DUALITIES IN ABELIAN STATISTICAL MODELS

By

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Abstract

Various aspects of duality in a series of Abelian lattice models defined on topologically non-trivial lattices are investigated. The dual theories on non-trivial spaces are found to contain extra topological degrees of freedom in addition to the usual local ones. By exploiting this fact, it is possible to introduce topological modes in the defining partition function such that the dual model contains a reduced set of topological degrees of freedom. Such a mechanism leads to the possibility of constructing self-dual lattice models even when the naive theory fails to be self-dual. After writing the model in field-strength formalism the topological modes are identified as being responsible for the quantization of global charges. Using duality, correlators in particular dimensions are explicitly constructed, and the topological modes are shown to lead to inequivalent sectors of the theory much like the inequivalent $\theta$-sectors in non-Abelian gauge theories. Furthermore, duality is applied to the study of finite-temperature compact $U(1)$, and previously unknown source terms, which arise in the dual Coulomb gas representation and consequently in the associated Sine-Gordon model, are identified. Finally, the topological modes are demonstrated to be responsible for the maintenance of target-space duality in lattice regulated bosonic string theory and automatically lead to the suppression of vortex configurations which would otherwise destroy the duality.
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Chapter 1

Introduction and Overview

1.1 Introduction

The word duality has many meanings and implications, and in this section an overview of duality in several of its guises will be given. In quantum mechanics, for example, particle-wave duality is the term used to describe the fact that a quantum mechanical object can be represented in a position or momentum basis. Given a wavefunction in position basis, that is highly localized, one finds that in the momentum basis, it is delocalized, and vice versa. This property of a small parameter (the width of the wavefunction) in one representation becoming large in a second is a key ingredient in strong-weak dualities. Duality however, does not necessarily have such a meaning attached to it. Consider a high-energy electron-positron annihilation into hadrons,

\[ e^+ + e^- \rightarrow \text{hadrons} \]

\[\begin{array}{c}
\text{e}^+ \\
\text{\gamma} \\
\text{e}^-
\end{array} \quad \begin{array}{c}
\text{\gamma} \\
\text{\bar{q}} \\
\text{q}
\end{array} \rightarrow \text{hadrons} \]

Figure 1.1: The Feynman diagram contributing to the total cross section of the process \( e^+e^- \rightarrow \text{hadrons} \) in the quark model. Notice that in the high energy limit the interactions in the box, which form hadrons from quarks, can be neglected.
In the theory of the strong interactions, quantum chromodynamics (QCD), the fundamental quarks which make up the hadrons have the property of asymptotic freedom [27, 49]. This is the observation that, at high energies, the quarks behave as essentially free point-like particles, and exposes itself under the guise of Björken scaling [8]. Thus, the above process proceeds through quark-antiquark pair production, via a virtual photon, followed by the formation of hadrons from the quarks after final-state interactions. For annihilations at very high energies, the quark final-state interactions are negligible, due to asymptotic freedom, and the $e^+e^-$ annihilation total cross section is obtained by ignoring them. This ability to ignore the interactions which bind the quarks into hadrons allows the computation of the cross-section to be easily carried out and is denoted quark-hadron duality (see fig. 1.1). This furnishes a second use of the term duality.

A third type of duality came about through the study of scattering matrices of hadronic matter (before QCD was known to be the correct model). Experiments in the 1960's uncovered an enormous proliferation of hadrons. Resonances of rather high spin were observed, and they were found to obey a linear relation with their squared mass, $m^2 = J/\alpha'$, where the slope, $\alpha' \sim 1 (GeV)^{-2}$, became known as the Regge slope, and the linear relation the Regge trajectory. Resonances of spin as high as $J = 11/2$ were experimentally observed and found to lie on the Regge trajectory. Furthermore, there appeared to be no indication that there was an upper bound on their spin. Consequently, in a tree-level diagrammatic expansion of an elastic process the mediating particle can be of any spin and there should in principle be two contributions to the amplitude, one from the $s$-channel and a second branch from the $t$-channel (see fig. 1.2),

$$A(s, t) = \sum_J g_J^2 (-s)^J \frac{1}{t - M_J^2}, \quad A'(s, t) = \sum_J g_J^2 (-t)^J \frac{1}{s - M_J^2}$$  \hspace{1cm} (1.1)

The observed high energy behaviour of the amplitudes is in fact softer than any individual term in the above series; however, the infinite summation can produce precisely such a
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Figure 1.2: The $s$ and $t$-channels in the elastic scattering amplitude. Only the $s$ or the $t$-channel must be included. This duality was seen to be a result of the fact that the amplitude has an interpretation in terms of scattering of string states, in which there is no distinction between $s$ and $t$-channels.

softening of the high energy behaviour. The infinite summation has a second consequence: it may produce poles in $A(s,t)$ as a function of $s$, in addition to the poles at $t = M_J^2$ (similarly for $A'(s,t)$). In a usual quantum field theory, where the spin is cut-off at some value, the only poles in the $t$-channel occur at $t = M_J^2$, and the only poles in the $s$-channel occur at $s = M_J^2$; however, the total amplitude must contain resonances at both $t = M_J^2$ and $s = N_J^2$, consequently, a summation over both $s$ and $t$-channels is required. Here, however, a summation over both $s$ and $t$-channels need not be carried out, because each channel may have poles in both $s$ and $t$. This led to the conjecture that the $s$ and $t$-channel diagrams gave alternative or "dual" descriptions of the same physics, i.e. the amplitudes in the separate channels were in fact equal. Two then young physicists, Veneziano and Suzuki [71, 67], noticed that the Euler-Beta function, $\beta(\cdot, \cdot)$, satisfied precisely this duality relation,

$$A(s,t) = A'(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} = \beta(-\alpha(s), -\alpha(t)) \tag{1.2}$$

where $\alpha(x) = \alpha(0) + \alpha'x$ is the equation of a straight line with parameters chosen to reproduce the correct Regge trajectory, and $\Gamma(x)$ is the Gamma-function. These "dual models" were formulated completely in terms of $S$-matrix amplitudes until Nambu and Goto [46, 24] realized that these amplitudes can be derived from a relativistic string model, at which point the duality was clear from the fact that the diagrams should really
be string world-sheets which have identical topologies in either channels (see fig. 1.2). It should be noted that the duality hypothesis mentioned here had very little experimental support, and the Veneziano model was merely an attempt to satisfy this hypothesis. Of course, other aspects of the dual models, which will not be discussed here, were found to be in disagreement with experiment and the correct parton model, paving the way for QCD, was seen to produce the proper picture.

Continuing with our overview of dualities, attention is now shifted to some more abstract notions that lead to powerful results. Although abstract, the formulation is not restricted to quantum mechanical situations; indeed, duality is an important ingredient in certain classical models as well. An illustrative case is supplied by a two-dimensional field theory in which the dynamical fields, $z_a$, live in the complex projective manifold $CP^1$. It can be demonstrated that a type of self-duality equation,

$$D_\mu z = \pm i \epsilon_{\mu \nu} D_\nu z$$

(1.3)

saturates a lower bound on the action, and leads to non-trivial instanton solutions (the name self-duality stems from the notion of Poincare-duality of differential forms, and simply refers to the fact that the right hand side is contracted with the anti-symmetric tensor, $\epsilon_{\mu \nu}$, in a such a manner as to produce a vector). In the above, $D_\mu = \partial_\mu + A_\mu$ represents a covariant derivative in which the gauge potential $A_\mu = z \partial_\mu z$. In lieu of the details involved in the two-dimensional case, it is instructive to consider a related situation in four-dimensional physics. The Yang-Mills action with $SU(2)$ gauge group in four-dimensions has action,

$$S[\vec{A}] = \frac{1}{4g^2} \int d^4 x |\vec{F}_{\mu \nu}|^2$$

(1.4)

with components of the curvature two-form being,

$$\vec{F}_{\mu \nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu.$$
Before discussing the instanton solutions, the structure of the semi-classical vacua needs to be understood. The classical minima of the action corresponds to gauge fields which are pure gauge at infinity. If the space-time is thought of as being bounded by a large three-dimensional sphere, $S^3$, then such configurations give a map, $g(x) : S^3 \rightarrow G$, from the three-sphere to the group space. Any two maps which are continuously deformable to one another induce small gauge transformations, and hence should be identified. Such a collection of maps, with the above mentioned identifications, are classified by the third homotopy group of the gauge group which is isomorphic to the integers for any semi-simple Lie group,

$$\pi_3(G) = \mathbb{Z}$$

(1.6)

where $\mathbb{Z}$ is the set of integers. This labeling of vacua is denoted the winding number, $n$, (also known as the Pontrjagin index) and given a gauge field configuration, $A_\mu$, its value is,

$$n = \frac{1}{32\pi^2} \int d^4x \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}$$

(1.7)

where,

$$\tilde{F}_{\mu\nu} \equiv \epsilon^{\alpha\beta} F_{\alpha\beta}$$

(1.8)

In the quantum theory, the path-integral is well-defined only within each sector, and the integration over the gauge fields must be chosen so that the winding number is fixed, $n = \text{const}$. The partition function with winding number $n$ is defined as,

$$Z(n) \equiv \int [\mathcal{D}\tilde{A}_\mu] \delta \left( n - \frac{1}{32\pi^2} \int d^4x \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right) \exp\{-S[\tilde{A}_\mu]\}$$

(1.9)

The so-called $\theta$-angle is related to winding number via a Fourier transformation,

$$Z(\theta) = \sum_{n=-\infty}^{\infty} Z(n) e^{in\theta}$$

$$= \int [\mathcal{D}\tilde{A}_\mu] \exp \left\{-S[\tilde{A}_\mu] + \frac{i\theta}{32\pi^2} \int d^4x \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \right\}$$

(1.10)
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The advantage of the Fourier transformed basis is clear: the integration over gauge fields is unrestricted, as all winding numbers contribute. Furthermore, only in the sector in which a summation over all winding numbers is included does the theory admit cluster decomposition of correlators.

Now that the vacuum structure is understood, instantons will now be introduced. The action can be bounded from below,

\[ S[\tilde{A}_\mu] \geq \frac{1}{4g^2} \left| \frac{1}{2} \epsilon_{\mu\nu\sigma\rho} \int d^4x \tilde{F}_{\mu\nu} \tilde{F}_{\sigma\rho} \right| \equiv \frac{1}{4g} Q[\tilde{A}_\mu] = \frac{8\pi^2 n}{g} , \quad n \in \mathbb{Z} \quad (1.11) \]

where \( n \) is the winding number. The lower bound is saturated by field configurations which satisfy the self-duality equation,

\[ \tilde{F}_{\mu\nu} = \pm \epsilon_{\mu\nu\sigma\rho} \tilde{F}_{\sigma\rho} = \pm \tilde{F}_{\mu\nu} \quad (1.12) \]

Once again the meaning of duality here is in the context of Poincaré duality: the object is contracted with an antisymmetric tensor having as many indices as the space-time. This inequality was derived by Belavin, Polyakov, Schwartz and Tyupkin [7]. Solutions to these equations can be demonstrated to connect vacua with different winding numbers. Consequently, these instantons can only be well understood using \( Z(\theta) \) as all windings contribute in the functional integral. This example demonstrates that topology affects the system in a highly non-trivial manner. The appearance of multiple vacua has led to the introduction of a new parameter in the theory, the \( \theta \)-angle, the description in which \( \theta \) is added into the partition function leads to the most natural interpretation of the solutions to the self-duality equations.

Poincare-duality appears not only in the guise of the self-duality equations for instanton configuration, it appears as a symmetry of the equations of motion as well. Consider for the moment \( U(1) \) gauge theories and the associated Maxwell's equations,

\[ \partial_\nu F^{\mu\nu} = -j^\mu , \quad \partial_\nu \tilde{F}^{\mu\nu} = 0. \quad (1.13) \]
where $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\sigma\rho} F_{\sigma\rho}$. In the vacuum the four current, $j^\mu$, vanishes, and the equations are invariant under the interchange $F \rightarrow \tilde{F}$ and $\tilde{F} \rightarrow -F$ which amounts to $E \rightarrow B$ and $B \rightarrow -E$. This is referred to as electro-magnetic duality. Notice that the Yang-Mills action is also invariant under such an interchange. In the presence of sources this transformation is clearly not a symmetry. However, a source term [16] is included into the Bianchi constraint (the second equation in (1.13)),

$$\partial_\nu F^{\mu\nu} = -j^\mu, \quad \partial_\nu \tilde{F}^{\mu\nu} = -k^\mu. \quad (1.14)$$

the new model is invariant under,

$$F \rightarrow \tilde{F}, \quad \tilde{F} \rightarrow -F, \quad j^\mu \rightarrow k^\mu, \quad k^\mu \rightarrow -j^\mu \quad (1.15)$$

This full invariance is much larger than the discrete invariance in the absence of sources, in fact, the fields are invariant under a continuous $SO(2)$ group which rotates the electric and magnetic quantities into one another. This modification of the theory, although trivial at the classical level, leads to highly non-trivial effects in the quantum theory. Since the field-strength no longer satisfies a Bianchi constraint, it need not be derived from a gauge potential. Nevertheless, it can be demonstrate that, for point magnetic sources, a gauge potential can be defined, however, it is not globally well-defined. Since these singularities are found to be gauge-dependent, their presence should not be observable. Forcing the singularities to be invisible in the quantum theory leads to the quantization of the electric charge. Consequently, modifying Maxwell's equations to satisfy a self-duality requirement has a highly non-trivial effect in the quantum theory and leads to the quantization of the electric charge.

Self-duality leads to non-trivial effects not only in field theories, but also in statistical models, the Ising model being the most well known example. Kramers and Wannier found that the model could be mapped into itself with a redefinition of the coupling
constant. This led to the identification of the second order phase-transition point as the fixed point of this mapping. Other examples, in which self-duality leads to the discovery of many interesting phases, are the so-called clock models (see the next section). A rich phase structure is observed in these models, and duality played a key role in the identification of the various phases of the theory and the in identification of the critical points. Statistical models will be revisited shortly.

The existence of a phase-transition at the self-dual coupling point in the Ising model can be viewed as the fact that the dual strong and weak coupling regimes cannot be consistently “glued” together. Similarly, Seiberg and Witten [60, 61] demonstrated that, in $\mathcal{N} = 2$ four dimensional supersymmetric (SUSY) Yang-Mills theory, the full effective action for the light degrees of freedom at any coupling constant can be constructed from duality considerations. The construction proceeds from a knowledge of the weak-coupling limit, and the behaviour at certain strong-coupling “singularities”, together with a holomorphy requirement that tells one how to patch together the different limiting regimes. One of the many interesting features that appears in these supersymmetric models is that the effective coupling constant is a complex number,

$$\tau = \frac{2\pi}{g^2} + \frac{i\theta}{4\pi} \quad (1.16)$$

Notice that there is an explicit dependence on the topological $\theta$-angle which was introduced earlier in our discussion of the semi-classical vacua of YM theories. The low energy action of this model has some interesting symmetries, in particular, a duality transformation can be constructed that replaces a gauge field which couples to an electric field to a dual gauge field which couples to a magnetic field, and at the same time maps the complexified coupling constant, $\tau \to \tau_D = -\frac{1}{\tau}$. The full duality group is in fact $SL(2, \mathbb{Z})$, and it can be demonstrated that a gauge field coupled to the complex combination $E - iB$
Figure 1.3: The grids of electric and magnetic charges in the SU(2) gauge theory. The crosses are charges of matter in the fundamental representation, while the circles are charges of matter in the adjoint representation. Diagram a) is the situation at $\theta = 0$ and the theory confines if the magnetic monopoles condense. At $\theta \neq 0$, diagram b), dyons can condense leading to the onset of oblique confinement. See text for further description.

is mapped to one which couples to $(c\tau + d)(E - iB)$ and at the same time,

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}.$$  \hspace{1cm} (1.17)

In SUSY models the non-trivial mixing of the topological $\theta$-angle and the Yang-Mills coupling constants leads to new phases in the theory (for a review see [62]). Indeed, Witten had demonstrated long ago [76] that the existence of Dyons is inevitable in any gauge theory with magnetic monopoles and non-zero $\theta$-angle: a magnetic monopole of charge $m$ necessarily acquires an electric charge $q$,

$$q = \frac{\theta q^2}{8\pi^2} m.$$  \hspace{1cm} (1.18)

In QCD it is believed that the condensation of magnetic monopoles leads to the confinement of the color degrees of freedoms. Analogously, the condensation of the dyons in the SUSY model leads to a new phase of the theory denoted oblique confinement. The situation is depicted in figure 1.3 and discussed below.

At zero $\theta$-angle (fig. 1.3a), the spectrum of possible electric/magnetic charges are shown. The elementary states with electric charge are along the $q$-axis (e.g. A), while
the magnetic monopole states (e.g. B) lie along the \( m \)-axis (higher charges correspond to pairs of monopoles etc.). States not on either axes (e.g. C) are bound states of quarks and monopoles. When the monopoles condense the usual situation of confinement, in which all states not on the vertical axis are confined by the formation of chromoelectric flux tubes between pairs of objects, occurs.

Raising \( \theta \) from zero slowly to an angle slightly larger than \( \pi \), leads to the situation depicted in fig. 1.3b (the oblique nature of the diagram is where the name oblique confinement comes from). The single monopole state (B) has moved far off the axis and acquired a large electric charge. It is therefore, unlikely that it will condense. However, the monopole with magnetic charge \( m = 2 \) near the vertical axis (D) has a (near) vanishing electric charge and is quite plausible that it condenses and all states that do not lie on the line connecting the origin to the state D will be confined. This is a peculiar type of confinement for the following reason: some states carrying the external quantum numbers of the fundamental quarks exist in the observable spectrum such as the bound state of a quark and a dyon (F). Since the dyon has no baryon charge, the state F has the same baryon number as the fundamental quarks - a rather surprising result. These bizarre and interesting features all rise through non-trivial dependence on the \( \theta \)-angle.

In the last several examples of duality the main theme is that topology (\( \theta \)-angle) plays a key role in introducing new physics. However, since the world appears to be in a \( \theta = 0 \) state, the begging question is, "Why is it necessary to understand the dependence on the \( \theta \)-angle?" There are several reasons why it is interesting, three reasons are listed here:

(i) Highly non-trivial physical effects arise in the presence of non-zero \( \theta \)-dependence, and this alone could be sufficient motivation for its study. Furthermore, even though the world is in a \( \theta = 0 \) state, several physical problems are solved by considering the dependence on this parameter and then setting it to zero. A notable example
is the $U(1)$ problem, which is resolved by noticing that the $\eta'$ mass depends on the curvature of the vacuum energy as a function of the $\theta$ parameter at $\theta = 0$,

$$m_{\eta'}^2 \sim \left. \frac{\partial^2 E_{\text{vac}}}{\partial \theta^2} \right|_{\theta = 0}$$

(ii) Although the world is currently in a $\theta = 0$ state, it is believed that at an earlier time of the evolution of the universe, when the temperature was above $150 \text{MeV}$, $\theta$ was in fact non-zero. It is believe that only through a dynamical relaxation process has it settled down to zero. The particle governing this dynamics is the so-called axion. Clearly, the dynamical properties of the theory depends non-trivially on the $\theta$-angle.

(iii) It is known that a non-vanishing $\theta$-angle leads to a rich spectrum of phases and phase transitions, such as the oblique confinement mentioned above. The possibility that such structures can exist in Abelian models motivates the work in this thesis. Such structures can be studied by analyzing self-dual models which at the critical points admit phase-transitions (see chapter 3) as in the Ising model case. Experimental studies in the high-energy phase structure of hadronic matter will soon be carried out in Brookhaven through heavy-ion collisions in the RHIC-experiments; consequently, gaining an understanding of the effects of topology on the system through any means possible is a worthwhile project.

The main goal of this work is to study the richness of physics related to the introduction of new topological parameters into the theory, such as the $\theta$-angle. This may help in the understanding of, and discovery of new phases in gauge theories.

Thus far, continuum models have been the focus of attention; however, statistical models also exhibit various aspects of duality. Lattices arise naturally in many physical contexts, such as crystalline structures in solids. The location of the nuclei involved act
as the points of a lattice on which spins can be localized (e.g. an outer shell electron). The structure of the lattice is clearly quite important on short distance scales; however, at large distances the details involved become irrelevant. This is one of the ideas behind renormalization, or in particular, block spin transformations. However, lattices can also appear from a more abstract point of view. A field theory typically has an ultra-violet cut-off, since so far all field theories are valid only up to some energy scale at which point a different theory should take over. The inverse of this ultra-violet cut-off has dimensions of length. Imposing an ultra-violet cut-off should then be equivalent to introducing a minimum length scale. This can be achieved by discretizing the space-time, so that the theory lives on a lattice rather than a continuous space-time, where the lattice spacings are of the order of the inverse ultra-violet cut-off. In this way a regulated version of the continuum theory has been introduced. At any finite lattice spacing the discrete and continuous theory may not even have the same form, however, at the critical point, where the correlation length diverges, it is expected that the lattice becomes irrelevant and the two partition functions yield the same physics. This is analogous to what happens in block-spin transformations, as the theory is described using larger and larger block sizes, the details of the underlying lattice becomes irrelevant and only its universal behaviour is seen.

The prototypical statistical model that is used as a testing ground for new ideas and techniques is the Ising model. The study of strong-weak duality was in fact initiated by the work of Kramers and Wannier [42] on the Ising model almost sixty years ago. This model consists of a two-dimensional (infinite) lattice in which the sites are occupied by a two-level system, it costs an energy of $+J$ to have neighboring sites in the same level, while neighboring sites that are in opposite levels contribute an energy of $-J$. This simple model has many interesting features, and can be mapped to the so-called lattice gas which was used to study the liquid-gas phase-transition. Kramers and Wannier [42]
demonstrated that this model can be re-written in terms of spins which are associated with the center of the plaquettes of the original lattice and have identical interactions when written in terms of a dual coupling constant: \( J* = -\frac{1}{2} \ln (\tanh J) \). Consequently, if the coupling constant in the original model was large, the coupling in the dual model is small and vice-versa. If the lattice happens to be a square lattice, then the dual lattice is identical to the original lattice (the fact that the spins are associated with sites shifted by one-half lattice spacing in each direction is of no consequence). Accordingly, the dual model and original model are completely equivalent with the exception of the mapping of the coupling constant. This allows the free energy of the system to be constrained in the following manner,

\[
F(J) = c(J^*) + F(J^*)
\]

(1.20)

here \( c(J^*) \) is a regular function of \( J \). Under the assumption that there is a unique phase transition point, it can only occur where \( J = J^* \) (otherwise there would be at least two phase transitions). This identified the critical value, \( J = \frac{1}{2} \ln(1 + \sqrt{2}) \), at which the phase-transition takes place, even though the free energy itself was not known. Of course, this was later confirmed by Onsager's exact solution of the model [48]. Since this model is defined on the infinite plane, and the relevant symmetry group is discrete, there are no topological effects and no analogs of \( \theta \)-angles appearing here. However, if the two directions are compactified, so that the lattice has the topology of a torus, many new features arise in the dual construction. It is here that a connection with non-trivial \( \theta \)-angles is seen. The models considered in the later chapters are higher dimensional generalizations of this simple model and the effect that topology plays on the dual construction and hence on the phase structure of the theories is studied. In the next section the example of a clock model on a torus is worked out in detail, paving the way for the generalizations made in the next chapter.
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1.2 The Clock Model

In this section the so-called “clock-models” on a torus will be discussed in some detail. The starting point is a square lattice, denoted by $\Omega$, placed on a toroidal surface, on which each site there is an angular variable, $\sigma_i = 0, \frac{2\pi}{N}, \ldots, \frac{2\pi(N-1)}{N}$; such variables are elements of the cyclic group of order $N$ which is denoted by $\mathbb{Z}_N$ (see figure 1.4). The “spins” favor being aligned with their nearest neighbour, as it costs energy to deviate from the aligned state (equally one could choose the favored configuration to be anti-aligned). The partition function is defined as the summation over all spin configurations weighted by a Boltzmann factor which induces the alignment,

$$Z = \sum_{\{\sigma_i \in \mathbb{Z}_N\}} \prod_{(ij) \in \Omega} B_{(ij)}(\sigma_i - \sigma_j), \quad (1.21)$$

where $(ij)$ denotes the link connecting the two neighboring sites $i$ to $j$; the Boltzmann factors, $B_{(ij)}(\cdot)$, will be chosen to be in Villain form[72],

$$B_{(ij)}(\sigma_i - \sigma_j) = \sum_{n=-\infty}^{\infty} e^{-J(\sigma_i - \sigma_j + 2\pi n)^2}; \quad (1.22)$$

and $J$ is the coupling constant characterizing the strength of the interactions. The summation over $n$ is included so that the Boltzmann weight is invariant under the shift $\sigma_i \rightarrow \sigma_i + 2\pi m$ where $m$ is an arbitrary integer.
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This model has been extensively studied in the literature, however, the lattice is usually taken to be a discretization of the plane, \( \mathbb{R}^2 \). The calculations in this section will demonstrate that the due to the topology of the lattice new topological modes appear in the dual theory. The existence of these modes will be demonstrated to destroy the self-dual property that the model enjoys in the flat case. However, through a modification of the model, in which some of the topological modes are included in the defining theory, self-duality will be restored. This feature of theories defined on lattices with non-trivial topology will be explored in a more generalized context in chapter 3.

A second feature of topology occurs when rewriting the theory in such a manner that the fundamental fields are defined on the links of the lattice. This seems more natural than using site variables, since the interactions only occur on the links. These new variables will be referred to as field-strengths. In rewriting the theory in terms of the field-strengths some extra constraints on the variables will be introduced. These constraints are remnants of the fact that the link variables were derived from site variables. The constraints that arise will be shown to factorize into local and global parts, and play the important role of imposing quantization conditions on the local and global "charges" in the system. When a similar procedure is applied to the modified theory some of the global constraints will be absent; consequently, the topological modes are seen to be associated with the quantization of the global charges in the system. More generalized analysis of this type will be investigated in chapter 5.

1.2.1 Duality Relation

The above results will now be derived using the duality transformations. The steps outlined here will appear many times in this thesis, with various levels of generalizations; this example should be referred back to when the need arises. The first step in any duality transformation is to perform a Fourier transformation, which respects the symmetry of
the variables, on the Boltzmann weights. The reason for this step is that this enables the introduction of variables that are defined on the links of the lattice, as opposed to the sites where the spins are defined, this is desirable because the product appearing in the partition function is over the links of the lattice and not the sites, also the argument of the Boltzmann factors are suggestively labeled by links. The Boltzmann weight, $B_{(ij)}(\cdot)$, in terms of its Fourier coefficients, $b_{(ij)}(\cdot)$, is given by,

$$B_{(kl)}(\sigma_k - \sigma_l) = \sum_{r_{(kl)}=0,2\pi/N,\ldots,2\pi N/(N-1)} b_{(kl)}(r_{(kl)}) e^{i\frac{2\pi}{N}r_{(kl)}(\sigma_k - \sigma_l)} \quad (1.23)$$

where $r_{(ij)} = 0, 2\pi N, \ldots, 2\pi N/(N-1)$ is a new variable labeled by the links of the lattice and, like the original spin variables $\sigma_i$, are elements of the cyclic group $\mathbb{Z}_N$; and the inverse transform is,

$$b_{(ij)}(r_{(ij)}) = \frac{1}{N} \sum_{\sigma=0,2\pi N,\ldots,2\pi N/(N-1)} B_{(ij)}(\sigma) e^{i\frac{N}{2\pi}r_{(ij)}\sigma} = \frac{1}{N} \sum_{\sigma=0}^{N-1} \sum_{n=-\infty}^{\infty} e^{-J\left(\frac{2\pi\delta - 2\pi n}{N}-ir_{(ij)}\delta\right)} \quad \text{(1.24)}$$

Notice that $b_{(ij)}(\cdot)$ has exactly the same form as the original Boltzmann weight, $B_{(ij)}(\cdot)$, it differs only by an overall constant and the replacement of the coupling $J$ by the “dual” coupling $J^* = \frac{N^2}{16\pi^2J}$. Consequently, when the coupling constant of the Boltzmann weight is large, the coupling appearing in its Fourier transform is small and is precisely the phenomena which implements the uncertainty principle. By replacing the Boltzmann weight by the above Fourier expansion, the partition function can be re-arranged into the form,

$$Z = \sum_{\{r_{(ij)} \in \mathbb{Z}_N\}} \prod_{(ij) \in \Omega} b_{(ij)}(r_{(ij)}) \sum_{\{\sigma_k \in \mathbb{Z}_N\}} \prod_{(kl)} e^{i\frac{2\pi}{N}r_{(kl)}(\sigma_k - \sigma_l)} \quad (1.25)$$

This expression does not appear to simplify the situation, but in fact it does. Consider
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Figure 1.5: a) Contributions to the product over links on the original lattice and its dual. b) The dual lattice consists of placing (dual) sites in the middle of every plaquette of the original lattice, and then joining those sites by (dual) links which then form new (dual) plaquettes. The site 0 on the original lattice is enclosed by the dual plaquette $p_0^*$. 

The subset of the product over links which have a single site, say $k = 0$ as in figure 1.5, in common; the contribution to the product is,

$$e^{i\frac{2\pi}{N} \sigma_0 \left(-r_{(10)} + r_{(02)} + r_{(03)} - r_{(40)}\right) + \sigma_1 r_{(10)} - \sigma_2 r_{(02)} - \sigma_3 r_{(03)} + \sigma_4 r_{(40)}}$$  \hspace{1cm} (1.26)

The site variable $\sigma_0$ does not appear in any other contributions to the product. Of course, the four other site variables will appear when the remaining links that touch them are included. This contribution can be re-written in a simplified manner by associating to every link on the original lattice a link on the dual lattice, to every site on the original lattice a plaquette on the dual lattice, and finally to every plaquette on the original lattice a site on the dual lattice; this is depicted in figure 1.5. The dual links that contribute to the $\sigma_0$ term form the dual plaquette, $p_0^*$, which enclose the site 0 on the original lattice; consequently, (1.26) can be written as,

$$\exp \left\{ i \frac{2\pi}{N} \sigma_0 \sum_{(ij)} r_{(ij)}^* \right\}$$  \hspace{1cm} (1.27)

where the other site variables have been ignored. The summation is over all links on the dual lattice that make up the dual plaquette $p_0^*$ which encloses the site 0 on the original lattice (see fig. 1.5). It is not difficult to see that all contributions to the second product
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over links appearing in (1.25) can be rewritten in a similar fashion, with the result,

\[ Z = \sum_{\{r_{(ij)}\in\mathbb{Z}_N\}} \prod_{\langle ij \rangle\in\Omega} b_{(ij)} (r_{(ij)}) \sum_{\{\sigma_k\in\mathbb{Z}_n\}} \prod_k \exp \left\{ \frac{i}{2\pi} N \sigma_k \sum_{(ij)^*\in\mathbb{Z}_n^*} r_{(kl)^*} \right\} \]  

(1.28)

Notice that the product is now over every site of the lattice, this allows the summation over the link variables to be performed, and the partition function is left with only link variables. It is important to realize that although the link variables, \( r_{(ij)} \), are labeled by links on both the original and dual lattice, there exist only one set of variables. The contribution to the partition function from the site variables, \( \sigma_k \) can now be written in the form,

\[ \sum_{\{\sigma_k\in\mathbb{Z}_N\}} \prod_k \exp \left\{ \frac{i}{2\pi} N \sigma_k \sum_{(ij)^*\in\mathbb{Z}_n^*} r_{(kl)^*} \right\} = \prod_k \sum_{\sigma_k\in\mathbb{Z}_N} \exp \left\{ \frac{i}{2\pi} N \sigma_k \sum_{(ij)^*\in\mathbb{Z}_n^*} r_{(kl)^*} \right\} \]

\[ \propto \prod_k \delta \left( \sum_{(ij)^*\in\mathbb{Z}_n^*} r_{(kl)^*} \right) \]  

(1.29)

where the proportionality constant is of order \( N \), and \( \delta(\cdot) \) denotes the Kronecker Delta-function; however, it is irrelevant and therefore does not appear in the remaining calculations. Consequently, for each site on the lattice, the summation over all \( r \)-variables on the dual links which enclose that site must vanish. The most generic solution of these constraints is constructed by introducing new variables which are defined on the dual sites, \( i^* \),

\[ r_{(ij)^*} = \tilde{\sigma}_{i^*} - \tilde{\sigma}_{j^*} \]  

(1.30)

where,

\[ \tilde{\sigma}_{i^*} \in \mathbb{Z}_N \]  

(1.31)

are completely arbitrary. It is clear that such configurations satisfy the constraints. However, due to the lattice having compact directions, there are two additional classes of solutions which cannot be written in such a manner (later on it will be seen that these special cases are related to the non-trivial homology of the lattice),
Figure 1.6: A portion of the toroidal lattice showing the special links. The set of links \( \gamma_{1,2} \) form closed loops, while \( \gamma^{1,2} \) do not. However, the links on the dual lattice which \( \gamma_{1,2} \) cross, which will be denoted \( \gamma^{*1,*2} \), do not form a closed loop, while the links on the dual lattice which \( \gamma^{1,2} \) cross, which will be denoted \( \gamma^{*1,2} \), are closed loops. Notice that the links with an asterisk have the same attributes as the links without the asterisk except that they are located on the dual lattice. Also, a label in the lower index indicates that those set of links for a closed loop.
\[ r_{ij}^* = h \varepsilon((ij)^*; \gamma^a) \] (1.32)

where \( \varepsilon(\cdot; \cdot) \) is defined as,
\[
\varepsilon(l; \gamma) = \begin{cases} 
1, & l \in \gamma \\
0, & \text{otherwise} 
\end{cases}
\] (1.33)

\( h \) is an arbitrary element of the cyclic group, \( \mathbb{Z}_N \); and the set of links \( \gamma^a \) are depicted in figure 1.6. They consist of those links on the dual lattice which are crossed by the set of links on the original lattice that form a closed loop in one of the compact directions (notice that only 2 such sets of links are necessary for the torus). The index on \( \gamma^a \) distinguishes between the two compact directions on the lattice.

