# A STUDY OF NUMERICAL ERRORS IN SIMULATIONS OF THE COSMIC MICROWAVE BACKGROUND POLARIZATION <br> By <br> Bell I.L. Chen <br> B. Sc. (Physics \& Astronomy), University of British Columbia 

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE
in

THE FACULTY OF GRADUATE STUDIES

PHYSICS \& ASTRONOMY

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
October 2000
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#### Abstract

There have been numerous studies of data analysis issues involving temperature anisotropies on the microwave sky, but far less attention paid to the polarization signals. The production of maps and their reduction to power spectra proceeds by choosing a particular way of dividing the sphere into pixels. The Equidistant Cylindrical Projection (ECP) is the geometrically simplest pixelization scheme and might have been sufficient in the early days of Cosmology, when the uncertainties from numerical calculations were ususally overshadowed by experimental errors. But with future satellite missions (MAP and Planck) on the horizon, we need to make sure that the pixelization scheme we choose does not add to the small experimental errors, in order to determine the cosmological parameters as accurately as possible. Numerical errors are a small part of the whole polarization data analysis process but one that is easily dealt with in comparison to other more complicated analyses of polarization. In this thesis, we begin by stating a consistent set of equations for calculating the CMB polarization sky and power spectrum. This will be implemented into an ECP scheme, which allows us to study the numerical errors introduced by various effetcs related to pixelization. We will show that all these errors can be optimally reduced by numerical techniques. And finally, we will apply these techniques to a HEALPix pixelization scheme to obtain the most accurate polarization power spectra.


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## 1. Introduction

The most fundamental aspect of the theoretical background for modern Cosmology was set in the early nineteenth century by Einstein's General Relativity. Together with facts like Olbers' Paradox and Hubble's Law, this theory led to the postulation of the Big Bang model. In simple terms, it states that our current Universe evolved from a primordial sea of photons kept in thermal equilibrium with the surrounding particles through electromagnetic interactions. This allowed the photons to remain in thermal equilibrium until the Universe was cool enough to become neutral (epoch of recombination). After that, matter and radiation evolved separately from each other (see e.g. Peebles, 1993). The Big Bang model was little more than a hypothesis before the discovery of the Cosmic Microwave Background (CMB, Penzias \& Wilson, 1965), which is the remnant of the primordial photons. Further measurements showed that the CMB is a nearly-isotropic radiation field with a Planckian spectrum at $2.725 \pm 0.001 \mathrm{~K}$ (COBE's FIRAS, Mather et al. 1990, 1998) .

There are physical mechanisms that redshift or redistribute the CMB photons after the recombination epoch. This generally creates some small features in the overall isotropy. For example, two photons at some angular separation at the epoch of recombination will be subject to gravitational lensing by the matter distribution along their path, resulting in a different angular separation (e.g. Seljak, 1996). Or the photons might scatter through hot electron gas in a galaxy cluster, resulting in a slightly distorted planckian spectrum (Sunyaev \& Zel'dovich, 1980). And evolving gravitational potential wells can redshift the CMB (Rees \& Sciama, 1968). Underlying these effects is the primordial anisotropy that carries the information of the quantum fluctuations from the Big Bang - the initial conditions that create the structures of the Universe - together with the physics of acoustic oscillations in the primordial plasma (White, Scott, \& Silk, 1994 and references therein).

The anisotropy is around $10^{-5} \mathrm{~K}$, and it was first mapped by $C O B E$ with an angular
resolution of around $10^{\circ}$ (Smoot et al, 1992). Present-day technology allows us to probe the anisotropy with higher angular resolution so that we have maps of parts of the sky with a resolution near $10^{\prime}$ (e.g. MAXIMA and BOOMERANG). In the future, MAP and Planck will extend these maps to the whole sky. In addition to the anisotropy, such technology also allows us to measure the CMB polarization, which is generally at the $\mu K$ level, or 5 to $10 \%$ of the temperature anisotropy.

Polarization is generated by CMB photons Thomson-scattering off the electrons at the last scattering surface just before recombination (Rees, 1968). The Thomson scattering differential cross-section is proportional to $\left|\hat{\epsilon} \cdot \hat{\epsilon}^{\prime}\right|$, the polarization directions before and after scattering (Chandrasekhar, 1960). If a radiation field scatters off an electron, the resulting polarization of the radiation will be largely parallel to the incident one. For an isotropic radiation field, there is no net polarization because the contributions from all directions add to zero. An incident dipole radiation field similarly gives the same result because of symmetry arguments (see e.g. Seljak \& Zaldarriaga, 1998, Hu \& White, 1996). A quadrupolar incident radiation, on the other hand, produces a net polarization pattern because of the excess radiation intensity in the polar lobes. Since polarization is produced by Thomson scattering, most of the polarization was created just before the epoch of recombination. This is known as the last scattering surface. After that free eletrons are too scarce to have much effect. Secondary polarization could also have been produced once the Universe was reionized, but this polarization signal will be quite weak and only on the largest scales (scales corresponding approximately to the horizon at the reionization epoch). The primary constituent of the CMB polarization will be that produced during last scattering. In this way, polarization is a more direct probe than the anisotropies of the physics at last scattering. Polarization is also important because it is an unavoidable prediction of the models for structure formation- hence its detection will provide firm confirmation of the Big Bang paradigm.

The signals returned by a satellite to measure the anisotropy and polarization are in the form of a time series, which can be processed into a full-sky map according to the satellite's rotation pattern, or the scan strategy employed for that particular mission. We expect to obtain most of the cosmological information by turning the map into a power spectrum. In general terms, the map is a linear combination of orthogonal polynomials called spinweighted spherical harmonics. A spin-weighted spherical harmonic of degree $\ell$ characterizes features with an angular scale of around $\frac{\pi}{\ell}$. If a map has many features with this angular size, its power spectrum as a function of $\ell$ will be strongly peaked near that $\ell$. The shape of the power spectrum, $C_{\ell}$, encodes all the essential cosmological information. For example, one of the most notable features of the temperature anisotropy power spectrum is the first acoustic peak produced by the oscillations of the coupled photon-baryon fluid (see Figure 1-1). While gravity tried to compress the fluid, photon pressure resisted the compression, setting up acoustic oscillations. Once the baryons and photons decoupled, those acoustic signatures remained in the CMB as hot and cold spots (e.g. Hu, Sugiyama, \& Silk, 1996).


Figure 1-1: An example of the southern microwave sky for an 1800 square degree patch taken by BOOMERANG. Here we can clearly see cold and hot spots of $1^{\circ}$ size, which indicates that the power spectrum will be strongly peaked at the corresponding multipole $\ell$. This picture is taken from the BOOMERANG homepage (see also de Bernardis et al., 2000).


Figure 1-2: This plot shows some power spectrum estimates from the map in Figure 1-1 together with data from the Maxima-1 experiment (Hanany et al., 2000) and a compilation of previous data by Pierpaoli et al., 2000.

Since various cosmological parameters define the shape of the $C_{\ell}$ spectrum, we would expect to be able to determine their values once we have completely mapped the temperature anisotropy. Things are, unfortunately, not so simple. Some combinations of parameters can produce the same features in the power spectrum that another set of combinations produces. This is commonly referred to as "parameter degeneracy". This degeneracy is breakable if we have some additional information, independent of $C_{\ell}$. We can use the CMB polarization power spectra just for this task. Except for its much lower amplitude, the polarization spectra contain just as much information as the anisotropy power spectrum. In fact, polarization provides a much cleaner snapshot of the last scattering than the anisotropy because fewer physical mechanisms can alter polarization after $z \sim 1000$ (redshift, $z$, at last scattering), thus constraining the parameters more precisely.

