g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}\right)_{\mu\nu} \tilde{\alpha}^\mu_{-n}\right]|B\rangle = 0, \tag{3.8}$$

$$\left[g_{\mu\nu}p^\mu - i\frac{\alpha'}{2}U_{\mu\nu}x^\mu\right]|B\rangle = 0. \tag{3.9}$$

To satisfy this it is clear that $|B\rangle$ must be a coherent state, and it is given by [4]

$$\begin{aligned} |B\rangle &= \mathcal{N}\prod_{n\geq 1} \exp\left(-\left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)_{\mu\nu} \frac{\alpha^\mu_{-n}\tilde{\alpha}^\nu_{-n}}{n}\right) \\ &\qquad \exp\left(-\frac{\alpha'}{4}x^\mu U_{\mu\nu} x^\nu\right)|0\rangle \\ &= \mathcal{N}\prod_{n\geq 1} \exp\left(-\Lambda^n_{\mu\nu}\alpha^\mu_{-n}\tilde{\alpha}^\nu_{-n}\right) \exp\left(-\frac{\alpha'}{4}x^\mu U_{\mu\nu}x^\nu\right)|0\rangle \end{aligned} \tag{3.10}$$

where $\mathcal{N}$ is a normalization constant which must be determined, and we define

$$\Lambda^n_{\mu\nu} = \frac{1}{n}\left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)_{\mu\nu} \tag{3.11}$$

for future convenience.

3.2.1 Conformal Transformation of Bosonic Boundary State

Clearly this boundary state is not conformally invariant due to the addition of the interaction with the tachyon field. The two cases where we expect conformal invariance are at the two fixed points of renormalization group flow, namely $U = 0$ and $U = \infty$, which correspond respectively to the case of Neumann or Dirichlet boundary conditions on the boundary of the string world sheet [68]. Note that in the case of Dirichlet boundary conditions the interaction with the background electromagnetic field is eliminated, as would be expected from the sigma model point of view. Because of this it is interesting to examine how the boundary state transforms under the PSL(2,R) symmetry that is broken by the presence of the U term in the boundary state. In the two conformally invariant cases this leaves the action invariant. The action of PSL(2,R) on the complex coordinates of the disk is to perform the mapping

$$z \to w(z) = \frac{az + b}{\bar{b}z + \bar{a}} \tag{3.12}$$

where a and b satisfy the relation

$$|a^2| - |b^2| = 1. \tag{3.13}$$

This transformation maps the interior of the unit disk to itself, the exterior to the exterior and the boundary to the boundary. Moreover, this transformation of the coordinates induces a mapping which intermixes the oscillator modes. To see this consider the definition of the oscillator modes

$$\alpha^\mu_m = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z) \tag{3.14}$$

where the contour is the boundary of the unit disk, and the mode expansion of X is

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_m \frac{\alpha^\mu_m}{z^{m+1}}. \tag{3.15}$$

Now, using the fact that X is a scalar, or equivalently the fact that ∂X is a (1,0) tensor, we see that

$$\alpha^\mu_m = \oint \frac{dz}{2\pi i} z^m \partial_w X^\mu(w) \frac{dw}{dz}. \tag{3.16}$$

Now, using the fact that a mode expansion for X exists in terms of w with coefficients α'_m in exactly the same way as (3.15), we see that

$$\alpha^\mu_m = M^{(a,b)}_{mn} \alpha'^\mu_n \tag{3.17}$$

where

$$M^{(a,b)}_{mn} = \oint \frac{dz}{2\pi i} z^m \frac{(\bar{b}z + \bar{a})^{n-1}}{(az+b)^{n+1}}. \tag{3.18}$$

The properties of the matrix M are interesting and facilitate further study. Some of the properties of $M^{(a,b)}_{mn}$ are examined in Appendix A.1. The matrix has a block diagonal form so that creation and annihilation operators are not mixed by the conformal transformation, and with appropriate normalization of the oscillator modes it can be seen to be Hermitian, or equivalently that it preserves the inner product on the space of operators. The exact form M as a function of its indices can be easily obtained, but for the purposes of this discussion it is easier to to simply note that with the rescaling $\mathcal{M}_{mp} = \sqrt{\frac{p}{m}} M_{mp}$ for either $m, p > 0$ or $m, p < 0$ then $\mathcal{M}^{-1}_{mp} = \mathcal{M}^\dagger_{mp}$. (The purpose of the rescaling is to normalize the creation and annihilation operators to have the standard simple harmonic oscillator commutation relations.)

Using this information we obtain that the modification of the boundary state associated with a particular conformal transformation is

$$|B_{a,b}\rangle = \mathcal{N}\exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^{\mu}_{-k} M^{(a,b)}_{-n-k} \Lambda^{n}_{\mu\nu} \bar{M}^{(a,b)}_{-n-j} \tilde{\alpha}^{\nu}_{-j}\right) \exp\left(-\frac{\alpha'}{4} x^{\mu} U_{\mu\nu} x^{\nu}\right)|0\rangle. \tag{3.19}$$

In this equation and all following equations we drop the $'$ associated with the transformed oscillators for notational simplicity. Due to the intuition gained from the conformally invariant cases we propose a boundary state

$$|B\rangle = \int d^2a d^2b \delta(|a^2| - |b^2| - 1)|B_{a,b}\rangle \tag{3.20}$$

which we will show is the state that reproduces the sigma model amplitudes. This is just the boundary state (3.19) integrated over the Haar measure of PSL(2,R).

3.2.2 Boundary State Single Particle Emission

Since we wish to show that the boundary state is an algebraic version of the action (3.3) we must calculate the emission probability for various particles from the boundary state above. This has been done in more detail in [70], (see also [31, 33]) but we recapitulate the results here for completeness.

The case of the tachyon is straightforward. To calculate the emission probability for this or any particle from the d-brane described by the boundary state we must evaluate the overlap of the Fock space ground state with the transformed boundary state (3.19). Here, and in subsequent formulae we omit the momentum conserving δ-functions, and the integration over the transformation parameters for the boundary state. For a tachyon with momentum p^{μ} we find that the probability for emission from the boundary state

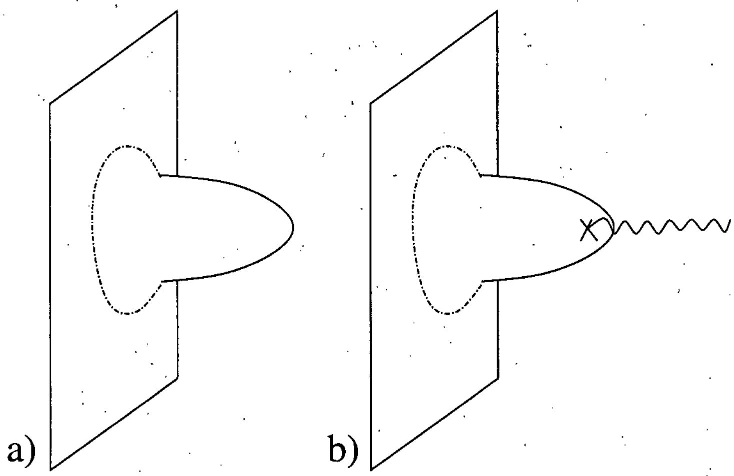

Figure 3.1: A schematic of the disk tadpole (a) and the emission of one particle by the boundary state (b).

is

$$\begin{aligned}\langle 0, p^\mu | B_{a,b}\rangle &= \mathcal{N}\exp\left(-p^\mu p^\nu \frac{\alpha'}{2}\sum_{n=1}^{\infty} M^{(a,b)}_{-n0}\Lambda^n_{\mu\nu}\bar{M}^{(a,b)}_{-n0}\right)\\ &= \mathcal{N}\exp\left(-p^\mu p^\nu \frac{\alpha'}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)_{\mu\nu}\frac{|b|^{2n}}{|a|^{2n}}\right).\end{aligned} \tag{3.21}$$

In the above expression we have used the previously defined form for $\Lambda^n_{\mu\nu}$, the fact that $M^{(a,b)}_{-n0} = \left(\frac{-\bar{b}}{\bar{a}}\right)^n$, and the conventional normalization $\alpha^\mu_o = \sqrt{\frac{\alpha'}{2}}p^\mu$.

Similarly, for an arbitrary massless state with polarization tensor $P_{\mu\nu}$ and momentum p^μ

$$|P_{\mu\nu}\rangle = P_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0, p^\mu\rangle \tag{3.22}$$

the overlap to be calculated is

$$\langle P_{\mu\nu}|B_{a,b}\rangle = \mathcal{N}\exp\left(-p_\mu p_\nu \frac{\alpha'}{2}\sum_{n=1}^{\infty} M^{(a,b)}_{-n0}\Lambda^n_{\mu\nu}\bar{M}^{(a,b)}_{-n0}\right)$$

$$P^{\mu\nu}\left[-\sum_{n=1}^{\infty} M^{(a,b)}_{-n-1}\Lambda^n_{\mu\nu}\bar{M}^{(a,b)}_{-n-1}\right.$$
$$\left.+p^\alpha p^\beta \frac{\alpha'}{2}\sum_{n=1}^{\infty} M^{(a,b)}_{-n-1}\Lambda^n_{\mu\alpha}\bar{M}^{(a,b)}_{-n0}\sum_{m=1}^{\infty} M^{(a,b)}_{-m0}\Lambda^m_{\beta\nu}\bar{M}^{(a,b)}_{-m-1}\right]$$
$$= \mathcal{N}\exp\left(-p^\mu p^\nu \frac{\alpha'}{2}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)_{\mu\nu}\frac{|b|^{2n}}{|a|^{2n}}\right)$$
$$P^{\mu\nu}\left[-\sum_{n=1}^{\infty} n\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\nu}\frac{|b|^{2(n-1)}}{|a|^{2(n-1)}}\frac{1}{|a^2|^2}\right.$$
$$+p^\alpha p^\beta\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{n}}\right)_{\mu\alpha}\frac{|b|^{2(n-1)}}{|a|^{2(n-1)}}\frac{-\bar{b}}{|a^2|\bar{a}}$$
$$\left.\times\sum_{m=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{m}}{g+2\pi\alpha' F+\frac{\alpha}{2}\frac{U}{m}}\right)_{\beta\nu}\frac{|b|^{2(m-1)}}{|a|^{2(m-1)}}\frac{-b}{|a^2|a}\right] \tag{3.23}$$

where again the explicit form of the matrices M has been used in the last equality.

This kind of argument can be repeated indefinitely on a state by state basis to determine the emission probability for that particular state, but we present here another more general calculation which will prove useful to consider. In particular the state A with momentum p^μ defined by

$$|A_{\mu\nu\delta}\rangle \;=\; A_{\mu\nu\delta}\alpha^\mu_{-a}\tilde{\alpha}^\nu_{-b}\tilde{\alpha}^\delta_{-c}|0,p^\mu\rangle, \tag{3.24}$$

has its overlap with the boundary state is given by

$$\langle A_{\mu\nu\delta}|B\rangle \;=\; \mathcal{N}\exp\left(-p_\mu p_\nu\frac{\alpha'}{2}\sum_{n=1}^{\infty}M^{(a,b)}_{-n0}\Lambda^n_{\mu\nu}\bar{M}^{(a,b)}_{-n0}\right)\times$$
$$A^{\mu\nu\delta}\sqrt{\frac{\alpha'}{2}}\left[\sum_n abM^{(a,b)}_{-n-a}\Lambda^n_{\mu\nu}\bar{M}^{(a,b)}_{-n-b}p^\alpha\sum_m cM^{(a,b)}_{-m0}\Lambda^m_{\alpha\delta}\bar{M}^{(a,b)}_{-m-c}\right.$$
$$+p^\alpha\sum_n acM^{(a,b)}_{-n-a}\Lambda^n_{\mu\delta}\bar{M}^{(a,b)}_{-n-c}\sum_m bM^{(a,b)}_{-m0}\Lambda^m_{\alpha\nu}\bar{M}^{(a,b)}_{-m-b}$$

$$-p^\alpha p^\beta p^\gamma \frac{\alpha'}{2} \sum_n a M^{(a,b)}_{-n-a} \Lambda^n_{\mu\alpha} \bar{M}^{(a,b)}_{-n0} \sum_m b M^{(a,b)}_{-m0} \Lambda^m_{\beta\nu} \bar{M}^{(a,b)}_{-m-b}$$

$$\times \sum_l c M^{(a,b)}_{-l0} \Lambda^l_{\gamma\delta} \bar{M}^{(a,b)}_{-l-c} \Bigg] . \tag{3.25}$$

The summation looks formidable, but we note that the contractions of the various matrices look suspiciously like those of Green's functions, which it will transpire that they are, but to see this requires a simple calculation. A special case of a more general formula proven in the next section shows that for $y = \frac{az+b}{\bar{b}z+\bar{a}}$ subject to $|a|^2 - |b|^2 = 1$ we have that

$$\frac{1}{(k-1)!} \partial^k z^d(y)|_{y=0} = k\bar{M}^{(a,b)}_{-d-k}. \tag{3.26}$$

Note that since the transformation from z to y is one-to-one the above equation makes sense and is appropriate for the mapping of a point to the origin. This completes the analysis for the emission of one particle from the boundary state $|B_{a,b}\rangle$, however the question becomes more interesting for the emission of more than one particle.

3.2.3 Boundary State Multiple Particle Emission

As in the case of emission of one particle by the boundary state it is perhaps the most instructive to consider the case of the emission of two tachyons first, and then specialize to more complicated correlators. Ordering the operators appropriately for radial (as opposed to anti-radial) quantization and noting that the PSL(2,R) transformation is not sufficient to fix the location of both closed string vertex operators. Therefore it is necessary to integrate over the position of the second vertex operator. The quantity that we will wish to compare with in the sigma model is the integration over insertion points of an arbitrary number of vertex operators, and in this language one, the 'bra' or 'ket' appearing in the overlap equations is singled out as being moved to the origin. We proceed to calculate, using the previous definitions and mode expansion

$$
\begin{aligned}
\langle B_{a,b}| : e^{(ik^\mu X_\mu)} : |_\omega |0, p^\mu\rangle \;=\;& \mathcal{N}\langle 0| \exp\left(-\sum \alpha_i^\mu M_{ni}^{(a,b)} \Lambda_{\mu\nu}^n \bar{M}_{nj}^{(a,b)} \tilde{\alpha}_j^\nu\right) \\
& \exp\left(k_\mu \sqrt{\frac{\alpha'}{2}} \sum_{l>0} \frac{1}{l}\left(\alpha_{-l}^\mu \omega^l + \tilde{\alpha}_{-l}^\mu \bar{\omega}^l\right)\right) \\
& \exp\left(ik^\mu x_\nu + \sqrt{\frac{\alpha'}{2}} k_\mu \alpha_0^\mu \ln|\omega|^2\right) |0, p^\mu\rangle \\
=\;& \mathcal{N} \exp\left(k^\mu p_\mu \frac{\alpha'}{2} \ln|\omega|^2\right) \\
& \exp\left(-p^\mu p^\nu \frac{\alpha'}{2} \sum_{n=1}^{\infty} M_{n0}^{(a,b)} \Lambda_{\mu\nu}^n \bar{M}_{n0}^{(a,b)}\right) \\
& \exp\left(-p^\mu k^\nu \frac{\alpha'}{2} \sum_{n=1,j=0}^{\infty} M_{n0}^{(a,b)} \Lambda_{\mu\nu}^n \bar{M}_{nj}^{(a,b)} \bar{\omega}^j\right) \\
& \exp\left(-k^\mu p^\nu \frac{\alpha'}{2} \sum_{n=1,i=0}^{\infty} \omega^i M_{ni}^{(a,b)} \Lambda_{\mu\nu}^n \bar{M}_{n0}^{(a,b)}\right) \\
& \exp\left(-p^\mu k^\nu \frac{\alpha'}{2} \sum_{n=1,i,j=0}^{\infty} \omega^i M_{ni}^{(a,b)} \Lambda_{\mu\nu}^n \bar{M}_{nj}^{(a,b)} \bar{\omega}^j\right).
\end{aligned}
\tag{3.27}
$$

Just as mentioned following (3.25), this result is reminiscent of a pair of exponentiated Green's functions.

The next natural quantity to calculate is the emission of a more general state in place of either, or both tachyons in the previous calculation. It is of course possible to demonstrate the overlap of an arbitrary string state explicitly, but the combinatorial nature of the result quickly renders the resulting expressions obscure. With this in mind we examine the slightly more general state that corresponds to the calculation done in the case of one particle emission (3.25).

$$\langle B_{a,b}| : e^{(ik^\mu X_\mu)} : |_\omega A_{\mu\nu\delta}\alpha_n^\mu\tilde{\alpha}_p^\nu\tilde{\alpha}_q^\delta|0,p^\mu\rangle = \mathcal{A}_{T2} \times A^{\mu\nu\delta}$$

$$\left[\sqrt{\frac{\alpha'}{2}}^3 \left(-\sum nM_{rn}^{(ab)}\Lambda_{\mu\gamma}^r\bar{M}_{rj}^{(ab)}\bar{\omega}^j k^\gamma - k_\mu\frac{1}{\omega^n} - \sum nM_{rn}^{(ab)}\Lambda_{\mu\gamma}^r\bar{M}_{r0}^{(ab)}p^\gamma\right)\right.$$
$$\left(-\sum k^\gamma\omega^i M_{ri}^{(ab)}\Lambda_{\gamma\nu}^r\bar{M}_{rp}^{(ab)}p - k_\nu\frac{1}{\bar{\omega}^p} - \sum p^\gamma M_{r0}^{(ab)}\Lambda_{\gamma\nu}^r\bar{M}_{rp}^{(ab)}p\right)$$
$$\left(-\sum k^\gamma\omega^i M_{ri}^{(ab)}\Lambda_{\gamma\delta}^r\bar{M}_{rq}^{(ab)}q - k_\nu\frac{1}{\bar{\omega}^q} - \sum p^\gamma M_{r0}^{(ab)}\Lambda_{\gamma\delta}^r\bar{M}_{rq}^{(ab)}q\right)$$
$$+\left(-\sum k^\gamma\omega^i M_{ri}^{(ab)}\Lambda_{\gamma\delta}^r\bar{M}_{rq}^{(ab)}q - k_\nu\frac{1}{\bar{\omega}^q} - \sum p^\gamma M_{r0}^{(ab)}\Lambda_{\gamma\delta}^r\bar{M}_{rq}^{(ab)}q\right)$$
$$\left.\left(-\sum nM_{rn}^{(ab)}\Lambda_{\mu\nu}^r\bar{M}_{rp}^{(ab)}p\right)\sqrt{\frac{\alpha'}{2}}\right] + (p \leftrightarrow q, \nu \leftrightarrow \delta)\,. \quad (3.28)$$

In the above $\mathcal{A}_{T2}$ is the result for the boundary state to emit two tachyons, which appears as a multiplicative factor and is calculated explicitly above (3.27).

Similarly it is possible to calculate the analogous expression for the vertex which emits the complicated state at the point ω on the disk, and using the standard commutation relationships as outlined previously we find

$$\langle B_{a,b}| : A_{\mu\nu\delta}\frac{\partial^n}{(n-1)!}X^\mu\frac{\bar{\partial}^p}{(p-1)!}X^\nu\frac{\bar{\partial}^q}{(q-1)!}X^\delta e^{(ik^\mu X_\mu)} : |_\omega|0,p^\mu\rangle = \mathcal{A}_{T2}A^{\mu\nu\delta}$$
$$\left[-\left(\sum\frac{1}{(n-1)!}\frac{1}{(p-1)!}\frac{m!}{(m-n)!}\omega^{m-n}M_{rm}^{(ab)}\Lambda_{\mu\nu}^r\bar{M}_{rj}^{(ab)}\frac{j!}{(j-p)!}\bar{\omega}^{j-p}\right)\right.$$
$$+\left\{\frac{\alpha'}{2}\left(-\sum\frac{1}{(n-1)!}\frac{m!}{(m-n)!}\omega^{m-n}M_{rm}^{(ab)}\Lambda_{\mu\gamma}^r\bar{M}_{rj}^{(ab)}\bar{\omega}^j k^\gamma\right.\right.$$
$$\left.-\sum\frac{1}{(n-1)!}\frac{m!}{(m-n)!}\omega^{m-n}M_{rm}^{(ab)}\Lambda_{\mu\gamma}^r\bar{M}_{r0}^{(ab)}p^\gamma + p_\mu(-1)^n\omega^{-n}\right)$$
$$\left(-\sum p^\gamma M_{r0}^{(ab)}\Lambda_{\gamma\nu}^r\bar{M}_{rj}^{(ab)}\frac{1}{(p-1)!}\frac{j!}{(j-p)!}\bar{\omega}^{j-p}\right.$$
$$\left.\left.\left.-\sum k^\gamma\omega^m M_{rm}^{(ab)}\Lambda_{\gamma\nu}^r\bar{M}_{rj}^{(ab)}\frac{1}{(p-1)!}\frac{j!}{(j-p)!}\bar{\omega}^{j-p} + p_\mu(-1)^p\bar{\omega}^{-p}\right)\right\}\right]$$

$$\times\left(-\sum p^\gamma M_{r0}^{(ab)}\Lambda^r_{\gamma\delta}\bar{M}_{rj}^{(ab)}\frac{1}{(q-1)!}\frac{j!}{(j-q)!}\bar{\omega}^{j-q}\right.$$
$$\left.-\sum k^\gamma\omega^m M_{rm}^{(ab)}\Lambda^r_{\gamma\delta}\bar{M}_{rj}^{(ab)}\frac{1}{(q-1)!}\frac{j!}{(j-q)!}\bar{\omega}^{j-q}+p_\mu(-1)^q\bar{\omega}^{-q}\right)$$
$$+\Big(p\leftrightarrow q,\nu\leftrightarrow\delta\Big) \tag{3.29}$$

The above expression can be seen to be the same as that of the emission with the complicated vertex at the center, as the case of two tachyon emission would suggest.

