A STUDY OF SOME RARE ETA-MESON DECAYS

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
Doctor of Philosophy

in

THE FACULTY OF GRADUATE STUDIES
DEPARTMENT OF PHYSICS

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

December 1993

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Date 17 December 1993

DE-6 (2/68)
Abstract

Previous calculations of the rate of the decay $\eta \rightarrow \pi^0 \gamma \gamma$ have fallen short of the experimental result, 0.84 ± 0.18 eV. One such calculation, that of vector meson dominance (VMD), is included herein (giving $0.29^{+0.16}_{-0.12}$ eV), and moreover is modified by the inclusion of an added ingredient, the scalar meson $a_0$. The added mechanism does not adequately fix the result; assuming constructive interference the calculated rate becomes $0.37^{+0.20}_{-0.15}$ eV. The rate is then calculated according to a quark-box mechanism, where the $\eta q\bar{q}$ and $\pi^0 q\bar{q}$ couplings are fixed by the rates of the $\eta \rightarrow \gamma \gamma$ and $\pi^0 \rightarrow \gamma \gamma$ decays. With constituent quark masses of 300 MeV for $u$ and $d$ quarks the calculated rate is $0.70 \pm 0.12$ eV, which is in good agreement with experiment.

Lower bounds (the unitarity limits) for the decays $\eta \rightarrow \pi^0 e^+ e^-$ and $\eta \rightarrow \pi^0 \mu^+ \mu^-$ are calculated, using the intermediate state $\pi^0 \gamma \gamma$. The available experimental information is inadequate for purposes of a model-free calculation of these unitarity limits; therefore, the models considered for $\eta \rightarrow \pi^0 \gamma \gamma$ are used in these calculations. Using the quark-box mechanism as above, the unitarity limits are $2.9 \pm 0.5 \mu$eV and $4.3 \pm 0.7 \mu$eV, for $\eta \rightarrow \pi^0 e^+ e^-$ and $\eta \rightarrow \pi^0 \mu^+ \mu^-$, respectively. These results show that the measurement of these (as yet unseen) modes at $\eta$ factories such as that of Saturne can be expected.
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Acknowledgements

Many people have contributed positively to my experience as a graduate student, and I wish to thank them all here, with apologies to any whom I neglect to mention specifically.

First, there is Dr. John Ng, who suggested this project, and provided guidance throughout it. Then, my fellow students who provided frequent interesting or useful discussions. Notable among them are Scott Hayward, Roger Kemp, Jeff Lange, Henry Lee, Gail Meagher, Steve Patitsas, Jacob Sagi, Bill Scott, Wolfe Wall, and Glenn Wells.

I am grateful for the continual encouragement and moral support from my parents. The same goes for my four brothers, each of whom contributed to my sanity over the past few years, all in different ways. Also, I am not convinced that I did not have some intangible divine help, over the last year in particular; so at risk of sounding eccentric to those whose views on “ultimate questions” conflict with mine, I wish to thank God here.
Chapter 1

Introduction

It is the aim of this thesis to examine three of the rare decays of the $\eta$ meson.

The word "meson" signifies, among other things, that this particle is believed to have substructure; i.e. it is not considered to be a fundamental particle. Those particles which are considered (for now, at least) to be fundamental fall into three categories; quarks, leptons, and the so-called "gauge particles" which mediate the various kinds of forces. (See, for example, [1].)

The most familiar example of a particle in this last category is the photon, symbolized by $\gamma$, which is associated with electromagnetic interactions. The others are the $W^\pm$ and $Z^0$, and the eight kinds of gluons, which are involved in (respectively) the weak and the strong interactions. The gluons, like the photon, are massless, but the $W^\pm$ and $Z^0$ have masses of respectively 80 GeV and 91 GeV.

Gravitational interactions have not been mentioned; indeed, the role they play, if any, on the scale of elementary particles remains obscure. It is assumed that any successful microscopic theory of gravity will involve a mediating particle analogous to those named above; it is called the graviton.

The most familiar lepton is the electron, $e^-$, with a mass of 0.51 MeV. There are two other kinds of charged leptons, the $\mu^-$ (muon) and the $\tau^-$ (simply called the tau). These have basic interactions which are similar to those of the electron, but they have much greater masses: 106 MeV for the $\mu^-$ and 1.78 GeV for the $\tau^-$. Corresponding to the three charged leptons are three neutral leptons ("neutrinos"), the $\nu_e$, the $\nu_\mu$, and the $\nu_\tau$. 
The neutrinos are, as far as we know, massless.

The quarks, like the leptons, come in six types (or "flavours"). The quark flavours are (in increasing order of mass) "up", "down", "strange", "charm", "bottom", and "top", symbolized by the letters u, d, s, c, b, t. The u, c, and t quarks have electric charge $+\frac{2}{3}$ (in the units in which the electron's charge is $-1$) and the d, s, and b quarks have electric charge $-\frac{1}{3}$.

For each particle there is a corresponding antiparticle, with the same mass but opposite electric charge. In the case of the leptons and quarks, the symbols are $e^+$, $\mu^+$, $\tau^+$, $\bar{e}$, $\bar{\mu}$, $\bar{\tau}$, $\bar{u}$, $\bar{d}$, $\bar{s}$, $\bar{c}$, $\bar{b}$, $\bar{t}$. The photon is its own antiparticle; similarly for the $Z^0$ and two of the eight gluons. The $W^+$ and $W^-$ form a particle-antiparticle pair, and three such pairs are formed by the other six gluons.

All these particles fit into a mathematical framework known as the "Standard Model" (SM). The SM is an extraordinarily successful scheme for describing the particles and their basic interactions. One of its weaknesses is that, in its most basic form, it does not allow for any of the fundamental particles to have mass. This can be fixed in several ways; the simplest scheme requires, as a side effect, the addition of another particle. This is known as the Higgs particle ($H^0$).

The quarks are always bound by the strong interactions into composite particles called "hadrons". These are subdivided into two categories, "baryons" and "mesons". The baryons consist of three quarks. The most familiar baryons are the proton (two u quarks and one d quark) of mass 938 MeV, and the neutron (two d quarks and one u quark) of mass 940 MeV. (Of course there are also antibaryons, made of three antiquarks.) The mesons are made of a quark and an antiquark. The $\eta$ is an example of a meson, as are the three pions ($\pi^0$ and $\pi^\pm$). More will be said about their structure in the next section.

There are over a hundred known types of each of the two categories of hadrons. This
diversity is due to the fact that a hadron's characteristics are determined not only by the flavours of the constituent quarks, but also by the way in which the quarks are put together. The following section will provide some specific examples of this, in the case of mesons.

That part of the SM which describes the strong force is known as quantum chromodynamics (QCD). The basic interactions of QCD are between quarks and gluons, and between gluons and other gluons. It might be expected that QCD would provide a framework for theoretical descriptions of hadronic processes, since hadrons are considered to be strongly-bound systems of quarks. In practice, however, QCD is useless for purposes of discussing low-energy hadronic processes. This is because (1) at low energies the coupling constant which describes the strength of the fundamental interactions in QCD is of order unity, thus negating the validity of perturbation theory (wherein quantities to be calculated are described as power series in the coupling constants), and (2) the mathematics involved in describing a strongly bound state in terms of QCD is prohibitively complex. In light of this, it is natural to look for “effective” (non-fundamental) theories to account for the results of experiments involving low-energy hadronic interactions.

Section 1.1 will give a more detailed description of the mesons which are featured in this thesis, section 1.2 will provide motivation for the decays to be studied as well as a description of the experimental outlook, and section 1.3 will give an overview of the rest of the thesis.

1.1 A description of the $\eta$

This section mainly consists of a discussion of the structure of the $\eta$. This particle has spin 0, parity $-1$, and charge-parity $+1$. Its mass is $M_\eta = (547.45 \pm 0.19)$ MeV, and its full width is $\Gamma(\eta \rightarrow \text{all}) = (1.19 \pm 0.11)$ keV (corresponding to a lifetime of $(5.5 \pm 0.5) \times 10^{-19}$
Mesons are well described as bound states of quark-antiquark pairs. They can be characterized by the flavours of the quark and antiquark, and by the values of various quantum numbers: $S$, the summed spin of the quarks; $L$, the $q\bar{q}$ system orbital angular momentum; $J$, the meson’s total spin ($\tilde{J} = \tilde{S} + \tilde{L}$); and $n$, the quantum number of radial excitations. Other properties of the meson follow from these.

There are, as mentioned previously, six flavours of quarks. If we consider $N$ of these flavours, this gives us $N^2$ mesonic states for any given values of $S$, $L$, $J$, and $n$, i.e. $|u\bar{u}\rangle$, $|u\bar{d}\rangle$, etc. The physical states are not, in general, those which can be written the most simply, but rather are superpositions of those states. For example, there is no meson corresponding to the combination $|u\bar{u}\rangle$. However, the physical states are somewhat restricted in terms of which simple states may be mixed.

The parity of a $|q\bar{q}\rangle$ state is $P = (-1)^{L+1}$. If the quark and antiquark are of the same flavour, the state is also an eigenstate of charge-conjugation, with $C = (-1)^{L+S}$. The electric charge of a meson is simply the sum of the charges of the quark and antiquark. For mesonic states to mix, they must have the same $J$, $P$, and electric charge. The states which are $C$-eigenstates must also have the same value of $C$ in order to mix.

The strong interactions do not distinguish between flavours. A more precise statement of this is that, for $N$ flavours, the strong interactions obey an $SU(N)$ symmetry. This means that, for given values of $S$, $L$, $J$, and $n$, all flavour combinations, except the singlet $\frac{1}{\sqrt{N}}(|u\bar{u}\rangle + |d\bar{d}\rangle + ...)$, are equivalent as far as the strong force is concerned. In the absence of other considerations, one would expect that one of the physical meson states would be the singlet, and that all of the mesons which share the same $S$, $L$, $J$, and $n$, with the exception of the singlet, would have the same mass. But the $SU(N)$ symmetry is broken by the weak and electromagnetic interactions, and by the differences in the masses of the quarks.
Chapter 1. Introduction

The quark mass differences among the heavy quarks \( m_c \approx 1.5 \text{ GeV}, m_b \approx 5 \text{ GeV}, m_t > 91 \text{ GeV} [2] \), and between the heavy and light quarks (masses \( \leq 500 \text{ MeV} \) as discussed in section 2.3), are dramatic enough that the mesonic states which include a given heavy flavour are separated from the states which include only lighter flavours. For example, pure \(|c\bar{c}\rangle\) mesons (such as the \( J/\psi \)) and pure \(|b\bar{b}\rangle\) mesons (such as the \( \Upsilon \)) have been seen. (The \( t \)-quark has not been seen at all.) Also, the quark mass differences cause considerable differences in the meson masses which would, according to the \( SU(N) \) scheme, be the same. So the \( SU(N) \) description of the mesons is not particularly useful if any of the heavy quarks are included.

From this point on only the three lightest flavours, \( u, d, \) and \( s \), will be considered. The \( SU(3) \) scheme works quite well. Mesons which consist only of the light quarks fall into nonets (sets of nine) with \( S, L, J, \) and \( n \) in common, and (for most of the nonets) similar masses. Indeed, historically the existence of several particles was successfully predicted using this model. However, the \( SU(3) \) symmetry is not good enough to prevent mixing between the \( SU(3) \) singlet state \( \frac{1}{\sqrt{3}}(|u\bar{u}| + |d\bar{d}| + |s\bar{s}|) \) and the other (i.e. octet) states.

The \( SU(2) \) symmetry of the \( u \) and \( d \) works almost exactly; this ensures that within the \( SU(3) \) nonet there is no noticeable mixing between states which belong to \( SU(2) \) representations of different sizes. Thus the state \( \frac{1}{\sqrt{2}}(|u\bar{u}| - |d\bar{d}|) \), which is connected by \( SU(2) \) transformations to the states \(|u\bar{d}| \) and \(|d\bar{u}| \) and is said to be of “isospin 1”, does not mix appreciably with the (“isospin 0”) \( SU(2) \) singlets \( \frac{1}{\sqrt{2}}(|u\bar{u}| + |d\bar{d}|) \) and \(|s\bar{s}| \), or with \(|d\bar{s}| \) (which is connected by an \( SU(2) \) transformation to \(|u\bar{s}| \) so it is of “isospin \( \frac{1}{2} \)).

Table (1.1) matches the mesons of the two nonets of least mass (the pseudoscalars with \( S = 0, L = 0, J = 0, n = 1 \) and the vectors with \( S = 1, L = 0, J = 1, n = 1 \)) with their quark flavour configurations.

The mixing of the neutral strange states \(|d\bar{s}| \) and \(|s\bar{d}| \) is an interesting topic itself, but it is beyond the scope of this thesis. For the present it suffices to say that the \( K^0 \)
Table 1.1: The quark-content of the mesons in the two nonets of least mass.

<table>
<thead>
<tr>
<th>quark configuration</th>
<th>mesons</th>
<th>vectors</th>
<th>notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>ud\rangle, \frac{1}{\sqrt{2}}(</td>
<td>u\bar{u}</td>
<td>-</td>
</tr>
<tr>
<td>(</td>
<td>u\bar{s}\rangle,</td>
<td>s\bar{u}\rangle)</td>
<td>(K^+, K^-)</td>
</tr>
<tr>
<td>(</td>
<td>d\bar{s}\rangle,</td>
<td>s\bar{d}\rangle)</td>
<td>(K^0, K^0)</td>
</tr>
<tr>
<td>(\frac{1}{\sqrt{6}}(</td>
<td>u\bar{u}</td>
<td>+</td>
<td>d\bar{d}</td>
</tr>
</tbody>
</table>

(1): The mesons with these names correspond respectively to the quark configurations given in the first column.

