Green Schwarz Superstring on a pp-wave Ramond-Ramond background.

by

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Abstract

The discovery of the dualities between the large N gauge theories and the string theories on the particular curved backgrounds has made the later subject of much study over the last decade. Of particular interest is the string theory on $AdS_5 \times S^5$ which was shown by Maladacena to be dual to the conformal $\mathcal{N} = 4$ super Yang-Mills theory. This string theory, however, is difficult to quantize and it proved useful to work in the specific Penrose limit of the $AdS_5 \times S^5$. String theory on this limit, called pp-wave background, proved to be explicitly quantizable and furthermore it turned out to be itself a dual of the particular limit of the corresponding Conformal Field Theory. It became therefore an important case for testing the principles of the AdS/CFT correspondence. The properties of the string theory on this background have been studied extensively by Metsaev and Metsaev and Tseytlin in the papers hep-th/0112044 and hep-th/0202109 and this thesis is mostly a review of their results. The type IIB superstring action is constructed and quantized on the background and the supergravity spectrum of the theory is found. Finally some results concerning two-point functions and vertex operators, not given in Metsaev, are derived.

Contents

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Chapter 1

Introduction

1.1 AdS/CFT Correspondence

The common motivation for string theories is the need to incorporate the quantum gravity into a consistent quantum theory. The way this is accomplished is by replacing the point-like particles with the Planck scale extended one-dimensional objects called Strings as the fundamental objects of the theory. Strings are allowed to oscillate and thus there will exist a spectrum of energies (and masses) associated with different modes of the oscillations. At the low-energy scale (and most observational scales will be low-energy compared with the Planck-scale of the strings) such oscillating strings will look like the point-like particles they replaced. In this way, single string will be able to give rise to a full spectrum of particles depending on its state of oscillation. An important fact is that every String theory includes an oscillation state with zero mass and the spin two, the expected characteristics of a Graviton. Furthermore, the only consistent interaction of this state is gravity, indicating that gravity is indeed present in every string theory. This makes a string theory a serious candidate for the much desired Theory of Quantum Gravity [1].

This motivation, however, was not the original source of the string theory. It was developed to answer less fundamental problem, that of the certain periodicity in the spectrum of the large number of Mesons and Hadrons that were discovered in the 1960's. True to the modern string theory, those particles were supposed to be represented as the different oscillation modes of a fundamental string. This model was initially successful, explaining, among other things, the simple relation between the spin and the mass of the lightest hadron of that spin. Ultimately, however, it was found incomplete and was abandoned with the advent of the QCD which provided much more thorough understanding of the Mesons and Hadrons. [2]

QCD is an $SU(3)$ based gauge theory. It is also a non-Abelian gauge theory, which makes it asymptotically free. As such, it has the coupling constant that is inversely proportional to energy. The resulting difficulties in working in the low-energy limit of the QCD have motivated the search for the simplified model of that theory. It was suggested by T'Hooft [3] that the $SU(N)$ based gauge theories simplify as the N is allowed to become very large. Specifically the $N = \infty$ case was supposed to be exactly solvable. It would then be potentially possible to do the expansion in $1/N = 1/3$. An interesting feature of the $N = \infty$ theories was observed that helped in this project and also explained the limited success the string theory had in describing the spectrum of the $SU(3)$ theory.

Namely, large N theory appeared to be a dual of a free string theory with the string coupling constant equal to $1/N$.

Duality phenomenon was known for many years in two dimensional field theories and has been discovered in many instances of string theories. It refers to the theories that have more then one different description such that when one description is weakly coupled the other is strongly coupled and vice versa. Mostly, both sides of the duality were string theories, but there was a hope that a dual theory of the QCD could be found. If it were found, analytical study of the low-energy regime, where the standard gauge-theory description is strongly coupled, would become possible. Supporting the idea that such dual theory might exist, and that it might be a string theory, is the fact that the QCD-like gauge theories have their own string-like objects. Those are the Wilson lines (or flux tubes) that appear between the quark and an anti-quark that separate from each other. Being extended one dimensional objects, Wilson lines have a some string-like properties and a number of attempts have been made to express the gauge theories in terms of a string theory of Wilson lines.

The original case for the duality between the t'Hooft limit $SU(N)$ and the string theories came from the comparison between the vacuum diagrams of the $SU(N)$ theory in the t'Hooft limit and those of the perturbative theory with closed oriented strings with string coupling $g_{string} = 1/N$ [2]. That analysis gives strong indication that there indeed is a connection between the gauge theories and the string theories but it does not specify which string theory is dual to the particular gauge theory. The matching was done with considerable success for two dimensional gauge theories [4], but for the four-dimensional cases no appropriate duals were found until the work of Maldacena [2] in the 1990's.

Attempts to construct the dual of the $4 - D$ gauge theory were dogged by the fact that the planar diagram expansion in this case was prohibitively complicated. Alternative approach was to attempt to directly construct the relevant string theory based on the loop equations of the Wilson loop observables of the gauge theory [5]. This approach ran into the standard problem of attempting to construct the string theory in four dimensions.

The fundamental property of the string theory is conformal invariance. The derivation of the string action requires the metric of the world-sheet (which is a string theory equivalent of the world line of the particle) to be conformaly invariant. If it is not, in quantizing the theory, the anomaly will appear. For the flat space only theories that posses the requisite conformal invariance are the theory of bosonic strings in 26 dimensions and supersymmetric theory of bosons and fermions in 10 dimensions. Neither of these theories, nor the hybrid "heterotic" string theory are the candidates for the string theory of Wilson loops in 4D for the simple reason that none of them carries a set of symmetries that would be associated with the supersymmetric $SU(N)$ gauge theory in four dimensions. The conformal group for the theory of interest in four dimensions ends up being $SO(4,2)$ including the 4-dimensional Poincare transformations, scale transformations and special conformal transformations. There is also a global $SU(4)$ R-symmetry that rotates the six scalars and four fermions that form the spectrum of the theory. Only one geometry exists that contains $SO(4,2)$ isometries:

Five dimensional Anti-deSitter space or AdS_5. In order to have the proper flat space limit however, any theory of superstrings has to live on exactly ten dimensions and therefore we require an additional five dimensional geometry. A natural choice for the remaining dimensions, given the $SU(4) = SO(6)$ is the five sphere. It can therefore be expected that the appropriate dual for the $SU(N)$ gauge theory on four dimensions will be a string theory on $AdS_5 \times S^5$ ten dimensional space.[2]

Much less heuristic argument, that firmly established this conjecture, was produced by Maldacena.[2] The exact nature of this argument is beyond the scope of this review. It relies on the properties of the string-theoretic objects called D-branes (Dirchlet Membranes: regions where an open string can end) and the black holes in string theory. The argument establishes the $SU(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry with the strong coupling as describing the near horizon region of the black-brane, black-hole like objects, whose geometry is $AdS_5 \times S^5$. A number of calculations, such as graviton absorption cross-section, on both sides of the duality, further confirm the conjecture.

1.2 $AdS_5 \times S^5$ and the pp-wave limit

With the advent of the Maldacena conjecture the String Theories on the curved spaces in general and on $AdS_5 \times S^5$ in particular became ever more important. A number of papers was written developing the string theory on this background. The difficulty here was the fact that while it was possible to formulate the the superstring action on the $AdS_5 \times S^5$ background [6] quantizing it in the light-cone gauge proved complicated.

The root cause of this difficulty is that the relevant $AdS_5 \times S^5$ contains, in addition to the gravitational field represented by the curved metric, further Ramond-Ramond background fields which couple to the strings and have to be included in the action. The presence of the Ramond-Ramond fields makes the use of the standard Ramond-Neveu-Schwartz (RNS) formulation of string theory extremely inconvenient [7]. It is therefore preferable to use the Green-Schwartz (GS) formalism with its manifest supersymmetry. Quantization of the GS string in the flat space is most effectively done in the light-cone gauge. The backgrounds in question here, however, have no globally defined light-like direction and therefore no light-cone gauge. Much effort is used in determining the appropriate gauge for the GS quantization and even the most successful choices still retain non-trivial interaction terms. [6]

The question then rises whether it is possible to find some sort of a limit of the $AdS_5 \times S^5$ space that would still be appropriate background of string theory, preferably one that can be explicitly quantized. The answer [8],[9],[10] turns out to be yes and this will be the main subject of this review. The background in question is given by the plane-wave metric supported by a Ramond-Ramond 5-form background.

$$ds^2 = 2dx^+dx^- - f^2 x_I^2 dx^+dx^+ dx^I dx^I \quad I = 1, ..., 8 \tag{1.1}$$

$$F_{+1234} = F_{+5678} = 2f \tag{1.2}$$

This background has several desirable properties. It is a Penrose limit of the $AdS_5 \times S^5$ obtained by focusing on the single trajectory within it. Furthermore this limit has the natural light cone direction thus making the gauge choice simple. As a consequence, not only is there a string theory on this background but it can also be explicitly quantized with not much greater difficulty then the flat space one. Finally, as the $AdS_5 \times S^5$ pp-wave background has the simple flat-space limit and the superstring action reduces to the standard GS action in that limit. In addition, it was shown [10] that this string theory has its own dual in the large N limit at fixed g^2_{YM} of maximally supersymmetric Super Yang-Mills $U(N)$ which itself is a limit[1] of the dual theory of the strings on $AdS_5 \times S^5$. This made the pp-wave background one of the best places to test the principles of the AdS/CFT correspondence.

This review will have the following structure: In Chapter 2. we consider the geometry and symmetries of the Anti-deSitter space and how they carry over into the pp-wave limit and derive the isometries of that background. Supersymmetry superalgebra is also considered and the argument is given as to why both $AdS_5 \times S^5$ and pp-wave background have maximal number of Killing spinors. The exact superalgebra is given at the end of the chapter. In Chapter 3. we will follow Metsaev [8] and derive the Superstring action on pp-wave using the formalism of Cartan forms defined on the coset superspace. In Chapter 4. we will derive the equations of motion from the Lagrangian and review the Metsaev and Tseytlin [9] canonical quantization of this system. The symmetries of the superalgebra will be made manifest in the light-cone gauge through their generators, and those generators will be given in terms of the creation and annihilation operators. Hamiltonian and stress-energy tensor will also be calculated. In Chapter 5. there will be a discussion of the spectrum of type IIB supergravity near the pp-wave background, in particular lowest energy eigenvalues will be found for the supergravity fields and the equations of motion of those fields will be derived. Finally in the Chapter 6. some of our own results concerning the two point functions and string vertex operators on this background will be presented.

[1] so called "double scaling limit"

Chapter 2

$AdS_5 \times S^5$ and pp-wave geometry

2.1 Geometry of $AdS_5 \times S^5$ and their Killing vectors

The AdS_5 and S^5 are maximally symmetric spaces. This means that they are homogenous (symmetric under the translations) and isotropic (symmetric under rotations) and also that they have maximum number of killing vectors. As those killing vectors will be of interest to us in the pp-wave limit we derive them here from the basic equations for these spaces.

Out requirement for the AdS_5 was the $SO(2,4)$ isometry which corresponds to the conformal symmetry of the flat four dimensional Minkowski space. Such a space would be a solution of the following equation of the hyperboloid:

$$X_0^2 + X_5^2 - \sum_{i=1}^{4} X_i^2 = R^2 \tag{2.1}$$

in the flat six dimensional space. It has metric:

$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^{4} dX_i^2 \tag{2.2}$$

This equation can be solved by setting:

$$X_0 = R\cosh(\rho)\cos(\tau) \tag{2.3}$$

$$X_5 = R\cosh(\rho)\sin(\tau) \tag{2.4}$$

$$X_i = R\sinh(\rho)\Omega_i \quad \sum_{i=1}^{4} \Omega_i^2 = 1 \tag{2.5}$$

Substituting these into the 2.2 we obtain the metric given by:

$$ds^2 = R^2(-\cosh^2(\rho)d\tau^2 + d\rho^2 + \sum_{i=1}^{4} \sinh(\rho)d\Omega_i^2) \tag{2.6}$$

Where τ, ρ and Ω_is are called global coordinates of the AdS because for $0 \leq \rho$ and $0 \leq \tau \leq 2\pi$ the solution covers the entire hyperboloid once.

In finding the Killing Vectors we use the method for maximally symmetric spaces from Weinberg [11].

$$\varepsilon_{i,j} = X_i \frac{d}{dX_j} \mp X_j \frac{d}{dX_i} \tag{2.7}$$

where the sign is given by the sign of the appropriate X in the metric. To perform this calculation in terms of global coordinates we will need:

$$\tau = \sin^{-1} \frac{X_5}{\sqrt{X_0^2 + X_5^2}} = \cos^{-1} \frac{X_0}{\sqrt{X_0^2 + X_5^2}} \tag{2.8}$$

$$\rho = \sinh^{-1} \frac{\sqrt{\sum_{i=1}^4 X_i^2}}{\sqrt{X_0^2 - \sum_{i=1}^4 X_i^2 + X_5^2}} = \cosh^{-1} \frac{\sqrt{X_0^2 + X_5^2}}{\sqrt{X_0^2 - \sum_{i=1}^4 X_i^2 + X_5^2}} \tag{2.9}$$

$$\Omega_i = \frac{X_i}{\sqrt{\sum_{i=1}^4 X_i^2}} \tag{2.10}$$

From these we get:

$$\frac{d\tau}{dX_5} = \frac{X_0}{R^2} \cosh^2 \rho = \frac{\cos \tau}{R \cosh^2(\rho)}, \qquad \frac{d\tau}{dX_0} = \frac{-\sin \tau}{R \cosh^2 \rho}, \qquad \frac{d\tau}{dX_i} = 0 \tag{2.11}$$

$$\frac{d\rho}{dX_0} = \frac{-\cos \tau \sinh \rho}{R}, \qquad \frac{d\rho}{dX_5} = \frac{-\sin \tau \sinh \rho}{R}, \qquad \frac{d\rho}{dX_i} = \frac{\cosh \rho \Omega_i}{R} \tag{2.12}$$

$$\frac{d\Omega_i}{dX_i} = \frac{1 - \Omega_i^2}{R \sinh \rho}, \qquad \frac{d\Omega_i}{dX_j} = \frac{-\Omega_i \Omega_j}{R \sinh \rho}, \qquad \frac{d\Omega_i}{dX_0} = \frac{d\Omega_i}{dX_5} = 0 \tag{2.13}$$

Having this, we can now write:

$$\varepsilon_1 = X_0 \frac{d}{dX_5} - X_5 \frac{d}{dX_1} = \tag{2.14}$$

$$X_0 [\frac{\cos(\tau)}{R \cosh \rho} \frac{d}{d\tau} - \frac{\sin(\tau) \sinh(\rho)}{R} \frac{d}{d\rho}] - X_5 [\frac{-\sin(\tau)}{R \cosh \rho} \frac{d}{d\tau} - \frac{\cos(\tau) \sinh(\rho)}{R} \frac{d}{d\rho}] = \frac{d}{d\tau}$$

$$\varepsilon_{2...5} = X_0 \frac{d}{dX_i} + X_i \frac{d}{dX_0} = \tag{2.15}$$

$$X_0 [\frac{\cosh(\rho) \Omega_i}{R} \frac{d}{d\rho} + \frac{1 - \Omega_i^2}{R \sinh \rho} \frac{d}{d\Omega_i} -$$

$$\sum_{j \neq i} \frac{\Omega_i \Omega_j}{R \sinh(\rho)} \frac{d}{d\Omega_j}] + X_i [\frac{-\sin(\tau)}{R \cosh(\rho)} \frac{d}{d\tau} - \frac{\cos(\tau) \sinh(\rho)}{R} \frac{d}{d\rho}] =$$

$$\cos(\tau) \Omega_i \frac{d}{d\rho} + \coth(\rho) \cos \tau (1 - \Omega_i^2) \frac{d}{d\Omega_i} - \sum_{j \neq i} \coth(\rho) \cos \tau \Omega_i \Omega_j \frac{d}{d\Omega_j} -$$

$$\tanh(\rho)\sin(\tau)\Omega_i\frac{d}{d\tau}$$

$$\varepsilon_{6..11} = X_i\frac{d}{dX_j} + X_j\frac{d}{dX_i} = \tag{2.16}$$

$$X_i[\frac{\cosh(\rho)\Omega_j}{R}\frac{d}{d\rho} + \frac{1-\Omega_j}{R\sinh(\rho)}\frac{d}{d\Omega_j} - \sum_{k\neq j}\frac{\Omega_k\Omega_j}{R\sinh(\rho)}\frac{d}{d\Omega_k} -$$

$$X_j[\frac{\cosh(\rho)\Omega_i}{R}\frac{d}{d\rho} + \frac{1-\Omega_i}{R\sinh(\rho)}\frac{d}{d\Omega_i} - \sum_{l\neq i}\frac{\Omega_l\Omega_i}{R\sinh(\rho)}\frac{d}{d\Omega_l}$$

$$= \Omega_i\frac{d}{d\Omega_j} - \Omega_j\frac{d}{d\Omega_i}$$

$$\varepsilon_{12...15} = X_5\frac{d}{dX_i} + X_i\frac{d}{dX_5} = \tag{2.17}$$

$$X_5[\frac{\cosh(\rho)\Omega_i}{R}\frac{d}{d\rho} + \frac{1-\Omega_i^2}{R\sinh\rho}\frac{d}{d\Omega_i}$$

$$-\sum_{j\neq i}\frac{\Omega_i\Omega_j}{R\sinh(\rho)}\frac{d}{d\Omega_j}] + X_i[\frac{-\cos(\tau)}{R\cosh(\rho)}\frac{d}{d\tau} - \frac{\sin(\tau)\sinh(\rho)}{R}\frac{d}{d\rho}] =$$

$$\sin(\tau)\Omega_i\frac{d}{d\rho} + \coth(\rho)\sin\tau(1-\Omega_i^2)\frac{d}{d\Omega_i} - \sum_{j\neq i}\coth(\rho)\sin\tau\Omega_i\Omega_j\frac{d}{d\Omega_j} -$$

$$\tanh(\rho)\cos(\tau)\Omega_i\frac{d}{d\tau}$$

Those killing vectors represent all isometries of the AdS_5 given in the global variables.

It is useful to express the 5-sphere in the global coordinates as well. It solves the equation:

$$X_0^2 + X_5^2 + \sum_{i=1}^{4} X_i^2 = R^2 \tag{2.18}$$

in the flat six dimensional space. It's metric is:

$$ds^2 = +dX_0^2 + X_5^2 + \sum_{i=1}^{4} dX_i^2 \tag{2.19}$$

The equation solves in the same way as the hyperboloid above:

$$X_0 = R\cos(\theta)\cos(\psi) \tag{2.20}$$

$$X_5 = R\cos(\theta)\sin(\psi) \tag{2.21}$$

$$X_i = R\sin(\theta)\Omega_i' \quad \sum_{i=1}^{4}\Omega_i'^2 = 1 \tag{2.22}$$

with the metric:

$$ds^2 = R^2(-\cos^2(\theta)d\psi^2 + d\theta^2 + \sum_{i=1}^{4} \sinh(\theta)d\Omega_i'^2) \tag{2.23}$$

Once again we get the global coordinates:

$$\psi = \sin^{-1}\frac{X_5}{\sqrt{X_0^2+X_5^2}} = \cos^{-1}\frac{X_0}{\sqrt{X_0^2+X_5^2}} \tag{2.24}$$

$$\theta = \sin^{-1}\frac{\sqrt{\sum_{i=1}^4 X_i^2}}{\sqrt{X_0^2+\sum_{i=1}^4 X_i^2 + X_5^2}} = \cos^{-1}\frac{\sqrt{X_0^2+X_5^2}}{\sqrt{X_0^2+\sum_{i=1}^4 X_i^2 + X_5^2}} \tag{2.25}$$

$$\Omega_i' = \frac{X_i}{\sqrt{\sum_{i=1}^4 X_i^2}} \tag{2.26}$$

$$\frac{d\psi}{dX_5} = \frac{X_0}{R^2}\cos^2\rho = \frac{\cos\psi}{R\cos^2(\theta)}\,, \quad \frac{d\psi}{dX_0} = \frac{-\sin\psi}{R\cos^2\theta}\,, \quad \frac{d\psi}{dX_i} = 0 \tag{2.27}$$

$$\frac{d\theta}{dX_0} = \frac{-\cos\psi\sin\theta}{R}\,, \quad \frac{d\theta}{dX_5} = \frac{-\sin\psi\sin\theta}{R}\,, \quad \frac{d\theta}{dX_i} = \frac{\cos\theta\Omega_i'}{R} \tag{2.28}$$

$$\frac{d\Omega_i'}{dX_i} = \frac{1-\Omega_i'^2}{R\sin\theta}\,, \quad \frac{d\Omega_i'}{dX_j} = \frac{-\Omega_i'\Omega_j'}{R\sin\theta}\,, \quad \frac{d\Omega_i'}{dX_0} = \frac{d\Omega_i}{dX_5} = 0 \tag{2.29}$$

We produce the Killing Vectors in the same fashion as for the AdS_5

$$\varepsilon_1 6 = X_0\frac{d}{dX_5} - X_5\frac{d}{dX_1} = \frac{d}{d\psi} \tag{2.30}$$

$$\varepsilon_{17\ldots20} = X_0\frac{d}{dX_i} - X_i\frac{d}{dX_0} = \tag{2.31}$$

$$\cos(\psi)\Omega_i'\frac{d}{d\theta} + \cot(\theta)\cos\psi(1-\Omega_i'^2)\frac{d}{d\Omega_i'} - \sum_{j\neq i}\cot(\theta)\cos\psi\Omega_i'\Omega_j'\frac{d}{d\Omega_j'} -$$

$$\tan(\theta)\sin(\psi)\Omega_i'\frac{d}{d\psi}$$

$$\varepsilon_{21..26} = X_i\frac{d}{dX_j} - X_j\frac{d}{dX_i} = \Omega_i'\frac{d}{d\Omega_j'} - \Omega_j'\frac{d}{d\Omega_i'} \tag{2.32}$$

$$\varepsilon_{27\ldots30} = X_5\frac{d}{dX_i} - X_i\frac{d}{dX_5} = \tag{2.33}$$

$$\sin(\psi)\Omega_i'\frac{d}{d\theta} + \cot(\theta)\sin\psi(1-\Omega_i'^2)\frac{d}{d\Omega_i'} - \sum_{j\neq i}\cot(\theta)\sin\psi\Omega_i'\Omega_j'\frac{d}{d\Omega_j'} -$$

$$\tan(\theta)\cos(\psi)\Omega_i' \frac{d}{d\psi}$$

Now we have all 30 Killing vectors of the $AdS_5 \times S^5$ These however, are not all the symmetries of the space. While $SO(2,4)$ is indeed the same as the conformal group on the four dimensional flat space, supersymmetry on both sides of the duality still needs to be accounted for. The superconformal group on 4 dimensional space is $SU(2,2 \mid 4)$ [6] and ultimately so is the supersymmetric generalization of the above isometry called AdS supergroup. In trying to understand the nature of the AdS Supergroup we follow Aharony at al. [2] and Nahm [12] and start with the simple supergravity with a cosmological constant Λ. In the simple four dimensional case of $\mathcal{N} = 1$ theory the action is:

$$S = \int d^4x(-\sqrt{g}(R - 2\Lambda) + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma^5\gamma_\nu\tilde{D}_\rho\psi_\sigma) \tag{2.34}$$

where

$$\tilde{D}_\mu = D_\mu + \frac{i}{2}\sqrt{\frac{\Lambda}{3}}\gamma_\mu \tag{2.35}$$

and the D_μ is a standard covariant derivative. Here gravitno ψ_μ transforms under supersymmetry as:

$$\delta\psi_\mu = \tilde{D}_\mu\epsilon(x) \tag{2.36}$$

The global supersymmetry of the background is determined by the requirement that the gravitino remains unchanged under the ϵ transformation. This leads to the Killing Spinor equation:

$$\tilde{D}_\mu\epsilon = [D_\mu + \frac{i}{2}\sqrt{\frac{\Lambda}{3}}\gamma_\mu]\epsilon = 0 \tag{2.37}$$

and the ϵ spinors that solve it are the Killing Spinors, playing the same role in the Supergroup as the Killing Vectors.