Now inserting the most general solution of the constraints (1.29), which is sum of all three solutions mentioned above, into the partition function, (1.28), the dual model is obtained,
\[
Z = \sum_{h_1, h_2 \in \mathbb{Z}_N} \sum_{\{\tilde{\sigma}_i^* \in \mathbb{Z}_N\} (ij)^*} \prod_{(ij)^*} b_{(ij)^*} \left( \tilde{\sigma}_i^* - \tilde{\sigma}_j^* + \sum_{a=1,2} h_a \varepsilon((ij)^*; \gamma^a) \right) 
\] (1.34)

The product over links which appeared in (1.28) has been replaced by a product over dual links, this should cause no confusion, since there is a one-one mapping between these as described earlier. This is the final form of the dual model, it is clearly not self-dual - there are two additional topological fields. Their affect, however, is felt only on those dual links which are “cut” by a closed path on the original lattice which winds around a compact direction. This leads to the following intuitive picture for building the dual model:

(i) cut the torus along the two closed paths on the original lattice so that the lattice now looks rectangular

(ii) perform the usual duality transformation on this flat lattice neglecting boundary effects
Figure 1.7: Pictorial representation of the dual clock model on a torus (solid lines are links on dual lattice, dotted lines are links on the original lattice). Cutting the torus along two closed loops on the lattice yields a rectangular topology. Performing the usual dual transformations that work for flat topologies and then sewing the lattice back together along the dual links which were cut, and applying all possible twists in the coupling along those links, yields the dual model on the torus.

(iii) sew the lattice along one direction to form a cylinder. Along those dual links just sewn, sum over all relative phases between the site variables at their end points.

(iv) sew the remaining ends together to form a torus. Along those dual links just sewn, sum over all relative phases between the site variables at their end points.

This procedure is depicted in figure 1.7 and is very reminiscent of summing over spin structures. The variables $h_1, h_2$ will be referred to as topological modes, since their presence is solely due to the non-trivial topology of the lattice, and are completely absent in the flat case. The physical content of these modes will now be investigated.

1.2.2 Bianchi Constraints

Since the Boltzmann weights appearing in the partition function, (1.21), contain arguments which are labeled by links of the lattice, yet the fundamental variables reside on the sites of the lattice, it is natural to introduce dynamical link variables which take the place of the site variables. This can be achieved by making use of a Fadeev-Popov like
trick and inserting the following identity into the partition function,

\[ 1 = \sum_{\{v_l \in \mathbb{Z}_N\}} \prod \delta\left(v_{ij} - (\sigma_i - \sigma_j)\right) \]  

(1.35)

here \( v_l \) are variables which reside on the links, \( l \), of the lattice. The partition function can then be rewritten in the form,

\[ Z = \sum_{\{v_l \in \mathbb{Z}_N\}} \prod B_l(v_l) \Pi(\{v_l\}) \]  

(1.36)

where \( \Pi(\{v_l\}) \) excludes configurations of link variables which are not derivable from a configuration of site variables; explicitly,

\[ \Pi(\{v_l\}) = \sum_{\{\sigma_i \in \mathbb{Z}_N\}} \prod \delta\left(v_{ij} - (\sigma_i - \sigma_j)\right) \]  

(1.37)

Notice that this constraint has exactly the form of the original partition function (1.21), where the Boltzmann weights are now delta-functions,

\[ B_l(g) = \delta(v_l - g) \]  

(1.38)

and the link variables \( \{v_l\} \) behave as some external source. Its Fourier modes are trivial,

\[ b_l(r) = e^{-\frac{ir}{\Delta} v_l} \]  

(1.39)

It is then possible to use the dual partition function, (1.34), with Boltzmann weights and Fourier coefficients given by (1.38) and (1.39) respectively, to rewrite the constraint in a more palatable form. In the following equation the dual constraint has been written in terms of variables which are defined on the lattice, as opposed to the dual lattice as in (1.34), and the relevant changes are explained below.

\[ \Pi(\{v_l\}) = \sum_{h_1, h_2 \in \mathbb{Z}_N} \sum_{\{p \in \mathbb{Z}_N\}} \prod \exp\left\{ \frac{iN}{2\pi} v_{ij} \left( \Delta p_{ij} + \sum_{a=1,2} h_a \varepsilon((ij); \gamma_a) \right) \right\} \]  

\[ = \sum_{\{p \in \mathbb{Z}_N\}} \prod e^{i\frac{N}{2\pi} v_{ij} \Delta p_{ij}} \sum_{h_1, h_2 \in \mathbb{Z}_N} \prod \exp\left\{ \frac{iN}{2\pi} v_{ij} \sum_{a=1,2} h_a \varepsilon((ij); \gamma_a) \right\} \]  

\[ = \Pi_1(\{v_l\}) \Pi_2(\{v_l\}) \]  

(1.40)
Here, $\Delta p(ij)$ denotes the difference between the two plaquettes which have the link $(ij)$ in common. There are two differences in this representation compared with (1.34), the first is that rather than having the dynamical variables defined on the sites of the dual lattice, $\sigma_i$, we have chosen to label them by the plaquettes of original lattice, $p_i$, which is allowed since there is a simple one-one correspondence between the sites of the dual lattice and the plaquettes of the original lattice. The second difference is that the set of links which appear in the topological term is now $\gamma_a$, a set of links forming a closed path on the original lattice ($a$ labels the two choices). Notice that the constraint factorizes into two parts; the first, $\Pi_1(\{v_i\})$, contains only the local degrees of freedom, $p_{(ij)}$; while the second constraint, $\Pi_2(\{v_i\})$, contains only the topological degrees of freedom, $h_a$. These two constraints will now be considered in turn.

The local constraint, $\Pi_1(\{v_i\})$, can be rewritten by collecting terms which have a single plaquette in common. Consider the contribution to the product depicted in figure 1.8, it
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can be written as,

$$\prod_{p_0 \in \mathbb{Z}_N} \sum_{p_1} e^{iN_p} \sum_{l \in \square} v_l = \prod_{l \in \square} \delta \left( \sum_{l \in \square} v_l \right) \quad (1.42)$$

Here the product is over all plaquettes $\square$ on the lattice. These constraints can be interpreted as suppressing local vortex excitations, since the $v_l$ variables must sum to zero around every elementary plaquette. In the case of gauge theories a similar constraints, which must hold on the cubes of the lattice rather than the plaquettes, are referred to as the Bianchi constraint, or flatness condition.

Turning now to the global constraints,

$$\Pi_2(\{v_l\}) = \sum_{h_1, h_2 \in \mathbb{Z}_N} \prod_{\langle ij \rangle} \exp \left\{ \frac{iN_h}{2\pi} \sum_{a=1,2} h_a \varepsilon(\langle ij \rangle; \gamma_a) \right\}$$

$$= \prod_{a=1,2} \sum_{h_a} \exp \left\{ \frac{iN_h}{2\pi} \sum_{\langle ij \rangle} \varepsilon(\langle ij \rangle; \gamma_a) v_{\langle ij \rangle} \right\}$$

$$= \prod_{a=1,2} \delta \left( \sum_{l \in \gamma_a} v_l \right) \quad (1.43)$$

it is clear that these constraints also force the global vortex excitations to vanish, and in the case of gauge theories are referred to as holonomy constraints, or global Bianchi constraints.

The model (1.21) rewritten in terms of variables which reside on the link of the lattices rather than the sites is therefore,

$$Z = \sum_{\{v_l \in \mathbb{Z}_N\}} \prod_i B_i(v_l) \prod_{\square} \delta \left( \sum_{l \in \square} v_l \right) \prod_{a=1,2} \delta \left( \sum_{l \in \gamma_a} v_l \right) \quad (1.44)$$
The two global modes in the dual construction lead to the appearance of the global constraints which would have otherwise been absent in the flat case. These extra global constraints play a very destructive role in the context of string theory dualities. In particular, when the world-sheet of the string is discretized and the target-space is circular, which corresponds to the $X - Y$ model on a compact surface, Gross and Klebanov [26] noticed that the theory apparently looses one of its symmetries known as target space duality. To restore it they found, by guessing, a model which turns out to be equivalent to (1.44) with softened global constraints. In chapter 4 I will discuss how such a model can be systematically constructed from a proper treatment of the topological modes in the continuum model and a subsequent straightforward discretization of the world-sheet.

1.2.3 Motivation for Self-Dual Theories

In this subsection we give some reasons why the property of self-duality is desirable, and what consequences it leads to. This follows closely the work of [13]. Consider a system in which there is one coupling constant $K$ and assume that the model is self-dual so that the under the mapping $K \to \tilde{K}$ where

$$\tilde{K} = t(K)$$

(1.45)

the interactions in the model remain invariant. Self-duality is also taken to mean that the operation is its own inverse so that $t(t(K)) = K$. The self-dual point of the model is clearly defined as $t(K^*) = K^*$. The goal is to obtain a condition on the Gell-Mann-Low $\beta$-function of the model at the critical point. For lattice models it can be defined as,

$$\beta(K) = -a \frac{\partial K}{\partial a}$$

(1.46)
where $a$ is the lattice spacing and $\beta(K)$ indicates how the coupling constant transforms under a change of scale. By acting on eq.(1.45) with $-a\partial/\partial a$ one finds,

$$-a\frac{\partial \tilde{K}}{\partial a} = \beta(K)t'(K) \quad (1.47)$$

The left hand side of this equation is now identified with the $\beta$-function at the dual point. Consequently given the duality relation, knowing the $\beta$-function in, say, the weak coupling regime is sufficient to know the strong coupling behaviour of the $\beta$-function,

$$\beta(\tilde{K}) = \beta(K)t'(K) \quad (1.48)$$

This places strict constraints on the $\beta$-function at the self-dual point. In particular, since the function $t(K)$ satisfies $t(t(K)) = 1$, its derivative at the self-dual point must be $t'(K^*) = \pm 1$. As an example, if the system is the two-dimensional Ising model it is easily check that the derivative is negative. In fact, as will be shown in the later chapters, the function $t(K)$ always has a negative derivative. Consequently, near the self-dual point eq.(1.48) implies that

$$\lim_{K \to K^-} \beta(K) = -\lim_{K \to K^+} \beta(K) \quad (1.49)$$

which can hold only if $\beta(K^*)$ vanishes or is discontinuous. The vanishing of the $\beta$-function implies that a second (or higher) order phase transition takes place. The case of a discontinuity corresponds to a first order phase transition. Therefore, due to the connection between the flows of $K$ and $\tilde{K}$ the self-dual point is found to be associated with a phase-transition. This simple analysis is what motivates the generation of self-dual models. Notice that this analysis makes one main assumption: the renormalization flow commutes with the duality transformation. In other words, the block-spin transformation is assumed to produce only irrelevant operators. If relevant operators appear, then the conclusions of this section may fail.
1.2.4 Generation of Self-Dual Model

Attention is now placed on modifying the topological modes somewhat and studying the effects that they have on both the dual model and the link representation. Since these topological modes appear in the construction of the dual model, what would happen if they were included in the defining theory? The answer is that they would disappear in the dual construction, this must happen because the dual construction must give the identity under performing it twice. However, a second interesting question is, "What would happen if only one topological mode, say $h_1$, was included in the defining model?"

This is a much more interesting question, and rather than simply stating the answer, a few illustrative points in the calculation will be given. The defining model is now,

$$Z = \sum_{h_1 \in \mathbb{Z}_N} \{\sigma_k \in \mathbb{Z}_N\} \prod B(\langle kl \rangle) \left(\sigma_k - \sigma_l + h_1 \epsilon(\langle kl \rangle; \gamma^l)\right)$$  \hspace{1cm} (1.50)

The set of links, $\gamma^l$, are the equivalent of $\gamma^s$ except that they appear on the original lattice (see figure 1.6). They are the set of links on the original lattice which are "cut" by the set of dual links, $\gamma^*$, which form a closed loop in the compact direction. Performing the Fourier transformation as before leads to a new contribution,

$$\sum_{h_1 \in \mathbb{Z}_N} \prod e^{\frac{i}{2\pi} \tau(\langle kl \rangle) h_1 \epsilon(\langle kl \rangle; \gamma^l)} \propto \delta\left(\sum_{\langle kl \rangle} \tau(\langle kl \rangle) \epsilon(\langle kl \rangle; \gamma^l)\right) = \delta\left(\sum_{\langle kl \rangle} \gamma^l \right)$$  \hspace{1cm} (1.51)

The other contributions remain as in (1.25). Performing the summation over $\sigma_k$ then leads to the same solutions for $r_{ij}$ as outlined above (1.30,1.32); however, now the additional constraint, (1.51), must be applied. By inserting the most general solution, $r_{(kl)} = \sigma_{k^*} - \sigma_{l^*} + \sum_{a=1,2} h_a e^{(\langle kl \rangle^*; \gamma^s_a)}$ into the above constraint, one immediately realizes that the site valued fields are irrelevant. This is because the links on the dual lattice, $\gamma^*_i$, which correspond to the set of links $\gamma^l$ forms a closed loop, and therefore the summation over the site variables will be identically zero. In addition, the special solution labeled by $\gamma^s_{11}$ has a zero contribution to (1.51), this is because the set of links $\gamma^*_i$ contains no
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links in common with $\gamma^*$. Finally, the contribution of the special solution labeled by $\gamma^*$ sums to $h_2$, since it contains exactly one link in common with $\gamma^*$. Consequently, the constraint (1.51) implies that $h_2 = 0$, while $h_1$ and the spin configurations, $\{\sigma_{i*}\}$, are completely free. The dual model to (1.50) is then,

$$Z = \sum_{h_1 \in \mathbb{Z}_N} \sum_{\{\sigma_{i*} \in \mathbb{Z}_N\}} \prod b_{(kl)^*} \left( \sigma_{k^*} - \sigma_{l^*} + h_1 \epsilon(\langle kl \rangle^*; \gamma^*) \right)$$

Notice that this is identical to the original model (1.50) with the exception that the couplings have been replaced by the dual couplings and the lattice by the dual lattice. The exchange of the lattice for its dual has no physical consequence for the square lattice, and hence the model is self-dual. The special point $J = J^*$ corresponds to the self-dual point of the theory, and one clearly sees that the weak and strong coupling limits of the modified model (1.50) are mapped into one another through the duality relations. The same conclusion would have arose if instead the other topological term was kept in the partition function from the start - that model would also be self-dual.

It is interesting to write this model in terms of the link variables introduced in the previous subsection. Rather than carrying out the calculation we simply quote the result, which can be derived following the procedure in the previous subsection,

$$Z = \sum_{\{v_i \in \mathbb{Z}_N\}} \prod_{i} B_i (v_i) \delta \left( \sum_{j \in \gamma_1} v_i \right) \prod \delta \left( \sum_{j \in \square} v_i \right)$$

Notice that the local constraints are unaltered from their form in (1.44); however, only one of the global constraints are present, and global vortex configurations along the closed loop, $\gamma_1$, are allowed. Consequently, the topological modes are seen to control the existence of constraints which either allow or disallow the appearance of vortices. In fact, if $h_1$ was restricted to a proper subgroup of $\mathbb{Z}_N$, i.e. a another cyclic group of order $M$ where $M$ is a factor of $N$, then the second constraint which is absent in the above equation would appear; however, it would not restrict vortices fully. Detailed discussions
of such affects will be given in chapter 5 where such models will be constructed explicitly.

The purpose of this section was to give a brief over view of some of the general features of the duality transformation on topologically non-trivial lattices. The existence of topological modes has been demonstrated, and their role in the quantization of global charges has been pointed out. With the help of the inclusion of topological modes in the defining theory, we were able to render a model self-dual. All of these features, and more, will be explored in much more detail in the subsequent chapters of this thesis. To make the presentation more concise we make heavy use of the language of simplicial homology which is briefly reviewed in the second chapter. The next section discusses how the remainder of this thesis is organized.

1.3 Organization

This thesis is organized into six additional chapters. Chapter 2 addresses some of the mathematical background on simplicial homology and simplicial language which is used throughout the remainder of the thesis. In addition, it provides the reader with a reference point upon which to build an intuition of the simplicial notation, as there are several examples of well-known and well-studied models written first in terms of continuum notation, then in terms of the standard lattice regularized versions, and finally written in terms of the simplicial language introduced in the first section. The role of simplicial notation becomes clear in the final section where it is pointed out that although many continuum models appear to be considerably different, once written in simplicial language they can all be treated at the same time, obviating any need to analyze each model case by case. There is a clear advantage to this type of generalized notation, and it is exploited considerably in analyzing the duality transformations on the models which appear in the later chapters.
Chapter 3 is organized into two sections: the first section deals with models that contain a single dynamical degree of freedom, for instance a pure gauge theory or scalar field model, and in the second section the global symmetries present in the models containing a single dynamical degree of freedom is lifted to a local one, which necessarily introduces interactions with a second field which has extra space-time indices, i.e. a tensor of higher rank, or on the lattice, a dynamical degree of freedom living on cells of one higher dimension. In each of these sections the dual model on topologically non-trivial manifolds is derived. Reference [70] is a classical paper on this subject, although the analysis there is restricted to flat lattices, they do derive the topological modes that appear when some directions are compactified. The existence of new topological modes, which were absent in the original model, and are absent in the case of trivial topology, are explored through several examples. The two-dimensional Ising model on the torus is treated in some detail, and the similarity between the topological modes and spin-structures is pointed out. Furthermore, finite temperature gauge theories in $\mathbb{R}^4$ are shown to be explicitly self-dual, and it is possible to compute the location of phase transition points in the limit of zero temperature and infinite temperature. The Coulomb gas representation of three-dimensional compact $U(1)$ gauge theory is also discussed at finite temperature, and the topological modes are found to act as external source which interact with the charges. Next the models are generalized to include some of the topological modes that appear in the dual model directly in the defining model. This has the effect of canceling some of the would-be topological degrees of freedom, and leads to the possibility of constructing self-dual models on topologically non-trivial manifolds. Some of the work in this chapter is drawn from my work in [22]. The introduction of these topological modes in the defining theory bears a strong resemblance to 't Hooft's discussion of twisted boundary conditions in gauge theories [69].

In chapter 4 the analysis turns to a study of target-space duality. In the first section
target-space duality is described both from the point of view of the spectrum of the string and from the path-integral angle. In the path-integral formalism it is pointed out that if the target-space contains non-trivial homology, then the scalar field can be multi-valued. Then using what was learnt from quantum mechanics on a circle, the scalar field is lifted to the cover of the space, so that it is rendered single-valued, and topological fields which project the fields onto the target-space itself are introduced. Such a procedure has non-trivial physical implications. Firstly, it explicitly renders the fields to be single-valued functions throughout, while at the same time making target-space duality manifest. Secondly, on discretizing the string world-sheet so that matrix model techniques can be applied, it leads to an adjustment of the lattice action. This adjustment is seen to be necessary for the lattice model, which is simply a version of an X-Y model, to maintain target-space duality. The topological modes serve to suppress vortex excitations and lead to constraints that are usually introduced by hand. Next the procedure it applied to discrete target-spaces and it is demonstrate that a continuous target-space can be mapped into a discrete one. This chapter extends from my work in [32].

Chapter 5 focuses on the close relationship between target-space duality and strong-weak duality. Attention is drawn to the fact that to construct self-dual models, topological modes had to be introduced into the theories in both cases. This allows the construction of general models which reduces to the cases of strong-weak duality and target-space duality for particular choices of the parameters. The dual models are then constructed and examples provided. The role that the summation over topological modes play in these generalized models is illuminated by deriving the field-strength formalism. Models defined on spaces with non-trivial topology obtain additional global Bianchi constraints along with the usual local Bianchi constraints that arise in pure gauge theories, spin models etc... These global constraints are then interpreted, and imply the existence
(or lack) of global fractional charges. Finally, several self-dual models containing these fractional charges are constructed.

In chapter 6 the focus is on the computation of correlation functions, and order and dis-order correlators in these models are defined. The outcome is simply that order correlators are constructed by considering the boundary of membranes of particular dimensions, while the dis-order correlators are defined by constructing an order correlator in the dual model and then inverting the duality relation to obtain a correlator in the original model which is then identified as a disorder correlator. After deriving this mapping it is demonstrated that certain classes of correlators vanish identically in any model. This class consists of those correlators which form representations of certain homology groups, and can be thought of as those correlators which wrap around non-trivial cycles of the lattice. The next section illustrates how explicit expressions, in terms of a finite number of sums, can be obtained from duality when the dimension of the lattice is equal to the dimension of the cells on which interactions take place. Chapter 6 is concluded by carrying out several explicit computations for the spin model and gauge theory cases, where the presence topological modes are demonstrated to lead to existence of inequivalent sectors akin to those that appear in non-Abelian models. My work in chapters five and six are published in [33].

The final chapter in the thesis summarizes the main results and some suggestions for possible future work are provided.
Chapter 2

Statistical Models: Continuum to the Lattice

Statistical models appear throughout this thesis. This chapter is devoted to motivating the models that are study and their relation to continuum theories. In particular, starting with well known continuum models their lattice versions will be derived. The advantage of using lattice models is several fold. Firstly, there is a natural ultra-violet regulator, the lattice spacing, and this leads to well defined objects with no obvious singularities. Secondly, lattice models are typically easier to study than their continuum cousins since local symmetries are trivial to implement as the relevant objects that appear are groups rather than their Lie-algebra. Thirdly, the models studied here all have an identical underlying structure attached to them when observed from the point of view of a lattice regularization. Finally, exact transformations can be made mathematically precise using the language of simplicial homology under which Abelian lattice models are most eloquently formulated. This chapter begins with an overview of simplicial language and give several illustrative examples of how this language is used to attach an intuitive picture to the abstract notations. In the second section scalar field theories are considered and its lattice formulations derived. Similarly the third section is devoted to deriving the Wilsonian lattice action associated with Yang-Mills theory. In the forth section, the scalar field theory is revisited and the global group symmetry is elevated to a local one. This requires, as in continuum models, the introduction of a gauge field. This chapter is then concluded by extrapolating the lattice actions derived in the earlier sections to higher dimensional tensor fields. Although the language of simplicial homology is used to
perform the duality transformations and other operations, in the forthcoming chapters several examples using standard language will be given so that it is not necessarily need to absorb all the details in the general analysis.

2.1 Mathematical Preliminaries

We now give a short review of simplicial homology (for more details see e.g. [65, 44]). Consider a lattice $\Omega$ and associate to every $k$-dimensional cell of the lattice an oriented generator $c_k^{(i)}$ where $i$ indexes the various cells of dimension $k$. These objects generate the $k$-chain group, denoted by $C_k(\Omega, G)$,

$$\sum_{i=1}^{N_k} g_i c_k^{(i)} = g \in C_k(\Omega, G) \quad , \quad g_i \in G$$  \hspace{1cm} (2.1)

Here $G$ is an arbitrary Abelian group with group multiplication implemented through addition and $N_k$ is the number of $k$-cells in the lattice $\Omega$. An element $g \in C_k(\Omega, G)$ is called a $G$-valued $k$-chain or simply a $k$-chain. Clearly $C_k(\Omega, G) = \oplus_{i=1}^{N_k} G$.

Two homomorphisms, the boundary $\partial$ and the coboundary $\delta$, define the chain complexes $(C_*(\Omega, G), \partial)$ and $(C_*(\Omega, G), \delta)$ where $C_*(\Omega, G) \equiv \oplus_{k=0}^{d} C_k(\Omega, G)$:

$$0 \xrightarrow{\partial_{k+1}} C_d(\Omega, G) \xrightarrow{\partial_d} \ldots \xrightarrow{\partial_k} C_k(\Omega, G) \xrightarrow{\partial_{k-1}} \ldots \xrightarrow{\partial_0} C_0(\Omega, G) \xrightarrow{\delta_0} 0$$

$$0 \leftarrow \delta_d C_d(\Omega, G) \leftarrow \delta_{d-1} \ldots \leftarrow \delta_k C_k(\Omega, G) \leftarrow \delta_{k-1} \ldots \leftarrow \delta_0 C_0(\Omega, G) \leftarrow \delta_0 0$$

These homomorphisms are defined by their actions on the generators $c_k^{(i)}$ (the dimension subscripts on $\partial_k$ and $\delta_k$ are displayed only when essential),

$$\partial c_k^{(i)} = \sum_{j=1}^{N_{k-1}} [c_k^{(i)} : c_{k-1}^{(j)}] c_{k-1}^{(j)} \quad , \quad \delta c_k^{(i)} = \sum_{j=1}^{N_{k+1}} [c_k^{(j)} : c_k^{(i)}] c_{k+1}^{(j)}$$  \hspace{1cm} (2.2)

where the incidence number is given by,

$$[c_k^{(i)} : c_{k-1}^{(j)}] = \begin{cases} 
  \pm 1 & \text{if the } j^{th} (k-1)\text{-cell is contained in the } i^{th} k\text{-cell} \\
  0 & \text{otherwise} 
\end{cases}$$  \hspace{1cm} (2.3)
The plus or minus sign reflects the relative orientation of the cells. If two cells have non-zero incidence number the cell of lower dimension is called the face of the higher dimensional one. The incidence numbers are required to satisfy one additional constraint,

$$\sum_{i=1}^{N_k} [c_k^{(j)} : c_k^{(i)}] [c_k^{(i)} : c_{k-1}^{(k)}] = 0$$

This enforces the nilpotency of the boundary and coboundary operators: $\partial \partial = \delta \delta = 0$.

The boundary (coboundary) chains and the exact (coexact) chain groups (also referred to as cycles and cocycles) are defined as,

$$B_k(\Omega, G) = \text{Im } \partial_{k+1} \quad B^k(\Omega, G) = \text{Im } \delta_{k-1}$$

$$Z_k(\Omega, G) = \ker \partial_k \quad Z^k(\Omega, G) = \ker \delta_k$$

These sets inherit their group structure from the chain complex. The quotient groups,

$$H_k(\Omega, G) = Z_k(\Omega, G) / B_k(\Omega, G) \quad H^k(\Omega, G) = Z^k(\Omega, G) / B^k(\Omega, G)$$

are the homology and cohomology groups respectively. The elements of the homology group are those chains that have zero boundary and are themselves not the boundary of a higher dimensional chain; while the elements of the cohomology group are those chains that have zero coboundary and are themselves not the coboundary of a lower dimensional chain. Since these groups will appear many times in the remaining chapter, several explicit examples are given below.

The spaces of interests have topologies of $\mathbb{R}^n$ ($n$-dimensional Euclidean space), $S^n$ ($n$-dimensional sphere), or $T^n$ ($n$-dimensional torus) and possibly products of these spaces. These spaces are torsion free, i.e. their (co)homology groups with integer coefficients are products of $\mathbb{Z}$ or trivial. To begin the homology groups are recorded below, and the chains which generate the respective groups are described in the next paragraph.

$$H_n(\mathbb{R}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$$
\[ H_n(S^m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & 0 < n < m \\ \mathbb{Z} & n = m \end{cases} \]  \hspace{1cm} (2.8)

\[ H_n(T^m, \mathbb{Z}) = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \]  \hspace{1cm} (2.9)

of course the homology groups of dimensions larger than that of the space-time vanish.

The homology groups of lowest and highest dimensions of any compact orientable lattice is independent of the lattice. Explicitly, \( H_0(\Omega, \mathbb{Z}) \) is generated by the chain \( h = c_0^{(1)} \), of course any other cell of dimension 0 is a valid generator, while \( H_d(\Omega, \mathbb{Z}) \) (where \( d \) is the dimension of \( \Omega \)) is generated by the chain \( h = \sum_{i=1}^{N_d} c_d^{(i)} \). However, the generators of the homology groups of other dimensions depend on the lattice. For the above examples only the \( n \)-tori case warrants further explanation. The generators of \( H_n(T^m) \) are given by the set of \( \binom{n}{m} \) possible \( n \)-tori sub-spaces made out of the \( S^1 \)'s making up \( T^m \), and \( h_a = \sum_{i \in T_a} c_n^{(i)} \) where \( a \) labels the various surfaces. For example consider the torus \( T^2 \), the first homology group has 2 generators given by the two circles making up the torus and can be thought of as non-contractable loops. The case of \( T^3 \) gives a slightly non-trivial case: the first homology group has three generators - the three non-contractable loops; the second homology group also has three generators - the tori made from circles 1&2, 1&3 or 2&3; and the third homology group consists of the 3-cells making up the entire \( T^3 \).

The final case to consider is the product group. For spaces like the ones considered here, which have no torsion (the integer homology group are all free Abelian), there is a natural isomorphism between the homology of the product space and the individual homologies. It is known as the K"{u}nneth formula,

\[ H_n(\Omega_1 \times \Omega_2, \mathbb{Z}) \cong \oplus_{i+j=n}(H_i(\Omega_1, \mathbb{Z}) \otimes H_j(\Omega_2, \mathbb{Z})) \]  \hspace{1cm} (2.10)
Since the spaces considered in this work have freely generated homology groups, i.e. are products of \( \mathbb{Z} \), the tensor products are rather trivial: \((G_1 \oplus \ldots \oplus G_n) \otimes \mathbb{Z} \cong G_1 \oplus \ldots \oplus G_n\).

Notice that the \( n \)-tori result can be easily recovered from this formula and the result for the circle. These relationships are almost sufficient to obtain all the homology groups that will appear in this work.

The last remaining isomorphism allows the computation of homologies with arbitrary coefficient group. It is called the coefficient theorem,

\[
H_n(\Omega, \mathcal{G}) \cong H_n(\Omega, \mathbb{Z}) \otimes \mathcal{G}
\]

(of course this holds only for torsion free homologies, the case of torsion requires some modifications which are not of interest here).

Thus far, only the homology groups have been dealt with, this is in fact all that is necessary to compute the cohomology groups. This is because the cohomology group for these spaces are in fact isomorphic the homology groups. There are a set of natural relationships among the \((co)\)homology groups of different dimensions,

\[
H_k(\Omega, \mathbb{Z}) \cong H^{d-k}(\Omega, \mathbb{Z}) \cong H^k(\Omega, \mathbb{Z}) \cong H_{d-k}(\Omega, \mathbb{Z})
\]

The isomorphisms between cohomology and homology is induced by the operator which projects a chain onto cell (a sort of inner product),

\[
\langle c^{(i)}_k, c^{(j)}_l \rangle = \delta_{k,l} \delta^{ij}
\]

which is linear in both arguments. The boundary and coboundary operators are dual to each other with respect to this operation,

\[
\langle \partial c^{(i)}_k, c^{(j)}_{k-1} \rangle = \langle c^{(i)}_k, \delta c^{(j)}_{k-1} \rangle
\]

Denote the generators of the \( k \)th homology group by \( \{h_a : a = 1, \ldots, A_k\} \) and the \( k \)th cohomology group by \( \{h^a : a = 1, \ldots, A^k\} \). The isomorphisms, (2.12), imply the following.
orthogonality relations for the generators,

$$\langle h_a, h^b \rangle = \delta_a^b$$  \hspace{1cm} (2.15)$$

and \( h_a \) is said to be dual to \( h^a \), not to be confused with interpreting on the dual lattice (to be defined shortly). The isomorphisms also imply that \( A_k = A^{d-k} = A^k = A_{d-k} \).

Let us work out a few illustrative examples of how this language is used. Let \( \sigma \in C_0(\Omega, G) \), consider the following inner product,

$$\langle \delta \sigma, c^{(l)}_1 \rangle = \langle \sigma, \partial c^{(l)}_1 \rangle = \left( \sum_{i=1}^{N_0} \sigma_i c^{(l)}_0, (c^{(l)}_0 - c^{(l)}_1) \right) = \sum_{i=1}^{N_0} \sigma_i \langle c^{(l)}_0, (c^{(l)}_0 - c^{(l)}_1) \rangle = \sigma_{l_1} - \sigma_{l_2}$$  \hspace{1cm} (2.16)$$

where the link \( l \) points from the site \( l_1 \) to the site \( l_2 \). This is precisely the form of the interaction for a nearest neighbour spin model. As another example take \( \sigma \in C_1(\Omega, G) \) and repeat the above,

$$\langle \delta \sigma, c^{(p)}_2 \rangle = \langle \sigma, \partial c^{(p)}_2 \rangle = \left( \sum_{l=1}^{N_1} \sigma_l c^{(l)}_1, \sum_{l' \in \mathcal{P}} c^{(l')}_1 \right) = \sum_{l=1}^{N_1} \sigma_l \sum_{l' \in \mathcal{P}} \langle c^{(l)}_1, c^{(l')}_1 \rangle = \sum_{l' \in \mathcal{P}} \sigma_l$$  \hspace{1cm} (2.17)$$

where \( p \) labels a plaquette, and it is assumed that the links are oriented consistently with that plaquette so that no relative signs appear. Not surprisingly this reproduces the form of the Wilson action for lattice gauge theory.

It will be necessary at some point during the analysis in the next few sections to introduce the notion of the dual lattice. To obtain the dual lattice, a new chain group is introduced which is generated by a new set of generators. These new generators are constructed from the generators of \( C_*(\Omega, G) \): a generator of \( C_*(\Omega^*, G) \), \( c^{*(i)}_k \), is obtained from the generators of \( C_*(\Omega, G) \), \( c^{(i)}_k \), by making the following identifications:

$$c^{(i)}_k \leftrightarrow c^{*(i)}_{d-k}$$
$$[c^{(i)}_k : c^{(j)}_{k-1}] \leftrightarrow [c^{*(j)}_{d-k+1} : c^{*(i)}_{d-k}]$$  \hspace{1cm} (2.18)$$
In general this identification does not produce a lattice, however in case the original lattice was a triangulation of a closed orientable manifold it does. Under this identification the following mappings occur,

\[
\langle \delta g, c_k^{(i)} \rangle \leftrightarrow \langle \partial g^*, c_{d-k}^{(i)} \rangle \\
\langle \partial g, c_k^{(i)} \rangle \leftrightarrow \langle \delta g^*, c_{d-k+2}^{(i)} \rangle
\]

where \( C_k(\Omega, g) \in g = \sum_{i=1}^{N_k} g_i c_k^{(i)} \) and \( C_{d-k+1}(\Omega^*, g^*) \in g^* = \sum_{i=1}^{N_{k-1}} g_i c_{d-k+1}^{(i)} = \sum_{i=1}^{N_{k-1}} g_i c_{d-k+1}^{(i)} \). Consequently, the homology (cohomology) generators on the lattice \( \Omega \) become cohomology (homology) generators on the dual lattice \( \Omega^* \).

