

**INVERSION OF
THREE-DIMENSIONAL DIRECT CURRENT RESISTIVITY DATA**

by

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ABSTRACT

A direct current (d.c.) resistivity experiment investigates subsurface geo-electrical structures by measuring the electric field set up by introducing current into the earth. Information about geo-electrical structures is extracted by inverting the observed data to generate an image of the conductivity or to construct a conductivity model. The goal of this thesis is to develop efficient inversion techniques for the interpretation of three-dimensional ($3d$) d.c. resistivity data. The study assumes data consisting of pole-pole potentials measured over a regular grid on the surface for many current locations. The Born approximation is employed to linearize the inverse problem.

The source of the electric field measured in the d.c. resistivity is the accumulated electric charges. Different aspects of the charge accumulation are reviewed, enlarged with new insights and presented in a unified notation. This provides the basis for understanding the fundamentals of d.c. resistivity experiments. Two algorithms are developed to image simple $2d$ conductivities. The first constructs a structural image by combining the charge density images obtained by inverting multiple sets of common current potentials. The second constructs a conductivity image directly. Processing and displaying the apparent conductivity, and constructing equivalent sources from secondary potentials are studied as the means of imaging. Assuming a multiplicative perturbation to a uniform half-space, the potential anomaly of pole-pole arrays is expressed as a depth integral of the logarithmic perturbation convolved with a kernel function in the horizontal directions. Applying the Fourier transform decomposes the data equation for a $3d$ problem into a set of $1d$ equations. A rapid approximate $3d$ inversion is developed based upon this decomposition by solving a sequence of $1d$ inversions in the wavenumber domain. The approximate $3d$ inversion is used to construct iterative inversion algorithms using the AIM (Approximate Inverse Mapping) formalism. The approximate inversion and an exact forward mapping are used to update the model successively so that the final result reproduces the observed data. The AIM inversion is applied to analyse a set of field data.

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献给我亲爱的爸爸妈妈:

在那红色的黑暗中 ——

您历经辛酸苦楚养育了我,

给我毅力开始今日的追求,

教我带着希望面向明天...

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CHAPTER 1

INTRODUCTION

The direct current (d.c.) resistivity experiment is one of the oldest geophysical methods devised to investigate subsurface structures. An electric field in the earth is set up by putting steady current into the ground through a pair of electrodes. A separate pair of electrodes is used to measure the field in the form of an electrical potential difference. When an electric current field is introduced into the earth, it sets up a distribution of accumulated electric charges both on and beneath the earth's surface. These charges exist in the region where there is a gradient of conductivity and a non-zero electric field parallel to it. It is these accumulated charges which give rise to the electric field whose potentials are being measured. The distribution of the charge is determined by the conductivity structure. Thus different conductivity structures will result in different distribution of accumulated charges and, hence, different distributions of electric field. The measurement of the field, therefore, contains information about the conductivity.

Many different configurations of electrode placement can be used. Electrodes can be placed on the ground or some can be in boreholes. This difference leads to the division between surface and borehole d.c. resistivity methods. The surface experiment has been the most commonly used, but recent years have seen a rapid development in borehole d.c. resistivity methods. Measurements are generally one of the three types (Keller and Frischknecht 1966). The first is the potential in reference to a point at "infinity". An example of this type is the data measured by a pole-pole array. The second is the potential gradient or the electric field intensity. The data from pole-dipole, Schlumberger, and Wenner arrays are of this type. The third is the curvature of the electric potential generated by a point source. A dipole-dipole array collects such data. Among these, the potential data measured by a pole-pole array are the most basic since the other types can be synthesized from the pole-pole data by the principle of superposition.

Due to the decay of the field away from the current source, the measurement will only be sensitive to a limited region surrounding the active electrodes (those which are not placed at

“infinity”). Each measurement represents an integrated effect of the conductivity within this region. The extent of this effective region increases with both the separation of the current electrodes and the separation between source and potential electrode pairs and for this reason the d.c. resistivity experiment involves a series of spatially distributed measurements with different array separations. This series of measured data collectively carries the information about the subsurface conductivity structure. Successful processing and inversion of the data yields such information by generating an image of the conductivity structure or by constructing a conductivity model.

The conductivity of the material in the earth's crust depends upon a number of factors such as the composition, structure and the physical processes involved. Within the regime of d.c. resistivity experiments, the conduction of electricity is mainly of two types, namely, electronic conduction and electrolytic conduction (Ward and Fraser 1971, Telford *et al.* 1976). The former is by the free electrons in the material while the latter is by the ions in the electrolyte. Only a small portion of the naturally occurring minerals is a good electrical conductor. Among these are native metals, graphite, magnetite and most of the sulphides. However, the majority of the rock building minerals are very poor conductors and the propagation of electricity in rocks is mainly electrolytic and depends upon the interstitial fluid. Thus the conductivity of the material is influenced by its porosity, connectivity of the pores, fluid saturation and the concentration of ions. Since temperature changes the concentration and mobility of the ions in the fluids, the conductivity of the rock will increase with the temperature under most near-surface conditions. These dependencies make it possible for the d.c. resistivity experiment to have a wide range of applications.

The first, and perhaps the most prominent area of application is in mineral exploration. Many metallic mineral deposits such as massive sulphides have conductivities much higher than the surrounding rock and form good targets to be located by d.c. resistivity experiments. More often, the local geological environment or the host of the ore deposit has anomalous conductivities. Identifying such geological settings serves to predict the localities of possible ore deposits. For

instance, many hydrothermal deposits occur on the boundaries of intrusive structures, or they may be associated with the vein formed in fault structures which are usually highly resistive. Another example is the mineralization resulting from alteration processes in the hydrothermal deposits. The ascending hot fluid precipitates precious metals by altering the iron rich rock. The process eventually forms a halo containing a significant amount of pyrite surrounding the ore deposits. The conductive halo thus forms a rather unique signature. Much of the literature on d.c. resistivity methods focus upon this application. Another application for the d.c. resistivity surveys is in the exploration for geothermal resources which are structurally controlled and often are associated with conductive anomalies (e.g., Wright *et al.* 1985).

The second area of application for d.c. surveys is in environmental, groundwater and geotechnical problems. Recent years have witnessed the growing application of d.c. resistivity methods in such problems. This is evidenced by extensive publications (e.g., Ward 1990). Most applications of this type are in the characterization and monitoring of waste disposal sites, the assessment of existing contamination sites, and the study of the transportation of groundwater contaminants. Conductivity variations are diagnostic to answering specific questions in such environments and d.c. resistivity experiments provide an indirect but effective means to detect these changes. In contrast to mineral and geothermal exploration, these problems are usually carried out on a much smaller spatial scale and consequently the electrode separations are much smaller.

The third area of application is in archeological problems and a considerable amount of work has been published. An example is mapping man-made structures which often exhibit highly resistive signatures in the more conductive sedimentary settings. Scollar *et al.* (1990) present a rather good overview of the application of the d.c. resistivity method in archeology.

The applications above are presented in the order of decreased spatial scale of the problem. However, there are also applications in the extreme ends of the scale. One experiment was conducted to investigate the crustal structures in South Africa (Van Zijl and Joubert 1975), where the current electrode separation exceeded 400 km. On the small scale, techniques have

been developed which use direct currents to perform a medical scan of the human body (e.g., Brown 1986, Yorkey and Webster 1987). There are also techniques which use d.c. fields to reconstruct tomographically the conductivity of core-samples (Dines and Lytle 1981). The electrode separation in these applications can be as small as 1 cm.

In exploration problems, the data of surface d.c. resistivity experiments are displayed in the form of apparent resistivity, which is the resistivity of a uniform half-space which produces the same potential difference as that acquired by the field electrodes. In keeping with the recently adopted convention, I choose to work with the apparent conductivity throughout this thesis (conductivity is the reciprocal of resistivity). The display of data in such a form constitutes the crudest way of extracting information about subsurface conductivity structures, since the apparent conductivity is only a convoluted representation of the true structure. Any quantitative information can only be obtained by inverting the data to obtain conductivity models which can reproduce the observed data.