The integrability of the above requires:

$$[\tilde{D}_\mu, \tilde{D}_\nu]\epsilon = \frac{1}{2}(R_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} - \frac{2}{3}\Lambda\sigma_{\mu\nu})\epsilon \tag{2.38}$$

If AdS is a classical solution of the supergravity above then cosmological constant has to be $\Lambda = \frac{3}{R^3}$ where R is the size of the hyperboloid from the 2.1 [2]. Also, because AdS is maximally symmetric:

$$R_{\mu\nu\rho\sigma} = \frac{1}{R^2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \tag{2.39}$$

and therefore integrability equation is obeyed for every spinor ϵ. Given that the Killing Spinor equation is a first order equation this implies that there is as many solutions to the equation as there are independent spinor components and therefore AdS is maximally supersymmetric as well as maximally symmetric. It is this supersymmetry that introduces the Ramond-Ramond background fields which force use to use GS formalism when attempting to quantize this theory.

The Ramond-Ramond five-form flux that gives us the required symmetry in $AdS_5 \times S^5$ case is given by: [14]

$$F_5 = 4R^4[\cosh(\rho)\sinh^3(\rho)d\tau \wedge d\rho \wedge d\Omega_3 + \cos(\theta)\sin^3(\theta)d\psi \wedge d\theta \wedge d\Omega_3' \quad (2.40)$$

it can be easily checked that the Killing spinor equations using the covariant differential constructed from this 5-form behave much as the sample ones listed above. This property will carry through to the pp-wave limit and we will discuss it some more there.

In the next section we are going to see how the $AdS_5 \times S^5$ metric, as well as the 5-form R-R flux, transform into those of the pp-wave in the appropriate Penrose limit. We are also going to apply the same limit on the Killing vectors to see how the isometries of the space are preserved. With respect to the supersymmetries our approach will be slightly different. We will present the group theoretic argument as to form of the superalgebra and will use the commutation relations therefrom together with the form of the kinematical charges derived from Killing vectors to obtain supercharges.

2.2 PP-wave as the Penrose limit of the $AdS_5 \times S^5$

Penrose Limit [14] is a limit describing the neighborhood of a certain geodesic in the curved space. In this case we are interested in the light-like equatorial geodesic on the S^5. subspace. To see how we go from the full $AdS_5 \times S^5$ to this limit [10] consider once again the full $AdS_5 \times S_5$ metric as given above:

$$\begin{aligned} ds^2 = R^2[-d\tau^2\cosh^2(\rho) + d\rho^2 + \sinh^2(\rho)d\Omega_3^2 + \\ d\psi^2\cos^2(\theta) + d\theta^2 + \sin^2(\theta)d\Omega_3'^2] \end{aligned} \quad (2.41)$$

the geodesic of interest is a trajectory of the particle moving along the ψ direction while sitting at 0 in ρ and θ coordinates. To observe its neighborhood we first introduce the coordinates $\tilde{x}^{\pm} = \frac{\tau \pm \psi}{2}$ and then re-scale:

$$x^+ = \mu * \tilde{x}^+, \quad x^- = \frac{1}{\mu}R^2\tilde{x}^-, \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R}. \quad R \to \infty \quad (2.42)$$

Here we can express the Ω and Ω' as the components of the vectors r and y whose magnitudes are given by ρ and θ as follows:

$$r_i = R\rho\Omega_i, \quad y_i = R\theta\Omega_i' \quad (2.43)$$

With those we can now write the metric as:

$$\begin{aligned} ds^2 = \lim_{R\to\infty}(R^2[-(\sinh^2(\frac{r}{R}+1)(\mu dx^+ + \frac{dx^-}{\mu R})^2 + \frac{dr^2}{R^2} + \sinh^2(\frac{r}{R})\frac{dr_i^2}{r^2} + \\ (1-\sin^2(\frac{y}{R})(\mu dx^- + \frac{dx^-}{\mu R})^2 + \frac{dy^2}{R^2} + \sin^2(\frac{y}{R})\frac{dy_i^2}{y^2} \end{aligned} \quad (2.44)$$

taking the limit and using the:

$$\lim_{R\to\infty}(R^2\sinh^2(r/R)) = \lim_{R\to\infty}(R^2\sin^2(r/R)) = r^2 \tag{2.45}$$

we get:

$$ds^2 = -4dx^+dx^- - \mu^2\vec{x}^2 {dx^+}^2 + d\vec{x}^2 \tag{2.46}$$

where the vector $\vec{x}$ is an 8 dimensional vector composed of the components r_i and y_i. This has the form of the metric of the pp-wave.

Now we can use the same limit to see what is happening with the killing vectors. First we need the derivatives:

$$\frac{\partial}{\partial\tau} = \frac{\partial x^+}{\partial\tau}\frac{\partial}{\partial x^+} + \frac{\partial x^-}{\partial\tau}\frac{\partial}{\partial x^-} = \frac{\mu^2}{2}\frac{\partial}{\partial x^+} + \frac{R^2}{2\mu^2}\frac{\partial}{\partial x^-} \tag{2.47}$$

$$\frac{\partial}{\partial\psi} = \frac{\mu^2}{2}\frac{\partial}{\partial x^+} - \frac{R^2}{2\mu^2}\frac{\partial}{\partial x^-} \tag{2.48}$$

$$\frac{\partial}{\partial\rho} = \sum_i \frac{Rr_i}{r}\frac{\partial}{\partial r_i} \tag{2.49}$$

$$\frac{\partial}{\partial\theta} = \sum_i \frac{Ry_i}{y}\frac{\partial}{\partial y_i} \tag{2.50}$$

$$\frac{\partial}{\partial\Omega_i} = r\frac{\partial}{\partial r_i} \tag{2.51}$$

$$\frac{\partial}{\partial\Omega'_i} = y\frac{\partial}{\partial y_i} \tag{2.52}$$

The simple Killing vectors such as ε_1 or $\varepsilon_{6..11}$ have obvious limits:

$$\varepsilon_1 = \frac{\partial}{\partial\tau} = \frac{\mu^2}{2}\frac{\partial}{\partial x^+} + \frac{R^2}{2\mu^2}\frac{\partial}{\partial x^-} \tag{2.53}$$

$$\varepsilon_{6..11} = \Omega_i\frac{d}{d\Omega_j} - \Omega_j\frac{d}{d\Omega_i} = r_i\frac{d}{dr_j} - r_j\frac{d}{dr_i} \tag{2.54}$$

Here we calculate one of the more difficult ones, others being obtained in the exactly the same manner:

$$\varepsilon_{17..20} = \cos(\psi)\Omega'_i\frac{d}{d\theta} + \cot(\theta)\cos\psi(1-\Omega_i'^2)\frac{d}{d\Omega'_i} -$$

$$\sum_{j\neq i}\cot(\theta)\cos\psi\Omega'_i\Omega'_j\frac{d}{d\Omega'_j} - \tan(\theta)\sin(\psi)\Omega'_i\frac{d}{d\psi}$$

$$= \cos(\psi)[\frac{y_i}{y}\sum_j\frac{Ry_j}{y}\frac{\partial}{\partial y_j} + \frac{R}{y}(1-\frac{y_i^2}{y^2})y\frac{\partial}{\partial y_i} -$$

$$\sum_{j\neq i} R\frac{y_i y_j}{y^2}\frac{\partial}{\partial y_j}] + \sin(\psi)[\tan(\frac{r}{R})\frac{r_i}{r}\frac{\partial}{\partial\psi}]$$

$$= R\cos(\psi)[\frac{y_i^2}{y^2} + \frac{y^2 - y_i^2}{y^2}]\frac{\partial}{\partial y_i} + \sin(\psi)[\tan(\frac{r}{R})\frac{r_i}{r}(\frac{\mu^2}{2}\frac{\partial}{\partial x^+} - \frac{R^2}{2\mu^2}\frac{\partial}{\partial x^-})]$$

$$= R[\cos(\mu x^+ - \frac{x^-}{\mu R^2})\frac{\partial}{\partial y_i} + \sin(\mu x^+ - \frac{x^-}{\mu R^2})\tan(\frac{r}{R})\frac{r_i}{r}(\frac{\mu^2}{2}\frac{\partial}{\partial x^+} - \frac{R^2}{2\mu^2}\frac{\partial}{\partial x^-})]$$

And taking the limit:

$$\varepsilon_{17..20} = \cos(\mu x^+)\frac{\partial}{\partial x^i} - \frac{\mu}{2}\sin(\mu x^+)x_i\frac{\partial}{\partial x^-}$$

We can now write entire set of Killing vectors for the pp-wave and therefore obtain the symmetries of the metric. Uncoupling the linear combinations and renumbering them they are:

$$\varepsilon_1 = \frac{\partial}{\partial x^+}, \quad \varepsilon_2 = \frac{\partial}{\partial x^-} \tag{2.55}$$

$$\varepsilon_{3..10} = \cos(\mu x^+)\frac{\partial}{\partial x^i} - \frac{\mu}{2}\sin(\mu x^+)x_i\frac{\partial}{\partial x^-}$$

$$\varepsilon_{11..16} = x_i\frac{d}{dx_j} - x_j\frac{d}{dx_i} \quad i,j \in (1,2,3,4)$$

$$\varepsilon_{17..22} = x_i\frac{d}{dx_j} - x_j\frac{d}{dx_i} \quad i,j \in (5,6,7,8)$$

$$\varepsilon_{23..30} = \sin(\mu x^+)\frac{\partial}{\partial x^i} + \frac{\mu}{2}\cos(\mu x^+)x_i\frac{\partial}{\partial x^-}$$

and therefore the symmetries will be:

$$\delta x^+ = a^+ \tag{2.56}$$

$$\delta x^- = b^- + c_i^- \sin(\mu x^+)x_i + d_i^- \cos(\mu x^+)x^i \tag{2.57}$$

$$\delta x^i = e_j^i + c_-^i \cos(\mu x^+) + d_-^i \sin(\mu x^+) \tag{2.58}$$

with i,j in the last line being both either from the first or from the second set of 4.

Already, a very important property of the pp-wave geometry becomes apparent. In contrast with both flat and the $AdS_5 \times S^5$ cases, there is no symmetry mixing the x^+ light cone direction with any other. This will have several important consequences. First one is that it is not possible to obtain the critical dimension using the usual operator formalism argument [1]. Second, and more important, consequence is that the usual way of defining the Vertex Operators for the Green-Schwarz string [15] will not work. This method consists of constructing the vertices in a frame where the formulas are particularly simple (momentum along the x^+ axis) and then rotating to a generic case using the symmetry. Some of the problems this causes will be discussed in Chapter 6.

Also, it would appear from the form of the metric that in addition to the 30 symmetries inherited from $AdS_5 \times S_5$ the new ones appear that mix the $[x_1, x_2, x_3$ and $x_4]$ coordinates with the $[x_5, x_6, x_7$ and $x_8]$ ones. We did not list those symmetries in the above for, while they would indeed exist in the purely geometric sense they are precluded by the form of the R-R flux as seen bellow.

Güven has shown that the Penrose limit generalizes to the background fields beyond metric [16]. To see what happens with the 5-form R-R flux we simply take the limit using the same coordinate transformations:

$$
\begin{aligned}
F_5 &= 4R^4[\cosh(\rho)\sinh^3(\rho)d\tau \wedge d\rho \wedge d\Omega_3 + \cos(\theta)\sin^3(\theta)d\psi \wedge d\theta \wedge d\Omega'_3 \qquad (2.59)\\
&= 4R^4[(\sinh^3(\frac{r}{R}) + \sinh^4(\frac{r}{R}))(\mu dx^+ + \frac{dx^-}{\mu R}) \wedge \frac{dr}{R} \wedge \frac{dr_i}{r}\\
&\quad +(\sin^3(\frac{y}{R}) - \sin^4(\frac{y}{R}))(\mu dx^+ - \frac{dx^-}{\mu R}) \wedge \frac{dy}{R} \wedge \frac{dy_i}{y}\\
&= 4\mu dx^+ \wedge [dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8]
\end{aligned}
$$

or, writing alternatively:

$$F_{+1234} = F_{+5678} = 4\mu \tag{2.60}$$

with the other components being equal to zero.

As predicted by Güven [16] the supersymmetries of the background are preserved in the limit. This means that there are exactly 32 supersymmetries in addition to 30 isometries. Once again, this can easily be checked using the Killing spinor equation $\mathcal{D}\epsilon = 0$. Here we follow Metsaev [8] and adopt the normalization in which the Einstein equations take the form:

$$R^{\mu\nu} = (\frac{1}{24}F^{\mu\rho_1\cdots\rho_4}F^{\nu\rho_1\cdots\rho_4}) \tag{2.61}$$

and take the 5-form to be anti-self dual:

$$F\mu_1\ldots\mu_5 = -(\frac{1}{120})\epsilon^{\mu_1\cdots\mu_5\rho_1\cdots\rho_5}F^{\rho_1\cdots\rho_5} \tag{2.62}$$

Raising the indices in 2.60 and expressing the anti-symmetry explicitly we have:

$$F^{-i_1\cdots i_4} = 2\mu\epsilon^{i_1 \cdot\cdot i_4}, \quad F^{-i'_1\cdots i'_4} = 2\mu\epsilon^{i'_1 \cdot\cdot i'_4} \tag{2.63}$$

In the equation above and from here on we adopt the convention:

$$i, j \in [1, 2, 3, 4], \quad i', j' \in [5, 6, 7, 8],$$

$$I, J \in [1, 2, 3, 4, 5, 6, 7, 8], \quad \mu, \nu, \rho \in [+, -, I]$$

With this:

$$\mathcal{D} = d + \frac{1}{4}\omega^{\mu\nu}\gamma^{\mu\nu} - \frac{i}{960}\gamma^{\mu_1\cdots\mu_5}\bar{\gamma}^{\rho}e^{\rho}F^{\mu_1\cdots\mu_5} \tag{2.64}$$

where e^ρ are 10-beins of the plane wave space and $\omega^{\mu\nu}$ is a Lorentz connection. Inserting the above forms for the 5-forms field 2.63 it can be shown [17] that the integrability equation is satisfied for all ϵ in the same way as in 2.38 with $\mathcal{D}^2 = 0$, so, once again, we have 32 Killing spinors.

We now proceed to give the full superalgebra of the pp-wave R-R background. It can be seen from the above argument that this algebra will be divided into the even, or Bosonic, part (coming from Isometries) and the odd, or Fermionic, part (coming from supersymetries). The bosonic part will include the ten translations P^μ, two $SO(4)$ rotational sub-algebras J^{ij} and $J^{i'j'}$ and eight rotational generators in the (x^-, x^I) plane J^{+I}. The fermionic part will consist of the 16-component spinor Q_α $\alpha = 1, ..., 16$ which is a half of a 32-component negative chirality spinor.

The commutators of the bosonic part can be obtained directly from the Killing vectors as given in 2.55. Obtaining the commutators between the bosonic and fermionic generators and the anti-commutators between the fermionic ones is more difficult as it would require actually calculating the exact form of Killing Spinors from 2.64. While this can be done, we choose to follow Metsaev [8] and just present it as a known algebra with the representation given above. As a confirmation of its form we will check that in the flat-space $\mu \to \infty$ limit it reduces to the standard d=10, IIB Poincare superalgebra.

The commutation relations between even generators are as follows:

$$[P^-, P^I] = -\mu^2 J^{+I} \tag{2.65}$$

$$[P^I, J^{+J}] = -\delta^{IJ} P^+, \qquad [P^-, J^{+J}] = P^I \tag{2.66}$$

$$[P^i, J^{jk}] = \delta^{ij} P^j - \delta^{ik} P^j, \qquad [P^{i'}, J^{j'k'}] = \delta^{i'j'} P^{j'} - \delta^{i'k'} P^{j'} \tag{2.67}$$

$$[J^{+i}, J^{jk}] = \delta^{ij} J^{+k} - \delta ik J^{+j}, \qquad [J^{+i'}, J^{j'k'}] = \delta^{i'j'} J^{+k'} - \delta^{i'k'} J^{+j'} \tag{2.68}$$

$$[J^{ij}, J^{kl}] = \delta^{jk} J^{il} + \delta^{jl} J^{ik} + \delta^{ik} J^{jl} + \delta^{il} J^{jk} \tag{2.69}$$

$$[J^{i'j'}, J^{k'l'}] = \delta^{j'k'} J^{i'l'} + \delta j'l' J^{i'k'} + \delta^{i'k'} J^{j'l'} + \delta i'l' J^{j'k'}$$

Between the odd and even generators they are:

$$[J^{ij}, Q_\alpha] = \frac{1}{2} Q_\beta (\gamma^{ij})^\beta_\alpha, \qquad [J^{i'j'}, Q_\alpha] = \frac{1}{2} Q_\beta (\gamma^{i'j'})^\beta_\alpha \tag{2.70}$$

$$[J^{+I}, Q_\alpha] = \frac{1}{2} Q_\beta (\gamma^{+I})^\beta_\alpha, \tag{2.71}$$

$$[P^\mu, Q_\alpha] = \mu^2 \frac{i}{2} Q_\beta (\Pi \gamma^+ \bar{\gamma}^\mu)^\beta_\alpha \tag{2.72}$$

and the commutators that come from the complex conjugations of the above.

The anti-commutator is of the form:

$$\{Q_\alpha, \bar{Q}_\beta\} = -2i\gamma^\mu_{\alpha\beta} P^\mu + \mu[-2(\bar{\gamma}^i \Pi)_{\alpha\beta} J^{+i} - (\bar{\gamma}^{i'} \Pi')_{\alpha\beta} J^{+i'} \tag{2.73}$$

$$+2(\bar{\gamma}^+ \gamma^{i,j} \Pi)_{\alpha\beta} J^{ij} + 2(\bar{\gamma}^+ \gamma^{i,j} \Pi)_{\alpha\beta} J^{ij}]$$

Here and hereafter the γ^μ are 16×16 gamma matrices. $\gamma^{\mu_1 \cdots \mu_k}$ are anti symmetrized products of k gamma matrices and the Π are given as follows:

$$\Pi^\alpha_\beta = (\gamma^1\bar{\gamma}^2\gamma^3\bar{\gamma}^4)^\alpha_\beta), \qquad \Pi'^\alpha_\beta = (\gamma^5\bar{\gamma}^6\gamma^7\bar{\gamma}^8)^\alpha_\beta \tag{2.74}$$

It should be noted here that the (anti) commutation relationships of the superalgebra are invariant under U(1) transformation of supercharges: $Q \to e^{1\phi}Q, \bar{Q} \to e^{1\phi}\bar{Q}$ which is the original U(1) symmetry of the type IIB supergravity realized on the pp-wave.

It is trivial to see that the above reduces to the subalgebra of the IIB superalgebra in the flat space limit. The transformation group G above, then, has the generators: $P^\mu, Q_\alpha, \bar{Q}_\alpha, J^{ij}, J^{i'j'}, J^{+I}$ it is easy to see that this group has a subgroup H, also called stability subgroup, with generators: $J^{ij}, J^{i'j'}, J^{+I}$ We can then define the pp-wave R-R superspace as the coset superspace G/H

Chapter 3

The superstring action

3.1 Cartan 1-forms

Looking back at the derivation of the superstring action in the GS formalism for the flat space [1] we see that the Lagrangian was fully determined by its symmetries, those being isometries, global supersymmetries and the local κ - supersymmetry that helps simplify the equations of motion. Having defined the global symmetries of the pp-wave R-R background we will now follow Metsaev [8] and try to use them to construct the action.

The most convenient way to express these symmetries is through the Cartan 1-forms. The Cartan forms are the objects that map the tangent vectors on the Lie group onto the elements of the appropriate Lie algebra in linear way.

On our coset superspace we define these in the following fashion:

$$L^A = dX^M L^A_M, \qquad X^M = (x^\mu, \theta^\alpha, \bar{\theta}^\alpha) \tag{3.1}$$

they are then given in terms of members of the (super)symmetry group G by:

$$G^{-1}dG = L^\mu P^\mu + L^\alpha \bar{(Q)}_\alpha + \bar{L}^\alpha Q_\alpha + \frac{1}{2} L^{\mu\nu} J^{\mu\nu} \tag{3.2}$$

The fact that the generators $J^{-\mu}$ and $J^{ij'}$ do not exist means that we need to impose the following condition on the above:

$$L^{+\mu} = 0, \qquad L^{ij'} = 0 \tag{3.3}$$

in the above, the L^μ are the 10-beins, L^α and $\bar{L}^\alpha = (L^\alpha)^\dagger$ are spinor 16-beins, and the $L^{\mu\nu}$ are the tensors called "Cartan H connections".