### 2.2 Scalar Field Theories

Consider the free field theory,

\[
S = \frac{1}{\alpha'} \int_{\Sigma^2} d\varphi \wedge *d\varphi
\]

Here \( \Sigma^2 \) represents an arbitrary orientable Riemann surface and \( \varphi \) is a scalar field defined on \( \Sigma^2 \) and generally take values in some one-dimensional target manifold. However, the present discussion will be restricted to the case in which the field is real-valued. The bosonic string theory is governed by this action. To obtain the full string partition function a summation over all Riemann surfaces \( \Sigma^2 \) weighted by a factor of \( g_{\Sigma^2}^\chi \), where \( \chi \) is the Euler character of the surface, and an integration over all metrics (modulo diffeomorphisms) on each surface must be introduced. Upon discretizing, the world-sheet derivatives become difference operators and the partition function (without the sum over topologies or metric) takes on the form,

\[
Z = \prod_{i=1}^{N} D\varphi_i \prod_{(ij)} \exp \left\{ -\frac{1}{\alpha'} (\varphi_i - \varphi_j)^2 \right\}
\]
here \((ij)\) denotes a link on the discretized surface with end points sites \(i\) and \(j\). The objects \(\varphi_i\) labeled by sites can be thought of as spins and since the continuum action depended only on \(d\varphi\) the spins interact solely with their nearest neighbours. To reproduce the summations over surfaces and metrics, matrix model techniques can be taken advantage of (more about this will be said in chapter 4). At this point the model will be written using the language introduced in the previous section. Upon discretizing the world-sheet, the field \(\varphi\) have been replaced with the site valued objects, \(\{\varphi_i\}\), the information contained in this configuration of spins is more economically encoded by introducing a chain. Consider the 0-chain defined by \(\varphi = \sum_{i=1}^{N_0} \varphi_i c_{0}^{(i)}\) this is an element of the 0-chain complex with coefficient group \(\mathbb{R}\). Clearly, given any configuration \(\{\varphi_i\}\) the above construction of an element in \(C_0(\Omega, \mathbb{R})\) can be carried out, conversely given any element of \(\varphi \in C_0(\Omega, \mathbb{R})\) a spin configuration can be constructed via:

\[
\varphi_i = \langle \varphi, c_{(i)}^0 \rangle \tag{2.22}
\]

That is, the projection of \(\varphi\) onto the \(i^{th}\) 0-cell gives the values of the spin on that site. Consequently, there is an isomorphism between configurations and elements of the 0-chain complex. This implies that the functional integral over \(\varphi\) can be replaced by a sum over elements in the relevant chain complex. In addition, using the fact that the projection of the coboundary of a 0-chain onto a link is the difference operator on that link (see (2.16)) the sigma model partition function is reduced to the form,

\[
Z = \sum_{\varphi \in C_0(\Omega, \mathbb{R})} \frac{N_1}{i=1} B_{l} \left( \langle \delta \varphi, c_{l}^{(i)} \rangle \right) \tag{2.23}
\]

where the Boltzmann weights are defined as,

\[
B_{l}(g) = \exp \left\{ -\frac{g^2}{\alpha'} \right\} \tag{2.24}
\]
A familiar example stems from replacing the group $\mathbb{R}$ with $\mathbb{Z}_2$, and taking the Boltzmann weights to be,

$$B_1(g) = \begin{cases} 
  e^{-\beta}, & g = 0 \\
  e^{+\beta}, & g = 1
\end{cases} \quad (2.25)$$

This clearly reduces to the standard Ising model, where the $\mathbb{Z}_2$ group multiplication is implemented via addition. Furthermore, if the $\mathbb{Z}_2$ group is replaced with $\mathbb{Z}_N$ and the Boltzmann weights are choosing as,

$$B_1(g) = e^{-\kappa \cos(g)} \quad (2.26)$$

the standard clock-models are recovered. Including some direct products of $\mathbb{Z}_N$, e.g. taking the group to be $\mathbb{Z}_N \oplus \mathbb{Z}_N$, the generalized Ashkin-Teller models are obtained. Choosing the group to be $U(1)$, with an appropriate choice of the Boltzmann weight, leads to the $X - Y$ model. Hopefully, these few examples serve to demonstrate the power of this formalism - all of these very different models can be treated on equal footing and dealt with all at once.

### 2.3 Pure Gauge Theories

In this section, gauge theories will be written in the language introduced in section 1. To begin consider the partition function of pure gauge theory - the Yang-Mills action [77],

$$Z = \int \left[ \frac{DA}{\text{Vol}(G)} \right] \exp \left\{ -\frac{1}{4g^2} \int d^d x \, \text{Tr} \, F \wedge \ast F \right\} \quad (2.27)$$

where the components of the curvature two-form (field-strength) are,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + if_{abc} A_\mu^b A_\nu^c \quad (2.28)$$

with structure constants $f_{abc}$ and $\text{Vol}(G)$ is the volume of the gauge group. The gauge group will not be assumed Abelian until later on this section. In 1973 Wilson [75]
Figure 2.1: Yang-Mills theory on a lattice has dynamical variables living on the links of the lattice and take on values in the gauge group (not the algebra). The action is a function of the plaquette valued object \( U_p = U_{ij}U_{jk}U_{ki}U_{ii} \).

proposed a lattice analog of the Yang-Mills action in hopes to clarify quark confinement in quantum chromodynamics. Two years earlier Wegner [73] had derived the same action in the Abelian case. Lattice implementations of local gauge invariance leads to a simple geometric picture of the gauge fields - they are defined on the links of the lattice and take values in the Lie group rather than the Lie algebra. In lieu of deriving the discrete model from the continuum version, we will take the route that Wilson took, and postulate a lattice version and then demonstrate that in the continuum limit it reproduces the above partition function. As mentioned, Wilson’s choice is to take group valued objects living on the links of the lattice. This information can be encoded (much like the spins in the previous section) into a 1-chain \( U \equiv \sum_{l=1}^{N_1} U_l \ c_1^{(l)} \) where \( \{U_l\} \) denotes the group valued field on the link \( l \). This is clearly an element of the one-chain complex \( C_1(\Omega, \mathcal{G}) \) where \( \mathcal{G} \) is the relevant gauge group. The proposed partition function is,

\[
Z = \int \prod_{\text{links}} [dU_{ij}] \exp \left\{ -\kappa \sum_{p=1}^{N_2} \text{Re} \left( \text{Tr} \left( U_p + U_p^\dagger \right) \right) \right\} \tag{2.29}
\]
where \( U_p = U_{ij} U_{jk} U_{kl} U_{li} \) and the sites \( lijk \) make up the corners of the plaquette \( p \) (see figure 2.1). Here the lattice spacing is denoted by \( a \) and the coupling \( \kappa \) will be shown to be related to the gauge coupling \( g \), furthermore, the integration measure is the invariant Haar measure for the gauge group \( G \). As in the case of spin models, the gauge field configurations are in one-one correspondence with elements of the first chain complex, \( U \in C_1(\Omega, G) \). If \( G \) happens to be Abelian then the plaquette valued objects can be defined as the coboundary of \( U \) (see equation 2.17),

\[
U_p = \langle \delta U, c_2^{(p)} \rangle,
\]

although in this notation the group multiplication is implemented through addition.

The lattice (2.29) and continuum (2.27) definitions will now be demonstrated to be equivalent. Firstly, it should be clear that the extremum of the lattice action \( \text{Tr} (U_p + U_p^\dagger) \) is attained by those configurations in which the gauge fields are pure gauge,

\[
U_{ij} = g_i g_j^\dagger
\]

where, \( g_i \) are elements of the gauge group and are associated with the sites of the lattice. The dynamics of the fluctuations about this extremum will lead to the equivalence of the two models. In particular, consider an element of the gauge group that is close to the identity, such an element can be written as the exponential of a Lie-algebra valued field, \( A(x_{ij}) \). In continuum language, parallel transport between two space-time points is carried out by the path-ordered exponential of these Lie-algebra valued fields and in fact make up the link valued objects \( U_{ij} \),

\[
U_{ij} = \text{P} \exp \left\{ i \int_{x_i}^{x_j} A_\mu(x) \, dx^\mu \right\}
\]

On the lattice, since \( x_i \) and \( x_j \) differ by a single lattice spacing \( a \), the above expression can be approximated by,

\[
U_{ij} = g_i \exp \{ i a A_\mu(x_{ij}) \hat{\mu} \} g_j^\dagger
\]
where $U_{ij}$ is now an element of the gauge group which is perturbatively close to a pure gauge element. Inserting this ansatz into the definition of the plaquette valued group elements, $U_p$, one finds,

$$U_p = g_i e^{i a A_\mu(x_{ij}) \mu} e^{i a A_\nu(x_{jk}) \nu} e^{-i a A_\mu(x_{kl}) \mu} e^{-i a A_\nu(x_{li}) \nu} g_i^\dagger$$  \hfill (2.34)

By applying the Baker-Cambell-Hausdorff [74] formula, and ignoring terms of order $a^3$ and higher, this expression simplifies to,

$$U_p = g_i \exp \left\{ i a (A_\mu(x_{ij}) \mu + A_\nu(x_{jk}) \nu - A_\mu(x_{kl}) \mu - A_\nu(x_{ij}) \mu + \ldots) \right\}$$ \hfill (2.35)

A further simplification occurs upon noting that $x_{kl} = x_{ij} + a \nu$ and $x_{li} = x_{jk} - a \mu$. Expanding once again to order $a^2$ gives,

$$U_p = g_i \exp \left\{ i a^2 \left( \partial_\mu A_\nu(x_{ij}) - \partial_\nu A_\mu(x_{jk}) + i[A_\mu(x_{ij}), A_\nu(x_{ij})] \mu \nu + \ldots \right) \right\} g_i^\dagger$$ \hfill (2.36)

By noting that the commutator term can be written as $[A^b_\mu, A^c_\nu]^a = i f^{abc}_{\mu \nu} A^b_\mu A^c_\nu$, the argument of the exponential is recognized as the curvature two-form (2.28). Consequently, upon expanding the real part of the trace of $U_p$ to leading order in lattice spacing the following equality holds,

$$\kappa \sum_p \text{Tr}(U_p + U_p^\dagger) = a^{4-d} \kappa g^2 \int \left\{ \frac{1}{g^2 a^2} - \frac{1}{g^2} \text{Tr}(F_{\mu \nu}^2) + O(a^2) \right\}$$ \hfill (2.37)

By choosing $\kappa = a^{4-d}/g^2$ the continuum model is recovered in the limit of zero lattice spacing after rescaling the partition function to get rid of the overall irrelevant (infinite) constant.

Dimensional counting implies the following scaling dimensions: $[A_\mu/g] = \frac{1}{2}(d - 2)$ while $[g^2] = 4 - d$. Accordingly, the continuous theory is renormalizable in four dimensions, of course local gauge invariance is assumed to be preserved under renormalization.
Chapter 2. Statistical Models: Continuum to the Lattice

The higher order corrections left out of the analysis involve higher derivative terms and as such their canonical dimensions are greater than four. This implies that they should only serve to perform a finite renormalization on the coupling constant without changing qualitative effects. Thus the lattice action has been demonstrated to be equivalent to the continuum model.

Before proceeding to the next section, the lattice action will be written in terms of simplicial homology language. As already mentioned the group valued link variables can be encoded into a 1-chain, and performing the summation of all elements of the 1-chain complex replaces the summation over configurations. The argument of the Wilson action, in the Abelian case, was seen to reduce to the projection of the coboundary of this 1-chain onto a particular plaquette (see equation (2.30)). Consequently, the following simplicial model follows,

$$Z = \sum_{U \in C_1(\Omega, \mathcal{G})} N_2 \prod_{p=1}^{N_2} B_p \left( \langle \delta U, c_p^{(p)} \rangle \right)$$

(2.38)

The Boltzmann weights can be any arbitrary function which takes elements of $\mathcal{G}$ into the reals and is invariant under conjugation, i.e. the Boltzmann weights should be class functions. One clear choice for the Boltzmann weight is the Wilson action as in (2.29), however, there are other choices which lead to the identical continuum theories. Since only fluctuations around the identity element are relevant, the most important terms are quadratic in the fluctuations of the field. An interesting choice for such a function is the geometrical distance of $U_p$ from the identity element on the group manifold. A distance measure is induced on $\mathcal{G}$ by the invariant quadratic form, i.e. the Casimir, on the Lie algebra. For $\mathcal{G} = U(1)$, $U_p = e^{i\theta_p}$ and it seems natural to choose the Boltzmann weights as $\kappa \sum_p \theta_p^2$. Unfortunately this action does not respect the symmetry of the group, namely it is not invariant under $\theta \rightarrow \theta + 2\pi$. This defect can be fixed using a an idea due to Villain [72]. It amounts to writing a character expansion for the Boltzmann weights (this
Chapter 2. Statistical Models: Continuum to the Lattice

will be utilized many times in the next chapters),

$$B_p(U_p) = \sum_{r \in G^*} \chi_r(U_p) \exp(-\kappa C_2(r))$$

(2.39)

where $C_2(r)$ is the quadratic Casimir in representation $r$ and $\chi_r(U)$ is character of the element $U$ in representation $r$, i.e. $\chi_r(U) = \text{Tr}_r U$; the summation is over all irreducible representations, $G^*$, of the group $G$. Such a Villain action is valid for both Abelian and non-Abelian groups $G$. In case $G$ is Abelian, the set of irreducible representations of $G$ is in fact a group itself with group multiplication implemented through the tensor product of representations. This simplifies the analysis of Abelian groups significantly. Such a modified form for the action can be demonstrated to yield the same continuum results as the Wilsonian choice.

2.4 Gauge Theory with Matter Content

Thus far, the case of either a pure matter content theory (the scalar fields) or pure gauge content has been discussed. It is, however, possible to describe interacting theories in simplicial language as well. The general form of the lattice action for arbitrary Lie groups will be discussed first and the specialization to Abelian groups made later. In the continuum, interactions are introduced through replacing the derivative operators in the matter content with covariant derivatives. Consider the model introduced earlier for the scalar field (2.23),

$$Z = \sum_{x \in C_0(n, \mathbb{R}^n)} \prod_{i=1}^{N_i} B_i \left( \langle \delta x, c_i^{(0)} \rangle \right)$$

(2.40)

This is invariant under a global $G$ rotation since such a rotation can be absorbed into the definition of $x$. In analogy to continuum models, gauge fields can be introduced to lift this global symmetry to a local one. This is achieved by introducing the gauge fields $U_{ij} \in G$ which are associated with the links of the lattice. The difference between
this and the continuum version is that the relevant mathematical structure is the group themselves rather than their Lie-algebra. The advantage of this is that local symmetry is manifest and simple compared to the continuum version. The other advantages are clear, local discrete symmetries, which have no direct interpretation in the continuum limit, yet provide an arena for computation and analysis are easily accommodated. The modified version of the scalar field action given by

\[ S = \langle \delta x + U, c^{(l)}_1 \rangle, \quad (2.41) \]

where \( U \) is a 1-chain, is clearly invariant under local \( G \) transformations as shifts in \( x \) can be absorbed by a change in \( U \). At this point there is still no dynamics in the gauge fields, however, that problem has been dealt with in the previous section, and it is possible to simply add the kinetic term found there. The full model for a scalar field interacting with a gauge field then has the following general form,

\[ Z = \sum_{U \in C_1(G)} \sum_{x \in C_0(G)} \prod_{l=1}^{N_1} B^{(l)}_1 \left( \langle \delta x + U, c^{(l)}_1 \rangle \right) \prod_{p=1}^{N_2} B^{(2)}_p \left( \langle \delta U, c^{(p)}_2 \rangle \right) \quad (2.42) \]

where the Boltzmann weights \( B^{(1,2)} \) denote the interacting part of the theory and the kinetic term for the gauge fields respectively. The local symmetry is not spoiled by the kinetic term, since under a shift \( x \rightarrow x + x' \) where \( x' \) is an arbitrary element of the 0-chain complex the argument of the first term becomes, \( \delta x + U + \delta x' \). Under a further shift of \( U \rightarrow U - \delta x' \) the interacting part of the model is restored, while the argument of the kinetic term for the gauge fields becomes \( \delta U - \delta \delta x' \) this of course reduces to \( \delta U \) since the coboundary operator is nilpotent, and the original action is recovered demonstrating that the theory has local \( G \) symmetry.
2.5 The General Models: Putting it all together

In the previous sections, the cases of scalar fields and gauge fields along with possible minimal couplings between them have been discussed separately. In the language of simplicial homology these ideas are easily generalized to higher dimensional fields, for example Kalb-Ramond fields[35]. Comparing the two partition functions (2.23) and (2.38) a clear similarity is apparent: both fields are defined in the chain complex of a particular dimension and although the interactions occur on links in the case of the scalar fields, and on plaquettes in the case of the gauge fields, the Boltzmann weights depend solely on the coboundary of the chain in both cases. This suggest that the two models can be treated simultaneously by suitable choices of the functional form for the Boltzmann weights, and dimensions of the chain complex. Other than that, the two partition functions behave identically - a stark contrast to the continuum models. This suggests that the following general partition function to describe fields of higher rank be used,

\[ Z = \sum_{\sigma \in C_{k-1}} \prod_{l=1}^{N_k} B_p (\delta \sigma, c^{(l)}_k) \]  

(2.43)

These models can be easily demonstrated to have continuum limits, the particular model attained in this limit simply depends on the choice of the Boltzmann weights. Duality can be studied using this general model, and the analysis in the next chapter is devoted to understanding what happens under the strong-weak mapping mentioned in the introduction.

The model (2.43) just like the pure scalar and pure gauge theories enjoys only global symmetry with respect to \( \mathcal{G} \). Following the line of arguments mentioned in the previous section, it is possible to lift this symmetry to a local one, in doing a second field must be introduced, and new interactions appear. The natural generalization of (2.42) to higher
dimensions then leads to the partition function,

\[ Z = \sum_{U \in C_k(\Omega, g)} \sum_{x \in C_{k-1}(\Omega, g)} \prod_{l=1}^{N_k} B_l^{(k)}(\langle \delta x + U, c_k^{(l)} \rangle) \prod_{p=1}^{N_{k+1}} B_p^{(k+1)}(\langle \delta U, c_{k+1}^{(p)} \rangle) \]  

(2.44) 

the model is invariant under the transformations:

\[ x \to x + x' \quad , \quad U \to U - \delta x' \] 

(2.45) 

and expresses the expected local invariance. The duality transformations of this model will also be studied in the next chapter.

This completes the overview of the underlying models used throughout this thesis. Further generalizations will be made along the way, as the duality relations will clearly indicate that these models do not pose the property of self-duality. Instead, they must be modified by the inclusion of topological terms. The relevant modifications will be derived and its utility demonstrated in the later chapters, applying it to cases such as target-space duality, and the computation of correlation functions.
Chapter 3

Strong-Weak Duality

Strong-Weak duality was originally used by Kramers and Wannier [42] over half a century ago to obtain the critical temperature of the two-dimensional Ising model. Since this pioneering work there has been much interest in duality relations that map the weak coupling limit of one theory to the strong coupling of another, possible identical, theory. In this chapter, the duality transformations for models of the general type displayed in (2.43) and (2.44) will be derived. The complication that occurs here is that topology adds new physics into the problem, and leads to new types of self-dual models. Even in the case where the model is not self-dual, duality can be very useful. For example, duality will be applied to the case of a three dimensional gauge theory at finite temperature, on a compact spatial manifold, to derive the effective Coulomb gas picture. Topology modifies the usual Coulomb gas that arises and the physical implications for a such a modification are investigated. Since the duality transformations developed here will be used many times throughout the remainder of this thesis, much detail is included so that if any confusion occurs later on this chapter can be used as reference guide. Although topology obstructs models from being self-dual, certain modifications of the theories which restore the self-duality will be constructed. In particular, the duality mapping gives rise to disorder defects on the canonical cocycles of the lattice appear in either the original or dual model. These defects are similar to the familiar t'Hooft loops which are used to characterize the phase structure of gauge theories. The defects can be classified systematically and this the identification of self-dual models on spaces
with non-trivial topology. For example, given a model that is self-dual on an infinite space, the modification of the statistical sum which renders the model self-dual when some of the dimensions are compactified can be constructed systematically. The above discussion only applies to the case of single fields (i.e. models stemming from (2.43)), once interactions among fields of consecutive dimensions are included, it no longer becomes possible to define self-dual models when topological obstructions occur. However, it is still possible to apply the naive modifications that are expected to render the model self-dual. This will lead to other interesting effects in the dual theory which will be discussed in the latter half of this chapter.

3.1 Single Fields

3.1.1 Construction of the Dual Model

Using the language introduced in the previous chapter, the statistical models considered here can be stated as follows: the degrees of freedom take values in an Abelian group $\mathcal{G}$ and are defined on the $(k-1)$-cells of a lattice $\Omega$, and the Boltzmann weights are defined on the $k$-cells of the lattice, with argument equal to the coboundary of a $(k-1)$-chain,

$$Z = \sum_{g \in C_{k-1}(\Omega, \mathcal{G})} \prod_{i=1}^{N_k} B_i \left( (\delta g, c_k^{(i)}) \right)$$  \hspace{1cm} (3.1)

As mentioned in the previous chapter, this amounts to a generalization of nearest neighbour interactions. Here, and throughout our discussion, the Boltzmann weights, $B_i$, are allowed to differ on each $k$-cell. This allows a large class of models to be considered simultaneously, for example some of our results are easily generalized to random bond models. Also, it enables the partition function to be used as a generating function for certain correlators. Of particular interest is the correlator of a disorder operator and an anti-disorder operator. A detailed discussion of these operators will be deferred until
In the continuum, duality is usually performed by replacing the \((k-1)\)-form with a \(k\)-form (field-strengths) which satisfy a flatness condition, however, on topologically non-trivial lattices there are additional constraints which must be added. A priori the extra constraints are not known, consequently, here an alternative route will be taken. Once the duality transformations have been performed, the additional constraints which arise in the field-strength formulation will be explicitly derived in chapter 5.

The first step in the derivation is to expand the Boltzmann weights in (3.1) in terms of the characters, \(\chi_R(g)\) of the irreducible representations \(R \in G^*\),

\[
B_i(g) = \sum_{R \in G^*} b_i(R) \chi_R(g) \quad , \quad b_i(R) = \frac{1}{|G|} \sum_{h \in G} \overline{\chi}_R(h) B_i(h) \quad (3.2)
\]

where \(|G|\) is the order of the group. In the case of a continuous group the normalized sum over group elements is replaced by the Haar integration measure. Since \(G\) is Abelian, \(G^*\) inherits an Abelian group structure, where the product (taken to be addition) is implemented via the tensor product of representations of \(G\). The characters are trivial for Abelian groups, for example in the case of \(G = \mathbb{Z}_N\),

\[
\chi_r(h) = e^{ir \frac{2\pi h}{N}} \quad (3.3)
\]

where \(r = 0, \ldots, N-1\) labels the representations, and \(h = 0, \ldots, N-1\) the elements of the group. The characters then satisfies the following factorization properties,

\[
\begin{align*}
\chi_R(h_1 + h_2) &= \chi_R(h_1) \chi_R(h_2) \quad &R \in G^* \text{ and } h_1, h_2 \in G \\
\chi_R(a h_1) &= \chi_{aR}(h_1) \quad &a \in \mathbb{Z}
\end{align*} \quad (3.4)
\]

and satisfy the orthogonality relations,

\[
\begin{align*}
\sum_{g \in G} \chi_r(g) \overline{\chi}_{r'}(g) &= \delta_{G^*}(r - r') \quad , \quad r, r' \in G^* \\
\sum_{r \in G^*} \chi_r(g) \overline{\chi}_r(g') &= \delta_G(g - g') \quad , \quad g, g' \in G
\end{align*} \quad (3.5)
\]
Figure 3.1: Performing a character expansion on the Boltzmann weights in a two-dimensional spin model, \( k = 1 \), is achieved by introducing a representation of the group \( \mathcal{G}, \mathcal{G}^* \), on every link of the lattice.

where \( \bar{\chi}_r(g) \) is the character of \( g \) in the representation conjugate to \( r \), i.e. \( \chi_{-r}(g) \). The subscript on the delta function reminds us that it is \( \mathcal{G} \) or \( \mathcal{G}^* \) invariant delta-function.

Using these properties and on inserting (3.2) into the partition function (3.1) one finds,

\[
Z = \sum_{g \in C_{k-1}(\Omega, \mathcal{G})} \prod_{i=1}^{N_k} b_i(r_i) \chi_{r_i} \left( \langle \delta g, c_k^{(i)} \rangle \right)
\]

\[
= \sum_{g \in C_{k-1}(\Omega, \mathcal{G})} \sum_{r \in C_k(\Omega, \mathcal{G}^*)} \prod_{i=1}^{N_k} \left\{ b_i \left( \langle r, c_k^{(i)} \rangle \right) \chi_{(r, c_k^{(i)})} \left( \langle \delta g, c_k^{(i)} \rangle \right) \right\} \tag{3.6}
\]

Here, the product over \( k \)-cells has been interchanged with the sum over group representations. Every \( k \)-cell has been associated with a representation \( r_i \in \mathcal{G}^* \) and this information is encoded in the \( \mathcal{G}^* \) valued \( k \)-chain, \( r = \sum_{i=1}^{N_k} r_i c_k^{(i)} \) (see figure 3.1). Applying the factorization properties (3.4) to the product of characters in (3.6) leads to the following,

\[
\prod_{i=1}^{N_k} \chi_{(r, c_k^{(i)})} \left( \langle \delta g, c_k^{(i)} \rangle \right) = \prod_{j=1}^{N_{k-1}} \chi_{(\partial r, c_{k-1})} \left( \langle g, c_{k-1}^{(j)} \rangle \right) \tag{3.7}
\]

which follows directly from linearity of the inner product and the definition of the
coboundary operator. Including the sum over $g$ which appears in (3.6) is now a trivial matter,

$$\sum_{g \in C_{k-1}(\Omega, G)} \prod_{j=1}^{N_{k-1}} \chi_{(\partial r, c_k^{(j)})} \left(\langle g, c_k^{(j)} \rangle\right) = |G|^{N_{k-1}} \prod_{j=1}^{N_{k-1}} \delta_{G^*} \left(\langle \partial r, c_k^{(j)} \rangle\right)$$  \hspace{1cm} (3.8)

This equality follows from the orthogonality of the characters (3.5). The partition function now depends on $k$-chains with constraints,

$$Z = |G|^{N_{k-1}} \sum_{r \in C_k(\Omega, G^*)} \prod_{i=1}^{N_k} b_i \left(\langle r, c_k^{(i)} \rangle\right) \prod_{j=1}^{N_{k-1}} \delta_{G^*} \left(\langle \partial r, c_k^{(j)} \rangle\right)$$  \hspace{1cm} (3.9)

However, the constraints simply force $r$ to be an exact chain ($\partial r = 0$). The most general exact chain can be expanded as a sum of two terms: a boundary of a chain of one dimension higher and an element of the homology group under inclusion into the chain group,

$$r = \partial r' + \sum_{a=1}^{A_k} h_a \gamma_a , \quad r' \in C_{k+1}(\Omega, G^*)$$  \hspace{1cm} (3.10)

where $\{\gamma_a : a = 1, \ldots, A_k\}$ are the generators of $H_k(\Omega, G^*)$. This group consists of a direct product of groups, i.e. $H_k(\Omega, G^*) \cong \bigoplus_{a=1}^{A_k} H_{k,a}(\Omega, G^*)$ and the coefficients occurring in (3.10) are taken to be elements of these sub-groups: $h_a \in H_{k,a}(\Omega, G^*)$. Inserting the above ansatz for $r$ into the partition function allows us to remove the constraints to obtain,

$$Z = |G|^{N_{k-1} - d_k} \sum_{h \in H_k(\Omega, G^*)} \sum_{r \in C_{k+1}(\Omega, G^*)} \prod_{i=1}^{N_k} b_i \left(\langle h + \partial r, c_k^{(i)} \rangle\right)$$  \hspace{1cm} (3.11)

Here the prime on the dummy index $r'$ has been removed, and $d_k \equiv \dim \ker \partial_k$.

The final step in the duality transformation is to re-interpret $c_k^{(i)}$ as elements on the dual lattice (see discussion at the end of section 2.1). This reinterpretation reduces the dual partition function to the compact expression,

$$Z = |G|^{N_{d-1} - d_{d-k-1}} \sum_{h \in H_{d-k}(\Omega^*, G^*)} \sum_{r \in C_{d-k-1}(\Omega^*, G^*)} \prod_{i=1}^{N_{d-k}} b_i \left(\langle h + \delta r, c_{d-k}^{(i)} \rangle\right)$$  \hspace{1cm} (3.12)
where $D_{d-k}^* = d_k$ and is the dimension of the kernel of $\delta_{d-k}$ on the dual lattice. Notice that $h$ is now an element in the $(d-k)^{th}$ cohomology group of the dual lattice, which is isomorphic to the $k^{th}$ homology group of the lattice. There is an interesting symmetry in this partition function which stems from making a different choice of a representative element of the cohomology group: $h \rightarrow h + \delta r'$, this change can be absorbed into a redefinition of $r$: $r \rightarrow r - \delta r'$. This symmetry represents the choice in choosing a representative "surface" of the homology and hence of the dual "surface" of the cohomology. It is reminiscent of the fact that in 3-d compact \text{U}(1), the computation of a Wilson loop correlator is supplemented by an additional sum over all surfaces whose boundary is the Wilson loop you are computing. However, in the present context it is explicitly obtained by going to the dual model, while in the original work of Polyakov [50] it had to be put in by hand in order to have invariance under the choice of the surface.

The above analysis shows that in general, the dual theory has additional topological degrees of freedom which be viewed as disorder terms corresponding to the generators of the cohomology group of the dual lattice. By comparing (3.12) and (3.1), it is clear that, in order for (3.1) to be self-dual, $H_k(\Omega, G^*)$ has to be trivial (the appearance of cocycles in duality transformations was also appreciated by Rakowski and Sen [55, 54]). Of course, in addition, the lattice must be self-dual, $k$ must equal $d - k$ so that the original and dual degrees of freedom are defined on the same type of cells and $G$ must be isomorphic to $G^*$.\footnote{$G \cong G^*$ for any finite Abelian group. An example of a group where this does not hold is $U(1)$ where $U(1)^* \cong \mathbb{Z}$.} This duality is interpreted in the general sense where the Boltzmann factors $B_i(g)$ for each group element $g$ can be regarded as independent coupling constants and the dual Boltzmann factors $b_i(r)$ for each $r \in G^* \cong G$ are the images of these constants under the duality transformation. In the normal sense of duality a more specific manner in which $B(g)$ transforms into $b(r)$ is required, in particular, they are required to have the same
functional form.

3.1.2 Examples

In this section, a few uses of the duality transformation will be illustrated. For the most part, the models discussed in this section are well-known and will be presented in standard language as opposed to the simplicial language used in the previous section. This allows the focus to be on the outcome of the duality relations rather than the technical details in deriving them. In the first subsection the case of a two-dimensional Ising model on a torus will be discussed. The second subsection will deal with another model that is self-dual in the flat case, a four dimensional $\mathbb{Z}_N$ gauge theory. Finally, in the last subsection a discussion of the three dimensional $U(1)$ gauge theory, its Coulomb gas representation and subsequent Sine-Gordon model will be given.

The Two Dimensional Ising Model

Of course, the Ising model on an infinite plane satisfies all the requirements of self-duality. However, suppose both directions are compactified, so that the model is now defined on a torus. Topology is the only obstruction to that model being self-dual. To see this explicitly the partition function for the dual model will be given shortly. Firstly note that the generators of $H^1(\Omega^*, G^*)$ are the cocycles $h^1 = \sum_{\ell \in \gamma^1} c_1^{(\ell)}$ and $h^2 = \sum_{\ell \in \gamma^2} c_1^{(\ell)}$, where the sums are over the links in $\gamma^i$ shown in figure 1.4. These cocycles are the duals to the cycles $\gamma_i$ which generate the homology group on the original lattice. Using (3.12), the dual of the Ising model on the torus has partition function,

$$ Z = c(\beta^*) \sum_{h_{1,2} = \pm 1} \sum_{\{r_i = \pm 1\}} \exp \left\{ \beta^* \sum_{\langle ij \rangle} \left[ h_1^{\varepsilon(\gamma^1;ij)} h_2^{\varepsilon(\gamma^2;ij)} r_i r_j \right] \right\} \quad (3.13) $$
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here \( i \) labels the sites on the dual lattice, \((ij)\) are nearest neighbours forming a link and,

\[
\varepsilon(\gamma^k, ij) = \langle \gamma^k, c^*(ij) \rangle = \left\{ \begin{array}{ll}
1 & \text{if } (ij) \text{ is in } \gamma^k \\
0 & \text{otherwise}
\end{array} \right.
\]

(3.14)

In (3.13) the dual coupling constant \( \beta^* = -\frac{1}{2} \ln \tanh \beta \) was introduced and the constant in front of the summations \( c(\beta^*) = (2 \sinh(2\beta^*))^{-\frac{\kappa^1}{2}} \) appears as a result of the character expansion. The dual partition function is then the sum of four Ising models in which the couplings along the links in \((\gamma^1, \gamma^2)\) are taken to be \((\beta^*, \beta^*), (-\beta^*, \beta^*), (\beta^*, -\beta^*) \) and \((-\beta^*, -\beta^*)\). There is a close resemblance between these four “twists” in the coupling and the periodic/anti-periodic boundary conditions that are applied to a fermion on the torus - the spin-structure. However, there is another interpretation of these extra twists, and that is as disorder defects [34, 20] along the canonical cocycles. Of course, the partition function is not the expectation value of a dis-order variable, since the coboundary of the set of links which make up the cocycle vanish identically (if it did not then the partition function would look like an expectation value of a dis-order defect located where the coboundary had support, more about this will be said in chapter 6).

