FIELD THEORETICAL QUANTITIES IN THE FRACTIONAL QUANTUM HALL EFFECT

By

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Abstract

This thesis studies two models of the fractional quantum Hall effect (FQHE), the bosonic (Chern-Simons-Landau-Ginzburg) description and the fermionic (composite fermion gauge theory) description. The bosonic theory attempts to describe the FQHE states at filling fractions \( \nu = \frac{1}{2n+1} \) while the fermionic theory attempts to describe the states at \( \nu = \frac{2np \pm 1}{2np} \) and the metallic states in between.

Within the bosonic theory, the fractionally charged quasiparticles of the FQH system are vortices which appear during the breakdown of the uniform quantum Hall state. The energetics of a single vortex state are studied whereby it is shown how the system may become unstable to the formation of vortices. Numerical vortex profiles are computed by minimising the Hamiltonian.

Using the fermionic theory of composite fermions interacting with gauge fluctuations, we consider two important field theoretic quantities, the self-energy and the thermodynamic potential in a finite magnetic field. We find that the conventional Luttinger-Ward treatment of the oscillatory behaviour of the thermodynamic potential is not applicable in two dimensions, for any kind of interaction. Instead we propose a new formulation which omits all crossed graphs and which necessarily includes the oscillatory self-consistent self-energy.

To second order in perturbation theory, the oscillatory self-energy is calculated by retaining Landau level quantisation on the internal fermion line. The low energy form of the self-consistent self-energy is obtained by means of a new iterative procedure which is introduced here. This procedure makes use of the structure introduced by Landau level quantisation. We also investigate the structure induced in the analogous two dimensional electron-phonon problem, in order to assist our understanding of the composite fermion
self-energy. In the low energy limit, it is found that the renormalised form of the composite fermion Green's function is of the same form as the unrenormalised Green's function. Therefore we argue that the principal effects of interactions may be accounted for using a field-dependent renormalised mass.

The iterative procedure for finding the self-consistent self-energy is used to evaluate the renormalised gap between the Fermi energy and the first excited states, which rapidly converges in a few iterations. We find a significant departure from the asymptotic result obtained by ignoring Landau level quantisation in the regime of experimentally relevant values of the parameters. We compare our findings with measurements of the gap in fractional Hall states near $\nu = 1/2$. 
# Table of Contents

Abstract ii  
Table of Contents iv  
List of Tables viii  
List of Figures ix  
Acknowledgements xiv  

1 Introduction 1  
1.1 Physics in Two Dimensions 3  
1.2 The Quantum Hall Effect 5  
1.2.1 The Integer Effect 6  
1.2.2 The Fractional Effect and the Laughlin Theory 9  
1.3 Chern-Simons Field Theories and the Transmutation of Statistics 11  
1.4 Overview 13  
1.4.1 Vortices 14  
1.4.2 A Novel State of Matter? 14  
1.4.3 Magnetic Oscillations - the Old Meets the New 15  

2 Vortices and the Bosonic Description of the FQHE 17  
2.1 Superfluid Analogy 18  
2.1.1 Off-Diagonal Long Range Order 19
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1.2</td>
<td>The Excitation Spectrum</td>
<td>19</td>
</tr>
<tr>
<td>2.1.3</td>
<td>Experiments</td>
<td>20</td>
</tr>
<tr>
<td>2.2</td>
<td>Vortices</td>
<td>22</td>
</tr>
<tr>
<td>2.3</td>
<td>Numerical Vortex Configurations</td>
<td>28</td>
</tr>
<tr>
<td>2.4</td>
<td>Discussion of Results</td>
<td>35</td>
</tr>
<tr>
<td>2.5</td>
<td>Summary</td>
<td>38</td>
</tr>
<tr>
<td>3.1</td>
<td>Composite Fermions</td>
<td>42</td>
</tr>
<tr>
<td>3.1.1</td>
<td>Experimental Evidence for Composite Fermions</td>
<td>42</td>
</tr>
<tr>
<td>3.1.2</td>
<td>The Composite Fermion Gauge Theory</td>
<td>44</td>
</tr>
<tr>
<td>3.2</td>
<td>The Self-Energy at $\Delta B = 0$</td>
<td>48</td>
</tr>
<tr>
<td>3.3</td>
<td>The Self-Energy for $\Delta B \neq 0$</td>
<td>50</td>
</tr>
<tr>
<td>3.4</td>
<td>The Renormalised Gap</td>
<td>55</td>
</tr>
<tr>
<td>3.5</td>
<td>Phonon Interactions</td>
<td>59</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Debye Phonons</td>
<td>60</td>
</tr>
<tr>
<td>3.5.2</td>
<td>Einstein Phonons</td>
<td>65</td>
</tr>
<tr>
<td>3.6</td>
<td>Iterative Self-Consistent Results for $\Sigma(e)$ and $\Delta \omega_c$</td>
<td>68</td>
</tr>
<tr>
<td>3.7</td>
<td>The Effective Mass: Comparison to Experiment</td>
<td>72</td>
</tr>
<tr>
<td>3.8</td>
<td>Summary</td>
<td>76</td>
</tr>
<tr>
<td>4.1</td>
<td>Non-Interacting Approximation</td>
<td>79</td>
</tr>
<tr>
<td>4.1.1</td>
<td>The Chemical Potential</td>
<td>79</td>
</tr>
<tr>
<td>4.1.2</td>
<td>The Compressibility Measurement</td>
<td>84</td>
</tr>
<tr>
<td>4.1.3</td>
<td>The Average Energy</td>
<td>85</td>
</tr>
<tr>
<td>4.1.4</td>
<td>Measuring the Average Energy</td>
<td>88</td>
</tr>
</tbody>
</table>
### List of Tables

2.1 Energy and size ($r_0$) of vortices and antivortices for $n = 3, 5$ and 7. $\mu_{cr}$ is a naive estimate of the value of the chemical potential at which a condensate of these configurations is expected to form ($\mu_0 = 10^{-2}eV$). $\mu_{cr}$ is more carefully described in the text. ............................................. 36

2.2 The density $\rho_{v1}$ and the corresponding filling fraction $\nu_1$ at which the condensate of vortices is expected to become dense. (See text for a precise definition.) .................................................. 36

2.3 Summary of numerical results for quasi-particle and quasi-hole creation energies and gap energy. Units are $e^2/\xi$. .......................................................... 37

3.1 Four experiments which measure the SdH oscillations in the fractional quantum Hall regime and their corresponding self-energy coefficients. $K_2$ is determined from the given experimental values for $\epsilon$, $n_e$ and $m_e$ using Eq. (3.27). .......................................................... 72
List of Figures

2.1 The function $h(r)$ corresponding to a vortex for $\nu = 1/3, 1/5$ and 1/7. 

2.2 Density profile of the vortex configuration for $\nu = 1/3, 1/5$ and 1/7.

2.3 The function $h(r)$ corresponding to an antivortex for $\nu = 1/3, 1/5$ and 1/7.

2.4 Density profile of the antivortex configuration for $\nu = 1/3, 1/5$ and 1/7.

3.1 Schematic drawing of the geometry used in the magnetic focusing experiment of Goldman, Su and Jain. Each arc is a cyclotron orbit (after Goldman et al. [63]).

3.2 Feynman diagrams representing the first order corrections, $K^{(1)}(q, \omega)$, to the gauge propagator.

3.3 Feynman diagram representing the self-energy. $n$ and $n'$ are Landau level indices. The straight line represents the fermion (either electron or composite fermion) and the wavy line represents the boson (either a phonon or a gauge fluctuation).

3.4 Real part of the self-energy of composite fermions. $s = 2$ corresponds to Coulombic electron-electron interactions; $s = 3$ corresponds to short-ranged screened interactions (see text). The energies are in units of $\Delta \omega_c$.

3.5 Imaginary part of the self-energy of composite fermions. $s = 2$ corresponds to Coulombic electron-electron interactions; $s = 3$ corresponds to short-ranged interactions and the energies are in units of $\Delta \omega_c$, as before.

3.6 Graphical solution to (3.34) for different values of $n$. The crosses show the poles between which lies the renormalised gap.
3.7 First order perturbation results for the effective gap, $\Delta \omega_e^*$, as a function of coupling, $K_2$, for $p = 50$. The solid curve is $|1 - \frac{2K_2}{\pi} \log(2p + 1)|^{-1}$ and points are numerical solutions to Eq. (3.34). The inset shows the same data plotted as a function of $n_e$, which is related to $K_2$ via eq. (3.27).

3.8 Imaginary part of the Green function of composite fermions in a finite magnetic field with $n = 0$ and $p = 50$. The upper figure has a coefficient $K_2 = 0.31$ and the lower one has $K_2 = 6.3$. The energy is in units of $\Delta \omega_e$.

3.9 The real and imaginary parts of the self-energy for electrons interacting with phonons with a Debye spectrum ($\omega_D = 20\omega_c$) in a small magnetic field. All energies are in units of $\omega_c$; the coupling is $K_D = 1$.

3.10 The derivative with respect to energy of the real part of the self-energy of electrons interacting with Debye phonons is a finite magnetic field ($\omega_D = 20\omega_c$). The energy is in units of $\omega_c$, and $K_D = 1$.

3.11 The imaginary part of the Green function of electrons interacting with Debye phonons for $K_D = .001, .1$ and .25 with $n = 0$ (see the text for definition of the coupling $K_D$).

3.12 Real part of the self-energy of electrons interacting with phonons with an Einstein spectrum ($\omega_E = 20\omega_c$). Both axes are in units of $\omega_c$; the coupling is $K_E = 1$.

3.13 Rainbow diagram contributions to the self-energy.

3.14 The effective mass, $m^*$ as a function of $B$ and $\nu$ for composite fermions. The lower $x-$ axis has been obtained by assuming an electron density $n_e = 2.23 \times 10^{11} \text{cm}^{-2}$. The crosses are numerical calculations which include corrections coming from $\Sigma^{(2)}$ (the self-consistent self-energy). The curves are a guide for the eye.
3.15 Four experiments which measure the effective mass from SdH measurements. a) Du et al. [30], b) Coleridge et al. [32], c) Leadley et al. [57] d) Manoharan et al. [31]. Note that in all cases the vertical axis is in units of $m_e$, whereas in Fig. 3.14 it is drawn in units of $m_b$. 

3.16 The effective mass $m^*$ as a function of electron density. These are the same results as in Fig. 10 except that the $x$-axis has been obtained using $n_e = \frac{1}{4\pi} \left( \frac{2\hbar^2 m_e^2}{\epsilon} \right)^2$ with $\epsilon = 13$ and $m = 0.07m_e$. The crosses are numerical calculations which include corrections coming from $\Sigma^{(2)}$ (the self-consistent self-energy).

3.17 Experimental results of Leadley et al. [57] showing the effective mass as a function of electron density.

4.1 The chemical potential of composite fermions as a function of filling fraction calculated for $m^* = 20m_b$ and $B = 13T$ using Eqs. (4.7) and (4.8). This plot shows only the oscillatory component of $\mu$, which oscillates about its values at $\nu = \frac{1}{2}$.

4.2 Experimental results of Eisenstein et al. [55] showing a) the compressibility of a 2-D electron gas and b) the chemical potential, which is the integral of the compressibility. Note that the horizontal axis is the same on both plots.

4.3 The compressibility as a function of filling fraction calculated for $m^* = 20m_b$ and The $y$-axis has been scaled by the interaction strength, $\frac{e^2\lambda}{\epsilon}$ (see text).
4.4 A schematic illustration of the splitting of the lowest Landau level. When there are no interactions the lowest Landau is not split. The mean field theory of composite fermions splits the lowest Landau level into sublevels separated by $\Delta \omega_c$. Gauge fluctuations renormalise the mass, so that in the end the lowest Landau is subdivided into levels separated by $\Delta \omega^*_c$, but still centered around $\frac{\omega_c}{2}$.

4.5 The average energy as a function of magnetic field at $T = 0$. The scaling of the $y$-axis has been chosen to match the experimental parameters of Kukushkin et al. (see text).

4.6 The experimental results of Kukushkin et al. [54] showing b) The average energy as a function of magnetic field at $T = 0$ and c) its derivative.

4.7 The derivative of the average energy with respect to the magnetic field as a function of magnetic field at $T = 0$. The discontinuities are related to the gap as explained in the text.

4.8 The self-energy when the highest composite fermion Landau level is half-filled at $T = 0$ (solid) and for a finite temperature with a small, positive shift of the chemical potential (dashed). $\epsilon$ is measured with respect to the chemical potential. The vertical scale is arbitrary (it depends on the coupling $K_2$).

4.9 Feynman diagram representing the second order correction to the thermodynamic potential. The straight lines represent composite fermions and the wavy line represents a gauge fluctuation.

4.10 The zeroth, first and second order oscillatory contributions to $\Omega^{(2)}$. The wiggly cuts indicate that the oscillatory component of the fermion line is to be evaluated.
C.1 Some Feynman diagrams representing fourth order contributions to $\Omega - \Omega_0$.
The first shows a disconnected graph, the second is the $O(\lambda^4)$ skeleton graph and the third is an example of a skeleton graph with a self-energy insertion.
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Chapter 1

Introduction

A system of electrons confined to two dimensions in the presence of a strong magnetic field has many of the features that characterise problems in modern condensed matter physics. It is a strongly correlated system, thanks to the presence of strong Coulombic electron-electron interactions. It is an ideal example of a lower dimensional system, where its confinement to two dimensions gives rise to fractional quantum numbers. Its phase diagram consists of states with vanishing resistivity which are alternated with insulating phases as a function of magnetic field, giving rise to the phenomenon known as the fractional quantum Hall effect. Finally, the system appears to be describable as a gauge theory with a novel quasiparticle, but is essentially non-perturbative thanks to the strong electron-electron interactions.

The fractional quantum Hall effect (FQHE) is a phenomenon that occurs at very low temperatures when electrons confined to two dimensions at the interface between two semiconductors are exposed to a strong magnetic field. The magnetic field is applied perpendicular to the two-dimensional electron gas. The current response to an electric field is measured, from which the resistivity is determined. The resistivity is, in fact, a two-dimensional tensor, by virtue of the usual Hall effect, since the electrons experience a force due to the magnetic field which is perpendicular to their velocity. The FQHE is the experimental observation of plateaux in the transverse component of the resistivity occurring when the lowest energy level of the system has a fractional occupancy [1]. The longitudinal component of the resistivity is observed to vanish at these same values.
This behaviour is exactly analogous to the integer quantum Hall effect (IQHE), which is the observation of the same phenomenon when an integer number of energy levels are completely occupied \([2]\). While the features of the IQHE may be accounted for with a theory of non-interacting electrons in the presence of a magnetic field, the same does not hold for the features of the FQHE, which are contingent on the presence of strong electron-electron interactions. The ongoing theoretical challenge of the FQHE has been the search for a description that is both physically correct and that illuminates the essential physics of the problem.

Laughlin solved this system for the cases when the fractional occupancy of the lowest energy level is \(\nu = \frac{1}{2\nu+1}\) by his discovery of a variational ground state wavefunction for a system of interacting electrons in a magnetic field \([3]\). Extensive numerical work on small systems has proven that Laughlin’s wavefunction has an extremely large overlap with numerical solutions \([4]\), and is an exact solution for point interactions \([5, 6]\). Laughlin’s discovery goes a long way towards illuminating the nature of the ground state at these special occupancies. In particular, it explains the presence of a gap in the excitation spectrum. The main shortcoming of the Laughlin solution is that it is not readily extendible to the other filling fractions. Furthermore, the strongly interacting theory from whence the Laughlin wavefunction is derived is essentially non-perturbative, and thus its utility for making physical predictions is limited by the lack of viable approaches to attack this non-perturbative problem.

There have been two other approaches to understanding the FQHE, both of which were motivated by the desire to gain a better physical insight into the problem. The bosonic theory (also known as the Chern-Simons-Landau-Ginzburg (CSLG) theory) was invented in order to exploit the similarities between the FQHE and superfluidity \([7]\). The fermionic theory (composite fermion (CF) theory) was originally conceived by Jain as an explanation of the remarkable similarity between the FQHE and the IQHE \([8]\). This
Chapter 1. Introduction

approach provides an explanation for much of the phenomenology associated with the FQHE and also serves as a starting point for a field-theoretic approach.

This thesis is concerned with several aspects of these models of the FQHE. In this Introduction we describe the physics of two dimensions, followed by a discussion of basic ideas of the integer and fractional QHE. In Chapter 2 we elaborate on the system at the filling fractions \( \nu = \frac{1}{2n+1} \) using the CSLG theory. The special role of vortices is described and numerical solutions for vortex profiles are presented. Chapters 3 and 4 are concerned with the composite fermion theory of the FQHE. In these Chapters we examine two important field theoretic quantities, the self-energy and the thermodynamic potential. The self-energy yields information about the low energy single particle excitations, in particular, the gap. The thermodynamic potential is the theoretical origin of several quantities of both theoretic and experimental interest, including the compressibility, chemical potential and magnetisation. In the presence of a magnetic field, these quantities are "oscillatory", that is, they oscillate as a function of an applied magnetic field. In Chapter 3 the self-energy is calculated self-consistently in full oscillatory form and is used to determine the effective mass of composite fermions for a range of physically realistic parameters. In Chapter 4 we discuss the oscillatory quantities associated with the FQHE and compute the thermodynamic potential.

1.1 Physics in Two Dimensions

"I call our world Flatland, not because we call it so, but to make its nature clear to you, my happy readers, who are privileged to live in space." [9]

This thesis will make extensive use of ideas that are especially pertinent to the physics of lower-dimensional systems. In particular, we need to understand the characteristics of a system of identical particles in two dimensions. It is well known that in three or
more dimensions the symmetry of the wavefunction is required to be $S = +1$ or $-1$, corresponding to bosons or fermions respectively. As an example of this we consider exchanging two particles and then returning them to their original configuration; single valuedness implies that $S^2 = 1$. This is a consequence of the fact that in $D \geq 3$ all the paths in configuration space that connect two identical initial and final states may be continuously deformed into a single path (i.e., they belong to a single homotopy class). This is not necessarily so in two dimensions; in general there will be many homotopy classes, corresponding to the number of times the particles encircle each other. In this case the homotopy classes are isomorphic to elements of the braid group [10]. In general this leads to a symmetry $S^n = 1$ and the possibility of fractional statistics.

Given the possible existence of fractional statistics, we proceed to investigate the scenarios which may give rise to it. Using a differential geometry approach, Leinaas and Myrheim showed that paths on a generalised 2-dimensional surface enclosing a singularity (such as the tip of a cone) in general acquire an additional phase [11]. Our interests lie with another particular case, where phases are acquired due to the presence of a gauge field.

Gauge fields always appear in the context of charged particles as the mediators of the force between them. They are subject to a symmetry known as gauge invariance which is related to conservation of charge current. This symmetry is

$$\psi(x) \rightarrow \psi(x) \exp[i\Lambda(x)]$$

$$A(x) \rightarrow A(x) + \frac{1}{e} \nabla \Lambda(x)$$

where $\psi(x)$ is the electron wavefunction and $A(x)$ is the gauge field. It is obvious from these relations that gauge fields may also influence the statistics of particles. The most famous example of this is the Aharonov-Bohm effect, in which particles in a 1-D closed geometry acquire a phase when magnetic flux is adiabatically introduced through the
Chapter 1. Introduction

ring [12]. However, we are concerned with the 2-D case. In this situation a term of the form

$$\frac{\alpha e_{\mu,\nu,\lambda} A_\mu \partial_\nu A_\lambda}{\pi}$$

(1.3)

may be added to the Lagrangian, which is gauge invariant up to a surface term. This term gives rise to particles which upon exchange acquire a phase of $S = \exp\left(\frac{i\pi}{\alpha}\right)$ [13]; such particles are known as “anyons”.

There is one other feature related to the fact that we are confined to two dimensions that is of considerable importance to the Hall effect. In the special situation where a magnetic field is applied perpendicular to the plane, we find that the kinetic energy is quenched by the magnetic field. The energy spectrum collapses into highly degenerate Landau levels and the momenta $k_x, k_y$ together are no longer good quantum numbers. The energy is determined completely from the Landau level index number. It is the large gaps between the Landau levels and their large degeneracy that set the stage for the integer quantum Hall effect.

1.2 The Quantum Hall Effect

The integer quantum Hall effect was discovered in 1980 by von Klitzing, Dorda and Pepper [2], following a suggestion of Ando, which was to examine the Hall conductivity of a two dimensional electron gas in a large magnetic field when the Fermi level was between two Landau levels [14]. As predicted, they observed that electrons in the impurity bands between Landau levels are localised and that the Hall (transverse) current arises from electrons in all of the Landau levels. However, no one had anticipated the most astonishing feature of the results: a robustly quantised Hall conductance whenever $\nu$, the number of filled Landau levels, is an integer.

The fractional quantum Hall effect was discovered only two years later by Tsui,
Stormer and Gossard [1], completely unanticipated by theorists. They originally observed a plateau in the Hall conductance at $\nu = \frac{1}{3}$. Improved samples have led to observations of the FQHE at filling fractions with denominators as large as $\nu = \frac{9}{19}$ [15].

The device used to measure the QHE is a semiconductor heterojunction, typically GaAs-AlGaAs. A thin layer of electrons is trapped at the interface, when excess electrons from AlGaAs move into the lower energy conduction band of GaAs. An electron gas of thickness $\approx 50\text{Å}$ and carrier density $n_e \approx 10^{11}\text{cm}^{-2}$ is formed. With this density, a magnetic field $B \approx 4T$ is required to achieve a filling factor of $\nu \approx 1$. With a band mass $m_b \approx .07m_e$, this corresponds to a cyclotron frequency $\omega_c \equiv \frac{eB}{mc} \approx 6 \times 10^{-3}\text{eV} \approx 70\text{K}$, which sets the temperature limit for observation of the IQHE.

1.2.1 The Integer Effect

The integer effect is now well understood. The system consists of electrons confined to two dimensions inside a magnetic field which points in the third direction, and is described by the Hamiltonian

$$H = \frac{1}{2m}(i\nabla - eA)^2$$

where $A$ satisfies $\nabla \times A = B$. The solution to the single particle Hamiltonian consists of harmonic oscillator eigenstates with an energy spectrum of Landau levels at $E_n = (n + \frac{1}{2})\frac{eB}{mc}$. The spacing between Landau levels is given by the cyclotron energy $\omega_c \equiv \frac{eB}{mc}$. In an infinite, disorderless system the energy density of states is a series of delta functions, $\delta(E - E_n)$. Disorder and finite size cause the delta functions to spread out and introduce localised states between the Landau levels. The localised states between Landau levels do not contribute to the conductivity. In order to see the IQHE a large $B$-field must be used, so that the spacing between Landau levels is large and the levels do not overlap even when they are broadened by disorder.
The filling fraction $\nu$ is defined as

$$\nu \equiv \frac{n_e \Phi_0}{B}$$

(1.5)

where $n_e$ is the electron density (per unit area) and $\Phi_0 = \frac{\hbar}{e}$ is the magnetic flux quantum. $\nu$ is the number of filled Landau levels; it is also the number of electrons per magnetic flux quantum of the external magnetic field. The IQHE effect occurs whenever $\nu$ is an integer. When this happens the system behaves as an insulator and the longitudinal conductivity falls to zero exponentially as a function of temperature. At the same time, when an electric field is applied, every free electron gains a net momentum in the transverse direction, creating a Hall current that is proportional to the density of electrons, and a conductivity

$$\sigma = \begin{pmatrix} 0 & -\frac{n_e e^2}{B} \\ \frac{n_e e^2}{B} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{e^2}{h} \\ \frac{e^2}{h} & 0 \end{pmatrix}.$$  

(1.6)

The resistivity is the inverse of this matrix,

$$\rho = \begin{pmatrix} 0 & \frac{h}{e^2 \nu} \\ -\frac{h}{e^2 \nu} & 0 \end{pmatrix},$$

(1.7)

which tells us that when a current is injected through the sample a Hall voltage (perpendicular to the current) appears but there is no voltage in the direction of the current.

Looking at the above form for the conductivity, it is not at all apparent why the conductivity should increase in quantised steps as a function of $\nu$. The plateaux in the conductivity are due to the presence of impurities which trap electrons into localised states which occur in between the Landau levels. These states can accommodate extra electrons as they are added to the system without contributing to the current, and in this way the system stays in the quantum Hall state even as $\nu$ is varied away from the integer values.
The single particle eigenstates may be written as linear combinations of either extended or localized harmonic oscillator wavefunctions which are the solutions to (1.4). The set of solutions depends on the gauge: the gauge $A_x = 0, A_y = Bx$ gives extended eigenstates, and the gauge $A_x = -\frac{By}{2}, A_y = \frac{Bx}{2}$ gives localized eigenstates. Clearly one solution may be expressed as a linear combination of solutions of the other type. The (unnormalised) extended states are written as

$$\psi_{k,n}(x,y) = e^{iky}H_n(x)e^{-x^2/2}$$ (1.8)

where $H_n(x)$ is the $n$th order Hermite polynomial. (One could equally well interchange $x$ and $y$ here). The localized states are written (using complex coordinates $z = x + iy$) as

$$\psi_{m,n}(z) = \left(\frac{\partial}{\partial z}\right)^m \left(\frac{\partial}{\partial \bar{z}}\right)^n e^{-|z|^2}.$$ (1.9)

In both cases $n$ is the Landau level index. The important difference between the two is that the extended wavefunctions are also $y$-momentum eigenstates and the localised states are angular momentum eigenstates (with quantum number $m$). Therefore, it is preferable to use the localised eigenstates when the electron is in the vicinity of an impurity since the perturbation caused by the impurity is localised. Likewise, near the edges of the sample it is more appropriate to perturb about the extended solutions.

When an electric field is applied (in the $x$-direction) the Hamiltonian is still solvable if the gauge yielding extended eigenstates is used. Laughlin [16] showed that the electric field causes the centre of the wavefunctions to be shifted by an amount $\Delta x = mc^2E/eB^2$. The current is calculated for each eigenfunction using the covariant derivative, $D\psi = (i\bar{\partial} - e\bar{A})\psi$ which yields a gauge invariant result. The current in the $x$-direction, $J_x = \frac{e<\rho_x>}{m} = \frac{e<i\partial_x(i\partial_x+eA_x)\psi(x,y)>}{m}$ remains zero (i.e., $\sigma_{xx} = 0$). However, $J_y$ gets shifted by an amount $\Delta J_y = ne^2E/h$, which corresponds to a Hall conductivity $\sigma_{xy} = ne^2/h$, where $n$ is the Landau level index.
1.2.2 The Fractional Effect and the Laughlin Theory

The fractional quantum Hall effect owes its existence to interactions between the electrons. A new ground state is formed, but only at very low temperatures, lower than the interaction energy between the electrons. When this happens the conductivity of the system behaves in a manner that is remarkably similar to the integer system, except that it occurs when the lowest Landau level is fractionally filled. This cannot be understood in terms of individual electrons; it must be considered as a many-body phenomenon.

The $N$-particle Hamiltonian is given by

$$H = \sum_{j=1}^{N} \left( \frac{|\mathbf{A}_j|^2}{2m} + V(|z_j|) \right) + \sum_{j<k} \frac{e^2}{|z_j - z_k|}$$

where the last two terms are the background potential and the electron-electron interaction respectively. For a single particle ($N = 1$) the last term vanishes and the background is assumed to be uniform, and so the second term is constant. Then the solutions are the same as those found for the integer effect. There is no analytic many-body solution when $N > 1$; the solution discovered by Laughlin [3] is a variational solution which has been shown numerically to be a very good approximation to the actual solution.

The Laughlin many-body solution minimizes $H$ at filling fractions $\nu = \frac{1}{m}$,

$$\psi_m(z_1, \cdots, z_N) = \prod_{j<k} (z_j - z_k)^m \exp \left( -\frac{1}{4} \sum_{l=1}^{N} |z_l|^2 \right)$$

where $m$ is an odd integer. The oddness of $m$ forces $\psi$ to be antisymmetric in the $z_i$. This wavefunction is comprised of states from the lowest Landau level. This may be seen by comparing it with (1.9) - it only depends on $z$, not and $\bar{z}$, therefore the Landau level index is $n = 0$. The wavefunction is also an eigenstate of angular momentum, with eigenvalue $\frac{N(N-1)m}{2}$, as required by the symmetry of $H$ with respect to a rotation about $\hat{z}$. A notable feature of this wavefunction is that it is incompressible [17]. The incompressibility is a consequence of the fact that, in order to expand or contract the system, particles must be
Chapter 1. Introduction

injected into the system, and this costs a finite amount of energy. The gap is the energy needed to produce pairs of fractionally charged quasiparticles (QP’s) and quasiholes (QH’s). The size of the gap also indicates the relative stability of the state. The many-body form of the QH wavefunction was also conjectured by Laughlin [18] by considering how the ground state wavefunction would evolve as one magnetic flux quantum was added to the system adiabatically, and then removed by a gauge transformation. The form of the QH wavefunction located at \( z_0 \) is

\[
\psi^+(z_1, z_2, \ldots, z_N) = \prod_{l} (z_l - z_0) \prod_{j<k} (z_j - z_k)^{m-1} \exp \left( -\frac{1}{4} \sum_{l} |z_l|^2 \right) \tag{1.12}
\]

which Laughlin argued had a charge of \( \frac{1}{m} \).

Laughlin’s form of the QH wavefunction is generally accepted; however, several different forms for a QP have been proposed [4, 18, 19, 20, 21]. Here we simply quote two of the trial wavefunctions, and we note that all of them have a different form from that which will be considered in Chapter 2. The QP wavefunction proposed by Laughlin is

\[
\psi_m^+ = \prod_{l} \left( \frac{\partial}{\partial z_l} - \bar{z}_0 \right) \prod_{j<k} (z_j - z_k)^m \exp \left( -\frac{1}{4} \sum_{l} |z_l|^2 \right). \tag{1.13}
\]

Jain’s QP wavefunction located at the origin is [20]

\[
\psi_m^- = \begin{vmatrix}
\bar{z}_1 & \bar{z}_2 & \bar{z}_3 & \ldots \\
1 & 1 & 1 & \ldots \\
z_1 & z_2 & z_3 & \ldots \\
z_1^2 & z_2^2 & z_3^2 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{vmatrix} \prod_{j<k} (z_j - z_k)^{m-1} \exp \left( -\frac{1}{4} \sum_{l} |z_l|^2 \right). \tag{1.14}
\]

Like the ground state and the QH state, the Laughlin QP state does not include any mixing into higher Landau levels. In contrast, this mixing appears explicitly in Jain’s proposed form, as indicated by the presence of \( \bar{z}_1, \bar{z}_2, \ldots \). Jain argued that the nature
of the FQHE was fundamentally related to the IQHE and that this relation could be exploited by explicitly including structure from the higher Landau levels into his proposed many-body states [23]. More will be said about Jain’s theory in Chapter 3.

1.3 Chern-Simons Field Theories and the Transmutation of Statistics

Although Laughlin solved the FQHE for $\nu = \frac{1}{2n+1}$, it soon became evident that a different approach was needed to account for other aspects of the theory. Girvin [7] first expounded the need for a theory based on an order parameter in order to formalise what was already evident, i.e., that the FQH system at fractional fillings shares with superfluids the important property of vanishing resistivity [22]. This led to the development of the “Chern-Simons-Landau-Ginzburg” (CSLG) approach to the FQHE in which the order parameter is derived from bosonic wavefunctions [19]. Another unresolved aspect of the theory was the so-called “hierarchy” problem, the experimental observation that certain series of filling fractions ($\nu = \frac{p}{2p+1} = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots$) had greater stability than others ($\nu = \frac{p}{4p+1} = \frac{1}{5}, \frac{2}{9}, \frac{3}{13}, \ldots$). This problem was addressed by several authors [6, 24] who proposed various schemes for the generation of the “daughter” states by condensation of the quasiparticles of the Laughlin states, and also by Jain [8] in his theory of “composite fermions” (CF’s). Jain’s idea may also be extended to a theory of fermions interacting with gauge fluctuations [25, 26]. Despite the fact that they both describe the same phenomenon, the CSLG and the CF theories have completely different perspectives. However, both theories are derived using the same trick, which makes use of the properties of two dimensional systems, allowing for the addition of a Chern-Simons term in the Lagrangian.