In the CMB polarization convention, polarization is separated according to its parity, $(-1)^{\ell}$ : that with even parity $(+1)$ is referred to as E-type and that with odd parity $(-1)$ as the B-type (analogous with the E and B fields of electromagnetism, Seljak \& Zaldarriaga, 1998). In additionl to these two, there are also cross-correlations of $E$ and $B$ with the temperature, which are usually referred to as T-E and T-B cross-correlations. The T-B cross-correlation is zero because the functions that generate $T$ and $B$ have opposite parities (Kamionkowski et al., 1996). Thus there are four non-zero power spectra we can obtain: temperature anisotropy (T), E- and B-type polarization, and the T-E cross-correlation. Standard theories also suggest that the amplitude of the $B$-type spectrum may be quite small, in agreement with the current limits of tensor perturbations which could produce B-type polarization (Zibin et al., 1999). Of the three polarization-related spectra, the T-E cross-correlation will be the most easily measurable because of its higher overall amplitude. Figure 1-3 shows an example of these four power spectra (produced by CMBFAST).


Figure 1-3: Examples of the T, E, B, and T-E power spectra produced by CMBFAST.

In a more realistic scenario, the map will not just contain the CMB alone. There will be the Galactic and extragalactic foregrounds, perhaps Zodiacal light, and of course some instrumental noise. In the case of CMB polarization, we need to worry about Galactic synchrotron radiation (Smoot, 1999), polarized dust emission (Lazarian \& Prunet, 1999), and Faraday rotation induced by the Galactic magnetic field (Giovannini, 1995), among others. It is not an easy task to remove all these to obtain the underlying CMB. Before the age of space missions, most CMB measurements only focussed on small patches of the sky. There are some advantages to this, of course. For example, we can choose a field of view away from dusty regions of the Galaxy, and the subsequent data analysis to obtain the power spectrum will only involve a simple Fast Fourier Transform (FFT). Staying away from the Galactic plane means we only have to deal with instrumental (and usually atmospheric) noise, which can be modelled independently, and the FFT is reasonably fast and accurate (for estimating the power spectrum). The most obvious drawback to only
focussing on a small piece of the sky is that we are limited to probing the part of the power spectrum with upper $\ell$ set by angular resolution and lower $\ell$ set by the map's dimension. Future space missions are full-sky surveys, so we can probe the whole CMB spectrum under the constraint of how well we can remove the foregrounds and how well we can calculate the $C_{\ell}$ 's from the spin-weighted harmonics. The goal of this thesis is to study the numerical effects associated with the latter issue.

In section 2, we will briefly summarize the mathematics for simulating the anisotropy sky and focus on the polarization and T-E cross-correlation. We will also describe how we can implement a simple HEALPix algorithm from an Equidistant Cylindrical Projection based on their symmetry properties. In section 3, we will discuss how we obtain the CMB spectra from the full-sky maps. In section 4, we will describe in detail the method we derived to remove the numerical errors caused by pixelization, and how well it can do so. And finally, section 5 lists some of the methods that further reduce numerical errors.

## 2. Simulation of Temperature and Polarization Maps

The goal of this section is to establish a numerical routine to calculate the CMB temperature anisotropy and polarization maps from some given power spectra. We will also summarize a few geometrical characteristics of the two pixelization methods we will consider. This routine, together with the inversion we will establish in the next section, will form a basis for our discussion of the numerical uncertainties which arise from pixelization.

Making temperature or polarization maps is based on the same principle, known as orthogonal polynomial expansion, and the polynomials in question are strongly related to each other. We will begin with the simpler of the two: making temperature anisotropy maps. The anisotropies in CMB ( $\Delta T$ or sometimes $\frac{\Delta T}{T}$ ) are scalar quantities, in other words, they remain invariant on a sphere (the sky) under rotation. Thus, naturally, we can calculate $\frac{\Delta T}{T}$ with an expansion in spherical harmonics for a given $C_{\ell}$ :

$$
\begin{equation*}
\frac{\Delta T}{T}(\theta, \phi)=\sum_{\ell=2}^{\ell_{\max }} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \phi) \tag{1}
\end{equation*}
$$

The $Y_{\ell m}$ 's are the spherical harmonics defined in terms of the Legendre polynomials, $P_{\ell m}$ :

$$
\begin{equation*}
Y_{\ell m}(\theta, \phi)=\sqrt{\frac{2 \ell+1}{4 \pi} \frac{(\ell-m)!}{(\ell+m)!}} P_{\ell m}(\cos \theta) e^{i m \phi} \tag{2}
\end{equation*}
$$

The $a_{\ell m}$ coefficients are related to $C_{\ell}$ (variance of $a_{\ell m}$ ) by

$$
\begin{equation*}
C_{\ell}=\frac{1}{2 \ell+1} \sum_{m=-\ell}^{\ell}\left\langle a_{\ell m}^{2}\right\rangle \tag{3}
\end{equation*}
$$

where the angular brackets denote an expectation value. In equation (1), the lower summation limit for $\ell$ is 2 because we do not include the CMB blackbody temperature, $\ell=0$ and its dipole term, $\ell=1$ for the anisotropy map. The upper limit is some $\ell_{\max }$ set by computational limitations. Generally this makes the expansion an approximation, since we lose the information contained in modes above $\ell_{\max }$. In our case, however, the primordial $C_{\ell}$ spectrum is usually damped to zero around $\ell \sim 3000$ (except for some eccentric
cosmological models). Thus if our $\ell_{\max }$ is high enough, the expansion will be essentially exact.

A direct implementation of equation (1) makes the expansion an extremely time-consuming process. There have been a few suggestions for speeding up the computation, the most notable being the separation of variables (Muciaccia, Natoli, and Vittorio, 1997, hereafter MNV; also in Mohlenkamp, 1997). In summary, the summations

$$
\sum_{\ell=2}^{\ell_{\max }} \sum_{m=-\ell}^{\ell} \text { and } \sum_{m=-\ell_{\max }}^{\ell_{\max }} \sum_{\ell=|m|}^{\ell_{\max }}
$$

cover the same part of the $\ell m$ plane; therefore, equation (1) is equivalent to

$$
\frac{\Delta T}{T}(\theta, \phi)=\sum_{m=-\ell_{\max }}^{\ell_{\max }} \sum_{\ell=|m|}^{\ell_{\max }} a_{\ell m} Y_{\ell m}(\theta, \phi)
$$

Together with equation (2), we can factor out the only $\phi$-dependent part, $e^{i m \phi}$, so that

$$
\begin{equation*}
\frac{\Delta T}{T}(\theta, \phi)=\sum_{m=-\ell_{\max }}^{\ell_{\max }} e^{i m \phi} b_{m}(\theta) \tag{4}
\end{equation*}
$$

where

$$
b_{m}(\theta)=\sum_{\ell=|m|}^{\ell_{\max }} \sqrt{\frac{2 \ell+1}{4 \pi} \frac{(\ell-m)!}{(\ell+m)!}} a_{\ell m} P_{\ell m}(\cos \theta)
$$

MNV suggested using an FFT on the $e^{i m \phi}$ part since $b_{m}(\theta)$ 's are essentially coefficients of $e^{i m \phi}$. In this way, the computing time for equation (4) is reduced from $N_{\phi}^{2}$ to $N_{\phi} \log N_{\phi}$, and the whole routine requires computing time proportional to $N_{\phi}^{3} \log N_{\phi}$ (usually stated as $N_{\phi}^{3}$ since $\log N_{\phi}$ for any reasonable resolution is small). This FFT method requires that the pixels be equally spaced in the $\phi$ direction, which every reasonable pixelization scheme does. The only place where this might not be applicable is when we want to include some points along each latitude to find a better estimate of an integral for inversion (sections 3 and 5 ). In such cases, we might choose to sacrifice speed for accuracy, since we want the inversion to be as accurate as possible. And we must also note that the most time consuming part
of the algorithm is the generation of spherical harmonics and its subsequent calculations, so not performing an FFT really does not lengthen the process too much (although it is still a noticeable delay). Figure 2-1 is an example of the anisotropy map we have simulated using the model in Figure 1-3.