3.2.4 Bosonic Sigma Model

Having performed an the calculations from the point of view of the raising and lowering operators it is now instructive to compare with what should be analogous results from sigma model calculations. We fix our convention that the functional integral is in all cases the average over the action given in (3.3),

$$\langle\mathcal{O}(X)\rangle=\int\mathcal{D}Xe^{-S(X)}\mathcal{O}(X). \tag{3.30}$$

In addition, the Green's function on the unit disk with Neumann boundary conditions is determined to be [60]

$$G^{\mu\nu}(z,z')=-\alpha' g^{\mu\nu}\left(-\ln|z-z'|-\ln|1-z\bar{z}'|\right), \tag{3.31}$$

and it will be useful also to know the bulk to boundary propagator which is

$$G^{\mu\nu}(\rho e^{i\phi},e^{i\phi'})=2\alpha' g^{\mu\nu}\sum_{m=1}^{\infty}\frac{\rho^m}{m}\cos[m(\phi-\phi')]. \tag{3.32}$$

The boundary to boundary propagator can be read off from (3.32) as the limit in which $\rho\to 1$. We use $z=\rho e^{i\phi}$ as a parameterization of the points within the unit disk, so $0\le\rho\le 1$ and $0\le\phi<2\pi$. Using the bulk to boundary propagator it is possible to integrate out the quadratic interactions on the boundary [40] and to obtain an exact propagator, which is given by

$$\begin{aligned}
G^{\mu\nu}(z,z') &= -\alpha' g^{\mu\nu}\ln|z-z'| \\
&+\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\{\mu\nu\}}\frac{(z\bar{z}')^n+(\bar{z}z')^n}{n} \\
&+\alpha'\sum_{n=1}^{\infty}\left(\frac{2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{[\mu\nu]}\frac{(z\bar{z}')^n-(\bar{z}z')^n}{in} \\
&= -\alpha' g^{\mu\nu}\ln|z-z'| \\
&+\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\{\mu\nu\}}\frac{(z\bar{z}')^n+(\bar{z}z')^n}{n} \\
&+\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{[\mu\nu]}\frac{(z\bar{z}')^n-(\bar{z}z')^n}{in}. \qquad (3.33)
\end{aligned}$$

Note that this expression is appropriately symmetric because the antisymmetry of Lorentz indices in the final term is compensated by the antisymmetry of the coordinate term.

The first calculation that must be done to determine the normalization of the sigma model amplitudes is the partition function. In this approach the oscillator modes of X must be integrated out with the contributions from F and U treated as perturbations. Since both perturbations are quadratic, all the Feynman graphs that contribute to the free energy can be written and evaluated, and explicitly the free energy is given by

$$\mathcal{F} = -\sum_{m=1}^{\infty} Tr\ln\left(g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{m}\right). \qquad (3.34)$$

See [40, 71] for further calculations done in this spirit. From (3.34) we immediately obtain the partition function

$$Z = e^{-T_0}\prod_{m=1}^{\infty}\frac{1}{\det\left(g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{m}\right)}\int dx_0 e^{-\frac{U_{\mu\nu}}{4}x_0^\mu x_0^\nu}$$

$$= \frac{1}{\det\left(\frac{U}{2}\right)} e^{-T_0} \prod_{m=1}^{\infty} \frac{1}{\det\left(g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{m}\right)}. \tag{3.35}$$

This expression is divergent, but using ζ-function regularization [66] it can be reduced to

$$Z = e^{-T_0} \sqrt{\det\left(\frac{g + 2\pi\alpha' F}{U/2}\right)} \det\Gamma\left(1 + \frac{\alpha' U/2}{g + 2\pi\alpha' F}\right), \tag{3.36}$$

where $\Gamma(g)$ is the Γ function and the dependence of all transcendental functions on the matrices U and F is defined through a Taylor expansion.

3.2.5 Conformal Transformation in the Sigma Model

We now wish to calculate the expectation value for vertex operators that correspond to different closed string states, however this is a process that must be done with some care. To calculate the emission of a closed string in the world-sheet picture one generally considers a disk emitting an asymptotic closed string state. This is really a closed string cylinder diagram. The standard method is to use conformal invariance to map the closed string state to a point on the disk, namely the origin, where a corresponding vertex operator is inserted. On the other hand it has been cogently argued that it is necessary to have an integrated vertex operator for closed string states to properly couple [28], in particular that the graviton must be produced by an integrated vertex operator to couple correctly to the energy momentum tensor. There is no distinction between a fixed vertex operator and an integrated vertex operator in the conformally invariant case because the integration will only produce a trivial volume factor, however in the case we consider more care must be taken. We wish to consider arbitrary locations of the vertex operators on the string world sheet, and the natural measure to impose is that of the conformal transformations which map the origin to a point within the unit disk on the complex plane.

In other words we propose to allow the vertex operator corresponding to the closed string state to be moved from the origin by a conformal transformation that preserves the area of the unit disk, namely a PSL(2,R) transformation. The method to accomplish this is to go to a new coordinate system

$$y = \frac{az+b}{\bar{b}z+\bar{a}}, \quad |a^2| - |b^2| = 1, \tag{3.37}$$

where a vertex operator at the origin $y = 0$ would correspond to an insertion of a vertex operator at the point $z = \frac{-b}{a}$. It is worth noting that in the case of conformal invariance, that is when $U \to 0$ or $U \to \infty$ the Green's function remains unchanged in form, the y dependence coming from the replacement $z \to z(y)$. Even in the case of finite U the only change to the Green's function is the addition of a term that is harmonic within the unit disk. The parameter of the integration over the position of the vertex operator would be to the measure on PSL(2,R), giving an infinite factor in the conformally invariant case [28, 75, 97]. From this argument we have a definite prescription for the calculation of vertex operator expectation values, which is to use the conformal transformation to modify the Green's function, and calculate the expectation values of operators at the origin with this modified Green's function.

3.2.6 Sigma Model Single Particle Emission

Now we will use this prescription to calculate the sigma model expectation values of some operators, and we will start with the simplest, that of the closed string tachyon. The vertex operator for the tachyon is $: e^{ip_\mu X^\mu(z(y))} :$, and it is inserted at the point $y = 0$. The normal ordering prescription for all such operators is that any divergent pieces will be subtracted, but finite pieces will remain and by inspection we see that the appropriate subtraction from the Green's function is

$$: \mathcal{G}^{\mu\nu}(z, z') : \; = \; G^{\mu\nu}(z, z') - g^{\mu\nu} \alpha' \ln |z - z'| \tag{3.38}$$

Using (3.38) we see that

$$\begin{aligned}\langle : e^{ip_\mu X^\mu(y=0)} :\rangle &= Ze^{-\frac{1}{2}p_\mu p_\nu :\mathcal{G}^{\mu\nu}(z(y),z'(y)):}\Big|_{y=0} \\ &= Z\exp\left(-\frac{\alpha'}{2}p_\mu p_\nu \sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}\frac{1}{n}\frac{|b^{2n}|}{|a^{2n}|}\right).\end{aligned} \tag{3.39}$$

We recall that our procedure will necessitate an integration over the the parameters of the PSL(2,R) transformation, but comparison with (3.21) reveals that the normalization is fixed by

$$\mathcal{N} = Z. \tag{3.40}$$

Having obtained this result fixing the normalization it is natural to check the expectation value for other vertex operators to see if the relation persists. We perform a similar analysis for the massless closed string excitations. In particular the graviton insertion at $y = 0$ is given by

$$\langle \mathcal{V}_h \rangle = \langle : -\frac{2}{\alpha'}h_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ip_\mu X^\mu(y=0)} :\rangle \tag{3.41}$$

where h is a symmetric traceless tensor and the normalization follows the conventions of [78]. This can be analyzed by the same techniques as for the tachyon, noting that there will be cross contractions between the exponential and the X-field prefactors. Explicitly we obtain

$$\begin{aligned}\langle \mathcal{V}_h \rangle &= -\frac{2}{\alpha'}Zh_{\mu\nu}\Big(\partial\bar{\partial}' : \mathcal{G}^{\mu\nu}\left(z(y),z'(y)\right) : + \partial : \mathcal{G}^{\mu\alpha}\left(z(y),z'(y)\right) : \\ &\quad \times\bar{\partial} : \mathcal{G}^{\mu\beta}\left(z(y),z'(y)\right) : (ip_\alpha)(ip_\beta)\Big)\, e^{-\frac{1}{2}p_\mu p_\nu :\mathcal{G}^{\mu\nu}(z(y),z'(y)):}\Big|_{y=0} \\ &= Zh_{\mu\nu}\left(-\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu} n\frac{|b^{2(n-1)}|}{|a^{2(n-1)}|}\frac{1}{|a^2|^2}\right. \\ &\quad + \frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\alpha}\frac{|b^{2(n-1)}|}{|a^{2(n-1)}|}\frac{-b}{|a^2|a}\end{aligned}$$

$$\times \sum_{m=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{m}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{m}} \right)^{\nu\beta} \frac{|b^{2(m-1)}|}{|a^{2(m-1)}|} \frac{-\bar{b}}{|a^2|\bar{a}} p_\alpha p_\beta \Bigg)$$

$$\exp\left(-\frac{\alpha'}{2} p_\mu p_\nu \sum_{n=1}^{\infty} \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}} \right)^{\mu\nu} \frac{1}{n}\frac{|b^{2n}|}{|a^{2n}|} \right). \quad (3.42)$$

and by comparing (3.23) and (3.42) we see that the relation $\mathcal{N} = Z$ holds and that the form of the two expectation values is identical in detail.

Finally, we can perform the same kind of calculation for a more general closed string state, like the one considered in (3.25). We consider a state which may be off shell in the sense that it not annihilated by the positive modes of the σ-model energy momentum tensor (the Virasoro generators), may not satisfy the mass shell condition, and may not be level matched. Our explicit choice is to consider the operator

$$\langle \mathcal{V}_A \rangle \;=\; \langle : -i \left(\frac{2}{\alpha'}\right)^{3/2} A_{\mu\nu\delta} \frac{\partial^a}{(a-1)!} X^\mu \frac{\bar{\partial}^b}{(b-1)!} X^\nu \frac{\bar{\partial}^c}{(c-1)!} X^\gamma e^{ip_\mu X^\mu} :\rangle \quad (3.43)$$

which is an arbitrary state involving three creation operators. We find that

$$\begin{aligned}
\langle \mathcal{V}_A \rangle \;=\; & Z A_{\mu\nu\delta} \left(\frac{2}{\alpha'}\right)^{3/2} \Bigg(\frac{\partial^a}{(a-1)!} \frac{\bar{\partial}'^b}{(b-1)!} : G^{\mu\nu}(z,z') : \\
& \qquad \frac{\bar{\partial}^c}{(c-1)!} : G^{\delta\alpha}(z,z') : p_\alpha \\
& + \frac{\partial^a}{(a-1)!} \frac{\bar{\partial}'^c}{(c-1)!} : G^{\mu\delta}(z,z') : \frac{\bar{\partial}^b}{(b-1)!} : G^{\nu\alpha}(z,z') : p_\alpha \\
& - \frac{\partial^a}{(a-1)!} : G^{\mu\alpha}(z,z') : \frac{\bar{\partial}^b}{(b-1)!} : G^{\nu\beta}(z,z') : \\
& \qquad \frac{\bar{\partial}^c}{(c-1)!} : G^{\delta\gamma}(z,z') : p_\alpha p_\beta p_\gamma \Bigg) \\
& \times e^{-\frac{1}{2} p_\mu p_\nu : G^{\mu\nu}(z,z') :} \Big|_{y=0} .
\end{aligned} \quad (3.44)$$

Comparing (3.44) with (3.25) and (3.26) we observe that the two coincide.

3.2.7 Sigma Model Multiple Particle Emission

In the previous section we demonstrated that the sigma model calculation of the particle emission coincides with that calculated using the boundary state, so we now look at the emission of two particles. We expect that the two particle emission amplitude will depend upon the relative position of the two vertex operators since, even in the conformally invariant case, there are not enough free parameters to fix two closed string vertex operators on the disk world sheet. We first calculate the expectation value of the emission of two tachyons, with momenta p and k.

$$\begin{aligned}
\langle : e^{ik_\mu X^\mu} : \Big|_\omega : e^{ip_\nu X^\nu} : \Big|_0 \rangle &= Z \exp\left(-\frac{k_\mu k_\nu}{2} G^{\mu\nu}\left(z(\omega), z(\omega)\right)\right) \\
&\quad \times \exp\left(-\frac{p_\mu p_\nu}{2} G^{\mu\nu}\left(z(0), z(0)\right)\right) \\
&\quad \times \exp\left(-\frac{k_\mu p_\nu}{2} G^{\mu\nu}\left(z(\omega), z(0)\right)\right) \\
&= \mathcal{A}_{T2\sigma}
\end{aligned} \tag{3.45}$$

This is the necessary first step in determining a more arbitrary amplitude. To make contact with the more complicate amplitudes calculated in (3.28) and (3.29) we consider the expression

$$\begin{aligned}
&\langle A_{\mu\nu\delta} : \frac{\partial^n}{(n-1)!} X^\mu \frac{\bar\partial^p}{(p-1)!} X^\nu \frac{\bar\partial^q}{(q-1)!} X^\delta e^{ik_\mu X^\mu} : \Big|_\omega : e^{ip_\mu X^\mu} : \Big|_0 \rangle \\
&= \mathcal{A}_{T2\sigma} A_{\mu\nu\delta} \left[\left(\frac{ik_\alpha}{(n-1)!} \partial^n G^{\mu\alpha}(z(\omega), z'(\omega)) + \frac{ip_\alpha}{(n-1)!} \partial^n G^{\mu\alpha}(z(\omega), z'(0)) \right) \right. \\
&\qquad \times \left(\frac{ik_\beta}{(p-1)!} \bar\partial^p G^{\nu\beta}(z(\omega), z'(\omega)) + \frac{ip_\beta}{(p-1)!} \bar\partial^p G^{\nu\beta}(z(\omega), z'(0)) \right) \\
&\qquad \times \left(\frac{ik_\gamma}{(q-1)!} \bar\partial^q G^{\delta\gamma}(z(\omega), z'(\omega)) + \frac{ip_\gamma}{(q-1)!} \bar\partial^q G^{\delta\gamma}(z(\omega), z'(0)) \right) \\
&\qquad\qquad + \frac{\partial^n}{(n-1)!} \frac{\bar\partial'^p}{(p-1)!} G^{\mu\nu}(z(\omega), z'(\omega)) \\
&\qquad \times \left(\frac{ik_\gamma}{(q-1)!} \bar\partial^q G^{\delta\gamma}(z(\omega), z'(\omega)) + \frac{ip_\gamma}{(q-1)!} \bar\partial^q G^{\delta\gamma}(z(\omega), z'(0)) \right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial^n}{(n-1)!}\frac{\bar{\partial}'^q}{(q-1)!}G^{\mu\delta}(z(\omega),z'(\omega)) \\
&\times\left(\frac{ik_\beta}{(p-1)!}\bar{\partial}^p G^{\nu\beta}(z(\omega),z'(\omega))+\frac{ip_\beta}{(p-1)!}\bar{\partial}^p G^{\nu\beta}(z(\omega),z'(0))\right)\Bigg]
\end{aligned}
\tag{3.46}
$$

Also note that if we consider

$$
\langle : e^{ik_\mu X^\mu} : \Big|_\omega A_{\mu\nu\delta} : \frac{\partial^n}{(n-1)!}X^\mu \frac{\bar{\partial}^p}{(p-1)!}X^\nu \frac{\bar{\partial}^q}{(q-1)!}e^{ip_\mu X^\mu} : \Big|_0 \rangle
$$

we see that it gives the above expression (3.46) with $\omega \leftrightarrow 0$.

To demonstrate the general equivalence of the boundary state approach with that of the sigma model the sums that appear in the general expressions of boundary state matrix elements must be shown to coincide with the expressions that appear above. To this end consider first the sum that appears in (3.27),

$$
\begin{aligned}
\sum_{m=0}^{\infty}\omega^m M_{nm}^{(a,b)} &= \sum_{m=0}^{\infty}\oint\frac{dz}{2\pi i}\omega^m z^n\frac{(\bar{b}z+\bar{a})^{m-1}}{(az+b)^{m+1}} \\
&= \oint\frac{dz}{2\pi i}z^n\frac{1}{(\bar{b}z+\bar{a})(a-\omega\bar{b})}\left(z-\frac{\bar{a}\omega-b}{-\bar{b}\omega+a}\right)^{-1} \\
&= \left(\frac{\bar{a}\omega-b}{-\bar{b}\omega+a}\right)^c.
\end{aligned}
\tag{3.47}
$$

This derivation uses the normalization condition on a and b, and can be seen to be equal to $z^n(y)$ which is the inverse transform of (3.12).

The other sum that appears generally in this analysis is

$$
\sum_{m=0}^{\infty}\frac{m!}{(m-n)!}\omega^{m-n}M_{rm}^{(ab)}
$$

as seen in (3.29). In the case $n > m$ we have used the shorthand

$$
\frac{m!}{(m-n)!} = m(m-1)\ldots(m-n+1).
$$

Now consider

$$\begin{aligned}
\sum_{m=0}^{\infty}\frac{m!}{(m-n)!}\omega^{m-n}M_{rm}^{(ab)} &= \sum_{m=0}^{\infty}\oint\frac{dz}{2\pi i}\frac{m!}{(m-n)!}\omega^{m-n}z^r\frac{(\bar{b}z+\bar{a})^{m-1}}{(az+b)^{m+1}} \\
&= n!\oint\frac{dz}{2\pi i}z^r\frac{(\bar{b}z+\bar{a})^{n-1}}{(a-\bar{b}\omega)^{n+1}}\left(z-\frac{\bar{a}\omega-b}{-\bar{b}\omega+a}\right)^{-n-1} \\
&= \partial^n z^r(\bar{b}z+\bar{a})^{n-1}(a-\bar{b}\omega)^{-(n+1)}\Big|_{z=\frac{\bar{a}\omega-b}{-\bar{b}\omega+a}} \\
&= \partial^n\left(\frac{\bar{a}z-b}{-\bar{b}z+a}\right)^r\Bigg|_{z=\omega}. \qquad (3.48)
\end{aligned}$$

The last equality in this can be shown by induction, the case for $n=1$ is trivial, and so we demonstrate the induction. First note that use of the Leibnitz rule gives

$$\partial^k z^n(\bar{b}z+\bar{a})^{k-1}(a-\bar{b}\omega)^{-(k+1)}\Big|_{z=\frac{\bar{a}\omega-b}{-\bar{b}\omega+a}} =$$

$$\frac{(\bar{a}\omega-b)^{n-k}}{(-\bar{b}\omega+a)^{n+k}}\sum_{j=0}^{k}\binom{k}{j}\frac{n!}{(n-(k-j))!}\frac{(k-1)!}{(k-j-1)!}\bar{b}^j(\bar{a}\omega-b)^j. \qquad (3.49)$$

Now consider for $k=k_0+1$ the expression can be manipulated

$$\begin{aligned}
\partial^k\left(\frac{\bar{a}z-b}{-\bar{b}z+a}\right)^n\Bigg|_{z=\omega} &= \partial\left(\partial^{k_0}\left(\frac{\bar{a}z-b}{-\bar{b}z+a}\right)^n\right) \\
&= \partial\frac{(\bar{a}z-b)^{n-k_0}}{(-\bar{b}z+a)^{n+k_0}}\sum_{j=0}^{k_0}\binom{k_0}{j}\frac{n!}{(n-(k_0-j))!} \\
&\qquad \frac{(k_0-1)!}{(k_0-j-1)!}\bar{b}^j(\bar{a}z-b)^j \\
&= \frac{(\bar{a}z-b)^{n-(k_0+1)}}{(-\bar{b}z+a)^{n+(k_0+1)}}\sum_{j=0}^{k_0}\binom{k_0}{j}\frac{n!}{(n-(k_0-j))!} \\
&\qquad \times\frac{(k_0-1)!}{(k_0-j-1)!}\bar{b}^j(\bar{a}z-b)^j \\
&\qquad \times\left((n+j-k_0)+(2k_0-j)\bar{b}(\bar{a}z-b)\right)
\end{aligned}$$

$$= \frac{(\bar{a}z - b)^{n-(k_0+1)}}{(-\bar{b}z + a)^{n+(k_0+1)}} \sum_{j=0}^{k_0+1} \binom{k_0+1}{j} \frac{n!}{(n-(k_0+1-j))!} \frac{k_0!}{(k_0-j)!} \bar{b}^j (\bar{a}z - b)^j \quad (3.50)$$

as desired. This demonstrates the induction step, and the validity of (3.48). Note that similar results can be obtained for expressions with negative indices on $M_{nk}^{(a,b)}$ and negative powers of ω. These are obtained from considering the boundary state on the right of the matrix elements. This have to be interpreted as a dual description of the boundary states presented. This is because radial quantization and the operator state correspondence imply that in this case the domain of interest is the complex plane with the unit disk excluded. This is equally a fundamental region of the plane, and the conformal transformation between the two is $\omega \to \frac{1}{\bar{\omega}}$, a fact which is intimated at by the fact that (for $n, k > 0$) $M_{-n-k}^{(a,b)} = \bar{M}_{nk}^{(a,b)}$.

Now we have demonstrated that the results obtained from the boundary state calculations exactly match those of the sigma model after the propagator including the boundary perturbations has been obtained, and the resulting expression has been transformed into a new coordinate system. This shows that the boundary state renders all matrix elements that would otherwise be calculated in the sigma model obtainable by algebraic manipulations. This observation will be important as we generalize these results to higher genus surfaces. We also remark that the result explicitly presented for the emission of two closed string states clearly generalizes to the emission of an arbitrary number of such particles. Mechanically this can be seen because the commutation of two such vertex operators to produce a normal-ordered expression produces the familiar logarithmic term, and the boundary state gives the F and U dependence within the inner product.

3.2.8 Bosonic Boundary State Summary

In the preceding sections we have developed the bosonic boundary state. It is a coherent state involving the holomorphic and antiholomorphic creation operators which satisfies the boundary conditions associated with the boundary conditions (3.8) and (3.9), and is given by (3.20), the content of which we repeat for convenience.

$$|B\rangle = \int d^2a d^2b \delta(|a^2| - |b^2| - 1)|B_{a,b}\rangle$$

with

$$\begin{aligned} |B_{a,b}\rangle &= Z \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^{\mu}_{-k} M^{(a,b)}_{-n-k} \Lambda^{n}_{\mu\nu} \bar{M}^{(a,b)}_{-n-j} \tilde{\alpha}^{\nu}_{-j} \right) \\ &\exp\left(-\frac{\alpha'}{4} x^{\mu} U_{\mu\nu} x^{\nu} \right) |0\rangle. \end{aligned}$$

We have shown that this state gives the overlap with an arbitrary number of closed string states in the sense that it reproduces the string sigma model calculation of those same amplitudes. The reason for the integration over the PSL(2,R) group is that in the operator state correspondence (a pedagogical overview of which is given in [78]) the external state $\langle a|$ at the end of the overlap $\langle a|B\rangle$ is defined by a limiting process which takes it to infinite world-sheet time, thereby fixing it at the origin. Since the object to which this must be compared is an amplitude with integration over the positions of the inserted vertex operators it is necessary to mimic this with an integration over PSL(2,R).