We first consider the situation described by the CSLG theory, i.e., $\nu = \frac{1}{2n+1}$. In this
case we have a magnetic field satisfying

\[
\frac{eB}{\Phi_0} = (2n + 1)n_e
\]

which means that there are exactly \(2n + 1\) magnetic flux quanta of the external magnetic field for each electron. This is described by a Lagrangian

\[
\mathcal{L} = -i\psi^\dagger (ch\partial_0)\psi + \frac{1}{2m} |(\hbar \nabla - \frac{ie}{c} A)|^2 + \int dy \psi^\dagger(y)\psi(y)V(x - y)\psi^\dagger(x)\psi(x)
\]

where \(c\partial_0 = \partial_t\). The trick is to attach to each electron \(2n+1\) magnetic flux quanta oriented oppositely to the external applied field by means of a (singular) gauge transformation [27]:

\[
\psi(x) \rightarrow \psi(x) \exp \left( i(2n + 1) \sum_{i \neq j} \theta_{ij} \right) \equiv \tilde{\psi}(x)
\]

\[
A(x) \rightarrow A(x) + \frac{(2n + 1)}{e} \sum_{i \neq j} \nabla \theta_{ij} \equiv A(x) + a(x)
\]

where \(\theta_{ij}\) is the angle between electrons at positions \(i\) and \(j\). \(a(x)\) is known as the "statistical" gauge field. Let us evaluate the average value of \(\nabla \times a(x)\) using Stokes's theorem,

\[
\int \nabla \times \sum_{i \neq j} \nabla \theta_{ij} dA = \int \sum_{i \neq j} \nabla \theta_{ij} \cdot dS.
\]

The angle may also be expressed as \(\text{Im} \log(z_i - z_j)\). The only non-zero contributions to the integral come from angles associated with particles inside the closed path. Thus the result is simply \(2\pi N\), and so the average value of \(\nabla \times a\) is just \(b \equiv \nabla \times a = (2n + 1)\Phi_0 n_e\).

The new many-body wavefunction \(\tilde{\psi}(x)\) is bosonic, as may be seen by exchanging any two indices, which corresponds to exchanging particles: \(\psi(x)\) is fermionic, therefore exchange of two particles yields a minus sign; as for the phase, the angle transforms as \(\theta_{ij} \rightarrow \theta_{ij} + \pi\), which yields \(\exp i(2n + 1)\pi = -1\). It is clear that if \(2n + 1\) was replaced by \(2n\), \(\tilde{\psi}\) would be fermionic.
Chapter 1. Introduction

The constraint that there are $2n + 1$ flux quanta attached to each particle is implemented by the introduction of a Chern-Simons term in the Lagrangian,

$$\mathcal{L} = -i\psi^\dagger (\hbar \partial_0 - ie(a_0 + A_0))\psi + \frac{1}{2m} |(\hbar \nabla - \frac{ie}{c}(A + a))\psi|^2$$

$$+ \frac{e}{2\Phi_0(2n + 1)} \epsilon_{\mu \nu \lambda} a^\mu \partial^\nu a^\lambda + \int dy \psi^\dagger(y)\psi(y)V(x - y)\psi^\dagger(x)\psi(x).$$

Varying with respect to $a_0$ yields the equation of motion

$$\psi^\dagger \psi \equiv n_e = \frac{\nabla \times a}{\Phi_0(2n + 1)}$$

which is the required constraint.

The important thing to notice is that when the system is at $\nu = \frac{1}{2n+1}$ there is an exact cancellation between $b$ and $B$, so that what remains is a system of free bosons in zero magnetic field. The fermionic case is similar except that the cancellation occurs at even fillings, $\nu = \frac{1}{2n}$. The consequences of this will described in much greater detail in the following chapters.

1.4 Overview

The FQHE has attracted a great deal of theoretical interest in recent years as a practical application of the intriguing mathematical ideas just presented. The bosonic theory gave physicists an alternative view of the nature of the quantum Hall states. The fermionic theory described the quantum Hall states observed at all filling fractions and the metallic states in between. Very recently it has been shown [76] that the metallic states (for example, at $\nu = \frac{1}{2}$) are the first physical example of a non-Fermi liquid. The fermionic theory also serves naturally as a mean field starting point for the describing the oscillatory phenomenology of the FQHE. This thesis explores some of these issues.
1.4.1 Vortices

We have just shown that the Hall system at $\nu = \frac{1}{2n+1}$ can be mapped into a system of interacting bosons in zero magnetic field. It was shown by Laughlin [17] that the primary excitations are pairs of fractionally charged QP's and QH's which are associated with the local addition of a single magnetic flux quantum. Both the QH and the QP occur naturally in the bosonic description, and the fact that each electron is tied to $2n+1$ flux quanta ensures that there is a charge $\pm \frac{e}{2n+1}$ in the vicinity of a single flux quantum. In Chapter 2 we show that within the bosonic theory the charge configurations of both the QP and QH are vortices.

The fractional charge associated with the vortices has the important property that it is local. If the vortices are pinned, then the fractional charge does not contribute to the bulk current. This is the mechanism by which the fractional quantum Hall effect is maintained even as the filling fraction is varied away from $\nu = \frac{1}{2n+1}$, analogous to the role played by impurities in the integer effect. Given the size of a vortex, we may determine a criterion for the breakdown of the quantum Hall state. From this we set limits on the size of the plateaux in the Hall conductivity.

1.4.2 A Novel State of Matter?

Landau [28] conjectured that a system of interacting fermions (a Fermi liquid) could be mapped into a system of weakly interacting fermions with renormalised parameters. The dressed electrons are known as "quasiparticles," and the primary excitations are just quasiparticles with a momentum greater than the Fermi momentum. The parameters in question include quantities such as the effective mass and the interaction parameters (the famous f-function), which are defined in perturbation theory, and may be measured in experiments.
Using the composite fermion theory, the Hall system at $\nu = \frac{1}{2n}$ is described as a system of fermions in zero magnetic field. They are coupled through a long-range Coulomb force which may be expressed as an interaction with a gauge field. The transverse component of the gauge field gives rise to perturbative corrections which appear to invalidate the Landau quasiparticle picture [26]. This is apparent in several closely related features, the most striking of which is an apparent divergence of the effective mass.

Slightly away from $\nu = \frac{1}{2}$ the cancellation between the external magnetic field and the fictitious "statistical" field is no longer exact. In this situation the composite fermions are in a small magnetic field and so naively we can predict that the system will have a gap associated with the cyclotron frequency of the effective magnetic field. The gap can thus be parametrised in terms of an effective mass, $\Delta \omega_c \equiv \frac{eAB}{m^*}$. Perturbative calculations show that this mass also diverges in the limit $\nu \to \frac{1}{2}$ [29]. Even more astonishing, this result has been qualitatively confirmed by experiment [30, 31, 32]. Until now, a quantitative comparison has been lacking because of the extreme limits analysed in the initial calculations. Furthermore, quantitative results based on a perturbative approach are questionable because the coupling in this system is not small. In Chapter 3 we address both of these issues in our search for a self-consistent determination of the effective mass away from $\nu = \frac{1}{2}$.

1.4.3 Magnetic Oscillations - the Old Meets the New

Magnetic oscillations occur in two or three dimensions as the filling factor of a system is varied. This may be achieved by either varying $n_e$ or $B$; in most experiments it is the latter. For example, consider what happens to the system as the magnetic field is increased. The spacing between Landau levels grows and the number of states in each Landau level increases until there is a sudden drop in the chemical potential. This produces oscillations in the density of states near the Fermi surface and every other
measurable quantity (such as the resistivity, magnetisation and compressibility) will vary in the same manner. The existence of such oscillations was first noted in theory by Landau in 1930 [33] and observed later that year by Shubnikov and de Haas [34] in resistivity measurements and by de Haas and van Alphen [35] in magnetisation measurements. The connection between theory and experiment was established in 1933 by Peierls [37].

The dHvA effect, which is an oscillation of the magnetisation, proved to be a very useful tool for mapping out the Fermi surface. Theoretical activity in this area reached its prime in the 1960's when different techniques were used to obtain formulae describing the amplitudes of the various types of oscillations [38, 39, 40, 41].

What is surprising about the fractional Hall system is that the period of oscillations depends on $\frac{1}{\Delta B}$, not $\frac{1}{B}$. This suggests that the fractional Hall system may be analysed using known techniques with a simple substitution of $\Delta B$ for $B$, and this appears to be the case when a non-interacting system of composite fermions with a renormalised, field-dependent mass is used. However, when we attempt to account for interactions in perturbation theory, we find that approximations developed in the 1960's are not readily applicable to the Hall system because of the confinement to two dimensions instead of three. This is true for any interactions, not just gauge fluctuations. In Chapter 4 we discuss the oscillatory quantities of non-interacting composite fermions with an effective mass, and we re-examine the formulae derived for three dimensions, and show why they are not valid in two dimensions.

\footnote{An interesting historical account of magnetic oscillations may be found in [36].}
Chapter 2

Vortices and the Bosonic Description of the FQHE

In the bosonic theory of the FQHE the electrons in a quantum Hall state at filling fraction $\nu = \frac{1}{2n+1}$ are transformed into bosons by attaching an odd number of magnetic flux quanta to each one. Historically, there have been two different motivations for this. First, there was the desire to express the theory with an order parameter, to make a formal analogy between the Hall system and other systems with vanishing resistivity, namely, superconductivity and superfluids [7]. Secondly, the introduction of the phase associated with the bound flux quanta was precisely what was needed to give the theory off-diagonal long range order [43]. This important physical property is not at all evident in the Laughlin theory. In both cases a very specific number of bound flux quanta is necessary - it must be chosen so that on average it cancels the background magnetic field. In other words, the electrons are bound to $\frac{1}{\nu} = 2n + 1$ flux quanta. For the purposes of this Chapter, our interests lie with the first motivation since a theory with an order parameter opens the possibility of performing viable calculations for large numbers of particles and hence possesses predictive capabilities.

In the first section of this Chapter we describe the construction of the order parameter and the calculation of the excitation spectrum. The lowest energy excitations have been the subject of intense experimental scrutiny; some of these experiments are discussed here.

Vortices are configurations in which the order parameter has a singularity at some point (i.e., it vanishes) and are thus topologically distinct from the uniform ground state
associated with the fillings $\nu = \frac{1}{2n+1}$. As we shall see, the vortices of this theory possess the same properties as the Laughlin quasiparticles and quasiholes which are described in the Introduction. They have a localised fractional charge $\pm \frac{1}{2n+1}$ positioned where an additional magnetic flux quantum has been introduced. These properties are explained in a natural and intuitive way within the bosonic theory of the FQHE. The vortex states are of interest for two reasons. First, they may account for the existence of the plateau in the resistivity during the transitions between Hall states in the fractional regime. Second, a vortex/anti-vortex pair is the lowest energy neutral excitation occurring on at least part of the excitation curve.

The final sections of this Chapter are a study of single vortex states using a simple Landau-Ginzburg model. By establishing appropriate boundary conditions and minimising the Hamiltonian, we numerically construct vortex and anti-vortex profiles. Using these profiles, the creation energy of a vortex/anti-vortex pair is calculated and compared with results obtained by other methods. Within our model, which neglects interactions between vortices, we also discuss the energetics of the formation of a vortex lattice and its role in the transition between FQHE states.

2.1 Superfluid Analogy

The bosonic theory is an attempt to capture the essential physics of the FQHE at and near the fillings $\nu = \frac{1}{2n+1}$, where the vanishing resistivity suggests that the ground state is a novel kind of condensate. The first step in formulating the superfluid analogy is the construction of an order parameter which follows from the bosonisation of the theory.
2.1.1 Off-Diagonal Long Range Order

In this Section we review the arguments given by Girvin and MacDonald [43, 42] which demonstrated that the bosonic wavefunctions, formed by attaching $2n + 1$ flux quanta to each electron at $\nu = \frac{1}{2n+1}$, possess off-diagonal long range order (ODLRO). In superfluids the presence of this property indicates that there is a long range correlation of the one-body density matrix, i.e.

$$\rho(z, z') = N \int d^2z_2 \ldots d^2z_N \psi^*(z, z_2, \ldots z_N) \psi(z', z_2, \ldots z_N)$$

$$\rightarrow |z - z'|^{-\theta} \text{ for large } |z - z'|$$

for some power $\beta > 0$. The Laughlin wavefunction (1.11) does not possess ODLRO. Using the singular gauge transformation (1.17) and (1.18) which attaches $2n + 1$ magnetic flux quanta to each electron, Girvin and MacDonald showed that that Laughlin wavefunction becomes

$$\tilde{\psi}(z_1, z_2, \ldots, z_N) = \prod_{i<j} |z_i - z_j|^{2n+1} \exp \left( -\frac{1}{4} \sum_k |z_k|^2 \right)$$

and that for large $|z - z'|$ the one-body density matrix goes as

$$\rho(z, z') \rightarrow |z - z'|^{-\frac{2}{2n+1}}.$$ 

This demonstrates that the system of electrons with $2n + 1$ flux quanta attached possesses algebraic ODLRO.

2.1.2 The Excitation Spectrum

Using Feynman’s theory of superfluid He$^4$ [44] as a starting point, Girvin, MacDonald and Platzman [22] derived the excitation spectrum of the quantum Hall states at $\nu = \frac{1}{2n+1}$. By direct application of Feynman’s method, they found that in the long wavelength limit the mode at the cyclotron energy, $\Delta(0) = \hbar \omega_c$, completely saturates the oscillator
weight, in agreement with Kohn's theorem\(^1\). In the FQHE it is expected that the lowest energy excitations should occur within the lowest Landau level; i.e., that there should be intra-Landau level excitations. Girvin, MacDonald and Platzman sought the spectrum of this type of excitation by projecting their result onto the lowest Landau level. Because of Kohn's theorem, the weight of this lower energy mode must vanish faster than \(k^2\) as \(k \to 0\).

Their results showed that the system has a gap for any value of \(k\). In the limit \(k \to \infty\) the excitations are free Laughlin quasiparticle quasihole pairs. There is a minimum which occurs at the inverse magnetic length \((l = \frac{1}{\sqrt{eH}} \approx 10^{-6}\text{cm})\) known as the magnetoroton minimum which is associated with a bound state of a QP-QH pair. As \(k \to 0\) there is a crossover to a magnetoplasmon mode\(^2\). Similar curves were derived within the composite fermion theory (described in the next Chapter) for the filling fractions \(\nu = \frac{p}{2n \pm 1}\) \(^{[46]}\). These curves are characterised by having \(p\) minima.

\subsection{2.1.3 Experiments}

A great deal of effort has been devoted to the experimental verification of the excitation spectrum. The earliest experiments that measured the gap were performed using thermal activation \(^{[47]}\). In these experiments the longitudinal resistivity is measured as a function of temperature and the gap \(\Delta\) is determined using the relation \(^{[48]}\)

\[\rho_{xx} \propto \exp \left(\frac{-\Delta}{k_B T} \right)\]  

\(\Delta\) is always measured to be less than predicted gap energies, even when finite sample width and L.L. mixing have been included in the calculations, because of impurities which

\(^{[45]}\)Kohn's theorem states that the cyclotron mode has all of the weight of the oscillator strength in the limit \(k \to 0\) and that all other modes vanish faster than \(k^2\) \(^{[45]}\).

\(^{[19]}\)Zhang has suggested another possibility: that in the limit \(k \to 0\) the lowest energy excitation may consist of a bound state of two QP's and two QP's. \(^{[19]}\)
Chapter 2. Vortices and the Bosonic Description of the FQHE

broaden the energy levels. Later experiments [31, 49] showed that there exist a series of filling fractions (such as \( \nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots \)) with decreasing gap sizes, implying decreasing stability. This was the basis for complicated “hierarchy” schemes [6, 24] and for Jain’s theory of composite fermions [8].

There are two types of experiment that probe the \( k \rightarrow 0 \) limit of the excitation spectrum. The first is light-scattering measurements [50]. Typically, light with wavevector \( k \approx 10^4 \text{ cm}^{-1} \ll \frac{1}{\lambda} \) is used in a back-scattering geometry. A temperature activated peak is observed at the energy of the excitation. Another version of this experiment [51] was performed using samples in which there was a density modulation imposed on the electron gas with wavevector \( q_0 \approx \frac{1}{\lambda} \) (corresponding to the magnetoroton minimum). This provides a means for photons to couple to the 2-D electron gas at wavevectors \( mq_0 \), for \( m \) an integer. The resulting spectra are a series of peaks at different energies corresponding to wavevectors \( mq_0 \).

The other type of experiment that probes the \( k \rightarrow 0 \) limit is thermodynamic measurements in which the gap is determined indirectly from some kind of discontinuity as one of the parameters (either the magnetic field or the electron density) is varied to bring the system through a quantum Hall state. These measurements include photoluminescence [52, 53, 54], which measures the free energy of the electron gas, compressibility [55] from which the chemical potential may be determined, and resistivity (SdH oscillations) [30, 31, 32, 56, 57]. Much more will be said about these measurements in Chapters 3 and 4.

Finally, ballistic phonons have been used to probe the excitation spectrum at \( k = \frac{1}{\lambda} \) [58]. The point where the phonons have the strongest interaction with the electron gas is where the structure factor is a maximum, which occurs at the magnetoroton minimum.
2.2 Vortices

Vortices play an essential role in the transition between quantised states. By breaking the translational symmetry of the system, their role is analogous to the role played by impurities in the integer case. Physically, they are singularities which consist of a localised excess/deficit of charge. In this way the system can accommodate slight local deviations of charge density away from the special filling fractions \( \nu = \frac{1}{2n+1} \), while maintaining the bulk characteristics. This leads to a finite width in the steps of the quantised Hall conductivity.

It has been shown that single vortex solutions exist and can be found by the addition (or removal) of particles to the system [19]. On the other hand, we can consider varying the filling fraction by making a small, uniform change in the external magnetic field \( B \). In this case a finite number of vortices per unit area would be needed to accommodate the excess magnetic field and maintain the quantum Hall state. Thus, in this scenario we might reasonably expect that the ground state would be a lattice of vortices. However, each vortex costs a finite amount of energy to create. Because of this fact, it is obvious that a lattice of vortices would not be the ground state of a system with an infinitesimal change in \( B \), since an infinitesimal change in \( B \) would give rise to an infinitesimal change in the energy of the uniform state. A ground state consisting of a vortex lattice is only possible when the change in the external magnetic field is finite, as will be shown below\(^3\).

As described in Chapter 1, the theory of interacting electrons in an external magnetic field \( B = \nabla \times A \) is transformed into a theory of bosons via a unitary (singular) transformation which attaches an odd number of flux quanta to each electron. The boson field is represented by a complex function \( \psi \). The theory is described by the Lagrangian

\[
\mathcal{L} = -i\psi^\dagger [c\hbar \partial_0 - ie(a_0 + A_0)]\psi + \frac{1}{2m} [\hbar \nabla - \frac{ie}{c} (A + a)]^2 \psi^2
\]

\(^3\)The details of this section appear in [59].
Chapter 2. *Vortices and the Bosonic Description of the FQHE*

\[ + \frac{e}{2\Phi_0(2n+1)} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda + \frac{\lambda}{2} (\psi^\dagger \psi)^2 - \mu \psi^\dagger \psi \]  

(2.6)

where \( A \) is the externally applied gauge field, \( a \) is the statistical magnetic field (the flux attached to each electron), \( \lambda \) is a repulsive potential which is used as an approximation to the Coulomb interaction and \( \mu \) is the chemical potential.

Following the approach of [60], we begin by writing down the equations of motion obtained from (2.6) by varying \( \psi \), \( a_0 \) and \( a_i \) respectively:

\[ i(\partial_0 + iea_0)\psi + \frac{D^2_i}{2m} \psi - \lambda(\psi^\dagger \psi)\psi + \mu \psi = 0 \]  

(2.7)

\[ \psi^\dagger \psi \frac{\varepsilon_{ij} \partial_i a_j}{\Phi_0(2n+1)} \equiv \frac{b}{\Phi_0(2n+1)} \]  

(2.8)

\[ \frac{i}{2m} [\psi^\dagger D_i \psi - (D_i \psi)^\dagger \psi] = \frac{e}{\Phi_0(2n+1)} \varepsilon_{ij} (-\partial_0 a_j + \partial_j a_0) \equiv \frac{e}{\Phi_0(2n+1)} \varepsilon_{ij} e^j \]  

(2.9)

where \( D_i = \partial_i - i e (A_i + a_i) \) is the covariant derivative and \( e_j = -\partial_0 a_j + \partial_j a_0 \) is the statistical electric field. The first equation (2.7) is just the Schroedinger equation. Eq. (2.8) is the mean field constraint that the electron density \( \rho = \psi^\dagger \psi \) is proportional to the number of attached flux, \( b = \nabla \times a \). The third equation (2.9) relates the current to the statistical electric field. We seek solutions to these equations for \( a(x) \) and \( \rho(x) \).

In the special case where

\[ B = -(2n+1) \Phi_0 \frac{\mu}{\lambda} \]  

(2.10)

is constant the equations of motion have a constant solution

\[ \rho = \frac{b}{\Phi_0(2n+1)} = \frac{-B}{\Phi_0(2n+1)} = \frac{\mu}{\lambda}. \]  

(2.11)

Another approach to solving for the eigenstates of the system is to minimize the Hamiltonian,

\[ H = \int d^2 x \mathcal{H} = \int d^2 x \left[ \frac{1}{2m} |D_i \psi|^2 - \mu |\psi|^2 + \frac{\lambda}{2} |\psi|^4 \right] \]  

(2.12)
subject to the constraint Eq. (2.8) that

$$\nabla \times a = \Phi_0(2n + 1)\rho. \quad (2.13)$$

It is clear that (2.11) minimises $H$. The first term of $H$ is zero and the solution $\rho = \frac{\mu}{\lambda}$ lies in the minimum of the potential given by the last two terms. This solution corresponds to the special filling fraction $\nu = \frac{1}{2n+1}$.

Now we consider the case where $B$ does not satisfy (2.10). It is straightforward to show that there are no covariantly constant solutions, such that $D_\mu \psi = 0$. The proof is as follows: first write $\psi = \xi e^{i\Omega}$. Now $D_\mu \psi = 0$ implies that $\partial_\mu \xi = 0$ and that $\partial_\mu \Omega = e(A_\mu + a_\mu)$, so that the combination $A + a$ is a pure gauge. It follows that $B = -b$.

Now Eqs. (2.7) and (2.8) with $D_\mu \psi = 0$ imply that either $\psi = 0$ or that $\frac{\mu}{\lambda} = \frac{b}{\Phi_0(2n+1)}$, so that $\frac{\kappa}{\lambda} = \frac{-B}{\Phi_0(2n+1)}$. This completes the proof that nonzero covariantly constant solutions only exist when $\frac{\mu}{\lambda} = \frac{-B}{\Phi_0(2n+1)}$.

In addition to the lack of solutions with $D_\mu \psi = 0$ there are also no nonzero covariantly static solutions (i.e. solutions with $D_0 \psi = 0$) unless $\frac{\mu}{\lambda} = \frac{-B}{\Phi_0(2n+1)}$. Consider $\psi = \eta e^{i\theta}$. Then $D_0 \psi = 0$ implies that $\partial_0 \eta = 0$ and $\partial_0 \theta + e\alpha_0 = 0$. The third equation of motion (2.9) then implies that

$$\frac{(ea_i + eA_i - \partial_0 \theta)\eta^2}{m} = -\varepsilon_{ij} \partial_0 (ea_j + eA_j + \partial_j \theta) \eta \frac{\Phi_0}{\Phi_0(2n+1)}. \quad (2.14)$$

There are three possibilities to consider. First, $\eta = 0$, in which case $\psi = 0$. Secondly, we could have that $e(a_i + A_i) = \partial_0 \theta$. Then $(b + B) = \nabla \times (a + A) = \frac{1}{\lambda} \varepsilon_{ij} \partial_i \partial_j \theta = 0$, which, using (2.8), implies that $B = -(2n + 1)\Phi_0 \frac{\mu}{\lambda}$. Finally we consider the possibility that $e(a_i + A_i) + \partial_i \theta$ varies with time. $\partial_0 A_i = 0$ by assumption and $\partial_0 a_i \neq 0$ would imply that $\partial_0 \eta \neq 0$, which contradicts the main assumption that $D_0 \psi = 0$, so all the time dependence would have to be in $\partial_j \theta$. Then $\partial_i \theta \propto \varepsilon_{ij} \partial_0 \partial_j \theta$ which has no solution. Thus there are no nonzero covariant static solutions unless $B = -(2n + 1)\Phi_0 \frac{\mu}{\lambda}$. 
So instead we seek other static solutions with $\partial_0 \psi = 0$ and $a_0 \neq 0$. This is done by minimizing the Hamiltonian (2.12) subject to the constraint (2.8). If we consider the case $-B = (2n + 1)\Phi_0 f_\lambda^e \neq (2n + 1)\Phi_0 f_\lambda^a$, then clearly the constant configuration $\rho = \frac{\mu}{\lambda}$, which minimises the potential in (2.12), is no longer a solution since then the gauge fields $A$ and $a$ will not cancel and the first term in (2.12) will be infinite.

On the other hand, a solution of the form $\rho = \frac{\mu}{\lambda}$ does satisfy the equations of motion (2.7-2.9) with

$$ea_0 = \frac{\mu^2}{\lambda} f(1 - f).$$

(2.15)

This corresponds to a to shift of the chemical potential, effectively shifting the position of the minimum of the potential in (2.12). A cancellation of the gauge fields $A$ and $a$, occurs, and the solution lies at the new potential minimum. Since this solution does not minimize the old potential, it leaves open the possibility that a lower energy solution exists. However, we will show that this can only occur for a finite change of $B$.

In seeking solutions to the equations of motion our final alternative is to examine configurations where $\rho$ deviates locally towards the direction of the minimum of the potential. If the deviation is infinitesimal, then it will cause an infinitesimal flux $f dS \cdot (B + b)$ to pierce the system. Since it is not an integer amount of flux, it cannot be “gauged away” and thus it gives a diverging contribution to the first term of (2.12).

At last we consider the case when there is an integer amount of flux threading the system. This scenario necessarily corresponds to vortex configurations. At large distances there must be a phase associated with $\psi$ that will cancel $e(A + a)$, otherwise the first term in (2.12) is logarithmically divergent. If the wavefunction is to be single-valued, it follows that $\psi$ must vanish at the origin.

In the case where $B = -\Phi_0(2n+1)\frac{\mu}{\lambda}$, vortex solutions will always have a greater energy than the constant configuration, since, as we have shown, the constant configuration
minimises $H$. We wish to know the vortex energy in the situation when $B \not\equiv -\Phi_0(2n + 1)\frac{e}{\hbar}$. Since the chemical potential has been moved away from the special value, we might conclude that the vortices would lower the energy of the system and thus describe the FQHE away from the special filling as a collection of these vortices. However, in what follows, we demonstrate that this is not generally the case.

The point is that the configurations which are extrema (but not necessarily minima) of the Hamiltonian (2.12) subject to the constraint (2.13) are completely independent of the value of the chemical potential $\mu$! We have, in fact, proven this already. Configurations are extrema of (2.12) subject to (2.13) if and only if they satisfy the equations of motion (2.7–2.9). Thus if a particular $\psi$ is such an extremum (it may be a constant or, more generally, a multivortex configuration) with a given value $\mu_1$ of $\mu$, then it will satisfy the equations of motion for some function $a_0^{(1)}$. The same $\psi$ will satisfy the equations of motion for any other value $\mu_2$ of $\mu$ but this time with a new $e\epsilon_{a_0^{(2)}} = e\epsilon_{a_0^{(1)}} + \mu_2 - \mu_1$. It is thus also an extremum of the Hamiltonian with this new value $\mu_2$ of $\mu$.

Even though the extremal configurations for differing values of $\mu$ are the same, the energetics may differ for different values of $\mu$. Notice, for example, that for $\mu$ such that $\frac{\mu}{\lambda} = \frac{-B}{\Phi_0(2n+1)}$ (let us call this value of the chemical potential $\mu_0$), the configuration $\rho = \frac{\mu_0}{\lambda}$ clearly has the lowest possible energy. A single vortex configuration $\psi_v(x)$ which solves the equations of motion (see [61]) will have a larger energy than the ground state. Let $\epsilon_v$ be the excess energy of the vortex with respect to the ground state of the Hamiltonian with $\mu = \mu_0$. Now consider an alternate Hamiltonian with $\mu = \mu_1 \neq \mu_0$. The same configuration $\psi_v(x)$ will still be an extremum of this Hamiltonian but, its energy $\epsilon_v^{(1)}$ (which is the difference in energy between the vortex configuration and the configuration with $\rho = \frac{\mu_0}{\lambda}$) will differ:

$$\epsilon_v^{(1)} = \epsilon_v - (\mu_1 - \mu_0) \int d^2x \left( |\psi_v(x)|^2 - \frac{\mu_0}{\lambda} \right) = \epsilon_v - (\mu_1 - \mu_0)N_v,$$  \hspace{1cm} (2.16)
where \( N_v = \pm \frac{1}{n} \) is the "particle number" of the vortex. Note that when \( \mu \) is decreased it is preferable to form a vortex \((N_v < 0)\) whereas if \( \mu \) is increased an antivortex is preferred.

Equation (2.16) has the following consequences. For small values of \( \delta \mu = \mu_0 - \mu_1 \) the vortex configuration increases the energy of the system at \( \mu = \mu_1 \). We thus expect that the constant configuration with \( \rho = \frac{\mu_0}{\lambda} \) will be the configuration of lowest energy despite the fact that the potential energy is not at its minimum. Thus the system remains at the special filling fraction even after \( \mu \) has been shifted from \( \mu_0 \) to \( \mu_1 \). (We emphasize again that this occurs for small shifts \( \mu_0 - \mu_1 \).) It follows that in the mean field approximation

\[
\frac{d\rho}{d\mu} \biggr|_{\text{eq}} = 0 \tag{2.17}
\]

for a range of \( \mu \) near \( \mu = \mu_0 \). This equation is familiar from the integer quantum Hall effect and is due to the presence of a gap in the spectrum. It implies that a finite change in the chemical potential is required before the density can be modified. Returning to Eq. (2.16) we see that when \( \delta \mu = |\mu_0 - \mu_1| \geq \epsilon_v/N_v \), a single vortex has lower energy than the constant configuration \( |\psi|^2 = \mu_0/\lambda \). A gas of such vortices will have an even lower energy. We thus expect that near this value \((\epsilon_v/N_v)\) of \( \delta \mu \) the lowest energy configuration of the Hamiltonian (2.12) subject to the constraint (2.13) will be a collection of vortices. The vortices are charged, therefore the interactions between the vortices will be repulsive, which implies a higher energy. Thus we expect a lattice of vortices to be formed only when the energy lost by forming vortices is greater than the interaction energy.

The situation is similar when the magnetic field is modified instead of \( \mu \). If we begin at the special filling fraction with \( B = -\Phi_0(2n + 1)\frac{\mu_0}{\lambda} \) and change \( B \) at fixed \( \mu \) by a small amount, the lowest energy configuration of the Hamiltonian will occur at a new density for which \( \rho \) is still equal to \( \frac{\rho}{\Phi_0(1+2n)} \) but which will now not be at the minimum of the potential. As the magnetic field is increased (or decreased) further (again at fixed \( \mu = \mu_0 \)) the cost in energy of a single vortex (or antivortex) becomes progressively smaller until at
some critical value of the field it becomes negative. At that point the lowest energy mean field configuration is no longer a constant but rather a finite density of vortices in which case $\rho$ is no longer equal to $\frac{B}{\Phi(2n+1)}$. (If the magnetic field is increased then the vortices will "condense", whereas if it is decreased the antivortices will "condense".) Thus if $\frac{\rho}{\mu}$ is plotted either as a function of $\mu$ or as a function of $B$ there is a plateau surrounding the value $\mu_0$ for which this ratio is constant and equal to $\frac{1}{2n+1}$. If, on the other hand, $B$ is varied at fixed density then we move off the plateau and the lowest energy mean field configuration consists of a finite density of vortices.

2.3 Numerical Vortex Configurations

In the remainder of this Chapter we look more closely at the vortices of this model which, as we have discussed, will be solutions both for $\mu = \mu_0$ and for values of $\mu$ differing from $\mu_0$. We present some numerical solutions for these vortices which will allow us to estimate the value of $\mu$ at which a lattice of vortices starts to form. The shape of the vortices will also lead to an estimate of the density at which the collection of vortices becomes non-dilute. The method we have chosen for finding vortices is by considering the Hamiltonian and constraint given by Eqs. (2.12–2.13) at a value of $\mu = \mu_0 = \frac{B\lambda}{\Phi(2n+1)}$. We then look for radially symmetric configurations (which necessarily carry an integer number of flux quanta of $B + b$) which minimize the Hamiltonian.

Anticipating the fact that our solution will be a vortex with an integer number of flux quanta we organize a radial ansatz as follows: First write

$$\psi(r, \theta) = \xi(r)e^{-ik\theta}$$

(2.18)

with $k$ an integer. The Hamiltonian (2.12) can now be written as:

$$H = \int_0^\infty 2\pi rdr \left[ \frac{1}{2m} \left\{ \left( \frac{d\xi(r)}{dr} \right)^2 + \left( \frac{k}{r} + \frac{eBr}{2} + \frac{e}{r} \int_r^\infty \tilde{b}(\tilde{r})d\tilde{r} \right)^2 \right\} \xi^2(r) \right]$$
with the constraint
\[ \xi^2(r) = \frac{b(r)}{\Phi_0(2n + 1)}. \] (2.20)
(Note that if we were considering a value of \( \mu \) not equal to \( \mu_0 \) we would still use the above equation but with \( \mu = \mu_0/f \) so that the mean field solution would be \( \xi^2 = f \mu/\lambda \).)