The computational difficulty in inverting a data set increases with the number of the dimensions of the problem. The *1d* and *2d* problems have been studied extensively. The methods of automated *1d* inversion generally fall under the category of direct techniques or iterative approaches. Direct methods usually employ the properties of the *1d* field to construct directly a depth-conductivity profile from the given observations (e.g., Koefoed 1976, Szaraniec 1976, Szaraniec 1980, Coen and Yu 1981, Parker 1984). The most rigorous is the bilayer expansion method by Parker (1984). On the other hand, iterative approaches start from a initial model and iteratively update the model to produce a final solution based upon linearization of the mapping from conductivity to potential (e.g., Inman 1975, Jupp and Vozoff 1975, Oldenburg 1978) or upon some *ad hoc* methods of matching the apparent resistivity curve (e.g., Zohdy 1974, Goldberg *et al.* 1982). The most rigorous solution is that of Oldenburg (1978), who derives an analytic form of Fréchet derivative and treats both model construction and appraisal problem using linearized inverse theory of Backus and Gilbert. The *2d* inversion is carried out mostly in parametric form and employs iterative techniques (e.g., Pelton *et al.* 1978, Smith and Vozoff 1984, Tripp *et al.* 1984,

McGillivray and Oldenburg 1990). This usually involves, in each iteration, the generation of the sensitivity matrix of data with respect to the model parameters and the solution of the matrix equation by an optimization technique to obtain an updated model. Some work is also done to invert for $2d$ interfaces (e.g., Lee 1972). This uses the asymptotic expansion of the $1d$ kernel function in a $2d$ environment.

In contrast, the work regarding the inversion of $3d$ d.c. resistivity data is rather limited. Alfano (1959) proposed an approach for a general $3d$ model based upon a rectangularly gridded model and the theory of charge accumulation. That work provides much insight into the nature of the $3d$ d.c. resistivity problems. Vozoff (1960) applied a similar formulation to simple $3d$ problems and proposed an approximate algorithm to invert for a conductivity model consisting of a number of conductivity anomaly blocks in a uniform half-space. The geometry of the block must be specified in advance. Petrick *et al.* (1988) devised a method which uses the concept of alpha centres (Stefanescu and Stefanescu 1974) to invert for the central locations of the conductive anomalies. However, this method is not capable of inverting for a general $3d$ conductivity model. Recently Ellis and Oldenburg (unpublished but presented at 10th workshop on EM induction, Ensenada 1990) formulated a linearized, iterative $3d$ algorithm. Park and Van (1991) presented a similar iterative $3d$ inversion algorithm which handles a rather limited number of parameters.

One of the reasons for the delayed development of $3d$ algorithms may be the lack of practical data for tackling $3d$ problems. In his study of the d.c. resistivity inverse problems via the integral equation approach, Stevenson (1934) concludes that a unique $3d$ conductivity solution exists in theory if the surface potential measurements form a complete $3d$ data set. One example of such a data set consists of the potential measured continuously over a $2d$ surface for all the current sources continuously positioned along a curve across the surface. This conjecture is proven rigorously by Backus (see Vozoff 1960). Although this uniqueness theorem is not directly applicable to practical problems with incomplete and inaccurate data, it does however place a requirement on the data. The number of dimensions of the data must be at least equivalent to, or no less than, the number of dimensions sought for the model. For $3d$ studies, such data have not been available

until recent advancements in acquisition techniques. The E-SCAN®* data set seems to be the first of such data sets.

E-SCAN® d.c. resistivity data are usually acquired on a regular grid over a target area. The current source is placed at each grid point in turn with the current sink at “infinity”. For each source location, the surface potential is measured at the remaining grid points in reference to a second potential electrode placed at “infinity” (The details of the acquisition procedure will be discussed in Chapter 6). Thus the data set consists of common source pole-pole potentials over the grid for many different source locations. It provides the necessary data obtainable in practice for attempting to construct a $3d$ conductivity model. It is the availability of E-SCAN® data which prompted the research work in this thesis.

In principle, standard approaches used in $2d$ inversion can be applied to $3d$ problems. However, calculating and inverting a sensitivity matrix for a realistic $3d$ problem demands far more computing power than in the $2d$ cases. No matter what approach one takes to compute the sensitivity matrix, it always requires that the forward modelling be performed a number of times. For a $3d$ problem, this is very demanding computationally. The problem is further compounded if the sensitivity matrix is actually inverted, since it is large and dense. Therefore, alternative methods are needed to cope with the size of the problem in $3d$ cases.

The final goal of the research in this thesis is to develop efficient techniques to facilitate the interpretation of $3d$ d.c. resistivity data such as the E-SCAN® data. To put this in perspective, when the research was initiated in 1987, only $2d$ algorithms for inverting d.c. resistivity data (e.g., Pelton *et al.* 1978, Smith and Vozoff 1984) existed. Moreover these algorithms only inverted for a small number of parameters involving conductivities and dimensions of blocks. The inversions were formulated as an over-determined problem and generally suffered from problems of poor convergence. There was no algorithm for recovering a general $3d$ conductivity model characterized by many cells of unknown conductivity. My research, therefore, began with interpreting $2d$ data and this led to the formulation of $2d$ imaging algorithms. I parameterized

* E-SCAN® is the registered trade mark of Premier Geophysics Inc.

the problem with a large number of cells and obtained the solution by minimizing an objective function comprised of the model norm and the data misfit. Thus the type of the model obtained was determined by the norm of the model rather than *a priori* parameterization. As a natural extension of the $2d$ imaging algorithms, I formulated a rapid approximate inversion algorithm for imaging complex $3d$ conductivity structures. It uses the data from fixed pole-pole arrays and solves the $3d$ problem by decomposing it into a sequence of $1d$ inversions in the wavenumber domain. It is capable of inverting a large number of data to recover conductivities of a large number of cells. This algorithm is then used to construct iterative inversions that generate conductivity models reproducing the observed data.

In Chapter 2, I present the theory of charge accumulation in the d.c. resistivity experiment. This is a different point of view from the more commonly used approach which focuses upon current flow. In this chapter, different aspects of charge accumulation are reviewed, enlarged with some new insight obtained through this research and presented in a unified notation. This provides the basis for understanding the fundamentals of d.c. resistivity experiments. The algorithms developed later in the thesis all hinge upon the material in this chapter.

The direct application of the theory developed in Chapter 2 results in the $2d$ imaging algorithm using charge accumulation. This method generates a structural image of the subsurface $2d$ conductivity by constructing a sparse representation of the charge density accumulated on the boundaries of conductivity prisms. The final image is a composite of many charge density images obtained by inverting multiple sets of common source potential data. Thus the algorithm requires potential data acquired along a traverse over the $2d$ structure. A variation of this approach is to construct a conductivity image directly, utilizing the relationship between the charge accumulation and the surface potential. This results in the second method for imaging $2d$ structures. Both algorithms depend upon an approximate (Born) equation for the surface potential. In the Born approximation an estimated background conductivity is assumed and the charge accumulation is generated only by the primary electric field and the internal interactions are neglected. These

two methods, together with brief discussions on data display and an equivalent source imaging method from Chapter 3.

In Chapter 4, I present a rapid approximate inversion for the $3d$ problems. Using the Born equation, I derive an approximate integral equation for the potential data of a surface pole-pole array. This equation states that the percentage potential anomaly of a surface pole-pole array, relative to a known uniform half-space, is equal to a depth integral of the logarithmic conductivity anomaly convolved with a kernel function in horizontal directions. Fourier transforming the equation, and applying the convolution theorem, decomposes it into a sequence of $1d$ equations in the wavenumber domain. By such a decomposition, each component of the surface potential data is given by a single depth integral of the corresponding component of conductivity anomaly multiplied by a kernel in the wavenumber domain. The number of discrete (complex) data for each $1d$ equation is equal to the number of data maps for different pole-pole arrays. Solving these $1d$ inversions yields the Fourier transform of the conductivity anomaly, and applying an inverse Fourier transform produces the $3d$ conductivity model in the spatial domain. The solution of the $3d$ inversion is, therefore, obtained by solving a sequence of $1d$ inversions in the wavenumber domain. Various aspects of the algorithm are examined in this chapter. These include the preparation of data, calculation of kernel functions, and the solution of each $1d$ problem. The $1d$ inverse problem can be formulated either in continuous or in discrete form. Both approaches are presented.

The above solution for the conductivity model is only approximate and does not provide an acceptable model. Because of the way it is formulated, the algorithm itself cannot be further developed to yield a full solution which reproduces the observed data within the statistical tolerance of the error. However, it can be used as an approximate inverse mapping in AIM (Approximate Inverse Mapping) inversion (Oldenburg and Ellis 1991).