A cartoon picture that can be attached to the dual construction is as follows:

(i) cut the torus along the two closed paths on the original lattice so that the lattice now looks rectangular

(ii) perform the usual duality transformation on this flat lattice neglecting boundary effects

(iii) sew the lattice along one direction to form a cylinder. Along those dual links just sewn, sum over all relative phases between the site variables at their end points.

(iv) sew the remaining ends together to form a torus. Along those dual links just sewn, sum over all relative phases between the site variables at their end points.
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Figure 3.2: The phase structure of a four dimensional $\mathbb{Z}_N$ gauge theory. Here $x = 1 - e^{-kT}$ so that zero temperature corresponds to $x = 0$ and infinite temperature $x = 1$. The phases of the theories in these three regimes from left to right are: a) a phase which confine electric charges b) free phase with no confinement and c) a phase which confines magnetic charges.

This procedure is depicted in figure 1.6 and is very reminiscent to summing over spin structures. This picture generalizes quite easily to any spin model on a Riemann surface.

Four Dimensional Gauge Theories

Gauge theories in four dimensional infinitely flat space-times are also known to be self-dual. Now consider compactifying the imaginary time direction so that the topology is $\mathbb{R}^3 \times S^1$, this corresponds to treating the system at finite temperature. Using the dual partition function (3.12), the relevant cohomology group for a gauge theory is seen to be $H^2(\Omega^*, \mathcal{G}^*)$ which, since all two-surfaces are contractable in this topology, is trivial. Therefore, a gauge theory at finite temperature\(^2\) is in fact self-dual! Consider the case of $\mathbb{Z}_2$-gauge theory, with the same Boltzmann weights as in the Ising model; of course, the

\(^2\)With $\mathbb{Z}_N$ gauge group - which is required so that the group of representations of the gauge group is isomorphic to the gauge group itself.
dual model will have the same dual coupling as in the Ising case. Consequently, strong and weak coupling limits of the model are interchanged under duality. It is possible to construct a qualitative picture of the finite-temperature phase diagram for this gauge theory. At zero temperature there is only one fixed point under duality, \( \beta_c = \beta_c^* = \frac{1}{2} \ln(1 + \sqrt{2}) \). As in the two-dimensional Ising model, this fixed point corresponds to the point where a phase transition takes place. In the opposite limit, infinite temperature, the partition function factorizes into a product of a three-dimensional Ising model and a three-dimensional \( \mathbb{Z}_2 \) gauge theory,

\[
Z \xrightarrow{T \to \infty} Z_{IG}(\beta) \cdot Z_I(\beta)
\]  

(3.15)

Under duality these two terms in the partition function map into one another. However, it is possible to perform the duality transformation on only one of the factors, say \( Z_{IG} \). This is then mapped into the three-dimensional Ising model with dual coupling, so that,

\[
Z \xrightarrow{T \to \infty} c(\beta^*) \cdot Z_I(\beta^*) \cdot Z_I(\beta)
\]  

(3.16)

where \( c(\beta^*) \) is the same factor as in the previous section. It is not clear where the phase transition point occurs, however, from this form, it is clear that it must occur at the same point as the three-dimensional Ising model. Through strong coupling expansions this number can be computed numerically, \( \beta_2 \). However, since the two terms are interchanged under duality, a second phase transition point exists: \( \beta_3 = \beta_2^* \). Now, using the fact that the full model remains self-dual for all radii of compactification, the qualitative picture of the finite-temperature four-dimensional \( \mathbb{Z}_2 \) gauge model given in figure 3.2 is arrived at. The dotted lines at finite temperature are only guides, as they are not known exactly. However, once one branch is known, the second is obtained via the duality map. It is quite plausible that these lines do not cross, as one would expect them to be monotonic functions \( \beta_c \) with \( \beta_{2,3} \).
As second gauge theory example, consider a lattice which contains topological ob­structions to duality. For instance, take the topology of the lattice to be $S^2 \times S^2$. The generators of the second cohomology group on the dual lattice are the set of plaquettes dual to the generators of the second homology on the original lattice. There are two such generators, the set of plaquettes enclosing one of the spheres and the set of plaques enclosing the second sphere. This is very much like the 2d-Ising case, since there are two generators of the relevant cohomology group. Since this is a gauge theory on a four-dimensional lattice, the model would be self-dual if there was trivial cohomology. However, the dual model now contains extra topological fields and the dual partition function is (take the gauge group to be $\mathbb{Z}_2$),

$$Z = c(\beta^*) \sum_{h_1, h_2 = \pm 1} \sum_{U_i = \pm 1} \exp \left\{ -\beta^* \sum_{\square} h_1^c(\gamma^{*1}; \square) h_2^c(\gamma^{*2}; \square) \prod_{l \in \square} U_l \right\}$$

(3.17)

where $\square$ label the plaquettes on the lattice, $l \in \square$ are the oriented links in that plaquette and,

$$c(\gamma^{*k}; \square) = \langle \gamma^{*k}, c_1^\square \rangle = \left\{ \begin{array}{ll} 1 & \text{if } \square \text{ is in } \gamma^{*k} \\ 0 & \text{otherwise} \end{array} \right.$$  

The dual coupling is identical to the Ising case, and the overall constant, $c(\beta^*) = (2\sinh(2\beta^*))^{-N_2/2}$, differs only in the power. It is a little more difficult to visualize this case, however, it is quite similar to the Ising model. The cartoon picture is: take the plaquettes forming a sphere enclosing one of the $S^2$'s, these plaquettes cut a series of plaquettes on the dual lattice, sew the dual lattice back together with all possible twists in the coupling coupling constants sign.

**Gauge Theory in Three Dimensions**

As an application of the duality transformations to a model in which the dual variables are not associated with cells of the same dimension as the original ones, consider compact
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$U(1)$ gauge theory at finite temperature. As usual, finite temperature implies that the time direction is compactified, and hence the topology of the lattice is chosen to be $S^2 \times S^1$. The $S^2$ should be considered as the spatial directions. Since this is a three dimensional gauge theory the cohomology group which appears in the dual model is $H^1(\Omega, \mathbb{Z})$. In this topology there is only one generator: $\gamma = \sum_{\gamma \in \gamma} c^{*}_{1}(\gamma)$ where the set of links, $\gamma$, are the links pointing outward from the $S^2$ into the compact time direction. Of course, they can be thought of as the set of links dual to the generator of $H_2(\Omega, \mathbb{Z})$ which is given by the set of plaquettes enclosing the $S^2$. Then $\gamma$ above consists of those links which pierce these plaquettes. The dual model is straightforward to write down,

$$Z = \sum_{h \in \mathbb{Z}} \sum_{x \in \mathbb{Z}} \exp \left\{ \frac{1}{2\beta} \sum_{(ij)} (x_i - x_j + h \cdot \varepsilon (\gamma; ij))^2 \right\}$$

$$= \sum_{h \in H^1(\Omega, \mathbb{Z})} \sum_{x \in \mathbb{Z}} \exp \left\{ \frac{1}{2\beta} \left\{ \langle x, \Delta x \rangle + 2\langle x, \partial h \rangle + \langle h, h \rangle \right\} \right\} \quad (3.18)$$

where the Villain form [72] for the Boltzmann weight has been chosen,

$$B(\theta) = \sum_{\tau \in \mathbb{Z}} e^{-\beta(\theta - 2\pi \tau)^2} \quad (3.19)$$

as usual $\varepsilon$ has support on those links contained in $\gamma$; and $\Delta = \partial \delta$ is the lattice Laplacian. Compact $U(1)$ admits a formulation in terms of a Coulomb gas of monopoles, at finite temperature this is no different, however, the topological term serves as an external source which interacts with the gas. To place the partition function in its Coulomb gas form, a Poisson resummation of (3.18) must be performed. By allowing $x$ to range over all real numbers, and constraining it to be integer via a Lagrange multiplier field and then performing the integration over $x$, a second dual formulation of (3.18) known as the Coulomb gas representation [50] is obtained,

$$Z = \sum_{h \in H^1(\Omega, \mathbb{Z})} \sum_{\lambda \in \mathbb{Z}} \sum_{x \in \mathbb{Z}} \exp \left\{ \frac{1}{2\beta} \left\{ \langle x, \Delta x \rangle + \langle x, (2\partial h + 2\pi i \lambda) \rangle + \langle h, h \rangle \right\} \right\}$$

$$= \sum_{h \in H^1(\Omega, \mathbb{Z})} \sum_{\lambda \in \mathbb{Z}} \exp \left\{ \frac{1}{2\beta} \left\{ \langle \partial h + \pi i \lambda, \Delta^{-1}(\partial h + \pi i \lambda) \rangle - \langle h, h \rangle \right\} \right\} \quad (3.20)$$
Ignoring the topological terms (setting \( h = 0 \)) the action takes on the form,

\[
S_{CG} = -\frac{\pi^2}{2\beta} \sum_{x,y} \lambda_x \Delta^{-1}(x, y) \lambda_y
\]  

(3.21)

In three dimensions the inverse Laplacian is \( \sim |x - y|^{-1} \), and it is apparent that the point-like objects, \( \lambda_i \), interact through a Coulombic force. In the weak coupling regime only the terms with \( \lambda_i = \pm 1 \) contribute heavily to the sum, and the summations over all charges can be truncated (keeping them amounts to keeping higher harmonics in the effective Sine-Gordon theory to appear below). In that case, the summation over \((x, y)\) can be separated into those terms with \( x \neq y \) and those with \( x = y \). The latter contains a factor of the inverse Laplacian with arguments at the same point. This naively diverges, however, since there is a natural ultraviolet cut-off provided by the lattice its contribution is finite (but large). Denote this factor by \( \mu \), then (with the restriction \( \lambda_i = \pm 1 \)) the partition function is,

\[
Z = \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \int d^3 x_1 \ldots \int d^3 x_N \sum_{\lambda_i = \pm 1} \exp \left\{ -\frac{\pi^2}{\beta} \sum_{i<j} \lambda_i \Delta^{-1}(x_i - x_j) \lambda_j \right\}
\]  

(3.22)

The terms in the above expression need a little explanation, \( \xi \equiv e^{-\text{const.} / \beta} \) and acts as the fugacity for the Coulomb gas, \( \xi^N \) takes place of the terms in the action with \( x = y \); the factor of \( N! \) appears because the charges are bosonic; while the integrals arise in the limit of zero lattice spacing (\( x_i \) is the location of the \( i^{\text{th}} \) charge \( \lambda_i \)). It is now possible to introduce an auxiliary field the represent the inverse Laplacian,

\[
Z = \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \int d^3 x_1 \ldots \int d^3 x_N \sum_{\lambda_i = \pm 1} \int D\phi \exp \left\{ -\beta \int d^3 x \phi \Delta \phi - \beta i \sum_{i=1}^{N} \lambda_i \phi(x_i) \right\}
\]  

\[
= \int D\phi \ e^{-\beta \int d^3 x \phi \Delta \phi} \sum_{N=0}^{\infty} \frac{\xi^N}{N!} \int d^3 x_1 \ldots \int d^3 x_N \cos(\phi(x_1)) \ldots \cos(\phi(x_N))
\]  

\[
= \int D\phi \ e^{-\beta \int d^3 x \left\{ (\partial_i \phi)^2 - \frac{\xi}{2} \cos(\phi) \right\}}
\]  

(3.23)
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This is dual Sine-Gordon (SG) action. Including the topological term causes no problems and the effective SG action dual to finite temperature compact $U(1)$ is found to be,

$$Z = \sum_{h \in H^1(\Omega, \mathbb{Z})} \int \mathcal{D}\phi \ e^{-\beta \int \left\{ (d\phi + h)/\star (d\phi + h) + \frac{\delta}{2} \cos(\phi) \sqrt{-g} \right\} \ d^3x}$$

(3.24)

The action has been written in terms of differential forms for convenience. To understand what effects the topological fields $h$ have on this action let us consider for the moment the case in which the cosine interaction is absent. Then it is possible to introduce a one-form in place of $d\phi + h$,

$$Z = \int \mathcal{D}A \ \delta (\star DA) \ \sum_{n \in \mathbb{Z}} \delta \left( \Omega A - 2\pi n \right) \ e^{-\beta \int A^\star A}$$

(3.25)

The first constraint implies that $A$ is a flat one-form, and is therefore the sum of the exterior derivative of a zero-form and an element of the cohomology both with coefficients in $\mathbb{R}$. However, the cohomology coefficients must be forced to be integer in order for integration over $A$ to yield (3.24), hence the appearance of the holonomy constraints. Therefore, if $d\phi + h$ is replaced by $d\phi'$, then the scalar field $\phi'$ is defined on the group $\mathbb{R}/\mathbb{Z} = U(1)$ - that is, it is a scalar field living in a circular target-space. This is the required result, the dual Sine-Gordon picture of compact $U(1)$ is unaltered from the usual form, with the exception that the scalar field are now defined in a circular target-space.

3.1.3 Introducing Topological Terms

The dual theory on a lattice with non-trivial cohomology has been demonstrated to contain extra topological terms. Consequently, models defined on spaces with non-trivial topology are hindered from being self-dual. The fact that topological modes appear in the dual theory suggests that including some of them in the original model might perhaps cancel some would-be modes in the dual theory. This could lead to a restoration of self-duality. It is not difficult to check that this picture is in fact correct, and in this
section, the analysis carried out previously will be repeated, however, it will be applied
to a modified partition function,

\[ Z = \sum_{h \in H^k_{(A)}(\Omega, \mathcal{G})} \sum_{g \in C_{k-1}(\Omega, \mathcal{G})} \prod_{i=1}^{N_k} B_i \left( \left( (\delta g + h), c^{(i)}_k \right) \right) \]  

(3.26)

Here \( H^k_{(A)}(\Omega, \mathcal{G}) \) is a subgroup of \( H^k(\Omega, \mathcal{G}) \) generated by a subset of the generators of the
full cohomology group on the original lattice, \( \{ \gamma^a : a \in A \subseteq \{1, \ldots, A_k\} \} \). An element
\( h \in H^k_{(A)}(\Omega, \mathcal{G}) \) is written as, \( h = \sum_{a \in A} h^a \gamma^a \) with \( h^a \in H^{k,a}(\Omega, \mathcal{G}) \) and the generating
cocycles are \( \gamma^a = \sum_{t \in \tau^a} c^{(t)}_k \). It should be clear that a sum over a subset (and not the
entire group) of the dis-order loops, which appear in the dual model, has been introduced
into the original defining partition function. For example, consider the case of the Ising
model on the torus and take \( A \) to be the single element which labels one of the two
canonical cocycles on the original lattice. Then this partition function represents cutting
the lattice along the canonical cycle on the dual lattice, and sewing it back together with
the two twists in the coupling constant. Of course, both cocycles could be placed in the
defining model; that would, however, correspond to the dual of the model without any
topological terms, consequently, the duality transformation will reproduce the partition
function without any topological terms (since performing duality twice is the identity
operation). It then seems likely that under the duality transformation the model with a
single cocycle will be mapped into a model with a single cocycle. As suggested earlier
this intuition is correct, however, it leaves open the questions as to which cocycle appears
in the dual theory. Let us now perform the duality transformations explicitly and obtain
the exact mapping.

Once again the first step in the duality transformation is to perform a character
expansion of the Boltzmann weights,

\[ Z = \sum_{r \in C_k(\Omega, \mathcal{G}^*)} \sum_{h \in H^k_{(A)}(\Omega, \mathcal{G})} \sum_{g \in C_{k-1}(\Omega, \mathcal{G})} \prod_{i=1}^{N_k} \left\{ b_i \left( (r, c^{(i)}_k) \right) \frac{1}{\chi_{(r, c^{(i)}_k)}} \left( \left( (\delta g + h), c^{(i)}_k \right) \right) \right\} \]  

(3.27)
In the above, the representations that are defined on the \( k \)-cells has been encoded in the \( k \)-chain \( r \), and introduced the character coefficients \( b_i(\langle r, c_{k}^{(i)} \rangle) \) as in equation (3.2). The factorization properties of the characters (3.4) allows the partition function to be factorized,

\[
Z = \sum_{r \in C_k(\Omega, \mathcal{G}^*)} \frac{N_k}{\prod_{i=1}^{N_k} b_i(\langle r, c_{k}^{(i)} \rangle)} \times \sum_{g \in C_{k-1}(\Omega, \mathcal{G})} \prod_{j=1}^{N_k} \chi_{(r, c_{k}^{(j)})} \left( \langle \delta g, c_{k}^{(j)} \rangle \right) \sum_{h \in H_{(\gamma)}^k(\Omega, \mathcal{G})} \prod_{i=1}^{N_k} \chi_{(r, c_{k}^{(i)})} \left( \langle h, c_{k}^{(i)} \rangle \right)
\]  

(3.28)

The sum over \( g \) was performed previously and produced a delta function forcing \( r \) to be an arbitrary exact \( k \)-chain (see (3.7) and (3.8)). The sum over the cohomology elements will force additional constraints on the representations. Using the factorization properties of the characters and the explicit representation \( h = \sum_{a \in \mathcal{A}} h^a \gamma^a \) one finds,

\[
\chi_{(r, c_{k}^{(i)})} \left( \langle h, c_{k}^{(i)} \rangle \right) = \prod_{a \in \mathcal{A}} \chi_{(r, c_{k}^{(i)})} \left( h^a \langle \gamma^a, c_{k}^{(i)} \rangle \right) = \prod_{a \in \mathcal{A}} \chi_{(r, \gamma^a)} \left( h^a \right)
\]

(3.29)

Performing the sum over \( H_{k,a} \) and product over \( k \)-cells yields,

\[
\sum_{h \in H_{(\gamma)}^k(\Omega, \mathcal{G})} \prod_{i=1}^{N_k} \chi_{(r, c_{k}^{(i)})} \left( \langle h, c_{k}^{(i)} \rangle \right) = \prod_{a \in \mathcal{A}} |H_{k,a}(\Omega, \mathcal{G})| \delta_{(H_{k,a}(\Omega, \mathcal{G})^*)} \left( \langle r, \gamma^a \rangle \right)
\]

(3.30)

Therefore, \( \langle r, \gamma^a \rangle \in \mathcal{G}^* \) is forced to be a trivial element in the group dual to \( H_{k,a}(\Omega, \mathcal{G}) \). Inserting the two constraints into the partition function yields,

\[
Z \propto \sum_{r \in C_k(\Omega, \mathcal{G}^*)} \left( \prod_{i=1}^{N_k} b_i(\langle r, c_{k}^{(i)} \rangle) \right) \delta_{\mathcal{G}^*} (\partial_\gamma) \prod_{a \in \mathcal{A}} \delta_{H_{k,a}(\Omega, \mathcal{G})^*} \left( \langle r, \gamma^a \rangle \right)
\]

(3.31)

where the constant of proportionality is: \( |\mathcal{G}|^{N_{k-1}-d_{k+1}} |H| \) with \( |H| = \prod_{a \in \mathcal{A}} |H_{k,a}(\Omega, \mathcal{G})| \).

It is possible to solve the constraints and remove the delta functions. Since \( r \) is forced to be exact, take it to be of the form (3.10). The other constraints then allows some of the coefficients \( h_b \) to be determined. On lattices with no boundaries or torsion, there exists an isomorphism between cocycles and cycles induced by the inner product. The
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generators can be paired in the following way: \( \langle \gamma_a, \gamma^b \rangle = \delta_a^b \) (the inner product is unity since the canonical generators can be chosen to intersect on only one cell; compare Figure 1.6). The remaining constraints can then be easily solved,

\[
(\partial r', \gamma^a) + \sum_{b=1}^{A_k} h_b(\gamma_b, \gamma^a) = 0 \quad \Rightarrow \quad h_a = -(\partial r', \gamma^a) = -\langle r', \delta \gamma^a \rangle = 0 \quad \text{for } a \in A \quad (3.32)
\]

the last equality follows since \( \{ \gamma^a \} \) are cocycles and therefore have zero coboundary. Since \( H_{k,a}(\Omega, \mathcal{G}^*) \cong (H^{k,a}(\Omega, \mathcal{G}))^* \) for lattices with no torsion, the zero element need not be carefully defined. Consequently, the additional constraints force the coefficients of the cycles "dual" to the cocycles to vanish. Using this solution to the constraints renders the partition function in the following form,

\[
Z = |G|^N_{k-1-d_{k+1}} |H| \sum_{h \in H^{d_{k+1}}(\Omega, \mathcal{G}^*)} \sum_{r \in C_{k+1}(\Omega, \mathcal{G}^*)} \prod_{i=1}^{N_k} b_i \left( \langle (\partial r + h), c^{(i)}_k \rangle \right) \quad (3.33)
\]

In the above \( \overline{A} \) denotes the compliment of the set \( A \), so that the homology subgroup appearing here is generated by those cycles which are not dual to the cocycles appearing in the defining model (3.26). Interpreting this partition function on the dual lattice by making the relevant associations leads to,

\[
Z = |G|^N_{d_{k+1}} |H| \sum_{h \in H^{d_{k}}(\Omega^*, \mathcal{G}^*)} \sum_{r \in C_{d_{k}-1}(\Omega^*, \mathcal{G}^*)} \prod_{i=1}^{N_{d_{k}}^*} b_i \left( \langle (\delta r + h), c^{(i)}_{d_{k}-1} \rangle \right) \quad (3.34)
\]

The generators of \( H^{d_{k}-k}_{(\overline{A})}(\Omega^*, \mathcal{G}^*) \) are the cocycles on the dual lattice that are associated with the set of cycles \( \{ \gamma_b : b \in \overline{A} \} \) on the original lattice. This is the final form of the dual model. After motivating the generation of self-dual models, several examples that illustrate the abstract formalism derived here will be given. Particular choices of the set \( A \) will be shown to lead to models which are explicitly self-dual and certain linear combinations of models will be introduced which form self-dual models.
3.1.4 Self-Dual Clock Model on a Torus

As demonstrated in a previous subsection, some of the topological terms are eliminated in going from the original model to the dual model. The current goal is then to obtain explicitly self-dual models on topological non-trivial lattices. In order to illustrate the physical content of the dual model the $\mathbb{Z}_N$ model on the torus with topological terms will be discussed in some detail. Denoting elements of $\mathbb{Z}_N$ by $n = 0, \ldots, N - 1$, and the representations of $\mathbb{Z}_N$, which is isomorphic to $\mathbb{Z}_N$, are labeled by the integers $r = 0, \ldots, N - 1$. Furthermore, the characters of $\mathbb{Z}_N$ are simply $\chi_r(n) = \exp(i rn2\pi/N)$.

The most general case on the torus is given by taking $A$ to be one of the following subsets of the generators of the first cohomology group: $s_1 = \{0\}$, $s_2 = \{\gamma^1\}$, $s_3 = \{\gamma^2\}$ or $s_4 = \{\gamma^1, \gamma^2\}$ (compare figure 1.6). For a given choice of $s_a$ the model is defined by the following partition function,

$$Z(s_a, \beta) = \sum_{\{n_i=0\}} \left( \prod_{\gamma \in s_a} \sum_{h_\gamma=0}^{N-1} \right) \prod_{(ij)} B(n_i - n_j + \epsilon(\gamma; ij)h_\gamma)$$ (3.35)

for $a = 1, 2, 3$ or $4$, $\epsilon(\gamma; ij)$ has support only on $\gamma$ and is defined in (3.14). The Boltzmann weights are $\mathbb{Z}_N$ invariant (which is generated by a shift of $N$ in its argument) and is given in Villain form [72] by (equal on all links)

$$B(n) = \sum_{j \in \mathbb{Z}} \exp \left\{ -\frac{\beta}{2} \left( \frac{n}{N} - j \right)^2 \right\}.$$ (3.36)

The coefficients $b(r)$ of the character expansion are found via the character decomposition,

$$b(r) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{j \in \mathbb{Z}} e^{-\frac{\beta}{N}\left(\frac{n}{N} - j\right)^2} e^{-i \frac{2\pi}{N} rn}$$

$$= \frac{1}{N} \sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2N^2} n^2} e^{-i \frac{2\pi}{N} rn} = \sqrt{\frac{2\pi}{\beta}} \sum_{m \in \mathbb{Z}} e^{-\frac{\beta^*}{4} \left(\frac{n}{N} - m\right)^2}$$ (3.37)

Here $\beta^* = (2\pi)^2 N^2 \beta^{-1}$ is the dual coupling constant. Consequently, $b(r)$ has the same functional form as $B(n)$. The four partition functions defined here have a rather simple
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Figure 3.3: The transformation of the topological terms in 2-d models on a torus. (a) depicts those cocycles on the original lattice included in the defining model. (b) displays the cycles dual to the cocycles displayed in (a), the constraint (3.32) forces the coefficients of these cycles to vanish. (c) displays the cycles which have non-vanishing coefficients allowed by the constraints. (d) displays the cocycles on the dual lattice which are dual to the cycles in (c).

interpretation. As was pointed earlier, the topological sums can be viewed as summing over spin structures on the lattice, as such (in the case $N = 2$) the four partition functions introduced here can be viewed as applying periodic or anti-periodic boundary conditions on an Ising model defined on a finite two-dimensional lattice.

Applying the rules derived in the previous section on the cancellation of the cycles, the following transformation properties of $Z(s_a, \beta)$ under duality are found,

$$Z(s_a, \beta) = \left(\frac{4\pi}{\beta}\right)^{N_0} \sum_{b=1}^{4} M_{ab} Z(s_b, \beta^*)$$  

(3.38)
where the transformation matrix $M_{ab}$ is given by

$$
M = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
4 & 0 & 0 & 0
\end{pmatrix}
$$

(3.39)

The matrix $M$ obeys $M^2 = 1$, as it should, since applying the duality transformation twice should result in the original model. As an illustrative case, consider the partition function $Z(\gamma^1, \beta)$. The constraint (3.32) implies that the coefficient of the generator dual to $\gamma^1$, which is $\gamma_1$, vanishes. Consequently, only the generator $\gamma_2$ remains in the dual construction. Interpreting $\gamma_2$ on the dual lattice leads to a set of links which is identical to those in $\gamma^1$, with the exception that these links are now on the dual lattice. Hence the model is equivalent to itself, up to the factors that appear in the dual transformation (see eq. (3.33)) and the exchange of coupling constant. This procedure is depicted in figure 3.3 for the three cases $s_{2,3,4}$.

A similar structure for the case of the homogeneous Ising model in a different approach was discussed in [9, 10, 11]. However, their analysis was unable to account for inhomogeneous couplings while the formalism developed here is not sensitive to such a generalization. In addition, the fact that self-dual models can be constructed once this map is found was not discussed.

Self-dual and anti-self-dual models can be constructed using the eigenvectors of $M$. Since $M^2 = 1$, the eigenvalues are either $+1$ or $-1$. In particular three eigenvectors have eigenvalue 1 giving rise to the self-dual models $Z(s_2, \beta)$, $Z(s_3, \beta)$ and $Z(s_1, \beta) + 2Z(s_4, \beta)$. The fourth eigenvector with eigenvalue $-1$ corresponds to the anti-self-dual model $-Z(s_1, \beta) + 2Z(s_4, \beta)$. Notice that two of the models map into themselves under duality, while the third self-dual model is obtained by adding the model to its own dual.
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Starting with the four possible deformed partition functions duality simply mixed them around and included various numerical factors. By solving for the eigenvectors of the matrix which connects the original vector of partition functions to the dual partition functions, self-dual models can be identified. This strategy of choosing a complete set of cycles, computing the corresponding dual theories and finding eigenvectors of the transformation matrix $M$ is a generic method for finding self-dual theories and can easily be implemented for other models and in higher dimensions. In general, $M$ contains a single non-zero entry in each row. Since $M$ is non-singular (in fact $M^2 = 1$), this implies that every column also has only one non-zero entry. As such it is possible to find self-dual models for any genus surface in the case of spin models, any four-surface for gauge models, etc... There is one question that begs to be asked here, namely why is it interesting that self-dual models of such generality can be identified? There is also one answer that comes to mind immediately - simply the fact that such general statements can be made is in itself quite interesting on a mathematical level. Physically, however, this is lacking. There are of course good physical reasons too. For instance, the two-dimensional Ising model on an infinite lattice was identified as having a phase-transition at some critical value of the coupling constant. This constant was not known for quite some time, until Kramers and Wannier [42] demonstrated that the Ising model was self-dual. Assuming that the free energy had only one singular point lead to the conjecture that the phase-transition point occurs at the self-dual coupling value. This was a natural choice, since above this critical temperature the system is in a highly disordered phase while below the critical temperature it is in an ordered phase. Since these two phases are mapped into one another through duality, it is quite plausible that the fixed point of this map is the point at which high and low temperature features become fuzzy and a phase-transition occurs. In light of the discussion in the previous subsection, the self-dual points of these models can be identified as a critical point in theory. Unfortunately, it is not possible
to determine whether it is of first or second order, without an explicit construction
the renormalization group flow for example. In chapter 6 correlation functions will be
demonstrated to undergo particular nice transformations under duality, and connections
with phase-transitions can be made more precise.

3.2 Interacting Fields

In the last section, theories defined on topologically non-trivial manifolds were made self-
dual by introducing topological terms in the defining partition function. This allowed, for
example, the construction of modified self-dual $\mathbb{Z}_N$ models on the torus. However, the
models considered there contained only a single dynamical degree of freedom. Just as the
introduction of gauge fields into a scalar field theory can alter the parameter regime in
which the theory undergoes phase transitions[31], it can also alter the duality relations.
For instance, if the scalar field theory is self-dual it may destroy that duality, however, it
can also render a theory which does not posses such symmetry to become self-dual. In this
section, the duality relations for models which contain two types of dynamical variables
living on cells of consecutive dimensions will be derived. In contrast to the case of single
fields, self-dual models of the type considered in this section will be restricted to odd
dimensions rather than even ones. Furthermore, the introduction of topological terms in
the defining model will not lead to the simple cancellation of would-be topological modes
as in the single field case. In the first subsection, the model and its dual model will
be given, and in the following subsection the model with scalar fields having local $U(1)$
symmetry will be treated in some detail. Finally, topological terms will be introduced
into the defining model, and the modifications of the dual models explored.
3.2.1 The Model and its Dual

In section 2.4, a gauge theory interacting with a scalar field was trivially written in simplicial language. Furthermore, in section 2.5 any theory in which there are two dynamical variables living on cells of consecutive dimensions was demonstrated to have a simple form when written in simplicial language. The partition function of interest is,

\[ z = \sum_{U \in C_k(\Omega, G)} \sum_{x \in C_{k-1}(\Omega, G)} \prod_{l=1}^{N_k} B_l^{(k)} \left( (\delta x + U, c_k^{(l)}) \right) \prod_{p=1}^{N_k+1} B_p^{(k+1)} \left( (\delta U, c_{k+1}^{(p)}) \right) \]  

(3.40)

In the case of \( k = 1 \), the field \( x \) is a scalar while the field \( U \) is a dynamical gauge field which lifts the global \( G \)-symmetry to a local one.

The only complicating factor in obtaining the dual formulation here is that there are multiple fields, however, this does not pose any real barrier to the construction. As before, a character expansion for both Boltzmann weights is carried out by introducing a representation of \( G \) on every \( k \)-cell and on every \( (k+1) \)-cell, encoding this information into the \( k \)-chain \( r \) and \( (k+1) \)-chain \( R \) leads to the following form of the partition function,

\[ Z = \sum_{r \in C_k(\Omega, G^*)} \sum_{R \in C_{k+1}(\Omega, G^*)} \prod_{l=1}^{N_k} b_l^{(k)} \left( (r, c_k^{(l)}) \right) \prod_{p=1}^{N_k+1} b_p^{(k+1)} \left( (R, c_{k+1}^{(p)}) \right) \]

\[ \times \sum_{U \in C_k(\Omega, G)} \sum_{x \in C_{k-1}(\Omega, G)} \prod_{l=1}^{N_k} \chi_{(r, c_k^{(l)})} \left( (\delta x + U, c_k^{(l)}) \right) \prod_{p=1}^{N_k+1} \chi_{(R, c_{k+1}^{(p)})} \left( (\delta U, c_{k+1}^{(p)}) \right) \]  

(3.41)

Using the factorization properties (3.4) and the orthogonality of characters (3.5) it is not difficult to demonstrate that the expression appearing on the second line above reduces to a product of delta function constraints,

\[ \prod_{i=1}^{N_{k-1}} \delta_{G^*} \left( (\partial r, c_{k-1}^{(i)}) \right) \prod_{l=1}^{N_k} \delta_{G^*} \left( (r + \partial R, c_k^{(l)}) \right) \]  

(3.42)

This is similar to the constraints which arose in the previous section, however, there is now an additional complication in that \( (k+1) \)-chain \( R \) depends explicitly on the \( k \)-chain
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As before, the constraint on \( r \) simply forces it to be the sum of an exact chain and an element of the \( k^{th} \) homology group,

\[
  r = h + \partial r' \quad \text{where,} \quad r' \in C_{k+1}(\Omega, G^*) \quad h \in H_k(\Omega, G^*)
\]  

(3.43)

The second constraint forces,

\[
  \partial(R + r') + h = 0
\]  

(3.44)

Since an element of the homology group is by definition not the boundary of a chain, this constraint forces \( h = 0 \). Then, \( R + r' \) can be the sum of an exact chain and an element of the \( (k + 1)^{th} \) homology group,

\[
  R = \partial R' - r' + H \quad \text{where,} \quad R' \in C_{k+2}(\Omega, G^*) \quad H \in H_{k+1}(\Omega, G^*)
\]  

(3.45)

This leaves a total of three degrees of freedom, one of which is purely topological, the second would-be topological field has been eliminated by the constraints. In the case where there is no minimal coupling between the two fields, there would be four degrees of freedom, this demonstrates one of the roles that the minimal coupling plays – it fixes some of the would be topological degrees of freedom. To see the other effects let us insert the above ansatz into (3.41). On re-interpreting the objects on the dual lattice (see section 2.1) the final form for the dual model is,

\[
  Z = \sum_{H \in H^{d-k-1}(\Omega^*, G^*)} \sum_{R \in C_d - h - 2(\Omega^*, G^*)} \prod_{i=1}^{\mathcal{N}_{d-k}} \prod_{p=1}^{\mathcal{N}_{d-k-1}} b_i^{(k)} \left( \langle \delta r, c_{d-k}^*(i) \rangle \right) b_p^{(k+1)} \left( \langle \delta R - r + H, c_{d-k-1}^*(p) \rangle \right)
\]  

(3.46)

Here the dummy prime indices on \( r' \) and \( R' \) have been removed. In addition to the reduction in the number of expected degrees of freedom, the minimal coupling (ignoring the topological field for the moment) has moved from one Boltzmann weight the other. That is, the interactions that the old gauge fields felt are now felt by the scalar field
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(Along with its gauge invariant partner), and the new gauge fields interact as the old scalars did (of course, the fact that the Boltzmann weight is replaced by its character coefficients must also be taken into account). Furthermore, the dual model is invariant under the following transformations:

\[ R \rightarrow R + \tilde{R}, \quad r \rightarrow r + \delta \tilde{R} \]  \hspace{1cm} (3.47)

which is essentially the same as the local symmetry of the original model (2.45). This is a welcomed feature of duality, it respects the local symmetries of the problem.