We now define the function \( h(r) \) via the formula
\[ b(r) + B = \frac{h'(r)}{r} \] (2.21)
where \( h'(r) = dh/dr \). \( h(0) \) can be chosen equal to 0 without loss of generality. The second term in Eq. (2.19) is then proportional to:
\[ \left( \frac{k}{r} + \frac{B}{2} + \frac{1}{r} \int_0^r r'b(r')dr' \right)^2 \xi^2(r) = \left( \frac{k}{r} + \frac{h(r)}{r} \right)^2 \xi^2(r). \] (2.22)

In order for the integral to be finite at large \( r \) we require \( h(\infty) = k \) (or, more precisely, we require \( h(\infty) \) to be an integer and we choose \( k \) in Eq. (2.18) to be that integer). (We only consider the cases \( k = \pm 1 \) here, since those configurations have the lowest energy.) Furthermore, as is standard for all vortices, \( \xi^2(r) \) must vanish at the origin in order for the energy to be finite. The density constraint Eq. (2.20) then implies
\[ \xi^2(r) = \frac{b(r)}{\Phi_0(2n + 1)} = \frac{-B + h'(r)/r}{\Phi_0(2n + 1)} \geq 0 \text{ and } \to 0 \text{ as } r \to 0. \] (2.23)

This is the most difficult condition to implement in a numerical scheme in which the function \( h(r) \) is varied to minimize the energy.

With the above definitions the Hamiltonian is given by
\[ H = 2\pi \int_0^\infty r dr \left[ \frac{1}{2m} \left( \frac{d\xi(r)}{dr} \right)^2 + \frac{1}{2m} \left( \frac{k}{r} - \frac{h(r)}{r} \right)^2 \xi^2(r) - \mu \xi^2(r) + \frac{\lambda}{2} \xi^4(r) \right]. \] (2.24)
with \( h(r) \) chosen so that \( h(0) = 0, \ h(\infty) = k \) and \( \xi^2(r) \), defined by Eq. (2.23), is \( \geq 0 \). Notice that these conditions guarantee that the total flux of the vortex is

\[
\int d^2x (b + B) = \Phi_0 k.
\] (2.25)

The final step is to subtract, from the energy of the vortex solution of Eq. (2.24), the energy of the mean field solution \( \xi^2 = \frac{\mu}{\lambda} \). This results in a vortex energy given by:

\[
H_v = 2\pi \int_0^\infty rdr \left[ \frac{1}{2m} \left( \frac{d\xi(r)}{dr} \right)^2 + \frac{1}{2m} \left( \frac{k}{r} - \frac{h(r)}{r} \right)^2 \xi^2(r) - \mu(1 - f) \left( \xi^2(r) - \frac{\mu}{\lambda} \right) + \frac{\lambda}{2} \left( \xi^2(r) - \frac{\mu}{\lambda} \right)^2 \right].
\] (2.26)

The procedure at this stage is to search, numerically, through the space of such functions \( h(r) \) until the Hamiltonian is minimized.

One point which is clear is that the form of the vortex solution (for which \( f dA(B+b) > 0 \)) is quite different from that of the antivortex solution. The reason is that \( \xi(r) \) and thus \( b(r) \) must vanish at the origin. As a consequence the density \( \psi^2(r) \propto b(r) \) for the vortex solution can be a monotonic function of \( r \) which increases from zero at the origin and reaches \( B \) at infinity. The antivortex solution must however be zero at the origin then increase to a value greater than \( B \) (so that \( f(B+b) < 0 \)) and then decrease again to attain its asymptotic value \( B \) as \( r \to \infty \). We see this behaviour clearly in the numerical solutions shown in Figs. 2.1-2.4 below.

For the numerical work we chose some representative values for the parameters of the model\(^4\):

\[
\begin{align*}
\mu_0 &= 0.010 \text{ eV} \\
\frac{\mu_0}{\lambda} &= 10^{11} \text{ cm}^{-2} \\
m &= 0.08 m_e.
\end{align*}
\] (2.27)

\(^4\)These values were chosen to achieve a density of carriers, chemical potential and effective mass similar to those found in experimental conditions [57, 62, 63].
Note that $\mu_0/\lambda$ is the density of carriers.

In Figs. 2.1 and 2.2 we present the numerical solution for the vortex (QP) configurations (in which $b$ is lowered relative to $B$ and $h(r) \to -1$ as $r \to \infty$). We plot the functions $h(r)$ and the density $\rho(r)$ for $\nu = 1/3, 1/5$ and $1/7$ respectively. When changing $\nu$, the density $\rho$ remains fixed as the magnetic field $B$ is varied. Figs. 2.3 and 2.4 contain plots of $h(r)$ and $\rho(r)$ for the same values of $\nu$ but now for the antivortex (QH) configurations. In Table 2.1 we present the energies and a measure $r_0$ of the size of each vortex and antivortex. We have arbitrarily chosen the size of the vortex as the value of $r$ at which the energy density has reached 99% of its total value.

We are now ready to describe quantitatively (within this model) what happens when the chemical potential is varied from $\mu_0$. As discussed in great detail in this Chapter there is no change in the density unless $\mu - \mu_0$ is approximately equal to the energy of a vortex times $2n + 1$ (i.e. the energy per particle of the vortex). We can now see, quantitatively, how this works from Eq. (2.26). If $f \neq 1$ so that $\mu = \mu_0/f \neq \mu_0$ then the energy of the vortex is simply

$$\epsilon_v(\mu) = \epsilon_v(\mu_0) - (\mu - \mu_0) \times \left(\mp \frac{1}{2n + 1}\right)$$

where the minus sign is for a vortex and the plus sign for an antivortex. Thus for $\mu < \mu_0$ the vortex configuration has lower energy than the antivortex configuration. We naively expect that when $\mu_0 - \mu = n\epsilon_v(\mu_0)$ (or near this point) the mean field ground state should be a condensate (possibly a lattice) of vortices. Conversely, when $\mu > \mu_0$, the antivortex has lower energy, and when $\mu - \mu_0 = n\epsilon_v(\mu_0)$, we naively expect a condensate of antivortices. Unfortunately, for our choice of parameters, $\epsilon_v$ is quite large. Thus the value $\mu_{cr}$ or $\mu$ at which this condensate occurs in the above naive calculation and which is shown in Table 2.1 differs from $\mu_0$ by an unreasonably large amount. This leads, in particular, to a negative value of $\mu_{cr}$ for the $2n + 1 = 5$ and $2n + 1 = 7$ vortex
Figure 2.1: The function $h(r)$ corresponding to a vortex for $\nu = 1/3, 1/5$ and $1/7$.

configurations. In fact, as $\mu$ is varied from $\mu_0$ towards $\mu_{cr}$, another critical value $\hat{\mu}$ of $\mu$ is reached at which $\Phi_0\hat{\mu}/\lambda B = 1/(2n + 3)$ well before $\mu_{cr}$ is reached. At $\hat{\mu}$ the system is better described by a Chern–Simons theory with the new value $\hat{\nu} = 1/(2n + 3)$ of the filling fraction.

In light of the above remarks we should try to understand whether in fact one does form a vortex condensate in our model at our chosen values of the parameters. Certainly the vortices must overlap well before $\hat{\mu}$ or $\mu_{cr}$ is reached. This can be better understood by first supposing that such a condensate is formed as $\mu$ is lowered. The approximate
density $\rho_1$ at which a description of this condensate in terms of the single vortex solutions presented above fails depends most prominently on the size of a vortex. In Table 2.2 we show the approximate density of vortices $\rho_{v1}$ and chemical potential $\mu_1$ at which the vortices begin to “touch”. For a hexagonal lattice of vortices this will occur when $\rho_{v1} = 1/(2\sqrt{3}d^2)$, i.e., where $d$ (the distance between vortices) equals the size of the vortices. Table 2.2 also shows the corresponding values of the filling fraction. Notice that in most cases the vortex lattice becomes dense well before the “next” value of $n$ (i.e. well before $\frac{ge}{\hbar} = 1/(2(n \pm 1) + 1)$). We conclude from this that an approximation
Figure 2.3: The function $h(r)$ corresponding to an antivortex for $\nu = 1/3, 1/5$ and $1/7$. 

in terms of a dilute gas of vortices breaks down well before $\mu = \mu_{cr}$. It is thus likely that even for our chosen values of the parameters a lattice of vortices will form in the mean field description. The formation of pinned vortices is what gives rise to the Hall plateaux by allowing the system to continue to behave as if it were in a state $\nu = \frac{1}{2n+1}$ even after the magnetic field or the chemical potential has been changed to move it away from that value. This is because the (anti)vortices accommodate the localised excess (deficit) of charge.
2.4 Discussion of Results

The vortex antivortex configurations described in the preceding section are identically the Laughlin quasiparticles and quasiholes, which are described in Section 1.6. The energies of these configurations have also been determined by other numerical methods. The usual method involves calculating the energy of a Laughlin type many-body solution for a finite number of particles. These energies are often given in terms of the interaction energy, $\frac{e^2}{d}$. A summary of the results of various calculations are shown in Table 2.3. We
Chapter 2. Vortices and the Bosonic Description of the FQHE

<table>
<thead>
<tr>
<th>vortex/antivortex</th>
<th>n</th>
<th>size (μm)</th>
<th>energy (×10^2 eV)</th>
<th>μ_{cr}/μ₀</th>
</tr>
</thead>
<tbody>
<tr>
<td>vortex</td>
<td>3</td>
<td>0.027</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.018</td>
<td>0.21</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>0.015</td>
<td>0.20</td>
<td>-0.4</td>
</tr>
<tr>
<td>antivortex</td>
<td>3</td>
<td>0.035</td>
<td>0.83</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>5</td>
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<td>0.80</td>
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</tr>
<tr>
<td></td>
<td>7</td>
<td>0.020</td>
<td>0.79</td>
<td>6.5</td>
</tr>
</tbody>
</table>

Table 2.1: Energy and size (r₀) of vortices and antivortices for n = 3, 5 and 7. μ_{cr} is a naive estimate of the value of the chemical potential at which a condensate of these configurations is expected to form (μ₀ = 10^−2 eV). μ_{cr} is more carefully described in the text.

<table>
<thead>
<tr>
<th>vortex/antivortex</th>
<th>n</th>
<th>ρ_{v1} (μm^−2)</th>
<th>ν_1⁻¹</th>
</tr>
</thead>
<tbody>
<tr>
<td>vortex</td>
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<td>400</td>
<td>2.6</td>
</tr>
<tr>
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<td></td>
<td>7</td>
<td>1280</td>
<td>5.7</td>
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</tr>
<tr>
<td></td>
<td>5</td>
<td>460</td>
<td>5.5</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>720</td>
<td>7.7</td>
</tr>
</tbody>
</table>

Table 2.2: The density ρ_{v1} and the corresponding filling fraction ν_1 at which the condensate of vortices is expected to become dense. (See text for a precise definition.)

have assumed ε = 13 in the conversion of our results, so that ε/ε^* = 0.015 eV.

There is a close agreement of the QH creation energies determined by Laughlin and by Morf and Halperin, because they used the same many-body wavefunction to describe the QH. However, in the time between the Laughlin theory and the emergence of the Chern-Simons theories, there was a great deal of confusion about the form of the quasiparticle state, although it was realised by Laughlin that both the QP’s and QH’s would be formed as the result of a local adiabatic addition of one flux quantum. Within the Laughlin theory, it was straight-forward to hypothesize that the QH would have a zero at the
position of the hole. In the discussions of vortices presented above, we have shown that the addition of one flux quantum to create a QP or a QH necessarily implies that the order parameter vanishes at the position of the excitation. The node in the electron density of the QP is an artifact of the Landau Ginzburg theory and it is one source of the discrepancy with the energies found using many-body wavefunctions. In fact, we have not included effects arising from the binding of QP’s to the vortices, (which is beyond the Landau Ginzburg theory); such effects would modify the vortex profile (the local charge density) as well as the energy of the configuration.

Another source of discrepancy with our results originates from the simplification of the Coulombic interaction to a point interaction. The chemical potential is assumed to be roughly equal in each system and the interaction strength $\lambda$ we have chosen produces the right number of particles (i.e., the correct filling). This means that for a uniform configuration, our interaction is chosen so that the system will have the same amount of energy per particle as a realistic system, whatever its interaction may be. The other results considered in Table 2.3 used a Coulombic type interaction, which is non-local. If a small amount of localised charge is removed (as for a QH), the energy loss of the local interaction will be higher than the loss of a non-local interaction, which yields the result that our energies are expected to be higher than the energies obtained using a Coulomb interaction.

<table>
<thead>
<tr>
<th>author</th>
<th>quasi-hole</th>
<th>quasi-particle</th>
<th>gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laughlin [18]</td>
<td>0.025</td>
<td>0.022</td>
<td>0.047</td>
</tr>
<tr>
<td>Haldane and Rezayi [4]</td>
<td></td>
<td></td>
<td>1.05 ±.005</td>
</tr>
<tr>
<td>Morf and Halperin [64]</td>
<td>0.026</td>
<td>0.073 ±.008</td>
<td>0.099</td>
</tr>
<tr>
<td>Curnoe and Weiss [59]</td>
<td>.16</td>
<td>.55</td>
<td>.71</td>
</tr>
</tbody>
</table>

Table 2.3: Summary of numerical results for quasi-particle and quasi-hole creation energies and gap energy. Units are $e^2/\epsilon_1$. 
2.5 Summary

In this Chapter we have studied the mean field behavior of the CSLG description of the FQHE when the filling fraction deviates from the special filling fractions $\nu = \frac{1}{2n+1}$. We have shown how the field theoretic description of this model at a fixed chemical potential $\mu$ and magnetic field $B$ can be studied for a range of $\mu$ surrounding the value $\mu_0$ corresponding to the special filling fraction. For small values of $|\mu - \mu_0|$ (and at zero temperature) the density is independent of $\mu$. As $\mu$ is decreased (increased) beyond some critical value we have shown how the homogeneous mean field configuration is unstable to the formation of fractionally charged vortices (antivortices). We have presented a numerical example of these vortex and antivortex configurations and we estimated the densities and filling fractions at which the description in terms of a noninteracting system of vortices breaks down.
Chapter 3

Composite Fermions, Gauge Fluctuations and the Gap

Despite the success of the bosonic theory and the appealing physical insight it provided, it was unable to describe filling fractions other than $\nu = \frac{1}{2n+1}$ in any matter at all. Clearly a similar mechanism is at work for all the filling fractions, as evidenced by the oscillatory behaviour observed in many physical properties, including resistivity [30, 31, 32], magnetisation (extrinsic photoluminescence) [53], intrinsic photoluminescence [65] and compressibility [55]. Composite fermions provide such a mechanism.

The composite fermion-gauge theory of the fractional quantum Hall effect is an effective field theory which, at a mean field level, explains the hierarchy of observed FQH states in a natural and intuitive way. In addition to this, it has the structure of an ordinary fermionic gauge theory and hence lends itself to the possibility of performing calculations of real physical quantities using standard methods of quantum field theory. The catch is that such methods are usually perturbative, while this theory is plagued with a rather large coupling between the fermions and the gauge field. Hence perturbative methods may be applied and do in fact reveal qualitative features of the system, but the real challenge has been to seek and apply other methods to understand this theory.

This Chapter studies the self-energy of composite fermions and also the gap, which is a physical quantity derived from the self-energy. The self-energy is the correction coming from interactions to the free 1-particle Green's function, and yields corrections to the particles' dispersion relation, in particular the effective mass. The perturbative calculation of this quantity for a gapless system with gauge interactions was first performed
some time ago in a different context [66]; it has also appeared in studies of the gauge theory of high temperature superconductors [67] and a similar calculation is applicable to the FQHE at the gapless state, \( \nu = \frac{1}{2} \) [26]. We extend this calculation to find the gaps of filling fractions away from \( \nu = \frac{1}{2} \) and improve the results using a self-consistent approach.

As we shall see, the composite fermion theory of filling fractions away from \( \nu = \frac{1}{2} \) requires an effective magnetic field, which is the difference between the real external field and the field due to the magnetic flux attached to each electron. The study of interacting systems in a magnetic field has a long history, but had until the QHE been mostly limited to three dimensions. In two dimensions we find that the highly degenerate density of states introduces singularities in the spectral functions, even for ordinary Fermi liquids. For fermions interacting with gauge fluctuations the effects are even more pronounced. A study of an interacting system which accounts for singularities has not been done, even for Fermi liquids. We make use of the singular structure of the self-energy to employ a new, iterative, self-consistent method to compute the gap of the FQH system at fillings \( \nu = \frac{p}{2p+1} \).

As discussed in Chapter 2, there has been a considerable number of experiments which have measured the gap, using in most cases fairly direct methods. However, because the gap decreases rapidly for higher denominator filling fractions, these methods have not been able to study the gap beyond \( \nu = \frac{3}{7} \). In recent years an innovative set of experiments has emerged which claim to be able to determine the effective mass through a large range of fractions, up to \( \nu = \frac{7}{15} \), by parametrising the SdH oscillations of the resistivity in terms of an effective mass. These experiments show an apparent divergence of the effective mass for filling fractions \( \nu = \frac{p}{2p+1} \) in the limit \( p \rightarrow \infty \) [30, 31, 32]. This corresponds to a vanishing of the gap in this limit, in agreement with theoretical predictions [29]. One of our objectives has been to compute the effective mass for a range of physically relevant
parameters, in order to achieve a direct, quantitative comparison between the composite fermion gauge theory and experiment.

We approach this problem with a second order perturbative calculation of the self-energy which retains the Landau level structure of the internal fermion line. This calculation is the first step of a new iterative procedure used to find the self-consistent form of the self-energy, which makes use of the structure introduced by Landau level quantisation. We also investigate the analogous two dimensional electron-phonon problem, which has never been looked at before, in order to better understand the peculiar features of the composite fermion self-energy. This procedure is used to evaluate the gap, which rapidly converges after a few iterations. Our results are in sharp disagreement with experiments; however analysis of the various experiments shows that their results do not agree with each other either. Therefore it remains unclear as to whether or not the composite fermion gauge theory is a quantitatively correct description of the FQHE.

This Chapter is organised as follows. We begin in Section 3.1 with a review of composite fermions, the key experiments that justify the composite fermion construction and the Lagrangian formulation of the composite fermion gauge theory. This is followed in Section 3.2 by a discussion of the self-energy calculation at \( \nu = \frac{1}{2} \) performed by Halperin, Lee and Read [26]. Then in Section 3.3 we show how to do the analogous calculation away from \( \nu = \frac{1}{2} \), taking into account the Landau level structure arising from the finite effective field \( \Delta B \). We use the self-energy to calculate the lowest order in perturbation corrections to the gap in Section 3.4. In Section 3.5 we compare the composite fermion gauge theory results for the self-energy and the gap to a calculation of the self-energy for a normal Fermi liquid in the presence of a magnetic field, interacting with phonons with Debye and optical spectra. Finally in Section 3.6 we present an iterative self-consistent method for determining the gap and in Section 3.7 we compare our results with experiments.
3.1 Composite Fermions

Composite fermions are electrons to each of which is attached an even number \((2n)\) of magnetic flux quanta. The composite fermions experience a magnetic field \(\Delta B\) that is the difference between the external magnetic field \(B\) and the field associated with the attached flux, \(B_\perp\). When the filling fraction is \(\nu = \frac{p}{2np\pm 1}\) the effective magnetic field is

\[
\Delta B = B_{\frac{p}{2np\pm 1}}
\]

\[
= \Phi_0 n_e \left( \frac{2np \pm 1}{p} - 2np \right)
\]

\[
= \frac{\pm \Phi_0 n_e}{p}
\]

\[
= \frac{B}{1 \pm 2pn}.
\]

Eq. (3.3) tells us that composite fermions are in a magnetic field \(\Delta B\) with a filling \(p\), i.e., there are \(p\) filled CF Landau levels. This is the basis of Jain’s observation that the FQHE of electrons could be interpreted as the IQHE of composite fermions. Eq. (3.4) tells us that the effective magnetic field \(\Delta B\) is a fraction of the external field, as is the associated cyclotron frequency \(\Delta \omega_c = \frac{\Delta B}{m}\). Thus we can picture that the lowest Landau level has been subdivided into levels separated by an energy \(\Delta \omega_c\), of which \(p\) are filled. In the special cases when \(p\) is an integer, the system has a gap, which equals \(\Delta \omega_c\). This provides a qualitative explanation for the significance of the special filling fractions \(\nu = \frac{p}{2np\pm 1}\) and also for the decreasing stability of the incompressible quantum Hall states for higher values of \(m\) and \(p\) [8].

3.1.1 Experimental Evidence for Composite Fermions

The FQHE is the observation of SdH oscillations of the longitudinal resistivity and plateaux occurring in the Hall resistivity at the special filling fractions \(\nu = \frac{p}{2np\pm 1}\). In
addition to the resistivity measurements and other experiments which observe oscillatory behaviour as a function of $\Delta B$, there is other evidence that composite fermions exist. Various methods have been used to determine the cyclotron radius of the charged quasiparticles away from $\nu = \frac{1}{2}$. These experiments have confirmed that the quasiparticles do in fact behave as if they were in a magnetic field $\Delta B$, with a diverging cyclotron radius as $\nu \to \frac{1}{2}$.

The experiment of Goldman, Su and Jain [63] is an elegant measurement of the cyclotron radius using magnetic focusing. The 2-D electron system is exposed to a magnetic field while a current is injected at one point of the sample. The current is picked up at a point some distance away (see Fig. 3.1). As the field is varied on one side of $\nu = \frac{1}{2}$ the intensity of the current oscillates with maxima occurring when the distance is an even integral multiple of the cyclotron radius,

$$R_c = \frac{\hbar k_F}{e \Delta B}.$$  \hspace{1cm} (3.5)

On the other side of $\nu = \frac{1}{2}$ there are no oscillations observed since the sense of the cyclotron orbit is reversed. The behaviour of the composite fermions is completely analogous to the behaviour of electrons in an external magnetic field.

Figure 3.1: Schematic drawing of the geometry used in the magnetic focusing experiment of Goldman, Su and Jain. Each arc is a cyclotron orbit (after Goldman et al. [63]).

Willet et al. [68] observed the cyclotron radius by finding geometric resonances using surface acoustic waves (SAW's). SAW's traversing the sample are attenuated by piezoelectric interactions with the electrons in the 2-D gas, causing a shift in the velocity of the
SAW. The amount of shift decreases with increasing longitudinal conductivity, which is wave-vector dependent. As a function of magnetic field, a minimum in the velocity shift corresponds to a resonance of the conductance which occurs at a wave-vector \( q = \frac{1}{4\pi R_L} \). Two minima are observed symmetrically on either side of \( \nu = \frac{1}{2} \), corresponding to resonances at \( \pm \Delta B \). This interpretation of the data is consistent with having \( \Delta B = 0 \) at \( \nu = \frac{1}{2} \). Furthermore, using (3.5) to extract \( k_F \), close agreement was found with the gauge theory prediction, \( k_F = (4\pi n_e)^{\frac{1}{2}} \). This is proof that a Fermi surface does exist at \( \nu = \frac{1}{2} \).

Finally we consider the experiment of Kang et al. [69], who observed cyclotron resonance of electron transport through an antidot superlattice. An antidot lattice is an array of holes superimposed on 2-D electron gas surface. The dots are \( \approx 100 - 200 \text{nm} \) in size and separated by a period \( d \) of about five dots. Minima in the longitudinal resistivity are associated with a localisation of the electrons which occurs when the electronic cyclotron orbits precisely encircle a square array of dots (1, 4, 9 etc). The results show well defined minima at fields \( B = \frac{\hbar k_F}{e} \) and \( \Delta B = \frac{\sqrt{2}\hbar k_F}{e} \), corresponding to the cyclotron resonances of electrons and of composite fermions.

### 3.1.2 The Composite Fermion Gauge Theory

The first theory to describe the FQHE, formulated by Laughlin, is one of strongly interacting electrons in the presence of a magnetic field. In the Introduction we have shown how to transform this theory into one in which the electrons have been replaced by composite fermions in the presence of a field \( \Delta B \). Experiments have shown that this theory gives a good description of many of the features of the FQHE. Thus it seems as if the strong electron-electron interactions have manifested themselves primarily through stabilising the binding of the magnetic flux to each electron. Therefore we describe the
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

mean field composite fermion system by the Lagrangian

\[ \mathcal{L} = \psi^\dagger (x) (-i \partial_t + a_0) \psi(x) + \psi^\dagger (x) \left( i \partial_t - \frac{e \Delta A_j}{2m} \right) \psi(x) + \frac{a_0 \nabla \times A}{2n \Phi_0} + \int d^2 y \frac{V(x-y) : \rho(x) \rho(y) :}{2} \]

where \( \Delta A = A - a \) and \( \nabla \times \Delta A = \Delta B \). The constraint that there are \( 2n \) flux quanta attached to each electron comes from the equation of motion that is found by varying \( a_0 \), which is \( \psi^\dagger (x) \psi(x) \equiv \rho = \frac{\nabla \times a}{2n \Phi_0} \). The second term is the Hamiltonian for "free" composite fermions and has the usual solutions of a charged particle in a magnetic field. The fourth term is the strong electron-electron interaction, without which there would be no FQHE. This term may be rewritten in terms of the gauge field \( a \) using the constraint. In the fourth term, \( V(x-y) = \frac{e^2}{\epsilon |x-y|} \) is a Coulombic interaction, with Fourier transform

\[ v(q) = \frac{2 \pi e^2}{\epsilon q} \]  

We also consider a slightly different case where the Coulombic electron-electron interactions are screened (which may be realised by placing the 2-D electron gas near to a conducting plate); the case when \( v(q) \approx v \) arises when the interaction is effectively zero-range (of strength \( v \), and range less than the magnetic length \( l \)). Note that there is no \( F_{\mu \nu} F^{\mu \nu} \) term in \( \mathcal{L} \) since in the low energy-momentum limit which we will always consider, the Chern-Simons term and the corrections \( K^{(0)} \) (below) will dominate in all components of the gauge propagator.

The next step in the theory is to allow for gauge fluctuations around the mean field solution, \( \Delta A = 0 \), which we may write as \( a_i = A_i + \delta a_i \). The fermion field \( \psi \) interacts with the gauge fluctuations in the usual way, via the covariant derivative. The quadratic parts in \( \delta a \) yield a bare propagator for the gauge fluctuations; the Fourier transform of
this is
\[ D^{(u)} = \begin{pmatrix} 0 & \frac{iq}{n\Phi_0} \\ -\frac{iq}{n\Phi_0} & \frac{q^2 + (\omega)^2}{(n\Phi_0)^2} \end{pmatrix}, \tag{3.8} \]

where the matrix elements are the longitudinal and transverse components respectively.

The effective action for the gauge fluctuations may be determined by integrating out the composite fermion field \( \psi \) [25, 26]. In the low energy limit, this is determined precisely using the corrections \( K^{(0)}(q, \omega) \) from the two 1-loop graphs shown in Fig. 3.2 [70]. These corrections are also calculated at \( \Delta B = 0 \), and in the limit of small \( q, \omega \) are

\[
K_{00}^{(0)} = \frac{m^*}{2\pi} \left( 1 + \frac{i\omega}{qv_F} \right) \tag{3.9}
\]

\[
K_{11}^{(0)} = -\frac{q^2}{12\pi m^*} + \frac{2n_\omega\omega}{k_Fq}. \tag{3.10}
\]

Then the renormalised gauge fluctuation propagator takes the form

\[
D = (D^{(0)} - 1 + K^{(0)})^{-1}. \tag{3.11}
\]

The most singular component is the transverse component, calculated in the Coulomb gauge at \( \Delta B = 0 \) for small \( \omega \) and \( q \):

\[
D_{11}(q, \omega) = \frac{q}{i\gamma\omega + q^*\chi}. \tag{3.12}
\]
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

The exponent $s$ can have values between 2 and 3. $s = 2$ when there are unscreened Coulombic electron-electron interactions and $s > 2$ corresponds to screening those interactions (as discussed above). The constants are $\gamma = \frac{2\pi}{k_F}$ and $\chi = \frac{e^2}{8\pi\epsilon}$ for $s = 2$ and $\chi = \frac{1}{24\pi m} + \frac{\nu}{(4\pi)^2}$ for $s = 3$ in this random phase approximation [26]. $\epsilon$ is the dielectric constant.

This form of the gauge propagator has a long history which predates the QHE. It was first derived in 1973 by Holstein, Norton and Pincus [66] for the physically quite different problem of metallic electrons coupled to ordinary transverse photons. They showed that it causes non-Fermi liquid exponents in the temperature dependence of the resistivity, heat capacity and NMR relaxation rate in metals. However, there is an important difference between the scenario studied by Holstein et al. and the one presented in this thesis. In the earlier case the gauge fluctuations are fluctuations of the regular electromagnetic field - i.e. photons, and therefore the self-energy corrections which give rise to the non-Fermi liquid exponents are of order $\left(\frac{ue}{\epsilon}\right)^2$. In the case we are considering the gauge field is introduced to enforce a constraint on the density of particles; in the gauge theory of high temperature superconductors [67] it is introduced to prevent double site occupancy. In both cases it causes a strong coupling between the gauge fluctuations and density fluctuations. The result is that corrections coming from this gauge propagator are not small, in fact they are usually of order unity.

This seemed to be a very plausible origin for non-Fermi liquid behaviour generally and for a while it seemed like gauge fluctuations could be a universal mechanism responsible for the unusual behaviour of many strongly correlated systems [71]. However, this theory still awaits experimental confirmation and a consistent non-perturbative treatment. For this reason, one of the goals of the Section 3.3 is to derive quantitative predictions of the mass renormalisation arising from gauge fluctuations. But first we consider the situation when $\Delta B = 0$. 
3.2 The Self-Energy at $\Delta B = 0$

The expression (3.12) was first used to calculate self-energy corrections arising from interactions with gauge fluctuations by Holstein et al. [66] and by others [67, 72, 73] in the context of high $T_c$ superconductors. Similar results were obtained by Halperin, Lee and Read at $\nu = \frac{1}{2}$, i.e., at $\Delta B = 0$ [26]. The self-energy is given in perturbation theory as [74]

$$\Sigma(k, \epsilon) = \int \frac{d^2 q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} \left| \frac{k \times \hat{q}}{m} \right|^2 \text{Im}D(q, \omega) \times \left( \frac{1 + n_B(\omega) - n_f(\xi_{k-q})}{\epsilon - \omega - \xi_{k-q} + i\delta} + \frac{n_B(\omega) + n_f(\xi_{k-q})}{\epsilon + \omega - \xi_{k-q} + i\delta} \right)$$

(3.13)

where $\xi_k \equiv \frac{(k-k_F)^2}{2m}$ and

$$\text{Im}D(q, \omega) = \frac{q\gamma\omega}{\gamma^2\omega^2 + q^2\chi^2}.$$  

(3.14)

At zero temperature, in the limit of small $\epsilon$ evaluated at $\epsilon = \xi_k$ Eq. (3.13) yields [26] for $s > 2$

$$\Sigma'(k, \epsilon) \sim \epsilon^{2/s}$$

(3.15)

$$\Sigma''(k, \epsilon) \sim \epsilon^{2/s}$$

(3.16)

and for $s = 2$

$$\Sigma'(\epsilon) \sim \epsilon \log \epsilon$$

(3.17)

$$\Sigma''(\epsilon) \sim \epsilon.$$  

(3.18)

$\Sigma'(\epsilon)$ may be used to obtain the effective mass [75]

$$\frac{m^*}{m} = \lim_{k \to k_F} \left( \frac{1 - \partial\Sigma(k, \epsilon)/\partial\epsilon}{1 + \partial\Sigma(k, \epsilon)/\partial\xi_k} \right)$$

(3.19)

$$\sim \log(\epsilon) \quad \text{for } s = 2.$$  

(3.20)
This result indicates that there is a logarithmically divergent correction to the effective mass as the Fermi surface is approached as a function of $\varepsilon$. An analogous behaviour is predicted to occur as $\nu$ approaches $\frac{1}{2}$, which will be explained more in Section 3.3.

The experiment of Jiang et al. [76] revealed the first physical example of a metallic state with unusual, non-Fermi liquid properties by studying the temperature dependence of the resistivity at $\nu = \frac{1}{2}$. In mathematical terms, Fermi and non-Fermi liquids are distinguished by the analytic properties of the Green’s functions. Fermi liquids are characterised by the presence of a pole (or, more generally, a branch cut) in the propagator which occurs at the dispersion of a physical particle. In the bare fermion Green’s function the pole is a $\delta$-function,

$$G^{(0)}(\varepsilon, k) = \frac{1}{\varepsilon - \xi_k - i\delta} \quad (3.21)$$

indicating that the particle has a well defined energy. The position and strength of the pole are modified by interactions, which is described by the self-energy appearing in the renormalised Green’s function,

$$G(\varepsilon, k) = \frac{1}{\varepsilon - \xi_k + \Sigma(\varepsilon, k)} \quad (3.22)$$

but the essential feature of Fermi liquid theory is that the analytic properties remain the same. This means that the dispersion relation will have the same functional form, but with the bare mass replaced by an effective mass which is given by (3.19).