(2): These names refer respectively to the quark configurations given in the first column; the real mesons are combinations of these.

(3): The mesons with these names are combinations of the quark configurations given in the first column.

and \(K^0\) mix by about 45\(^\circ\) to form the \(K_L^0\) and the \(K_S^0\).

The quark configurations of the \(\eta\) and \(\eta'\) (see table (1.1)) are traditionally expressed in terms of the mixing angle \(\theta_\eta\), as follows:

\[
\begin{align*}
|\eta\rangle &= \cos \theta_\eta |\eta_8\rangle - \sin \theta_\eta |\eta_1\rangle \\
|\eta'\rangle &= \sin \theta_\eta |\eta_8\rangle + \cos \theta_\eta |\eta_1\rangle,
\end{align*}
\]

where

\[
\begin{align*}
|\eta_8\rangle &\equiv \frac{1}{\sqrt{6}}(|u\bar{u}| + |d\bar{d}| - 2|s\bar{s}|) \\
|\eta_1\rangle &\equiv \frac{1}{\sqrt{3}}(|u\bar{u}| + |d\bar{d}| + |s\bar{s}|).
\end{align*}
\]

Many methods have been used to determine the mixing angle \(\theta_\eta\). For a long time the usual method was to use the relationships between the masses of the \(\eta\), \(\pi\) and \(K\) mesons. The starting point is the Gell-Mann–Okubo mass formula

\[
M_\eta^2 = \frac{1}{3}(4M_K^2 - M_\pi^2).
\]
To see where this comes from, assume that the mass differences in the $SU(3)$ octet arise solely from the differences in quark masses, and use the approximation that $m_u = m_d$ (i.e. that the flavour-$SU(2)$ is an exact symmetry). By this assumption, with $\theta_\eta = 0$, one would be tempted to write

\begin{align*}
M_\pi &= m_0 + 2m_u \\
M_K &= m_0 + m_u + m_s \\
M_\eta &= m_0 + \frac{2}{3}(m_u + 2m_s),
\end{align*}

(1.4)

using equation (1.1) and table (1.1), where $m_0$ is an underlying flavour-independent mass which is the same for all of the mesons of the octet. This would lead to a formula similar to (1.3), but with linear rather than quadratic masses. However, mesons are bosons, so their masses must appear quadratically in the effective mesonic Lagrangian (in terms such as $M^2 \Phi^2$) whereas the quark masses appear linearly in the quark Lagrangian (in terms such as $m \bar{\Psi} \Psi$) since quarks are fermions. This suggests that the squared meson masses should be related directly to the (linear) quark masses, as in

\begin{align*}
M^2_\pi &= \mu(m_0 + 2m_u) \\
M^2_K &= \mu(m_0 + m_u + m_s) \\
M^2_\eta &= \mu(m_0 + \frac{2}{3}(m_u + 2m_s)),
\end{align*}

(1.5)

where $\mu$ is a parameter with units of mass. A better justification for this kind of relationship arises from Chiral Perturbation Theory; see section 2.2. The formula (1.3) follows from (1.5). If we remove the restriction $\theta_\eta = 0$, then (1.3) should be replaced by

\begin{equation}
M^2_{88} = \frac{1}{3}(4M^2_K - M^2_\pi),
\end{equation}

(1.6)

where $M^2_{88}$ is the octet-octet component of the mass-squared matrix

\begin{equation}
M = \begin{pmatrix}
M^2_{11} & M^2_{18} \\
M^2_{18} & M^2_{88}
\end{pmatrix}
\end{equation}

(1.7)
which is diagonalized by the states $|\eta\rangle$ and $|\eta\rangle'$. Using (1.1) and

$$M_{\eta}^2\eta^2 + M_{\eta}^2\eta'^2 = (\begin{array}{c} \eta_1 \\ \eta_8 \end{array})M(\begin{array}{c} \eta_1 \\ \eta_8 \end{array})$$

(1.8)

one finds that

$$\tan \theta_\eta = \frac{M_{8\eta}^2 - M_{8\eta}'^2}{M_{8\eta}^2}, \quad \text{and}$$

$$\tan^2 \theta_\eta = \frac{M_{8\eta}^2 - M_{8\eta}'^2}{M_{8\eta}^2 - M_{8\eta}'^2}. \quad (1.9)$$

(1.10)

Using the present best values for $M_{K^\pm}$, $M_{\pi^\pm}$, $M_\eta$, and $M_{\eta'}$, (1.10) gives $\tan^2 \theta_\eta = 0.031$, so $\theta_\eta = -10^\circ$. The sign of $\theta_\eta$ is chosen to agree with the sign taken from other determinations of the mixing angle, of which one example is given below. Also, the quark configurations of $|\eta_1\rangle$ and $|\eta_8\rangle$, and the fact that $m_s > m_u$, suggest that $M_{8\eta}^2 < 0$, hence (1.9) implies that $\tan \theta_\eta < 0$.

One weakness of this approach is that the result is highly sensitive to imperfections in (1.6). For example, if $M_{8\eta}^2$ differs from the (1.6) result by 10\%, the mixing angle becomes $-17^\circ$.

Another way of determining the mixing angle is based on the assumption that the amplitudes for the decays of the $\pi^0$, $\eta$, and $\eta'$ into two photons are proportional to $\sum_i c_i Q_i^2$, where $c_i$ is the coefficient of the $|q\bar{q}\rangle$ term of the $i^{th}$ flavour in the meson's wavefunction and $Q_i$ is the electric charge of that flavour of quark, and otherwise independent of the meson type. Defining $H(\eta)$ by

$$\mathcal{M}(\eta \to \gamma\gamma) = H(\eta)\epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\nu k_1^\rho k_2^\sigma, \quad (1.11)$$

where $\epsilon_i^\mu$ and $k_i^\mu$ are respectively the polarization vector and 4-momentum of the $i^{th}$ photon, and similarly for $H(\pi)$ and $H(\eta')$, the ratios of these quantities are.
\[
\frac{H(\eta)}{H(\pi)} = \frac{\cos \theta_\eta - 2\sqrt{2} \sin \theta_\eta}{\sqrt{3}} \quad (1.12)
\]
\[
\frac{H(\eta')}{H(\pi)} = \frac{2\sqrt{2} \cos \theta_\eta}{\sqrt{3}} + \frac{\sin \theta_\eta}{\sqrt{3}} \quad (1.13)
\]

Using the experimental rates of these decays, (1.12) implies that \( \theta_\eta = (-16 \pm 4)^\circ \), and (1.13) implies that \( \theta_\eta = (-23 \pm 3)^\circ \).

Gilman and Kauffman [3] have examined a variety of determinations of this mixing angle, including more sophisticated versions of the two methods described here. They point out that there are questionable elements in each of the analyses, but conclude that the total weight of evidence points to a mixing angle in the vicinity of \(-20^\circ\).

It is interesting that the \( \eta-\eta' \) mixing angle differs greatly from that of the vector mesons \( \phi \) and \( \omega \). If we repeat the discussion of mixing, but use the mesons of the vector nonet, the mass-formula argument given above leads to a mixing angle of \( \theta_\omega = 40^\circ \). This is close to the "ideal" mixing \( (\approx 35.3^\circ) \) for which the \( \phi \) meson would be a pure \( |s\bar{s}\rangle \) state. This difference in behaviour between the vector and pseudoscalar mesons is not understood.

1.2 The Rare Decays

One aspect of hadronic physics which has received considerable attention, both experimentally and theoretically, in the last decade, is the spectroscopy of light mesons and the study of their decays [4]. New experiments have been performed due to the availability of intense hadron beams and detectors with wide acceptances. The \( e^+e^- \) colliders have also opened up new areas of investigation involving light mesons, including studies of meson production via the two-photon reaction and in decays of the \( J/\psi \) meson.
This thesis is concerned with the decays

\[
\eta \rightarrow \pi^0\gamma\gamma, \quad (1.14)
\]

\[
\eta \rightarrow \pi^0e^+e^-, \quad \text{and} \quad (1.15)
\]

\[
\eta \rightarrow \pi^0\mu^+\mu^- . \quad (1.16)
\]

Of these, only (1.14) has been observed. The measurements [5] were made at the IHEP accelerator (Serpukhov, Russia). A liquid-hydrogen target was hit by incident \( \pi^- \) mesons of momentum 30 GeV. Photons from the decays of neutral mesons produced in reactions of the type \( \pi^- p \rightarrow (X^0)n \), \( (X^0) \rightarrow \gamma' \) 's were recorded in the hodoscope Cerenkov spectrometer GAMS-2000. \( (X^0) \) represents a neutral meson or a set of neutral mesons.) Four-photon events were selected. Those events which satistified the kinematics of the meson-pair-production reactions \( \pi^- p \rightarrow \pi^0\pi^0n \) and \( \pi^- p \rightarrow \pi^0\pi^0n \) were rejected. The background from \( \pi^- p \rightarrow K_S^0\Lambda, K_S^0 \rightarrow \pi^0\pi^0 \), was dealt with similarly, taking into account the fact that the \( K_S^0 \) can travel a few cm before decaying. (The \( \Lambda \) is an isospin-0 \( |uds \) baryon.) Also excluded were those events for which no two photons could be reconstructed kinematically into a \( \pi^0 \).

After these (and a few other) subtractions, the mass spectrum of the remaining four-photon events (see figure (1.1)) has a noticeable peak at the \( \eta \) mass; these events are attributed to the decay (1.14). The background which is visible around the \( \eta \) peak in this figure is mostly from the decay \( \eta \rightarrow 3\pi^0 \), where two of the six resulting photons are lost. After subtracting this background (interpolating in the region of the \( \eta \) peak), the number of events of the decay (1.14) is determined.

Considering both experiments in ref. [5], the rate of the decay (1.14) is [2]

\[
\Gamma_{\text{exp}}(\eta \rightarrow \pi^0\gamma\gamma) = 0.84 \pm 0.18 \text{eV.} \quad (1.17)
\]

The rare decays (1.15) and (1.16) have not yet been seen, but the outlook for finding
them is favourable. A facility dedicated to the production of $\eta$ mesons has been installed at the Saturne accelerator (Saclay, France) for the purpose of investigating the production and properties of the $\eta$-meson [6]. As such it is ideal for the examination of its rare decay modes. The $\eta$’s are produced by $pd \rightarrow \eta^3\text{He}$ near threshold, which was found in 1988 to have an unexpectedly large cross-section [7]. The $^3\text{He}$ are detected in a magnetic spectrometer which can easily separate them from the incident proton beam. This clean detection of the $^3\text{He}$ provides a reliable way to tag and identify the $\eta$ events. This facility presently produces $10^8 \eta$’s per day, but it is claimed that this rate can easily be increased by a factor of 50 [8]. (The branching ratios for (1.15) and (1.16) should be greater than $3 \times 10^{-9}$; see chapter 4. Therefore, the observation of these decays at Saturne can be expected, and is eagerly awaited.)

Other possibilities exist for high-intensity $\eta$ production. Proposed expansions to experimental facilities at Brookhaven or at TRIUMF could make use of the reaction

Figure 1.1: Effective mass spectrum of $\pi^0\gamma\gamma$ from the reaction $\pi^- p \rightarrow (X^0)n, (X^0) \rightarrow \gamma's$ (taken from Alde et al.[5]).
π−p → η n to produce over 10^{11} η's per day, although the difficulty of detecting the neutrons would cause many η events to be missed, thus cutting down the effective production of η's by perhaps a factor of 10 [8].

Studies of np → η d and γ p → η p [9] have shown that such η production near threshold is dominated by an intermediate state which includes the \(N(1535)\) baryon (i.e. the |uud\rangle or |udd\rangle state of isospin \(\frac{1}{2}\), \(J = \frac{1}{2}\), \(P = -1\), and mass near 1535 MeV). The large cross-section for pd → η ^3\text{He} near threshold is presumably related to this.

A measurement of (1.15) or (1.16) would provide a good direct test of C- and CP-conservation in electromagnetic interactions [10]. The decay through a one-photon intermediate state (figure (1.2)),

\[
\eta \rightarrow \pi^0 \gamma \rightarrow \pi^0 l^+ l^- \quad (l = e, \mu),
\]

is C- and CP-violating [11], so it does not occur according to standard QED, so the two-photon mechanism (figure (1.3))

\[
\eta \rightarrow \pi^0 \gamma \gamma \rightarrow \pi^0 l^+ l^- \quad (l = e, \mu)
\]

which respects C and CP should therefore dominate [12]. A measured rate significantly greater than that implied by the dominance of (1.19) might indicate a contribution from (1.18). A measurement of an asymmetry in the energy spectra of the two leptons, however, would be a definite indication of C-violation [10]. This could arise, for example, from interference between the C-violating amplitude of (1.18) and the C-conserving amplitude of (1.19).

A test of CPT-invariance based on these decays has also been suggested [13]. CPT-invariance implies that the decay spectrum of \(\eta \rightarrow \pi^0 l^+ l^-\) can have only even powers of \(\cos \theta\) where \(\theta\) is the angle between the \(\pi^0\) and the \(l^+\) in the \(l^+ l^-\) centre-of-mass frame.