It is also useful to note that the construction parallels that of [61], which has as a boundary condition that the two dimensional conformal symmetry is not broken, and this can be stated as

$$\left(L_n - \tilde{L}_{-n}\right)|B\rangle = 0 \tag{3.51}$$

where the Ls are the Virasoro generators. For oscillator number $n = 0$ this is nothing but the level matching condition. Our results show that the level matching condition is not satisfied without the integration over the conformal group. This can be seen in (3.25) because there was no condition on the indices a, b, c. The properties of the conformal transformation matrices $M_{mn}^{(a,b)}$ are such that upon integration over PSL(2,R) the level matching condition is enforced. This is because the matrices depend upon the phases of b and a, and if the numbers bs and $\bar{b}$s are not matched in a particular overlap an integral of the form $\int d\phi e^{in\phi}$ results and vanishes. Upon integration at each level (3.51) is satisfied, as can be seen using the properties derived in Appendix A for the matrices $M_{mn}^{(a,b)}$.

3.3 Bosonic Amplitudes in the Euler Number Expansion

Since the overlap of the boundary state with either single or multiple particle states has been shown to coincide with that calculated in the sigma model, we have the tools that are needed to proceed and determine higher order contributions in the string loop sense to the vacuum energy of the object described by the boundary state. We will proceed by utilizing a sewing construction to relate higher order amplitudes to products of lower order amplitudes. The procedure outlined is envisioned to produce an arbitrary number of interactions with the boundary state at the oriented tree level, and an arbitrary number of handles and interactions with the boundary state in the unoriented sector. As is well known, the description of higher genus orientable surfaces is a more difficult subject and the construction will produce results that are implicit rather than explicit. The final result, excluding terms with-

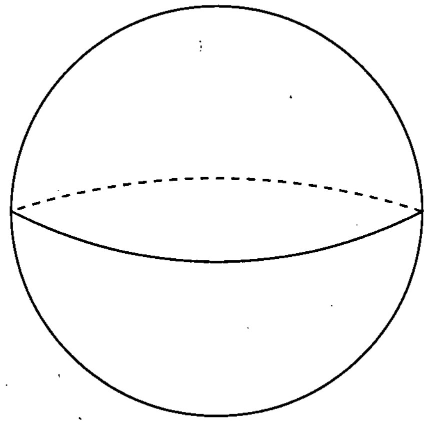

Figure 3.2: The sphere presented schematically. The sphere's contribution to the partition function is not included because it has no boundary.

out boundary, will be several terms in the Euler number expansion so that

$$\mathcal{Z} = Z_{disk} + Z_{ann} + Z_{MS} + \dots \tag{3.52}$$

where Z_{ann} refers to the annulus partition function, Z_{MS} refers to that of the Mobius strip, and each term carries the appropriate power of the open string coupling constant. The results in this section will be organized by Euler number, and where appropriate compared with other similar results in the literature.

3.3.1 $\chi = 1$

There are two surfaces with $\chi = 1$, the disk and RP^2. The non-orientable surface RP^2, see [62] for details in a similar context, has no interaction with the fields F and U and so is not of interest for this analysis. The disk by contrast has been analyzed previously in this work and the contribution to the partition function for the boundary state is given by its overlap with the

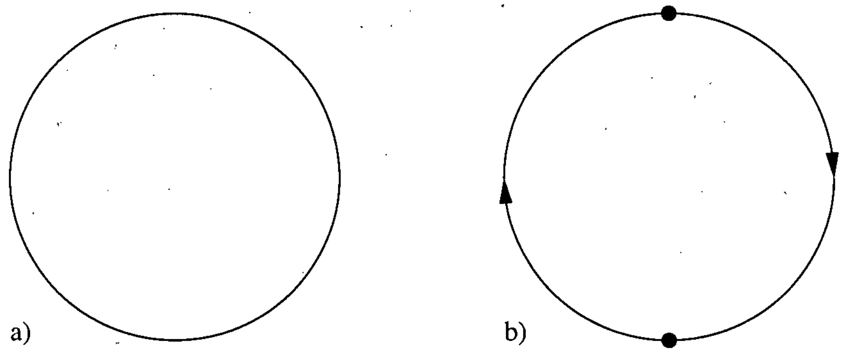

Figure 3.3: A drawing of the two surfaces with $\chi = 1$. Both the disk (a) and the surface RP^2 (b) are shown.

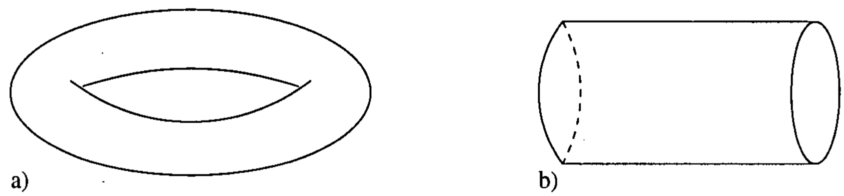

Figure 3.4: The two orientable surfaces with $\chi = 0$: the annulus (a) and the annulus (b). The annulus is shown in a manner that reminds its role as a closed string propagator.

unit operator (equivalently the tachyon with zero momentum), as given in (3.35). Both the disk and RP^2 are illustrated in figure 3.3.

3.3.2 $\chi = 0$

There are several surfaces that have an Euler number of 0. The easiest to discuss in this is the torus, which is immaterial for the same reason that RP^2 was among the surfaces with $\chi = 1$, namely that it has no interactions with F or U. Similarly the Klein bottle, the unoriented equivalent of the torus,

a) b)

Figure 3.5: The two non-orientable surfaces with $\chi = 0$. The Klein bottle (a) has no boundary, while the Mobius strip (b) has both a cross-cap and a boundary.

will not contribute to the partition function. We are left with the annulus and with the Mobius strip as the nontrivial contributions at this level. The annulus can be thought of as the tree level closed string exchange channel. The Mobius strip is the non-orientable analogue of the disk.

We consider first the annulus that was analyzed in detail in [70], we recapitulate some of the salient results. Suppressing for brevity the integrations over the parameters of the conformal transformations we have that

$$Z_{ann} = \langle B_{a,b}|\frac{1}{\Delta}|B_{a',b'}\rangle. \tag{3.53}$$

Using the integral representation of the closed string propagator

$$\frac{1}{\Delta} = \frac{1}{4\pi}\int \frac{d^2z}{|z|^2} z^{L_0-1}\bar{z}^{L_0-1}$$

and suppressing the z integrals we obtain

$$\begin{aligned}
Z_{ann} &= Z_{disk}^2\langle 0|\exp\left(-\alpha_i^\mu M_{ni}^{(a,b)}\Lambda_{\mu\nu}^n \bar{M}_{nj}^{(a,b)}\tilde{\alpha}_j^\nu\right) z^{\sum\alpha_{-n}\alpha_n}\bar{z}^{\sum\tilde{\alpha}_{-n}\tilde{\alpha}_n} \\
&\quad \exp\left(-\alpha_{-k}^\gamma M_{-m-k}^{(a',b')}\Lambda_{\gamma\delta}^m \bar{M}_{-m-l}^{(a',b')}\tilde{\alpha}_{-l}^\delta\right)|0\rangle \\
&= Z_{disk}^2\langle 0|\exp\left(-\alpha_i^\mu M_{ni}^{(a,b)}\Lambda_{\mu\nu}^n \bar{M}_{nj}^{(a,b)}\tilde{\alpha}_j^\nu\right) \\
&\quad \exp\left(-z^k\alpha_{-k}^\gamma M_{-m-k}^{(a',b')}\Lambda_{\gamma\delta}^m \bar{M}_{-m-l}^{(a',b')}\tilde{\alpha}_{-l}^\delta \bar{z}^l\right)|0\rangle \\
&= Z_{disk}^2 F(p)\exp\left(\sum_{k=1}^\infty \frac{1}{k} g^{\mu\nu}\delta_{rs}\left\{\left[r M_{nr}^{(a,b)}\Lambda_{\mu\alpha}^n \bar{M}_{nj}^{(a,b)} j\bar{z}^j g^{\alpha\delta}\right.\right.\right.
\end{aligned}$$

$$\bar{M}^{(a',b')}_{-m-j}\Lambda^{m}_{\nu\delta}M^{(a',b')}_{-m-s}z^{s}\Big]^{k}\Big\}^{rs}_{\mu\nu}\Bigg) \qquad (3.54)$$

Verifying the first equality requires the use of the Baker-Hausdorff formula for commutators of exponentials, and the second equality is an application of Wick's theorem. The term in the last exponential is understood to have its powers defined with contraction of both the Lorentz and oscillator indices. The number $F(p)$ is a Gaussian factor dependent on the (otherwise implicit) momentum of each boundary state, which can be read off from the boundary conditions (3.9). Explicitly the form of $F(p)$ is given by

$$\begin{aligned} F(p) &= \exp\Big\{p^{\mu}p^{\nu}\Big[\Big(\delta_{0j}g_{\mu\delta} - M^{(a,b)}_{n0}\Lambda^{n}_{\mu\delta}\bar{M}^{(a,b)}_{nj}\Big) \\ &\quad \left(\frac{1}{\delta_{jk}g_{\delta\gamma} - j\bar{z}^{j}\bar{M}^{(a',b')}_{-m-j}\Lambda^{m}_{\eta\delta}M^{(a',b')}_{-m-l}g^{\eta\zeta}z^{l}lM^{(a,b)}_{rl}\Lambda^{r}_{\zeta\gamma}\bar{M}^{(a,b)}_{rk}}\right)^{\delta\gamma}_{jk} \\ &\quad \Big(\delta_{k0}g_{\gamma\nu} - k\bar{z}^{k}\bar{M}^{(a',b')}_{sk}\Lambda^{s}_{\nu\gamma}M^{(a',b')}_{s0}\Big) - g_{\mu\nu}\Big]\Big\}. \end{aligned} \qquad (3.55)$$

In addition this is multiplied by terms coming from the zero mode part of the propagator. In the preceding equations the oscillator index has been chosen as positive or zero to make the negative signs meaningful. In all cases, repeated indices indicate summation.

Equation 3.54 is a concrete realization of the proposal of [28] for the calculation of loop corrections to the tachyon action. This proposal calculates the tree level couplings to closed strings for off-shell boundary interactions and shows that the correct procedure is to use integrated vertex operators to calculate these couplings. It further argues that otherwise the vertex operator does not couple correctly to the background fields, for instance in the case of the graviton a non-integrated vertex operator does not couple to the standard energy momentum tensor. By demanding closed string factorization of the one loop amplitudes [28] determine that the partition function for the string amplitude with two boundaries is

$$Z(S_{bdy}) = \sum_I \int dp Z(V_I(p), S_{bdy}) \frac{f(p^2+m_I^2)^2}{p^2+m_I^2} Z(V_I(-p), S_{bdy}). \qquad (3.56)$$

In (3.56) $V_I(p)$ is a vertex operator for particle I with momentum p and $f(p^2+m_I^2)$ is a function which goes to 1 when the exchanged particle is on mass-shell, producing the expected poles in the particle exchange, and

$$Z(V_I(p), S_{bdy}) = \int dX e^{-S} V_I(p). \qquad (3.57)$$

For the quadratic tachyon background we have shown that $\langle V_I|B\rangle$ gives $Z(V_I(p), S_{bdy})$ so the result (3.54) completes the summation over I in (3.56).

Considering (3.54) we note that the cases of $U \to 0$ and $U \to \infty$ give a particularly simple form for the matrices $M\Lambda\bar{M}$. We have

$$\begin{aligned} M^{(a,b)}_{km}\Lambda^k_{\mu\nu}\bar{M}^{(a,b)}_{kn}\Big|_{U\to 0} &= M^{(a,b)}_{km}\frac{1}{k}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)_{\mu\nu}\bar{M}^{(a,b)}_{kn} \\ &= \left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)_{\mu\nu}\frac{1}{m}\delta_{mn}, \end{aligned} \qquad (3.58)$$

and similarly

$$M^{(a,b)}_{km}\Lambda^k_{\mu\nu}\bar{M}^{(a,b)}_{kn}\Big|_{U\to\infty} = -g_{\mu\nu}\frac{1}{m}\delta_{mn}. \qquad (3.59)$$

These results can be obtained by explicit contour integration using the definition of M and are derived in Appendix A. We can see that the $U=0$ case gives the boundary state of a background gauge field [73] and when $U=\infty$ a localized object appears. In fact this parameter U interpolates between Neumann and Dirichlet boundary conditions [66].

It is worthwhile to check the result obtained in (3.54) in the known case where only the field F is present. Then the boundary conditions enforce that $p=0$, and with the above simplification we find

$$Z_{ann}(F) \;=\; \exp\left(\sum_{k=1}^{\infty}\frac{1}{k}g^{\mu\nu}\delta_{rs}\left\{\left[r\delta_{rj}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)_{\mu\alpha}\frac{1}{j}g^{\alpha\delta}\right.\right.\right.$$

$$
\begin{aligned}
& j\bar{z}^j \delta_{js} \left(\frac{g - 2\pi\alpha' F}{g + 2\pi\alpha' F} \right)_{\delta\nu} z^s \Big]^k \Big\}^{rs}_{\mu\nu} \Bigg) Z^2_{disk} F(0) \\
= & \exp \Bigg(-\sum_{r=0}^{\infty} Tr \ln \left(g - \frac{1+|z|^{2r}}{1-|z|^{2r}} 4\pi\alpha' F + 4\pi^2\alpha'^2 F^2 \right) \\
& - \sum_{r=0}^{\infty} Tr \ln \left(g(1-|z|^{2r}) \right) \\
& - \frac{1}{2} Tr \ln \left(g + 4\pi\alpha' F + 4\pi^2 \alpha'^2 F^2 \right) \Bigg) Z^2_{disk} \\
= & \prod_{r=1}^{\infty} (1-|z|^{2r})^{-D} \prod_{r=1}^{\infty} \det \left(g - \frac{1+|z|^{2r}}{1-|z|^{2r}} 4\pi\alpha' F + 4\pi^2\alpha'^2 F^2 \right)^{-1} .
\end{aligned}
\tag{3.60}
$$

This result agrees upon the inclusion of the ghost contribution with that obtained in [40]. Note that the partition function for the disk is cancelled by the term constant in r which is then summed using ζ function regularization, mimicking the calculation of [40] that produced the Born-Infeld action at disk level.

Now, considering the fact that, as mentioned, the field U governs the interpolation between Neumann and Dirichlet boundary conditions and that we expect the space filling branes to be unstable, it is also interesting to examine how these expressions for Z_{ann} vary with U around the two fixed points. In particular, ignoring the linear terms in U in the normalization, which can be seen (3.36) to be divergent, the expression for Z_{ann} near $U = 0$ is

$$
Z_{ann} = Z_{ann}(U=0) + Tr \left(U \frac{\partial}{\partial U} Z_{ann}(U=0) \right) + \ldots \tag{3.61}
$$

Immediately upon differentiation we see that the linear term will be given by

$$
Tr \left(U \frac{\partial}{\partial U} Z_{ann}(U=0) \right) = \int d^2a d^2b \delta(|a^2| - |b^2| - 1) \mathcal{V}_{PSL(2,R)} \mathcal{N}^2 e^{2t}
$$

$$\frac{1}{\det\left(1-e^{-2tb}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\right)}$$
$$\times Tr\left(\frac{(g-2\pi\alpha' F)/(g+2\pi\alpha' F)}{(g+2\pi\alpha' F)^2-(g-2\pi\alpha' F)^2}U\right)$$
$$\times\frac{-4be^{-2b}}{1-e^{-2b}}\bar{M}^{(a,b)}_{-n-b}\frac{1}{n^2}M^{(a,b)}_{-n-b}. \tag{3.62}$$

The factor of $\frac{e^{-2b}}{1-e^{-2b}}$ comes from the fact that all the other Λ terms become trivial because we have evaluated them at $U = 0$ which was noted to be conformally invariant, and from summing the terms e^{-b} which stand between these. Likewise note that the factor $1/n^2$ instead of $1/n$ between $\bar{M}$ and M comes from the fact that U enters always as U/n. Also, one of the integrals over the PSL(2,R) groups becomes trivial, and relabeling gives the factor $\mathcal{V}_{PSL(2,R)}$ and only one integral. Now, we evaluate

$$\begin{aligned}\sum_{n\geq 1}\bar{M}_{-n-b}\frac{1}{n^2}M_{-n-b} &= \sum_{n\geq 1}\oint\frac{dz}{2\pi i}\frac{d\bar{z}}{-2\pi i}\frac{1}{n^2}\frac{1}{z^n\bar{z}^n}\\ &\quad\frac{(\bar{a}\bar{z}+\bar{b})^{b-1}}{(b\bar{z}+a)^{b+1}}\frac{(az+b)^{b-1}}{(\bar{b}z+\bar{a})^{b+1}}\\ &\quad\frac{1}{n!^2}\partial_z^{n-1}\partial_{\bar{z}}^{n-1}\left.\frac{(\bar{a}\bar{z}+\bar{b})^{b-1}}{(b\bar{z}+a)^{b+1}}\frac{(az+b)^{b-1}}{(\bar{b}z+\bar{a})^{b+1}}\right|_{z,\bar{z}=0}\end{aligned} \tag{3.63}$$

and we find that when we include the integration over PSL(2,R) the expression becomes

$$\begin{aligned}&\int d^2ad^2b\delta(|a^2|-|b^2|-1)\times\sum_{n\geq 1}\bar{M}_{-n-b}\frac{1}{n^2}M_{-n-b} =\\ &\qquad\int d^2ad^2b\delta(|a^2|-|b^2|-1)\sum_{n\geq 1}\sum_{q=0}^{min(n-1,b-1)}\frac{1}{(nb)^2}\\ &\qquad\left(\frac{b+n-q-1!}{q!n-q-1!b-q-1!}\right)^2\left(\frac{|b^2|}{|a^2|}\right)^{b+n-2q-2}\frac{1}{|a^2|^2}\end{aligned} \tag{3.64}$$

and we have used the fact that upon integration over the phase of a and b we will have orthogonality in the sum. We find that the contribution is

$$
\begin{aligned}
Tr\left(U\frac{\partial}{\partial U}Z_{ann}(U=0)\right) &= \int d^2ad^2b\delta(|a^2|-|b^2|-1)\mathcal{V}_{PSL(2,R)}\mathcal{N}^2e^{2t} \\
&\times\frac{1}{\det\left(1-e^{-2tb}\left(\frac{g-2\pi\alpha'F}{g+2\pi\alpha'F}\right)^2\right)} \\
&\times Tr\left(\frac{(g-2\pi\alpha'F)/(g+2\pi\alpha'F)}{(g+2\pi\alpha'F)^2-(g-2\pi\alpha'F)^2}U\right) \\
&\sum_{n,m\geq 1}\frac{-4me^{-2m}}{1-e^{-2m}}\sum_{q=0}^{min(n-1,m-1)}\left(\frac{1}{(nm)^2}\frac{1}{|a^2|^2}\right. \\
&\left.\left(\frac{(m+n-q-1)!}{q!(n-q-1)!(m-q-1)!}\right)^2\left(\frac{|b^2|}{|a^2|}\right)^{m+n-2q-2}\right).
\end{aligned}
\tag{3.65}
$$

A similar calculation can be done around the condensate ($U\to\infty$ with $1/U$ the natural expansion parameter) and it is found that

$$
\begin{aligned}
Tr\left(\frac{1}{U}\frac{\partial}{\partial(1/U)}Z_{ann}\left(\frac{1}{U}=0\right)\right) &= \int d^2ad^2b\delta(|a^2|-|b^2|-1) \\
&\mathcal{V}_{PSL(2,R)}\frac{\mathcal{N}^2e^{2t}}{\det\left(1-e^{-2tb}\right)} \\
&\times 4Tr\left(\frac{1}{U}\right)\sum_{n,m\geq 1}\frac{me^{-2m}}{1-e^{-2m}}\bar{M}^{(a,b)}_{-n-m}M^{(a,b)}_{-n-m}.
\end{aligned}
\tag{3.66}
$$

Because the natural coefficient for $\frac{1}{U}$ is n the n dependence between the matrices M is suppressed. Evaluations show that $\bar{M}_{-n-a}M_{-n-b}$ has zero entries on diagonal, so this variation vanishes about the condensate. This comparison between (3.65) and (3.66) shows that the case of Neumann boundary conditions, (corresponding to $U=0$) is unstable with respect to variations of the tachyon condensate since the linear variation does not vanish, but that Dirichlet boundary conditions, obtained as $U \to \infty$ are stable. This illustrates the well known phenomenon of tachyon condensation and gives a mechanism to see explicitly how the open string tachyon has been removed from the excitations of the condensed state.

In a similar method we can obtain the partition function for the Mobius strip in this background as well. We use the crosscap state elaborated on in [62] .

$$|\mathcal{C}\rangle = \exp\left(-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\alpha_{-n}\tilde{\alpha}_{-n}\right)|0\rangle. \tag{3.67}$$

Using this in analogy with the development of (3.54) we find that the

$$\begin{aligned} Z_{Mobius} &= \langle B_{a,b}|\frac{1}{\Delta}|\mathcal{C}\rangle \\ &= Z_{disk}Z_{RP^2}\exp\left(\sum_{k=1}^{\infty}\frac{1}{k}g^{\mu\nu}\delta_{rs}\left\{\left[rM_{nr}^{(a,b)}\Lambda_{\mu\nu}^{n}\bar{M}_{nj}^{(a,b)}\right.\right.\right. \\ &\qquad \left.\left.\left. j\bar{z}^j\frac{(-1)^s}{s}\delta_{js}z^s\right]^k\right\}_{\mu\nu}^{rs}\right). \end{aligned} \tag{3.68}$$

As in the case of the annulus, we find that the contributions Z_{disk} cancel explicitly when we go to the $U = 0$ limit, where conformal invariance is restored. In that limit we find

$$Z_{Mobius}(F) = \prod_{m=1}^{\infty}\det\left(g + F\frac{1+(-1)^m|z|^{2m}}{1-(-1)^m|z|^{2m}}\right)^{-1}. \tag{3.69}$$

Finally it is amusing to check and make sure that an analogous calculation will go through and reproduce the known partition function for the Klein bottle. Instead of the two copies of the boundary state two crosscaps are inserted, and the resulting expression

$$Z_{K^2} = Z_{RP^2}^2\exp\left(-\sum_{r=1}^{\infty}g^{\mu\nu}\ln\left(g_{\mu\nu}(1-|z|^{2r})\right)\right) \tag{3.70}$$

which can be seen to reduce to Dedekind η-functions [51], in agreement with the known result [78].