It is possible to place constraints on self-dual models. The simplest property to check is that the dimensions of the cells in which the dual fields are defined are equivalent to the original model. This imposing the two constraints (which are of course linearly dependent): \( d - k - 1 = k \) for the higher dimensional field and \( d - k - 2 = k - 1 \) for the lower dimensional one. Consequently, self-dual models can be defined only in odd dimensions. This is in stark contrast to the case of a single field in which the dimension of the lattice was forced to be even. Furthermore, there are stricter constraints on the Boltzmann weights, in the single field case the Boltzmann weight simply had to have the same form as its character coefficients. However, in this situation there are two Boltzmann factors, and they must map into one another’s character expansion. Of course, this can be arranged to occur by choosing the two weights to be the same and then this restriction reduces to that of the single field case. The final restriction on self-dual models is, as usual, that the group on which the variables are defined must be isomorphic to its group of representations \( (G \cong G^*). \)

An interesting example is furnished by a scalar field theory with local \( U(1) \) symmetry, i.e. a scalar field coupled to a gauge field. This model was discussed in section 2.4. In three dimensions, the pure gauge theory and pure scalar field theory map into one another, and as expected when minimal coupling is introduced between the fields
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the model appears to be self-dual (with appropriate choices of Boltzmann weights and lattices). This case satisfies the fundamental criteria derived above namely, $k = 1$ and $d = 2k + 1 = 3$. However, since the dual group to $U(1)$ is the integers, the model cannot be self-dual. Nevertheless, the dual description is valid.

3.2.2 Introducing Topological Terms

Just as in the case of single fields, the multi-field models are obstructed from being self-dual by topological terms. Employing the same technique as in the previous section here, and introducing a summation over some of the cohomology classes in the defining partition function in hopes of killing the would-be topological modes which appear in the dual model, implies the following modifications,

$$
Z = \sum_{H \in H_{(A)}^{k+1}(\Omega, \mathcal{G})} \sum_{U \in C_k(\Omega, \mathcal{G})} \sum_{x \in C_{k-1}(\Omega, \mathcal{G})} \prod_{i=1}^{N_k} B_i^{(k)} \left( \langle \delta x + U, c_{k,i}^{(l)} \rangle \right) \prod_{p=1}^{N_{k+1}} B_p^{(k+1)} \left( \langle \delta U + H, c_{k+1,p}^{(l)} \rangle \right)
$$

(3.48)

Where, as previously, $H_{(A)}^{k+1}(\Omega, \mathcal{G})$ denotes the subgroup of the cohomology group generated by a subset of the full generators. Repeating the procedure in the previous subsection, the following constraints on the representations $r$ and $R$ will be found,

$$
\prod_{i=1}^{N_{k-1}} \delta_{g^*} \left( \langle \delta r, c_{k-1}^{(i)} \rangle \right) \prod_{l=1}^{N_k} \delta_{g^*} \left( \langle \delta R + r, c_{k}^{(l)} \rangle \right) \prod_{\alpha \in \mathcal{A}} \delta_{(H^{k,a}(\Omega, \mathcal{G}))^*} \left( \langle R, \gamma^{\alpha} \rangle \right)
$$

(3.49)

The first two constraints arose without the topological terms, now they are supplemented by the additional condition on the global behaviour of $R$. These additional constraints are the analogs of the global constraints appearing in equation (3.31) for the case of a single field. The solution to the first two constraints was given in the previous subsection where it was found that,

$$
r = \partial r' \quad \text{and} \quad R = \partial R' + H - r'
$$

(3.50)
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with

\[ r' \in C_{k+1}(\Omega, \mathcal{G}^*), \quad R' \in C_{k+2}(\Omega, \mathcal{G}^*) \quad \text{and} \quad H \in H_{k+1}(\Omega, \mathcal{G}^*) \] (3.51)

The additional global constraint forces the homology coefficients of \( R \) to obey,

\[ \langle \partial R' + H - r', \gamma^a \rangle = \text{trivial in } (H^{k+1}(\Omega, \mathcal{G}))^* \quad \forall \ a \in A \] (3.52)

Consequently,

\[ H = \sum_{a \in A} \langle r', \gamma^a \rangle \gamma_a + \sum_{a \notin A} h_a \gamma_a \] (3.53)

where \( h_a \in H_{k,a}(\Omega, \mathcal{G}^*) \) are elements of the subgroup of the homology group generated by the representative \( \gamma_a \) and the projection \( \langle r', \gamma^a \rangle \) must be interpreted as an element of \( H_{k,a}(\Omega, \mathcal{G}^*) \). This construction is quite different from what was found in the case of a single degree of freedom, the extra global constraints do not force a vanishing of particular coefficients of \( H \). Instead, it complicates those coefficients making them depend on the scalar field itself. Nevertheless, it is possible to write down the dual model, after reinterpreting everything on the dual lattice one finds,

\[ Z = \sum_{H \in H^{4-k-l}(\Omega^*, \mathcal{G}^*)} \sum_{R \in C_{d-k-2}(\Omega^*, \mathcal{G}^*)} \sum_{r \in C_{d-k-1}(\Omega^*, \mathcal{G}^*)} \prod_{l=1}^{N_{d-k}^l} b_l^{(k)} \left( \langle \delta r, c_{d-k}^{(l)} \rangle \right) \times \prod_{p=1}^{N_{d-k-1}^p} b_p^{(k+1)} \left( \langle \delta R - r + \sum_{a \in A} \langle r, \gamma_a^* \rangle \gamma_a^* + \sum_{a \notin A} h_a \gamma_a^*, c_{d-k-1}^{(p)} \rangle \right) \] (3.54)

This rather complicated looking dual model is invariant under the following transformations:

\[ R \rightarrow R + R', \quad r \rightarrow r + \delta R' \] (3.55)

the only term that seems suspect is the coefficients: \( \langle r, \gamma_a^* \rangle \rightarrow \langle r, \gamma_a^* \rangle + \langle \delta R, \gamma_a^* \rangle \), however the second term simply vanishes since \( \gamma^* \) is a generator of the homology group and therefore \( \partial \gamma_a^* = 0 \). This is a comforting feature of locally invariant models, although the dual fields may be of different dimensions than the original fields, there is a sort of
memory, and the dual model maintains local symmetry. It should be pointed out that local symmetry in the dual model is not local symmetry in original model and vice versa. Rather, under a local transformation in the dual model the original variables transform under a large gauge transformation.
Chapter 4

Target-Space Duality

Target space duality [37, 57] is a symmetry of string theory which maps models defined on classically distinct target manifolds into one another. This is a rather surprising result when observed from the point of view of the target space. Since the classical equations of motion are completely different, this implies that only the full quantum corrections to those solutions can restore the equivalence. However, it has been known for some time now that the underlying principle of $T$-duality is intimately connected with the Hodge duality of forms on the world-sheet and is manifest in the sigma model that defines the theory [43, 19]. An interesting and useful question to ask is whether this duality survives lattice regularization of the world-sheet. The authors of [64] determined the potential, for a $D = 0$ matrix model, which preserved $T$-duality at the level of Feynman graphs. The question of whether this duality survives when spins are included on the sites of the graph was first studied by Gross and Klebanov [26] where a string partition function for a discretized world-sheet with circular target space was formulated. They noticed that when the Hodge duality was applied on the lattice, which amounts to performing a Kramers-Wannier $S$-duality transformation [42], the $T$-duality of the continuum model was lost due to the existence of vortices, and the model undergoes a Kosterlitz-Thouless phase transition [40] at a critical radius of the target space. This loss of $T$-duality was seen as a lattice artifact and was solved by altering the string partition function to forbid all vortex configurations. An ansatz which implemented this restriction was inserted by hand and the model regained its self-dual nature. In this chapter, it will be demonstrated that
by writing the partition function in terms of single-valued fields living in the cover of the
target space, while a sum over harmonic forms induces the projection down to the target
space, a straightforward lattice regularization of the model immediately implements such
a constraint. The sum over harmonic forms can be thought of as a sum over large gauge
transformations and is analogous to the schemes implemented in refs. [63, 41, 28, 29] in
which sums over the different theta sectors of (Super)-Yang-Mills theory was introduced
in order to produce the correct $2\pi$ periodicity of the various correlators. This study is of
course motivated by the appearance of the sum over topological sectors in the dual to spin
models in the previous chapter and an explicit connection will be seen here. Furthermore,
the method applied to the string theory on a circle case can be applied to other systems
with little effort but with interesting consequences. Defining the model to contain sums
over topological sectors from the start, which renders the fields single-valued, is a new
definition of the model which has direct physical consequences.

The idea of lifting the field from the target-space, $S^1$, to the covering space, $\mathbb{R}$, is
quite similar to what occurs when quantizing on a circle or any other multiply connected
space [59]. This general strategy will be applied to the sigma-models with target-space
$G/H$ where the world-sheet has been regulated by a lattice. The spins will be taken to
be elements of the natural cover of $G/H$, which is $G$, and the projection onto the target-
space will be implemented through a sum over $H$-valued cohomology. On the lattice, a
generalized idea of duality is observed in which the target-space and the coefficient group
of the cohomology are interchanged. This is the analog of the momentum and winding
modes being interchanged under duality in the continuum theory. Choices of $G$ and
$H$, which renders the model explicitly self-dual will be determined. In addition to the
usual self-dual circular target space, the target-spaces $\mathbb{Z}_N$ with cover $\mathbb{Z}_{N^2}$ are identified
as new self-dual models. These models are the discrete versions of the circular target-
space. Furthermore, generalized models in which a $G/H$-valued "spin" is associated with
the \((k - 1)\)-dimensional cells of a \((p + 1)\)-dimensional triangulated world-volume will be constructed. These models have the interpretation of a \(p\)-brane on which a \(G/H\)-valued \((k - 1)\)-form field is defined. If, however, the world-volume is viewed as space-time, then the models correspond to modified theories of spins, gauge fields, antisymmetric tensors fields, etc., the various topological sectors of the theory are summed over.

4.1 Target-Space Duality in Continuum

4.1.1 The Spectrum of a Closed String

To begin, a discussion of \(T\)-duality in the continuum (for a review see for instance [1, 23]) from the point of view of the spectrum of the closed string will be given. For more details on string theory see for instance [25, 53]. The Nambu-Goto [46, 47] form of the string partition function is defined as follows,

\[
Z = \sum_{\Sigma^2} g_s^{\chi(\Sigma^2)} \int \mathcal{D}X^\mu \exp \left\{ -\frac{1}{2\pi \alpha'} \int_{\Sigma^2} d^2\sigma \sqrt{-\det h_{\alpha\beta}(X^{\mu}(\sigma))} \right\}
\]  

(4.1)

where \(h_{\alpha\beta}\) is the induced metric on the world-sheet of the string,

\[
h_{\alpha\beta} = G^{\nu\mu}(X^{\nu}(\sigma)) \partial_\alpha X^{\mu}(\sigma) \partial_\beta X^{\nu}(\sigma),
\]

(4.2)

\(G^{\nu\mu}\) is the target-space metric, \(g_s\) is the string coupling constant, and \(\chi(\Sigma^2)\) is the Euler character of the surface. The fields \(X^{\mu}\) are the coordinates of the string in the target-space and is parameterized (at least locally) by the world-sheet coordinates \((\sigma^1, \sigma^2)\). The action is the area of the world-sheet which the string sweeps out in target-space. This particular form for the action is difficult to work with since the dynamical fields appear under a square root. This defect can be fixed by rewriting the model in Polyakov form [51, 52],

\[
Z(\Sigma^2) = \int \frac{\mathcal{D}h_{\alpha\beta}}{\text{Diff}(h)} \mathcal{D}X^\mu \exp \left\{ -\frac{1}{4\pi \alpha'} \int_{\Sigma^2} d^2\sigma \sqrt{h} h_{\alpha\beta}(X^{\mu}(\sigma)) \partial^\alpha X^{\mu} \partial^\beta X^{\nu} \right\}
\]

(4.3)
where $h = \det h_{\alpha\beta}$ and the summation over surfaces has been suppressed. In this expression the world-sheet metric, $h_{\alpha\beta}$, is independent of the target-space coordinates $X^\mu$. The measure on the world-sheet metric is defined modulo diffeomorphisms, i.e. different functions describing the same geometry in different coordinates are not to be counted twice. Through the equations of motion, the Polyakov form of the partition function reduces to the Nambu-Goto form. The advantage of this formulation is clear - the target-space coordinates appear only quadratically in the action. Furthermore, the action appearing in (4.3) is conformally invariant, hence, going into a conformal gauge [14, 17] the world-sheet metric, $h_{\alpha\beta}$, can be removed completely leaving only an integral over the moduli (there is also the possibility of using light-cone gauge [39] but this will not be discussed here). Attention will now be restricted to the closed string sector of the model by excluding surfaces with boundaries from the summation. In that case, the classical equations of motion for the string coordinates, $X^\mu$, must satisfy the Laplace equation,

$$\partial \bar{\partial} X^\mu(z, \bar{z}) = 0$$

(4.4)

where the complex coordinates $z = e^{\sigma^1-i\sigma^2}$ and $\bar{z} = e^{\sigma^1+i\sigma^2}$ have taken the place of the world-sheet coordinates (time now runs radially). This implies that $X(z, \bar{z})$ is the sum of a holomorphic and anti-holomorphic function, which can be viewed as left and right movers on the world-sheet,

$$X(z, \bar{z}) = X_L(z) + X_R(\bar{z})$$

(4.5)

These fields have the following mode expansion,

$$X_L^\mu(z) = x^\mu - i\alpha' \alpha_0^\mu \ln(z) + i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m^\mu}{m} z^{-m}$$

(4.6)

$$X_R^\mu(\bar{z}) = \bar{x}^\mu - i\alpha' \bar{\alpha}_0^\mu \ln(\bar{z}) + i \sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\bar{\alpha}_m^\mu}{m} \bar{z}^{-m}$$

(4.7)
where $x^\mu$ is the position of the center mass. Under $\sigma^2 \rightarrow \sigma^2 + 2\pi$ the oscillator terms are periodic, however, the zero mode contributions are not:

$$X^\mu(z, \bar{z}) \rightarrow 2\pi \alpha' (\alpha^\mu_0 - \bar{\alpha}^\mu_0)$$  \hfill (4.8)

In case the target-space direction $\mu$ is non-compact, $X^\mu(z)$ must be single valued which implies that

$$\alpha^\mu_0 = \bar{\alpha}^\mu_0 \equiv \sqrt{\alpha'/2} p^\mu$$  \hfill (4.9)

However, suppose one field is forced to be identified as follows: $X \sim X + 2\pi w R$, i.e. this direction in target-space is periodic, then the momentum corresponding to this coordinate is quantized in units of $1/R$. Furthermore, under $\sigma^2 \rightarrow \sigma^2 + 2\pi$ this coordinate (say $\mu = 25$) can change by $2\pi w R$, and corresponds to the string wrapping around the target-space direction in such a way that it cannot unravel itself without breaking apart. Consequently,

$$\alpha^{25}_0 + \bar{\alpha}^{25}_0 = \frac{2n}{R} \sqrt{\frac{\alpha'}{2}} , \quad \alpha^{25}_0 - \bar{\alpha}^{26}_0 = \sqrt{\frac{2}{\alpha'} w R}$$  \hfill (4.10)

which implies,

$$\alpha^{25}_0 = \left( \frac{n}{R} + \frac{w R}{\alpha'} \right) \sqrt{\frac{\alpha'}{2}} , \quad \bar{\alpha}^{25}_0 = \left( \frac{n}{R} - \frac{w R}{\alpha'} \right) \sqrt{\frac{\alpha'}{2}}$$  \hfill (4.11)

The mass spectrum for this string is then,

$$M^2 = -p_\mu p^\mu = \frac{2}{\alpha'} (\alpha^{25}_0)^2 + \frac{4}{\alpha'} (N - 1) = \frac{2}{\alpha'} (\bar{\alpha}^{25}_0)^2 + \frac{4}{\alpha'} (\bar{N} - 1)$$  \hfill (4.12)

In the above expression $\mu$ runs over the non-compact directions only, and $N(\bar{N})$ is the total level of the left(right) moving excitations. This spectrum has a very peculiar behavior which is not observed in the spectrum of field theories. In particular, as the radius of compactification becomes very large ($R \rightarrow \infty$) the quantization on the momentum is relaxed and its eigenvalues form a continuum, on the other hand, the winding modes with non-zero winding number become infinitely massive and therefore only the zero winding
mode, \( w = 0 \), survives in this limit. In the opposite limit when the radius of compactification becomes extremely small \( (R \to 0) \) the momentum modes are widely spaced and only the zero mode remains. This is quite typical in field theories and in fact is behind the dimensional reduction that occurs in field theories at infinite temperature. However, in string theory, due to the existence of the winding modes, as \( R \) tends toward zero a new continuum arises since the energy spacing between winding modes becomes infinitely small. Consequently, in the limit of \( R \to 0 \) an extra dimension reappears. These limits are displayed in Figure 4.1. Furthermore, as can be seen from the above expressions, the spectrums at radius \( R \) and \( \alpha'/R \) are identical when the winding and momentum modes are interchanged \( n \leftrightarrow w \). This was first noticed in [37]. Such a mapping has a non-trivial effect on the zero mode operators,

\[
\alpha_0^{25} \to \alpha_0^{25}, \quad \hat{\alpha}_0^{25} \to -\hat{\alpha}_0^{25}
\]  

(4.13)

Not only is the spectrum of the zero modes identical, but all the interactions are as well [45] which can be seen by introducing the dual field,

\[
\tilde{X}^{25}(z, \bar{z}) = X_L^{25}(z) - X_R^{25}(\bar{z})
\]  

(4.14)
It can be demonstrated that $T$-duality is an exact symmetry of closed string theory to all orders in perturbation theory [15].

Thus far, target-space duality has only been demonstrate to hold on world-sheets which have the topology of a cylinder. It is, however, possible to demonstrate, through path integral methods, that the duality holds on two-manifolds with arbitrary topologies. This is subject of the next section. The connection with the work carried out in the previous chapter will also be more apparent within that setting.

4.1.2 Hodge-Duality and Lifting to the Cover

The sigma model action is given by,

$$ S = \frac{1}{\alpha'} \int_{\Sigma} d^2 \sigma \ G_{\mu \nu} \partial X^\mu \partial X^\nu $$

(4.15)

here $\Sigma$ is a fixed, but arbitrary, orientable two-dimensional world-sheet of genus $g$, $G^{\mu \nu}$ is the target-space metric and the world-sheet metric has been trivialized. Consider the case in which the target-space is a circle and write $X^0 = \theta$. The mode expansion of the $\theta$ co-ordinate contains, in addition to the vibrational modes, winding modes corresponding to the string wrapping around the target-space. Hence, $\theta$ can be multi-valued; which implies that under transport along a non-contractable loop $\theta$ can pick up an additional $2\pi \times$ integer phase, i.e.,

$$ \oint_{\gamma_a} d\theta \in 2\pi \mathbb{Z} $$

(4.16)

where $\{\gamma_a : a = 1, \ldots, 2g\}$ are the canonical set of cycles which generate the first singular homology group of $\Sigma$. Defining the partition function in terms of multi-valued fields is undesirable both from a pedagogical point of view, and since multi-valued fields do not exist in lattice regularization. Lifting $\theta$ to the covering space of the circle, i.e. the real line, allows a natural decomposition into a smooth single-valued function and an element of the integer cohomology of the world-sheet: $d\theta \rightarrow d\theta + 2\pi h$. The cohomology elements
are the analogs of the winding modes in the mode expansion. It is important to realize that this ansatz for introducing single-valued fields is quite general. If the $\theta$ co-ordinate takes values in $\mathcal{G}$ and the cohomology in $\mathcal{H}$, then the target-space is the quotient space $\mathcal{G}/\mathcal{H}$, which has $\mathcal{G}$ as a natural cover. To make contact with the work of [26], the case of a circular target-space will be analyzed first. This leads to the choice $\mathcal{G} = \mathbb{R}$ and $\mathcal{H} = \mathbb{Z}$.

The new partition function [32] (for a fixed surface $\Sigma$) is then written as,

$$Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \int D\theta \exp \left\{ -\frac{1}{\alpha'} \int_{\Sigma} G^{00}(d\theta + 2\pi h) \wedge *(d\theta + 2\pi h) \right\}$$  \hspace{1cm} (4.17)

This will be the defining continuum theory, and all models introduced in this chapter are straightforward lattice regularizations of this partition function and simple modifications of the coefficient groups. Notice that there is an explicit sum over several partition functions each defined on the cover of the target-space and from the outset multi-valued fields are absent. Although this is a simple re-writing of the model, its lattice regularization leads to an explicitly self-dual theory without the insertion of any extra constraints, as was previously thought necessary.

The partition function (4.17) will now be demonstrated to be explicitly self-dual. The strategy is a familiar one, first introduce a one-form $V$ which satisfies a Bianchi constraint plus holonomy constraints,

$$Z = \int DV \delta (\ast dV) \left( \prod_{a=1}^{2g} \delta_{2\pi} \left( \oint_{\gamma_a} V \right) \right) \exp \left\{ -\frac{1}{\alpha'} \int_{\Sigma} G^{00}(V \wedge \ast V) \right\}$$  \hspace{1cm} (4.18)

There are two distinct types of delta-functions here, the first constraint implies that $V$ is the sum of an exact form plus cohomology elements with real coefficients, while the second periodic constraint forces the coefficients of the cohomology to be elements of $2\pi \mathbb{Z}$. Solving the constraints on $V$ leads back to the original model (4.17). Alternatively, Lagrange multipliers can be introduced to implement the constraints,

$$Z = \int DV \mathcal{D} \tilde{\theta} \sum_{h \in H^1(\Sigma, \mathbb{Z})} \exp \left\{ -\frac{1}{\alpha'} \int_{\Sigma} G^{00}(V \wedge \ast V) + i \ dV \wedge \tilde{\theta} + i \ V \wedge \tilde{h} \right\}$$  \hspace{1cm} (4.19)
To represent the holonomy constraints the equality: \( \int_{\gamma_a} V = \int_{\Sigma} V \wedge h^a \) (where \( \{ h^a \} \) are the canonical set of cohomology elements dual to the cycles \( \{ \gamma_a \} \): \( \int_{\gamma_a} h^b = \delta^b_a \) ) has been used. The one-form \( V \) now appears only quadratically in the action, and it can be eliminated via its equations of motion to give the dual partition function (the determinant factor only serves to shift the dilaton which is ignored here),

\[
Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \int D\bar{\theta} \exp \left\{ -\alpha' \int_{\Sigma} \frac{1}{4G_{00}} (d\bar{\theta} + \bar{h}) \wedge * (d\bar{\theta} + \bar{h}) \right\} \tag{4.20}
\]

Here the replacement of the old fields with the Lagrange multiplier fields is the analog of interchanging the winding and momentum modes in the mode expansion of the string coordinates. Of course, in addition, the target-space metric transformed as \( G^{00} \to \alpha' / G^{00} \).

### 4.2 Target-Space Duality on the Lattice

#### 4.2.1 Discrete Random Surfaces: The Model

The previous sections have dealt with the continuum theory, however, the lattice versions will also prove to have interesting properties which are missed in continuum models. Furthermore, the connection with the dualities derived in the previous chapter will become evident. Performing a lattice regularization of a two-dimensional world-sheet is best motivated through a discussion of matrix models. The description here most closely follows ref. [38] and the interested reader is referred to that work for more details.

As discussed in the first section of this chapter, string theory in Polyakov form contains two dynamical fields, the world-sheet metric, \( h_{\alpha\beta} \), and the embedding fields, \( X^\mu(\sigma) \). In one dimension, it is equivalent to 2-d quantum gravity coupled to a scalar field. The full path-integral for the partition function was already given in eq.(4.3). As described in the last two sections, much progress can be made when the world-sheet metric is placed in conformal gauge. However, it does not seem possible to perform the summation
over all possible surfaces and metrics in the continuum. The dynamical random surfaces paradigm is to approximate the world-sheet by a lattice and replace the sum over metrics by a summation over all possible discretizations of that surface built out of particular elementary cells (triangles, squares, pentagons etc...). In the continuum this choice should be immaterial and, in principle, various combinations of elementary shapes can be taken as the building blocks. Each elementary shape is flat - the curvature of the surface (and hence the effect of the metric) is contained at the vertices of the lattice. Consider the case in which the lattice is made out of equilateral triangles, then each vertex of the lattice has a conical singularity with deficit angle $\pi(5 - q_i)$, where $q_i$ is the number of triangles that meet at the vertex $I$. Clearly this deficit angle can be positive zero or negative, and hence mimics the effect of positive, negative or zero curvature. As the continuum limit is taken the singular nature of these vertices will be washed out and a smooth surface with appropriate locally varying curvatures is generated.

The main assumption of the discretized approach is that the summation over world-sheet geometries may be defined as the sum over all distinct triangulations of the manifold (of course in the limit of zero lattice spacing). For each given triangulation, the integral over the scalar field, $\int dX$, is replaced by a summation over all possible configurations of site valued fields. For each fixed lattice the string action is simply that of a sigma model and its discretization was discussed in chapter 1. The essential result was that,

$$\frac{1}{4\pi\alpha'} \int d^2 \sigma \sqrt{h} \ h^{\alpha\beta} \partial_{\alpha} X \partial_{\beta} X \sim \sum (X_i - X_j)^2$$  \hspace{1cm} (4.21)

where $(ij)$ are nearest neighbours on the lattice and only a single scalar field has been included. The full partition function is then given by,

$$Z = \sum_h g^x \sum_{\Omega} \sum_{X \in C_3(\Omega, \mathbb{R})} \prod_{i=1}^{N_i} B \left( \langle \delta X, c_i^{(ij)} \rangle \right)$$ \hspace{1cm} (4.22)

The sum over $h$ represents the sum over topologies and the sum over $\Omega$ represents the
sum over all triangulations of a genus $h$ surface $\Sigma^2$. The Boltzmann weight can be chosen to match the discretized action, however, it is suspected that this is just one choice of many which lie in the same universality class, and upon taking the continuum limit the particular choice of the Boltzmann weight becomes irrelevant.

There is a very beautiful tool for performing summations over triangulations of two-dimensional surfaces which was introduced by Kazakov and Migdal [36]. They noticed that the diagrammatic expansion of a quantum mechanics problem involving $N \times N$ matrix valued fields precisely produces a summation over all triangulations of arbitrary two-dimensional surfaces. To demonstrate this consider the path integral,

$$Z = \int \mathcal{D}\Phi \exp \left\{ -\beta \int dx \text{ Tr} \left[ \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + V(\Phi) \right] \right\}$$  \hspace{1cm} (4.23)

Various choices of the potential correspond to the shapes which are used to triangulate the surface. For example, $V(\Phi) = \frac{1}{2a^2} \Phi^2 - \frac{1}{3!} \Phi^3$ corresponds to using triangles as the building blocks as will be explained below (see Fig. 4.2). By a suitable scaling on the fields the partition function can be placed in the form,

$$Z = \int \mathcal{D}\Phi \exp \left\{ -N \int dx \text{ Tr} \left[ \frac{1}{2} \left( \frac{\partial \Phi}{\partial x} \right)^2 + \tilde{V}(\Phi) \right] \right\}$$  \hspace{1cm} (4.24)

The new potential is $\tilde{V}(\Phi) = \frac{1}{2a} \Phi^2 - \frac{\kappa}{3!} \Phi^3$ and the cubic coupling constant is $\kappa = \sqrt{N/\beta}$. The connection with the statistical model (4.22) appears in the expansion of $Z$ as a power
Figure 4.3: Basic diagram appearing in the expansion of the partition function in powers of \( \kappa \). The diagrams correspond to the dual of a triangular lattice. The external propagators should be connected to form a vacuum diagram. Consequently, the first diagram corresponds to a spherical topology while the second to a toroidal topology.

The equivalent diagrammatic expansion contains vertices with three legs, and is therefore in one-one correspondence with the dual of a triangular lattice (see Fig. 4.3). Every diagram contains an overall factor of \( N \) raised to some power. It is possible to demonstrate that this power is precisely the Euler character of the surface. Therefore, upon expanding the partition function in powers of \( \kappa \) one finds,

\[
\lim_{T \to \infty} \ln Z = \sum_h N^x \sum_{\Omega} \kappa^V \prod_{i=1}^{V} \int_{-\infty}^{\infty} dx_i \prod_{(ij)} \exp \left\{ -\frac{1}{\alpha} |x_i - x_j| \right\}
\]

(4.25)

Notice that this is precisely the form of the statistical model (4.22). The Euclidean time \( x \) plays the role of the string coordinate \( X \), and the exponential of the propagator, \( 1/|x_i - x_j| \), plays the role of the Boltzmann weights. Notice that it is not exactly the same Boltzmann weight, however, it is suspected that this choice lies in the same universality class as the quadratic form in (4.22). The factor of \( N^x \) appears since each vertex contributes \( \sim N \), each edge (link or propagator) \( \sim 1/N \), and each face \( N \) since each face contains one closed loop. Consequently, each diagram (lattice) is weighted by \( N^{V-E+F} = N^x \). Therefore, in addition to performing a power series in \( \kappa \), terms with same powers of \( N \) can be collected together. This naturally separates the free energy into a sum over topologies (see Fig. 4.3) and also leads to the identification of \( \sim \frac{1}{N} \) as
the string coupling constant.

### 4.2.2 The Dual Model

Now that the motivation has been given, in this section the model will be demonstrated to be invariant under target-space duality. Furthermore, the existence of the topological term appearing in (4.17) will be seen to eliminate vortex configurations and leads directly to the constrained model in [26]. The partition function of interest is,

$$ Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \sum_{\sigma \in C_0(\Sigma, \Re)} \prod_{i=1}^{N_1} B \left( \left( \delta \sigma + 2\pi h, c_1^{(l)} \right) \right) $$

(4.26)

Here the spins $\sigma$ are the analog of $\theta$ in the continuum and the Boltzmann weight is defined by $B(g) \equiv \exp\{-R^2/\alpha'g\}$. It is possible to introduce a real-valued one-chain in place of the spins and cohomology, much like introducing the real-valued one-form $V$ in the continuum. This leads to the following representation,

$$ Z = \sum_{v \in C_1(\Sigma, \Re)} \left( \prod_{i=1}^{N_1} \delta_{\mathbb{R}} \left( \left( \delta v, c_1^{(l)} \right) \right) \right) \left( \prod_{a=1}^{2g} \delta_{U(1)} \left( \left( v, h_a \right) \right) \right) \prod_{i=1}^{N_1} B \left( \left( v, c_1^{(l)} \right) \right) $$

(4.27)

where $h_a \equiv \sum_{l \in \gamma_a} c_1^{(l)}$ are the generators of the first homology group. In the above $\delta_G$ represents a $G$ invariant delta function. This form of the partition function is the lattice analog of (4.18), the first constraints are the Bianchi constraints forcing $v$ to be the sum of a co-exact chain and a cohomology element both with real coefficients; while the second constraints forces the coefficients of the cohomology to be elements of $2\pi \mathbb{Z}$. Consequently, solving the constraints on $v$ reproduces the original model much like in the continuum. There exists a slightly different representation of the model which makes direct contact with the work of [26]. This involves decomposing $\sigma$ into an integer valued chain, $\tilde{\sigma}$, and a $U(1)$ valued chain, $\theta$. Decomposing $\sigma$ in this manner allows the introduction of an integer-valued one-chain in place of $\tilde{\sigma}$ and $h$, while leaving $\theta$ untransformed. In such a representation there are dynamical variables on both the sites
and links of the lattice and the partition function in this decomposition is given by the following,

\[
Z = \sum_{h \in \mathcal{H}(\Sigma, \mathbb{Z})} \sum_{\theta \in C_0(\Sigma, U(1))} \sum_{\delta \in C_0(\Sigma, \mathbb{Z})} \prod_{l=1}^{N_1} B \left( \left( \langle \delta \theta + 2\pi (\delta \alpha + h), c_1^{(l)} \rangle \right) \right)
\]

\[
= \sum_{\theta \in C_0(\Sigma, U(1))} \sum_{\nu \in C_1(\Sigma, \mathbb{Z})} \left( \prod_{l=1}^{N_1} \delta_{\mathbb{Z}} \left( \langle \delta \nu, c_1^{(l)} \rangle \right) \right) \prod_{l=1}^{N_1} B \left( \left( \langle \delta \theta + 2\pi \nu, c_1^{(l)} \rangle \right) \right) \quad (4.28)
\]

Notice that here there are no lattice analogs of the holonomy constraints as in (4.27). The advantage in writing the model in its present form is that the spin variables take values in the target-space itself rather than its covering space. This is a desirable prescription, however, the decomposition which leads to this model is not well-defined in the continuum as integer valued fields are problematic, and is thus only valid on the lattice. In this representation the winding modes are implemented through the action of the link-valued objects \( \nu \). The winding number, given by \( \sum_{l \in \mathcal{P}} \nu_l \) (\( \mathcal{P} \) is an arbitrary elementary plaquette), must vanish due to the Bianchi constraint, while along the canonical cycles there is no restriction on the winding number. A non-zero winding number around elementary plaquettes indicates the presence of vortices and if not projected out of the model they can destabilize and lead to a Kosterlitz-Thouless phase transition [40]. Such a constraint was inserted by hand in [26] into the discrete version of (4.15), which they wrote as an \( X - Y \) model on \( \Sigma \), in order to suppress vortex configurations. It is, however, clear that these constraints follow directly from a lattice regularization of (4.17) and there is no need to insert it by hand. This feature is a direct consequence of writing the continuum variables as single valued fields in the covering space of the circle and then projecting onto the target-space through a sum over cohomology elements. It is stressed that such an ansatz leads to the correct physical picture, and will restore the lost T-duality which Gross and Klebanov found[26].