Because of the dominance of the mechanism (1.19) in these decays, a theoretical description of them must rely on knowledge of the decay (1.14). But (1.14) is interesting
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Figure 1.2: The C- and CP-violating one-photon intermediate state for $\eta \rightarrow \pi^0 l^+ l^-$. 

Figure 1.3: The C- and CP-conserving two-photon intermediate state for $\eta \rightarrow \pi^0 l^+ l^-$. 

in its own right. The calculations of its rate based upon the effective theories known as Vector Meson Dominance (VMD) and Chiral Perturbation Theory (ChPT) do not agree particularly well with the data: A recent calculation [14] combining these effects in the most favourable way gives

$$\Gamma_{\text{ChPT+VMD}}(\eta \rightarrow \pi^0 \gamma \gamma) = 0.42 \pm 0.20 \text{ eV},$$

whereas the experimental result is given by (1.17). Strictly, one cannot claim that these results are in disagreement, since the error ranges cause them nearly to meet, but neither do they inspire confidence in the validity of ChPT and VMD for the description of the decay (1.14). Thus, a study of this decay will be an important consideration in constructing an effective theory for the decays of light mesons.

1.3 Overview

Chapter 2 will give a description of VMD and ChPT (the two theories which were mentioned at the end of section 1.2), as well as a brief discussion of models which, in contrast to VMD and ChPT, make explicit use of quarks, thus setting the stage for the calculations of chapter 3.

Chapter 3 will begin by showing how a result similar to (1.20) can be obtained by a simple phenomenological approach, essentially VMD with the addition of other effects. Then the rate will be recalculated using a quark model, providing a result which is compatible with (1.17).

Chapter 4 will be concerned with the rare decays (1.15) and (1.16). A detailed description will be given of the use of the decay chain (1.19) to find a lower bound for the rates of (1.15) and (1.16) via the imaginary part of the amplitude for this decay chain. It will also be shown that knowledge of the rate of (1.14) alone is insufficient for the purpose of a model-free calculation of this unitarity bound. Additional information such
as the energy spectrum for the pion or one of the photons would be needed, but since only the overall rate has been measured, we must resort to the use of models to calculate the two form factors involved in the decay (1.14). The unitarity bound will be given for (1.15) and (1.16), using both the VMD model of section 3.2 and the quark loop model of section 3.3.

Chapter 4 will end with a discussion of the real part of the decay chain amplitude and an estimate of the actual decay rates for (1.15) and (1.16). Chapter 5 will consist of concluding comments.
Chapter 2

Theoretical Preliminaries

The purpose of this chapter is to describe the main theoretical models which are used in calculations of light meson decays, and thus to establish the theoretical background for the approaches used in the following chapters. Throughout this thesis $c = \hbar = 1$.

2.1 Vector Meson Dominance

The principle of Vector Meson Dominance (VMD) was introduced in 1962 by Gell-Mann, Sharp, and Wagner [15] as an explanation for the branching ratio $\Gamma(\omega \rightarrow \pi \gamma)/\Gamma(\omega \rightarrow 3\pi)$, and was subsequently applied to a variety of other pseudoscalar and vector meson decays, with a fair amount of success (Baracca and Bramon [16] compiled a list of results of VMD calculations for many light meson decays).

The main assumption is that the basic meson-meson and meson-photon couplings are all of the form $VV P$, $PP V$, and $V \gamma$, as shown in figure (2.1), where $P$ and $V$ respectively denote pseudoscalar and vector mesons. The strengths of these couplings are not specified by the theory.

In the approach of ref. [16], relationships between the various coupling constants were derived from a second assumption, namely, that of an exact flavour $SU(3)$ symmetry for the strong interactions. These relationships are
Figure 2.1: The basic vertices in the VMD theory

\[ g_{\eta_8 PP} = g_{\eta_8 \omega_8 \omega_8} = g_{\pi \omega_8} \]
\[ g_{\eta_1 PP} = g_{\eta_1 \omega_8 \omega_8} \]
\[ g_{\pi \omega_1} = g_{\eta_8 \omega_8 \omega_8} \] \hspace{1cm} (2.1)

Here \( \eta_8 \) and \( \eta_1 \) refer to the octet and singlet components of the wavefunction of the \( \eta \), and similarly \( \omega_8 \) and \( \omega_1 \) refer to the octet and singlet components of the wavefunction of the \( \omega \). These couplings are related to the real-particle couplings via the mixing formulae:

\[ |\eta\rangle = \cos \theta \eta |\eta_8\rangle - \sin \theta \eta |\eta_1\rangle \]
\[ |\eta'\rangle = \sin \theta \eta |\eta_8\rangle + \cos \theta \eta |\eta_1\rangle \] \hspace{1cm} (2.2)
\[ |\phi\rangle = \cos \theta \omega |\omega_8\rangle - \sin \theta \omega |\omega_1\rangle \]
\[ |\omega\rangle = \sin \theta \omega |\omega_8\rangle + \cos \theta \omega |\omega_1\rangle \] \hspace{1cm} (2.3)

For the couplings \( g_{\rho \gamma} \) and \( g_{\omega \gamma} \), the authors of ref. [16] make use of the arguments of Gell-Mann and Zachariasen [17] to get

\[ g_{\rho \gamma} = \frac{e M_{\rho}^2}{g_{\rho \pi \pi}} \]
\[ g_{\omega \gamma} = \frac{e M_{\omega_8}^2}{\sqrt{3} g_{\rho \pi \pi}} \] \hspace{1cm} (2.4)
and the measured rate of $\rho \rightarrow \pi\pi$ is used to get $g_{\rho\pi\pi}$. (Alternatively, $g_{\rho\gamma}$ and $g_{\omega\gamma}$ could be determined from the measured rates of the decays $\rho \rightarrow e^+e^-$ and $\omega \rightarrow e^+e^-$, which proceed through the intermediate state of one virtual photon.)

Given the mixing angles $\theta_\eta$ and $\theta_\omega$ (see section 1.1) the coupling constants in (2.1) are then determined from the measured rates of the decays $\pi^0 \rightarrow \gamma\gamma$, $\eta \rightarrow \gamma\gamma$, and $\omega \rightarrow \pi^0\gamma$.

In chapter 3 it will be shown that VMD is not adequate to account for the rate of (1.14) (the result is too low by a factor of almost 3), but the approach taken will differ from that outlined above. In particular the $SU(3)$ assumption will not be used; rather, all the relevant couplings will be determined directly from experiments. It will also be noticed that $PV\gamma$ vertices, which fall outside of the usual framework of VMD, will be included. This is because such a vertex is equivalent to the combination of a $PVV$ vertex and a $V\gamma$ vertex, connected by an internal vector meson line, while the fact that the photon is real fixes the value of the vector meson propagator. (See figure (2.2).)

Whereas VMD was introduced without theoretical justification, it is interesting that other models can provide some support for it; see the following two sections.
2.2 Chiral Perturbation Theory

Chiral Perturbation Theory is a low-energy effective field theory which is based directly on the Standard Model [18]. It proceeds from the observation that the QCD Lagrangian $\mathcal{L}_{\text{QCD}}$ is invariant under $C$, $P$, and the chiral group $G \equiv SU(3)_L \times SU(3)_R$. We can write the full light-quark Lagrangian as

$$
\mathcal{L} = \mathcal{L}_{\text{QCD}} + \bar{q} \gamma^\mu (v_\mu + \gamma^5 a_\mu) q - \bar{q} (\mathcal{S} - i\gamma^5 \mathcal{P}) q,
$$

where $q = \text{column}(u, d, s)$ (we neglect the heavy quarks), and the fields $\mathcal{S}, \mathcal{P}, v_\mu, a_\mu$ are matrix-valued and represent all non-QCD quark interactions. For calculational purposes, $v_\mu$ and $a_\mu$ take the values dictated by the electromagnetic and weak interactions,

$$
\begin{align*}
    r_\mu &= v_\mu + a_\mu = -e QA_\mu \\
    l_\mu &= v_\mu - a_\mu = -e QA_\mu - \frac{e}{\sqrt{2} \sin \theta_W} (W^+_\mu T_+ + \text{h.c.})
\end{align*}
$$

The quark masses are introduced by

$$
\mathcal{S} = \text{diag}(m_u, m_d, m_s),
$$

and $\mathcal{P}$ is set to 0. Naturally, when $v_\mu$, $a_\mu$, and $\mathcal{S}$ take these values, this will break the chiral symmetry of (2.5). But for purposes of constructing the effective Lagrangian of ChPT, the fields $v_\mu$, $a_\mu$, $\mathcal{S}$, and $\mathcal{P}$ are considered to be unspecified, and to preserve the chiral symmetry.

The effective Lagrangian is the most general chirally-invariant Lagrangian written in terms of meson fields instead of quark fields; the pseudoscalar mesons are written in
terms of $U = \exp(i\sqrt{2}(\Phi_1 + \Phi_8)/F)$ where

$$
\Phi_8 = \begin{pmatrix}
\pi^0 + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\
-\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 & \\
K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}}
\end{pmatrix}, \quad \Phi_1 = \frac{1}{\sqrt{3}} \eta_3 I_3,
$$

(2.8)

$I_3$ is the $3 \times 3$ identity matrix, and $F$ is a parameter about which more will be said shortly. The Lagrangian, as just described, will have an infinite number of terms. The “lowest order” part, in the sense described below, is

$$
\mathcal{L}_2 = \frac{F^2}{4} \text{Tr}(D_\mu U D^\mu U^\dagger + \chi U^\dagger + \chi^\dagger U)
$$

(2.9)

where $D_\mu U = \partial_\mu U - i\pi_\mu U + iU\pi_\mu$, $\chi = 2B_0(S + iP)$, and $B_0$ is another parameter.

The Lagrangian (2.9) provides the information needed to make calculations to the lowest order in ChPT. A calculation of pion decay shows that $F = F_\pi = 93.2\text{MeV}$, the pion decay constant, while an expansion of $\mathcal{L}_2$ to second order in $\Phi$ allows us to relate $B_0$ to the masses of the pseudoscalar meson octet:

$$
B_0 = \frac{M_{K^+}^2}{m_u + m_s} = \frac{M_{\pi^+}^2}{m_u + m_d}
$$

(2.10)

Now we address the question of in what sense this is the lowest order. ChPT is often described as an expansion in orders of momentum, but no specific momentum is referred to in such a description. The fields which appear in the Lagrangian, with the exception of $S$ and $P$, are simply counted according to their mass dimension, i.e. $U$ is zeroth order, whereas $D_\mu U$, $v_\mu$ and $a_\mu$ are first order. On the other hand, the fields $S$ and $P$ are counted as second order, rather than first, because, as (2.10) shows, the squares of the meson masses are proportional to the quark masses, which appear in the field $S$, to which the field $P$ is tied in the combined field $\chi$. So the lagrangian in (2.9) is entirely second order, hence the name $\mathcal{L}_2$. 

The full effective Lagrangian of ChPT is

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots \]  

where the terms are labeled by order in the sense defined above. \( \mathcal{L}_2 \) has already been discussed; the next piece is

\[ \mathcal{L}_4 = \frac{1}{\sqrt{2}} \tr(D_\mu U^\dagger D^\mu U) + \frac{1}{2} \tr(D_\mu U^\dagger D_\nu U^\dagger D^\nu U) \]

\[ + L_3 \tr(D_\mu U^\dagger D_\nu D_\sigma U^\dagger D^\nu U) + L_4 \tr(D_\mu U^\dagger D^\mu U) \tr(\chi^\dagger U + \chi U^\dagger) \]

\[ + L_5 \tr(D_\mu U^\dagger D^\mu U(\chi^\dagger U + U^\dagger \chi)) + L_6 \tr(\chi^\dagger U + \chi U^\dagger)^2 + L_7 \tr(\chi^\dagger U - \chi U^\dagger)^2 \]

\[ + L_8 \tr((\chi^\dagger U)^2 + (\chi U^\dagger)^2) - i L_9 \tr(F_{\mu \nu}^R D_\mu U D_\nu U^\dagger + F_{\mu \nu}^L D_\mu U^\dagger D_\nu U) \]

\[ + L_{10} \tr(U^\dagger F_{\mu \nu}^R U F_{L \mu \nu}) + L_{11} \tr(F_{R \mu \nu} F_{R}^\mu \nu + F_{L \mu \nu} F_{L}^\mu \nu) + L_{12} \tr(\chi^\dagger \chi) \]

where

\[ F_{\mu \nu}^R = \partial_{\mu} r_{\nu} - \partial_{\nu} r_{\mu} - i [r_{\mu}, r_{\nu}] \]

\[ F_{\mu \nu}^L = \partial_{\mu} l_{\nu} - \partial_{\nu} l_{\mu} - i [l_{\mu}, l_{\nu}] \]  

The \( L_{11} \) and \( L_{12} \) terms contain no meson fields so they do not concern us. The other ten parameters are determined phenomenologically, and it is found that they can be accounted for by considering the four-particle interactions implied by \( \mathcal{L}_4 \) to arise from the exchange of mesons. Also, vector mesons dominate in the cases in which they can contribute at all, thus providing some support for the principle of VMD.