The analytic properties of the Green’s functions of this composite fermion gauge theory are modified by the self-energy given in Eqs. (3.15) - (3.18). The appearance of a logarithm indicates the onset of a branch cut singularity instead of a pole; the fractional power which appears in (3.15) is a continuation of this trend. The important point is that in the low energy limit, the self-energy correction is the dominant term in the renormalised Green’s function. In addition, the large imaginary part of the self-energy
gives rise to a broad peak in the spectral functions which indicates that the quasiparticle is not well-defined in the usual sense of a Fermi liquid theory.

The considerations outlined above show that any theory which attempts to describe the \( \nu = \frac{1}{2} \) scenario must be able to address the non-Fermi liquid nature in a non-perturbative way. There have been several different approaches to this problem for the gauge theories, including renormalisation group analysis [70, 77, 78, 79], 1/\( N \) expansion [80, 81, 82], eikonal expansion [83], and bosonisation [79, 84], but these various approaches do not yield equivalent results. It is not even resolved as to whether or not Fermi liquid theory actually breaks down in two dimensional systems with short-ranged interactions\(^1\). Using renormalisation group analysis various authors [70, 77, 79] have been able to show that systems with long-ranged interactions of the form \( s = 2 \) (discussed above) are driven by a novel, non-Fermi liquid fixed point of the action. These results lend credence to the initial approach of Halperin, Lee and Read [26], which, although perturbative, appears to capture the essential physics of the problem.

3.3 The Self-Energy for \( \Delta B \neq 0 \)

\(^2\)For a finite magnetic field the self-energy expression (3.13) becomes

\[
\Sigma_n(\epsilon) = \frac{2E_F}{m} \int \frac{d^2q}{(2\pi)^2} \int_0^{\infty} \frac{d\omega}{\pi} \text{Im} D(q, \omega) \sum_{n' = 0} |\Lambda(n, n', q)|^2 \int_{-\infty}^{\infty} \frac{d\epsilon'}{\pi} \text{Im} G^{0*}(\epsilon')
\]

\[
\times \left( \frac{1 + n_B(\omega) - n_f(\epsilon')}{\epsilon - \epsilon' - \omega + i\delta} + \frac{n_B(\omega) + n_f(\epsilon')}{\epsilon - \epsilon' + \omega + i\delta} \right).
\]

(3.23)

We assume that there are \( |p \) filled composite fermion Landau levels (\( |p \) the greatest integer less than \( p \)) and that the chemical potential lies in the centre of the gap, at \( \mu = \Delta \omega_c |p \). We will consider a system of CF's in a field \( \Delta B \) which is small, using the

\(^1\)Opinions have shifted back and forth through the years. Refs. [85] argue that there is a breakdown of Fermi liquid theory, while Refs. [86] argue that there is not.

\(^2\)The results of this Section appear in [87].
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

gauge fluctuation propagator at $\Delta B = 0$, given by (3.12) (this is justified in Appendix D). Within this same semi-classical limit, we adopt a well known approximation for the overlap matrix element between plane waves and Landau level states [88, 89] (see Appendix A for more details):

$$|\Lambda(n, n', q)|^2 \approx \frac{1}{q\pi} \left( \frac{e\Delta B}{2p} \right)^{\frac{3}{2}}$$

(3.24)

where $n, n' \approx p$ are Landau level indices, as shown in Fig. 3.3.

Figure 3.3: Feynman diagram representing the self-energy. $n$ and $n'$ are Landau level indices. The straight line represents the fermion (either electron or composite fermion) and the wavy line represents the boson (either a phonon or a gauge fluctuation).

It is useful to rewrite these equations shifting the sum to start at $-p$ and then enforcing particle-hole symmetry by truncating the sum at $p - 1$, the number of filled Landau levels. (In reality the problem is not particle-hole symmetric, but we are only interested in energies within $\omega_c$ of the Fermi surface, where particle-hole symmetry is almost exact. In the case of composite fermions, this corresponds to having an upper cutoff at the true cyclotron frequency).

Using (3.12) and $\text{Im} G_{nn}(\epsilon) = \pi \delta(\epsilon + \mu - (n + \frac{1}{2})\Delta \omega_c)$ we find the real and imaginary parts of the self-energy at $T = 0$ [89] (see Appendix B for more details). For $s = 2$ one has:

$$\Sigma'(\epsilon) = \frac{K_2\Delta \omega_c}{\pi} \sum_{m=0}^{p-1} \log \left| \frac{(m + 1/2)\Delta \omega_c - \epsilon}{(m + 1/2)\Delta \omega_c + \epsilon} \right|$$

(3.25)

$$\Sigma''(\epsilon) = K_2\Delta \omega_c \sum_{m=0}^{p-1} (\theta[-m - 1/2]\Delta \omega_c - \epsilon] + \theta[-m - 1/2]\Delta \omega_c + \epsilon])$$

(3.26)
\[ K_2 = \frac{E_F}{e^2} = \frac{\epsilon \sqrt{4\pi n_e}}{2me^2} \] (3.27)

whilst for \(3 \geq s > 2\):

\[
\Sigma'(\epsilon) = \text{sgn}(\epsilon)\Delta \omega_c \cot\left(\frac{2\pi}{s}\right) \sum_{m=0}^{p-1} \left( \left[(m+1/2)\Delta \omega_c - \epsilon\right]^{-\alpha} \theta[(m+1/2)\Delta \omega_c - \epsilon] \right. \\
+ \left. \left[(m+1/2)\Delta \omega_c + \epsilon\right]^{-\alpha} \theta[(m+1/2)\Delta \omega_c + \epsilon] \right)
\] (3.28)

\[
\Sigma''(\epsilon) = K_s \Delta \omega_c \sum_{m=0}^{p-1} \left( \left[(-m-1/2)\Delta \omega_c - \epsilon\right]^{-\alpha} \theta[(-m-1/2)\Delta \omega_c - \epsilon] \right) \\
\] (3.29)

\[
K_s = \frac{E_F}{48\pi \gamma^2/s \sqrt{4\pi n_e}} \csc \left(\frac{\pi}{s}\right)
\] (3.30)

where \(\alpha = (s-2)/s\) is positive. Note that the coefficient \(K_2\) is the same as the one calculated by Stern and Halperin [29]. Plots of \(\Sigma'(\epsilon)\) and \(\Sigma''(\epsilon)\) for \(s = 2, 3\) are shown in Figs. 3.4 and 3.5.

The results shown in the figures display divergences at every multiple of \(\Delta \omega_c\). Later in Section 3.5 we find singular behaviour in the derivative of the self-energy coming from interactions with phonons, but the self-energy itself stays finite. Therefore the divergences appearing in the self-energy shown in Fig. 3.4 are due to both the singular density of states (Landau levels) and the transverse gauge interaction.

Stern and Halperin performed similar calculations but with the sum over Landau levels replaced by an integral, effectively smearing out all of the structure shown here. While this is a valid approximation for energies \(\epsilon \ll \Delta \omega_c\), it is clear that these structures are crucial in any discussion of energies \(\epsilon \sim \Delta \omega_c\). This energy range will be important when we consider the renormalised gap.

A rather extraordinary feature of the CF propagator appears when we closely examine \(\Sigma''(\epsilon)\) and \(\Sigma'(\epsilon)\) around the divergences - we notice that \(\Sigma'(\epsilon)\) shows only positive
divergence on both sides of each Landau level. The paradox is that typically one would expect $\Sigma(\epsilon)$ to have the form

$$\Sigma(\epsilon) \sim \sum_r \frac{|V_r|^2}{\epsilon - E_r + i\delta}$$

(3.31)

in simple perturbation theory if our starting fermion spectrum is composed of discrete (albeit degenerate) levels at energies $E_r$, and $V_r$ is some perturbation. This leads to divergences in $\Sigma''$ each time one crosses the energies $E_r$.

This sign change occurs for $\Sigma'(\epsilon)$ in the electron-phonon problem (to follow in Section
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

Figure 3.5: Imaginary part of the self-energy of composite fermions. \( s = 2 \) corresponds to Coulombic electron-electron interactions; \( s = 3 \) corresponds to short-ranged interactions and the energies are in units of \( \Delta \omega_c \), as before.

3.5): this is most clearly seen by plotting \( \partial \Sigma'(\epsilon)/\partial \epsilon \) for this problem, where one sees a series of positive divergences around each Landau level, with no sign change, as shown in Fig. 3.10. This is exactly what we would expect from (3.31), since it yields

\[
\frac{\partial \Sigma'(\epsilon)}{\partial \epsilon} \sim -P \sum_r \frac{|V_r|^2}{(\epsilon - E_r)^2}.
\]

(3.32)

The behaviour of \( \Sigma_{CF}(\epsilon) \) is thus exactly the opposite of what one expects. The explanation of this paradox is that although \( \Sigma''_{CF}(\epsilon) \) is strongly peaked (indeed divergent) around each Landau level, and does not change sign as one crosses a Landau level,
nevertheless the peak has such a peculiar shape that its Hilbert transform $\Sigma_{CF}'(\epsilon)$ also does not change sign as one crosses a Landau level. Each Landau level $r$ contributes a term $\sim (\epsilon - r)^{-\alpha} \theta(\epsilon - r)$ to $\Sigma''_{CF}(\epsilon)$, with long tails for $(\epsilon - r) \gg 1$. The Hilbert transform of such a function does not change sign as $\epsilon$ crosses $r$, unless $\alpha > 1/2$; however in the CF gauge theory, $0 \leq \alpha \leq 1/3$.

This explains the paradoxical form of $\partial \Sigma / \partial \epsilon$ for composite fermions - it comes from the very long “tails” which extend out from each Landau level. These have no counterpart in the self-energy of a Fermi liquid, such as the electron-phonon problem to be discussed in Section 3.5.

3.4 The Renormalised Gap

The self-energy derived above is the first-order correction to the one-particle composite fermion Green’s function, yielding the full Green’s function,

$$G_n(\epsilon) = \frac{1}{\epsilon - (n + 1/2)\Delta \omega_c + \Sigma(\epsilon)}. \quad (3.33)$$

We wish to find the poles of $G_n(\epsilon)$, which are the solutions to

$$\Sigma'(\epsilon) + \epsilon = (n + 1/2)\Delta \omega_c, \quad (3.34)$$

which correspond to peaks in the spectral function. For each value of $n$ there can be many solutions to (3.34); in particular we note that for each value of $n$ there is exactly one pole within the gap $|\epsilon| < \frac{\Delta \omega_c}{2}$. This is shown in Fig. 3.6. Since the imaginary part is zero in this region these poles appear as $\delta$-functions in the spectral function.

The gap may be determined by approximating

$$\Sigma'(\epsilon) = \frac{\partial \Sigma'(0)}{\partial \epsilon} \epsilon \quad (3.35)$$

3The results of Sections 3.4-3.6 appear in [89].
and then solving (3.34) for $n = -1$ and $n = 0$ (for $s = 2$):

$$
\frac{\partial \Sigma'(0)}{\partial \epsilon} = -\frac{K_2 \Delta \omega_c}{\pi} \sum_{m=0}^{p-1} \frac{2}{m + 1/2}
$$

$$
\approx -\frac{2K_2 \Delta \omega_c}{\pi} \log(2p + 1)
$$

(3.36)

(3.37)

where the sum over $m$ has been replaced by an integral. Then the solution to (3.34) is

$$
\epsilon_{\pm} \left(1 - \frac{2K_2}{\pi} \log(2p + 1)\right) = \pm \frac{\Delta \omega_c}{2}.
$$

(3.38)

We find that for $s = 2$ the renormalised gap is

$$
\Delta \omega_c^* \approx \Delta \omega_c \left|1 - \frac{2K_2}{\pi} \log(2p + 1)\right|^{-1}
$$

(3.39)

and for $s > 2$

$$
\Delta \omega_c^* \approx \Delta \omega_c \left|1 - \frac{4K_2}{\sqrt{3}} p^{-\alpha}\right|^{-1}.
$$

(3.40)

These results were obtained by Kim et al. [90] and by Stern and Halperin [29]. Using $\frac{\Delta \omega_c}{\Delta \omega_c^*} = \frac{m^*}{m}$, it is pointed out by these authors that the divergence in the effective mass $m^*$ as $p \to \infty$ is completely analogous to the divergence near the Fermi surface at $\nu = 1/2$. 

Figure 3.6: Graphical solution to (3.34) for different values of $n$. The crosses show the poles between which lies the renormalised gap.
However the use of Eq. (3.35) to get a solution for (3.34) is only valid if $\Sigma'(\epsilon) + \epsilon$ varies slowly over $\epsilon$ such that $|\Sigma'(\epsilon) + \epsilon| < \frac{\Delta \omega_c}{2}$, and thus (3.35) is a poor approximation when $K_2$ is in an intermediate range ($\approx 0.5$). Moreover, the divergence in (3.39) at $K_2 = \pi/(2 \log(2p + 1))$ is unphysical; instead we expect the actual gap $\Delta \omega_c^*$ to decrease monotonically with $K_2$. In fact, we estimate that in the actual experiments so far done

![Figure 3.7: First order perturbation results for the effective gap, $\Delta \omega_c^*$, as a function of coupling, $K_2$, for $p = 50$. The solid curve is $|1 - \frac{2K_2}{\pi} \log(2p + 1)|^{-1}$ and points are numerical solutions to Eq. (3.34). The inset shows the same data plotted as a function of $n_e$, which is related to $K_2$ via eq. (3.27).](image)

[30], $K_2 \approx 0.8$, which places it right in the intermediate range, so it is necessary to go beyond the estimates in (3.39) and (3.40). We do this first by solving (3.34) numerically;
as an example we have done this for \( p = 50 \), as shown in Fig. 3.7. The numerical results show that for small values of the coefficient the gap is reduced by a very small amount. For larger values of \( K_2 \) the gap decreases rapidly. Turning our attention to the spectral functions shown in Fig. 3.8 we see that there is a simple physical interpretation for this. Small values of the coefficient give rise to very narrow double peaks at each cyclotron energy in the spectral function. This is the result of multiple solutions to (3.34); in fact there are two for each logarithmic peak in \( \Sigma'(\epsilon) \). The extreme narrowness of these

![Graph](image)

Figure 3.8: Imaginary part of the Green function of composite fermions in a finite magnetic field with \( n = 0 \) and \( p = 50 \). The upper figure has a coefficient \( K_2 = 0.31 \) and the lower one has \( K_2 = 6.3 \). The energy is in units of \( \Delta \omega_c \).
peaks suggests that these excitations are merely Landau level mixing. This conclusion is supported by the fact that for small values of $K_2$ the gap is only very slightly reduced. As $K_2$ is increased the double peaks are reduced in size and eventually give rise to incoherent structures. In the example we have been considering, $p = 50$, this crossover occurs when $K_2 \approx 0.5$ which is also where the gap starts to decrease. When the coefficient becomes large there is no longer any evidence of simple Landau level mixing.

3.5 Phonon Interactions

In this Section we do an analogous calculation for electrons in a finite magnetic field interacting with phonons, which, surprisingly, has never been previously investigated. Our main reason for looking at the electron-phonon problem is that it is a well-understood example of a Fermi liquid, which nevertheless acquires a non-trivial structure when Landau quantisation is introduced. We find structure associated with the Landau level degeneracies, although it is not quite as spectacular as the CF-gauge interaction result. The following results are useful for understanding the results of Section 3.3.

In zero field, the spectral functions of the coupled electron-phonon system were first studied by Engelsberg and Schreiffer \[91\]. Using optical and Debye phonon spectra they calculated the real and imaginary parts of the self-energy using conserving approximations (obeying the Ward identities). They found that the spectral function,

$$ A(\epsilon, p) = \frac{1}{\pi} \text{Im} G(\epsilon, p) $$

$$ = \frac{1}{\pi} \frac{\Sigma''(\epsilon, p)}{(\Sigma'(\epsilon, p) + \epsilon - \xi_p)^2 + \Sigma''(\epsilon, p)^2} $$

(3.41)

yielded structures that they could identify as well defined QP's for two cases: when $p$ is close to the Fermi surface, and when it is very large. In the intermediate regime there exists only an incoherent “smeared” structure.
The self-energy expression is given in perturbation theory by

$$\Sigma_n(\epsilon) = \int \frac{d^2q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} \text{Im} U(q, \omega) \sum_{n',0}^\infty |A(n, n', q)|^2 \int_{-\infty}^\infty \frac{d\epsilon'}{\pi} \text{Im} G_n^{(0)}(\epsilon')$$

$$\times \left( \frac{1 + n_B(\omega) - n_f(\epsilon')}{\epsilon - \epsilon' - \omega + i\delta} + \frac{n_B(\omega) + n_f(\epsilon')}{\epsilon - \epsilon' + \omega + i\delta} \right), \quad (3.42)$$

where $G_n^{(0)}$ is the bare electron Green function, i.e.,

$$\text{Im} G_n^{(0)}(\epsilon) = \pi \delta(\epsilon + (p - 1/2 - n)\omega_c). \quad (3.43)$$

and $U(q, \omega)$ depends on the electron-phonon coupling (see below). The electronic frequency $\epsilon$ is measured from the Fermi energy, $\mu = p\omega_c$, and the energy of the highest filled Landau level is $(p - 1/2)\omega_c$; since we are doing a quasi-classical calculation, we assume that the Fermi energy is halfway between Landau levels [90, 87].

In the rest of this section we first calculate the self-energy and the renormalised cyclotron gap energy for the case of a deformation coupling to Debye phonons; we then do the same for coupling to optical phonons.

### 3.5.1 Debye Phonons

In our analysis we first consider a two dimensional system of electrons in a small magnetic field $B$ which interacts with phonons having a Debye spectrum. In this case we will use the phonon propagator calculated at $B = 0$,

$$D(q, \omega) = \frac{q c_s}{\omega^2 - c_s^2 q^2} \quad (3.44)$$

which is assumed to have a cutoff at the Debye frequency $\omega_D$ ($c_s$ is the speed of sound). We will use the same approximation for the overlap matrix element between plane waves and Landau level states as in Section 3.3, given by (3.24).
The function $U(q, \omega)$ in (3.42) is just the effective interaction between electrons; in the Debye model it is given by

$$U(q, \omega) = \frac{q \Xi_D^2 \hbar}{2 \omega \rho a} D(q, \omega)$$

(3.45)

where $D(q, \omega)$ is the phonon Green function (3.44). The electron-phonon interaction is parametrised by the deformation potential, $\Xi_D$, and $\rho$ is the ion mass density of the material (see Appendix A for more details). Evaluating the integrals in Eq. (3.42) yields,

Figure 3.9: The real and imaginary parts of the self-energy for electrons interacting with phonons with a Debye spectrum ($\omega_D = 20 \omega_c$) in a small magnetic field. All energies are in units of $\omega_c$; the coupling is $K_D = 1$. 
at temperature $T = 0$:

\[
\Sigma'_n(\epsilon) = \frac{K_D}{\pi} \sum_{n'=p}^{\infty} \left( \omega_D - [(n' - p + 1/2)\omega_c - \epsilon] \log \left| \frac{\omega_D + [(n' - p + 1/2)\omega_c - \epsilon]}{(n' - p + 1/2)\omega_c - \epsilon} \right| \right)
+ \sum_{n'=0}^{p-1} \left( \omega_D + [(n' - p + 1/2)\omega_c - \epsilon] \log \left| \frac{(n' - p + 1/2)\omega_c - \epsilon - \omega_D}{(n' - p + 1/2)\omega_c - \epsilon} \right| \right)
\]

\[
\Sigma''_n(\epsilon) = K_D \left( \sum_{n'=p}^{\infty} (\omega_D - \epsilon + [(n' - p + 1/2)\omega_c - \epsilon]\theta[(n' - p + 1/2)\omega_c])
+ \sum_{n'=0}^{p-1} (\omega_D + \epsilon + [(-n' + p - 1/2)\omega_c + \epsilon]\theta[(-n' + p - 1/2)\omega_c]) \right). \tag{3.47}
\]

The right hand side is independent of $n$, and so we will omit the subscript on $\Sigma(\epsilon)$ henceforth. $\Sigma(\epsilon)$ is in units of $\omega_c$ and $K_D$ is a dimensionless constant,

\[
K_D = \frac{\Xi_D^2 m_{\omega_c}}{4\pi c^2 a\rho} \left( \frac{1}{4\pi n_{\epsilon}} \right)^{1/2}. \tag{3.48}
\]

Shifting the sum to start at $-p$ and truncating the sum at $p - 1$, we find

\[
\Sigma'(\epsilon) = \frac{K_D}{\pi} \sum_{m=0}^{p-1} \left( [(m + 1/2)\omega_c - \epsilon] \log \left| \frac{(m + 1/2)\omega_c - \epsilon + \omega_D}{(m + 1/2)\omega_c - \epsilon} \right| 
- [(m + 1/2)\omega_c + \epsilon] \log \left| \frac{(m + 1/2)\omega_c + \epsilon + \omega_D}{(m + 1/2)\omega_c + \epsilon} \right| \right)
\]

\[
\Sigma''(\epsilon) = K_D \sum_{m=0}^{p-1} \left( [(m + 1/2)\omega_c - \epsilon]\theta[(m + 1/2)\omega_c - \epsilon + \omega_D]\theta[(-m - 1/2)\omega_c + \epsilon]
+ [(m + 1/2)\omega_c + \epsilon]\theta[(m + 1/2)\omega_c + \epsilon + \omega_D]\theta[(-m - 1/2)\omega_c - \epsilon] \right). \tag{3.50}
\]

These functions are shown together in Fig. 3.9.

The structure caused by the Landau levels is very weak; it can just be seen in $\Sigma''(\epsilon)$ (if $\omega_c/\omega_D$ is larger, it is much more obvious). $\Sigma''(\epsilon)$ is simply a sum of ramp functions,
Figure 3.10: The derivative with respect to energy of the real part of the self-energy of electrons interacting with Debye phonons is a finite magnetic field \( (\omega_D = 20\omega_c) \). The energy is in units of \( \omega_c \), and \( K_D = 1 \).

coming from each Landau level; as \( \omega_c \to 0 \), \( \Sigma''(\epsilon) \) becomes parabolic for \( \epsilon \ll \omega_D \). The logarithmic singularities in \( \Sigma'(\epsilon) \) are quite invisible in Fig. 3.9, but they may be seen in the derivative, shown in Fig. 3.10.

We wish to find the poles of \( G_n(\epsilon) \). Generally there are 0, 1, 2 or 3 roots to the equation

\[
\Sigma'(\epsilon) + \epsilon = (n + 1/2)\omega_c, \tag{3.51}
\]

which correspond to peaks in the spectral function, \( \text{Im}G_n(\epsilon) \). In this expression both
Figure 3.11: The imaginary part of the Green function of electrons interacting with Debye phonons for $K_D = .001, .1$ and $.25$ with $n = 0$ (see the text for definition of the coupling $K_D$).

$n$ and $\epsilon$ are measured with respect to the Fermi surface. We are only concerned with the low energy behaviour which corresponds to small $n$ and $\epsilon$. When $n$ is small and $K_D$ is small there is one solution occurring near $\epsilon = (n + 1/2)\omega_e$, which appears as a well defined quasiparticle peak (a $\delta$-function for $n = 0$). When $K_D$ is large there are three solutions. Two occur well beyond the Debye frequency (outside of the low energy regime) and the third occurs close to $\epsilon = 0$ yielding a $\delta$-function. Intermediate values of $K_D$ do not in general yield the very narrow peaks characteristic of a dressed particle, since if
solutions to (3.51) do occur they occur at finite $\epsilon$ where the imaginary part of $\Sigma$ is also finite. There may also be features arising from incoherent contributions. Some of these are shown in Fig. 3.11. We remark that the magnetic field causes slight discontinuities in the slope of the spectral functions at the Landau level energies.

### 3.5.2 Einstein Phonons

We now consider the interaction with the more singular Einstein phonon spectrum, with phonon propagator $D(q,\omega) = \omega_E/(\omega^2 - \omega_E^2)$; $\omega_E$ is the "optical" phonon frequency. In zero field, this was first considered by Engelsberg and Schreiffer who found the self-energy to be

$$
\Sigma'(\epsilon) = -\frac{K_E\omega_c}{\pi} \log \left| \frac{\epsilon + E_F + \omega_E}{\epsilon + E_F - \omega_E} \right| \quad (3.52)
$$

$$
\Sigma''(\epsilon) = K_E\omega_c \quad \text{for} |\epsilon| > \omega_E
$$

$$
= 0 \quad \text{for} |\epsilon| < \omega_E. \quad (3.53)
$$

With $\Sigma$ expressed in units of $\omega_c$, $K_E$ is the dimensionless coupling:

$$
K_E \approx \frac{g^2a(4\pi n_e)^{1/2}}{4E_F\omega_c}. \quad (3.54)
$$

The effect of the magnetic field is to modify these equations to

$$
\Sigma'(\epsilon) = -\frac{K_E\omega_c^2}{\pi} \sum_{m=0}^{p-1} \left( \frac{1}{(m+1/2)\omega_c + \omega_E - \epsilon} - \frac{1}{(m+1/2)\omega_c + \omega_E + \epsilon} \right) \quad (3.55)
$$

$$
\Sigma''(\epsilon) = K_E\omega_c^2 \sum_{m=0}^{p-1} (\delta[(m+1/2)\omega_c + \omega_E - \epsilon] + \delta[(m+1/2)\omega_c + \omega_E + \epsilon]). \quad (3.56)
$$

$\Sigma'(\epsilon)$ is shown in Fig. 3.12. There is no virtually no effect of a small magnetic field on the self-energies for $|\epsilon| < \omega_E$; beyond this region the smooth functions are replaced by functions with divergences at each Landau level energy. All poles in the spectral function
appear as \( \delta \)-functions since the imaginary part of the self-energy vanishes everywhere except at discrete points, but the weights of the poles vary. The new electron spectrum is determined by summing over all \( n \). Since the \( n \) are discrete a renormalised gap can be defined as the energy difference between the highest occupied and lowest unoccupied states, i.e., by the difference between the positions of the \( \delta \)-function peaks of \( G_0(\epsilon) \) and \( G_{-1}(\epsilon) \).

![Figure 3.12: Real part of the self-energy of electrons interacting with phonons with an Einstein spectrum \((\omega_E = 20\omega_c)\). Both axes are in units of \( \omega_c \); the coupling is \( K_E = 1 \).](image)

The amount of renormalization of the gap depends on the size of the coefficient of the self-energy, \( K_E \). When \( K_E \) is small the poles lie near \((n + 1/2)\omega_c\) for \( n\omega_c < \omega_E \), as in the Debye case. The renormalized gap is approximately \( \omega_c \), only slightly increased due to the interactions. As \( K_E \) is increased the gap increases further. We determine the gap using the same approximation as in Section 3.4, (3.35) and then solving (3.51) for \( n = -1 \) and \( n = 0 \):

\[
\frac{\partial \Sigma'(0)}{\partial \epsilon} = -\frac{K_E}{\pi} \sum_{m=0}^{n-1} \frac{2}{(m + 1/2)\omega_c + \omega_E}
\]  

(3.57)
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

\[ \approx -\frac{2K_E}{\pi} \log \left( \frac{(p + 1/2)\omega_c + \omega_E}{\omega_c/2 + \omega_E} \right) \]  

(3.58)

where the sum over \( m \) has been replaced by an integral. Then the solution to (3.51) is

\[ \epsilon_\pm \left( 1 - \frac{2K_E}{\pi} \log \left( \frac{(p + 1/2)\omega_c + \omega_E}{\omega_c/2 + \omega_E} \right) \right) = \pm \frac{\omega_c}{2} \]  

(3.59)

and the gap is

\[ \omega^*_c = |\epsilon_+ - \epsilon_-| \]

\[ = \omega_c \left| 1 - \frac{2K_E}{\pi} \log \left( \frac{(p + 1/2)\omega_c + \omega_E}{\omega_c/2 + \omega_E} \right) \right|^{-1}. \]

(3.60)

This procedure is also applicable to the Debye case when there are well defined quasi-particles for \( n = -1, 0 \). In this case the renormalized gap is

\[ \omega^*_{c_D} = \omega_c \left| 1 - \frac{K_D}{\pi} \log(2\omega_D + \omega_c) \right|^{-1} \]

\[ -(2p + 1) \log \left( \frac{(p + 1/2)\omega_c + \omega_D}{(p + 1/2)\omega_c} \right) \]  

(3.61)

We may draw some general conclusions about the size of the gap for different coupling strengths using the forms above. Considering first the Einstein case, for small values of the couplings the gap is renormalized to a larger value. This continues to be the case as the coupling is increased. The divergence in the gap seen in Eq. (3.60) is avoided because the gap is always bounded by \( 2\omega_E \). This is due to the presence of the logarithmic peaks in the real part of \( \Sigma(\epsilon) \) which ensures that there will always be a solution to (3.51) for \( |\epsilon| < \omega_E \). For large couplings the gap is reduced. In the Debye case, for small \( K_D \) there are well defined quasi-particle peaks outside the gap but they are not strictly \( \delta \)-functions thanks to small but non-zero \( \text{Im} \Sigma(\epsilon) \). The finite width maintains the size of the gap to be \( \omega_c \), as shown in Fig. 3.11. For larger values of \( K_D \) the divergence of the gap in (3.61) is avoided for the same reason, as shown in Fig. 3.11b. For large couplings the
gap is reduced because of QP poles that lie within $\omega_c$, as shown in Fig. 3.11c. As can be seen from Figs. 3.9 and 3.11, the effect of Landau quantization on the quasi-particle properties, for realistic values of $\omega_D, \omega_E$ and $\omega_c$, is very small, at least at low energies (it is however worth noting that once $\epsilon > \omega_E$ in the case of Einstein phonons, a rather obvious singular structure appears in the self-energy, see Fig. 3.12). Nevertheless the calculation of these effects may be useful in systems for which $\omega_c$ can be made as large as $\omega_D$ or $\omega_E$.

In this Section we have examined the structure induced by Landau level quantisation on the two dimensional electron-phonon problem. We find singular, but weak structure which has never previously been investigated. The analysis of this well understood problem has been useful for improving our understanding of the composite fermion Green's function.

3.6 Iterative Self-Consistent Results for $\Sigma(\epsilon)$ and $\Delta \omega_c^*$

As already explained in Refs. [77, 80, 82, 84], the CF problem is essentially a non-perturbative one. The results these references address the $\nu = 1/2$ gapless system; an analysis of the $s = 2$ case for $\nu \neq 1/2$ was also given by Stern and Halperin [29], but taking no account of the structure described above.

Here we attempt to improve the results just given, by solving iteratively for the self-energy. We have found a way to do this which handles the coherent pole contributions to $\Sigma(\epsilon)$ in an exact rainbow summation; the incoherent parts must still be treated approximately.

We start from the iterative equation

$$\Sigma^{(i+1)}(\epsilon) = \sum_s \int d\epsilon' D(\epsilon - \epsilon') G_m^{(i)}(\epsilon')$$

(3.62)
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

with

\[ G^{(i)}_m(\epsilon) = \frac{1}{\epsilon - (m + 1/2)\Delta \omega_c + \Sigma^{(i)}(\epsilon) - i\delta}. \]  (3.63)

When the iterations converge (i.e., when \( \Sigma(\epsilon) = \Sigma_m \int d\epsilon' D(\epsilon - \epsilon')G_m(\epsilon') \)) we will have succeeded in summing over all of the self-energy graphs which have no crossed lines, as shown in Fig. 3.13. These graphs are known as the “rainbow” graphs.

\[ \text{Figure 3.13: Rainbow diagram contributions to the self-energy.} \]

This kind of calculation is usually difficult but in this case the analysis is simplified by the following. First, because the real part of \( \Sigma^{(1)}(\epsilon) \) is bounded on either side of the gap \( |\epsilon| < \frac{\Delta \omega_c}{2} \) by logarithmic peaks, we are guaranteed that there are solutions to (3.51) for all \( n \) within the gap. That the imaginary part of \( \Sigma^{(1)}(\epsilon) \) is zero in the gap implies that there are actually \( 2p \) \( \delta \)-function peaks in \( \sum_{m=-p}^{m=p-1} G^{(1)}_m(\epsilon) \) within the gap (exactly one for each \( m \)). The distance between the peaks resulting from \( m = 0 \) and \( m = -1 \) gives a renormalized gap, \( \Delta \omega_c^{(1)} \).