Let us now perform the duality transformation on this model. On the lattice it
is easiest to perform the transformations directly on (4.26) rather than on (4.27) or (4.28). Inserting a character expansion (in this case a character expansion amounts to a Fourier transformation) of the Boltzmann weights in the partition function introduces a representation on every link, encoding this information into a one-chain, denoted by \( r \), leads to,

\[
Z = \sum_{h \in H^1(\Sigma, \mathbb{Z})} \sum_{\sigma \in C_0(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} \sum_{r_l \in \mathbb{R}} b(r_l) \chi_{r_l} \left( \left< (\delta \sigma + 2\pi h), c^{(l)}_1 \right> \right)
\]

\[
= \sum_{r \in C_1(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} b \left( \left< r, c^{(l)}_1 \right> \right) \sum_{h \in H^1(\Sigma, \mathbb{Z})} \sum_{\sigma \in C_0(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} \chi_{(r, c^{(l)}_1)} \left( \left< (\delta \sigma + 2\pi h), c^{(l)}_1 \right> \right) \tag{4.29}
\]

The character coefficients of the Boltzmann weights are given by

\[
b(r) = \sum_{g \in \mathbb{R}} \overline{\chi}_r(g) B(g)
\]

(throughout overall constants will be ignored). Using the factorization properties of the characters (3.4) to re-arrange the sums over \( h \) and \( \sigma \) in (4.29), one finds,

\[
\sum_{h, \sigma} \ldots = \left( \prod_{i=1}^{N_0} \sum_{\sigma_i \in \mathbb{R}} \chi_{(\partial r, c^{(i)}_1)}(\sigma_i) \right) \left( \prod_{a=1}^{2g} \sum_{m_a \in \mathbb{Z}} \chi_{(r, h^a)}(2\pi m_a) \right) = \delta_{\mathbb{R}}(\partial r) \prod_{a=1}^{2g} \delta_{U(1)}(2\pi \langle r, h^a \rangle) \tag{4.30}
\]

The orthogonality of the characters (3.5) was used to obtain the last equality and \( h^a \) is the generator of the cohomology dual to homology generator \( h_a \): \( \langle h_a, h^b \rangle = \delta^b_a \). The first constraint forces \( r \) to be closed and is therefore a sum of an exact chain and an element of the homology group both with real coefficients: \( r = \partial \sigma + h \); while the second constraint forces the coefficients of the homology to be integers. This differs slightly from the analysis in the previous chapter, in that the constraints in that chapter were on identical groups for both the homology elements and the one-chain. This is the key difference here, and prevents some of the topological modes from being annihilated as they were in the previous chapter. Inserting this solution of the constraints into (4.29) yields,

\[
Z = \sum_{\bar{h} \in H_1(\Sigma, \mathbb{Z})} \sum_{\delta \in C_0(\Sigma, \mathbb{R})} \prod_{l=1}^{N_1} b \left( \left< \partial \sigma + \bar{h}, c^{(l)}_1 \right> \right) \tag{4.31}
\]
Interpreting the generators, \( \{ c^{(i)}_k \} \), of the chain complex on the dual lattice transforms the boundary operator to a co-boundary operator and homology to cohomology, the dual partition function then reads,

\[
Z = \sum_{\tilde{h} \in H^1(\Sigma, \mathbb{Z})} \sum_{\tilde{\delta} \in C_0(\Sigma^*, \mathbb{R})} \mathcal{N}_i \prod_{l=1}^{N_i} b \left( \langle \delta \tilde{\sigma} + \tilde{h}, c^{(i)}_l \rangle \right)
\]  

(4.32)

where the starred objects are on the dual lattice. This is clearly equivalent to the original model (4.26) with the Boltzmann weights being replaced by their character coefficients,

\[
B(g) = \exp \left\{ -\frac{R^2}{\alpha'} g^2 \right\}, \quad b(\tilde{g}) = \sqrt{\frac{\pi \alpha'}{R^2}} \exp \left\{ -\frac{\alpha'}{4R^2} \tilde{g}^2 \right\}
\]

(4.33)

Thus, the lattice has been replaced by the dual lattice and \( R \to \alpha'/R \) which recovers the \( T \)-duality transformation of the continuum model.

To summarize, the arguments in this section demonstrated that a straightforward lattice regularization of a continuum model, in which the dynamical variables are defined in the covering space of the circular target-space, leads to an automatic suppression of vortex configurations, and to an explicit restoration of self-duality.

### 4.2.3 \( \mathcal{G}/\mathcal{H} \) Target-Spaces

As with most formalisms it is useful to generalize the results as much as possible in order to see what the key ingredient of the analysis and results are. With this in mind, the target-spaces will be generalized to the quotient of two arbitrary Abelian groups, \( \mathcal{G}/\mathcal{H} \), in which \( \mathcal{H} \) acts freely on \( \mathcal{G} \). Regulating the world sheet by a lattice, the partition function is given by a trivial extension of (4.26),

\[
Z = \sum_{h \in H^1(\Sigma, \mathcal{H})} \sum_{\sigma \in C_0(\Sigma, \mathcal{G})} \mathcal{N}_i \prod_{l=1}^{N_i} B \left( \langle \delta \sigma + h, c^{(i)}_l \rangle \right)
\]

(4.34)

Here elements of \( \mathcal{H} \) are written in such a way so that addition in the Boltzmann weight is well-defined. For example, if \( \mathcal{G} = U(1) \) and \( \mathcal{H} = \mathbb{Z}_N \), then the argument of the
Chapter 4. Target-Space Duality

Table 4.1: Transformations of the various groups under duality.

<table>
<thead>
<tr>
<th>Original Model</th>
<th>Spin Variable</th>
<th>Cohomology Coefficient</th>
<th>Target-Space</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$G$</td>
<td>$H$</td>
<td>$G/H$</td>
</tr>
<tr>
<td>Dual Model</td>
<td>$G^*$</td>
<td>$G^* \cdot H^*$</td>
<td>$(G^* \cdot H^<em>)/H^</em>$</td>
</tr>
</tbody>
</table>

Boltzmann weight should be: $\delta \sigma + (2\pi/N)h$. These models are very similar to those considered in the previous chapter, where a sum over a subset of the generators of the cohomology was introduced to generate self-dual spin models. The difference here is that the sum extends over the entire cohomology and its coefficient group differs from the spins coefficient group, while in [22] they were taken to be identical.

Performing the dual transformations on (4.34) is a simple generalization of the previous calculation, and in lieu of repeating the steps the relevant points are mentioned here. A character expansion of the Boltzmann weights is carried out and the one-chain $r$ carries an element of $G^*$ (the group of irreducible representations of $G$, for Abelian groups $G^*$ inherits the groups Abelian structure) on every link. The factorization properties of the characters allows the sum over $h$ and $\sigma$ to be performed and constrains $\partial r$ to vanish in $G^*$ and $(r, h^a)$ to vanish in $H^*$. The first constraint forces $r$ to be a sum of an exact chain and an element of the homology group with $G^*$ coefficients, while the second set of constraints forces the coefficient group of the homology to be $G^* \cdot H^*$ (elements in $G^*$ that are trivial in $H^*$). Interpreting the objects on the dual lattice leads to the dual model,

$$Z = \sum_{h \in H^*(\Sigma, G^* \cdot H^*)} \sum_{\sigma \in C_0(\Sigma, G^*)} \prod_{l=1}^{N_1^*} b\left(\left((\delta \sigma + h), c^{(l)}\right)\right)$$

(4.35)

The effects of the duality transformation on the various coefficient groups and the target-space are shown in Table 4.1. This table illustrates an interesting generalized version of $T$-duality. Unfortunately, if either the original or dual spin variables take values in a discrete group, then these models can only be defined on the lattice. This is simply because a
continuum theory cannot have discrete valued fields. Nevertheless, the lattice models are perfectly well-defined, and the implications of the duality should be investigated. It is interesting to identify the groups which lead to explicitly self-dual models. Certainly a necessary condition is that \( G \cong G^* \) (spin models on 2-d infinite lattices also have this self-dual restriction). In that case, under duality the coefficient group of the cohomology and the target-space are interchanged and then replaced by their dual group. This is the analog of the interchanging of the momentum and winding modes and \( R \leftrightarrow \alpha'/R \) in the mode expansion of the string co-ordinates. It is also the analog of the cohomology group being replaced by the Lagrange multipliers which implement the holonomy constraints in the path integral.

Consider the model defined in eq. (4.26). Its dual model will be obtained using Table 4.1. In that case \( G = \mathbb{R} \) and \( \mathcal{H} = 2\pi R \mathbb{Z} \) so that the target-space is \( \mathbb{R}/2\pi R \mathbb{Z} \cong S^1_R \) where the subscript identifies the radius of the circle. The dual model has \( G' = \mathbb{R}^* \cong \mathbb{R} \), \( \mathcal{H}' = G'^{-1}(2\pi R \mathbb{Z})^* \cong R^{-1} \mathbb{Z} \) and target-space \( G'/\mathcal{H}' \cong S^1_{(2\pi R)^{-1}} \). The earlier result, that the duality transformation only serves to invert the radius of the target-space, has thus been recovered. Notice that it is straightforward to write down the result for a toroidal target-space \( T^n = S^1 \times \ldots \times S^1 \). In that case the groups are chosen as follows: \( G = \mathbb{R} \oplus \ldots \oplus \mathbb{R} \) and \( \mathcal{H} = 2\pi R_1 \mathbb{Z} \oplus \ldots \oplus 2\pi R_n \mathbb{Z} \) the target-space is obviously the \( n \)-tori with compactification radii \( R_i \) in the \( i \)-th direction. The dual model leaves \( G \) invariant as \( G' = G^* \cong G \) while \( \mathcal{H}' = G'^{-1}\mathcal{H}^* = R_1^{-1} \mathbb{Z} \oplus \ldots \oplus R_n^{-1} \mathbb{Z} \) and the dual target-space is an \( n \)-tori with radii \( (2\pi R_i)^{-1} \) in the \( i \)-th direction. This case corresponds to taking the target-space metric to be diagonal. Of course it is possible to consider metrics with off diagonal elements. To incorporate this into our formalism the Boltzmann weights should
be defined as follows,

\[ B((g_1, \ldots, g_n)) = \exp \left\{ -G^{ij} g_i g_j \right\}, \quad b((g_1, \ldots, g_n)) = \frac{1}{4\pi \sqrt{G}} \exp \left\{ -\frac{1}{4} (G^{-1})^{ij} g_i g_j \right\} \]

(4.36)

here the \( n \)-tuple \((g_1, \ldots, g_n)\) and \((g_1', \ldots, g_n')\) represent elements of \( \mathcal{G} = \mathbb{R} \oplus \ldots \oplus \mathbb{R} \) and \( \mathcal{G}' \cong \mathcal{G} \) respectively. Also, choose \( \mathcal{H} = 2\pi \mathbb{Z} \oplus \ldots \oplus 2\pi \mathbb{Z} \) so that the metric information is contained solely in the Boltzmann weight. This demonstrates that under duality the target-space metric is replaced by its inverse and reduces to one of the Buscher formulae \[12\] in the case of vanishing torsion.

The toroidal compactifications are the simplest example of a self-dual model. There are other groups which satisfy the necessary condition \( \mathcal{G} \cong \mathcal{G}^* \) namely the cyclic groups \( \mathbb{Z}_p \). For this choice of \( \mathcal{G} \) the coefficient group of the cohomology is forced to be cyclic as well, \( \mathcal{H} = \mathbb{Z}_N \) where \( N \) is a factor of \( P \) (let \( P = NM \)). This is necessary so that the action of \( \mathcal{H} \) on \( \mathcal{G} \) (identifying elements of \( \mathcal{G} \) which differ by angle of \( 2\pi/N \)) is well-defined. These choices lead to the discrete target-space \( \mathbb{Z}_M \), i.e. \( M \) points on a circle. The coefficient group of the dual model is \( \mathbb{Z}_M \) while the dual target-space is \( \mathbb{Z}_{M} \). Consequently, under duality the number of points in the target-space is interchanged with the number of points in the coefficient group of the cohomology. This is a novel feature of these models. In addition to this interchanging, the radius of the target-space undergoes a transformation. Rather than including the radius in the defining group, it appears in the Boltzmann weights. Displayed below are the Boltzmann weights and character coefficients for the case of a direct product of discrete groups: \( \mathcal{G} = \mathbb{Z}_{P_1} \oplus \ldots \oplus \mathbb{Z}_{P_n} \) \((P_i = N_i M_i)\) and \( \mathcal{H} = \mathbb{Z}_{N_1} \oplus \ldots \oplus \mathbb{Z}_{N_n} \),

\[ B((g_1, \ldots, g_n)) = \sum_{m_1, \ldots, m_n \in \mathbb{Z}} \exp \left\{ -G^{ij} \left( \frac{g_i}{P_i} + m_i \right) \left( \frac{g_j}{P_j} + m_j \right) \right\} \]

\[ b((g_1, \ldots, g_n)) = \sqrt{\frac{\pi}{G}} \sum_{m_1, \ldots, m_n \in \mathbb{Z}} \exp \left\{ -\frac{1}{4} (\bar{G}^{-1})^{ij} \left( \frac{g_i}{P_i} + m_i \right) \left( \frac{g_j}{P_j} + m_j \right) \right\} \]

(4.37)
where $\tilde{G}^{ij} = G^{ij}/P_iP_j$ is the normalized "metric" on the original target-space. This demonstrates that even in these models an inversion of the "metric" occurs, much like in the toroidal compactifications, as expected. For the model to be self-dual the number of points in both the original and dual theory should be identical. This is achieved if $N_i = M_i$ so that $P_i$ is a perfect square. Notice that in the limit of very large $N_i$ there are a large number of points on the target-space while the number of elements in the spin group is of order $N_i^2$. The limit of infinite $N_i$ can then roughly be viewed as the limit in which the target-space becomes a continuous circle while the spin variable reduces to the reals, thus recovering the $S^1$ case. The existence of self-dual models where the spin group is of finite order should be of interest to numerical calculations as these models can be written in matrix model form. They can then serve as testing grounds of a model which is explicitly invariant under target-space duality while such computations are difficult in the case of infinite order groups.

Of course it is possible to choose $G$ to be products of $\mathbb{Z}_{N_i}$ and $\mathbb{R}$, and $H$ to be products of $\mathbb{Z}$ and $\mathbb{Z}_N$, to obtain mixed target-spaces which are explicitly self-dual. Some other choices for $G$ and $H$ which are interesting on there own, but are not self-dual, are $G = U(1)$ and $H = \mathbb{Z}_N$. With such target-spaces the string is allowed to wrap around the space only a finite number of times before it is homotopically equivalent to zero windings. In this case, the dual target-space is a discrete space, $\mathbb{Z}_N$, even though the original target-space was continuous. This nicely demonstrates that even for Abelian isometries the dual need not have the same fundamental group as the original target-space (for the non-Abelian case see [2]).

It is possible to generalize these results to the case where the lattice $\Sigma$ is a triangulation of an arbitrary $(p+1)$-dimensional orientable manifold. There is one technical constraint on $\Sigma$: $H^k(\Sigma, \mathbb{Z})$ must be free Abelian, this is automatic for the case of two-dimensional
orientable manifolds but not for higher dimensional spaces. The models in this case are written as,

$$Z = \sum_{h \in H^k(\Sigma, \mathcal{H})} \sum_{\sigma \in C_{k-1}(\Sigma, \mathcal{G})} N_k^h \prod_{l=1}^{N_k^h} B \left( \langle (\delta\sigma + h), c_k^{(l)} \rangle \right)$$

These are models in which a $G/H$-valued field are associated with the $(k-1)$-dimensional cells of a $(p+1)$-dimensional world lattice. For example, if $k = 2$ this describes a gauge theory on a $(p + 1)$-dimensional world-volume. The duality transformations can be applied to this model with very little effort. Simply replace the 0-dimensional objects with $(k - 1)$-dimensional ones and 1-dimensional objects with $k$-dimensional ones. The dual model is a trivial extension of the previous dual model,

$$Z = \sum_{h \in H^{p+1-k}(\Sigma, \mathcal{G}/\mathcal{H})} \sum_{\sigma \in C_{p-k}(\Sigma, \mathcal{G}^*)} N_{p+1-k}^h \prod_{l=1}^{N_{p+1-k}^h} b \left( \langle (\delta\tilde{\sigma} + h), c_{p+1-k}^{*(l)} \rangle \right)$$

Self-dual models exist only when the previous relations among the groups are satisfied and $p + 1 = 2k$. Clearly $p = 1, k = 1$ is among those and reproduces the string-theory case. The next case is $k = 2$ and $p = 3$. This is a gauge theory, with gauge group $\mathcal{G}$ defined on the 4-dimensional world-volume $\Sigma$ and the sum over $\mathcal{H}$-valued cohomology is akin to summing over the topological sectors of the theory. A continuum example is given by,

$$Z = \sum_{h \in H^2(\Sigma, \mathcal{Z})} \int DA \exp \left\{ -g^2 \int (dA + 2\pi h) \wedge * (dA + 2\pi h) \right\}$$

where the field $A$ is a real-valued one-form on $\Sigma$. There are of course many higher-dimensional analogs of such self-dual models.
In the previous two chapters, duality relations for lattice models with dynamical variables living on cells of arbitrary dimensions have been constructed. The effects of interactions between fields of consecutive dimensions was also given. In chapter 3, topological terms in the defining model which canceled would-be topological modes in the dual theory were introduced along some of the cycles of the lattice. Furthermore, in chapter 4 topological modes were introduced along all non-contractable cycles of the two-dimensional world sheet; however, the group in which the site valued fields are defined was taken to be different from the coefficient group of the topological modes — in contrast with the situation in chapter 3. In each of these cases the duality transformation were carried out in much the same manner, the only difference being how the constraints that arise after integrating out (summing over) the original dynamical fields were solved. In the case where the topological modes carried the same symmetries as the dynamical variables certain modes were found to cancel, while in the case where the topological modes are defined in a subgroup of the group in which the dynamical fields are defined, the constraints forced elements to be trivial within the smaller group. In this chapter, another generalization of the models which, for particular choices of groups, include the two cases studied in the previous chapters will be given. The generalization consists of inserting topological modes which carry (possibly) different symmetries for each relevant cycle of the lattice. In the first section the generalized models are introduced and the duality relations are derived. Several examples will then be given, and in particular the reduction to the
earlier cases demonstrated.

The models introduced in the earlier chapters, and the ones introduced in this chapter, all have one common theme: the interactions occur on cells one-dimension higher than the cells on which the dynamical fields are associated. In light of this it is tempting to replace the dynamical fields with fields defined on the cells on which the interactions take place. In gauge theories this corresponds to writing the model in terms of its field-strength (curvature two-form), and replacing the functional integration over gauge fields by a functional integral over curvature two-forms. In spin models a similar procedure can be carried out: the spin variables are replaced by link-valued fields. These fields that are defined on cells of one dimension higher than the original fields will be collective called field-strengths, and in section 5.2 the generalized models will be rewritten in terms of these variables. Furthermore, the physical content of the summation over topological sectors will be illuminated in this setting. It will be seen that a choice of topological modes in the defining model corresponds to a choice of quantization conditions on the global charges in the system. This chapter then closes with several examples of models containing fractionally charged global modes.

5.1 Generalized Abelian Lattice Models

5.1.1 The Models

As mentioned above the models considered are directly motivated by the study of the statistical models in chapter 3 and the lattice regularized string theory in chapter 4. As usual, the lattice will be restricted to be orientable with freely generated $k$-th cohomology group: $H^k(\Omega, \mathbb{Z}) \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, and let $\{h^a : a = 1, \ldots, A^k\}$ denote its generators. The
defining partition function is,

\[ Z = \sum_{\{ n_a \in \mathcal{H}_a \}} \sum_{\sigma \in C_{k-1}(\Omega, \mathcal{G})} \prod_{p=1}^{N_k} B_p \left( \left( \delta \sigma + n_a h^a \right), c^{(p)} \right) \]  

(5.1)

The groups \( \mathcal{H}_a \) are in general subgroups of \( \mathcal{G} \), and one must define addition in the Boltzmann weights to mean addition in the group \( \mathcal{G} \). For example, if \( \mathcal{G} = U(1) \) and \( \mathcal{H}_a = \mathbb{Z}_N \) the addition should be written as \( \delta \sigma + (2\pi / N) n_a h^a \). In the case \( \mathcal{G} \) and/or \( \mathcal{H} \) are continuous the sums should be replaced by the appropriate integrals. These models correspond to standard statistical models when \( \mathcal{H}_a \) are taken to be identity elements in \( \mathcal{G} \). For example, if \( k = 1 \), (5.1) is a spin model with \( \mathcal{G} \)-valued spins on the sites of a lattice interacting with their nearest neighbour. For \( k = 2 \), (5.1) is a gauge theory with gauge group \( \mathcal{G} \) and Boltzmann weight having arguments which depend on the sum of the links that make up a plaquette. When \( k = 3 \) the model consists of an anti-symmetric tensor field where the arguments of the Boltzmann weights depend on the sum over plaquettes in an elementary cube. The role of the extra summations over \( \{ n_a \} \), which will be termed topological sectors of the theory, will be explained in section 5.2 once the model is written in terms of the field-strength variables. Particular choices of the groups \( \{ \mathcal{H}_a \} \) reduces to the cases studied in the earlier chapters.

### 5.1.2 The Dual Model

In this section, the duality transformations on (5.1) will be carried out. By now this procedure should be of second nature. The first step is as always to introduce a character expansion for the Boltzmann weights,

\[ B(g) = \frac{1}{|\mathcal{G}|} \sum_{r \in \mathcal{G}^*} b(r) \chi_r(g), \quad b(r) = \sum_{g \in \mathcal{G}} B(g) \chi_r(g) \]  

(5.2)

where \( |\mathcal{G}| \) is the order of the group and \( \mathcal{G}^* \) denotes the group of irreducible representations of \( \mathcal{G} \), and inherits \( \mathcal{G} \)'s Abelian structure (for example \( \mathbb{R}^* = \mathbb{R}, U(1)^* = \mathbb{Z}, \mathbb{Z}_N^* = \mathbb{Z}_N \)). In
the case $G$, or $G^*$, are continuous groups the normalized sum over group elements should be replaced by the appropriate integral. Throughout the remainder of this chapter, the irrelevant overall constants will be ignored to avoid clutter in the equations. The character expansion can be thought of as a Fourier transformation which respects the group symmetries.

On inserting the character expansion (5.2) into the partition function, (5.1) the sum over representations and product over $k$-cells can be re-ordered. This introduces a representation, $r_p$, on every $k$-cell of the lattice, which can be encoded into a $k$-chain denoted by $r \equiv \sum_{p=1}^{N_k} r_p c_k^{(p)}$. The partition function then becomes,

$$Z = \sum_{r \in C_k(\Omega, G^*)} \prod_{p=1}^{N_k} b_p \left( \langle r, c_k^{(p)} \rangle \right) \sum_{\{n_a \in \mathbb{H}_a\}} \sum_{\sigma \in C_{k-1}(\Omega, G^*)} \prod_{p=1}^{N_k} \chi_{(r, c_k^{(p)})} \left( \langle (\delta \sigma + n_a h^a), c_k^{(p)} \rangle \right)$$

Using the factorization properties of the characters, (3.4), and the definition of the (co)-boundary operator, (2.2), one can rewrite the product over characters in (5.3),

$$\prod_{p=1}^{N_k} \chi_{(r, c_k^{(p)})} \left( \langle (\delta \sigma + n_a h^a), c_k^{(p)} \rangle \right) = \prod_{l=1}^{N_{k-1}} \chi_{(\partial r, c_{k-1}^{(l)})} \left( \langle \sigma, c_{k-1}^{(l)} \rangle \right) \prod_{a=1}^{A^k} \chi_{(r, h^a)} (n_a)$$

In the above the generators of the cohomology group were written as $h^a = \sum_{p \in \gamma^a} c_k^{(p)}$. The sum over $\sigma$ and $\{n_a\}$ in (5.3) can then be performed with use of the orthogonality relations, (3.5),

$$\sum_{\{n_a \in \mathbb{H}_a\}} \sum_{\sigma} \ldots = \prod_{l=1}^{N_{k-1}} \sum_{\sigma_l \in G} \chi_{(\partial r, c_{k-1}^{(l)})} (\sigma_l) \prod_{a=1}^{A^k} \sum_{n_a \in \mathbb{H}_a} \chi_{(r, h^a)} (n_a)$$

$$= \prod_{l=1}^{N_{k-1}} \delta_{G^*} \left( \langle \partial r, c_{k-1}^{(l)} \rangle \right) \prod_{a=1}^{A^k} \delta_{H^*} (\langle r, h^a \rangle)$$

The first constraint implies that the boundary of the $k$-chain $r$ vanishes identically. The most general chain which satisfies that constraint is the sum of a boundary of a higher dimensional chain and an element of the $k^{th}$ homology both carrying $G^*$ coefficients:

$$r = \partial r' + h, \quad \text{where,} \quad r' \in C_{k+1}(\Omega, G^*), \quad h \in H_k(\Omega, G^*)$$
The homology part of $r$ is written in terms of the generators of the homology group,

$$h = \sum_{a=1}^{A_k} \tilde{n}_a h_a.$$  

On inserting (5.6) into the remaining constraints in (5.5), and using the orthogonality (2.15), one finds that coefficients of the homology are further constrained,

$$\prod_{a=1}^{A_k} \delta_{\mathcal{H}_a^*} (\tilde{n}^a)$$  

(5.7)

Since $\{\tilde{n}^a\}$ are elements of $G^*$ this constraint forces $\tilde{n}^a \in G^* \cap \mathcal{H}_a^*$ (those elements in $G^*$ which are trivial in $H^*$) where $\tilde{n}_a$ is the coefficient of $h_a$ dual to $h^a$. Inserting (5.6) and (5.7) into the partition function (5.3) leads to,

$$Z = \sum_{\{\tilde{n}^a \in G^* \cap \mathcal{H}_a^*\}} \sum_{r \in C_{k+1}(\Omega, G^*)} N_r \prod_{p=1}^{N_r} b_p \left( \langle \delta r + \tilde{n}_a h_a, c_k^{(p)} \rangle \right)$$  

(5.8)

where the dummy prime index on $r'$ from (5.6) has been removed. As usual, this representation of the model is not of the same form as original one since the argument of the Boltzmann weight contains a boundary operator acting on $r$ rather than a coboundary operator and since $\{\tilde{n}_a\}$ are coefficients of homology rather than cohomology. This problem is circumvented by interpreting the objects on the dual lattice,

$$Z = \sum_{\{\tilde{n}_a \in G^* \cap \mathcal{H}_a^*\}} \sum_{r \in C_{d-k-1}(\Omega^*, G^*)} N_{r^*} \prod_{p=1}^{N_{r^*}} b_p \left( \langle \delta r^* + \tilde{n}_a h^{*a}, c_{d-k}^{(p)} \rangle \right)$$  

(5.9)

Here $\{h^{*a} : a = 1, \ldots, A^{d-k}\}$ are the generators of $H^{d-k}(\Omega^*, \mathbb{Z})$ which are obtained by considering the generators of $H_k(\Omega, \mathbb{Z})$ on the dual lattice. Notice that this final form of the dual theory has precisely the same form as the original model (5.1). Clearly the following criteria are necessary conditions for the model to be self-dual: the lattice must be self-dual, $k$ must be equal to $d - k$ so that the dynamical degrees of freedom are on the same types of cells and the coefficient group $G$ must be isomorphic to its group of representations $G^*$. Additional constraints on obtaining self-dual models are imposed by the cohomology coefficients $\{\mathcal{H}_a\}$. The set of $\mathcal{H}_a$ must reproduce itself under duality, consequently every group appearing in $\{\mathcal{H}_a\}$ must appear in $\{G^* \cap \mathcal{H}_a^*\}$ with the same
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5.1.3 Illustrative Examples

As mentioned earlier these models are generalizations of the models considered in [22, 32]. This connection will now be made explicit; furthermore several examples which are unique to the present discussion will be given. Consider the situation in which $k = 1, \mathcal{G} = \mathbb{Z}_2$ and $\Omega$ is a square lattice triangulation of a two handled surface. Since the dynamical variables are defined on the $(k - 1) = 0$-dimensional cells of the lattice and interact through a nearest neighbour coupling this corresponds to the Ising model with appropriate choice of Boltzmann weights. The relevant cohomology here is $H^1(\Omega, \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

To understand what the generators of this group are it is best to consider the homology group of dimension $d - k = 1$ on the dual lattice. The generators of that group are well known: $h_a = \sum_{l \in \gamma_a^*} c_1^{(l)}$ where $\gamma_a^*$ are the set of links which wrap around the $a$-th handle of the 2-tori. The cohomology generators of dimensions $k = 1$ are then obtained by interpreting each $\gamma_a^*$ back on the original lattice; denote that set of links by $\gamma_a$. One way to visualize $\gamma_a$ is to view of them as "steps of a ladder" wrapping around the $a$-th handle of the 2-tori. This construction is quite general: if the lattice is orientable, first construct the generators of $H_{d-k}$ on the dual lattice, and then interpret back on the original lattice that will yield the generators of $H^k$. Consider the model in which $\mathcal{H}_1 = \mathbb{Z}_2$ and all
the rest are the identity element. The set of links \( \gamma_1 \) are depicted in the first diagram of Figure 5.1 by the solid lines forming steps around the handle. The model (5.1) in standard language is then given by,

\[
Z = \sum_{n_1=\pm 1} \sum_{\{\sigma_i\}=\pm 1} \prod_{(ij) \in \Omega} \exp \left\{ \beta \sigma_i \sigma_j \left( n_1 \right)^{\epsilon \left( \gamma_1; ij \right)} \right\} \tag{5.10}
\]

here \( i \) labels sites on the lattice, \( (ij) \) are nearest neighbours forming a link and,

\[
\epsilon(\gamma_1; ij) = \langle \gamma_1, c_1^{(ij)} \rangle = \left( \sum_{l \in \gamma_1} c_1^{(l)} \right)^{\langle c_1^{(ij)} \rangle} = \begin{cases} 
1 & \text{if } (ij) \text{ is in } \gamma_1 \\
0 & \text{otherwise}
\end{cases} \tag{5.11}
\]

This model is the sum of two Ising models: one which has coupling constant \( \beta \) on every link and another with coupling \( \beta \) on all links except for those contained in \( \gamma_1 \) where it is taken to be \( -\beta \). Another view is to take the Ising model on a 2-tori and slice the lattice along one canonical cycle on the dual lattice and then sew the lattice back together with the two possible "twists" in the coupling.

Let us now demonstrate how the dual model is obtained. The first step is to find the homology cycle which is dual, in the sense of (2.12), to the generator \( h_1 = \sum_{l \in \gamma_1} c_1^{(l)} \). The dual cycle, \( h_1 \), is depicted by the dotted line in the first diagram in Figure 5.1. According to the analysis in the previous sub-section the coefficient group of \( h_1 \), in the dual theory before interpretation on the dual lattice (5.8), is \( G^\perp \mathcal{H} = \mathbb{Z}_2^\perp \mathbb{Z}_2 = \{e\} \). Consequently, it is not summed over in the dual theory. The remaining cycles, depicted in the second diagram in Figure 5.1, have weight \( G^\perp \{e\} = \mathbb{Z}_2^\perp \{e\} = \mathbb{Z}_2 \) and are summed over in the dual theory although they were absent in the original model. The final step is to interpret the groups associated with the cycles \( h_1, \ldots, h_4 \), as groups for the cocycles on the dual lattice (see the third diagram in Figure 5.1). Denote these cocycles by \( h_1^*, h_2^*, h_3^* \) (notice that \( h_1^* \) is similar to \( h_1 \), it is simply shifted by \( \frac{1}{2} \) lattice spacing in both directions). The
dual theory is then given by,

$$Z = c(\beta^*) \left( \prod_{a=1}^{3} \sum_{\tilde{n}_a = \pm 1} \right) \sum_{\{\sigma_i^*\} = \pm 1} \prod_{(i,j) \in \Omega^*} \exp \left\{ \left( \prod_{a=1}^{3} (n_a e^{(\gamma^* : \tilde{n})}) \right) \beta^* \sigma_i^* \sigma_j^* \right\} \tag{5.12}$$

where $\beta^* = -\frac{1}{2} \ln \tanh \beta$ is the well known dual coupling constant and the overall nonsingular piece $c(\beta^*) = (2 \sinh(2\beta^*))^{-\frac{N_1}{2}}$ appears when performing the character expansion of the Boltzmann weight. Also, $\gamma^a$ denotes the set of links which define $h^a$. This particular example is clearly not self-dual. However, it is not difficult to convince oneself that if a sum over two cocycles, each in a separate handle, was included in the original model, then the dual model would be identical to the original one. Thus on the 2-tori there are 4 inequivalent self-dual theories: $\{1,3\}, \{1,4\}, \{2,3\}$ and $\{2,4\}$ where $\{i,j\}$ is intended to mean that a sum over the cocycle labeled by $i$ and $j$ was included (the cocycles are ordered so that 1 and 2 are around the first handle and 3 and 4 are around the second). There are of course more self-dual models formed by summing models of type (5.1). For example, a trivial way to obtain a self-dual model is to take a model, which is not self-dual, and add it to its dual (one could also subtract it from its dual to obtain anti-self-dual models). This however, does not exhaust all possible self-dual models that one can write down with $\mathcal{H}_a = \mathbb{Z}_2$ or the identity. It is possible to find all of them by constructing a vector of partition functions where the index labels all possible partition functions, then carry out the duality transformation on each component of the vector, to obtain a new vector of partition functions. Each element of this "dual" vector must be a linear combination of the original vector of partition functions since it contained all possible partition functions. Consequently, one can construct a matrix which acts on the original vector to give to the dual vector. This matrix is a matrix representation of the duality transformation, and its eigenvectors with eigenvalues plus one (it could have eigenvalues minus one as well, since only its square must be the identity) label the possible self-dual models. An example of this was given in chapter 3 and the construction
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will not be repeated here.