Figure 2-1: A simulated map of CMB temperature anisotropy for a particular cold dark matter model normalized to $C O B E$. Here we have generated modes up to $\ell=1500$.

### 2.1 Spin-Two Spherical Harmonics Transform

The CMB sky is fully characterized by its anisotropy and polarization, which can be described in terms of the four Stokes Parameters, $I, Q, U$, and $V$. The parameter $I$ gives the radiation intensity, which is the anisotropy in this context. Thomson scattering does not produce circular polarization, so $V$ is expected to be zero. The other two parameters that describe polarization are spin-2 quantities, i.e., they transform under a rotation through an angle $\alpha$ by

$$
\begin{equation*}
(Q \pm i U)^{\prime}=e^{ \pm i 2 \alpha}(Q \pm i U) \tag{5}
\end{equation*}
$$

So this naturally demands the spin- 2 spherical harmonics, ${ }_{ \pm 2} Y_{\ell m}$, if we want to calculate the $Q$ and $U$ parameters on the sky. These functions are also defined in terms of the Legendre polynomials:

$$
\begin{equation*}
{ }_{ \pm 2} Y_{\ell m}(\theta, \phi)=\left(W_{\ell m}(\theta) \pm i X_{\ell m}(\theta)\right) e^{ \pm i m \phi} \tag{6}
\end{equation*}
$$

with $W_{\ell m}(\theta)$ and $X_{\ell m}(\theta)$ defined below (Kosowsky, 1999; Kamionkowski, 1996; Zaldarriaga, 1997):

$$
\begin{align*}
W_{\ell m} & =-\lambda_{\ell m}\left(\left(\frac{\ell-|m|^{2}}{\sin ^{2} \theta}+\frac{\ell(\ell-1)}{2}\right) P_{\ell m}(\cos \theta)-(\ell+|m|) \frac{\cos \theta}{\sin ^{2} \theta} P_{\ell-1, m}(\cos \theta)\right) \\
X_{\ell m} & =-\lambda_{\ell m} \frac{|m|}{\sin ^{2} \theta}\left((\ell-1) \cos \theta P_{\ell m}(\cos \theta)-(\ell+|m|) P_{\ell-1, m}(\cos \theta)\right)  \tag{7}\\
\lambda_{\ell m} & =\sqrt{8} \sqrt{\frac{2 \ell+1}{4 \pi} \frac{(\ell-|m|)!}{(\ell+|m|)!} \frac{(\ell-2)!}{(\ell+2)!}} .
\end{align*}
$$

Equations (6) and (7) say that $e^{ \pm i m \phi}$ is the only part of the function that depends on $\phi$, so the separation of variables that applies to the spherical harmonics also applies here. In other words, we calculate

$$
\begin{equation*}
Q \pm i U=\sum_{m=-\ell_{\max }}^{\ell_{\max }} \sum_{\ell=|m|}^{\ell_{\max }} a_{\ell m \pm 2} Y_{\ell m} \tag{8}
\end{equation*}
$$

Since the $Q$ and $U$ parameters must necessarily be real (Stokes parameters are time averages of physical quantities), we must be able to write equation (8) in terms of real functions only. We do this for computational simplicity and possibly efficiency on some computers. So let us define another set of coefficients in place of the ${ }_{ \pm 2} a_{\ell m}$ (Zaldarriaga, 1997):

$$
\begin{aligned}
& a_{E, \ell m}=-\frac{1}{2}\left(+2 a_{\ell m}+{ }_{-2} a_{\ell m}\right) \\
& a_{B, \ell m}=-\frac{1}{2 i}\left(+2 a_{\ell m}-{ }_{-2} a_{\ell m}\right) .
\end{aligned}
$$

Both of these coefficients can be calculated for a given cosmological model, and they are more often used in the literature because of the parity properties associated with the E
and $B$ mode polarization. And although in principle we can fully expand equation (8), i.e.,

$$
Q \pm i U=\sum_{m=-\ell_{\max }}^{\ell_{\max }} \sum_{\ell=|m|}^{\ell_{\max }}\left(a_{E, \ell m} \pm a_{B, \ell m}\right)\left(W_{\ell m}(\theta) \pm i X_{\ell m}(\theta)\right) e^{ \pm i m \phi}
$$

we can simplify this a little by assigning $\cos m \phi$ to $m \geq 0$ and $\sin m \phi$ to $m<0$. This is a valid operation because $\left(a_{E} \pm a_{B}\right)\left(W_{\ell m} \pm i X_{\ell m}\right)$ is always real. This is the same process as changing the complex Fourier series into a series of sines and cosines. Therefore, we can write

$$
\begin{align*}
& Q=\left(\sum_{m=0}^{\ell_{\max }} \cos m \phi+\sum_{m=-\ell_{\max }}^{-1} \sin m \phi\right) \sum_{\ell=|m|}^{\ell_{\max }}\left(a_{E} W_{\ell m}-a_{B} X_{\ell m}\right) \\
& U=\left(\sum_{m=0}^{\ell_{\max }} \cos m \phi+\sum_{m=-\ell_{\max }}^{-1} \sin m \phi\right) \sum_{\ell=|m|}^{\ell_{\max }}\left(a_{E} X_{\ell m}+a_{B} W_{\ell m}\right) \tag{9}
\end{align*}
$$

where every term is explicitly real. We also tried implementing the fully expanded $Q$ and $U$ of equation (8). The result gives the same map but takes approximately twice the computing time of equation (9). This is because the latter contains twice as many arithmetic operations.


Figure 2-2: Amplitude of $Q$ for the standard cold dark matter model in Figure 1-3. We have reduced $\ell_{\text {max }}$ to 200 for this map so that some features become more apparent.


Figure 2-3: Amplitude of $U$ produced by the same simulation for the same model at $\ell_{\max }=200$.

Figures 2-2 and 2-3 show polarization in the form of the two Stokes parameters. An alternative way is to show maps of polarization amplitude, $P$, and angle, $\alpha$, which are defined as (e.g. Kosowsky, 1996)

$$
P=\sqrt{Q^{2}+U^{2}}
$$

and

$$
\alpha=\frac{1}{2} \tan ^{-1} \frac{U}{Q}
$$

We do not show these maps here because both contain the same amount of information, and our eyes will not see any significant difference between the two sets.

The last of the four maps relevant in the analysis of CMB data is the T-E cross-correlation map. In the case of Gaussian statistics (what we have assumed so far for all the temperature and polarization maps), making realizations of temperature and polarization is easy since

$$
a_{\ell m}^{\mathrm{T}}=r_{1} \sqrt{C_{\ell}^{\mathrm{T}}}
$$

and

$$
a_{\ell m}^{\mathrm{E}}=r_{2} \sqrt{C_{\ell}^{\mathrm{E}}}
$$

for any Gaussian random number $r_{1}$ and $r_{2}$ with unit variance. However, the presence of a T-E cross-correlation means that $r_{1}$ and $r_{2}$ are constrained such that

$$
\left\langle a_{\ell m}^{\mathrm{T} *} a_{\ell m}^{\mathrm{E}}\right\rangle=C_{\ell}^{\mathrm{X}}
$$

where " X " denotes cross-correlation. Thus, independently-chosen T and E realizations such as those in Figures $2-1,2-2$, and $2-3$ will not give a T-E map predicted by the same cosmological model. Since cross-correlation is a scalar quantity, making a T-E map requires only the spherical harmonics used for the anisotropy map. We have ignored the cross-correlation here, since there is no effect on numerical error analysis.