Next we use \( G^{(1)} \) in (3.62) to generate \( \Sigma^{(2)} \). We are particularly interested in the form of \( \Sigma^{(2)} \) within the gap because this is where further corrections to the renormalized gap originate. The contributions to \( \text{Im}G^{(1)} \) may be split into two parts: a coherent piece, coming from the \( \delta \)-function peaks within the gap, and an incoherent piece in the region outside the gap (we use this terminology for convenience- there may in fact be solutions to (3.51) in the latter region which correspond to poles in \( G^{(0)} \)). Upon evaluating \( \text{Re}\Sigma^{(2)} \) we notice that the coherent parts of \( \text{Im}G^{(1)} \) give rise to exactly the same logarithmic
divergences as seen in $\Sigma^{(1)}$, except that the peaks are located at the poles of $\text{Im}G^{(1)}$ and each peak is weighted by a factor

$$z_p(\epsilon_m) = \left|1 + \frac{\partial \Sigma' (\epsilon_m)}{\partial \epsilon}\right|^{-1} \tag{3.64}$$

where $\epsilon_m$ are the positions of the poles. However, in most cases the weight coming from the poles within the gap is small compared to the weight of the incoherent part of $\text{Im}G^{(1)}_m$, which means that there may be significant contributions to $\text{Re} \Sigma^{(2)}$ that do not come from the poles of $\text{Im}G^{(1)}_m$. Only the coherent parts of $\text{Im}G^{(1)}$ contribute to $\text{Im}\Sigma^{(2)}(\epsilon)$ for $|\epsilon| < \frac{\Delta \omega_c}{2}$, which vanishes within the renormalized gap, $|\epsilon| < \frac{\Delta \omega_c^{(1)}}{2}$.

The corrections arising from the coherent parts may be calculated exactly. However we are forced to examine the incoherent corrections using approximations, beginning with the use of (3.35). This approximation has the effect of smearing out the logarithmic peaks at the Landau levels. As discussed above, these features are a crucial element within the gap; however outside the gap their role is not as important. They may generate quasiparticles peaks as solutions to (3.51) but the weight of these peaks is very small. The artificial smoothing of the Landau level structures would greatly simplify a numerical integration over these parts. Instead of doing this, we make use of the fact that the weight of the incoherent parts may be determined exactly, since

$$\int_{|\epsilon| > \Delta \omega_c/2} d\epsilon \text{Im}G^{(1)}_m(\epsilon) = 1 - z_p(\epsilon_m).$$

Therefore we approximate (3.62) as

$$\Sigma^{(2)}(\epsilon) = \sum_m \int dq D(q, \epsilon - \epsilon_{max})[1 - z_p(\epsilon_m)] \tag{3.65}$$

where the peak of $\text{Im}G_m(\epsilon)$ is at $\epsilon_{max}$. This approximation is valid for $\epsilon_{max} > \Delta \omega_c/2$, which occurs when $\frac{2\Delta \omega_c}{\pi} \log(2p + 1) \approx 1$.

We have used this procedure to calculate the gap as a function of $p$, keeping only results which appear to have converged after the second step. We find corrections that are as large as 30% of the first order result. These calculations are shown in Fig. 3.14,
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

Figure 3.14: The effective mass, $m^*$ as a function of $B$ and $\nu$ for composite fermions. The lower $x-$ axis has been obtained by assuming an electron density $n_e = 2.23 \times 10^{11} \text{cm}^{-2}$. The crosses are numerical calculations which include corrections coming from $\Sigma^{(2)}$ (the self-consistent self-energy). The curves are a guide for the eye.

The effective mass is plotted as $\frac{m^*}{m} = \frac{\Delta \omega_c}{\Delta \omega_c^*}$ for various couplings. We estimate the actual coupling, in the experiment of Du et al. [30] to be $K_2 \approx 0.8$ (see Table 3.1). We emphasize that the results for small values of $p$ are not meaningful because of the approximation (3.24), which assumes that $p$ is large, thus there is no overlap with experimental results. Therefore, we draw no conclusions about the exact relation between $m^*$ and $p$. However, there is a range of $p$ where $p$ may be considered to be large and $\log(2p + 1)$ is not. In this range we expect our results to be valid but to have not yet reached the asymptotic limit.
of $m^*/m = K_2 \log(2p + 1)$. This calculation shows the effective mass to be far more sensitive to the coupling than to $p$, thus it is difficult to determine what relation these results have to experiments without knowing the coupling exactly.

### 3.7 The Effective Mass: Comparison to Experiment

In recent years a small number of groups have attempted to measure the effective mass of composite fermions indirectly using the SdH oscillations of the longitudinal resistivity as a function of the magnetic field $\Delta B$. This is achieved by fitting the extrema of the oscillations to the Dingle formula which is parametrised in terms of an effective mass:

$$\frac{\Delta \rho_{xx}}{\rho_0} = 4 \exp\left(\frac{-\pi}{\Delta \omega_c^* \tau}\right) \frac{\xi}{\sinh \xi} \cos[\pi(2p - 1)]$$

$$(3.66)$$

$$\xi = \frac{2\pi^2 k_B T}{\hbar \Delta \omega_c^*}.$$  

$$(3.67)$$

The only other unknown parameter is the scattering time $\tau = \frac{1}{2\Gamma}$, where $\Gamma$ is the broadening of the CF Landau levels. $\Gamma$ is the difference between the true gap and the gap determined from thermal activation measurements; the true gap is inferred from the SdH measurements for small values of $p$. There have been four such measurements, with corresponding self-energy coefficients as listed in Table 3.1.

<table>
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<th>author</th>
<th>$\varepsilon$</th>
<th>$n_e \times 10^{11}$ cm$^{-2}$</th>
<th>$m_b (m_e)$</th>
<th>$K_2$</th>
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<td>13.1</td>
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<td>0.63-4.5</td>
<td>0.068</td>
<td>0.45-1.20</td>
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<td>Coleridge et al. [32]</td>
<td>13</td>
<td>1.27, 1.39</td>
<td>0.067</td>
<td>0.65, 0.68</td>
</tr>
</tbody>
</table>

Table 3.1: Four experiments which measure the SdH oscillations in the fractional quantum Hall regime and their corresponding self-energy coefficients. $K_2$ is determined from the given experimental values for $\varepsilon$, $n_e$ and $m_b$ using Eq. (3.27).
Figure 3.15: Four experiments which measure the effective mass from SdH measurements. 

- a) Du et al. [30], b) Coleridge et al [32], c) Leadley et al. [57] d) Manoharan et al. [31].

Note that in all cases the vertical axis is in units of $m_e$, whereas in Fig. 3.14 it is drawn in units of $m_b$.

The semi-classical approximation we have used ($p \gg 1$) prevents us from making a direct, quantitative comparison with any of the experimental results, which are all in the range $p < 8$. However, all of the experiments exhibit definite trends so that we may extrapolate their results to larger values of $p$.

The information shown in Fig. 3.14 has been calculated using the fillings shown on the upper axis for the different couplings shown on the plots. This information may be interpreted in two ways. First, we may assume that the density, $n_e$, is fixed and that the
different couplings arise by variations of the other parameters in (3.27). The values of $B$ shown in the lower $x$-axis have been determined using $n_e = 2.2 \times 10^{11} \text{cm}^{-2}$. This allows us to compare to the experimental data of Du et al. [30] - their data is consistent with a coupling of $K_2 \approx 1$, which is slightly larger than the coupling $K_2 = 0.83$ calculated using (3.27). This corresponds to a theoretical prediction for the effective mass that is nearly ten times smaller than the experimental observations. Coleridge et al. [32] also find an effective mass that is $\approx 10$ times larger than our prediction. Our calculations do
not agree with the results of Leadley et al. either, who do not observe a mass divergence [57], but their data do not go beyond $\nu = 3/7$, which is not within the range of our approximations. (Obviously, we cannot compare the results of Du et al. at these values either.)

The disagreement with the results of Du et al. [30] may be attributable to corrections coming from diagrams not included in the sum over rainbow graphs, or more generally, from the non-perturbative nature of this calculation, thanks to the large coupling $K_2$. In general the coupling constant will be of order unity, but the experimental scenario investigated by Manoharan et al. [31] is a notable exception. In their experiment the carriers are holes with a rather large band mass $m_h = 0.38m_e$. Thus the coupling coefficient is reduced to $K_2 = 0.13$, which is within the perturbative regime for interactions with gauge fluctuations. This is clearly evident from our calculations, and in Fig. 3.14 we see that for $K_2 \leq 0.25$ there is essentially no mass renormalisation up to $p = 50$. The results of Manoharan et al. are in stark disagreement with this. Their results suggest a mass divergence as $\nu \rightarrow 1/2$ and in the range of their observations the mass is up...
to five times larger than the theoretically determined one. While our results for small values of $p$ may be objectionable because of the semiclassical approximations we have used, we do expect that our results are correct for small coupling and $p$ of order 50. A divergence as $\nu \to \frac{1}{2}$ is still predicted, but the divergence would not become apparent until $\frac{2K^2}{\pi} \log(2p + 1) \approx 1$, or $p \approx 10^5$ ($\nu = \frac{10^5}{2 \times 10^7 + 1}$). It seems extremely unlikely that the measurements of Manoharan et al. would reverse their divergent tendency to meet with our predictions at $p = 50$ or the asymptotic result, therefore we are forced to conclude that gauge fluctuations are not entirely responsible for the mass divergences observed in the SdH measurements. In support of this, we note that preliminary investigations indicate that density fluctuations strongly affect the measured values of $m^*$.\footnote{P. T. Coleridge, private communication.}

Alternatively, we may assume that each curve in Fig. 3.14 is associated with a different density, $n_e$. Using $\varepsilon = 13$ and $m = 0.07m_e$ in (3.27) we replot the results as shown in Fig. 3.16. The curves show that the effective mass increases with electron density, which agrees with the observations by Leadley et al., but there is no way to make a quantitative comparison for the range of data we have calculated, due to the fact that our approximations are not valid for the range of $\nu$ studied in the experiment.

### 3.8 Summary

In this Chapter we have computed the second order self-energy of composite fermions interacting with gauge fluctuations while retaining the full oscillatory form coming from Landau level quantisation. We find that Landau level quantisation introduces divergent structures in the composite fermion self-energy. The better understood two dimensional electron-phonon problem is analysed in the same manner, in order to compare with the composite fermion self-energy, and is found to have a weaker but still singular structure.
Chapter 3. Composite Fermions, Gauge Fluctuations and the Gap

The inclusion of Landau level quantisation in the self-energy of composite fermions yields a significant departure from the results of calculations which ignore this structure in the regime of experimentally relevant parameters. The results are then improved using a new iterative procedure. This procedure is used to compute the effective mass for a range of filling fractions and physically realistic parameters. These results are compared with experiments, three of which do show an enhancement of the effective mass as $\nu \to 1/2$. However, of these experiments, all display a much larger enhancement than our theoretical predictions and in one case an enhancement is observed that is not predicted.
Chapter 4

Oscillatory Quantities

In system of fermions, all experimental quantities are observed to oscillate as a function of applied magnetic field. The best known quantities include the resistivity (SdH oscillations), magnetisation (dHvA effect) and the compressibility. This Chapter is concerned with the latter two quantities, which are both derived from the thermodynamic potential.

In Section 4.1 we examine the chemical potential, average energy and thermodynamic potential by considering free composite fermions with a field dependent renormalised mass. The various thermodynamic quantities oscillate as a function of \(\Delta B\). This model assumes that the only effect of interactions is to renormalise the effective mass, which greatly simplifies the analysis of these quantities. Under this assumption, we find qualitatively similar behaviour to experiments; this enables us to extract effective masses from the experimental results and cross-compare them.

The effects of gauge fluctuations are discussed in Section 4.2 in order to provide justification of the approach of Section 4.1. We argue that the discontinuity in the chemical potential of composite fermions, which occurs when the topmost composite fermion Landau level is completely filled, is the same as the renormalised gap. The simple model of Section 4.1 is justified by a finite temperature calculation of the compressibility when the filling of CF Landau levels is integral. We also examine the chemical potential when the top CF Landau level is half-filled; this is further confirmation of the simple model of Section 4.1.
In Section 4.3 we examine the thermodynamic potential $\Omega$ for two dimensional interacting systems. We seek a non-perturbative form for $\Omega$ since the composite fermion gauge theory is non-perturbative. We examine a construction given by Luttinger [39] which expands $\Omega$ in powers of $\frac{\omega}{\mu}$ and find that it is not applicable in two dimensions, for any kind of interaction. This means that standard techniques for evaluating oscillation amplitudes in three dimensions are not valid in two dimensions. However, we do find a non-perturbative expression for $\Omega$ in two dimensions which is valid when all "crossed" diagrams are neglected. This form is used to evaluate the magnetisation of composite fermions interacting with gauge fluctuations.

4.1 Non-Interacting Approximation

In this Section we analyse various thermodynamic quantities of free fermions in a magnetic field. The corresponding quantities for composite fermions are found by a simple substitution of $m^*$ for $m$ and $\Delta B$ for $B$. In addition, some care is required to properly account for energy offsets - this is important in the calculation of the average energy. The results of this Section are compared with experiments that measure the compressibility and average energy.

4.1.1 The Chemical Potential

The chemical potential is closely related to the problem of the gap, since in the non-interacting system at $T = 0$ the gap is the same as the discontinuity in the chemical potential of composite fermions. As the density is varied through the series $\nu = \frac{p}{2p+1}$, the chemical potential jump of electrons is $2p + 1$ times higher than that of CF's. To see this, consider what happens when we add one electron to a system where there are exactly $p$ filled CF Landau levels ($p$ an integer). Before adding the electron there are
Adding the electron is the same as adding one CF and subtracting two flux quanta, so that there are \( Np + 1 \) CF’s and \( N - 2 \) flux quanta. Dividing yields a remainder of \( 2p + 1 \), which is the number of CF’s that have been promoted to the next CF Landau level. This costs an energy \( (2p+1)\Delta \omega_c^* \). Similarly, when the magnetic field is varied instead of the density, the chemical potential jump of electrons is \( p\Delta \omega_c^* \).

To begin, we first consider the chemical potential at \( T = 0 \) for electrons in a magnetic field. The density of states is simply a set of degenerate Landau levels, \( g^{(0)}(\epsilon) = \sum_n \delta(\epsilon - (n + \frac{1}{2})\omega_c) \), of which the topmost one may be only partially filled. As a function of the filling fraction (or density, when the magnetic field is held fixed), the chemical potential is simply a set of steps, beginning at \( \frac{\mu_c}{2} \) and increasing in jumps of \( \omega_c \) each time the filling fraction passes through an integer value (i.e., each time a Landau level is completely filled):

\[
\mu = \left(\lfloor p + 1/2 \rfloor\right)\omega_c. \quad (4.1)
\]

The \( T \neq 0 \) corrections are derived under the assumption that the chemical potential yields the correct number of particles in the integral over the density of states:

\[
p = \int_{-\infty}^{\infty} d\epsilon g^{(0)}(\epsilon)n_f(\epsilon - \mu) \quad (4.2)
= \sum_{n=0}^{\infty} n_f(\lfloor n + 1/2 \rfloor\omega_c - \mu) \quad (4.3)
\]

\( p \) is the filling fraction, and may be a non-integer number. Writing \( p \equiv \lfloor p + \delta p \rfloor \) and \( \mu = \left(\lfloor p + 1/2 \rfloor\omega_c + \delta \mu \right) \) and shifting the sum over \( n \) to start at \( -\lfloor p \rfloor \), we find

\[
p + \delta p = \sum_{n=-\lfloor p \rfloor}^{\infty} n_f(n - \delta \mu). \quad (4.4)
\]

Finally, we make the assumption that \( T \ll \omega_c \). Within this approximation, we take the value of the Fermi function to be either exactly one or exactly zero for all values of \( n \),
except for \( n = 0 \). This yields

\[
[p + \delta p = [p + n_f(-\delta \mu)]
\]

which has the solution

\[
\delta \mu = T \log \left( \frac{\delta p}{1 - \delta p} \right). \tag{4.6}
\]

This result gives rise to undesirable singularities as \( \delta p \to 1, 0 \), which happens to be the region of greatest interest. To avoid this, we keep two terms in the sum (4.4) instead of only one. The second term is chosen depending on which limit of \( \delta p \) we are nearest to.

For \( \delta p \to 0 \) we keep \( n_f(-1 - \delta \mu) \) and for \( \delta p \to 1 \) we keep \( n_f(1 - \delta \mu) \), which yields \(^1\)

\[
\delta \mu = -T \log \left( \frac{-[1 + f(T)]\delta p + \sqrt{[1 + f(T)]^2 \delta p^2 + 4f(T)(1 + \delta p)(1 - \delta p)}}{2(1 + \delta p)} \right)
\]

for \( \delta p \to 0 \)

\[
= -T \log \left( \frac{-[f(T) + 1](\delta p - 1) + \sqrt{[f(T) + 1]^2(\delta p - 1)^2 - 4\delta pf(T)\delta (\delta p - 2)}}{2f(T)\delta p} \right)
\]

for \( \delta p \to 1 \)

\[
f(T) = \exp \left( \frac{\omega_c}{T} \right). \tag{4.9}
\]

We patch these solutions together, switching between (4.7) and (4.8) at \( \delta p = 0.5 \). This gives rise to a discontinuity at \( \delta p = 0.5 \) which becomes large at high temperatures. One could solve (4.4) with more terms included, but this does not remedy the problem for high temperatures and the expression becomes unwieldy even when only three terms are included. Using the notion of composite fermions, the extension to the FQHE is straightforward. All the arguments given above also apply here, except that \( \omega_c \) is replaced by \( \Delta \omega_c \) everywhere. The important differences arise from the dependence of \( \Delta B \) on the density of electrons. When we fix the external magnetic field and vary the density, \( \Delta B \) also varies according to Eq. (3.3).

\(^{1}\)Eq. (4.7) was derived in [92].
Fig. 4.1 shows the chemical potential of \textit{composite fermions} as a function of filling fraction $\nu$. The plot reveals oscillatory patterns for various temperatures. In this plot we have subtracted a contribution $p\Delta\omega_c$, so that what remains is just the oscillatory component. Below $\nu = 1/3$ we have made an abrupt change to composite fermions that are electrons with four flux quanta attached. Both types of CF’s lead to exactly the same discontinuity at $\nu = 1/3$ at the mean field level (and so do the bosonic particles that have three flux quanta). It is evident from the Figure that at the higher temperatures Eq. (4.9) begins to break down, especially for the
smaller gaps. The corresponding plot of electrons has a chemical potential jump that is $2p + 1$ times larger than that of composite fermions at $T = 0$.

The effective mass of $m^* = 20m_b$, corresponding to a gap $\Delta \omega^*_c = 3.9K$ at $\nu = \frac{1}{3}$, used in these plots was chosen to achieve a good match to the experimental results of Eisenstein, Pfeiffer and West [55] (shown in Fig. 4.2). We have used this value of $m^*$ for the entire curve, although it should actually depend on $\nu$. Experimental results such as those obtained by Du et al. [30] show only a small variation in $m^*$ for the first few incompressible states after $\nu = \frac{1}{3}$. It is not possible to obtain $m^*$ from our calculation in Chapter 3 of the gap at $\nu = \frac{1}{3}$ because the approximations used are not valid for the low denominator fractions. In principle it is possible to obtain this value from other experiments, such as measurements of the SdH oscillations performed by Leadley et al. [57], where they observe $m^*$ at fixed $\nu$ as a function of $n_e$. However, they find $m^* \approx 11m_b$ at $\nu = \frac{1}{3}$, which is considerably smaller. On the other hand, the measurements of Du et al. [30] found an effective mass $m^* \approx 20m_b$ at $\nu = \frac{1}{3}$ which is in rough agreement with the compressibility measurement. However, their experiment was performed with a

![Figure 4.2: Experimental results of Eisenstein et al. [55] showing a) the compressibility of a 2-D electron gas and b) the chemical potential, which is the integral of the compressibility. Note that the horizontal axis is the same on both plots.](image-url)
density \( n_e = 2.25 \times 10^{11}\text{cm}^{-2} \), which is roughly twice as large as the density at \( \nu = \frac{1}{3} \) for the compressibility measurement \( (n_e = 1.1 \times 10^{11}\text{cm}^{-2}) \). Our calculations in Chapter 3 (see Figure 3.15) show that the effective mass does depend on the density (this sensitivity was also observed by Leadley et al.), therefore a direct comparison between the effective masses obtained from the compressibility measurement and the measurement of Du et al. may not be valid.

For the discussions of the oscillatory quantities to come, it will be helpful to understand the behaviour of the chemical potential for free composite fermions as a function of the external magnetic field. For fixed \( n_e \) and an unrenormalised effective mass, it is:

\[
\mu = (|p + 1/2|) \Delta \omega_c = (|p + 1/2|) \frac{n_e \Phi_0}{p m}.
\]  

(4.10)

This quantity oscillates about \( \mu = \frac{n_e \Phi_0}{m} \), which is precisely the value of the chemical potential at \( \nu = \frac{1}{2} \) in the non-interacting case. Interactions change the effective mass and as before we can guess that there will be a corresponding shift in the chemical potential, so that it will reach the same limit as \( \nu \to \frac{1}{2} \) as the non-interacting case. Thus we find that

\[
\mu = (|p + 1/2|) \Delta \omega_c^* + \frac{\omega_c^*}{2} - \frac{\omega_c^*}{2}.
\]  

(4.11)

where \( \omega_c^* = \frac{e B}{m} \).

### 4.1.2 The Compressibility Measurement

Intrinsic magnetoluminescence experiments measure the strength of the luminescence lines resulting from the recombination of electrons in the 2-D gas with holes [65]. Unlike the situation for extrinsic photoluminescence (see Section 4.3.1) the holes are located in the vicinity of the 2-D gas. The strength of the line varies with the compressibility of the 2-D gas. The line is strongest when the electrons are in an incompressible FQH state.
since then the electrons are completely unable to screen the hole and the recombination rate is a maximum.

The relation between screening and incompressibility is the basis of a quantitative measurement by Eisenstein et al. [55]. In this case a voltage is applied to a gate above a double layer Hall system. The first layer partially screens the gate voltage from the second layer. Current detected at the second layer is indicative of the screening ability of the first, and hence its compressibility may be determined.

We have derived the results shown in Fig. 4.3 using \( \frac{1}{\kappa} \equiv \frac{\partial \mu}{\partial n_e} \), which is the derivative of Fig. 4.1. The y-axis has been scaled by the interaction strength, \( \frac{e^2 I}{\epsilon} \) which is 368 K, for \( B = 13T \) and \( \epsilon = 12.8 \); these are the values used by Eisenstein et al.

A significant difference between our theoretical results using composite fermions and the experimental results is a negative offset in the compressibility that appears in the experiments. This is a higher energy effect which originates from the strong electron-electron interactions at low densities [93]. In contrast, the composite fermion theory is a low energy theory and as such, gives rise to the fine scale structure associated with the FQHE but information concerning higher energy scales is lost in the treatment that we have used, which has a cut-off at the true cyclotron frequency \( \omega_c \).

### 4.1.3 The Average Energy

Using the result that the composite fermions seem to behave as free particles with a renormalised mass, we may proceed to calculate the average energy in the same way as the chemical potential. The \( T = 0 \) result for \( \rho \) filled CF Landau levels with an unrenormalised mass is

\[
E_{av}(\rho) = \left( \sum_{n=0}^{(\rho-1)} (n + 1/2)\Delta\omega_c + \delta\rho(p + 1/2)\Delta\omega_c \right) \frac{n_e}{\rho} \tag{4.12}
\]

The factor \( \frac{n_e}{\rho} \) is the degeneracy of each Landau level.
Figure 4.3: The compressibility as a function of filling fraction calculated for $m^* = 20m_b$ and the y-axis has been scaled by the interaction strength, $\frac{\omega_1}{e}$ (see text).

At $T \neq 0$ some of the CF's will be thermally activated to the next Landau level. In this case the average energy is

$$E_{av}(\nu, T) = \sum_{n=0}^{\infty} n_f (n + 1/2 - \mu) \Delta \omega_c \frac{n_c}{p}. \quad (4.13)$$

In the formulae given above there are implicit assumptions about the splitting of the lowest Landau level into sub-levels. In particular, for an unrenormalised effective mass it is consistent to assume that the energy of the lowest sublevel is $\frac{\Delta \omega_c}{2}$. This picture breaks down when the effective mass increases. As has been discussed at length in previous
sections, an increase of the effective mass is associated with a decrease in the gap. The chemical potential compensates for this by smoothly increasing while the topmost CF Landau level is being filled, resulting in the formula (4.11) when \( n_e \) is held fixed. Likewise, the average energy receives an upward shift which reflects the fact that when there is a renormalised effective mass the CF Landau levels are bunched up closely near \( \frac{\omega_c}{2} \) and the energy of the lowest CF Landau level is somewhat larger than \( \frac{\Delta\omega_c}{2} \). This is illustrated in Fig. 4.4. There is also an offset coming from the \( g \)-factor of the electrons, which shifts

\[ \omega_c \]

\[ \omega \neq 2 \]

\[ \Delta\omega_c \]

\[ \Delta\omega^* \]

Figure 4.4: A schematic illustration of the splitting of the lowest Landau level. When there are no interactions the lowest Landau is not split. The mean field theory of composite fermions splits the lowest Landau level into sublevels separated by \( \Delta\omega_c \). Gauge fluctuations renormalise the mass, so that in the end the lowest Landau is subdivided into levels separated by \( \Delta\omega^* \), but still centered around \( \frac{\omega_c}{2} \).

the lowest electronic Landau level down by \( \frac{\omega_c}{2} \). Taking all of this into account, we plot the average energy as a function of \( B \) in Fig. 4.5. The offsets used guarantee that the crossover between the integer and fractional regimes is smooth. The crossover occurs at \( \nu = 1 \), which is considered to be part of both the integer series and the series \( \nu = \frac{p}{2p-1} \).

This figure has been drawn in order to compare with the results of Kukushkin et al. [54] shown in Figure 4.6. An electron density \( n_e = 2.4 \times 10^{11} \text{ cm}^{-2} \) and \( g = 0.42 \) [94] have been used to scale the \( y \)-axis. In the fractional regime, an effective mass \( m^* = 20m_b \) has been used to achieve a reasonable agreement with the experimental results. This is in close agreement with the compressibility measurement at \( \nu = \frac{1}{3} \), although it should be noted that the compressibility measurement had about half the
Figure 4.5: The average energy as a function of magnetic field at $T = 0$. The scaling of the $y$-axis has been chosen to match the experimental parameters of Kukushkin et al. (see text).

electron density ($n_e = 1.1 \times 10^{11}\text{cm}^{-2}$). It also agrees well with the SdH measurements of Du et al., which were performed at a similar density ($n_e = 2.25 \times 10^{11}\text{cm}^{-2}$).

4.1.4 Measuring the Average Energy

In a series of papers [52, 53, 54] Kukushkin et al. have reported measurements of magnetoluminescence spectra as a function of magnetic field. In these experiments luminescence
Chapter 4. Oscillatory Quantities

Figure 4.6: The experimental results of Kukushkin et al. [54] showing b) The average energy as a function of magnetic field at $T = 0$ and c) its derivative.

Lines are the result of the recombination of electrons with holes bound to remote acceptors. The acceptors are in a monolayer with density $\sim 5 \times 10^9$ cm$^{-2}$ approximately 40 nm away from the 2-D electron gas. The large distances ensures that the hole wave-functions do not overlap the electron gas which means that the electrons will not be able to screen the holes.

The position of the luminescence line is the integral over energy of the intensity of the line. This is related to twice the average energy of the 2-D electron gas [95]. As expected, our results display the same qualitative behaviour as the experimental results. In the fractional regime we see that the average energy increases linearly with the energy of the lowest Landau level ($\frac{\alpha_e}{2}(1-g)$) with small perturbations arising from the fractional states. One important feature that has been omitted from these plots is the weighting of the Landau levels. From our discussions in Sections 3.4 and 3.6 we know that a large fraction of the weight of the spectral function lies beneath the the $\delta$-function peaks that are the renormalised Landau levels. This would account for an overall decrease of the slope of the average energy in the fractional regime. Note that the experimental results display a slope that is a little less than two times larger. An accurate measurement of
the slope would help to determine the distribution of the weight in the density of states.

Fig. 4.7 shows the derivative of the average energy of composite fermions (Fig. 4.5)

![Graph showing the derivative of the average energy with respect to the magnetic field as a function of magnetic field at T = 0. The discontinuities are related to the gap as explained in the text.](image)

**Figure 4.7:** The derivative of the average energy with respect to the magnetic field as a function of magnetic field at $T = 0$. The discontinuities are related to the gap as explained in the text.

with respect to the magnetic field. The discontinuities are related to the gap $\Delta$ in both fractional and integer regimes as

$$
\delta \left( \frac{dE_{CF}}{dB} \right) = \frac{2\Delta}{B\nu}.
$$

(4.14)

Note that in the fractional regime $\nu = \frac{p}{2p \pm 1}$. Considering real electrons instead of composite fermions, the discontinuity is a factor of $p$ larger, in agreement with the formula
Chapter 4. Oscillatory Quantities

derived by Apal’kov and Rashba [95],

\[ \delta \left( \frac{dE_{el}}{dB} \right) = \frac{2\Delta}{Bq} \]  \hspace{1cm} (4.15)

where the filling is \( \nu = \frac{2}{q} \). Taking this into account, an effective mass of \( m^* = 20m_b \) was chosen to match the relative size of the discontinuity at \( \nu = \frac{2}{3} \) and \( \nu = 1 \).

4.1.5 Thermodynamic Potential and its Derivatives

The thermodynamic potential is yet another quantity that can be evaluated using the idea of non-interacting composite fermions with renormalised effective mass. The expression for this quantity at finite temperature is \( (k_B = 1) \)

\[ \Omega^{(0)} = -T \sum_{n=0}^{\infty} \log \left[ 1 + \exp \left( \frac{-(n + 1/2)\Delta\omega^*_c + \mu}{T} \right) \right] \frac{n_c}{p} \]  \hspace{1cm} (4.16)

where \( \frac{n_c}{p} = \frac{\Delta B}{\Phi_0} \) is the Landau level degeneracy. From this we may derive the temperature dependent magnetisation,

\[ M^{(0)} = -\frac{\partial \Omega}{\partial B} \]  \hspace{1cm} (4.17)

\[ \begin{align*}
M^{(0)} &= \frac{-n_c}{pm^*} \sum_{n=0}^{\infty} \left[ n + 1/2 \right] n_f \left[ n + 1/2 \right] \Delta\omega^*_c - \mu \\
&\quad + \frac{T}{\Phi_0} \sum_{n=0}^{\infty} \log \left[ 1 + \exp \left( \frac{-(n + 1/2)\Delta\omega^*_c + \mu}{T} \right) \right].
\end{align*} \]  \hspace{1cm} (4.18)

Note that the first term dominates in the low temperature limit. The compressibility is

\[ \kappa^{(0)} = -\frac{\partial^2 \Omega}{\partial \mu^2} \]  \hspace{1cm} (4.19)

\[ \kappa^{(0)} = \frac{m^* \Delta\omega^*_c}{2\pi T} \sum_{n} n_f \left( -\left( n + 1/2 \right) \Delta\omega^*_c + \mu \right) n_f \left( n + 1/2 \right) \Delta\omega^*_c - \mu. \]  \hspace{1cm} (4.20)

Later in this Chapter we shall argue that Eq. (4.16) does in fact yield the leading order contributions to the oscillatory behaviour of the thermodynamic potential, and consequently the related thermodynamic quantities.
Chapter 4. Oscillatory Quantities

In contrast to the approximation using free particles given above, the full thermodynamic potential is derived from the Hamiltonian which includes interactions with gauge fluctuations,

\[ \Omega - \Omega^{(0)} = -T \log \left( \frac{Z}{Z^{(0)}} \right) \]  
\[ Z = \text{Tr} \exp \left( \frac{-K}{T} \right) \equiv \text{Tr} \exp \left( \frac{-(H - \mu N)}{T} \right). \]  

The trace is taken over all configurations and numbers of particles. In the non-interacting case this reduces to [96]

\[ Z^{(0)} = \prod_i \left[ 1 + \exp \left( \frac{-\epsilon_i + \mu}{T} \right) \right]. \]  

The thermodynamic potential is thus

\[ \Omega^{(0)} = -T \sum_i \log \left[ 1 + \exp \left( \frac{-\epsilon_i + \mu}{T} \right) \right] \]
\[ \equiv -T \text{Tr} \log \mathcal{G}^{(0)}(\epsilon_i, \xi_i) \exp(\xi_i 0^+). \]  

(4.21) follows from this formula for a system of free particles in a magnetic field \( \Delta B \) with an effective mass \( m^* \). \( \Omega^{(0)} \) in the form (4.25) will be useful later on. The proof of (4.25) is given in Appendix C.

