VACUUM STRUCTURE AND THERMODYNAMICS OF TWO DIMENSIONAL GAUGE THEORIES COUPLED TO MATTER

By

Lori Dean Paniak

B. Sc. (Mathematical Physics), The University of Alberta, 1992

M. Sc. (Physics), The University of British Columbia, 1994

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

THE FACULTY OF GRADUATE STUDIES

DEPT. OF PHYSICS

We accept this thesis as conforming

to the required standard

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Physics and Astronomy

The University of British Columbia

2075 Wesbrook Place

Vancouver, Canada

V6T 1W5

Date: September 18, 1998

Abstract

The physics of two-dimensional Yang-Mills theories both with and without heavy, static matter is investigated. In particular, we focus on the relationship of these systems to the mathematical structures of Lie group theory. From this point of view explicit calculations of the vacuum structure and thermodynamics of finite N $SU(N)$ gauge theory with adjoint representation matter are carried out. The limit $N \to \infty$ of these systems is also constructed, making connection with the well-known formalism of unitary matrix models. An example of adjoint and fundamental matter interacting via $SU(N)$ gauge fields is considered explicitly and shown to have non-trivial phase structure which is qualitatively similar to what is expected to arise in higher dimensional gauge theories. This phase structure is interpreted in terms of group theoretic quantities. Finally, the case of fundamental representation matter interacting via $SU(N)$ fields and confined to a one-dimensional box is examined. The formalism for detailed exploration of this system is developed and used to show that there is a phase transition as a function of particle density and size of the box. The consequences of this observation for a string theory interpretation of a gas of Wilson loops on the two-dimensional space-time sphere are briefly discussed.

Table of Contents

List of Figures

Acknowledgements

Over the past few years it has been my pleasure to be in contact with a large group of researchers in high energy theoretical physics. The discussions and projects resulting from these contacts have stimulated my growth as a young investigator in the field. While thanking every single person that has had a hand in this process would take a prohibitive amount of space, there are some individuals that need to be singled out.

First, I must express my gratitude and appreciation to my research supervisor Gordon Semenoff. His direction and never-ending supply of ideas has provided me with a model to strive for. In addition his generosity in supporting his students' appearances at various schools, conferences and research institutes has served to broaden my research horizons and contacts with the broader community. I must also acknowledge my research collaborators and their contributions to my professional development. In particular I would like to thank Ariel Zhitnitsky, Christof Gattringer, Sebastian Jaimungal, Gianluca Grignani, Pasquale Sodano, Richard Szabo and Nathan Weiss. From each of these researchers I have learned a unique aspect of the art of answering, and asking, questions in the often abstract world of high energy theoretical physics. As well I would like to thank the heretofore unmentioned members of my supervisory committee, Janis McKenna and David Austin for their contributions to the content of this thesis.

I would like to thank those who have provided support for my endeavours both inside and outside Hennings. Especially I am grateful to my office-mates: Todd Fugleberg, Stephanie Curnoe and Ken Kiers for making discussions of physics and other topics much more enlightening. Of course the support of friends and family is a necessary ingredient for the successful conclusion of any project of this scale and I am blessed to have the best

of both. Last, but certainly not least, I must record my heart-felt gratitude to Elissa Kraus for her love and unflinching confidence in me and this thesis in the difficult time of its production.

Finally I am gratefully to the Natural Sciences and Engineering Research Council of Canada, the University of British Columbia and my parents for financial support during my graduate education. Without their help none of this would have been possible.

Chapter 1

Introduction and Overview

1.1 Yang-Mills Theory and Strings

The study of Yang-Mills [110] theories, or gauge theories as they are often referred to, has occupied much of the attention of high energy theoretical physicists for almost 30 years now. While this thesis is about one aspect of Yang-Mills theories, namely those that are defined in two spacetime dimensions, this Introduction is meant to give an (admittedly incomplete) overview how this study has proceeded over a generation. We will begin with a short description of the Yang-Mills theory from the point of view of mathematics and the fundamental definitions we will use throughout. Of course there is a need for details of the representation theory of compact Lie groups, which are most conveniently left to the Appendix. We begin with the long story of how, and why the study of Yang-Mills fields became important to high energy physics, in particular the description of the 'asymptotic freedom' of quarks in hadrons. Once the physical basis for studying such structures is established, we will give a lightning review of the evidence for the connection of Yang-Mills theory to another theory of modern high energy physics - string theory. In order to understand more about the details of how these two theories are related, we are immediately led to two space-time dimensions where the correspondence can be formulated in a precise manner. From there we will introduce the models on which the rest of the Thesis is based and give an overview of the results of the investigations.

Yang and Mills [110] originally introduced the concept of dynamical gauge theory as a

1

modification of the $SU(2)$ global isospin symmetry of nuclear physics. Taking motivation from the successful theory of quantum electrodynamics (QED) they promoted the global non-Abelian symmetry to a local one. As in the case of electrodynamics, in order to maintain invariance of the equations of motion under local gauge transformations , it is necessary to introduce a space-time dependent gauge field $A_\mu(\mathrm{x})$. The only exception here is that A_μ in the present case is an element of the algebra of the non-Abelian symmetry (Lie) group. If one considers a field theory without any sources whatsoever, then the theory can be written entirely in terms of A_μ, or more precisely, the anti-symmetric field strength tensor $F_{\mu\nu}$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] \tag{1.1}$$

This is the same definition that arises in electrodynamics with the exception of the final commutator term which takes into account the non-Abelian nature of the local symmetry group. The physical consequences of this structure are contained in the D-dimensional action $S[A]$

$$S[A] = -\frac{1}{2} \int_{\mathcal{M}} d^D x \mathrm{Tr}\, F^{\mu\nu}(x) F_{\mu\nu}(x) \tag{1.2}$$

and the partition function which gives the definition the functional measure in the physical theory

$$Z = \int \mathcal{D}A \; \mathrm{e}^{-S[A]} \tag{1.3}$$

The physical picture of a gauge theory as one that arises when one considers a local, space-time dependent symmetry also has a mathematical interpretation. From this point of view we can consider a principal bundle $P(\mathcal{M}, G)$ over a simply connected manifold $\mathcal{M}$ with Lie structure (gauge) group G. For a theory without sources, the configuration space is the space $\mathcal{A}(P)$ of gauge connection one-forms. If $\omega \in \mathcal{A}(P)$ is a connection on P then on an open patch U of $\mathcal{M}$ a local trivialization t of the bundle recovers the local

gauge potential A_μ on the manifold via the pull-back

$$A_\mu dx^\mu = t^* \omega|_U \tag{1.4}$$

In the bundle the connection ω naturally defines a unique curvature two-form F_ω on $\mathcal{M}$ through the pull-back of the curvature on the bundle.

$$F = dA + eA \wedge A \tag{1.5}$$

As we have seen before the equations of motion for these structures follow upon extremizing the action

$$S[A] = -\frac{1}{2} \int_{\mathcal{M}} \text{Tr}\, F \wedge *F \tag{1.6}$$

where $*$ is the Hodge dual operation on the Lorentzian manifold $\mathcal{M}$. In this notation the equations of motion read

$$dF = 0 \quad , \quad d*F = 0 \tag{1.7}$$

In fact, the second of these follows from the equations of motion and the first is the Bianchi identity which reiterates the fact that the field strength is locally derived from a potential. While one can now study the details of such mathematical structures, we will instead concentrate on the physical applications of the generalization of these classical mechanical systems to the quantum realm.

The main reason for studying this type of quantum field theory is that it gives a framework for understanding the results of high-energy particle collider experiments. In the late 1960's experiments carried out at the Stanford Linear Collider (SLAC) produced evidence that protons and neutrons were actually composite objects composed of three constituents. The presence of these constituents was consistent with the quark-parton model of matter and, hence were given the title 'quarks'. Surprisingly, in these experiments it appeared that the force which bound the quarks into protons and neutrons became increasingly weak at short distances, or equivalently at high energies. This

'asymptotic freedom' was somewhat counter-intuitive from a QED point of view since the same force appears to be sufficiently strong at long distances as to make isolated quarks unobservable. An explanation of this effect can be found uniquely among renormalizable quantum field theories in Yang-Mills theories both with or without matter. The full theory of quarks interacting via Yang-Mills gauge fields retains this property for less than 16 different 'flavours' of quarks and is known as quantum chromodynamics (QCD).

In order to see that one has asymptotic freedom in a Yang-Mills theory, we must consider the effects of quantum corrections. In a quantum field theory one must carry out a procedure of regularization and renormalization. This means that the perturbative expansion of the partition function contains infinities that must be controlled and absorbed into the coupling constants of the theory. Classical Yang-Mills theory has a single 'bare' coupling constant which we have called e. If we carry out this procedure of regulating the field theory by cutting off high energy effects at some scale Λ, and low energy effects at a second scale μ, then the renormalized coupling e_R can be calculated, in this case for $SU(N)$ Yang-Mills theory

$$e_R = e + \frac{e^3}{(4\pi)^2} \frac{11N}{3} \log \Lambda/\mu + O(e^5) \tag{1.8}$$

This most important thing to notice here is that the renormalized coupling constant is larger than the bare one. This 'anti-screening' of colour-electric charge by the quantum corrections of Yang-Mills theory lies at the heart of the phenomena of asymptotic freedom and is very different from QED, for example which screens and reduces bare electric charge.

These observations can be formalized and are contained in the framework of 'renormalization group' [24, 6, 95, 97]. This powerful philosophy of modern physics stresses the dis-entanglement of different energy scales in physical processes. The philosophy is that the physics at a particular energy only depends on the details of the system at the

same or lower energies and so parameters (like coupling constants and masses) vary with energy scale. For the case of $SU(N)$ Yang-Mills theory, renormalization group techniques allow one to calculate the coupling constant for a particular energy E referenced to some energy scale E_0

$$e_E = \frac{e_{E_0}}{\sqrt{1 + \frac{11}{24\pi^2} N e_{E_0}^2 \log E/E_0}} \tag{1.9}$$

The most important and interesting feature of this result is that in the limit $E \to \infty$, the renormalized coupling is independent of the reference and vanishes

$$\lim_{E \to \infty} e_E = \frac{1}{\sqrt{\frac{11}{24\pi^2} N \log E}} \tag{1.10}$$

This shows that in Yang-Mills theory with $SU(N)$ gauge group (this calculation can be easily generalized to other semi-simple groups) the coupling constant goes to zero at high energies and is consistent with the experimental observations of free quarks in protons and neutrons. The realization that Yang-Mills theory has the property of asymptotic freedom [33, 79] leads to the widespread belief that it is the correct theory for describing the physics of the strong force. It also is important since, in the absence of other masses coupled to the gauge field, the bare coupling is irrelevant and can be absorbed into the definition of the energy scale of the system. In other words, pure Yang-Mills theory has no free parameters to adjust!

The fact that pure Yang-Mills theory has no adjustable parameters is certainly remarkable from a theoretical point of view, but it makes practical calculations in the theory nearly impossible. For instance the use of perturbation theory in QED has led to spectacular agreement with experiment, but it is of no use in the present case of QCD. The search for a handle on Yang-Mills theory led t'Hooft [46] and later Witten [104] to consider the expansion of $U(N)$ Yang-Mills theory in powers of $1/N$ for large N. The power and elegance of this idea is easy to demonstrate. If we write out the Lagrangian

density associated with the action in (1.2) we can read off the interactions in the theory

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu} \\
&= \frac{1}{2}\mathrm{Tr}\, (\partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu])^2
\end{aligned}
\tag{1.11}
$$

Since the gauge field A_μ is an element of the Lie algebra associated with the gauge group $(U(N))$, we can consider it to be represented by a Hermitean matrix $(A_\mu)^i_j$ where i and j range over the dimension of the defining representation, in this case N. As is typical in quantum field theory, each term in this Lagrangian density can be represented diagrammatically, and a perturbative expansion of the partition function can be systematically constructed. For instance, the terms in (1.11) which contain two derivatives are of the form

$$
\partial_\mu (A_\nu)^i_j \partial^\mu (A^\nu)^j_i
\tag{1.12}
$$

The Feynman diagram corresponding to this term is a double line graph - a propagator - as shown in Figure 1.1(a). The indices are displayed with and without bars to distinguish lower (barred) and upper (un-barred) indices in (1.12). Likewise the arrows on the lines run from un-barred to barred indices. This is to keep track of the index structure when we join the propagator with the interaction vertices. In Yang-Mills theory there are two distinct self-interactions of the gauge fields. The first is the cubic interaction which is due to terms in (1.11) of the form

$$
e(A_\mu)^l_j (A_\nu)^i_l \partial_\mu (A_\nu)^j_i
\tag{1.13}
$$

Following the formalism we described for the propagator term, the diagrammatic representation of this term is given in Figure 1.1(b). Finally, the quartic interaction of gauge fields is due to terms of the form

$$
e^2 (A_\mu)^j_i (A_\nu)^k_j (A_\mu)^l_k (A_\nu)^i_l
\tag{1.14}
$$
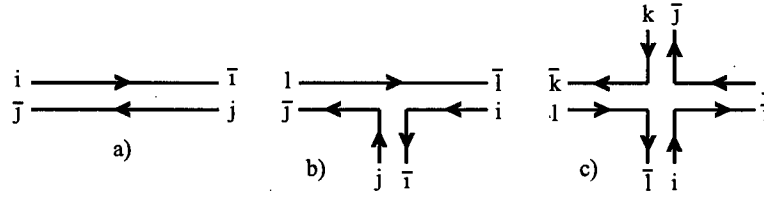
Figure 1.1: The diagrammatic representation of the interactions in Yang-Mills field theory. These pictorially represent (a) the propagator corresponding to Equation 1.12, (b) the interaction of three gauge fields as in Equation 1.13 and (c) the interaction of four gauge fields as given in Equation 1.14.

with the associated diagram shown in Figure 1.1(c). These self-interactions between the components of the Yang-Mills field are the most interesting and difficult aspect of non-Abelian gauge theory. In the case where the gauge group is Abelian, as in quantum electrodynamics where the gauge group is $U(1)$, these terms are absent and the resulting theory is linear and hence much easier to deal with. Physically this simplification occurs because the gauge field quantum (the photon) couples only to electric charge and the photon is electrically neutral. For non-Abelian gauge theories though, the gauge field quantum (the gluon) couples to representations of the gauge group of which the gluon carries a definite (adjoint) representation and so there are self-interactions.

The standard technique of the Feynman diagram expansion for the partition function in a quantum field theory involves combining the three different elements of Figure 1.1 to create composite diagrams. For example, one of the lowest order corrections to the propagator in the theory, the gluon polarization, is given by combining two three point vertices with a propagator as in Figure 1.2. The joining together of elements is carried out by summing over common barred and un-barred indices. For example in Figure 1.2 (a) we sum over l, k, n, and m with their barred counterparts. The result is to connect the two interaction vertices into a correction to the propagator (Figure 1.2 (b)). It is useful to keep track of the powers of N and the coupling constant e in such diagrams. Each three-point interaction vertex carries a power of e and when we sum over the index
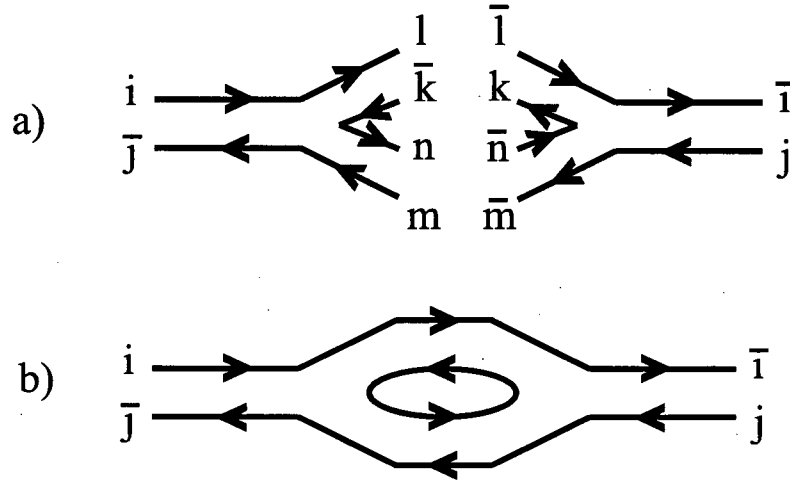
Figure 1.2: One of the simplest corrections to the propagator comes from the joining of two three point interactions. The contribution of this diagram to the full quantum mechanical gluon propagator is of order $e^2 N$.
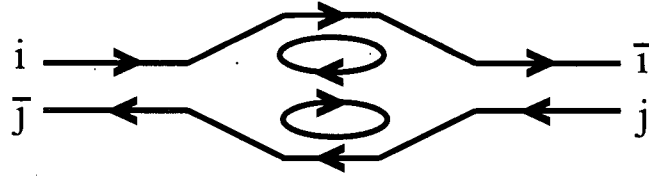
Figure 1.3: A contribution to the gluon polarization from four-point vertices. This quantum correction is of order $e^4 N^2$.

structure the closed loop that occurs provides a combinatoric factor of N. Consequently the whole diagram contributes to the gluon polarization with a factor of $e^2 N$. Similarly there is a contribution from the four-point interaction which is depicted in Figure 1.3. Here there are two closed loops which are summed over and so a factor of N^2 arises. The factors of coupling constant from the four-point vertices give the whole diagram a contribution of $e^4 N^2$ to the full quantum gluon propagator. Finally there is a diagram contributing to the gluon polarization which involves two four-point vertices combined in a different way (Figure 1.4). Here, due to the crossed nature of the graph, there are no closed loops and hence no combinatoric factors of N and the contribution of this graph

Figure 1.4: A contribution to the gluon polarization from four-point vertices which is sub-leading in the large N limit. Here the lack of closed index loops reduces the powers of N associated with the diagram. This diagram is of order e^4.

is of order e^4.

On the surface it appears that this exercise in perturbative expansions in quantum field theory is somewhat empty. However, in the limit of large N, this expansion for Yang-Mills theory takes on a remarkable topological interpretation. As we have seen, the diagrams in Figures 1.2 and 1.3 represent corrections to the gluon propagator of order $e^2 N$ and $e^4 N^2$, respectively. For these contributions to be finite in the limit $N \to \infty$ we must consider a rescaled coupling constant $\tilde{e} = e^2 N$ which is held constant at large N. Consequently the contribution of the diagram in Figure 1.4 in the rescaled coupling is $\tilde{e}^2/N^2$, and so is suppressed in the large N limit. This is a generic feature of the large N expansion. Most importantly it has a topological interpretation. If we consider the diagrams to be drawn on two dimensional surfaces, we see that while the first two can be drawn on spheres without overlapping themselves, the last one cannot as shown in Figure 1.5. In fact, in order for it to be drawn on a two dimensional surface without overlaps the surface must have at least one hole- the simplest example of which is a torus. This generalizes and it can be shown that the factors of N in the perturbative expansion go like N^{-2g} where g is the genus of the lowest genus surface on which the diagram can be drawn.

For pure gauge theory, the result is that we have a theory of two dimensional surfaces which are classified according to their genus - the number of handles on the surface. This kind of expansion has a strong resemblance to another theory of modern theoretical
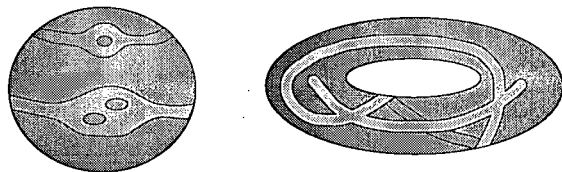
Figure 1.5: The topology of a double lined Feynman graph is defined as the lowest genus surface it can be drawn on without overlapping itself. The diagrams of Figures 1.2 and 1.3 are said to have spherical topology while the sub-leading diagram of Figure 1.4 must wind around a torus to avoid self-intersection.

physics, namely that of string theory [27]. Here we are concerned with the surfaces traced out by the propagation of one dimensional strings in some background. For string theory the expansion parameter that controls the genus expansion is the string coupling constant. The fact that large N Yang-Mills theory and string theory have similar perturbation expansions has lead to the conjecture that Yang-Mills theory is some sort of string theory. Recently this conjecture has received strong backing [64].

String theory naturally has the ability to describe all known forces of nature, including gravity and so is thought by many to be the so-called theory of everything. It is based on the idea that the fundamental objects in nature (at least to lowest, perturbative order) are one-dimensional objects which are unsurprisingly called strings. In the following example of the simplest of such theories, we will define the position of the string $X^\mu(\sigma, \tau)$ as parameterized by the two 'world sheet' variables σ and τ. The action associated with such objects is the two-dimensional area that they sweep out as they move in a background (Minkowski) space-time with metric $\eta_{\mu\nu}$. This is most compactly stated in terms of the Nambu-Goto action for the string of tension T

$$S_{NG} = T \int d\sigma d\tau \sqrt{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2} \tag{1.15}$$

with

$$X \cdot Y \equiv X^\mu Y^\nu \eta_{\mu\nu} \quad , \quad \dot{X}^\mu = \frac{\partial X^\mu(\sigma, \tau)}{\partial \tau} \quad , \quad X'^\mu = \frac{\partial X^\mu(\sigma, \tau)}{\partial \sigma} \tag{1.16}$$

Figure 1.6: The splitting and joining of closed strings leads to a genus expansion of the world sheet. Here each splitting and joining operation creates a hole in the surface.

On minimizing S_{NG}, this elegant formula 1.15 generalizes the idea that point objects follow a path of least action or minimum length. Unfortunately this form is very difficult to deal with when it comes to the quantum mechanical generalizations of the theory because of the inherently non-linear form of 1.15. Consequently one defines an auxiliary metric $g_{\alpha\beta}$ on the two-dimensional world sheet and defines the Polyakov action

$$S_P = -\frac{T}{2} \int d\sigma d\tau \sqrt{\det g_{\alpha\beta}} g^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \qquad (1.17)$$

Here, in addition to the position of the string X^μ, we also consider the world sheet metric $g_{\alpha\beta}$ to be dynamical. Hence the partition function is given by

$$Z = \int \mathcal{D}X \, \mathcal{D}g_{\alpha\beta} \, e^{-S_P} \qquad (1.18)$$

The evaluation of this functional integral for even the simplest string theories occupied many physicist- (and mathematician) years and is fraught with pitfalls related to the technicalities of conformal symmetries and the vagaries of integrating over two dimensional metrics. We shall not dwell on such details but only note that the partition function has a natural expansion in terms of topologically distinct two-dimensional world sheets.

In particular, for closed strings or loops, we would like to consider the possibility of the creation and annihilation of strings in the full quantum mechanical theory. If we consider 'Feynman diagrams' for string interactions with no external strings (the vacuum diagrams of Figure 1.6) we see that the splitting and joining of strings leads to a world sheet with non-trivial topology. As in the case with regular point particle field theories, we can associate a factor, the string coupling constant $e^{-1/G}$ with each splitting or joining

of a string. Consequently we have a perturbative expansion for the string theory over two-dimensional surfaces with different genus

$$Z = \sum_g (\, e^{-1/G})^{2-2g} \int_g \mathcal{D}X \, \mathcal{D}g_{\alpha\beta} \, e^{-S_p} \tag{1.19}$$

This genus expansion suggests that with the identification $e^2 N \sim e^{-1/G}$, large N Yang-Mills theory has a definite connection to closed string theory. Of course this evidence is only circumstantial and much effort has been put into formalizing this identification. This endeavor is more than an academic exercise in relating the two most investigated models of modern theoretical physics. A string theory of Yang-Mills theory suggests certain simplifications that can actually be of calculational value.

The hints as to a connection between Yang-Mills theory and strings is not limited to theoretical constructions like we have seen in the large N limit, but also encompasses phenomenological observations. By the middle of the 1960's experimental physics had identified a zoo of strongly interacting particles - hadrons - and theoreticians spent much of their time trying to give a simple coherent framework into which to fit all these particles. For example, the existence of hadrons of seemingly arbitrary high spin suggested that there must be an underlying theory. This is because the quantum field theory of point-like high spin particles is non-renormalizable and hence cannot be made consistent. Supporting a universality of such an underlying theory was the observation that the mass squared m^2 of the lightest particles with spin J were proportional to J:

$$m^2 = J/\alpha' \tag{1.20}$$

This relationship suggests the existence of a fundamental 'Regge trajectory' in hadron physics. In addition it was observed that, in general, there was a symmetry in the amplitudes for interactions between hadrons. Typically, in a field theory one must consider in the elastic interaction between two particles through some intermediate field. This results in two possible outcomes: one where the interacting particles annihilate each other,

producing a virtual intermediate particle which then 'decays' into two of the original particles and the other where the particles simply exchange an intermediate particle. In field theory amplitudes for these contributions must be added together since they are both possible outcomes. It was noticed though that for high energy strong interactions the contributions from each type of interaction to the total amplitude are equal, and so one need only consider one or the other. This feature of QCD was called duality. In the late 1960's it was realized that a theory of spinning strings with matter at the ends had both the properties of a mass squared spectrum linear with spin (Regge trajectory) and duality. These observations, in addition to the later motivation from large N gauge theory led many to believe that Yang-Mills theory is some type of string theory.

1.2 Why Two-Dimensional Yang-Mills Theory?

Yang-Mills theory definitely appears to be a good description of hadron physics, at least at high energies, and string theory has the potential to be physically relevant so understanding the connections between these two theories will almost certainly be of use in discerning the mysteries of high energy physics. The idea that the two most important theories of modern high energy physics are related at some level is the ultimate conjecture in unifying our understanding of fundamental physics. The interchange between the two theories would almost certainly change the way one, or both, is viewed. Recently there have been connections made between 10-dimensional string theory and supersymetric generalizations of of Yang-Mills theory at large N [64] but here we will take a different route. While the conjecture of an equivalence between large N Yang-Mills and string theory in higher dimensions remains unproven, it is known explicitly how to carry out the identification in two space-time dimensions. Hence, two dimensional Yang-Mills theory marks a cross-roads of modern high energy theoretical physics.

Soon after the discovery of asymptotic freedom, Wilson [102] proposed to consider gauge theories in the context of discretized space-time lattices. These lattice calculations, while breaking Lorentz invariance, make clear the confining nature of the gauge theory and recover the continuum theory in the limit of vanishing lattice spacing. Numerical simulations based on this formalism, though computationally expensive, promise a way to gain non-perturbative information about the system. Apart from numerical studies it was pointed out by Migdal [65] that in the special instance of a gauge theory in two dimensional space-time, the discretized model was equivalent to the continuum theory for all values of the lattice spacing. Moreover, the partition function for the gauge theory could be written completely in terms of a sum over the representations R of the gauge group. For Yang-Mills theory on a closed surface of genus g and area A we have the partition function:

$$Z_g[e, A] = \sum_R d_R^{2-2g} \; e^{-e^2 A C_2(R)/2} \tag{1.21}$$

Here d_R is the dimension and $C_2(R)$ the eigenvalue of the quadratic Casimir operator for the irreducible representation R. We will give more detail on the derivation of (1.21) in the next Chapter, but for now we note important result that the quantum Yang-Mills theory has been reduced to a problem in the representation theory of compact Lie groups, a well understood field of mathematics. This observation lies at the foundation of all results in this Thesis. Moreover, such expansions on the lattice give further evidence to the existence of a connection between string theory and Yang-Mills theory, since the strong coupling perturbation expansion can be put into the form of a genus expansion over two dimension surfaces [52, 57].

The interpretation of the partition function of Yang-Mills theory (1.21) in the large N limit as a string theory is the result of work by Gross and Taylor [36, 37] and Cordes, Moore and Ramgoolam [12]. In particular, it was shown how to interpret (1.21) in terms

of a particular string theory with string coupling constant $1/N$ and some extra factors to take into account the combinatorics of covering the genus g surface on which the Yang-Mills theory is defined with higher genus h string world sheets. This procedure can be sketched [20] by considering the first few terms of (1.21) for $SU(N)$ Yang-Mills theory on a sphere $g = 0$

$$Z_0 = 1 + 2N^2 \, e^{-A/2} + \left(2N^4 + (\frac{A^2}{2} - 2A - 1)N^2 + (1 + \frac{A^2}{2} - \frac{A^3}{3} + \frac{A^4}{24}) \right) \, e^{-A} + \cdots$$

(1.22)

Here we have absorbed the coupling constant into the definition of the area A to simplify the notation. Consequently we see that this is an expansion, in the large N limit, in powers of $N^{2-2g} \, e^{-n\tilde{e}^2 A/2}$. As shown by Gross and Taylor each term of (1.22) can be interpreted in terms of a string theory. In particular, the term $2N^2 \, e^{-A/2}$ is a contribution from the sector of the string theory which contains no splitting of strings (genus zero). The associated factor $e^{-A/2}$ signifies that the mappings from the string world sheet cover the sphere on which the Yang-Mills theory is defined only once. Likewise, mappings which are double covers of the sphere are enumerated by the coefficient of the e^{-A} term in (1.22). Again the genus zero sector of the string theory contributes the coefficient $(\frac{A^2}{2} - 2A - 1)$. Since this term represents double covers of a sphere by a sphere, it is not surprising that the factors of A can be seen to count branch points of the mapping [36]. We will not go into the details here but move on to the contribution from the sector of the string theory where there is a splitting and reconnection of a string (genus one). Again the factors of A in the N^0 term of (1.22) represent branch singularities due to mappings from, in this case, the torus to the sphere. In passing we note that the N^4 term arises from disconnected Feynman diagrams, and can be ignored.

While the details of this construction are not going to be discussed here, it is important to note the string theory as presented by Gross and Taylor is essentially a strong coupling

expansion in terms of small $e^{-\tilde{e}^2 A}$ where $\tilde{e}^2 = e^2 N$ is the effective large N coupling constant. Consequently after the seminal work of Gross and Taylor it became more important to find any phase transitions that may arise in the large N theory. In order for the string picture to be completely equivalent to Yang-Mills theory, we need to test its validity for all values of the coupling e, including small e. If there exists a phase transition for some e_c, then a continuation from strong coupling to weak may not be possible and the string picture will break down. In fact soon after the work in the 1970's of Migdal it was discovered by Gross and Witten [34] that for a lattice gauge theory with $U(N)$ gauge group there is a phase transition as a function of coupling constant in the limit $N \to \infty$. This result did not immediately mean that a global string theoretic interpretation of two-dimensional Yang-Mills theory could not be found. Essentially the question was whether this phase transition was an artifact of non-commutativity in taking the large N limit and the continuum limit $a \to 0$. What was needed was an investigation of large N Yang-Mills theory on compact manifolds.

This investigation was based on the work of Rusakov [84] who set up the framework for the investigation of large N Yang-Mills theory on compact manifolds. It was Douglas and Kazakov [19] who discovered a phase transition on the two-dimensional sphere for a particular value of $e^2 A$. This phase transition removed doubts about the Gross-Witten transition, since it appeared in the continuum formulation and was completely free of the ambiguities of lattice calculations. We will discuss this further in the next Chapter but, in short, the phase transition on the sphere showed that the string picture was only good at large couplings/ large spheres and broke down as the coupling/ area are made small. Similar transitions were noted on the sphere with a single point removed (disk) [38] and the sphere with two points removed (cylinder) [8].

The failure of the Gross-Taylor string theory to describe Yang-Mills theory for all

values of the coupling is a set-back in the construction of a gauge/string theory. The existence of a string interpretation of two-dimensional Yang-Mills theory, even if incomplete, leads one to consider extensions of the formalism to higher space-time dimensions, or the inclusion of physical matter in the string picture of Yang-Mills theory. The physically relevant use for Yang-Mills theory is to describe the interactions of quarks in hadrons such as protons, neutrons and pions in QCD , so that exploring gauge fields coupled to matter in two dimensions is of interest as a toy model of nature. Moreover some of the original evidence (such as the existence of Regge trajectories) for the belief that string theory is related to QCD is still only qualitatively understood. Even the most basic issues in such a picture remain unresolved including fundamental concepts such as the mechanism for including particles with spin on the ends of a string [78].

The most interesting extension of pure Yang-Mills theory involves the addition of fermionic matter which transforms in the adjoint representation of the gauge group. There are a number of reasons for this. First, since two dimensional Yang-Mills theory contains no propagating degrees of freedom, adjoint matter could mimic the effects of adjoint representation gluons which are the dynamical fields in higher dimensional Yang-Mills theory hence this model will give a more realistic approximation to the full 4-dimensional theory. In fact if one considers dimensional reduction of Yang-Mills theory, then there is typically an adjoint representation matter field induced in the lower dimensional theory. The action which we will consider throughout this entire thesis, in different limits, is

$$S = \int d^2x \mathrm{Tr} \left[\Psi^T \gamma^0 (i\gamma^\mu D_\mu - m)\Psi - \frac{1}{2e^2} F^{\mu\nu} F_{\mu\nu} \right] \qquad (1.23)$$

Here Ψ is a single adjoint, Majorana fermion field and the derivative operator is defined

$$D_\mu \Psi = \partial_\mu \Psi + i[A_\mu, \Psi] \qquad (1.24)$$

In the large N limit initial analytic [58, 55] and numerical [13, 2, 18] investigations with adjoint representation matter show that there is reason to think that the string picture can be extended to include matter. In this model the lowest bound states of adjoint particles lie on Regge trajectories as in the four-dimensional theory. In addition the number and density of bound states is consistent with a string theory. A consequence of this behaviour is the existence of a Hagedorn temperature [43, 22, 47, 77, 13, 58] whereby the system makes a first order phase transition to a phase where bound quarks are deconfined in a quark-gluon plasma. In addition to having the same qualitative dynamical behaviour as we expect in full QCD, Yang-Mills theory with (or without) adjoint matter also exhibits the feature of vacuum degeneracy in which the theory admits multiple vacua. The choice of vacua is an external parameter and can have an effect on the physics of the system.

1.3 Contributions From This Thesis

Many of the issues we will consider here emerge from the study of two-dimensional QCD with adjoint representation matter. Instead of embarking on numerical studies of the full theory, we prefer to look at the aspects of the theory in simpler, more controlled settings. The basis of these investigations is the group theoretic definition of two-dimensional Yang-Mills theory, an example of which we have already seen in (1.21). In this setting it is a well-defined and straightforward operation to add heavy, static quarks to the system at finite temperature. We will develop this formalism in the first and second Chapters and provide the tools for providing definite answers about some questions about Yang-Mills theory with matter transforming not only under the adjoint representation but the general case as well.

The first use of this formalism is an investigation in Chapter three of the vacuum

structure of two-dimensional Yang-Mills theory with finite numbers of adjoint represen-
tation particles in the system. Our techniques not only reproduce the somewhat ab-
stract topological enumeration of the acceptable vacua for the theory but also provides a
method for examining the effect of the choice of vacuum on physically relevant quantities
[105, 70, 71]. This formalism is completely general and we use it enumerate the vacua of
Yang-Mills theory for all compact Lie gauge groups. These considerations are extended
in Chapter four to the case of not pure Yang-Mills theory, but one which includes a ther-
modynamic gas of heavy adjoint representation charges [21, 50]. This model is simply
the large mass limit of (1.23) at finite temperature. Here we can directly investigate the
effects of multiple vacua on the bound states of particles interacting via colour-electric
forces. In particular we find that the system exhibits confinement at low densities of
adjoint particles and crosses over to a screening phase at high densities. This behaviour
is a simple model of what is expected in (1.23) [83, 40, 14].

Chapter five marks the beginning of the second part of the thesis where we change
emphasis from Yang-Mills with finite rank gauge group to $SU(N)$ theory in the limit
$N \to \infty$. This change of direction is required since the simplicity of the finite rank
two-dimensional Yang-Mills theories also leaves them thermodynamically sterile. Here
the general formalism of the non-Abelian Coulomb gas is developed by extending group
theoretic constructions of the first two Chapters to unitary matrix models which are
much more convenient in the large N limit. This framework is used in Chapter six
in a detailed investigation of the interactions of a gas of heavy adjoint representation
charges with a gas fundamental representations on the line. Again this particular model
is the large mass limit of (1.23) with the addition of a large number (N) of fundamental
representation quark fields. This system is solved exactly in terms of elliptic functions and
the solution is shown to have a qualitatively similar behaviour to full four-dimensional
QCD. In particular it exhibits a line of phase transitions which generalize the Hagedorn

deconfinement transition in (1.23). Using this exact solution as a test-bed, we introduce a novel order parameter for quantifying the differences between screened/deconfined and confined phases of gauge theories coupled to fundamental representation matter [72].

Finally, using the formalism of the non-Abelian Coulomb gas we consider the case of Yang-Mills theory coupled to fundamental matter and confined to a box in Chapter Six. This system is the finite temperature equivalent of considering a gas of Wilson loops on a space-time sphere and so generalizes the system considered by Douglas and Kazakov [13]. We argue that the phase transition of pure Yang-Mills theory is smoothly connected to the deconfinement phase transition seen in Chapter five, effectively unifying these two apparently different phenomena. In particular this result shows that the breakdown of the string picture of Gross and Taylor [36] for small space-time spheres persists for arbitrarily large spheres with the addition of sufficient amounts of fundamental matter.

As is common in theoretical physics, there are issues and topics which are left partly, or completely unexplored in this thesis. This is certainly the case in the final Chapter of this thesis where the connection of the models studied here to the string theory picture of two-dimensional Yang-Mills theory is not fully elucidated. This is particularly frustrating since the constructions of Chapter six provide a definite way to answer questions about the inclusion of matter into the string theory of Gross and Taylor [36]. The difficulty lies with the fact that the answers are hidden in the (asymptotic) solution of a non-linear integral equation which, up to this point, has evaded quantitative analysis. Consequently, the most substantial of the results to be gained from the work done here are yet to be explored. Hence the results of our labours are not immediately evident and we are left to chop wood with Einstein.

Chapter 2

Pure Yang-Mills Theory in Two Dimensions: Group Theory

We begin by demonstrating explictly the equivalence of two-dimensional Yang-Mills theory defined in the common field theoretic approach to a pure group theoretic interpretation. The latter approach will be of great utility in the next few Chapters when discussing pure Yang-Mills theory on space-times of different topologies and when introducing external colour-electric charges into the theory. A reduction to group theory allows one to calculate physically relevant quantities in terms of invariants of the gauge group: irreducible representations, their dimensions and (quadratic) Casimir operators. The details of the mathematical structures involved are relegated to the Appendix.

2.1   From Lattice Field Theory to Group Theory

The partition function of two dimensional Yang-Mills theory is

$$Z = \int \mathcal{D}A_\mu \exp\left(-\frac{1}{2e^2}\int_M d^2x \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})\right) \tag{2.1}$$

where the Lie algebra valued field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \tag{2.2}$$

This continuum definition can be represented on a two dimensional square lattice of spacing a [102, 48] by the partition function

$$Z_{\mathrm{lat}} = \int \prod_{\mathrm{links}} dU_{ij} \exp\left(\frac{1}{a^2e^2}\sum_P \mathrm{Tr}\left(U_P + U_P^\dagger\right)\right) \tag{2.3}$$

21
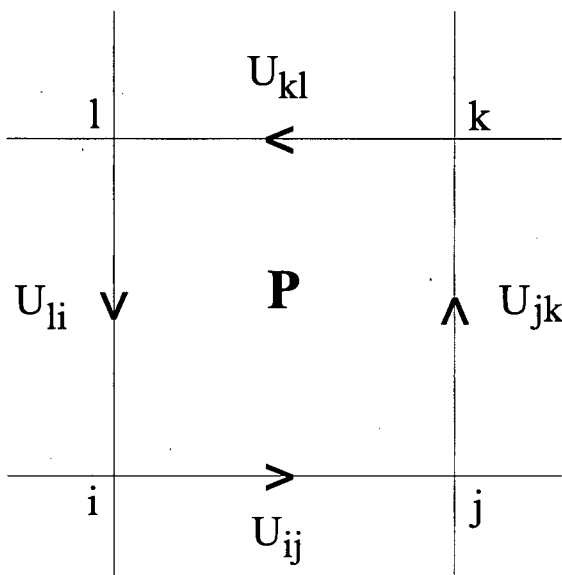
Figure 2.1: Yang-Mills theory can be defined on a discrete lattice of space and time. Each cell of the lattice $(ijkl)$ is referred to as a plaquette (P). The gauge field degrees of freedom are unitary matrices U_{ij} that live on the oriented links of the lattice (ij).

where the $N \times N$ group matrices $\{U_{ij}\}$ live on the directed links of the lattice labeled by the end points i and j. The integration is with respect to the invariant Haar measure on the Lie group, dU_{ij}. The sum in the action is over all (elementary) plaquettes P. Additionally,

$$U_P = U_{ij}U_{jk}U_{kl}U_{li} \tag{2.4}$$

where $\{ijkl\}$ are the corners of the plaquette P.