The Ising model example only considered the degenerate case of $\mathcal{H}_a = \mathcal{G}$ or the identity. It is, however, possible to relax this condition and still obtain self-dual theories. Consider the case of a spin model on a 2-dimensional Riemann surface and choose $\mathcal{G} = \mathbb{R}$ and $\{\mathcal{H}_a = 2\pi R \mathbb{Z}\}$. Since the coefficient group is identical for all cocycles, the dual theory is obtained by simply replacing $\mathcal{G}$ by $\mathcal{G}^* = \mathcal{G} = \mathbb{R}$ and $\{\mathcal{H}_a\}$ by $\{\mathcal{G}^* \mathcal{H}_a^* = R^{-1} \mathbb{Z}\}$. This case was studied in chapter 4 in the context of string theory on a circle, the change in the coefficient $2\pi R \rightarrow R^{-1}$ is a manifestation of target space duality. Other examples with $\mathcal{H}_a = \mathcal{H} \neq \mathcal{G}$ were also studied in the context of string theory on discrete target spaces in chapter 4.

One further example of a self-dual spin-model on a Riemann surface is furnished by the case $\mathcal{G} = \mathbb{Z}_{NM}$ and $\mathcal{H}_{i(a)} = \mathbb{Z}_N$ and $\mathcal{H}_{i(b)} = \mathbb{Z}_M$ where $(a)$ indicates an $a$-cycle (the short non-trivial cycle) and $(b)$ a $b$-cycle (the long non-trivial cycle) on the surface, and $i$ labels the handles. A description of this case will be deferred to section 5.1.3 where they will be interpreted as fractionally charged models with different fractional charges around the various cycles of the lattice.

5.2 Field Strength Formulation and Bianchi Constraints

In pure gauge theories the action depends on the gauge fields only through their field-strengths. It seems more natural then to rewrite the theory so that the field-strengths are the dynamical degrees of freedom. Various authors [30, 3, 5, 6] have shown that the price of doing this is the appearance of Bianchi constraints on the field-strengths. On the lattice the gauge fields are associated with links, while their field-strengths are defined on the plaquettes of the lattice. The Bianchi constraints are realized, in this situation, by forcing the group product of the field-strength variables in every elementary cube to be
the identity element. In this section the generalized problem where the models are defined by the partition function in (5.1) will be studied. A $k$-chain will replace the coboundary of the $(k-1)$-chain $\sigma$. The variables which are defined on the $k$-dimensional cells will collectively be referred to as field-strengths. The construction of the partition function in terms of the field-strength variables on topologically non-trivial lattices will be given. The usual Bianchi identities that arise in the flat case will be found to be supplemented by constraints along the homology cycles of the lattice as a direct consequence of the duality transformations derived in the previous section. This will be construction will be carried out in an explicitly gauge-invariant manner. One of the main advantages of this rewriting is that it illuminates the role that the topological summations play. In addition, this type of formulation of the lattice models leads to a simple interpretation of the strong coupling expansion and is amenable to mean-field techniques.

5.2.1 Computation of The Constraints

A counting of the degrees of freedom in the problem leads to a good estimate of what the constraints in the field-strength formulation would be. However, it is much more satisfying to construct them explicitly. To do so, a Fadeev-Popov trick is used by inserting the following identity into the partition function (5.1),

$$1 = \sum_{f \in C_k(\Omega,\sigma)} \delta_g(f - (\delta \sigma + n_a h^a))$$

(5.13)

On insertion into (5.1) and re-ordering the summations one finds,

$$Z = \sum_{f \in C_k(\Omega,\sigma)} \prod_{p=1}^{N_k} B_p \left( \langle f, c_k^{(p)} \rangle \right) \Pi(f)$$

(5.14)

where $\Pi(f)$ is the constraint sum,

$$\Pi(f) = \sum_{\{n_a \in H_a\}} \prod_{\sigma \in C_{k-1}((\Omega,\sigma)} \delta_g \left( \langle f - (\delta \sigma + n_a h^a), c_k^{(p)} \rangle \right)$$

(5.15)
Notice that \( \Pi(f) \) is itself a partition function of the form (5.1) where the Boltzmann weights are \( G \)-invariant delta functions, the presence of the chain \( f \) can be interpreted as an external field acting on the effective statistical system. Since the Boltzmann weights are particularly simple the character decomposition is trivial,

\[
\beta_p(r) = \chi_r((f, c_k^{(p)})) \tag{5.16}
\]

The dual representation of \( \Pi(v) \), before interpretation on the dual lattice, is straightforward to write down and is given by (5.8),

\[
\Pi(f) = \sum_{\{\tilde{n} \in e^* \mathcal{H}_a^*\}} \sum_{r \in C_{k+1}(\Omega, \theta^*)} \prod_{p=1}^{N_k} \chi_{(\tilde{n}+n^a_h, c_k^{(p)})}((f, c_k^{(p)})) \tag{5.17}
\]

The factorization properties (3.4) were used to obtain the second equality. This form of the constraints demonstrate that they are two distinct classes: a topologically trivial class \( \Pi_1(f) \) and a purely topological class \( \Pi_2(f) \). These two sectors will be treated separately.

Using (5.4), \( \Pi_1(f) \) can be written in the following form,

\[
\Pi_1(f) = \sum_{r \in C_{k+1}(\Omega, \theta^*)} \frac{N_{k+1}}{c=1} \chi_{(r, \partial c_k^{(c)})}((\delta f, c_k^{(c)})) = \prod_{c=1}^{N_{k+1}} \delta_{\theta^*}((\delta f, c_k^{(c)})) \tag{5.18}
\]

Once again, the orthogonality of the characters, (3.5), was used to obtain the second equality. These constraints are the usual local Bianchi constraints which, in the gauge theory case, force the gauge connection to be flat modulo harmonic pieces. Another interpretation of these constraints is that within every \((k+1)\)-cell a topological charge can appear and these constraints force the charges to be quantized. For example, in
a spin model this corresponds to the quantization of the vortex fluxes (the sum of the
vector field around elementary plaquettes is quantized). For a gauge theory it is the
quantization of the monopole charge (the sum of the field-strength around an elementary
cube is quantized). This interpretation will also apply to the global constraints, to be
derived shortly, and will be exploited in the next section.

Let us now turn to the topological constraints. The factorization properties, (3.4), and
orthogonality relations, (3.5), once again lead to a drastic simplification of the expression,

\[ \Pi_2(f) = \prod_{a=1}^{A_k} \sum_{n^a \in \mathcal{G}^a \setminus \mathcal{H}_a^2} \langle f, h_a \rangle = \prod_{a=1}^{A_k} \delta_{(g^a \setminus \mathcal{H}_a^2)^*} (\langle f, h_a \rangle) \] (5.19)

These constraints are the lattice analogs of holonomy constraints in the continuum, to
be more specific it forces the integral of the field-strength around the canonical cycles
of the manifold to be quantized. These constraints will be referred to as global Bianchi
constraints. These constraints force quantization conditions on the global charges around
the non-contractable \( k \)-cycles of the lattice. Such constraints were already seen to arise
in the continuum \( \sigma \)-model action used for the bosonic string theory which appeared in
the first order formalism, see equation (4.18). It is precisely in this context that these
constraints should be interpreted.

As mentioned previously, the usual statistical models correspond to choosing \( \{ \mathcal{H}_a = \{ e \} \} \),
under these conditions the local and global charges satisfy the same quantization condi­
tions. By allowing \( \{ \mathcal{H}_a \} \) to be non-trivial it is possible to introduce fractional charges
in the system. This will be discussed in detail in the next section. Before closing this
section let us write down the full field-strength formulation of the model by inserting the
two constraints, (5.19) and (5.18), into (5.14),

\[ Z = \sum_{f \in C_k(\Omega, G)} \prod_{p=1}^{A_k} B_p \left( \langle f, c_k^{(p)} \rangle \right) \prod_{c=1}^{A_{k+1}} \delta_g \left( \langle \delta f, c_{k+1}^{(c)} \rangle \right) \prod_{a=1}^{A_k} \delta_{(g^a \setminus \mathcal{H}_a^2)^*} (\langle f, h_a \rangle) \] (5.20)
Notice that if \( d = k \) the local Bianchi constraints are absent and the field-strength variables are constrained only through the global constraints. In next chapter this property will be used to obtain explicit expressions for arbitrary correlators when the lattice has dimension \( k \).

### 5.2.2 The Role of Summing Over Topological Sectors

In section 5.1.3 several self-dual spin models were found by making various choices of the groups \( \mathcal{G} \) and \( \{ \mathcal{H}_a \} \). In particular consider the example where \( \Omega \) is a square triangulation of an orientable two-manifold of genus \( g \), \( \mathcal{G} = \mathbb{Z}_2 \) and \( \mathcal{H}_a = \mathbb{Z}_2 \) for the \( a \) cocycles (the generators which wrap around the handles "vertically") and \( \mathcal{H}_a = \{ e \} \) for the \( b \) cocycles (the generators which wrap around the handles "horizontally"), also choose the standard Ising Boltzmann weights. To obtain the field-strength formulation, the cycles dual to the cocycles which are summed over must be found. Then a constraint which satisfies \( (\mathcal{G}^\perp \mathcal{H}_a)^* \) symmetry must be imposed on these cycles. Firstly, consider the \( b \) cocycles, the \( a \) cycles are dual to these and thus have \( (\mathcal{G}^\perp \mathcal{H}_a)^* = (\mathbb{Z}_2^\perp \{ e \})^* = \mathbb{Z}_2 \) constraints on the coefficient group. The \( a \) cocycles are dual to the \( b \) cycles, consequently the constraints on those cycles will be \( \mathbb{Z}_2^\perp \mathbb{Z}_2 = \{ e \} \), i.e. there are no constraints along those cycles. The partition function can then be written in terms of the field-strengths as follows,

\[
Z = \sum_{\{f_i=\pm1\}} \prod_{(ij) \in \Omega} e^{\beta f_{ij}} \prod_{\partial \in \mathcal{E}} \frac{1}{2} \left( 1 + \prod_{l \in \partial} f_l \right) \prod_{a=1}^{g} \frac{1}{2} \left( 1 + \prod_{l \in \gamma_a} f_l \right)
\]  

(5.21)

The explicit form of a \( \mathbb{Z}_2 \)-invariant delta function was used here. Recall that \( \gamma_a \) are the set of \( k \)-cells which define the generator of the homology group. In this instance \( \gamma_a \) corresponds to the set of links forming the \( a \)-th non-contractable loop around the handles of the lattice \( (a = 1, \ldots, 2g) \). The set \( \{ \gamma_{2a} : a = 1, \ldots, g \} \) contains the \( a \) cycles, and \( \{ \gamma_{2a-1} : a = 1, \ldots, g \} \) contains the \( b \) cycles. The constraints in (5.21) imply certain restrictions on the vortices in the model. The local Bianchi constraints force the vortex
hold that the flux within each elementary plaquette to vanish and the global Bianchi constraints force
the global vortex flux to vanish. However, since the global Bianchi constraints are only
present for the $a$ cycles, arbitrary vortex configurations around the $b$ cycles are allowed
while the vortex flux must vanish around the $a$ cycles. This example serves to illustrate
the fact that including a sum over the entire group along a particular cocycle removes
the constraints on the global charges which wrap around its dual cycle. In this case the
global charges are completely free of quantization, and can take on any charge allowed
by their symmetry group.

As a second illustration, consider the same lattice, but with groups $G = \mathbb{R}$ and
$H_a = \mathbb{Z}$ for all cocycles. This model was also demonstrated to be self-dual in section
5.1.3, and corresponds to the sigma model of the bosonic string theory with target space
the circle (see chapter 3). In terms of field-strengths it is clear that since all cocycles
have the same group then the global Bianchi constraints force the global vortex fluxes to
respect $(\mathbb{R}^* \mathbb{Z}^*)^* = U(1)$ constraints. The partition function reads,

$$Z = \left( \prod_{(ij)} \int df_{(ij)} \right) \prod_{(ij) \in \Omega} B_{(ij)} \left( f_{(ij)} \right) \prod_{\square \in \Omega} \delta \left( \sum_{l \in \square} f_l \right) \prod_{a=1}^{2g} \sum_{m_a \in \mathbb{Z}} \delta \left( 2\pi m_a - \sum_{l \in \gamma_a} f_l \right)$$

The local Bianchi constraints imply that the vortex flux must be identically zero around
every elementary plaquette, while the global Bianchi constraints forces the vortex fluxes
around the cycles of the lattice to be quantized in units of $2\pi \mathbb{Z}$. This illustrates that
when the topological sectors are summed over a subgroup of the group in which the
dynamical variables take values in, then the global charges that arise are not completely
free or completely constrained, but rather they satisfy a relaxed quantization condition.

As a final example consider the case of a $U(1)$ gauge theory on the lattice $\Omega$. In the
present language this model is obtained by choosing $G = U(1)$ and $H_a = \{e\}$. The field-
strengths in this case are dynamical plaquette valued fields. The local Bianchi constraints
force the sum of the field-strength around each elementary cube to be $2\pi \times$ integer, and
correspond to the quantization of the monopole charges. Since \((U(1)^\ast \{-e\})^\ast = U(1)\)
the global Bianchi constraints force the sum of the field-strength around all two-cycles
of the lattice to be \(2\pi x\) integer as well, and corresponds to the quantization of the
global charges. Consequently in a standard gauge theory all topological excitations are
quantized by the same fundamental charge, as expected. Now consider the case where
\(\mathcal{H}_1 = U(1)\), and the remaining \(\mathcal{H}_a = \{e\}\). In this case the local charges remain
quantized in units of \(2\pi \mathbb{Z}\), while the global charge around the canonical two-cycle dual to the
cocycle \(h^1\) is completely free of constraints since \((U(1)^\ast - U(1)^\ast)^\ast = \{e\}\). Clearly, if more
than one sector is summed over the result simply generalizes. Let
\[
\mathcal{H}_a = \begin{cases} U(1) & a \in B \subset 1, \ldots, A^k \\ \{e\} & \text{otherwise} \end{cases}
\]
then using (5.20) the model in field-strength form is,
\[
Z = \prod_{p=1}^{N_2} \int_{-\pi}^{\pi} \frac{df_p}{2\pi} B_p(f_p) \prod_{c=1}^{N_3} \sum_{m \in \mathbb{Z}} \delta \left(2\pi m - \sum_{p \in c} f_p \right) \prod_{a \in B} \sum_{m' \in \mathbb{Z}} \delta \left(2\pi m' - \sum_{p \in \gamma_a} f_p \right)
\]
In this case \( \gamma_a \) consists of the set of plaquettes which wrap around the \(a\)-th two-dimensional
hole in the lattice. In this form, it is clear that including a sum over a particular cocycle
removes the quantization condition on the charges which are enclosed by the cycle dual to
that cocycle, much like the liberation of the vortex quantization earlier. These examples
have been restricted to the case in which the topological symmetry was taken to be the
same as the symmetry group of the dynamical variables. It is interesting to ask what
happens if a sum over a proper subgroup of the full group, i.e. \(\mathcal{H}_a < \mathcal{G}\), was included
instead? Continuing with the example of \(U(1)\) gauge theory, take \(\mathcal{H}_a = \mathbb{Z}_{N_a}\). Then
the topological charge around the canonical two-cycle dual to the cocycle \(h^a\) can contain
a fractional charge \(2\pi/N_a \times \text{integer}\) while the local charges are still quantized in units
of \(2\pi \mathbb{Z}\). To see this notice that \((U(1)^\ast \mathbb{Z}_{N_a})^\ast = (\mathbb{Z}_{N_a})^\ast = (N_a)^{-1} \mathbb{Z}\), so that the
field-strength formulation of this model is,

\[ Z = \prod_{p=1}^{N_2} \int_{-\pi}^{\pi} \frac{df_p}{2\pi} B_p(f_p) \prod_{c=1}^{N_3} \sum_{m_c \in \mathbb{Z}} \delta \left( 2\pi m_c - \sum_{p \in c} f_p \right) \prod_{a=1}^{A_2} \sum_{m'_a \in \mathbb{Z}} \delta \left( \frac{2\pi}{N_a} m'_a - \sum_{p \in \gamma_a} f_p \right) \]  

(5.25)

It is the hoped that these examples amply illustrate that the global charges and local charges in the system need not satisfy the same quantization conditions. Due to this large freedom in the choice of symmetry group for the topological modes, it is possible to construct models that contain fractional charges along the various cycles of the lattice which satisfy distinct quantization conditions. The study of such models is topic of the next section.

5.3 Fractionally Charged Self-Dual Models

In this section, self-dual models which contain several distinct fractional global charges will be constructed. It will be assumed that the Boltzmann weights are chosen so that its character coefficients have the same functional form as the Boltzmann weight itself. This allows us to focus on the set of \( \{ H_a \} \) which lead to self-dual models. In section 5.1.3 it was demonstrated that on a genus \( g \) surface, if \( H_a = \mathcal{G} \) for all \( a \) cycles then a spin model on that surface is explicitly self-dual. In section 5.2.2 it was pointed out that introducing such a sum over the entire group releases the quantization condition on the topological charge around that cycle (see Eq.(5.21)). It was also demonstrated in section 5.1.3 that the model with \( \mathcal{G} = \mathbb{R} \) and \( H_a = \mathbb{Z} \) for every cycle on the surface was self-dual. In this case the local charges are forced to vanish, while the global charges are quantized in units of \( 2\pi \mathbb{Z} \) (see Eq.(5.22)). The natural question to ask is whether one can construct a self-dual model which has different global charges around the various generators. In the next subsection, a collection of spin models in two-dimensions which have such a distribution of fractional charges will be constructed. In addition, gauge theories in four
Chapter 5. Combining The Models

5.3.1 Spin Models

As a first example consider a spin model on a torus with $\mathcal{G} = \mathbb{Z}_P$, this forces $\mathcal{H}_1 = \mathbb{Z}_M$ and $\mathcal{H}_2 = \mathbb{Z}_N$ for some integers $N$ and $M$. It is clear that in order for $\mathcal{H}_a$ to act freely on $\mathcal{G}$, $N$ and $M$ must be factors of $P$. Table 5.2 contains the transformations of the coefficient groups under duality. It is obtained as follows, let $\{h_1, h_2\}$ be the generators of $H^1(\Omega, \mathcal{G})$ (see Figure 5.2), $h_1$ consists of the set of links wrapping around the $a$-cocycle (short direction) in the first diagram, and $h_2$ is the set of links wrapping around the $b$-cocycle in the second diagram. According to the dual construction: the dual to $h_1$, $h_1$, is forced to have coefficient group $\mathcal{G}^* \mathcal{H}_1^* = \mathbb{Z}_{P/M}$. Interpreting $h_1$ on the dual lattice leads to an object which wraps around the $b$-cocycle of the dual lattice (much like the generator $h_2$ on the original lattice, see Figure 5.2). Repeat the above, starting with the generator $h_2$ and combining the two results imply that under duality $\mathcal{H}_1 = \mathbb{Z}_M \rightarrow \mathbb{Z}_{P/N}$ and $\mathcal{H}_2 = \mathbb{Z}_N \rightarrow \mathbb{Z}_{P/M}$. Requiring the theory to be self-dual implies that,

\[
\frac{P}{N} = M \quad \text{and} \quad \frac{P}{M} = N \tag{5.26}
\]

<table>
<thead>
<tr>
<th>Original Model</th>
<th>Spin Variable</th>
<th>Coefficient of $h_1$</th>
<th>Coefficient of $h_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dual Model</td>
<td>$\mathcal{G}^*$</td>
<td>$\mathcal{G}^* \mathcal{H}_2^*$</td>
<td>$\mathcal{G}^* \mathcal{H}_1^*$</td>
</tr>
</tbody>
</table>

Table 5.2: Mixing of the coefficient group under duality for a spin model on a torus.
Figure 5.2: Constraints on the cocycles in the first diagram force constraints on the dual cycle (shown by the dotted line), interpreting on the dual lattice leads to constraints on the cocycle in the second diagram.

clearly $P = MN$ imposes the self-duality condition. The partition function in terms of field-strengths is easily written down,

$$Z = \sum_{\{f_i=0,\ldots,MN-1\}} \prod_{\langle i,j \rangle \in \Omega} B(f_{\langle i,j \rangle}) \prod_{p=\Box \in \Omega} \sum_{m_p \in \mathbb{Z}} \delta \left( 2\pi m_p MN - \sum_{l \in \Box} f_l \right) \left( \sum_{m_{1} \in \mathbb{Z}} \delta \left( 2\pi m_{1} M - \sum_{l \in \gamma_1} f_l \right) \right) \left( \sum_{m_{2} \in \mathbb{Z}} \delta \left( 2\pi m_{2} N - \sum_{l \in \gamma_2} f_l \right) \right)$$

Consequently, the model has local charges which are quantized in units of $2\pi MN$ while the global charge around the cycle $h_1$ is quantized in units of $2\pi M$ and around the cycle $h_2$ is quantized in units of $2\pi N$. This nicely illustrates that even models containing global charges which satisfy different quantization conditions along the cycles of the lattice can be self-dual. There is an interesting interpretation in light of the discussion of target-space duality (see chapter 4). Namely, since the target space is given by the quotient of the group in which the dynamical spins are defined and the group in which the topological modes are defined, along the time slices the string sees a $\mathbb{Z}_N$ target-space, but if one follows a fixed point on the string and let it evolve in time, it can wrap around a $\mathbb{Z}_M$ target-space ($N$ and $M$ are completely unrelated). This idea can be generalized to the case of an orientable surface of genus $g$. Simply choose $G = \mathbb{Z}_{NM}$ and $H_a = \mathbb{Z}_N$ for
all $a$ cocycles and $\mathcal{H}_a = \mathbb{Z}_M$ for all $b$ cocycles. This choice is invariant since starting with an $a$ cocycle ($b$ cocycle) and then imposing the constraints on the dual $b$ cycle ($a$ cycle) and finally interpreting the cycle on the dual lattice leads to a $b$ cocycle ($a$ cocycle) around the same handle as the original $a$ cocycle ($b$ cocycle). In other-words the duality transformations only serve to mix the groups within each handle. This follows directly from Poincare-duality. Since the case of a single handle has just been proven to be self-dual, this observation implies that the model on the genus $g$ surface is also invariant under duality.

5.3.2 Gauge Theories

For gauge theories a similar analysis can be carried out. In this case the dimension of the lattice has to be four. Two lattices which produce self-dual models with multiple fractional charges will be considered here, there are of course many more. The first example is a lattice with the topology of $S^2 \times S^2$. Firstly, the relevant cohomology has group structure $H^2(\Omega, \mathcal{G}) \cong \mathcal{G} \oplus \mathcal{G}$ and the generators are given by the sets of plaquettes which encase one of the two-spheres. The generators of $H^2(\Omega, \mathcal{G})$ are then the plaquettes which are dual to those surfaces, and can be thought of as plaquettes perpendicular to the two-sphere (of course this is in four dimensions). Table 5.2 still holds in this situation, where $h^1$ and $h^2$ are taken to be the appropriate generators, and $\mathcal{G}$ is now the gauge-group rather than the spin group. Consequently, as in the last section, choosing $\mathcal{G} = \mathbb{Z}_{NM}$, $\mathcal{H}_1 = \mathbb{Z}_M$ and $\mathcal{H}_2 = \mathbb{Z}_N$ leads to an explicitly self-dual model containing distinct fractional charges around the two cycles of the lattice.

As a second example consider the lattice with the topology of $T^4 = S^1 \times S^1 \times S^1 \times S^1$. In this case there are six generators of the homology and cohomology: $H^2(\Omega, \mathcal{G}) \cong H_2(\Omega, \mathcal{G}) \cong \oplus_{s=1}^{6} \mathcal{G}$. The homology generators are the plaquettes which make up the $\binom{4}{2}$ possible tori constructed out of the four circles, and as usual the cohomology are the dual
to those. Six coefficient groups must then be specified to define the model. However, just as in the spin model on the genus \( g \) surface, the generators form pairs, which under duality only communicate within each pair. This feature is a result of Poincare-duality. For example, the cocycle corresponding to the torus formed by the first two circles will force the coefficient group of the cycle corresponding to the torus formed by the last two circles to be \( G^* \times TL \). Interpreting this cycle on the dual lattice leads to a cocycle which is isomorphic to the generator corresponding to the torus formed by the last two circles. Consequently, the transformations of the coefficient group can be summarized as in Table 5.3 where the cohomology generators \( h^i \) are dual to the following generators of the homology: \( h_1 \sim \{1,2\}, h_2 \sim \{3,4\}, h_3 \sim \{1,3\}, h_4 \sim \{2,4\}, h_5 \sim \{1,4\} \) and \( h_6 \sim \{2,3\} \). The notation \( h_a \sim \{i,j\} \) means that two-cycle consists of plaquettes which wrap around the torus formed by the \( i^{th} \) and \( j^{th} \) circle. It should be clear from the examples that under the duality \( h_2a (h_2a-1) \) forces constraints on \( h_{2a+1} (h_{2a}) \) and then interpretation on the dual lattice implies that constraints are forced on \( h_{2a-1} (h_{2a}) \) and one finds the situation depicted in Table 5.3. At the level of the coefficient groups, the present case is identical to the spin model on a surface of genus 3. This implies that choosing \( G = \mathbb{Z}_{MN}, H_{2a-1} = \mathbb{Z}_N \) and \( H_{2a} = \mathbb{Z}_M, a = 1,2,3 \) (one could also mix \( N \) and \( M \) for different values of \( a \) ) leads to self-dual models with different fractional charges around the various cycles of the lattice.

<table>
<thead>
<tr>
<th>Model</th>
<th>( G )</th>
<th>( \mathcal{H}_1 )</th>
<th>( \mathcal{H}_2 )</th>
<th>( \mathcal{H}_3 )</th>
<th>( \mathcal{H}_4 )</th>
<th>( \mathcal{H}_5 )</th>
<th>( \mathcal{H}_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dual</td>
<td>( G^* )</td>
<td>( G^* \times \mathcal{H}_2^* )</td>
<td>( G^* \times \mathcal{H}_3^* )</td>
<td>( G^* \times \mathcal{H}_4^* )</td>
<td>( G^* \times \mathcal{H}_5^* )</td>
<td>( G^* \times \mathcal{H}_6^* )</td>
<td>( G^* \times \mathcal{H}_5^* )</td>
</tr>
</tbody>
</table>

Table 5.3: Mixing of the coefficient group under duality for a gauge theory on \( T^4 \). See the text for description of \( h^i \).
These examples do not by any means exhaust the possible models which exhibit self-duality with several fractional charges. They do serve to illustrate the canonical situations in which they appear however. It is not entirely clear under what physical situations such models would arise, however, the fact that they are explicitly self-dual coupled with the topological nature in which duality is maintained warrants them being mentioned.
In the previous chapters, the transformation of the partition function under duality was the focus. The dual models have been explicitly constructed, and several examples of models which transformed trivially under duality were given. The cases in which duality transformed the models into completely different theories, for example a spin model in four dimensions transforms into a model in which the dynamical variables are Kalb-Ramond fields, were mostly ignored. However, duality can give interesting insights in those models as well. In particular, duality can be used to get a handle on correlators of one theory in terms of correlators in the dual theory. If the coupling constant happens to large in the original model, and perturbation theory is not valid, the problem can, in principle, be transformed into a question about the dual theory in which the coupling constant is small and carry out a perturbative treatment there. One of the goals of this chapter is to derive the mapping between correlators in the original model and its dual. As a consequence, when the dimension of the lattice is the same as the dimension of the cells on which the interactions take place ($d = k$), arbitrary correlators can be reduced to a finite number of topological sums. This was alluded to in the previous chapter where it was noticed that in these dimensions the field-strength variables acquire only topological constraints, the local constraints are absent. Furthermore, it will be demonstrate that the fractional global charges introduced in chapter 5 lead to interesting transformation properties of the correlation functions. On a compact manifold, what is denoted by inside and outside of a submanifold is ill-defined, and as such any correlator on such surfaces
Chapter 6. Correlators

is expected to be independent of this assignment. In the case where the global charges are absent from the picture this is indeed the case, however, if global charges are present there will appear an explicit dependence on this choice. It will be demonstrated that there exists \( N \) inequivalent sectors of the theory (here \( 1/N \) denotes the fractional global charge) which bears a strong resemblance to the appearance of inequivalent sectors of a non-Abelian theory in the presence of a \( \theta \)-term. In addition, correlators which wrap non-trivially around the cycles of the lattice will be shown to vanish identically. To begin this chapter a short discussion of order and dis-order correlators follows.

An order or disorder correlator is defined through the following statistical sum,

\[
\left\langle \prod_{p \in \gamma} \chi_s \left( \left\langle \sigma, \partial c_k^{(p)} \right\rangle \right) \right\rangle \equiv \frac{1}{Z} \sum_{\{n_a \in \mathbb{N}_a\}} \sum_{\sigma \in C_{k-1}(\Omega, \mathcal{G})} \prod_{p \in \gamma} \chi_s \left( \left\langle \sigma, \partial c_k^{(i)} \right\rangle \right) \times \prod_{p=1}^{N_b} B_p \left( \left\langle (\delta \sigma + n_a h^a), c_k^{(p)} \right\rangle \right) \quad (6.1)
\]

where \( Z \) is the partition function (5.1), and \( \gamma \) is a collection of \( k \)-cells on the lattice. Particular choices of \( \gamma \) lead to the order or disorder correlators.

The order correlator is defined by choosing \( \gamma \), on the original lattice, such that it spans an arbitrary arc-wise connected \( k \)-dimensional surface with boundary. Define the chain \( \Gamma \equiv \sum_{p \in \gamma} c_k^{(p)} \), then consider the boundary of this chain \( \partial \Gamma = \sum_{l \in \partial \gamma} c_k^{(l)} \). In general it will consist of several disconnected components, denote this set of boundaries by \( \{ B_i = \sum_{l \in \partial \gamma, i} c_k^{(l)} \} \) with \( i = 1, \ldots, b \). Then (6.1) is the order correlator of \( \{ B_i \} \). By making use of the factorization properties of the characters, (3.4), this is explicitly seen to be the case,

\[
\prod_{p \in \gamma} \chi_s \left( \left\langle \sigma, \partial c_k^{(p)} \right\rangle \right) = \chi_s \left( \left\langle \sigma, \partial \Gamma \right\rangle \right) = \prod_{i=1}^{b} \chi_s \left( \left\langle \sigma, B_i \right\rangle \right) = \prod_{i=1}^{b} \chi_s \left( \sum_{l \in \partial \gamma} \sigma_l \right) \quad (6.2)
\]

As a simple example, consider the case of a spin model (\( k = 1 \)) and take \( \gamma \) to be a set of
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links running between the site \(i\) and the site \(j\). Then the correlator, equation (6.1), is,

\[
\left\langle \prod_{p \in \gamma} \chi_s \left( \langle \sigma, \partial c_{x}^{(p)} \rangle \right) \right\rangle = \langle \chi_s(\sigma_i) \chi_s(\sigma_j) \rangle
\]

which is the usual two-point function between sites \(i\) and \(j\).

Disorder correlators are defined by considering a correlator in the dual model and then re-interpreting it back on the original lattice. Let \(\gamma^*\) be a collection of \((d - k)\)-cells on the dual lattice (notice that \(d - k \geq 0\) since the models currently under consideration interact on the \(k\)-cells of the lattice) which forms an arc-wise connected \((d - k)\)-dimensional surface with boundary. Such a surface could be used to define a correlator in the dual theory. However, instead consider the collection of \(k\)-cells on the original lattice which are dual to these \((d - k)\)-cells, denote this collection by \(\gamma\). It should be clear that \(\gamma\) does not form an arc-wise connected \(k\)-dimensional surface, rather it forms a collection of disconnected \(k\)-cells. Let \(\{B^*_i\}\) be the collection of \((d - k - 1)\)-cells which form the boundaries of \(\gamma^*\) on the dual lattice, and denote by \(B_i\) the collection of \((k + 1)\)-cells on the original lattice dual to \(B^*_i\). The correlation function (6.1) with the above choice for \(\gamma\) is the disorder correlator of the collection of \((k + 1)\)-cells \(\{B_i\}\). It is possible to obtain \(\{B_i\}\) directly from \(\gamma\) since \(\{B^*_i\}\) are the boundaries of \(\gamma^*\) on the dual lattice then \(\{B_i\}\) are the set of \((k + 1)\)-cells forming the coboundary of \(\gamma\) on the original lattice. Notice that there is no simplifying interpretation of these correlators in terms of the \(n\)-point functions as there was for the order correlators (6.2).

As an example, consider the case of the Ising model in two dimensions. A correlation function is labeled by a set of links which start at site \(i\) and end at site \(j\) (see the top left diagram in Figure 6.1). A disorder operator on the dual lattice is obtained by interpreting the set of links joining site \(i\) to site \(j\) on the dual lattice. This corresponds to the links displayed in the bottom left diagram in Figure 6.1. Inserting the characters along each link in that diagram into the partition function computes the correlator of
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Figure 6.1: The relation between disorder and order correlators in a 2-D spin system. The top diagram shows a curve which defines the 2-point correlation function on the dual lattice, while the bottom diagram is the collection of links which give the disorder correlator.