A calculation to the next (fourth) order in ChPT must consider both tree-level diagrams derived from \( \mathcal{L}_4 \) and one-loop diagrams derived from \( \mathcal{L}_2 \). But that is not all; there is also the “chiral anomaly” represented by the Wess-Zumino term [19], an additional piece of the effective Lagrangian which violates the chiral symmetry and arises because of complications in the quantization of the classical chirally-invariant Lagrangian. This
Chapter 2. Theoretical Preliminaries

is very complex in its general form; as far as the \( \pi^0 \) and \( \eta \) are concerned it can be written [20]

\[
\mathcal{L}_{WZ} = \frac{\alpha}{8\pi F} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} (\pi^0 + \frac{\eta}{\sqrt{3}}).
\]

The Wess-Zumino term gives a good description of the two-photon decays of the pseudoscalar mesons \( \pi^0, \eta, \eta' \).

Chiral perturbation theory has success in other areas, including \( \pi\pi \) scattering and semileptonic \( K \) decays. However, as noted already in chapter 1, it runs into trouble with the decay (1.14). A calculation to fourth order [21] in ChPT gives

\[
\Gamma_{\text{ChPT}}(\eta \to \pi^0\gamma\gamma) = 0.0035\text{eV},
\]

more than two orders of magnitude below the experimental result. Adding the amplitude from ChPT to that of VMD (the VMD terms are sixth-order in the order-counting scheme of ChPT) gives a better result, but even this is too low. (The result (1.20) comes from these terms, plus a partial eighth-order ChPT calculation using two vertices from \( \mathcal{L}_{WZ} \).)

### 2.3 Quark-Model-Motivated Theories

Given the fundamental nature of quarks, it is natural to try to describe hadronic behaviour by making effective theories which make explicit use of quarks. These fall into two categories. In the first category are models in which quarks, instead of hadrons, are used for intermediate states, but they couple to the hadrons in the initial and/or final states. The meson-quark couplings are treated as experimental quantities. The second category (see, for example, [22]) consists of models in which the hadrons of the initial and/or final states are described as systems of quarks. These include (a) models in which the quarks are treated as nonrelativistic (or slightly relativistic) particles in a simple potential, (b) bag models, and (c) the "mock meson" method where the quarks are treated
as free, but with momenta which just happen to coincide. In the approach (a), some explicit form of wavefunction such as that of a harmonic oscillator or of a hydrogen atom is used; hence this approach is more explicit, in terms of hadron dynamics, than the calculations of the first category.

An early compilation of the results of using a simple quark model to describe hadronic properties is that of Van Royen and Weisskopf [23]. Using only the quark contents of the hadrons and the assumption of additivity, the magnetic moments of many hadrons were successfully calculated. Electromagnetic and weak decays of light mesons were also well described by making simple assumptions about the behaviour of the $q\bar{q}$ wavefunction at zero distance, in calculations of type (a) of the second category described above.

Calculations of the first category are more recent; an example of this is the use of the quark-triangle for the description of $P \rightarrow \gamma\gamma$ and $P \rightarrow \gamma l^+l^-$ ($P = \pi^0, \eta, \eta', l = e, \mu$) [24]. Good agreement with experiment is also obtained here. In addition the $P\gamma\gamma$ form-factor found by the quark-triangle agrees with that of the VMD model, for appropriate assumed values of the quark masses. Agreement with VMD is indeed a surprisingly frequent result of these quark-model-motivated effective theories; thus, VMD is given some further indirect justification.

More will be said about the quark-triangle in chapter 3, where a calculation of the rate of (1.14) will be made in an analogous model.

In successful calculations of both the first category and type (a) of the second, the quark masses which are used are the so-called "constituent" masses. These are to be differentiated from the "current" quark masses which presumably originate from symmetry breaking in the Standard Model. For the $u, d$ quarks the constituent masses are generally considered to be in the vicinity of $300$ MeV, approximately one third of the proton mass, whereas for the $s$ quark the mass used is always close to $500$ MeV [22],[24],[25].
Chapter 3

The Decay $\eta \rightarrow \pi^0\gamma\gamma$

This chapter is concerned with the calculation of the rate of the decay $\eta \rightarrow \pi^0\gamma\gamma$. The first section will describe some general characteristics of these calculations and establish notation. The second section will give the results for this rate, firstly under the assumption of VMD and then with the addition of the $a_0$ exchange; and the third section will give the quark-box calculation.

3.1 General Analysis

We begin by finding the most general form of the amplitude for $\eta \rightarrow \pi^0\gamma\gamma$ respecting parity and gauge invariance. The effective Lagrangian must be of the form

$$L_{\eta\pi\gamma} = \frac{A}{2} \eta \pi^0 F_{\mu\nu} F^{\mu\nu} - \frac{B}{M_\eta^2} (\partial^\mu \partial_\nu \eta) \pi^0 F_{\mu\rho} F^{\nu\rho} \quad (3.1)$$

Gauge invariance dictates that the photon field $A_\mu$ must appear in the combination $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and that derivatives such as $\partial^\mu F_{\mu\nu}$ must vanish. Momentum conservation implies that the momenta of the particles are not all independent; this, along with the fact that the Feynman rule corresponding to a derivative in the interaction term is a momentum, allows us to choose one field in the effective Lagrangian for which derivatives will not be considered. Here, the $\pi^0$ field has been chosen for this role. The antisymmetry of $F_{\mu\nu}$ means that expressions such as $(\partial_\mu \partial_\nu \eta) F^{\mu\nu}$ will vanish, while parity invariance disallows terms such as $\eta \pi^0 \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$. With these restrictions, it can be seen that (3.1) is the most general effective lagrangian for (1.14). $M_\eta$ is the mass of the $\eta$; it is
Chapter 3. The Decay $\eta \rightarrow \pi^0 \gamma \gamma$

included in the $B$ term so that $A$ and $B$ will have the same dimensions, i.e. mass$^{-2}$. The corresponding amplitude is (using the Feynman rule $\partial_\mu \rightarrow ip_\mu$)

$$\mathcal{M}(\eta \rightarrow \pi^0 \gamma \gamma) = -i \frac{A}{2} (k_1^{\mu} \epsilon_{1\nu} - k_1^{\nu} \epsilon_{1\mu}) (k_2^{\mu} \epsilon_2^{-\nu} - k_2^{\nu} \epsilon_2^{-\mu}) - i \frac{B}{M_\eta^2} P_\mu P_\nu (k_1^{\mu} \epsilon_{1\rho} - k_1^{\rho} \epsilon_{1\mu}) (k_2^{\nu} \epsilon_2^{-\mu} - k_2^{\mu} \epsilon_2^{-\nu})$$

(3.2)

where the $k$'s and $\epsilon$'s are respectively the 4-momenta and polarizations of the outgoing photons, and $P_\mu$ is the 4-momentum of the $\eta$. It is convenient to rewrite this as

$$\mathcal{M}(\eta \rightarrow \pi^0 \gamma \gamma) = i \epsilon_{1\mu} \epsilon_{2\nu} T^{\mu\nu},$$

(3.3)

where

$$T^{\mu\nu} = A(x_1, x_2) \left[ -k_1 \cdot k_2 g^{\mu\nu} + k_2^{\mu} k_1^{\nu} \right] + B(x_1, x_2) \left[ -M_\eta^2 x_1 x_2 g^{\mu\nu} - \frac{k_1 \cdot k_2}{M_\eta^2} P_\mu P_\nu + x_1 k_2^{\mu} P_\nu + x_2 P_\mu k_1^{\nu} \right],$$

(3.4)

and $x_i \equiv P \cdot k_i / M_\eta^2$. Here it is made explicit that the two form factors, $A$ and $B$, are in general unknown scalar functions of the momenta of the particles involved. The interdependence of the momenta in question implies that the form factors can be expressed as a function of $x_1$ and $x_2$ with no loss of generality; hence $A$ and $B$ are written in this way.

Using the fact that $\sum_{\text{polarizations}} \epsilon_{i\mu} \epsilon_{i\nu} = -g_{\mu\nu}$ it is easily seen that

$$|\mathcal{M}|^2 \equiv \sum_{\text{polarizations}} |\mathcal{M}|^2 = \frac{M_\eta^4}{8} \left\{ \left| 2A + B \right|^2 \left[ 2(x_1 + x_2) + \frac{M_\pi^2}{M_\eta^2} - 1 \right]^2 
+ \left| B \right|^2 \left[ 2(x_1 + x_2) - 4x_1 x_2 + \frac{M_\pi^2}{M_\eta^2} - 1 \right]^2 \right\}.$$  

(3.5)

The differential decay rate with respect to the two photons is given by

$$\frac{d^2\Gamma}{dx_1 dx_2} = \frac{M_\eta}{128\pi^3} |\mathcal{M}|^2.$$ 

(3.6)

The total decay rate is obtained by integrating this throughout the region of $(x_1, x_2)$ space allowed by momentum and energy conservation, i.e. the region bounded by

$$x_1 + x_2 - 2x_1 x_2 \leq \frac{M_\pi^2 - M_\eta^2}{2M_\eta^2} \leq x_1 + x_2$$

(3.7)
The next two sections will give results for this rate based upon two different models. It will be seen that the quark-box model gives a result consistent with experiment whereas VMD does not; however, it should be noticed that these models give different predictions for the form of the functions $A$ and $B$, and therefore a high-statistics measurement of this decay would place us in a better position to judge the validity of these models. For example, the differential decay rate in the region near the linear part of the Dalitz boundary (the second inequality in (3.7)) is highly sensitive to $B$, as can be seen from (3.5) and (3.6).

3.2 Phenomenological Intermediate-Meson Approach

The background to the principle of Vector Meson Dominance (VMD) has been described in the previous chapter. This principle is applied to the decay $\eta \to \pi^0\gamma\gamma$ as shown in figure (3.1). This calculation follows older ones, such as in refs. [16] and [26] (for another recent calculation, see [27]). The vector-pseudoscalar-photon couplings are taken from the measured rates of the decays $V \to \eta\gamma$ and $V \to \pi^0\gamma$, where $V$ refers to the vector meson.

By arguments similar to those in section 3.1, the effective Lagrangian for $V \to \eta\gamma$ must be of the form

$$\mathcal{L}_{V\eta\gamma} = \frac{g_{V\eta\gamma}}{2}\eta(\partial^\mu V^\nu)F_{\mu\nu}^{\rho\sigma}\epsilon_{\rho\sigma}, \quad (3.8)$$

so the amplitude is

$$\mathcal{M}(V \to \eta\gamma) = ig_{V\eta\gamma}p^\mu q^\nu\epsilon_{\mu\nu\rho\sigma}\epsilon_\rho^{(V)}\epsilon_\sigma^{(\gamma)} \quad (3.9)$$

where $p^\mu$ and $q^\mu$ are the 4-momenta of the vector meson and the photon, respectively,
Chapter 3. The Decay $\eta \to \pi^0\gamma\gamma$

Figure 3.1: The VMD model applied to the decay $\eta \to \pi^0\gamma\gamma$.

and $\epsilon^{\mu}_\gamma$ and $\epsilon^{\nu}_\eta$ are their polarization vectors. The decay rate is

$$\Gamma(V \to \eta\gamma) = \frac{|g_{V\eta\gamma}|^2 M_V^3}{96\pi} (1 - \frac{M_\eta^2}{M_V^2})^3,$$

(3.10)

and similar results hold for $V \to \pi^0\gamma$.

From the most recent data [2], the couplings are

$$g_{\rho\eta\gamma} = 5.7^{+0.5}_{-0.6} \times 10^{-4} \text{ MeV}^{-1},$$
$$g_{\rho\pi\gamma} = (3.0 \pm 0.4) \times 10^{-4} \text{ MeV}^{-1},$$
$$g_{\omega\eta\gamma} = (1.4 \pm 0.3) \times 10^{-4} \text{ MeV}^{-1},$$
$$g_{\omega\pi\gamma} = 7.0^{+0.3}_{-0.2} \times 10^{-4} \text{ MeV}^{-1},$$
$$g_{\phi\eta\gamma} = 2.12^{+0.06}_{-0.07} \times 10^{-4} \text{ MeV}^{-1},$$
$$g_{\phi\pi\gamma} = 4.2^{+0.2}_{-0.3} \times 10^{-5} \text{ MeV}^{-1}. \hspace{1cm} (3.11)$$

Note that the decay data say nothing about the phases of the couplings. For purposes of applying VMD to the decay (1.14), it is assumed that the product $g_{V\pi\gamma}g_{V\pi\gamma}$ has the same phase for each vector meson type. It is simplest, then, to consider all of these couplings to be real and positive.
For the decay (1.14), the first diagram of figure (3.1) provides an amplitude of

$$M_1 = -ig_{\eta \gamma}g_{\pi \gamma}p^{\alpha}k_1^{\beta}e_{\alpha \beta \mu} \left( \frac{g_{\rho \rho} - p^\rho p^\rho / M_V^2}{p^2 - M_V^2} \right) p^\lambda k_2^\epsilon \epsilon_{\lambda \sigma \nu} e^{\epsilon} e_2^\nu,$$

$$= ig_{\eta \gamma}g_{\pi \gamma} \left[ \frac{(M_n^2 - P \cdot k_1)(k_1 \cdot k_2 g_{\mu \nu} - k_{2\mu} k_{1\nu})}{M_n^2 - 2P \cdot k_1 - M_V^2} \right. - \left. \frac{P \cdot k_1 P \cdot k_2 g_{\mu \nu} + k_1 \cdot k_2 p_\mu p_\nu - P \cdot k_1 k_{2\mu} p_\nu - P \cdot k_2 p_\mu k_{1\nu}}{M_n^2 - 2P \cdot k_1 - M_V^2} \right],$$

using $p = P - q_1$. The second diagram is the same, but with the exchanges $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$. Comparing this with (3.4) shows that the two form factors for (1.14) are given by

$$A_{VMD} = -\sum_V g_{\eta \gamma}g_{\pi \gamma} \left( \frac{1 - x_1}{1 - 2x_1 - M_V^2 / M_n^2} + \frac{1 - x_2}{1 - 2x_2 - M_V^2 / M_n^2} \right),$$

$$B_{VMD} = \sum_V g_{\eta \gamma}g_{\pi \gamma} \left( \frac{1}{1 - 2x_1 - M_V^2 / M_n^2} + \frac{1}{1 - 2x_2 - M_V^2 / M_n^2} \right),$$

where the sum is over the various types of vector mesons. As can be seen from the $M_V^2 / M_n^2$ term in each denominator, only the lowest-mass vector mesons contribute significantly. Here we include the first three: $\rho, \omega, \phi$.