To show the equivalence of the lattice (2.3) and the continuum (2.1) definition we must first find the stationary point of the lattice action. It is easy to see that $\mathrm{Tr}\,(U_P + U_P^\dagger)$ is maximized for the link variables taking the pure gauge form

$$U_{ij} = g_i g_j^\dagger \tag{2.5}$$

for g some element of the gauge group which implements gauge transformations on the lattice. Now we would like to investigate the effect of fluctuations about this extremum.
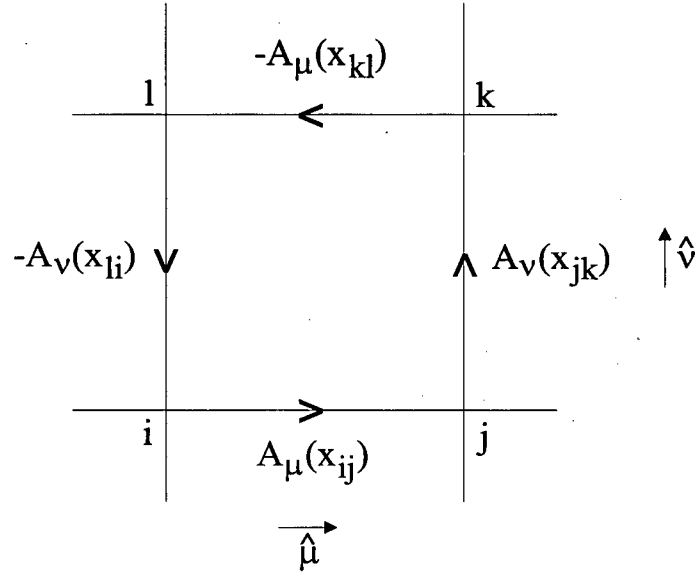
Figure 2.2: The gauge field $A_\mu(x_{ij})$ describes the fluctuations of the unitary field U_{ij} about an extremum of the action and characterizes the system in the limit of vanishing small lattice spacing - the continuum limit. Vector directions on the lattice are defined by $\hat\mu$ and $\hat\nu$.

To calculate these fluctuations we introduce the algebra valued gauge field $A(x_{ij})$. In the continuum, the path ordered exponential of these gauge fields defines the group element that carries out parallel transport between two space-time points

$$U_{ij}^{\text{cont}} = \mathrm{P}\exp\left(i\int_{x_i}^{x^j} A_\mu(x)dx^\mu\right) \tag{2.6}$$

On the lattice, over the infinitesimal lattice spacing a, we can approximate this by

$$U_{ij} = g_i\exp\left(iaA_\mu(x_{ij})\hat\mu\right)g_j^\dagger \tag{2.7}$$

Calculating the product of these link variables around a particular plaquette gives

$$U_P = g_i\exp\left(iaA_\mu(x_{ij})\hat\mu\right)\exp\left(iaA_\nu(x_{jk})\hat\nu\right)\exp\left(-iaA_\mu(x_{kl})\hat\mu\right)\exp\left(-iaA_\mu(x_{li})\hat\nu\right)g_i^\dagger \tag{2.8}$$

This product can be carried out via successive application of the formula of Baker, Campbell and Hausdorff [98]

$$\mathrm{e}^A\,\mathrm{e}^B = \mathrm{e}^{A+B+[A,B]/2+\cdots} \tag{2.9}$$

in which we have neglected contributions from nested commutators of A and B which are of higher order in the infinitesimal lattice spacing. The result is

$$U_P = g_i \exp\left(iaA_\mu(x_{ij})\hat{\mu} + iaA_\nu(x_{jk})\hat{\nu} - iaA_\mu(x_{kl})\hat{\mu} - iaA_\mu(x_{li})\hat{\nu}\right. \tag{2.10}$$

$$\left. -a^2[A_\mu(x_{ij}), A_\nu(x_{ij})] + \cdots\right)g_i^\dagger$$

This can be simplified by noting that $x_{kl} = x_{ij} + a\hat{\nu}$ and $x_{li} = x_{jk} - a\hat{\mu}$. Expanding to leading order in the lattice spacing gives:

$$U_P = g_i \exp\left(ia^2[\partial_\mu A_\nu(x_{ij}) - \partial_\nu A_\mu(x_{jk}) + i[A_\mu(x_{ij}), A_\nu(x_{ij})]]\hat{\mu}\hat{\nu} + \cdots\right)g_i^\dagger \tag{2.11}$$

In the brackets we recognize the field strength tensor (2.2). Expanding the real part of the trace of U_P to leading order we find that a sum over plaquettes weighted by the area of each two-dimensional plaquette gives

$$a^2 \sum_P \mathrm{Tr}\left(U_P + U_P^\dagger\right) = \int d^2x[1 - \frac{a^4}{2}\mathrm{Tr}\, F^2(x) + o(a^4)] \tag{2.12}$$

Hence, up to an irrelevant (infinite) multiplicative factor, (2.3) recovers the continuum result (2.1) as we take the lattice spacing to zero.

The usefulness of the lattice action rests with the fact that it is invariant under gauge transformations, so that:

$$\mathrm{Tr}\, U_P \to \mathrm{Tr}\, g_i U_P g_i^\dagger = \mathrm{Tr}\, U_P \tag{2.13}$$

Because of this invariance, the lattice action only depends on the eigenvalues of the plaquette variables and can be expanded in terms of class functions, or group characters

$$\exp\left(\frac{1}{a^2e^2} \sum_P \mathrm{Tr}\left(U_P + U_P^\dagger\right)\right) = \sum_R d_R \lambda_R(a^2e^2)\chi_R(U_P) \tag{2.14}$$

which is generalization of the Fourier decomposition of a function extended to compact Lie groups. The coefficients λ_R of this expansion can be calculated via heat kernel methods [48] with the result

$$\lambda_R(a^2e^2) = 1 - \frac{a^2e^2}{2}C_2(R) + o(a^4) \tag{2.15}$$
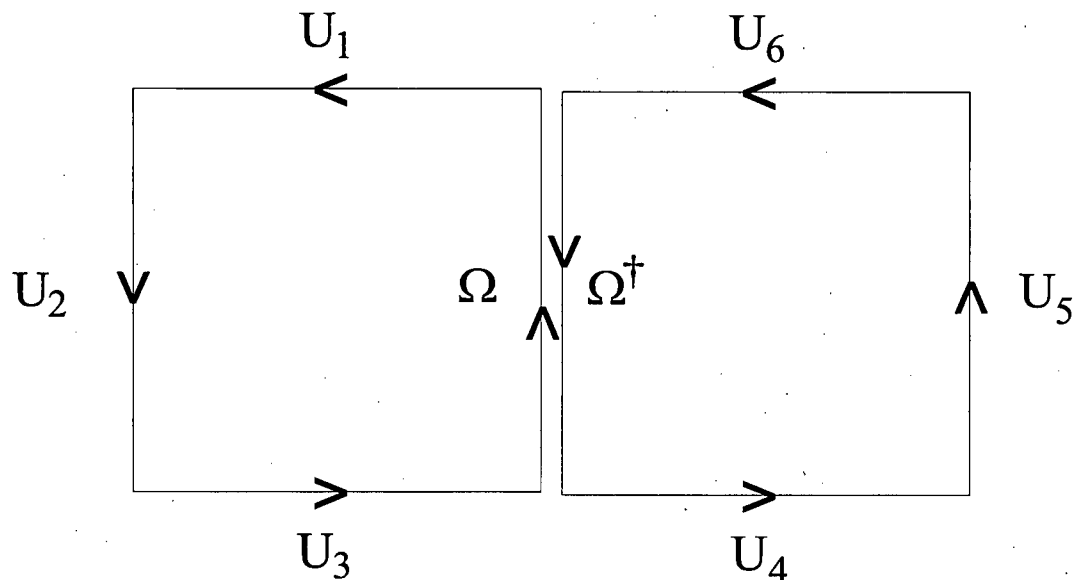
Figure 2.3: Yang-Mills theory on two separate plaquettes is equivalent to Yang-Mills theory on the union after an integration over the unitary degree of freedom Ω which live on the common lattice link. This welding procedure is referred to as the sewing principle.

where $C_2(R)$ is the eigenvalue of the quadratic Casimir operator for the irreducible representation R. The reader is referred to the Appendix, Section A.7 for details on these operators for various gauge groups.

In this group theoretic form it is now simple to complete the program of calculating the partition function of Yang-Mills theory on any two dimensional manifold. We start by considering the integration over common links of two neighbouring plaquettes in the partition function (2.3). The boundary of the pair of plaquettes is defined by the product of group elements $U_1 \ldots U_6$ with the common (oriented) links labeled by $\Omega, \Omega^\dagger$. The integration over these variables effectively 'sews' the plaquettes together.

$$K[U_1 \ldots U_6] = \sum_{SR} d_R d_S \lambda_R \lambda_S \int d\Omega \chi_R(\Omega U_1 U_2 U_3) \chi_S(U_4 U_5 U_6 \Omega^\dagger) \qquad (2.16)$$

By the orthogonality of group characters (A.60), this is simply

$$K[U_1 \ldots U_6] = \sum_R d_R \lambda_R^2 \chi_R(U_1 \ldots U_6) \tag{2.17}$$

We see that the 'sewing together' of plaquettes in the lattice theory leads to an equivalent functional form. Consequently, it is an easy task to iterate this procedure to build up an area A with boundary group element Γ. Using the definition of the Fourier coefficient λ_R in the limit of vanishing lattice spacing (2.15), the continuum partition function for Yang-Mills theory on a disk is

$$\begin{aligned} Z[A, \Gamma] &= \lim_{a \to 0} \sum_R d_R \lambda_R^{A/a^2} \chi_R(\Gamma) \\ &= \sum_R d_R \chi_R(\Gamma) \, e^{-\frac{e^2 A}{2} C_2(R)} \end{aligned} \tag{2.18}$$

With the partition function for the disk it is now a straightforward operation to calculate the partition function for Yang-Mills theory on any oriented compact two dimensional manifold [84]. Classifying these surfaces by genus g we can calculate the partition function by choosing the boundary in the disk partition function to be of the form

$$\Gamma = \Omega_1 \Omega_2 \Omega_1^\dagger \Omega_2^\dagger \ldots \Omega_g \Omega_{g+1} \Omega_g^\dagger \Omega_{g+1}^\dagger \tag{2.19}$$

Integration over the group elements $\Omega_1 \ldots \Omega_{g+1}$ effectively sews the disk into a surface of genus g. These can be carried out by successive application of the key formulas (A.59) and (A.60) for group integration of characters. For example if $g = 0$ we have the partition function of Yang-Mills theory on a sphere of area A

$$\begin{aligned} Z_{S^2}[A] &= \sum_R d_R \, e^{-\frac{e^2 A}{2} C_2(R)} \int d\Omega \chi_R(\Omega \Omega^\dagger) \\ &= \sum_R d_R^2 \, e^{-\frac{e^2 A}{2} C_2(R)} \end{aligned} \tag{2.20}$$

As well for the torus ($g = 1$)

$$Z_{T^2}[A] = \sum_R d_R \, e^{-\frac{e^2 A}{2} C_2(R)} \int d\Omega_1 d\Omega_2 \, \chi_R(\Omega_1 \Omega_2 \Omega_1^\dagger \Omega_2^\dagger) \tag{2.21}$$

$$= \sum_R e^{-\frac{e^2 A}{2} C_2(R)}$$

In general, Yang-Mills theory on a surface of genus g has partition function

$$Z_g[A] = \sum_R d_R^{2-2g} \ e^{-\frac{e^2 A}{2} C_2(R)} \tag{2.22}$$

2.2 Yang-Mills Theory on the Cylinder

For now we will redirect our attention from the definition of the theory on a manifold without boundary to Yang-Mills on a space-time cylinder with two free boundaries. As we shall see, this construction is of great utility in calculations of the interactions of static colour-electric charges. Let us begin with the partition function of the disk with boundary defined by

$$\Gamma = U_1^\dagger \Omega U_2 \Omega^\dagger \tag{2.23}$$

Integrating over Ω leads directly to Yang-Mills theory on a cylinder

$$K[U_1, U_2; A] = \sum_R \chi_R^*(U_1) \ e^{-\frac{e^2 A}{2} \ C_2(R)} \ \chi_R(U_2) \tag{2.24}$$

The simplicity of this formula deserves some comment. The group characters as we have seen are the basis elements of the linear vector space of irreducible representations to the gauge group hence (2.24) is seen to be a diagonal operator on this vector space. In more physical terms, we see that colour-electric fields in irreducible representations do not interact as they move along a space-time cylinder, Fig. (2.4). Since the propagator is diagonal in this basis it is convenient to consider (2.24) as an operator

$$K[A] = \ e^{-\frac{e^2 A}{2} \ C_2} \tag{2.25}$$

The convolution of two propagators to form a single propagator follows from the sewing property [65] (see Figure 2.5),
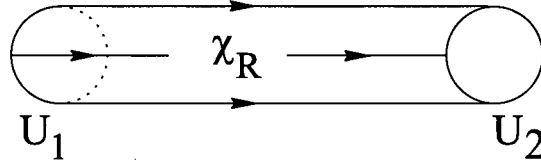
Figure 2.4: The propagator between two open ends on the cylinder transports irreducible representations with quadratic Casimir dependent exponential damping.
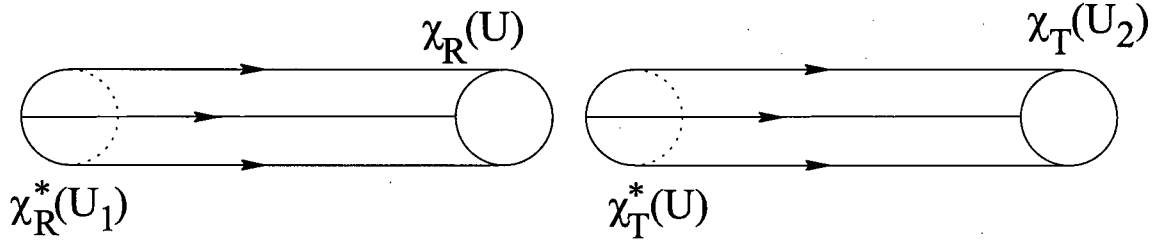
Figure 2.5: The sewing principle allows one to join space-time cylinders by integrating over the unitary element on the common boundary. Integration over U results in a single propagator of the form (2.24) .

$$\int dU \ K[U_1, U; A_1] K[U, U_2; A_2] = K[U_1, U_2; A_1 + A_2] \tag{2.26}$$

This relation follows directly from the orthogonality of group characters (A.51), which has the effect of multiplying the exponents of (2.25) in a naive way.

$$K[A_1]K[A_2] = K[A_1 + A_2] \tag{2.27}$$

It is the continuum equivalent to the sewing of lattice plaquettes which we have seen in the previous Section. Using this property one can identify the group elements at each end of the cylinder and easily recover the partition functions for Yang-Mills theory on the torus and sphere

$$Z_{T_2}[A] = \int d\Omega \ K[A; \Omega, \Omega] \tag{2.28}$$
$$Z_{S^2}[A] = K[A, \mathbb{1}, \mathbb{1}]$$

As the case of Yang-Mills on the sphere will be the topic of interest in the next Section, it deserves further attention. To understand the form of the partition function on S^2 we
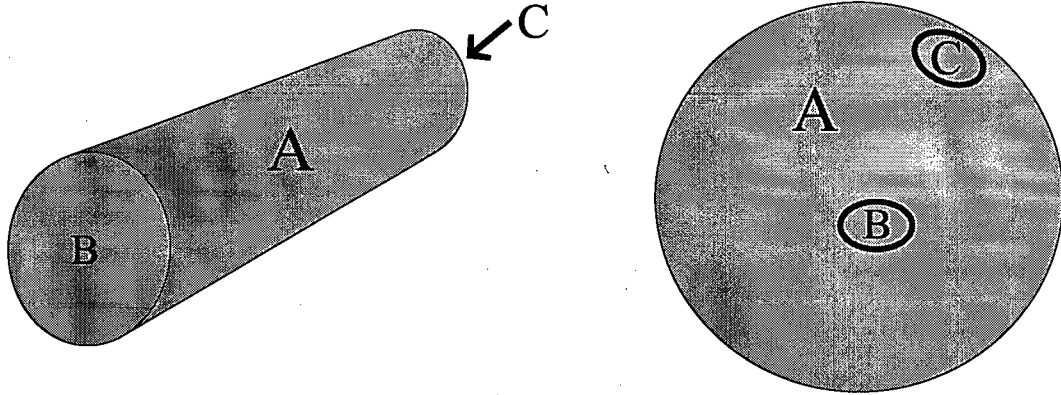
Figure 2.6: The partition function for Yang-Mills on the sphere can be constructed from a cylinder of area A with disks of areas B and C attached at either end.

can think of a sphere as a cylinder with disks attached at either end as in Figure 2.6. From the expression for Yang-Mills theory on a disk we have an alternate expression for the partition function on the sphere

$$Z_{S^2}[A] = \sum_R d_R \, e^{-e^2 B C_2(R)/2} \, e^{-e^2 A C_2(R)/2} \, e^{-e^2 C C_2(R)/2} d_R \qquad (2.29)$$
$$= \sum_R d_R^2 \, e^{-e^2(A+B+C)C_2(R)/2}$$

Consequently the contribution from the disks at the ends is trivial and only serves to supply dimension factors to the calculation. So long as the total area $A + B + C$ remains constant, the result is the same regardless the size of the end disks. Due to the simplicity of two dimensional theory there are a number of interpretations of the physical situation which is described by the propagation of colour-electric flux on the space-time cylinder. In each case we will take the area $A = \tau L$ where τ denotes an interval of time and L is a spatial interval. The two different situations we will consider correspond to taking the temporal or spatial dimension to be compactified. Compactifying the temporal direction is useful when considering the classical thermodynamics of Yang-Mills theory on the spatial line. In this situation the compactified temporal direction will be identified with

the inverse of the temperature $\tau = 1/T$ of the system. If one takes this point of view then Yang-Mills theory on the space-time torus is equivalent to the finite temperature theory on the circle. Moreover, Yang-Mills theory on the sphere is equivalent to the finite temperature theory on a spatial line segment. The other option is to consider the spatial direction to lie in the periodic direction on the cylinder. This corresponds to the quantum mechanical system of Yang-Mills theory on a circle. Formally, calculations are the same regardless of which point of view we assume but the notations are different in the two cases as we will explain shortly.

2.3 Phase Transition in Large N Theory on the Sphere

It is often said that two dimensional Yang-Mills theory is trivial. While this may be true for the theory defined on the open plane, on a compact manifold the situation is quite different. There non-trivial effects can, and do arise due to boundary conditions. To end this Chapter we return to case of Yang-Mills theory on the sphere with $U(N)$ gauge group in the limit of large rank ($N \to \infty$). As demonstrated by Douglas and Kazakov [19] this theory has a phase transition as one varies the area. Since this phase transition is similar to what we will see later with static charges coupled to Yang-Mills theory on the line at finite temperature, in this Section we will outline the calculation of the free energy and critical behaviour of this model.

Recall the partition function of Yang-Mills theory on the sphere (2.20)

$$Z_{S^2}[A] = \sum_R d_R^2 \, \mathrm{e}^{-\frac{e^2 A}{2} C_2(R)} \tag{2.30}$$

For the unitary group we can express the quadratic Casimir eigenvalue in terms of the standard Young table row variables $\{l_i\}$ associated with each irreducible representation,

(A.82):

$$C_2(R) = \sum_{i=1}^{N} l_i(l_i - 2i + N + 1) \tag{2.31}$$

and the dimension

$$d_R = \prod_{i<j} \left(1 - \frac{l_i - l_j}{i - j}\right) \tag{2.32}$$

The sum over all irreducible representations R in (2.30) is now a sum over non-negative integer row variables $\{l_i\}$ satisfying the dominance condition

$$l_1 \geq l_2 \geq \ldots l_N \tag{2.33}$$

In the limit of large N it is convenient to define continuum variables

$$l(x) = \frac{l_i}{N} \quad , \quad x = \frac{i}{N} \tag{2.34}$$

and then the shifted continuum row variable $\lambda(x)$

$$\lambda(x) = -l(x) + x - 1/2 \tag{2.35}$$

where the dominance constraint (2.33) is now stated

$$\frac{\lambda(x) - \lambda(y)}{x - y} \geq 1 \tag{2.36}$$

The quadratic Casimir eigenvalue, $C_2(R)$ and dimension, d_R take on particularly simple form when written in terms of the new variables $h(x)$ in the limit of large N

$$C_2[\lambda(x)] = N^3 \int_0^1 dx \, \lambda^2(x) - \frac{N^3}{12} \tag{2.37}$$

$$\log d[\lambda(x)] = \frac{N^2}{2} \int_0^1 dx \int_0^1 dy \, \log |\lambda(x) - \lambda(y)| + \text{constant} \tag{2.38}$$

Combining the results for the Casimir eigenvalue and the dimension we find an effective action S_{eff} in the variable $\lambda(x)$

$$Z_{S^2}[A] = \int d\lambda(x) \, e^{-N^2 S_{eff}[\lambda(x)]} \tag{2.39}$$

where

$$S_{eff}[\lambda(x)] = -\int_0^1 dx \int_0^1 dy \, \log|\lambda(x) - \lambda(y)| + \frac{\tilde{e}^2 NA}{2} \int_0^1 dx \, \lambda^2(x) - \frac{\tilde{e}^2 NA}{24} \quad (2.40)$$

Note that in the large N limit of a gauge theory the effective coupling is $\tilde{e}^2 = e^2 N$ (see Section 1.1 for an example). The key step in calculating the large N limit of (2.39) is the introduction of the density of Young table variables

$$\rho(\lambda) = \frac{\partial x(\lambda)}{\partial \lambda} \quad (2.41)$$

which is a normalized distribution

$$\int d\lambda \, \rho(\lambda) = 1 \quad (2.42)$$

The dominance condition (2.36) is easily stated in terms of the distribution

$$\rho(\lambda) \leq 1 \quad (2.43)$$

The form of (2.39) is such that in the limit of large N a saddle-point approximation is appropriate. What we will be calculating in this approximation is the particular irreducible representation, the Young tableau variables of which are encoded in the distribution ρ. The stationarity condition for the action

$$\left. \frac{\delta S_{eff}[\lambda(\eta)]}{\delta \lambda(\eta)} \right|_{\eta_0} = 0 \quad (2.44)$$

leads to an integral equation of Riemann-Hilbert type

$$\frac{e^2 NA}{2} \lambda = \int \frac{d\eta \, \rho(\eta)}{\lambda - \eta} \quad (2.45)$$

The solution of this equation is known to be given by the semi-circle distribution

$$\rho(\lambda) = \frac{e^2 NA}{2\pi} \sqrt{\frac{4}{e^2 NA} - \lambda^2} \quad (2.46)$$

The free energy of the system is easily calculated for this saddle-point

$$\frac{dF(A)}{dA} = -\frac{\partial S_{eff}[\lambda_0]}{\partial A} = \frac{\tilde{e}^2}{2A} - \frac{\tilde{e}^2}{24} \tag{2.47}$$

hence

$$F(A) = \frac{\tilde{e}^2}{24}A - \frac{\tilde{e}^2}{2}\log A \tag{2.48}$$

Up to now we have ignored the constraint posed on the distribution ρ by the dominance condition. From (2.43) we require that

$$1 \geq \rho(\lambda) = \frac{\tilde{e}^2 A}{4\pi}\sqrt{\frac{8}{\tilde{e}^2 A} - \lambda^2} \tag{2.49}$$

This inequality is just saturated when the maximum of ρ at $h = 0$ reaches unity. This occurs when the area of the two-dimensional sphere is:

$$A_c = \frac{\pi^2}{\tilde{e}^2} \tag{2.50}$$

For A greater than this critical value, A_c we need to impose the constraint (2.43) by hand when solving the saddle-point condition (2.45). Consequently we look for a distribution of Young table variables

$$\rho(\lambda) = \begin{cases} 1 & \text{for } -a \leq \lambda \leq a \\ \tilde{\rho}(\lambda) & \text{otherwise} \end{cases} \tag{2.51}$$

There is a remarkable interpretation of this ansatz in terms of group representation theory. When the dominance condition is saturated, here for $-a \leq \lambda_c \leq a$, we have

$$\frac{1}{\rho(\lambda_c)} = \frac{\partial \lambda_c}{\partial x} = 1 \tag{2.52}$$

From the definition of the shifted row variable (2.35), (2.52) implies irreducible representation which dominates the large N limit of the theory has a particular form. The

associated Young tableau has vanishing occupation number for those rows which saturate the dominance condition

$$l_k = l_{k+1} = \cdots = l_{N-k} = 0 \qquad (2.53)$$

In other words, the Young tableau which saturates the large N saddle-point evaluation of the partition function, develops a gap precisely at the point of the phase transition where the dominance condition comes into play. For $A < A_c$, this gap disappears and all rows of the dominate Young tableau are occupied. This concept of a gap opening in the group theoretic variables describing a two-dimensional Yang-Mills system will be utilized in a more complex situation in Chapter seven to discern the phases of a gauge system with matter.

With this ansatz (2.51) for the solution of the large N saddle-point equation we have to solve the following singular integral equation for $\tilde\rho$

$$\frac{\tilde{e}^2 A}{2\lambda} - \log \frac{\lambda - a}{\lambda + b} = \fint d\eta \frac{\tilde\rho(\eta)}{\lambda - \eta} \qquad (2.54)$$

The solution of such an equation is known [19] and is expressible in terms of elliptic functions of the third kind, $\Pi[u, v]$

$$\tilde\rho(\lambda) = \frac{1}{\pi} \sqrt{\frac{(a + \lambda)(b + \lambda)}{(a - \lambda)(b - \lambda)}} \Pi \left[\frac{2a}{a + b} \frac{\lambda - b}{\lambda + a}, 2\frac{\sqrt{ab}}{a + b} \right] \qquad (2.55)$$

It is useful to define the constants a, b in the solution in terms of the parameters of the elliptic functions $K(k)$ and $E(k)$ (see [89] for example) and the area A and the effective coupling constant $\tilde{e}^2$

$$\tilde{e}^2 A b = 4K(k) \quad , \quad k = a/b \qquad (2.56)$$

and

$$\tilde{e}^2 A = 4K(k) \left(2E(k) - (1 - k^2)K(k) \right) \qquad (2.57)$$

In terms of these parameters the free energy of the system can be calculated in closed form

$$F'(A) = \frac{1}{6}b^2 - \frac{1}{12}b^2(1 - k^2) - \frac{1}{24} + \frac{1}{96}b^4\tilde{e}^2 A(1 - k^2)^2 \qquad (2.58)$$

The existence of a closed form for the free energy in the large area phase of Yang-Mills theory on the sphere is a remarkable achievement of the group theoretic interpretation of the problem. The strength of this result will become clear by the end of this Thesis when we will return to this system but from the point of view of unitary matrix theory [67, 38]. In that language we will find that the large area phase of the model is determined by a particular non-linear integral equation and is currently beyond explicit solution.

The free energy 2.58 can be compared with the expression for the free energy of the system 2.47 without regard for the constraint on the eigenvalue distribution to determine the order of the phase transition which occurs at A_c. Expanding the elliptic functions we find [19]

$$F'_{\text{strong}} - F'_{\text{weak}} = \frac{\tilde{e}^4}{\pi^6}(A - A_c)^2 + \cdots \qquad (2.59)$$

and conclude that the phase transition is of third order, much like the Gross-Witten transition [34]. In Chapter seven we will come across another third order phase transition, this one when considering the thermodynamics of a gas of fundamental representation 'quarks' interacting via two-dimensional Yang-Mills fields. These two seeming different third order phase transitions will be related to each other in a precise way in Chapter eight.

Finally we note that the large area free energy (2.58) can be expanded in a double series of $e^{\tilde{e}^2 A/2}$ and $A\, e^{\tilde{e}^2 A/2}$ and be shown to agree completely with a term-by-term expansion of the partition function (1.21). As we have seen in the Introduction, this expansion has a definite string-theoretic interpretation due to Gross and Taylor [36, 37, 12]. The extension of the model of Yang-Mills theory on the space-time sphere to include

matter and its subsequent string theory interpretation will be the main motivation of the final Chapter of this Thesis.

Chapter 3

Adding Sources to 2D Yang-Mills Theory

3.1 Correlators of Group Characters: Static Charges

With the partition function for the cylinder, it is an easy task in principle to calculate the correlation function of any number of group characters in arbitrary representations. Such correlators are of great interest in gauge theories since they give gauge invariant -and hence physical- information about the system.

The first correlator we will consider is that of a pair of group characters in conjugate representations $S, \bar{S}$ separated by a space-time area of A (Figure 3.1). The calculation of the correlator uses the sewing property where one integrates over the unitary degrees of freedom at each end of the space-time cylinders with a character in a particular representation included

$$Z[S, \bar{S}; A] = \sum_T \left[\int dU_1 \; \chi_S(U_1)\chi_T^*(U_1) \right] \; e^{-e^2 A C_2(T)/2} \left[\int dU_2 \; \chi_{\bar{S}}^*(U_2)\chi_T(U_2) \right] \qquad (3.1)$$

The quantities in the square brackets integrate to delta-functions in the representation index by the orthogonality of group characters (See Section A.5). Hence the pair correlator simplifies to the very simple expression

$$Z[S, \bar{S}; A] = \; e^{-e^2 A C_2(S)/2} \qquad (3.2)$$

The physical interpretation of this calculation depends on the choice of direction we have used for compactification. If the compact direction is the spatial one then Equation
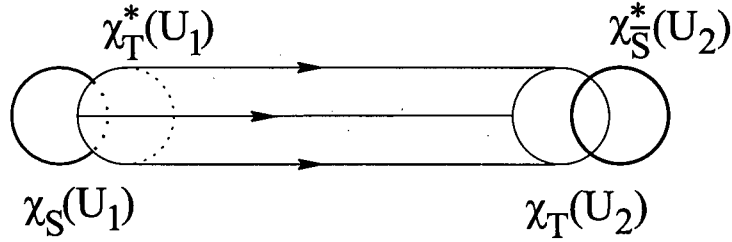
Figure 3.1: Sewing characters to the ends of a cylinder gives the pair correlator.

(3.2) gives the correlation of a Wilson loop in representation S on a circle of circumference L with a Wilson loop in the conjugate representation $\bar{S}$ at time A/L. This point of view was adopted in [32], on which our presentation here is based.

Much more interesting is the case in which the temporal direction is taken to be compactified with the circumference equal to the inverse temperature of the system $\beta = 1/T$ and the space-time area $A = \beta L$ where L is the spatial separation of the Polyakov loops. Calculating the energy F associated with such a such a configuration is an easy exercise in basic statistical mechanics

$$F = -\frac{\partial}{\partial \beta} \log Z = e^2 L C_2(S)/2 \qquad (3.3)$$

Again this is another sign of the simplicity of Yang-Mills theory in two dimensions. Here the temperature serves only to rescale the measure of distance L or the gauge coupling constant e^2. In fact one can absorb the space-time dependence of the entire theory by rescaling the gauge coupling $e^2 A \to e^2$, or vice-versa.

The form of the energy F and its dependence on the separation distance between the two Polyakov loops immediately suggests that the energy per unit length of the colour-electric field between the two loops is a constant. In fact this energy can also be interpreted as the binding energy of the two loops and it is useful to define the 'string' tension σ

$$\sigma = F/L = e^2 C_2(S)/2 \qquad (3.4)$$

which has the interpretation of binding energy per unit length. The fact that the binding energy between a pair of Polyakov loops is a linear function of separation is characteristic of interacting colour-electric charges in one space dimension. We will prove this statement in a precise way from the field theoretic development of Yang-Mills theory at the end of this Chapter. In anticipation we will naturally associate a Polyakov loop in representation S with a static, infinitely massive source of colour-electric charge in representation S at the same point in space. Consequently, the correlation function for a configuration of group characters on the cylinder is the partition function for a system of stationary, static sources in the same configuration. This equivalence will be of great utility for investigating Yang-Mills theory with matter content in subsequent Chapters, but for now we concentrate on the calculation of character correlation functions from a purely abstract point of view.

The calculation of the correlator of a pair of characters is easily generalized. It will be useful for our purposes here, and in the Chapters to follow to consider the correlators of arbitrarily many group characters on a toroidal surface. In light of the previous discussion of Polyakov loops, we see that such character correlators correspond to the partition function of static charges on a spatial circle (Figure 3.2). Using the operator notation introduced in the last Section, the correlator of n characters χ_{R_i} each in irreducible representations R_i and separated by a distance $L_1 \cdots L_n$, is given by

$$P[R_1, \ldots, R_n; L_1, \ldots, L_n] = \tag{3.5}$$

$$\mathrm{Tr}\, \chi_{R_1}\, \mathrm{e}^{-e^2\beta L_1 C_2/2} \cdots \chi_{R_n}\, \mathrm{e}^{-e^2\beta L_n C_2/2} = \mathrm{Tr} \prod_{k=1}^{n} \left(\chi_{R_k}\, \mathrm{e}^{-e^2\beta L_k C_2/2} \right)$$

The trace here closes the space-time cylinder into a torus with length $L = \sum L_i$. We can at any time recover the correlation function for a number of characters on a space-time cylinder with boundaries say χ_{R_1} and χ_{R_n} by taking the length of the intermediate section of propagator to infinity. This suppresses the transfer of colour-electric flux on
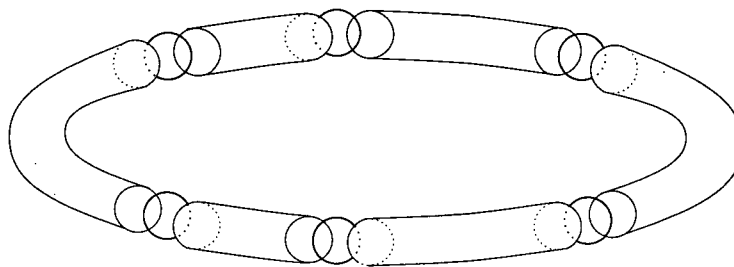
Figure 3.2: Sewing n cylinders together with frustration and identifying the ends, leaves one with a torus corresponding to n charges on a circle.

this section and effectively breaks the toroidal configuration.

A special case of (3.5) is the correlator of a pair of characters on the space-time torus. We will consider the most general case with the representations of the characters R and R'.

$$
\begin{aligned}
P[R, R'; L_1, L_2] &= \mathrm{Tr}\, \chi_R\, e^{-e^2\beta L_1 C_2/2} \chi_{R'}\, e^{-e^2\beta L_2 C_2/2} \\
&= \sum_{S,T} N_{SR}^T N_{TR'}^S\, e^{-e^2\beta L_1 C_2(S)/2}\, e^{-e^2\beta L_2 C_2(T)/2}
\end{aligned}
\tag{3.6}
$$

where the fusion numbers N_{SR}^T enumerate the occurences of the representation T in the tensor product of S and R (see Section A.6).

From this general formula one can immediately make quantitative statements about the binding between pairs of characters in the case where one side of the torus becomes large ($L_2 \to \infty$). In this limit the sum over representations T in (3.6) is exponentially damped and the only surviving term in the correlator is

$$
P[R, R'; L_1, L_2 \to \infty] = \delta_{RR'}\, e^{-e^2\beta L_1 C_2(R)/2}
\tag{3.7}
$$

Let us end this Section with the particular case of $SU(2)$ where the fusion numbers are known explicitly. We will label representations by the number of boxes, l in the single allowed row of Young tableaux for $SU(2)$. From calculations for the quadratic Casimir operator (A.81), $C_2(n) = ((n+1)^2 - 1)/2$. The fusion numbers in this case can

be deduced from the Littlewood-Richardson procedure [90, 107] for calculating tensor products of representations described in Section A.6 and are given by

$$N_{jl}^i = \begin{cases} 1 & \text{when } i = j+l, j+l-2, \ldots, |j-l| \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

Substituting this information into the general pair correlator (3.6) we have the pair correlator of characters with $SU(2)$ Young labels m and n

$$P[m,n; L_1, L_2] = \tag{3.9}$$

$$\sum_{j,l=0}^{\infty} e^{-e^2\beta L_1[(l+1)^2-1]/2} e^{-e^2\beta L_2[(j+1)^2-1]/2} \sum_{r=0}^{\min l,m} \sum_{s=0}^{\min j,n} \delta_{m+n,2(s+r)} \delta_{2l+2s+m,2j+2r+n}$$

The first delta function serves to enforce the condition that $m + n$ must be even. This is a particular example of the general fact in $SU(N)$ that in order for the correlator of any system of characters to be non-vanishing the total charge of the characters must be vanishing mod N. This restriction ensures that the system contains a charge singlet.

Two special cases that are of interest are the pair correlators of fundamental and adjoint Polyakov loops. For the fundamental case $n = m = 1$ and

$$P[1,1; L_1, L_2] = \sum_{l=0}^{\infty} e^{-e^2\beta L_2[(l+1)^2-1]/2}\left[e^{-e^2\beta L_1(2l+3)/4} + e^{-e^2\beta(L_2-L_1)(2l+3)/2}\right] \tag{3.10}$$

Likewise for the correlator of a pair of adjoint loops $m = n = 2$

$$P[2,2; L_1, L_2] = \left(e^{-4e^2L_1} + e^{-4e^2(L_2-L-1)} \right. \tag{3.11}$$
$$\left. + \sum_{l=1}^{\infty} e^{-e^2 L_2[(l+1)^2-1]/2}\left[e^{-2e^2 L_1(l+2)} + 1 + e^{-2e^2(L_2-L-1)(l+2)}\right] \right)$$

This ends our discussion on correlation functions of group characters. We will make use of these results in our analysis of the vacuum structure and thermodynamics of Yang-Mills theory with adjoint matter in Chapters four and five.

3.2 Hamiltonian Formulation of Yang-Mills Theory at Finite Temperature

In this Section we shall review the Hamiltonian formulation of two-dimensional Yang-Mills theory and show that the calculation of correlators of group characters is equivalent to the standard methods for including external charges in a gauge theory. We begin with the partition function as a function of temperature [1]

$$Z[T] = \int \mathcal{D}A_\mu \; e^{-S[A]} \qquad . \qquad (3.12)$$

with action

$$S[A] \;=\; \int_0^{1/T} d\tau \int dx \; \mathcal{L} \;=\; -\int_0^{1/T} d\tau \int dx \; \frac{1}{2e^2} \mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) \qquad (3.13)$$

where x is a spatial coordinate, τ is a complex time coordinate and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \qquad (3.14)$$

Here the only non-vanishing component is $F_{01} = -F_{10}$. The gauge fields A_μ are taken in the fundamental representation of the Lie algebra of the gauge group, and can be expanded in a canonical set of generators as

$$A_\mu(x,\tau) = \sum_a T^a A_\mu^a(x,\tau) \qquad (3.15)$$

For the compact semi-simple Lie algebras we will be considering

$$\mathrm{Tr}\,T^a T^b = \frac{1}{2}\delta^{ab} \quad , \quad \left[T^a, T^b\right] = if^{abc}T^c \qquad (3.16)$$

Consequently the field strength tensor in component form reads

$$F_{01}^a = \partial_0 A_1^a - \partial_1 A_0^a - f^{abc} A_0^b A_1^c \qquad (3.17)$$

---

[1]Please note that Planck's constant, $\hbar$ is set equal to unity here and throughout this Thesis

Canonical momenta conjugate to the gauge field components are

$$(\pi^0)^a = \frac{\partial \mathcal{L}}{\partial \dot{A}_0^a} = 0 \tag{3.18}$$

$$(\pi^1)^a = \frac{\partial \mathcal{L}}{\partial \dot{A}_1^a} = \frac{1}{e^2} F_{01}^a \equiv E^a \tag{3.19}$$

Hence the electric field, E^a is the canonical variable conjugate to the spatial component of the gauge field $A_1^a \equiv A^a$, and consequently we have the non-vanishing equal time commutation relation

$$\left[A^a(x, \tau), E^b(y, \tau) \right] = i\delta^{ab}\delta(x - y) \tag{3.20}$$

The phase space variables E^a and A^a have periodic boundary conditions in the complex time direction with period equal to the inverse temperature of the system

$$E(x, \tau) = E(x, \tau + 1/T) \ , \ A(x, \tau) = A(x, \tau + 1/T) \tag{3.21}$$

These boundary conditions and the action (3.13) are preserved by local gauge transformations under which the phase space variables transform as

$$E(x, \tau) \quad \to \quad E^g(x, \tau) = g(x, \tau)E(x, \tau)g^\dagger(x, \tau) \tag{3.22}$$

$$A(x, \tau) \quad \to \quad A^g(x, \tau) = g(x, \tau)\left(A(x, \tau) - i\nabla\right)g^\dagger(x, \tau)$$

The Hamiltonian of the dynamical system can be formed by the Legendre transformation

$$
\begin{aligned}
H &= \int dx \left[(\pi^0)^a \dot{A}_0{}^a + (\pi^1)^a \dot{A}_1{}^a - \mathcal{L} \right] \\
&= \int dx \left[E^a(e^2 E^a + \nabla A_0^a + f^{abc} A_0^b A^c) - \frac{e^2}{2} E^a E^a \right] \\
&= \int dx \left[\frac{e^2}{2} E^a E^a - A_0^a(\nabla E^a - f^{abc} A^b E^c) \right]
\end{aligned}
\tag{3.23}
$$

where summation over repeated indices is assumed. The equation of motion for A_0^a

$$\frac{\delta H}{\delta A_0^a} \sim \frac{\partial \mathcal{L}}{\partial \dot{A}_0^a} = (\pi^0)^a = 0 \tag{3.24}$$

generates a constraint which generalizes Gauss' law for the non-Abelian gauge theory

$$\mathcal{G}^a(x) \equiv \left(\nabla E^a(x) - f^{abc} A^b(x) E^c(x)\right) \sim 0 \qquad (3.25)$$

The standard way to introduce static external charges into a gauge theory is via a coupling of the charge density $\rho^a(x)$ to the temporal component of the gauge potential

$$H_{int} = \int dx \, \rho^a(x) A_0^a(x) \qquad (3.26)$$

This modifies Gauss' law (3.25) for Yang-Mills theories interacting with external charges to

$$\mathcal{G}_{int}^a(x) \equiv \left(\nabla E^a(x) - f^{abc} A^b(x) E^c(x) - \rho^a(x)\right) \sim 0 \qquad (3.27)$$

For concreteness, we will consider ρ^a to be the density corresponding to discrete particles with color charges in representations $R_1, \ldots, R_K$ located at positions $x_1, \ldots, x_K$. If $T_{R_i}^a$ are generators in the representation R_i operating on the color degrees of freedom of the i'th particle, the charge density can be written as

$$\rho^a(z) = \sum_{i=1}^{K} T_{R_i}^a \delta(z - x_i) \qquad (3.28)$$

There are two options for imposing the Gauss' law constraint (3.27). The first is to impose a gauge fixing condition such as $A \sim 0$, and use the constraint to solve for the colour-electric field E

$$E^a(x) = \frac{1}{2} \int dy \, \epsilon(x - y) \rho^a(y) \qquad (3.29)$$

where $\epsilon(x)$ is the step function

$$\epsilon(x) = \begin{cases} -1 \text{ if } x < 0 \\ \ 1 \text{ if } x > 0 \end{cases} \qquad (3.30)$$

If we insist that physical configurations have finite energy then we need to impose one further condition which restricts one to colour-electric singlet configurations. This is

most easily imposed by requiring the total charge Q of the system vanish

$$Q^a \equiv \int dx \; \rho^a(x) \sim 0 \tag{3.31}$$

With this restriction and the solution for the electric field (3.29) the resulting Hamiltonian for a system of colour-electric charges is

$$H = \sum_{i<j,a} \frac{e^2}{2} T_{R_i}^a \otimes T_{R_j}^a \, |x_i - x_j| \; , \tag{3.32}$$

which was considered in [69]. It is the energy of an infinite range spin model where the spins take values in the Lie algebra of $U(N)$.