4.2 Gauge Interaction Effects

In this Section we provide some justification for the approach of Section 4.1 by evaluating the effects of gauge fluctuations to second order in perturbation theory. We consider the special cases when the filling of CF Landau levels is integral and half integral. The results show that at these fillings the dominant effect of gauge fluctuations is to renormalise the effective mass. The chemical potential is examined in Section 4.2.1 followed by the compressibility in Section 4.2.2.
4.2.1 Chemical Potential

We now consider the effects of gauge fluctuations on the chemical potential. \( \mu \) satisfies the same criterion, \( (4.2) \), but in this case the density of states is no longer simply a set of \( \delta \)-functions; instead it has the form

\[
g(\epsilon) = \sum_n \frac{\Sigma''(\epsilon)}{[\epsilon - (n + \frac{1}{2}) \Delta \omega_c + \Sigma'(\epsilon)]^2 + \Sigma''(\epsilon)^2},
\]

and in general must be evaluated at finite temperature. \( \Sigma(\epsilon) \) is the self-energy calculated in Chapter 3. We begin by computing the discontinuities in \( \mu \) at \( T = 0 \). The goal is to determine what range of \( \mu \) will satisfy \( (4.2) \) for \( \mu = \int p \),

\[
|\mu = \int_{-\infty}^\mu d\mu(\epsilon).
\]

This is obviously true for all \( \mu \) that lie within the gap. When \( \mu \) lies outside the gap both the limit of the integral and the form of the density of states change. These two effects enhance each other. For example, a lower value of \( \mu \) will cause the weight of the spectrum to be shifted upwards and the lower cutoff ensures that even less weight will be integrated. Thus at \( T = 0 \) the discontinuities in \( \mu \) will be the same as the gap. In between discontinuities we expect \( \mu \) to increase smoothly and monotonically\(^2\).

For \( T \neq 0 \) there are corrections coming from the considerations described in Section 4.1.1 and also from the temperature dependence of the spectral function. In our calculations we neglect the latter, so that the final result merely substitutes \( \Delta \omega_c^\prime \) for \( \Delta \omega_c \), with a smooth increase of \( \mu \) between the incompressible states.

We have just shown that the \( T = 0 \) jump in the chemical potential corresponds to the gap as determined in Chapter 3. We now consider the effect of gauge fluctuations at finite temperature when the topmost CF Landau level is approximately half filled. In

\(^2\text{This picture was described in Ref. [29].}\)
this case the chemical potential must satisfy
\[ p \equiv |p + \delta p| = \int_{-\infty}^{\infty} \text{den}_f(e) g(e + \mu) \]
\[ = \sum_{n=-|p|}^{\infty} \int_{-\infty}^{\infty} \text{den}_f(e) \frac{\Sigma''(\epsilon)}{[\epsilon + \delta \mu - n\Delta \omega_c + \Sigma'(\epsilon)]^2 + \Sigma''(\epsilon)^2} \] (4.29)

where \( \delta \mu \) and \( \delta p \) have the same meaning as before. The self-energy looks slightly different than the form considered when solving for the gap (compare Fig. 4.8 to Fig. 3.4). When

Figure 4.8: The self-energy when the highest composite fermion Landau level is half-filled at \( T = 0 \) (solid) and for a finite temperature with a small, positive shift of the chemical potential (dashed). \( \epsilon \) is measured with respect to the chemical potential. The vertical scale is arbitrary (it depends on the coupling \( K_2 \)).

the topmost Landau level is half filled we expect there to be particle-hole symmetry just as in the case when the topmost Landau level is completely filled and the chemical potential lies halfway between the filled and unfilled levels. Therefore, at half filling the sum that appears in the expression for the self-energy must be truncated at \( |p| \). Particle-hole symmetry will not be conserved once the system moves away from half filling; this is true when the cut-off at \( |p| \) is used. For a given value of \( n \) there will be one pole
of $\text{Im}G_n(\epsilon)$ near $\epsilon = 0$ which will carry some fraction of the weight; the rest of the weight is divided between the upper and lower parts of the spectral function. When the spectral functions of $n$ and $-n$ are considered together the remaining "incoherent" weight is divided approximately equally between the upper and lower parts of the sum of the two spectral functions, since particle-hole symmetry is approximately conserved. The spectral function for $n = 0$ will have a pole close to $\epsilon = 0$ and we assume that its incoherent weight will be distributed symmetrically between positive and negative energies.

For the purposes of this Section, we consider the limit of large $\log p$, where we know that an approximation that smears the Landau level structure of the self-energy is valid for finding the renormalised gap (based on the discussion in Section 3.4). In this limit, the renormalised pole positions are found by linearising the self-energy in the vicinity of $\epsilon = 0$; therefore the new poles are evenly spaced (with spacing that is $\Delta \omega_c^*$ by definition) and they all have the same weight.

As before we consider the limit $T \ll \Delta \omega_c^*$. In this limit we may assume that the Fermi function is either one or zero except at the Landau level nearest to the Fermi energy. Now all that remains is to compute the positions of the pole nearest to the Fermi energy and its weight. To do this we solve

$$\epsilon + \delta \mu - n\Delta \omega_c + \Sigma'(\epsilon) = 0$$

(4.30)

as before, except that the self-energy must be evaluated for a more general chemical potential. This is easily determined from the results derived in Chapter 3:

$$\Sigma'(\epsilon) = \frac{K_2}{\pi} \sum_{n=-[\rho]}^{\rho} \sum_{n=-[\rho]}^{\rho} \left[ -n_f(n - \delta \mu) \log(\epsilon + \delta \mu - n) + n_f(\delta \mu - n) \log(n - \epsilon - \delta \mu) \right].$$

(4.31)

This is shown in Fig. 4.8.
Linearising Eq. (4.31) about $\epsilon + \delta \mu$ gives

$$
\Sigma'(\epsilon) = \frac{K_2}{\pi} \sum_{m=-|p|}^{|p|} \left( \frac{n_f(m - \delta \mu) - n_f(\delta \mu - m)}{m} \right) (\epsilon + \delta \mu). \quad (4.32)
$$

Clearly the pole $\epsilon_p$ occurs at $\epsilon_p = -\delta \mu$ with weight

$$
z_p = \left( 1 + \frac{K_2}{\pi} \sum_{m=-|p|}^{|p|} \left( \frac{n_f(m - \delta \mu) - n_f(\delta \mu - m)}{m} \right) \right)^{-1}. \quad (4.33)
$$

The spacing between the poles is given by

$$
\Delta \omega_c^* = \left[ 1 + \frac{K_2}{\pi} \sum_{m=-|p|}^{|p|} \left( \frac{n_f(m - \delta \mu) - n_f(\delta \mu - m)}{m} \right) \right] \Delta \omega_c. \quad (4.34)
$$

With all of these considerations Eq. (4.29) becomes

$$
|p + \delta p = |p + \frac{1 - w}{2} + z_p n_f(\epsilon_p) \quad (4.35)
$$

$$
\approx |p + \frac{1 - w}{2} + z_p \left( \frac{1}{2} - \frac{\epsilon_p}{4T} \right) \quad (4.36)
$$

Using $\epsilon_p = -\delta \mu$ we find that

$$
\delta \mu = \frac{4T(\delta p - 1/2)}{z_p}. \quad (4.37)
$$

This is equivalent to the expression (4.6) (evaluated for $\delta p \approx 1/2$), except for an overall scaling by a factor of $z_p$. This is exactly what is expected, since the spacing between the poles, which is equivalent to the renormalised gap, is $z_p \Delta \omega_c \equiv \Delta \omega_c^*$.

### 4.2.2 The Second Order Corrections to $\Omega$

In Section 4.1.5 we wrote an expression for $\Omega$ under the assumption that the only effect of interactions was to renormalise the effective mass. We now wish to determine the effect of interactions on the thermodynamic potential in a more rigorous fashion. It is a well known result from quantum field theory that the diagrammatic contributions to Eq. (4.21) consist of all closed, connected diagrams [97]. In this Section we examine the
Figure 4.9: Feynman diagram representing the second order correction to the thermodynamic potential. The straight lines represent composite fermions and the wavy line represents a gauge fluctuation.

Second order corrections to the thermodynamic potential arising from interactions with gauge fluctuations, represented by the diagram shown in Fig. 4.9. We will rederive some of the results of Kim et al. [90] in a slightly different way, in order to demonstrate that the behaviour of the temperature dependent compressibility at a filled CF Landau level is consistent with the picture of free composite fermions with an effective mass.

The second order correction to the thermodynamic potential is given in perturbation theory as

\[
\Omega^{(2)} \sim T \sum_{\omega_n} G^{(0)}(i\omega_n) \Sigma^{(2)}(-i\omega_n) 
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \text{den}_f(\epsilon) \left[ G_R(\epsilon) \Sigma_A(\epsilon) - G_A(\epsilon) \Sigma_R(\epsilon) \right] 
\]

\[
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \text{den}_f(\epsilon) \left[ G_R^\ast(\epsilon) \Sigma_R^\ast(\epsilon) - G_R^\ast(\epsilon) \Sigma_R(\epsilon) \right] 
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \text{den}_f(\epsilon) \left[ G_R''(\epsilon) \Sigma_R'(\epsilon) - G_R'(\epsilon) \Sigma_R''(\epsilon) \right] 
\]

\[
= \Omega_a^{(2)} + \Omega_b^{(2)} 
\]

where

\[
G_R(\epsilon) = \frac{1}{\epsilon + \mu - (n + 1/2 + |p|)\Delta\omega_c - i\delta} 
\]

and \(\Sigma_R(\epsilon)\) is defined by Eqs. (3.25)-(3.27). Using \(\delta\mu = \mu - (|p| + 1/2)\Delta\omega_c\) as before,

\[
\Omega_a^{(2)} = \frac{n_e}{p\pi} \sum \int \text{den}_f(\epsilon) G_R''(\epsilon) \Sigma_R'(\epsilon) 
\]
\[ \Delta \omega_c^2 = \frac{n_e}{p} \sum_n n_f(n - \delta\mu)\Sigma_R(n - \delta\mu) \] (4.45)

\[ = \frac{m\Delta \omega_c^2 K_2}{2\pi^2} \sum_{n,m} n_f(n - \delta\mu) \log|m - n|[n_f(\delta\mu - m) - n_f(m - \delta\mu)]. \] (4.46)

Next we evaluate the derivative to find the density of particles, which is proportional to \( p \) when \( B \) is held fixed.

\[ p_n^{(2)} = -\frac{\partial \Omega_n^{(2)}}{\partial \mu} \] (4.47)

\[ = \frac{-mK_2\Delta \omega_c^2}{2\pi^2T} \sum_{m,n} \log|m - n|[2n_f(\delta\mu - m)n_f(m - \delta\mu)n_f(n - \delta\mu)

\[ - n_f(\delta\mu - m)n_f(n - \delta\mu)n_f(\delta\mu - n)

\[ + n_f(m - \delta\mu)n_f(n - \delta\mu)n_f(\delta\mu - n)]. \] (4.48)

The compressibility is

\[ \kappa_n^{(2)} = \frac{\partial p_n^{(2)}}{\partial \mu} \] (4.49)

\[ = \frac{mK_2\Delta \omega_c^2}{2\pi^2T^2} \sum_{m,n} \log|m - n|

\[ \times [2n_f(\delta\mu - m)n_f^2(m - \delta\mu)n_f(n - \delta\mu)

\[ - 4n_f(\delta\mu - m)n_f(m - \delta\mu)n_f(n - \delta\mu)n_f(\delta\mu - n)

\[ - n_f(\delta\mu - m)n_f^2(n - \delta\mu)n_f(\delta\mu - n)

\[ + n_f(\delta\mu - m)n_f(n - \delta\mu)n_f^2(\delta\mu - n)

\[ + n_f(m - \delta\mu)n_f^2(n - \delta\mu)n_f(\delta\mu - n)

\[ - n_f(m - \delta\mu)n_f(n - \delta\mu)n_f^2(\delta\mu - n)

\[ - 2n_f^2(\delta\mu - m)n_f(m - \delta\mu)n_f(n - \delta\mu)]. \] (4.50)

Taking the usual limit \( T \ll \Delta \omega_c \), we evaluate this expression at \( \delta\mu = \frac{\Delta \omega_c}{2} \), keeping only
Chapter 4. Oscillatory Quantities

the dominant terms (all but the second), which go as \( \exp(-\frac{\Delta \omega_c}{2T})/T^2 \).

\[
\kappa^{(2)}_a = \frac{2mK_2\Delta \omega_c^2}{\pi^2T^2} \exp \left( -\frac{\Delta \omega_c}{2T} \right) \sum_{m=0}^{N-1} \log \left| \frac{m+1}{m} \right| \tag{4.51}
\]

Next we evaluate the second term in (4.42).

\[
\Omega^{(2)}_b = -\frac{1}{\pi} \sum_n \int \text{den}_f(\epsilon)G_R'(\epsilon)\Sigma_R''(\epsilon) \tag{4.53}
\]

\[
= -\frac{2K_2\Delta \omega_c^2}{\pi^2T^2} \exp \left( -\frac{\Delta \omega_c}{2T} \right) \log p \tag{4.52}
\]

Shifting \( \epsilon \) by \( \delta \mu \) we find

\[
\Omega^{(2)}_b = -\frac{n_eK_2\Delta \omega_c}{p\pi} \sum_{m,n} \int d\epsilon \frac{n_f(\epsilon - \delta \mu)}{\epsilon - n} [n_f(\delta \mu - m)\theta(\epsilon + \delta \mu - m) + n_f(m - \delta \mu)\theta(m - \epsilon - \delta \mu)]. \tag{4.55}
\]

The density of particles is

\[
p^{(2)}_b = -\frac{\partial \Omega^{(2)}_b}{\partial \mu} \tag{4.56}
\]

\[
= mK_2\Delta \omega_c^2 \sum_{m,n} \int d\epsilon \left[ n_f(\mu - \delta \mu)n_f(\delta \mu - m)n_f(m - \delta \mu)[\theta(\epsilon - m) - \theta(m - \epsilon)]
\right.

\[
+ n_f(\epsilon - \delta \mu)n_f(\delta \mu - \epsilon)[n_f(\delta \mu - m)\theta(m - \epsilon) + n_f(m - \delta \mu)\theta(m - \epsilon)] \right]. \tag{4.57}
\]

The compressibility is

\[
\kappa^{(2)}_b = \frac{\partial p^{(2)}_b}{\partial \mu} \tag{4.56}
\]

\[
= mK_2\Delta \omega_c^2 \sum_{m,n} \int d\epsilon \left[ -2n_f(\epsilon - \delta \mu)n_f(\delta \mu - \epsilon)n_f(\delta \mu - m)n_f(m - \delta \mu)[\theta(\epsilon - m) - \theta(m - \epsilon)]
\right.

\[
+ n_f(\epsilon - \delta \mu)n_f^2(\delta \mu - \epsilon) - n_f^2(\epsilon - \delta \mu)n_f(\delta \mu - \epsilon)]\right]. \tag{4.57}
\]
\[ x[n_f(\delta\mu - m)\theta(\epsilon - m) + n_f(m - \delta\mu)\theta(m - \epsilon)] + [n_f(\epsilon - \delta\mu)[n_f^2(m - \delta\mu)n_f(\delta\mu - m) - n_f(m - \delta\mu)n_f^2(\delta\mu - m)] x[\theta(\epsilon - m) - \theta(m - \epsilon)]]]. \tag{4.58} \]

Once again, taking the limit \( T \ll \Delta\omega_c \) and evaluating at \( \delta\mu = \frac{\Delta\omega_c}{2} \),

\[ k_b^{(2)} = \frac{mK_2\Delta\omega_c^2}{2\pi^2T^2} \sum_{m=-1}^{p-1} \int \frac{d\epsilon}{\epsilon - n} \left[ -2 \exp \left( \frac{-\Delta\omega_c}{2T} \right) n_f(\epsilon - 1/2)n_f(1/2 - \epsilon)\theta(\epsilon) - \theta(-\epsilon) + \theta(\epsilon - 1) - \theta(1 - \epsilon) \right] \]

\[ + \exp \left( \frac{-\Delta\omega_c}{2T} \right) (n_f(\epsilon - 1/2)n_f^2(1/2 - \epsilon) - n_f^2(\epsilon - 1/2)n_f(1/2 - \epsilon))\theta(\epsilon - 1) + \theta(-\epsilon) \]

\[ + \exp \left( \frac{-\Delta\omega_c}{2T} \right) n_f(\epsilon - 1/2)\theta(1 - \epsilon) - \theta(-\epsilon) \]

\[ + \exp \left( \frac{-\Delta\omega_c}{2T} \right) n_f^2(\epsilon - 1/2)n_f(1/2 - \epsilon)\theta(\epsilon - 1) - \theta(-\epsilon) \]. \tag{4.59} \]

There are four terms in this expression. The first and second terms will gain additional exponentially small factors upon integration. The fourth term survives the integration, yielding \( \sim \log \frac{n-1/2}{n} \) which vanishes upon summation in the limit of large \( p \). Only the the third term (which has no exponentially small factor) remains. The largest contributions to the integral come from \( \epsilon = 0 \) and \( \epsilon = 1 \), so the leading order contribution to \( k_b^{(2)} \) is

\[ k_b^{(2)} = \frac{mK_2\Delta\omega_c^2}{2\pi^2T^2} \exp \left( \frac{-\Delta\omega_c}{2T} \right) \sum_{n=-1}^{p-1} \log \left| \frac{n}{1 - n} \right| \]

\[ = \frac{mK_2\Delta\omega_c^2}{2\pi^2T^2} \exp \left( \frac{-\Delta\omega_c}{2T} \right) \sum_{n=0}^{p-1} \log \frac{n + 1}{n - 1} \]

\[ \approx \frac{K_2\Delta\omega_c^2}{\pi^2T^2} \exp \left( \frac{-\Delta\omega_c}{2T} \right) \log p. \]

The total compressibility to second order is

\[ \kappa^{(2)} = \kappa_a^{(2)} + k_b^{(2)} \]

\[ = \frac{-mK_2\Delta\omega_c^2}{\pi^2T^2} \exp \left( \frac{-\Delta\omega_c}{2T} \right) \log p. \]

\[ \tag{4.64} \]
Chapter 4. Oscillatory Quantities

This is added to the free particle result given by Eq. (4.20), except that here we use $\Delta \omega_c$ instead of $\Delta \omega^*$. The low temperature limit of this expression evaluated for a chemical potential half way between Landau levels is

$$\kappa^{(0)} = \frac{m \Delta \omega_c}{\pi T} \exp \left( -\frac{\Delta \omega_c}{2T} \right). \quad (4.65)$$

Eq. (4.64) gives the corrections to this, so the compressibility to second order is

$$\kappa = \frac{m \Delta \omega_c}{\pi T} \exp \left( -\frac{\Delta \omega_c}{2T} \right) \left( 1 - \frac{K_2 \Delta \omega_c}{\pi T} \log p \right) \quad (4.66)$$

$$\approx \frac{m \Delta \omega_c}{\pi T} \exp \left( -\frac{\Delta \omega_c}{2T} \left[ 1 + \frac{2K_2}{\pi} \log p \right] \right). \quad (4.67)$$

Kim et al. [90] argued that this result is consistent with particles having an effective mass

$$\frac{m^*}{m} = \left| 1 + \frac{2K_2}{\pi} \log p \right|^{-1} \quad (4.68)$$

$$\approx \left| 1 - \frac{2K_2}{\pi} \log p \right|. \quad (4.69)$$

in the limit of large $p$.

4.3 The Oscillatory Behaviour of $\Omega$

In this Section we examine the oscillatory behaviour of the thermodynamic potential. We seek a non-perturbative form, such as that discovered by Luttinger, which may be applied to the two dimensional problem of composite fermions interacting with gauge fluctuations. It is found that Luttinger's form, which is an expansion that neglects the oscillatory part of the self-energy, is not valid in two dimensions for any kind of interaction. Instead of this, we have found a different form which includes the oscillatory part of the self-energy and is valid when all "crossed" diagrams are neglected.
In Section 4.3.1 we examine the oscillatory contributions to the second order perturbative correction to $\Omega$ followed by in Section 4.3.2 an analysis of the non-perturbative expression for $\Omega$.

### 4.3.1 The Oscillatory Behaviour of $\Omega^{(2)}$

In any diagram contributing to the thermodynamic potential the oscillatory component arises through the sum over Landau levels associated with each fermion line. In general the contribution from each fermion line may be split into oscillatory and non-oscillatory parts. The non-oscillatory part is obtained by converting the sum over Landau levels into an integral. This operation has already been discussed in Section 3.4, where it was employed to give an approximation of the self-energy by essentially smoothing out the singular structure associated with the Landau levels. The remaining part may be expressed as a Fourier series by using the Poisson sum formula,

$$
\sum_{n=0}^{\infty} f(n\omega_c) = \int_{0}^{\infty} dx f(x) \left[ 1 + 2 \sum_{k=1}^{\infty} (-1)^k \cos \left( \frac{2\pi kx}{\omega_c} \right) \right]. \tag{4.70}
$$

By examining the second order contribution to $\Omega$, Engelsberg and Simpson [40] showed that the leading oscillatory behaviour of $\Omega^{(2)}$ could be determined by neglecting the oscillatory part of one of the fermion lines. However their results hold in three dimensions only; we show below that their proof does not apply in two dimensions. In three dimensions there is an extra degree of freedom, the momentum in the $z$-direction, which tends to smear the density of states. In two dimensions the magnetic field quenches all of the degrees of freedom and the density of states has a very singular structure, which in a general treatment may not be “smeared out”.

In three dimensions, the second order correction to $\Omega$ is

$$
\Omega^{(2)}_{3D} = \frac{T^2 \lambda^2}{2} \sum_{\omega_{n_1}, \omega_{n_2}} \sum_{m, n, k, k', q} G_n(i\omega_{n_1}, k_x)G_m(i\omega_{n_2}, k'_x)D(i\omega_{n_2} - i\omega_{n_1}, q) \tag{4.71}
$$
where $D(i\omega_n)$ is any general interaction. We change variables by defining $n\omega_c = \frac{p^2}{2m}$, 
$\epsilon = \frac{p_1^2 + p_1^2}{2m} - \mu$ and $\cos^2 \theta = \frac{p_0^2}{p_1^2 + p_1^2}$ \footnote{This is described in detail in \[40\].}. Using (4.70) we get

$$
\Omega^{(2)}_{3D} = \Omega^{(2)}_{0,3D} + \Omega^{(2)}_{1,3D} + \Omega^{(2)}_{2,3D}.
$$

where

$$
\Omega^{(2)}_{0,3D} = \frac{T^2 \lambda^2}{2} \sum_{\omega_{n1},\omega_{n2}} \sum_{\bar{q}} \int_{-\mu}^{\mu} d\epsilon d\epsilon' \int_{-1}^{1} d(\cos \theta) d(\cos \theta')(\epsilon + \mu)^{1/2}(\epsilon' + \mu)^{1/2} 
\times G^{(0)}(i\omega_{n1}, \epsilon)G^{(0)}(i\omega_{n2}, \epsilon')D(i\omega_{n2} - i\omega_{n1}, \bar{q})
$$

$$
\Omega^{(2)}_{1,3D} = 2T^2 \lambda^2 \sum_{\omega_{n1},\omega_{n2}} \sum_{\bar{q}} \int_{-\mu}^{\mu} d\epsilon d\epsilon' \int_{-1}^{1} d(\cos \theta) d(\cos \theta')(\epsilon + \mu)^{1/2}(\epsilon' + \mu)^{1/2} \sum_{k=1}^{\infty} (-1)^k
\times \cos \left( \frac{2\pi k}{\omega_c} (\epsilon + \mu)(1 - \cos^2 \theta) \right) \cos \left( \frac{2\pi k'}{\omega_c} (\epsilon + \mu)(1 - \cos^2 \theta') \right)
\times G^{(0)}(i\omega_{n1}, \epsilon)G^{(0)}(i\omega_{n2}, \epsilon')D(i\omega_{n2} - i\omega_{n1}, \bar{q})
$$

$$
\Omega^{(2)}_{2,3D} = 2T^2 \lambda^2 \sum_{\omega_{n1},\omega_{n2}} \sum_{\bar{q}} \int_{-\mu}^{\mu} d\epsilon d\epsilon' \int_{-1}^{1} d(\cos \theta) d(\cos \theta')(\epsilon + \mu)^{1/2}(\epsilon' + \mu)^{1/2} \sum_{k,k'}^{\infty} (-1)^{k+k'}
\times \cos \left( \frac{2\pi k}{\omega_c} (\epsilon + \mu)(1 - \cos^2 \theta) \right) \cos \left( \frac{2\pi k'}{\omega_c} (\epsilon + \mu)(1 - \cos^2 \theta') \right)
\times G^{(0)}(i\omega_{n1}, \epsilon)G^{(0)}(i\omega_{n2}, \epsilon')D(i\omega_{n2} - i\omega_{n1}, \bar{q})
$$

This is shown diagrammatically in Fig. 4.10.

\[\begin{align*}
\Omega &= \begin{array}{c}
\text{Fig. 4.10: The zeroth, first and second order oscillatory contributions to } \Omega^{(2)}. \text{ The wiggly cuts indicate that the oscillatory component of the fermion line is to be evaluated.}
\end{array}
\end{align*}\]
Chapter 4. Oscillatory Quantities

on one of the fermion lines. The third term takes the oscillatory contribution from both lines. Engelsberg and Simpson argued that the \( \cos \theta \) integrals could be extended to \( \pm \infty \) (the stationary phase approximation) since the largest contributions to the \( \epsilon \)-integral come from the vicinity \( \epsilon \sim 0 \) (near the Fermi surface). Then integrating over \( \cos \theta \) yields a factor \( \left( \frac{\omega_0}{\mu} \right)^{1/2} \) in the second term and \( \left( \frac{\omega_0}{\mu} \right) \) in the third, leading to the conclusion that the third term may be neglected because it is of order \( \left( \frac{\omega_0}{\mu} \right)^{1/2} \) with respect to the second.

In the 2-D case the \( \cos \theta \) integrals are absent and so is the Jacobian factor \( (\epsilon + \mu)^{1/2}(\epsilon' + \mu)^{1/2} \):

\[
\Omega_{2D}^{(2)} = \Omega_{0,2D}^{(2)} + \Omega_{1,2D}^{(2)} + \Omega_{2,2D}^{(2)}
\]

where

\[
\Omega_{0,2D}^{(2)} = \frac{T^2 \lambda^2}{2} \sum_{\omega_{n_1}, \omega_{n_2}, q_1, q_2} \int_{-\mu}^{\infty} d\epsilon d\epsilon' G^{(0)}(i\omega_{n_1}, \epsilon)G^{(0)}(i\omega_{n_2}, \epsilon')D(i\omega_{n_1} - i\omega_{n_2}, q)
\]

\[
\Omega_{1,2D}^{(2)} = 2T^2 \lambda^2 \sum_{\omega_{n_1}, \omega_{n_2}, q_1, q_2} \int_{-\mu}^{\infty} d\epsilon d\epsilon' \sum_{k=1}^{\infty} (-1)^k \cos \left( \frac{2\pi k(\epsilon + \mu)}{\omega_c} \right)
\times G^{(0)}(i\omega_{n_1}, \epsilon)G^{(0)}(i\omega_{n_2}, \epsilon')D(i\omega_{n_1} - i\omega_{n_2}, q)
\]

\[
\Omega_{2,2D}^{(2)} = 2T^2 \lambda^2 \sum_{\omega_{n_1}, \omega_{n_2}, q_1, q_2} \int_{-\mu}^{\infty} d\epsilon d\epsilon' \sum_{k, k'=1}^{\infty} (-1)^{k+k'} \cos \left( \frac{2\pi k(\epsilon + \mu)}{\omega_c} \right) \cos \left( \frac{2\pi k'(\epsilon' + \mu)}{\omega_c} \right)
\times G^{(0)}(i\omega_{n_1}, \epsilon)G^{(0)}(i\omega_{n_2}, \epsilon')D(i\omega_{n_1} - i\omega_{n_2}, q)
\]

Therefore it is obvious that the arguments of Engelsberg and Simpson are not applicable in this case. The second and third terms of the oscillatory expansion are evaluated in Appendix E for the 2-D composite fermion system where it is shown that the third term is the same order in \( \frac{\omega_0}{\mu} \) as the second.

Luttinger derived a much more general, non-perturbative form for \( \Omega \) by showing that the oscillatory part of the self-energy is a factor \( \left( \frac{\omega_0}{\mu} \right)^{3/2} \) smaller than the non-oscillatory part [39]. However this proof also depends crucially on the presence of a third dimension, as we discuss in the following Section.
4.3.2 Non-Perturbative evaluation of $\Omega$

The essentially non-perturbative nature of the composite fermion gauge theory necessitates a different approach to computing $\Omega$. It is useful to rewrite $\Omega$ in terms of the full composite fermion Green’s function and the proper self-energy, which satisfy the self-consistency relation

$$G^{-1} = G^{(0)}^{-1} + \Sigma.$$  \hfill (4.80)

Note that the proper self-energy has contributions from any closed connected graph that cannot be cut in two when a single fermion line is cut.

Following Luttinger and Ward [38], we write the thermodynamic potential as

$$\Omega = \tilde{\Omega} = \Omega_0 - T \sum_i \text{Tr}[\log[1 + G^{(0)}(\epsilon_i)\Sigma(\epsilon_i)] - \Sigma(\epsilon_i)G(\epsilon_i)] + \Omega' \quad \text{(4.81)}$$

$$= -T \text{Tr}[\log G^{-1}(\epsilon_i) - \Sigma(\epsilon_i)G(\epsilon_i)] + \Omega' \quad \text{(4.82)}$$

where the second line is derived from the first line using (4.25) and (4.80). $\Omega'$ consists of all closed connected skeleton diagrams (diagrams with no self-energy insertions) with the bare Green’s function $G^{(0)}$ replaced by the full one $G$. The proof that $\tilde{\Omega} = \Omega$ is complicated and the forthcoming arguments depend on this result, therefore we have provided the details in Appendix C.

It is also shown in Appendix C that $\Omega$ is stationary with respect to $\Sigma$,

$$\frac{d\Omega}{d\Sigma} = 0.$$ \hfill (4.83)

Luttinger used this to expand $\Omega$ in powers of the oscillatory part of the self-energy $\Sigma_{osc}$,

$$\Omega = -T \text{Tr}[\log G_0^{-1}(\epsilon_i) - \Sigma_0(\epsilon_i)G_0(\epsilon_i)] + \Omega'(\Sigma_0) + O(\Sigma_{osc}^2) \quad \text{(4.84)}$$

where

$$\Sigma = \Sigma_0 + \Sigma_{osc} \quad \text{(4.85)}$$
and
\[ G_0^{-1}(\epsilon) \equiv \epsilon - (n + 1/2)\omega_c + \Sigma_0 \]  

(4.86)

It was shown by Luttinger [39] that there is a cancellation of the second and third terms in (4.84). This is reviewed in Appendix C. In 3-D Lifshitz and Kosevich [98] showed (ignoring dynamic interactions) that

\[ T\text{Tr}[\log G^{-1}] \sim \left( \frac{\omega_c}{\mu} \right)^{5/2} \]  

(4.87)

and Luttinger [39] showed that

\[ \Sigma_{osc} \sim \left( \frac{\omega_c}{\mu} \right)^{3/2} \Sigma_0. \]  

(4.88)

Therefore to \( O \left( \left( \frac{\omega_c}{\mu} \right)^{5/2} \right) \) the last term in (4.84) may be neglected and the leading oscillatory behaviour of \( \Omega \) is

\[ \Omega_{osc} = -T \sum_i \text{Tr}[\log G_0^{-1}(\epsilon_i)] \]  

(4.89)

\[ = -T \sum_{i,\alpha} \log[\epsilon_i - (n + 1/2)\omega_c + \mu + \Sigma_0(\epsilon_i)] \]  

(4.90)

This is the standard expression used to evaluate the magnetisation [91, 99]. However, in 2-D we find that

\[ T\text{Tr}[\log G^{-1}] \sim \left( \frac{\omega_c}{\mu} \right)^2 \]  

(4.91)

and

\[ \Sigma_{osc} \sim \left( \frac{\omega_c}{\mu} \right) \Sigma_0. \]  

(4.92)

This is shown in Appendix F. This is very similar to the results of Section 4.4.3, since in both cases the factor \( \left( \frac{\omega_c}{\mu} \right)^{1/2} \) is lost by the omission of the stationary phase integral coming from the third direction. Therefore, in 2-D Eq. (4.84) is no longer a good expansion for \( \Omega \) since the fourth term is of the same order as the first, so we must
reconsider Eq. (4.82). In general, the last two terms of Eq. (4.82), \( \Sigma G + \Omega' \), cancel only up to second order in the coupling constant \( (O(\lambda^2)) \). However, if the self-energy satisfying

\[
\Sigma = \lambda^2 \int GD
\]

(4.93)

(i.e., it is the self-consistent sum over rainbow graphs) is used, then \( \Sigma G = \lambda^2 GDG \) cancels with the term in \( \Omega' \) originating from the \( O(\lambda^2) \) skeleton graph (see Fig. 4.9). This term is simply \( \lambda^2 G^{(0)} DG^{(0)} \), with \( G^{(0)} \) replaced by \( G \). Then the only parts remaining in \( \Omega' \) originate from higher order skeleton graphs, which in general have “crossed” interaction lines, such as the one shown in Fig. C.1b). Such contributions are important for vertex renormalisation, but we have always ignored them. Therefore, it is consistent with our calculations of Chapter 3 to try as a first approach the following expression for \( \Omega \):

\[
\Omega = -T \sum_{i,n} \log[\epsilon_i - (n + 1/2)\omega_c + \mu + \Sigma(\epsilon_i)]^{\eta e}_p
\]

(4.94)

where the self-consistent self-energy is used.