The AIM inversion algorithm is a general formalism by which an approximate inverse mapping operator can be used in conjunction with an exact forward mapping to successively improve the model and yield a full solution reproducing the observed data. There are two ways

the AIM formalism can be applied. In the first, a perturbation in model space is sought so that the updated model reproduces the observed data. This is the AIM-MS. In the second, a perturbation in data space is sought so that the updated data will, upon the application of the approximate inverse mapping, produce an acceptable model. This is the AIM-DS. In Chapter 5, I construct an AIM inversion method for $3d$ d.c. resistivity problem using the approximate $3d$ inversion developed in the preceding chapter. I first present a brief review of the AIM formalism. I then outline the application to the d.c. resistivity problem of AIM formalism and the approximate $3d$ inversion. The AIM algorithm is demonstrated with accurate and noisy data sets from two synthetic models. The first model consists of 5 prismatic anomalies in a uniform half-space while the second one is made of correlated random conductivity perturbations. The performance of the algorithm is excellent in all cases.

The combined application of the $3d$ approximate inversion and the AIM formalism results in the iterative inversion scheme. It has several unique characteristics. First, the method is very fast. The algorithm constructs a $3d$ conductivity model by solving a sequence of $1d$ inversion problems. Thus the computational effort is drastically reduced compared to solving the $3d$ problem directly. This approach of decomposing a large $3d$ problem into a sequence of smaller problems may prove to be advantageous in other geophysical problems. The method of decomposition, of course, will be problem dependent. Utilizing the AIM formalism, the iterative model construction is very efficient. At each iteration, only one application of the approximate inverse mapping and the forward mapping is used. The algorithm reduces the data misfit to an acceptable level within a very few iterations. This is in sharp contrast to linearized approach where many forward modellings have to be performed in each iteration, either in computing the sensitivity matrix or in search of a proper step-size for the model perturbation.

Second, the method is capable of solving problems with a large number of unknown parameters. The storage requirement and the CPU time for the solution increase approximately linearly with the number of parameters. This is important in practical applications, where even simple geologies demand a great number of the model parameters to describe the possible variations so

that meaningful information can be drawn from the inversion. The algorithm is also capable of inverting large data sets. This is closely related to the capability of dealing with large number of parameters.

Third, the method is stable and has less possibilities of generating spurious structures in the model. By virtue of the decomposition, the solution is sought through a sequence of scales. Structures in the model on different scales horizontally are obtained by inverting the data component on the same scale. The variation is controlled by limiting the smallest scale component of the data used in the inversion. At the same time, a minimum norm model is constructed in the vertical direction. Thus only the features required by the data appear in the model. Variations beyond the data resolution are excluded.

In Chapter 6, a set of field data from an epithermal regime is analysed. An epithermal deposit is typically formed in an area with well developed fracture and fault systems. It is usually close to intrusive structures, which provide the hot fluid for the hydrothermal system. The faults and fractures form the pathway for the fluid, which alters the surrounding rocks and deposits the precious metals. The deposits are often associated with silicified structures with deep “roots”, which usually exhibit strong resistive anomalies. The E-SCAN[®] experiment is employed to map the silicified zones and fault structures in an attempt to predict the localities of possible ore bodies. I present the detailed field procedure of the data acquisition and the format of the data set. I outline the different stages of processing applied to the data. First, the data set is corrected for the finite distance of the “infinite” electrodes based upon an estimated regional conductivity value. The pole-pole data maps are gathered from the corrected data set. Each map is then smoothed using a generalized cross-validation (GCV) technique to suppress the outliers in the data. I then apply the 3d AIM-DS inversion to construct a conductivity model. The validity of the features in the model is discussed, with the emphasis on the algorithmic performance.

CHAPTER 2

THEORETICAL BASIS OF d.c. RESISTIVITY EXPERIMENTS

In d.c. resistivity experiments, the information about the underground geo-electrical structure is contained in observed potential differences measured at the earth's surface or at depth. The measured potential arises from two sources: (1) the potential due to a current source embedded in a half-space of conductivity equal to that of the region immediately surrounding the current electrode; and (2) the potential that arises from the secondary charge distributions which are set up in the medium by the current input into the ground. The existence and distribution of these charges in the earth and on the earth's surface are fundamental to d.c. potential signals and it follows that a thorough understanding of charge accumulation should provide enhanced insight into the forward modelling of d.c. potentials and in formulating new approaches to the inversion of d.c. data.

Alpin (1947) was perhaps the first to point out that the source of potential in d.c. experiments is accumulated charge. That seminal paper presents the physical interpretation and basic mathematical description of the electric field in a conductive medium using charge accumulation and it also presents an integral equation for the surface charge density. Alfano (1959, 1960, 1961) provides a comprehensive treatment of modelling and interpreting d.c. resistivity data using charge accumulation. His presentation gives a basic understanding of the field behavior that arises in the presence of conductivity discontinuities in an orthogonally gridded $3d$ environment. Basic physical equations relating to the accumulation of electric charge in a conductive medium and evaluation of electric potentials from boundary charges are lucidly developed in the excellent work of Kaufman and Keller (1985, p 11-44). Kaufman (1985) presents a tutorial dissertation regarding the role of charge buildup in time varying problems; that work contains insight and physical understanding about the d.c. limit which is of interest here. The use of charge density in correcting for the effects of topographic distortion is examined by Oppliger (1984). Jiracek (Lecture at 9th workshop on EM induction, Sochi, 1988) investigates the distortion of

electromagnetic measurements due to the charge accumulation resulting from topography and near-surface conductivity variation. Finally, integral equations for the *d.c.* problem have been derived and used in forward modelling by numerous authors (e.g., Dieter *et al.* 1969, Snyder 1976, Okabe 1984). Integral equation forward modelling approaches have two steps. The first computes the charge density on boundaries of blocks across which there is a conductivity contrast, and the second uses Coulomb's law to compute the potential at the observer site that arises from this charge. This dissection tends to emphasize the importance and the physical role of charge accumulation.

The goal of this chapter is to assimilate some of the diverse information on the subject into a single document, enlarge it with new insight, and present results in a uniform notation. It is hoped that this will be of use to others who are involved with interpreting *d.c.* measurements.

The chapter begins with the governing equations and boundary conditions for *d.c.* potentials in a conducting medium; particular emphasis is paid to terms which involve charge density. The application of an electric field to a polarizable medium produces a polarization charge. Although it is generally well known that this charge does not alter the observed potentials, the precise role of the polarization charge is sometimes not appreciated and hence is a source of confusion. I treat this aspect explicitly. I next look at the relationship between refraction of currents at a boundary and the change in the electric field caused by the surface charges. This emphasizes the fundamental role played by charge accumulation in current channelling problems. An integral equation for charge density is developed and solved with the aid of the whole-space Green's function. This illustrates how boundary element forward modelling results can be obtained and also suggests how topographic problems can be handled. The effect of the charges accumulated on the earth's surface is studied with the aid of the integral equation and it is shown that formulating the problem in the half-space and in the whole-space are equivalent. Analytic expressions for charge density resulting from a buried current source in a layer over a half-space is derived using the integral equations. Since the image method is often used to solve simple *d.c.* problems, I show how this mathematical approach relates to the physical aspects of charge accumulation. Numerical

examples are presented to quantify the charge density buildup on a simple $2d$ topography and on a $3d$ body buried in a half-space. Lastly, the Born equation of the d.c. potential is outlined; this constitutes the basis for imaging and inversion techniques in this thesis.

2.1 Basic Equations

In a d.c. resistivity experiment, current is put into the ground and potential differences are measured away from the source. The electric potential at any point in the medium is dependent upon the distribution of the conductivity within the earth and that potential may be evaluated by using Maxwell's equations, conservation laws, constitutive relations and the boundary conditions. In this section I outline the basic equations and boundary conditions required to understand and to solve the d.c. problem. Equations which involve charge density are emphasized. Additional material can be found in Stratton (1941), Grant and West (1965), Jeans (1966), Keller and Frischknecht (1966), Telford *et al.* (1976), Kaufman and Keller (1985, chapter 2), and Ward and Hohmann (1988).

For a steady state problem, only two of Maxwell's equations are needed:

$$\nabla \times \vec{E} = \vec{0}, \quad (2.1)$$

$$\nabla \cdot \vec{D} = \rho_f, \quad (2.2)$$

where $\vec{E}$ is the electric field, $\vec{D}$, the electric displacement and ρ_f , the volumetric free charge density. ∇ is the del operator and is defined in a cartesian coordinate system as $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$, where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the unit vectors in the x -, y -, and z -directions respectively.