The width for (1.14) from VMD is found by numerically integrating (3.6) as discussed in section 3.1, and using (3.13) for the form factors. The result is

$$\Gamma_{VMD}(\eta \to \pi^0 \gamma \gamma) = 0.29^{+0.18}_{-0.12} \text{eV}. \quad (3.14)$$

This shows that VMD is insufficient to account for the experimental rate of (1.14).

It is useful to take a closer look at what has gone into this result. One could have obtained a similar result without reference to the principle of VMD, simply by looking for any mesons which couple to $\pi^0$'s, $\eta$'s, or photons in such a way that a simple (tree) Feynman diagram which contributes to (1.14) can be constructed from them. The vector mesons $\rho$ and $\omega$ are the two lightest mesons with this property, and it is natural to expect the lightest intermediate mesons to contribute the most to the amplitude. Indeed, if we
use the first two vector mesons alone, the calculated rate is $0.28^{+0.15}_{-0.11}\text{eV}$, which is not much less than (3.14). So mesons of mass greater than $M_\phi$ are not expected to contribute significantly. However, there is one other meson which has not yet been considered, whose mass is less than $M_\phi$ and which has the desired property. This is the scalar $a_0$ meson, which in its neutral version has been observed to decay into $\eta\pi^0$ and into two photons, thus providing another mechanism for (1.14) as shown in figure (3.2).

The $a_0$ has $J^{PC} = 0^{++}$; it also has isospin 1 and comes in three charges, $a_0^\pm$ and $a_0^0$. Its mass is $(982.7 \pm 2.0)\text{ MeV}$ and its full width is $(57 \pm 11)\text{ MeV}$ [2]. The observed decay modes are $a_0 \rightarrow \eta\pi$, $a_0 \rightarrow K\bar{K}$, and $a_0^0 \rightarrow \gamma\gamma$. The quantum numbers of the $a_0$ are such that it is most natural to assume that it is the $S = 1$, $L = 1$, $J = 0$ counterpart of the $\pi$ or $\rho$ mesons (see section 1.1). This description of the $a_0$ is not without controversy; some argue that it is an exotic $qq\bar{q}\bar{q}$ state (see, for example, [28]). One variation on the $qq\bar{q}\bar{q}$ theme is that of a $K\bar{K}$ “molecule” [29].

Arguments similar to those which were used for the vector mesons show that the $a_0$ mechanism contributes to $\eta \rightarrow \pi^0\gamma\gamma$ only by adding to $A$ the term

$$A_{a_0} = \frac{g_{a_0\pi}g_{a_0\gamma}/M_\eta^2}{M_\eta^2/M_\pi^2 - M_\rho^2/M_\eta^2 + 1 - 2(x_1 + x_2)}. \quad (3.15)$$
(It does not change $B$.) The couplings are taken from the measured rates of the decays $a_0 \rightarrow \gamma\gamma$ and $a_0 \rightarrow \eta \pi^0$, by

\[
\Gamma(a_0 \rightarrow \gamma\gamma) = \frac{|g_{a\gamma\gamma}|^2 M_a^3}{64\pi} \tag{3.16}
\]

\[
\Gamma(a_0 \rightarrow \eta \pi^0) = \frac{|g_{a\eta\pi}|^2}{16\pi M_a^3} \sqrt{\lambda(M_{a_0}^2, M_{a_0}^2, M_{\pi}^2)} \tag{3.17}
\]

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$. Neither of these rates has been measured directly, but conveniently we have the full width (given above) as well as the quantity $[2]$

\[
\frac{\Gamma(a_0 \rightarrow \eta \pi^0) \Gamma(a_0 \rightarrow \gamma\gamma)}{\Gamma(a_0 \rightarrow \text{all})} = 0.24^{+0.08}_{-0.07} \text{keV}. \tag{3.18}
\]

This indicates that the product of the couplings is of the magnitude

\[
g_{a\eta\pi}g_{a\gamma\gamma} = 0.015 \pm 0.004. \tag{3.19}
\]

Again, we do not know the phases of these couplings. If we consider the couplings to be real and positive as written in (3.19), and write

\[
A = A_{\text{VMD}} + e^{i\delta}A_{a_0} \tag{3.20}
\]

(introducing the unknown phase $\delta$), then we can work out the width of (1.14) for various assumed values of $\delta$. The most interesting cases are $\delta = 0$ (maximally constructive interference) and $\delta = \pi$ (maximally destructive interference). The following results are obtained:

\[
\Gamma_{\delta=0}(\eta \rightarrow \pi^0\gamma\gamma) = 0.37^{+0.20}_{-0.15} \text{eV}, \tag{3.21}
\]

\[
\Gamma_{\delta=\pi}(\eta \rightarrow \pi^0\gamma\gamma) = 0.23^{+0.15}_{-0.11} \text{eV}. \tag{3.22}
\]

Even the maximally constructive interference case gives a result lower than the experiment by a factor of 2.3. It appears, then, that this approach of inserting all possible intermediate mesons is inadequate. An alternate mechanism is proposed in the next section.
3.3 The Quark-Box Diagram

In this section the rate of the reaction $\eta \to \pi^0 \gamma \gamma$ is calculated according to the quark box diagram of figure (3.3). The calculation follows that of ref. [30]. The parameters which need to be considered are the quark masses and the meson-quark-quark couplings.

The masses of the $u$ and $d$ quarks have been varied from 280 MeV to 330 MeV, a span which covers the range of previously successful quark-model descriptions of meson processes (see section 2.3). For simplicity we keep $m_d = m_u$, although the calculation could be easily generalized to the case of unequal masses. The results are quite sensitive to these masses, as will be seen, but comparatively insensitive to the $s$ quark mass, which affects the result only through its effect on the $\eta$-quark-quark coupling (see below). We have held the $s$ quark mass at 500 MeV.

The effective Lagrangian term for the $\eta$-quark-quark interaction is assumed to be of the form

$$\mathcal{L}_{\eta q \bar{q}} = g_{\eta q \bar{q}} \gamma^5 q,$$

and similarly for the $\pi^0$-quark-quark interaction. The couplings are fixed by the measured values of the decays $\eta \to \gamma \gamma$ and $\pi^0 \to \gamma \gamma$. For this purpose we assume that the quark triangle diagram of figure (3.4) is the mechanism for these decays, and that the couplings for each meson type depend on the quark flavour only by being proportional to the coefficient of the $q\bar{q}$ term corresponding to that flavour in the quark wavefunction.

So the pion,

$$|\pi^0\rangle = \frac{1}{\sqrt{2}} (|u\bar{u}\rangle - |d\bar{d}\rangle),$$

has equal and opposite couplings to the two lightest flavours of quarks, i.e. $g_{\pi uu} = -g_{\pi dd}$.

In the case of the $\eta$ we must take into account the $\eta - \eta'$ mixing. The $\eta$ wavefunction is

$$|\eta\rangle = \frac{\cos \theta_\eta}{\sqrt{6}} (|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) - \frac{\sin \theta_\eta}{\sqrt{3}} (|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle)$$
Figure 3.3: The quark box as a mechanism for $\eta \rightarrow \pi^0\gamma\gamma$. 
which, using $\theta_\eta = -20^\circ$ (see section 1.1) gives

$$|\eta| \approx 0.58(|u\bar{u}| + |d\bar{d}|) - 0.57|s\bar{s}|$$

(3.26)

so $g_{\eta u\bar{u}} = g_{\eta d\bar{d}} = -\frac{0.58}{0.57}g_{\eta s\bar{s}}$.

The contribution of each quark type to the amplitude of $P \rightarrow \gamma\gamma$ ($P = \eta, \pi^0$) is

$$A_q \equiv H_q \epsilon_{\mu\nu\rho\sigma} \epsilon_1^\nu \epsilon_2^\rho k_1^\sigma k_2^\sigma,$$

(3.27)

where $\epsilon_i$ and $k_i$ ($i = 1, 2$) are respectively the polarization vectors and 4-momenta of the photons and $H_q$ is the form factor given by

$$H_q = -\frac{2\alpha Q_\pi^2 g_{P\pi}}{\pi} \frac{m}{M_P^2} \int_0^1 dt \frac{1 - 2t}{a - t + t^2} \ln(1 - t)$$

(3.28)

(where $a = \frac{m^2}{M_P^2}$). Using the (experimental) decay rates [2]

$$\Gamma(\pi^0 \rightarrow \gamma\gamma) = 7.7^{+0.6}_{-0.3} \text{ eV and}$$

$$\Gamma(\eta \rightarrow \gamma\gamma) = 0.46 \pm 0.05 \text{ keV},$$

(3.29)

(3.30)

we get

$$g_{\eta u\bar{u}} = 3.19 \pm 0.11$$

(3.31)

$$g_{\eta d\bar{d}} = 1.26 \pm 0.06$$

(3.32)
Table 3.1: The effect of varying the constituent quark mass \( m \) on the meson-quark-quark couplings and the rate of \( \eta \to \pi^0\gamma\gamma \).

<table>
<thead>
<tr>
<th>( m ) (MeV)</th>
<th>( g_{\pi^0qq} )</th>
<th>( g_{\eta qq} )</th>
<th>( \Gamma(\eta \to \pi^0\gamma\gamma) ) (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>280</td>
<td>2.96±0.11</td>
<td>0.95±0.04</td>
<td>0.97±0.16</td>
</tr>
<tr>
<td>300</td>
<td>3.19±0.11</td>
<td>1.26±0.06</td>
<td>0.70±0.12</td>
</tr>
<tr>
<td>330</td>
<td>3.52±0.13</td>
<td>1.62±0.07</td>
<td>0.60±0.10</td>
</tr>
</tbody>
</table>

using a \( u, d \) constituent quark mass of 300 MeV. The values of these couplings for other values of the quark mass are given in Table (3.1). The coupling \( g_{\eta uu} \) depends, of course, on the \( \eta - \eta' \) mixing angle. The number given is for \( \theta_{\eta} = -20^\circ \). Using \( \theta_{\eta} = -10^\circ \) will increase \( g_{\eta uu} \) by 4%.

Turning now to the box diagram, the tensor \( T^{\mu\nu} \) (see (3.3)) is given by

\[
T^{\mu\nu} = \sum_f \sum_{j=1}^6 T_{(f)j}^{\mu\nu},
\]

where

\[
T_{(f)j}^{\mu\nu} = i3e^2Q_q^2g_{\eta qq}g_{\pi qq} \int \frac{d^4q}{(2\pi)^4} U_{(f)j}^{\mu\nu},
\]

\[
U_{(f)1}^{\mu\nu} = \text{Tr}[\gamma_5(\not{q}+m)\gamma_5(\not{q}+\not{P}-\not{k}_1-\not{k}_2+m)\gamma^\nu(\not{q}+\not{P}-\not{k}_1+m)\gamma^\mu(\not{q}+\not{P}+m)]
\]
\[
(\not{q}^2-m^2)((q+P-k_2)^2-m^2)((q+P-k_1)^2-m^2)((q+P)^2-m^2),
\]

\[
U_{(f)2}^{\mu\nu} = \text{Tr}[\gamma_5(\not{q}+m)\gamma^\nu(\not{q}+\not{k}_2+m)\gamma_5(\not{q}+\not{P}-\not{k}_1+m)\gamma^\mu(\not{q}+\not{P}+m)]
\]
\[
(\not{q}^2-m^2)((q+k_2)^2-m^2)((q+P-k_1)^2-m^2)((q+P)^2-m^2),
\]

\[
U_{(f)3}^{\mu\nu} = \text{Tr}[\gamma_5(\not{q}+m)\gamma^\nu(\not{q}+\not{k}_1+\not{k}_2+m)\gamma_5(\not{q}+\not{P}-\not{m})\gamma_5(\not{q}+\not{P}+m)]
\]
\[
(\not{q}^2-m^2)((q+k_1+k_2)^2-m^2)((q+P-k_1)^2-m^2)((q+P)^2-m^2),
\]

\[
U_{(f)4}^{\mu\nu} = U_{(f)1}(k_1 \leftrightarrow k_2),
\]

\[
U_{(f)5}^{\mu\nu} = U_{(f)2}(k_1 \leftrightarrow k_2),
\]

\[
U_{(f)6}^{\mu\nu} = U_{(f)3}(k_1 \leftrightarrow k_2),
\]

\[
(3.35)
\]

\( m \) is the quark mass, and the index \( f \) refers to the quark flavours. (The slash notation refers to contraction with the Dirac matrices \( \gamma^\mu \); thus, \( \not{q} \equiv q_\mu \gamma^\mu \).)
These integrals need not be calculated in their entirety. The tensor $T_{\mu\nu}$ must, by gauge invariance, be of the form shown in (3.4). The form factors $A$ and $B$ contain all the information we need, and in order to extract these from equations (3.35) we need only find the coefficients of $P_{\mu}P^{\nu}$ and $g_{\mu\nu}$. Therefore, in calculating the traces, we need only keep the $g_{\mu\nu}$, $P_{\mu}P^{\nu}$, $P_{\mu}q^{\nu}$, $q_{\mu}P^{\nu}$, and $q_{\mu}q^{\nu}$ terms. For those who are interested, the mathematical details of this procedure are provided in an Appendix A.