Up to this point we have only discussed the separation of variables method, also known as the summation exchange, into the routine. There are also other symmetries in the Legendre polynomials that we can exploit to save additional computing time. One of them is a symmetry in $m$ (Abramowitz \& Stegun, 1972):

$$
\begin{equation*}
P_{\ell,+m}=P_{\ell,-m} \tag{10}
\end{equation*}
$$

This saves quite a significant amount of time (close to a factor of two), since generating the Legendre polynomials is the slowest part of the algorithm. Another symmetry is (MNV, 1997; Turok \& Crittenden, 1998)

$$
P_{\ell m}(\cos \theta)= \begin{cases}+P_{\ell m}(\cos (\pi-\theta)), & \text { if } \ell+m \text { is even } ;  \tag{11}\\ -P_{\ell m}(\cos (\pi-\theta)), & \text { if } \ell+m \text { is odd }\end{cases}
$$

This means we only have to calculate the Legendre polynomials above the equator. Those below just differ by a minus sign for odd $\ell+m$. These symmetries can be directly implemented into equation (2). But because of the more complicated $\theta$ and $P_{\ell m}$ dependence of $W_{\ell m}$ and $X_{\ell m}$ (equation (7)), this gives us another exercise in keeping track of minus signs.

### 2.2 Recurrence Relation of Legendre Polynomials

Now we come to the question of calculating the Legendre polynomials. This is usually done using recurrence relations. There are several of these recurrence relations available, but most of them involving $m$ contain a factor of $\left(1-x^{2}\right)^{-\frac{1}{2}}$ with $x=\cos \theta$, which is numerically unfavourable because of the singularity (Press et al., Chapter 6.8). For our calculations, we adopt the relation

$$
\begin{aligned}
Y_{\ell}^{m} & =(2 \ell-1) x Y_{\ell-1}^{m}-\sqrt{\frac{(\ell+m-1)(\ell-m-1)}{(\ell+m)(\ell-m)}} Y_{\ell-2}^{m} \\
Y_{m-1}^{m-1} & =\left(1-x^{2}\right)^{\frac{1}{2}} Y_{m}^{m}\left(1-\frac{1}{2 \ell}\right)^{\frac{1}{2}} \\
Y_{\ell}^{m} & =\sqrt{2 \ell-1} x Y_{\ell-1}^{m} .
\end{aligned}
$$

This recurrence relation is the one mentioned in Zaldarriaga (1997) and seems to be the most accurate and fastest subroutine available (downloadable at the CMBFAST homepage). In section 4.1, we will show that the recurrence relation is stable when we discuss numerical uncertainties.

For the spin-weighted harmonics, we simply evaluate equation (7) for each $\ell$ and $m$. However, there are places where $\sin ^{2} \theta$ in the denominator is near zero. We can avoid this by assigning the functions' values at these points. Specifically,

$$
\lim _{\theta \rightarrow 0} \frac{P_{\ell m}(\cos \theta)}{\sin ^{2} \theta}=\lim _{\theta \rightarrow 0} \frac{-\sin \theta P_{\ell m}^{\prime}(\cos \theta)}{2 \sin \theta \cos \theta}=-\frac{1}{2} P_{\ell m}^{\prime}(1) .
$$

$P_{\ell m}^{\prime}(1)$ is only non-zero for $m=2$ because

$$
P_{\ell m}^{\prime}(x)=-2 x\left(1-x^{2}\right)^{\frac{|m|}{2}-1}\left(\frac{d}{d x}\right)^{|m|} P_{\ell}(x)+\left(1-x^{2}\right)^{\frac{|m|}{2}}\left(\frac{d}{d x}\right)^{|m|+1} P_{\ell}(x)
$$

and the $\left(1-x^{2}\right)$ that makes $P_{\ell m}^{\prime}(x)$ zero for $x=1$ disappears from the first term when $|m|=2$. So

$$
\lim _{\theta \rightarrow 0} \frac{P_{\ell m}(\cos \theta)}{\sin ^{2} \theta}= \begin{cases}-\frac{1}{4}(\ell-1) \ell(\ell+1)(\ell+2), & |m|=2 \\ 0, & \text { all others } .\end{cases}
$$

If we apply this to equations (7), we find, for all $\ell$, the values of $W_{\ell 2}(\theta)$ and $X_{\ell 2}(\theta)$ where $\sin \theta$ is zero:

$$
\begin{aligned}
\left.W_{\ell 2}(\theta)\right|_{\sin \theta=0} & =-\sqrt{2} \sqrt{\frac{2 \ell+1}{4 \pi}} ; \\
\left.X_{\ell 2}(\theta)\right|_{\sin \theta=0} & =+\sqrt{2} \sqrt{\frac{2 \ell+1}{4 \pi}} .
\end{aligned}
$$

So how important is this? In Figure 2-4, we plotted $\Delta W_{\ell 2}\left(\sin \theta=10^{-10}\right)$ which is defined as

$$
W_{\ell 2}\left(\sin \theta=10^{-10}\right)-W_{\ell 2}^{\text {analytic }}(\sin \theta=0)
$$

We also plotted $\Delta W_{\ell 2}$ at two other values of $\sin \theta$ for reasons that will become apparent shortly. Since $10^{-10}$ is quite small, naively one would explain the difference as the numerical
inaccuracy as the result of dividing by something close to zero. This is why we also plotted the other two curves in Figure 2-4: as $\sin \theta$ approaches zero, $\Delta W_{\ell 2}$ also approaches zero asymptotically. So what we see in $\Delta W_{\ell 2}\left(\sin \theta=10^{-10}\right)$ is not a numerical error, but a real difference. We can further justify this statement by the limits of double-precision floating points $-1.7 \mathrm{E} \pm 308$. So $10^{-10}$ is well within this range. The analytic formula is useful when we need to calculate the value of the pixels precisely at the poles ( $\sin \theta=0$ exactly). And there are times when this is necessary.


Figure 2-4: A plot showing $W_{\ell 2}$ at various values of $\sin \theta$. This also demonstrates the recurrence relation's ability to accurately generate $W_{\ell m}$ and $X_{\iota m}$ at very small angles

### 2.3 Pixelization

Throughout this section we have always been using the separation of variables method. This is because for large simulations, the direct implementation becomes too slow to be useful. We have also seen that for this summation exchange to work properly, the pixelization scheme has to contain equal-latitude rings - rings at constant $\phi$ (Crittenden \& Turok, 1998). This is an essential feature for fast spherical harmonic transforms. There
are many pixelization schemes that have such a feature, and in this thesis we will focus on two of them: the Equidistant Cylindrical Projection (ECP), because it is simple and obviously flawed, and the HEALPix pixelization, because it has been commonly adapted as the standard spherical harmonics analysis package for its numerous advantages. Other pixelization schemes that have been proposed include the quadrilateralized spherical cube (Chan \& O'Neill, 1975), which was used in the COBE data analysis; the icosahedronbased method (Tegmark, 1996), which has a certain degree of spherical symmetry (not just azimuthal); and IGLOO pixelization (Crittenden \& Turok, 1998), which is rather similar to HEALPix in terms of geometrical properties. These share some, but not all of the advantages of HEALPix.

To implement this routine into an ECP pixelization is a trivial matter because $\frac{\Delta T}{T}(\theta, \phi)$ will simply form a rectangular array. The $\theta$ coordinate is incremented evenly from 0 to $\pi$ and the $\phi$ coordinate evenly from 0 to $2 \pi$. When we project this array onto a sphere, we see that the pixel concentration is much higher near polar areas than near the equator. This introduces a few problems: pixel shape distortion, unequal pixel area, and unnecessarilywasted computating time. In section 4 , we will focus on these problems directly and find methods to remedy them as much as possible.