The law of conservation of charge states that

$$\nabla \cdot \vec{j} = -\frac{\partial \rho_f}{\partial t}, \quad (2.3)$$

where $\vec{j}$ is the current density due to free charges. A positive current is defined to be related to the flow of positive charge carriers. In steady state conditions $\nabla \cdot \vec{j} = 0$ is satisfied everywhere

except at locations of sources or sinks of electric charges. If a current I is injected at a location $\vec{r}_s$,

$$\nabla \cdot \vec{j} = I\delta(\vec{r} - \vec{r}_s). \quad (2.4)$$

In addition to field equations we also require constitutive relations. For the purpose of this study, I assume that the medium is linear and isotropic. Thus

$$\vec{D} = \epsilon \vec{E}, \quad (2.5)$$

$$\vec{j} = \sigma \vec{E}, \quad (2.6)$$

where σ is the electrical conductivity and ϵ is the permittivity.

Finally, at an interface separating media of different conductivities, the tangential component of the electric field and the normal component of current density are continuous. Thus

$$E_{1t} = E_{2t}, \quad (2.7)$$

$$j_{1n} = j_{2n}, \quad (2.8)$$

where subscripts 1 and 2 refer to the respective media and subscripts t and n refer to tangential and normal components. The unit normal vector $\hat{n}$ is chosen to point outward from medium-1 at the interface (see Fig. 2.1). The normal components of $\vec{D}$ and $\vec{E}$ satisfy

$$D_{2n} - D_{1n} = \tau_f, \quad (2.9)$$

$$E_{2n} - E_{1n} = \frac{\tau_t}{\epsilon_0}, \quad (2.10)$$

where τ_f and τ_t are respectively the surface densities of free and total charge. Thus the normal components of $\vec{D}$ and $\vec{E}$ can be discontinuous if there is a surface charge distribution on the boundary. Using equation (2.1) and the vector identity $\nabla \times \nabla \phi = \vec{0}$, we can express the electric

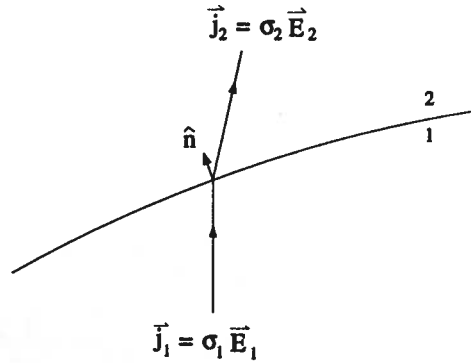


Figure 2.1 Basic geometry for current flow at a boundary.

field as the gradient of a scalar potential ϕ ,

$$\vec{E} = -\nabla\phi. \quad (2.11)$$

The electric field is bounded away from sources. Consequently, the potential is continuous and we have the boundary condition $\phi_1 = \phi_2$.

Any steady state conduction problem can be solved using the above equations and boundary conditions. In the most general approach, substitution of (2.6) and (2.11) into (2.4) yields

$$\nabla \cdot (\sigma \nabla \phi) = -I\delta(\vec{r} - \vec{r}_s), \quad (2.12)$$

which can be expanded to produce

$$\nabla^2 \phi = -\frac{\nabla \sigma \cdot \nabla \phi}{\sigma} - \frac{I}{\sigma} \delta(\vec{r} - \vec{r}_s). \quad (2.13)$$

Equation (2.13) is recognized formally as Poisson's equation. The two terms on the right hand side have units of ρ/ϵ_0 and therefore each can be thought of a charge density. For the purpose of calculating the potential, the term involving the source current can be replaced by an

effective point charge of magnitude $Q = \epsilon_0 I / \sigma_s$, where σ_s is the conductivity at the location of the current source. The first term on the right hand side is of prime interest because it expresses the charge buildup that exists as a result of changes in the electrical conductivity structure. It is non-zero whenever there is a component of the electric field parallel to the conductivity gradient. Under these conditions there will be a physical buildup of electric charge with volumetric density

$$\rho_t = \epsilon_0 \frac{\nabla \sigma \cdot \nabla \phi}{\sigma}. \quad (2.14)$$

In the limit that the conductivity gradient approaches infinity, i.e., when the medium suffers a discontinuous change in conductivity, the volumetric charge density in (2.14) becomes a surface charge density confined to the boundary separating the two regions. It is this charge which creates a discontinuity in the normal components of $\vec{D}$ and $\vec{E}$ as in equations (2.9) and (2.10).

By combining (2.7) and (2.10) and using Ohm's law, the charge density can be written as

$$\frac{\tau_t}{\epsilon_0} = (1 - \frac{\sigma_2}{\sigma_1}) E_{2n}. \quad (2.15)$$

Equations (2.14) and (2.15) show that the accumulated charge is negative when current flows from a resistive into a conductive region. Conversely, positive charge accumulates when a current flows from a conductive to a resistive region. This rule delineating the signs of the accumulated charge is very useful for predicting the general character of d.c. fields resulting from simple geological structures.

Equation (2.13) indicates that conductivity is the governing parameter for a stationary electric field and that permittivity does not play a part. Yet, application of an electric field to a polarizable material creates polarization charges which produce an electric potential. This apparent contradiction is sometimes a source of confusion and I therefore address it here. It is best illustrated by examining what is happening on the boundary between two media.

An electric field applied to a polarizable medium generates an electric polarization or dipole moment per unit volume. The polarization is proportional to the applied total electric field, and

is given by

$$\vec{P} = \chi \epsilon_0 \vec{E}, \quad (2.16)$$

where $\vec{P}$ is the polarization vector and $\chi = (\epsilon/\epsilon_0) - 1$ is the electric susceptibility. At the boundary of the medium there will be an polarization charge $\tau_b = \vec{P} \cdot \hat{n}$, where $\hat{n}$ is the normal vector. The net polarization charge density at a boundary separating media with permittivities ϵ_1 and ϵ_2 is

$$\tau_b = -(P_{2n} - P_{1n}). \quad (2.17)$$

and the total charge is the sum of this polarization charge plus the free charge. Substituting (2.5) into (2.9), and (2.16) into (2.17) and summing yields

$$\begin{aligned} \tau_t &= \tau_f + \tau_b \\ &= \epsilon_0 (E_{2n} - E_{1n}). \end{aligned}$$

This result is precisely the same equation as (2.10). It is this total charge density on the boundary which is responsible for the continuity of the normal component of the current density and this charge is not affected by variations in permittivity. The permittivity however, does determine how much free charge has to be accumulated at the boundary so that boundary conditions are satisfied.

In general, when the physical properties are changing continuously, a volumetric charge density of

$$\frac{\rho_t}{\epsilon_0} = -\frac{\nabla \sigma \cdot \vec{E}}{\sigma}$$

is set up. Again, this charge density is determined only by the conductivity, but if the medium is polarizable, then ρ_t can be explicitly formed as the sum of free and polarization charge densities. The expressions for these are

$$\rho_f = -\frac{\epsilon}{\sigma} \nabla \sigma \cdot \vec{E} + \vec{E} \cdot \nabla \epsilon, \quad (2.18)$$

$$\rho_b = (\epsilon - \epsilon_0) \frac{\nabla \sigma \cdot \vec{E}}{\sigma} - \vec{E} \cdot \nabla \epsilon. \quad (2.19)$$

The sum of ρ_f and ρ_b gives the total charge density which determines the final electric field.

In summary, in a *d.c.* experiment, electric charges accumulate whenever there is a gradient of conductivity and a non-zero component of electric field parallel to it. The final electric field is produced by the primary source and by surface and volumetric charge distributions. When the medium is polarizable, both free and polarization charges contribute to the total accumulated charge. However, the total accumulated charge is controlled only by electrical conductivity; the polarization plays no part other than to supply a portion of the charges needed to satisfy the boundary conditions.

2.2 Current Flow in *d.c.* Resistivity Problems

In this chapter I focus upon the accumulated electric charges in order to provide insight to the *d.c.* resistivity problem. A commonly used description of *d.c.* fields, however, is through the concept of current flow. The variations in conductivity structure alter the flow of electric charges and the final distribution of current is such that the energy loss due to ohmic dissipation is minimized. Physically, this results in current being channelled into the regions of high conductivity and deflected away from resistive regions. But knowledge of the current distribution, or even its direction, does not imply immediate knowledge of the potentials. In particular, current direction at any point is determined by the gradient of the potential. Current magnitude also requires specification of an electrical conductivity.