3.3.3 $\chi = -1$

To extend it beyond $\chi = 0$ the boundary state formalism requires careful contemplation. We propose the following method which allows the construction of states of arbitrary Euler number, and for the non-orientable sector in principle a complete description of the dynamics. The procedure proposed is as follows; using the sewing construction for higher genus amplitudes (described in [78] among others) and armed with the result proved earlier in this paper that the emission of any particle from the bosonic boundary state corresponds with the expectation of a vertex operator inserted at a definite position on the disk, we propose to add any number of interactions with the brane described by the boundary state and any number of cross caps.

To recapitulate, the idea motivating the sewing construction is to create a higher genus amplitude by joining two lower genus amplitudes by inserting a vertex operator on each of the lower genus amplitudes and summing over the vertex operator. Explicitly the construction is

$$\langle : A_1 : \ldots : A_n : \rangle_M = \int_\omega \sum_V \omega^{h_V} \bar{\omega}^{\tilde{h}_V} \langle : A_1 : \ldots : V : \rangle_{M_1} \langle : V : \ldots : A_n : \rangle_{M_2} \tag{3.71}$$

with $M = M_1 \# M_2$ and $\ldots$ represents arbitrary vertex insertions. This construction is tantamount to adding a closed string propagator between the two manifolds with vertex operators on them. Since we have shown that the emission of one particle from the disk with F and U on its boundary matches the overlap obtained from the boundary state

$$\langle V | B_{a,b} \rangle = \langle : V : \rangle_{T_0,U,F} \tag{3.72}$$

we can then use this to obtain the contribution of a boundary with the fields U and F at it. This sort of construction was considered in [26].

The novel feature presented here is the generalization of the boundary state and cross-cap operators through the state operator correspondence.

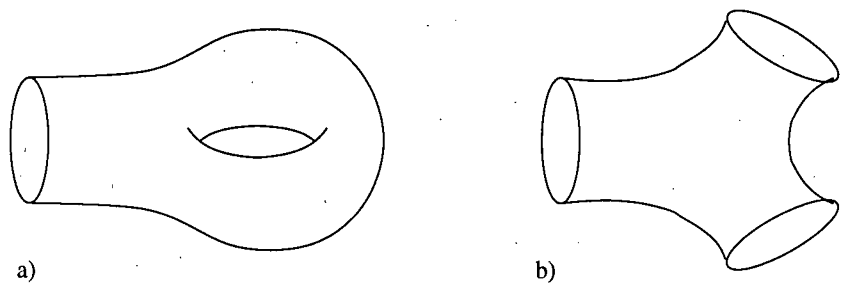

Figure 3.6: The two orientable surfaces with $\chi = -1$. They are a one-loop correction to the disk amplitude (a) and a surface with three interactions (b), topologically equivalent to a pair of pants.

The fact that sphere amplitude for three string scattering is conformally invariant is used, in combination with the fact that both $|\mathcal{C}\rangle$ and $|B_{a,b}\rangle$ both have a well defined overlap with any closed string state allows us to take the expression

$$\frac{1}{\Delta}|B_{a,b}\rangle = \int \frac{dzd\bar{z}}{|z|^4}|z|^{p^2} \exp\left(-z^k\alpha^{\gamma}_{-k}M^{(a',b')}_{-m-k}\Lambda^m_{\gamma\delta}\bar{M}^{(a',b')}_{-m-l}\tilde{\alpha}^{\delta}_{-l}\bar{z}^l\right)|0\rangle \tag{3.73}$$

and its equivalent using $|\mathcal{C}\rangle$ to (suppressing prefactors)

$$\exp\left(-z^k\frac{\partial^k}{(k-1)!}X^{\gamma}M^{(a',b')}_{-m-k}\Lambda^m_{\gamma\delta}\bar{M}^{(a',b')}_{-m-l}\frac{\bar{\partial}^l}{(l-1)!}X^{\delta}\bar{z}^l\right) \tag{3.74}$$

by use of the operator state correspondence. These states are inserted within expectation values to give higher genus contributions.

There are several different states with $\chi = -1$. The most obvious are the four possible insertions of boundary states and cross caps, and the addition of a handle to either a boundary state or cross cap (thereby going from $\chi = 1$ to $\chi = -1$ because increasing the genus by 1 decreases the Euler number by 2). Note that the state with three cross caps and the state with a cross cap and a handle are topologically equivalent.

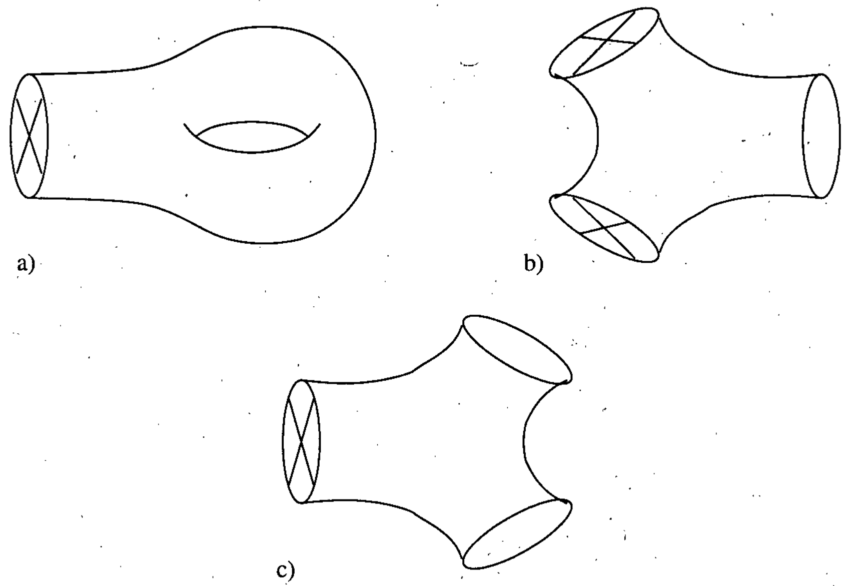

Figure 3.7: The three non-orientable surfaces with $\chi = -1$.

To obtain the amplitude for three boundaries we calculate

$$Z_{'pants'} = \langle B_{a,b}|\frac{1}{\Delta} : B_{a',b'} : \frac{1}{\Delta}|B_{a'',b''}\rangle \tag{3.75}$$

where $: B_{a',b'} :$ is as given in (3.74). Noting that the coefficient of α_m in $\frac{\partial^n}{(n-1)!}X$ is

$$\frac{\partial^n}{(n-1)!}X = \sum_{a=-\infty}^{\infty} D_{na}\alpha_a, \tag{3.76}$$

$$D_{na} = (-1)^{n-1}\frac{(a+1)\dots(n+a-1)}{(n-1)!}, \tag{3.77}$$

we proceed to calculate

$$Z_{'pants'} = Z^3_{disk}F_0(p)\exp\left(\sum_k \frac{1}{k}\delta_{na}\left(nC_{nm}(1)mC_{am}(3)\right)^k\right)$$
$$\exp\left(\sum_k \frac{1}{k}\delta_{na}\left(nC_{nm}(1)mD_{n'-a}C_{n'm'}(2)\bar{D}_{m'-m}\right)^k\right)$$

$$\exp\left(\sum_k \frac{1}{k}\delta_{na}\left(nC_{nm}(3)mD_{n'a}C_{n'm'}(2)\bar{D}_{m'm}\right)^k\right)$$
$$\exp\left(\sum_k \frac{1}{k}\delta_{na}\left(nC_{nm}(1)mD_{n'j}C_{n'm'}(2)\bar{D}_{m'-m}\right.\right.$$
$$\left.\left. jC_{jk}(3)kD_{n''-a}C_{n''m''}(2)\bar{D}_{m''k}\right)^k\right) \qquad (3.78)$$

Where as in (3.54) $F_0(p)$ is a complicated function which is Gaussian in the momentum of the boundary state, the integrals are implicit, and the expression $C_{nm}(i)$ is an abbreviation

$$C_{nm}(i) = z_i^n M_{kn}^{(a,b)} \Lambda_{\mu\nu}^k \bar{M}_{km}^{(a,b)} \bar{z}_i^m \qquad (3.79)$$

with i an index indicating the integration from which the closed string propagator z_i came from.

From this we see immediately that the contributions for the genus expansion become increasingly complicated as χ increases. In the particularly simple case of a vanishing tachyon we obtain a product of exponentials of hypergeometric functions. In particular for the case of the constant F field we obtain

$$Z_{'pants'}(F) = Z_{disk}^3 \exp\left(-\sum_n Tr\ln\left(1-|z_1z_3|^{2n}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\right)\right)$$
$$\exp\left(-\sum_{na} Tr\ln\left(1-n|z_1|^{2n}|z_2|^2\right.\right.$$
$$\left.\left. F(-n+1,-a+1;2;|z_2|^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\right)\right)$$
$$\exp\left(-\sum_{na} Tr\ln\left(1-n|z_3|^{2n}|z_2|^2\right.\right.$$
$$\left.\left. F(n+1,a+1;2;|z_2|^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\right)\right)$$

$$\exp\left(-\sum_{nma} Tr \ln \left(1 - n|z_1|^{2n}|z_2|^2 F(-n+1, m+1; 2; |z_2|^2)\right.\right.$$
$$\left.\left. m|z_3|^{2m}|z_2|^2 F(m+1, -a+1; 2; |z_2|^2)\left(\frac{g - 2\pi\alpha' F}{g + 2\pi\alpha' F}\right)^2\right)\right). \tag{3.80}$$

In the above $F(a,b;c;x)$ is the hypergeometric function defined by its series expansion

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a+n-1)!(b+n-1)!(c-1)!}{n!(a-1)!(b-1)!(c+n-1)!} x^n, \tag{3.81}$$

and the logarithm is interpreted as its series expansion, and both Lorentz and oscillator indices are summed over. Note that this expression has many of the properties that we expect for the partition function on a twice punctured disk. In particular this depends on three parameters (the z_i terms arising from the integration over the propagators to the various boundary states) which can be identified as the Teichmuller parameters for this surface [88]. In the limit of any of these parameters going to zero the dominant contribution is from the annulus amplitude. The analogous amplitude with any number of cross-caps gives a similar expression with the following modifications, for each cross-cap the argument in the hypergeometric expression acquires a negative sign, and the corresponding matrix of Lorentz indices undergoes the substitution $\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F} \to g$.

The other two diagrams that must be calculated are the corrections to the disk and to RP^2 which come from the addition of a handle. This addition is achieved by taking the trace, weighted by a factor exponentiated to the level number (coming from the propagator within the handle), which is an identical operation to taking the expectation value of this operator on the torus. For this calculation it is necessary to take the trace of an operator that generically has the normal ordered form

$$: \exp\left(-\alpha_n \mathcal{M}_{nm} \tilde{\alpha}_m\right) :$$

where the indices on $\mathcal{M}$ can be either positive or negative, with $\mathcal{M}$ defined by

$$\mathcal{M}_{mn} = D_{n'm} C_{n'm'} \bar{D}_{m'n}. \tag{3.82}$$

After a considerable amount of algebra we find by summing over all states in the Fock space that

$$\begin{aligned} Tr\left(\omega^h \tilde{\omega}^{\tilde{h}} : \exp\left(-\alpha_n \mathcal{M}_{nm} \tilde{\alpha}_m\right) :\right) &= \prod_{n=1}^{\infty} \frac{1}{|1-\omega^{h_i}|^2} \times \\ & \prod_{n=1}^{\infty} \frac{1}{1-\left(\frac{|a|\omega^{h_a}}{1-\omega^{h_a}} \mathcal{M}_{ab} \frac{|b|\tilde{\omega}^{\tilde{h}_b}}{1-\tilde{\omega}^{\tilde{h}_b}} \mathcal{M}_{-c-b}\right)^n}. \end{aligned} \tag{3.83}$$

This expression uses the convention that the sums within the denominator run over positive and negative indices. This suppresses the contribution from the momentum of the loop which is given by a Gaussian,

$$\begin{aligned} F(p) &= \exp\left\{pp\left[\left(\delta_{0j} - \mathcal{M}_{0j}\frac{|j|\tilde{\omega}^{h_j}}{1-\tilde{\omega}^{h_j}}\right)\left(\frac{1}{\delta_{kj} - \mathcal{M}_{kj}\frac{|k|\omega^{h_k}}{1-\omega^{h_k}}\mathcal{M}_{kl}}\right)\right.\right. \\ & \left.\left.\left(\delta_{0l} - \mathcal{M}_{0l}\frac{|l|\tilde{\omega}^{h_l}}{1-\tilde{\omega}^{h_l}}\right) - 1\right]\right\}. \end{aligned} \tag{3.84}$$

The specialization to the case of only interactions with a background F field is given by substituting $|z|^2 F(a+1, b+1; 2; |z|^2)$ for $\mathcal{M}_{ab}$.

It is interesting at this point to compare the results for this procedure with those obtained by the standard method of constructing the Green's function on an arbitrary surface [88], and then integrating out the boundary interaction as described previously (3.33). The Green's function of a unit disk with Neumann boundary conditions with a puncture of radius ϵ at $z=0$ and a puncture of radius δ at $z=re^{i\psi}$ is given by

$$\begin{aligned}
G'(z,z') &= G(z,z') + (\ln\epsilon)^{-1} G(z,0)G(z',0) \\
&\quad + (\ln\delta)^{-1} G(z,re^{i\psi})G(z',re^{i\psi}) \\
&\quad - Re\left(4\delta^2\left(\frac{1}{z-re^{i\psi}} + \frac{\bar{z}}{1-\bar{z}re^{i\psi}}\right)\right. \\
&\qquad\qquad \left.\times\left(\frac{1}{\bar{z}'-re^{-i\psi}} + \frac{z'}{1-z're^{-i\psi}}\right)\right) \\
&\quad - Re\left(4\epsilon^2\left(z^{-1}+\bar{z}\right)\left(\bar{z}'^{-1}+z'\right)\right) + O(\epsilon^2) + O(\epsilon^2\delta^2) + O(\delta^2).
\end{aligned} \tag{3.85}$$

In the above the explicit form of the Green's function for the disk (3.31) has been substituted into the last two lines. Integrating out the background field F can be done by recasting this as a one dimensional 3×3 matrix model. When this is done the interaction with a field on the boundary can be integrated out, much as was done for the 2×2 case in [40], and the resulting expression contains the lowest order terms (in the Teichmuller parameter) of the hypergeometric functions obtained previously. Similarly there is a procedure for obtaining the Green's function for the disk with a handle added between balls of radius ϵ centered at $z=0$ and $z=re^{i\psi}$. This gives

$$\begin{aligned}
G'(z,z') &= G(z,z') + (\ln\epsilon)^{-1}\left(G(z,0)-G(z,re^{i\psi})\right)\left(G(z',0)-G(z',re^{i\psi})\right) \\
&\quad - Re\left[4\epsilon^2\left(z^{-1}+\bar{z}\right)\left(\frac{1}{z'-re^{i\psi}} + \frac{\bar{z}'}{1-\bar{z}'re^{i\psi}}\right)\right. \\
&\quad \left. + \left(z'^{-1}+\bar{z}'\right)\left(\frac{1}{z-re^{i\psi}} + \frac{\bar{z}}{1-\bar{z}re^{i\psi}}\right)\right] + O(\epsilon^2).
\end{aligned} \tag{3.86}$$

As in the case of the disk with holes removed, this Green's function can be then used to integrate out the quadratic perturbation, obtaining results that are consistent with those presented in (3.83).

(b,h)	Z
(3,0)	$\int \exp\Big[-\sum_n Tr \ln\Big(1 - \lvert z_1 z_3\rvert^{2n}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\Big)$ $-\sum_{na} Tr\ln\Big(1 - n\lvert z_1\rvert^{2n}\lvert z_2\rvert^2 F(-n+1,-a+1;2;\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\Big)$ $-\sum_{na} Tr\ln\Big(1 - n\lvert z_3\rvert^{2n}\lvert z_2\rvert^2 F(n+1,a+1;2;\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\Big)$ $-\sum_{nma} Tr\ln\big(1 - n\lvert z_1\rvert^{2n}\lvert z_2\rvert^2 F(-n+1,m+1;2;\lvert z_2\rvert^2)$ $m\lvert z_3\rvert^{2m}\lvert z_2\rvert^2 F(m+1,-a+1;2;\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\Big)\Big]$
(1,1)	$\int \prod_{n=1}^{\infty}\frac{1}{\lvert 1-\omega^{h_i}\rvert^2}$ $\prod_{n=1}^{\infty}\frac{1}{1-\left(\frac{\lvert a\rvert\omega^{h_a}}{1-\omega^{h_a}}\lvert z\rvert^2 F(a+1,b+1;2;\lvert z\rvert^2)\frac{\lvert b\rvert\bar{\omega}^{\bar{h}_b}}{1-\bar{\omega}^{\bar{h}_b}}\lvert z\rvert^2 F(-c+1,-b+1;2;\lvert z\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\right)^n}$ $\exp\Big[pp\Big[\Big(\delta_{0j} - \lvert z\rvert^2 F(1,j+1;2;\lvert z\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)\frac{\lvert j\rvert\tilde{\omega}^{h_j}}{1-\tilde{\omega}^{h_j}}\Big)$ $\left(\frac{1}{\delta_{kj} - \lvert z\rvert^2 F(k+1,j+1;2;\lvert z\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\frac{\lvert k\rvert\omega^{h_k}}{1-\omega^{h_k}}\lvert z\rvert^2 F(k+1,l+1;2;\lvert z\rvert^2)}\right)$ $\Big(\delta_{0l} - \lvert z\rvert^2 F(1,l+1;2;\lvert z\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)\frac{\lvert l\rvert\tilde{\omega}^{h_l}}{1-\tilde{\omega}^{h_l}}\Big) - 1\Big]\Big]$

Table 3.1: The partition functions for the orientable surfaces with $\chi = -1$ in the case of $U = 0$. The number of boundaries and handles (b,h) is listed.

(b,c)	Z
(2,1)	$\int \exp\Big[-\sum_n Tr\ln\Big(1-\lvert z_1z_3\rvert^{2n}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\Big)$
	$-\sum_{na} Tr\ln\Big(1+n\lvert z_1\rvert^{2n}\lvert z_2\rvert^2F(-n+1,-a+1;2;-\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)\Big)$
	$-\sum_{na} Tr\ln\Big(1-n\lvert z_3\rvert^{2n}\lvert z_2\rvert^2F(n+1,a+1;2;-\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)\Big)$
	$-\sum_{nma} Tr\ln(1-n\lvert z_1\rvert^{2n}\lvert z_2\rvert^2F(-n+1,m+1;2;-\lvert z_2\rvert^2)$
	$m\lvert z_3\rvert^{2m}\lvert z_2\rvert^2F(m+1,-a+1;2;-\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)^2\Big)\Big]$
(1,2)	$\int \exp\Big[-\sum_n Tr\ln\Big(1-(-1)^n\lvert z_1z_3\rvert^{2n}\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)\Big)$
	$-\sum_{na} Tr\ln\Big(1+n\lvert z_1\rvert^{2n}\lvert z_2\rvert^2F(-n+1,-a+1;2;-\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)\Big)$
	$-\sum_{na} Tr\ln(1-n(-1)^n\lvert z_3\rvert^{2n}\lvert z_2\rvert^2F(n+1,a+1;2;-\lvert z_2\rvert^2))$
	$-\sum_{nma} Tr\ln(1-n\lvert z_1\rvert^{2n}\lvert z_2\rvert^2F(-n+1,m+1;2;-\lvert z_2\rvert^2)$
	$m(-1)^m\lvert z_3\rvert^{2m}\lvert z_2\rvert^2F(m+1,-a+1;2;-\lvert z_2\rvert^2)\left(\frac{g-2\pi\alpha' F}{g+2\pi\alpha' F}\right)\Big)\Big]$

Table 3.2: The partition functions for the non-orientable surfaces with $\chi = -1$ in the case of $U = 0$. The number of boundaries and crosscaps (b,c) is listed.

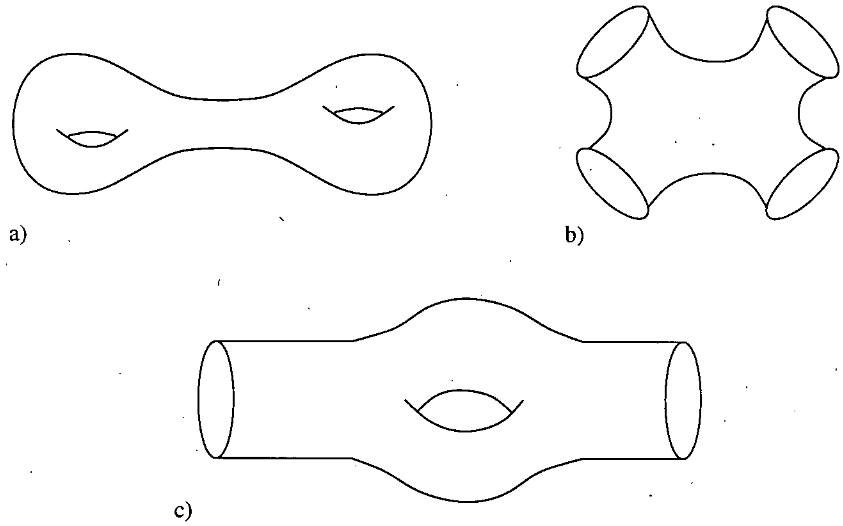

Figure 3.8: The three orientable surfaces with $\chi = -2$. The double torus (a) has no boundary, the surface with two boundaries (c) can be thought of as a one loop correction to the annulus, and the surface with four boundaries (b) is topologically a shirt (in the same spirit that 3.6b is a pair of pants)

3.3.4 $\chi = -2$

As the surfaces increase in complexity there are an increasing number of orientable and non-orientable surfaces with boundary at each Euler number, and consequently a larger number of amplitudes to calculate. The surfaces in question are illustrated in Figures 3.8 and 3.9 and can be described as follows. Among orientable surfaces there are three, one with two loops and no boundaries, one with one loop and two boundaries, corresponding to a 'one-loop' modification of a string propagator, and a surface with four boundaries and no loops, which by analogy with the exposition in previous sections can be thought of as a tree level interaction between 4 separate D-branes. Similarly in the non-orientable sector there are a number of different possibilities, and

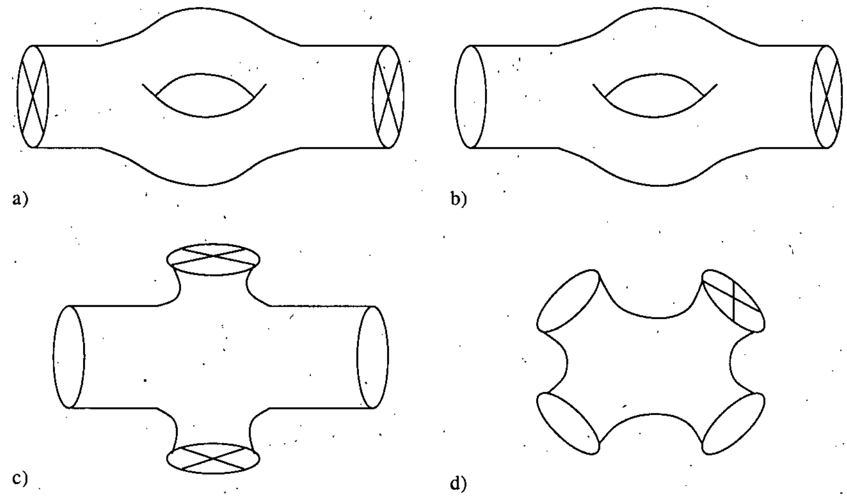

Figure 3.9: The four non-orientable surfaces with $\chi = -2$.

they are most simply classified keeping in mind the fact that the insertion of two cross-caps can be exchanged for a loop on a non-orientable surface. There is a surface with four cross-caps, the non-orientable analog of the two loop orientable graph, a surface with three cross-caps and one boundary, which will be the higher loop generalization of the interaction with the Mobius strip, the surface with two boundaries and two cross-caps which is the non-orientable contribution to the one-loop modification of the closed string propagator, and a surface with three boundaries and one cross-cap. We will examine each of these surfaces in turn, as in the previous sections.