As a second example consider a spin model in three dimensions. The correlator is as usual given by a set of links joining site $i$ and site $j$. A disorder correlator on the other hand is obtained by consider a set of plaquettes forming a surface on the dual lattice. Consider the example of a cylindrical like surface (see the top left diagram in Figure 6.2). Interpreting the surface back on the original lattice leads to a set of links on the original lattice which pierce those plaquettes (see the bottom left diagram in Figure 6.2). Inserting a set of characters along these links into the partition function leads to the correlator of the set of disorder variables shown in the bottom right diagram of Figure 6.2. Once again they are dual to the correlator on the dual lattice, which in this case is
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Figure 6.2: The relation between disorder correlators in a 3-D spin system and Wilson loops in a 3-D gauge theory. The top diagram is the surface which defines the two Wilson loops at its ends, and the lower diagram is its dual - a collection of links whose coboundary produces the disorder variables in the spin model.

the correlator of two Wilson loops.

6.1 Correlators From Duality

6.1.1 General Formalism

The general introduction considered correlators in which characters, in a particular fixed representation, are inserted along a collection of \( k \)-cells, \( \gamma \), which correspond to order and dis-order correlators. Here, a general correlator defined by an arbitrary collection of \( k \)-cells denoted by \( \gamma \) and a set of representations \( \{ S_p \in G^* \} \) for all \( p \in \gamma \) will be considered. These correlators include, but are not limited to the order and disorder correlators mentioned above. The correlator is defined much like (6.1),

\[
W(\{S_p\}) = \left\langle \prod_{p \in \gamma} \chi_{S_p} \left( \langle \delta \sigma, c_k^{(p)} \rangle \right) \right\rangle
\]

\[
= \frac{1}{Z} \sum_{\{n_a \in \mathbb{N}_a\}} \sum_{\sigma \in C_{k-1}(\Omega, G)} \prod_{p \in \gamma} \chi_{S_p} \left( \langle \delta \sigma, c_k^{(p)} \rangle \right) \prod_{p=1}^{N_k} B_p \left( \langle \delta \sigma + n_a h^a, c_k^{(p)} \rangle \right)
\]

(6.4)

where \( Z \) is of course the partition function (5.1). The dual transformations on this correlator can be carried out in a rather trivial manner. Firstly, let \( \gamma_o \) denote the set of \( k \)-cells
with representation \( s \), so that \( \cup_s \gamma_s = \gamma \). If \( \gamma_s \) has no boundary the correlation function of \( \gamma_s \) reduces to the partition function. This can be seen from (6.2). Consequently, the discussion will be restricted to the case where \( \gamma_s \) has a boundary for every \( s \). In that case, the product over characters can be trivially absorbed in a redefinition of the Boltzmann weight,

\[
B_p \left( (\delta\sigma + n_a h^a, c_k^{(p)}) \right) \rightarrow B_p \left( (\delta\sigma + n_a h^a, c_k^{(p)}) \right) \chi_{S_p} \left( (\delta\sigma, c_k^{(p)}) \right)
\]

\[
= \frac{1}{|G|} \sum_{r_p \in G^*} b_p(\langle r, c_k^{(p)} \rangle) \chi_{\langle r, c_k^{(p)} \rangle} \left( (\delta\sigma + n_a h^a, c_k^{(p)}) \right) \chi_{\langle r, c_k^{(p)} \rangle} \left( (\delta\sigma, c_k^{(p)}) \right)
\]

\[
= \frac{1}{|G|} \sum_{r_p \in G^*} b_p(\langle r - S, c_k^{(p)} \rangle) \chi_{\langle r, c_k^{(p)} \rangle} \left( (\delta\sigma + n_a h^a, c_k^{(p)}) \right) \tag{6.5}
\]

for all \( p \in \gamma \) and the \( k \)-chain \( S \equiv \sum_s S_s = \sum_{p \in \gamma} S_p c_k^{(p)} \) has been introduced. The last equality in (6.5) was obtained by using the factorization properties of the characters, (3.4), and then shifting \( r \). Although the two arguments of the characters in the second line appear to be different, this difference vanishes for \( p \in \gamma \) because it is always possible to arrange for the generators of the cohomology to have no overlap with the cells of \( \gamma \). To be precise \( \langle S, h^a \rangle = 0 \) since a non-vanishing inner product implies that \( S \) contains an element of the homology group, but this contradicts the assumption that all \( \gamma_s \) have a boundary.

With this shift in the Boltzmann weights the dual correlator is trivial to obtain, simply use (5.9) with the appropriate character coefficients.

\[
W(\{S_p\}) = \frac{1}{Z} \sum_{\tilde{\gamma} \in G^*} \sum_{r \in C_{d-k-1}(\Omega^*, G^*)} \prod_{p=1}^{N_{d-k}} b_p \left( (\tilde{r} - S^* + \tilde{n}_a h^{*a}, c_{d-k}^{*}) \right) \tag{6.6}
\]

The normalizing partition function, \( Z \), should be evaluated in its dual form (5.9) so that the factors of the group volume that were left out to avoid clutter throughout cancel. Also, \( S^* \) is the dual to \( S \), obtained by interpreting the set of \( k \)-cells, \( \gamma \), on the original lattice as a set of \( (d-k) \)-cells, \( \gamma^* \), on the dual lattice and defining \( S^* = \sum_{p \in \gamma^*} S_p c_{d-k}^{*} \). It
should be clear that if γ was chosen to define an order correlator, then (6.6) is a disorder correlator in the dual variables thus establishing the connection between strong and weak coupling, and order and disorder variables.

There is a class of correlators which this formalism does not include. These correlators correspond to inserting characters along a set of \((k - 1)\)-cells which form a closed sheet wrapping around a non-trivial \((k - 1)\)-cycle of the lattice. For example, a gauge theory on \(S^1 \times \mathcal{M}\), where \(\mathcal{M}\) is a contractable space, has a Wilson loop which wraps once around the compact direction. This set of links is not the boundary of a two-dimensional sheet and hence cannot be written in the form (6.4). However, in the next section, such correlators will be demonstrated to vanish identically.

### 6.1.2 The Vanishing Correlator

A powerful theorem, concerning the vanishing of certain correlators in a statistical model, will now be proven. The physical interpretation of the theorem will be explained, and some interesting applications will be given shortly.

**Theorem:** Let \(\gamma\) be a collection of \((k - 1)\)-cells with zero boundary and \(\{S_l \in \mathcal{G}^*\}\) a collection of representations labeled by \(l \in \gamma\). If \(S = \sum_{l \in \gamma} S_l c_{k-1}^{(l)}\) is a representative element of \(H_{k-1}(\Omega, \mathcal{G}^*)\) then the following correlator vanishes identically,

\[
\langle \gamma \rangle = \sum_{l \in \gamma} \prod_{i} \chi_{S_l} (\sigma_i) \left( \langle \sigma, c_{k-1}^{(l)} \rangle \right) \prod_{p=1}^{N_k} B_p \left( \langle \delta \sigma + n_a h^a, c_{k}^{(p)} \rangle \right)
\]

where \(Z\) is given by Eq. (5.1).

**Proof:** Introduce the character expansion of the Boltzmann weights, so that every \(k\)-cell carries an irreducible representation, \(\mathcal{G}^*\), of the group \(\mathcal{G}\). Encode this information
into a \( k \)-chain denoted by \( r \) and re-order the sum. The result is,

\[
W(\{S_p\}) = \sum_{r \in C_k(\Omega, G^*)} \prod_{p=1}^{N_k} b_p(\langle r, c_{k}^{(p)} \rangle) 
\times \sum_{\{n_a \in \mathcal{H}_a\}} \sum_{\sigma \in C_{k-1}(\Omega, G)} \prod_{i \in \gamma} \chi_{S_i} \left( \langle \sigma, c_{k-1}^{(l)} \rangle \right) \prod_{p=1}^{N_k} \chi_{\langle r_c^{(p)} \rangle} \left( \langle \delta \sigma + n_a h^a, c_{k}^{(p)} \rangle \right)
\]

where \( b(r) \) are the character coefficients of the Boltzmann weights given in (5.2). Using (5.4) and the factorization properties of the characters, (3.4), the summation over \( \sigma \) and \( \{n_a\} \) reduces,

\[
\sum_{n_a, \sigma} \cdots = \sum_{\sigma \in C_{k-1}(\Omega, G)} \prod_{i \in \gamma} \chi_{S_i} \left( \langle \sigma, c_{k-1}^{(l)} \rangle \right) \prod_{l=1}^{N_{k-1}} \chi_{\langle \partial r, c_{k-1}^{(l)} \rangle} \left( \langle \sigma, c_{k-1}^{(l)} \rangle \right) \sum_{\{n_a \in \mathcal{H}_a\}} \prod_{a=1}^{A_k} \chi_{\langle \partial r, h^a \rangle} (n_a)
\]

\[
= \sum_{\sigma \in C_{k-1}(\Omega, G)} \prod_{l=1}^{N_{k-1}} \chi_{\langle \partial r + S, c_{k-1}^{(l)} \rangle} \left( \langle \sigma, c_{k-1}^{(l)} \rangle \right) \prod_{a=1}^{A_k} \sum_{n_a \in \mathcal{H}_a} \chi_{\langle \partial r, h^a \rangle} (n_a)
\]

\[
= \prod_{l=1}^{N_{k-1}} \delta_{g^*} (\langle \partial r + S, c_{k-1}^{(l)} \rangle) \prod_{a=1}^{A_k} \delta_{\mathcal{H}_a^*} (\langle \partial r, h^a \rangle)
\]

where the orthogonality of the characters, (3.5), was used to obtain the last equality.

If \( S \) is a representative element of \( H_{k-1}(\Omega, G^*) \) then there are no configurations which solve the first constraint, since by definition there exists no \( k \)-chain \( r \) whose boundary equals an element of the homology group. Consequently, the correlator (6.7) vanishes, and completes the proof of the theorem.

This rather general theorem has some interesting physical consequences. Consider for example a spin model \( (k = 1) \) in arbitrary dimensions and compute the correlator of an odd number of characters. This vanishes identically from symmetry reasons alone, however, from the point of view of the above theorem it vanishes since an odd number of points is homologous to a single point, and hence is a representative element of \( H_0(\Omega, G^*) \) and thus must vanish on topological grounds. A more interesting application of the theorem is to consider a gauge theory \( (k = 2) \) on a manifold which has one compact direction, and compute the Wilson loop around that compact direction in representation
Figure 6.3: This diagram depicts the set of plaquettes which are used in the computation of the correlator between the two Wilson loops on its boundary.

This loop is clearly a representative of $H_1(\Omega, G^*)$ and thus the Wilson loop correlator must vanish identically. Although this result can be reasoned from symmetry arguments, it seems somewhat more illuminating to see how it follows from purely topological requirements. It is interesting to note that the correlator of two Wilson loops wrapped once around the compact direction, with conjugate representations (i.e. one with representation $s$ and the other with representation $-s$), does not vanish. The reason is because the $k$-chain $S = s(\Sigma_{l \in \gamma_1} - \Sigma_{l \in \gamma_2})$ is the boundary of a sheet formed between the two loops, $S = s \partial \Sigma_{p \in \Gamma} \epsilon_2^{(p)}$ where $\Gamma$ is set of plaquettes connecting one loop to the other (see Figure 6.3), and is hence not a representative of $H_1(\Omega, G^*)$. In fact, since the correlator is reduced to a correlator for plaquette valued objects, the results of section 6.1.1 can be used to complete the duality transformation. This is mentioned since when the distance between the two loops is taken to zero such a correlator corresponds to the computation of a single Wilson loop correlator in the adjoint representation which in a non-Abelian theory does not vanish. In general there are only two fundamentally different types of correlators: those that can be written in terms of the boundary of a higher dimensional surface, for which a systematic construction has been given in the previous
section; and those that form representative elements of the homology group, which have been demonstrated to vanish.

6.1.3 Correlators in $d = k$ dimensions

Since it has been established that a non-vanishing correlator is of the general form (6.4), and its dual formulation (6.6), the discussion will be restricted to those cases. It is possible to obtain an explicit form for these correlators in $d = k$ dimensions solely in terms of the topological properties of the lattice. The reason is as follows: in performing the duality transformations the constraints (5.5) must be solved, however, in dimensions $d = k$ the most general solution to the first set of constraints in (5.5) is $r = h \in H_d(\Omega, G^*)$ because there are no $(d + 1)$-chains on a $d$-dimensional lattice. Consequently, the dual theory contains only topological fluctuations. This sort of trivialization of the problem can also be seen from the point of view of the field-strength formulation (5.20). In dimensions $d = k$ the local Bianchi constraints are absent rendering the dynamical fields locally free while constraining them only globally. Of course the form of the dual correlator (6.6) implies this trivialization as well, since when $d = k$ the dynamical field $\sigma$ is a $(-1)$-chain, which of course has only the identity element, the only sums that remain are those over $\{\tilde{n}_a\}$. It is trivial to write down the correlators in these dimensions, and they consist of only a finite number of sums (or integrals if the dual group is continuous),

\[
W(\{S_p\}) = \frac{\sum_{\{\tilde{n}_a\} \in G^* \cdot N^k} \prod_{p=1}^{N^k_0} b_p \left( (\tilde{n}_a h^{*a} - S^*, c^{(p)}_0) \right)}{\sum_{\{\tilde{n}_a\} \in G^* \cdot N^k_0} \prod_{p=1}^{N^k_0} b_p \left( (\tilde{n}_a h^{*a}, c^{(p)}_0) \right)}
\]

(6.10)

here $\{h^{*a}\}$ are the set of generators of $H^0(\Omega^*, G^*)$ which are dual to the generators of the homology group $H_d(\Omega, G^*)$. In the next two sections this formula will be used to compute some non-trivial correlators in a spin model on an arbitrary graph and a gauge theory on an arbitrary orientable two-dimensional Riemann surface. These results have straightforward generalizations to the higher dimensional cases.
Chapter 6. Correlators

6.2 Examples

6.2.1 Spin System Case

In this section the computation of correlators in a spin model on a 1-dimensional graph using (6.10) will be carried out. Let $\Omega$ be a one dimensional graph, and let $\{h^a\}$ denote the generators of $H^0(\Omega^*, G^*)$. To attach a physical meaning to these generators consider the generators of $H_1(\Omega, G^*)$ and then re-interpret them on the dual lattice. It should be obvious that $H_1(\Omega, G^*)$ is generated by the set of links which form closed loops on the graph (take only an independent set of these generators). Figure 6.4 is an illustrative example of how this works. In general the generators of $H^0(\Omega^*, G^*)$ are $h^a = \sum_{i \in \Gamma^*_a} C^{(i)}_0$ where $\Gamma^*_a (a = 1, \ldots, A)$ is a collection points, on the dual lattice, which are dual to a set of links that form a closed loop on the original lattice. It is easy to convince oneself that the coboundary of $h^a$ vanishes, and since there are no lower dimensional chains this is an element of the cohomology. As long as the set of loops are independent then these generators form a complete set.

The correlation function is defined via a set of links on the original lattice, $\gamma$, and representations, $\{S_i\}$, for every link. When $\gamma$ is interpreted on the dual lattice it corresponds to a set of points, $\gamma^*$. Let $\gamma^*_a$ denote the set of points in $\gamma^*$ that are contained in...
\( \Gamma_a^* \) (i.e. the set of points dual to the set of links contained in the \( a \)-th loop of the original lattice). Also, let \( \{ S_i^* \} \) denote the collection of representations on the dual sites, and define the 0-chain \( S_a^* \equiv \sum_{i \in \gamma_a^*} S_i^* \). Applying the dual formulation of the correlator (Eq.(6.10)), represented by \( \gamma \) gives,

\[
W(\{ S_p \}) = \prod_{a=1}^{A} \left\{ \frac{\sum_{\bar{n}_{a} \in \mathcal{O}^*} \prod_{i \in \Gamma_a^*} b_i \left( n_a - \langle S_i^*, c_0^{(i)} \rangle \right)}{\sum_{\bar{n}_{a} \in \mathcal{O}^*} \prod_{i \in \Gamma_a^*} b_i \left( \bar{n}_{a} \right)} \right\}
\]  

(6.11)

It has been assumed that the generators have no overlap, i.e. \( \Gamma_a \cap \Gamma_b = \emptyset \) if \( a \neq b \). It is not difficult to include the cases where there are overlaps, however, this only serves to complicate the final sums over \( \{ n_a \} \). It is possible to simplify the summations in the above expression only once the Boltzmann weights are specified. A convenient example is furnished by choosing \( \mathcal{G} = U(1) \) and the Villain form for the Boltzmann weight, so that,

\[
b_i(\bar{n}) = \frac{1}{\sqrt{2\pi \beta}} \exp \left\{ -\frac{1}{2\beta} \bar{n}^2 \right\}
\]  

(6.12)

The sums over \( \bar{n}_a \) can now be performed in terms of well known special functions, the Jacobi Theta function [66]. To perform the sum it is also necessary to specify \( \{ \mathcal{H}_a \} \). Since \( \mathcal{G} \) was taken to be \( U(1) \) this forces \( \mathcal{H}_a \) to be a cyclic group, \( \mathbb{Z}_{N_a} \). Inserting the above ansatz into Eq.(6.11) leads to,

\[
W(\{ S_p \}) = \prod_{a=1}^{A} \left[ e^{-\frac{1}{2\beta} \sum_{i \in \gamma_a^*} S_i^2} \sum_{\bar{n}_{a} \in \mathbb{Z}} e^{-(2\beta)^{-1} N_0^a N_0^a \bar{n}_{a}^2 - \frac{1}{2\beta} \sum_{i \in \gamma_a^*} S_i^2} \right] \left[ e^{-(2\beta)^{-1} N_0^a N_0^a \bar{n}_{a}^2} \right]
\]  

(6.13)

Here \( N_0^a \) denotes the number of sites on the dual lattice contained in \( \Gamma_a^* \). This general formula is not extremely illuminating, however, it reduces to a reasonable form for the two point function. In that case \( \gamma \) consists of a set of links which form a single continuous
path from point $i$ to $j$ on the lattice and the representations on those links are all taken to be labeled by $s$. Then the above expression reads,

$$ W(\{s\}) = \prod_{a=1}^{A} e^{-\frac{s^2}{2\beta} \frac{L(\gamma^*_a)(N_{0,a}^s - L(\gamma^*_a))}{N_{0,a}^s}} \frac{\theta_3 \left( \frac{2\beta}{N_{0,a}^s}; -\frac{s L(\gamma^*_a)}{N_{0,a}^s} \right)}{\theta_3 \left( \frac{2\beta}{N_{0,a}^s}; 0 \right)} $$

(6.14)

where $L(\gamma^*_a)$ is the number of dual sites (links) contained in $\gamma^*_a$ ($\gamma_a$) and counts the number of links that the path $\gamma$ crossed in going through the $a$-th loop of the graph. Notice that the overall exponential factor is explicitly invariant under,

$$ L(\gamma^*_a) \rightarrow N_{0,a}^s - L(\gamma^*_a) $$

(6.15)

This symmetry implies that the overall factor is not affected by which path one took in going through a loop. However, the theta-function contributions are invariant only under an integer shift in the second parameter. Write $s = s_1 + N s_2$ where $s_1 \in \{0, \ldots, N - 1\}$, $s_2 \in \mathbb{Z}$ and $N$ is the lowest common factor of $\{N_a\}$. It is clear that under the transformation (6.15), the theta function is invariant only if $s_1 = 0$. In general for a fixed $s_2$ there are $N$ inequivalent sectors of the model, each sector transforms differently under the transformation (6.15). The appearance of these sectors can be traced back to the discussion in section 5.2.2. It was pointed out there that performing the sum over topological sectors, $\{n_a\}$, in a subgroup of the gauge group introduces fractional global charge into the system. In computing the two-point function one is forced to introduce a set of links forming a path with two end points. Naively, the choice of path has no effect on the correlator, however, due to the existence of fractional charge in the system there are lines of flux which pierce through the path. Consequently, a different choice of path carries a different amount of fractional charge, leading to a non-trivial transformation of the correlator under the shift (6.15). Only when the representation of the path respects the $n$-ality of the global charges can the system be symmetric under
6.2.2 Gauge Theory Case

Now consider an Abelian gauge theory \((k = 2)\), with gauge group \(G\), on an arbitrary orientable two-dimensional Riemann surface. Let \(\Omega\) be a discretization of this manifold. Since the generator of the second homology group is the entire surface, this implies that \(H^0(\Omega, G^*)\) has one generator which is given by \(h = \sum_{i \in \Omega} c_0^{(i)}\). The correlator (6.4) then reduces to,

\[
W(^{\{S_p\}}) = \frac{\sum_{\vec{n} \in G^* \cdot \mathcal{H}^*} \prod_{i=1}^{N_0^*} b_i \left( \vec{n} - (S^*, c_0^{(i)}) \right)}{\sum_{\vec{n} \in G^* \cdot \mathcal{H}^*} \prod_{i=1}^{N_0^*} b_i \left( \vec{n} \right)}
\]  

(6.16)

As in the last section consider \(G = U(1)\), \(\mathcal{H} = \mathbb{Z}_{N}^*\) and the Boltzmann weight in Villain form (6.12). Then Eq.(6.16) becomes,

\[
W(^{\{S_p\}}) = e^{-\frac{1}{\beta} \left( \sum_{i \in \mathcal{E}^*} S_i^* \right)^2 - \frac{1}{\beta} \left( \sum_{i \in \mathcal{E}^*} S_i^* \right)^2} \frac{\theta_3 \left( \frac{2\beta N^2 N_0^*}{N_0^*}; - \frac{1}{N_0^*} \sum_{i \in \mathcal{E}^*} S_i^* \right)}{\theta_3 \left( \frac{2\beta N^2 N_0^*}{N_0^*}; 0 \right)}
\]  

(6.17)

The simplest correlator in a gauge theory is a (filled) Wilson loop, in which \(\gamma\) is a collection of adjacent plaquettes and \(\{S_p = s\}\) (the two loops in Figure 6.3 is one such example). With this choice the correlator, Eq.(6.17), is simply,

\[
W(\{s\}) = e^{-\frac{2}{\beta} \frac{A(\gamma^*) N_0^* - A(\gamma^*)^2}{N_0^*} \theta_3 \left( \frac{2\beta N^2 N_0^*}{N_0^*}; - \frac{A(\gamma^*)}{N_0^*} \right)} \frac{\theta_3 \left( \frac{2\beta N^2 N_0^*}{N_0^*}; 0 \right)}{\theta_3 \left( \frac{2\beta N^2 N_0^*}{N_0^*}; 0 \right)}
\]  

(6.18)

Here \(A(\gamma^*)\) denotes the number of dual sites (plaquettes) contained in \(\gamma^* (\gamma)\). Just as in the spin model case the exponential is invariant under the following transformation,

\[
A(\gamma^*) \to N_0^* - A(\gamma^*)
\]  

(6.19)

This symmetry implies that the exponential part of the correlation function does not distinguish between what is consider the inside or outside area of the Wilson loop. Notice
that in the limit in which the area of the surface becomes infinite while the area of the Wilson loop remains finite \((\mathcal{A}(\gamma^*)/\mathcal{N}_0^* \to 0)\) the exponential reduces to the familiar area law. However, if the ratio of the number of plaquettes in the Wilson loop and the total number of plaquettes on the manifold is kept fixed \((\mathcal{A}(\gamma^*)/\mathcal{N}_0^* = \text{const.})\) as the number of plaquettes is taken to infinity, so that one reproduces the continuum limit, then there are finite size corrections given both by the theta-function and the overall exponential.

As in the spin model scenario, the theta-function does not respect the symmetry (6.19) in general. The generic case has \(s = s_1 + Ns_2\) where \(s_1 \in \{0, \ldots, N-1\}\) and \(s_2 \in \mathbb{Z}\), if \(s_1 \neq 0\) the system distinguishes between inside and outside of the loop, just as in the one-dimensional spin model. The reasoning follows through much like it did there. The fractional charge induces a flux which is incompatible with the representation (unless \(s = NZ\)) and as such a different choice of surface carries a different amount of flux. The \(N\) different sectors are once again seen to give rise to \(N\) different transformations of the correlator under (6.19). This bears a strong resemblance to the appearance of theta sectors in non-Abelian gauge theories. It indicates that there is a strong connection between theta sectors in a non-Abelian theory, and the topological sectors in an appropriate Abelian model on a topologically non-trivial manifold.

Before closing, we would like to illustrate how a direct product of groups can be included in a rather trivial manner. As an example consider the case in which the gauge group \(G = \bigoplus_{a=1}^m U(1)\) and \(H = \bigoplus_{a=1}^m \mathbb{Z}_{N_a}\). Also choose the heat kernel action,

\[
 b_t((\tilde{n}^1, \ldots, \tilde{n}^m)) = \prod_{a=1}^m \frac{1}{\sqrt{2\pi\beta_a}} \exp \left\{ -\frac{1}{2\beta_a} (n^a)^2 \right\}
\]

(6.20)

here \(\{\beta_a\}\) are independent coupling constants. Since the gauge group is a product group it is necessary to specify multiple representations of \(U(1)\) on every plaquette in \(\gamma\), label these representations by \(\{S_p^a\}\). Since the Boltzmann weights have no cross terms the
problem factorizes and the correlation function is straightforward to write down,

\[
W(\{S_p^a\}) = e^{-\frac{1}{g^2} \sum_{a=1}^{m} \left[ \sum_{\tau \in \Gamma^*} (S_{\tau}^a)^2 \frac{1}{N_{\gamma}^0} \left( \sum_{\tau \in \Gamma^*} S_{\tau}^a \right)^2 \right]} \prod_{a=1}^{m} \frac{\theta_3 \left( \frac{2\beta_a}{N_{\gamma}^0}; -\frac{1}{N_{\gamma}^0} \sum_{\tau \in \Gamma^*} S_{\tau}^a \right)}{\theta_3 \left( \frac{2\beta_a}{N_{\gamma}^0}; 0 \right)}
\]  

(6.21)

Once again consider the simplifying case of a filled Wilson loop in which \( \{S_p^a = s\} \) for all \( p \in \gamma \). The above expression then reduces to a product of factors like that appearing in (6.18),

\[
W(\{s\}) = e^{-\frac{2}{g^2 N_{\gamma}^0} A(\gamma)(N_{\gamma}^* - A(\gamma))} \prod_{a=1}^{m} \left( \frac{\theta_3 \left( \frac{2\beta_a}{N_{\gamma}^0}; -\frac{s A(\gamma)}{N_{\gamma}^*} \right)}{\theta_3 \left( \frac{2\beta_a}{N_{\gamma}^0}; 0 \right)} \right)
\]  

(6.22)

where \( \tilde{\beta} = \sum_{a=1}^{m} \beta_a^{-1} \). The invariance of this expression under \( A(\gamma) \rightarrow N_{\gamma}^* - A(\gamma) \) depends upon whether \( s \) is a representation which respects the \( n \)-ality of the fractional charge. In this case only if \( s \in N \mathbb{Z} \), where \( N \) is the least common multiple of \( \{N_a\} \), will the expression be invariant under that symmetry.
In this work, the effects that topology plays in various aspects of Abelian lattice models have been studied. The dual construction on lattices with non-trivial topology was found to require the introduction of topological modes in the dual models. Consequently, the phase transition which occurs at the self-dual point of a self-dual model defined on a topologically flat lattice is lost when some of the directions are compactified. Of course, the new topological modes are responsible for the destruction of phase transitions. The physical interpretation of the topological modes were illuminated through rewriting the models in terms of their field-strength variables. It was found that they are responsible for the quantization of the global charges in the system, and the specific quantization condition is intimately connected with the group on which the topological modes are defined. Furthermore, since these modes appear in the dual theory, we have carried out the obvious generalization and included them in the defining model. Under such an inclusion some of the would-be topological degrees of freedom in the dual theory are modified or even eliminated. Cancellations among the topological modes allow for the creation of self-dual models even on lattices with non-trivial topology. Given a model that is self-dual on flat topologies, a straightforward and systematic construction has been developed which renders the model self-dual when some directions are compactified.

The case of a three-dimensional finite temperature $U(1)$ gauge theory on a lattice with topology $S^1 \times S^2$ has been studied. It was found that the dual Coulomb gas picture is altered due to the existence of topological modes that appear under the duality
transformation. The compactification of the imaginary time direction was shown to alter the associated Sine-Gordon model only minimally: the scalar field now lived in a circular target-space as opposed to the real line. There are, in fact, several systems which are not affected when some directions are compactified, a notable example being finite temperature gauge theories in four dimensions with topology $S^1 \times S^3$. Since the relevant cohomology group vanishes for this topology, topological modes do not appear in the dual model. As such, the model is self-dual since it is self-dual in flat topologies. This observation allows one to obtain a qualitative description of the phase diagram as a function of both the temperature and the coupling constant.

The link between target-space duality and strong-weak duality on the lattice was exploited to demonstrate that, in the continuum model, one must take care to define the partition function in terms of single-valued fields. This led to the insertion of topological modes in the defining continuum partition function, and the Hodge-duality transformation led to an explicitly self-dual model. Upon discretizing the world-sheet manifold, the associated statistical model was shown to exhibit this self-duality only if the topological modes were introduced into the defining model. Subsequently, this procedure led to precisely the type of constraints on the statistical model which were placed in by hand in order to restore target-space duality in the lattice model by other authors. The connection between this type of construction and the insertion of topological modes to maintain strong-weak duality was discussed, and it was emphasized that the main difference is the group in which the topological modes were defined. In the present case the group in which the topological modes live in differs from the group in which the spins live. Models which are explicitly invariant under target-space duality, and which also have a finite order group as the fundamental spins, were constructed. Such models can be written in matrix model language, and can serve as a testing ground for numerical calculations of an explicitly self-dual string theory. Finally, through the use of the duality transformations,
two target-spaces with different homotopy groups were shown to have identical string excitation spectrums.

Both types of duality can be seen to follow from a generalized lattice model in which the topological modes around the various cycles of the lattice are completely independent. The case of target-space duality resulted when all cycles were chosen to have identical groups, with the further restriction that the quotient of the spin group and the topological modes group is the target-space itself. The earlier forms of duality were achieved by including the topological modes in only some of the cycles, while the group was forced to be identical to the group in which the local dynamical degrees of freedom lived.

Using these generalized models the connections between dis-order and order correlators of the model and its dual were illuminated. The correlation function of a set of variables which form a representation of the homology group of a particular dimension was demonstrated to explicitly vanish on topological grounds alone. More generally, given a correlation function, a mapping to its dual form was constructed. When the dimension of the lattice equals the dimension in which the local degrees of freedom interact, duality rendered the problem trivial by expressing the answer in terms of a finite number of summations. Using this result, the two-point function of a $\mathbb{Z}_N$ model defined on an arbitrary one-dimensional closed lattice was derived. Since the computation requires the introduction of a path connecting the two points, the result showed invariance under a change in this path only if there was no inclusion of topological modes in the defining model. The inclusion of topological modes destroyed the invariance under a choice of path, and led to inequivalent models labeled by the choice of the topological group. This is explained by the fact that the global charges around any loop which contain such topological modes are quantized differently from the local charges. One can view these fractional charges as lines of flux which pierce the lattice; consequently, choosing one path over another leads to a different amount of flux and to a different sector of the theory. Similar results
were found in the case of two-dimensional $U(1)$ gauge theory. Here the symmetry being destroyed is the invariance under a choice of surface whose boundary is the Wilson loop. Once again inequivalent sectors of the theory are found; however, since there is only one possible topological mode, the sectors are labeled by an integer and resemble the labeling of the semi-classical vacua in non-Abelian gauge theories.

Hopefully, the idea that topology plays an important role in duality transformations has been conveyed, and that considering only local degrees of freedom misses many interesting physical consequences. The existence of topological modes leads to the restoration of symmetries which one would naively believe to be broken on discretizing the model, such as in the case of target-space duality. Furthermore, the extra freedom presented by the inclusion of these modes into the defining model allows for the building of self-dual models. Topology plays a large role in the computation of correlators, as we have seen, and can lead to phenomena which are fairly easily derived in these Abelian models but bear strong resemblance to phenomena seen in non-Abelian versions of the models and may shed light on the highly non-trivial phase structures found there.
### Glossary

<table>
<thead>
<tr>
<th>Term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary</td>
<td>The operator which maps a $k$-chain into its boundary $(k-1)$-chain. Its action on elementary cells is given by $\partial c_k^{(i)} = \sum_j [c_k^{(i)} : c_{k-1}^{(j)}] c_{k-1}^{(j)}$, where $[\cdot : \cdot]$ is the incidence number.</td>
</tr>
<tr>
<td>Cell</td>
<td>An oriented generator of the Chain group, physically corresponding to a cell of a particular dimension on the lattice.</td>
</tr>
<tr>
<td>Chain</td>
<td>An element of the chain group.</td>
</tr>
<tr>
<td>Chain Group</td>
<td>The chain group of order $k$ is defined to be the set ${ g = \sum_i g_i c_k^{(i)} : g_i \in G }$, where $G$ is the symmetry group, and $c_k^{(i)}$ is the $i^{th}$ cell of dimension $k$.</td>
</tr>
<tr>
<td>Coboundary</td>
<td>The operator which maps a $k$-chain into its coboundary $(k+1)$-chain. Its action on elementary cells is given by $\delta c_k^{(i)} = \sum_j [c_k^{(j)} : c_{k+1}^{(i)}] c_{k+1}^{(j)}$, where $[\cdot : \cdot]$ is the incidence number.</td>
</tr>
<tr>
<td>Cocycles</td>
<td>Chains which have zero coboundary.</td>
</tr>
<tr>
<td>Cohomology</td>
<td>Elements of the chain group which have vanishing coboundary, and cannot be written as the coboundary of a lower dimensional chain.</td>
</tr>
<tr>
<td>Cycles</td>
<td>Chains which have zero boundary.</td>
</tr>
<tr>
<td>Homology</td>
<td>Elements of the chain group which have vanishing boundary, and cannot be written as the boundary of a higher dimensional chain.</td>
</tr>
<tr>
<td>Incidence #</td>
<td>The incidence # determines the relative orientation of two cells. It vanishes unless one cell is contained within the other, and the cells are of consecutive dimensions.</td>
</tr>
</tbody>
</table>
Bibliography


[76] E. Witten, "Dyons of Charge $e\theta/2\pi$", *Physics Letters* B86 (1979) 283.