The relationship between current flow and charge accumulation is best illustrated by considering the refraction, or change in direction, of currents impinging upon a plane interface separating two media. This refraction is a direct effect of surface charges which accumulate on the interface. This is quantified here but a similar discussion can be found in Kaufman and Keller (1985). Consider, as in Fig. 2.2(a), a point on the boundary and let $\vec{E}_b$ be the base electric field which is produced by all sources away from this point. Continuity of the normal component of the current density requires the existence of a charge density τ_t (given by equation (2.10)). This charge

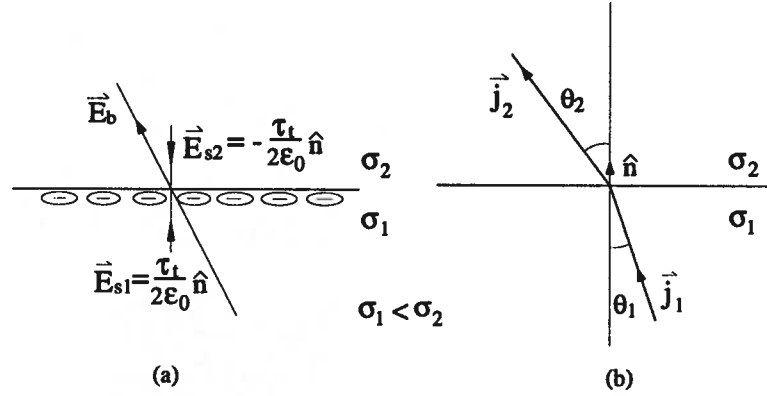


Figure 2.2 (a) Electric charges accumulate on the boundary separating two conductive media and produce perturbation fields $\vec{E}_{s1}$ and $\vec{E}_{s2}$ normal to the boundary. These fields are added to the base field $\vec{E}_b$ to produce a total field whose direction changes at the boundary. The resultant current flow shown in (b) is equivalently refracted at the boundary.

affects the normal components of the electric field so that

$$\begin{aligned} E_{1n} &= \vec{E}_b \cdot \hat{n} - \frac{\tau_t}{2\epsilon_0}, \\ E_{2n} &= \vec{E}_b \cdot \hat{n} + \frac{\tau_t}{2\epsilon_0}. \end{aligned} \quad (2.20)$$

An expression for the total accumulated charge, obtained by combining equation (2.20) and (2.8) using Ohm's law, is

$$\frac{\tau_t}{\epsilon_0} = 2 \frac{\sigma_1 - \sigma_2}{\sigma_2 + \sigma_1} \vec{E}_b \cdot \hat{n}. \quad (2.21)$$

The normal components of the electric field in (2.20) are then

$$\begin{aligned} E_{1n} &= \frac{2\sigma_2}{\sigma_1 + \sigma_2} \vec{E}_b \cdot \hat{n}, \\ E_{2n} &= \frac{2\sigma_1}{\sigma_1 + \sigma_2} \vec{E}_b \cdot \hat{n}. \end{aligned} \quad (2.22)$$

The tangential component of the electric field at the boundary is unaltered by the charge density and is equal to E_{bt} . Using equation (2.22), we obtain

$$\tan \theta_1 = \frac{\sigma_1 + \sigma_2}{2\sigma_2} \frac{E_{bt}}{E_{bn}},$$

$$\tan \theta_2 = \frac{\sigma_1 + \sigma_2}{2\sigma_1} \frac{E_{bt}}{E_{bn}},$$

and therefore

$$\frac{1}{\sigma_1} \tan \theta_1 = \frac{1}{\sigma_2} \tan \theta_2. \quad (2.23)$$

This shows that a current line refracts as it crosses an interface; it bends towards the normal when entering a resistive medium and, conversely, it bends away from normal when entering a conductive medium. Equation (2.23) is identical to the usual refraction formula (e.g., Keller and Frischknecht 1966, Telford *et al.* 1976) which is derived directly from the continuity conditions of the electric field and current density. The derivation here shows that the charge buildup on the boundary causes a change in the normal component of the electric field so that the direction of current flow is altered as it passes into a medium with different conductivity. In general, the electric field at any point within the medium is a vectorial sum of the primary field and the field produced by the accumulated charge. It is this additional field which changes the direction of current flow in the medium and results in the channelling of the current into conductive regions and the deflection away from resistive regions.

2.3 An Integral Equation for the Charge Density

Charge accumulation plays a fundamental role in d.c. resistivity problems and we therefore need a technique to explicitly quantify that role and to illustrate how it can be applied in practice. An integral equation is a desirable tool because it enables us to examine quantitatively the charge that is accumulated and it also provides an effective way to carry out the forward modelling of d.c. responses. This latter aspect has been studied by many authors (e.g., Dieter *et al.* 1969, Snyder 1976, Okabe 1984). In the following an integral equation is developed for charge densities which closely follows that presented by Snyder (1976).

I begin with the initial differential equation (2.13) and Green's second identity,

$$\iiint_V (\phi \nabla^2 G - G \nabla^2 \phi) dv = \oint_S \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n} \right) ds, \quad (2.24)$$

where S is the surface bounding the volume V . Green's identity is valid for any functions ϕ and G that are continuous and have derivatives up to second order (Morse and Feshbach 1953). I choose ϕ to be the potential function, satisfying (2.13), and G to be the whole-space Green's function

$$G(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|},$$

which satisfies

$$\nabla^2 G(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}').$$

Let V be the whole space. Then within V , both ϕ and G decrease as inverse distance from the source and $\partial\phi/\partial n$ and $\partial G/\partial n$ decrease at least as inverse distance squared. Therefore, if S is taken as the surface of a sphere with radius approaching infinity, the right hand side of (2.24) vanishes. Substituting (2.13) into (2.24) yields

$$\phi(\vec{r}) = \frac{I}{4\pi\sigma_s} G(\vec{r}, \vec{r}_s) + \frac{1}{4\pi} \iiint_V \frac{\nabla\sigma(\vec{r}') \cdot \nabla\phi(\vec{r}')}{\sigma} G(\vec{r}, \vec{r}') dv, \quad (2.25)$$

where σ_s is the conductivity at the current source location. The first term in (2.25) is recognized as the potential due to the point source in a uniform background of conductivity σ_s . The second term is the potential due to the accumulated charge distribution.

Although equation (2.25) can be used directly, it is simpler to assume that the conductivity structure is piecewise constant. Then $\nabla\sigma$ is zero everywhere except at boundaries between regions of different conductivities. The volume integral reduces to a set of surface integrals and the integrands contain the surface charge density.

Consider the geo-electrical model shown in Fig. 2.3. The background medium has a conductivity σ_s . There are $n + 1$ inhomogeneities embedded in this background; each has a constant conductivity σ_i and boundary Γ_i . The 0th inhomogeneity represents the volume of air above the earth and is bounded by the surface Γ_0 . All boundaries are assumed to be piecewise smooth and

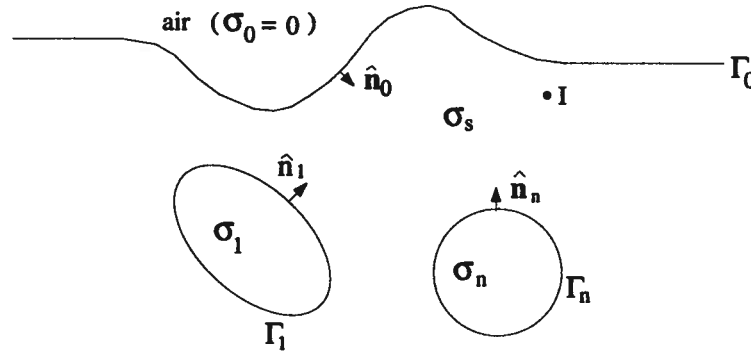


Figure 2.3 The geo-electrical model consisting of $n + 1$ bodies in a uniform background. n bodies are beneath topography and the air above constitutes the 0th body.

$\hat{n}_i$ is the outward pointing normal vector. The point current source is located in the background medium.