Once again the contributions of first listed surfaces, both in the orientable and nonorientable sectors can be ignored in our investigation of the partition function for the tachyon field. As before this is because these surfaces do not interact with this because they have no boundaries.

Next we consider the surfaces with no handles and at least one boundary. (This is actually all of the surfaces except for Figure 3.8c because surfaces with two crosscaps are equivalent to a non-orientable surface with a handle.)

In analogy with (3.75) we calculate for Figure 3.8b with the partition function given by

$$Z_{'shirt'} = \int \langle B_{a_1,b_1}|\frac{1}{\Delta} : B_{a_2,b_2} : \frac{1}{\Delta} : B_{a_3,b_3} : \frac{1}{\Delta}|B_{a_4,b_4}\rangle. \tag{3.87}$$

Using the convention from the $\chi = -1$ case we implicitly assume that the internal Bs have a propagator inside of them, and can be then written using 3.74 which gives

$$: B_{a_i,b_i} := \exp\left(-\alpha_p D_{np} C_{nm}(i) \bar{D}_{mq} \tilde{\alpha}_q\right), \tag{3.88}$$

with

$$C_{nm}(i) = z_i^n M_{kn}^{(a_i,b_i)} \Lambda_{\mu\nu}^k \bar{M}_{km}^{(a_i,b_i)} \bar{z}_i^m \tag{3.89}$$

as in (3.79), and D is as described in (3.76). Using this input, and noting that there is one factor of $\frac{1}{\Delta}$ that is not accounted for and whose parameter z we give the subscript 5 to, we can recast the expression for this amplitude as

$$Z_{'shirt'} = \langle \tilde{B}(1)| : \tilde{B}(2) :: \exp\left(-\alpha_k z_5^{-k} D_{-nk} C_{nm}(3) \bar{D}_{-mj} \bar{z}_5^{-j} \alpha_j\right) : |\tilde{B}(5 \times 4)\rangle \tag{3.90}$$

where the term $\tilde{B}$ denotes the inclusion of the propagator, and 5×4 in this case denotes the multiplication of z_5 and z_4 which only appear in the last term in the combination $z_5 z_4$, so we rescale, absorbing z_4 into the normalization of z_5. Upon performing the calculations we find

$$\begin{aligned} Z_{'shirt'} = \; & Z_{disk}^4 F_0'(p) \exp\left(\sum_k \delta_{na} \left(nC_{nm}(1) m C_{am}(5)\right)^k\right) \\ & \exp\left(\sum_k \delta_{na} \left(nC_{nm}(1) m D_{n'-a} C_{n'm'}(3) \bar{D}_{m'-m}\right)^k\right) \\ & \exp\left(\sum_k \delta_{na} \left(nC_{nm}(1) m D_{n'-a} C_{n'm'}(2) \bar{D}_{m'-m}\right)^k\right) \end{aligned}$$

$$
\begin{aligned}
&\exp\left(\sum_k \delta_{na}\left(nD_{n''n}C_{nm}(2)\bar{D}_{m''m}mD_{n'-a}C_{n'm'}(3)\bar{D}_{m'-m}\right)^k\right)\\
&\exp\left(\sum_k \delta_{na}\left(nD_{n''n}C_{nm}(2)\bar{D}_{m''m}mC_{am}(5)\right)^k\right)\\
&\exp\left(\sum_k \delta_{na}\left(nD_{n''n}C_{nm}(3)\bar{D}_{m''m}mC_{am}(5)\right)^k\right)\\
&\exp\Big(\sum_k \frac{1}{k}\delta_{na}\big(nD_{qn}C_{qp}(2)\bar{D}_{pm}mD_{n'j}C_{n'm'}(3)\bar{D}_{m'-m}\\
&\quad jC_{jk}(4)kD_{n''-a}C_{n''m''}(3)\bar{D}_{m''k}\big)^k\Big)\\
&\exp\Big(\sum_k \frac{1}{k}\delta_{na}\big(nC_{nm}(1)mD_{n'j}C_{n'm'}(2)\bar{D}_{m'-m}\\
&\quad jD_{j'j}C_{j'k'}(3)\bar{D}_{k'k}kD_{n''-a}C_{n''m''}(2)\bar{D}_{m''k}\big)^k\Big)\\
&\exp\Big(\sum_k \frac{1}{k}\delta_{na}\Big(nC_{nm}(1)m\\
&\quad \left(D_{n'j}C_{n'm'}(2)\bar{D}_{m'-m}+D_{n'j}C_{n'm'}(3)\bar{D}_{m'-m}\right)jC_{jk}(4)k\\
&\quad \left(D_{n''-a}C_{n''m''}(2)\bar{D}_{m''k}+D_{n''-a}C_{n''m''}(3)\bar{D}_{m''k}\right)\Big)^k\Big)\\
&\exp\Big(\sum_k \frac{1}{k}\delta_{na}\big(nC_{nm}(1)mD_{n'j}C_{n'm'}(2)\bar{D}_{m'-m}D_{n'j}C_{n'm'}(3)\bar{D}_{m'-m}\\
&\quad jC_{jk}(4)kD_{n''-a}C_{n''m''}(3)\bar{D}_{m''k}D_{n''-a}C_{n''m''}(2)\bar{D}_{m''k}\big)^k\Big) \qquad (3.91)
\end{aligned}
$$

This is clearly a lot more complicated than the corresponding result for the three boundary case. This can be generalized to the case of any number of crosscaps by substituting into the expression for C the term $\sum_n \frac{(-1)^n}{n}\delta^{\mu\nu}$ in the place of $\sum_n \frac{1}{n}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\mu\nu}$.

There remains only one general type of diagram to be concerned with, and that is the torus amplitude with two boundaries. This can be also constructed in the following manner, which we choose to emphasize the factorization properties [26, 28], because the torus can be thought of as the exchange of two closed string propagators. With this in mind we propose the following construction, each of the two arms of the torus is thought of as a closed string propagator occupying its own Fock space, and the operators that make up the two boundary states that form the two ends of this surface are allowed to be in either of the Fock spaces, and we average over all possible choices. This can be thought of as allowing the excitations from the boundary state to propagate along either of the two closed string propagators, and averaging over all possible choices in in analogy with the general spirit of path integrals. Then the partition function will be

$$\begin{aligned} Z &= \int Z_{disk}^2 \langle 0, p| \prod_{i,j=1,2} \exp\left(-\alpha_{n'}^{\mu i} z_1^{n'} M_{nn'}^{(a,b)} \Lambda_{\mu\nu}^n \bar{M}_{nm'}^{(a,b)} \bar{z}_1^{m'} \tilde{\alpha}_{m'}^{\nu j}\right) \\ &\quad y_1^{\sum \alpha_{-n}^1 \alpha_n^1} \bar{y}_1^{\sum \tilde{\alpha}_{-n}^1 \tilde{\alpha}_n^1} y_2^{\sum \alpha_{-n}^2 \alpha_n^2} \bar{y}_2^{\sum \tilde{\alpha}_{-n}^2 \tilde{\alpha}_n^2} \\ &\quad \prod_{i',j'=1,2} \exp\left(-\alpha_{-k}^{\gamma i'} z_2^k M_{-m-k}^{(a',b')} \Lambda_{\gamma\delta}^m \bar{M}_{-m-l}^{(a',b')} z_2^l \tilde{\alpha}_{-l}^{\delta j'}\right) |0, p\rangle \end{aligned} \tag{3.92}$$

where the superscript on the α operators in addition to the Lorentz index indicates the Fock space to which it belongs. Now, (3.92) can be evaluated giving

$$\begin{aligned} Z &= \int Z_{disk}^2 \langle 0, p| \prod_{i,j=1,2} \exp\left(-\alpha_{n'}^{\mu i} z_1^{n'} M_{nn'}^{(a,b)} \Lambda_{\mu\nu}^n \bar{M}_{nm'}^{(a,b)} \bar{z}_1^{m'} \tilde{\alpha}_{m'}^{\nu j}\right) \\ &\quad \prod_{i',j'=1,2} \exp\left(-\alpha_{-k}^{\gamma i'} y_i^k z_2^k M_{-m-k}^{(a',b')} \Lambda_{\gamma\delta}^m \bar{M}_{-m-l}^{(a',b')} z_2^l y_j^l \tilde{\alpha}_{-l}^{\delta j'}\right) |0, p\rangle \end{aligned} \tag{3.93}$$

3.3.5 $\chi = -3$

As for $\chi = -2$ there are a number of different surfaces of this genus that can be obtained with the insertion of handles, cross-caps, and boundaries. The method presented above provides a concrete proposal for the construction of these higher genus amplitudes for all $\chi \leq -1$. The construction is particularly appropriate for what can be interpreted as tree level scattering amplitudes for an arbitrary number of closed strings emitted from the brane described by the boundary state.

3.3.6 Beyond the Born-Infeld Action

In the preceding we have further explored the bosonic boundary state formalism [26] and discussed its extension to the off-shell case including interaction with a tachyon field of quadratic profile. The boundary state has been shown to reproduce the σ model calculations for emission of any number of closed string states, as detailed in the correspondence

$$\langle V_1| : V_2 : \ldots |B_{a,b}\rangle \;=\; \langle : V_1 :: V_2 : \ldots \rangle_{T_0,U,F}. \tag{3.94}$$

This can be restated as the fact that the boundary state encodes the bosonic string propagator in an algebraic manner.

It has been shown that the inner product of two of the boundary states also reproduces the σ model calculations for a world-sheet of the appropriate genus. We also present a generalization of this to higher genus, the results of which become progressively more complicated. In the case of vanishing tachyon field we obtain the following expansion in the open string coupling constant g_o

$$\begin{aligned}
Z_F &= \sum_\chi g_o^\chi Z_\chi \\
&= g_o^{-1}\sqrt{\det\left(g+2\pi\alpha' F\right)} \\
&\quad + \int \prod_r \left(1-|z^2|^r\right)^{-D} \prod_r \det\left(g - \frac{1+|z^2|^r}{1-|z^2|^r} 4\pi\alpha' F + 4\pi^2\alpha'^2 F^2\right)^{-1} \\
&\quad + \int \prod_r \left(1-(-1)^r|z^2|^r\right)^{-D} \prod_r \det\left(g - \frac{1+(-1)^r|z^2|^r}{1-(-1)^r|z^2|^r} 2\pi\alpha' F\right)^{-1} \\
&\quad + g_o^1\left(Z_{o30}+Z_{o11}+Z_{n21}+Z_{n12}\right) \\
&\quad + O(g_o^2)
\end{aligned} \tag{3.95}$$

where Z_{xij} is the partition function given in Table 3.3.3 or 3.3.3 for orientable $x = o$ and non-orientable $x = n$ surfaces with i, j boundaries and handles (or boundaries and cross-caps if appropriate). This is a generalization of the Born Infeld action taking into account higher loop stringy corrections, specifically including contributions from Euler number $\chi = -1$ in addition to the $\chi = 1$ and $\chi = 0$ terms previously in the literature, and including the contributions from non-orientable surfaces such as the Mobius strip. The construction presented can be generalized to higher genus with particular success in the case of the sphere with a number of boundaries and cross-caps added. It quickly becomes apparent that the simplifications obtained by the method of encoding the Green's function in the boundary state are overwhelmed by the increase in the parameters associated with the various boundary states.

3.4 Fermionic Boundary State

Despite the details shown in the previous sections the fermionic contribution to the boundary states is in fact more involved than that for the X fields. This stems in part from the fact that the fermions have a more involved world-sheet action, involving the Ramond and Neveu-Schwarz sectors corresponding to different boundary conditions for the fermions [79]. Another complication that will appear briefly is that there are branch cuts in the integrals that define the matrices relating the the oscillators before and after a conformal transformation, and this introduces some subtlety of treatment, however, that is for the Ramond sector fermions, whose zero modes make them inappropriate for the study of tachyon condensation [67], especially considering that the lowest lying states in that sector are bosonic.

We start as in the bosonic case with the consideration of the world-sheet action [6, 15, 68]

$$S_{ferm} = \int_M \left(\psi_+^\mu \partial_- \psi_+^\nu + \psi_-^\mu \partial_+ \psi_-^\nu\right) + \oint_{\partial M} F_{\mu\nu} \left(\psi_+^\mu \psi_+^\nu - \psi_-^\mu \psi_-^\nu\right) +$$
$$U_{\mu\nu} \left(\psi_+^\mu \frac{1}{\partial_\phi} \psi_+^\nu - \psi_-^\mu \frac{1}{\partial_\phi} \psi_-^\nu\right) \quad (3.96)$$

where as in the case of the X fields there is a boundary interaction with a constant gauge field, and the term involving the tachyon profile U is a simple generalization of the result in [76], and is appropriate to the NS sector since that is the sector with the tachyon, as well as that the fermions not having zero modes renders the inverse integral well defined

$$\frac{1}{\partial_\phi} \psi(\phi) = \frac{1}{2} \int d\phi' \epsilon(\phi - \phi') \psi(\phi') \quad (3.97)$$

where ϵ is a step function: $\epsilon(x) = 1$ for $x > 0$ and $\epsilon(x) = -1$ for $x < 0$. Combining the previously mentioned expansion for ψ_+ and ψ_- with and expanding as in [32, 34] we obtain the boundary conditions which must be satisfied

$$\left(g + 2\pi\alpha' F + \frac{\alpha'}{2} \frac{U}{n}\right)_{\mu\nu} \psi_n^\nu + i\eta \left(g - 2\pi\alpha' F - \frac{\alpha'}{2} \frac{U}{n}\right)_{\mu\nu} \tilde{\psi}_{-n}^\nu = 0 \quad (3.98)$$

where the factor of i comes from the conformal rotation of the world-sheet coordinates, and $\eta = \pm 1$ will accomplish the GSO projection with the selections [32, 34]

$$2|B_\psi\rangle = |B_+\rangle - |B_-\rangle. \quad (3.99)$$

where $|B_\pm\rangle$ are the coherent states that satisfy the boundary conditions with the corresponding positive or negative value for η, explicitly

$$|B_\pm\rangle = \mathcal{N}_f \exp\left[\pm i \sum \psi_{-n}^\mu \chi_{\mu\nu}^n \tilde{\psi}_{-n}^\nu\right] |0\rangle \quad (3.100)$$

with

$$\chi_{\mu\nu}^n = \left(\frac{g - 2\pi\alpha' F - \frac{\alpha'}{2}\frac{U}{n}}{g + 2\pi\alpha' F + \frac{\alpha'}{2}\frac{U}{n}}\right)_{\mu\nu} \quad (3.101)$$

Note that in the case of the Ramond sector the above difference becomes a sum and there is no contribution from the tachyon profile U, and also that χ is closely related to the bosonic term Λ. We specialize this discussion to the Neveu-Schwarz sector, as that is the case which can draw the parallel with the discussion in the bosonic sector.

Now we examine how the conformal transformation which redefines world-sheet coordinates (3.12) acts on the $(\frac{1}{2}, 0)$ degrees of freedom. Using the standard mode expansion [51, 79] the relationship between modes before and after transformation is

$$\begin{aligned} \psi_m &= N_{mn}\psi'_n \\ N_{mn} &= \oint \frac{dz}{2\pi i} z^{m-1/2} \frac{(\bar{b}z + \bar{a})^{n-1/2}}{(az+b)^{n+1/2}} \end{aligned} \tag{3.102}$$

The expression for N also contains an arbitrary phase that comes from the choice of branch for the square root of the Jacobean for the transformation, which can be ignored because in all cases we deal with bilinears in this, and also a relative sign can be absorbed in the definition of η. An examination of the properties of N, as well as its bosonic partner are found in Appendix A.

3.4.1 Particle Emission from Fermionic Boundary State

Now, in analogy with the development we can calculate the emission amplitude for a state in the NS-NS sector by taking the overlap with the appropriate element of the Fock space. As the development here is very similar to that in the bosonic case we only present a representative sample of the possible overlaps. First the massless state, corresponding to among other things the graviton, which is given by

$$|P_{\mu\nu}\rangle = P_{\mu\nu}\psi^{\mu}_{-1/2}\tilde{\psi}^{\nu}_{-1/2}|0\rangle$$

within the Fock space. The overlap of this with the boundary state given above is then

$$\begin{aligned}
\langle P_{\mu\nu}|B\rangle &= \int d^2a d^2b \delta(|a^2| - |b^2| - 1) i\mathcal{N}_f P^{\mu\nu} N^{(a,b)}_{m1/2} \chi^m_{\mu\nu} \bar{N}^{(a,b)}_{m1/2} \\
&= \int d^2a d^2b \delta(|a^2| - |b^2| - 1) i\mathcal{N}_f P^{\mu\nu} \chi^m_{\mu\nu} \frac{|b|^{2m-1}}{|a|^{2m+1}} \qquad (3.103)
\end{aligned}$$

with the bosonic part implicit and calculated previously (3.21). By contrast the overlap with a state with higher number of excitations is somewhat longer. An example is to consider the state

$$|\mathcal{P}_{\mu\nu\alpha\beta}\rangle = \mathcal{P}_{\mu\nu\alpha\beta} \psi^{\alpha}_{-1/2} \psi^{\mu}_{-1/2} \tilde{\psi}^{\beta}_{-1/2} \tilde{\psi}^{\nu}_{-1/2} |0\rangle$$

where there are the obvious symmetry and antisymmetry relations between the indices. The overlap is then

$$\begin{aligned}
\langle \mathcal{P}_{\mu\nu\alpha\beta}|B\rangle &= -\int d^2a d^2b \delta(|a^2| - |b^2| - 1) \mathcal{P}^{\mu\nu\alpha\beta} \\
&\quad \Big(N^{(a,b)}_{m1/2} \chi^m_{\mu\alpha} \bar{N}^{(a,b)}_{m1/2} N^{(a,b)}_{n1/2} \chi^n_{\nu\beta} \bar{N}^{(a,b)}_{n1/2} \\
&\quad - N^{(a,b)}_{m1/2} \chi^m_{\nu\alpha} \bar{N}^{(a,b)}_{m1/2} N^{(a,b)}_{n1/2} \chi^n_{\mu\beta} \bar{N}^{(a,b)}_{n1/2} \Big) \qquad (3.104)
\end{aligned}$$

The expressions for the matrices $N^{(a,b)}_{m1/2}$ can be found in Appendix A.2.

3.4.2 Particle Emission in the Superstring Sigma Model

Now, we pursue the analogy with the bosonic case further by calculating the disk amplitude for emission of the corresponding particle. We start by mentioning the two point functions for the NS fermions on the disk in the free case, which are respectively (with $G_\psi(z,w) = \langle \psi(z)\psi(w)\rangle$ and the obvious notation for the conjugate fields

$$G_\psi(z,w) = \frac{\alpha'}{i}\left(\frac{\sqrt{zw}}{z-w} - \frac{\sqrt{z\bar{w}}}{1-z\bar{w}}\right) \qquad (3.105)$$

$$\tilde{G}_\psi(z,w) = \frac{\alpha'}{i}\left(-\frac{\sqrt{\bar{z}\bar{w}}}{\bar{z}-\bar{w}} + \frac{\sqrt{\bar{z}w}}{1-\bar{z}w}\right) \tag{3.106}$$

Note in passing that these reproduce the well known expression [67] for the correlators of fermions on the boundary of the world-sheet, and when we parameterize $z = e^{i\phi}$ and $w = e^{i\phi'}$ we obtain as the sum of the holomorphic and antiholomorphic propagators

$$\langle\psi(\phi)\psi(\phi')\rangle + \langle\tilde{\psi}(\phi)\tilde{\psi}(\phi')\rangle = \frac{-2}{\sin\left(\frac{\phi-\phi'}{2}\right)} \tag{3.107}$$

in agreement with [67]. Just as in the case of the bosonic fields it is possible to integrate out the boundary interactions and obtain the modified propagator which satisfies the boundary conditions, obtaining, now including the Lorentz indices,

$$\begin{aligned}\frac{i}{\alpha'}G_\psi(z,w)^{\mu\nu} &= \frac{\sqrt{zw}}{z-w}g^{\mu\nu}\\ &- \sum_{r\in\mathbb{Z}+1/2>0}\left(\frac{g-2\pi\alpha'F-\frac{\alpha'}{2}\frac{U}{r}}{g+2\pi\alpha'F+\frac{\alpha'}{2}\frac{U}{r}}\right)^{\{\mu\nu\}}\mathrm{Im}\,(z\bar{w})^r\\ &- \sum_{r\in\mathbb{Z}+1/2>0}\left(\frac{g-2\pi\alpha'F-\frac{\alpha'}{2}\frac{U}{r}}{g+2\pi\alpha'F+\frac{\alpha'}{2}\frac{U}{r}}\right)^{[\mu\nu]}\mathrm{Re}\,(z\bar{w})^r\end{aligned} \tag{3.108}$$

and the corresponding expression for the conjugate fields

$$\begin{aligned}\frac{-i}{\alpha'}G_\psi(z,w)^{\mu\nu} &= \frac{-\sqrt{\bar{z}\bar{w}}}{\bar{z}-\bar{w}}g^{\mu\nu}\\ &+ \sum_{r\in\mathbb{Z}+1/2>0}\left(\frac{g-2\pi\alpha'F-\frac{\alpha'}{2}\frac{U}{r}}{g+2\pi\alpha'F+\frac{\alpha'}{2}\frac{U}{r}}\right)^{\{\mu\nu\}}\mathrm{Im}\,(\bar{z}w)^r\\ &+ \sum_{r\in\mathbb{Z}+1/2>0}\left(\frac{g-2\pi\alpha'F-\frac{\alpha'}{2}\frac{U}{r}}{g+2\pi\alpha'F+\frac{\alpha'}{2}\frac{U}{r}}\right)^{[\mu\nu]}\mathrm{Re}\,(\bar{z}w)^r\end{aligned} \tag{3.109}$$

which reproduce the results from [15, 102]. In a similar way the partition function from the fermions can be evaluated to obtain

$$Z_\psi = \prod_{r\in\mathbb{Z}+1/2>0}\det\left(g+2\pi\alpha'F+\frac{\alpha'}{2}\frac{U}{r}\right) \tag{3.110}$$

We note that in the case of vanishing tachyon profile U the disk level partition functions of the bosons and the fermions on the world-sheet the two partition functions cancel each other in agreement with [17].