As a particular example we can consider the system with only two, conjugate representations: R at x_1 and $\bar{R}$ at x_2. The energy in this instance is

$$E = \frac{e^2}{2} T_R^a \otimes T_{\bar{R}}^a \, |x_1 - x_2| = \frac{e^2}{2} C_2(R) \, |x_1 - x_2| \tag{3.33}$$

where we have used the definition of the quadratic Casimir operator (A.74). This result is precisely what we had found previously for the energy associated with the interaction of a pair of group characters in conjugate representations (3.3) where $L = |x_1 - x_2|$.

A second option for imposing (3.27) is to impose the constraint as a physical state condition where $\mathcal{G}$ annihilates physical wavefunctions

$$\mathcal{G}^a(x) \, \Psi_{\text{phys}} = 0 \; . \tag{3.34}$$

To do this, it is most convenient to work in the functional Schrödinger picture, where the states are functionals of the gauge field, $\psi[A]$ and the electric field is the functional derivative operator

$$E^a(x)\Psi[A] = \frac{1}{i} \frac{\delta}{\delta A^a(x)} \; \Psi[A] \tag{3.35}$$

The Hamiltonian is

$$H = -\frac{e^2}{2} \sum_{a=1}^{N^2} \int dx \frac{\delta^2}{(\delta A^a(x))^2} \tag{3.36}$$

and hence the time-independent functional Schrödinger equation is

$$H \ \Psi^{a_1 \cdots a_K} [A; x_1, \ldots, x_K] \ = \ \mathcal{E} \ \Psi^{a_1 \cdots a_K} [A; x_1, \ldots, x_K] \ . \tag{3.37}$$

Gauss' law implies that the physical states, i.e. those which obey the gauge constraint (3.25), transform as

$$\Psi^{a_1 \cdots a_K} [A^g; x_1, \ldots, x_K] \ = \ g^{R_1}_{a_1 b_1}(x_1) \ldots g^{R_K}_{a_K b_K}(x_K) \ \Psi^{b_1 \cdots b_K} [A; x_1, \ldots, x_K] \ , \tag{3.38}$$

where

$$A^g \ \equiv \ g A g^\dagger - i g \nabla g^\dagger \tag{3.39}$$

is the gauge transform of A.

For a fixed number of particles, the quantum mechanical problem is exactly solvable. The path ordered exponential of the gauge field in the appropriate representation generates Wilson lines (strings) of colour-electric flux between the particles in the system. These lines of flux connect group theoretic indices to form states which satisfy the physical state condition. For example, the wavefunction of a colour-electric charge in representation R at point x_1 interacting with a conjugate charge at point x_2 is

$$\Psi^{ij}_R[A; x_1, x_2] \ = \ \left(\mathcal{P} e^{i \int_{x_1}^{x_2} dy A_R(y)} \right)^{ij} \tag{3.40}$$

where A_R is the gauge potential taken in the representation R. Operating on this state with the Hamiltonian (3.36), the energy is found to be

$$E = \frac{e^2}{2} C_2(R) |x_1 - x_2| \tag{3.41}$$

Again, comparing to the gauge fixed result (3.33) and the energy associated with the interaction of group characters (3.3) we find complete agreement.

In general, for a fixed distribution of quarks, a state-vector is constructed by connecting them with appropriate strings of electric flux so that the state is gauge invariant.

The number of ways of doing this fixes the dimension of the quantum Hilbert space. If the flux strings overlap, the Hamiltonian can mix different configurations, so the energy eigenstates are superpositions of string configurations. Calculating these eigenstates is a purely group theoretic problem which involves finding tensor products of irreducible representations. This task is quite difficult in general and in the remainder of this Thesis we will be interested in useful approximations to the full computations.

Chapter 4

Vacuum Structure of 2D Yang-Mills Theories

We are now in a position to begin to investigate some of the properties of Yang-Mills theories in two dimensions. We will begin by examining the existence of so-called topological parameters in the theory which were only discovered in the seventies while considering the non-perturbative nature of quantum field theories. The existence of topological parameters in certain quantum field theories, such as the θ - angle QCD [28, 49, 7, 11] or the Schwinger model of two-dimensional QED [9, 10] is an interesting aspect which can have a profound effect on the physical properties of the models. In these cases the existence of an extra, previously unrecognized parameter in the theory stems from the contribution from boundary terms in the action. For Yang-Mills theory defined on a four-dimensional space-time, finite action configurations of the system require that the gauge field tend to pure gauge on the boundary of space-time, which can be taken to be a three-sphere. The pure gauge configurations that we are interested in are of the form

$$A_\mu(x) = ig^\dagger(x)\partial_\mu g(x) \tag{4.1}$$

where $g(x)$ is an element of the gauge group and x is some point on the boundary S^3. At this point on may ask whether all such pure gauge configurations are related to one another through continuous gauge transformation. Mathematically, the equivalent question is how many topologically inequivalent mappings are there from S^3 into the gauge group G? If there are inequivalent mappings, then we can expect there to be a number of admissible gauge-inequivalent ways to satisfy the finite action condition. In this way the Yang-Mills theory can have multiple vacua, and in general it does.

The classification of mappings from S^3 to the gauge group is given by the third homotopy group, $\pi_3(G)$ of the manifold defined by the Lie group G. For any compact Lie group we have $\pi_3(G) = \mathbb{Z}$ so that there are a countable infinity of topologically distinct boundary conditions for Yang-Mills theory in four dimensions. Conveniently, the equivalence class n to which a given gauge field A_μ belongs can be calculated in terms of an integral over the space-time manifold of the Chern-Pontryagin density

$$n = \frac{1}{32\pi^2} \int d^4x \; \epsilon^{\mu\nu\rho\eta} \mathrm{Tr}\, F_{\mu\nu} F_{\rho\eta} \tag{4.2}$$

where $n \in \mathbb{Z}$.

Hence we have shown that there are a number of equivalent boundary conditions or topological sectors in four-dimensional Yang-Mills theory. It is interesting to examine how these different sectors make their presence known and effects felt in the physics of the theory. A priori, we will assume that all sectors contribute to the calculation of a particular expectation value with different weights $\chi(n)$[97]

$$\langle X \rangle = \frac{\sum_n \chi(n) \int_n \mathcal{D}A \; X[A] \; e^{-S_V}}{\sum_n \chi(n) \int_n \mathcal{D}A \; e^{-S_V}} \tag{4.3}$$

The subscripts on the integrals denote that only contributions from the n topological sector are to be included and the subscript V denotes that the action S is calculated over the large space-time volume V. Now divide this volume into two, $V = V_1 + V_2$ so that the contribution to the expectation value $\langle X \rangle$ from V_2 is negligible. This can always be accomplished for sufficiently 'large' V_1 by the principle of locality. Now the topological number n for the whole of V is split into contributions n_1 and n_2 from each volume so that (4.3) can be written

$$\langle X \rangle = \frac{\sum_{n=n_1+n_2} \chi(n_1 + n_2) \int_{n_1} \mathcal{D}A \; X[A] \; e^{-S_{V_1}} \int_{n_2} \mathcal{D}A \; e^{-S_{V_2}}}{\sum_{n=n_1+n_2} \chi(n_1 + n_2) \int_{n_1} \mathcal{D}A \; e^{-S_{V_1}} \int_{n_2} \mathcal{D}A \; e^{-S_{V_2}}} \tag{4.4}$$

But since the contribution from the volume V_2 is, by construction, small it should factorize

completely out of the calculation. This can only occur if

$$\chi(n_1 + n_2) = \chi(n_1)\chi(n_2) \tag{4.5}$$

and so we see that locality forces us to choose the form of the weighting for each topological sector

$$\chi(n) = e^{i\theta n} \tag{4.6}$$

where θ is a periodic but otherwise unspecified parameter. Using this result with the definition of the topological number from (4.2) we see that the partition function for a four-dimensional gauge theory actually contains an extra θ term from the contributions of topologically non-trivial configurations

$$Z = \int \mathcal{D}A \, e^{-S_\theta} \tag{4.7}$$

where the path integral is over all gauge field configurations and the modified action S_θ is given

$$S_\theta = -\frac{1}{2e^2} \int d^4x \mathrm{Tr} \, F^{\mu\nu} F_{\mu\nu} + \frac{i\theta}{32\pi^2} \int d^4x \, \epsilon^{\mu\nu\rho\eta} \mathrm{Tr} \, F_{\mu\nu} F_{\rho\eta} \tag{4.8}$$

The existence of an arbitrary parameter θ in four-dimensional Yang-Mills theory may well be the answer to some outstanding questions. In part these include the so-called $U(1)$ problem of QCD where the expected light Goldstone-like boson associated with axial symmetry breaking is missing in nature. The existence of a θ term in the theory explicitly breaks the axial symmetry and hence there really is no spontaneous symmetry breakdown. Moreover, the θ angle gives a parameterization of the vacua of Yang-Mills theory in four dimensions and in this Chapter we will give a detailed account of the admissible vacua in the two-dimensional case.

In the two-dimensional, theory we will be directly interested in the counting [105] and physical properties of the allowable vacua [70, 71] for different gauge groups (see [62] for a

different approach). We will begin with a topological classification of the vacua based on the symmetries of the theory at hand. We will restrict ourselves to the case of Yang-Mills theory with heavy external adjoint representation charges. These give maximal symmetry under gauge transformations since the gauge fields themselves transform in the adjoint representation. After the topological classification, we will proceed with a classification which follows from the group theoretic structure of the gauge group. Starting with the case of $SU(N)$ Yang-Mills theory we show that the counting of vacua via this method gives the same results as the topological classification but also physical information about the interactions of external charges in the different backgrounds.

4.1 θ-vacua in 2D: Topological Calculation

The standard method of classifying the multiplicity of vacua in a particular gauge theory hinges on identifying the effective gauge group. This process considers the transformation properties of all fields in the theory. For example a field ϕ in the fundamental representation of the gauge group, under a gauge transformation by the group element g, would transform as

$$\phi \rightarrow g\phi \qquad (4.9)$$

Likewise a field ψ in the adjoint representation transforms by the adjoint action of the gauge group

$$\psi \rightarrow g\psi g^{\dagger} \qquad (4.10)$$

The identification of the effective gauge group is now reduced to a question about the triviality of a gauge transformation on the fields of the theory. In particular we will be interested in the transformation properties of fields under elements in the center of the gauge group, Z. Recall the center is the subgroup of a Lie group with the property that its elements commute with all other elements of the group. Consequently, the elements

of the center are proportional to the identity. Hence if $h \in Z$ then the transformation of fundamental and adjoint representation fields are, respectively

$$\phi \to h\phi \quad , \quad \psi \to h\psi h^\dagger = hh^\dagger \psi = \psi \tag{4.11}$$

Here we see that while the fundamental field transforms non-trivially, a center transformation on the adjoint field is just an identity operation. Consequently, if we were dealing with a gauge theory coupled to adjoint matter, since all gauge fields transform in the adjoint representation, a gauge transformation in the center of the gauge group is an identity operation and we say it is not an effective gauge transformation. In order to obtain the effective gauge group for a theory with adjoint representation, one should identify gauge transformations which only differ by elements of the center of gauge group. Here, since gauge transformations operate by adjoint action on all fields, the true gauge group is the quotient of the gauge group and its center. This quotient is multiply connected. For simply connected semi-simple gauge group G with discrete center Z we have the exact sequence

$$\cdots \to \pi_1(G) \to \pi_1(G/Z) \to \pi_0(Z) \to \pi_0(G) \to \cdots \tag{4.12}$$

For $\pi_1(G) = \pi_0(G) = 0$ the exact sequence to the right gives the isomorphism

$$\pi_1(G/Z) \sim \pi_0(Z) = Z \tag{4.13}$$

This suffices for the simply connected Lie groups and for $SO(2N+1)$ we have $\pi_1(SO(2N+1)) = \mathbb{Z}_2$ and a trivial center so we immediately have the result

$$\pi_1(SO(2N+1)/Z) = \mathbb{Z}_2 \tag{4.14}$$

For the even rank special orthogonal groups $SO(2N)$ we need a more sophisticated approach. We record the results in Table 4.1.

G	Z	$\pi_1(G)$	$\pi_1(G/Z)$	Number of vacua
SU(N)	$\sim \mathbb{Z}_N$	0	$\mathbb{Z}_N$	N
Sp(2N)	$\sim \mathbb{Z}_2$	0	$\mathbb{Z}_2$	2
SO(2N+1)	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	2
SO(2N)	$\sim \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (N even) $\mathbb{Z}_4$ (N odd)	4
E_6	$\sim \mathbb{Z}_3$	0	$\mathbb{Z}_3$	3
E_7	$\sim \mathbb{Z}_2$	0	$\mathbb{Z}_2$	2
E_8	1	0	0	1
F_4	1	0	0	1
G_2	1	0	0	1

Table 4.1: Relevant topological information and counting of multiple vacua in two dimensional Yang-Mills theories.

Since G/Z is a symmetry of the Hamiltonian, we expect that all physical states carry a representation of $\pi_1(G/Z)$. In the case where the center of the group is Abelian all of its irreducible representations are one dimensional and further, when $Z \sim \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_j}$, we are lead to a classification of all physical states in terms of j generators of Z, $\{z_1, \cdots, z_j\}$. If $\mathcal{Z}$ is a unitary realization of Z and $|\psi>$ is a physical state we have

$$\mathcal{Z}|\psi> = z_1 \cdots z_j |\psi> \tag{4.15}$$

For example, if $G = SU(N)$ then its center is $Z = \mathbb{Z}_N$, the cyclic group of order N and

$$\pi_1\left(SU(N)/\mathbb{Z}_N\right) = \mathbb{Z}_N \tag{4.16}$$

There are N one-dimensional irreducible unitary representations or elements z of the center which satisfy $z^N = 1$. Consequently the eigenvalues of the center operator $\mathcal{Z}$ are of the form $z = e^{ik/N}$ where k is an integer between 1 and N. Hence for each k the physical states of the system carry a different representation and can be thought of as belonging to one of N different vacua of the theory.

If one considers a theory with matter in a representation other than the adjoint then the construction above can be easily adapted. Since adjoint matter has the maximal

symmetry under center transformations then the addition of other representations will serve to reduce the order of the subgroup. That leaves all fields in the theory invariant. Then, by the previous arguments we would find that the number of vacua are reduced as compared to the adjoint case with the remaining vacua a subset of the original ones.

4.2 θ-vacua in 2D: Group Theoretic Calculation

While the topological argument of the previous section gives a precise and elegant solution to the problem of enumerating the number of vacua we can expect in two dimensional Yang-Mills theory with (or without) matter, a more constructive approach may be useful. The topological results for instance do not contain any information about the properties of each of the distinct vacua of the theory. The approach we will take in this Section relies solely on the group theoretic description of Yang-Mills theory that we have considered in the second Chapter. Our constructions for the case of $SU(N)$ gauge group will recover the results of the previous section for the enumeration of vacua but also the properties of the theory in each of the vacua. In particular we will find the binding energy between a pair of static adjoint representation charges as a function of the vacuum state.

We begin by returning to the correlator of a pair of group characters on a space-time torus for the gauge group $SU(2)$, Equation (3.9)

$$P[m, n; L_1, L_2] = \tag{4.17}$$
$$\sum_{j,l=0}^{\infty} e^{-e^2\beta L_1[(l+1)^2-1]/2}\, e^{-e^2\beta L_2[(j+1)^2-1]/2} \sum_{r=0}^{\min l,m} \sum_{s=0}^{\min j,n} \delta_{m+n,2(s+r)}\delta_{2l+2s+m,2j+2r+n}$$

We begin by recalling that for $SU(2)$ all irreducible representations are labeled by a single, non-negative integer. If we consider the case of adjoint representation characters $m = n = 2$ then the first delta function is of the form $\delta_{2,s+r}$. Hence either s and r are both odd or both even. In turn, this information in the second delta function requires

that either both l and j are even or both are odd. Consequently we see that the sums over the variables l and m separate into distinct sums over odd and even integers

$$P[2,2;L_1,L_2] = \tag{4.18}$$

$$\sum_{j,l \text{ even}} e^{-e^2\beta L_1[(l+1)^2-1]/2} \, e^{-e^2\beta L_2[(j+1)^2-1]/2} \sum_{r=0}^{\min l,m} \sum_{s=0}^{\min j,n} \delta_{2,s+r}\delta_{l+s,j+r}$$

$$+ \sum_{j,l \text{ odd}} e^{-e^2\beta L_1[(l+1)^2-1]/2} \, e^{-e^2\beta L_2[(j+1)^2-1]/2} \sum_{r=0}^{\min l,m} \sum_{s=0}^{\min j,n} \delta_{2,s+r}\delta_{l+s,j+r}$$

The interpretation of this phenomena is clear. Each series over 'odd' or 'even' representations defines an independent sector of the theory which does not communicate with the other. It is not unreasonable then to conclude that $SU(2)$ gauge theory with adjoint matter, as we have defined it, is the direct sum of a theory where only even representations exist and a theory with only odd representations. We say that the theory has two different vacua - as expected from the topological considerations of the previous Section.

Next, we establish the corresponding result for arbitrary $SU(N)$ Yang-Mills theory. The correlator of a pair of adjoint ($\mathcal{A}$) representation charges (characters) can be easily determined from the general formula (3.6) calculated in Chapter two:

$$P[\mathcal{A},\mathcal{A};L_1,L_2] = \sum_{S,T} N_{S\mathcal{A}}^T N_{T\mathcal{A}}^S \, e^{-e^2\beta L_1 C_2(S)/2} \, e^{-e^2\beta L_2 C_2(T)/2} \tag{4.19}$$

The identification of the property of this configuration that results in the sum over irreducible representations S and T splitting into independent sub-sums leads us to investigate the transformation properties of representations under the center Z of the gauge group $SU(N)$. If $\mathcal{Z}$ generates center transformations then, as shown in Section A.5 of the Appendix

$$\mathcal{Z}\chi_S = e^{2i\pi C_1(S)/N}\chi_S \tag{4.20}$$

where $C_1(S)$ is the total number of boxes in the Young table associated with S. In

particular, for the adjoint representation it was shown that this transformation is trivial

$$\mathscr{Z}\chi_{\mathcal{A}} = \chi_{\mathcal{A}} \tag{4.21}$$

What we need to do here is evaluate the transformation properties of the product of representations under the center. From the Littlewood-Richardson tensor product rules established in Section A.6 of the Appendix it is clear that the number of boxes is conserved mod N in the product of irreducible representations

$$\mathscr{Z}(\chi_R \chi_S) = e^{2i\pi(C_1(R)+C_1(S))/N}\chi_R\chi_S \tag{4.22}$$

in particular, if R is the adjoint representation

$$\mathscr{Z}(\chi_{\mathcal{A}}\chi_S) = e^{2i\pi C_1(S)/N}\chi_{\mathcal{A}}\chi_S \tag{4.23}$$

Consequently, the product of S with the adjoint results in representations that have the same transformation properties under the center Z as S. In other words, the fusion number $N_{S\mathcal{A}}^T$ is vanishing unless the representation T transforms under the center as S ie.

$$\mathscr{Z}\chi_T = e^{2i\pi C_1(S)/N}\chi_T \tag{4.24}$$

Equivalently, the Young table associated with T must have the same number of boxes, mod N as the Young table associated with S. As we have seen in the previous Section, and as is clear from the form of the generator of the center, $z = e^{2i\pi/N}$, there are N distinct transformations under the center and hence the calculation of the pair correlator splits into N independent sums over irreducible representations

$$
\begin{aligned}
P[\mathcal{A}, \mathcal{A}; L_1, L_2] &= \sum_{k=1}^{N} \sum_{S_k, T_k} N_{S_k\mathcal{A}}^{T_k} N_{T_k\mathcal{A}}^{S_k} \, e^{-e^2\beta L_1 C_2(S_k)/2} \, e^{-e^2\beta L_2 C_2(T_k)/2} \\
&= \sum_{k=1}^{N} P_k[\mathcal{A}, \mathcal{A}; L_1, L_2]
\end{aligned}
\tag{4.25}
$$

Here the sum over k labels the N sectors of the theory by the 'N-ality' of the representations in each sub-series under the center. For example for a representation S_k with N-ality equal to k

$$\mathcal{Z}\chi_{S_k} = e^{2i\pi k/n}\chi_{S_k} \tag{4.26}$$

So by direct calculation we have shown that for $SU(N)$ Yang-Mills theory with adjoint matter [1] in two dimensions there are N different sectors or vacua. These results are, unsurprisingly, in agreement with the topological calculations of the last Section. The advantage here is that we can continue with this construction and identify the sector dependence of physical quantities like string tension between the adjoint charges.

We begin by defining the normalized partition function of the kth sector of the theory with a pair of adjoint charges on a space-time torus

$$Z_k[\mathcal{A}, \mathcal{A}; L_1, L_2] = \frac{P_k[\mathcal{A}, \mathcal{A}; L_1, L_2]}{P_k[0, 0; L_1, L_2]} \tag{4.27}$$

Here $P_k[\mathcal{A}, \mathcal{A}; L_1, L_2]$ is the contribution to the full correlator of a pair of adjoint charges from the kth sum in (4.25). We normalize this by the correlator in the same sector where the adjoint charges are removed

$$P_k[0, 0; L_1, L_2] = \sum_{S_k} e^{-e^2\beta(L_1+L_2)C_2(S_k)/2} \tag{4.28}$$

which is just the partition function of pure Yang-Mills theory on the torus in the kth sector of the theory.

In order to find the effect of multiple vacua on the physics of a pair of adjoint charges interacting on an open line at finite temperature we would like to take the limit $L_2 \to \infty$ and open the spatial circle. Due to the exponential damping of the propagator, only the

---

[1]These results for a pair of adjoint representation charges can be extended to any finite system of charges. The case of a thermodynamic gas (infinite number) of charges will be considered in the next Chapter

representation F_k with least eigenvalue of the quadratic Casimir operator will contribute to the sum over representations, T_k in the partition function (4.27)

$$Z_k[\mathcal{A}, \mathcal{A}; L_1, L_2 \to \infty] = \sum_{S_k} N_{S_k,\mathcal{A}}^{F_k} N_{F_k,\mathcal{A}}^{S_k} \, e^{-e^2\beta L_1[C_2(S_k)-C_2(F_k)]/2} \left(1 + O(\frac{1}{L_2^3})\right) \quad (4.29)$$

The representation of the kth sector with least Casimir eigenvalue has been suggestively labeled F_k which is notation we have used previously for the kth fundamental representation of $SU(N)$. As shown in the Appendix (Section A.5), each fundamental representation belongs to a different class under center transformation and it can be shown in a straightforward manner that they are the minimum Casimir representatives of each class.

Now we are in a position to calculate a physical quantity in the theory as a function of vacuum sector. Using the notation for irreducible representations and the Littlewood-Richardson rules of Section A.6, we have the tensor product for an adjoint ($[N - 1, 1]$) representation with the kth fundamental ($[k]$) representation:

$$[k] \otimes [N - 1, 1] = [k] \oplus [N - 1, k + 1] \oplus [k - 1, 1] \oplus [N - 1, k, 1] \quad (4.30)$$

from which the fusion numbers $N_{F_k\mathcal{A}}^{S_k}$ and $N_{S_k\mathcal{A}}^{F_k}$ can be easily deduced. Using this result along with the general formula for the quadratic Casimir eigenvalues of $SU(N)$ representations (A.81), we have the partition functions for each sector of the theory on the open line

$$Z_0[\mathcal{A}, \mathcal{A}; L] = e^{-e^2\beta LN} \quad (4.31)$$

$$Z_1[\mathcal{A}, \mathcal{A}; L] = Z_{N-1}[A, A; L] = 1 + e^{-e^2\beta L(N+1)} + e^{-e^2\beta L(N-1)}$$

$$Z_k[\mathcal{A}, \mathcal{A}; L] = 1 + e^{-e^2\beta L(N-k)} + e^{-e^2\beta Lk} + e^{-e^2\beta L(N+1)} \quad (4.32)$$

We see that only a small number of configurations contribute to the partition function for a pair of adjoint charges in each sector of the theory. Considering each configuration separately we can calculate the binding energy per unit separation or string tension for

each state. For the general case $1 < k < N - 1$, four configurations contribute with string tensions (3.4)

$$\sigma = 0, e^2(N - k), e^2 k, e^2(N + 1) \tag{4.33}$$

Consequently we see that the interactions between adjoint charges can depend on the vacuum k in which we do the measurement. The most important observation here is that the first configuration has vanishing string tension, and hence the binding energy between the static adjoint charges is vanishing in the non-trivial $k \neq 0$ sectors of the theory. This might suggest that the adjoint charges are not interacting with each other-a picture which is more consistent is one in which we have zero tension strings. Here the adjoint charges do interact with each other but the energy density of the string of colour-electric flux that runs between them is exactly that of the background which is factored out by the normalization in (4.27).

In this simple system of pure Yang-Mills interactions in two dimensions we see that multiple vacua can have a considerable effect on the physics of external charges introduced into the system. For the vacua which correspond to the classes of representations that transform non-trivially under elements of the center we have found that the interactions between adjoint charges are effectively screened. These interactions with zero tension strings are similar to those conjectured to exist in Yang-Mills theory coupled to massless dynamical adjoint matter [40]. Here we have a simple model where the physics is clear, whereas adjoint QCD requires the numerical solution of an infinite set of integral equations [13, 2, 18, 41]. Admittedly the situation we are dealing with here is somewhat different than the dynamical case-we will see in the next Chapter that the addition of a gas of heavy adjoint representation charges provides an excellent model for discussing screening and confinement in addition to the effect of multiple vacua on a system with gauge interactions.

G	minimal weights	meta-stable vacua	stable vacua
SU(N)	N-1	N	N
Sp(2N)	1	N+1	2
SO(2N+1)	1	3	2
SO(2N)	3	4	4
E_6	2	3	3
E_7	1	2	2
E_8	0	1	1
F_4	0	2	1
G_2	0	2	1

Table 4.2: Counting of vacua in Yang-Mills theories with adjoint matter. Details on the minimal (dominant) weights for each group can be found in the section 4.3.

4.3 The Classification of Vacua for General Lie Groups

In order to extend the results of the previous section to the case of general Lie groups it is clear that we need to extend the idea of N-ality - the transformation properties of different representations under transformations in the center of the gauge group. Since the adjoint representation transforms trivially in such cases, we can build up chains of representations which have like transformation properties by taking tensor products of a particular representation with many adjoint representations. This process has already been hinted at in Equation (4.23). The physical picture of what is going on is clear: we are interested in knowing the chains of representations that can decay into one another via interaction with adjoint representations. In particular, the minimum energy representatives of these chains will determine the details of the interaction between external charges in the different vacua of the theory just as the fundamental representations did for the case of $SU(N)$.

Mathematically this idea is formalized in the concept of minimal representations which have been utilized before in the physics literature especially by Goddard and Olive [26]

in discussions of monopole stability. As noted there, if we have the lattice of weights $\Lambda(G)$ for a simply connected compact Lie group G, then the lattice of roots (weights of the adjoint representation) $\Lambda_{Ad}(G)$ is a subgroup of it. Consequently we can form the cosets Λ/Λ_{Ad} by identifying weights which differ by integral linear combinations of roots and in this way we identify precisely which representations lie in like decay chains. These cosets form a finite dimensional Abelian group which is isomorphic to the center Z of the group G and now the importance of the center in classifying vacuum states becomes obvious. Clearly the notion of N-ality for the unitary groups is generalized here where to each coset one can assign a (conserved) element of an Abelian group- typically, but not always, an integer.

Once we have determined the classes to which different representations belong a question still remains: Is there a representative of each class which is stable? The answer to this question is yes. First we deal with the identity coset where the representation with minimum quadratic Casimir eigenvalue (energy) is just the trivial representation. Clearly this coset includes the adjoint representation and all other representations which transform trivially under the center of the group. As for the other cosets, we can equivalently define the minimal weights θ as those which lie closest to the origin of the weight diagram or have the least Casimir or where all weights $\{\theta\}$ of a representation lie on the same orbit under the action of the Weyl group. The positive (dominant) elements of the minimal weights then identify the minimal representations of the group and identify the complete set of stable charges which label non-trivial vacuum states for any Lie group.

It is now an easy task to enumerate the minimal representations for all simple, compact Lie groups using information contained in standard References [107, 88, 73, 66]. In the case of the special unitary group SU(N) these are of course the N-1 fundamental representations as we have seen. For the symplectic group Sp(2N) only the 2N dimensional fundamental is stable in interactions with adjoint charges. The other classical Lie group

is the orthogonal group SO(N) and here we have two different cases. For SO(2N+1) only the 2^N dimensional spinor representation is minimal, but SO(2N) is interesting, as both of the spinor representations are minimal in addition to the N dimensional vector representation. As for the exceptional groups, both 27 dimensional fundamentals of E_6 are minimal as is the 56 dimensional fundamental of E_7. The remaining compact Lie groups E_8, F_4 and G_2 have no non-trivial minimal representations, that is all representations can decay to the trivial one via adjoint charges. Now from the previous calculations for the case of $SU(N)$ one can use this information about the minimal representations to calculate the string tensions of different adjoint charge configurations as a function of vacua for different Lie groups. We will not delve into such detail hear but only reiterate that the number of vacua in the theory is equal to the number of minimal representations and the classification which we have completed here (see Table 4.2) are in complete agreement with the results of the pure topological classification (see Table 4.1).

Chapter 5

Non-Abelian Coulomb Gases on the Open Line: Finite N

In the previous Chapter we saw that the choice of vacuum had a definite effect on the physics of external test charges introduced into a two-dimensional Yang-Mills theory. In particular the effective binding, or lack thereof, between charges depended greatly on the choice of vacuum. Our purpose here is to extend this analysis to the situation of two dimensional $SU(N)$ gauge fields coupled to a gas of infinitely many classical adjoint charges [50] and explicitly determine the thermodynamics of the system as function of vacuum sector.

The $SU(N)$ non-Abelian Coulomb gas with adjoint charges in two dimensions is a useful model for investigating the topological and symmetric properties of gauge theories in higher dimensions. In the limit of infinite N, which we will dicuss later, the model has a non-trivial phase structure which is reminiscent of that in higher dimensional QCD. Unfortunately, for the case of finite N which we will be studying here there is no such phase transition in two dimensions. However at finite N the vacuum structure due to the topological structure of the theory is clearly apparent. Moreover the model is explicitly solvable in the limits of high and low particle density and in these limits we will investigate the thermodynamics of the adjoint gas in each sector of the theory. This explicit solvability will be of particular utility in answering questions about the differences between screening and confinement in the system.

Analogous calculations can be carried out for a gas of fundamental representation charges but at finite N this system has neither a non-trivial phase or vacuum structure.

This system for $N = 2, 3$ was considered in [69] and similar results can be shown to hold for arbitrary N. The thermodynamics of classical charges interacting via Abelian and non-Abelian electric forces in one spacial dimension has been considered previously [61, 69]. Additionally the effect of multiple vacua in the $SU(2)$ adjoint gas has been considered previously [21]. We will consider constant pressure ensembles as these authors have done, but we will use a different formalism to construct the partition function of the system. This formalism can easily be extended to configurations other than that of the open line and will be of use in subsequent Chapters.

We begin in the next Section with a short description of our methods for constructing the model of $1 + 1$ dimensional non-Abelian Coulomb gas, which are based on the developments of the third Chapter. Restricting ourselves to the case of the adjoint charges, we proceed with an analysis of the low density/pressure limit of the model. Here using group theoretic techniques the explicit dependence of the equation of state of the adjoint gas on the vacuum parameter k is established. Converting to the Fourier domain we find the high density/pressure limit of the model is equivalent to solving a system of coupled quantum oscillators. In this limit the equation of state is shown to be independent of k.

5.1   The Classical Non-Abelian Coulomb Gas

Using the general formula for the correlator of n group characters on the space-time torus (3.5) and the identification of characters in a particular representation with static, stationary physical colour-electric charges we can calculate the thermodynamics of a gas of adjoint representation charges. Using the operator notation introduced in the last section, the correlator of n characters in the same representation χ_S each in separated by a distance $L_1 \cdots L_n$, is given by

$$P[S; L_1, \ldots, L_n] = \qquad\qquad\qquad\qquad (5.1)$$

$$\operatorname{Tr} \chi_S \, \mathrm{e}^{-e^2 \beta L_1 C_2/2} \cdots \chi_S \, \mathrm{e}^{-e^2 \beta L_n C_2/2} = \operatorname{Tr} \prod_{k=1}^{n} \left(\chi_S \, \mathrm{e}^{-e^2 \beta L_k C_2/2} \right)$$

The trace here closes the space-time cylinder into a torus with length $L = \sum L_i$. Integrating over the separations of the charges and dividing by a combinatorial factor appropriate for indistinguishable particles, we have the classical partition function for n charges on the circle

$$Z_n = \int dL_1 \cdots \int dL_n \; \delta(L - \sum L_i) \operatorname{Tr} \prod_{k=1}^{n} \left(\chi_S \, \mathrm{e}^{-e^2 \beta L_k C_2/2} \right) \tag{5.2}$$

Here the delta function restricts the circumference of the spacial circle to L. Letting L go to infinity, we recover the partition function for n adjoint representation charges on the open line. Carrying out the now unrestricted integrations over $\{L_i\}$ we find that the partition function takes on a rather simple transfer matrix form

$$Z_n = \frac{1}{n!} \operatorname{Tr} \prod_{i=1}^{n} \left[\frac{1}{e^2 \beta C_2/2} \chi_S \right] \equiv \frac{1}{n!} \operatorname{Tr} T^n \tag{5.3}$$

Hence, the calculation of the thermodynamics of n static charges on the open line reduces to solving the eigenvalue problem for the operator T acting on the state-vector of the system, Ψ:

$$T\Psi = \frac{1}{e^2 \beta C_2/2} \chi_S \Psi = \lambda \Psi \tag{5.4}$$

For the purpose of finding thermodynamic quantities, it is convenient to deal not with the constant volume ensemble as we have up to now, but rather a constant pressure ensemble. The change to a constant pressure ensemble can be carried out in a straightforward manner by introducing a pV term into the energy of the system with the result of shifting the energy per unit length. The resulting eigenvalue problem reads

$$T_p \Psi = \frac{1}{\beta(e^2 C_2/2 + p)} \chi_S \Psi = \lambda \Psi \tag{5.5}$$

In the thermodynamical limit where $n \to \infty$, all information of the system is contained in the largest eigenvalue λ_0 of the operator T_p. The remainder of this Chapter will involve finding λ_0 for the case of a gas of adjoint charges.

5.2 Low Density Limit: Group Theory

The effective eigenvalue problem for the non-Abelian gas (5.5) was previously derived via a different approach by Nambu et al [69]. As in that case, the transfer matrix problem (5.5) is equivalent to the following linear equation,

$$H\Psi \equiv (\alpha C_2 - q\chi_S)\Psi = -p\Psi \tag{5.6}$$

where H is an operator which acts on the space of irreducible representations, so that

$$\Psi = \sum_R a_R \chi_R \tag{5.7}$$

with $\alpha = e^2/2$ and $q = 1/(\beta\lambda)$. The structure of this equation is the same as one would find in a quantum mechanics problem. The quadratic Casimir operator is diagonal on the space of irreducible representations

$$C_2\, \chi_R = C_2(R)\, \chi_R \tag{5.8}$$

and corresponds to the kinetic term. The role of the potential is played by the character χ_S which mixes the eigenvectors of the kinetic term. This can be easily seen by the multiplication rule

$$\chi_R\,\chi_S = \chi_{R\otimes S} = \sum_T N_{RS}^T \chi_T \tag{5.9}$$

Here N_{RS}^T is the fusion number which enumerates the occurrence of the irreducible representation T in the Kronecker product of representations R and S (See Section A.6). The only difference between quantum mechanics and the current situation is that we would like to solve for the eigenvalue of the transfer matrix problem $\lambda = 1/(\beta q)$ as a function of the pressure p as opposed to solving for the energy of the system as a function of the potential.

As in the case of quantum mechanics one can begin to solve the eigenvalue problem by considering the symmetries of the system which will lead to conserved quantities.

Here we are most interested in the symmetric properties of the 'Hamiltonian', H, under transformations which lie in the center of the gauge group. As we have seen, the presence of such a symmetry immediately leads to the phenomena of multiple vacua. The action of a transformation under the center of the gauge group is defined as

$$\mathcal{Z}\chi_R = z_R\chi_R \tag{5.10}$$

Here z_R is a representation of the center of the gauge group. Since for all compact Lie groups, the center forms an Abelian subgroup, we can take z_R to be a complex phase factor. The details of this phase factor depend on the structure of the center Z. For $U(N)$, Z is isomorphic to $U(1)$ hence $z_R = e^{iaC_1(R)}$ where $a \in \mathbb{R}$ and $C_1(R)$ is the $U(N)$ linear Casimir operator. $C_1(R)$ is simply the number of boxes in the Young table associated with R. For the case of interest, $SU(N)$, $Z \sim \mathbb{Z}_N$ and consequently $z_R = e^{2\pi iC_1(R)/N}$. This follows from the $U(N)$ case with the restriction that $z_R^N = 1$. The question of whether H commutes with $\mathcal{Z}$ is reduced, since Casimir operators commute amongst themselves, to determining the irreducible representations S which satisfy

$$[\chi_S, \mathcal{Z}] = (1 - e^{2\pi iC_1(S)/N})\chi_S = 0 \tag{5.11}$$

The solutions to such a condition are clearly $C_1(S) = 0 \pmod{N}$. In terms of irreducible representation S of the matter content of the theory, this means that the full vacuum degeneracy is apparent only when $C_1(S) = 0 \pmod{N}$. The simplest examples of such representations are the trivial representation in which case the theory reduces to that of pure Yang-Mills in $1 + 1$ dimensions and the case where all matter is in the adjoint representation. This final case is the one we will be considering in this Chapter.

In the case of the adjoint gas where the center operator commutes with (5.6), in analogy with the conservation of eigenvalues of commuting operators in quantum mechanics we see that the eigenvalue of the linear Casimir operator is conserved (mod N) and is a

good 'quantum number' (mod N). Consequently for the $SU(N)$ adjoint gas there exists a family of N distinct solutions to the eigenvalue problem, each of which we will label by $k = 0, \cdots, N - 1$. For each value of k we have an isolated sector of the theory complete with a stable vacuum and an infinite tower of excited states. These are precisely the discrete 'θ-vacua' of the model.

In the limit where $q \to 0$, the eigenvalue problem in (5.6) is reduced to that of the free $1 + 1$ dimensional Yang-Mills theory and we can easily identify the vacuum states of each sector. In this case the eigenvectors, Ψ of the transfer matrix, T_p are simply the irreducible representations of the gauge group. Since we are interested in the case with adjoint $SU(N)$ charges, the labeling of vacua introduced above with $k = 0, \cdots, N-1$ will be followed, although strictly speaking, the free theory has a countably infinite vacuum degeneracy. We denote by $[k]$ the kth vacuum state which is the $k(k-1)/2$ dimensional completely antisymmetric fundamental representation of $SU(N)$ (See Section A.5). Each of these fundamental representations is the lowest lying energy state of each of the k sectors and will serve as a starting point for a perturbative calculation of the eigenvalue problem for small q, or equivalently, small p in each sector.