The important feature of the Cartan forms that makes them convenient way to express the symmetries of the group is s Maurer-Cartan equation that gives their external derivative. The external derivative d takes the Cartan 1-forms into the 2-forms as follows:

$$dL^A = \frac{1}{2} \sum_{BC} f^A_{BC} L^B \wedge L^C \tag{3.4}$$

where f^A_{BC} are the structure constants that determine the commutation relations on the Lie algebra. We can then write:

$$dL^\mu = -L^{\mu\nu} \wedge L^\nu - 2i\bar{L}^\alpha \bar{\gamma}^\mu_{\alpha\beta} \wedge L^\beta \tag{3.5}$$

$$dL^{\alpha} = -\frac{1}{4}L^{\mu\nu}(\gamma^{\mu\nu})^{\alpha}_{\beta} \wedge L^{\beta} + \frac{i\mu^2}{2}L^{\mu}(\Pi\gamma^{+}\bar{\gamma}^{\mu})^{\alpha}_{\beta} \wedge L^{\beta} \tag{3.6}$$

$$d\bar{L}^{\alpha} = -\frac{1}{4}L^{\mu\nu}(\gamma^{\mu\nu})^{\alpha}_{\beta} \wedge \bar{L}^{\beta} - \frac{i\mu^2}{2}L^{\mu}(\Pi\gamma^{+}\bar{\gamma}^{\mu})^{\alpha}_{\beta} \wedge \bar{L}^{\beta} \tag{3.7}$$

$$dL^{-i} = L^{+} \wedge L^{i} + L^{-j} \wedge L^{jk} + 2L^{\alpha}(\bar{\gamma}^{i}\Pi)_{\alpha\beta} \wedge L^{\beta} \tag{3.8}$$

$$dL^{ij} = L^{ik} \wedge L^{kj} + L^{\alpha}(\bar{\gamma}^{+}\gamma^{ij}\Pi)_{\alpha\beta} \wedge L^{\beta} \tag{3.9}$$

with the similar ones for $L^{-i'}$ and $L^{i'j'}$

From here on we can suppress the wedge product symbols keeping in mind the standard properties of the wedge product between the vectors and spinors:

$$L^{\mu}l^{\nu} = -L^{\nu}L^{\mu}, \qquad L^{\mu}L^{\alpha} = -L^{\alpha}L^{\mu}, \qquad L^{\alpha}L^{\beta} = L^{\beta}L^{\alpha} \tag{3.10}$$

from here also we temporarily suppress the factor of μ. We will eventually recover it by the following re-scaling:

$$L^{\mu} \to \mu^2 L^{\mu}, \qquad L^{\mu\nu} \to L^{\mu\nu}, \qquad L^{\alpha} \to \mu L^{\alpha}, \qquad x^{\mu} \to \mu^2 x^{\mu}, \qquad \theta \to \mu\theta \tag{3.11}$$

3.2 Action on Flat Space

We want to draw the analogy between the pp-wave R-R case and the flat space case. To do that lets recall the form of the flat space GS Lagrangian [1]:

$$\mathcal{L}_{flat} = \mathcal{L}_{1flat} + \mathcal{L}_{2flat} \tag{3.12}$$

$$\mathcal{L}_{1flat} = -\frac{1}{2}\sqrt{g}g^{ab}\Pi^{\mu}_{a}\Pi^{\mu}_{b} \tag{3.13}$$

where

$$\Pi^{\mu}_{a} = \partial_a X^{\mu} - i\bar{\theta}\bar{\gamma}^{\mu}\partial_a\theta - i\theta\bar{\gamma}^{\mu}\partial_a\bar{\theta} \tag{3.14}$$

is not to be confused with our Π matrix.

$$\mathcal{L}_{2flat} = -i\epsilon^{ab}(\partial_b X^{\mu} - i\bar{\theta}\bar{\gamma}^{\mu}\partial_a\theta - i\theta\bar{\gamma}^{\mu}\partial_a\bar{\theta})\theta\bar{\gamma}^{\mu}\partial_b\theta + h.c \tag{3.15}$$

It is obvious that the Π^{μ} from equation 3.14 was constructed to possess all the reparametrization invariances and the global supersymmetries. As such it is nothing more then the vielbein L^{μ}_{flat} analogous to our Cartan form L^{μ}. and given by:

$$(L^{A}_{a})_{flat} = \partial_a X^{M}(L^{A}_{M})_{flat} \tag{3.16}$$

To see that the Π is indeed a Cartan form for the flat space we can then use the commutators and anti-commutators of flat space:

$$[P^{\mu}, P^{\nu}] = 0, \qquad \{Q_{\alpha}, \bar{Q}_{\beta}\} = -2i\gamma^{\mu}_{\alpha\beta}P^{\mu} \tag{3.17}$$

to write the Maurer-Cartan equation for the L^{μ}_{flat} and L^{α}_{flat}

$$dL^{\mu}_{flat} = 0 \wedge dx^{\mu} - i\bar{L}^{\alpha}_{flat}\bar{\gamma}^{\mu}_{\alpha\beta} \wedge L^{\beta}_{flat} \tag{3.18}$$

$$dL^{\alpha}_{flat} = 0 \wedge d\theta \tag{3.19}$$

from which we get the:

$$L^{\mu}_{flat} = dx^{\mu} - i\bar{\theta}\bar{\gamma}^{\mu}d\theta - i\theta\bar{\gamma}^{\mu}da\bar{\theta} \tag{3.20}$$

which is equivalent to the equation 3.14, and

$$L^{\alpha}_{flat} = d\theta^{\alpha} \tag{3.21}$$

Being subject to all the (super)symmetries of the background, the $\mathcal{L}_{2flat}$ term also has to be a closed form constructed from the flat L forms. From the equations 3.15, 3.20 and 3.21 it is not hard to see that it is in effect a 3-form given by:

$$\mathcal{L}_{2flat} = d^{-1}\mathcal{H}_{flat} \tag{3.22}$$
$$\mathcal{H}_{flat} = iL^{\mu}_{flat}L^{\alpha}_{flat}\bar{\gamma}^{\mu}_{\alpha\beta}L^{\beta}_{flat} + h.c.$$

With the coefficient of the $\mathcal{H}_{flat}$ being given by the condition of κ invariance.

What we attempt to do in the rest of this section is find the analogous action for the pp-wave case with the Cartan forms as defined above.

3.3 Action on the pp-wave R-R space

Metsaev [8] gives four conditions (some of which are not completely independent) that the action that the superstring action must satisfy:

a) Its bosonic part is the standard σ-model in the pp-wave background.

b) It has global super-invariance with respect to supersymmetry given above.

c) It is invariant under local κ-symmetry.

d) It reduces to the standard Green-Schwarz type IIB superstring action in the flat space ($\mu \to 0$) limit.

Such an action would have its leading fermionic term contain required coupling to the R-R five form background field.

As in the flat space this action will be given by the sum of the σ-model term $\mathcal{L}_1$ and the term responsible to ensure the κ-symmetry. $\mathcal{L}_2 = d^{-1}\mathcal{H}$. Again, in order to satisfy the invariance with respect to the symmetry superalgebra both of those will have to be constructed in terms of the Cartan 1-forms L^{μ} and L^{α}. The strategy will be to find the action in terms of those forms and then find the forms themselves using the Maurer-Cartan equations 3.5 - 3.9.

The first thing to note is that under the action of an arbitrary element of the group G these forms transform as the tangent vectors (and spinors) of the stability subgroup H. This means that any combination of the L^μs and L^αs that is invariant under the stability group transformations will automatically be invariant under the full transformations of G.

The conditions a) and b) fix the structure of the σ-model part of the Lagrangian. It can be obtained by replacing the L^μ_{flat} in the $\mathcal{L}_{1flat}$ of the flat space action by the Cartan 1-forms of the pp-wave R-R background L^μ. As in the flat space the $\mathcal{L}_2$ part is more difficult but we deduce that, like in the flat case it will be a closed 3-form built out of the L^μs and L^αs that is also invariant under the transformations generated by the elements of the stability subgroup.

It turns out that the only relevant 3-form satisfying these requirements is one given by:

$$\mathcal{H} = \mathcal{H}^q + \mathcal{H}^{\bar{q}}, \qquad \mathcal{H}^q = (\mathcal{H}^{\bar{q}})^\dagger \tag{3.23}$$

where

$$\mathcal{H}^q = iL^\mu L^\alpha \bar{\gamma}^\mu_{\alpha\beta} L^\beta$$

While it is difficult to actually prove that this is the only such form we can demonstrate that it indeed satisfies the requirements. Furthermore the $\mathcal{L}_2$ composed using this form provides the necessary κ-symmetry thus making the action satisfy the condition c).

First we demonstrate that the form is indeed closed:

$$d\mathcal{H}^q = i[dL^\mu L^\alpha \bar{\gamma}^\mu_{\alpha\beta} L^\beta + L^\mu dL^\alpha \bar{\gamma}^\mu_{\alpha\beta} L^\beta + L^\mu L^\alpha \bar{\gamma}^\mu_{\alpha\beta} dL^\beta]$$

which we can write using the Maurer-Cartan equations 3.5 - 3.9 as:

$$\begin{aligned} d\mathcal{H}^q = & -iL^{\mu\nu} L^\nu L^\alpha \bar{\gamma}^\mu_{\alpha\beta} L^\beta + 2L^\gamma \bar{\gamma}^\mu_{\gamma\delta} L^\delta L^\alpha \bar{\gamma}^\mu_{\alpha\beta} L^\beta \\ & -\frac{i}{4} L^\mu L^{\rho\sigma} (\gamma^{\rho\sigma})^\alpha_\gamma L^\gamma \bar{\gamma}_{\alpha\beta} L^\beta + \frac{1}{2} L^\mu L^\nu (\Pi\gamma^+\bar{\gamma}^\nu)^\alpha_\gamma L^\gamma \bar{\gamma}_{\alpha\beta} L^\beta \\ & -\frac{i}{4} L^\mu L^\alpha \bar{\gamma}_{\alpha\beta} L^{\rho\sigma} (\gamma_{\rho\sigma})^\beta_\gamma L^\gamma + \frac{1}{2} L^\mu L^\alpha \bar{\gamma}_{\alpha\beta} L^\nu (\Pi\gamma^+\bar{\gamma}^\nu)^\beta_\gamma L^\gamma \end{aligned}$$

Splitting this into the parts proportional to $L^{\mu\nu}$ and the rest, and using the following relationship:

$$\bar{\gamma}^\mu \gamma^{\rho\sigma} = \bar{\gamma}^{\mu\rho\sigma} + \eta^{\mu\rho} \bar{\gamma}^\sigma - \eta^{\mu\sigma} \bar{\gamma}^\rho \tag{3.24}$$

We can write the part of $\mathcal{H}$ proportional to $L^{\mu\nu}$ as:

$$\begin{aligned} d\mathcal{H}^q_1 = i[& -L^{\mu\nu} L^\nu L^\alpha L^\beta \bar{\gamma}^\mu_{\alpha\beta} + \frac{1}{4} L^\mu L^{\rho\sigma} [\bar{\gamma}^{\mu\rho\sigma} + \eta^{\mu\rho} \bar{\gamma}^\sigma - \eta\mu\sigma \bar{\gamma}^\rho]_{\gamma\beta} L^\gamma L^\beta \\ & -\frac{1}{4} L^\mu L^\alpha [\bar{\gamma}^{\mu\rho\sigma} + \eta^{\mu\rho} \bar{\gamma}^\sigma - \eta\mu\sigma \bar{\gamma}^\rho]_{\alpha\beta} L^{\rho\sigma} L^\beta \end{aligned}$$

Commuting the L s according to 3.10 and changing the dummy variables to match, we get:

$$d\mathcal{H}^q_1 = iL^{\mu\nu} L^\nu L^\alpha L^\beta [-\bar{\gamma}^\mu_{\alpha\beta} + \bar{\gamma}^\mu_{\alpha\beta}] + \frac{1}{2} L^\mu L^{\rho\sigma} L^\alpha L^\beta (\bar{\gamma}^{\mu\rho\sigma})_{\alpha\beta}$$

$$= \frac{1}{2} L^\mu L^{\rho\sigma} L^\alpha L^\beta (\bar{\gamma}^{\mu\rho\sigma})_{\alpha\beta}$$

The remainder is given by:

$$d\mathcal{H}_2^q = L^\mu L^\nu L^\alpha (\bar{\gamma}^\mu \Pi \gamma^+ \bar{\gamma}^\nu)_{\alpha\beta} L^\beta$$

We know from 3.10 that $L^\alpha L^\beta$ which is present n all the remaining terms is symmetric under the exchange of indices. We now have to check the matrices $(\bar{\gamma}^{\mu\rho\sigma})_{\alpha\beta}$ and $(\bar{\gamma}^\mu \Pi \gamma^+ \bar{\gamma}^\nu)_{\alpha\beta}$ As said before, $\gamma^{\mu_1 \cdots \mu_k}$ are anti symmetrized products of k gamma matrices so we can write:

$$(\bar{\gamma}^{\mu\rho\sigma})_{\alpha\beta} = [\gamma^\mu \bar{\gamma}^\nu \gamma^\rho]_{\alpha\beta} - [\gamma^\mu \bar{\gamma}^\rho \gamma^\nu]_{\alpha\beta} + [\gamma^\rho \bar{\gamma}^\mu \gamma^\nu]_{\alpha\beta} - [\gamma^\rho \bar{\gamma}^\nu \gamma^\mu]_{\alpha\beta} + [\gamma^\nu \bar{\gamma}^\rho \gamma^\mu]_{\alpha\beta} - [\gamma^\nu \bar{\gamma}^\mu \gamma^\rho]_{\alpha\beta}$$

$$= (\gamma^\mu)_{\alpha\delta} (\bar{\gamma}^\nu)^{\delta\gamma} (\gamma^\rho)_{\gamma\beta} + (\gamma^\rho)_{\alpha\delta} (\bar{\gamma}^\mu)^{\delta\gamma} (\gamma^\nu)_{\gamma\beta} + (\gamma^\nu)_{\alpha\delta} (\bar{\gamma}^\rho)^{\delta\gamma} (\gamma^\mu)_{\gamma\beta}$$

$$-(\gamma^\rho)_{\alpha\gamma} (\bar{\gamma}^\nu)^{\gamma\delta} (\gamma^\mu)_{\delta\beta} - (\gamma^\nu)_{\alpha\gamma} (\bar{\gamma}^\mu)^{\gamma\delta} (\gamma^\rho)_{\delta\beta} - (\gamma^\mu)_{\alpha\gamma} (\bar{\gamma}^\rho)^{\gamma\delta} (\gamma^\nu)_{\delta\beta}$$

where in the last line we have used the fact that the γ matrices themselves are symmetric in the indices. It is obvious that the above is anti-symmetric in the α and β and that therefore this matrix multiplying the Cartan forms will be equal to zero. For the $(\bar{\gamma}^\mu \Pi \gamma^+ \bar{\gamma}^\nu)_{\alpha\beta}$ we have three possible cases: $(\mu = \nu = +)$, $(\mu = i, \nu = +)$ and $(\mu = i, \nu = j)$. The first will clearly be zero due to anti-symmetry in μ and ν in Cartan form part. second we can express as the: $(\bar{\gamma}^i \Pi)_{\alpha\beta}$ because of dirac algebra and the third we can anti-symmetrize and write as: $(\Pi \gamma^+ \bar{\gamma}^{ij})_{\alpha\beta}$ using the anti-commutator: $\gamma^i \Pi = -\Pi \gamma^i$. That same anti-commutator allows us to write:

$$\gamma^i_{\alpha\delta} \Pi^\delta_\beta = -\Pi^\delta_\alpha \gamma^i_{\delta\beta}$$

thus explicitly showing the anti-symmetry in α and β. Similarly, in the $(\Pi \gamma^+ \bar{\gamma}^{ij})_{\alpha\beta}$ there will be three anti-commutations (Π with γ^i, Π with γ^j and γ^i with γ^j) ensuring the sign change with the flip of indices.

Therefore these symmetry properties combined with the Maurer-Cartan equations ensure that in the end $d\mathcal{H}^q = 0$ and the form is indeed closed. Because all Cartan 1-forms in the pp-wave R-R background reduce to the corresponding Cartan 1-forms in the flat space background in the appropriate limit the 3-form $\mathcal{H}$ given in 3.23 will also reduce to the second part of the Lagrangian given in 3.22 thus ensuring that the action given in terms of these forms satisfies the condition d).

We can therefore write the Lagrangian in terms of the Cartan forms as follows:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \tag{3.25}$$

$$\mathcal{L}_1 = -\frac{1}{2} \sqrt{g} g^{ab} L^\mu_b L^\mu_a, \qquad \mathcal{L}_2 = d^{-1} \mathcal{H} \tag{3.26}$$

The κ-symmetry invariance is established in a way similar to that in the flat-space case. First the variations are defined in terms of Cartan forms and then the careful algebraic process is followed to make sure of the symmetry including the fitting of the variation of the metric. The sketch of the process can be found in Metsaev [8] and here we only list the actual variations.

$$\widehat{\delta x}^{\mu} = 0, \qquad \widehat{\delta\theta} = 2L^{\mu}_{a}\gamma^{\mu}\kappa^{a} \tag{3.27}$$

$$\delta(\sqrt{g}g^{ab}) = -8i\sqrt{(}L^{a}\bar{\kappa}^{b} + L^{b}\bar{\kappa}^{a} - \frac{1}{2}g^{ab}L_{c}\bar{\kappa}^{c}) + h.c. \tag{3.28}$$

With the self-duality constraints of the κ-parameter in the complex notation being given by:

$$\frac{\epsilon^{ab}}{\sqrt{g}}\kappa_{b} = -\bar{\kappa}^{a}, \qquad \frac{\epsilon^{ab}}{\sqrt{g}}\bar{\kappa}_{b} = -\kappa^{a} \tag{3.29}$$

The importance of the κ-symmetry for determining the exact form of the action is among other things in fixing the coefficient in front of the $\mathcal{L}_2$ to 1. We will also use the fixing of the κ-symmetry to simplify the differential equations from which the explicit 2 dimensional form of the action is obtained.

3.4 Explicit form of the action

It is now our goal to obtain the explicit form of the action on pp-wave R-R background in terms of the fields x and θ that generalizes the equations 3.14 and 3.15. In particular, we would like to express the $\mathcal{L}_2 = d^{-1}\mathcal{H}$ part with the integration given explicitly.

First of all, we choose a particular parametrization of the group element G from the equation 3.2:

$$G(x,\theta) = \mathrm{g}(x)g(\theta), \qquad g(\theta) = exp(\theta^{\alpha}\bar{Q}_{\alpha} + \bar{\theta}^{\alpha}Q_{\alpha}) \tag{3.30}$$

This parametrization is by no means unique. What it does is specializes the choice of fermionic coordinates. This particular parametrization is called "Wess-Zumino" or WZ parametrization.

To represent the $\mathcal{L}_2 = d^{-1}\mathcal{H}$ term as a density over the two-dimensional space we rescale $\theta \to \theta_t \equiv t\theta$

$$\mathcal{L}_2 = \mathcal{L}_2(t=1), \qquad \mathcal{L}_2 = d^{-1}[\mathcal{H}^{q}(0) + \int_0^1 dt\partial_t\mathcal{H}^{q}(t)] \tag{3.31}$$

From the Maurer-Cartan equations 3.5 to 3.9 we can then read out the differential equations for the "shifted" Cartan 1-forms $L^{A}_{t} = L^{A}(x, t\theta)$

$$\partial_t L^{\alpha}_{t} = d\theta + 1/4L^{\mu\nu}_{t}\gamma^{\mu\nu}\theta^{\alpha} - \frac{i}{2}L^{\mu}_{t}(\Pi\gamma^{+}\bar{\gamma}^{\mu}\theta)^{\alpha} \tag{3.32}$$

$$\partial_t L^{\mu}_{t} = -2i\theta^{\alpha}\bar{\gamma}^{\mu}_{\alpha,\beta}\bar{L}^{\beta}_{t} - 2i\bar{\theta}^{\alpha}\bar{\gamma}^{\mu}_{\alpha,\beta}L^{\beta}_{t} \tag{3.33}$$

$$\partial_t L^{-i}_{t} = 2\theta^{\alpha}(\bar{\gamma}^{i}\Pi)_{\alpha\beta}\bar{L}^{\beta}_{t} - 2\bar{\theta}^{\alpha}(\bar{\gamma}^{i}\Pi)_{\alpha\beta}L^{\beta}_{t} \tag{3.34}$$

$$\partial_t L^{ij}_{t} = -2\theta^{\alpha}(\bar{\gamma}^{+}\gamma^{ij}\Pi)_{\alpha\beta}\bar{L}^{\beta}_{t} + 2\bar{\theta}^{\alpha}(\bar{\gamma}^{+}\gamma^{ij}\Pi)_{\alpha\beta}L^{\beta}_{t} \tag{3.35}$$

with the similar ones for the $\partial_t L_t^{-i'}$ and $\partial_t L_t^{i'j'}$. These equations should be supplemented by the initial conditions:

$$L^{\mu}_{t=0} = e^{\mu}, \qquad L^{\mu\nu}_{t=0} = \omega^{\mu nu}, \qquad L^{\alpha}_{t=0} = 0 \tag{3.36}$$

Where e^{μ} is a set of orthogonal unit vectors on the pp-wave geometry (10-bein) and the $\omega^{\mu\nu}$ is a Lorentz connection between them. Putting these directly into the equation 3.23 and using the Maurer-Cartan equations again we get:

$$\partial_t \mathcal{H}^q(t) = -2id(L^{\mu}_t \theta^{\alpha} \bar{\gamma}^{\mu}_{\alpha\beta} L^{\beta}_t) \tag{3.37}$$

and can write:

$$\mathcal{L}_2 = -2i \int_0^1 dt L^{\mu}_t \theta^{\alpha} \bar{\gamma}_{\alpha\beta} L^{\beta}_t + h.c. \tag{3.38}$$

and therefore, finally:

$$\mathcal{L} = -\frac{1}{2}\sqrt{g} g^{ab} L^{\mu}_b L^{\mu}_a - 2i \int_0^1 dt \epsilon^{ab} L^{\mu}_{at} (\theta^{\alpha} \bar{\gamma}_{\alpha,\beta} L^{\beta}_{bt} + \bar{\theta}^{\alpha} \bar{\gamma}_{\alpha,\beta} \bar{L}^{\beta}_{bt}) \tag{3.39}$$

What remains to be done now is expressing the Lagrangian 3.39 in terms of some set of bosonic and fermionic vielbeins related to the fields x and θ and then proceeding as in the flat space case to fix the light-cone gauge in which we can hope to obtain simple equations of motion. This light-cone gauge fixing is done in two parts [1], first fixing the fermionic light-cone gauge through the use of κ-symmetry to set $\bar{\gamma}^+\theta = 0$ and then using the conformal invariance to set $\sqrt{g} g^{ab} = \eta^{ab}$ and the residual diffeomorphism symmetry to set $x^+ = p^+\tau$.

While it is possible to do things in this order in this case as well [8], the system of differential equations 3.32 to 3.35 simplifies considerably once we impose the fermionic light cone gauge. What we begin with, therefore, is solving these equations in the fermionic gauge $\bar{\gamma}^+\theta = 0$.

First of all we can write 3.33 for $\mu = +$

$$\partial_t L^+_t = -2i\theta^{\alpha} \bar{\gamma}^+_{\alpha,\beta} \bar{L}^{\beta}_t - 2i\bar{\theta}^{\alpha} \bar{\gamma}^+_{\alpha,\beta} L^{\beta}_t \tag{3.40}$$

which is obviously proportional to the $\bar{\gamma}^+\theta$ and is therefore equal to zero. Together with the initial condition from 3.36 this means that:

$$L^+_t = e^+ \tag{3.41}$$

We then multiply the equation 3.32 by the $\bar{\gamma}^+$ getting:

$$\partial_t(\bar{\gamma}^+ L) = \bar{\gamma} d\theta + 1/4 L^{\mu\nu}_t \bar{\gamma}^+ \gamma^{\mu\nu} \theta - \frac{i}{2} L^{\mu}_t \gamma^+ \Pi \gamma^+ \bar{\gamma}^{\mu} \theta \tag{3.42}$$

$$= 1/4 L^{\mu\nu}_t \{\bar{\gamma}^+, \gamma^{\mu\nu}\}\theta - \frac{i}{2} L^{\mu}_t \{\gamma^+, \Pi\gamma^+\bar{\gamma}^{\mu}\}\theta$$

which is equal to zero because $L^{+\mu} = 0$ by construction, and those are the multipliers of the only terms with the non zero anti-commutators. We therefore have $\partial_t(\bar{\gamma}^+ L^{\alpha}_t) = 0$ and therefore, with initial conditions:

$$\bar{\gamma}^+ L^{\alpha}_t = 0 \tag{3.43}$$

We can use this in the equation 3.33 for $\mu = I$ inserting the decomposition of unity $1 = \frac{1}{2}(\gamma^+\bar{\gamma}^- + \gamma^-\bar{\gamma}^+)$ between the θs and L^αs:

$$\partial_t L_t^I = -2i\theta^\alpha \bar{\gamma}^I_{\alpha,\beta}\frac{1}{2}(\gamma^+\bar{\gamma}^- + \gamma^-\bar{\gamma}^+)\bar{L}_t^\beta - 2i\bar{\theta}^\alpha\bar{\gamma}^I_{\alpha,\beta}\frac{1}{2}(\gamma^+\bar{\gamma}^- + \gamma^-\bar{\gamma}^+)L_t^\beta \quad (3.44)$$

Because γ^+ anti-commutes with γ^I this then disappears as γ^+ annihilates, either θ using the light-cone gauge condition, or the L_t^β using the equation 3.43. Once again, $\partial_t L_t^I = 0$ together with initial conditions gives us:

$$L_t^I = e^I \quad (3.45)$$

So far we have only chosen parametrization of the fermionic coordiantes. To simplify our expressions even further we can choose the bosonic bodies of the Cartan H connections L^{ij} and $L^{i'j'}$ to be equal to zero.

$$\omega^{i,j} = 0, \qquad \omega^{i',j'} = 0 \quad (3.46)$$

Together with the light-cone gauge condition, in the manner similar to above this leads to:

$$L_t^{ij} = L_t^{i'j'} = 0 \quad (3.47)$$

We can then go back to the equation 3.32 and write:

$$\partial_t L_t = d\theta - \frac{i}{2}L_t^\mu \Pi\{\gamma^+, \bar{\gamma}^\mu\} = d\theta - ie^+\Pi\theta \quad (3.48)$$

or:

$$L_t = t(d\theta - ie^+\Pi\theta) \quad (3.49)$$

finally, using this in the equation 3.33 for $\mu = -$ we obtain:

$$\partial_t L_t^- = -2it(\bar{\theta}\bar{\gamma}^- d\theta + \theta\bar{\gamma}^- d\bar{\theta}) - 4te^+\bar{\theta}\bar{\gamma}^-\Pi\theta \quad (3.50)$$

Solving for t and setting $t = 1$ we get:

$$L^- = e^- - i(\bar{\theta}\bar{\gamma}^- d\theta + \theta\bar{\gamma}^- d\bar{\theta}) - 2e^+\bar{\theta}\bar{\gamma}^-\Pi\theta \quad (3.51)$$

Only thing that remains is to find the appropriate expression for the bosonic 10-beins e^μ consistent with the 3.46. The choice that parallels the flat space case most closely is:

$$e^+ = dx^+, \qquad e^- = dx^- - \frac{1}{2}x_I^2 dx^+, \qquad e^I = dx^I \quad (3.52)$$

Simply inserting the thus obtained Cartan forms into the expressions for $\mathcal{L}$ and restoring the μ dependance we get:

$$\mathcal{L}_1 = -1\frac{1}{2}\sqrt{g}g^{ab}(2\partial_a x^+\partial_b x^- - \mu^2 x_I^2\partial_a x^+\partial_b x^+ + \partial_a x^I\partial_b x^I) \quad (3.53)$$

$$-i\sqrt{g}g^{ab}\partial_b x^+(\bar{\theta}\bar{\gamma}^-\partial_a\theta + \theta\bar{\gamma}^-\partial_a\bar{\theta} + 2i\partial_a x^+ 2\mu i\partial_a x^+\bar{\theta}\bar{\gamma}^-\Pi\theta)$$

$$\mathcal{L}_2 = i\epsilon^{ab}\partial_a x^+ \theta\bar{\gamma}^- \partial_b \theta + h.c \tag{3.54}$$

where the second term is straightforwardly obtained by the integration of the expression 3.38.