Preliminaries aside we may now calculate the expectation value of the vertex operator corresponding to the state discussed previously. From the point of view of a purely sigma model calculation, the vertex operator from the $(-1,-1)$ picture

$$|P_{\mu\nu}\rangle \rightarrow : P_{\mu\nu}\psi^{\mu}\tilde{\psi}^{\nu}e^{ikX} : \tag{3.111}$$

will vanish under path integral averaging. In contrast to this however, the fact that this system is annihilated by the BRST charge operator Q suggests that the $(0,0)$ picture is more appropriate in any case, and the corresponding vertex operator is

$$|P_{\mu\nu}\rangle \rightarrow : P_{\mu\nu}\left(\partial X^{\mu} + ik_{\alpha}\psi^{\alpha}\psi^{\mu}\right)\left(\bar{\partial}X^{\nu} + ik_{\beta}\tilde{\psi}^{\beta}\tilde{\psi}^{\nu}\right)e^{ikX} : \tag{3.112}$$

and so averaging we find

$$\begin{aligned}
&\langle : P_{\mu\nu}\left(\partial X^{\mu} + ik_{\alpha}\psi^{\alpha}\psi^{\mu}\right)\left(\bar{\partial}X^{\nu} + ik_{\beta}\tilde{\psi}^{\beta}\tilde{\psi}^{\nu}\right)e^{ikX} :\rangle = \\
&P_{\mu\nu}\Big(\partial\bar{\partial}G_X^{\mu\nu} - k_{\alpha}k_{\beta}\partial G_X^{\mu\alpha}G_X^{\beta\nu} + ik_{\alpha}\partial G_X^{\mu\alpha}ik_{\beta}\tilde{G}_{\psi}^{\beta\nu} + ik_{\beta}\bar{\partial}G_X^{\mu\beta}ik_{\alpha}G_{\psi}^{\alpha\nu} \\
&\qquad\qquad - k_{\beta}\tilde{G}_{\psi}^{\beta\nu}k_{\alpha}G_{\psi}^{\alpha\nu}\Big)\exp\left(-\frac{1}{2}k_{\mu}k_{\nu}G_X^{\mu\nu}\right)
\end{aligned} \tag{3.113}$$

where the Green's functions can be evaluated from expressions (3.108) and (3.109). For the bosonic parts of this expression it has already been shown that the boundary state encodes the interaction content of the sigma model, and for the fermionic degrees of freedom there are relevant calculations that can be found in Appendix B. The coincidence of this with the calculation from the boundary state is another independent check of the boundary state giving the correct overlap with closed string states, which is now for the perturbative superstring.

3.4.3 Euler Number Expansion for Fermions

To follow the same analysis for the fermions in Euler number expansion as for the bosons, the same steps are necessary. First the observation is repeated that the sphere and RP^2 do not have any interactions with these boundary fields and so are not of interest in constructing a stringy action for these fields on the brane, and as before the disk case has been calculated explicitly. Paralleling the development before we can interpret the annulus amplitude as either a tree level self interaction diagram for the brane fields, or in the case of distinct branes as a single particle exchange.

So, for the case $\chi = 0$ we take the fermion boundary state for the NS sector and the overlap given by

$$\begin{aligned}\langle B_\psi|\frac{1}{\Delta}|B_\psi\rangle &= \exp\left(\sum_k \frac{1}{k} Tr\left(\left[N_{nr}^{(a,b)}\chi_\mu^n \alpha \bar{N}_{nj}^{(a,b)}\bar{z}^j g^{\alpha\gamma}\right.\right.\right. \\ &\qquad \left.\left.\left.\bar{M}_{-m-j}^{(a',b')}\chi_{\nu\delta}^m M_{-m-s}^{(a',b')} z^s\right]^k\right)^{rs}_{\mu\nu}\right) \qquad (3.114)\end{aligned}$$

which in the case of vanishing tachyon gives the opposite contribution to (3.60) as seen explicitly in the calculations [17].

3.5 Ghosts and Antighosts

For completeness, we now mention the ghost and antighost systems, but since they do not couple to the boundary interactions, the discussion will be brief. (see [32, 34] for a more detailed discussion) It has been mentioned before that the boundary state is annihilated by the operator $Q_{BRST} + \tilde{Q}_{BRST}$. Expanding Q in terms of ghosts b and c, this condition together with the known form of the boundary state for the bosonic coordinates leads to the conditions

$$(c_n + \tilde{c}_{-n})|B_{bc}\rangle = 0 \qquad (3.115)$$

$$\left(b_n - \tilde{b}_{-n}\right)|B_{bc}\rangle = 0. \tag{3.116}$$

Due to the anticommutation relationships between b and c,

$$\{b_n, c_m\} = \delta_{m+n,0} \tag{3.117}$$

$$\{b_n, b_m\} = \{c_n, c_m\} = 0 \tag{3.118}$$

it is immediately possible to see that this coherent state is given by

$$|B_{bc}\rangle = \exp\left(\sum_n c_{-n}\tilde{b}_{-n} + \tilde{c}_{-n}b_{-n}\right)\frac{1}{2}(c_0 + \tilde{c}_0)|0\rangle \tag{3.119}$$

where $|0\rangle$ is a state which is annihilated by c_n for $n \geq 1$ and by b_n for $n \geq 0$.

Similarly the antighosts arise for the case of the superstring, and we mention here those appropriate for the NS sector, as that was where the tachyon field caused interest. The superghosts contribute to the energy momentum tensor of the string as do all the other fields, and by decomposing the Q_{BRST} into its components and defining η as in the fermionic case the $\beta\gamma$ modes relate according to

$$(\gamma_n + i\eta\tilde{\gamma}_{-n})|B_{\beta\gamma}\rangle = 0 \tag{3.120}$$

$$\left(\beta_n + i\eta\tilde{\beta}_{-n}\right)|B_{\beta\gamma}\rangle = 0 \tag{3.121}$$

and this gives a superghost boundary state as

$$|B_{\beta\gamma}\pm\rangle = \exp\left(\pm i \sum_{n\in\mathbb{Z}+1/2>0} \gamma_{-n}\tilde{\beta}_{-n} - \beta_{-n}\tilde{\gamma}_{-n}\right) \tag{3.122}$$

because the commutators for the $\beta\gamma$ system are

$$[\beta_n, \gamma_m] = \delta_{m+n,0} \tag{3.123}$$

$$[\beta_n, \beta_m] = [\gamma_n, \gamma_m] = 0. \tag{3.124}$$

3.6 Summary

While the boundary states for the ghosts and antighosts are well known [31, 32], we have developed in this chapter the boundary states for both bosons and fermions corresponding to the boundary string field theory actions

$$\begin{aligned} S(g,F,T_0,U) &= \frac{1}{4\pi\alpha'}\int_M \partial^a X^\mu \partial_a X_\mu + \psi_+^\mu \partial_- \psi_+^\nu + \psi_-^\mu \partial_+ \psi_-^\nu \\ &+ \oint_{\partial M} \left(\frac{1}{2}F_{\mu\nu}X^\nu \partial_\phi X^\mu + \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu\right), \\ &+ \oint_{\partial M} F_{\mu\nu}\left(\psi_+^\mu \psi_+^\nu - \psi_-^\mu \psi_-^\nu\right) + U_{\mu\nu}\left(\psi_+^\mu \frac{1}{\partial_\phi}\psi_+^\nu - \psi_-^\mu \frac{1}{\partial_\phi}\psi_-^\nu\right) \end{aligned} \tag{3.125}$$

which is given by

$$|B\rangle = Z|B_X\rangle|B_\psi\rangle|B_{bc}\rangle|B_{\beta\gamma}\rangle \tag{3.126}$$

with the normalization determined by the comparison of the overlap with closed string states to the analogous calculation in the world-sheet sigma model, and the integration over PSL(2,R) implicit. $|B_X\rangle$, $|B_\psi\rangle$, $|B_{bc}\rangle$, and $|B_{\beta\gamma}\rangle$ are given respectively by equations 3.19, 3.99, 3.119, and 3.122. This boundary state correctly reproduces the emission of particles by the brane described by the boundary interaction, and can be thought of as a state interpolating between the renormalization group fixed points of tachyon condensation.

Chapter 4

Generalized Boundary Interactions

In the previous chapter we considered exclusively the states in which the background consisted of a tachyon field with a quadratic boundary interaction, and also a constant antisymmetric gauge field. While these are interesting and have generated a great deal of investigation and study [4, 17, 68, 101] they clearly cannot be the whole story, because they do not exhaust the possible interactions on the boundary of the string world-sheet, including the possibility of interactions higher than quadratic. The programme in string theory is to regard these as coupling constants that generate higher order interactions on the string world-sheet and on the boundary; a famous example of which is the spacetime metric tensor which appears in the string action, when it is expanded around Minkowski spacetime it gives a massless two dimensional theory plus interaction terms,

$$\int \partial X^{\mu}\bar{\partial}X^{\nu}G_{\mu\nu}\left(X^{\alpha}\right) \to \int \partial X^{\mu}\bar{\partial}X^{\nu}\left(\eta_{\mu\nu} + \left(\partial^{\alpha}_{X}G_{\mu\nu}\left(X^{\beta}_{0}\right)\right)X^{\alpha} + \dots\right).$$

A general theory which has such arbitrary interactions is difficult to solve analytically without a great deal of symmetry [51]. While the study of general world-sheet actions is beyond the scope of this thesis, a much more general class of boundary interactions is available for investigation.

There are two directions, not mutually exclusive, that this can take, the first is to investigate additional constant fields and couplings in the context of the boundary state. These will add to the spectrum of possible fields and charges on the hyper-surface that spans the edge of the string world sheet. The second is to have non-quadratic interactions on the boundary, which will, in general, make it difficult to write out a nice and compact expression for the boundary state, but none the less, it is possible to recover some kind of dynamics for the strings from this. In all of this consideration there are still two general principles that govern the analysis, that the boundary states algebrize the world-sheet action, and that the effect of conformal transformations is accounted for.

4.1 Additional Boundary State Fields

We wish to demonstrate that the boundary state is not applicable only to the case of a tachyon field and gauge field, but also to fields of different world-sheet dimension. We will first examine the addition of different fields into the boundary state, which amounts to, in the bosonic case adding fields and interactions in the form

$$S = \int_M \partial X \bar{\partial} X + \int_{\partial M} T(X^\mu) + A_\mu(X)\partial X^\mu + B_\mu(X)\partial^2 X^\mu + C_\mu(X)\partial^3 X^\mu + \dots \tag{4.1}$$

For the purely quadratic case, each of those fields is expanded to linear order with the exception of the tachyon discussed before. The partition function and world-sheet two point functions are straightforward generalizations of the case examined in (3.33). In particular we find that the disk propagator becomes

$$\begin{aligned} G^{\mu\nu}(z, z') &= -\alpha' g^{\mu\nu} \ln |z - z'| \\ &\quad + \frac{\alpha'}{2} \sum_{n=1}^{\infty} \frac{(z\bar{z}')^n + (\bar{z}z')^n}{n} \\ &\quad \left(\frac{g - \frac{\alpha'}{2}\frac{U}{n} - 2\pi\alpha' F - \alpha' n B - \alpha' n^2 C + \dots}{g + \frac{\alpha'}{2}\frac{U}{n} + 2\pi\alpha' F + \alpha' n B + \alpha' n^2 C + \dots} \right)^{\{\mu\nu\}} \\ &\quad + \frac{\alpha'}{2} \sum_{n=1}^{\infty} \frac{(z\bar{z}')^n - (\bar{z}z')^n}{in}. \end{aligned}$$

$$\left(\frac{g - \frac{\alpha'}{2}\frac{U}{n} - 2\pi\alpha' F - \alpha' n B - \alpha' n^2 C + \dots}{g + \frac{\alpha'}{2}\frac{U}{n} + 2\pi\alpha' F + \alpha' n B + \alpha' n^2 C + \dots}\right)^{\{\mu\nu\}} \tag{4.2}$$

Here we have used the expansion (3.4) for the tachyon $T(X)$, assumed a prefactor of $\frac{1}{4\pi}$ for all the additional fields, and used the convention that B, C, and the higher terms in this expansion are the field strength associated with the corresponding field in (4.1), so

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(X) \\ B_{\mu\nu} &= \partial_\mu B_\nu(X) + \partial_\nu B_\mu(X) \\ C_{\mu\nu} &= \partial_\mu C_\nu(X) - \partial_\nu C_\mu(X) \\ &\dots \end{aligned} \tag{4.3}$$

In this background the disk partition function becomes

$$Z = \frac{1}{\det\left(\frac{U}{2}\right)} e^{-T_0} \prod_{m=1}^{\infty} \frac{1}{\det\left(g + \frac{\alpha'}{2}\frac{U}{m} + 2\pi\alpha' F + \alpha' n B + \alpha' n^2 C + \dots\right)}. \tag{4.4}$$

Additional fields such as those described above were discussed in [74], with boundary interaction

$$S_{bdy} = a + \frac{1}{8\pi}\oint d\theta d\theta' X^\mu(\theta) u_{\mu\nu}(\theta - \theta') X^\nu(\theta') \tag{4.5}$$

where θ parameterizes the boundary. The boundary coupling u in (4.5) preserves locality in the sense that the Taylor expansion consists of derivatives of a δ-function

$$u^{\mu\nu}(\theta - \theta') = \sum t^{\mu\nu} \frac{\partial^r}{\partial_\theta^r} \delta(\theta - \theta'). \tag{4.6}$$

The boundary coupling u can also be Fourier analyzed as

$$u_k^{\mu\nu} = u_{-k}^{\nu\mu} = \oint d\theta u^{\mu\nu}(\theta) e^{-ik\theta}. \tag{4.7}$$

In the language of (4.1) this corresponds to having the couplings

$$u_k^{\mu\nu} = U^{\mu\nu} + kF^{\mu\nu} + k^2 B^{\mu\nu} + k^3 C^{\mu\nu} + \dots \tag{4.8}$$

and the partition function is determined to be [74]

$$Z = \frac{1}{\det\left(\frac{u_0}{2}\right)} e^{-a} \prod_{k=1}^{\infty} \det\left(1 + \frac{u_k}{k}\right)^{-1} \tag{4.9}$$

in agreement with (4.4) with the identifications $a = T_0$, $u_0 = U$ and (4.8). It is also shown in [74] that using a point splitting regularization it is possible to introduce a short distance cut-off and truncate the expansion (4.1) and then renormalize with respect to this cut-off. For our purposes it is sufficient to note, using the formal relationship

$$X(\theta)X(\theta+\epsilon) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} X(\theta) \frac{\partial^n}{\partial_\theta^n} X(\theta) \tag{4.10}$$

it is possible to use a boundary state to describe a non-local boundary interaction. The boundary state corresponding to the action (4.1) is, just as in (3.19)

$$\begin{aligned} |B_{a,b}\rangle \; = \; & Z \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^{\mu}_{-k} M^{(a,b)}_{-n-k} \Xi^{n}_{\mu\nu} \bar{M}^{(a,b)}_{-n-j} \tilde{\alpha}^{\nu}_{-j} \right) \\ & \exp\left(-\frac{\alpha'}{4} x^{\mu} U_{\mu\nu} x^{\nu} \right) |0\rangle. \end{aligned} \tag{4.11}$$

with

$$\Xi^n_{\mu\nu} = \frac{1}{n} \left(\frac{g - \frac{\alpha'}{2}\frac{U}{n} - 2\pi\alpha' F - \alpha' n B - \alpha' n^2 C + \dots}{g + \frac{\alpha'}{2}\frac{U}{n} + 2\pi\alpha' F + \alpha' n B + \alpha' n^2 C + \dots} \right)^{\mu\nu}. \tag{4.12}$$

This will reproduce all the σ model amplitudes [70, 74]. We can also consider the case of interaction terms that are explicitly non-local. Due to the (anti)symmetry requirement (4.7) still holds and so using equation 4.8 the non-local can be recast into a set of local boundary interactions, potentially infinite in number.

This generalization has the following interesting property, it was noted that the transition from $U = 0$ to $U = \infty$ was characterizing the transition between Neumann and Dirichlet boundary conditions. This could be seen in the boundary state expression because for large values of U the coefficient in the exponential simplified to the well known expression for the boundary state for a D brane. Now, restricting attention to the case of the additional field B which appeared as $\oint_{\partial M} B_\mu \partial^2 X^\mu$ or equivalently the first term in the Taylor expansion $\oint_{\partial M} B_{\mu\nu} \partial X^\mu \partial X^\nu$ where $B_{\mu\nu} = \partial_\mu B_\nu + \partial_\nu B_\mu$, as detailed in equation 4.3 we see that the statement is still the same. In the case that $B \to \infty$ the strings will satisfy the regular Dirichlet boundary conditions, but they will not have a condition upon their zero mode. We can contrast the effects of the tachyon's U with this B. World-sheet excitations with large enough mode number will overcome the effect of U, since it appears as $U/n \to 0$ as $n \to \infty$, but by contrast B appears as nB which grows with increasing n and makes the system 'more' Dirichlet in the UV. Of course, this is just another way of saying that the coupling U is irrelevant in world-sheet power counting, and that B is relevant. To speculate what the effects of this kind of background field are, consider the case of a region of space where B is non-zero. An end of a string in that region will not leave for the same reason that it would not leave a brane's surface. A large region like that could model an extended object which traps strings near its boundaries.

Another similar point is that if there are large U and B couplings on the string boundary, there will be a finite number of modes which dominate the partition function, equivalently the action for these background fields, eliminating the need to regularize the expression. We now examine this thought somewhat more systematically.

4.2 Time Dependent Tachyons

It is also possible to study tachyons of a more general profile, as discussed in [43, 85, 86, 93, 94]. The motivation for this kind of study is to examine the time dependence of the tachyons described by a boundary state as opposed to simply the static solutions that describe either the tachyon vacuum or static D-branes of lower dimension. This allows a more sophisticated analysis of the dynamics that describe the decay of a space-filling brane into one of smaller dimension.

A model for the study of this process is the boundary interaction term

$$\delta S = \tilde{\lambda} \oint_{\partial M} \cosh X^0 \tag{4.13}$$

where the bulk action is the standard bosonic string action and the term $\tilde{\lambda}$ is written to conform with the conventions of [86, 95]. It has been previously shown [85] that this type of deformation is amenable to study. In fact, a compact expression for the boundary state for this type of perturbation is known to be

$$|\mathcal{B}\rangle = \mathcal{N} \sum_j \sum_m D^j_{m,-m}(R) |j, m, m\rangle\rangle \tag{4.14}$$

where j runs over non-negative integer and half integers and can be interpreted as a spin, m stands in the roll of projection of spin j, R is a rotation matrix in $SU(2)$ which can be parameterized as $R = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, and $|j, m, m\rangle\rangle$ is a Virasoro Ishibashi state [61] associated with the primary state $|j, m, m\rangle$ with momentum $2m$ and conformal weight (j^2, j^2). The matrix elements of D are defined, for the parameterization of R given, by the formula [43]

$$D^j_{m,n}(R) = \sum_{k=\max(0,n-m)}^{\min(j-m,j+n)} \frac{\sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}}{(j-m-k)!(j+n-k)!k!(m-n+k)!} a^{j+n-k} \bar{a}^{j+n-k} b^k (-\bar{b})^{m-n+k} \tag{4.15}$$

and also the primary state can be expressed, up to a phase as

$$|j, m, m\rangle = k\mathcal{O}_{j,m} \exp(2imX)\,|0\rangle \tag{4.16}$$

where $\mathcal{O}_{j,m}$ are a combination of oscillators with left- and right-moving level of $j^2 - m^2$. Since the potential has been specialized to be in the X^0 direction it is possible to pick out the coefficient of the part of the boundary state that has no dependence on the α^0 or $\tilde{\alpha}^0$ oscillators. Note that the other 25 bosonic directions are given by a boundary state like (3.10) but with all external fields vanishing giving

$$|B_{25}\rangle = \exp\left(-\sum_{i=1}^{25}\sum_{n\geq 1}\frac{\alpha^i_{-n}\alpha^i_n}{n}\right) \tag{4.17}$$

as in [34]. Now, fixing the phases by comparison with known configurations [93, 95] we are able to find that for the hyperbolic cosine perturbation (4.13) it is

$$|\mathcal{B}_0\rangle = \mathcal{N}\left[1 + 2\sum_{n\geq 1}\left(-\sin(\tilde{\lambda}\pi)\right)^n \cosh\left(nX^0\right)\right]|0\rangle \tag{4.18}$$

which can be explicitly summed to give a time dependent constant in front of the X^0 mode independent part of $|\mathcal{B}_0\rangle$, and obtain $|\mathcal{B}_0\rangle = \mathcal{N}f(x^0)|0\rangle$ with

$$F(x^0) = \frac{1}{1 + e^{x^0}\sin\left(\tilde{\lambda}\pi\right)} + \frac{1}{1 + e^{-x^0}\sin\left(\tilde{\lambda}\pi\right)} - 1. \tag{4.19}$$

In all of this, $\sin\left(\tilde{\lambda}\pi\right)$ is a parameter from the $SU(2)$ transformation necessary to put the boundary state in this form.