Of the three problems we stated for the ECP pixelization, the HEALPix pixelization avoids the latter two while having only minimal pixel distortion (Gorski, 2000). At base resolution, it has 12 equal-area pixels: 4 around the north pole, 4 along the equator, and 4 around the south pole. Each step up in resolution divides each pixel into 4 equal-area pixels. The hierarchy allows for an efficient way to locate neighboring pixels, which is essential if a quick analysis demands only a lower resolution map. A more detailed description of HEALPix can be found at the HEALPix home page at http://www.eso.org/~kgorski/healpix/. The webpage provides a numerical implementation of the HEALPix pixelization scheme in Fortran 90 . Here we will show that we can modify the ECP code only sightly so that it
can produce maps in the HEALPix scheme (i.e. we use the HEALPix algorithm in our code).

HEALPix is a curvilinear partition of a sphere into equal-area quadrilaterals with varying shapes. The ECP is a simple-minded pixelization that happens to have a few nice geometrical properties which are essential for fast spherical harmonic transform and for exploiting the various symmetries of the spherical harmonics. On first observation, the pixel shapes look very different for the two. However, we must note that unless we perform an exact integration in the inversion (see section 3), all that the program "sees" are the pixel centers, whose values are each taken as an approximation for the entire pixel. So in both pixelizations, we have points that lie on rings of constant latitude. The only difference is that HEALPix tries to conserve pixel area by having fewer and fewer points near polar caps (see Figure 2-5). We can integrate this fact into the ECP by calculating only the necessary $e^{i m \phi}$ around the poles. For example, Figure $2-5$ shows the two pixelizations at the same resolution: $22.5^{\circ}$, with HEALPix at its first partition (each base pixel divided into four) and the ECP being an $8 \times 16$ rectangular array ( 8 divisions along equator). In the context of modifying the ECP, all we need to do is evaluate $e^{i m \phi}$ at the four locations in the first HEALPix ring, the eight in the second ring, and so on, instead of the same number of $e^{i m \phi}$ for every latitude (ring). This is only a matter of coding. In addition, the $e^{i m \phi}$ 's need not be calculated whenever we start with a different latitude. The $\phi$ angles are the same for the northern and southern hemispheres, and we can pre-compute all the $e^{i m \phi}$ 's as soon as we decide what resolution we choose. So all the symmetry properties are applicable to this pixelization.

This modification results in a slightly faster and more accurate algorithm because of the fewer number of polar pixels. The issue of speed depends very much on the computing facility the routines are performed, but generally the HEALPix routine requires about $10 \%$ less time than the ECP routine. The speed is not so dramatically different because we
still have to calculate the same number of $\theta$-intervals, which are the most time-consuming. The most notable improvement is in the accuracy of inversion. We performed an inversion for some maps using the resolution seen in Figure 2-5, the resulting residual (difference between obtained spectrum and input) for the HEALPix routine was 100 times smaller then that for the ECP. Thus, if we can reduce the numerical error found in the ECP routine substantially, the numerical error in the HEALPix routine will become even smaller.


Figure 2-5: This figure shows that all we need to do is modify the $\phi$-summation a little in order to implement the HEALPix pixelization. The one to the left is HEALPix, and to the right is ECP.

## 3. Inverse Spin-Two Spherical Harmonics Transform

Now we turn our attention to an equally important issue - the inversion to the power spectrum. CMB anisotropy data have been available for almost a decade, so not surprisingly, the spherical harmonics inversion has been investigated several times, the most notable being the Quadrilateralized Spherical Cube of $C O B E$, the fast ECP inversion utilizing FFT, and HEALPix. In this section we will mainly focus on the inversion of CMB polarization data, in other words, inversion involving the spin- 2 spherical harmonics. This is an otherwise trivial matter if not for the various sign conventions seen throughout the literature.

In section 2 we have written down an explicitly real expression for calculating the $Q$ and $U$ Stokes parameters by defining a set of coefficients from ${ }_{ \pm 2} a_{\ell m}$ that can be predicted by various cosmological models. Our goal now is to do the same for the $a_{E}$ and $a_{B}$ coefficients.

The functions ${ }_{ \pm 2} Y_{\ell m}$ 's are mutually orthonormal on a spherical surface,

$$
\begin{equation*}
\iint_{S}{ }^{ \pm} Y_{\ell m \pm 2}^{*} Y_{\ell^{\prime} m^{\prime}} \sin \theta d \theta d \phi=\delta_{\ell \ell^{\prime}} \delta_{m m^{\prime}} \tag{13}
\end{equation*}
$$

We apply this to equation (8) and solve for the coefficients ${ }_{ \pm 2} a_{\ell m}$ :

$$
{ }_{ \pm 2} a_{\ell m}=\iint(Q \pm i U)(W \pm i X)^{*}\left(e^{ \pm i m \phi}\right)^{*} \sin \theta d \theta d \phi
$$

For numerical calculations, we must convert the integral into a summation with a certain "grid size" (resolution) and consider $\ell$ up to some limit of $\ell_{\text {max }}$. In general, if we are inverting a numerically simulated map, the $\ell_{\max }$ will be set by the $\ell_{\max }$ of the map since the simulated map itself contains no information above that. For the inversion of an actual data set, $\ell_{\text {max }}$ is limited only by the angular resolution of the experiment. In complex notation, we perform

$$
\begin{equation*}
{ }_{ \pm 2} a_{\ell m}=\sum_{j=1}^{N_{\theta}} \sin \theta_{j}(W \pm i X)_{j}^{*}\left(\sum_{i=1}^{N_{\phi}}(Q \pm i U)_{i j}\left(e^{ \pm i m \phi_{i}}\right)^{*} \Delta \phi\right) \Delta \theta \tag{14}
\end{equation*}
$$

The $(W \pm i X)_{j}^{*}$ and $e^{i m \phi}$ are usually approximated by their values at the pixel center, but at times we might need to integrate over the entire pixel or perform a better approximation instead of just using its central value. This is the only approximation necessary in the mapping and inversion processes. Equations (1) and (8) are still exact whether $\ell_{\max }$ is finite or infinite.

After some algebra we obtain $a_{E}$ and $a_{B}$ from equation (13). And as before, we choose to assign $\cos m \phi$ to $m \geq 0$ and $\sin m \phi$ to $m<0$ instead of fully expanding equation (14):

$$
\begin{align*}
& a_{E}=\sum_{j=1}^{N_{\theta}} \sin \theta_{j} \sum_{i=1}^{N_{\phi}}\left(Q_{i j} W_{j}-U_{i j} X_{j}\right) \begin{cases}\cos m \phi, & m \geq 0 \\
\sin m \phi, & m<0\end{cases}  \tag{15}\\
& a_{B}=\sum_{j=1}^{N_{\theta}} \sin \theta_{j} \sum_{i=1}^{N_{\phi}}\left(Q_{i j} X_{j}+U_{i j} X_{j}\right) \begin{cases}\cos m \phi, & m \geq 0 \\
\sin m \phi, & m<0\end{cases}
\end{align*}
$$

The expressions above are written in variable-separated form in which the $\phi$-summation is outside of the $\theta$-summation. So again we can implement this with the ECP and HEALPix pixelizations.

## 4. Numerical Errors

The purpose of all these expansion and inversion routines is that we need to know how much numerical error is contained in the obtained spectra, T, E, B, and T-E, in the absence of experimental (instrumental) errors. In the hypothetical case where the inversion routine leaves absolutely no numerical error, the only part we need to worry about is the recurrence relation, which gives errors independent of the pixelization scheme. So for all purposes, if we can reduce the overall numerical error to a magnitude comparable to that made by the recurrence relation, our work is done. That is why we test the recurrence relation first in this section.