5.3 Calculation of Fusion Numbers

Having identified the ground states of each sector of the theory, all we require to calculate the solution of the eigenvalue problem (5.6), to lowest order in the pressure p, are the fusion numbers N_{RS}^T (5.9). As we have seen, these are pure group theoretic quantities which detail the mixing effect of the potential on irreducible representations and, in particular, the anti-symmetric ground states $[k]$. In order to carry out this calculation, we need to know the matrix elements of the potential in the basis of the irreducible representations. This information follows from the Kronecker product of the adjoint

representation with our chosen basis

$$\chi_{\mathcal{A}}\chi_R = \chi_{\mathcal{A}\otimes R} = \sum_T N^T_{\mathcal{A}\ R}\chi_T \qquad (5.12)$$

Here $N^T_{\mathcal{A}\ R}$ is the fusion number enumerating the occurrence of the irreducible representation T in the product of the adjoint representation $\mathcal{A}$ and R. As in quantum mechanics we can easily calculate the corrections to p up to third order in q using the unperturbed basis of irreducible representations.

$$
\begin{aligned}
p_{[k]} \;=\;& qN^{[k]}_{\mathcal{A}\ [k]} + q^2 \sum_{R\neq[k]} \frac{\left(N^R_{\mathcal{A}\ [k]}\right)^2}{C_2([k]) - C_2(R)} \\
&+ q^3\Bigg[\sum_{R,S\neq[k]} \frac{N^{[k]}_{\mathcal{A}\ S}N^S_{\mathcal{A}\ R}N^R_{\mathcal{A}\ [k]}}{\left(C_2([k]) - C_2(R)\right)\left(C_2([k]) - C_2(S)\right)} \\
&+ N^{[k]}_{\mathcal{A}\ [k]} \sum_{R\neq[k]} \left(\frac{N^R_{\mathcal{A}\ [k]}}{C_2([k]) - C_2(R)}\right)^2 \Bigg] + O(q^4)
\end{aligned}
\qquad (5.13)
$$

It should be noted that we have left out a constant, sector-dependent, background contribution to the pressure.

In order to explain details of the calculation, we need to adopt a notation to label the irreducible representations. The one we will use is given by the column variables $[m_1, m_2, \ldots]$ of the Young diagram associated with the representation (See Section A.6). For example the antisymmetric combination of k, N dimensional fundamental representations in $SU(N)$ - the ground state of the kth sector - corresponds to a Young diagram with a single column of k boxes: $[k]$. Another example which appears in all calculations is that of the adjoint representation ($\mathcal{A}$) which in column variables is given by: $[N-1, 1]$. In this notation the quadratic Casimir for a representation $[m_1, \ldots]$ is given by

$$C_2(R) = \qquad (5.14)$$

$$V \begin{pmatrix} 1-\frac{1}{N} & 1-\frac{2}{N} & 1-\frac{3}{N} & \cdots & 1-\frac{N-1}{N} \\ 1-\frac{2}{N} & 2(1-\frac{2}{N}) & 2(1-\frac{3}{N}) & \cdots & 2(1-\frac{N-1}{N}) \\ 1-\frac{3}{N} & 2(1-\frac{3}{N}) & 3(1-\frac{3}{N}) & \cdots & 3(1-\frac{N-1}{N}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-\frac{N-1}{N} & 2(1-\frac{N-1}{N}) & 3(1-\frac{N-1}{N}) & \cdots & (N-1)(1-\frac{N-1}{N}) \end{pmatrix} V^T - \frac{N}{12}(N^2-1)$$

where

$$V = (m_1 + 1, m_2 + 1, \ldots, m_{N-1} + 1)$$

The remaining task is to compute the relevant fusion numbers. We begin by presenting the results of the calculations in Table 5.1. Here we record the fusion numbers $N^S_{\mathcal{A}\,R} = N^R_{\mathcal{A}\,S}$ for the representations R and S of importance in the calculation (5.13) of the pressure. Each sub-table corresponds to a different background k for $SU(N)$ since the details of the calculation of fusion numbers in general depends on k and N. These results are only good for $k \leq N/2$ where the remainder of the cases can be found via the symmetry of the eigenvalue problem under conjugation $k \to N - k$. For completeness we present the details of the second table for $k = 1$ and $N = 3$. This is the familiar case of $SU(3)$ and via common tensor or Young diagram methods the Kronecker products of the 8-dimensional adjoint representation ($\mathcal{A}$) with the lowest lying representations can be calculated. In dimension notation we have

$$\begin{aligned} 8 \otimes 3 &= 3 \oplus 6 \oplus 15 \\ 8 \otimes 6 &= 3 \oplus 6 \oplus 15 \oplus \cdots \qquad (5.15) \\ 8 \otimes 15 &= 3 \oplus 6 \oplus 2 \times 15 \oplus \cdots \end{aligned}$$

In the last two products we have ignored higher representations which do not contribute

$k = 0,\ N \geq 2$	$[0]$	$[N-1,1]$
$[0]$	0	1
$[N-1,1]$	1	2

$k = 1,\ N = 3$	$[1]$	$[2,2]$	$[2,1,1]$
$[1]$	1	1	1
$[2,2]$	1	1	1
$[2,1,1]$	1	1	2

$k = 1,\ N \geq 4$	$[1]$	$[N-1,2]$	$[N-1,1,1]$
$[1]$	1	1	1
$[N-1,2]$	1	2	1
$[N-1,1,1]$	1	1	2

$k = 2,\ N \geq 3$	$[2]$	$[N-1,3]$	$[1,1]$	$[N-1,2,1]$
$[2]$	1	1	1	1
$[N-1,3]$	1	2	0	1
$[1,1]$	1	0	1	1
$[N-1,2,1]$	1	1	1	3

$k \geq 3,\ N \geq 3$	$[k]$	$[N-1,k+1]$	$[k-1,1]$	$[N-1,k,1]$
$[k]$	1	1	1	1
$[N-1,k+1]$	1	2	0	1
$[k-1,1]$	1	0	2	1
$[N-1,k,1]$	1	1	1	3

Table 5.1: Table of relevant fusion numbers $N^S_{\mathcal{A}\,R} = N^R_{\mathcal{A}\,S}$ for the calculation of the pressure of the adjoint gas. Note these results hold only for $k \leq N/2$ with the other cases given by the symmetry $k \to N - k$.

to $O(q^3)$ in (5.13). Converting to our column notation

$$3 \equiv [1] \tag{5.16}$$

$$6 \equiv [2,2]$$

$$8 \equiv [2,1] \tag{5.17}$$

$$15 \equiv [2,1,1]$$

we have the results of the second sub-table in Table 5.1.

5.4   Equation of State of the Adjoint Gas

Now we would like to develop the equation of state for the two-dimensional adjoint Coulomb gas. As is familiar from more physical gauge theories, the number of microscopic degrees of freedom n may not be the number of macroscopic degrees of freedom, n^*. For example it is believed that in QCD pairs and triples of quarks are bound into observable mesons and baryons, respectively. The equation of state per microscopic degree of freedom for the constant pressure ensemble is

$$\frac{p<V>}{n} \equiv p<v> = \frac{n^*}{n}T \tag{5.18}$$

where $<V>$ is the expectation value of the total volume of the system which is canonically conjugate to p. It can be determined by inverting the relation $p(q)$ and using the relationships between the thermodynamic variables

$$<V> = -T\frac{\partial \log Z}{\partial p} = -nT\frac{\partial \log \lambda}{\partial p} = nT\frac{\partial \log q(p)}{\partial p} \tag{5.19}$$

It is convenient to define ρ as the ratio of macroscopic to microscopic degrees of freedom

$$\rho = \frac{n^*}{n} = \frac{\partial \log q}{\partial \log p} \tag{5.20}$$

$N > 2$ $k = 0$	$\tilde{p} = \tilde{q}^2 + \tilde{q}^3 + O(\tilde{q}^4)$ $\rho = \frac{1}{2} - \sqrt{\frac{\tilde{p}}{2}} + O(\tilde{p}^2)$
$N = 3$ $k = 1$	$\tilde{p} = \tilde{q} + \frac{9}{4}\tilde{q}^2 + \frac{45}{16}\tilde{q}^3 + O(\tilde{q}^4)$ $\rho = 1 - \frac{9}{4}\tilde{p} + \frac{153}{16}\tilde{p}^2 + O(\tilde{p}^3)$
$N > 3$ $k = 1$	$\tilde{p} = \tilde{q} + 2\frac{N^2}{N^2-1}\tilde{q}^2 + 4\frac{N^4}{(N^2-1)^2}\tilde{q}^3 + O(\tilde{q}^4)$ $\rho = 1 - 2\frac{N^2}{N^2-1}\tilde{p} + 4\frac{N^4}{(N^2-1)^2}\tilde{p}^2 + O(\tilde{p}^3)$
$N > 3$ $k = 2$	$\tilde{p} = \tilde{q} + (\frac{N}{2} + \frac{N}{N+1} + \frac{N}{N-2})\tilde{q}^2 + (\frac{N^3}{(N-2)(N+1)} + \frac{N^2}{(N-2)^2} + \frac{2N^2}{(N+1)^2}+)\tilde{q}^3 + O(\tilde{q}^4)$ $\rho = 1 - (\frac{N}{2} + \frac{N}{N+1} + \frac{N}{N-2})\tilde{p} + \frac{3N^4+10N^3-13N^2-8N-24}{4(N-2)^2(N+1)^2}N^2\tilde{p}^2 + O(\tilde{p}^3)$
$N > 3$ $k > 2$	$\tilde{p} =$ $\tilde{q} + (\frac{N}{k} + \frac{N}{N+1} + \frac{N}{N-k})\tilde{q}^2 + (\frac{N^2}{(N-k)^2} + \frac{N^2}{k^2} + \frac{2N^2}{(N+1)^2} + \frac{2N^3}{k(N-k)(N+1)})\tilde{q}^3 + O(\tilde{q}^4)$ $\rho =$ $1 - (\frac{N}{k} + \frac{N}{N+1} + \frac{N}{N-k})\tilde{p} + (\frac{N^2}{(N-k)^2} - \frac{N^2}{(N+1)^2} + \frac{N^2}{k^2}\frac{5k+N+7Nk+N^2}{(N-k)(N+1)})\tilde{p}^2 + O(\tilde{p}^3)$

Table 5.2: Table of the pressure p and the ratio of macroscopic to microscopic degrees of freedom ρ for the various vacuum states, k of the $SU(N)$ adjoint gas. The primed variables are defined as $\tilde{x} \equiv \frac{x}{\alpha N}$. Note the conjugation symmetry $k \to N - k$.

Comparing with (5.18) we see the fundamental importance of the quantity ρ. The dependence of ρ on p is tabulated for the different sectors of the $SU(N)$ gas in Table 5.2.

The results of these calculations deserve some comment. The most striking between the different sectors of the theory is the configuration of adjoint charges in the limit of vanishing pressure. For the $k = 0$ sector we find that $\rho = 1/2 - \cdots$ and hence the adjoint charges in the system are bound pairwise in the low pressure limit. This behaviour is not surprising and is seen in both the $U(1)$ [61] and $SU(N)$ [69] one dimensional (fundamental representation) Coulomb gases. What is different here in the $k = 0$ sector is that the first corrections in pressure to this pair-wise binding come about with a negative sign and so the adjoint charges begin to form macroscopic configurations where the number

of constituents is three of more. This is possible since adjoint charges are of course self-adjoint and an arbitrary number of them can form an observable charge singlet.

When one moves to the cases when $k > 0$ we see a distinct change in the vanishing pressure macroscopic structure of the theory. As explained previously for the Yang-Mills case [105, 70, 71], different sectors of the a $1 + 1$ dimensional gauge theory are equivalent to considering a the theory with different constant background colour electric fields. For $SU(N)$, each admissible background is given by one of the N fundamental representations which we label by the parameter k. In Table 5.2 we see that for a non-trivial background ($k > 0$) the adjoint charges of the system can interact with the background electric field and form stable, colour singlet configurations where they are the macroscopic degrees of freedom. In other words they act as free particles.

5.5   High Density Limit: Fermions on the Circle

The eigenvalue problem of (5.6) can also be solved exactly in the limit of large values of q which corresponds to the limit of high pressure. This is most conveniently carried out by converting the group theoretic equation of (5.6) to a linear differential equation with periodic coefficients. In this Section the gauge group will be taken to be $U(N)$ as this will simplify calculations. Recovering the results for $SU(N)$ is a trivial step which will be noted at the appropriate point in the calculation.

The starting point for converting the eigenvalue problem of (5.6) to a differential equation is to consider the eigenvector Ψ as a linear combination of irreducible representations each labeled by N integers $\{n_1, \cdots, n_N\}$. These integers correspond to reduced row variables for the Young diagram associated with the irreducible representation. They are related to the usual row variables l_i for a Young table by the equation

$$n_i = l_i + N - i \tag{5.21}$$

Because of the restrictions on the variables l_i, n_i satisfy the dominance condition

$$\infty > n_1 > n_2 > \cdots > n_N > -\infty \tag{5.22}$$

The quadratic Casimir operator (A.81) is diagonal in this basis with the action

$$C_2 \psi_{n_1, \cdots, n_N} = \frac{1}{2} \left(\sum_{i=1}^{N} n_i^2 - \frac{N(N^2-1)}{12} \right) \psi_{n_1, \cdots, n_N} \tag{5.23}$$

The action of the character χ_S on these states is, however, more complicated. This is partly due to the dominance restriction on the Young diagram. If one relaxes this restriction then the action of the character is simplified somewhat. In this case the action on a general state of a character in the adjoint representation is given by,

$$\chi_A \psi_{n_1, \cdots, n_N} = \sum_{r,s=1}^{N} \psi_{n_1 + \delta_{r,1} - \delta_{s,1}, \cdots, n_N + \delta_{r,N} - \delta_{s,N}} \tag{5.24}$$

The result of this operation is to add unity to n_r and then subtract unity from n_s and sum over all r and s. Consequently the eigenvalue problem of (5.6) is a difficult recurrence type equation. This type of equation is most successfully dealt with by introducing the periodic function $\tilde{\psi}(x_1, \cdots, x_N)$, which is completely symmetric under the exchange of arguments, via a Fourier transform

$$\psi_{n_1, \cdots, n_N} = \int_0^{2\pi} dx_1 \cdots \int_0^{2\pi} dx_N \ e^{i \sum_i n_i x_i} \tilde{\psi}(x_1, \cdots, x_N) \Delta(x_1, \cdots, x_N) \tag{5.25}$$

The factor $\Delta(\{x_i\})$ is a Vandermonde determinant defined as $\prod_{i<j}(x_i - x_j)$ and as such is completely antisymmetric under the exchange of any two x_i's. This factor is included to force the integration to vanish identically for $n_i = n_j$, when $i \neq j$. In this way we can effectively impose the dominance condition (5.22) of the reduced Young diagram variables in the Fourier domain.

Acting with the Casimir operator and the adjoint character on (5.25) we find the transfer matrix problem of (5.6) is equivalent to the second order linear partial differential

equation with periodic coefficients

$$\left[-\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \frac{N}{24}(N^2 - 1) - \frac{q}{\alpha} \left[N + 2 \sum_{i<j} \cos(x_i - x_j) \right] \right] \Delta\tilde{\psi} = -\frac{p}{\alpha} \Delta\tilde{\psi} \qquad (5.26)$$

In this form some of the features of the adjoint non-Abelian Coulomb gas are more apparent. For example, the center operator $\mathcal{Z}$ has a simple interpretation in the Fourier domain

$$\mathcal{Z} = e^{iaC_1} = \exp\left[ia \sum_{k=1}^{N} n_k \right] = \exp\left[a \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \right] \qquad (5.27)$$

This is exactly the same structure as the translation operator in quantum mechanics. Here $\mathcal{Z}$ generates uniform shifts of the coordinates $\{x_i\} \to \{x_i + a\}$. It is easy to verify that (5.26) has this symmetry and from the arguments of the previous Section, we expect a continuum of vacua for the adjoint gas with $U(N)$ gauge group. Additionally with periodic coefficients and the restriction to completely anti-symmetrized wavefunctions, $\Delta\tilde{\psi}$, the eigenvalue problem is equivalent to that of non-relativistic fermions in a periodic potential. This correspondence is familiar from matrix models [5] and is exploited in the solution of the large N non-Abelian Coulomb gas [86, 87, 23].

In the limit of high densities, or equivalently high pressure, the wave functions are localized about the minima of the potential. Expanding the potential about the local minimum at $\{x_i - x_j = 0\}$ leads to the coupled harmonic oscillator:

$$\left\{ -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \frac{N}{24}(N^2 - 1) + \frac{q}{\alpha} \sum_{i<j} (x_i - x_j)^2 - \frac{N^2 q}{\alpha} \right\} \Delta\tilde{\psi} = -\frac{p}{\alpha} \Delta\tilde{\psi} \qquad (5.28)$$

Performing the change of variables to the orthonormal basis $\{u_i\}$ given by

$$u_n = \frac{((N - n)\, x_n - (x_{n+1} + x_{n+2} + \ldots + x_N))}{\sqrt{(N - n)(N - n + 1)}} \qquad n = 1 \ldots N - 1 \qquad (5.29)$$

$$u_N = \frac{1}{\sqrt{N}} (x_1 + x_2 + \ldots + x_N)$$

diagonalizes the system. In this basis the decoupled oscillator is:

$$\left[-\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial u_i^2} + \frac{1}{2} \left(\frac{2Nq}{\alpha} \right) \sum_{i=1}^{N-1} u_i^2 \right] \psi = E\psi \qquad (5.30)$$

where,

$$E = \frac{N}{24}(N^2 - 1) - \frac{p - N^2 q}{\alpha} \qquad (5.31)$$

At this point we note that there is no potential for the u_N coordinate, which is to be expected as it describes a center of mass coordinate in the change of variables (5.29) and the original potential in (5.26) depends only on relative not absolute positions. Consequently the u_N dependence of the system is only through a phase. In the case of $SU(N)$ this center of mass coordinate is restricted but otherwise behaves exactly as in $U(N)$, entering only as a phase. Regardless of the details of this phase, we will see it does not contribute to the high pressure equation of state of the adjoint gas. The other modes corresponding to the coordinates $\{u_1 \cdots u_{N-1}\}$ have degenerate frequencies which are easily read off the diagonal form

$$\omega_N = \sqrt{\frac{2Nq}{\alpha}} \qquad (5.32)$$

Knowing the normal modes of the eigenvalue problem we are now in a position to find the ground state solution of (5.26) which will correspond to the dominant eigenvalue of the transfer matrix problem. A solution which satisfies the requirement of antisymmetry with respect to permutation of the coordinates is given by,

$$\Delta \tilde{\psi} \sim \prod_{i<j}(x_i - x_j) \, \exp\left\{i\frac{M}{\sqrt{N}}\sum_i x_i\right\} \exp\left\{-\frac{1}{2}\frac{\omega_N}{N}\sum_{i<j}(x_i - x_j)^2\right\} \qquad (5.33)$$

The parameter M is an integer associated with the center of mass coordinate and contributes a constant to the energy eigenvalue. Notice that the potential exponentiated, this is a direct consequence of the normal modes having degenerate energy. The above state has $N^2 - 1$ quanta of energy,

$$E = (N^2 - 1)\frac{\omega_N}{2} + \frac{1}{2}M^2 \qquad (5.34)$$

so that the pressure is given, up to an irrelevant constant by,

$$\frac{p}{\alpha} = \frac{N^2 q}{\alpha} - (N^2 - 1)\sqrt{\frac{Nq}{2\alpha}} \tag{5.35}$$

Inverting this relation to find $q(p)$ and using the definition (5.20) we find, to leading order, the ratio of macroscopic to microscopic degrees of freedom

$$\rho = 1 - \frac{(N^2 - 1)}{2}\sqrt{\frac{\alpha}{2Np}} + O(\frac{1}{p}) \tag{5.36}$$

Consequently, in this high pressure limit the adjoint charges are the macroscopic degrees of freedom and, as in the $k > 0$ low pressure cases, can be interpreted as being free. The most striking feature of (5.36) is the absence of any k dependence. This follows from the fact that the information of the center of mass coordinate appears only as a phase contributing the additive constant M^2 to the eigenvalue problem which is negligible in the limit of large pressure. In terms of physics the high pressure adjoint gas effectively screens all colour electric fields over large distances and so the fundamental colour electric fields associated with the different vacuum sectors are washed out by the adjoint degrees of freedom.

5.6 Discussion

In conclusion, from consideration of the thermodynamics of a system of static adjoint representation charges interacting via $SU(N)$ colour electric fields in $1 + 1$ dimension we have shown that the physics depends on the discrete vacuum index k. We have solved the model in two regions: low and high pressure. In the low pressure regime, which is equivalent to low particle density, the constant pressure equation of state was shown to have strong dependence on k. In the limit of high pressures/particle densities the dependence on k was shown to become trivial, and does not enter into the equation of

state for the adjoint gas. This is attributed to the screening nature of the high pressure limit, which washes out any global structure like a vacuum index.

Currently there are ongoing investigations as to the nature of the differences between confinement and screening especially in two-dimensional Yang-Mills theory coupled to adjoint matter [40, 14, 41]. The simple model we have considered here can be seen to capture the qualitative features and gives a clear way to discuss confinement and screening.

Confinement arises from the fact that all basic constituents of the theory are permanently trapped in bound states with a well defined, finite number of other basic constituents. The mesons of particle physics are expected to be examples of this- they are constructed of pairs of quarks which are the basic matter constituents of QCD. Here, with adjoint quarks this would mean that the low energy spectrum consists only of string-like bound states where the colour-electric flux provides a linear confining potential for all particle separations. As we have seen in Section 5.4 the low density limit of the adjoint non-Abelian Coulomb gas in the $k = 0$ vacuum state gives a working example of this behaviour. There the ratio of macroscopic to microscopic degrees of freedom ρ went to a limit of $1/2$ at zero density showing that the adjoint charges are bound pair-wise.

Screening occurs when the such string-like bound states can be broken by the spontaneous pair production of intermediate particles. The result is that isolated constituents are surrounded by a cloud of indefinitely many virtual particles. A good example of such a situation is an electron in a classical plasma. The high density limit of the non-Abelian adjoint Coulomb gas gives another example of this behaviour. The absence of any vacuum dependence on the physics is equivalent to the system screening any fundamental representation external charges that one may introduce. This external charge would bind to the background gas of adjoint charges and, on interaction with infinitely many of them, be neutralized. Similar screening of fundamental representation charges with

massless adjoint representation matter has recently been investigated in [83, 40, 14, 41] and it appears that the high density limit we have considered here gives a qualitatively similar, solvable example of such behaviour.

While confinement and screening are adequate to describe the $k = 0$ low density and high density aspects of the adjoint non-Abelian Coulomb gas, the $k \neq 0$ vacua do not fit into either picture. As we have seen, the adjoint charges of the gas are not bound to each other in these vacua and act as free particles. This behaviour is consistent with the tension-less strings we have seen in the previous Chapter for non-trivial vacua. In the gases of heavy, static charges that we consider throughout this Thesis it appears that the only way to construct such 'bleached' states is via the introduction of non-trivial vacua but it may be possible, and interesting to see them arise in a model with dynamical matter. In particular it would be interesting to find a model that exhibited a smooth transition from the confined to screened phase and to the 'bleached' phase as well. This is not possible here since due to the discrete nature of the vacuum index k.

This is the end of our analysis of non-Abelian Coulomb gases with finite rank gauge groups. In subsequent Chapters we will be interested in the limit $N \to \infty$ for $U(N)$ and $SU(N)$ gauge theories. The advantage of these theories are that they have interesting phase structure but, unfortunately the vacuum structure is washed out. As we will see, the natural energy scale Λ of these models at large N is $\Lambda \sim N^2$. In the previous two Chapters we have shown that the vacua for finite N $SU(N)$ gauge theory are determined by the background fundamental representations which we have labeled here by k. The characteristic energy densities of these backgrounds are given by the product of the gauge coupling e^2 with the quadratic Casimir of the background representation

$$\Lambda_k \sim e^2 C_2(F_k) = (e^2 N)\frac{k}{N}(1 - \frac{k}{N}) \tag{5.37}$$

Since k is of at most order N, we see that the energy splittings between the different

vacua are of order N^0 and hence sub-leading in large N. This phenomenon has been noted previously in more general circumstances (see [103] for example) and shows that vacuum structure is a sub-leading detail in a large N expansion of gauge theories.

Chapter 6

The Large N Non-Abelian Coulomb Gas

Up to this point we have considered two-dimensional Yang-Mills theory almost completely in the basis of irreducible representations. This is particularly convenient since, as we have seen, the propagator of Yang-Mills theory on the space-time cylinder is diagonal in this basis. The drawback is that the inclusion of external charges is a complicated process which requires the calculation of fusion numbers for products of irreducible representations. In this Chapter we would like to consider Coulomb gases of arbitrary representation charges in more general settings than the open line as we have in the previous Chapter. Changing the basis of states in which we operate from irreducible representations to traces of a group element $g(x)$, much as we did via Fourier transform methods in Section 5.5, is the key to carrying out this plan. As we will see the advantage of this change is two-fold. First the inclusion of external charges is a diagonal operation in this basis and second, the Yang-Mills theory in this basis can be interpreted as a problem in unitary matrix quantum mechanics which is familiar from models of two-dimensional quantum gravity. Using the well-known techniques for dealing with the large N limit of such models we will develop the equations of motion for the non-Abelian Coulomb gas at finite temperature on a cylinder with arbitrary boundary conditions.

6.1   The Basis of Class Functions

The correlator of a pair of group characters (in representations R and S) or static colour-electric charges on a line at finite temperature (see Equation (3.6) for example) can be

represented in the basis of irreducible representations $|R\rangle$

$$\langle R_f | P[R, S; L_1, L_2, L_3] \| R_i \rangle \tag{6.1}$$

$$= \langle R_f | \; \mathrm{e}^{-e^2\beta L_1 C_2/2} | R_4 \rangle \langle R_4 | \chi_R | R_3 \rangle \langle R_3 | \; \mathrm{e}^{-e^2\beta L_2 C_2/2} | R_2 \rangle \langle R_2 | \chi_S | R_1 \rangle \langle R_1 | \; \mathrm{e}^{-e^2\beta L_3 C_2/2} | R_i \rangle$$

Here we have explicitly inserted identity operators in the basis of irreducible representations $|R_i\rangle$ and suppressed sums over $R_1 \cdots R_4$.

As we have seen before, a convenient basis for the vector space of irreducible representations is given by the group characters χ_R. By their definition, there is a close connection to the elements g of the gauge group

$$\chi_R(g) = \mathrm{Tr}\,_R g \tag{6.2}$$

Moreover, since the trace function is invariant under gauge rotations, we see that (6.2) gives a mapping from class functions of g to the irreducible representations R. We will denote the set of class functions of g as $|g\rangle$ and will use this basis to investigate the non-Abelian Coulomb gas in a more general setting.

The usefulness of this new basis is the fact that the character operator is diagonal in this basis.

$$\chi_R|g\rangle = \chi_R(g)|g\rangle \tag{6.3}$$

The price of having the characters diagonal in this basis is that the quadratic Casimir operator, and hence the propagator, is not diagonal. From our previous calculations, the matrix element of the propagator in the $|g\rangle$ is known

$$\langle g_f | \; \mathrm{e}^{-e^2\beta L C_2/2} | g_i \rangle = \sum_R \chi_R^*(g_f) \; \mathrm{e}^{-e^2\beta L C_2(R)/2} \chi_R(g_i) \tag{6.4}$$

This Hamiltonian form of the propagator, while compact and transparent, is not very useful when considering the correlations of large numbers of characters/ charges. For

this purpose it is much more convenient to use a Lagrangian representation of the gauge propagator in terms of a path integral for a one-dimensional unitary matrix model

$$\langle g_f |\, e^{-e^2 \beta L C_2 / 2} | g_i \rangle = \int_{\mathcal{P}} \mathcal{D}g(x)\, e^{\frac{1}{e^2 \beta} \int_0^L dx \text{Tr}\, |\nabla g(x)|^2} \tag{6.5}$$

where the integration is over all paths $\mathcal{P}$ connecting the configurations g_i and g_f. In the new basis of class functions $|g\rangle$ the pair correlator (6.1) has a very useful form

$$\langle g_f | K[R, S; L1, L_2, L_3] \| g_i \rangle \tag{6.6}$$

$$= \int \prod dg \langle g_f |\, e^{-e^2 \beta L_1 C_2 / 2} | g_4 \rangle \langle g_4 | \chi_R | g_3 \rangle \langle g_3 |\, e^{-e^2 \beta L_2 C_2 / 2} | g_2 \rangle \langle g_2 | \chi_S | g_1 \rangle \langle g_1 |\, e^{-e^2 \beta L_3 C_2 / 4} | g_i \rangle$$

$$= \int_{\mathcal{P}} \mathcal{D}g(x)\, e^{\frac{N}{2\gamma} \int_0^L dx \text{Tr}\, |\nabla g(x)|^2}\, \text{Tr}_R(g(x_1))\, \text{Tr}_S(g(x_2))$$

Where $L = L_1 + L_2 + L_3$ and x_1 and x_2 lie in the interval $(0, L)$. Note that we have introduced an effective coupling constant

$$\gamma = \frac{e^2 N \beta}{2} \tag{6.7}$$

In the large N limit in which we are interested, the gauge coupling e and the inverse temperature β will adjusted so that γ will be of $O(1)$. This will effectively set the energy scale in all subsequent discussions. The result (6.6) for the correlator of a pair of charges can be easily generalized to the correlator for an arbitrary number of charges in arbitrary representations $\{R_i\}$ at positions $\{x_i\}$

$$K[R_i, x_i; L] = \int_{\mathcal{P}} \mathcal{D}g(x)\, e^{-\frac{N}{2\gamma} \int_0^L dx \text{Tr}\, |\nabla g(x)|^2}\, \text{Tr}_{R_1}(g(x_1)) \cdots \text{Tr}_{R_K}(g(x_K)) \tag{6.8}$$

We are interested in the thermodynamics of gases of non-Abelian charges and so it is useful to form the grand canonical partition function. We will do this by summing over all possible configurations of an open system of charges including a fugacity factor, λ_R, for each particle in each representation where

$$\lambda_R = e^{\beta \mu_R} \tag{6.9}$$

Here the parameter μ_R is the chemical potential for charges in the representation R. By tuning μ_R or, equivalently, λ_R we can control the density of R charges in the system. For a gas of charges in a single representation the sum over number and position can be explicitly carried out

$$Z[\gamma; \lambda_R] \tag{6.10}$$
$$= \frac{1}{Z_0} \sum_{k=0}^{\infty} \frac{\lambda_R^k}{k!} \int_{\mathcal{P}} \mathcal{D}g(x) \; e^{-\frac{N}{2\gamma} \int_0^L dx \mathrm{Tr} \, |\nabla g(x)|^2} \int_0^L dx_1 \cdots dx_k \mathrm{Tr}_R(g(x_1)) \cdots \mathrm{Tr}_R(g(x_k))$$
$$= \frac{1}{Z_0} \int_{\mathcal{P}} \mathcal{D}g(x) \; e^{-\int_0^L dx \left[\frac{N}{2\gamma} \mathrm{Tr} \, |\nabla g(x)|^2 - \lambda_R \mathrm{Tr}_R(g(x))\right]}$$

where we have divided by a background factor $Z_0 = Z[\gamma, 0]$ to normalize the ensemble. Again it is straightforward generalize to the case with arbitrarily many different types of charges in the system. If we define a potential for the charges in the system

$$V(g(x)) = - \sum_R \lambda_R \mathrm{Tr}_R g(x) \tag{6.11}$$

then the most general grand canonical partition function for a system of static colour-electric charges at finite temperature interacting via 2D Yang-Mills gauge fields is

$$Z = \int_{\mathcal{P}} \mathcal{D}g(x) \; e^{-S[g]} \tag{6.12}$$

where the action $S[g]$ is given

$$S[g] = \int_0^L dx \left[\frac{N}{2\gamma} \mathrm{Tr} \, |\nabla g(x)|^2 - V(g(x)) \right] \tag{6.13}$$

Here we stress that this action and the equations of motion we will soon derive from it are incomplete unless we implement the boundary conditions which are prescribed by the path ordering $\mathcal{P}$ in (6.5). The effective action (6.13) with vanishing potential was discussed in [29, 30, 31] and was solved explicitly in the limit $N \to \infty$ by Zarembo [111, 112]. The model with adjoint representation charges was solved in [86] and [87] and with both adjoint and fundamental representation charges in [23]. Most generally,

(6.13) defines the **non-Abelian Coulomb gas**, which we will focus on throughout the remainder of this Thesis.

6.2 Matrix Quantum Mechanics

If we re-interpret x as Euclidean time, the partition function that we have derived has the form of a Euclidean space representation of the partition function for matrix quantum mechanics, where the free energy is identical to the ground state energy of the matrix quantum mechanics. We can study the latter model by mapping the problem to real time τ by setting $x = i\tau$. The action in real time is then

$$S_{QM} = \int_0^{-iL} d\tau \left(\frac{N}{2\gamma} \text{Tr} \, |\dot{g}|^2 - V(g) \right) . \tag{6.14}$$

We remark that this action must not be confused with the action (6.13). S_{QM} is the action for a 0+1-dimensional system (quantum mechanics), while (6.13) is the action for Yang Mills theory in 1+1 dimensions. This remark also holds for the Hamiltonian below. In order to avoid confusion, we label the quantum mechanical quantities with the subscript QM.

The canonical momentum conjugate to the group valued position variable g is the Hermitean Lie algebra element

$$\Pi = \frac{iN}{\gamma} g^\dagger \dot{g} , \tag{6.15}$$

and the Hamiltonian is

$$H_{QM} = \frac{\gamma}{2N} \text{Tr} \Pi^2 + V(g) . \tag{6.16}$$

We can expand the canonical momentum as

$$\Pi = \sum_a \Pi^a T^a , \tag{6.17}$$

Then, the components satisfy the Lie algebra

$$\left[\Pi^a, \Pi^b\right] \;=\; i f^{abc} \Pi^c \tag{6.18}$$

$$\left[\Pi^a, g\right] \;=\; g T^a$$

$$\left[\Pi^a, g^\dagger\right] \;=\; -T^a g^\dagger$$

It follows that in the Schrödinger picture the components of the canonical momentum are represented as

$$\Pi^a \;=\; \mathrm{Tr} g T^a \frac{\partial}{\partial g} \;=\; g_{ij} T^a_{jk} \frac{\partial}{\partial g_{ik}} \;. \tag{6.19}$$

Denoting the gauge group Laplacian in components

$$\triangle \;\equiv\; \sum_{a=1}^{N^2} (\Pi^a)^2 \;, \tag{6.20}$$

the Hamiltonian reads

$$H_{QM} \;=\; \frac{\gamma}{4N} \triangle \;+\; V(g) \;. \tag{6.21}$$

As we have mentioned previously, the mapping from irreducible representations requires that the physical states are class functions of g and hence depend only on its eigenvalues. In a basis where g is diagonal,

$$g_{rs}(x) \;=\; \mathrm{e}^{i\alpha_r(x)} \delta_{rs} \tag{6.22}$$

the wavefunctions in the Schrödinger picture are 2π periodic functions of each α_i,

$$\psi_{\mathrm{phys}}(\alpha_1, \ldots, \alpha_N) = \psi_{\mathrm{phys}}(\alpha_1, \ldots, \alpha_i + 2\pi, \alpha_N) \;. \tag{6.23}$$

The physical states must be symmetric functions of α_i. There is a residual gauge invariance [59, 60] under the Weyl group which permutes the eigenvalues. The physical state condition requires that the physical states be symmetric under these permutations.

This invariance is clear from the fact that the trace in (6.2) does not depend on the ordering of the eigenvalues. The normalization integral for the wavefunction is

$$\int [dg]\psi^\dagger(g)\psi(g) = 1 . \tag{6.24}$$

Since the integrand depends only on the eigenvalues of g, It is convenient to write the Haar measure [90] as an integral over eigenvalues of g with a Jacobian factor which is the Vandermonde determinant,

$$\int (\prod_i d\alpha_i)|\tilde{J}(\alpha)|^2\psi^\dagger(\alpha)\psi(\alpha) = 1 . \tag{6.25}$$

where

$$\tilde{J}(\alpha) = \prod_{i<j} 2\sin\frac{1}{2}(\alpha_i - \alpha_j) = \frac{1}{(2i)^{N(N-1)/2}}\frac{J(\alpha)}{\prod_i z_i^{(N-1)/2}} , \tag{6.26}$$

and

$$J(z) = \prod_{i<j}(z_i - z_j) \quad , \quad z_i \equiv e^{i\alpha_i} . \tag{6.27}$$

Since the potential $V(g)$ is also a class function and depends only on the eigenvalues, when operating on the physical states, the Hamiltonian can be expressed in terms of eigenvalues and derivatives by eigenvalues

$$H_{QM} = \frac{\gamma}{4N}\frac{1}{\tilde{J}(\alpha)}\left(\sum_1^N -\frac{\partial^2}{\partial\alpha_i^2} - N(N^2-1)/12\right)\tilde{J}(\alpha) + V(\alpha) , \tag{6.28}$$

The Hamiltonian and inner product have a particularly simple form when we redefine the wavefunction as

$$\tilde{\psi}(\alpha_1,\ldots,\alpha_N) \equiv \tilde{J}(\alpha)\psi(\alpha_1,\ldots,\alpha_N) . \tag{6.29}$$

Since $\tilde{J}$ is antisymmetric, $\tilde{\psi}$ is a completely antisymmetric function of the eigenvalues, which we can think of as the coordinates of fermions. The Hamiltonian is that of an interacting Fermi gas

$$\left\{\frac{\gamma}{4N}\left(\sum_1^N -\frac{\partial^2}{\partial\alpha_i^2} - N(N^2-1)/12\right) + V(\alpha)\right\}\tilde{\psi}(\alpha) = \mathcal{E}\tilde{\psi}(\alpha) . \tag{6.30}$$

This correspondence of a c=1 matrix model with a Fermi gas was first pointed out in [5].

6.3 Large N: Collective Variables

In this section we shall examine the collective field formulation of the large-N limit of the theory discussed in the previous Section [111, 51, 96, 17, 23]. The Hamiltonian obtained in the previous Section reads

$$H_{QM} = \frac{\gamma}{4N} \sum_{a=1}^{N^2} (\Pi^a)^2 + V(g) , \tag{6.31}$$

with (6.11)

$$V(g) \equiv - \sum_R \lambda_R \mathrm{Tr}_R \left(g(\tau) \right) . \tag{6.32}$$

It was shown (compare (6.22),(6.23)) that the wavefunction depends only on the eigenvalues $e^{i\alpha_j}$ of g and thus the density of eigenvalues

$$\rho(\theta, \tau) \equiv \frac{1}{N} \sum_{i=1}^{N} \delta(\theta - \alpha_i(\tau)) , \tag{6.33}$$

completely characterizes the properties of the system. Interpretation of the physics of the system at large N is more convenient when one considers the Fourier transform of the eigenvalue distribution

$$\rho(\theta, \tau) = \frac{1}{2\pi} + \frac{1}{2\pi} \sum_{n \neq 0} c_n(\tau) e^{-in\theta} , \tag{6.34}$$

where we have defined the coefficients c_n,

$$c_n(\tau) \equiv \frac{1}{N} \mathrm{Tr}\, g^n(\tau) \quad , \quad c_{-n}(\tau) = \bar{c}_n(\tau) . \tag{6.35}$$

We now turn our attention to developing the collective field theory formulation of the (thermo-) dynamical problem given by the Hamiltonian (6.31). Since the wavefunction depends only on the eigenvalues of g, we would like a Hamiltonian equivalent to (6.31) but written in terms of the eigenvalue density ρ and a conjugate momentum Π. At large N we will find this Hamiltonian and write equations of motion for ρ and Π. So far we

have not imposed any restriction on the potential $V(g)$, but from now on we assume, that it can be expressed as a functional of the eigenvalue density $\rho(\theta)$.