Including the crossed graphs leads to terms that are of \( O(\lambda^4) \) or greater. These contributions will in general not be classified in powers of \( \frac{\omega_c}{\mu} \), instead their order is determined by the scale of the interactions. For example, in the case of Debye phonons the expansion parameter is \( \frac{\omega_c}{\theta_D} \) [99], while for spin fluctuations the fluctuation energy determines the relevant scale [91, 100].

Eq. (4.94) is used to evaluate the magnetisation by following most of the steps in the derivation given in [91] and [99]. We begin by rewriting (4.94) as a contour integral:

\[
\Omega = \frac{m \Delta \omega_c}{\Phi_0} \sum_{n=0} \int \frac{dx}{\pi} n_f(x) \tan^{-1} \left( \frac{\Sigma''(x)}{x - (n + 1/2)\Delta \omega_c + \Sigma'(x)} \right)
\]

(4.95)

Next we use the Poisson sum formula (4.70) and change variables by defining \( \epsilon = n \Delta \omega_c - \mu \),

\[
\Omega = \frac{m}{\Phi_0} \int_{-\mu}^{\infty} d\epsilon \int \frac{dx}{\pi} n_f(x) \tan^{-1} \left( \frac{\Sigma''(x)}{x - \epsilon + \Sigma'(x)} \right)
\]
Extending the limits on \( \epsilon \) to \( \pm \infty \) we integrate by parts and neglect the end-point contributions to get

\[
\times \left[ 1 + 2 \sum_{k=1}^{\infty} (-1)^k \cos \left( \frac{2\pi k(\epsilon + \mu)}{\Delta \omega_c} \right) \right].
\]  

(4.96)

This expression is the same as what would be found for the non-interacting case, except that the full spectral function replaces the free one. In Section 3.4 we discussed the form of the spectral function for the case when the self-energy is determined to second order and in Section 3.6 we discussed the case when the self-energy is determined self-consistently. In both cases we showed that in the vicinity of the Fermi surface when the system is gapped the density of states consists of a series of weighted \( \delta \)-functions which are the renormalised Landau levels. The oscillatory behaviour originates from the parts of the integral close to the Fermi surface; in essence this is the reason why the description using free particles with a field-dependent renormalised mass works.

Performing the \( \epsilon \) integral in Eq. (4.97) yields

\[
\Omega_{osc} = \frac{-m \Delta \omega_c}{\pi \Phi_0} \int_{-\infty}^{\infty} d\epsilon \int \frac{dx}{n_f(x)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin \left( \frac{2\pi k(\epsilon + \mu)}{\Delta \omega_c} (x + \Sigma(x) + \mu) \right).
\]  

(4.97)

This expression has slight differences compared to the one derived by Engelsberg and Simpson because it has been done for two dimensions instead of three. The most important difference is that the full oscillatory self-energy appears in the spectrum \( x + \Sigma(x) \).

The magnetisation is

\[
M_{osc} = \frac{\partial \Omega_{osc}}{\partial B}
\]

(4.99)

\[
= \frac{1}{\Phi_0} \int dx \frac{n_f(x)}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin \left( \frac{2\pi k(x + \Sigma(x) + \mu)}{\Delta \omega_c} \right).
\]
Chapter 4. Oscillatory Quantities

\[ (-1)^k 2\pi (x + \Sigma(x) + \mu) \cos \left( \frac{2\pi k}{\Delta \omega_c} (x + \Sigma(x) + \mu) \right) \]

\[ -(-1)^k 2\pi \frac{m}{e} \frac{\partial \Sigma(x)}{\partial B} \cos \left( \frac{2\pi k}{\Delta \omega_c} (x + \Sigma(x) + \mu) \right) \]. \quad (4.100)

The second term dominates the first by a factor \( \frac{\mu}{\Delta \omega_c} \). It is the leading oscillatory contribution to the magnetisation derived by Engelsberg and Simpson. In the 2-D case the third term is of the same order as the second. To show this, we consider the real part of the self-energy for composite fermions interacting with gauge fluctuations, but the result is general.

\[ \Sigma'(x) = \frac{K_2 \Delta \omega_c}{\pi} \sum_{n=0} \log \left| \frac{(n + 1/2) \Delta \omega_c - \mu - x}{\Delta \omega_c} \right| 
\quad \left[ n_f(\mu - (n + 1/2) \Delta \omega_c) - n_f((n + 1/2) \Delta \omega_c - \mu) \right] \quad (4.101) \]

\[ = \frac{K_2}{\pi} \int d\epsilon \left[ 1 + 2 \sum_{k'=1} \cos \left( \frac{2\pi k'(\epsilon + \mu)}{\Delta \omega_c} \right) \right] \]
\[ \times \log \left| \frac{\epsilon - x}{\Delta \omega_c} \right| [n_f(-\epsilon) - n_f(\epsilon)] \quad (4.102) \]

\[ \frac{\partial \Sigma'(x)}{\partial B} = \frac{K_2}{\pi} \sum_{k'=1} \int d\epsilon \frac{-2\pi k'(\epsilon + \mu)}{m(\Delta \omega_c)^2} \sin \left( \frac{2\pi k'(\epsilon + \mu)}{\Delta \omega_c} \right) \]
\[ \times \log \left| \frac{\epsilon - x}{\Delta \omega_c} \right| [n_f(-\epsilon) - n_f(\epsilon)]. \quad (4.103) \]

Putting this result in Eq. (4.100) we see that the term with \( \frac{\mu}{\Delta \omega_c} \) is the same order as the second term in Eq. (4.100). If there had been a third dimension then the third term would have been suppressed by a factor \( \left( \frac{\Delta \omega_c}{\mu} \right)^{1/2} \). This result shows that the conventional treatment of interaction effects on dHvA oscillations must be modified when applied in two dimensions and terms involving the oscillatory self-energy should be retained in any perturbative calculation.
4.4 Summary

In this Chapter we have studied the thermodynamic quantities of composite fermions interacting with gauge fluctuations which oscillate as a function of $\Delta B$. The chemical potential and related quantities, namely the compressibility and average energy, have been analysed using non-interacting composite fermions with a field dependent effective mass and compared to experiment. It was found that this approximation provides a reasonable description of the experiments, and that the effective masses obtained by analysing the compressibility, free energy and SdH measurements are in rough agreement. Explicit second order corrections coming from gauge fluctuations have also been calculated for the compressibility and chemical potential and were found to agree with the approximation using non-interacting particles with an effective mass.

We have also examined the oscillatory behaviour of the thermodynamic potential for interacting systems in two dimensions. Considering both perturbative and non-perturbative forms for $\Omega$, we have shown explicitly that an approximation which neglects the oscillatory part of the self-energy is valid in three dimensions but not in two, which means that the conventional Luttinger-Ward treatment of quantum oscillations is not applicable in two dimensions. Instead, we have proposed an alternative, non-perturbative form for $\Omega$ which neglects crossed graphs. This form contains the self-consistent, oscillatory self-energy discussed in Chapter 3.
Chapter 5

Conclusion

This thesis studies two models of the fractional quantum Hall effect, namely the bosonic (Chern-Simons-Landau-Ginzburg) and the fermionic (composite fermion gauge theory) descriptions. The bosonic theory attempts to describe the states at $\nu = \frac{1}{2n+1}$ using an order parameter and a $|\psi|^4$ point interaction. Using this theory we have found numerical fractionally charged vortex and anti-vortex profiles for $\nu = \frac{1}{3}, \frac{1}{5}$ and $\frac{1}{7}$. The size and energy of each vortex may be extracted from these results. We describe the role that these parameters play in the breakdown of the uniform Hall state within this theory. Our results are compared with results obtained from other numerical methods.

In contrast to the bosonic theory, the fermionic theory strives to describe the fractional quantum Hall states at all filling fractions as well as the states in between. The mean field theory, which describes the states at $\nu = \frac{1}{2n}$ as Fermi liquids in zero magnetic field, is readily extended to an interacting gauge theory. Lowest order perturbative calculations of the self-energy arising from interactions with the gauge field show that the renormalised fermion Green's function has a distinctly different analytic structure compared to its mean field (Fermi liquid) form, which is characterised by a diverging effective mass.

Near $\nu = \frac{1}{2n}$ the interacting fermions exist in a field $\Delta B = B - B_{1/2}$ and undergo oscillations as a function of this field. We compute the self-energy of composite fermions while retaining Landau level quantisation of all internal fermions lines. The gap may be extracted from this quantity; in lowest order perturbation theory we find agreement with previous calculations of the self-energy [29] (which have ignored this structure) in

111
the limit $\nu \to \frac{1}{2}$ where one finds a divergence of the effective cyclotron mass. However our results using Landau level quantisation give rise to different predictions for the gap for filling factors away from $\nu = 1/2$. We also clarify the structure of the self-energy and Green’s function by calculating the analogous quantities for the electron-phonon system in two dimensions.

We then go beyond perturbation theory by means of a new iterative procedure which computes the self-energy self-consistently (summing over all rainbow graphs). We calculate the effective mass resulting from this procedure for a range of physically relevant parameters in the limit of large $p$ ($\nu = \frac{p}{2p+1}$). In this limit our results cannot be directly compared to existing experimental data. However, three experiments do observe an enhancement of the effective mass as $\nu \to 1/2$ but the enhancement is much larger than predicted and in one case an enhancement is observed which is not predicted.

Within the fermionic theory, we have also examined the oscillatory behaviour of the thermodynamic potential and related quantities. Our first observation is that a theory of free composite fermions with an effective mass (arising from interactions) gives a good description of the experimentally observed compressibility and free energy. Leading order corrections to the unperturbed theory indicate that the principal effect of interactions is to renormalise the effective mass. This is a reflection of the fact that the renormalised Green’s function maintains the same low energy structure as the unrenormalised one.

Finally we demonstrate the consistency of our entire approach for treating quantum oscillations in two dimensional interacting systems, by examining the gauge invariant thermodynamic potential $\Omega$. We find that well-known arguments for three dimensional systems, which indicate that the leading oscillatory behaviour comes from the sum over Landau levels of a single fermion line in a Feynman diagram contributing to the thermodynamic potential do not apply in two dimensions. This implies that in second order perturbation theory it is necessary to keep explicit sums over Landau levels in both
Chapter 5. Conclusion

fermion lines when computing $\Omega^{(2)}$. This is consistent with our calculation of the second order self-energy of composite fermions.

We then go on to examine the general structure to all orders in perturbation theory of the oscillatory thermodynamic potential. Our results show that the standard non-perturbative expansion of the thermodynamic potential used to evaluate dHvA amplitudes is not valid in two dimensions either. However, we have found an alternative form which is valid when crossed diagrams are neglected. This form contains the same oscillatory self-consistent self-energy, discussed above, thereby justifying our previous work on the self-consistent composite fermion gap. This form is valid for the discussion of any two dimensional system in which oscillations in the crossed graphs do not contribute in leading order to the quantum oscillations. We apply this result to a calculation of the magnetisation, where it is shown explicitly that the conventional treatment of this quantity (which neglects the oscillatory part of the self-energy) is not valid in two dimensions.
Symbols

\( a(x) \) statistical gauge field (due to attached flux)

\( A(x) \) external gauge field

\( b \) statistical magnetic field (due to attached flux)

\( B \) external magnetic field

\( \Delta \) gap

\( \Delta B \) effective magnetic field (\( \Delta B = B - b \))

\( \Delta \omega_c \) cyclotron frequency (\( \Delta \omega_c = \frac{e\Delta B}{m_e} \))

\( \Delta \omega_c^* \) effective cyclotron frequency (\( \Delta \omega_c^* = \frac{e\Delta B}{m^*_e} \))

\( \varepsilon \) dielectric constant (\( \varepsilon = 13 \) for GaAs)

\( g \) spin splitting factor (\( g \approx 0.42 \) for GaAs)

\( \gamma \) (\( \gamma = \frac{2n_e}{k_F} \))

\( K_2, K_s, K_D, K_E \) coupling constants (defined by Eqs. (3.27, 3.30, 3.48, 3.54))

\( l \) magnetic length (\( l = \sqrt{\frac{\hbar \varepsilon}{eB}} \))

\( m_b \) band mass (\( m_b = 0.067m_e \) for GaAs)

\( m^* \) effective (renormalised) mass

\( n_e \) electron density (\( n_e \approx 10^{11} \) cm\(^{-2} \) for GaAs)

\( \nu \) filling fraction (\( \nu = \frac{\Phi_0 n_e}{h} \))
Chapter 5. Conclusion

\( \rho_{xx}, \rho_{xy} \) longitudinal, transverse resistivity

\( \Phi_0 \) magnetic flux quanta (\( \Phi_0 = \frac{\hbar c}{e} = 4.13 \times 10^{-11} \text{T cm}^2 \))

\( \chi \) susceptibility (defined on page 46)

\( \omega_c \) cyclotron frequency (\( \omega_c = \frac{eB}{mc} \))

\( \omega_c^* \) effective (renormalised) cyclotron frequency (\( \omega_c^* = \frac{eB}{m^*c} \))

\( \omega_D \) Debye (acoustic) phonon frequency

\( \omega_E \) Einstein (optical) phonon frequency

\( \lfloor x \rfloor \) greatest integer less than \( x \)
Bibliography


Appendix A

Phonon Interactions in Finite Magnetic Field

A.1 Debye Phonons

In the self-energy calculations presented in the text we have assumed a model similar to the one described by Ezawa et al. [101]. In this model it is assumed that phonons can propagate throughout the semiconductor layers without reflection at the boundaries. The electrons are confined to a very thin region which we approximate to be two-dimensional. The usual forms of the electron-phonon interactions for both Debye and Einstein models may be used. For the Debye model, the electron-phonon interaction has the form

\[ H_{int} = \Xi_D \int d^3r \psi_e^\dagger(\vec{r})\psi_e(\vec{r}) \nabla \cdot \vec{u}(\vec{r}) \]  \hspace{1cm} (A.1)

where \( \Xi_D \) is the deformation potential,

\[ \vec{u}(\vec{r}) = \sum_q \hat{q} \sqrt{\frac{\hbar}{2\omega_q \rho V}} (a_q e^{i\vec{q}\cdot\vec{r}} + a_q^\dagger e^{-i\vec{q}\cdot\vec{r}}) \]  \hspace{1cm} (A.2)

is the displacement operator, with ion mass density \( \rho \), cell volume \( V = a^3 \) and dispersion \( \omega_q = c_s q \); and \( \psi_e \) is the electron wavefunction expressed in a Landau level basis,

\[ \psi_e(\vec{r}) = \sum_{n,k} c_{n,k} \left( \frac{1}{2\pi m l} \right)^{1/2} \left( \frac{m \omega_e}{\pi \hbar} \right)^{1/4} H_n(y/l - kl) \exp \left( -\frac{(y/l - kl)^2}{2} + ikx \right) \delta(z) \] \left( A.3 \right)

In these expressions \( \omega_e = eB/m \) is the cyclotron frequency, \( m \) is the electron mass and \( l = (\hbar/eB)^{1/2} \) is the magnetic length. This leads to

\[ H_{int} = \Xi_D \sum_{n,n',k,k',\vec{q}} \sqrt{\frac{\hbar}{2\omega_q \rho V}} \hat{q} (a_{\vec{q}} + a_{-\vec{q}}^\dagger) c_{n,k}^\dagger c_{n',k'} \Lambda(n,n',k,k',\vec{q}) \] \hspace{1cm} (A.4)
where \([41]\)

\[
\Lambda(n, n', k, k', \bar{q}) = \left( \frac{1}{2n+n'!n''!} \right)^{1/2} \left( \frac{m\omega_c}{\pi\hbar} \right)^{1/2} \frac{1}{l} \int d^3r H_n(y/l - kl)H_{n'}(y/l - k'l)\delta(z)
\]

\[
\times \exp \left( -\frac{(y/l - kl)^2}{2} - \frac{(y/l - k'l)^2}{2} \right) + iq_y y + i(-k + k' + q_z) x + iq_z z \right)
\]

\[\times \left( \frac{-q_z l - iq_y l}{2} \right)^{n-n'} L_{n-n'}^2 \left( \frac{-q_z l^2}{2} \right) \]

(A.5)

for \(n \leq n'\) and \(q^2 = q_x^2 + q_y^2\). The variables \(k, k'\) are of no concern as far as the properties of the electron Green's functions are concerned because the energy of the non-interacting electrons does not depend \(k\) and the final form of \(H_{int}\) does not depend on \(k, k'\) or \(q_z\). Furthermore, in what follows we assume that the phonon operator is isotropic with respect to the \(x\)-\(y\) plane. In the limit of large \(n \approx n'\) and \(ql \gg (n' - n)/n\) the matrix element \(|\Lambda|^2\) reduces to

\[
|\Lambda(n, q)|^2 = \frac{1}{\pi q} \left( \frac{eB}{2n} \right)^{1/2}
\]

(A.7)

\[
\approx \frac{1}{\pi q} \left( \frac{eB}{2p} \right)^{1/2}
\]

(A.8)

The final form of \(H_{int}\) is

\[
H_{int} = \Xi_D \sum_{n, n', q} \sqrt{\frac{h\alpha}{2c_s\rho V}} (a_q + a_{-q}^\dagger) c_{n'}^\dagger c_n \Lambda(n, n', q).
\]

(A.9)

The self-energy is derived from the second order contribution of \(H_{int}\)

\[
\Sigma_n(\epsilon) = \Xi_D^2 \left( \frac{\hbar}{2c_s\rho V} \right) \sum_{n'=0} \int \frac{q d^2q}{(2\pi)^2} \int_{\omega_D}^{\omega_D} \frac{d\omega}{\pi} |\Lambda(n', q)|^2 \text{Im} D(q, \omega)
\]

\[
\times \left( 1 + n_B(\omega) - n_f[(n + \frac{1}{2})\omega_c + i\delta] + \frac{n_B(\omega) + n_f[(n + \frac{1}{2})\omega_c + i\delta]}{\epsilon - \omega - (n + \frac{1}{2})\omega_c + i\delta} \right)
\]

(A.10)
where the factor $1/a$ comes from the $q_z$-integral. We now shift the energies by an amount $p\omega_c$ which is the chemical potential when it lies halfway between Landau levels. The index is shifted by $p$. Performing the integrals at $T = 0$ using $\text{Im}D(q, \omega) = \frac{1}{\omega^2 - \omega_c^2}$ yields the result (3.49) and (3.50). When the self-energy and the energy $\epsilon$ are expressed in units of the Fermi energy, $E_F = p\omega_c$ the dimensionless coefficient is

$$K_D = \frac{\Xi_D^2 m E_F}{4\sqrt{2}\pi c_s^3 \rho a} = \frac{3\pi \Xi_D^2 E_F m}{2\omega_D^3 \sqrt{4\pi n_e a M}}. \quad (A.11)$$

Using typical values for GaAs, i.e., $\omega_D = (6\pi)^{1/3} c_s / a = 345K$, $\Xi_D = -4.4 eV$, $m = 0.07 m_e$, $M \approx 10^5 m_e$, $a = 3.6 \text{Å}$, $n_e = 2.25 \times 10^{11} \text{cm}^{-2}$ and $E_F \approx 1.5 \times 10^{-3} \text{eV}$, one has $K_D \approx 0.23$.

Equation (A.11) shows that there is no magnetic field dependence in the coefficient. The relation between $K_D$, which is given by (3.48), and $\tilde{K}_D$ is $K_D = \tilde{K}_D / p$. The difference arises because $K_D$ is used when the self-energy and the energy $\epsilon$ are explicitly given in units of $\omega_c$, which is appropriate for solving (3.51).

### A.2 Einstein Phonons

Following the approach of Engelsberg and Schreiffer we choose a coupling that is independent of $q$

$$H_{\text{int}} = g \sum_{n,n',q} (a_q + a_{-q}^\dagger) c_n^\dagger c_{n'}^\dagger \Lambda(n, n', q). \quad (A.12)$$

This leads to a self-energy

$$\Sigma_n(\epsilon) = g^2 \sum_{n'} |\Lambda(n, n', q)|^2 a^2 \int \frac{d^2 q}{(2\pi)^2} \int_0^{\infty} \frac{d\omega}{\pi} \text{Im}D(q, \omega)$$

$$\times \left( \frac{1 + n_B(\omega) - n_f([n + \frac{1}{2}]\omega_c)}{\epsilon - \omega - (n + \frac{1}{2})\omega_c + i\delta} + \frac{n_B(\omega) + n_f([n + \frac{1}{2}]\omega_c)}{\epsilon + \omega - (n + \frac{1}{2})\omega_c + i\delta} \right) \quad (A.13)$$

where we have assumed that the normalization of the $q$-integral is $(2\pi)^2 / a^2$. Evaluating at $T = 0$ and using $\text{Im}D(q, \omega) = \frac{1}{\omega - \omega_E}$ yields Eqs. (3.55) and (3.56). With $\Sigma$ and the
energy $\epsilon$ expressed in units of $E_F$, the dimensionless coefficient is

$$K_E = \frac{g^2 (2p)^{1/2} a}{4E_F^2 l} = \frac{g^2 a (4\pi n_e)^{1/2}}{4E_F^2}.$$  \hspace{1cm} (A.14)

In this form there is no dependence of the coupling on the magnetic field.
Appendix B

Self-Energy Corrections in a Finite Magnetic Field

The original theory of strongly interacting electrons in a magnetic field takes the form

\[ \mathcal{L} = \psi^\dagger(x)(-i\partial_t + a_0)\psi(x) + \psi^\dagger(x)\frac{(i\partial_t - eA_i)^2}{2m}\psi(x) + \int d^2y \psi^\dagger(x)\psi(x)V(x-y)\psi^\dagger(y)\psi(y) \]

(B.1)

where \( B \equiv \nabla \times A \) is the external magnetic field.

The electrons are transformed into composite fermions by attaching two magnetic flux quanta to each. The constraint that this additional magnetic field, \( b \), is proportional to the electron density is

\[ b \equiv \nabla \times a = 4\pi \psi^\dagger(x)\psi(x) \]  

(B.2)

and is implemented with the use of a Chern-Simons term in \( \mathcal{L} \). In addition, the CF's experience a gauge field that is the difference between the external field \( A \) and the "statistical" field \( a \), \( A - a = \Delta A \):

\[ \mathcal{L} = \psi^\dagger(x)(-i\partial_t + a_0)\psi(x) + \psi^\dagger(x)\frac{(i\partial_t - eA_i)^2}{2m}\psi(x) + \frac{a_0 \nabla \times a}{4\pi} + \int d^2y \frac{\nabla \times a(x)V(x-y)\nabla \times a(y)}{16\pi^2} \]

(B.3)

where the third term is the Chern-Simons term and we have used the constraint (B.2) to rewrite the last term. In this expression \( \psi \) is a fermionic operator representing composite fermions. The theory is completed by allowing fluctuations of the gauge field, \( \delta a \),

\[ \mathcal{L} = \psi^\dagger(x)(-i\partial_t + a_0)\psi(x) + \psi^\dagger(x)\frac{(i\partial_t - eA_i - \delta a)^2}{2m}\psi(x) + \delta a_i D_{ij}^{-1} \delta a_j. \]

(B.4)
The last term is the effective action of the gauge fluctuations. The second term yields the free CF Hamiltonian,

\[ H_{CF} = \frac{\psi^\dagger(x) \left( i \partial_t - e \Delta A_t \right)^2 \psi(x)}{2m} \quad \text{(B.5)} \]

and the CF-gauge fluctuation interaction,

\[ H_{int} = \frac{\delta a_t(x) \psi^\dagger(x) \left( i \partial_t - e \Delta A_t \right) \psi(x)}{m} \quad \text{(B.6)} \]

in the Coulomb gauge, \( q \cdot \delta a = 0 \). The eigenfunctions of \( H_{CF} \) are the same as (A.3) except that \( \omega_c \) is replaced by \( \Delta \omega_c \). The operator \( V_t \equiv \left( i \partial_t - e \Delta A_t \right) / m \) introduces additional complications in the determination of the final form of \( H_{int} \). We are only considering interactions with the transverse component of \( \delta a = \delta a_t \), corresponding to the transverse component of \( V_t = \cos \theta V_x + \sin \theta V_y \). \( V_x \) and \( V_y \) can both be expressed in terms of creation and annihilation operators which act on the harmonic oscillator part of \( \psi \):

\[ V_x = \sqrt{\frac{\Delta \omega_c}{2m}} (a + a^\dagger) \quad \text{(B.7)} \]
\[ V_y = i \sqrt{\frac{\Delta \omega_c}{2m}} (a - a^\dagger) \quad \text{(B.8)} \]
\[ V_1 = \sqrt{\frac{\Delta \omega_c}{2m}} (ae^{i\vartheta} + a^\dagger e^{-i\vartheta}) \quad \text{(B.9)} \]

In this case the matrix element will be

\[ \Lambda_{CF}(n, n', q) = \sqrt{\frac{\Delta \omega_c}{2m}} (\sqrt{n'} \Lambda(n, n' - 1, q)e^{i\vartheta} + \sqrt{n'} + 1 \Lambda(n, n' + 1, q)e^{-i\vartheta}) \quad \text{(B.10)} \]

which, in the limit of large \( n \approx n' = p \), yields

\[ |\Lambda_{CF}(n, n', q)|^2 \approx \frac{4p \Delta \omega_c}{2m} |\Lambda(n, q)|^2. \quad \text{(B.11)} \]

The coefficient is

\[ \frac{4p \Delta \omega_c}{2m} \approx \frac{2E_F}{m}. \quad \text{(B.12)} \]
Thus we are left with

$$H_{\text{int}} = \sqrt{\frac{2E_F}{m}} \sum_n \sum_n \Lambda(n, n', q)(a_n + a_{-n})c_n c_{n'}^\dagger,$$  \hspace{1cm} (B.13)

therefore the self-energy is

$$\Sigma_n(\epsilon) = \frac{2E_F}{m} \int \frac{d^2q}{(2\pi)^2} \int_0^\infty \frac{d\omega}{\pi} \int_{-\infty}^\infty \frac{d\epsilon'}{\pi} \sum_{n'=0}^\infty |\Lambda(n, n', q)|^2 \text{Im} D(q, \omega) \text{Im} G_{n'}^0(\epsilon') \times \left( \frac{1 + n_B(\omega) - n_f(\epsilon')}{\epsilon - \omega - \epsilon' + \text{i}\delta} + \frac{n_B(\omega) + n_f(\epsilon')}{\epsilon + \omega - \epsilon' + \text{i}\delta} \right)$$  \hspace{1cm} (B.14)

Just as in the phonon case, a cut-off in the sum at $n' = 2p$ is needed. Using $\text{Im} G_{n'}^0(\epsilon) = \pi \delta(\epsilon + \mu - (n + \frac{1}{2})\Delta \omega_c)$ and $\text{Im} D(q, \omega) = \frac{\gamma \omega^2}{\gamma \omega^2 + q^2 \chi^2}$ we calculate the imaginary self-energies at finite temperature [87]. For $s > 2$

$$\Sigma''(\epsilon) = \frac{E_F}{2\pi^2 m} \left( \frac{e\Delta B}{2p} \right)^{1/2} \int_0^\infty dq \int_0^\infty d\omega \int_{-\infty}^\infty d\epsilon' \sum_{n=0}^\infty \frac{q^2 \gamma \omega}{\gamma^2 \omega^2 + q^2 \chi^2} \delta(\epsilon' + \mu - (n + 1/2)\Delta \omega_c) \times [[n_B(\omega) + n_f(-\epsilon')]\delta(\epsilon - \omega - \epsilon') + [n_B(\omega) + n_f(\epsilon')\delta(\epsilon + \omega - \epsilon')].$$  \hspace{1cm} (B.15)

The $q$-integral may be done by scaling:

$$\int_0^\infty dq \frac{q^2 \gamma \omega}{\gamma^2 \omega^2 + q^2 \chi^2} = \frac{\pi}{\chi^{2s} (\gamma \omega)^{1-2/s} 2^s \csc \left( \frac{\pi}{s} \right)}.$$  \hspace{1cm} (B.16)

Using $\left( \frac{e\Delta B}{2p} \right)^{1/2} = \frac{e\Delta B}{\sqrt{4\pi n_s}}$ we find

$$\Sigma''(\epsilon) = K_s \Delta \omega_c \sum_{n=0}^\infty \int_0^\infty \frac{d\omega}{\omega^{1-2/s}} \int_{-\infty}^\infty d\epsilon' \delta(\epsilon' + \mu - [n + 1/2]\Delta \omega_c) \times [[n_B(\omega) + n_f(-\epsilon')]\delta(\epsilon - \omega - \epsilon') + [n_B(\omega) + n_f(\epsilon')\delta(\epsilon + \omega - \epsilon')]$$  \hspace{1cm} (B.17)

$$= K_s \Delta \omega_c \sum_{n=0}^\infty \left( \frac{n_B(\epsilon - [n + 1/2]\Delta \omega_c + \mu) + n_f(\mu - [n + 1/2]\Delta \omega_c)}{\epsilon - [n + 1/2]\Delta \omega_c} \right)^{1-2/s} \times \theta(\epsilon - [n + 1/2]\Delta \omega_c + \mu) + n_B([n + 1/2]\Delta \omega_c - \epsilon - \mu) + n_f([n + 1/2]\Delta \omega_c - \mu)$$  \hspace{1cm} (B.18)
Likewise for \( s = 2 \)

\[
\int_0^\infty dq \frac{q^\gamma \omega}{q^2 \omega^2 + q^4 \chi^2} = \frac{\pi}{4\chi} \tag{B.19}
\]
therefore

\[
\Sigma''(\epsilon) = K_2 \Delta \omega_c \sum_{n=0}^\infty \int_0^\infty d\omega \int_{\infty}^\infty d\epsilon' \delta(\epsilon' + \mu - [n + 1/2] \Delta \omega_c)
\]

\[
\times \left[ [n_B(\omega) + n_f(-\epsilon')]\delta(\epsilon - \omega - \epsilon') + [n_B(\omega) + n_f(\epsilon')]\delta(\epsilon + \omega - \epsilon') \right] \tag{B.20}
\]

\[
= K_2 \Delta \omega_c \sum_{n=0}^\infty \left( [n_B(\epsilon - [n + 1/2] \Delta \omega_c + \mu) + n_f(\mu - [n + 1/2] \Delta \omega_c)]
\right.
\times \theta(\epsilon - [n + 1/2] \Delta \omega_c + \mu)
\]

\[
+ [n_B([n + 1/2] \Delta \omega_c - \epsilon - \mu) + n_f([n + 1/2] \Delta \omega_c - \mu)]
\times \theta([n + 1/2] \Delta \omega_c - \epsilon - \mu). \tag{B.21}
\]

The real parts are determined using the Kramers-Kronig relation

\[
G'_R(\epsilon) = \frac{1}{\pi} \int_{-\infty}^\infty d\epsilon' \frac{G''_R(\epsilon')}{\epsilon' - \epsilon} \tag{B.22}
\]
which for \( s > 2 \) is

\[
\Sigma'(\epsilon) = \frac{K_s \Delta \omega_c}{\pi} \sum_{n=0}^\infty \int_{-\infty}^\infty d\epsilon' \frac{|\epsilon'|^{2/s-1}}{\epsilon' + (n + 1/2) \Delta \omega_c - \mu - \epsilon}
\]

\[
\times \left[ n_f(\mu - [n + 1/2] \Delta \omega_c)\theta(\epsilon') + n_f([n + 1/2] \Delta \omega_c - \mu)\theta(-\epsilon') \right] \tag{B.23}
\]

\[
= K_s \Delta \omega_c \sum_{n=0}^\infty \left( [n + 1/2] \Delta \omega_c - \mu - \epsilon \right)^{2/s-1}
\]

\[
\times \left( n_f(\mu - [n + 1/2] \Delta \omega_c) \csc \left( \frac{2\pi}{s} \right) \theta([n + 1/2] \Delta \omega_c - \mu - \epsilon)
\right.
\]

\[
- n_f([n + 1/2] \Delta \omega_c - \mu) \cot \left( \frac{2\pi}{s} \right) \theta(\epsilon + \mu - [n + 1/2] \Delta \omega_c)
\]

\[
- n_f([n + 1/2] \Delta \omega_c - \mu) \csc \left( \frac{2\pi}{s} \right) \theta(\epsilon + \mu - [n + 1/2] \Delta \omega_c)
\]

\[
+ n_f([n + 1/2] \Delta \omega_c - \mu) \cot \left( \frac{2\pi}{s} \right) \theta([n + 1/2] \Delta \omega_c - \mu - \epsilon) \tag{B.24}
\]
Appendix B. Self-Energy Corrections in a Finite Magnetic Field

\[
\sum'(\epsilon) = \text{sgn}(\epsilon) \cot \left( \frac{2\pi}{s} \right) \sum_s \left( - \frac{1}{2} \right)^s \sum_{n=0}^{\infty} \left| \epsilon + \mu - n - 1/2 \right|^2 \frac{1}{s-1} \\
+ K_s \csc \left( \frac{2\pi}{s} \right) \sum_{n=0}^{\infty} \left| \epsilon + \mu - n - 1/2 \right|^{2/3-1} \left( n_f(\mu - (n + 1/2)\Delta\omega_c)\theta((n + 1/2)\Delta\omega_c - \epsilon) - n_f((n + 1/2)\Delta\omega_c - \mu - \epsilon) \right) \\
- n_f((n + 1/2)\Delta\omega_c - \mu)\theta(\epsilon + \mu - (n + 1/2)\Delta\omega_c)) 
\] (B.25)