This lagrangian can be rewriten in such a way as to separate the bosonic and fermionic components.

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \tag{3.55}$$

where:

$$\mathcal{L}_B = -1\frac{1}{2}\sqrt{g}g^{ab}(2\partial_a x^+ \partial_b x^- - \mu^2 x_I^2 \partial_a x^+ \partial_b x^+ + \partial_a x^I \partial_b x^I) \tag{3.56}$$

is a standard bosonic sigma model on the pp-wave geometry. And

$$\mathcal{L}_F = -i\sqrt{g}g^{ab}\partial_b x^+ (\bar{\theta}\bar{\gamma}^- \partial_a \theta + \theta\bar{\gamma}^- \partial_a \bar{\theta} + 2i\partial_a x^+ 2\mu i \partial_a x^+ \bar{\theta}\bar{\gamma}^- \Pi\theta) + \tag{3.57}$$

$$i\epsilon^{ab}\partial_a x^+ (\theta\bar{\gamma}^- \partial_b \theta + \bar{\theta}\bar{\gamma}^- \partial_b \bar{\theta})$$

We can now proceed to fix the bosonic light-cone gauge

$$\sqrt{g}g^{ab} = \eta^{ab}, \qquad -\eta^{\tau\tau} = \eta^{\sigma\sigma} = 1 \tag{3.58}$$

Again we have a residual diffeomorphism freedom that we can use to set the τ direction along one of the x coordinates. We choose x^+ and write:

$$x^+(\tau, \sigma) = p^+\tau \tag{3.59}$$

Writing the Lagrangians in this gauge we get:

$$\mathcal{L}_B = -\frac{1}{2}[\partial_a x^I \partial^a x^I + (\mu p^+)^2 (x^I)^2] \tag{3.60}$$

$$\mathcal{L}_F = i(\bar{\theta}\bar{\gamma}^- \partial_\tau \theta + \theta\bar{\gamma}^- \partial_\tau \bar{\theta} + \theta\bar{\gamma}^- \partial_\sigma \theta + \bar{\theta}\bar{\gamma}^- \partial_\sigma \bar{\theta}) - 2(\mu p^+)\bar{\theta}\bar{\gamma}^- \Pi\theta \tag{3.61}$$

With the

$$x^\pm = \frac{1}{\sqrt{2}}(x^\sigma \pm x^\tau) \tag{3.62}$$

and the complex Weyl spinor θ split into two real Majorana-Weyl spinors given by:

$$\theta = \frac{1}{\sqrt{2}}(\theta^1 + i\theta^2), \qquad \bar{\theta} = \frac{1}{\sqrt{2}}(\theta^1 - i\theta^2) \tag{3.63}$$

Lagrangian can also be written as:

$$\mathcal{L}_B = \frac{1}{2}(\partial_+ x^I \partial_- x^I - m^2 x_I^2), \qquad m \equiv p^+\mu \tag{3.64}$$

$$\mathcal{L}_F = i(\theta^1\bar{\gamma}^- \partial_+ \theta^1 + \theta^2\bar{\gamma}^- \partial_- \theta^2 - 2m\theta^1\bar{\gamma}^- \Pi\theta^2) \tag{3.65}$$

Which is the way they are given in [9].

Chapter 4

Canonical quantization

4.1 Equations of motion

First we get the equations of motion from the Lagrangain through the standard Euler-Lagrange method:

$$\frac{\partial \mathcal{L}}{\partial q^\alpha} - \frac{d}{da}\frac{\partial \mathcal{L}}{\partial(\partial_a q^\alpha)} = 0 \tag{4.1}$$

which in our case translate to:

$$\partial_+\partial_- x^I + m^2 x^I = 0 \tag{4.2}$$

$$\partial_+\theta^1 - m\Pi\theta^2 = 0, \qquad \partial_-\theta^2 + m\Pi\theta^1 = 0 \tag{4.3}$$

for the bosonic and fermionic fields respectively. In addition, given that we are dealing with the closed string, we also have periodicity condition:

$$x^I(\sigma+1,\tau) = x^I(\sigma,\tau), \qquad \theta^I(\sigma+1,\tau) = \theta^I(\sigma,\tau) \tag{4.4}$$

The solutions of these are:

$$x^I(\sigma,\tau) = \cos(m\tau)x_0^I + \frac{1}{m}\sin(m\tau)p_0^I + i\sum_{n\neq 0}\frac{1}{\omega_n}(\varphi_n^1(\sigma,\tau)\alpha_n^{1I} + \varphi_n^2(\sigma,\tau)\alpha_n^{2I}) \tag{4.5}$$

$$\theta^1(\sigma,\tau) = \cos(m\tau)\theta_0^1 + \sin(m\tau)\Pi\theta_0^2 + \sum_{n\neq 0} c_n(\varphi_n^1(\sigma,\tau)\theta_n^1 + i\frac{\omega_n - k_n}{m}\varphi_n^2(\sigma,\tau)\Pi\theta_n^2) \tag{4.6}$$

$$\theta^2(\sigma,\tau) = \cos(m\tau)\theta_0^2 - \sin(m\tau)\Pi\theta_0^1 + \sum_{n\neq 0} c_n(\varphi_n^2(\sigma,\tau)\theta_n^2 - i\frac{\omega_n - k_n}{m}\varphi_n^1(\sigma,\tau)\Pi\theta_n^1) \tag{4.7}$$

where:

$$\varphi_n^1(\sigma,\tau) = \exp[-i(\omega_n\tau - k_n\sigma)], \qquad \varphi_n^2(\sigma,\tau) = \exp[-i(\omega_n\tau + k_n\sigma)] \tag{4.8}$$

and

$$\omega_n = \sqrt{k_n^2 + m^2}, \quad n > 0; \qquad \omega_n = \sqrt{k_n^2 + m^2}, \quad n < 0 \tag{4.9}$$

$$k_n = 2\pi n, \qquad c_n = \frac{m}{\sqrt{m^2 + (\omega_n - k_n)^2}} = \frac{m}{\sqrt{2\omega_n(\omega_n - k_n)}}, \quad n = \pm 1, \pm 2, ... \tag{4.10}$$

4.2 Stress-Energy tensor and the x^-

Before we proceed with quantizing the action, let us derive the relationship between the x^- and the other x fields in the light cone gauge. This calculation will be of use in a number of different places, starting with the derivation of the canonical equations.

In both, bosonic and the GS theory on the flat space, going to the light-cone gauge not only simplifies the x^+ coordinate considerably but also defines the more complicated x^- embedding function, fully as a quadratic combination of the remaining x^i functions [1]. This relationship comes from the Virasoro constraint conditions, which demand the vanishing of the stress-energy tensor

$$T_{\alpha\beta} = -\frac{1}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}} \tag{4.11}$$

Given that the diagonal components of T wanish automatically as a consequence of the Weyl symmetry the remaining Virasoro constraints are:

$$T_{++} = T_{--} = 0 \tag{4.12}$$

The vanishing of the Stress Energy Tensor is a requirement for the equivalence between Nambu and the Poincare formulations of the string theory and therefore of great importance. In flat space the Virasoro constraints give the equation:

$$(\dot{x} \pm \acute{x})^2 = 0 \tag{4.13}$$

and therefore:

$$(\dot{x}^- \pm \acute{x}^-) = \frac{1}{2p^+}(\dot{x}^i \pm \acute{x}^i)^2 \tag{4.14}$$

The Virasoro conditions hold in more or less the same form in the pp-wave R-R background as well. The difference is that the relatively simple formula 4.13 is replaced by a more general one derived from 4.12.

In order to use 4.12 we need to derive the stress-energy tensor. It is easiest to do that from the form of the Lagrangian with the explicit g^{ab} dependance 3.53. Performing the functional derivative 4.11 we get the:

$$T_{++} = 2\partial_+ x^+ \partial_+ x^- - \mu^2 x_i^2 \partial_+ x^+ \partial_+ x^+ + \partial_+ x^i \partial_+ x^i \tag{4.15}$$

$$-i\partial_+ x^+ [\bar{\theta}\bar{\gamma}^- \partial_+ \theta + \theta\bar{\gamma}^- \partial_+ \bar{\theta} + 2i\mu\partial_+ x^+ \bar{\theta}\bar{\gamma}^- \Pi \partial_+ \theta] = 0$$

which can also be written as:

$$T_{++} = 2p^+ \partial_+ x^- - m^2 x_i^2 + (\partial_+ x^i)^2 + 2i\theta^A \bar{\gamma}^- \partial_+ \theta^A - 4im\theta^1 \bar{\gamma}^- \Pi \theta^2 \tag{4.16}$$

with A going over 1 and 2. Here one factor of p^+ has been absorbed into each pair of θs in order to remain consistent to Metsaev notation [9]. In the same fashion we get:

$$T_{--} = 2p^+ \partial_- x^- - m^2 x_i^2 + (\partial_- x^i)^2 + 2i\theta^A \bar{\gamma}^- \partial_- \theta^A - 4im\theta^1 \bar{\gamma}^- \Pi \theta^2 \tag{4.17}$$

We can therefore get the following constraints on the derivatives on x^-:

$$\partial_+ x^- = -\frac{1}{2p^+}[-m^2 x_i^2 + (\partial_+ x^i)^2 + 2i(\theta^A \bar{\gamma}^- \partial_+ \theta^A - 2m\theta^1 \bar{\gamma}^- \Pi\theta^2)] \quad (4.18)$$

$$\partial_+ x^+ = -\frac{1}{2p^+}[-m^2 x_i^2 + (\partial_- x^i)^2 + 2i(\theta^A \bar{\gamma}^- \partial_- \theta^A - 2m\theta^1 \bar{\gamma}^- \Pi\theta^2)] \quad (4.19)$$

or, in terms of the σ and τ derivatives:

$$\acute{x}^- = -\frac{1}{p^+}[\dot{x}^i \acute{x}^i + i(\theta^A \bar{\gamma}^- \acute{\theta}^A)] \quad (4.20)$$

$$\dot{x}^- = -\frac{1}{p^+}[\frac{1}{2}(\dot{x}^i \dot{x}^i + \acute{x}^i \acute{x}^i - m^2 x^i x^i) + i(\theta^A \bar{\gamma}^- \dot{\theta}^A - 2m\theta^1 \bar{\gamma}\Pi\theta^2)] \quad (4.21)$$

These will be used later to give the exact value of the x^- in terms of the oscillators but for now they will be used in calculating the phase space Lagrangian.

4.3 Hamiltonian and the canonical equations

Next step in quantizing the action is to derive the classical Poisson-Dirac brackets for the oscillators in the above equations and promote them to the equal-time (anti)commutators of quantum coordinates. We follow Metsaev [8] and derive the classical brackets by going to the phase-space formulation. We will use this formulation to derive the Hamiltonian and the Noether charges of the superalgebra as well.

Given that we will be making a Legendre transformation (transformation into the frame governed by the canonical momentum) with respect to the bosonic coordinates only, it simplifies our calculation to separate out the parts of the Lagrangian that do not depend on the τ derivative of the bosonic fields.

We go back to the equations 3.55 - 3.57 with the bosonic light-cone gauge still not fixed and rewrite Lagrangian in the following form:

$$\mathcal{L} = -h^{ab}\partial_a x^+ \partial_b x^- - \frac{1}{2}h^{ab}\partial_a x^I \partial_b x^I + \frac{1}{2}h^{ab}\partial_a x^+ \partial_b x^+ \mathrm{B} + \partial_a x^+ \mathrm{A}^a + \mathrm{C} \quad (4.22)$$

where:

$$h^{ab} = \sqrt{g} g^{ab}, \qquad h^{\tau\tau} h^{\sigma\sigma} - (h^{\tau\sigma})^2 = -1 \quad (4.23)$$

and

$$\mathrm{A}^a = -ih^{ab}(\bar{\theta}\bar{\gamma}^- \partial_b \theta + \theta\bar{\gamma}^- \partial_b \bar{\theta}) + i\epsilon^{a\sigma}(\theta\bar{\gamma}^- \acute{\theta} + \bar{\theta}\bar{\gamma}^- \acute{\bar{\theta}}) \quad (4.24)$$

$$\mathrm{B} = \mu^2 x_I^2 + 4\mu\bar{\theta}\bar{\gamma}^- \Pi\theta \quad (4.25)$$

$$\mathrm{C} = -i\acute{x}^+(\theta\bar{\gamma}^- \dot{\theta} + \bar{\theta}\bar{\gamma}^- \dot{\bar{\theta}}) \quad (4.26)$$

With dot and prime being the derivatives over τ and σ respectively. We can now easily get the canonical momenta for the bosonic coordinates:

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \quad (4.27)$$

gives:

$$\mathcal{P}^+ = -h^{\tau\tau}\dot{x}^+ - h^{\tau\sigma}\acute{x}^+ \tag{4.28}$$

$$\mathcal{P}^I = -h^{\tau\tau}\dot{x}^I - h^{\tau\sigma}\acute{x}^I \tag{4.29}$$

$$\mathcal{P}^- = -h^{\tau\tau}\dot{x}^- - h^{\tau\sigma}\acute{x}^- + A^\tau - B\mathcal{P}^+ \tag{4.30}$$

We can then write out the phase-space Lagrangian:

$$\mathcal{L} = \mathcal{P}^+\dot{x}^- + \mathcal{P}^-\dot{x}^+ + \frac{1}{2h^{\tau\tau}}(2\mathcal{P}^+\mathcal{P}^- + 2\acute{x}^+\acute{x}^- + (\mathcal{P}^+\mathcal{P}^+ - \acute{x}^+\acute{x}^+)\mathrm{B}) \tag{4.31}$$

$$+\frac{h^{\tau\sigma}}{h^{\tau\tau}}(\mathcal{P}^+\acute{x}^- + \mathcal{P}^-\acute{x}^+) - \frac{1}{h^{\tau\tau}}(\mathcal{P}^+ + h^{\tau\sigma}\acute{x}^+)\mathrm{A}^\tau + \acute{x}^+\mathrm{A}^\sigma + \mathrm{C}$$

$$+\mathcal{P}^I\dot{x}^I + \frac{1}{2h^{\tau\tau}}(\mathcal{P}_I^2 + \acute{x}_I^2) + \frac{h^{\tau\sigma}}{h^{\tau\tau}}\mathcal{P}^I\acute{x}^I$$

Now, we can impose the light cone gauge:

$$x^+ = \tau, \qquad \mathcal{P}^+ = p^+, \qquad h^{\tau\tau} = -p^+ \tag{4.32}$$

Inserting this into the equation above we get the light-cone gauge form of the phase-space Langrangian:

$$\mathcal{L} = -\frac{1}{2p^+}(p^+)^2\mathrm{B} - \frac{h^{\tau\sigma}}{p^+}(p^+\acute{x}^-) + A^\tau \tag{4.33}$$

$$+\mathcal{P}^I\bar{x}^I - \frac{1}{2p^+}(\mathcal{P}_I^2 + \acute{x}_I^2) - \frac{h^{\tau\sigma}}{p^+}P^I\acute{x}^I$$

and

$$\mathrm{C} = 0 \tag{4.34}$$

We now reinsert the A and B functions back into the Lagrangian and get:

$$\mathcal{L} = \mathcal{P}^I\dot{x}^I + ip^+(\bar{\theta}\bar{\gamma}^-\dot{\theta} + \theta\bar{\gamma}\dot{\bar{\theta}}) - \frac{1}{2p^+}(\mathcal{P}_I^2 + \acute{x}_I^2 + p^{+2}(x_I^2 + 4\bar{\theta}\bar{\gamma}^-\Pi\theta)) \tag{4.35}$$

$$+i(\theta\bar{\gamma}^-\acute{\theta} + \bar{\theta}\bar{\gamma}\acute{\bar{\theta}})$$

$$-\frac{h^{\tau\sigma}}{p^+}(p^+\acute{x}^- + \mathcal{P}^I\acute{x}^I + ip^+(\bar{\theta}\bar{\gamma}^-\acute{\theta} + \theta\bar{\gamma}\acute{\bar{\theta}})$$

Where the entire last line is equal to zero due to the constraint 4.20 leaving the:

$$\mathcal{L} = \mathcal{P}^I\dot{x}^I + ip^+(\bar{\theta}\bar{\gamma}^-\dot{\theta} + \theta\bar{\gamma}\dot{\bar{\theta}}) - \frac{1}{2p^+}(\mathcal{P}_I^2 + \acute{x}_I^2 + p^{+2}(x_I^2 + 4\bar{\theta}\bar{\gamma}^-\Pi\theta)) \tag{4.36}$$

$$+i(\theta\bar{\gamma}^-\acute{\theta} + \bar{\theta}\bar{\gamma}\acute{\bar{\theta}})$$

Which gives us the equation for the Hamiltonian density:

$$\mathcal{H} = -\frac{1}{2p^+}(\mathcal{P}_I^2 + \acute{x}_I^2 + p^{+2}(x_I^2 + 4\bar{\theta}\bar{\gamma}^-\Pi\theta)) + +i(\theta\bar{\gamma}^-\acute{\theta} + \bar{\theta}\bar{\gamma}\acute{\bar{\theta}}) \tag{4.37}$$

which together with 4.21 confirms that as in the flat case we have:

$$\mathcal{H} = \mathcal{P}^- \tag{4.38}$$

From the above Lagrangian we can also easily derive the the canonical momenta for the fermionic coordinates θ^α and $\bar{\theta}^\alpha$:

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\theta}^\alpha} = ip^+ \bar{\theta}^\beta \bar{\gamma}^-_{\beta\alpha} \tag{4.39}$$

and

$$\bar{p}_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\theta}}^\alpha} = ip^+ \theta^\beta \bar{\gamma}^-_{\beta\alpha} \tag{4.40}$$

We can now write down the classical Poisson-Dirac brackets:

$$[\mathcal{P}^I(\sigma), x^J(\sigma')] = \delta^{IJ}\delta(\sigma,\sigma') \tag{4.41}$$

$$\{p_\beta(\sigma), \theta^\alpha(\sigma')\} = \frac{1}{2}(\gamma^+\bar{\gamma}^-)^\alpha_\beta \delta(\sigma,\sigma') \tag{4.42}$$

$$\{\bar{p}_\beta(\sigma), \bar{\theta}^\alpha(\sigma')\} = \frac{1}{2}(\gamma^+\bar{\gamma}^-)^\alpha_\beta \delta(\sigma,\sigma') \tag{4.43}$$

where $\frac{1}{2}(\gamma^+\bar{\gamma}^-)$ has been inserted to maintain the light-cone gauge condition on θs and $\bar{\theta}$s. Using the equations for the fermionic momenta 4.39 and 4.40 and the properties of the γ matrices we can rewrite the last two as:

$$\{\bar{\theta}^\alpha(\sigma), \theta^\beta(\sigma')\} = \frac{i}{2p^+}(\gamma^+)^{\alpha\beta}\delta(\sigma,\sigma') \tag{4.44}$$

or in the alternative notation:

$$\{\theta^{A,\alpha}, \theta^{B,\beta}\} = \frac{i}{2p^+}(\gamma^+)^{\alpha\beta}\delta^{AB}\delta(\sigma,\sigma') \tag{4.45}$$

Using those, the Equations for the fields 4.5 - 4.10 and the expression for the canonical momentum that comes directly from 4.5:

$$\mathcal{P}^I(\sigma,\tau) = \cos(m\tau)p_0^I - m\sin(m\tau)x_0^I + \sum_{n\neq 0}(\phi_n^1(\sigma\tau)\alpha_n^{1I} + \phi_n^2(\sigma\tau)\alpha_n^{2I}) \tag{4.46}$$

we can obtain the commutation relations for the modes α_n and θ_n. Equation 4.41 can be written as:

$$[\mathcal{P}^I(\sigma), x^J(\sigma')] = \cos^2(m\tau)[p_0^I, x_0^J] - \sin^2(m\tau)[x_0^I, p_0^J] \tag{4.47}$$

$$+\sin(m\tau)\cos(m\tau)(\frac{1}{m}[p_0^I, p_0^J] - m[x_0^I, x_0^J])$$

$$+i\sum_{n,m}\frac{1}{\omega_m}(e^{-i[(\omega_n+\omega_m)\tau-(k_n\sigma+k_m\sigma')]}[\alpha_n^{1I}, \alpha_m^{1J}] + e^{-i[(\omega_n+\omega_m)\tau-(k_n\sigma-k_m\sigma')]}[\alpha_n^{1I}, \alpha_m^{2J}]$$

$$e^{-i[(\omega_n+\omega_m)\tau+(k_n\sigma-k_m\sigma')]}[\alpha_n^{2I},\alpha_m^{1J}]e^{-i[(\omega_n+\omega_m)\tau+(k_n\sigma+k_m\sigma')]}[\alpha_n^{2I},\alpha_m^{2J}]$$
$$=\delta^{IJ}\delta(\sigma,\sigma')=\delta^{IJ}\sum_{n=0}^{\infty}e^{2i\pi n(\sigma-\sigma')}$$

by simply matching the terms we obtain the following Poisson-Dirac brackets:

$$[p_0^I x_0^J]=\delta^{IJ}, \qquad [\alpha_m^{\mathcal{I}I},\alpha_n^{\mathcal{J}J}]=\frac{i}{2}\omega_m\delta_{m+n}\delta^{IJ}\delta^{\mathcal{I}\mathcal{J}} \tag{4.48}$$

and in the same fashion, Equation 4.45 leads to the bracket:

$$\{\theta_m^{\mathcal{I}\alpha},\theta_n^{\mathcal{J}\beta}\}=\frac{i}{4}(\gamma^+)^{\alpha\beta}\delta^{\mathcal{I}\mathcal{J}\delta_{m+n}} \tag{4.49}$$

4.4 Noether charges and the superalgebra

Another thing that we can do before proceeding with quantization is deriving the Noether charges that generate the symmetries of the action. Given that the symmetries are not manifest in the light-cone gauge, expressing the Noether charges explicitely is the best way to demonstrate them. In addition, those charges are important in formulating the superstring field theory in the light-cone gauge [18].