For this model, equation (2.25) becomes

$$\phi(\vec{r}) = \frac{I}{4\pi\sigma_s} G(\vec{r}, \vec{r}_s) + \frac{1}{4\pi} \sum_{i=0}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} G(\vec{r}, \vec{r}') ds, \quad (2.26)$$

where τ_i is the charge density on the i^{th} boundary. Using equation (2.15) and writing the normal component of the electric field in the background as $E_n = -\hat{n}_j \cdot \nabla \phi$, I have

$$\frac{\tau_j}{\epsilon_0} = \frac{\sigma_s - \sigma_j}{\sigma_j} \hat{n}_j \cdot \nabla \phi. \quad (2.27)$$

Equation (2.26) is valid everywhere, so I can substitute (2.26) into (2.27) to eliminate ϕ and obtain the integral equation for τ_j :

$$\begin{aligned} \frac{\sigma_j}{\sigma_s - \sigma_j} \cdot \frac{\tau_j(\vec{r})}{\epsilon_0} &= \frac{I}{4\pi\sigma_s} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}_s) + \\ &\frac{1}{4\pi} \sum_{i=0}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') ds, \quad j = 0, \dots, n. \end{aligned} \quad (2.28)$$

Here $\vec{r} \in \Gamma_j$, $\hat{n}_j = \hat{n}_j(\vec{r})$, ∇ operates on the field point $\vec{r}$, and the integrals operate on the secondary source points $\vec{r}'$.

The integral over Γ_j is not straightforward to compute because the integrand in equation (2.28) is infinite when $\vec{r}' \rightarrow \vec{r}$. However, this singularity is removable. A circle of radius δ centred at $\vec{r}$ divides Γ_j into two parts: let Γ_δ denote the area inside the circle and let Γ'_j be the area outside. This is always possible if δ is sufficiently small since I assume that the boundaries are all piecewise smooth. As $\delta \rightarrow 0$, the charge density in Γ_δ can be taken as constant and therefore,

$$\lim_{\delta \rightarrow 0} \iint_{\Gamma_\delta} \frac{\tau_j(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') ds = -2\pi \frac{\tau_j(\vec{r})}{\epsilon_0}$$

Thus (2.28) becomes

$$\begin{aligned} \frac{\tau_j(\vec{r})}{\epsilon_0} &= \frac{I k_j}{2\pi \sigma_s} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}_s) + \\ &\quad \frac{k_j}{2\pi} \sum_{\substack{i=0 \\ i \neq j}}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') ds + \\ &\quad \frac{k_j}{2\pi} \iint_{\Gamma'_j} \frac{\tau_j(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') ds, \quad j = 0, \dots, n \end{aligned} \quad (2.29)$$

where $k_j = (\sigma_s - \sigma_j)/(\sigma_s + \sigma_j)$ and Γ'_j is the j^{th} boundary with a small area around point $\vec{r}$ excluded. All integrals in (2.29) are now proper.

Equation (2.29) is a Fredholm equation of the second kind and is solved using standard techniques to generate the charge density on all boundaries. The potential is then evaluated by substituting these charge densities into (2.26).

2.4 Effect of Charge Density at the Earth's Surface

The formulation thus far has been to solve for the potential in a whole-space. This generality is advantageous because we ultimately wish to solve for the potential in the presence of surface topography. With the whole-space formulation the undulating surface may be considered to be the boundary of another body and the surface charge due to a point source can be evaluated directly.

Most studies assume a half-space model with a flat upper surface. Under such an assumption, the charge density at the earth's surface does not appear in the formulation and is equivalently

taken into account by a zero flux boundary condition. Although an integral equation which incorporates the zero vertical flux of current at the earth's surface can be derived (Snyder 1976), further insight regarding the fundamental nature of the surface charges is obtained by using equation (2.29) to compute the charge density on the surface of a flat earth and then to compute its secondary potential and its effect on the charges of buried bodies. I have carried out this study and shown that the effect of the charge distribution at the earth's surface can be completely reproduced by a distribution of fictitious images.

Adopt a right handed Cartesian coordinate system with z positive downward and $z = 0$ denoting the earth's surface. The coordinates of the current source are (x_s, y_s, z_s) . From equation (2.29), the charge density on the earth's surface is

$$\begin{aligned} \frac{\tau_0(\vec{r})}{\epsilon_0} = & \frac{I}{2\pi\sigma_s} \frac{z_s}{((x - x_s)^2 + (y - y_s)^2 + z_s^2)^{3/2}} + \\ & \frac{1}{2\pi} \sum_{i=1}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \frac{z'}{((x - x')^2 + (y - y')^2 + z'^2)^{3/2}} ds. \end{aligned} \quad (2.30)$$

The integral term over the earth's surface (Γ_0) does not appear in (2.30) because $\nabla G(\vec{r}, \vec{r}')$ is perpendicular to the surface normal. The first term in (2.30) is the portion of the secondary charge produced by the primary field at the earth's surface, which is equal to the surface charge density in the absence of any underground inhomogeneities. The terms in the summation represent the portion due to the existence of the buried inhomogeneities as a result of interaction among the charges on all interfaces.

Denote the first part by τ_0^p ,

$$\frac{\tau_0^p(\vec{r})}{\epsilon_0} = \frac{I}{2\pi\sigma_s} \frac{z_s}{((x - x_s)^2 + (y - y_s)^2 + z_s^2)^{3/2}}. \quad (2.31)$$

We see that $\tau_0^p(x, y)$ attains its maximum at a point directly over top of the current source and that its amplitude falls off as $|\vec{r} - \vec{r}_s|^{-3}$. As the current source moves closer to the surface, the

charge density becomes more concentrated. In the limit,

$$\lim_{z_s \rightarrow 0} \frac{1}{2\pi} \frac{z_s}{((x - x_s)^2 + (y - y_s)^2 + z_s^2)^{3/2}} = \delta(x - x_s)\delta(y - y_s),$$

and therefore

$$\lim_{z_s \rightarrow 0} \frac{\tau_0^p(x, y)}{\epsilon_0} = \frac{I}{\sigma_s} \delta(x - x_s)\delta(y - y_s). \quad (2.32)$$

Insertion of this charge density into equation (2.26) yields, in the absence of other inhomogeneities, the total potential

$$\phi(\vec{r}) = \frac{I}{2\pi\sigma_s |\vec{r} - \vec{r}_s|}. \quad (2.33)$$

This potential is double the whole-space primary potential and shows that, for a current source at the surface of a flat earth, the surface charge density provides an additional potential which is of equal strength as the whole-space primary potential. In problems formulated explicitly as half-space problems, it is convenient to regard (2.33) as the primary potential.

The second part in (2.30), denoted by τ_0^s , is given by

$$\tau_0^s(\vec{r}) = \frac{1}{2\pi} \sum_{i=1}^n \iint_{\Gamma_i} \tau_i(\vec{r}') \frac{z'}{((x - x')^2 + (y - y')^2 + z'^2)^{3/2}} ds. \quad (2.34)$$

Integrating (2.34) over the surface to obtain the total charge, Q_s , yields,

$$Q_s = \frac{1}{2\pi} \sum_{i=1}^n \iint_{\Gamma_i} \tau_i(\vec{r}') \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z'}{((x - x')^2 + (y - y')^2 + z'^2)^{3/2}} dx dy \right) ds.$$

The integral in the bracket is recognized as the solid angle subtended by the earth's surface at a point $\vec{r}'$ underground and equal to 2π . Substituting in and carrying out the integrals over Γ_i yields,

$$Q_s = \sum_{i=1}^n Q_i, \quad (2.35)$$

where Q_i is the net secondary charge on i^{th} inhomogeneity. Therefore, the sum of the secondary charge on the surface due to the buried inhomogeneities is equal to the sum of the net secondary

charges underground. A similar conclusion was arrived by Alfano (1959) through a different approach.

Using Coulomb's law to compute the secondary potential produced by $\tau_0^s(\vec{r})$ yields,

$$\Delta\phi_s^s(\vec{r}) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau_0^s(\vec{r}'') G(\vec{r}, \vec{r}'') dx'' dy''.$$

Exchanging the integral over Γ_0 and Γ_i and applying the integral identity (Stevenson 1934)

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z_1}{((x_1 - x)^2 + (y_1 - y)^2 + z_1^2)^{3/2}} \frac{1}{((x_2 - x)^2 + (y_2 - y)^2 + z^2)^{1/2}} dx dy \\ &= \frac{2\pi}{((x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 + |z|)^2)^{1/2}}, \quad \text{for } z_1 > 0 \end{aligned} \quad (2.36)$$

yields

$$\Delta\phi_s^s(\vec{r}) = \frac{1}{4\pi} \sum_{i=1}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \frac{1}{((x - x')^2 + (y - y')^2 + (|z| + z')^2)^{1/2}} ds. \quad (2.37)$$

Above the earth's surface where $z < 0$, this is equal to the potential produced by the subsurface charges. Beneath the earth's surface where $z > 0$, $\Delta\phi_s^s$ is equal to the potential produced by the subsurface charges residing at their image positions above the surface. It follows that at any point in space the effect of this part of the surface charge is completely represented by a set of image bodies coinciding with the buried inhomogeneities or at their mirror positions about the earth's surface, depending upon whether the field point in consideration is above or beneath the surface. This theoretically justifies the common practice in which the actual secondary potential on a flat earth's surface is obtained by doubling the secondary potential computed only from the subsurface charge densities.