Similarly, it is interesting to find the coefficient of the term associated with the purely time (00) component of the graviton, that is the coefficient of the state

$$\alpha^0_{-1}\tilde{\alpha}^0_{-1}|k\rangle \in |\mathcal{B}\rangle$$

which is found to be

$$g(x^0) = \cos\left(2\tilde{\lambda}\pi\right) + 1 - f(x^0) \tag{4.20}$$

The sum of g and f is conserved, independent of x^0, and can be interpreted as the conserved energy density on an unstable d-brane by observing that the sum goes, in the small $\tilde{\lambda}$ regime as

$$f(x^0) + g(x^0) \to 2\left(1 - \tilde{\lambda}^2\pi^2\right) \tag{4.21}$$

but on the other hand the d-brane tension is given as $\frac{1}{2\pi^2 g^2}$ with g the open string coupling constant, and from the point of view of string field theory the potential energy for the tachyon field deformed by $\tilde{\lambda}$ is $-\frac{\tilde{\lambda}^2}{2g^2}$ and summing the two, one obtains the total energy [86, 93, 95]

$$\frac{1}{2\pi^2 g^2}\left(1 - \tilde{\lambda}^2\pi^2\right)$$

which is proportional to (4.21), and this then shows that it is correct to interpret the sum $f(x^0) + g(x^0)$ as the total energy density of the system of branes. This kind of construction will give the evolution of the normalization of the boundary states which describe the spatial d-brane. The explicit form of $f(x^0)$ is such that for boundary perturbations where $\sin\left(\tilde{\lambda}\pi\right) > 0$ we have

$$f(x^0) \to 0 \text{ as } x^0 \to \infty \tag{4.22}$$

which can be interpreted as a decay of the states with Neumann boundary conditions in all spatial directions.

Following [86, 95] it is possible to generalize this sort of construction to something that has spatial inhomogeneities rather than just some time dependence. A natural candidate in the spirit of (4.13) is the boundary interaction

$$\delta S = \tilde{\lambda} \oint_{\partial M} \cosh\frac{X^0}{\sqrt{2}} \cos\frac{X^1}{\sqrt{2}}. \tag{4.23}$$

As is apparent from the form of the interaction, this will be solvable in the same sense that (4.13) was, and further it can be seen to decompose into boundary states that are purely functions of $X^0 \pm iX^1$, and so the analysis above can be repeated. The boundary state describing this decouples as

$$|B\rangle = |B_{X^0,X^1}\rangle \otimes |B_{X^\mu,\mu\neq 0,1}\rangle \otimes |B_{b,c}\rangle \tag{4.24}$$

where

$$|B_{X^0,X^1}\rangle = |B_+\rangle \otimes |B_-\rangle \tag{4.25}$$

and following [86] it is possible to find that

$$\begin{aligned}
|B_\pm\rangle &= f(x^0 \pm ix^1)|0\rangle \\
&+\frac{1}{2}g(x^0 \pm ix^1)\left(\alpha^0_{-1} \pm i\alpha^1_{-1}\right)\left(\tilde{\alpha}^0_{-1} \pm i\tilde{\alpha}^1_{-1}\right)|0\rangle \\
&+\frac{1}{4}h_1(x^0 \pm ix^1)\left(\alpha^0_{-2} \pm i\alpha^1_{-2}\right)\left(\tilde{\alpha}^0_{-2} \pm i\tilde{\alpha}^1_{-2}\right)|0\rangle \\
&+\frac{1}{4}h_2(x^0 \pm ix^1)\left(\alpha^0_{-1} \pm i\alpha^1_{-1}\right)^2\left(\tilde{\alpha}^0_{-1} \pm i\tilde{\alpha}^1_{-1}\right)^2|0\rangle \\
&+\frac{i}{4}h_3(x^0 \pm ix^1)\left(\alpha^0_{-1} \pm i\alpha^1_{-1}\right)^2\left(\tilde{\alpha}^0_{-2} \pm i\tilde{\alpha}^1_{-2}\right)|0\rangle \\
&+\frac{i}{4}h_3(x^0 \pm ix^1)\left(\alpha^0_{-2} \pm i\alpha^1_{-2}\right)\left(\tilde{\alpha}^0_{-1} \pm i\tilde{\alpha}^1_{-1}\right)^2|0\rangle + \ldots
\end{aligned} \tag{4.26}$$

and similarly the implicit coefficient functions are determined to be

$$\begin{aligned}
f(x^0 \pm ix^1) &= \frac{1}{1+\exp\left(\frac{x^0 \pm ix^1}{\sqrt{2}}\right)\sin(\tilde{\lambda}\pi/2)} + \\
&\quad \frac{1}{1+\exp\left(-\frac{x^0 \pm ix^1}{\sqrt{2}}\right)\sin(\tilde{\lambda}\pi/2)} - 1 &(4.27)\\
g(x^0 \pm ix^1) &= 1+\cos(\tilde{\lambda}\pi) - f(x^0 \pm ix^1) &(4.28)\\
h_1(x^0 \pm ix^1) &= \Big(1+\cos(\tilde{\lambda}\pi)\Big)\Big(1-\sin(\tilde{\lambda}\pi/2)\Big)\cosh\left(\frac{x^0 \pm ix^1}{\sqrt{2}}\right) \\
&\quad - f(x^0 \pm ix^1) &(4.29)
\end{aligned}$$

$$\begin{aligned} h_2(x^0 \pm ix^1) &= 2\left(1+\cos(\tilde{\lambda}\pi)\right)\sin(\tilde{\lambda}\pi/2)\cosh\left(\frac{x^0 \pm ix^1}{\sqrt{2}}\right) \\ &\quad + f(x^0 \pm ix^1) \end{aligned} \tag{4.30}$$

$$h_3(x^0 \pm ix^1) = -\left(1+\cos(\tilde{\lambda}\pi)\right)\sin(\tilde{\lambda}\pi/2)\sinh\left(\frac{x^0 \pm ix^1}{\sqrt{2}}\right) \tag{4.31}$$

Now, using these expressions it is possible to expand the boundary states for arbitrary oscillators, and we see that, for instance, the coefficient of the tachyon mode (that is to say the Fock space vacuum $|0\rangle\rangle$) is the same as the coefficient of some of the purely spatial components of the graviton (for instance $(\alpha^3_{-1}\tilde{\alpha}^4_{-1} + \alpha^4_{-1}\tilde{\alpha}^3_{-1})|0\rangle\rangle$, with similar relationships occurring among all oscillator combinations with the same holomorphic and antiholomorphic levels in $X^0 \pm iX^1$. This can be briefly compared to the case of the quadratic tachyon profile considered in section 3.2. The similarity to the current consideration is that the states appearing in the boundary state expansion do not necessarily have equal numbers of creation operators at the same level as they would in the case of pure Neumann or Dirichlet boundary conditions (compare $\alpha_{-1}\alpha_{-1}\tilde{\alpha}_{-2}|0\rangle$ with $\alpha_{-1}\alpha_{-1}\tilde{\alpha}_{-1}\tilde{\alpha}_{-1}|0\rangle\rangle$)

The interesting point that arises from this analysis is that, as in the purely time dependent case there is time evolution of the coefficient functions. This evolution is analyzed in detail in [86] and it is found that the energy density evolves off a brane and becomes localized, showing the decay of a space filling brane into something smaller.

4.3 Spherically Symmetric Tachyon Condensation

Another possible generalization within the study of tachyon condensation is to consider, as in [54], a more symmetric case in which the symmetry renders the analysis more tractable. The problem considered was that of the condensation of open string tachyon fields which have an $O(D)$ symmetric profile. In the context of the quadratic tachyon profile studied in 3.2 this is simply the problem of condensation from a space-filling brane to a spherical symmetric state by decay of the radial direction.

This problem is investigated by using the observation of [27] that the bulk excitations can be integrated out of the partition function to get an effective non-local field theory which lives on the boundary. The problem is then reduced to a boundary conformal field theory with D scalar fields on a disc perturbed by relevant boundary operators with $O(D)$ symmetry. The model is exactly solvable in the large D limit and admits a tractable $1/D$ expansion, which only is consistent for tachyon fields that are polynomials. In the case of tachyon fields that are polynomial the theory is renormalizable by normal ordering, but in the case of non-polynomial tachyon potentials it is possible to have large anomalous dimensions for the operators and that these may require non-perturbative renormalization which could make the β-function nonlinear. This nonlinearity combined with the vanishing of the β-function as a field equation for the tachyon profile gives terms that describe tachyon scattering [64, 65]. However when the tachyon profile, and the other fields are adjusted so that the sigma model that they define is at an infrared fixed point of the renormalization group, these background fields are a solution of the classical equation of motion of string theory. Witten and Shatashvili [97, 104] have argued that these equations of motion can be derived from an action which is derived from the disc partition function.

We start with the world-sheet action

$$S = \int_M \partial X \bar{\partial} X + \int_{\partial M} T(X)$$

and breaking the field X into classical and quantum parts in the standard way

$$X = X_c + X_q$$

and X_c satisfies the wave equation on the entire surface and $X_q \to 0$ on the boundary. Integrating out one obtains the action decouples between the classical and quantum parts as

$$S = \int_M \partial X_q \bar{\partial} X_q + \int_{\partial M} \left(\frac{1}{2} X_c |\partial| X_c + T(X_c) \right) \tag{4.32}$$

where the term $|\partial|$ gives a non-local contribution to the kinetic term, defined by its Fourier transform

$$|\partial| \delta(\phi - \phi') = \sum_{n>0} \frac{n}{\pi} \cos n(\phi - \phi') \tag{4.33}$$

The quantum term is nothing but the partition function of the string with Dirichlet boundary conditions, and in the absence of the tachyon field the integration over the classical fields on the boundary will give the terms to convert from the Dirichlet to Neumann boundary conditions on the partition function.

Investigating the large D limit of this $O(D)$ invariant model we reparameterize

$$T(X) \to DT\left(X^2/D\right). \tag{4.34}$$

We introduce auxiliary fields and a source to the partition function of the boundary field theory, as

$$Z = Z_0 \int dX d\chi d\lambda \exp(-S) \tag{4.35}$$

with

$$S = \int \frac{d\phi}{2\pi} \left(\frac{1}{2} X^i |\partial + 2i\lambda| X^i + DT(\chi) - Di\lambda\chi - J^i X^i \right) \tag{4.36}$$

where λ is a scalar that enforces the condition $\chi = \frac{X^i X^i}{D}$ and J^i is a source term. As was apparent from the initial form of the action the zero modes of X are special in that they naively contribute a constant term in the two dimensional partition function (4.35) and also since in the above action all the X terms appear quadratically it is convenient to integrate out the non-zero modes (X_0) which then gives the effective action

$$\begin{aligned} S_{\text{eff}} &= \frac{D}{2} Tr \ln\left(|\partial| + 2i\lambda\right) + D\int \frac{d\phi}{2\pi}\left(T(\chi) - i\lambda\left(\chi - \frac{X^i X^i}{D}\right) - \frac{X^i J^i}{D}\right. \\ &\left. -\frac{1}{2D}\int \frac{d\phi}{2\pi}\left(J(\phi) - 2i\lambda X_0\right) G_X(\phi, \phi', 2i\lambda)\left(J(\phi') - 2i\lambda X_0\right)\right] \end{aligned} \quad (4.37)$$

where both the trace and the boundary greens function for X are only defined on non-zero modes as those were the ones integrated out. In the large D limit the integrals over χ and λ can be done using a saddle point method obtaining the equations

$$T'(\chi) = i\lambda \quad (4.38)$$

and

$$\chi = \frac{1}{D}\left(X + x(\phi)\right)^i \left(X + x(\phi)\right)^i + G_X(\phi, \phi, 2T'(\chi)). \quad (4.39)$$

In this $x(\phi)$ is the induced classical field

$$x(\phi) = \int \frac{d\phi'}{2\pi} G_X(\phi, \phi', 2T'(\chi))\left(J(\phi') - 2T'(\chi(\phi'))X_0\right), \quad (4.40)$$

and the saddle point relation for λ has already been used to simplify the expressions. Thus to leading order in the large D limit, the partition function is given by

$$Z = Z_0 \int dX_0 \exp -S_{\text{eff}}[\chi_0, \lambda_0, X_0] \quad (4.41)$$

where χ_0 and λ_0 are solutions of (4.39) and (4.38) respectively. This analysis can be extended to higher orders in $\frac{1}{D}$ as well.

When considering the effective action (4.37) we note that there are divergences that must be renormalized. It was argued in [54] that while the logarithm of the mode number contributes a divergence that can be regularized by ζ function regularization there remains a truly divergent term which multiplies the tachyon, and which can be subtracted by the renormalization transformation

$$T(\chi) \rightarrow: T(\chi - 2\zeta(1) - 2c): \tag{4.42}$$

where c is an arbitrary constant that should be fixed by a renormalization prescription. With the substitution of this into the effective action above, one can obtain an expression for an effective action that is finite, up to an arbitrary parameter that was discussed in [101].

Interpreting the ζ function as being involved with the cutoff of the theory at large world-sheet momentum we can see that taking the logarithmic derivative of $:T:$ will give a linear β-function for the tachyon field at this order [64, 65]

$$\beta(:T:) = -:T: -2:T':\tag{4.43}$$

which is the large D limit of the tachyon wave operator.

A transparent way to understand the content of the classical partition function is to consider the limit where $T(X)$ is a smooth function and to expand in derivatives of T. To do this, we set the source J to zero. Then, we expect that the condensate χ is a constant, independent of ϕ. Then, the Green function can easily be evaluated. It is most useful to consider an expansion of (4.39) (after renormalization)

$$\chi = \frac{\hat{X}^2}{D} - 2c_1 + 2\sum_{p=1}^{\infty}\zeta(p+1)(-2T'(\chi))^p \tag{4.44}$$

and sums of this type appear in the analysis in [54] Substituting into (4.37) we find

$$Z = Z_0 \int dX_0 e^{-DT(X_0^2/D)}$$

$$\times \left(1 - 2c_1 DT'(\frac{X_0^2}{D}) + 2D\zeta(2)\left[T'\left(\frac{X_0^2}{D}\right)\right]^2 + \dots\right) \tag{4.45}$$

and the omitted terms are of higher orders in derivatives of T by its argument. Now calculating the action, as discussed in [54] and around (2.24) given by

$$\begin{aligned} S &= \left(1 + \int \beta(T)\frac{\delta}{\delta T}\right) Z \\ &= Z_0 \int dX_0 e^{-DT\left(\frac{X_0^2}{D}\right)} \left\{1 + DT\left(\frac{X_0^2}{D}\right) + 2DT'\left(\frac{X_0^2}{D}\right)\right. \\ &\qquad \left.\left[1 - c_1 DT\left(\frac{X_0^2}{D}\right)\right]\right\} \end{aligned} \tag{4.46}$$

Which exactly coincides with the result of [101].

Chapter 5

Conclusions and Future Directions

In this thesis we investigated the interplay between interactions on the string world-sheet boundary, conformal invariance, and tachyon condensation. We have reviewed some of the background and developments which motivate the study of tachyon condensation. We developed a boundary state appropriate for non-conformally invariant boundary interactions [4, 70], used this boundary state to calculate higher genus string diagrams [69]. Where possible we verified that the amplitudes we obtained coincide with the known results calculated with other methods. We have commented on the applicability of our boundary state to other boundary interactions, including ones that violate world-sheet locality, and explored other ways to analyze tachyon condensation in Chapter 4 [54].

The boundary state

$$|B\rangle = \int d^2a d^2b \delta(|a^2| - |b^2| - 1)|B_{a,b}\rangle$$

with

$$\begin{aligned} |B_{a,b}\rangle &= Z \exp\left(\sum_{n=1,j,k=-\infty}^{\infty} \alpha^{\mu}_{-k} M^{(a,b)}_{-n-k} \Lambda^{n}_{\mu\nu} \bar{M}^{(a,b)}_{-n-j} \tilde{\alpha}^{\nu}_{-j} \right) \\ &\quad \exp\left(-\frac{\alpha'}{4} x^{\mu} U_{\mu\nu} x^{\nu} \right) |0\rangle. \end{aligned}$$

has been shown to correctly reproduce sigma model particle emission amplitudes, and thus describes a brane in the process of tachyon condensation. As the parameter U runs under RG flow from 0 to ∞ the string world-sheet undergoes a change from Neumann to Dirichlet boundary conditions, and this boundary state gives a smooth interpolation between the two. The normalization coefficient Z has been shown elsewhere [68] to correctly reproduce the expected [91] ratios between brane tensions during tachyon condensation, and this strengthens the interpretation of $|B\rangle$ as a brane. A similar boundary state was found in the superstring case as well, with the same properties.

We also use the boundary state to calculate higher genus amplitudes. For the case of a conformally invariant boundary we exactly reproduce the known results at $\chi = 0$ for a constant background gauge field. We also provide a concrete realization of the proposal [28] for the string loop corrections to tachyon condensation, manifestly reproducing the closed string factorization properties in the off-shell case considered.

Chapter 4 examines other boundary interactions, and details several different methods of probing their structure. We review the construction of boundary state for time dependent backgrounds. It exhibits many similarities to the conformally integrated boundary state defined above which suggests that these boundary states are also appropriate for the examination of the time dependent structure of tachyon decay. Also, we examined the $1/D$ expansion as an additional way of probing the properties of tachyon condensation.

This work highlights several opportunities for future research and investigation. The boundary state constructed in Chapter 3 is well understood in the context of boundary string field theory. As this state represents a tachyon in the process of condensing, it would be very interesting to study its representation in cubic string field theory. It would similarly be interesting to extend the analysis in section 3.3 to higher genus, and also to attempt cross-checks on the quantities calculated there. Also, as alluded to in Chapter 4 there is a natural connection between the Ishibashi states [61] used to describe the time dependent tachyon condensation and the boundary state

$|B\rangle$, and it is possible to include time dependent coefficients for the spatial directions in analogy with [86, 93].

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Appendix A

Properties of the Conformal Transformation Matrices

In this appendix we examine some of the properties of the matrices that perform the conformal transformation which maps

$$\omega = \frac{az+b}{\bar{b}z+a}$$

on the degrees of freedom in the bosonic and fermionic sectors respectively.

A.1 Bosonic Matrix $M_{mn}^{(a,b)}$

As discussed in chapter 3 the matrix that maps the bosonic degrees of freedom to one another under the conformal transformation above is

$$M_{mn}^{(a,b)} = \oint \frac{dz}{2\pi i} z^m \frac{(\bar{b}z+\bar{a})^{n-1}}{(az+b)^{n+1}} \tag{A.1}$$

with the contour for the integral around the unit circle, as seen in (3.18). This matrix has a simple block structure, and the elements in each block can be evaluated and are enumerated below. There are a total of nine cases.

First, $m > 0$, $n > 0$ has a pole of order $n+1$ at $-\frac{b}{a}$, and can be evaluated as

$$M_{mn}^{(a,b)} = \frac{1}{n!}\partial^n \frac{1}{a^{n+1}} z^m (\bar{b}z+\bar{a})^{n-1}\Big|_{-b/a} \tag{A.2}$$

and some of the elements of this are given explicitly as

$$\begin{aligned} M_{1n}^{(a,b)} &= \frac{1}{n}\frac{\bar{b}^{n-1}}{a^{n+1}} \\ M_{m1}^{(a,b)} &= m\frac{(-b)^{n-1}}{a^{n+1}} \end{aligned} \tag{A.3}$$

The case $m > 0$, $n = 0$ is immediately evaluated as

$$M_{m0}^{(a,b)} = \left(\frac{-b}{a}\right)^m \tag{A.4}$$

Examining $m > 0$, $n < 0$ there are no poles within the contour so the matrix vanishes.

The case $m = 0$, $n > 0$ can be obtained from the residue theorem as

$$M_{0n}^{(a,b)} = \frac{1}{n!}\partial^n \frac{1}{a^{n+1}}(\bar{b}z + \bar{a})^{n-1}\Big|_{-b/a} = 0 \tag{A.5}$$

Similarly we determine $M_{00}^{(a,b)} = 1$, and in the case of $m = 0$, $n < 0$ there are again no poles within the integration contour so the matrix elements vanish.

Now, for the case of $m < 0$, $n > 0$ we have poles at both zero and $-b/a$.

$$M_{mn}^{(a,b)} = \oint \frac{dz}{2\pi i} z^{-|m|} \frac{(\bar{b}z + \bar{a})^{n-1}}{(az + b)^{n+1}} \tag{A.6}$$

but with the transformation $z \to \omega = 1/z$ we can rewrite the integral as

$$M_{mn}^{(a,b)} = \oint \frac{d\omega}{2\pi i} w^{|m|} \frac{(\bar{b} + \bar{a}\omega)^{n-1}}{(a + b\omega)^{n+1}} \tag{A.7}$$

and the negative sign from the differential is compensated for by the switch of integration directions. This new expression can be seen, as for the $m > 0$, $n < 0$ case to have no poles within the contour and thus to vanish.

For the case $m < 0$, $n = 0$ there are again two poles, a pole of order m at 0 and a simple pole at $z = -b/a$. This can be evaluated by either performing the redefinition above $z \to \omega = 1/z$ in which case it is obvious that the expression is just the complex conjugate of the $m > 0$, $n = 0$ case, or it can be evaluated directly which we do for illustrative purposes here in the case $m = -2$.

$$M_{20}^{(a,b)} = \oint \frac{dz}{2\pi i} z^{-2} \frac{1}{(\bar{b}z + \bar{a})(az + b)}$$

$$= \left(\frac{-b}{a}\right)^{-2} + \partial \frac{1}{(\bar{b}z+\bar{a})(az+b)}\Bigg|_0$$
$$= \frac{a^2}{b^2} - \frac{1}{b^2\bar{a}} - \frac{\bar{b}}{b\bar{a}^2}$$
$$= \frac{a^2\bar{a}^2 - a\bar{a} - b\bar{b}}{b^2\bar{a}^2} = \frac{\bar{b}^2}{\bar{a}^2} \tag{A.8}$$

exactly as expected from the previous considerations.

Finally for the case $m < 0, n < 0$ the redefinition $z \to \omega = 1/z$ gives the equality immediately

$$\bar{M}^{(a,b)}_{|m|\;|n|} = M^{(a,b)}_{-|m|\;-|n|} \tag{A.9}$$

This analysis confirms a kind of block diagonal structure, and ensures that, as advertised, there is no mixing between creation and annihilation operators. There is however a flow to the zero mode which reflects a natural redefinition of the momentum after a conformal transformation. This was important in the work on the bosonic degrees of freedom to ensure that the overlap between the boundary state $|B\rangle$ and a particle matched the sigma model expectation value for the corresponding vertex operator.