### 4.1 Stability of the Recurrence Relation of Legendre Polynomials

We can test the stability of the $P_{\ell m}$ 's recurrence relation by looking at their values at certain $\theta$ where analytic expressions exist. For example, when $\cos \theta$ is 0 (Abramowitz \& Stegun, 1972),

$$
\begin{equation*}
P_{\ell m}(0)=\frac{2^{m}}{\sqrt{\pi}} \cos \left((\ell+m) \frac{\pi}{2}\right) \frac{\Gamma\left(\frac{\ell}{2}+\frac{m}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{\ell}{2}-\frac{m}{2}+1\right)} \tag{16}
\end{equation*}
$$

Since we start the recurrence relation (equation (12)) at $P_{\ell \ell}$, the error will be the largest for $P_{\ell 0}$. Figure 4-1 is a plot of $P_{\ell 0}(0)$ for even $\ell$ up to 3000 which we generated, overlaid with the values given by equation (16). Those for odd $\ell$ are zero due to the cosine term. We intentionally add 0.01 to those produced by equation (16) because the two differ only by a small amount. (We do not want the reader to have the impression that we forgot to plot one of them.) Figure 4-2 is the difference of the two. The increasing error is expected because each $P_{\ell 0}(0)$ is the result of $2 \ell+1$ recursions. Other than this fact, the errors are small and randomly distributed near $10^{-7}$. The spin- 2 harmonics are computed from the Legendre polynomials, so the uncertainty is of the same order of magnitude. For example, the $W_{\ell 1}$ and $X_{\ell 1}$ generated recursively and through analytical expressions (equations (7) and (16)) behave in the same way as the $P_{\ell 0}$ we have discussed. We do not want to fill the
page with identical figures, so they are shrunk and put into Figure $4-3$. We chose $m=1$ because $X_{\ell 0}=0$ for all $m$, which cannot tell us anything about errors. The error in $X_{\ell 1}$ is much smaller because the non-zero term in $X_{\ell 1}$ is a factor of $\ell$ less than the non-zero term in $W_{\ell 1}$ (refer back to equation (7)). The routine Gammln( x ) of Numerical Recipes (Press et al., Chapter 6.1) we used to calculate Gamma functions is based on an approximation derived by Lanczos in 1964, which looks like

$$
\Gamma(z+1)=\sqrt{2 \pi}\left(z+\gamma+\frac{1}{2}\right)^{z+\frac{1}{2}} e^{-\left(z+\gamma+\frac{1}{2}\right)}\left(C_{0}+\sum_{i=1}^{N} \frac{C_{i}}{z+i}\right)
$$

for certain integer $\gamma$ and $N$ and coefficients $C_{i}$. The routine we chose includes up to $C_{6}$ and gives errors less than $2 \times 10^{-10}$, which is sufficient for our test where errors are near $10^{-7}$. In Figures 4-2 and 4-3, there are some features around $\ell \sim 1400,2100,2700 \ldots$ etc. We believe this pattern appears in the recurrence relation since Press et al. claims that the accuracy of $\operatorname{Gammln}(\mathrm{x})$ is within $2 \times 10^{-10}$. This pattern does not affect the overall accuracy in any way because the magnitude is so low.


Figure 4-1: $P_{\ell 0}(0)$ 's for even $\ell$ up to 3000 . The blue, dashed, lines are from equation (16) shifted vertically by 0.01 , and the black lines are from the recurrence relation.


Figure 4-2: The difference between equation (15) and the $P_{\ell 0}(0)$ 's generated by the recurrence relation (without that constant we added).


Figure 4-3: The same plot as Figure 4-2 but for the functions $W_{\ell 1}$ and $X_{\ell 1}$.

### 4.2 Orthogonality Issues Related to Pixelization

Since the errors from the recurrence relation define a limit on how accurate the inversion can be, we can set our goal to reducing the other numerical errors to a comparable level. One of them comes from the fact that we only consider up to a finite $\ell_{\text {max }}$ of the orthogonal polynomials. This issue is described in detail in Crittenden \& Turok (1998) for spherical harmonics. In summary, by imposing a pixelization scheme on the sky, we break the orthogonality of spherical harmonics. A degree- $\ell$ spherical harmonic becomes an approximate linear combination of the rest of the spherical harmonics of the set $\left\{0, \ell_{\max }\right\}$. And in the process of inversion, the coefficient of each mode picks up something extra from this "linear combination". This effect on spherical harmonics can be efficiently reduced by fine resolution, since generating the spherical harmonics and summing over all $\ell$-modes have become a relatively fast procedure. After this all we see are the errors from the recurrence relation. Thus, at high enough resolution, the orthogonality issue for spherical harmonics is of little concern. To correct for this effect at low resolution, we can apply the unbiased $C_{\ell}$ spectrum estimator shown in Crittenden \& Turok (1998), which depends only on the pixelization scheme:

$$
\text { estimated } C_{\ell}=\mathrm{M}_{\ell \ell^{\prime}} C_{\ell^{\prime}}
$$

where

$$
\mathrm{M}_{\ell \ell^{\prime}}=\frac{1}{2 \ell+1} \sum_{m m^{\prime}}\left(\sum_{P} A_{P} W_{P, \ell m}^{*} W_{P, \ell m}\right)^{-2} \sum_{P} A_{P} W_{P, \ell m}^{*} W_{P, \ell^{\prime} m^{\prime}} \sum_{Q} A_{Q} W_{Q, \ell m}^{*} W_{Q, \ell^{\prime} m^{\prime}}
$$

Letter $A$ denotes pixel area, $P$ and $Q$ are pixel indices, and the $W_{\text {pixel, } \ell m}$ here is the window function of the pixel ( $\frac{1}{A}$ within the pixel and 0 elsewhere) integrated with $Y_{\ell m}$ over the entire pixel. This removes the majority of the leakage between multipoles, but it ignores the off-diagonal correlations between estimated $a_{\ell m}$. However, because of this fact, calculating $\mathrm{M}_{\ell \ell^{\prime}}$ is relatively fast (Crittenden \& Turok, 1998).

The above issue could have been called something along the lines of "orthogonality-breaking as a result of pixelization", but the word "leakage" has become the description of such an effect. We can expect to see this same kind of error in the $Q$ and $U$ maps. Equations (9) and (15) say that $a_{E}$ and $a_{B}$ together determine $Q$ and $U$, which in turn are required to calculate $a_{E}$ and $a_{B}$, so there are effectively twice as many multipoles to leak into each other. Combined with the fact that both $W_{\ell m}$ and $X_{\ell m}$ depend on $P_{\ell m}$ and $P_{\ell-1, m}$, it is not surprising that the leakage is as much as that seen in Figure 4-4. In reality, the amplitudes of the B-mode spectrum is likely to be small (or zero), which greatly reduces the leakage in the E-mode. We exaggerated Figure $4-4$ so that we can obtain a worstcase estimate of the errors. This is an important consideration particularly for trying to measure any weak B-mode signal in the presence of leakage from the E-mode.


Figure 4-4: A plot on the leakage problem for the polarization spectra. The blue lines are the spectra as the direct result of inversion. The red, dashed lines are the spectra we used to calculate polarization. The normalization for both spectra is arbitrary.