The difficulties introduced by the light-cone gauge manifest themselves in the symmetries involving the x^- embedding function and their fermionic equivalents. We can therefore divide the generators into the two groups, one whose generators not change significantly by the introduction of the light-cone gauge, and are thus quadratic in the string fields, and the other whose generators receive higher order interaction dependent corrections. Those two groups are:

$$P^+,\ P^I,\ J^{+I},\ J^{ij},\ J^{i'j'},\ Q^+,\ \bar{Q}^+ \tag{4.50}$$

the simple ones, which Metsaev [8] calls kinematical generators and:

$$P^-,\ Q^-,\ \bar{Q}^- \tag{4.51}$$

referred to as the dynamical generators. In the above the Q generator has been split into kinematical and dynamical part according to:

$$Q^+\equiv\frac{1}{2}\bar{\gamma}^-\gamma^+Q, \qquad Q^-\equiv\frac{1}{2}\bar{\gamma}^+\gamma^-Q \tag{4.52}$$

We follow the standard way of obtaining the Noether charges stating with the conserved currents [19]. The method consists of obtaining the currents based on the localization of the parameters of the associated global transformation and then obtaining the charges as the integrals of those currents.

Noether equation for the currents is:

$$j^\mu=\sum_\phi\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi \tag{4.53}$$

Where ϕ are the fields related by the symmetry. We will use this formula to find those currents that are related by the symmetries that do not involve compensating κ-symmetries to preserve the light-cone gauge. We will find the remaining currents by using the form of the action with both light-cone gauge and the κ-symmetries fixed. We start with the translations and rotations derived originally from the Killing vectors and given in the equations 2.56 - 2.58. Once again, those symmetries are:

$$\delta x^{\pm} = a^{\pm} \tag{4.54}$$

associated with the translations $P^{\pm}$,

$$\delta x^{-} = c_I^{-}\mu \sin(\mu x^{+})x_I, \quad \delta x^I = c_I^{-} cos(\mu x^{+}) \tag{4.55}$$

associated with "translations" P^I,

$$\delta x^{-} = d_I^{-}\cos(\mu x^{+})x_I, \quad \delta x^I = d_I^{-}\frac{1}{\mu} sin(\mu x^{+}) \tag{4.56}$$

associated with the rotations J^{+I}

$$\delta x^i = e^{ij}x^j, \quad \delta\theta_\alpha = \frac{1}{4}e^{ij}(\gamma^{ij})^{\alpha\beta}\theta_\beta \tag{4.57}$$

with e^{ij} antisymmetric, associated with $SO(4)$ rotations J^{ij}. And exactly same ones for the $SO(4)$ rotations $J^{i'j'}$.

Using the equation 4.53 and the above we then get:

$$\mathcal{P}^{+a} = -\sqrt{g}g^{ab}\partial_b x^{+} \tag{4.58}$$

$$\mathcal{P}^{Ia} = -\sqrt{g}g^{ab}[\cos(\mu x^{+})\partial_b x^{+} + \mu \sin(\mu x^{+})x^I\partial_b x^{+} \tag{4.59}$$

$$\mathcal{P}^{-a} = -\sqrt{g}g^{ab}[\partial_b x^{-} + i\bar{\theta}\bar{\gamma}^{-}\partial_b\theta + i\theta\bar{\gamma}^{-}\partial_b\bar{\theta} - x_I^2\partial_b x^{+} - 4\mu\bar{\theta}\bar{\gamma}^{-}\Pi\theta\partial_b x^{+}] \tag{4.60}$$
$$+\epsilon^{ab}[i\theta\bar{\gamma}^{-}\partial_b\theta + h.c.]$$

$$\mathcal{J}^{+Ia} = -\sqrt{g}g^{ab}[\frac{1}{\mu}\sin\mu x^{+}\partial_b x^I - \cos(\mu x^{+})x^I\partial_b x^{+}] \tag{4.61}$$

$$\mathcal{J}^{ija} = -\sqrt{g}g^{ab}[x^i\partial_b x^j - x^j\partial_b x^i - i\bar{\theta}\bar{\gamma}^{-}\gamma^{ij}\theta\partial_b x^{+}] - \epsilon^{ab}[\frac{i}{2}\theta\bar{\gamma}^{-}\gamma^{ij}\theta + h.c.] \tag{4.62}$$

$$\mathcal{J}^{i'j'a} = -\sqrt{g}g^{ab}[x^{i'}\partial_b x^{j'} - x^{j'}\partial_b x^{i'} - i\bar{\theta}\bar{\gamma}^{-}\gamma^{i'j'}\theta\partial_b x^{+}] \tag{4.63}$$
$$-\epsilon^{ab}[\frac{i}{2}\theta\bar{\gamma}^{-}\gamma^{i'j'}\theta + h.c.]$$

The invariances with respect to the super-transformations that do not require compensating κ transformations are:

$$\delta\theta = e^{-ix^{+}\Pi}\epsilon, \quad \delta\bar{\theta} = e^{ix^{+}\Pi}\bar{\epsilon}, \tag{4.64}$$

$$\delta x^{-} = -i\epsilon e^{-ix^{+}\Pi}\bar{\theta} - i\bar{\epsilon}e^{-ix^{+}\Pi}\theta \tag{4.65}$$

which lead to conserved super-currents:

$$\mathcal{Q}^{+a} = -2\bar{\gamma}^{-} e^{ix^{+}\Pi}(\sqrt{g}g^{ab}\partial_b x^{+}\theta + \epsilon^{ab}\partial_b x^{+}\bar{\theta}) \tag{4.66}$$

$$\bar{\mathcal{Q}}^{+a} = -2\bar{\gamma}^{-} e^{-ix^{+}\Pi}(\sqrt{g}g^{ab}\partial_b x^{+}\bar{\theta} + \epsilon^{ab}\partial_b x^{+}\theta) \tag{4.67}$$

We can now use those to calculate the actual charges using the equation:

$$\mathrm{G} = \int d^{\sigma}\mathcal{G}^{\tau} \tag{4.68}$$

Writing those and including the light-cone gauge conditions we get:

$$P^{+} = p^{+} \tag{4.69}$$

$$P^{I} = \int \cos(m\tau)\mathcal{P}^{I} + m\sin(m\tau)x^{I}p^{+} \tag{4.70}$$

$$J^{+I} = \int \frac{1}{\mu}\sin(m\tau)\mathcal{P}^{I} - \cos(m\tau)x^{I}p^{+} \tag{4.71}$$

$$Q^{+} = \int 2p^{+}\bar{\gamma}^{-}e^{im\Pi}\theta \tag{4.72}$$

$$\bar{Q}^{+} = \int 2p^{+}\bar{\gamma}^{-}e^{-im\Pi}\bar{\theta} \tag{4.73}$$

$$J^{ij} = \int x^{i}\mathcal{P}^{j} - x^{j}\mathcal{P}^{i} - i\bar{\theta}\bar{\gamma}^{-}\gamma^{ij}\theta \tag{4.74}$$

$$J^{i'j'} = \int x^{i'}\mathcal{P}^{j'} - x^{j'}\mathcal{P}^{i'} - i\bar{\theta}\bar{\gamma}^{-}\gamma^{i'j'}\theta \tag{4.75}$$

Out of the three remaining,"dynamical" charges, $\mathcal{P}^{-}$ has already been derived as a Hamiltonian in the equation 4.37 and is given by:

$$P^{-} = \int -\frac{1}{2p^{+}}(\mathcal{P}_I^2 + \acute{x}_I^2 + p^{+2}(x_I^2 + 4\mu\bar{\theta}\bar{\gamma}^{-}\Pi\theta)) + +i(\theta\bar{\gamma}^{-}\acute{\theta} + \bar{\theta}\bar{\gamma}\acute{\bar{\theta}}) \tag{4.76}$$

The remaining two supercharges Q^{-} and $\bar{Q}^{-}$ are somewhat more complicated to derive from the symmetries because we would need to use the κ-symmetry as well. Instead we can derive them from the commutation relations of the superalgebra in the following way:

First we re-express the commutators from the equations 2.70 - 2.73 in terms of Q^{+} and Q^{-} as given in 4.52:

$$[J^{ij}, Q_{\alpha}^{\pm}] = \frac{1}{2}Q_{\beta}^{\pm}(\gamma^{ij})_{\alpha}^{\beta} \tag{4.77}$$

$$[J^{i'j'}, Q_{\alpha}^{\pm}] = \frac{1}{2}Q_{\beta}^{\pm}(\gamma^{i'j'})_{\alpha}^{\beta} \tag{4.78}$$

$$[J^{+I}, Q_{\alpha}^{-}] = \frac{1}{2}Q_{\beta}^{+}(\gamma^{+I})_{\alpha}^{\beta} \tag{4.79}$$

$$[J^{+I}, Q_{\alpha}^{-}] = \frac{1}{2} Q_{\beta}^{+} (\gamma^{+I})_{\alpha}^{\beta} \tag{4.80}$$

$$[P^{I}, Q_{\alpha}^{-}] = \frac{i}{2} Q_{\beta}^{+} (\Pi \gamma^{+I})_{\alpha}^{\beta} \tag{4.81}$$

$$[P^{-}, Q_{\alpha}^{+}] = i Q_{\beta}^{+} \Pi_{\alpha}^{\beta} \tag{4.82}$$

$$\{Q_{\alpha}^{+}, \bar{Q}_{\beta}^{+}\} = -2i\gamma_{\alpha\beta} P^{+} \tag{4.83}$$

$$\{Q_{\alpha}^{+}, \bar{Q}_{\beta}^{-}\} = -i(\bar{\gamma}^{-}\gamma^{+}\bar{\gamma}^{I})_{\alpha\beta} P^{I} - (\bar{\gamma}^{-}\gamma^{+}\bar{\gamma}^{i})_{\alpha\beta} J^{+i} - (\bar{\gamma}^{-}\gamma^{+}\bar{\gamma}^{i'})_{\alpha\beta} J^{+i'} \tag{4.84}$$

$$\{Q_{\alpha}^{-}, \bar{Q}_{\beta}^{-}\} = -2i\gamma_{\alpha\beta}^{+} P^{-} + (\bar{\gamma}^{+}\gamma^{ij}\Pi)_{\alpha\beta} J^{ij} + (\bar{\gamma}^{+}\gamma^{i'j'}\Pi)_{\alpha\beta} J^{i'j'} \tag{4.85}$$

Because we know the form of all the other charges, we can now derive the $Q^{-} = \int \mathcal{Q}^{-\tau}$. We start with the general anzats:

$$\mathcal{Q}^{-\tau} = \mathcal{P}^{I} A_1^{I} \theta + x^{I} A_2^{I} \theta + \acute{x}^{I} B^{I} \bar{\theta} \tag{4.86}$$

and can, using the equations 4.71, 4.72, and the above commutators write:

$$[\sin(m\tau)\mathcal{P}^{J} - m\cos(m\tau)x^{J}, \mathcal{P}^{I} A_1^{I}\theta + x^{I} A_2^{I}\theta + \acute{x} B^{I}\bar{\theta}] = 2m\bar{\gamma}^{I} e^{im\Pi}\theta \tag{4.87}$$

$$\sin(m\tau) A_2^{I} [\mathcal{P}^{J}, x^{I}] - m\cos(m\tau) A_1^{I} [x^{J}, \mathcal{P}^{I}] = 2m\bar{\gamma}^{I} e^{im\Pi}$$

we can then use the Poisson-Dirac brackets 4.41 - 4.45to write:

$$\sin(m\tau) A_2^{I} + m\cos(m\tau) A_1^{I} = 2m\bar{\gamma}^{I} e^{im\Pi} \tag{4.88}$$

In the exact same fashion, equation 4.70 leads to:

$$-m\sin(m\tau) A_1^{I} + \cos(m\tau) A_2^{I} = 2im\bar{\gamma}^{I} \Pi e^{im\Pi} \tag{4.89}$$

The two above equations are solved by:

$$A_1^{I} = 2\bar{\gamma}^{I}, \qquad A_2^{I} = 2im\bar{\gamma}^{I}\Pi \tag{4.90}$$

And we thus have:

$$\mathcal{Q}^{-\tau} = 2\mathcal{P}^{I}\bar{\gamma}^{I}\theta + 2imx^{I}\bar{\gamma}^{I}\Pi\theta + \acute{x}^{I} B^{I}\bar{\theta} \tag{4.91}$$

and a very similar expression for $\mathcal{Q}^{-\sigma}$.

We use the conservation law for the super-current $\mathcal{Q}^{-a}$:

$$\partial_{\tau}\mathcal{Q}^{-\tau} + \partial_{\sigma}\mathcal{Q}^{-\sigma} = 0 \tag{4.92}$$

to determine the $B^{I} = -2\bar{\gamma}^{I}$ and thus we can write:

$$Q^{-} = \int 2\mathcal{P}^{I}\bar{\gamma}^{I}\theta - 2\acute{x}^{I}\bar{\gamma}^{I}\bar{\theta} + 2imx^{I}\bar{\gamma}^{I}\Pi\theta \tag{4.93}$$

$$\bar{Q}^{-} = \int 2\mathcal{P}^{I}\bar{\gamma}^{I}\bar{\theta} - 2\acute{x}^{I}\bar{\gamma}^{I}\theta + 2imx^{I}\bar{\gamma}^{I}\Pi\bar{\theta} \tag{4.94}$$

4.5 Quantization

What remains is to actually perform the quantization. In this we follow Metsaev and Tseytlin [9]. The quantization is done in the standard way by promoting the coordinates and the momenta of the Fourier components in the equations: 4.5 - 4.10 to operators and replacing the poison (anti)brackets given in 4.48-4.49 by the appropriate equal-time (anti)commutators using the rules:

$$\{.,.\}_{classical} \to i\{.,.\}_{quantum}, \qquad [.,.]_{classical} \to i[.,.]_{quantum} \tag{4.95}$$

With that, we can write:

$$[p_0^I, x_0^J] = -i\delta^{IJ} \tag{4.96}$$

$$[\alpha_m^{\mathcal{I}I}, \alpha_n^{\mathcal{J}J}] = \frac{1}{2}\omega_m\delta_{m+n}\delta^{IJ}\delta^{\mathcal{I}\mathcal{J}} \tag{4.97}$$

$$\{\theta_m^{\mathcal{I}\alpha}, \theta_n^{\mathcal{J}\beta}\} = \frac{1}{4}(\gamma^+)^{\alpha\beta}\delta^{\mathcal{I}\mathcal{J}}\delta_{m+n} \tag{4.98}$$

Hamiltonian is already given by in the equation 4.37:

$$H = \frac{1}{p^+}\int d\sigma[\frac{1}{2}(\mathcal{P}_I^2 + \acute{x}_I^2 + m^2x_I^2) + 2im\theta^1\bar{\gamma}^-\Pi\theta^2 - i(\theta^1\bar{\gamma}^-\acute{\theta}^1 - \theta^2\bar{\gamma}^-\acute{\theta}^2)] \tag{4.99}$$

which can be rewritten with the use of the fermionic equations of motion 4.3 as:

$$H = \frac{1}{p^+}\int d\sigma[\frac{1}{2}(\mathcal{P}_I^2 + \acute{x}_I^2 + m^2x_I^2) + i(\theta^1\bar{\gamma}^-\dot{\theta}^1 + \theta^2\bar{\gamma}^-\dot{\theta}^2)] \tag{4.100}$$

before we express this in terms of the quantum coordinates it makes sense to reintroduce those in terms of the creation and annihilation operators:

$$a_0^I = \frac{1}{\sqrt{2m}}(p_0^I + imx_0^I), \qquad \bar{a}_0^I = \frac{1}{\sqrt{2m}}(p_0^I - imx_0^I) \tag{4.101}$$

$$\alpha_{-n}^{\mathcal{I}I} = \sqrt{\frac{\omega_n}{2}}a_n^{\mathcal{I}I}, \qquad \alpha_n^{\mathcal{I}I} = \sqrt{\frac{\omega_n}{2}}\bar{a}_n^{\mathcal{I}I} \tag{4.102}$$

$$\theta_0 = \frac{1}{\sqrt{2}}(\theta_0^1 + i\theta_0^2), \qquad \bar{\theta}_0 = \frac{1}{\sqrt{2}}(\theta_0^1 - i\theta_0^2) \tag{4.103}$$

$$\theta_{-n}^{\mathcal{I}} = \frac{1}{\sqrt{2}}\eta_n^{\mathcal{I}}, \qquad \theta_n^{\mathcal{I}} = \frac{1}{\sqrt{2}}\bar{\eta}_n^{\mathcal{I}} \tag{4.104}$$

In terms of those we can write the commutation relations:

$$[\bar{a}_0^I, a_0^J] = \delta^{IJ}, \qquad [\bar{a}_m^{\mathcal{I}I}, a_n^{\mathcal{J}J}] = \delta_{mn}\delta^{\mathcal{I}\mathcal{J}}\delta^{IJ} \tag{4.105}$$

$$\{\bar{\theta}_0^\alpha, \theta_0^\beta\} = \frac{1}{4}(\gamma^+)^{\alpha\beta}, \qquad \{\bar{\eta}_m^{\mathcal{I}\alpha}, \eta_n^{\mathcal{J}\beta}\} = \frac{1}{2}(\gamma^+)^{\alpha\beta}\delta_{mn}\delta^{\mathcal{I}\mathcal{J}} \tag{4.106}$$

Where $\alpha = 1, ..., 16$ and the light-cone gauge condition on fermions reads:

$$\bar{\gamma}^+\theta_0^{\mathcal{I}} = 0, \qquad \bar{\gamma}^+\eta_n^{\mathcal{I}} = 0 \tag{4.107}$$

We can now write the light-cone energy operator while taking care of the normal-ordering:

$$E = E_0 + E^1 + E^2 \tag{4.108}$$

where:

$$E_0 = \mu \mathcal{E}_0, \qquad \mathcal{E}_0 = a_0^I \bar{a}_0^I + 2\bar{\theta}_0 \bar{\gamma}^- \Pi \theta_0 + 4 \tag{4.109}$$

and:

$$E^{\mathcal{I}} = \frac{1}{p^+} \sum_{n=1}^{\infty} \omega_n (a_n^{\mathcal{I}I} \bar{a}_n^{\mathcal{I}I} + \eta_n^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_n^I) \tag{4.110}$$

In the $E^{\mathcal{I}}$ there is no normal-ordering constant because there is an equal number of the bosonic and fermionic operators and the constants due to the two cancel out as they do in the flat space case. In the E_0 the factor that is obtained from the commutator of θs: $Tr(\gamma^+ \bar{\gamma}^- \Pi)$ is equal to zero and therefore the fermionic zero modes do not contribute to the normal ordering constant. What is left is the bosonic term: $(\frac{1}{2} \times 8 = 4$ which is added.

As in the flat-space we can now define the vacuum as the direct product of the zero-mode vacuum and the Fock vacuum for the string oscillations, or in other words:

$$\bar{a}_0^I |0\rangle = 0, \qquad \bar{\theta}_0^\alpha |0\rangle = 0, \qquad \bar{a}_n^{\mathcal{I}I} |0\rangle = 0, \qquad \bar{\eta}_n^{\mathcal{I}\alpha} |0\rangle = 0 \tag{4.111}$$

With the generic states being then obtained by acting with the creation operators $a_0^I, a_n^{\mathcal{I}I}, \theta_0^\alpha, \eta_n^{\mathcal{I}\alpha}$ on the vacuum.

$$|\Phi\rangle = \Phi(a_0, a_n, \theta_o, \eta_n)|0\rangle \tag{4.112}$$

As in the flat space case for the GS formalism the physical state condition is automatically satisfied. Only remaining condition is the level matching condition that is present for all closed string theories. In this case it can be written as:

$$N^1 |\Phi_{phys}\rangle = N^2 |\Phi_{phys}\rangle, \qquad N^{\mathcal{I}} = \sum_{n=1}^{\infty} k_n (a_n^{\mathcal{I}I} \bar{a}_n^{\mathcal{I}I} + \eta_n^{\mathcal{I}} \bar{\gamma}^- \bar{\eta}_n^I) \tag{4.113}$$

It is possible to derive these constraints from the stress-energy tensor but here it suffices to see that in the flat-space limit $m \to 0$ it reduces to the standard level-matching condition

$$E^1 |\Phi_{phys}\rangle = E^2 |\Phi_{phys}\rangle \tag{4.114}$$

with E given by 4.110.

It is now possible to perform the integrals in the equations 4.70 - 4.75 and 4.93 - 4.94 and obtain explicitly the generators of symmetries. The example of how this is done can be 4.70:

$$P^I = \int d\sigma \cos(m\tau) \mathcal{P}^I + m \sin(m\tau) x^I p^+ \tag{4.115}$$

$$= \int d\sigma[\cos^2(m\tau)p_0^I + \sin^2(m\tau)p_0^I + m\sin(m\tau)\cos(m\tau)x_0^I - m\sin(m\tau)\cos(m\tau)x_0^I$$

$$+ \sum_{n\neq 0}[\cos(m\tau)(e^{-i(\omega_n\tau - k_n\sigma)}\alpha_n^{1I} + e^{-i(\omega_n\tau + k_n\sigma)}\alpha_n^{2I})$$

$$+ \frac{im}{\omega_n}\sin(m\tau)(e^{-i(\omega_n\tau - k_n\sigma)}\alpha_n^{1I} + e^{-i(\omega_n\tau + k_n\sigma)}\alpha_n^{2I})]]$$

We can see immediately that all the terms in the sum will disappear as we are integrating over the entire length of string, which is to say the full period of σ. What is left simplifies to:

$$P^I = p_0^I \tag{4.116}$$

in the exactly same fashion we have:

$$J^{+I} = -ix_0^I p^+ \tag{4.117}$$

$$Q^+ = 2\sqrt{p^+}\bar{\gamma}^-\theta_0 \tag{4.118}$$

$$\bar{Q}^+ = 2\sqrt{p^+}\bar{\gamma}^-\bar{\theta}_0 \tag{4.119}$$

In the remaining ones we have terms with the products of the exponentials. In some of those the σ dependent term will disappear and therefore there will be terms dependant on the non-zero modes. Performing those integrals we get:

$$J^{IJ} = J_0^{IJ} + \sum_{\mathcal{I}=1,2}\sum_{n=1}^{\infty}(a_n^{\mathcal{I}I}\bar{a}_n^{\mathcal{I}J} - a_n^{\mathcal{I}J}\bar{a}_n^{\mathcal{I}I} + \frac{1}{2}\eta_n^{\mathcal{I}}\bar{\gamma}^-\gamma^{IJ}\bar{\eta}_n^{\mathcal{I}}) \tag{4.120}$$

where:

$$J_0^{IJ} = a_0^I\bar{a}_0^J - a_0^J\bar{a}_0^I + \frac{1}{2}\sum_{\mathcal{I}=1,2}\theta_0^{\mathcal{I}}\bar{\gamma}^-\gamma^{IJ}\theta_0^{\mathcal{I}} \tag{4.121}$$

and finally we have:

$$Q^{-1} = \frac{1}{\sqrt{p^+}}[2p_0^I\bar{\gamma}^I\theta_0^1 - 2mx_0^I\bar{\gamma}^I\Pi\theta_0^2 \tag{4.122}$$

$$+ \sum_{n=1}^{\infty}(2\sqrt{\omega_n}c_n a_n^{1I}\bar{\gamma}^I\bar{\eta}_n^1 + \frac{im}{\sqrt{\omega_n}c_n}a_n^{2I}\bar{\gamma}^I\Pi\bar{\eta}_n^2 + h.c.)]$$

$$Q^{-2} = \frac{1}{\sqrt{p^+}}[2p_0^I\bar{\gamma}^I\theta_0^2 + 2mx_0^I\bar{\gamma}^I\Pi\theta_0^1 \tag{4.123}$$

$$+ \sum_{n=1}^{\infty}(2\sqrt{\omega_n}c_n a_n^{2I}\bar{\gamma}^I\bar{\eta}_n^2 - \frac{im}{\sqrt{\omega_n}c_n}a_n^{1I}\bar{\gamma}^I\Pi\bar{\eta}_n^1 + h.c.)]$$

It is a trivial if somewhat tedious exercise to demonstrate that the commutation relations of the super-algebra can be derived from the above and the commutations of the oscillators in the equations 4.105-4.106.