In the appendix, $A$ and $B$ are written as double (at most) integrals of Feynman parameters. To evaluate them analytically will involve hundreds of Spence functions and this is deemed to be of dubious value. Instead they were evaluated numerically and $A$ and $B$ were fitted by third degree polynomial functions (using Mathematica). Within the Dalitz boundary for this decay, which is given by (3.7), a good fit to $A$ and $B$ is found to be given by

$$\frac{Q^2}{M_\eta^2} A(x_1, x_2) = -0.616 + 2.14(x_1 + x_2) - 2.509(x_1^2 + x_2^2) - 4.184x_1x_2 + 1.5896(x_1^3 + x_2^3) + 2.936x_1x_2(x_1 + x_2)$$

$$B(x_1, x_2) = -0.866 + 1.674(x_1 + x_2) - 3.260(x_1^2 + x_2^2) - 1.781x_1x_2 + 2.370(x_1^3 + x_2^3) + 1.089x_1x_2(x_1 + x_2)$$

where the quantities $A$ and $B$ are expressed in units of $10^{-6}\text{MeV}^{-2}$. ($Q = k_1 + k_2$ as in the Appendix.) Expressions (3.36) and (3.37) are valid for a $u, d$ constituent quark mass of $m = 300\text{MeV}/c^2$. Since $x_1$ and $x_2$ are both small numbers, it is not necessary to go beyond the third degree in the numerical fit.

As usual, the cross section is given by integrating (3.6) within the region bounded by (3.7). Using the functions for $A$ and $B$ given in (3.36) and (3.37) greatly reduces the amount of computer time required to perform the final phase space integrals over $x_1$ and $x_2$. With this quark-model approach, the decay rate for $\eta \rightarrow \pi^0\gamma\gamma$ is found to be

$$\Gamma(\eta \rightarrow \pi^0\gamma\gamma) = 0.70 \pm 0.12\text{eV}$$
for $m = 300\text{MeV}/c^2$. This result is somewhat sensitive to $m$; the results for other reasonable values of the constituent quark mass are given in table (3.1).

Throughout this calculation, the masses $m_d$ and $m_u$ have been kept equal. In fact, $m_d > m_u$, since the neutron ($|uud\rangle$) is more massive than the proton ($|uud\rangle$) by 1.3 MeV. If these quarks had equal mass, then the proton would be more massive than the neutron, due to the difference in electromagnetic binding energy. This difference would be 1 or 2 MeV; the calculated value depends on what assumptions one uses regarding the distribution of charge. This indicates that the $d$ quark is more massive than the $u$ by roughly 3 MeV.

For purposes of the quark-box calculation, this mass difference is unimportant. Since the $u$ and $d$ quark contributions are added to each other with a relative weight of 4:1, it follows that at small $\Delta m$, $\partial \Gamma / \partial (\Delta m) = -\frac{3}{10} \partial \Gamma / \partial m_{av}$, where $m_{av} = \frac{1}{2}(m_d + m_u)$ and $\Delta m = m_d - m_u$. Judging from table (3.1), a change of 3 MeV in $m_{av}$ causes at most a 5% change in $\Gamma$, so a change of 3 MeV in $\Delta m$ will change the result by less than 2%.

In principle, there are further tests one can use to distinguish between this model and the VMD model. One such test is the measurement of the dependence of $A$ and $B$ on $x_1$ and $x_2$. In figure (3.5) the form factor $A$ is shown as a function of $x_2$ for fixed values of $x_1$. The slope of $A$ is greater, in comparison with the magnitude of $A$, for VMD than for the quark-box model. Similar comparisons are given for $B$ in figure (3.6), but it is harder to distinguish between VMD and the quark-box model in the case of $B$. 
Figure 3.5: The form-factor $A$ as a function of $x_2$ for various values of $x_1$: (a) $x_1 = 0.1$, (b) $x_1 = 0.2$, (c) $x_1 = 0.3$, (d) $x_1 = 0.4$. The line which comes from VMD is marked as such; the others are from the quark-box model, with the value of the mass $m$ (in MeV) marked.
Chapter 3. The Decay $\eta \rightarrow \pi^0\gamma\gamma$

(c)
Chapter 3. The Decay $\eta \rightarrow \pi^0\gamma\gamma$
Figure 3.6: The form-factor $-B$ as a function of $x_2$ for various values of $x_1$: (a) $x_1 = 0.1$, (b) $x_1 = 0.2$, (c) $x_1 = 0.3$, (d) $x_1 = 0.4$. The line which comes from VMD is marked as such; the others are from the quark-box model, with the value of the mass $m$ (in MeV) marked.
Chapter 3. The Decay $\eta \rightarrow \pi^0 \gamma \gamma$
Chapter 3. The Decay $\eta \rightarrow \pi^0 \gamma \gamma$

(d)
Chapter 4

The Decays $\eta \rightarrow \pi^0 e^+ e^-$ and $\eta \rightarrow \pi^0 \mu^+ \mu^-$

It has been pointed out already that the rare decays (1.15) and (1.16) are dominated by the C-conserving two-photon mechanism (1.19), assuming no new physics manifests itself in these modes. One might, then, consider using the Feynman diagram of figure (4.1) to calculate the amplitudes for these decays. The difficulty with this is that the amplitude, as calculated this way, diverges, as can be seen from power-counting. (Notice that (3.4) implies that the $\eta \pi^0 \gamma \gamma$ vertex includes two powers of $k$.) Furthermore, there is no obvious way to deal with this divergence. Nor should we expect there to be one; for we are dealing with an effective theory and not a fundamental one.

It happens that the imaginary part of the amplitude converges, i.e. the divergence is contained entirely in the real part. In section 4.1 the imaginary part of the amplitude will be calculated, and rates for (1.15) and (1.16) will be derived from this, treating the imaginary part of the amplitude as if it is the whole amplitude. This will naturally give a lower bound (the "unitarity limit"), since

$$(\text{Im} \mathcal{M})^2 \leq \mathcal{M}^2 = (\text{Im} \mathcal{M})^2 + (\text{Re} \mathcal{M})^2.$$  \hfill (4.1)$$

Naturally the true physical amplitude must be finite; in section 4.2 a rough estimate will be made of the relative sizes of the real and imaginary parts using a dispersion integral with a cutoff. The calculations in this chapter follow ref. [31].
4.1 The Unitarity Limit for the Decays $\eta \to \pi^0 l^+ l^-$

It is well known (see, for example, [32]) that the imaginary part of an amplitude can be found by constraining the particles of the intermediate state to be on their respective mass-shells. In this case the two photons are constrained by replacing the usual photon propagator $-i g_{\mu\nu}/(p_\gamma^2 + i\epsilon)$ by $-2\pi g_{\mu\nu}\delta(p_\gamma^2)$. By this method the imaginary part of the amplitude is found to be

$$\text{Im}\mathcal{M}(\eta \to \pi^0 l^+ l^-)|_{2\gamma} = -\alpha^3 \int \frac{d^4 k}{(2\pi)^4} \delta(k^2)\delta((k - Q)^2) \left[ -2\lambda m_{l} s f_+ \bar{u} \gamma^\nu v 
\right. \\
-\frac{B}{2} \left( \frac{4}{M_\eta^2} f_+ k \cdot P (s + M_\pi^2 - M_\eta^2 - 2k \cdot P) m_{l} \bar{u} \gamma^\nu v 
\right. \\
+ \frac{1}{M_\eta^2} \left( f_+(t_- - t_+) (s + M_\eta^2 - M_\pi^2 - 4k \cdot P) - f_- [8(k \cdot P)^2 
-4k \cdot P (s + M_\eta^2 - M_\pi^2) + (M_\eta^2 - M_\pi^2)^2 + s(s - 2M_\pi^2)] \right) \bar{u} \gamma^\nu \gamma^\rho v \\
- \frac{s}{M_\eta^2} \left( (t_- - t_+) f_+ + f_- [4k \cdot P - (s + M_\eta^2 - M_\pi^2)] \right) \bar{u} \gamma^\rho v \right)$$

(4.2)

Figure 4.1: The two-photon intermediate state used in the calculation of the unitarity limit for $\eta \to \pi^0 l^+ l^-$. 
where
\[ s = Q^2, \]
\[ Q = p_+ + p_-, \]
\[ t_{\pm} = (P - p_{\pm})^2, \]
\[ f_{\pm} = \frac{1}{k \cdot p_-} \pm \frac{1}{k \cdot p_+}, \] (4.3)
and the kinematics are defined in figure (4.1). A and B are the form factors which were discussed in chapter 3.

Equation (4.2) shows that knowledge of the form factors A and B is necessary for this calculation. It would be ideal to have detailed experimental information, such as the Dalitz plot density, which is related to the form factors by (3.6). Such information could realistically become available at Saturne (see chapter 1). Presently, the only experimental knowledge we have of the decay (1.14) is its overall rate, so this unitarity limit calculation can only proceed if we make some assumptions about A and B. The models of chapter 3 will be used.

These form factors depend nontrivially on k, but for the present purpose it is sufficiently accurate to approximate them as being independent of the integrated momentum. Performing the integration of (4.2) gives

\[
\text{Im}\mathcal{M} = -\frac{\alpha}{4} \left\{ A M \bar{u}v + \frac{B}{2} \left[ \left( \frac{1}{3\beta^4} [-2 + 20b + 3(1 - 8b + 8b^2)L] \right) m_t \bar{u}v \right. \right. \\
\left. \left. + \frac{4(P \cdot Q)^2}{3s M_n^2 \beta^6} [1 + 6b + 20b^2 - 6b(1 + 4b^2)L] \right) m_t \bar{u}v \right. \right. \\
\left. \left. - \frac{8(P \cdot p_+)(P \cdot p_-)}{3s M_n^2 \beta^6} \left[ 1 + 26b - 12b(1 + b)L \right] m_t \bar{u}v \right. \right. \\
\left. \left. + \frac{4P \cdot (p_- - p_+)}{3M_n^2 \beta^4} [1 - b - 3b(1 - 2b)L] \bar{u} \not{r} \not{v} \right) \right\} \] (4.4)

where
\[ b = \frac{m_i^2}{s}, \quad \beta = \sqrt{1 - 4b}, \quad L = \frac{1}{\beta} \ln \frac{1 + \beta}{1 - \beta}. \] (4.5)
Chapter 4. The Decays $\eta \to \pi^0 e^+e^-$ and $\eta \to \pi^0 \mu^+\mu^-$

The differential cross-section with respect to the $\pi^0$ energy in the centre-of-mass frame of the $\eta$ is obtained by squaring (4.4) and doing some phase-space integrals. It is found that

$$\frac{d\Gamma}{dE_\pi} = \frac{\alpha_\pi^2}{256M_\eta\pi^3} |p_\pi| \left[ C_1 \frac{m^2_\pi^2 \beta^2}{2} - \frac{Xm^2_\eta}{2M^2_\eta} C_1 C_3 \right]$$

$$-\frac{sX}{8M^2_\eta} C_3^2 + \frac{1}{12} \left( \frac{\beta^2 s}{M^2_\eta} + \frac{X}{2M^4_\eta} (1 + 2b) \right)$$

$$\times \left( C_1 C_2 m^2_\pi^2 \beta^2 + 8C_1 C_3 m^2_\eta M^2_\eta - \frac{Xm^2_\eta}{2M^2_\eta} C_2 C_3 + 2s M^2_\eta C_2^2 + \frac{X}{2} C_3^2 \right)$$

$$+ \frac{1}{240} \left( \frac{3\beta^4 s^2}{M^4_\eta} + \frac{\beta^2 sX}{M^6_\eta} (1 + 6b) + \frac{X^2}{2M^8_\eta} (1 + 2b + 6b^2) \right)$$

$$\times \left( \frac{C_2 m^2_\pi^2 \beta^2}{2} + 8C_2 C_3 m^2_\eta M^2_\eta - 8M^4_\eta C_3^2 \right), \quad (4.6)$$

where

$$C_1 = AL + B \left( \frac{1}{6\beta^4} [-2 + 20b + 3(1 - 8b + 8b^2)L] \right)$$

$$+ \frac{X}{6s M^2_\eta \beta^6} [1 + 6b + 20b^2 - 6b(1 + 4b^2)L],$$

$$C_2 = -\frac{4BM^2_\eta}{3s \beta^6} [1 + 26b - 12b(1 + b)L],$$

$$C_3 = \frac{2B}{3\beta^4} [1 - b - 3b(1 - 2b)L],$$

$$X = (s + M^2_\eta - M^2_\pi)^2. \quad (4.7)$$

The integration over $E_\pi$ is done numerically. In the case of VMD, the form factors $A$ and $B$ (given in (3.13)) are expanded in terms of $x_1$, $x_2$ and $M^2_\eta/M^2_\pi$; at next-to-leading order, whenever $x_1$ and $x_2$ appear, they appear together in the combination $x_1 + x_2 = 1 - E_\pi/M_\eta$, so at this order the form-factors are indeed independent of $k$ (which was assumed in the calculation of (4.4)). The expanded form-factors are
Table 4.1: The unitarity limit for $\eta \to \pi^0e^+e^-$ and $\eta \to \pi^0\mu^+\mu^-$ according to the quark-box mechanism, for various values of the quark mass $m$.