Using the algebra (6.18) and the definition of the eigenvalue distribution (6.34), the canonical momentum can be seen to operate on the wavefunction as

$$
\begin{aligned}
\Pi^a \psi[\rho] &= \int d\theta \, [\Pi^a, \rho(\theta)] \frac{\delta}{\delta \rho(\theta)} \, \psi[\rho] \\
&= \frac{1}{2\pi N} \int d\theta \sum_K e^{-iK\theta} K \mathrm{Tr}(T^a g^K) \frac{\delta}{\delta \rho(\theta)} \, \psi[\rho] \,,
\end{aligned}
\tag{6.36}
$$

and the Laplacian (6.20) is

$$
\begin{aligned}
\Delta \psi[\rho] &= \left(\frac{1}{4\pi N} \int d\theta \sum_K e^{-iK\theta} |K| \left(\sum_{L=0}^{K} \mathrm{Tr} g^L \mathrm{Tr} g^{K-L} - N \mathrm{Tr} g^K \right) \frac{\delta}{\delta \rho(\theta)} \right. \\
&\quad \left. + \frac{1}{8\pi^2 N^2} \int d\theta d\theta' \sum_{KL} KL e^{-iK\theta - iL\theta'} \mathrm{Tr} g^{K+L} \frac{\delta^2}{\delta \rho(\theta) \delta \rho(\theta')} \right) \psi[\rho] \,,
\end{aligned}
\tag{6.37}
$$

which can be written as

$$
\begin{aligned}
\Delta \psi[\rho] &= -\frac{1}{2N} \int d\theta \rho(\theta) \left\{ \left(\frac{\partial}{\partial \theta} \frac{\delta}{\delta \rho(\theta)} \right)^2 - N^2 \mathcal{P} \int d\theta' \rho(\theta') \cot \left(\frac{\theta - \theta'}{2} \right) \frac{\partial}{\partial \theta} \frac{\delta}{\delta \rho(\theta)} \right\} \psi[\rho] \\
&= -\frac{1}{2N} \int d\theta \rho(\theta) \left(\left(\frac{\partial}{\partial \theta} \frac{\delta}{\delta \rho(\theta)} + \mathcal{V}(\theta) \right)^2 - \mathcal{V}^2(\theta) \right) \psi[\rho] \,,
\end{aligned}
\tag{6.38}
$$

where

$$
\mathcal{V}(\theta) = \frac{N^2}{2} \mathcal{P} \int d\theta' \rho(\theta') \cot \left(\frac{\theta - \theta'}{2} \right) \,.
\tag{6.39}
$$

$\mathcal{P}$ indicates principal value integral.

The transformation of the wavefunction

$$
\psi[\rho] = \tilde{\psi}[\rho] \exp \left(-\frac{N^2}{2} \int d\theta d\theta' \ln \sin \frac{|\theta - \theta'|}{2} \rho(\theta) \rho(\theta') \right) \,,
\tag{6.40}
$$

transforms the derivative in the Schrödinger equation so that it has the form

$$
\left\{ -\frac{\gamma}{8N^2} \int d\theta \rho(\theta) \left\{ \left(\frac{\partial}{\partial \theta} \frac{\delta}{\delta \rho(\theta)} \right)^2 - \mathcal{V}^2(\theta) \right\} + V[\rho] \right\} \tilde{\psi}[\rho] = E \, \tilde{\psi}[\rho] \,.
\tag{6.41}
$$

The second term involving $\mathcal{V}^2$ on the left-hand-side of this equation has a simple form. The variation by $\rho(\theta)$ of this term is given by (up to a factor)

$$\mathcal{W}(\theta) = \left(\mathcal{P}\int d\theta'\rho(\theta')\cot\frac{\theta - \theta'}{2}\right)^2 - 2\mathcal{P}\int d\theta'\rho(\theta')d\theta''\rho(\theta'')\cot\frac{\theta - \theta'}{2}\cot\frac{\theta' - \theta''}{2} .$$

(6.42)

In the integrals, we change variables to the complex variable

$$t = e^{i\theta} \quad, \quad t' = e^{i\theta'} \quad, \quad t'' = e^{i\theta''} ,$$

(6.43)

so that the integrals are over an interval on the unit circle and

$$\mathcal{P}\int d\theta'\rho(\theta')\cot\frac{\theta - \theta'}{2} = \mathcal{P}\int \frac{dt'}{t'}\frac{t + t'}{t - t'}\rho(t') = i + 2\mathcal{P}\int dt'\frac{1}{t - t'}\rho(t') .$$

(6.44)

We obtain

$$\mathcal{W}(t) = 4\left(\mathcal{P}\int dt'\frac{\rho(t')}{t - t'}\right)^2 - 8\mathcal{P}\int dt'dt''\frac{\rho(t')}{t - t'}\frac{\rho(t'')}{t' - t''} + 1 - 4\mathcal{P}\int dt'dt''\frac{\rho(t')\rho(t'')}{t'(t' - t'')} .$$

(6.45)

The last two terms are constants (which because of the normalization condition the density $\rho(t)$ must satisfy, must be irrelevant), and the first term can be found by the following argument. We consider the function

$$G(z) = \int dt\frac{\rho(t)}{t - z} .$$

(6.46)

This function is analytic everywhere in the complex plane except on the arc of the unit circle where the eigenvalue density has support. Obviously

$$G(z) \to 0 \quad \text{as } |z| \to \infty .$$

(6.47)

Also, by letting z approach the support of $\rho(t)$ from outside and inside the unit circle, we obtain $(\varepsilon > 0)$

$$\lim_{\varepsilon \to 0} G(t(1 + \varepsilon)) \equiv G_+(t) = \mathcal{P}\int dt'\frac{\rho(t')}{t' - t} - i\pi\rho(t) ,$$

(6.48)

and

$$\lim_{\varepsilon \to 0} G(t(1-\varepsilon)) \equiv G_-(t) = \mathcal{P} \int dt' \frac{\rho(t')}{t'-t} + i\pi\rho(t) , \tag{6.49}$$

respectively. The function

$$K(z) = G^2(z) - 2 \int dt \frac{\rho(t)}{t-z} \mathcal{P} \int dt' \frac{\rho(t')}{t'-t} , \tag{6.50}$$

is obviously analytic everywhere except eventually on the support of ρ. Using (6.48) and (6.49) one finds that it is continuous across the support of ρ since

$$\lim_{\varepsilon \to 0} K(t(1+\varepsilon)) - \lim_{\varepsilon \to 0} K(t(1-\varepsilon)) = 0 . \tag{6.51}$$

Thus K is an entire function of z. Furthermore, since it vanishes at infinity, and is analytic everywhere

$$K(z) = 0 . \tag{6.52}$$

Then, taking the real part of K on the support of ρ gives

$$\mathcal{W}(t) = 4\pi^2 \rho^2(t) + \text{const.} \quad . \tag{6.53}$$

This is the functional derivative of the term

$$\frac{4\pi^2}{3} \int d\theta \rho^3(\theta) + \text{const.} \quad , \tag{6.54}$$

which is proportional to the second term in the Hamiltonian in the Schrödinger equation (6.55). Thus, the Schrödinger equation has the form

$$\left\{ -\frac{\gamma}{8N^2} \int d\theta \left(\rho(\theta) \left(\frac{\partial}{\partial \theta} \frac{\delta}{\delta\rho(\theta)} \right)^2 - N^4 \frac{\pi^2}{3} \rho^3(\theta) \right) + V[\rho] - \frac{N^2 \gamma}{96} \right\} \tilde{\psi}[\rho] = E \, \tilde{\psi}[\rho] , \tag{6.55}$$

where we have subtracted a constant to maintain equivalence with the original Hamiltonian (6.31).

In the large-N limit the wavefunction is dominated by the eikonal approximation. In this approximation, we make the ansatz

$$\tilde{\psi}[\rho] = \exp\left(iN^2 S[\rho]\right) .$$ (6.56)

The eikonal, S then obeys the equation

$$\frac{H_{QM}[\rho, \Pi]}{N^2} = \frac{\gamma}{8} \int d\theta \left\{ \rho(\theta) \left(\frac{\partial}{\partial\theta} \frac{\delta S}{\delta\rho(\theta)}\right)^2 + \frac{\pi^2}{3}\rho^3(\theta) \right\} + \frac{1}{N^2}V[\rho] - \frac{\gamma}{96} = \frac{E}{N^2} .$$ (6.57)

Here, we have ignored a term which is of subleading order in N^2. We have also assumed that $V[\rho]$ will be of order N^2 and that the natural magnitude of the energy eigenvalue is of order N^2.

To solve this equation for the ground state, we must find its minimum by varying ρ and the canonical momentum

$$\Pi = \delta S/\delta\rho ,$$ (6.58)

subject to the condition that ρ is normalized. This leads to Hamilton's equations of motion for collective field theory

$$\frac{\partial}{\partial\tau}\rho(\tau, \theta) = \frac{\delta H_{QM}/N^2}{\delta\Pi(\tau, \theta)} ,$$ (6.59)

$$\frac{\partial}{\partial\tau}v(\tau, \theta) = -\frac{\partial}{\partial\theta} \frac{\delta H_{QM}/N^2}{\delta\rho(\tau, \theta)} ,$$

where

$$v(\tau, \theta) \equiv \frac{\partial}{\partial\theta} \Pi(\tau, \theta) .$$ (6.60)

Taking the derivative of the second equation with respect to θ eliminates a Lagrange multiplier which must be introduced on order to enforce the normalization condition for ρ. Using (6.57) one finds

$$\frac{\partial}{\partial\tau}\rho + \frac{\gamma}{4}\frac{\partial}{\partial\theta}(\rho v) = 0 ,$$ (6.61)

$$\frac{\partial}{\partial\tau}v + \frac{\gamma}{8}\frac{\partial}{\partial\theta}\left(v^2 + \pi^2\rho^2\right) + \frac{1}{N^2}\frac{\partial}{\partial\theta}\frac{\delta}{\delta\rho}V[\rho] = 0 .$$

It is interesting to note that these are nothing but Euler's equations for a compressible fluid with equation of state $P = \pi^2 \rho^3 / 3$ on a cylinder with coordinates (θ, τ). The first is the equation of continuity while the second one states the conservation of momentum in the system. Here the eigenvalue density ρ is the density distribution of the fluid and v is its velocity. The inclusion of a potential $V(\theta, \tau)$ corresponding to non-Abelian charges is equivalent to subjecting the fluid to an external force which is derived from $V(\theta, \tau)$. It is useful to consider the problem of the non-Abelian Coulomb gas in terms of the more intuitive fluid picture. In the final Chapter we will give a qualitative analysis of the non-Abelian Coulomb gas in a finite box at finite temperature based on this point of view.

Chapter 7

Non-Abelian Coulomb Gases on the Open Line: Infinite N

In Chapter five we discussed the thermodynamics of the adjoint non-Abelian Coulomb gas for finite rank $SU(N)$ and $U(N)$ gauge groups and we found that the vacuum structure of the model has a definite effect of the physics of the system. Here, using the constructions of the previous Chapter, we will consider the somewhat more interesting cases of static charges in different representations in the limit $N \to \infty$. This limit is interesting since the non-Abelian Coulomb gas exhibits non-trivial phase structure which for fundamental and adjoint matter has many qualitative similarities to four dimensional QCD with fundamental representation quarks and adjoint representation gluons.

Using the machinery of matrix models which has been developed over the last decade to address issues in theories of low-dimensional quantum gravity, we will be able to completely solve the large N saddlepoint equations of motion (6.62) for a particular potential in terms of elliptic functions. The exact solution allows one to formulate, and answer, precise questions about the phase structure of the system. In particular we will completely identify the order and position of all phase transitions and develop a new order parameter for discerning the phases of the model. This order parameter, which may be of relevance in more physical situations, is related to the free energy of a single higher fundamental representation being introduced into the system.

The study of systems where interactions are mediated by non-Abelian gauge fields is of direct relevance to the physically interesting case of quantum chromodynamics. At high temperature or density these systems are expected to undergo a phase transition where

the character of the effective degrees of freedom changes dramatically. For example, in the low temperature phase of four-dimensional QCD, quarks and gluons carrying colour charge are not observed but rather confined into composite baryons and mesons. It is expected, and can be shown in numerical simulations on the lattice, that at sufficiently high temperatures this confinement is relaxed and the fundamental degrees of freedom become mobile in a quark-gluon plasma. Quantifying the differences between these phases has been a subject of study for some time now [100, 82, 101, 53, 15, 3, 4] and is adequately understood only in the case of pure Yang-Mills theory without quarks. Here the Polyakov loop operator [80, 92, 81]

$$\langle \mathrm{Tr}\ g(x) \rangle \equiv \langle \mathrm{Tr}\ P\ e^{i \int_0^{1/T} A_0(x,\tau)d\tau} \rangle \tag{7.1}$$

provides an effective order parameter [93, 94] for the transition from the confined to the deconfined phase by testing to see if the symmetries of the action are realized faithfully in the observables. As we will show, it is useful to consider the trace of the group element g in group theoretic terms as defining a group character. Taking g in different irreducible representations will allow us to unambiguously define the strong and weak coupling regimes of a two-dimensional model, even in the presence of fundamental matter.

Even though two dimensional Yang-Mills theory is dynamically trivial, as the rank of the symmetry group is taken to infinity, group structure can drive phase transitions. Transitions of this type were first noted long ago in lattice theory [34] and more recently such phase transitions have been noted in the continuum with the Douglas-Kazakov transition [19] on the sphere and the related transition on the cylinder [38] being prime examples. In these cases, the theory is solved for large rank symmetry group in terms of a single irreducible representation which saturates the evaluation of the partition function in a saddle-point approximation. The phase transition corresponds to a point where the distribution of occupation numbers for the rows of the associated Young table develops

a gap [19, 38] as we have seen in the second Chapter.

In the present case under consideration the situation is somewhat different. The saddle-point is not determined in general by a single irreducible representation of the gauge group but by a linear combination of irreducible representations. This feature is also shared by Abelian and non-Abelian Coulomb gases in two dimensions with $U(N)$ and $SU(N)$ finite rank gauge groups [61, 69, 50] and can be generalized to the case of any compact Lie gauge group. In each of these cases the state vector of the system, Ψ is a class function (compare with Equation 5.7) and therefore can be represented by a linear combination of characters, χ_R of the irreducible representations, R of the gauge group with coefficients a_R that depend on the parameters of the model (temperature, pressure, gauge coupling constant...)

$$\Psi[g(x)] = \sum_R a_R \chi_R(g(x)) \tag{7.2}$$

Consequently we see that there are two different points of view to take in solving these models in two dimensions. One is to find a dominant configuration of the gauge group g_0, as we will do in the next Section by finding the distribution (ρ) of eigenvalues of g_0. The other is to find the dominant linear combination of irreducible representations, Ψ. The main objective of Section 7.4 is to quantify the connection between these two views and use it to characterize the differences between the strong and weak coupling regimes of the non-Abelian Coulomb gas. As we will see, the characters of the gauge group are completely determined by traces of powers of the gauge matrices, Tr g^n. In the two-dimensional model under consideration we will show that the vanishing of particular coefficients a_R provides a convenient way to characterize the different phases of the model. Clearly, if a particular coefficient a_R is vanishing then the system does not have excitations which can effectively screen a charge in irreducible representation R interacting with its conjugate $\bar{R}$. In this way we will be able to identify an order

parameter for the transition from strong to weak coupling and give a physical definition of the confinement-deconfinement transition with fundamental matter present towards the end of this Chapter.

7.1 Static Solutions to the Collective Field Equations

In this Section we will find static solutions to the collective field equations (6.62). The most simple potentials involve only the lowest representations, the fundamental, its conjugate and the adjoint. We shall consider a slight generalization of these and use powers of the lowest representations to include multiple windings of the Polyakov loop operator. Consequently, the potential reads

$$V(g) \equiv -\sum_{n=1}^{\infty} \left(\kappa_n N \mathrm{Tr}(g^n) + \bar{\kappa}_n N \mathrm{Tr}((g^\dagger)^n) + \lambda_n |\mathrm{Tr} g^n|^2 \right) , \qquad (7.3)$$

where we made use of the identity

$$\mathrm{Tr}_{\mathcal{A}} \, g^n = |\mathrm{Tr} \, g^n|^2 \qquad (7.4)$$

to relate the trace in the adjoint ($\mathcal{A}$) representation to the trace in the fundamental representation. The couplings for the fundamental representation charges (and their conjugates) were chosen to scale $\sim N$, to make the potential of order N^2. It should be remarked that this is not the most general form of potential one can consider, but rather one which is motivated by the high temperature effective theory of matter coupled to two-dimensional Yang-Mills theory. In the case of $U(N)$ gauge theory coupled to adjoint and $N \sim N_F$ flavours of fundamental fermionic matter with masses of m_A and m_F, respectively, the high temperature potential can be calculated [99, 35, 58] by integrating out the fermion content to one loop approximation

$$V_{eff}(g) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} [m_A K_1(m_A \beta n) \mathrm{Tr}_A g^n + 2N m_F K_1(m_F \beta n) \mathrm{Re} \mathrm{Tr}_F g^n] \qquad (7.5)$$

Here K_1 is a modified Bessel function. In the limit of large masses such that $m\beta$ is large it can be shown that the effective potential (7.5) reduces to the form of potential which we have considered previously in the transfer matrix problem (5.26) of Chapter five. The two different potentials (7.3) and (7.5), are one and the same if we make the identification of coefficients

$$
\begin{aligned}
\lambda_n &= \frac{2(-1)^{n+1}}{n\pi} m_A K_1(n\beta m_A) \\
\kappa_n &= \bar{\kappa}_n = \frac{2(-1)^{n+1}}{n\pi} m_F K_1(n\beta m_F)
\end{aligned}
\tag{7.6}
$$

The potential (7.3) indeed can be expressed as a functional of the eigenvalue density (6.34). The collective field Hamiltonian (6.57) then reads

$$
\begin{aligned}
\frac{H_{QM}}{N^2} &= \frac{\gamma}{8} \int d\theta \rho(\theta) \left[v^2(\theta) + \frac{\pi^2}{3} \rho^2(\theta) \right] \\
&\quad - \sum_{n=1}^{\infty} \left(\lambda_n \left| \int d\theta\, \rho(\theta) e^{in\theta} \right|^2 + \kappa_n \int d\theta\, \rho(\theta)\, e^{in\theta} + \bar{\kappa}_n \int d\theta\, \rho(\theta)\, e^{-in\theta} \right) - \frac{\gamma}{96} \, .
\end{aligned}
\tag{7.7}
$$

The corresponding collective field equations (6.62) read

$$
\frac{\partial \rho}{\partial x} + \frac{\gamma}{4} \frac{\partial}{\partial \theta}(\rho v) = 0 \, ,
\tag{7.8}
$$

$$
\frac{\partial v}{\partial x} + \frac{\gamma}{8} \frac{\partial v^2}{\partial \theta} - \frac{\pi^2 \gamma}{8} \frac{\partial \rho^2}{\partial \theta} + \frac{\partial}{\partial \theta} \sum_n \left[(\lambda_n c_{-n} + \kappa_n)\, e^{in\theta} + (\lambda_n c_n + \bar{\kappa}_n)\, e^{-in\theta} \right] = 0 \, .
\tag{7.9}
$$

Note that we have performed the change of variables, $\tau \to -ix$ and $v \to iv$ in these equations in order to invert the Wick rotation performed at the beginning of Section 6.2 prior to canonical quantization. Additionally, c_n are the x-dependent Fourier coefficients of ρ as introduced in (6.34). Here it is more convenient to express the c_n in terms of ρ

$$
c_n(x) = \int d\theta\, \rho(\theta, x) e^{in\theta} \, .
\tag{7.10}
$$

In order to solve the non-linear equations (7.9) we must impose boundary conditions which are consistent with the original formulation of the problem in terms of a unitary

matrix model (see (6.13) for example). Here we will be concerned with the case of the non-Abelian Coulomb gas on a circle of circumference L at finite temperature. Since we are imposing periodicity in all quantities under translation through the circumference (L) of the spatial circle, the boundary conditions on the eigenvalue density and velocity are

$$\rho(\theta, x) = \rho(\theta, x + L) \tag{7.11}$$
$$v(\theta, x) = v(\theta, x + L)$$

A simple, but somewhat degenerate way to implement the boundary conditions (7.11) is to consider a static solution of the equations of motion (7.9) where ρ and v are both independent of x. Applying such an ansatz reduces the differential equations of motion to an algebraic problem,

$$\rho v = \epsilon$$
$$\pi^2 \rho^2 - v^2 = E + \frac{8}{\gamma} \sum_n \left[(\lambda_n c_{-n} + \kappa_n) \, e^{in\theta} + (\lambda_n c_n + \bar{\kappa}_n) \, e^{-in\theta} \right] \tag{7.12}$$

here ϵ and E are integration constants. The constant E has a physical interpretation as the Fermi energy of a collection of N fermions [5] in the potential $V[\rho]$ and is fixed by the normalization condition

$$1 = \int d\theta \, \rho(\theta, x) . \tag{7.13}$$

To understand the role of the constant ϵ, consider the limiting case of pure Yang-Mills theory where all λ_n and κ_n vanish. In this case the potential term is absent and the only solution to (7.12), after solving the constraint (7.13)

$$v = 2\pi\epsilon , \qquad \rho = \frac{1}{2\pi} \tag{7.14}$$

Inserting this solution into the collective field Hamiltonian (7.7) one finds,

$$\langle \frac{H_{QM}}{N^2} \rangle = \frac{\pi^2}{2} \gamma \epsilon^2 \tag{7.15}$$

As it was computed in the absence of any matter, the natural interpretation of this result is that $\langle \frac{H_{QM}}{N^2} \rangle$ must correspond to the background energy density of the model much as we have seen in previous investigations of the vacuum structure of two-dimensional Yang-Mills theory in Section 4.2. There, for the case of pure Yang-Mills without matter, the energy density is proportional to the quadratic Casimir eigenvalue for the representation R background colour-electric flux

$$\frac{H_{YM}}{N^2} = \frac{e^2 N \beta}{2} \frac{C_2(R)}{N^3} \tag{7.16}$$

Equating (7.15) and (7.16) and using the definition of γ, (6.7) we find that the constant of integration ϵ is simply related to the energy density of the background flux in the system

$$\epsilon = \sqrt{\frac{2 C_2(R)}{\pi^2 N^3}} \tag{7.17}$$

We will now restrict ourselves to the case where ϵ, and hence v, are vanishing. The equations of motion are solved by

$$\rho_0(\theta) = \begin{cases} 2\sqrt{\frac{2}{\gamma \pi^2}} \sqrt{E + \sum(\lambda_n c_{-n} + \kappa_n) e^{in\theta} + \sum(\lambda_n c_n + \bar{\kappa}_n) e^{-in\theta}} & \text{where } \rho \text{ is real} \\ 0 & \text{otherwise} \end{cases} \tag{7.18}$$

The real support of the function $\rho_0(\theta)$ is the positive support of $\Lambda \equiv E + \sum(\lambda_n c_{-n} + \kappa_n) e^{in\theta} + \sum(\lambda_n c_n + \bar{\kappa}_n) e^{-in\theta}$. The zeros of Λ define the edges of the eigenvalue distribution and when these zeros condense, one has critical behaviour in the observables of the model as in general Hermitean and unitary matrix models [5, 16].

Inserting the static solution (7.18) in (7.7) we obtain the free energy

$$\frac{1}{N^2}\langle H_{QM} \rangle \equiv f = \frac{1}{3}E - \frac{1}{3}\sum_{n=1}^{\infty} \left[\lambda_n |c_n|^2 + 2(\kappa_n c_n + \bar{\kappa}_n c_{-n}) \right] - \frac{\gamma}{96}. \tag{7.19}$$

Note that f is the leading coefficient $(O(N^2))$ of the energy in matrix quantum mechanics,

but in considering the quark gas, it plays the role of the leading coefficient of the energy *density.*

In order to simplify calculations, we will subsequently be interested only in the case of two-dimensional QCD with heavy adjoint and fundamental matter ($m_A \sim m_F \to \infty$) at high temperature. In this limit the coefficients in the effective potential (7.5) are suppressed due to the asymptotic expansion for large x of the first modified Bessel function

$$K_1(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x} \left[1 + \frac{3}{8x} + \cdots\right] \tag{7.20}$$

Consequently, in the high mass limit we can approximate by the leading terms of the potential and consider the case with only $\lambda \equiv \lambda_1$ and $\kappa \equiv \kappa_1$ non-vanishing.

In the form of (7.18) it is evident we need to solve simultaneously for the normalization condition (7.13) and the Fourier coefficient (7.10) in order to have a self-consistent solution of the saddle-point equations. We begin by introducing an auxiliary parameter, μ defined by

$$2\mu(\lambda c_1 + \kappa) = E \tag{7.21}$$

and the integrals over the positive support of $\mu + \cos\theta$,

$$I_n(\mu) = \frac{2}{\pi} \int d\theta \cos n\theta \sqrt{\mu + \cos\theta} \tag{7.22}$$

In terms of μ, the solution of the normalization and moment conditions is given by

$$c_1 = \frac{I_1(\mu)}{I_0(\mu)} \tag{7.23}$$

and

$$\frac{\kappa}{\gamma} = \frac{1}{4I_0(\mu)^2} - \frac{\lambda}{\gamma}\frac{I_1(\mu)}{I_0(\mu)} \tag{7.24}$$

This last relation gives a family of lines in the $(\lambda/\gamma, \kappa/\gamma)$ plane parameterized by μ. As is shown in Figure 7.1, this family overlaps itself for lower densities of fundamental charges,
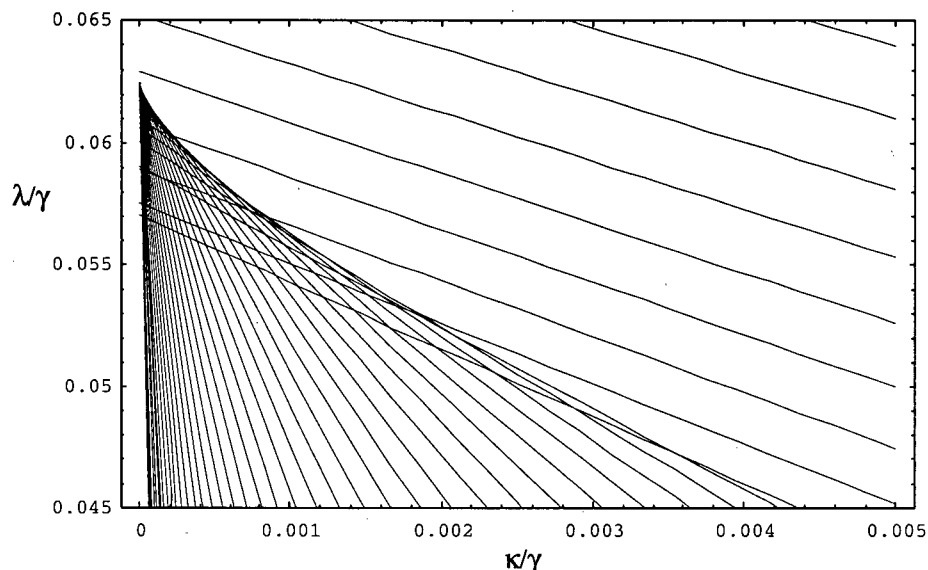
Figure 7.1: Plot of the lines (7.23) for μ ranging from 0.4 (upper right corner) to 75 (line at the extreme left). The region of overlapping lines corresponds to a region of first order phase transition.

κ/γ signaling the fact that there are multiple solutions to the equations of motion in this region of the phase diagram. This family of lines in the κ, λ-plane parameterized by μ represent a necessary condition which a solution of the normalization and moment equations must obey. Clearly the behaviour of the model depends crucially on $I_0(\mu)$ and $I_1(\mu)$ and in the next Section we will investigate the detailed structure of these integrals.

7.2 Analysis of the Parametric Integrals $I_n(\mu)$

Next we analyze the properties of the parametric integrals $I_n(\mu)$ originally defined in (7.22). As noted previously, the range of integration is over the positive support of the quantity $\mu + \cos\theta$. Hence for $\mu \geq 1$, the range of integration is $[0, \pi]$, while for $\mu < 1$, the range of integration is truncated to $[0, \arccos(-\mu)]$. All $I_n(\mu)$ can be expressed in terms

of complete and incomplete elliptic integrals [89]. Of immediate interest are

$$I_0(\mu) = \begin{cases} \frac{8\sqrt{\mu+1}}{\pi} E\left(\sqrt{\frac{2}{\mu+1}}\right) & \text{for } \mu \geq 1 \\ \frac{8\sqrt{\mu+1}}{\pi} E\left(\sqrt{\frac{2}{\mu+1}}; \arcsin\left(\sqrt{\frac{\mu+1}{2}}\right)\right) & \text{for } \mu \leq 1 \end{cases} . \tag{7.25}$$

$$I_1(\mu) = \begin{cases} \frac{8\sqrt{\mu+1}}{3\pi}\left[\mu E\left(\sqrt{\frac{2}{\mu+1}}\right) + (1-\mu)K\left(\sqrt{\frac{2}{\mu+1}}\right)\right] & \text{for } \mu \geq 1 \\ \frac{8\sqrt{\mu+1}}{3\pi}\left[\mu E\left(\sqrt{\frac{2}{\mu+1}}; \arcsin\left(\sqrt{\frac{\mu+1}{2}}\right)\right) \\ \qquad + (1-\mu)K\left(\sqrt{\frac{2}{\mu+1}}; \arcsin\left(\sqrt{\frac{\mu+1}{2}}\right)\right)\right] & \text{for } \mu \leq 1 \end{cases} . \tag{7.26}$$

We remark that $I_0(\mu)$ and $I_1(\mu)$ have first derivatives which diverge logarithmically as μ approaches 1. This fact can be seen from (7.25), (7.26) and known formulas for the derivatives of elliptic integrals [89] and, in fact is true for all $I_n(\mu)$. Naturally this non-analytic behaviour manifests itself as critical behaviour in the free energy of interacting non-Abelian electric charges. At $\mu = 1$ the parametric integrals can easily be evaluated

$$I_0 \equiv I_0(1) = \frac{8\sqrt{2}}{\pi} \quad , \quad I_1 \equiv I_1(1) = \frac{8\sqrt{2}}{3\pi} \tag{7.27}$$

Substituting into the solution of the equations of motion (7.24) we obtain the necessary condition for the critical ($\mu = 1$) values of λ and κ

$$\frac{\kappa_c}{\gamma} = \frac{\pi^2}{512} - \frac{1}{3}\frac{\lambda_c}{\gamma} . \tag{7.28}$$

Having identified a line in the phase space where we expect critical behaviour, we will now proceed to establish the details of this critical behaviour. Following [42, 91, 54] we begin by expanding about $\mu = 1$

$$\mu = 1 + \varepsilon, \varepsilon > 0 \quad , \quad I_0(1 + \varepsilon) = I_0 + \delta I_0 \quad , \quad I_1(1 + \varepsilon) = I_1 + \delta I_1 , \tag{7.29}$$

Due to the piecewise definition of the integrals (7.25) and (7.26) one has to distinguish the cases $\mu > 1$ and $\mu < 1$. In order to keep the formulas simple, we explicitly analyze only the case $\mu > 1$. The case $\mu < 1$ can be treated along the same lines and we denote the corresponding results in the end.

Using the well known formulas for the expansion of elliptic integrals [89] to obtain

$$\frac{\delta I_0}{I_0} = -\frac{\varepsilon}{8}\ln(\varepsilon) + \frac{\varepsilon}{8}[5\ln(2) + 1] + o(\varepsilon) . \tag{7.30}$$

The variation δI_1 can be related to δI_0 by using

$$I_1(\mu) + \mu I_0(\mu) = \frac{2}{\pi} \int [\mu + \cos\theta]^{3/2} d\theta . \tag{7.31}$$

Expanding the left hand side using (7.29) and the right hand side using Taylor expansion, one obtains

$$\frac{\delta I_1}{I_1} = -3\frac{\delta I_0}{I_0} + \frac{3}{2}\varepsilon . \tag{7.32}$$

Variations in the parameter μ correspond to variations about the critical line (7.28). We will analyze the variation of κ around κ_c while keeping λ fixed at its critical value

$$\kappa = \kappa_c + \delta\kappa \quad , \quad \lambda = \lambda_c . \tag{7.33}$$

The expansion now consists of two steps. We first expand the necessary condition (7.24) at $\mu = 1$ to obtain the relation between the variation $\delta\kappa$ and ε. In the second step we expand

$$\frac{df}{d\kappa} = -2\frac{I_1(\mu)}{I_0(\mu)} \tag{7.34}$$

at $\mu = 1$ and use the result of step one to express the variation of $df/d\kappa$ in terms of $\delta\kappa$. The latter result can then be used to analyze eventual singular behaviour of higher derivatives of the free energy.

Expanding the necessary condition (7.24) and using (7.28) we obtain for the variation of κ to lowest order

$$\frac{\delta\kappa}{\gamma} = -\frac{1}{2}\frac{\delta I_0}{(I_0)^3} - \frac{\lambda_c}{\gamma}\frac{I_1}{I_0}\left[\frac{\delta I_1}{I_1} - \frac{\delta I_0}{I_0}\right] = \left[\frac{-\pi^2}{256} + \frac{4}{3}\frac{\lambda_c}{\gamma}\right]\frac{\delta I_0}{I_0} - \frac{\lambda_c}{\gamma}\frac{\varepsilon}{2} , \tag{7.35}$$

where in the last step we made use of the relation (7.32) between the variations δI_0 and δI_1 and inserted the explicit results (7.27) for $I_0 = I_0(1)$ and $I_1 = I_1(1)$. Using the result

(7.30) for $\delta I_0/I_0$ to lowest order we obtain

$$\delta\kappa = -\varepsilon\ln(\varepsilon)\,\sigma\,, \tag{7.36}$$

where we introduced the abbreviation

$$\sigma \equiv \frac{1}{8}\left[-\frac{\gamma\pi^2}{256} + \frac{4\lambda_c}{3}\right]. \tag{7.37}$$

Inverting equation (7.35) (again taking into account only the leading order) gives

$$\varepsilon = -\sigma^{-1}\delta\kappa\left[\ln\left(\sigma^{-1}\delta\kappa\right)\right]^{-1}. \tag{7.38}$$

This equation is the relation between the variation $\delta\kappa$ and ε which is implied by the necessary condition (7.24). In the final step we expand the derivative of the free energy (7.34) at $\mu = 1$ and use the result (7.38) to obtain the variation of the derivative in terms of $\delta\kappa$

$$\frac{df}{d\kappa} = -2\frac{I_1}{I_0}\left[1 + \frac{\delta I_1}{I_1} - \frac{\delta I_0}{I_0}\right] = -\frac{2}{3} - \frac{1}{3}\varepsilon\ln(\varepsilon) - \varepsilon. \tag{7.39}$$

Using (7.38) we obtain

$$\frac{df}{d\kappa} = -\frac{2}{3} + \frac{1}{3}\sigma^{-1}\delta\kappa + \sigma^{-1}\delta\kappa\left[\ln\left(\sigma^{-1}\delta\kappa\right)\right]^{-1}. \tag{7.40}$$

The case $\mu < 1$ with expansion $\mu = 1 - \varepsilon$, $\varepsilon > 0$ changes only the sign of the argument of the logarithm. Differentiating the last result with respect to $\delta\kappa$ establishes the singular behaviour of the third derivative of the free energy with respect to κ. Thus we find a third order phase transition for $\mu = 1$. The critical line is a straight line given by (7.28).

It is important to notice, that at (see Equation (7.35))

$$\frac{\lambda_c}{\gamma} = \frac{3\pi^2}{1024}, \tag{7.41}$$

the leading term in the expression for $\delta\kappa$ vanishes. Equation (7.35) is reduced to the simpler relation

$$\frac{\delta\kappa}{\gamma} = -\varepsilon\frac{\lambda_c}{2\gamma}. \tag{7.42}$$

At this point the expansion of $df/d\kappa$ gives

$$\frac{df}{d\kappa} = -\frac{2}{3} - \frac{1}{3}\varepsilon \ln(\varepsilon) - \varepsilon = -\frac{2}{3} + \frac{2}{3\lambda_c}\delta\kappa \ln\left(-\frac{2}{\lambda_c}\delta\kappa\right) + \frac{2}{\lambda^2}\delta\kappa . \qquad (7.43)$$

Again the case $\mu < 1$ differs only by the sign of the argument of the logarithm. Differentiation with respect to $\delta\kappa$ shows, that the phase transition has turned to second order at that point. Using (7.28) one can compute also the κ/γ coordinate of the second order point giving $\kappa/\gamma = \pi^2/1024, \lambda/\gamma = 3\pi^2/1024$. In fact the more global analysis in the next Section will show, that the third order line terminates at the second order point $\kappa/\gamma = \pi^2/1024, \lambda/\gamma = 3\pi^2/1024$, and continues as a first order line.

7.3 Regime of the First Order Phase Transition

As pointed out at the end of the previous Section, at the point $\kappa/\gamma = \pi^2/1024, \lambda/\gamma = 3\pi^2/1024$, the third order transition along the $\mu = 1$ line (7.28) changes to second order. This unusual behaviour requires further investigation which we will carry out in this Section. To begin, a graphical analysis of the phase diagram is most useful and in Figure 7.1 we plot a number of representatives of the family of lines (7.24) for a range of values of μ. It is clear that in most of the κ, λ-plane points are in a one-to-one correspondence with values of the parameter μ. This correspondence breaks down though in a small region near the λ/γ axis between $\lambda/\gamma = 0.05$ and $\lambda/\gamma = 0.0625$. Due to the behaviour of the slope and intercept in the linear equation (7.28) lines begin to overlap for increasing μ starting at $\mu \sim 1$ and continuing as $\mu \to \infty$. In this overlap region the phase diagram is folded at the vertex $\kappa/\gamma = \pi^2/1024, \lambda/\gamma = 3\pi^2/1024$, and each point falls on three different lines of constant μ. Consequently the system simultaneously admits three configurations with different free energies in this region of the phase space. This circumstance allows for a first order phase transition to develop along a line where the free energies of the different phases are equal.
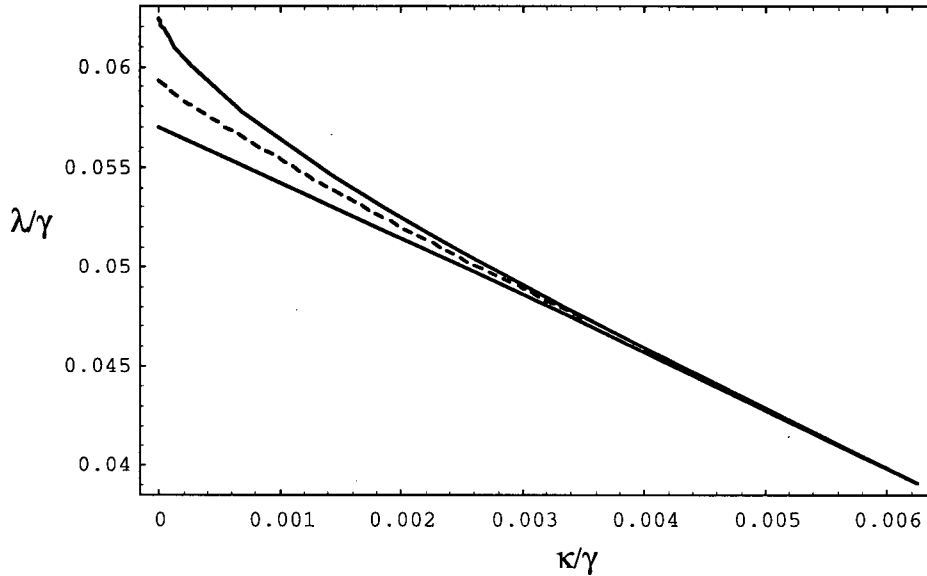
Figure 7.2: Plot of the boundary of the multiple phase region. The boundary (solid curve) is given by a caustic of lines in the one-parameter family as determined by Equation 7.42. The dotted curve shows the numerically determined first order line.

The edges of the triangular first order region in Figure 7.1 is given by a caustic of lines from the one parameter family (7.24). The boundary is defined by the curve where the family of curves is stationary with respect to μ. This condition can be used with (7.24) to give a definition of the boundary caustic. The stationary condition can be solved with the parametric result

$$\frac{\kappa}{\gamma} = \frac{1}{4I_0^2(\mu)} \frac{I_0'(\mu)I_1(\mu) + I_1'(\mu)I_0(\mu)}{I_1'(\mu)I_0(\mu) - I_0'(\mu)I_1(\mu)} . \tag{7.44}$$

As can be seen in Figure 7.2, the curve given by (7.44) intersects the λ/γ axis at two points: 0.057024 ($\mu = 0.95324$) and 1/16 ($\mu = \infty$) and reaches a singular maximum in the κ/γ direction for $\mu = 1$ at the point $\kappa/\gamma = \pi^2/1024$. The end of this region of first order transitions agrees with the position of the second order transition point which was determined by the analysis of critical behaviour in the previous Section.