Next, writing equation (2.29) to isolate $\tau_0(\vec{r})$, and specifying a flat earth's surface, yields

$$\begin{aligned} \frac{\tau_j(\vec{r})}{\epsilon_0} = & \frac{Ik_j}{2\pi\sigma_s} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}_s) + \frac{k_j}{2\pi} \sum_{\substack{i=1 \\ i \neq j}}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') ds + \\ & \frac{k_j}{2\pi} \iint_{\Gamma_j'} \frac{\tau_j(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') ds + \frac{k_j}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\tau_0(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') dx' dy'. \end{aligned} \quad (2.38)$$

Let β_j be the last term in the above equation. Substituting (2.30) into β_j and applying identity (2.36) yields

$$\beta_j = \frac{Ik_j}{2\pi\sigma_s} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}_s') + \frac{k_j}{2\pi} \sum_{i=1}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}'') ds, \quad (2.39)$$

where $\vec{r}_s'$ and $\vec{r}''$ are, respectively, the images of $\vec{r}_s$ and $\vec{r}'$ about the surface plane. The first term in (2.39) is recognized as the effect of a primary source at the image location $\vec{r}_s'$ and the second term is the effect of image bodies above the earth's surface, as has been demonstrated. The final expression, obtained by combining (2.38) and (2.39), is

$$\begin{aligned} \frac{\tau_j(\vec{r})}{\epsilon_0} = & \frac{Ik_j}{2\pi\sigma_s} \hat{n}_j \cdot \nabla \left(\frac{1}{|\vec{r} - \vec{r}_s|} + \frac{1}{|\vec{r} - \vec{r}_s'|} \right) + \\ & \frac{k_j}{2\pi} \sum_{\substack{i=1 \\ i \neq j}}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla \frac{1}{|\vec{r} - \vec{r}'|} ds + \\ & \frac{k_j}{2\pi} \iint_{\Gamma_j'} \frac{\tau_j(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla \frac{1}{|\vec{r} - \vec{r}'|} ds + \\ & \frac{k_j}{2\pi} \sum_{i=1}^n \iint_{\Gamma_i} \frac{\tau_i(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla \frac{1}{|\vec{r} - \vec{r}''|} ds. \end{aligned} \quad (2.40)$$

This is the same equation as that of Snyder (1976), which is derived through formulating the potential problem in a half-space and using the half-space Green's function.

The whole-space and half-space Green's function approaches to calculating the charge densities are essentially equivalent. They differ in that (2.29) is slightly more general and can be used

to treat topographic problems whereas (2.40) is specifically suited to plane surface problems and is numerically more efficient in such cases.

2.5 An Analytic Example of Charge Accumulation

Generally the solution of (2.40) demands numerical techniques but there are a few simple geometries in which analytic solutions for the charge density are available, i.e., a point current source buried in a half-space, and a point source in a layer which overlies a half-space. The former is discussed in the preceding section within the context of the effect of surface charges. I present the analysis for the latter example here. These examples provide enhanced insight into the manner in which charges are distributed on various surfaces and they also provide a foundation for understanding the relationship between the solution to potential problems via charge distribution and via the image method.

I evaluate the charge distributions when a point source is located at depth d inside a layer of thickness h overlying a half-space (see Fig. 2.4). I adopt the same coordinate system as before. To be consistent with the previous section, I take the layer in which the current source is located as the background medium and choose normals as shown in the Fig. 2.4. At the boundaries Γ_0 and Γ_1 ,

$$k_0 = 1,$$

$$k_1 = \frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2}.$$

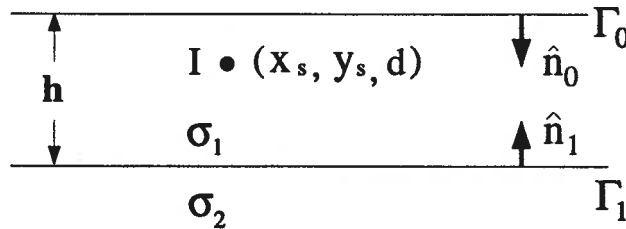


Figure 2.4 A point source in a layer over a half-space. The normal vector of the two boundaries are defined as pointing into background medium σ_1 .

To find the charge densities τ_0 and τ_1 on these surfaces, I employ (2.29) in an iterative manner. Equation (2.29) is first written as

$$\begin{aligned} \frac{\tau_j^{(l)}(\vec{r})}{\epsilon_0} &= \frac{Ik_j}{2\pi\sigma_1} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') + & i = 1 \quad \text{when} \quad j = 0 \\ & \frac{k_j}{2\pi} \iint_{\Gamma_i} \frac{\tau_i^{(l-1)}(\vec{r}')}{\epsilon_0} \hat{n}_j \cdot \nabla G(\vec{r}, \vec{r}') ds, & i = 0 \quad \text{when} \quad j = 1 \end{aligned} \quad (2.41)$$

$$l = 1, \dots, \infty$$

where l denotes the iteration number. When $l = 0$, the second term on the right hand side vanishes and the estimated charge densities are those due to the primary potential, i.e.,

$$\begin{aligned} \frac{\tau_0^{(0)}}{\epsilon_0} &= \frac{I}{2\pi\sigma_1} \frac{d}{(\eta^2 + d^2)^{3/2}}, \\ \frac{\tau_1^{(0)}}{\epsilon_0} &= \frac{Ik_1}{2\pi\sigma_1} \frac{d}{(\eta^2 + (h-d)^2)^{3/2}}, \end{aligned} \quad (2.42)$$

where $\eta = ((x - x_s)^2 + (y - y_s)^2)^{1/2}$ is the radial distance from the source point.

The presence of the primary charge distribution on Γ_0 affects the charge on Γ_1 (and vice versa); the next iteration attempts to account for this. Employing the integral identity (2.36) again, I obtain

$$\frac{\tau_0^{(1)}(\eta)}{\epsilon_0} = \frac{\tau_0^{(0)}(\eta)}{\epsilon_0} + \frac{Ik_1}{2\pi\sigma_1} \frac{2h-d}{(\eta^2 + (2h-d)^2)^{3/2}}, \quad (2.43)$$

$$\frac{\tau_1^{(1)}(\eta)}{\epsilon_0} = \frac{\tau_1^{(0)}(\eta)}{\epsilon_0} + \frac{Ik_1}{2\pi\sigma_1} \frac{h+d}{(\eta^2 + (h+d)^2)^{3/2}}. \quad (2.44)$$

Proceeding with this iterative process, I obtain the final solutions for charge density as series summations:

$$\begin{aligned} \frac{\tau_0(\eta)}{\epsilon_0} &= \frac{I}{2\pi\sigma_1} \frac{d}{(\eta^2 + d^2)^{3/2}} + \\ & \frac{I}{2\pi\sigma_1} \sum_{n=1}^{\infty} k_1^n \left(\frac{2nh-d}{(\eta^2 + (2nh-d)^2)^{3/2}} + \frac{2nh+d}{(\eta^2 + (2nh+d)^2)^{3/2}} \right), \end{aligned} \quad (2.45)$$

$$\frac{\tau_1(\eta)}{\epsilon_0} = \frac{Ik_1}{2\pi\sigma_1} \frac{h-d}{(\eta^2 + (h-d)^2)^{3/2}} +$$

$$\frac{I}{2\pi\sigma_1} \sum_{n=1}^{\infty} k_1^n \left(\frac{2nh - (h - d)}{(\eta^2 + (2nh - (h - d))^2)^{3/2}} + k_1 \frac{2nh + h - d}{(\eta^2 + (2nh + h - d)^2)^{3/2}} \right). \quad (2.46)$$

The charge densities are observed to have maximum magnitude at $\eta = 0$, i.e., just above and below the current electrode. Away from the central point the charge density decreases in magnitude but the rate of decrease depends upon the magnitude of k_1 . As $|k_1|$ becomes larger the charges are spread further over the interface. It is clear from the two formulae that the interaction between the charges on the upper and lower boundaries is primarily controlled by the magnitude of k_1 and the thickness of the layer.