Chapter 5

Supergravity Spectrum

In the last part of their paper, Metsaev and Tseytlin [9] derive the type IIB supergravity spectrum of the pp-wave background. The supergravity fields of this spectrum are in one to one correspondence with the lowest energy string states of the above model. Namely those string states obtained by acting with the bosonic and fermionic zero-mode creation operators on the vacuum defined in 4.111.

Represented as the products of Fermionic operators on the vacuum the lowest-lying states can be expressed as:

$|0\rangle$ - complex scalar

$\theta_0|0\rangle$ - spin 1/2 field

$\theta_0\theta_0|0\rangle$ - complex 2-form field

$\theta_0\theta_0\theta_0|0\rangle$ - spin 3/2 field

$\theta_0\theta_0\theta_0\theta_0|0\rangle$ - graviton and self dual 4-form field

... - complex conjugates of the above

We can then act with the bosonic zero mode operators to create the entire type IIB supergravity spectrum.

Here, we follow the Metsaev and Tseytlin and use the field theoretical approach based on the work of Schwarz [20] to explicitly derive this spectrum. We will take the results of the paper [20] as our starting point. To give structure to what will follow we present the decomposition of the 128+128 physical transverse supergravity degrees of freedom in the light-cone gauge using the $SO(8) \to SO(4) \times SO(4)$ decomposition:

First we have the **graviton** field. The breakdown of the $SO(8)$ into $SO(4) \times SO(4)$ means that the graviton is not wholly symmetric. Rather its h_{ij} and $h_{i'j'}$ parts will be symmetric (and traceless) but the $h_{ij'}$ will have no such constraints. The graviton therefore decomposes like:

$$h_{IJ}(35) = h_{ij}(9) \oplus h_{i'j'}(9) \oplus h_{ij'}(36) \oplus h(1) \tag{5.1}$$

with number in brackets giving the total degrees of freedom of that field.

Similarly we have the **4 - form field** which is one of the two fields that together form the equivalent, for this theory, of the antisymmetric B tensor. Again, antisymmetry is relaxed for the components mixing the i s with j' s so se have the decomposition:

$$a_{IJKL}(35) = a_{ij'ij'}(16) \oplus a_{iji'j'}(18) \oplus a(1) \tag{5.2}$$

where the number of the degrees of freedom of the second component is derived from the antisymmetry requirement: $a_i j i' j' = -\frac{1}{4}\epsilon_{ijkl}\epsilon i' j' k' l' a_{klk'l'}$

Second heir of the B tensor is the **complex 2-form field** which with the similar properties with regard to antisymmetry gives:

$$b_{IJ}(56) = b_{ij}(12) \oplus b_{i'j'}(12) \oplus b_{ij'}(32) \tag{5.3}$$

Finally last among the bosonic fields are Dilaton and the R-R scalar which together form what is referred to as the **complex scalar field**

$$\phi(2) \tag{5.4}$$

The fermionic fields are positive and negative chirality complex spinors.

Negative chirality spinor is λ whose light cone projection $\lambda^{\oplus} = \frac{1}{2}\bar{\gamma}^{-}\gamma^{+}\lambda$ is a **spin 1/2 field**

$$\lambda^{\oplus}(16) \tag{5.5}$$

Positive chirality spinor is a gravitino or the **spin 3/2 field** which splits into the γ-transverse and γ-parallel parts:

$$\psi(112) = \psi_i^{\oplus,\perp}(48) \oplus \psi_{i'}^{\oplus,\perp}(48) \oplus \psi^{\oplus||}(16) \tag{5.6}$$

5.1 Energy of the massless fields

One of the interesting features of the of this model is that, as in the case of the AdS supermultiplets and contrary to the flat space, the spectrum of the lowest eigenvalues of the light-cone energy operator is non-degenerate. The following method will give us the exact energy values for various fields of the massless spectrum.

We begin by deriving the Christoffel symbols, Riemann and Ricci tensors for our metric 1.1. From 1.1 we can read out:

$$g_{++} = \mu^2 x_I^2, \quad g_{+-} = 2, \quad g_{ij} = -\delta_{ij}, \quad g_{--} = 0 \tag{5.7}$$

$$g^{++} = 0, \quad g^{+-} = \frac{1}{2}, \quad g^{ij} = \delta^{ij}, \quad g^{--} = \mu^2 x_I^2 \tag{5.8}$$

The Christoffel symbols of the first kind are then given by:

$$[mn, r] = \frac{1}{2}(\partial_n g_{rm} + \partial_m g_{rn} - \partial r g_m n) \tag{5.9}$$

Clearly, the only non zero ones in our case will be:

$$[+i, +] = [i+, +] = -[++, i] = \mu^2 x_I \tag{5.10}$$

The Christofell symbols of the second kind are defined as:

$$\Gamma^r_{mn} = g^{rs}[mn, s] \tag{5.11}$$

Which, for our case gives:

$$\Gamma^{-}_{+i} = \Gamma^{-}_{i+} = -\Gamma^{i}_{++} = -\mu^2 x_I \tag{5.12}$$

The Riemman tensor is given by:

$$R_{rsmn} = \partial_m[sn, r] - \partial_n[sm, r] + \Gamma^p_{sm}[rn, p] - \Gamma^p_{sn}[rm, p] \tag{5.13}$$

which in our case gives:

$$R_{I++I} = -\mu^2 \tag{5.14}$$

with only other non zero ones being determined by the antisymmetric property of R. Finally, the Ricci tensor $R_{sm} = g^{rm} R_{rsmn}$ is here given by:

$$R_{+}+ = 8\mu^2 \tag{5.15}$$

We can then find the Laplacian of the geometry which will give us the massless scalar equation [1].

$$\nabla^2 \equiv g^{\mu\nu}\nabla_\mu \nabla_{nu} \tag{5.16}$$

$$= -\nabla_I\nabla_I + \mu^2 x_I x_I \nabla_-\nabla_- + \frac{1}{2}(\nabla_+\nabla_- + \nabla_-\nabla_+)$$

$$= -\nabla_I(\partial_I) + \mu^2 x_I x_I \nabla_-(\partial_-) + \frac{1}{2}(\nabla_+\partial_- + \nabla_-\partial_+)$$

$$= [-\partial_I\partial_I + \Gamma^m_{+-}\partial_m] + \mu^2 x_I x_I[\partial_-\partial_- - \Gamma^m_{--}\partial_m] + [\partial_+\partial_- - \Gamma^m_{+-}\partial_m]$$

$$= 2\partial^+\partial^- + \mu^2 x_I^2 \partial^{+2} + \partial_I^2$$

Where we have used the $\partial^+ = \partial_-$, $\partial^- = \partial_+$,$\partial^I = \partial_I$ notation in the last line.

We than write: the massless scalar equation:

$$\nabla^2\varphi = [2\partial^+\partial^- + \mu^2 x_I^2 \partial^{+2} + \partial_I^2]\varphi = 0 \tag{5.17}$$

We now perform the Fourier transform in the x^- and x^I corresponding to the light-cone description where the x^+ is an evolution parameter:

$$\varphi(x^+, x^-, x^I) = \int \frac{dp^+ d^8p}{(2\pi)^{\frac{9}{2}}} e^{i(p^+x^- + p^I x^I)} \tilde{\varphi}(x^+, p^+, p^I) \tag{5.18}$$

The equation now becomes:

$$(2p^+P^- - \mu^2 p^{+2}\partial^2_{p^I} + p_I^2)\tilde{\varphi} = 0 \tag{5.19}$$

As before, we interpret the quadratic P^- operator as a Hamiltonian for the free harmonic oscillator in 8 dimensions with mass p^+ and frequency μ:

$$H = -P^- = \frac{1}{2p^+}(p_I^2 - m^2\partial^2_{p^I}) \tag{5.20}$$

[1] Here and henceforth I use the symbol ∇ for the covariant derivative in breach with the Metsaev's notation.

If we introduce the standard creation and annihilation operators:

$$a^I \equiv \frac{1}{\sqrt{2m}}(p^I - m\partial_{p^I}), \qquad \bar{a}^I \equiv \frac{1}{\sqrt{2m}}(p^I + m\partial_{p^I}), \qquad [\bar{a}^I, a^J] = \delta^{IJ} \tag{5.21}$$

we can write the Hamiltonian, taking care of normal ordering:

$$H = \frac{1}{2}\mu(\bar{a}^I a^I + a^I \bar{a}^I) = \mu(a^I \bar{a}^I + 4) \tag{5.22}$$

where 4 comes from: $4 = \frac{D-2}{2}, D = 10$. Solution of the 5.17 is then found as usual by acting by a^I on a vacuum satisfying the $\bar{a}^I|0\rangle = 0$.

Not all the equations we will deal with are of the same form as 5.17. We will therefore need the following generalization:

$$(\nabla^2 + 2i\mu c\partial^+)\varphi(x) = 0 \tag{5.23}$$

where ∇^2 is same as defined above and the c is an arbitrary constant. The Hamiltonian then would be:

$$H = -P^- = \frac{p_I^2 - m^2\partial_{p^I}^2}{2p^+} + \mu c = \mu(a^I \bar{a}^I + 4 + c) \tag{5.24}$$

so the lowest energy value will be given by:

$$E_0 = \mu\mathcal{E}_0 = \mu(4 + c) \tag{5.25}$$

As we discuss the various fields of the supergravity we will be able to reduce their equations of motion to the form 5.23 and will therefore be able to obtain their lowest energy values.

5.2 Bosonic Fields

Complex scalar field

The dilaton and the R-R scalar do not interact with the 5-form background, so they satisfy the 5.17

$$\nabla^2\phi = 0 \tag{5.26}$$

and therefore:

$$\mathcal{E}_0(\phi) = 4 \tag{5.27}$$

Complex 2-form field

We begin with the result from [20] which gives the nonlinear equations determining the complex two form field B_{mn}:

$$\nabla^m G_{mm_1m_2} = P^m G^*_{mm_1m_2} - \frac{i}{3} F_{m_1\ldots m_5} G^{m_3m_4m_5} \tag{5.28}$$

where $G_{abc} = 3\partial_{[a}B_{bc]}$ We are interested in the equation for the small fluctuations $B_{mn} = b_{mn}$ in the pp-wave background, with $P_m = 0$, using the light-cone gauge:

$$b_{-m} = 0 \tag{5.29}$$

Based on 5.29 we can write:

$$G_{-IJ} = \partial_- b_{IJ}, \qquad G_{I--} = 0 \tag{5.30}$$

It is sufficient to analyze the equation 5.28 for the following values of the indices: (m_1, m_2): $(-, I)$ and (I, J). From the definition of the covariant derivative we have:

$$\nabla^\mu G_{\mu IJ} = g^{\nu\mu}[\partial_\nu G_{\mu IJ} - \Gamma^a_{\mu\nu} G_{aIJ} - \Gamma^a_{I\nu} G_{\mu aJ} - \Gamma^a_{J\nu} G_{\mu Ia}] \tag{5.31}$$

$$= g^{\nu\mu}(\partial_\nu G_{\mu IJ}) - (g^{++}\Gamma^a_{++} + 2g^{+i}\Gamma^a_{+i})G_{aIJ} - g^{+\mu}\Gamma^-_{I+}G_{\mu-J} - g^{+\mu}\Gamma^-_{J+}G_{\mu I-}$$

$$= g^{\nu\mu}(\partial_\nu G_{\mu IJ}) - 2g^{+-}\Gamma^-_{I+}G_{--J}$$

$$= g^{\nu\mu}(\partial_\nu G_{\mu IJ})$$

$$= \partial_+ G_{-IJ} + \partial_- G_{+IJ} + \partial_K G_{KIJ} + \mu^2 x_I^2 \partial_- G_{-IJ}$$

and in the very similar fashion:

$$\nabla^m G_{m-I} = \partial_- G_{+-I} + \partial_K G_{K-I} \tag{5.32}$$

where the remaining terms are missing because of the 5.30. Simply combining the equations 5.32 and the 5.28 we get:

$$\partial_- b_{+I} = \partial_K b_{KI} \tag{5.33}$$

Similarly, we can expand the equation 5.31 in terms of b.

$$\nabla^\mu G_{\mu IJ} = \partial_+\partial_- b_{IJ} + \partial_-\partial_+ b_{IJ} - \partial_-\partial_I b_{+J} - \partial_-\partial_J b_{I+} \tag{5.34}$$

$$+\partial_K\partial_K b_{IJ} + \partial_K\partial_I b_{IK} + \partial_K\partial_J b_{KJ} + \mu^2 x_I^2 \partial_-\partial_- b_{IJ} = \nabla^2 b_{IJ}$$

because middle four terms cancel out according to the 5.33 We now use the equation 5.28 and the fact that the only surviving R-R fields are the $F_{+1234} = F_{+5678} = 2\mu$ to derive:

$$F_{ij'm_3m_4m_5} = 0 \tag{5.35}$$

$$F_{ijm_3m_4m_5} G^{m_3m_4m_5} = 6\mu\epsilon_{ijkl}\partial^+ b_{kl} \tag{5.36}$$

and therefore we can write the equations for the physical modes b_{IJ}:

$$\nabla^2 b_{ij'} = 0, \qquad \nabla^2 b_{ij} + 2i\mu\epsilon_{ijkl}\partial_+ b_{kl} = 0, \qquad \nabla^2 b_{i'j'} + 2i\mu\epsilon_{i'j'k'l'}\partial_+ b_{k'l'} = 0 \tag{5.37}$$

Therefore, using the results of the previous section the lowest energy level for the bij' is $\mathcal{E}_0(b_{ij'} = 4$. To figure it out for the remaining fields we need to decompose the antisymmetric field b_{ij} into the irreducible tensors of the SO(4) algebra:

$$b_{ij} = b^\oplus_{ij} + b^\ominus_{ij}, \qquad b^\oplus_{ij} = \frac{1}{2}\epsilon_{ijkl}b^\oplus_{kl}, \qquad b^\ominus_{ij} = -\frac{1}{2}\epsilon_{ijkl}b^\ominus_{kl} \tag{5.38}$$

we then have:

$$(\nabla^2 + 4i\mu\partial_+)b^\oplus_{ij} = 0, \qquad (\nabla^2 - 4i\mu\partial_+)b^\ominus_{ij} = 0 \tag{5.39}$$

and the exact equivalent for the $b_{i'j'}$. We can therefore write the lowest energy values for all the complex 2-form fields:

$$\mathcal{E}_0(b_{ij}^{\ominus}) = \mathcal{E}_0(b_{i'j'}^{\ominus}) = 2, \qquad \mathcal{E}_0(b_{ij}^{\oplus}) = \mathcal{E}_0(b_{i'j'}^{\oplus}) = 6, \qquad \mathcal{E}_0(b_{ij'}) = 4 \tag{5.40}$$

Graviton and 4-form field

The calculation for the case of Graviton and the 4-form field is very similar to the one we just performed. The difference is that the fluctuation modes of the two are mixed and need to be considered together. Again, we begin with the full nonlinear equations:

$$R_{mn} = \frac{1}{24} F_{mm_2 \dots m_5} F_n^{m_2 \dots m_5} \tag{5.41}$$

$$F_{m_1 \dots m_5} = -\frac{1}{5!}\sqrt{-g}\,\epsilon_{m_1 \dots m_5 n_1 \dots n_5} F^{n_1 \dots n_5} \tag{5.42}$$

$$\nabla^m F_{mm_2 \dots m_5} = 0 \tag{5.43}$$

$$F_{m_1 \dots m_5} = 5\partial_{[m_1} A_{m2 \dots m_5]} \tag{5.44}$$

We can then expand in the neighborhood of the pp-wave R-R background, treating the graviton as the correction to the metric and the 4-form a_{ijkl} as similarly related to the 5-form field F:

$$g_{mn} \rightarrow g_{mn} + h_{mn} \tag{5.45}$$

$$A_{m1 \dots m_4} \rightarrow A_{m_1 \dots m_4} + a_{m_1 \dots m_4} \tag{5.46}$$

$$R_{mn} \rightarrow R_{mn} + r_{mn} \tag{5.47}$$

$$F_{m_1 \dots m_5} \rightarrow F_{m_1 \dots m_5} + f_{m_1 \dots m_5} \tag{5.48}$$

Where R is a Ricci tensor as calculated in 5.15 and pick the light-cone gauge for the h_{mn} and $a_{m_1 \dots m_4}$:

$$h_{-m} = 0, \qquad a_{-m_2 m_3 m_4} = 0 \tag{5.49}$$

We can then write the linearized form of the Einstein equation 5.41

$$r_{mn} = \frac{1}{24}(F_{mm_1 \dots m_4} f_n^{m_1 \dots m_4} + F_{mm_1 \dots m_4} f_n^{m_1 \dots m_4} \tag{5.50}$$
$$-4F_{mn_1 m_3 \dots m_5} F_{nn_2}^{m_3 \dots m_5} h^{n_1 n_5}$$

This follows directly from the 5.41 by taking the first order in fluctuations. In the similar fashion we can derive the fluctuation in the Ricci tensor straight from the Ricci tensor equation by considering the fluctuations in the metric:

$$r_{mn} = \frac{1}{2}(-\nabla^2 h_{mn} + \nabla_m \nabla^k h_{kn} + \nabla_n \nabla^k h_{km} - \nabla_m \nabla_n h_k^k \tag{5.51}$$
$$+2R_{mm_1 m_2 n} h^{m_1 m_2} + R_{mk} h_n^k + R_{nk} h_m^k)$$

We use the combination of the two last equations to establish the number of conditions for the modes of the graviton. First of all we can consider the $(mn) = (--)$ components. For those the equation 5.50 clearly gives zero because any F tensor with a $-$ index disappears. On the other hand, the light-cone gauge condition 5.49 and the known properties of the Riemman and Ricci tensors 5.14, 5.15 ensures that almost all elements in the equation 5.51 go to zero. Specifically, all the ones with a $-$ index either in the h or in either of Rs. The only survivor will be the 4th term that consists only of the trace of the transverse modes of the graviton. By combining these two results we get the zero-trace condition on the transverse modes:

$$h_{II} = 0 \tag{5.52}$$

The similar consideration can be applied to the components $(mn) = (-I)$ again, the $-$ will suffice to ensure $r_{-I} = 0$ from the equation 5.50 as well as most of the terms in 5.51. With trace term set to zero by the above consideration, the only remaining one term will be the $\nabla^m h_{mI}$ one leading to the constraint:

$$\nabla^m h_{mI} = 0 \tag{5.53}$$

In conjunction with the light-cone gauge conditions this leads to the constraint akin to the one for the 2-form field:

$$\partial_- h_{+I} = -\partial_J h_{JI} \tag{5.54}$$

We now need a similar expression for the field a. We derive it from the self-duality condition on 5-form F, equations 5.43,5.2. Considering the components $(I_1 I_2 I_3 I_4 -)$ of the 5-form we derive the:

$$\partial_- a_{+I_1 I_2 I_3} = -\partial_J a_{J I_1 I_2 I_3} \tag{5.55}$$

We can also rewrite the equation 5.42 in terms of the a_{IJKL} as:

$$a_{I_1 \ldots I_4} = -\frac{1}{4!}\epsilon_{I_1 \ldots I_4 J_1 \ldots J_4} a_{J_1 \ldots J_4} \tag{5.56}$$

Putting the above the results into the equations 5.50 and 5.51 and solving for the components $(mn) = (++)$ we obtain the expression for the h_{++} component of the gravition in terms of the transverse modes:

$$(\partial_-)^2 h_{++} = \partial_I \partial_J h_{IJ} \tag{5.57}$$

We now consider the transverse directions, four by four. The first thing we notice is that by using the constraints on h from equations 5.52,5.54 and 5.57, together with the light-cone gauge condition, we can eliminate all the terms in the equation 5.51 except the first, being left with:

$$r_{ij} = -\frac{1}{2}\nabla^2 h_{ij} \tag{5.58}$$

In the same way, the conditions on a, together with the self-duality condition give the following simplification for the equation 5.50:

$$r_{ij} = \mu\delta_{ij}\partial_- a, \qquad a = \frac{1}{6}\epsilon_{i_1..i_4}a_{i_1...i_4} \tag{5.59}$$

So we get the equation combining the SO(4) part of the graviton with pseudo-scalar part of the 4-form potential:

$$\nabla^2 h_{ij} + 2\mu\partial_- a = 0 \tag{5.60}$$

Using the constraints on a and the equation 5.43, in a very similar way we get the parallel equation:

$$\nabla^2 a - 8\mu h_{ii} = 0 \tag{5.61}$$

to get those equations in the requisite form 5.23 we need to diagonalize them by introducing the traceless graviton and the complex scalar:

$$h_{ij}^{\perp} = h_{ij} - \frac{1}{4}\delta_{ij}h_{kk} \tag{5.62}$$

$$h = h_{ii} + ia \tag{5.63}$$

$$\bar{h} = h_{ii} - ia \tag{5.64}$$

We can then write, again using the $\partial_- = \partial^+$ identity:

$$\nabla^2 h_{ij}^{\perp} = 0 \tag{5.65}$$

$$(\nabla^2 - 8i\mu\partial^+)h = 0 \tag{5.66}$$

$$(\nabla^2 + 8i\mu\partial^+)\bar{h} = 0 \tag{5.67}$$

So we get the lowest energy levels for the gravitons:

$$\mathcal{E}_0(h_{ij}^{\perp} = 4, \qquad \mathcal{E}_0(h) = 0, \qquad \mathcal{E}_0(\bar{h}) = 8 \tag{5.68}$$

With the same results for the other four transverse directions.