Once one has determined the region where there are multiple solutions of the equations

of motion the next issue to address is that of the position of the line of first order phase transitions where different solutions have the same free energy. In order to determine the line of first order phase transitions we begin by introducing the parameters μ and ν. Since the phase diagram is folded and allows for multiple solutions of the equations of motion, there exist points $(\kappa/\gamma, \lambda/\gamma)$ in the phase diagram for which distinct μ and ν are solutions. Equating (7.24) for each of μ and ν we have, for given κ/γ

$$\frac{I_0(\nu)I_1(\nu) - I_0(\mu)I_1(\mu)}{4\,I_0(\mu)I_0(\nu)} = \frac{\kappa}{\gamma}\Big[I_1(\nu)I_0(\mu) - I_0(\nu)I_1(\mu)\Big] , \qquad (7.45)$$

The line of first order transitions will occur for solutions of (7.45) when the free energy is equal in each branch of the phase diagram. Eliminating λ using the necessary condition (7.24) we can rewrite the free energy (7.19)) in terms of the parameters μ, κ and γ.

$$f = \frac{\gamma}{12 I_0(\mu)^2}\left[2\mu - \frac{I_1(\mu)}{I_0(\mu)}\right] - \kappa\frac{I_1(\mu)}{I_0(\mu)} - \frac{\gamma}{96} , \qquad (7.46)$$

Equating the free energy (7.46) in each branch we have the condition

$$\frac{1}{6}\left(\frac{\mu}{I_0^2(\mu)} - \frac{\nu}{I_0^2(\nu)}\right) - \frac{1}{12}\left(\frac{I_1(\mu)}{I_0^3(\mu)} - \frac{I_1(\nu)}{I_0^3(\nu)}\right) = \frac{\kappa}{\gamma}\frac{I_0(\nu)I_1(\mu) - I_1(\nu)I_0(\mu)}{I_0(\mu)I_0(\nu)} . \qquad (7.47)$$

Hence the line of first order phase transitions in the non-Abelian Coulomb gas with adjoint and fundamental representation charges is given by the simultaneous solution of (7.45) and (7.47).

Unfortunately, these equations are analytically intractable. Again we turn to a graphical analysis to gain further insight. In Figure 7.3 we plot the free energy of the system as a function of λ/γ for different values of fixed κ/γ. From here it is easy to see a number of features of the region of first order transitions. Increasing μ traverses these curves in a clock-wise rotation so that free energy increases for small values of μ, intersecting the nearly horizontal large μ free energy. This intersection point is a graphical demonstration of the first order transition which occurs here as the model jumps from 'weak'
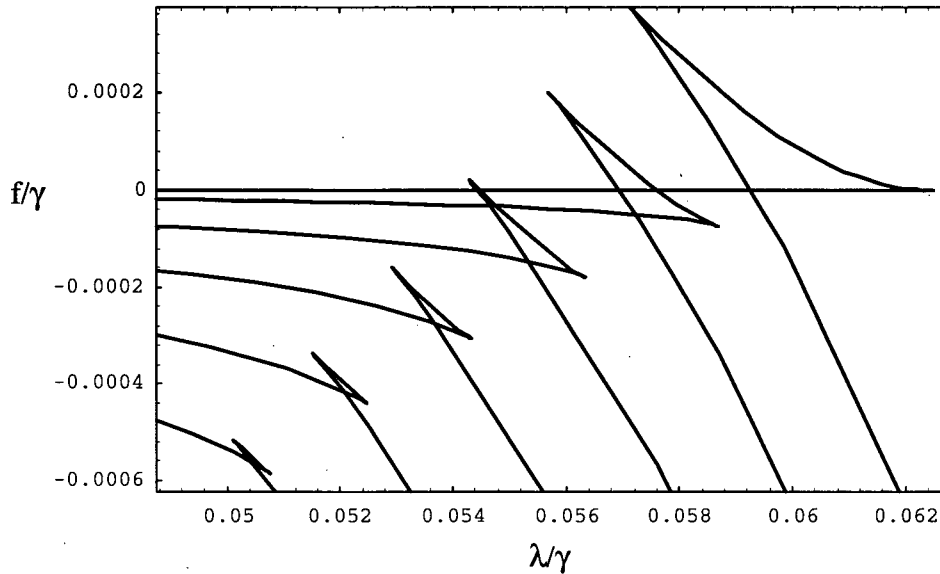
Figure 7.3: Free energy f/γ as a function of λ/γ in the region of first order phase transitions. Each curve is plotted for fixed κ/γ which from right to left is $\kappa/\gamma = 0$, 0.0005, 0.001, 0.0015, 0.002, 0.0025

($\mu < 1$) to 'strong' coupling (μ large). Each phase continues to exist after the transition point and may be reached by an adiabatic process until ending in cusps which mark the boundaries of the first order region in the λ/γ axis. It is interesting to note that there is an energetically infeasible intermediate 'medium coupling' phase which connects the weak and strong phases. Hence, for fixed κ/γ there exist three distinct configurations of the system for given λ/γ in the region of first order transitions. In the pure adjoint case with $\kappa = 0$ the equations can be solved numerically relatively easily and one finds the first order phase transition occurs at $\lambda/\gamma \sim 0.05925$ which is in agreement with the calculations of [87].

The true utility of the parameter μ in our solution has become apparent: first it characterizes the general structure of the phase diagram (Figure 7.4) where the 'strong coupling' regime is the region with $\mu > 1$ and the 'weak coupling' regime has $\mu < 1$. As well, and of more importance for our analysis, we find that the expectation values of

traces of powers of the group element g are given as a function of the single parameter μ

$$\langle \mathrm{Tr}\ g^n/N \rangle = c_n = \frac{I_n(\mu)}{I_0(\mu)} \tag{7.48}$$

Consequently, it makes sense for our purposes to re-define the eigenvalue distribution in terms of μ

$$\rho_0(\theta, \mu) = \frac{2}{\pi I_0(\mu)} \sqrt{\mu + \cos\theta} \tag{7.49}$$

In the remaining Sections of this Chapter, we will use this definition and its connection to the dominant configuration of the gauge element to analyze the phase diagram in terms of group theory.

The physics of this first order deconfinement phase transition certainly deserves some comment at this point. It is an example of a Hagedorn transition [43, 22, 47], an important feature of a gauge theory which may soon be tested experimentally through the production of quark-gluon plasmas. The Hagedorn phase transition occurs in a system with string-like excitations [22] due to the number of degrees of freedom available to absorb energy as one increases the temperature of the system. In a system where the number of bound states grows exponentially with the energy of the bound state and the mass spectrum grows exponentially with particle mass there are an infinite number of resonances that can be described by thermodynamics. In the limit of high energy and temperature these resonances consist of resonances and the hadron spectrum becomes so broad that it is impossible to distinguish nearby mass states. As the temperature is increased further the number and density of states accessible to the system increases exponentially and so it is thermodynamically favourable to produce more particle states instead of increasing the temperature. This leads to an upper limit for the temperature of the system [47]. When this temperature is reached it is generally believed that there should take place a phase transition to a phase where the interactions are weaker and the exponential dependence of the spectra are lost. In Yang-Mills theories and QCD
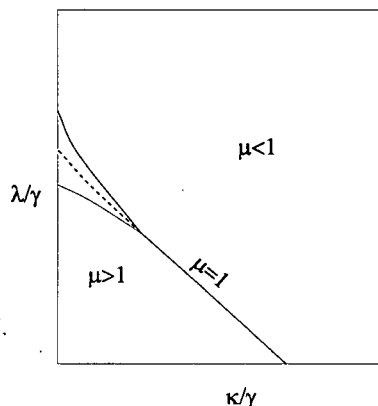
Figure 7.4: Schematic picture of the phase diagram for the adjoint and fundamental representation non-Abelian Coulomb gas. The dotted curve marks the first order part of the critical line. The solid curves above and below it are the boundaries of the area with two possible phases. They join at a point which shows second order behaviour. For larger κ/γ, we find a third order line ($\mu = 1$) marked by a solid line.

it is believed that this is the mechanism for the phase transition from a confining low energy theory to the screening quark-gluon plasma of high temperatures and densities. It appears that the non-Abelian Coulomb gas we have considered here is the only explicitly solvable example of the such behaviour in the presence of matter.

7.4 Interpretation of the Phase Diagram: An Order Parameter for the Generalized Deconfinement Transition

As is known, in the case of pure Yang-Mills theory, the realization of the center symmetry of the gauge group governs confinement [80, 92]. The Polyakov loop operator $\text{Tr}\, g(x) \sim \chi_F(g)$, as we have seen in the second Chapter, is related to the free energy $-T \log \langle \text{Tr} g(x) \, \text{Tr} g^\dagger(0) \rangle$ of a conjugate pair of static, external fundamental charges separated by a distance x. Consequently it can serve as an order parameter [93, 94] to test confinement. Since $\text{Tr}\, g(x)$ transforms under the center as $\text{Tr}\, g(x) \rightarrow z\, \text{Tr}\, g(x)$, the expectation value of the Polyakov loop operator must average to zero if the center

symmetry is preserved. Physically this suggests that an infinite amount of energy is required to introduce a single fundamental test charge into the system. The presence of a gas of fundamental charges ($\kappa \neq 0$) changes this situation though by explicitly breaking the center symmetry of the action. Consequently we lose the Polyakov loop operator as an order parameter for phase transitions in the system. In this Section we introduce a suitable generalization of the Polyakov loop operator which will allow us to identify a new order parameter.

As seen in previous Sections, the solution of the non-Abelian Coulomb gas with adjoint and fundamental representation charges is completely characterized by a Fourier sum of the traces $c_n = \langle \text{Tr } g^n / N \rangle$ - the higher winding Polyakov loops. As noted in [23] the character of these traces changes between the strong and weak coupling regimes. In particular, in the strong coupling ($\mu > 1$) phase, c_n is damped exponentially with n while in the weak coupling ($\mu < 1$) phase the damping follows a power law behaviour. This behaviour is our first hint that higher winding Polyakov loops (and higher irreducible representations) carry valuable information about the system.

7.5 Conversion of Matrix Variables to Irreducible Representations

Since the matrix g is an element of the special unitary group, its trace in an irreducible representation, R defines the group character for that representation

$$\chi_R(g) \equiv \text{Tr }_R g \tag{7.50}$$

For the N dimensional fundamental representation of $SU(N)$, F, the group character is just the Polyakov loop operator described above since we are considering group elements to be taken in the lowest fundamental representation

$$\chi_F(g) = \text{Tr } g \tag{7.51}$$

Further simple examples are the symmetric (S) and anti-symmetric (A) combinations of a pair of fundamental representations, where we have

$$\chi_S(g) = \frac{1}{2}[(\text{Tr } g)^2 + \text{Tr } g^2] \;\;,\;\; \chi_A(g) = \frac{1}{2}[(\text{Tr } g)^2 - \text{Tr } g^2] \tag{7.52}$$

A general relation between characters and the group elements is given by the Weyl formula (A.49) but is not necessary for the following. A complete discussion can be found in standard references (see [108, 113] for example).

The main idea is that the eigenvalues of the group matrices, which are the only relevant dynamical variables, are completely determined by the N quantities $\{\text{Tr } g^n\}$, $n = 1 \dots N$. In turn these traces form an algebraic basis equivalent to the characters of the N fundamental (completely anti-symmetric) irreducible representations of $SU(N)$ (including the trivial representation). Here we will explicitly demonstrate the relationship between the basis of traces and the basis of group characters. Ultimately it is the group theoretic variables which we will use to characterize the phases of the model with heavy adjoint and fundamental representation matter interacting via two-dimensional Yang-Mills fields.

The standard basis for general functions (of finite degree) of the eigenvalues of a matrix is the set of elementary symmetric functions $\{a_r\}$. In terms of the eigenvalues $\lambda_j = \text{e}^{i\theta_j}$ of the group element g they are given by

$$a_1 = \sum_j \lambda_j \tag{7.53}$$

$$a_2 = \sum_{j<k} \lambda_j \lambda_k$$

$$a_3 = \sum_{j<k<l} \lambda_j \lambda_k \lambda_l$$

$$\vdots$$

$$a_N = \prod \lambda_j = \det g = 1 \tag{7.54}$$

with $a_r \equiv 0$ for $r > N$. The relationship of the symmetric functions $\{a_r\}$ to the traces of the group elements, $S_n = \text{Tr } g^n$, is given [63] by the determinant

$$a_k = \frac{1}{k!} \begin{vmatrix} S_1 & 1 & 0 & \cdots & & & \\ S_2 & S_1 & 2 & 0 & \cdots & & \\ S_3 & S_2 & S_1 & 3 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & & 0 \\ S_{k-1} & S_{k-2} & S_{k-3} & \cdots & S_2 & S_1 & k-1 \\ S_k & S_{k-1} & S_{k-2} & \cdots & S_3 & S_2 & S_1 \end{vmatrix} \qquad (7.55)$$

Most importantly, it can be shown that the elementary symmetric functions are nothing more than the characters of the fundamental representations for the unitary group [63, 44]. That is, for the fundamental representation which is the anti-symmetric combination of k, N dimensional representations, $\chi_k(g) = a_k$.

The determinant (7.55) can be evaluated [68] in terms of a multinomial expansion most compactly stated in terms of a generating function

$$\chi_k(g) = \frac{(-1)^k}{k!} \frac{d^k}{dz^k} \exp\left[-\sum_{n=1}^{\infty} \frac{\text{Tr } g^n}{n} z^n\right]\bigg|_{z=0} \qquad (7.56)$$

For our purposes, it is useful to convert to a contour integral about the origin.

$$\chi_k(g) = \frac{(-1)^k}{2\pi i} \oint \frac{dz}{z^{k+1}} \exp\left[-\sum_{n=1}^{\infty} \frac{\text{Tr } g^n}{n} z^n\right] \qquad (7.57)$$

These last two expressions explicitly demonstrate the relationship between the group element g and the kth fundamental representation of the gauge group and are completely general results.

With these relations we see that there is a direct connection between the gauge group element g and the irreducible (fundamental) representations of the gauge group. It is natural to ask what configuration of irreducible representations correspond to the dominant g_0 which is described by the eigenvalue distribution ρ_0 (7.49). This is equivalent

to evaluating the expectation $\langle\chi_k(g)\rangle$ in the background of the non-Abelian gas. In principle this involves calculating expectations of the form $\langle\text{Tr }g^{n_1}\cdots\text{Tr }g^{n_r}\rangle$ but because of the factorization of gauge invariant objects in the limit $N\to\infty$, this reduces to a product of expectations, $\langle\text{Tr }g^{n_1}\rangle\cdots\langle\text{Tr }g^{n_r}\rangle$. Consequently $\langle\chi_k(g)\rangle$ is determined by replacing $\text{Tr }g^n$ by its expectation value in (7.57). Of course expectation values of the group element traces are intimately related to the eigenvalue density $\rho(\theta,\mu)$ (see (6.34) and (7.49)) hence, after performing an infinite sum, we obtain

$$\langle\chi_\alpha\rangle[\rho(\theta,\mu)]\equiv\frac{(-1)^{\alpha N}}{2\pi i}\oint\frac{dz}{z}\exp\left[\frac{N}{2}\int d\theta\rho(\theta,\mu)\log\left(\frac{1+z^2-2z\cos\theta}{z^{2\alpha}}\right)\right]\qquad(7.58)$$

Note that we have defined a new real parameter $\alpha=k/N$ on the unit interval that effectively labels the fundamental representations in the large N limit. Of course (7.58) now depends on a continuous variable and is of a slightly different functional form than the discrete case $\langle\chi_k\rangle$. In the remainder of this discussion we will consider only the character parameterized by α as defined in (7.58).

7.6 Calculating the Expectation Values of Fundamental Representations

In this Section we will concentrate on calculating $\langle\chi_\alpha\rangle$ with eigenvalue density (7.49) for the non-Abelian Coulomb gas. This calculation will give a clear picture of the group theoretic excitations present in different regions of the phase diagram and consequently will allow us to define an order parameter for the deconfinement transition, even in the presence of fundamental matter.

Since explicit evaluation of (7.58) is difficult, we begin with some special limiting cases. As $\mu\to-1$ the support of the eigenvalue distribution (7.49) vanishes at $\theta=0$. The distribution does not vanish though as it retains unit normalization and effectively becomes a delta function, $\delta(\theta)$. Consequently we find the gauge matrix g is just the

identity at $\mu = -1$, hence

$$\langle \chi_\alpha \rangle = \lim_{N \to \infty} \begin{pmatrix} N \\ \alpha N \end{pmatrix} = 2^N \sqrt{\frac{2}{N\pi}} \, e^{-2N(\alpha - 1/2)^2} \qquad (7.59)$$

In this limit we find that the distribution of characters is symmetric about $\alpha = 1/2$ as one would expect in a system where the total colour charge is vanishing. As well in this limit $\langle \chi_\alpha \rangle$ is non-vanishing and all fundamental representations are present in the large N background solution of the model. As we will see, this result is generic in the weak coupling phase $\mu < 1$.

In the opposite limit, as $\mu \to \infty$, it can be shown that the eigenvalue distribution (7.49) approaches a constant value $\rho = 1/2\pi$ with the eigenvalues of the group element g becoming uniformly distributed on the unit circle. Since expectation values of the traces of powers of the gauge matrix are essentially Fourier transforms of the eigenvalue distribution, it is easy to see that $\langle \text{Tr } g^n \rangle \to 0$ in this limit and

$$\langle \chi_\alpha \rangle \to \delta_{0,\alpha} \qquad (7.60)$$

This limit corresponds to the extreme strong coupling phase of the model where the Polyakov loop operator ($\langle \text{Tr } g \rangle \sim \langle \chi_{1/N} \rangle$) has vanishing expectation value and the standard analysis would point to a phase where colour charges are strictly confined into hadron-like structures.

In general the integral (7.58) can be evaluated by saddle-point methods in the large N limit in which we are interested. The relevant action in this limit is

$$S(\alpha, \mu, z) = \int d\theta \rho(\theta, \mu) \log \left(\frac{1 + z^2 - 2z \cos \theta}{z^{2\alpha}} \right) \qquad (7.61)$$

Solving the stationarity condition, $dS/dz|_{z_0} = 0$, for α in terms of z_0 we find the saddle-point condition for the large N behaviour of the integral (7.58) is given by the relationship

$$\alpha = \int d\theta \rho(\theta, \mu) \frac{z_0(z_0 - \cos \theta)}{1 + z_0^2 - 2z_0 \cos \theta} \qquad (7.62)$$
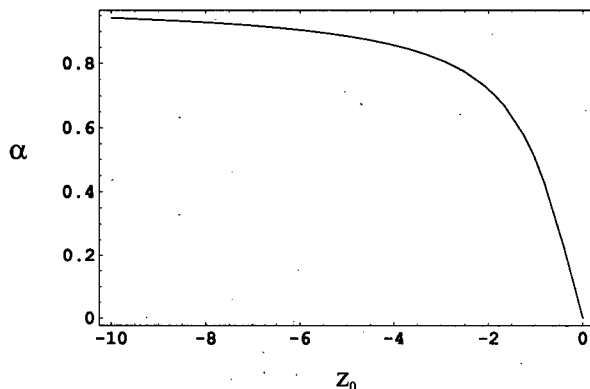
Figure 7.5: Plot of the solutions of the saddle-point relation (7.60) for $\mu = 0.5$.

Since α is a real parameter restricted to the unit interval $[0, 1]$ it can be shown that the saddle-point value of the parameter z_0 is real. Further, for $z_0 > 1$ and $0 < z_0 < 1$ Equation 7.62 returns values of $\alpha > 1$ and $\alpha < 0$, respectively. Consequently we need only consider real, negative values of the parameter z_0.

We now turn to an examination of the saddle-point approximation of (7.58) for different regions of the phase diagram of the model at hand beginning with the weak coupling phase, $\mu < 1$. In this case the support of the eigenvalue distribution (7.49) is bounded away from $\theta = \pm\pi$ and hence the denominator in (7.62) is non-singular for all values of z_0. Consequently, in this regime α varies smoothly and monotonically with z_0 and the relation (7.62) can in principle be inverted to obtain $z_0(\alpha)$. With this information, the large N asymptotic form of the expectation value of the characters $\langle\chi_\alpha\rangle$ can be determined by standard saddle-point methods. In Figure 7.5 we show a numerically calculated example of α as a function of z_0 for $\mu = 0.5$. For this same case we show a schematic diagram of the magnitude of the expectation value $|\langle\chi_\alpha\rangle|$ as a function of α in Figure 7.6. In particular we see that the system has excitations in all irreducible representations.

For $\mu > 1$ the situation is somewhat different. Now the support of the eigenvalue distribution (7.49) is the full interval $\theta \in [-\pi, \pi]$, and the denominator of (7.62) causes
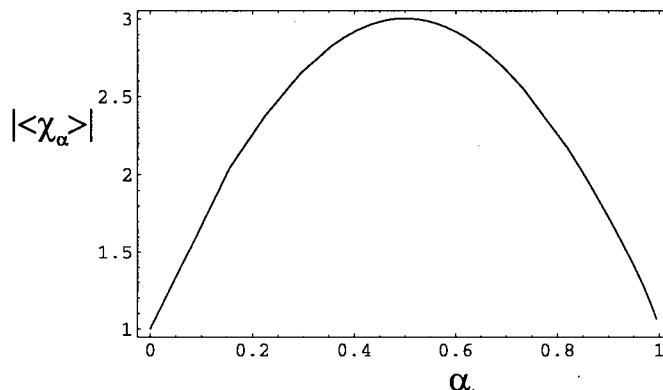
Figure 7.6: Schematic diagram of $|\langle \chi_\alpha \rangle|$ vs. α for $\mu = 0.5$. Note that all fundamental representations have non-vanishing expectation value.

non-analytic behaviour to appear. As one increases μ through unity the saddle-point relation for α shows this non-analytic behaviour as a discontinuity at $z_0 = -1$ (see Figure 7.7). The result is that an open interval of α values centered on $\alpha = 1/2$ are mapped into this discontinuity when the saddle-point relation (7.62) is inverted. Since this discontinuity occurs in the saddle-point relation, it is not surprising to find that the curvature associated with the Gaussian integration of the saddle-point approximation is divergent, effectively forcing the integral to vanish. In terms of the expectation values of different representations in the background of the non-Abelian Coulomb gas, we see that an open interval of fundamental representations centered about $\alpha = 1/2$ is missing from the spectrum in the large N limit. In Figure 7.8 we show an example of the behaviour of the expectation value $|\langle \chi_\alpha \rangle|$ with α for $\mu = 1.2$.

The main outcome of this analysis is that the expectation value of the central fundamental character $\langle \chi_{1/2} \rangle$ is vanishing if and only if $\mu \geq 1$. Consequently it may be considered an order parameter distinguishing between the strong and weak coupling phases of the model. Physically the situation is clear: in the weak coupling phase the system can effectively screen the interactions of any pair of charges regardless of their representation since the system contains excitations in all representations of the gauge
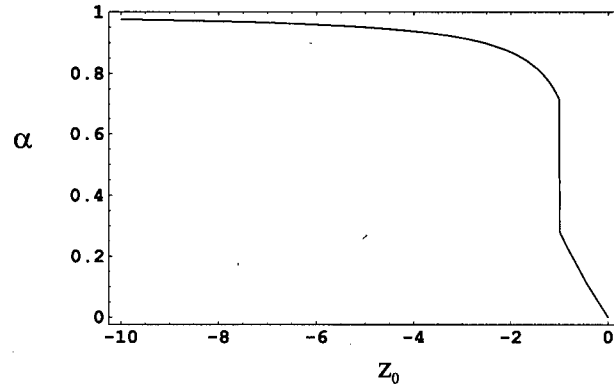
Figure 7.7: Plot of the solutions of the saddle-point relation (7.60) for $\mu = 1.2$.
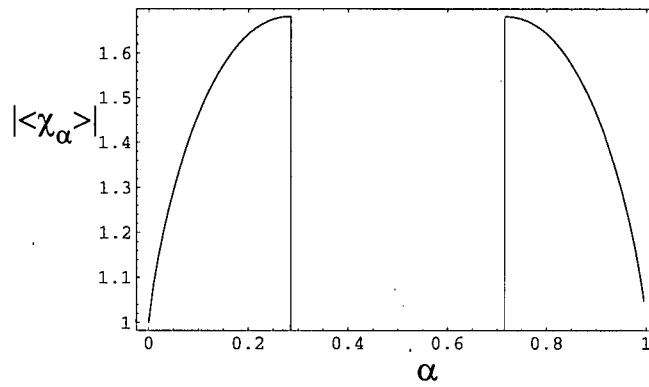

Figure 7.8: Schematic diagram of $|\langle \chi_\alpha \rangle|$ vs. α for $\mu = 1.2$. In this case the expectation value of representations with α between ~ 0.25 and ~ 0.75 is vanishing.

group. We conclude that the system looks much like a quark-gluon plasma where charges are effectively deconfined. At the phase transition line non-Abelian flux in the $\alpha = 1/2$ fundamental representation becomes too energetically costly to produce and the system can no longer screen the interaction between a pair of $\alpha = 1/2$ fundamental charges. In this strong coupling phase the interacting pair sees a linear confining potential (though somewhat reduced as compared to the empty background). As one further increases μ the gap in the spectrum of fundamental representations becomes larger and in the extreme limit $\mu = \infty$ the system contains only excitations in the trivial representation. This is precisely the confining phase of pure Yang-Mills theory.

7.7 Discussion

As we have shown, the generalization of the concept of the Polyakov loop operator to probe the group theoretic excitations of a system of non-Abelian electric charges provides a convenient and unified way to quantify the physics of phase transitions. While the details of our presentation have centered on a two dimensional model with an infinite number of colours, the general concepts developed here should be applicable to interacting gauge systems in arbitrary dimensions for both infinite and finite rank (N) gauge groups. The first obvious direction to proceed in is a two dimensional model with finite N gauge group. Unfortunately these models are too simple in that there are no phase transitions and they are always in the confined phase.

One immediate problem with using the fundamental representations to characterize the phase diagram arises when considering finite, odd rank groups. For example in the physically relevant case of $SU(3)$, there are only two fundamental representations $k = 1, 2$ and the order parameter $\langle \chi_{1/2} \rangle$ would naively denote the $k = 3/2$ fundamental representation. In terms of group theory this fractional representation is nonsense and strongly

suggests that direct application of the large N results is not prudent. Alternatively, we are free to use any independent set of irreducible representations to characterize the system. In particular the completely symmetric representations provide an equivalent algebraic basis to the fundamentals we have considered here. The strength of this approach is that there is no restriction on the number of symmetric representations for the unitary groups contrasting the $N-1$ fundamental representations for $SU(N)$. Unfortunately, repeating the calculations of the previous Section with symmetric representations one discovers that there is no evidence of the phase transition and in particular one cannot define an order parameter. This does not preclude the existence of an independent set of irreducible representations that spans the degrees of freedom of the system and contains an order parameter for the deconfinement transition in higher dimensional gauge theories with fundamental representation matter.

In higher dimensions one difficulty that may be insurmountable is that the phase diagram may not be divided into isolated regions by phase transitions, as in the case we have considered here. There is some numerical evidence that the line of first order transitions we have seen terminates at a second order transition point and does not continue as a higher transition across the phase diagram. Consequently the confining ($\mu > 1$) and screening ($\mu < 1$) phases we have been discussing are connected via paths along which all observables are analytic and hence there does not exist a quantitative way to distinguish the phases. If this is not the case then one should expect that some variant of the ideas presented here should provide an effective order parameter for the generalized deconfinement phase transition that occurs when fundamental representation matter is included in a pure gauge theory.

The only way to test these ideas, in lieu of analytic results for higher dimensional gauge theories with matter, is via numerical simulations. There it would be useful to calculate a large sample of multiply wound Polyakov loops in different regions of the phase

space and use these to reconstruct the expectation values for irreducible representations in the theory. In particular it would be interesting to investigate the role of higher representations in the setting of lattice calculations of gauge theories that are known to possess phase transitions. In fact, for the case of pure $SU(2)$ gauge theory, the $J = 1/2, 1$ and $3/2$ Polyakov loops have been calculated on the lattice [53] (see also [15]). It would certainly be very instructive to have more complete information about the higher representation Polyakov loops in this simple model both with and without fundamental representation matter.

Chapter 8

The Non-Abelian Coulomb Gas in a Box

In the previous Chapter we solved the model of a non-Abelian Coulomb gas of fundamental and adjoint representation charges on the spatial circle with circumference L including the case of the open line where $L \to \infty$. In this Chapter we will consider the same system confined to a segment of the spatial line of length L with impenetrable boundaries at finite temperature. Mathematically this is the same situation as including a gas of Wilson loops on the space-time sphere. In our discussion we will recover the results of [8, 38] and re-discover the Douglas-Kazakov phase transition on the sphere of the second Chapter (see Section 2.3 in the language of eigenvalue distributions of unitary matrices. We will find a solution of the fundamental gas in the box that interpolates between the high density/large L and low density/small L limits of the model which suggests a strong connection between the third order generalized deconfinement phase transition in the fundamental non-Abelian Coulomb on the open line and the third order Douglas-Kazakov phase transition for pure Yang-Mills theory in a box. Using the description of the model in terms of a one-dimensional fluid we will argue that the Douglas-Kazakov phase transition is a remnant of the generalized deconfinement transition we have investigated the previous Chapter when the non-Abelian Coulomb gas is restricted to a one-dimensional box.

The main motivation for considering the non-Abelian Coulomb gas in a box is the relation of this configuration to a gas of Wilson loops on the space-time sphere. From

the third Chapter we know that the inclusion of static colour-electric charges in two-dimensional Yang-Mills theory at finite temperature is formally equivalent to including Polyakov loops around the compactified temporal direction. Since there exists a string theory interpretation of Yang-Mills theory on compact two dimensional manifolds due to Gross and Taylor, we would like to know how this picture is modified by the inclusion of a gas of Wilson loops on the surface. Unfortunately we will have little to say about this connection due to calculational difficulties but the machinery introduced here should allow for eventual progress on this point.

8.1 The Fundamental Gas in a Box as an Integral Equation Problem

As we have seen previously, the effective large N Hamiltonian that governs a system of fundamental representation charges interacting via colour-electric forces in two dimensions is (compare 7.7)

$$\frac{H_{QM}}{N^2} = \frac{\gamma}{8} \int d\theta \rho(\theta) \left[v^2(\theta) + \frac{\pi^2}{3} \rho^2(\theta) \right] - 2\kappa \int d\theta \, \rho(\theta) \cos \theta - \frac{\gamma}{96} \tag{8.1}$$

Again the eigenvalue distribution $\rho(\theta, t)$ gives the large N saddlepoint estimation of the gauge fields in the theory and the velocity $v(\theta, t)$ is related to the associated canonical momentum. As in Chapter five, κ is simply related to the fugacity of fundamental representation charges (see 6.9) and, hence controls the density of the gas.

We begin our analysis by rescaling the space variable x which runs from 0 to L and introducing a dimensionless parameter τ by

$$x = \tau L \tag{8.2}$$

so that $\tau \in [0, 1]$. In these new variables, the equations of motion following from (8.1) are

$$\frac{\partial \rho}{\partial \tau} + \frac{\gamma L}{4} \frac{\partial}{\partial \theta} (\rho v) = 0 \tag{8.3}$$

$$\frac{\partial v}{\partial \tau} + \frac{\gamma L}{8} \frac{\partial}{\partial \theta} \left(v^2 - \pi^2 \rho^2 \right) - 2L\kappa \sin \theta = 0$$

These equations of motion are exactly the same as we had for the non-Abelian gas on the circle in the previous Chapter. The differences between that situation and the current one are the boundary conditions that supplement (8.3). As we have seen in the Chapter two, the space-time sphere can be obtained by setting the group elements that define the ends of a cylinder to the identity. At finite temperature, where the temporal direction is compactified, the space-time sphere has the interpretation of a one dimensional box. In terms of our eigenvalue distribution, we need to solve the equations of motion (8.3) with the following boundary conditions for $\rho(\theta, \tau)$,

$$\rho_0(\theta) \equiv \rho(\theta, 0) = \rho(\theta, 1) = \delta(\theta) \tag{8.4}$$

We still need to specify the velocity function v at the edges of the box. This is most easily done by recognizing that the physical situation is unchanged under the change of variable

$$\tau \to 1 - \tau \quad \text{or} \quad x \to L - x \tag{8.5}$$

which effectively rotates the system by 180 degrees. The equations of motion (8.3) are invariant under such a coordinate change if

$$\rho(\theta, \tau) = \rho(\theta, 1 - \tau) \tag{8.6}$$

and

$$v(\theta, \tau) = -v(\theta, 1 - \tau) \tag{8.7}$$

In particular, if $\tau = 0$ we have the boundary condition for the velocity field $v(\theta, \tau)$,

$$v_0(\theta) = v(\theta, 0) = -v(\theta, 1) \tag{8.8}$$

Note that $v_0(\theta)$ is an undetermined function and finding one that satisfies these boundary conditions constitutes the entire mathematical problem of this Chapter. In passing we

note that the boundary condition (8.8) demands that the velocity field vanish at $\tau = 1/2$. Consequently the eigenvalue density at this point, $\rho(\theta, 1/2)$ completely characterizes the system.

Boundary conditions such as (8.4) and (8.8) are difficult to implement in conjunction with the Euler equations (8.3) which are more amenable to initial value conditions. Hence it is much more convenient to convert the problem of finding $v_0(\theta)$ to an integral equation for which the boundary conditions are easily imposed. Introducing the new complex scalar function

$$f(\theta, \tau) = v(\theta, \tau) + i\pi\rho(\theta, \tau) \tag{8.9}$$

the equations of motion can easily be shown to reduce to the Hopf equation

$$8\frac{\kappa}{\gamma}\frac{\partial}{\partial\theta}\cos\theta + \frac{4}{\gamma L}\frac{\partial f}{\partial \tau} + f\frac{\partial f}{\partial\theta} = 0 \tag{8.10}$$

With $\kappa = 0$ this is the (complex) Hopf equation which is the prototypical one dimensional model of an equation which admits wave solutions complete with shocks and other inherently non-linear behaviour. The standard technique [109] for studying the Hopf equation is to consider a generalized version of (8.10) where we introduce a viscosity term which will serve to smooth out the solutions and allow one to calculate the details of the solution f in terms of initial conditions. With this addition, the original set of differential equations (8.3) are transformed into the (complex) Burgers' equation with an extra potential term

$$\epsilon\frac{\partial^2 f}{\partial\theta^2} = \frac{4}{\gamma L}\frac{\partial f}{\partial \tau} + f\frac{\partial f}{\partial\theta} + 8\frac{\kappa}{\gamma}\frac{\partial}{\partial\theta}\cos\theta \tag{8.11}$$

Obviously, in the limit as $\epsilon \to 0$ we recover the Hopf equation (8.10). The usefulness of this addition to the problem is that there exists a change of variable which will reduce (8.11) to a second order, linear differential equation. This transformation, known as the

Cole-Hopf transform, is given by

$$f(\theta, \tau) = -2\epsilon \frac{\partial}{\partial \theta} \log \psi(\theta, \tau) \qquad (8.12)$$

In terms of the new variable and ignoring an irrelevant constant of integration, the Burgers' equation becomes

$$H\psi[\theta, \tau] = \left[\epsilon^2 \frac{\gamma L}{4} \frac{\partial^2}{\partial \theta^2} + \kappa L \cos \theta \right] \psi[\theta, \tau] = \epsilon \frac{\partial}{\partial \tau} \psi[\theta, \tau] \qquad (8.13)$$

With the change of variable $\epsilon = \hbar$, $\tau = it$, this linear equation is simply the Schrödinger equation for the physical pendulum. A complete set of solutions to (8.13) are given by the Mathieu functions and consequently one can in principle form the appropriate Green's function $G(\theta, \phi; \tau)$ for solving the initial value problem. The difficulty is that we want to solve (8.13) subject to the boundary conditions (8.4) and (8.8) implemented at $\tau = 0$ and $\tau = 1$. This boundary value problem is most succinctly expressed for $\hbar = i\epsilon$ as the evolution of the wavefunction ψ by the 'Hamiltonian' H defined in (8.13)

$$\psi[\theta, 1] = \psi^*[\theta, 0] = e^{iH/\hbar} \psi[\theta, 0] \qquad (8.14)$$

Alternate to this operator equation, using an explicit form for the Green's function, one can express the boundary value problem for the non-Abelian gas in a box as a non-linear integral equation for the boundary velocity of the system v_0 in the limit $\epsilon \to 0$

$$\exp\left[-\frac{1}{2\epsilon} \int^\theta d\eta \left\{ i\pi\rho_0(\eta) - v_0(\eta) \right\} \right] \qquad (8.15)$$
$$= \int d\phi \, G(\theta, \phi; 1) \exp\left[-\frac{1}{2\epsilon} \int^\phi d\eta \left\{ i\pi\rho_0(\eta) + v_0(\eta) \right\} \right]$$

Unfortunately, even with explicit knowledge of the eigenfunctions and spectrum of the Green's function, the non-linearity of the problem makes it very difficult to solve in general. Consequently we will only be able to deal with (8.15) accurately in certain limits. In the next Section we will consider a related model that is exactly solvable and

provides a qualitative and, in certain limits, quantitative knowledge of the solutions of (8.15).

8.2 An Approximation to the Non-Abelian Coulomb Gas in a Box

In this Section we will consider a truncated version of the equations of motion (8.13) for the fundamental representation non-Abelian Coulomb gas in a box. The truncation we are going to consider involves replacing the periodic potential in (8.13) with a Gaussian potential plus a constant offset. This is equivalent to truncating a Taylor expansion of $\cos\theta$ at quadratic order. To be more precise, we are going to study the boundary value problem given by (8.4) and (8.8) and

$$\left[\epsilon^2\frac{\gamma L}{4}\frac{\partial^2}{\partial\theta^2}+\frac{\kappa L}{2}\theta^2\right]\psi[\theta,\tau]=\epsilon\frac{\partial}{\partial\tau}\psi[\theta,\tau] \qquad (8.16)$$

The relation of this differential equation to the Schrödinger equation for the quantum mechanical oscillator leads to the identifications

$$m=\frac{2}{\gamma L}\quad,\quad \omega^2=\frac{\kappa}{2\gamma}(\gamma L)^2 \qquad (8.17)$$

One issue which we need to deal with here is that of periodicity in the variable θ. The original potential was obviously periodic under shifts $\theta\to\theta+2\pi$ but our Gaussian truncation is not. In general periodicity of the Hamiltonian and wavefunctions in the Schrödinger picture is only important when the wavefunctions overlap themselves on non-contractible loops in the configuration space. The prototypical example of this is quantum mechanics on the circle (see [85] for instance) but also is of importance in more complex systems like Yang-Mills theory on a D-dimensional torus [56]. In the present situation we will see that there is a phase transition in the system precisely when the periodicity becomes important. We will not deal with the phase of the model where

the periodicity is important but it could be accessed here by considering the periodic extension of θ^2 from the interval $-\pi < \theta < \pi$.

The main advantage of truncating the periodic potential is that we can now solve the boundary value/non-linear integral equation problem (8.15) exactly. The Green's function associated with the differential operator in (8.16) is the familiar form

$$G(\theta, \phi; \tau) \sim \exp\left[\frac{a(\tau)}{\epsilon}\theta\phi - \frac{b(\tau)}{\epsilon}(\theta^2 + \phi^2)\right] \tag{8.18}$$

where the τ dependent coefficients are

$$a(\tau) = \frac{m\omega}{\sinh\omega\tau} \quad , \quad b(\tau) = \frac{m\omega}{2}\coth\omega\tau \tag{8.19}$$

Even with a simple and explicit form for the Green's function of the problem at hand solving the non-linear integral equation (8.15) for the boundary velocity $v_0(\theta)$ requires some guess-work. Following [8, 38] we make an ansatz for the functional form of v and ρ

$$v(\theta, \tau) = \alpha(\tau)\theta \tag{8.20}$$

$$\rho(\theta, \tau) = \frac{1}{\pi}\sqrt{\mu(\tau) - \frac{\mu^2(\tau)}{4}\theta^2}$$

where the eigenvalue density ρ is defined to be non-vanishing only when real-valued and α and μ are functions are chosen to solve the boundary value problem.

It is readily verified that, in the limit $\epsilon \to 0$, the ansatz (8.20) is a solution of the simple harmonic oscillator problem (8.16) respecting the boundary condition (8.8) when

$$\alpha(0) = -2m\omega\coth\omega + \sqrt{\frac{m^2\omega^2}{\sinh^2\omega} + \mu^2(0)} \tag{8.21}$$

Having solved the boundary value problem in terms of the free parameter $\mu(0)$, it is now a simple task to use the Green's function (8.18) to evolve the initial data and generate the solution of the equations of motion at an arbitrary point τ. Evolving we

find that the ansatz (8.20) retains functional form when the undetermined functions α and μ are given by

$$
\begin{aligned}
\alpha(\tau) &= -4b(\tau) + \frac{4a^2(\tau)(4b(\tau) + \alpha(0))}{(4b(\tau) + \alpha(0))^2 - \mu^2(0)/4} \\
\mu(\tau) &= \frac{4a^2(\tau)\mu(0)}{(4b(\tau) + \alpha(0))^2 - \mu^2(0)/4}
\end{aligned}
\tag{8.22}
$$

In order to match the boundary condition (8.4), all we require is that the eigenvalue distribution tend to a delta function at the endpoints of the sphere ($\tau = 0, 1$). This can be accomplished in the present case by taking the limit $\mu(0) \to \infty$. In this limit the solution of the large N eigenvalue problem is given by the ansatz (8.20) with

$$
\begin{aligned}
\alpha(\tau) &= -\frac{m\omega \sinh\left[\omega(1 - 2\tau)\right]}{\sinh\left[\omega(1 - \tau)\right] \sinh\left[\omega\tau\right]} \\
\mu(\tau) &= \frac{2m\omega \sinh\left[\omega\right]}{\sinh\left[\omega(1 - \tau)\right] \sinh\left[\omega\tau\right]}
\end{aligned}
\tag{8.23}
$$

Since we have ignored the periodicity of the variable θ, our solution of the boundary value problem is only valid as long as the support of the eigenvalue distribution is bounded away from $\theta = \pm\pi$. It is easy to see that the endpoints (θ_0) of the eigenvalue distribution for the ansatz (8.20) are given by

$$
\theta_0 = \pm\frac{2}{\sqrt{\mu(\tau)}}
\tag{8.24}
$$

From the solution of the boundary value problem (8.23) it is easily verified that the coefficient μ reaches a global minimum at $\tau = 1/2$. Consequently we have the self-consistency bound on our solution

$$
\pi^2 \geq \frac{4}{\mu(1/2)} = \frac{1}{m\omega}\tanh \omega/2 = \sqrt{\frac{\gamma}{2\kappa}}\tanh\left(\sqrt{\frac{\kappa}{8\gamma}}L\gamma\right)
\tag{8.25}
$$

Saturating this bound leads to a breakdown of our solution because the non-trivial topology of the θ direction of the configuration space becomes important. By examining the limits of the truncated model in the next sub-Sections we will find that this breakdown is associated with a familiar phase transition.