Now we will present a method to remove this leakage. Let us suppose we invert some
polarization map (data) which gives us overestimated E and B spectra. Since

$$
C_{\ell}^{\mathrm{E}, \mathrm{~B}}=\frac{1}{2 \ell+1} \sum_{m=-\ell}^{\ell}\left\langle a_{\ell m}^{\mathrm{E}, \mathrm{~B}}\right\rangle^{2}
$$

the $a_{\ell m}^{\mathrm{E}, \mathrm{B}}$ 's are more useful when we want to know how much extra contribution they get from other modes of ${ }_{ \pm 2} Y_{\ell m}$. And from here on we will use their complex notations, i.e., equation (8) for the expansion,

$$
\begin{equation*}
Q \pm i U=\sum_{\ell=0}^{\ell_{\max }} \sum_{m=-\ell}^{\ell} \pm 2 a_{\ell m \pm 2} Y_{\ell m} \tag{17}
\end{equation*}
$$

and a simpler form of equation (14) for the inversion,

$$
\begin{equation*}
{ }_{ \pm 2} a_{\ell m}=\sum_{\text {all pixels }}(Q \pm i U)_{ \pm 2} Y_{\ell m}^{*} \tag{18}
\end{equation*}
$$

We will also assume that the expansion is exact based on the reasons discussed earlier in section 2 (in short, due to that fact that the spectra are zero after some $\ell$ ). Thus the only significant numerical error here other than that made by recurrence relations is due to pixelization. For a particular pixelization, what equation (18) does is introduce an extra additive constant for each $\ell$ and $m$. Therefore, ${ }_{ \pm 2} a_{\ell m}$ can be expressed as

$$
\begin{equation*}
{ }_{ \pm 2} a_{\ell m}={ }_{ \pm 2} a_{\ell m}^{\text {True }}+\Delta_{ \pm 2} a_{\ell m} \tag{19}
\end{equation*}
$$

Now suppose we take this ${ }_{ \pm 2} a_{\ell m}$, which contains the true power spectrum and a little offset, and calculate $Q \pm i U$ using equation (17):

$$
\begin{align*}
(Q \pm i U)_{1} & =\sum_{\ell=0}^{\ell_{\max }} \sum_{m=-\ell}^{\ell}{ }_{ \pm 2} a_{\ell m \pm 2} Y_{\ell m} \\
& =\sum_{\ell=0}^{\ell_{\max }} \sum_{m=-\ell}^{\ell}\left({ }_{ \pm 2} a_{\ell m}^{\text {True }}+\Delta_{ \pm 2} a_{\ell m}^{\text {True }}\right)_{ \pm 2} Y_{\ell m}  \tag{20}\\
& =(Q \pm i U)+\sum_{\ell=0}^{\ell_{\text {max }}} \sum_{m=-\ell}^{\ell}\left(\Delta_{ \pm 2} a_{\ell m}^{\text {True }}\right)_{ \pm 2} Y_{\ell m}
\end{align*}
$$

Then we find that we can estimate $\Delta_{ \pm 2} a_{\ell m}$ by another inversion:

$$
\begin{equation*}
\Delta_{ \pm 2} a_{\ell m}^{\text {True }} \simeq \sum_{\text {all pixels }} \Delta(Q \pm i U)_{ \pm 2} Y_{\ell m}^{*} \tag{21}
\end{equation*}
$$

where $\Delta(Q \pm i U)$ is $(Q \pm i U)_{1}-(Q \pm i U)$. We say approximately equal in equation (21) because $\Delta_{ \pm 2} a_{\ell m}$ is also subject to the same pixelization effects as ${ }_{ \pm 2} a_{\ell m}$. This gives us a first-order approximation to ${ }_{ \pm 2} a_{\ell m}^{\text {True }}$. Examples of this are in Figures 4-5 and 4-6.


Figure 4-5: This shows the first-order approximation to ${ }_{ \pm 2} a_{\ell m}^{\text {True }}$ in blue. The purple line is the ${ }_{ \pm 2} a_{\ell m}$ in the discussion (without error reduction). The green line is the correct ${ }_{ \pm 2} a_{l m}^{\text {True }}$ we used as input. The blue and green lines almost coincide, so only two lines are clearly visible.


Figure 4-6: This plot shows the numerical error before and after the analysis. The purple line (with larger amplitude) is the residual without iteration. The green line is the residual after the first iteration. The red line marks the horizontal axis.

These plots are of the worst-case scenario we just mentioned, where both $C_{\ell}^{\mathrm{E}}$ and $C_{\ell}^{\mathrm{B}}$ have comparable amplitude. In this case, we fail to remove some leakage for small $\ell$. This is due to the fact that our best estimate of the first order corrections to ${ }_{ \pm 2} a_{\ell m}, \Delta_{ \pm 2} a_{\ell m}^{\text {True }}$, is based on the polarization maps produced by the ${ }_{ \pm 2} a_{\ell m}$ which already contains the leakage. There is really nothing we can do about that since ${ }_{ \pm 2} a_{\ell m}^{\text {True }}$ is unknown. However, we can cleverly avoid most of this before we calculate equation (20). We know the cross-leakage (i.e., leakage into one $C_{\ell}$ due to the non-zero other) is minimum when one of $C_{\ell}^{\mathrm{E}, \mathrm{B}}$ is zero. So for every step after equation (20), we can choose to set either $C_{\ell}^{\mathrm{E}}$ or $C_{\ell}^{\mathrm{B}}$ to zero. This ensures that the $\Delta_{ \pm 2} a_{\ell m}^{\text {True }}$ we get is closest to the original $\Delta_{ \pm 2} a_{\ell m}^{\text {True }}$ when we invert the data. Once we do that, the low- $\ell$ error which we failed to remove earlier goes away. The other $\ell$-modes' errors have also decreased as we expected. This is seen in Figure 4-7. The residual errors are of order $10^{-3}$, except for larger $\ell$ near the "Nyquist frequency"
determined by the number of pixels along any great circle on the map. We expect this to disappear for very large simulations because this will occur where $C_{\ell}$ is zero.


Figure 4-7: The difference between the recovered and the input $C_{\ell}^{\mathrm{E}}$ spectra after we use an iterative scheme to minimize the leakage (see text).

Another important issue is the leakage in the B-mode. Since the B polarization is likely to be very small, how well we can remove the leakage becomes an important issue as the experiments designed to measure B-polarization will be challenging. But in our case, smaller B-mode allows for better inversion accuracy. Without performing any error-removal procedures, we will still see some leakage just because there is a non-zero E polarization. But we can reduce the error to those shown in Figure 4-7 as long as the errors come from pixelization effects. Therefore, it should be possible to estimate the amplitudes correctly.

We can carry this method further since equation (21) also contains some leakage, which we did not consider previously. To find this second-order correction, we just go back to equation (19) and substitute in the appropriate terms. And let $\Delta^{2}{ }_{ \pm 2} a_{\ell m}$ denote $\Delta\left(\Delta_{ \pm 2} a_{\ell m}\right)$ and $\Delta^{2}(Q \pm i U)$ denote $\Delta(\Delta(Q \pm i U))$. Then we have the following:

$$
\begin{equation*}
\Delta_{ \pm 2} a_{\ell m}=\Delta_{ \pm 2} a_{\ell m}^{\text {True }}+\Delta^{2}{ }_{ \pm 2} a_{\ell m}, \tag{22}
\end{equation*}
$$

and

$$
\begin{align*}
(Q \pm i U)_{2} & =\sum_{\ell=0}^{\ell_{\max }} \sum_{m=-\ell}^{\ell} \Delta_{ \pm 2} a_{\ell m \pm 2} Y_{\ell m} \\
& =\sum_{\ell=0}^{\ell_{\max }} \sum_{m=-\ell}^{\ell}\left(\Delta_{ \pm 2} a_{\ell m}^{\text {True }}+\Delta^{2}{ }_{ \pm 2} a_{\ell m}^{\text {True }}\right)_{ \pm 2} Y_{\ell m}  \tag{23}\\
& =\Delta(Q \pm i U)+\sum_{\ell=0}^{\ell_{\max }} \sum_{m=-\ell}^{\ell}\left(\Delta^{2}{ }_{ \pm 2} a_{\ell m}^{\text {True }}\right)_{ \pm 2} Y_{\ell m}
\end{align*}
$$