We can now apply the exactly the same analysis (equations 5.50, 5.51 and 5.43) to obtain the equations for the components mixing the two SO(4)s:

$$\nabla^2 a_{ij'} - 4\mu\partial_- h_{ij'} = 0, \qquad a_{ij'} = \frac{1}{3}\epsilon_{ii_1i_2i_3}a_{ji_ii_2i_3} \tag{5.69}$$

$$\nabla^2 h_{ij'} + 4\mu\partial_- a_{ij'} = 0 \tag{5.70}$$

we diagonalize those in the same way, defining the complex tensor:

$$h_{ij'} = h_{ij'} + ia_{ij'} \tag{5.71}$$

$$\bar{h}_{ij'} = h_{ij'} - ia_{ij'} \tag{5.72}$$

with the attendant differential equations:

$$(\nabla^2 - 4i\mu\partial^+)h_{ij'} = 0 \tag{5.73}$$

$$(\nabla^2 + 4i\mu\partial^+)\bar{h}_{ij'} = 0 \tag{5.74}$$

So we get the lowest energy eigenvalues:

$$\mathcal{E}_0(h_{ij'}) = 2, \qquad \mathcal{E}_0(\bar{h}_{ij'}) = 6 \tag{5.75}$$

And finally we have:

$$\nabla^2 a_{iji'j'} = 0 \tag{5.76}$$

and consequently:

$$\mathcal{E}_0(a_{iji'j'}) = 4 \tag{5.77}$$

5.3 Fermionic Fields

Spin 1/2 fields

The calculations for the fermionic Spin 1/2 and Spin 3/2 fields are similar to the ones in Bosonic cases. The starting point non-linear supergravity equations of motion are still taken from the [20] and adopted to our case. The main difference is that instead of ∇^m we now deal with the spinor covariant derivative:

$$D_m = \partial_m + \frac{1}{4}\omega_m^{\mu\nu}\bar{\gamma}^{\mu\nu} \tag{5.78}$$

Again, we use the vielbein basis corresponding to the metric:

$$e^+ = dx^+, \qquad e^- = dx^- - \frac{\mu^2}{2}x_I^2 dx^+, \qquad e^I = dx^I \tag{5.79}$$

We can then write the Spinor covariant derivative explicitely:

$$D_- = \partial_-, \qquad D_I = \partial_I, \qquad D_+ = \partial_+ - \frac{\mu^2}{2}x^I\bar{\gamma}^{+I} \tag{5.80}$$

With this in mind we can write the complex 16-component version of the Schwarz's equation of motion for the Majorana-Weyl negative chirality spin 1/2 fields [20]:

$$(\gamma^m D_m - \frac{i}{480}\gamma^{m_1\ldots m_5}F_{m_1\ldots m_5})\lambda = 0 \tag{5.81}$$

Here the matrices γ^m are given by the:

$$\gamma^m = e_\mu^m \gamma^\mu \tag{5.82}$$

where e_μ^m are the inverses of the Vielbein matrix given above with:

$$e^\mu = e_m^\mu dx^m \tag{5.83}$$

From the form of the F field we know that the indicies $m_1...m_5$ in the above equation can only be $(+,1,2,3,4)$ or $(+5,6,7,8)$ in some antisymmetric combination. At the same time $\gamma^{+m_1...m_5}$ is an antisymmetized product of the γ^+ and the four γ matrices with number indices as above. We also know that the product of the numbered matrices is given by:

$$\Pi = (\gamma^1\gamma^2\gamma^3\gamma^4) \tag{5.84}$$

with anti-commutation being taken care with the antisymmetry in the $\gamma^{+m_1...m_5}$ we can write:

$$\gamma^{+m_1...m_5}F_{m_1...m_5} = 480\mu\gamma^+\Pi \tag{5.85}$$

Using that and the 5.82 equation above we can write the equation of motion as:

$$[\gamma^+(\partial^- + \frac{\mu^2}{2}x_I^2\partial^+ - i\mu\Pi) + \gamma^-\partial^+ + \gamma^I\partial^I]\lambda = 0 \tag{5.86}$$

We can then decompose the λ as:

$$\lambda = \lambda^\oplus + \lambda^\ominus \tag{5.87}$$

Where:

$$\lambda^\oplus = (1+\gamma^9)\lambda, \qquad \lambda^\ominus = (1-\gamma^9)\lambda \tag{5.88}$$

or:

$$\lambda^\oplus = \bar{\gamma}^-\gamma^+\lambda, \qquad \lambda^\ominus = \bar{\gamma}^+\gamma^-\lambda \tag{5.89}$$

The $\lambda^\oplus$ is the light-cone projection of the λ. $\lambda^\ominus$ is a non-dynamical mode that we need to express in terms of $\lambda^\oplus$. To do that we need the following properties of the γ matrices:

$$\gamma^-\bar{\gamma}^- = \frac{1}{2}(\gamma^9 - 1)(\gamma^9 + 1) = \frac{1}{2}(1-1) = 0 \tag{5.90}$$

$$\gamma^+\bar{\gamma}^- = \frac{1}{2}(\gamma^9 + 1)(\gamma^9 + 1) = \frac{1}{2}(2 + 2\gamma^9) = \gamma^+ \tag{5.91}$$

Using the above we can act with $\bar{\gamma}^-$ on the equation 5.86 from the right. The first term disappears and the last we write in terms of $\lambda^\oplus$ getting:

$$2\partial_-\lambda^\ominus = \bar{\gamma}^I\partial^I\gamma^+\lambda^\oplus \tag{5.92}$$

Now we can act with the $2\bar{\gamma}^-\partial+$ on the equation 5.86 from the right and, using the above result get:

$$(\nabla^2 - 2i\mu\Pi\partial^+)\lambda^\oplus = 0 \tag{5.93}$$

It is now trivial to further decompose the $\lambda^\oplus$ into:

$$\lambda^\oplus = \lambda_L^\oplus + \lambda_R^\oplus, \qquad \lambda_R = \frac{1+\Pi}{2}\lambda, \qquad \lambda_L = \frac{1-\Pi}{2}\lambda \tag{5.94}$$

getting the equations in the desired form:

$$(\nabla^2 - 2i\mu\partial^+)\lambda_R^\oplus = 0, \qquad (\nabla^2 + 2i\mu\partial^+)\lambda_R^\oplus = 0 \tag{5.95}$$

Thus finding the lowest energy values for the spin 1/2 fields:

$$\mathcal{E}_0(\lambda_R^{\oplus}) = 3, \qquad \mathcal{E}_0(\lambda_L^{\oplus}) = 5 \tag{5.96}$$

. **Spin 3/2 fields**

The approach for the Spin 3/2 fields is again very simmiar: we start with the equation for the gravitino of positive chirality:

$$\bar{\gamma}^n D_n \psi_m - D_m \psi - fraci960 \bar{\gamma}^n \gamma^{n_1 \ldots n_5} F_{n_1 \ldots n_5} \bar{\gamma}_m \psi_n = 0, \qquad \psi = \bar{\gamma}^n \psi_n \tag{5.97}$$

Which, due to the equation 5.85 is equal to:

$$\bar{\gamma}^n D_n \psi_m - D_m \psi - \frac{i\mu}{2} \bar{\gamma}^n \Pi \bar{\gamma}_m \psi_n = 0 \tag{5.98}$$

On this we impose the light-cone gauge condition:

$$\psi_- = 0 \tag{5.99}$$

We can then test the equation 5.98 for the case $m = -$. The first term disapears and in the last term $\bar{\gamma}_-$ commutes through leaving the entire equation as a multiple of the *psi* from which we can read:

$$\psi = \bar{\gamma}^+ \psi_+ + \bar{\gamma}^I \psi_I = 0 \tag{5.100}$$

we can multiply the above equation by γ^+ and use 5.91 to obtain:

$$\gamma^+ \bar{\gamma}^I \psi_I = 0 \tag{5.101}$$

We can use this to simplify the expression $\bar{\gamma}^J \Pi \gamma^+ \bar{\gamma}_i \psi_J$ by commuting the $\bar{\gamma}^J$ all the way to the right. We can then write:

$$\bar{\gamma}^J \Pi \gamma^+ \bar{\gamma}_i \psi_J = 2\Pi \bar{\gamma}^+ (\delta_{ij} - \gamma_i \bar{\gamma}_j) \psi_j \tag{5.102}$$

and the

$$\bar{\gamma}^J \Pi \gamma^+ \bar{\gamma}'_i \psi_J = -2\Pi \bar{\gamma}^+ (\delta_{i'j'} - \gamma'_i \bar{\gamma}'_j) \psi'_j \tag{5.103}$$

and use those in the 5.98 with $m = i$ component to write:

$$[\bar{\gamma}^+ (\partial^- + \frac{\mu}{2} x_I^2 \partial^+) + \bar{\gamma}^- \partial^+ + \bar{\gamma}^J \partial_J] \psi_i - i\mu \Pi \bar{\gamma}^+ (\delta_{ij} - \gamma_i \bar{\gamma}_j) \psi_j \tag{5.104}$$

and the equivalent for the $m = i'$.

Now we can divide the ψ into the $\psi^{\oplus}$ and $\psi^{\ominus}$ in the same way as in the Spin 1/2 case. Using the very same trick as in that case we end up with:

$$2\partial_- \psi_I^{\ominus} = \gamma^+ (\bar{\gamma}^J \partial_J) \psi_I^{\oplus} \tag{5.105}$$

and

$$\nabla^2 \psi_i^{\oplus} - 2i\mu \Pi (\delta_{ij} - \gamma_i \bar{\gamma}_j) \partial_- \psi_j^{\ominus} \tag{5.106}$$

we can now decompose the $\psi_I^{\oplus}$ into the γ-transverse and γ-parallel parts:

$$\psi_i^{\oplus\perp} = (\delta_i j - \frac{1}{4}\gamma_i\bar{\gamma}_j)\psi_j^{\oplus} \tag{5.107}$$

$$\psi^{\oplus||} = \bar{\gamma}_i\psi_i^{\oplus} \tag{5.108}$$

And for those we get:

$$(\nabla^2 - 2i\mu\Pi\partial^+)\psi_i^{\oplus\perp} = 0 \tag{5.109}$$

$$(\nabla^2 - 6i\mu\Pi\partial^+)\psi^{\oplus||} = 0 \tag{5.110}$$

splitting into the left and right modes as before we get:

$$(\nabla^2 - 2i\mu\partial^+)\psi_{iR}^{\oplus\perp} \tag{5.111}$$

$$(\nabla^2 + 2i\mu\partial^+)\psi_{iL}^{\oplus\perp} \tag{5.112}$$

$$(\nabla^2 - 6i\mu\partial^+)\psi_R^{\oplus||} \tag{5.113}$$

$$(\nabla^2 + 6i\mu\partial^+)\psi_L^{\oplus||} \tag{5.114}$$

and consequently:

$$\mathcal{E}_0(\psi_{iR}^{\oplus\perp}) = 3, \qquad \mathcal{E}_0(\psi_{iL}^{\oplus\perp}) = 5, \qquad \mathcal{E}_0(\psi_R^{\oplus||}) = 1, \qquad \mathcal{E}_0(\psi_L^{\oplus||}) = 7 \tag{5.115}$$

Similarly:

$$\nabla^2\psi_{i'}^{\oplus} + 2i\mu\Pi(\delta_{i'j'} - \gamma_{i'}\bar{\gamma}_{j'})\partial_-\psi_{j'}^{\ominus} \tag{5.116}$$

leading to the

$$\mathcal{E}_0(\psi_{i'R}^{\oplus\perp}) = 5, \qquad \mathcal{E}_0(\psi_{i'L}^{\oplus\perp}) = 3 \tag{5.117}$$

This ends the part of our paper in which we review the work of Metsaev [8] and Metsaev and Tseytlin [9]. In the last section we will present the results of our attempt to find the vertex operators for some of the lowest energy states listed above.

Chapter 6

Explicit form of x^- and its two-point function

6.1 Vertex operator of the graviton

It was our intention to follow up on the work of Metsaev and Tseytlin [9] by finding the vertex operators for some of the lowest energy states listed above. While we are able to present the candidate for the vertex operator of the graviton (and, in the same manner, most of the other lowest energy states) we can not prove that those are indeed proper vertex operators due to the difficulties in calculating their conformal dimension. We present the reasoning and some of the calculations we did in the attempt to calculate the conformal dimension.

As it can be seen from the above, the overall lowest energy state of the spectrum is one of the components of the graviton field label above as h given by the equations 5.66 and 5.68. In keeping with the [1] we request that the Vertex operator for h $V(h)$ satisfies the equation 5.66:

$$[2\partial_-\partial_+ + \mu^2 x_i^2 \partial_-^2 + \partial_i^2 - 8i\mu\partial_-]V(h) = 0 \tag{6.1}$$

We can see immediately that the solution of this equation will have to be a Gaussian so we can write the following anzac form for V(h):

$$V(h) = \exp(Ax_ix_i + k^+x^- + k^-x^+ + k^ix_i) \tag{6.2}$$

We can then put this into the above equation:

$$[2\partial_-\partial_+ + \mu^2 x_i^2 \partial_-^2 + \partial_i^2 - 8i\mu\partial_-]V(h) = \tag{6.3}$$

$$[k^+k^- + \mu^2 x_i^2 k^+k^+ + 16A + 4A^2x_i^2 + 4Ak^ix_i + k_i^2 - 8i\mu k^+]V(h) = 0$$

Working in orders of x_i we see immediately that the $k^i = k^- = 0$ and that we can express the A by:

$$A = \frac{i}{2}\mu k^+ \tag{6.4}$$

Putting this in 6.2 and writing k in terms of the momenta p we can write:

$$V(h) = \exp[p^+(\frac{i}{2}\mu x_i^2 + x^-)] = \exp[\frac{i}{2}mx_i^2 + p^+x^-] \tag{6.5}$$

Method used by Green and Schwarz [15] in calculating the light-cone gauge Vertex operators is to take a most convenient momentum direction possible to prove the properties of the operator and then use the rotation symmetries to obtain the general form. It is immediately obvious that this method will not work in our case. Not only are the rotations involving the x^+ direction absent from the symmetries of the action but also, the only simplifying choice of momenta $p^+ = 0$ makes the expression above entirely vacuous. We have therefore to work with the Gaussian forms such as the one given in 6.5 if we are to learn anything about the vertex operators.

6.2 Two point function of the Vertex Operators

Our main attempt to show that the 6.5 really gives the vertex operator for the lowest energy state of graviton was by trying to demonstrate that it is an operator of the anomalous dimension 2. As is well known [1] the anomalous dimension of the operator is determined by its two point function in the following way:

$$\langle V(z)V(0)\rangle = |z|^{-2d} \tag{6.6}$$

Where d is the anomalous dimension of the operator V.

The two point function for the Gaussian operator given in the 6.5, however, is difficult to calculate. What we chose to do is find the first terms of the Taylor expansion:

$$\langle e^{\frac{i}{2}mx_i^2(z)+p^+x^-(z)}|e^{\frac{i}{2}mx_i^2(0)+p^+x^-(0)}\rangle = \tag{6.7}$$

$$\langle 1+[\frac{i}{2}mx_i^2(z)+p^+x^-(z)]+...|...+[\frac{i}{2}mx_i^2(z)+p^+x^-(z)]+1\rangle$$

and see if we can extrapolate from those. Because both x_i^2 and x^{-2} are quadratic in the creation/anihilation operators it is obvious that this expansion will have a form:

$$\langle e^{\frac{i}{2}mx_i^2(z)+p^+x^-(z)}|e^{\frac{i}{2}mx_i^2(0)+p^+x^-(0)}\rangle = \tag{6.8}$$

$$1+\langle\frac{i}{2}mx_i^2(z)+p^+x^-(z)|\frac{i}{2}mx_i^2(0)+p^+x^-(0)\rangle+$$

$$\frac{1}{2!^2}\langle(\frac{i}{2}mx_i^2(z)+p^+x^-(z))^2|(\frac{i}{2}mx_i^2(0)+p^+x^-(0))^2\rangle+...$$

It can be seen that this can be rewritten as:

$$\langle e^{\frac{i}{2}mx_i^2(z)+p^+x^-(z)}|e^{\frac{i}{2}mx_i^2(0)+p^+x^-(0)}\rangle = \tag{6.9}$$

$$1+\langle\frac{i}{2}mx_i^2(z)+p^+x^-(z)|\frac{i}{2}mx_i^2(0)+p^+x^-(0)\rangle+$$

$$\frac{1}{2!}\langle\frac{i}{2}mx_i^2(z)+p^+x^-(z)|\frac{i}{2}mx_i^2(0)+p^+x^-(0)\rangle^2+...+c.t$$

Where c.t. refers to the cross-terms that come from the two point functions of the squares and higher powers of $(\frac{i}{2}mx_i^2(0)+p^+x^-(0))$ are not themselves included in the corresponding powers of the two point functions of the same.

We thus end with:

$$\langle e^{\frac{i}{2}mx_i^2(z)+p^+x^-(z)}|e^{\frac{i}{2}mx_i^2(0)+p^+x^-(0)}\rangle = \tag{6.10}$$

$$e^{\langle\frac{i}{2}mx_i^2(z)+p^+x^-(z)|\frac{i}{2}mx_i^2(0)+p^+x^-(0)\rangle} + c.t.$$

Some combinatorics on the cross-terms best done with the use of diagrams actually shows that, the above can actually be written as:

$$\langle e^{\frac{i}{2}mx_i^2(z)+p^+x^-(z)}|e^{\frac{i}{2}mx_i^2(0)+p^+x^-(0)}\rangle = \tag{6.11}$$

$$[1+c.t.]e^{\langle\frac{i}{2}mx_i^2(z)+p^+x^-(z)|\frac{i}{2}mx_i^2(0)+p^+x^-(0)\rangle}$$

The significant amount of information about the anomalous dimension of the V(h) then rests in the two point function:

$$\langle\frac{i}{2}mx_i^2(z)+p^+x^-(z)|\frac{i}{2}mx_i^2(z')+p^+x^-(z')\rangle \tag{6.12}$$

and in what follows we proceed to calculate this two point function. The hope was that the cross-terms will prove insignificant in comparison and that we will be able to extract the two point function of the operator from the exponential in the 6.11.

6.3 x^- two point function

We have found two ways for calculating the two point function in 6.12. The more elegant one involves using the defining properties of x^- in the equations 4.18-4.19 directly and expressing the 6.12 in terms of the two point functions $G(z-z') = \langle x_i(z)|x_i(z')\rangle$. We shall present the entire calculation using this method. More brute-force approach was to use the 4.18-4.19 and the known forms of x_i in terms of creation/anihilation operators to express the x^- in terms of αs and θs as well, and then calculate the two point function using the commutation relations. At the end of this chapter we calculate the x^- in terms of the cretion/anihilation operators and confirm our result using the brute-force method.

We begin by writing splitting the 6.12 into the components:

$$\langle\frac{i}{2}mx_i^2(z)+p^+x^-(z)|\frac{i}{2}mx_i^2(z')+p^+x^-(z')\rangle = \tag{6.13}$$

$$-\frac{m^2}{4}\langle x_i^2(z)|x_i^2(z')\rangle+\frac{imp^+}{2}\langle x_i^2(z)|x^-(z')\rangle+\frac{imp^+}{2}\langle x^-(z)|x_i^2(z')\rangle+p^{+2}\langle x^-(z)|x^-(z')\rangle$$

Using the $G(z-z') = \langle x_i(z)|x_i(z')\rangle$ we write the first term as:

$$-\frac{m^2}{4}\langle x_i^2(z)|x_i^2(z')\rangle = -8\times 2\frac{m^2}{4}G(z-z') \tag{6.14}$$

Where 8 comes from the number of components of x_i and 2 is a combinatorics factor Second and third term cancel each other: to see that we write their derivatives with respect to σ using the equation 4.20 for the σ derivative of the x^-:

$$\partial_\sigma \langle x_i^2(z)|x^-(z')\rangle = -\langle x_i^2(z)|\partial_{\sigma'} x^-(z')\rangle \tag{6.15}$$
$$= -\langle x_i(z)x_i(z)|\dot{x}_i(z')\acute{x}_i(z')\rangle = -\dot{G}(z-z')\acute{G}(z-z')$$

Where the minus sign comes from acting on the function of z' with the derivative. Each of the Gs will also carry the minus sign because they pick up derivatives with respect to the primed variable but there are two Gs in the product so those signs cancel. At the same time:

$$\partial_\sigma \langle x^-(z)|x_i^2(z')\rangle = \langle \partial_\sigma x^-(z)|x_i^2(z')\rangle \tag{6.16}$$
$$= \langle \dot{x}_i(z)\acute{x}_i(z)|x_i(z')x_i(z')\rangle = \dot{G}(z-z')\acute{G}(z-z')$$

so the derivatives of the two terms cancel out, leaving their sum without σ dependance. Because the τ derivative of x^- is also quadratic in the derivatives x_i exactly same sort of calculation sets the τ derivative of the sum to zero and therefore we conclude that the second and third terms cancel out.