8.2.1 High density and Small Boxes

The limits in which the truncated model we have solved exactly is a good approximation to the true model are easily characterized. Since in the limit of vanishing ϵ the wavefunction ψ in (8.15) is exponentially damped outside of the support of the eigenvalue density, only the details of the Green's function within the support affect the solution- including those of the potential. Consequently our small angle approximation is good so long as the endpoints, θ_0 of ρ remain near the origin

$$1 >> \theta_0 = \frac{1}{m\omega} \tanh \omega/2 \tag{8.26}$$

which implies that

$$\sqrt{\frac{2\kappa}{\gamma}} >> \tanh \left(\sqrt{\frac{\kappa}{8\gamma}} \gamma L \right) \tag{8.27}$$

Due to the behaviour of the hyperbolic tangent function we see that the truncated model (8.16) should be a good approximation for large fundamental charge density, κ. As well, in the limit of small L we expect the small angle approximation to hold for finite κ. Consequently we find that the solution (8.23) interpolates between the high density non-Abelian Coulomb gas model on the infinite line and pure Yang-Mills theory in a vanishingly small box. In fact we will argue that this connection in the full model holds for all densities of fundamental gas in all sizes of box, but first let us investigate another limit of the exact solution.

8.2.2 Pure Yang-Mills on the Sphere

In the limit where the density of fundamental charges, κ is taken to vanish we should expect to recover the results for pure Yang-Mills theory on a sphere of volume βL. Of course we have already investigated this system in Chapter two from a group-theoretic point of view but here we will recover those same results using the field-theoretic techniques of the two previous Chapters.

We can check what becomes of the self-consistency requirement in the limit $\kappa \to 0$. From (8.25) we have the restriction

$$\pi^2 \geq \frac{\gamma L}{4} \tag{8.28}$$

The saturation of this inequality occurs for

$$4\pi^2 = \gamma L = \frac{e^2 N \beta L}{2} \tag{8.29}$$

This is exactly the point of the Douglas-Kazakov transition (2.50) we have seen on the sphere with area

$$A = \beta L = \frac{8\pi^2}{e^2 N} \tag{8.30}$$

Hence we have found that the breakdown of the solution for the truncated model when the support of the eigenvalue distribution reaches $\pm\pi$ is an extension to non-vanishing κ of the Douglas-Kazakov phase transition on S^2.

This result provides a firm connection between the breakdown of truncated model (8.16) and the phase transition in pure Yang-Mills theory on the sphere, but we are still left short of a complete solution of the non-Abelian Coulomb gas in a box. Since the quadratic truncation leaves us with a finite potential on the interval $(-\pi, \pi)$, we expect that the mechanism for the phase transition will persist as we slowly deform the truncated potential back into the full, periodic one of Equation (8.13). In order to argue this point more effectively we now turn to the similarities between the eigenvalue problem at hand and a one-dimensional fluid mechanics problem with the same equations of motion, (8.3).

8.3 Qualitative Description of the Solution

Here we will use qualitative methods based on the picture of the eigenvalue problem for the non-Abelian Coulomb gas in a box as a one-dimensional fluid flow problem. In

this way we will be able to argue effectively that the Douglas-Kazakov phase transition as seen on the sphere is a remnant of the generalized deconfinement transition for the non-Abelian Coulomb gas in a box.

The equations of motion (8.3) for a fundamental representation gas in a box are of the same form as for a one-dimensional fluid moving in a background potential as described in Eulerian coordinates. In this interpretation the eigenvalue density, ρ plays the role of the density distribution of a drop of fluid which we will refer to as the 'eigen-fluid'. This fluid moves with velocity v through a potential energy trough generated by the $\cos\theta$ potential. The non-trivial topology of the configuration space is realized by identifying the $\theta = \pm\pi$ edges of the trough. The boundary condition (8.4) for the eigenvalue distribution states that the eigen-fluid begins its motion at $\tau = 0$ in a delta-function configuration centered at $\theta = 0$ and then evolves outward as τ increases. This flow is driven by a non-zero velocity field $v(\theta)$ at $\tau = 0$. By the symmetry of the problem (8.6), this expansion continues until $\tau = 1/2$ when the velocity changes sign and consequently the eigen-fluid begins to contract. At $\tau = 1$ the eigen-fluid returns to a delta-function configuration and the velocity field is of the same magnitude, but opposite sign, as it began with at $\tau = 0$.

This picture of the motion of the fluid drop is supported by the explicit solution for the truncated model of the last Section (8.16), at least until the self-consistency relation (8.25) is saturated. In order to understand this breakdown better we need to consider the details of the potential trough in which the fluid flows. First, we consider the limits where the truncated model is a good approximation to the full model (8.13)- when the density κ is large and/or the size of the box L is small. In the fluid picture these are instances when the trough in which the eigen-fluid moves is deep and/or short. In both cases it is difficult for the fluid to expand from its original delta-function configuration. When the potential trough is deep, the fluid does not have sufficient energy to climb out of the minimum at $\theta = 0$. In small boxes the velocity field cannot expand the fluid much before the midpoint
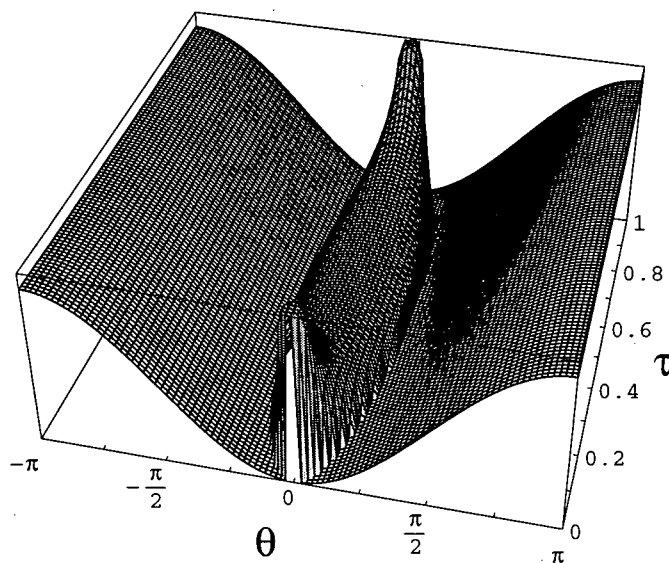
Figure 8.1: Schematic diagram of the τ-evolution of the eigen-fluid (dark central region) in the periodic background potential trough. In the high density and/or small box limit the eigen-fluid remains near the origin. The depth of the potential trough and/or boundary conditions restrict the expansion of the fluid.

is reached and contraction back to a delta function configuration begins. Consequently we see from the fluid drop picture that support of the eigenvalue distribution remains localized near the origin and the truncated model is a good approximation. In Figure 8.1 we give a graphical description of a typical example of the flow of the eigenvalue density in this regime.

As the density of the fundamental gas in the box decreases and/or the size of the box increases, the eigen-fluid is free to expand as it evolves in τ. In intermediate regions where the drop remains away from the identified points $\theta = \pm\pi$ the periodicity of the configuration space remains irrelevant and could be well described by truncating the periodic potential in (8.13) at higher order. In Figure 8.2 we give an example of this intermediate configuration.

Eventually, as κ decreases and/or L increases, the eigen-fluid will expand to reach
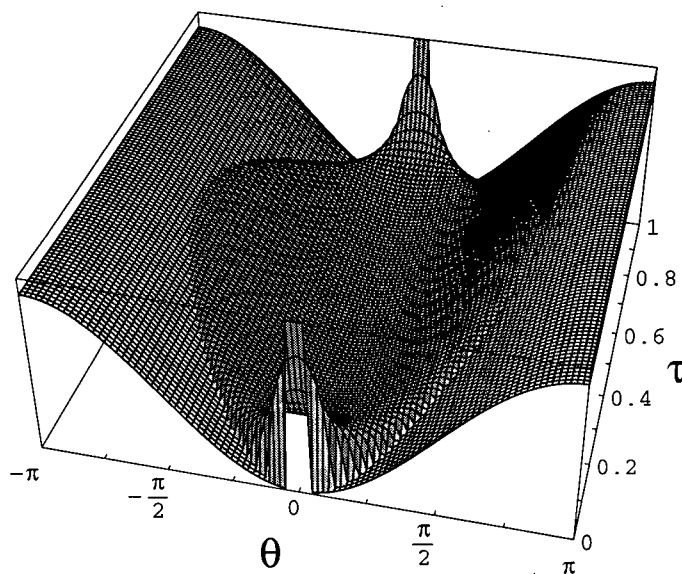
Figure 8.2: When the density of fundamental charges drops and/or the size of the box increases, the eigen-fluid is free to expand and rise up the side of the potential well.

$\theta = \pm\pi$ (see Figure 8.3). As we have previously described, this will lead to a breakdown of the solution for the truncated model since periodicity will become important. This breakdown results in a phase transition which is the extension of the Douglas-Kazakov phase transition for non-vanishing fundamental gas density. Here we would like to compare the details of where this breakdown occurs in both the truncated and full models. From (8.25), as $L \to \infty$ we find that the truncated model breaks down at

$$\kappa_{tc} = \gamma/2\pi^4 \sim \gamma/195$$

This should be compared with the value for the critical point in the pure fundamental non-Abelian gas on the open line (see Equation 7.28 with $\lambda_c = 0$) where a phase transition was seen to occur for

$$\kappa_c = \gamma\pi^2/512 \sim \gamma/52$$

Obviously the critical points for the full and truncated models on the infinite line are not the same. This is to be expected, since the potential well in which the eigen-fluid
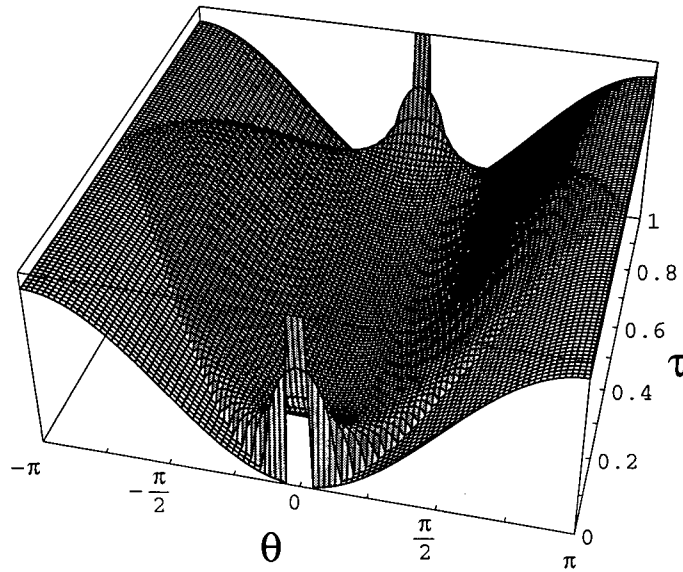
Figure 8.3: When the eigen-fluid reaches the top of the potential well, periodicity becomes important and there is a phase transition.

moves is different in each case. For the full model, the periodic potential is no longer increasing at $\theta = \pm\pi$ and the maximum value of the potential in (8.13) is κL. This is contrasted with the truncated model (8.16) for which the maxima occur at the periodic boundary with magnitude $\kappa L(\pi^2/2 - 1) \sim 3.93\kappa L$. Clearly the truncated potential is deeper and, by the arguments we have given, we should expect that it is less likely that the eigen-fluid will reach $\theta = \pm\pi$. The result is that, for fixed L, as κ is decreased the breakdown of the full model occurs before that of the truncated model. This is consistent with what we have seen at infinite L and since for $-\pi < \theta < \pi$

$$\cos\theta \geq 1 - \theta^2/2 \tag{8.31}$$

the logic holds for all L. As κ is decreased further the critical value of L tends to a non-vanishing limit due to the cohesive forces in the eigen-fluid. This limiting value of L is simply the size of box for which the Douglas-Kazakov transition occurs (2.50).

The conclusions of this analysis are most clearly summed up in a schematic phase
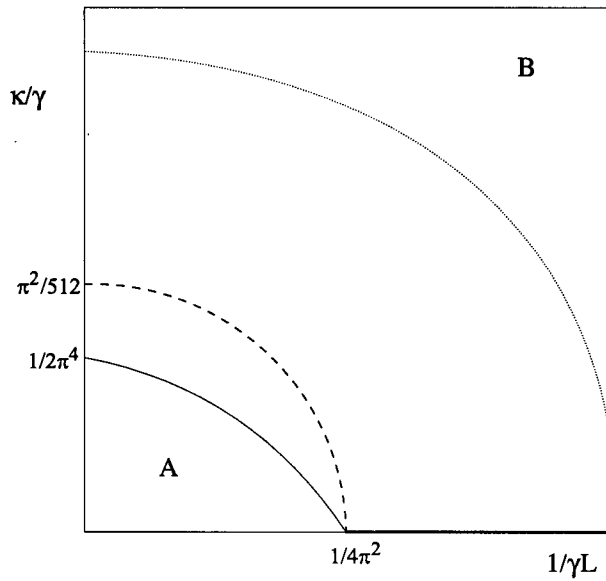
Figure 8.4: Phase diagram for the fundamental gas in a box. The solid curve denotes a line of phase transitions in the truncated model, above which the solution (8.16) is valid. The coarsely dashed curve represents the expected line of phase transitions for the full model (8.13) of the fundamental gas in a box. The region B of the diagram, above the finely dashed curve, is a region of the parameter space where the truncated model is a good approximation to the full model. The region A, which is currently inaccessible, is where we expect a string-theoretic picture of the model to hold.

diagram (Figure 8.4). The solid curve running from $\kappa/\gamma = 1/2\pi^4$ to $1/\gamma L = 1/4\pi^2$ is the line of phase transitions where the solutions of the truncated model we have been discussing (8.16) break down. The solution for the truncated model given in (8.23) is valid for all points above this curve. As we have argued, above this curve lies the (dashed) line of phase transition in the full model with periodic potential (8.13). Unfortunately, the precise equation for this curve is determined dynamically and appears to require the full solution of the model. In the upper part (B) of the phase diagram above the finely dotted line, the eigen-fluid is confined to a region near the origin (see Figure 8.1 for example) and the truncated model is an excellent approximation to the full model. The further one is above the dotted curve which represents the inequality (8.27), the

better the approximation. Also included is the $1/\gamma L$ axis above $1/4\pi^2$ where the pure Yang-Mills solution as given in (8.23) with $\kappa = 0$ is valid.

There is still one region of the phase diagram we have yet to discuss. The region A in the bottom corner is that of the strong coupling phase where the periodic nature of the configuration space plays an important role.

Finally, in the bottom corner of the phase diagram, Figure 8.4, the region labeled A is where we expect a string-theoretic interpretation of the fundamental non-Abelian Coulomb gas to hold. In terms of the fluid flow picture, it is in this region of parameters where the eigen-fluid rises over the edge of the potential (see Figure 8.5 for example) and the periodicity of the configuration space becomes crucial. From the point of view of computation, the solution in this region is difficult since the only large area solution we have at our disposal, the solution of the Coulomb gas on the open line from Chapter seven, does not have the proper boundary conditions. Hence we do not even have a starting point for perturbation theory in this phase. This is unfortunate since the strong coupling region is where we expect the string picture of Gross and Taylor to be valid. The inclusion of a gas of fundamental charges would certainly be an interesting situation to explore to gain a better understanding of the inclusion of matter into the string picture of pure Yang-Mills theory.
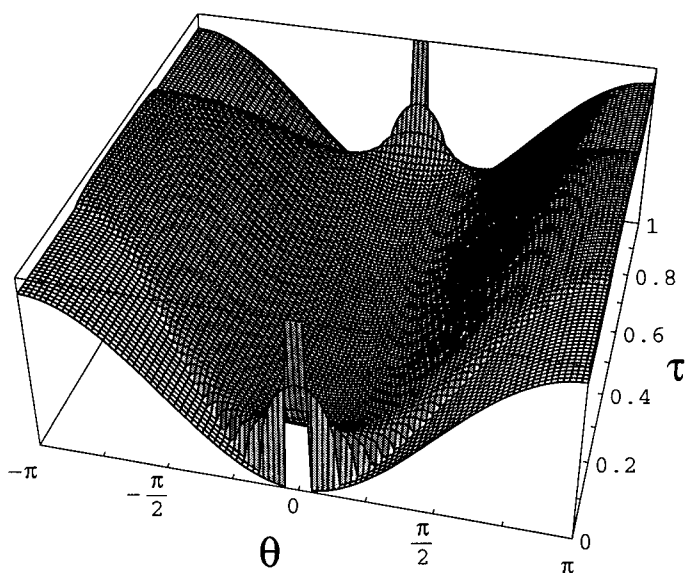
Figure 8.5: The strong coupling regime is where the eigen-fluid spills over the top of the potential. In these cases the periodicity of the configuration space becomes important and the solutions of the equations of motion we have constructed break down.

Chapter 9

Summary

Here we have investigated the physics of heavy, static matter interacting with two-dimensional Yang-Mills theory at finite temperature. In particular, the interpretation of these systems in terms of the structures of Lie group theory was exploited to provide a solid mathematical background for examining the properties of matter interacting with gauge fields. The Thesis consisted of two main sections, the first of which pertained to finite rank Yang-Mills theories both with and without matter content (Chapters two through five) on compact and open space-times. The second half of the Thesis was devoted to infinite rank gauge $U(N)$ gauge groups interacting with matter (Chapters six through eight). Here we provide a short summary of the developments of the Thesis following the initial introductory Chapters where the framework for our investigations was constructed.

In Chapter four we determined, via pure group theoretic methods, the number of vacua in pure Yang-Mills theories in two dimensions in agreement with topological arguments. Moreover, our methods allowed for the calculation of physical properties of each of these vacua. As an example, we explicitly calculated the binding energy (string tension) of external test charges as a function of vacuum. Using the same formalism, in Chapter five we were able to extend these results to a thermodynamical gas of heavy $SU(N)$ charges, a non-Abelian Coulomb gas. Such a model, with adjoint representation charges has the same symmetric properties as the pure Yang-Mills theory and consequently has the same vacuum structure. In the limits of low gas density we were able

141

to show the non-trivial dependence of the 'meson' spectrum on the vacuum state. In the high gas density limit, this dependence was shown to be washed out by screening of colour-electric flux. These results give some of the first explicit calculations of the effect of a choice of vacuum on the behaviour of a system of matter interacting via Yang-Mills fields.

In Chapter six we developed a formalism for extending the results of the first five Chapters to $U(N)$ gauge fields in the limit $N \to \infty$. Including heavy matter in this case leads to a connection with the well-known formalism of unitary matrix models and, in particular, the non-Abelian Coulomb gas is shown to have non-trivial critical behaviour in the large N limit. As an example, we considered the example of adjoint and fundamental matter in great detail in Chapter seven. The non-trivial phase structure in this model is qualitatively similar to what is expected to arise in higher dimensional gauge theories interacting with matter. In particular the model exhibits a transition from a phase where colour-electric charges are confined to one where they are screened. This phase structure was interpreted in terms of group theoretic quantities and from this point of view a novel order parameter for discerning the confined and screened phases in the presence of fundamental matter was developed.

Finally, in the previous Chapter, the case of fundamental representation matter interacting via $U(N)$, $(N \to \infty)$ fields confined to a one-dimensional box was examined. We developed the formalism for detailed exploration of this system by combining the equations of motion for the non-Abelian Coulomb gas from Chapter six with the boundary conditions appropriate to the physical situation. The result was a non-linear integral equation problem for the configuration of the system which is dominant in the large N limit. This problem was solved in the limit where the effective interactions between the matter is large, namely that of high particle density and/or small boxes. It was explicitly shown that the approximate solution exhibits a phase transition which can be thought

of as an extension of the Douglas-Kazakov transition on the space-time sphere. It was argued that this transition is present for the full solution of the model. Unfortunately, the solutions discussed were not valid in the region of large boxes at low particle density where an effective string picture of the model is expected to exist.

Appendix A

The Representation Theory of Compact Lie Groups

Yang-Mills theory is the quantum field theory of a dynamical symmetry group. Generally the symmetry group is one of the compact Lie groups. In two dimensional Yang-Mills theory, due to its dynamical simplicity, a comprehensive knowledge of Lie group representation theory is a powerful tool. This being the case, we begin with a short introduction of the elements which will prove to be crucial to the rest of our analysis.

A.1   The Cartan-Weyl Basis

In general we are interested in the Lie group G and its associated Lie algebra $\mathcal{G}$. The elements of the two are related by an exponential mapping of the form

$$g = e^{iT_R^a n^a} \qquad (A.1)$$

where g is a group element, $g \in G$ and T_R is an element of the algebra, $T_R \in \mathcal{G}$. In this way the algebra is seen to be the tangent space of the group at the point $n = 0$. The representation theory of a compact Lie group is based on analysis of the associated Lie algebra. Let $\{T_R^a\}$ be a set of n Hermitean generators satisfying the commutation relation

$$[T_R^a, T_R^b] = i \sum_{c=1}^{n} f^{abc} T_R^c \qquad (A.2)$$

The number of generators in the set is called the dimension of the algebra. Here f^{abc} are the structure constants of the group. We will take f to be real and completely anti-symmetric in its indices which is consistent for the compact, simple Lie algebras we will

be concerned with here. The commutation relation (A.2) leads immediately to the cyclic Jacobi identity for generators X, Y and Z in a particular representation

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \qquad (A.3)$$

In fact this relation can be taken to define a Lie algebra.

We will be concerned only with simple Lie algebras here which means there is no proper subset of the generators $\{T_R^a\}$ that forms a closed algebra under commutation with any T_R^a. The subscript R signifies that the generator belongs to an particular set which obeys the commutation relation (A.2) and is closed under it. There are many such sets of generators each of which defines a representation of the Lie algebra and hence Lie group. The identification of these representations and their properties is our objective in this Section.

Representations of Lie algebras are realized as finite linear operators $(T_R^a)_{ij}$ on a vector space V. The dimension of V is called the dimension of the representation and will be denoted by d_R. Any representation for which not all generators can be brought into block diagonal form by a similarity transformation on V is called irreducible. In general we will be concerned only with irreducible representations or the reduction of reducible representations.

For example, the fundamental representation of the Lie group $SU(2)$ is generated by the set of 2×2 matrices τ^a which are proportional to the familiar Pauli spin matrices σ^a

$$\tau^1 = \frac{\sigma^1}{2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (A.4)$$

$$\tau^2 = \frac{\sigma^2}{2} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$\tau^3 = \frac{\sigma^3}{2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For an arbitrary Lie group we define the adjoint representation (*ad*) to be that which is generated by the structure constants

$$(T_{ad})^a_{bc} = -if^{abc} \tag{A.5}$$

The analysis of the representation structure of a Lie algebra begins by identifying the maximal set of commuting generators H^i

$$[H^i, H^j] = 0 \tag{A.6}$$

where $i = 1, \ldots, r$ and r is the called the rank of the group. This set of generators forms the Cartan subalgebra h. Since all elements of the Cartan subalgebra are mutually commuting they can be simultaneously diagonalized. Taking all H^i to be diagonal we define the remaining generators of the group, E^α to satisfy the eigenvalue equation

$$[H^i, E^\alpha] = \alpha^i E^\alpha \tag{A.7}$$

where α^i are the components of the r-dimensional vector α which is called a root and E^α is the associated ladder operator. While we have assumed the original generators T^a_R to be Hermitian, the generators E^α are not in general but rather

$$(E^\alpha)^\dagger = E^{-\alpha} \tag{A.8}$$

Equation (A.7) shows that if α is a root then $-\alpha$ is as well.

In the adjoint representation where the components of the root vector are the eigenvalues of H^i, the vector space on which the generators act is the Lie algebra itself. This leads to an identification of the generators with the states of the representation

$$E^\alpha \quad \rightarrow \quad |E^\alpha\rangle \equiv |\alpha\rangle \tag{A.9}$$

$$H^i \quad \rightarrow \quad |H^i\rangle$$

The action of a generator X in the adjoint representation is given by $ad(X)$ which is defined as

$$ad(X)Y \equiv [X, Y] \tag{A.10}$$

In particular from the eigenvalue equation, (A.7) and the identifications, (A.9) we had

$$ad(H^i)E^\alpha = \alpha^i E^\alpha \rightarrow H^i|\alpha\rangle = \alpha^i|\alpha\rangle \tag{A.11}$$

It is interesting to note that the zero eigenvectors of the operator H^i are the states $|H^i\rangle$. It follows that the degeneracy of the zero eigenvalue is equal to the rank r of the group. Furthermore, from this construction, where the adjoint representation generators act on the Lie algebra, we deduce that the dimension of the adjoint representation d_{ad} is equal to the dimension of the algebra.

Having specified the commutators of H^i with itself and E^α, in order to complete the algebra we need to evaluate commutators of the form $[E^\alpha, E^\beta]$. From the Jacobi identity (A.3) we have

$$[H^i, [E^\alpha, E^\beta]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta] \tag{A.12}$$

If $\alpha + \beta$ is a root vector then $[E^\alpha, E^\beta] \sim E^{\alpha+\beta}$. If $\alpha = -\beta$, the right side vanishes and $[E^\alpha, E^{-\alpha}]$ is a linear combination of generators of the Cartan subalgebra. We will fix the normalization of the ladder operators by setting

$$[E^\alpha, E^{-\alpha}] = \frac{2}{(\alpha, \alpha)}\alpha^i H^i \tag{A.13}$$

where the repeated index is summed over. Finally, if $\alpha + \beta$ is not a root vector or vanishing then closure of the algebra requires $[E^\alpha, E^\beta]$ to vanish.

It is useful to record the preceeding construction of the Lie algebra in the so-called Cartan-Weyl basis

$$[H^i, H^j] = 0 \tag{A.14}$$

$$[H^i, E^\alpha] = \alpha^i E^\alpha \qquad\qquad (A.15)$$

$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha\beta} E^{\alpha+\beta} & \text{if } \alpha + \beta \text{ is a root} \\ \frac{2}{(\alpha,\alpha)} \alpha^i H^i & \text{if } \alpha = -\beta \\ 0 & \text{otherwise} \end{cases}$$

A.2   The Killing Form

Having constructed the generators for the adjoint representation we now move on to define the notion of a metric on the generators of a Lie algebra. This task is taken care of by the symmetric Killing form defined as

$$K(X, Y) = \frac{1}{\lambda} \text{Tr} \left(ad(X) ad(Y) \right) \qquad\qquad (A.16)$$

Here λ is a algebra dependent constant which will not be of concern to us here. We will always take generators of the algebra to be orthogonal with respect to K

$$K(T_R^a, T_R^b) = \delta^{ab} \qquad\qquad (A.17)$$

and this can be extended to the Cartan subalgebra in a trivial way

$$K(H^i, H^j) = \delta^{ij} \qquad\qquad (A.18)$$

In order to find the scalar product of ladder operators we need the property of the Killing form

$$K([X, Y], Z) + K(Y, [X, Z]) = 0 \qquad\qquad (A.19)$$

Taking $X = E^\alpha$, $Y = E^{-\alpha}$, $Z = H^i$ we find that

$$K(E^\alpha, E^{-\alpha}) = \frac{2}{(\alpha, \alpha)} \qquad\qquad (A.20)$$

It can be readily verified that the scalar product involving any other combination of ladder operators vanishes.

The Killing form is important because it induces a scalar product on the root vector space. If γ and ρ are root vectors we define their inner product by

$$(\gamma, \rho) \equiv K(\rho^i H^i, \gamma^j H^j) = \rho^i \gamma^i \qquad (A.21)$$

where repeated indices are summed over.

A.3 Simple Roots, Weights and Dynkin Diagrams

In this Section we will investigate the details of the adjoint representation and introduce more general representations. These considerations will lead to a comprehensive classification of all admissible simple Lie algebras in a convenient graphical form known as Dynkin diagrams.

The number of roots which is equal to the dimension of the algebra less the rank (= dimension of the Cartan subalgebra) is in general larger than the rank which is the dimension of the vector space on which the adjoint representation acts. Consequently the roots define an overdetermined system. In this Section we will define a linearly independent subset of the roots called the simple roots.

We begin by choosing an independent basis of r vectors β_i on which to expand any root

$$\alpha = \sum_{k=1}^{r} \omega_k \beta_k \qquad (A.22)$$

An ordering can be introduced in this basis. A root α is said to be a positive root if the first non-vanishing component in the expansion over the β basis, $(\omega_1, \ldots \omega_r)$, is positive. The choice of basis $\{\beta\}$ is completely arbitrary and the introduction of any ordering is sufficient to entirely characterize the group algebra.

Now we can define the simple roots of a Lie algebra. A simple root α_i is defined to be a positive root that cannot be written as the sum of two positive roots. There

are r simple roots, a proof of which is somewhat tedious and left to the references (see [25] for example). We will label the simple roots $(\alpha_1, \ldots \alpha_r)$. There are two immediate consequences of this definition of simple roots. First if α and β are simple roots then $\alpha - \beta$ is not a root. The proof of this statement is as follows: assume $\alpha - \beta$ is a positive root then $\alpha = \beta + (\alpha - \beta)$ is the sum of two positive roots which is a contradiction. We get a similar result if $\beta - \alpha$ is a positive root hence $\beta - \alpha$ cannot be a root. The second immediate result is that any positive root is the sum of simple roots. This follows from the fact that a positive root is either simple or can be written as the sum of two positive roots. If it is the sum of two positive roots then we can iterate the argument until there are only simple roots in the sum.

Having examined the adjoint representation in great detail it will be of great benefit to consider the case of general representations for a moment. For any representation one can always find a basis of states $\{|\lambda\rangle\}$ such that

$$H^i|\lambda\rangle = \lambda^i|\lambda\rangle \tag{A.23}$$

where the eigenvalues λ^i are the components of an r-dimensional vector λ which is called the weight vector. Comparing with (A.11) we see that the root vectors are the weight vectors of the adjoint representation.

Acting on a state with $H^i E^\alpha$ we begin to understand why E^α is called a ladder operator

$$H^i E^\alpha|\lambda\rangle = [H^i E^\alpha]|\lambda\rangle + E^\alpha H^i|\lambda\rangle = (\lambda^i + \alpha^i)E^\alpha|\lambda\rangle \tag{A.24}$$

Consequently, $E^\alpha|\lambda\rangle$ is proportional to the state $|\lambda + \alpha\rangle$ and we see that E^α shifts states by the root vector α. Since we are dealing with finite dimensional representations there are finite number of states in a representation and hence, there are limits to the number of times we can apply the ladder operator to a state and get a non-zero result. This is

an important realization and demands that there exist integers p and q such that

$$(E^\alpha)^{p+1}|\lambda\rangle \quad \sim \quad E^\alpha|\lambda + p\alpha\rangle = 0 \tag{A.25}$$

$$(E^{-\alpha})^{q+1}|\lambda\rangle \quad \sim \quad E^{-\alpha}|\lambda - q\alpha\rangle = 0$$

In order to further analyze the consequences of this observation we define the coefficients $N^\lambda_{\pm\alpha}$ by

$$E^{\pm\alpha}|\lambda\rangle = N^{\pm\alpha}_\lambda|\lambda \pm \alpha\rangle \tag{A.26}$$

Now we would like to consider the sum of expectation values of the commutator of ladder operators over states

$$
\begin{aligned}
\Gamma &= \sum_{s=-q}^{p} \langle\lambda + \alpha s|[E^\alpha, E^{-\alpha}]|\lambda + \alpha s\rangle \tag{A.27} \\
&= \sum_{s=-q}^{p} \langle\lambda + \alpha s|\alpha^i H^i|\lambda + \alpha s\rangle \\
&= \sum_{s=-q}^{p} (\alpha, \lambda + \alpha s) \\
&= (p + q + 1)(\alpha, \lambda) + (\alpha, \alpha)\left[\frac{p(p+1)}{2} - \frac{q(q+1)}{2}\right] \\
&= (p + q + 1)\left[(\alpha, \lambda) + (\alpha, \alpha)\frac{p - q}{2}\right]
\end{aligned}
$$

Note in the second line we have used the commutation relations for the Cartan-Weyl basis. Expanding the commutator we also have the alternate evaluation of Γ

$$
\begin{aligned}
\Gamma &= \sum_{s=-q}^{p} \langle\lambda + \alpha s|[E^\alpha, E^{-\alpha}]|\lambda + \alpha s\rangle \tag{A.28} \\
&= \sum_{s=-q}^{p} \langle\lambda + \alpha s|(E^\alpha E^{-\alpha} - E^{-\alpha}E^\alpha)|\lambda + \alpha s\rangle \\
&= \sum_{s=-q}^{p} \left[|N^{-\alpha}_{\lambda+\alpha s}|^2 - |N^\alpha_{\lambda+\alpha s}|^2\right]
\end{aligned}
$$

But by definition (A.26),

$$N^{-\alpha}_\lambda = \langle\lambda - \alpha|E^{-\alpha}|\lambda\rangle \tag{A.29}$$

$$= \langle \lambda - \alpha | (E^\alpha)^\dagger | \lambda \rangle$$

$$= \langle \lambda | E^\alpha | \lambda - \alpha \rangle^* = (N^\alpha_{\lambda - \alpha})^*$$

This useful relationship allows us to rewrite (A.28) as

$$\begin{aligned}
\Gamma &= \sum_{s=-q}^{p} \left[|N^\alpha_{\lambda + \alpha(s-1)}|^2 - |N^\alpha_{\lambda + \alpha s}|^2 \right] \\
&= \sum_{s=-q-1}^{p-1} |N^\alpha_{\lambda + \alpha s}|^2 - \sum_{s=-q}^{p} |N^\alpha_{\lambda + \alpha s}|^2 \\
&= \sum_{s=-q-1}^{p-1} |N^\alpha_{\lambda + \alpha(s)}|^2 - \sum_{s=-q}^{p} |N^\alpha_{\lambda + \alpha s}|^2 \\
&= |N^\alpha_{\lambda - \alpha(q+1)}|^2 - |N^\alpha_{\lambda + \alpha p}|^2
\end{aligned} \tag{A.30}$$

Since we are dealing with a finite dimensional representation, we are free to choose the integers p and q such that the coefficients $N^\alpha_{\lambda - \alpha(q+1)}$ and $N^\alpha_{\lambda + \alpha p}$ vanish. Consequently, comparing the two expressions (A.27) and (A.30) for Γ we find the important equality

$$2 \frac{(\alpha, \lambda)}{(\alpha, \alpha)} = -(p - q) \tag{A.31}$$

From this we see immediately that for any representation with weight λ, the appropriately normalized inner product of the weight with the root vector is constrained to be an integer.

If we take $\lambda = \beta$ to be a root vector this statement allows one to completely classify the compact Lie groups. In this case we have the two copies of (A.31)

$$\begin{aligned}
2 \frac{(\alpha, \beta)}{(\alpha, \alpha)} &= -(p - q) = k \\
2 \frac{(\beta, \alpha)}{(\beta, \beta)} &= -(m - n) = l
\end{aligned} \tag{A.32}$$

Multiplying these two expressions together gives the interesting relationship for the angle θ between root vectors

$$\frac{kl}{4} = \frac{(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = \cos \theta \tag{A.33}$$

Since k and l are integers the possibilities for angles between roots are greatly restricted. In fact if we limit ourselves to angles between 90° and 180° there are only four different choices for θ: 90°, 120°, 135°, 150°.

The only detail we are missing about the simple root structure of any Lie algebra is the length of the simple roots. It can be shown that for any given simple Lie algebra, there are only two different lengths of simple root possible, each generally termed long and short [25]. This detail is unimportant since the special unitary $su(n)$ algebra in which we are most interested is simply laced, meaning all roots are of the same length.

There is a convenient pictorial way for presenting this information about the root structure of a Lie algebra we have collected. The Dynkin diagram method assigns a node to every simple root, open if it is long, shaded if it is short. Connecting the nodes to their neighbours are single, double or triple lines depending on whether the angle between simple roots is 120°, 135° or 150°, respectively. Simple roots connected by 90° are orthogonal and form their own subalgebra and hence do not arise for simple algebras. The completion of the classification of the simple Lie algebras is now an exercise in the classification of Dynkin diagrams ([25]). The results are that there are four infinite families of simple Lie algebras (the classical algebras): the special unitary algebra $su(r+1)$, the orthogonal algebras $so(2r+1)$ and $so(2r)$ and the symplectic algebra $sp(2r)$ where r is the rank. Additionally there are five exceptional simple Lie groups with associated algebras: E_6, E_7, E_8, F_4 and G_2 where the subscripts give the rank. The Dynkin diagrams for these algebras are displayed in Figure (A.1).

A.4 Fundamental Weights and Young Tableaux

We have already seen weight vectors for arbitrary representations in the previous Section during our discussion of simple roots. Consequently it is natural to expand weight vectors
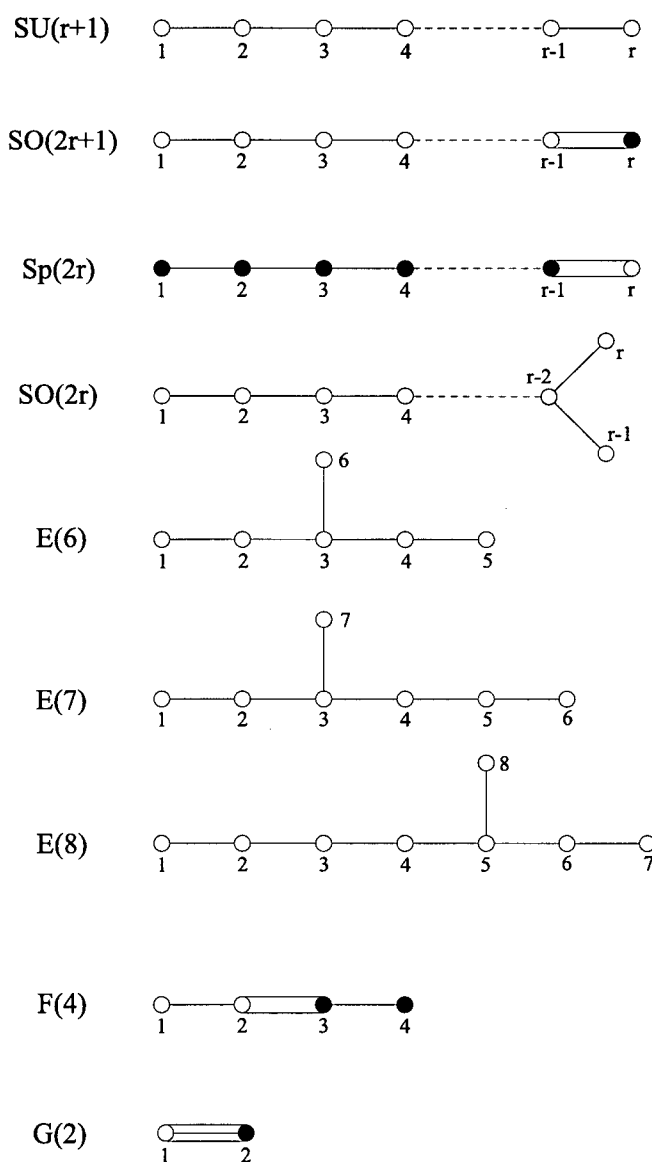
Figure A.1: The Dynkin diagrams for the root structure of the simple Lie groups. Here r is the rank of the algebra/group and shaded circles represent short roots while long roots are represented by open circles.

on a basis of simple roots. This is a perfectly valid expansion but one that suffers from the drawback that the components of the expansion are not necessarily integers for irreducible finite dimensional representations. This is a less than optimal situation since there exists a basis in which these components are integer valued - the basis of fundamental weights $\{\omega_j\}$. The fundamental weights are defined to be dual to the normalized simple roots $\{\alpha_i\}$

$$(\omega_j, \frac{2\alpha_i}{(\alpha_i, \alpha_i)}) = \delta_{ij} \tag{A.34}$$

The integer valued expansion coefficients λ_i of a weight λ on the basis of fundamental weights are called the Dynkin labels

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \tag{A.35}$$

We will denote the weight λ by these coefficients in the following component form

$$\lambda = \{\lambda_1, \ldots, \lambda_r\} \tag{A.36}$$

For any finite irreducible representation there is a unique highest weight state $|\kappa\rangle$ which is of course specified by its Dynkin labels $\{\kappa_1, \ldots \kappa_r\}$. The highest weight state is defined to be that for which the sum of its coefficients when expanded in a simple root basis is maximal. The result of this is that all positive ladder operators operate trivially on this state

$$E^\alpha |\kappa\rangle = 0 \quad , \quad \forall \alpha > 0 \tag{A.37}$$

The importance of this relation is that to each highest weight state we can unambiguously assign an irreducible representation of the algebra. Translation of the highest weight state by the action of the lowering ladder operators fills out the remaining states of the representation. We will not concern ourselves with the details of this procedure here but rather move on to a convenient method for describing irreducible representations which is more amenable to calculations.
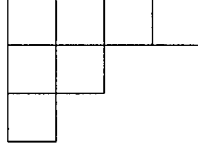
Figure A.2: An example of a Young tableau for a unitary group representation. The Dynkin labels associated with this representation are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$ and all others vanishing: $\{2, 1, 1, 0, \ldots, 0\}$. The row (partition) variables in this case are $l_1 = 4$, $l_2 = 2$, $l_3 = 1$ with all others vanishing: $(4, 2, 1, 0, \ldots, 0)$.