<table>
<thead>
<tr>
<th>$m$ (MeV)</th>
<th>$\Gamma(\eta \to \pi^0e^+e^-)$ ($\mu$eV)</th>
<th>$\Gamma(\eta \to \pi^0\mu^+\mu^-)$ ($\mu$eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>280</td>
<td>7.3±1.2</td>
<td>3.9±0.7</td>
</tr>
<tr>
<td>300</td>
<td>2.9±0.5</td>
<td>4.3±0.7</td>
</tr>
<tr>
<td>330</td>
<td>1.2±0.2</td>
<td>4.3±0.7</td>
</tr>
</tbody>
</table>

$$A_{\text{VMD}} \approx \sum_{\nu} g_{\nu\gamma} g_{\nu} \frac{M_{\nu}^2}{M_{\nu}^2} \left( 1 + \frac{E_\pi}{M_\eta} + 2 \frac{M_{\pi}}{M_\nu} \right),$$

$$B_{\text{VMD}} \approx 2 \sum_{\nu} g_{\nu\gamma} g_{\nu} \frac{M_{\nu}^2}{M_{\nu}^2} \left( 1 + \frac{M_{\nu}^2}{M_\nu^2} \right).$$ (4.8)

The numerical integration of (4.6) yields

$$\Gamma(\eta \to \pi^0e^+e^-)|_{\text{VMD}} \geq 2.4 \pm 0.8 \mu\text{eV} \quad \text{and}$$

$$\Gamma(\eta \to \pi^0\mu^+\mu^-)|_{\text{VMD}} \geq 3.5 \pm 0.8 \mu\text{eV}. \quad (4.9)$$

For the quark-box mechanism of section 3.3, the form-factors, in their approximate form as given by (3.36) and (3.37) (for which the $u, d$ quark masses were taken to be 300 MeV) were merely averaged over the region allowed by (3.7), i.e. the Dalitz boundary for $\eta \to \pi^0\gamma\gamma$. Thus, for purposes of integrating (4.6), $A$ and $B$ were treated as constants. The results are

$$\Gamma(\eta \to \pi^0e^+e^-)|_{\text{box}} \geq 2.9 \pm 0.5 \mu\text{eV} \quad \text{and}$$

$$\Gamma(\eta \to \pi^0\mu^+\mu^-)|_{\text{box}} \geq 4.3 \pm 0.7 \mu\text{eV}. \quad (4.10)$$

For other values of the quark mass, see table (4.1). The $\pi^0e^+e^-$ mode is seen to be much more sensitive to the constituent quark mass than the $\pi^0\mu^+\mu^-$ mode.
Chapter 4. The Decays $\eta \rightarrow \pi^0 e^+ e^-$ and $\eta \rightarrow \pi^0 \mu^+ \mu^-$

The $\pi^0 e^+ e^-$ mode is quite insensitive to the terms involving $A$, since $A$ contributes primarily to the part of the phase-space in which the $\pi^0$ is nearly stationary and the $e^+$ and $e^-$ are nearly back-to-back, which leads to helicity suppression. The $\pi^0 \mu^+ \mu^-$ mode, on the other hand, involves interference between $A$ and $B$, which turns out to overcome the phase-space suppression enough so that $\Gamma(\eta \rightarrow \pi^0 \mu^+ \mu^-) > \Gamma(\eta \rightarrow \pi^0 e^+ e^-)$, except in the case of the quark-box model where $m$ is in the lower part of the range which was considered (see table (4.1)).

The spectrum (4.6) is presented in figure (4.2), for both the $\pi^0 e^+ e^-$ and $\pi^0 \mu^+ \mu^-$ modes. In each case, one curve is for the VMD model and the other is for the quark-box model with $m = 300$ MeV. The two modes give noticeably different spectra; the peak occurs at a higher pion energy for the electron mode.
Figure 4.2: The $\pi^0$ energy spectrum in the decays (a) $\eta \rightarrow \pi^0 e^+ e^-$ and (b) $\eta \rightarrow \pi^0 \mu^+ \mu^-$, for VMD and for the quark-box model with $m = 300$ MeV.
Chapter 4. The Decays $\eta \rightarrow \pi^0 e^+ e^-$ and $\eta \rightarrow \pi^0 \mu^+ \mu^-$
4.2 The Real Part of the Amplitude

The real and imaginary parts of the decay amplitude are related, in principle, by the dispersion relation (see, for example, [33])

\[ \text{Re} \mathcal{M}(s) = \frac{1}{\pi} \text{PP} \int_{4m_t^2}^{\infty} \frac{\text{Im} \mathcal{M}(s')}{s' - s} ds' \]  

(4.11)

where \( \text{PP} f \) denotes the principal part of the integral. However, if one tries to use this to calculate the real part, the integral diverges. (This is equivalent to the fact, already noted, that the directly-calculated amplitude diverges, but its divergence is contained in the real part.)

Clearly a finite value for the real part of the amplitude would be obtained by replacing the infinite limit in the integral of (4.11) by some finite cutoff. The integration variable has dimensions of mass squared, so whatever value of the cutoff gives the correct (physical) value for the real part should, intuitively, be the square of the natural mass-scale for this calculation, which is \( M_\eta \). This is an \textit{ad hoc} procedure which should be considered to give rough results at best. Therefore it is inappropriate to calculate the integral in detail.

Instead, (4.11) was calculated (with the cutoff), with certain selected terms from (4.4) replacing the full imaginary part of the amplitude. This was done once with the \( A \) term, and once with \( \frac{1}{6b^2} [-2 + 20b + 3(1 - 8b + 8b^2)L] \) (i.e. the first \( B \) term). In both cases the result of the integral was less than half of the representative term of the imaginary part, throughout all values of \( s \). While this cannot be claimed to be conclusive, it is an indication that the actual rate of the rare decays (1.15) and (1.16) should not be greater than the unitarity limit by more than 25%.
Chapter 5

Conclusions

The decays (1.14), (1.15), and (1.16) have been studied in terms of two models; VMD and the quark-box mechanism. The prediction of the rate of (1.14) in the former model falls short of the experimental result, even when the $a_0$-exchange is added in. Similar calculations, in which the effects of Chiral Perturbation Theory are included, provide a result which is closer to that of experiment, but still not in good agreement with it. In order for these calculations to achieve their best results (with regard to the rate of (1.14)), it is necessary to assume that the various contributing effects all interfere constructively, but no a priori justification has been given for this assumption. The quark-box model, on the other hand, gives results which are in good agreement with experiment. A new experiment, with a greater number of events, would be helpful; if the measured rate persists in being greater than the rate calculated according to VMD and ChPT, this would place these models in some difficulty, unless modifications were made. One possible approach would be to generalize the couplings of VMD from constants (as they are generally assumed to be) to functions of the momenta of the particles being coupled (i.e., form factors). The detailed experimental information about the momentum behaviour of the VMD couplings which would be required for such a calculation is currently lacking. In principle, the Dalitz decays of pseudoscalar mesons (e.g., $\pi^0 \rightarrow \gamma e^+ e^-$) would be of great help. There is presently no theoretical reason for assuming any specific momentum dependence of these form factors.

The model dependence of the two $\eta \pi \gamma \gamma$ form factors $A$ and $B$ has also been examined.
In order to distinguish between the models experimentally, one would need to measure differential decay rates such as $d^2\Gamma/dx_1 dx_2$, thus providing information on the form factors via equations (3.5) and (3.6). Then the $x_1$- and $x_2$-dependence of the form factors would need to be examined. In particular, the behaviour in $x_1$ and $x_2$ of the form factor $A$ in the quark-box model differs from that in the VMD model. On the other hand, the difference is small for the form factor $B$, other than overall normalization. These studies would require high statistics measurements, which are feasible at $\eta$-factories.

The experimental determination of the $x_1$- and $x_2$-dependence of the form factors $A$ and $B$ would also be useful in that it would provide the input necessary for a model-free calculation of the unitarity bound for the decays (1.15) and (1.16). Until then, a calculation of the unitarity bound will have to rely on models, as was done in chapter 4. The fact that the quark-box model gives better results than VMD in the case of the decay (1.14) suggests that the results (4.10) which come from the quark-box model are to be taken more seriously than those of (4.9) which are from VMD. These results point to a branching ratio $> 3 \times 10^{-9}$ for the decays (1.15) and (1.16), which indicates that the experimental discovery of these decay modes is feasible. A detailed measurement of these decays would provide a test for electromagnetic C- and CP-invariance.

Estimations of the real parts of the amplitudes of (1.15) and (1.16) were made using a dispersion relation with a cutoff. The contribution of the real part to the rate is not expected to be more than 25% of the contribution of the imaginary part, but this cannot be said with certainty. Nevertheless, a measurement of a rate several times greater than the unitarity limit for these modes might signal new physics. For the sake of comparison, the measured branching ratios of the decays of neutral pseudoscalar mesons into lepton pairs are shown in table (5.1), along with the relevant unitarity limits. In all three cases, the measured branching ratio is close to the unitarity limit. These results indicate that the real parts of the amplitudes are small for these purely leptonic decays. It would be
Table 5.1: Unitarity limits and experimental branching ratios for the decays of neutral pseudoscalar mesons into lepton pairs.

<table>
<thead>
<tr>
<th>Decay mode</th>
<th>Experiment</th>
<th>Unitarity limit</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^0 \rightarrow e^+e^- )</td>
<td>((6.9 \pm 2.4) \times 10^{-8})</td>
<td>(4.7 \times 10^{-8})</td>
<td>[34], [35]</td>
</tr>
<tr>
<td>( \eta \rightarrow \mu^+\mu^- )</td>
<td>((5.1 \pm 0.8) \times 10^{-6})</td>
<td>(4.3 \times 10^{-6})</td>
<td>[8]</td>
</tr>
<tr>
<td>( K_L^0 \rightarrow \mu^+\mu^- )</td>
<td>((7.3 \pm 0.4) \times 10^{-9})</td>
<td>(6.8 \times 10^{-9})</td>
<td>[2], [36]</td>
</tr>
</tbody>
</table>

It is interesting to see whether this is also the case for the semileptonic decays (1.15) and (1.16).
Bibliography


Appendix A

Details of the Quark-Box Model Calculation

If we arrange the $g^{\mu\nu}$, $P^{\mu}P^{\nu}$, $P^{\mu}q^{\nu}$, $q^{\mu}P^{\nu}$, and $q^{\mu}q^{\nu}$ terms in (3.35) so as to provide as much cancellation with the denominators as possible, we find

\[
U^{\mu\nu}_{(f)1} = -4(g^{\mu\nu}\left[\frac{1}{2}D^{-1}_2(0, P-k_1) + \frac{1}{2}D^{-1}_2(0, -Q) + \frac{1}{4}(2P \cdot k_2 - Q^2)D^{-1}_3(0, P-Q, P-k_1) + \frac{1}{2}(P \cdot Q - P^2)D^{-1}_3(0, P-Q, P) - \frac{1}{2}P \cdot k_1D^{-1}_3(0, P-k_1, P) + \frac{1}{4}Q^2D^{-1}_3(0, -Q, -k_1) + \frac{1}{4}Q^2(2P \cdot k_1 - P^2)D^{-1}_4(0, P-Q, P-k_1, P)]
\]

\[+ q^{\mu}q^{\nu}\left[-2D^{-1}_3(0, -Q, -k_1) + 2(P^2 - P \cdot Q)D^{-1}_4(0, P-Q, P-k_1, P)]
\] + \[\frac{1}{2}(P^{\mu}q^{\nu} + q^{\mu}P^{\nu})[-D^{-1}_3(0, P-Q, P-k_1) - D^{-1}_3(0, P-k_1, P) + (Q^2 + 4P^2 - 4P \cdot Q)D^{-1}_4(0, P-Q, P-k_1, P) + P^{\mu}P^{\nu}[-D^{-1}_3(0, P-Q, P-k_1) - D^{-1}_3(0, P-k_1, P) + (2P^2 - 2P \cdot Q + Q^2)D^{-1}_4(0, P-Q, P-k_1, P)]) + ...
\]

\[
U^{\mu\nu}_{(f)2} = -4(g^{\mu\nu}\left[-\frac{1}{2}D^{-1}_2(0, P-k_1) - \frac{1}{2}D^{-1}_2(0, P-k_2) + \frac{1}{4}(Q^2 - 2P \cdot k_2)D^{-1}_3(0, k_2, P-k_1) + \frac{1}{2}P \cdot k_2D^{-1}_3(0, P, k_2) + \frac{1}{2}P \cdot k_1D^{-1}_3(0, P-k_1, P) + \frac{1}{4}(Q^2 - 2P \cdot k_1)D^{-1}_3(0, P-Q, P-k_2) + \frac{1}{4}(4P \cdot k_2 - P^2Q^2)D^{-1}_4(0, k_2, P-k_1, P)]
\]