And thus

$$
\begin{equation*}
\Delta^{2}{ }_{ \pm 2} a_{\ell m}^{\text {True }} \simeq \sum_{\text {all pixels }} \Delta^{2}(Q \pm i U)_{ \pm 2} Y_{\ell m}^{*} \tag{24}
\end{equation*}
$$

where all notations are defined accordingly. Essentially this can be iterated many times to remove the pixelization effect:

$$
\begin{equation*}
\Delta_{ \pm 2}^{\mathrm{i}} a_{\ell m}=\Delta_{ \pm 2}^{\mathrm{i}} a_{\ell m}^{\text {True }}+\Delta^{\mathrm{i}+1}{ }_{ \pm 2} a_{\ell m} \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
\Delta^{\mathrm{i}+1} \pm 2 a_{\ell m} \simeq \sum_{\text {all pixels }} \Delta^{\mathrm{i}+1}(Q \pm i U)_{ \pm 2} Y_{\ell m}^{*} \tag{26}
\end{equation*}
$$

However, there seems to be no need to go beyond the second iteration when the first can do as well as in Figure 4-7. Equation (26) is given for completeness purposes. All of this can be summarized by

$$
\begin{equation*}
{ }_{ \pm 2} a_{\ell m}^{\text {True }} \simeq \sum_{\mathrm{i}=0}^{\mathrm{n}} \Delta_{ \pm 2}^{\mathrm{i}} a_{\ell m}^{\text {True }} \tag{27}
\end{equation*}
$$

where $n$ is the number of iterations performed. Figure $4-8$ shows the residual of the second iteration. The residual error is so small that the spectra we obtained are almost identical to the input spectra (the plot of $C_{\ell}^{\mathrm{E}}$ and $C_{\ell}^{\mathrm{B}}$ are not shown for this reason).


Figure 4-8: The residual after the second iteration. The solid line is that for the first iteration, and the dashed is that for the second. For higher $\ell$, the residual is quite small. This really shows how well this method works at recovering the correct spectra.

## 5. Other Relevant Numerical Issues

The result in section 4 more or less ends the story of removing pixelization effects, but there are still a few places that might potentially improve the accuray. We will be looking for techniques that can reduce the numerical error without slowing down the inversion in the following sections.

### 5.1 Numerical Error as a Function of Map Resolution

Apparently we can increase resolution to achieve better accuracy, since in the limit of infinite resolution ( $\Delta \theta=\Delta \phi=0$, equation (14)), the summation becomes an integral. But how much can we increase the resolution before the increased computing time outweighs the gain in accuracy? In reality we can not increase the resolution indefinitely since each experiment has a beam size, which experimentally supresses information. And there are also computational limitations. But on a purely theoretical basis, we can ask what is a good compromise between accuracy and computing time, which is directly related to resolution.

To answer this question, we performed a test with a low $\ell_{\max }$ map to see how the accuracy improves with resolution. For the ECP at a fixed $\ell_{\max }$, the computing time generally increases linearly with the number of pixels. In Figure 4-4, we plotted the variance of the residual spectrum (difference of input and inversion output) versus sampling factor, where a sampling factor of 1 means we sample at the Nyquist frequency given by $\ell_{\max }$, and higher means we oversample the map by that factor times the Nyquist frequency. We see that sampling twice as often as the Nyquist factor gives a reasonably small numerical error while keeping the computational time short. This is precisely what is expected for MAP and Planck, which are planned to over-sample the beam size by about a factor of three.


Figure 5-1: A plot showing how the numerical error decreases with increasing number of pixels. The vertical scale is normalized to the point on the far right. Computing time scales with the square of the Nyquist frequency if $\ell_{\max }$ is fixed.

### 5.2 Approximation of Inverse Transform Integration

Basically, everything we described in section 4 can only be done after we have obtained the power spetra. We assumed that the pixel-center approximation for the integral in equation (14) was sufficiently accurate, but in most cases, it is not. This is especially true for low-resolution maps where the value of ${ }_{ \pm 2} Y_{\ell m}$ 's can vary greatly within a pixel (longitudinally). In this case, the pixel-center values are no longer sufficient, and we must look for better ways to calculate

$$
\begin{equation*}
{ }_{ \pm 2} a_{\ell m}=\iint(Q \pm i U)(W \pm i X)^{*}\left(e^{ \pm i m \phi}\right)^{*} \sin \theta d \theta d \phi \tag{28}
\end{equation*}
$$

which, in its variable-separated form, looks like

$$
\begin{equation*}
{ }_{ \pm 2} a_{\ell m}=\int_{\theta} \sin \theta(W \pm i X)^{*} \int_{\phi}(Q \pm i U)\left(e^{ \pm i m \phi}\right)^{*} d \phi d \theta \tag{29}
\end{equation*}
$$

One option is to consider averaging the ${ }_{ \pm 2} Y_{\ell m}$ from the four corners of each pixel (Figure 5-1). This is advantageous because it is a better representation of ${ }_{ \pm 2} Y_{\ell m}$ for each pixel, and it defines a pixel "boundary" which the computer does not know otherwise. However, to do this, we must resort to an inversion without separation of variables (equation (28)), since summing over the four $e^{ \pm i m \phi}$ values and then multiplying by the four $(W \pm i X)^{*}$ will produce twelve unwanted and unremovable cross terms. This fact alone prohibits the use of four-corner averaging. Any other similar n-point averaging technique will have the same problem as long as it has points lying on different latitudes.


Figure 5-2: This illustrates the four-corner averaging scheme. Unfortunately, its incompatibility with separation of variables renders it useless.

Another option is to only average points on the same latitude (points with subscripts 2 and 4 in Figure 5-1). This works in unison with equation (29)'s variable separation, and since it does not require any additional calculations of $(W \pm i X)$, the algorithm does not suffer any significant loss in speed. We performed a low-resolution simulation with $\ell=100$, and it showed that this averaging can reduce the overall error (sum of errors in each multipole) by $25 \%$. This is not much, considering it is only a reduction in overall error (the improvement in each individual multipole is approximately $2.5 \times 10^{-3}$ ), but it
is still better than nothing. The time for the small simulation was about $20 \%$ longer than a simulation without averaging.

### 5.3 ECP with Equal-Area Pixels

We know the ECP pixelization loses its accuracy partly because of varying pixel areas. Once the rectangular pixels on a plane are projected onto a spherical surface, each pixel's area is altered by a $\sin \theta$ factor. In an attempt to correct for this, we tried to calculate $\theta$ 's at intervals such that each pixel has the same area, regardless of shape. Apparently, this increases pixel shape distortion because now we have very-long polar pixels. We found that this neither improves the inversion accuracy nor computing time. Although this seems like a worthless study, we were able to conclude that both equal shape and area are necessary constituents of a good pixelization scheme, as previous studies have shown (e.g. Crittenden \& Turok, 1998). The majority of the pixels in the original ECP pixelization have similar shapes. The better result given means that pixel shape is a more important effect than pixel area. This is reasonable because pixel area effects can be removed by proper normalization, while the normalization for shape is more difficult to quantify.

In practice we can improve results of ECP (for the same total number of pixels) by reducing the number of polar pixels and increasing the number around the equator. This can be done without any unnecessary computing time because the number of operations involving ${ }_{ \pm 2} Y_{\ell m}$ remains the same. Continuing this process, we eventually would arrive at a pixelization scheme very similar to HEALPix or IGLOO pixelization.

## 6. Conclusion

The main numerical error in the calculation of CMB temperature and polarization power spectra comes from pixelization schemes which cannot fully describe spin-zero and spin-two spherical harmonics. We found that such numerical error can be removed to within $10^{-6}$ of the original spectra by an iterative method that calculates how much leakage one multipole receives due to a particular pixelization. We also investigated some other numerical issues that might be relevant for future polarization studies. Together with previous studies on some similar issues for temperature anisotropy and polarization, our results ensure that the numerical part of the polarization data analysis should not contribute any significant errors to the determination of the cosmological parameters.

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CMBFAST: http://www.sns.ias.edu/~matiasz/CMBFAST/cmbfast.html
HEALPix: http://www.eso.org/~kgorski/healpix/
MAXIMA: http://cfpa.berkeley.edu/group/cmb/