Fourth term involving the two point function of x^- is the most difficult one to calculate. This time we will use the derivatives of the x^- to calculate the second derivatives of the function and will then find the way to integrate the result. In this case it is more convenient to use the plus/minus derivatives of x^- given in 4.18 and 4.19. Writing those two equations again:

$$\partial_+ x^- = -\frac{1}{2p^+}[-m^2x_i^2 + (\partial_+ x^i)^2 + 2i(\theta^A\bar{\gamma}^-\partial_+\theta^A - 2m\theta^1\bar{\gamma}^-\Pi\theta^2)] \tag{6.17}$$

$$\partial_- x^- = -\frac{1}{2p^+}[-m^2x_i^2 + (\partial_- x^i)^2 + 2i(\theta^A\bar{\gamma}^-\partial_-\theta^A - 2m\theta^1\bar{\gamma}^-\Pi\theta^2)] \tag{6.18}$$

we can rewrite them using the equations of motion for fermions:

$$\partial_+ x^- = -\frac{1}{2p^+}[-m^2x_i^2 + (\partial_+ x^i)^2 + 2i(-\theta^1\bar{\gamma}^-\partial_+\theta^1 + \theta^2\bar{\gamma}^-\partial_+\theta^2)] \tag{6.19}$$

$$\partial_- x^+ = -\frac{1}{2p^+}[-m^2x_i^2 + (\partial_- x^i)^2 + 2i(\theta^1\bar{\gamma}^-\partial_-\theta^1 - \theta^2\bar{\gamma}^-\partial_-\theta^2)] \tag{6.20}$$

it is obvious that we can separate our calculations for the bosonic and fermionic part of the x^-. Doing that we can write:

$$\partial_-^2 \langle x^-(z)|x^-(z')\rangle_{bos} = -\langle \partial_- x^-(z)|\partial_{-'} x^-(z')\rangle_{bos} = \tag{6.21}$$

$$-\frac{1}{4p^{+2}}\langle(\partial_- x_i(z))^2 - (mx_i(z))^2|(\partial_{-'} x_j(z'))^2 - (mx_j(z'))\rangle$$

$$= -8\times 2\frac{1}{4p^{+2}}[(\partial_-^2 G(z-z')) - 2m^2(\partial_- G)^2 + m^4G^2]$$

To perform the similar calculation for the fermionic part of the function we need a brief detour to find the two point functions of θs. Going back to the Lagrangian:

$$\mathcal{L}_F = i(\theta^1\bar{\gamma}^-\partial_+\theta^1 + \theta^2\bar{\gamma}^-\partial_-\theta^2 - 2m\theta^1\bar{\gamma}^-\Pi\theta^2) \tag{6.22}$$

we can write this as:

$$\mathcal{L}_F = i\begin{pmatrix} \theta^1 & \theta^2 \end{pmatrix}\begin{pmatrix} \bar{\gamma}^-\partial_+ & -m\bar{\gamma}^-\Pi \\ +m\bar{\gamma}^-\Pi & \bar{\gamma}^-\partial_- \end{pmatrix}\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix} \tag{6.23}$$

We can then write the Matrix L as:

$$L = i\begin{pmatrix} \bar{\gamma}^-\partial_+ & -m\bar{\gamma}^-\Pi \\ +m\bar{\gamma}^-\Pi & \bar{\gamma}^-\partial_- \end{pmatrix} \tag{6.24}$$

We now look for projective inverse $L^{\#}$ which satisfies:

$$L = L^{\#}LL^{\#} \tag{6.25}$$

finding it to be:

$$\begin{aligned} L^{\#} &= -i\begin{pmatrix} \bar{\gamma}^{-\#}\partial_- & +m\bar{\gamma}^{-\#}\Pi \\ -m\bar{\gamma}^{-\#}\Pi & \bar{\gamma}^{-\#}\partial_+ \end{pmatrix}\frac{1}{\partial_+\partial_- - m^2} \\ &= -i\begin{pmatrix} \bar{\gamma}^{-\#}\partial_- & +m\bar{\gamma}^{-\#}\Pi \\ -m\bar{\gamma}^{-\#}\Pi & \bar{\gamma}^{-\#}\partial_+ \end{pmatrix}G(z-z') \end{aligned} \tag{6.26}$$

where $\bar{\gamma}^{-\#}$ satisfies:

$$\bar{\gamma}^- = \bar{\gamma}^-\bar{\gamma}^{-\#}\bar{\gamma}^- \tag{6.27}$$

and is therefore given by:

$$\bar{\gamma}^{-\#} = \gamma^+ \tag{6.28}$$

We therefore have:

$$L^{\#} = -i\begin{pmatrix} \gamma^+\partial_- & +m\gamma^+\Pi \\ -m\gamma^+\Pi & \gamma^+\partial_+ \end{pmatrix}G(z-z') \tag{6.29}$$

From which we can read out the two point functions:

$$\langle\theta^1_\alpha(z)|\theta^1_\beta(z')\rangle = -i\gamma^+_{\alpha\beta}\partial_-G(z-z') \tag{6.30}$$

$$\langle\theta^1_\alpha(z)|\theta^2_\beta(z')\rangle = -i(\gamma^+\Pi)_{\alpha\beta}mG(z-z') \tag{6.31}$$

$$\langle\theta^2_\alpha(z)|\theta^1_\beta(z')\rangle = +i(\gamma^+\Pi)_{\alpha\beta}mG(z-z') \tag{6.32}$$

$$\langle\theta^2_\alpha(z)|\theta^2_\beta(z')\rangle = -i\gamma^+_{\alpha\beta}\partial_+G(z-z') \tag{6.33}$$

With those we can now write the fermionic part of the two point function for the x^-

$$\partial_-^2\langle x^-(z)|x^-(z')\rangle_{fer} = -\langle\partial_-x^-(z)|\partial_{-'}x^-(z')\rangle_{fer} = \tag{6.34}$$

$$= \frac{1}{p^{+2}}[\langle\theta^1_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^1_\beta|\theta^1_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^1_\delta\rangle - \langle\theta^1_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^1_\beta|\theta^2_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^2_\delta\rangle$$

$$= -\langle\theta^2_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^2_\beta|\theta^1_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^1_\delta\rangle + \langle\theta^2_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^2_\beta|\theta^2_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^2_\delta\rangle]$$

where the fields in the bra part of the bra-ket are always assumed to be functions of z and the ones in ket functions of z'

We then write:

$$\langle\theta^1_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^1_\beta|\theta^1_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^1_\delta\rangle = \tag{6.35}$$

$$= \bar{\gamma}^-_{\alpha\beta}\bar{\gamma}^-_{\gamma\delta}[\frac{1}{2}\langle\theta^1_\alpha|\partial_-\theta^1_\delta\rangle\frac{1}{2}\langle\partial_-\theta^1_\beta|\theta^1_\gamma\rangle - \frac{1}{2}\langle\theta^1_\alpha|\theta^1_\gamma\rangle\frac{1}{2}\langle\partial_-\theta^1_\beta|\partial_-\theta^1_\delta\rangle]$$

where factors of 2 come from the properties of the Majorana spinors and the minus signs from the standard expectation value properties of the fermions [19]. Taking care of the sign change due to derivative on the z' variable we re-write this as:

$$\langle\theta^1_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^1_\beta|\theta^1_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^1_\delta\rangle = \tag{6.36}$$

$$= \frac{1}{4}\bar{\gamma}^-_{\alpha\beta}\bar{\gamma}^-_{\gamma\delta}[-\partial_-\langle\theta^1_\alpha|\theta^1_\delta\rangle\partial_-\langle\theta^1_\beta|\theta^1_\gamma\rangle + \langle\theta^1_\alpha\theta^1_\gamma\rangle\partial^2_-\langle\theta^1_\beta|\theta^1_\delta\rangle]$$

$$= \frac{1}{4}\bar{\gamma}^-_{\alpha\beta}\bar{\gamma}^-_{\gamma\delta}[\gamma^+_{\alpha\delta}\gamma^+_{\beta\gamma}(\partial^2_-G(z-z'))^2 - \gamma^+_{\alpha\gamma}\gamma^+_{\beta\delta}\partial^3_-G(z-z')G(z-z)]$$

$$= \frac{1}{4}Tr(\bar{\gamma}^-)[(\partial^2_-G(z-z'))^2 - \partial^3_-G(z-z')G(z-z)]$$

$$= \frac{8}{4}[(\partial^2_-G(z-z'))^2 - \partial^3_-G(z-z')G(z-z)]$$

Where the $Tr(\bar{\gamma}^-) = 8$ in the last line comes from the fact that the $\bar{\gamma}^-$ is a rank 8 16-dimensional matrix whose eigen-values can be 1 or 0.

In the same fashion we get:

$$\langle\theta^1_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^1_\beta|\theta^2_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^2_\delta\rangle = \langle\theta^2_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^2_\beta|\theta^1_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^1_\delta\rangle = \tag{6.37}$$

$$= \frac{8}{4}m^2[(\partial_-G(z-z'))^2 - \partial^2_-G(z-z')G(z-z')]$$

and also:

$$\langle\theta^2_\alpha\bar{\gamma}^-_{\alpha\beta}\partial_-\theta^2_\beta|\theta^2_\gamma\bar{\gamma}^-_{\gamma\delta}\partial_-\theta^2_\delta\rangle = \frac{8}{4}(\partial_+\partial_-G(z-z') - \partial_+G(z-z')\partial_+\partial^2_-G(z-z') = \tag{6.38}$$

$$= \frac{8}{4}((m^2G(z-z'))^2 + m^2\partial_+G(z-z')\partial_-G(z-z'))$$

Adding all the terms and putting them back into 6.34 we get:

$$\partial^2_-\langle x^-(z)|x^-(z')\rangle_{fer} = \frac{2}{p^{+2}}[(\partial^2_-G(z-z'))^2 - \partial^3_-G(z-z')G(z-z) \tag{6.39}$$

$$-2m^2(\partial_-G(z-z'))^2 + 2m^2\partial^2_-G(z-z')G(z-z')$$

$$+(m^2G(z-z'))^2 + m^2\partial_+G(z-z')\partial_-G(z-z')]$$

Adding the bosonic and fermionic parts we can write:

$$p^{+2}\langle x^-(z)|x^-(z')\rangle = -\frac{2}{\partial_-^2}[(\partial_-^2 G)^2 + \partial_-^3 G\partial_- G \tag{6.40}$$

$$-2m^2(\partial_- G)^2 - 2m^2\partial_-^2 GG + m^4G^2 - m^2\partial_+ G\partial_- G]$$

$$= -\frac{2}{\partial_-^2}[\partial_-(\partial_- G\partial_-^2 G) - 2m^2\partial_-(\partial_- GG) - m^2\partial_-(\partial_+ GG)]$$

$$= -\frac{1}{\partial_-^2}[\partial_-^2(\partial_- G)^2 - 2m^2\partial_-^2 G^2 - m^2\partial_-\partial_+ G^2]$$

or:

$$p^{+2}\langle x^-(z)|x^-(z')\rangle = -[(\partial_- G)^2 - 2m^2G^2 - \frac{\partial_+}{\partial_-}m^2G^2] \tag{6.41}$$

Following the exactly same method but starting from the $\partial_+^2\langle x^-|x^-\rangle$ we obtain:

$$p^{+2}\langle x^-(z)|x^-(z')\rangle = -[(\partial_+ G)^2 - 2m^2G^2 - \frac{\partial_-}{\partial_+}m^2G^2] \tag{6.42}$$

Finally we consider $\partial_+\partial_-\langle x^-|x^-\rangle$ and apply one of the derivatives on bra and one on the ket side. After the analysis much akin to the one above we get:

$$p^{+2}\langle x^-(z)|x^-(z')\rangle = [2m^2G^2 + \frac{\partial_-}{\partial_+}m^2G^2 + \frac{\partial_+}{\partial_-}m^2G^2] \tag{6.43}$$

With three equations with three unknowns as it were, integration becomes trivial and we can write the final answer for the x^- two point function:

$$p^{+2}\langle x^-(z)|x^-(z')\rangle = [2m^2G^2 - (\partial_+ G)^2 - (\partial_- G)^2] \tag{6.44}$$

and therefore:

$$\langle \frac{i}{2}mx_i^2(z) + p^+x^-(z)|\frac{i}{2}mx_i^2(z') + p^+x^-(z')\rangle \tag{6.45}$$

$$= -[2m^2G^2 + (\partial_+ G)^2 + (\partial_- G)^2]$$

All that remains now is to obtain the value for $G(z-z')$ from the expression for x_i in terms of creation/anihilation operators 4.5 we can write:

$$G(z-z') = \langle x_i(z)|x_i(z')\rangle = \tag{6.46}$$

$$\frac{1}{2m}\langle -i\cos(m\tau)[a_0-\bar{a}_0]+\sin(m\tau)[a_0+\bar{a}_0]|i\cos(m\tau')[a_0-\bar{a}_0]+\sin(m\tau')[a_0+\bar{a}_0]\rangle$$

$$-\sum_{nm}\frac{1}{\omega_n\omega_m}[\langle\alpha_n^1\alpha_m^1\rangle\varphi_n^1(z)\varphi_m^1(z') + \langle\alpha_n^2\alpha_m^2\rangle\varphi_n^2(z)\varphi_m^2(z')]$$

$$= \frac{1}{2m}[\cos(m|\tau|) + i\sin(m|\tau|)] + \sum_n\frac{1}{\omega_n}[\varphi_n^1(|z|) + \varphi_n^2(|z|)]$$

Taking derivatives we get:

$$\dot{G}(z-z') = \frac{1}{2}[-\sin(m|\tau|) + i\cos(m|\tau|)] + i\sum_n \frac{\omega_n}{\omega_n}[\varphi_n^1(|z|) + \varphi_n^2(|z|)] \quad (6.47)$$

$$\acute{G}(z-z') = -i\sum_n \frac{k_n}{\omega_n}[\varphi_n^1(|z|) - \varphi_n^2(|z|)] \quad (6.48)$$

Putting those into the equation 6.45 and simplifying we get:

$$\langle \frac{i}{2}mx_i^2(z) + p^+x^-(z)|\frac{i}{2}mx_i^2(z') + p^+x^-(z')\rangle \quad (6.49)$$

$$= [\cos(m|\tau|) + i\sin(m|\tau|)]\sum_n \frac{\omega_n - m}{\omega_n}[\varphi_n^1(|z|) + \varphi_n^2(|z|)] +$$

$$\sum_{nm}[\frac{\omega_n\omega_m + k_nk_m - m^2}{\omega_n\omega_m}(\varphi_n^1(|z|)\varphi_m^1(|z|) + \varphi_n^2(|z|)\varphi_m^2(|z|)) +$$

$$\frac{\omega_n\omega_m - k_nk_m - m^2}{\omega_n\omega_m}(\varphi_n^1(|z|)\varphi_m^2(|z|) + \varphi_n^2(|z|)\varphi_m^1(|z|))]$$

It is clear that the exponential of this does not satisfy the anomalous dimension requirement for the vertex operator, and therefore, if the V(h) as given by 6.5 is indeed to be the vertex operator, then so far neglected cross-terms must be taken into account. Even so, the above result will be necessary starting point for such more detailed calculation. It can also be useful in studying the string interactions on the pp-wave once forms of the vertex operators are known.

6.4 Explicit form of x^-

The above calculation can be performed in another manner as well. The exact form of the x^- in terms of creation/anihilation can be obtained by integration from the equations 4.20 and 4.21 and then the commutation relations can be used to calculate the two point function. This is a rather lengthy calculation and, being as it is the case that it just confirms the result of the previous section, we would need not perform it here. Reason we do, however, has to do with the other way in which it may be possible to confirm that the $V(h)$ is indeed a vertex operator for the graviton. In the original Green-Schwarz paper [15] the way the form of the vertex operator was determined was by imposing not the anomalous dimension condition 6.6 but rather the supersymmetry conditions through the (anti)commutation properties of the operators with the generators of the supersymmetries Q. Given that the expressions for Qs in terms of creation/anihilation operators are already derived in 4.118-4.119 and 4.122-4.123 the way to test those relations is to have the x^- also expressed in this way.

This time it is more convenient to start with the expressions 4.20 and 4.21 for the derivatives of the x^-. Writing their bosonic parts we have:

$$\acute{x}^-_{bos} = -\frac{1}{p^+}[\dot{x}^i\acute{x}^i] \quad (6.50)$$

$$\dot{x}^-_{bos} = -\frac{1}{p^+}[\frac{1}{2}(\dot{x}^i\dot{x}^i + \acute{x}^i\acute{x}^i - m^2x^ix^i)] \tag{6.51}$$

we then need the expression for x^i given by 4.5

$$x^i = \cos(m\tau)x^i_0 + \frac{1}{m}\sin(m\tau)p^i_0 + i\sum_{n\neq 0}\frac{1}{\omega_n}(\varphi^1_n(\sigma,\tau)\alpha^{1i}_n + \varphi^2_n(\sigma,\tau)\alpha^{2i}_n) \tag{6.52}$$

and the appropriate derivatives:

$$\acute{x}^i = \sum_{n\neq 0} -\frac{k_n}{\omega_n}(\varphi^1_n\alpha^{1i}_n - \varphi^1_n 2\alpha^{2i}_n) \tag{6.53}$$

$$\dot{x}^i = \cos(m\tau)p^i_0 - m\sin(m\tau)x^i_0 + \sum_{n\neq 0}(\varphi^1_n\alpha^{1i}_n + \varphi^1_n 2\alpha^{2i}_n) \tag{6.54}$$

We can then write:

$$\acute{x}^-_{bos} = [\cos(m\tau)p^i_0 - m\sin(m\tau)x^i_0]\sum_{n\neq 0} -\frac{k_n}{\omega_n}(\varphi^1_n\alpha^{1i}_n - \varphi^1_n 2\alpha^{2i}_n) \tag{6.55}$$

$$+\sum_{n\neq 0} -\frac{k_n}{\omega_n}[\alpha^{1i}_n\alpha^{1i}_{-n} - \alpha^{2i}_n\alpha^{2i}_{-n}]$$

$$+\sum_{n\neq 0}\sum_{m\neq 0 m\neq -n} -\frac{k_n}{\omega_n}[\varphi^1_n\varphi^1_m\alpha^{1i}_n\alpha^{1i}_m - \varphi^2_n\varphi^2_m\alpha^{2i}_n\alpha^{2i}_m]$$

$$+\sum_{n\neq 0}\sum_{m\neq 0, m\neq -n, m\neq n} -\frac{k_n}{\omega_n}[\varphi^1_n\varphi^2_m\alpha^{1i}_n\alpha^{2i}_m - \varphi^2_n\varphi^1_m\alpha^{2i}_n\alpha^{1i}_m]$$

In the above, we have separated out those terms where the products of the exponential functions φ no longer depend on the σ or τ derivatives. In the $\varphi^1_n\varphi^1_{-n}$ and the $\varphi^2_n\varphi^2_{-n}$ case the products simply disappear leaving the exponential of zero multiplying the appropriate factors. The exponentials of σ and τ resulting from the $\varphi^1\varphi^2$ products with $n = m$ and $n = -m$ cancel between themselves due to the minus sign between the $\varphi^1_n\varphi^2_m$ and $\varphi^2_n\varphi^1_m$ terms and the fact that we are summing over both positive and negative values of n.

We can then integrate:

$$\int d\sigma\acute{x}^-_{bosonic} = [\cos(m\tau)p^i_0 - m\sin(m\tau)x^i_0]\sum_{n\neq 0}\frac{i}{\omega_n}(\varphi^1_n\alpha^{1i}_n - \varphi^1_n 2\alpha^{2i}_n) \tag{6.56}$$

$$-\sum_{n\neq 0}\frac{k_n}{\omega_n}\sigma[\alpha^{1i}_n\alpha^{1i}_{-n} - \alpha^{2i}_n\alpha^{2i}_{-n}]$$

$$+i\sum_{n\neq 0}\sum_{m\neq 0 m\neq -n}\frac{k_n\omega_m + k_m\omega_n}{2\omega_n\omega_m(k_n + k_m)}[\varphi^1_n\varphi^1_m\alpha^{1i}_n\alpha^{1i}_m + \varphi^2_n\varphi^2_m\alpha^{2i}_n\alpha^{2i}_m]$$

$$+i\sum_{n\neq 0}\sum_{m\neq 0, m\neq -n, m\neq n}\frac{k_n\omega_m - k_m\omega_n}{2\omega_n\omega_m(k_n-k_m)}[\varphi_n^1\varphi_m^2\alpha_n^{1i}\alpha_m^{2i}+\varphi_n^2\varphi_m^1\alpha_n^{2i}\alpha_m^{1i}]$$

The same, but lengthier, kind of calculation can be done with respect to the $\dot{x}^i_{bos}$. We know already from the integrability of the 4.20 and 4.21 equations that the terms depending on both σ and τ will have to be same coming from both. Nevertheless it is a confirmation of our calculations to see that they indeed do come out that way in terms of the creation/anihilation operators.

Picking up a few τ dependant terms from the τ integration we can write:

$$x^-_{bos} = \frac{1}{2}[m^2x_0^2 + p_0^2]\tau + [\cos(m\tau)p_0^i - m\sin(m\tau)x_0^i]\sum_{n\neq 0}\frac{i}{\omega_n}(\varphi_n^1\alpha_n^{1i} - \varphi_n^1 2\alpha_n^{2i}) \tag{6.57}$$

$$+\sum_{n\neq 0}\frac{k_n^2}{\omega_n^2}\tau[\alpha_n^{1i}\alpha_{-n}^{1i}+\alpha_n^{2i}\alpha_{-n}^{2i}] - \frac{k_n}{\omega_n}\sigma[\alpha_n^{1i}\alpha_{-n}^{1i}-\alpha_n^{2i}\alpha_{-n}^{2i}] + \frac{2m^2 i}{\omega_n^3}e^{-i\omega_n\tau}[\alpha_n^{1i}\alpha_n^{2i}]$$

$$+i\sum_{n\neq 0}\sum_{m\neq 0 m\neq -n}\frac{k_n\omega_m + k_m\omega_n}{2\omega_n\omega_m(k_n+k_m)}[\varphi_n^1\varphi_m^1\alpha_n^{1i}\alpha_m^{1i}+\varphi_n^2\varphi_m^2\alpha_n^{2i}\alpha_m^{2i}]$$

$$+i\sum_{n\neq 0}\sum_{m\neq 0, m\neq -n, m\neq n}\frac{k_n\omega_m - k_m\omega_n}{2\omega_n\omega_m(k_n-k_m)}[\varphi_n^1\varphi_m^2\alpha_n^{1i}\alpha_m^{2i}+\varphi_n^2\varphi_m^1\alpha_n^{2i}\alpha_m^{1i}]$$

The calculation for the fermionic part follows the same steps, producing the following result:

$$x^-_{fer} = \sum_{n\neq 0}c_n k_n[\cos(m\tau) + i\frac{\omega_n+k_n}{m}\sin(m\tau)][\varphi_n^1(\theta_0^1\bar{\gamma}^-\theta_n^1) + \varphi_n^2(\theta_0^2\bar{\gamma}^-\theta_n^2)] \tag{6.58}$$

$$+c_n k_n[\sin(m\tau) - i\frac{\omega_n+k_n}{m}\cos(m\tau)][\varphi_n^1(\theta_0^2\bar{\gamma}^-\theta_n^1) - \varphi_n^2(\theta_0^1\bar{\gamma}^-\theta_n^2)]$$

$$+ik_n\sigma[\theta_n^1\bar{\gamma}^-\theta_{-n}^1 - \theta_n^1\bar{\gamma}^-\theta_{-n}^1] + [\omega_n - 2c_n^2(\omega_n-k_n)]\tau[\theta_n^1\bar{\gamma}^-\theta_{-n}^1 + \theta_n^1\bar{\gamma}^-\theta_{-n}^1]$$

$$+\sum_{n\neq 0}\sum_{m\neq 0 m\neq -n}c_m c_n\frac{k_m-k_n}{k_m+k_n}\frac{m^2-(\omega_n-k_n)(\omega_m-k_m)}{2m^2}$$

$$[\varphi_n^1\varphi_m^1(\theta_n^1\bar{\gamma}^-\theta_m^1) - \varphi_n^2\varphi_m^2(\theta_n^2\bar{\gamma}^-\theta_m^2)]$$

$$+i\sum_{n\neq 0}\sum_{m\neq 0, m\neq -n, m\neq n}c_m c_n\frac{k_m+k_n}{k_m-k_n}\frac{(\omega_m-\omega_n)-(k_m-k_n)}{2m}$$

$$[\varphi_n^1\varphi_m^2(\theta_n^1\bar{\gamma}^-\theta_m^2) - \varphi_n^2\varphi_m^1(\theta_n^2\bar{\gamma}^-\theta_m^1)]$$

and

$$x^- = x^-_{bos} + x^-_{fer} \tag{6.59}$$

By expressing the $x^i x^i$ in terms of the creation/anihilation operators and combining this result with the above it is actually possible to directly confirm the result from the equation 6.49 using the commutation relations of the operators.

The hope is that, following the Green and Schwarz [15], it may be possible to use the generators of supersymmetries 4.118-4.119 and 4.122-4.123 acting on the $V(h)$ to find the candidate for the vertex operator for the super-partner of the h particle and then acting with them again return to the $V(h)$, proving that it satisfies the supersymmetry requirement for the vertex operator. We have not yet been able to perform that calculation.

Chapter 7

Conclusion

The bulk of this thesis, Chapters 3-5, was a review of the work of Metsaev [8] and Metsaev and Tseytlin [9] on the string theory on the pp-waves. Results of this work are quite impressive. The light-cone gauge action for the type IIB superstring on this background was found and explicitly quantized. The generators of the superalgebra were also explicitly presented. The supergravity spectrum of the theory was derived.

The close relationship between this theory and the theory on the $AdS_5 \times S^5$ makes these results useful in investigations of the AdS/CFT correspondence, even more so after the discovery by Maldacena et al. [10] of the limit on the Conformal Field Theory side corresponding to the Güven-Penrose limit that produces the pp-wave out of $AdS_5 \times S^5$.

The original part of the thesis, Chapter 6, consists mostly of the investigation of the operator x^-, of this theory, which, in the light-cone gauge becomes quadratic in the creation/anihilation operators in the similar way it does in the flat space case. Our original intention of finding the vertex operators for the lowest energy states of the spectrum has not been fully realized. Although we have a plausible candidate for the vertex operator of the scalar component of the graviton we have yet to give definitive proof that it satisfies the requirements for the vertex operator. We have, however, found the exact forms of x^- operator and its two point function, which are necessary for the discussion of Vertex operators and can be useful in other calculations as well.

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