For the algebra of the special unitary group, $su(N)$, we have the Dynkin labels for a particular highest weight state and, hence, irreducible representation

$$\lambda = \{\lambda_1, \ldots, \lambda_{N-1}\} \tag{A.38}$$

It will be more convenient to describe the representation in terms of its partition

$$\lambda = (l_1, l_2, \ldots, l_{N-1}) \tag{A.39}$$

where

$$l_i = \lambda_i + \cdots + \lambda_{N-1} \tag{A.40}$$

To each set non-increasing sequence of integers $\{l_i\}$ we associate a Young tableaux $N - 1$ rows where the ith row from the top of the table contains l_i boxes. This graphical method of displaying irreducible representation is useful for calculations of tensor products of representations, a process which lies at the heart of all of the results in this Thesis. In Figure A.2 we give an example of these two labeling systems for a particular $su(n)$ representation. While we will only explicitly use Young tableaux for representations of $su(n)$, these methods can be extended in a straightforward way to encompass all non-spinor representations of any Lie group [45].

It is interesting to note the Young tableaux for the particular representations which we have been dealing with up to this point. First, since the Dynkin labels are the

coefficients of the weights of a representation in the basis of fundamental weights we have immediately that for the kth fundamental representation of $su(N)$, F_k

$$\lambda_{F_k} = \{0, \ldots, 0, 1, 0, \ldots, 0\} \tag{A.41}$$

where the only the kth component is non-vanishing. The corresponding Young table lengths are easy to deduce

$$\lambda_{F_k} = (1, \ldots, 1, 0, \ldots 0) \tag{A.42}$$

where the first k^{th} components are non-vanishing. Hence we see that the fundamental representations of $su(N)$ have Young tableaux that are nothing but a single column of boxes. Since we will be using the fundamental representations often it is convenient to define the column variables of the Young table to be the list of integers that gives the number of boxes in each column of the table starting from the left. We will denote the column variables by square brackets

$$\lambda = [n_1, n_2, \ldots] \tag{A.43}$$

For example, the representation shown in Figure A.2 is labeled in column notation as $[3, 2, 1, 1]$. Note that in contrast to the row variables where $N - 1$ l_i are enough to specify any $su(N)$ Young table, there is no limit to the number of column variables for a general table. On the other hand for the kth fundamental representation we have the concise form

$$\lambda_{F_k} = [k] \tag{A.44}$$

The one representation that has been front and center throughout this discussion is the adjoint representation and it too will play an important role in the future. The Young table for the adjoint representation in $su(N)$ can be determined from the Dynkin index

$$\lambda_{ad} = \{1, 0, \ldots, 0, 1\} \tag{A.45}$$

Figure A.3: The Young tableau for the kth fundamental representation of a unitary group. The Dynkin label associated with this representation are $\{0, 0, \ldots, 1, 0, \ldots, 0\}$ where the only non-vanishing element is in the kth position. The row (partition) variables in this case are $(1, 1, \ldots, 1, 0, \ldots, 0)$ where only the first k elements of the list are non-vanishing. The column label here is simply $[k]$.

By the definitions of the row and column variables we have

$$\lambda_{ad} = (2, 1, 1, \ldots, 1) = [N - 1, 1] \tag{A.46}$$

A.5 Group Characters

It will become useful throughout our calculations to have a way to encode the information carried by an irreducible representation in a functional way. The characters χ_R of a group serve this purpose and in this Section we will explain a few of their more important properties. One definition of a group character is as the trace of a group element in a particular representation R,

$$\chi_R(U) \equiv \text{Tr}_R U \tag{A.47}$$

Here U is an element of the group, which will be one of the unitary groups $SU(N)$ or $U(N)$. From this definition we see immediately that the character is invariant under group rotations of its argument. If g is an element of the group then

$$\chi_R(gUg^\dagger) = \text{Tr}_R(gUg^\dagger) = \chi_R(U) \tag{A.48}$$

due to the cyclic nature of the trace. Since a unitary matrix can be diagonalized by elements of the unitary group, we see that the character can only depend on the eigenvalues

of its argument. We will henceforth take U to be diagonal with $U_{ij} = \delta_{ij} \, e^{i\phi_j}$. We define any function of a group element that only depends on the eigenvalues of that element to be a class function.

Now we are in a position to give an explicit formula for the character of a group element in a particular representation due to Weyl [108]. If the irreducible representation R is associated with the Young table row variables $\{l_i\}$ then

$$\chi_R(U) = \frac{\det_{jk} | \, e^{i(l_k + N - k)\alpha_j} |}{\det_{jk} | \, e^{i(N-k)\alpha_j} |} \tag{A.49}$$

If we take the group element to be the identity element it is clear from the trace definition (A.47) that the character will count the number of elements in the representation and give the dimension as a function of the Young table variables

$$d_R = \chi_R(\mathbb{1}) = \prod_{i<j} \frac{l_i - l_j + j - i}{j - i} \tag{A.50}$$

The characters of a compact Lie group satisfy a number of important properties analogous to those of characters in discrete groups (see [90] for example). The first is the orthogonality condition

$$\int d\Omega \; \chi_R(\Omega)\chi_S(\Omega^\dagger) = \int d\Omega \; \chi_R(\Omega)\chi_S^*(\Omega) = \delta_{R,S} \tag{A.51}$$

Here we have integrated over the volume of Lie group with the Haar measure $d\Omega$. Integration over the group is a linear operation that has the properties of left and right invariance,

$$\int d\Omega \; f(g^{-1}\Omega) = \int d\Omega \; f(\Omega g) = \int d\Omega \; f(\Omega) \tag{A.52}$$

where g is any element of the Lie group, G. As well, if $f(\Omega) \geq 0 \; \forall g \in G$ then

$$\int d\Omega \; f(\Omega) \geq 0 \tag{A.53}$$

Finally, we will take the measure to be normalized so that

$$\int d\Omega \; 1 = 1 \tag{A.54}$$

The existence of such a measure and integration will not be discussed here but rather the reader is referred to standard texts on group theory (see [90] for example).

The second property of characters to discuss is that of completeness

$$\sum_R \chi_R(U)\chi_R^*(V) = \delta_c(U, V) \tag{A.55}$$

where $\delta_c(U, V)$ is a class function which is a δ-function on the eigenvalues of the group elements U and V up to permutations.

The fact that the group characters satisfy both orthogonality and completeness conditions suggests that they may be though of as a complete set of vectors spanning some abstract linear space. This is indeed the case and we will often use characters as basis elements in the space of irreducible representations. For example given some class function $f(U)$ we can expand it in a basis of characters

$$f(U) = \sum_R a_R \, \chi_R(U) \tag{A.56}$$

where the coefficients a_R can be calculated by the inverse expansion

$$\int dU \, \chi_S^*(U)f(U) = \sum_R a_R \int dU \, \chi_S^*(U)\chi_R(U) = a_R \tag{A.57}$$

where we have used the orthogonality property (A.51) of group characters. This situation may look familiar from standard Fourier analysis and for good reason. For the group $U(1)$ the irreducible representations, R are labeled by the integers n and the group elements are simply exponentials of a single angle: $U = e^{i\phi}$. Consequently, $\chi_R = e^{in\phi}$ and the character expansion (A.56) in the case of $U(1)$ is the discrete Fourier transform

$$f(\phi) = \sum_n a_n \, e^{in\phi} \tag{A.58}$$

In fact the notion of a group character naturally extends the concept of a Fourier or other orthogonal function expansion to an abstract setting. The details of this generalization are the contents of the Peter-Weyl theorem which states for a Lie group G

1) The group characters are dense in $L^2(G)$.

2) The space $L^2(G)$ decomposes into a Hilbert space direct sum of irreducible representations of G.

3) Every irreducible representation of G is finite dimensional.

4) Each irreducible representation of G occurs in $L^2(G)$ with a multiplicity equal to its dimension.

5) Any unitary representation of G on any Hilbert space decomposes into a Hilbert space direct sum of irreducible representations.

A proof of these statements can be found in [90] and we refer the reader there.

A pair of identities that follow from the orthogonality (A.51) and completeness (A.55) relations for group characters will prove to be of great utility in the analysis of Yang-Mills theory on the lattice (see Chapter two) are

$$\int d\Omega \ \chi_R(\Omega U \Omega^\dagger V) = \frac{1}{d_R}\chi_R(U)\chi_R(V) \qquad (A.59)$$

and

$$\int d\Omega \ \chi_R(\Omega U)\chi_S(\Omega^\dagger V) = \frac{1}{d_R}\chi_R(UV)\delta_{R,S} \qquad (A.60)$$

Each of these can be proved using the orthogonality and completeness relations of the group characters.

The last topic we will consider in this Section is the behaviour of the basis elements of the space of representations under a particular type of group transformation. For any compact Lie group there exists an Abelian subgroup (possibly trivial) that commutes with all elements of the group. This subgroup is called the center of the group and is denoted by Z. For example, the center of the special unitary group $SU(N)$ is generated

by an element z of the form

$$z = e^{2\pi i/N} \mathbb{1} \qquad (A.61)$$

and the remaining elements of the subgroup are integer powers of z, $\{z, z^2, \ldots z^N\}$. It is interesting to note how a general irreducible representation transforms under an element of Z. Defining the center transformation operator $\mathcal{Z}$ by

$$\mathcal{Z}\chi_R(U) \equiv \chi_R(zU) = \frac{\det_{jk} | e^{i(l_k+N-k)(2\pi/N+\alpha_j)}|}{\det_{jk} | e^{i(N-k)(2\pi/N+\alpha_j)}|} = e^{2\pi i l/N}\chi_R \qquad (A.62)$$

where $l = \sum l_i$ is the total number of boxes in the Young table associated with the representation R. Hence we see that transformation under the center divides the representation space into N different equivalence classes depending on the number of boxes in the Young table mod N. For future reference we note that the kth fundamental representation and the adjoint representation of the special unitary group transform as

$$\mathcal{Z}\chi_{F_k} = z^k \chi_{F_k} \quad , \quad \mathcal{Z}\chi_{ad} = \chi_{ad} \qquad (A.63)$$

In particular the adjoint representation transforms trivially under the center Z.

A.6 Young Tableaux and Tensor Products of Representations

As is shown in Chapter two, pure Yang-Mills theory in two dimensions is a very simple field theory that has a strong dependence on boundary conditions. The main point of this Thesis is to introduce and describe the interaction of classical colour-electric charges in this environment. The basis of such a program is the calculation of tensor products of irreducible representations of the compact Lie gauge group. Here we will give the rules of multiplying irreducible representations and decomposing the result into a linear combination of irreducible representations using the very convenient methods of Young tableaux for $SU(N)$.

In terms of group characters, the tensor products of irreducible representations R and S will be represented as

$$\chi_R \chi_S = \chi_{R \otimes S} = \chi_{T_1 \oplus \cdots \oplus T_k} = \sum_T N_{RS}^T \chi_T \qquad \text{(A.64)}$$

Here the first equality is the product identity for group characters and the last follows from linearity. The fusion number N_{RS}^T is a non-negative, integer-valued coefficient that counts the number of times the irreducible representation T appears in the product of R and S, or equivalently, the number of trivial representations in the triple product $R \otimes S \otimes T$.

The determination of the fusion numbers is a completely group theoretic calculation which we now describe for $SU(N)$ in terms of Young tableaux. The Littlewood-Richardson rule gives the process for re-arranging the Young tables of representations R and S into a linear combination of representations $\{T\}$. This process begins with the filling of the first row of one table with 1's, the second row with 2's and so on until the table is full. We then move boxes labeled with 1's and add them to the right side of the second table such that the following conditions are met

1) New tables must satisfy the dominance condition: the number of boxes in a row is less than or equal to the number of boxes in the row immediately above.

2) No column of a new table can contain more than one box marked 1.

Any table which does not meet these conditions is ignored. The process now repeats using the boxes in the second row marked with 2's. Again all resulting tables have to fulfill the conditions above where in 2), 1 is replaced by 2. In addition to these two constraints adding a second row introduces the new constraint

3) Counting from the top and right of a table, the number of 1's must always be greater or equal to the number of 2's.

This whole process is iterated until all rows of the first table have been added to the second. If there are more than three rows in the first table then the last condition is extended so that the number of 2's is greater than or equal to the number of 3's, and so on. Finally, to the remaining tables we impose two more restrictions

4) Any table with more than N boxes in the first column is to be ignored.

5) All columns with exactly N boxes are to be removed from their respective tables.

At the end of this process one has the graphical decomposition of the tensor product of R and S. For a representation T, the fusion number N_{RS}^{T} is the number of tables resulting from the Littlewood-Richardson process with the associated Young table and distinct patterns of 1's, 2's, 3's, and so on.

The first explicit example of an $SU(N)$ tensor product we will calculate is the simple one of the $k = 1$ fundamental representation with the $k = N - 1$ fundamental representation. In terms of Young column variables we have

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \vdots \\ \hline N\text{-}1 \\ \hline \end{array} \quad \otimes \quad \square$$

Applying the Littlewood-Richardson procedure we find this product can be decomposed into the direct sum of irreducible representations

The first representation is the trivial one, $\mathbb{1}$ and the second is the adjoint representation

for $SU(N)$. Consequently we see the connection between these two representations in Young table form. From the Littlewood-Richardson rules we deduce that a column of boxes are anti-symmetrized with respect to each other and so the anti-symmetric combination of the $k = 1$ and the $k = N - 1$ fundamental representations is the trivial one. In fact this a special case of the existence of a conjugate $\bar{R}$ for every irreducible representation R for which $\bar{R} \otimes R$ contains the trivial representation. In fact we can express the conjugate of any representation in $SU(N)$ easily in terms of the column Young table variables

$$\overline{[n_1, n_2, \ldots, n_k]} = [N - n_k, \ldots, N - n_2, N - n_1] \qquad (A.65)$$

Furthermore, from the general formula for dimensions of a representation, (A.50), it can be shown that the conjugate is of the same dimension as the original representation, $d_{\bar{R}} = d_R$.

The second example we will give is much less trivial and is of great usefulness in the Chapter five. There we will require the tensor product of any fundamental $[k]$ with the adjoint $[N - 1, 1]$ representation.

$$[N - 1, 1] \otimes [k] = [k] \oplus [N - 1, k, 1] \oplus [k - 1, 1] \oplus [N - 1, k + 1] \qquad (A.66)$$

The second last term is absent if $k = 1$ and likewise the last is absent if $k = N - 1$ in order to satisfy the conditions of the Littlewood- Richardson procedure.

A.7 Invariant Tensors and Casimir Operators

The last aspect of the theory of compact Lie groups which we will require is that of Casimir operators. Since Yang-Mills is a gauge theory the physically relevant quantities in the theory are all gauge invariant and in calculations of gauge invariant objects it is common to find traces of the form

$$\Lambda_R^{a_1 \cdots a_n} = \text{Tr } T_R^{a_1} \cdots T_R^{a_n} \tag{A.67}$$

Objects of this form are generally referred to as invariant tensors where the name follows from the fact that the trace is invariant under the operation

$$T_R^a \rightarrow U_R T_R^a U_R^\dagger = (U_{ad})^{ab} T_R^b \tag{A.68}$$

where U_R is an element of the group in the representation R and U_{ad} is taken in the adjoint representation. Hence

$$\Lambda_R^{a_1 \cdots a_n} = (U_{ad})^{a_1 b_1} \cdots (U_{ad})^{a_n b_n} \Lambda_R^{b_1 \cdots b_n} \tag{A.69}$$

If one takes an explicit parameterization of U_{ad} in terms of the exponential map and the structure constants for the adjoint representation, (A.5)

$$U(\mathbf{n})^{ij} = \exp\left[i\mathbf{T}_{ad} \cdot \mathbf{n}\right]^{ij} = \exp\left[f^{iaj} n^a\right] \sim \delta^{ij} + f^{iaj} n^a \cdots \tag{A.70}$$

In this infinitesimal form, the identity (A.69) takes on the form

$$\sum_k \sum_b f^{a_k cb} \Lambda_R^{a_1 \cdots a_{k-1} b a_{k+1} \cdots a_n} = 0 \tag{A.71}$$

This gives a generalization of the Jacobi identity (A.3) and may be used as a definition of an invariant tensor.

To every invariant tensor one can associate a Casimir operator $C_R(\Lambda)$

$$C_R(\Lambda) = \sum_{a_1 \ldots a_n} T_R^{a_1} \ldots T_R^{a_n} \Lambda_R^{a_1 \ldots a_n} \tag{A.72}$$

Casimir operators have the property that they commute with all the generators of the representation they are taken in (ie. $[C_R(\Lambda)), T_R^a] = 0$. This can be proved using the generalized Jacobi identity

$$
\begin{aligned}
[C_R(\Lambda), T_R^b] &= \sum_{a_1 \dots a_n} \Lambda_R^{a_1 \dots a_n} [T_R^{a_1} \dots T_R^{a_n}, T_R^b] \tag{A.73} \\
&= \sum_{a_1 \dots a_n} \sum_{b,k} \Lambda_R^{a_1 \dots a_n} i f^{a_k bc} T_R^1 \dots T_R^{a_{k-1}} T_R^b T_R^{a_{k+1}} \dots T_R^{a_n} = 0
\end{aligned}
$$

By Schur's Lemma [90], this commutativity property leads directly to the conclusion that all Casimir operators are proportional to the identity element.

Of particular interest in two-dimensional Yang-Mills theory is the quadratic Casimir operator C_2. From the general definition (A.72) with $n = 2$ we have for the basis element of irreducible representation R

$$
C_2 \chi_R = \operatorname{Tr} T_R^a T_R^a \chi_R = C_2(R) \chi_R \tag{A.74}
$$

where we have labeled the eigenvalue $C_2(R)$. Of course there are also higher Casimir operators but we will not need them here. More information on these can be found in the papers [74, 75, 76].

The actual evaluation of the quadratic Casimir operator for a given representation is most easily carried out when one works in the Cartan-Weyl basis that we have introduced at the beginning of this Appendix. In this basis the quadratic Casimir operator has the form

$$
C_2 = \sum_i H^i H^i + \sum_{\alpha > 0} \frac{(\alpha, \alpha)}{2} (E^\alpha E^{-\alpha} + E^{-\alpha} E^\alpha) \tag{A.75}
$$

where the second sum is over all positive roots. Since the Casimir operator commutes with all generators of the algebra, the evaluation of the eigenvalue is the same for any state of the representation. The most obvious, and convenient choice of state on which to evaluate (A.75) is the state of highest weight $|\lambda\rangle$. Recalling the action of H on a state

(A.11), the evaluation of the first term is straightforward

$$\sum_i H^i H^i |\lambda\rangle = \sum_i \lambda^i \lambda^i |\lambda\rangle = (\lambda, \lambda)|\lambda\rangle \tag{A.76}$$

Since the state is of highest weight, E^α annihilates it for $\alpha > 0$, which we are summing over. Hence the only contribution from the second sum is of the form

$$\sum_{\alpha>0} \frac{(\alpha, \alpha)}{2}[E^\alpha, E^{-\alpha}]|\lambda\rangle = \sum_{\alpha>0,i} \alpha^i H^i |\lambda\rangle = \sum_{\alpha>0,i} (\alpha, \lambda|\lambda\rangle \tag{A.77}$$

where in the second line we have used the previous result for the commutator $[E^\alpha, E^{-\alpha}]$ from (A.14). The sum over all the positive roots defines the Weyl vector ρ up to a factor of two [90]

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum_i \omega_i = \{1, \ldots, 1\} \tag{A.78}$$

where $\{\omega_i\}$ are the fundamental weights of the algebra. With this definition the Casimir operator acting on a highest weight state $|\lambda\rangle$, and hence the associated irreducible representation, R is given by

$$C_2|\lambda\rangle = (\lambda, \lambda + 2\rho)|\lambda\rangle \tag{A.79}$$

Consequently, the quadratic Casimir operator is diagonal on the basis of weights and, hence, irreducible representations as expected from earlier general arguments. The eigenvalue is simply the inner product of the highest weight of a representation with itself, shifted by twice the Weyl vector. The inner product here is defined, as always, by the Killing form of the algebra (see Equation (A.21)).

Explicit expressions for the eigenvalues of quadratic Casimir operators acting on tensor representations of the classical Lie groups can be given in terms of simple formulae of the Young table row variables (l_i). Using the definition of the row variables as a partition of the Dynkin weights,

$$l_i = \lambda_i + \cdots + \lambda_{N-1} \tag{A.80}$$

the eigenvalue of the Casimir operator (A.79) for a representation R of $SU(N)$ with associated Young table (l_i) is

$$C_2(R) = \sum_{i=1}^{N} l_i(N + l_i + 1 - 2i - l/N) \tag{A.81}$$

where we have denoted the total number of boxes in the Young table by $l = \sum l_i$. As well we have the Casimir operator for the unitary group

$$U(N) : C_2(R) = \sum_{i=1}^{N} l_i(N + l_i + 1 - 2i) \tag{A.82}$$

In a similar way the Casimir eigenvalues can be calculated in terms of Young tableaux variables for the other classical Lie groups [71]

$$Sp(2N) : C_2(R) = \frac{1}{2}\sum_{i=1}^{N} l_i(l_i + 2N + 2 - 2i) \tag{A.83}$$

$$SO(N) : C_2(R) = \sum_{i=1}^{N} l_i(l_i + N - 2i) \tag{A.84}$$

Finally, for the orthogonal groups $SO(N)$ there are representations that do not admit Young tableaux. The Casimir eigenvalues of these spinor representations consequently must be calculated from the general formula (A.79). Typically this is a tedious task. For our applications we have found that existing tables [107, 73, 88] are more than adequate.

Bibliography

[1] T. Banks and A. Ukawa: Deconfining and Chiral Phase Transitions in Quantum Chromodynamics at Finite Temperature, *Nuclear Physics* **B225** (1983), 145.

[2] G. Bhanot, K. Demeterfi and I. Klebanov: 1+1-Dimensional Large N QCD Coupled to Adjoint Fermions, *Physical Review* **D48** (1994), 4980; hep-th/9307111.

[3] M. Billo, M. Caselle, A. D'Adda and S. Panzeri: Finite Temperature Lattice QCD in the Large N Limit, *International Journal of Modern Physics* **A12** (1997), 1783; hep-th/9610144.

[4] O. Borisenko, M. Faber and G. Zinoviev: Deconfinement in QCD with Dynamical Quarks, *Modern Physics Letters* **A12** (1997), 949; hep-lat/9604020.

[5] E. Brezin, C. Itzykson, G. Parisi and J.-B. Zuber: Planar Diagrams, Communications in Mathematical Physics **59** (1978), 35.

[6] C.G. Callen: Broken Scale Invariance in Scalar Field Theory, *Physical Review* **D2** (1970), 1541.

[7] C.G. Callen, R.F. Dashen, D.J. Gross: The Structure of the Gauge Theory Vacuum, *Physics Letters* **63B** (1976), 334.

[8] M. Caselle, A. D'Adda, L. Magnea and S. Panzeri: Two-Dimensional QCD on the Sphere and on the Cylinder, *Proceedings of the 1993 Trieste Summer School in High Energy Physics and Cosmology*; hep-th/9309107.

[9] S. Coleman, R. Jackiw and L. Susskind: Charge Shielding and Quark Confinement in the Massive Schwinger Model, *Annals of Physics (N.Y.)* **93** (1975), 267.

[10] S. Coleman: More About the Massive Schwinger Model, *Annals of Physics (N.Y.)* **101** (1976), 239.

[11] S. Coleman: *Aspects of Symmetry*, Cambridge University Press, 1985.

[12] S. Cordes, G. Moore and S. Ramgoolam: Lectures on 2D Yang-Mills Theory, Equivariant Cohomology and Topological Field Theories, *Proceedings of the 1994 Trieste Spring School on Superstrings*; hep-th/9411210.

[13] S. Dalley and I. Klebanov: String Spectrum of 1+1-Dimensional Large N QCD with Adjoint Matter, *Physical Review* **D47** (1993), 2517; hep-th/9209049.

[14] S. Dalley: Adjoint QCD_2 and the non-Abelian Schwinger Mechanism, *Physics Letters* **B418** (1998), 160; hep-th/9708115.

[15] P.H. Damgaard and M. Hasenbusch: Screening and Deconfinement of Sources in Finite Temperature SU(2) Lattice Gauge Theory, *Physics Letters* **B331** (1994), 400; hep-lat/9404008.

[16] S.R. Das, A. Dhar, A.M. Sengupta and S.R. Wadia: Critical Behaviour in Some D=1 Large N Matrix Models, *Modern Physics Letters* **A5** (1990), 891.

[17] S.R. Das and A. Jevicki: String Field Theory and Physical Interpretation of D=1 Strings, *Modern Physics Letters* **A5** (1990), 1639.

[18] D. Demeterfi, G. Bhanot and I. Klebanov: Glueball Spectrum in a (1+1)-dimensional Model for QCD, *Nuclear Physics* **B418** (1994), 15; hep-th/9311015.

[19] M. Douglas and V. Kazakov; Large N Phase Transition in Continuum QCD_2, *Physics Letters* **319B** (1993), 219; hep-th/9305047.

[20] M. Douglas: Large N Gauge Theory - Expansions and Transitions, *Nuclear Physics Proceeding Supplement* **41** (1995), 66.

[21] M. Engelhardt: One-Dimensional Classical Adjoint SU(2) Coloumb Gas, *Physical Review* **D56** (1997), 1050; hep-th/9701105.

[22] S. Fubini and G. Veneziano: *Nuovo Cimento* **64A** (1969), 811.

[23] C.R. Gattringer, L.D. Paniak and G.W. Semenoff: Deconfinement Transition for Quarks on a Line, *Annals of Physics (N.Y.)* **256** (1997), 74; hep-th/9612030.

[24] M. Gell-Mann and F. Low: Quantum Electrodynamics at Small Distances, *Physical Review* **95** (1954), 1300.

[25] H. Georgi: *Lie Algebras in Particle Physics*, Addison-Wesley Publishing, 1982.

[26] P. Goddard and D. Olive: The Magnetic Charges of Stable Selfdual Monopoles, *Nuclear Physics* **B191** (1981), 528.

[27] M.B. Green, J.H. Schwarz and E. Witten: *Superstring Theory: Volume I*, Cambridge University Press, 1987.

[28] V. Gribov: Unpublished notes of the Lenningrad Winter School, 1976.

[29] G. Grignani, G. W. Semenoff and P. Sodano; Confinement-Deconfinement Transition in 3-Dimensional QED, *Physical Review* **D53** (1996), 7157; hep-th/9504105.

[30] G. Grignani, G. W. Semenoff, P. Sodano and O. Tirkkonen: Charge Screening in the Finite Temperature Schwinger Model, *International Journal of Modern Physics* **A11** (1996), 4103; hep-th/9511110.

[31] G. Grignani, G. W. Semenoff, P. Sodano and O. Tirkkonen: Charge Screening and Confinement in Hot 3-D QED, *Nuclear Physics* **B473** (1996), 143; hep-th/9512048.

[32] G. Grignani, L. Paniak, G. Semenoff and P. Sodano: Loop Correlators and Theta States in 2D Yang-Mills Theory, *Annals of Physics (N.Y.)* **260** (1997), 275; hep-th/9705102.

[33] D.J. Gross and F. Wilczek: Ultraviolet Behavior of Non-Abelian Gauge Theories, *Physical Review Letters* **30** (1973), 1343.

[34] D. Gross and E. Witten: Possible Third Order Phase Transition in the Large N Lattice Gauge Theory, *Physical Review* **D21** (1980), 446.

[35] D. Gross, R. Pisarski and L. Yaffe: QCD and Instantons at Finite Temperature, *Reviews of Modern Physics* **53** (1981), 43.

[36] D. Gross and W. Taylor: Two Dimensional QCD is a String Theory, *Nuclear Physics* **B400** (1993), 181; hep-th/9301068.

[37] D. Gross and W. Taylor: Twists and Wilson Loops in the String Theory of Two Dimensional QCD, *Nuclear Physics* **B403** (1993), 395; hep-th/9303046.

[38] D. J. Gross and A. Matytsin: Some Properties of Large N Two Dimensional Yang-Mills Theory, *Nuclear Physics*, **B437**, 541, (1995); hep-th/9410054.

[39] D. J. Gross and A. Matytsin: Instanton Induced Large N Phase Transitions in Two and Four Dimensional QCD, *Nuclear Physics* **B429** (1994,) 50; hep-th/9404004.

[40] D. Gross, I. Klebanov, A. Matytsin and A. Smilga: Screening vs. Confinement in 1+1 Dimensions, *Nuclear Physics* **B461** (1996), 109; hep-th/9511104.

[41] D. Gross, A. Hashimoto and I. Klebanov: The Spectrum of a Large N Gauge Theory Near Transition from Confinement to Screening, *Physical Review* **D57** (1998), 6420; hep-th/9710240.

[42] S. Gubser and I. Klebanov: A Modified c=1 Matrix Model with New Nritical Behavior, *Physics Letters* **B340** (1994), 35; hep-th/9407014.

[43] R. Hagedorn: Statistical Thermodynamics of Strong Interactions at High Energies, *Nuovo Cimento Supplement* **3** (1965), 147.

[44] C. R. Hagen and A. J. Macfarlane: Reduction of Representations of SU_{mn} with Respect to the Subgroup $SU_m \otimes SU_n$, *Journal of Mathematical Physics* **6** (1965), 1355.

[45] M. Hammermesh: Group Theory and its Application to Physical Problems, Dover Publishing, New York, 1964.

[46] G. t'Hooft: A Planar Diagram Theory for Strong Interactions, *Nuclear Physics* **B72** (1974), 461.

[47] K. Huang and S. Weinberg: Ultimate Temperature and the Early Universe, *Physical Review Letters* **25** (1970), 895.

[48] C. Itzykson and J.-M. Drouffe: *Statistical Field Theory*, Cambridge University Press, 1989.

[49] R. Jackiw and C. Rebbi: Vacuum Periodicity in a Yang-Mills Quantum Theory, *Physical Review Letters* **37** (1976), 172.

[50] S. Jaimungal and L. Paniak: Theta Sectors and Thermodynamics of a Classical Adjoint Gas, *Nuclear Physics* **B517** (1998), 622; hep-th/9710044.

[51] A. Jevicki and B. Sakita: Loop Space Representation and the Large N Behaviour of the One Plaquette Kogut-Susskind Hamiltonian, *Physical Review* **D22** (1980), 467.

[52] V. Kazakov: Connection Between Quantum Chromodynamics and a Free String on a Lattice, *Soviet Physics-JETP* **58** (1983), 1096.

[53] J. Kiskis: Behaviour of Higher Representation Wilson Lines in Finite Temperature SU(2) Lattice Gauge Theory, *Physical Review* **D41** (1990), 3204.

[54] I. Klebanov and A. Hashimoto: Non-Perturbative Solution of Matrix Models Modified by Trace-Squared Terms, *Nuclear Physics* **B434** (1995), 264; hep-th/9409064.

[55] I. Kogan and A. Zhitnitsky: Two dimensional QCD with matter in adjoint representation: What does it teach us?, *Nuclear Physics* **B465** (1996), 99; hep-ph/9509322.

[56] J. Koller and P. van Baal: A Nonpertubative Analysis in Finite Volume Gauge Theory, *Nuclear Physics* **B302** (1988), 1.

[57] I. Kostov: Strong Coupling Planar Diagrams, Random Surfaces, and the Large N Phase Transition in U(N) Lattice Gauge Theories, *Nuclear Physics* **B265** (1986), 223.

[58] D. Kutasov: Two Dimensional QCD Coupled to Adjoint Matter and String Theory, *Nuclear Physics* **B414** (1994), 33; hep-th/9306013.

[59] E. Langmann and G. Semenoff: Gauge Theories on a Cylinder, *Physics Letters* **B296** (1992), 117; hep-th/9210011.

[60] E. Langmann and G. Semenoff: Gribov Ambiguity and Non-Trivial Vacuum Structure of Gauge Theories on a Cylinder, *Physics Letters* **B303** (1993), 303; hep-th/9212038.

[61] A. Lenard: Exact Statistical Mechanics of a One-Dimensional System with Coulomb Forces, *Journal of Mathematical Physics* **2** (1961), 682.

[62] F. Lenz, M. Shifman and M. Thies: Quantum Mechanics of the Vacuum State in Two-Dimensional QCD with Adjoint Fermions, *Physical Review* **D51** (1995), 7060; hep-th/9412113.

[63] D. E. Littlewood: *The Theory of Group Characters*, Second Edition, University of Oxford Press, Oxford, 1958.

[64] J. Maldacena: The Large N Limit of Superconformal Field Theories and Supergravity; hep-th/9711200.
S. Gubser, I. Klebanov and A. Polyakov: Gauge Theory Correlators from Non-Critical String Theory; hep-th/9802109.
E. Witten: Anti-de Sitter Space And Holography; hep-th/9802150.

[65] A.A. Migdal: Recursion Equations in Gauge Theories, *Soviet Physics-JETP* **42** (1975), 413.

[66] M. Mimura and H. Toda: *Topology of Lie Groups I and II*, Translations of Mathematical Monographs **91**, American Mathematical Society, Providence, RI, 1991.

[67] J.A. Minahan and A.P. Polychronakos: Equivalence of Two-Dimensional QCD and the c=1 Matrix Model, *Physics Letters* **B312**, 155, (1993); hep-th/9303153.

[68] T. Muir and W. Metzler: *Theory of Determinants*, George Banta Publishing Company, Albany NY, 1930.

[69] Y. Nambu, B. Bambah and M. Gross: One Dimensional Quark Gas, *Physical Review* **D26** (1982), 2875.

[70] L.D. Paniak, G.W. Semenoff and A.R. Zhitnitsky: Vacuum Structure and θ-States of Adjoint QCD in Two Dimensions, *Nuclear Physics* **B487** (1997), 191; hep-th/9606194.

[71] L.D. Paniak, G.W. Semenoff and A.R. Zhitnitsky: Vacuum Structure of Two Dimensional Gauge Theories For Arbitrary Gauge Groups, *Nuclear Physics* **B506** (1997), 521; hep-ph/9701270.

[72] L.D. Paniak: Order Parameter for Confinement in Large N Gauge Theories with Fundamental Matter, *Physical Review* **D56** (1997), 6566; hep-th/9707092.

[73] J. Patera and D. Sankoff: *Tables of Branching Rules for Representations of Simple Lie Algebras*, Les Presses de l'Université de Montréal, Montréal, 1973.

[74] A.M. Perelomov and V.M. Popov: Casimir Operators for the Unitary Group, *JETP Letters* **1** (1965) ,160.

[75] A.M. Perelomov and V.M. Popov: Casimir Operators for the Orthogonal and Symplectic Groups, *JETP Letters* **2** (1965), 20.

[76] A.M. Perelomov and V.M. Popov: The Casimir Operators for the Unitary Group , *Soviet Journal of Nuclear Physics*, **5** (1967), 489.

[77] J. Polchinski: High Temperature Limit of the Confining Phase, *Physical Review Letters* **68** (1992), 1267; hep-th/9109007.

[78] J. Polchinski: Strings and QCD?, Presented at the Symposium on Black Holes, Wormholes, Membranes and Superstrings, H.A.R.C., Houston TX, January 1992; hep-th/9210045.

[79] H.D. Politzer: Reliable Perturbation Results for Strong Interactions?,*Physical Review Letters* **30** (1973), 1346.

[80] A. M. Polyakov: Thermal Properties of Gauge Fields and Quark Liberation, *Physics Letters* **72B** (1978), 477.

[81] A. M. Polyakov: *Gauge Fields and Strings*, Harwood Academic Publishers, Chur, Switzerland, 1987.

[82] A. Roberge and N. Weiss: Gauge Theories with Imaginary Chemical Potential and the Phases of QCD, *Nuclear Physics* **B275** (1986), 734.

[83] H. Rothe, K. Rothe and J Swieca: Screening versus Confinement, *Physical Review* **D 19** (1979), 3020.

[84] B.Ye. Rusakov: Loop Averages and Partition Functions in U(N) Gauge Theory on Two-Dimensional Manifolds , *Modern Physics Letters***A5** (1990), 693.

[85] L.S. Schulman: *Techniques and Applications of Path Integration*, Wiley, New York, 1981.

[86] G. W. Semenoff, O. Tirkkonen and K. Zarembo: Exact Solution for the One-Dimensional Non-Abelian Coloumb Gas at Large N, *Physical Review Letters* **77** (1996), 2174; hep-th/9605172.

[87] G. W. Semenoff and K. Zarembo: Adjoint Non-Abelian Coluomb Gas at Large N, *Nuclear Physics* **B480** (1996), 317; hep-th/9606117.

[88] R. Slansky: Group Theory for Unified Model Building , *Physics Reports* **79** (1981), 1.

[89] J. Spanier and K.B. Oldham: *An Atlas of Functions*, Hemisphere Publishing Company, New York, 1987.

[90] S. Sternberg: Group Theory and Physics, Cambridge University Press, New York, 1997.

[91] F. Sugino and O. Tsuchiya: Critical Behavior in c=1 Matrix Model with Branching Interactions, *Modern Physics Letters* **A9** (1994), 3149; hep-th/9403089.

[92] L. Susskind: Lattice Models of Quark Confinement at High Temperature, *Physical Review* **D20** (1979), 2610.

[93] B. Svetitsky and L. Yaffe: Critical Behaviour at Finite Temperature Confinement Transitions, *Nuclear Physics* **B210** (1982), 423.

[94] B. Svetitsky: Symmetry Aspects of Finite-Temperature Confinement Transitions, *Physics Reports* **132** (1986), 1.

[95] K. Symanzik, Small Distance Behavior in Field Theory and Power Counting, *Communications in Mathematical Physics* **18** (1970), 227.

[96] S.R. Wadia: The N=Infinity Phase Transition in a Class of Exactly Soluble Model Lattice Gauge Theories, *Physics Letters* **93B** (1980), 403.

[97] S. Weinberg: *The Quantum Theory of Fields II*, Cambridge University Press, New York, 1996.

[98] G.H. Weiss and A.A. Maradudin: The Baker-Hausdorff Formula and a Problem in Crystal Physics, *Journal of Mathematical Physics* **3** (1962), 771.

[99] N. Weiss: The Effective Potential for the Order Parameter of Gauge Theories at Finite Temperature, *Physical Review* **D24** (1981), 475.

[100] N. Weiss: Wilson Line in Finite Temperature Gauge Theories, *Physical Review* **D25** (1982), 2667.

[101] N. Weiss: How to Distinguish a Confining from Deconfining Phase in Gauge Theories with Fermions, *Physical Review* **D35** (1987), 2495.

[102] K. Wilson: Confinement of Quarks , *Physical Review* **D10** (1974), 2445.

[103] E. Witten: Current Algebra Theorems for the $U(1)$ 'Goldstone Boson', *Nuclear Physics* **B156** (1979), 269.

[104] E. Witten: Baryons in the $1/N$ Expansion, *Nuclear Physics* **B160** (1979), 57.

[105] E. Witten: Theta Vacua in Two-Dimensional Quantum Chromodynamics, *Nuovo Cimento* **51A** (1979), 325.

[106] E. Witten: Two Dimensional Gauge Theories Revisited, *J. Geom. Phys.* **9** (1992), 303.

[107] B. Wybourne and P. Butler: *Symmetry Principles and Atomic Spectroscopy*, Wiley Interscience, New York, 1970.

[108] H. Weyl: *The Classical Groups*, Princeton University Press, Princeton, N.J., 1946.

[109] G.B. Whitham: *Linear and Non-linear Waves*, Wiley, New York, 1974.

[110] C.N. Yang and R.L. Mills: Conservation of Isotopic Spin and Isotopic Gauge Invariance, *Physical Review* **96** (1954), 191.

[111] K. Zarembo: Collective Field Approach to Gauged Principal Chiral Field at Large N, *Modern Physics Letters* **A10** (1995), 677; hep-th/9405080.

[112] K. Zarembo: Lattice Gauged Principal Chiral Field at Large N, *Teor. i Mat. Fiz.* **104** (1995), 25.

[113] D.P. Zhelobenko: *Compact Lie Groups and Their Representations*, Translations of Mathematical Monographs **40**, The American Mathematical Society, Providence, RI, 1973.