While perhaps obvious, we now check that the expected composition laws hold for these matrices. So, calculating we find

$$M^{(a,b)}_{mn} M^{(a',b')}_{nk} = \sum_n \oint \frac{dz}{2\pi i}\frac{d\omega}{2\pi i} z^m \frac{(\bar{b}z+\bar{a})^{n-1}}{(az+b)^{n+1}} \omega^n \frac{(\bar{b}'\omega+\bar{a}')^{k-1}}{(a'\omega+b')^{k+1}} \tag{A.10}$$

Since we know that the positive and negative elements of this matrix are complex conjugates of each other, and further the structure on M_{m0} and M_{0n} we can restrict the sum to be from 1 to ∞ and concentrate only on the annihilation operators, knowing that the sum will also work for the creation operators, so

$$M^{(a,b)}_{mn} M^{(a',b')}_{nk} = \oint \frac{dz}{2\pi i}\frac{d\omega}{2\pi i} z^m \frac{\omega}{(az+b)} \frac{1}{(a-\bar{b}\omega)z + (b-\bar{a}\omega)} \frac{(\bar{b}'\omega+\bar{a}')^{k-1}}{(a'\omega+b')^{k+1}}$$

$$\begin{aligned}
&= \oint \frac{d\omega}{2\pi i}\frac{(\bar{b}'\omega+\bar{a}')^{k-1}}{(a'\omega+b')^{k+1}}\left[\left(\frac{-b}{a}\right)^m+\left(\frac{-(b-\bar{a}\omega)}{(a-\bar{b}\omega)}\right)^m\right] \\
&= \oint \frac{d\omega}{2\pi i}\omega^m\frac{(\bar{b}'(a\omega+b)+\bar{a}'(\bar{b}\omega+\bar{a}))^{k-1}}{(a'(a\omega+b)+b'(\bar{b}\omega+\bar{a}))^{k+1}} \\
&= M_{mk}^{(a'a+b'\bar{b},ba'+b'\bar{a})} \qquad \text{(A.11)}
\end{aligned}$$

and in the second to last line the redefinition $\omega \to \frac{a\omega+b}{\bar{b}\omega+\bar{a}}$ was used. It can immediately be seen that $|a'a+b'\bar{b}|^2 - |ba'+b'\bar{a}|^2 = 1$ so this is another conformal transformation of the same type, as expected. This also shows that the expected inverse matrix transformation for$M_{mn}^{(a,b)}$, which would be $M_{mn}^{(\bar{a},-b)}$ is in fact the inverse.

Finally, we check the claimed property that renders these matrices moot in the conformally invariant case, explicitly that,

$$M_{km}^{(a,b)}\frac{1}{k}\bar{M}_{kn}^{(a,b)} = \frac{1}{m}\delta_{mn} \qquad \text{(A.12)}$$

As before the stated property that $M_{0m}=0$ helps, and we restrict to positive k, finding

$$\begin{aligned}
M_{km}^{(a,b)}\frac{1}{k}\bar{M}_{kn}^{(a,b)} &= \oint \frac{dz}{2\pi i}\frac{d\omega}{2\pi i}z^k\frac{(\bar{b}z+\bar{a})^{m-1}}{(az+b)^{m+1}}\frac{1}{k}\omega^k\frac{(b\omega+a)^{n-1}}{(\bar{a}\omega+\bar{b})^{n+1}} \\
&= -\oint \frac{dz}{2\pi i}\frac{d\omega}{2\pi i}\ln(1-\omega/z)\frac{(\bar{b}z+\bar{a})^{m-1}}{(az+b)^{m+1}}\frac{(b\omega+a)^{n-1}}{(\bar{a}\omega+\bar{b})^{n+1}} \qquad \text{(A.13)}
\end{aligned}$$

where we have transformed $z \to 1/z$. Integrating along the branch cut which runs from $z=0$ to $z=\omega$, and redefining again $z \to \frac{az-\bar{b}}{-bz+\bar{a}}$ we obtain

$$\begin{aligned}
M_{km}^{(a,b)}\frac{1}{k}\bar{M}_{kn}^{(a,b)} &= \oint \frac{d\omega}{2\pi i}\frac{1}{m}\frac{(b\omega+a)^{n-1}}{(\bar{a}\omega+\bar{b})^{n+1}}\left[\left(\frac{\bar{a}\omega+\bar{b}}{b\omega+a}\right)^m-\left(\frac{\bar{b}}{a}\right)^m\right] \\
&= \frac{1}{m}\delta_{mn} \qquad \text{(A.14)}
\end{aligned}$$

We have now verified the salient points claimed within the text, and shown that these transformations in fact act as a group, and become trivial in the cases where $\Lambda(n)$ is independent of n.

A.2 Fermionic Matrix $N_{rm}^{(a,b)}$

We have also described previously the matrix in the fermion NS sector, $N_{rm}^{(a,b)}$ that describes the mappings between the various fermion creation and annihilation operators. This matrix was derived 3.102 to be

$$N_{rm}^{(a,b)} = \oint \frac{dz}{2\pi i} z^{r-1/2} \frac{(\bar{b}z+\bar{a})^{m-1/2}}{(az+b)^{m+1/2}} \qquad \text{(A.15)}$$

where $r \in \mathbb{Z} + \frac{1}{2}$ Since there are no zero modes for this, the number of possible options is significantly less, but as in the case of the bosonic matrix we enumerate them.

In the case $r > 0$, $m > 0$ we have poles of order $m + 1/2$ at $-\frac{b}{a}$. This can be evaluated to give

$$N_{rm}^{(a,b)} = \frac{1}{a^{m+1/2}} \frac{1}{(m-1/2)!} \partial^{m-1/2} z^{r-1/2} (\bar{b}z+\bar{a})^{m-1/2} \Big|_{-b/a} \qquad \text{(A.16)}$$

and some of the cases that are short to write are

$$\begin{aligned} N_{1/2\ m}^{(a,b)} &= \frac{\bar{b}^{m-1/2}}{a^{m+1/2}} \\ N_{3/2\ m}^{(a,b)} &= \frac{\bar{b}^{m-3/2}}{a^{m+3/2}} \left(1-|b|^2\right) \end{aligned}$$

For the case $r > 0$, $m < 0$ there are no poles within the integration contour, and so these elements vanish. Similarly, for the case $r < 0$, $m > 0$ there are apparently poles at both 0 and $-b/a$ but just as in the bosonic case it is possible to make the transformation $z \to \frac{1}{z}$ which results in a new function to be integrated with the poles outside of the contour, and hence vanishes. The same trick can be used for the case $r < 0$, $m < 0$ and we explicitly exhibit it for completeness

$$N_{rm}^{(a,b)} = \oint \frac{dz}{2\pi i} z^{r-1/2} \frac{(\bar{b}z+\bar{a})^{m-1/2}}{(az+b)^{m+1/2}}$$

$$
\begin{aligned}
& \text{with} z \to \frac{1}{\omega} \\
&= \oint \frac{d\omega}{2\pi i} \frac{1}{\omega} \frac{1}{\omega^{r-1/2}} \frac{(\bar{b}+\bar{a}\omega)^{m-1/2}}{(a+b\omega)^{m+1/2}} \\
&= \oint \frac{d\omega}{2\pi i} \omega^{|r|-1/2} \frac{(a+b\omega)^{|m|-1/2}}{(\bar{b}+\bar{a}\omega)^{|m|+1/2}} \qquad \text{(A.17)}
\end{aligned}
$$

Which shows that

$$
\bar{N}^{(a,b)}_{|m|\ |n|} = N^{(a,b)}_{-|m|\ -|n|} \qquad \text{(A.18)}
$$

as desired. This analysis shows as in the bosonic case that the creation and annihilation operators do not mix under these transformations.

To complete the parallel with the bosonic case it is necessary to show the composition law, and that in the case that the matrices are contracted through a $PSL(2,R)$ invariant exponent in the boundary state that they contract to a unit matrix. The first problem is to calculate

$$
\begin{aligned}
N^{(a,b)}_{rp} N^{(a',b')}_{pq} &= \sum_p \oint \frac{dz}{2\pi i} \frac{d\omega}{2\pi i} z^{r-1/2} \frac{(\bar{b}z+\bar{a})^{p-1/2}}{(az+b)^{p+1/2}} \omega^{p-1/2} \frac{(\bar{b}'\omega+\bar{a}')^{q-1/2}}{(a'\omega+b')^{q+1/2}} \\
&= \oint \frac{dz}{2\pi i} \frac{d\omega}{2\pi i} z^{r-1/2} \frac{1}{z(a-\bar{b}\omega)-(-b+\bar{a}\omega)} \frac{(\bar{b}'\omega+\bar{a}')^{q-1/2}}{(a'\omega+b')^{q+1/2}} \\
&= \oint \frac{d\omega}{2\pi i} \frac{(-b+\bar{a}\omega)^{r-1/2}}{(a-\bar{b}\omega)^{r+1/2}} \frac{(\bar{b}'\omega+\bar{a}')^{q-1/2}}{(a'\omega+b')^{q+1/2}} \\
& \quad \text{with} \omega \to \frac{a\omega+b}{\bar{\omega}+\bar{a}} \\
&= \oint \frac{d\omega}{2\pi i} \omega^{r-1/2} \frac{(\bar{b}''\omega+\bar{a}'')^{q-1/2}}{(a''\omega+b'')^{q+1/2}} \qquad \text{(A.19)}
\end{aligned}
$$

with $b'' = \bar{a}b' + a'b$ and $a'' = a'a + b'\bar{b}$ just as in the bosonic case.

Now we calculate in analogy to A.14 the quantity

$$
N^{(a,b)}_{rp} \bar{N}^{(a,b)}_{rq} = \sum_r \oint \frac{dz}{2\pi i} \frac{d\omega}{2\pi i} z^{r-1/2} \frac{(\bar{b}z+\bar{a})^{p-1/2}}{(az+b)^{p+1/2}} \omega^{r-1/2} \frac{(b\omega+a)^{q-1/2}}{(\bar{a}\omega+\bar{b})^{q+1/2}} \qquad \text{(A.20)}
$$

and by transforming $z \to \frac{1}{z}$ and summing we find

$$\begin{aligned} N_{rp}^{(a,b)} \bar{N}_{rq}^{(a,b)} &= \oint \frac{dz}{2\pi i} \frac{d\omega}{2\pi i} \frac{1}{z-\omega} \frac{(\bar{b}+\bar{a}z)^{p-1/2}}{(a+bz)^{p+1/2}} \omega^{r-1/2} \frac{(b\omega+a)^{q-1/2}}{(\bar{a}\omega+\bar{b})^{q+1/2}} \\ &= \oint \frac{dz}{2\pi i} \frac{(a+bz)^{q-p-1}}{(\bar{b}+\bar{a}z)^{q-p+1}} = \delta_{qp} \end{aligned} \tag{A.21}$$

These relations show that the matrix of transformations for the fermions in the NS sector has the analogous nice properties as that of the bosonic transformation.

Appendix B

Green's Functions

Here we present in some detail the calculations of the bosonic and fermionic Green's functions for the quadratic tachyon background under consideration. The construction presented below will make the generalization to the case of different (quadratic) boundary interactions that are mentioned in section 4.1 and the case of more complicated interactions, that is to say higher order than quadratic, while not presented explicitly because they are not amenable to exact expression in a compact manner can be dealt with through standard techniques of field theory.

B.1 Bosonic Tree level

The starting point for this calculation is the action (3.3) which is rewritten here for convenience

$$\begin{aligned} S\left(g,F,T_0,U\right) &= \frac{1}{4\pi\alpha'}\int_\Sigma d\sigma d\phi\ g_{\mu\nu}\partial^a X^\mu \partial_a X_\mu \\ &+ \int_{\partial\Sigma} d\phi \left(\frac{1}{2}F_{\mu\nu}X^\nu\partial_\phi X^\mu + \frac{1}{2\pi}T_0 + \frac{1}{8\pi}U_{\mu\nu}X^\mu X^\nu\right). \end{aligned} \tag{B.1}$$

Now, for a disk world-sheet the greens function satisfying Neumann boundary conditions is determined in [60] and we wrote it as (3.31)

$$G^{\mu\nu}(z,z') = -\alpha' g^{\mu\nu}\left(-\ln|z-z'| - \ln|1-z\bar{z}'|\right). \tag{B.2}$$

Clearly it is possible to either calculate exactly from the boundary conditions this greens function in the background of (3.3), or we can treat the boundary terms as perturbations and perform an explicit sum (an equivalent procedure).

For illustrative purposes we choose the second method, and since the interaction terms are quadratic there is one term at each order in perturbation theory. The final bulk to bulk propagator will be the sum of the propagator with no boundary terms and the increasing number of boundary interactions. For the parameterization of the world-sheet $z = \rho e^{i\phi}$ we have the bulk to boundary propagator (3.32) which is

$$G^{\mu\nu}(\rho e^{i\phi}, e^{i\phi'}) = 2\alpha' g^{\mu\nu} \sum_{m=1}^{\infty} \frac{\rho^m}{m} \cos[m(\phi - \phi')] \tag{B.3}$$

and also the boundary to boundary propagator

$$G^{\mu\nu}(\rho e^{i\phi}, e^{i\phi'}) = 2\alpha' g^{\mu\nu} \sum_{m=1}^{\infty} \frac{\cos[m(\phi - \phi')]}{m} \tag{B.4}$$

using the identities from [50].

Now, to first order in the perturbing terms the contribution to the propagator is

$$\begin{aligned} G_1^{\mu\nu}(\rho e^{i\phi}, \rho' e^{i\phi'}) &= \int d\theta G^{\mu\nu'}(\rho e^{i\phi}, e^{i\theta}) \left(F\partial_\theta + \frac{1}{4\pi}U\right)_{\nu'\mu'} G^{\mu'\nu}(e^{i\theta}, \rho' e^{i\phi'}) \\ &= (2\alpha')^2 \int d\theta \sum_{m,m'} \frac{\rho^m \rho'^{m'}}{mm'} \cos m(\phi - \theta) \\ &\quad \times \left(F\partial_\theta + \frac{1}{4\pi}U\right)^{\mu\nu} \cos m'(\theta - \phi') \\ &= (2\alpha')^2 \pi \sum_m \frac{\rho^m \rho'^m}{m^2} \\ &\quad \times \left(\frac{1}{4\pi}U \cos m(\phi - \phi') - mF \sin m(\phi - \phi')\right)^{\mu\nu} . \end{aligned} \tag{B.5}$$

Similarly the second order contribution can be read off from the concatenation of (B.5) with (B.3) to give

$$\begin{aligned} G_2^{\mu\nu}(\rho e^{i\phi}, \rho' e^{i\phi'}) &= (-2\alpha')^3 \pi \int d\theta \sum_m \frac{\rho^m}{m^2} \\ &\quad \left(\frac{1}{4\pi} U \cos m(\phi - \theta) - mF \sin m(\phi - \theta) \right)^{\mu\nu'} \\ &\quad \left(F\partial_\theta + \frac{1}{4\pi} U \right)_{\nu'\mu'} \sum_{m'} g^{\mu'\nu} \frac{\rho'^{m'}}{m'} \cos m'(\theta - \phi') \\ &= (-2\alpha')^3 \pi^2 \sum_m \frac{\rho^m \rho'^m}{m^3} \Big(\frac{U^2}{(4\pi)^2} \cos m(\phi - \phi') \\ &\quad -m \left(F\frac{U}{4\pi} + \frac{U}{4\pi} F \right) \sin m(\phi - \phi') \\ &\quad +m^2 F^2 \cos m(\phi - \phi') \Big)^{\mu\nu} \end{aligned} \tag{B.6}$$

with the obvious generalization to higher orders.

Now, in the above we note that all terms in this sum will naturally separate into terms with $\cos m(\phi - \phi')$ and $\sin m(\phi - \phi')$ and by inspection the dependence on F and U is such that the coefficient of $\sin m(\phi - \phi')$ is naturally combinations of F and U that are antisymmetric in Lorentz indices, just as those for $\cos m(\phi - \phi')$ are symmetric in F and U. Using the facts

$$\begin{aligned} \sin m(\phi - \phi') &= \frac{e^{i(\phi-\phi')} - e^{-i(\phi-\phi')}}{2i} \\ \cos m(\phi - \phi') &= \frac{e^{i(\phi-\phi')} + e^{-i(\phi-\phi')}}{2} \end{aligned}$$

and the identification $z = \rho e^{i\phi}$, the sum

$$G_0^{\mu\nu} + G_1^{\mu\nu} + G_2^{\mu\nu} + \ldots$$

can then be calculated as

$$G^{\mu\nu}(z, z') = -\alpha' g^{\mu\nu} \ln |z - z'|$$

$$+\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{\{\mu\nu\}}\frac{(z\bar{z}')^n+(\bar{z}z')^n}{n}$$
$$+\frac{\alpha'}{2}\sum_{n=1}^{\infty}\left(\frac{g-2\pi\alpha' F-\frac{\alpha'}{2}\frac{U}{n}}{g+2\pi\alpha' F+\frac{\alpha'}{2}\frac{U}{n}}\right)^{[\mu\nu]}\frac{(z\bar{z}')^n-(\bar{z}z')^n}{in}. \quad \text{(B.7)}$$

as noted in (3.33). This includes the $\ln|1-z\bar{z}'|$ term in the two F and U dependent terms as can be seen by the limit that as $F,U\to 0$ we recover the known expression (B.2), and in the case of $U\to\infty$, Dirichlet boundary conditions, we obtain

$$G^{\mu\nu}(z,z')=-\alpha' g^{\mu\nu}\left(-\ln|z-z'|+\ln|1-z\bar{z}'|\right), \quad \text{(B.8)}$$

which is the Dirichlet propagator on the disk.

B.2 Fermionic Tree level

As in the bosonic case we start with the fermionic action (3.96) which is

$$S_{ferm}=\int_M\left(\psi_+^\mu\partial_-\psi_+^\nu+\psi_-^\mu\partial_+\psi_-^\nu\right)+\oint_{\partial M}F_{\mu\nu}\left(\psi_+^\mu\psi_+^\nu-\psi_-^\mu\psi_-^\nu\right)+$$
$$U_{\mu\nu}\left(\psi_+^\mu\frac{1}{\partial_\phi}\psi_+^\nu-\psi_-^\mu\frac{1}{\partial_\phi}\psi_-^\nu\right). \quad \text{(B.9)}$$

The appropriate Green's functions for the free case have been determined to be [15, 102]

$$G_\psi(z,w)=\frac{\alpha'}{i}\left(\frac{\sqrt{zw}}{z-w}-\frac{\sqrt{z\bar{w}}}{1-z\bar{w}}\right) \quad \text{(B.10)}$$
$$\tilde{G}_\psi(z,w)=\frac{\alpha'}{i}\left(-\frac{\sqrt{\bar{z}\bar{w}}}{\bar{z}-\bar{w}}+\frac{\sqrt{\bar{z}w}}{1-\bar{z}w}\right). \quad \text{(B.11)}$$

As in the bosonic case we specialize to the bulk to boundary propagator, which upon imposition of the antisymmetry requirement on it becomes

$$G_\psi^{\mu\nu}\left(\rho e^{i\phi},e^{i\phi'}\right) = 2\alpha' g^{\mu\nu}\sum_{r\in\mathbb{Z}+1/2>0}\rho^r\sin r(\phi-\phi'). \quad \text{(B.12)}$$

Upon insertion of the interaction term associated with the gauge field, the first order modification to the bulk to bulk propagator is

$$\begin{aligned} G^{\mu\nu}_{1\psi}\left(\rho e^{i\theta}, \rho' e^{i\theta'}\right) &= (2\alpha')^2 F^{\mu\nu} \int d\phi \sum_{r,r'\in\mathbb{Z}+1/2>0} \rho^r \rho'^{r'} \\ &\quad \times \sin r(\theta-\phi) \sin r'(\phi-\theta') \\ &= (2\alpha')^2 \pi F^{\mu\nu} \sum_{r\in\mathbb{Z}+1/2>0} (\rho\rho')^r \cos r(\theta-\theta'). \end{aligned} \quad \text{(B.13)}$$

Similarly we can determine the order F^2 modification as

$$\begin{aligned} G^{\mu\nu}_{2\psi}\left(\rho e^{i\theta}, \rho' e^{i\theta'}\right) &= (2\alpha')^3 \pi (F^2)^{\mu\nu} \int d\phi \sum_{r,r'\in\mathbb{Z}+1/2>0} \rho^r \rho'^{r'} \\ &\quad \times \cos r(\theta-\phi) \sin r'(\phi-\theta') \\ &= (2\alpha')^3 \pi^2 (F^2)^{\mu\nu} \sum_{r\in\mathbb{Z}+1/2>0} (\rho\rho')^r \sin r(\theta-\theta') \end{aligned}$$

(B.14)

with higher order terms determined similarly.

For the insertion of the U interaction term associated with the tachyon field, it is important to remember the definition of $\frac{1}{\partial_\theta}$,

$$\frac{1}{\partial_\phi}\psi(\phi) = \frac{1}{2}\int d\phi' \epsilon(\phi-\phi')\psi(\phi') \quad \text{(B.15)}$$

where ϵ is a step function: $\epsilon(x) = 1$ for $x > 0$ and $\epsilon(x) = -1$ for $x < 0$. Using this the lowest order correction to the fermionic Green's function due to $U^{\mu\nu}$ is

$$\begin{aligned} G^{\mu\nu}_{1\psi}\left(\rho e^{i\theta}, \rho' e^{i\theta'}\right) &= (2\alpha')^2 U^{\mu\nu} \int d\phi \sum_{r,r'\in\mathbb{Z}+1/2>0} \rho^r \rho'^{r'} \\ &\quad \times \sin r(\theta-\phi) \frac{1}{\partial_\phi} \sin r'(\phi-\theta') \\ &= (2\alpha')^2 \pi U^{\mu\nu} \int d\phi \sum_{r,r'\in\mathbb{Z}+1/2>0} \rho^r \rho'^{r'} \end{aligned}$$

$$\times \sin r(\theta - \phi)\frac{\cos r'(\phi - \theta')}{r'}$$
$$= (2\alpha')^2 \pi U^{\mu\nu} \sum_{r \in \mathbb{Z}+1/2>0} (\rho\rho')^r \frac{\sin r(\theta - \theta')}{r}. \quad \text{(B.16)}$$

The steps in (B.16) can evidently be repeated indefinitely and so for the nth insertion of U into the bulk to bulk propagator we obtain

$$G^{\mu\nu}_{n\psi}\left(\rho e^{i\theta}, \rho' e^{i\theta'}\right) = (2\alpha')^{n+1}\pi^n (U^n)^{\mu\nu} \sum_{r \in \mathbb{Z}+1/2>0} (\rho\rho')^r \frac{\sin r(\theta - \theta')}{r^n}. \quad \text{(B.17)}$$

It is also clear that for interactions with combinations of F and U the resultant will depend on $\sin r(\theta - \theta')$ in the case of an even number of Fs, and on $\cos r(\theta - \theta')$ for an odd number. Summing the contributions of interactions with both U and F allows the verification of (3.108). The Green's function for the antiholomorphic coordinates $\tilde{\psi}$ can be obtained by an identical argument.