\[+ 2q^{\mu}q^{\nu}(P^2 - P \cdot Q)D^{-1}_4(0, k_2, P-k_1, P) + \frac{1}{2}(P^{\mu}q^{\nu} + q^{\mu}P^{\nu})[-D^{-1}_3(0, k_2, P-k_1) - D^{-1}_3(0, k_2, P) + D^{-1}_3(0, P-k_1, P) + D^{-1}_3(0, P-Q, P-k_2) + 2(P^2 - P \cdot Q)D^{-1}_4(0, k_2, P-k_1, P)]
\] + \[P^{\mu}P^{\nu}[D^{-1}_3(0, P-k_1, P) + D^{-1}_3(0, P-Q, P-k_2)] + ...
\]
Appendix A. Details of the Quark-Box Model Calculation

\[ U_{(f)}^{\mu \nu} = -4(g^\mu \nu \left[ \frac{1}{2} D_2^{-1}(0, Q) + \frac{1}{2} D_2^{-1}(0, P - k_2) + \frac{1}{4} Q^2 D_3^{-1}(0, k_2 Q) - \frac{1}{2} P \cdot k_2 D_3^{-1}(0, k_2 P) \right) \]
\[ + \frac{1}{2} (P \cdot Q - P^2) D_3^{-1}(0, Q, P) + \frac{1}{4} (2P \cdot k_1 - Q^2) D_3^{-1}(0, k_1, P - k_2) \]
\[ + \frac{1}{4} (2P \cdot k_2 - P^2) Q^2 D_4^{-1}(0, k_2, Q, P) \]
\[ + q^\mu q^\nu \left[ -2D_3^{-1}(0, k_2, Q) + 2(P^2 - P \cdot Q) D_3^{-1}(0, k_2, Q, P) \right] \]
\[ + \frac{1}{2} (P^\mu q^\nu + q^\mu P^\nu) [D_3^{-1}(0, k_2 P) + D_3^{-1}(0, k_3 P - k_2) - Q^2 D_4^{-1}(0, k_2 Q, P)] \] + ... \hspace{1cm} (A.1)

where

\[ D_n(p_1, ..., p_n) \equiv \prod_{i=1}^{n} ((q + p_i)^2 - m^2) \hspace{3cm} (A.2) \]

and \( Q = k_1 + k_2 \). The ellipses denote terms which do not, after integration, contribute to the \( g^\mu \nu \) and \( P^\mu P^\nu \) terms in \( T^{\mu \nu} \). In order to make the integrals simpler, we have used the substitutions \( q \rightarrow q - P, q \rightarrow q - k_1, q \rightarrow q - k_2 \) to ensure that each denominator contains \( m^2 \) and one factor of \( q^2 - m^2 \).

Some of the integrals in \( T^{\mu \nu} \) are divergent. We handle this by the standard dimensional regularizion method, where the number of spacetime dimensions is \( n = 4 - \epsilon \). We define

\[ \Delta \equiv \frac{i}{16\pi^2} \left( \frac{2}{\epsilon} - \gamma_E + \ln 4\pi \right) \hspace{3cm} (A.3) \]

where \( \gamma_E \) is Euler’s constant. The only kinds of integrals in \( T^{\mu \nu} \) which are divergent are

\[ \int \frac{d^n q}{(2\pi)^n} D_2^{-1}(a, 0) = \Delta + O(1), \quad \text{and} \]
\[ \int \frac{d^n q}{(2\pi)^n} q^\mu q^\nu D_3^{-1}(a, b, 0) = \frac{1}{4} g^{\mu \nu} \Delta + O(1) \hspace{1cm} (A.4) \]

so we can easily identify the divergences arising from integrating (A.1). The divergences all reside in the coefficients of the \( g^\mu \nu \) terms, and they all cancel out.

For the sake of keeping the integrated expressions short, we define the following
Appendix A. Details of the Quark-Box Model Calculation

quantities:

\[ I(a) \equiv \lim_{\epsilon \to 0} (-\Delta + \int \frac{d^3q}{(2\pi)^3} D_2^{-1}(a, 0)) = -\frac{i}{16\pi^2} \int_0^1 dx \ln(m^2 - a^2x(1 - x)) \]
\[ J(a, b) \equiv \int \frac{d^4q}{(2\pi)^4} D_3^{-1}(a, b, 0) = \frac{i}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \]
\[ \times (m^2 - a^2x(1 - x) - b^2y(1 - y) + 2a \cdot bxy)^{-1} \]
\[ J^\mu(a, b) \equiv \lim_{\epsilon \to 0} \left( \frac{1}{4} g^{\mu\nu} \Delta + \int \frac{d^3q}{(2\pi)^3} q^\nu q^\nu D_3^{-1}(a, b, 0) \right) = \frac{i}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \]
\[ \times ((a^\mu x + b^\nu y)(a^\nu x + b^\mu y)(m^2 - a^2x(1 - x) - b^2y(1 - y) + 2a \cdot bxy)^{-1} \]
\[ \times \frac{1}{2} g^{\mu\nu} \ln(m^2 - a^2x(1 - x) - b^2y(1 - y) + 2a \cdot bxy) \)]
\[ K(a, b, c) \equiv \int \frac{d^4q}{(2\pi)^4} D_4^{-1}(a, b, c, 0) = \frac{i}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \]
\[ \times (m^2 - a^2x(1 - x) - b^2y(1 - y) - c^2z(1 - z) + 2a \cdot bxy + 2a \cdot cxz + 2b \cdot cyz)^{-2} \]
\[ K^\mu(a, b, c) \equiv \int \frac{d^4q}{(2\pi)^4} q^\mu D_4^{-1}(a, b, c, 0) = \frac{i}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz (a^\mu x + b^\nu y + c^\nu z) \]
\[ \times (m^2 - a^2x(1 - x) - b^2y(1 - y) - c^2z(1 - z) + 2a \cdot bxy + 2a \cdot cxz + 2b \cdot cyz)^{-2} \]
\[ K^{\mu\nu}(a, b, c) \equiv \int \frac{d^4q}{(2\pi)^4} q^\mu q^\nu D_4^{-1}(a, b, c, 0) = \frac{i}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz ((a^\mu x + b^\nu y + c^\nu z) \]
\[ \times (a^\nu x + b^\mu y + c^\nu z)(m^2 - a^2x(1 - x) - b^2y(1 - y) - c^2z(1 - z) \]
\[ + 2a \cdot bxy + 2a \cdot cxz + 2b \cdot cyz)^{-2} - \frac{1}{2} g^{\mu\nu}(m^2 - a^2x(1 - x) - b^2y(1 - y) \]
\[ - c^2z(1 - z) + 2a \cdot bxy + 2a \cdot cxz + 2b \cdot cyz)^{-1} \). \]

We also define the functions \( J_{(i)} \), \( J_{(ij)} \), \( \widehat{J} \), \( \widehat{J}' \) by

\[ J^\mu(a_1, a_2) = - \sum_i a_i^{(i)} J_{(i)}(a_1, a_2), \quad \widehat{J}(a_1, a_2) = \sum_i J_{(i)}(a_1, a_2), \]
\[ J^{\mu\nu}(a_1, a_2) = \sum_{i,j} a_i^{(i)} a_j^{(j)} J_{(ij)}(a_1, a_2) + g^{\mu\nu} \widehat{J}(a_1, a_2), \quad \widehat{J}'(a_1, a_2) = \sum_{i,j} J_{(ij)}(a_1, a_2), \]

where the sums all go from 1 to 2. \( K_{(i)} \), \( K_{(ij)} \), \( \widehat{K} \), \( \widehat{K}' \) are defined similarly, but they have three arguments so their sums go from 1 to 3.
If we denote the finite contributions to the $g^{\mu\nu}$ and $P^\mu P^\nu$ terms in the integrals of $U^{\mu\nu}_{i}$ by $V^{\mu\nu}_{i}$, and define $A(f)$ and $B(f)$ by

$$\sum_{i=1}^{6} V^{\mu\nu}_{i} = iA(f)g^{\mu\nu} + iB(f)P^\mu P^\nu / M_n^2,$$  \hspace{1cm} (A.7)

then comparing (A.1) to the definitions (A.5) and (A.6) gives, after a bit of algebra,

$$A(f) = i[8I(Q) + 2Q^2(J(k_1, Q) + J(k_2, Q)) + 4(P \cdot Q - P^2)(J(P, Q) + J(P, P - Q))$$

$$Q^2(2P \cdot k_1 - P^2)(K(P, P - k_1, P - Q) + K(P, k_1, Q))$$

$$+ Q^2(2P \cdot k_2 - P^2)(K(P, P - k_2, P - Q) + K(P, k_2, Q))$$

$$+(4P \cdot k_1 P \cdot k_2 - P^2 Q^2)(K(P, k_1, P - k_2) + K(P, k_2, P - k_1))$$

$$- 16(J(k_1, Q) + J(k_2, Q))$$

$$+ 8(P^2 - P \cdot Q)(\overline{K}(P, P - k_1, P - Q) + \overline{K}(P, P - k_2, P - Q)$$

$$+ \overline{K}(P, k_1, P - k_2) + \overline{K}(P, k_2, P - k_1) + \overline{K}(P, k_1, Q) + \overline{K}(P, k_2, Q))]$$  \hspace{1cm} (A.8)

and

$$B(f) = 4iM_n^2[(2P^2 - 2P \cdot Q + Q^2)(K(P, P - k_1, P - Q) + K(P, P - k_2, P - Q))$$

$$+ 2P^2 - P \cdot Q)[-\sum_{i=2}^{3}(K(i)(k_1, P, P - k_2) + K(i)(k_2, P, P - k_1))$$

$$+ \sum_{i=2}^{3} \sum_{j=2}^{3}(K(i)(k_1, P, P - k_2) + K(i)(k_2, P, P - k_1))$$

$$+ \overline{K}(P, P - k_1, P - Q) + \overline{K}(P, P - k_2, P - Q) + K(33)(k_1, Q, P) + K(33)(k_2, Q, P)]$$

$$-(4P^2 - 4P \cdot Q + Q^2)(\overline{K}(P, P - k_1, P - Q) + \overline{K}(P, P - k_2, P - Q))$$

$$+ Q^2(K(3)(k_1, Q, P) + K(3)(k_2, Q, P))]$$  \hspace{1cm} (A.9)

We wish to write $A(f)$ and $B(f)$ as integrals of Feynman parameters, as in (A.5). It is easy to perform the first Feynman parameter integration in the cases where one of the
4-vectors in the expression is null. By using

\[ K(P, P - k, P - Q) = K(P, k, Q) \]

\[ \tilde{K}(P, P - k, P - Q) = -K_1(P, k, Q) + K(P, k, Q) \]

\[ \hat{K}(P, P - k, P - Q) = K(P, k, Q) - 2K_1(P, k, Q) + K(P, k, Q) \]

\[ \tilde{K}(P, P - k, Q) = \tilde{K}(P, k, Q) \] (A.10)

we can ensure that \( k_1 \) or \( k_2 \) appears as an argument in every \( K \)-type function. We get

\[ A_{(f)} = \frac{1}{4\pi^2} \left[ 1 - \frac{1}{2} (1 - \frac{4\rho}{\sigma}) \int_0^1 \frac{dx}{x} \ln \left( 1 - \frac{\sigma}{\rho} x(1-x) \right) + \int_0^1 dx \int_0^{1-x} dy \left( \rho - x(1-x) - ay(1-y) + 2(1-x_1-x_2)xy - \frac{1}{2} \rho(1-2x_1) \right) \right. \]

\[ \left. + \frac{1}{2} \frac{\rho-x(1-x) - \sigma y(1-y) + 2(x_1+x_2)xy}{\rho - x(1-x) + (1-a)xy + 2x_1x(1-x-y)} \right] \]

\[ + \frac{1}{4} \frac{\rho - x(1-x) - (1-2x_2)y(1-x-y)}{\rho - x(1-x) + (1-a)xy + 2x_1x(1-x-y)} \]

and

\[ B_{(f)} = \frac{1}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \left( \frac{2\sigma x + 4(1-x_1-x_2)x^2}{2x_1x + \sigma y} \left( \frac{1}{\rho-x(1-x) - \sigma y(1-y) + 2(x_1+x_2)xy} \right. \right. \]

\[ - \frac{1}{\rho - x(1-x) + (1-a)xy + 2x_1x(1-x-y)} \]

\[ + \frac{2(1-x_1-x_2)(x+y)(1-x-y)}{2x_1y + (1-a-2x_2)x} \left( \rho - (1-2x_2)x(1-x-y) - y(1-x-y) \right) \]

\[ \left. - \frac{1}{\rho - (1-2x_1)y(1-x-y) - ax(1-x-y)} \right) + (x_1 \leftrightarrow x_2), \] (A.12)
where $a \equiv \frac{M_e^2}{M_q^2}$, $\rho \equiv \frac{m^2}{M_q^2}$, and $\sigma \equiv \frac{Q^2}{M_q^2} = -(1 - a - 2(x_1 + x_2))$.

From (3.4), (3.35), (A.7), and the definition of $V^\mu_\nu(f)$, one sees that

\[ A = -\sum_f 3e^2 Q^2 g_{\eta qq} g_{\pi qq} \left( -\frac{1}{k_1 \cdot k_2} A(f) + \frac{M_e^2 x_1 x_2}{(k_1 \cdot k_2)^2} B(f) \right) \quad \text{and} \]
\[ B = -\sum_f 3e^2 Q^2 g_{\eta qq} g_{\pi qq} \left( -\frac{1}{k_1 \cdot k_2} B(f) \right), \quad (A.13) \]

where the sum is over the quark flavours and $A(f)$ and $B(f)$ are given by (A.11) and (A.12).