For \( s = 2 \) we have

\[
\sum'(\epsilon) = \frac{K_2\Delta\omega_c}{\pi} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} d\epsilon' \\
\times \frac{n_f(\mu - (n + 1/2)\Delta\omega_c)\theta(\epsilon') - n_f((n + 1/2)\Delta\omega_c - \mu - \epsilon)}{\epsilon' + (n + 1/2)\Delta\omega_c - \mu - \epsilon} \\
= \frac{K_2\Delta\omega_c}{\pi} \sum_{n=0}^{\infty} \log \left| \frac{(n + 1/2)\Delta\omega_c - \mu - \epsilon}{\Delta\omega_c} \right| \\
(n_f(\mu - (n + 1/2)\Delta\omega_c) - n_f((n + 1/2)\Delta\omega_c - \mu)) 
\] (B.26)

Equations (3.25), (3.26), (3.28) and (3.29) follow in the limit of \( T = 0 \). When the self-energy and the energy \( \epsilon \) are expressed in units of the Fermi energy, \( E_F = p\Delta\omega_c \), the dimensionless coupling is

\[
K_2 = \frac{E_F \epsilon}{e^2} \left( \frac{1}{4\pi n_e} \right)^{1/2} \approx \left( \frac{n_0}{n_e} \right)^{1/2}, 
\] (B.28)

where \( n_0 = \frac{E_F^2}{e^2} \frac{\sqrt{\pi}}{4\pi} \approx 1.5 \times 10^{11} \text{cm}^{-2} \) is the density at which \( K_2 = 1 \). For the experiment of Du et al. [30], one has \( E_F = 2\pi n_e/m, n_e = 2.3 \times 10^{11} \text{cm}^{-2}, m = .07m_e \) and \( \epsilon = 13 \), so one expects \( K_2 \approx 0.8 \). Note that equations (3.25)-(3.27) assume that the self-energy has units of \( \Delta\omega_c \), which is more appropriate for solving the self-consistent equation, (3.51).
Appendix C

Proofs Involving the Thermodynamic Potential

C.1 Proof of (4.25)

1 We will derive (4.24) from (4.25).

\[
\Omega_0 = -T \text{Tr} \log G^{(0)}(i\omega_n, \xi_i) \exp(i\omega_n 0^+) \tag{C.1}
\]

\[= -T \sum_{n,l} \log(i\omega_n - \xi_l) \exp(i\omega_n 0^+) \tag{C.2}
\]

We change the sum over \( n \) to a contour integral around the lower half plane and then integrate by parts:

\[
\Omega_0 = \frac{1}{2\pi i} \sum_i \int_{\gamma} d\epsilon \log(\epsilon - \xi_i) n_f(\epsilon - \mu) \exp(\epsilon 0^+) \tag{C.3}
\]

\[= -T \frac{1}{2\pi i} \sum_i \int_{\gamma} d\epsilon \frac{1}{\epsilon - \xi_i} \log \left( 1 + \exp \left( \frac{-\xi_i + \mu}{T} \right) \right) \exp(\epsilon 0^+) \tag{C.4}
\]

\[= -T \sum_i \log \left( 1 + \exp \left( \frac{-\xi_i + \mu}{T} \right) \right) \tag{C.5}
\]

C.2 Proof of (4.81)

2 The thermodynamic potential is the sum over all closed connected diagrams, which implicitly suggests a perturbative approach to computing \( \Omega \). We wish to rewrite \( \Omega \) in a different way. Referring to the connected diagrams shown in Figs. C.1 b) and c), we see

\[1\text{This proof is due to Luttinger and Ward [38]}
\[2\text{We refer the reader to the original paper of Luttinger and Ward [38] and to Section 16.2 of [74].}\]
that a general \( n \)th order connected diagram may be written as

\[
\sum_{r,l} \tilde{\Sigma}^{(n)}(\omega_r, \xi_l) G^{(0)}(\omega_r, \xi_l)
\]  

\( \tilde{\Sigma}^{(n)} \) denotes a general \( n \)th order self-energy term - it may be proper or improper. \( \tilde{\Sigma}^{(n)} \) is the sum over all such \( n \)th order contributions, proper and improper, so that the total \( n \)th order contribution to \( \Omega \) is

\[
\Omega_n = \frac{1}{n} \sum_{r,l} \tilde{\Sigma}^{(n)}(\omega_r, \xi_l) G^{(0)}(\omega_r, \xi_l)
\]  

The factor \( \frac{1}{n} \) compensates for an overcounting of diagrams in this procedure. This occurs because in a general \( n \)th order diagram for \( \Omega \) there are \( n \) fermion lines, therefore there are \( n \) ways to write that particular contribution to \( \Omega \) in the form (C.6).

![Feynman diagrams](image)

**Figure C.1:** Some Feynman diagrams representing fourth order contributions to \( \Omega - \Omega_0 \). The first shows a disconnected graph, the second is the \( O(\lambda^4) \) skeleton graph and the third is an example of a skeleton graph with a self-energy insertion.

Let us assume that the coupling at each vertex is \( \lambda \). A general \( n \)th order contribution is proportional to \( \lambda^n \), and when we consider the form (C.7), all of the dependence is in \( \tilde{\Sigma} \). Therefore we may write

\[
\Omega - \Omega_0 = \sum_{r,l} \int_0^\lambda d\lambda' G^{(0)}(\omega_r, \xi_l) \tilde{\Sigma}(\omega_r, \xi_l; \lambda') \frac{d\lambda'}{\lambda}
\]  

\( \frac{1}{n} \) compensates for an overcounting of diagrams in this procedure. This occurs because in a general \( n \)th order diagram for \( \Omega \) there are \( n \) fermion lines, therefore there are \( n \) ways to write that particular contribution to \( \Omega \) in the form (C.6).
Here \( \bar{\Sigma}(\omega_r, \xi_l; \lambda') \) is the total self-energy (proper plus improper), which depends on a variable coupling \( \lambda' \). Eq. (C.8) may be rewritten as
\[
\lambda \frac{d\Omega}{d\lambda} = -T \sum_{r,\ell} G^{(0)}(\omega_r, \xi_l) \bar{\Sigma}(\omega_r, \xi_l; \lambda)
\] (C.9)
\( \bar{\Sigma} \) satisfies a Dyson-like equation
\[
\bar{\Sigma} = \Sigma + \Sigma G^{(0)} \Sigma + \Sigma G^{(0)} \Sigma G^{(0)} \Sigma \ldots
\] (C.10)
\[
= \Sigma G^{(0)-1} G
\] (C.11)
where \( \Sigma \) is the proper self-energy. Therefore (C.9) is
\[
\lambda \frac{d\Omega}{d\lambda} = -T \sum_{r,\ell} \Sigma(\omega_r, \xi_l; \lambda) G(\omega_r, \xi_l; \lambda)
\] (C.12)

We wish to show that (4.82) also satisfies this differential equation. We begin by considering \( \Omega' \). Recall that \( \Omega' \) consists of the subset of connected diagrams for \( \Omega \) with no self-energy insertions (such as diagram b, but not c), and with all bare propagators \( G^{(0)} \) replaced by the full ones \( G \). Thus by definition \( \Omega' \) is a functional of \( G \), so that
\[
\delta \Omega' = -T \sum_{i,\ell} \Sigma(\epsilon_i, \xi_l) \delta G(\epsilon_i, \xi_l)
\] (C.13)
Furthermore, \( G \equiv (G^{(0)-1} + \Sigma)^{-1} \) which implies that
\[
\frac{\partial \Omega'}{\partial \Sigma} = \frac{\partial \Omega'}{\partial G} \frac{\partial G}{\partial \Sigma} = T \sum_{r,\ell} \Sigma(\epsilon_i, \xi_l) G^2(\epsilon_i, \xi_l)
\] (C.14)
Now we compute the derivative \( \frac{\partial \Omega}{\partial \Sigma} \), using (4.82)
\[
\frac{d\Omega}{d\Sigma} = -T \sum_{r,\ell} \Sigma(\omega_r, \xi_l) G^2(\omega_r, \xi_l) + \frac{d\Omega'}{d\Sigma}
\] (C.15)
\[
= 0
\] (C.16)
Now we turn to \( \frac{d\Omega}{d\lambda} \). Because of the stationarity property (C.16) and the fact that \( G \) depends on \( \lambda \) only through \( \Sigma \) we find that
\[
\frac{d\Omega}{d\lambda} = \frac{d\Omega'}{d\lambda}
\] (C.17)
The only other dependence on $\lambda$ comes from the vertices of the skeleton diagrams of $\Omega'$, which enables us to use the same reasoning as (C.7-C.12) to obtain

$$\lambda \frac{\partial \Omega'}{\partial \lambda} = -T \sum_{i,j} \Sigma(\epsilon_r, \xi_i; \lambda) G(\epsilon_r, \xi_i; \lambda) = \lambda \frac{\partial \Omega}{\partial \lambda} \quad (C.18)$$

Finally we note that

$$\Omega(0) = \Omega_0 = \Omega(0) \quad (C.19)$$

therefore

$$\Omega = \Omega. \quad (C.20)$$

### C.3 Proof of (4.84)

In this section we review Luttinger's derivation [39] of the leading order contributions to $\Omega_{osc}$. Given that $\Sigma_{osc} \sim \left(\frac{\omega_c}{\mu}\right)^{3/2} \Sigma_0$ (this is shown in Appendix F), where $\Sigma = \Sigma_0 + \Sigma_{osc}$ and that $T \text{Tr} \log G^{-1} \sim \left(\frac{\omega_c}{\mu}\right)^{5/2}$, Eq. (4.82) can be approximated as

$$\Omega_{osc} = -T \sum_i \text{Tr} \log(\epsilon_i + \Sigma_0(\epsilon_i) - (n + 1/2)\omega_c + \mu) - \Sigma_0(\epsilon_i) G_0(\epsilon_i) + \Omega'(\Sigma_0) \quad (C.21)$$

where

$$G_0(\epsilon_i) = \frac{1}{\epsilon_i + \mu - (n + 1/2)\omega_c + \Sigma_0(\epsilon_i)} \quad (C.22)$$

$\Omega'$ is evaluated by keeping oscillatory part of each fermion line once while ignoring the oscillatory parts from the rest of the lines. This gives the leading order in the oscillatory behaviour of $\Omega'$. This is the same as $\text{Tr} \sum_i G_0(\epsilon_i) F(\epsilon_i)$, where $F(\epsilon_i)$ is the sum of diagrams obtained by cutting each fermion line in the skeleton graphs once and ignoring oscillatory parts. But this is just $\Sigma_0(\epsilon_i)$, and so the second and third terms in Eq. (C.21) cancel.
Appendix D

Oscillatory Corrections to \( D(\omega, q) \)

In this Appendix we compute the oscillatory behaviour of the screened gauge propagator and show that it does not influence the oscillatory self-energy. We consider the Coulomb interaction (\( s = 2 \)) only. There is no oscillatory dependence in the original unscreened gauge propagator; it arises only as a result of screening due to the presence of the fermion lines which appear in the corrections \( K_{10}^{(0)} \) and \( K_{11}^{(0)} \) (see Fig. 3.2). Only \( \text{Im}K_{11}^{(0)} \) appears in the low energy limit of the \( s = 2 \) version of \( D(q, \omega) \). It appears in the first term of the denominator of \( D(q, \omega) \), which is \( i\gamma\omega \). Recall that \( \gamma = \frac{2n_c}{k_F} \). We will recalculate \( \text{Im}K_{11}^{(0)} \) keeping a sum over Landau level indices on the fermion lines at \( T = 0 \). In the absence of a magnetic field \( K_{11}^{(0)} \) is given by

\[
K_{11}^{(0)}(\omega, q) = \frac{-n_e}{m} + \int \frac{d^2k}{(2\pi)^2} \left( \frac{k_y}{m} \right)^2 \left( \frac{n_f(\xi_{k+q}) - n_f(\xi_k)}{\omega - \xi_{k+q} + \xi_k + i\delta} \right)
\]  

(D.1)

In a finite magnetic field at \( T = 0 \) we find

\[
\text{Im}K_{11}^{(0)} = \frac{2E_F}{m} \frac{1}{\pi q} \left( \frac{eB}{2p} \right)^{1/2} \frac{1}{2\pi l^2} \sum_{m,n=-|p|} \frac{\pi}{\omega_c} \delta(\omega/\omega_c - n + m)(\theta(-n + 1/2) - \theta(-m + 1/2))
\]

(D.2)

\[
= \frac{E_F}{\pi q} \left( \frac{eB}{2p} \right)^{1/2} \sum_{n=-|p|} (\theta(-n + 1/2) - \theta(-n + \omega/\omega_c + 1/2))\theta(n - \omega/\omega_c + |p|)
\]

(D.3)

\[
= \frac{E_F eB}{\pi q k_F} \left( \sum_{n=-|p|} - \sum_{n=-|p|} \right) \theta(n - \omega/\omega_c + |p|)
\]

(D.4)

\[
= \gamma \omega_c |(1/2 + \omega/\omega_c)|
\]

(D.5)
This shows how $\text{Im}K_{00}^{(0)}$ oscillates with $\omega$. The symbol $\lfloor x \rfloor$ denotes the greatest integer less than $x$, indicating that it is a stepwise function, so it is not very singular. It appears in the renormalised gauge propagator as

$$D(\omega, q) = \frac{q}{i \gamma \omega_c \left( \frac{1}{2} + \omega/\omega_c \right) - \tilde{\chi} q^2}$$  \hspace{1cm} (D.6)$$

In the self-energy expression the $q$-integral scales and there is no dependence left on $\lfloor \omega \rfloor$. Therefore, for $s = 2$ there is no oscillatory contribution from $D(\omega, q)$ on the self-energy.
Appendix E

Oscillatory Corrections to $\Omega^{(2)}$

In this Appendix we evaluate the second and third terms in the oscillatory expansion of $\Omega^{(2)}$

$$\Omega^{(2)} = \Omega_0^{(2)} + \Omega_1^{(2)} + \Omega_2^{(2)}. \quad (E.1)$$

These terms are defined by Eq. (4.76). We consider the two dimensional system of composite fermions with $s = 2$ (long-range interactions).

$$\Omega_1^{(2)} = \frac{T^22E_F}{m} \sum_{\omega_n, \nu_n} \frac{1}{\omega_n - \epsilon + \mu} \frac{1}{\nu_n - \epsilon' + \mu} \sum_{k=1}^{\infty} (-1)^k \int \frac{d^2q}{2\pi} |\Lambda(q, p)|^2 \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right) D(i\nu_n, q)$$

$$= \frac{2TE_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon - \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon - \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - i\delta} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon - \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - i\delta} \right) \left( \frac{D_A(x, q) - D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - x} \right) \left( \frac{D_A(x, q) - D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - x} \right) \left( \frac{D_A(x, q) - D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - x} \right) \left( \frac{D_A(x, q) - D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - x} \right) \left( \frac{D_A(x, q) - D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - x} \right) \left( \frac{D_A(x, q) - D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - x} \right) \left( \frac{D_A(x, q) + D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

$$= \frac{2E_F \Delta \omega_c}{\pi^2 \sqrt{4\pi \nu_n}} \int dq \sum_{\nu_n} \int_0^\infty d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon'}{\Delta \omega_c} \right) \cos \left( \frac{2\pi k \epsilon + \epsilon'}{\Delta \omega_c} \right)$$

$$\times \left( \frac{\left( n_f(\epsilon - \mu) - n_f(\epsilon' - \mu) \right)}{\epsilon - \epsilon' - x} \right) \left( \frac{D_A(x, q) + D_B(x, q)}{\epsilon - \epsilon' - x} \right)$$

137
Appendix E. Oscillatory Corrections to $\Omega^{(2)}$

\[ \times \left( \int \frac{d\pi}{\pi} \left( \frac{-\text{Im}D_R(q, x)}{\epsilon - \epsilon' - x} \right) \right) - \text{Re}D_R(\epsilon, q) n_\eta(\epsilon - \epsilon) \]  

\[ = \frac{2E_F\Delta \omega_c}{\pi^3 \sqrt{4\pi} n_e} \int dq \sum_{k=1}^{\infty} \int_0^{\infty} d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\Delta \omega_c} \right) \left( (n_f(\epsilon - \mu) - n_f(\epsilon' - \mu)) \right) \]

\[ \times \int_{-\infty}^{\infty} dx \text{Im}D_R(x, q) \left( \frac{-n_b(x) + n_b(\epsilon' - \epsilon)}{\epsilon' - \epsilon - x} \right) \]  

\[ \times \int_{-\infty}^{\infty} dx \text{Im}D_k(x, q) \left( \frac{n_b(x) + n_b(\epsilon' - \epsilon)}{\epsilon' - \epsilon + x} \right) \]  

These steps show the following operations. First the sums over Matsubara frequencies are converted to contour integrals; note that the Bose frequency $i\nu_n$ is evaluated along the line $i\nu_n = x$. Using $D_A(x) = D_R^*(x)$ and the Kramers-Kronig relations, the form $\text{Im}D_R(x, q)$ is placed everywhere. Finally, we change the limits of the $x$-integral so that the $q$-integral may now be done by scaling,

\[ \Omega^{(2)}_{1} = \frac{E_F\Delta \omega_c}{2\pi^2 \sqrt{4\pi} n_e x} \sum_{k=1}^{\infty} \int_0^{\infty} d\epsilon d\epsilon' \cos \left( \frac{2\pi k \epsilon}{\omega_c} \right) \left( (n_f(\epsilon - \mu) - n_f(\epsilon' - \mu)) \right) \]

\[ \times \int_{0}^{\infty} dx \left( \frac{-n_b(x) + n_b(\epsilon' - \epsilon)}{\epsilon' - \epsilon - x} - \frac{n_b(-x) + n_b(\epsilon' - \epsilon)}{\epsilon' - \epsilon + x} \right) \]  

This expression may be evaluated at $T = 0$,

\[ \Omega^{(2)}_{1} = \frac{4K_2}{\pi} \sum_{k=1}^{\infty} (-1)^k \left( \int_{0}^{\mu} d\epsilon \int_{0}^{\epsilon} d\epsilon' \int_{-\infty}^{\infty} dx + \int_{\epsilon}^{\mu} d\epsilon \int_{-\infty}^{\infty} dx \right) \]

\[ \cos(2\pi k \epsilon / \Delta \omega_c) + \cos(2\pi k \epsilon / \Delta \omega_c) \]  

\[ \epsilon - \epsilon' + x \]  

(E.9)

The third term in the oscillatory expansion has a similar form,

\[ \Omega^{(2)}_{2} = \frac{16K_2}{\pi} \sum_{k, k'} (-1)^{k+k'} \left( \int_{0}^{\mu} d\epsilon \int_{0}^{\epsilon} d\epsilon' \int_{-\infty}^{\infty} dx + \int_{\epsilon}^{\mu} d\epsilon \int_{-\infty}^{\infty} dx \right) \]

\[ \cos(2\pi (k \epsilon + k' \epsilon') / \Delta \omega_c) + \cos(2\pi (k \epsilon - k' \epsilon') / \Delta \omega_c) \]  

\[ \epsilon - \epsilon' + x \]  

(E.10)

There are no additional factors of $\Delta \omega_c / \mu$ appearing in the third term.
Appendix F

Oscillatory Expansion of $\Omega$ in 2-D

F.1 Proof of (4.86)

In this Section we evaluate $T \log G^{(0)-1}$ in 2-D at $T = 0$ for energy levels at $E_n = (n + \frac{1}{2})\omega_c$.

This is just

$$\Omega^{(0)} = \frac{m\omega_c}{\Phi_0} \sum_{n=0}^\infty -T \log \left[ 1 + \exp \left( \frac{\mu - (n + 1/2)\omega_c}{T} \right) \right]$$  \hspace{1cm} (F.1)

$$= \frac{-Tm}{\Phi_0} \int_0^\infty d\epsilon \left[ 1 + 2 \sum_{k=1} \left( -1 \right)^k \cos \left( \frac{2\pi k\epsilon}{\omega_c} \right) \right] \log \left[ 1 + \exp \left( \frac{\mu - \epsilon}{T} \right) \right]$$  \hspace{1cm} (F.2)

$$= \Omega_0^{(0)} + \Omega_{osc}^{(0)}$$  \hspace{1cm} (F.3)

Evaluated at $T = 0$, the first term is just

$$\Omega_0^{(0)} = \frac{\mu^2 m}{2\Phi_0}. \hspace{1cm} (F.4)$$

To evaluate the second term we begin by integrating by parts,

$$\Omega^{(0)}_{osc} = \frac{2m}{\Phi_0} \sum_{k=1} (-1)^k \frac{\omega_c}{2\pi k} \int_0^\infty d\epsilon \sin \left( \frac{2\pi k\epsilon}{\omega_c} \right) n_f(\epsilon - \mu), \hspace{1cm} (F.5)$$

and then evaluate at $T = 0$,

$$\Omega^{(0)}_{osc} = \frac{2m}{\Phi_0} \sum_{k=1} (-1)^k \left( \frac{\omega_c}{2\pi k} \right)^2 \left[ \cos \left( \frac{2\pi k\mu}{\omega_c} \right) - 1 \right]$$  \hspace{1cm} (F.6)

$$= \frac{m}{2\Phi_0} \left( \frac{\omega_c}{2\pi} \right)^2 \left[ \text{mod} \left( \frac{2\mu}{\omega_c}, 1 \right) \right]$$  \hspace{1cm} (F.7)

The last factor gives the oscillations (it is a number that is always less than one). Therefore,

$$\Omega^{(0)}_{osc} \sim \left( \frac{\omega_c}{\mu} \right)^2 \Omega_0^{(0)} \hspace{1cm} (F.8)$$
Appendix F. Oscillatory Expansion of $\Omega$ in 2-D

Luttinger [39] considered the more general case of $\text{Tr} \log G^{-1}$, which he showed could be evaluated using the renormalised quasiparticle spectrum,

$$\text{Tr} \log G^{-1} = \frac{-1}{T} \sum_r \log \left( 1 + \exp \left( \frac{\mu - \epsilon_r}{T} \right) \right)$$ (F.9)

This expression may be evaluated using the methods of Lifshitz and Kosevich [98] for a general Fermi surface.

F.2 Proof of (4.87)

In this Section we do a calculation in 2-D in analogy to Luttinger's calculation in 3-D [39] in which he demonstrated that $\Sigma_{osc} \sim \left( \frac{\omega_c}{\mu} \right)^{3/2}$. We are interested in the quantity $\rho(21)$ which is related to the self-energy

$$\rho(21) = \sum_r n_f (\epsilon_r - \mu) \psi^*(\vec{x}_2; r) \psi(\vec{x}_1; r)$$ (F.10)

$\psi(\vec{x}; r)$ is the normalised wave-function,

$$\psi_{3D}(\vec{x}; n, k_x, k_z) = \left( \frac{1}{2^n n!} \right)^{1/2} \left( \frac{m \omega_c}{\pi \hbar} \right)^{1/4} \frac{2\pi}{a} H_n \left( \frac{y}{l} - k_z l \right)$$

$$\times \exp \left( \frac{- (y/ l - k_z l)^2}{2} + ik_x x + ik_z z \right)$$ (F.11)

$$\psi_{2D}(\vec{x}; n, k_x) = \left( \frac{1}{2^n n!} \right)^{1/2} \left( \frac{m \omega_c}{\pi \hbar} \right)^{1/4} \left( \frac{2\pi}{a} \right)^{1/2} H_n \left( \frac{y}{l} - k_x l \right)$$

$$\times \exp \left( \frac{- (y/ l - k_x l)^2}{2} + ik_x x \right).$$ (F.12)

Use the identity

$$\frac{1}{e^x + 1} = \frac{1}{2\pi i} \int_{-i\infty+c}^{i\infty+c} dt \frac{e^{-tx}}{(1/\pi) \sin(\pi t)}$$

$$0 < c < 1$$ (F.13)

to get

$$\rho(21) = \frac{1}{2\pi} \int_{-i\infty+c}^{i\infty+c} dt \frac{e^{-t\mu/ T}}{(1/\pi) \sin(\pi t)} Q(\vec{x}_2, \vec{x}_1; t/T)$$ (F.14)
Appendix F. Oscillatory Expansion of $\Omega$ in 2-D

where

$$ Q(\vec{x}_2, \vec{x}_1; t/T) = \sum_r \psi^*(\vec{x}_2; r) e^{-i r/T} \psi(\vec{x}_1; r). \quad (F.15) $$

We evaluate $Q(\vec{x}_2, \vec{x}_1; t/T)$ separately for 2-D and 3-D:

$$ Q_{2D}(\vec{x}_2, \vec{x}_1; t/T) = \frac{\text{Area} \Phi_0}{2\pi} \sum_n \psi^*_{2D}(\vec{x}_2; k_x, n) \exp \left( \frac{-t(n + 1/2)\omega_c}{T} \right) \psi_{2D}(\vec{x}_1; k_x; \Phi) \quad (F.16) $$

$$ Q_{3D}(\vec{x}_2, \vec{x}_1; t/T) = \frac{\text{Area} \Phi_0}{2\pi} \sum_n \int dk_z \psi^*_{3D}(\vec{x}_2; k_x, k_z, n) \times \exp \left( \frac{-t(n + 1/2)\omega_c + k^2_z/2m}{T} \right) \psi_{3D}(\vec{x}_1; k_x, k_z, n) \quad (F.17) $$

$$ = \int dk_z \exp \left( ik_z(z_2 - z_1) - \frac{tk^2_z}{2mT} \right) Q_{2D}(\vec{x}_2, \vec{x}_1; t/T) \quad (F.18) $$

$$ = \left( \frac{Tm}{2\pi t} \right)^{1/2} \exp \left( \frac{Tm(z_1 - z_2)^2}{2t} \right) Q_{2D}(\vec{x}_2, \vec{x}_1; t/T) \quad (F.19) $$

$Q(\vec{x}_2, \vec{x}_1; t/T)$ is evaluated to give (see [39])

$$ Q(\vec{x}_2, \vec{x}_1; t/T) = \exp \left( \frac{i\omega_c(y_2 + y_1)(x_2 - x_1)}{2} \right) F(\vec{x}_2 - \vec{x}_1; \omega_c; t/T) \quad (F.20) $$

where

$$ F_{2D}(\vec{x}; \omega_c; t/T) = \frac{\omega_c}{4\pi \sinh(\omega_c t/2T)} \exp \left( -\frac{\omega_c}{4} \coth \left( \frac{\omega_c t}{2T} \right) (x^2 + y^2) \right) \quad (F.21) $$

$$ F_{3D}(\vec{x}; \omega_c; t/T) = \left( \frac{Tm}{2\pi t} \right)^{1/2} \exp \left( \frac{Tmz^2}{2t} \right) F_{2D}(\vec{x}; \omega_c; t/T). \quad (F.22) $$

This enables us to write $\rho(21)$ in the form

$$ \rho(21) = \exp \left( \frac{i\omega_c(y_2 + y_1)(x_2 - x_1)}{2} \right) g(\vec{x}_2 - \vec{x}_1; \omega_c; t/T) \quad (F.23) $$

where

$$ g(\vec{x}; \omega_c; t/T) = \frac{1}{2\pi i} \int_{-\infty + c}^{\infty + c} dt \frac{e^{\mu i T}}{(1/\pi) \sin(\pi t)} F(\vec{x}; \omega_c; t/T). \quad (F.24) $$

g(\vec{x}; \omega_c; t/T) has poles on the imaginary axis at

$$ t_l = \frac{2\pi i l T}{\omega_c} \quad l = \pm 1, \pm 2 \ldots \quad (F.25) $$
which are the poles of $F(x; \omega; t/T)$; it has also has poles along the real axis at $t = \pm n$ for $n = 1, 2, 3, \ldots$. Still following Luttinger, we deform the contour of the integral and divide it into two pieces,

$$g(x; \omega; t/T) = g_0(x; \omega; t/T) + g_{osc}(x; \omega; t/T)$$

(F.26)

$$g_0(x; \omega; t/T) = \frac{1}{2\pi i} \int_C dt \frac{e^{it/T}}{(1/\pi)\sin(\pi t)} F(x; \omega; t/T)$$

(F.27)

$$g_{osc}(x; \omega; t/T) = \sum_{l=-\infty}^{\infty} \frac{1}{2\pi i} \int_{C_l} dt \frac{e^{it/T}}{(1/\pi)\sin(\pi t)} F(x; \omega; t/T)$$

(F.28)

where $C$ is a contour that encircles the negative real axis counter-clockwise and the $C_l$ are counter-clockwise circles about the points $t_l$. The sum does not include the point $l = 0$. The first term of Eq. (F.26), given by (F.27), is the non-oscillatory contribution to the self-energy. It is evaluated by keeping the lowest order contribution in powers of $t$,

$$F_{2D}(x; \omega; t/T) = \frac{\mu}{2\pi T} \exp \left( \frac{-\mu}{2t} \right)$$

(F.29)

$$F_{3D}(x; \omega; t/T) = \left( \frac{T}{2\pi t} \right)^{1/2} \exp \left( \frac{-\mu}{2t} \right) F_{2D}(x; \omega; t/T)$$

(F.30)

Therefore,

$$g_{0,2D} = \frac{1}{2\pi i} \frac{\mu}{2\pi} \int_C \frac{dt}{t^2} \exp \left( \frac{-\mu}{2t} \right)$$

(F.31)

$$g_{0,3D} = \frac{1}{2\pi i} \left( \frac{\mu}{2\pi} \right)^{3/2} \int_C \frac{dt}{t^{3/2}} \exp \left( \frac{-\mu}{2t} \right)$$

(F.32)

The second term of Eq. (F.26) is the oscillatory contribution. It is evaluated by first shifting all the contours to $t' = t - t_i$, so that each contour is a little circle about $t' = 0$, and then taking the limit of small $t'$ as above,

$$F_{2D}(x; \omega; (t' + t_i)/T) = \frac{(-1)^t T}{2\pi t'} \exp \left( \frac{-\mu}{2t'} \right)$$

(F.33)

$$F_{3D}(x; \omega; (t' + t_i)/T) = \left( \frac{T}{2\pi t_i} \right)^{1/2} \frac{(-1)^t T}{2\pi t'} \exp \left( \frac{-\mu}{2t'} - \frac{\mu}{2t_i} \right)$$

(F.34)
Notice that the first term in the exponent of Eq. (F.34) dominates, therefore

\[ F_{3D}(\bar{x}; \omega_c; (t' + t_i)/T) = \left( \frac{T}{2\pi t_i} \right)^{1/2} F_{2D}(\bar{x}; \omega_c; (t' + t_i)/T) \]  \hspace{1cm} (F.35)

\[ \sim (\omega_c)^{1/2} F_{2D}(\bar{x}; \omega_c; (t' + t_i)/T). \]  \hspace{1cm} (F.36)

This leads to the result that

\[ g_{osc,2D} = \frac{1}{2\pi i} \sum_{l=-\infty}^{\infty} \frac{T}{2\pi t_i} \exp \left( \frac{t_l\mu}{T} \right) \int \frac{dt'}{t'} \exp \left( t' \mu - \frac{x^2 + y^2}{2t'} \right) \]  \hspace{1cm} (F.37)

\[ g_{osc,3D} = \frac{1}{2\pi i} \sum_{l=-\infty}^{\infty} \left( \frac{T}{2\pi t_i} \right)^{3/2} \exp \left( \frac{t_l\mu}{T} \right) \int \frac{dt'}{t'} \exp \left( t' \mu - \frac{x^2 + y^2}{2t'} \right). \]  \hspace{1cm} (F.38)

Substituting Eq. (F.25) for \( t_l \), we find that

\[ g_{osc,2D} \sim \frac{\omega_c}{\mu} g_{0,2D} \]  \hspace{1cm} (F.39)

\[ g_{osc,3D} \sim \left( \frac{\omega_c}{\mu} \right)^{3/2} g_{0,3D}. \]  \hspace{1cm} (F.40)

Luttinger argued that the origin of these relations lies in the Fermi function, and that a similar dependence occurs in higher skeleton graphs, yielding the general result that

\[ \Sigma_{osc,2D} \sim \frac{\omega_c}{\mu} \Sigma_{0,2D} \]  \hspace{1cm} (F.41)

\[ \Sigma_{osc,3D} \sim \left( \frac{\omega_c}{\mu} \right)^{3/2} \Sigma_{0,3D}. \]  \hspace{1cm} (F.42)