As the current source approaches the surface, i.e., as $d \rightarrow 0$, a delta-like charge distribution will appear on the surface coinciding with the current source. In addition, there is a charge density

$$\Delta\tau(\eta) = \frac{I}{\pi\sigma_1} \sum_{n=1}^{\infty} k_1^n \frac{2nh}{(\eta^2 + (2nh)^2)^{3/2}},$$

which is due to the interaction with the bottom of the layer. This charge distribution will be positive if the layer is more conductive than the underlying half-space and it will be negative, otherwise.

Examining equations (2.45) and (2.46), we notice that every term has the form of equation (2.31); i.e., each term is like the charge density on a single plane interface induced by a point source. Furthermore, applying equation (2.36) to calculate the potential due to each of these terms results in a potential which is the same as that produced by a point charge away from the interface. This leads to an understanding of the physical basis for image method solution of potential problems.

2.6 Image Method Explained by Charge Accumulation

Consider the example of a point source buried in the lower half-space of conductivity σ_1 overlain by an upper half-space of conductivity σ_0 . Let the source be located at (x_s, y_s, d) in the same coordinate system as before. The charge density along the interface, obtained by evaluating

equation (2.29), is

$$\frac{\tau(x, y)}{\epsilon_0} = \frac{Ik}{2\pi\sigma_1} \frac{d}{((x - x_s)^2 + (y - y_s)^2 + d^2)^{3/2}} \quad (2.47)$$

where $k = (\sigma_1 - \sigma_0) / (\sigma_1 + \sigma_0)$. As $\sigma_0 \rightarrow 0$ we get to the special case of a half-space whose charge density is given by equation (2.31).

The potential anywhere in the whole-space is the sum of the primary potential from the current source and the secondary potential due to the charges on the interface. Evaluating (2.25) and using identity (2.36) yields

$$\phi(x, y, z) = \frac{I}{4\pi\sigma_1} \frac{1}{R} + \frac{Ik}{4\pi\sigma_1} \frac{1}{((x - x_s)^2 + (y - y_s)^2 + (d + |z|)^2)^{1/2}}.$$

where $R = ((x - x_s)^2 + (y - y_s)^2 + (z - d)^2)^{1/2}$. Thus the potential in the lower medium where $z > 0$, is

$$\phi_1 = \frac{I}{4\pi\sigma_1} \left(\frac{1}{R} + \frac{k}{R'} \right), \quad (2.48)$$

where $R' = ((x - x_s)^2 + (y - y_s)^2 + (z + d)^2)^{1/2}$. This potential is equivalent to that arising from a current of strength I located at R and another current of strength kI located at R' . In the upper medium where $z < 0$, the potential may be written as

$$\begin{aligned} \phi_2 &= \frac{I}{4\pi\sigma_1} \frac{1+k}{R''} \\ &= \frac{I}{4\pi\sigma_0} \frac{t}{R''}, \end{aligned} \quad (2.49)$$

with $R'' = ((x - x_s)^2 + (y - y_s)^2 + (z - d)^2)^{1/2}$ and $t = 2\sigma_0 / (\sigma_1 + \sigma_0)$. The second form of the potential in equation (2.49) is equivalent to that produced by a current of strength tI at a distance R'' in a whole-space of conductivity σ_0 . Equations (2.48) and (2.49) are identical to those derived by the image method in Keller and Frischknecht (1966) and Telford *et al.* (1976) where an analogy is drawn between electric current flow and geometrical optics based on the fact that the intensity of both current density and light emanating from a point source varies as the inverse square of the distance. The plane interface between two media is viewed as a semi-transparent

mirror. Based on this analogy, t and k defined above are referred as transmission and reflection coefficients, respectively. Although the correct equations are derived using the geometrical optics formula, the physical understanding is not present. The derivation here has explicitly shown that the image method works because it was possible to find a fictitious point charge which produced the same potential as the true distribution of charges on the boundary.

Although the whole-space single interface example establishes the basic principles of the image method, a slightly more complicated model is required so that internal interactions can be modelled. Correspondingly, I consider the structure in Fig. 2.4 with the current source moved to the surface. From (2.32), (2.45) and (2.46), the charge densities on the upper and lower surfaces are

$$\frac{\tau_0(x, y)}{\epsilon_0} = \frac{I}{\sigma_1} \delta(x - x_s) \delta(y - y_s) + \frac{I}{\pi \sigma_1} \sum_{n=1}^{\infty} k^n \frac{2nh}{(\eta^2 + (2nh)^2)^{3/2}}, \quad (2.50)$$

and

$$\frac{\tau_1(x, y)}{\epsilon_0} = \frac{I}{\pi \sigma_1} \sum_{n=1}^{\infty} k^{n+1} \frac{(2n+1)h}{((\eta^2 + ((2n+1)h)^2)^{3/2}}. \quad (2.51)$$

where $k = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$. By evaluating (2.29), the potential at any point in the space is

$$\begin{aligned} \phi(x, y, z) = & \frac{I}{2\pi\sigma_1 R} + \\ & \frac{I}{2\pi\sigma_1} \sum_{n=1}^{\infty} \frac{k^n}{((\eta^2 + (2nh + |z|)^2)^{1/2}} + \\ & \frac{I}{2\pi\sigma_1} \sum_{n=0}^{\infty} \frac{k^{n+1}}{((\eta^2 + ((2n+1)h + |z-h|)^2)^{1/2}}, \end{aligned} \quad (2.52)$$

where $\eta = ((x - x_s)^2 + (y - y_s)^2)^{1/2}$. Equation (2.52) is identical to that derived from image methods. For that approach one needs to compute potentials due to two families of image sources. (see Keller and Frischknecht 1966, pp.108-111)

To summarize the statements concerning the relationship between the charge density and the image solution I return to equation (2.51). τ_1 represents the charge that is accumulated on the

interface between the layer and the underlying half-space. The potential due to this charge is obtained by integrating with the whole-space Green's function, i.e.,

$$\Delta\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iint_{\Gamma_1} \tau_1(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} ds \quad (2.53)$$

at any position $\vec{r}$ in the medium. The image method reproduces the same potential through a series solution (the second summation in equation (2.52)). Each term in the series is recognized as the potential due to a point charge located in the half-space and beneath the primary source location. Thus whereas $\Delta\phi$ in (2.53) is obtained by the integration of a real charge distribution on the boundary, the series summation produces the same result by employing a set of fictitious image sources. Therefore the essence of the image method solution to the potential problem is to derive a set of fictitious sources which produce the same potential as does the true charge distribution. It follows that the image method is viable only when the conductivity structure is such that the effect of the accumulated charge can be represented by a set of point images.

The image method is not applicable to all problems. In general, potentials from all *1d* conductivity functions can be found using images (Kunetz 1972, Szaraniec 1976, Levy *et al.* 1988). Applicability of the image method to *2d* or *3d* problems is more restricted. Alfano (1959) considered conductivity structures composed of rectangular prisms of constant conductivity bounded by two orthogonal plane interfaces. If $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ denote the conductivities in successive quadrants defining the prisms, Alfano proved that the image solution exists if and only if

$$\sigma_1\sigma_4 = \sigma_2\sigma_3. \quad (2.54)$$

Physically, this condition ensures that the charge density on each interface is continuous at the intersection point. When continuity of physical charge density exists then the potentials can be represented as arising from a set of fictitious image charges. However when (2.54) is not satisfied, the charge density is discontinuous at the intersection point, no representation of potentials by point charges is available, and the image method is not applicable.

This understanding, plus the previous derivations, allows me to make general summary statements regarding the image solution. Steady state current flow in the earth demands the physical existence of charges which reside on the surfaces of buried bodies and on the earth's surface. The secondary potential at any point in the medium is obtained by integrating these charges with the whole space Green's function. The secondary potential $\Delta\phi$ arising from a charge density τ on a boundary Γ is given by

$$\Delta\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \iint_{\Gamma} \tau(\vec{r}') G(\vec{r}, \vec{r}') ds. \quad (2.55)$$

Since τ is a surface function it can be expanded in terms of an arbitrary set of basis $\psi_i(\vec{r})$ as

$$\tau(\vec{r}) = \sum_{i=1}^{\infty} \alpha_i \psi_i(\vec{r}), \quad (2.56)$$

where α_i 's are expansion coefficients. Substituting (2.56) into (2.55) yields

$$\Delta\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{\infty} \alpha_i \iint_{\Gamma} \psi_i(\vec{r}') G(\vec{r}, \vec{r}') ds. \quad (2.57)$$

If the basis functions are chosen so that the integrals can be computed analytically, then (2.57) may be efficiently computed. The image method as presented in the literature is equivalent to choosing the basis functions as a charge distribution on a plane interface arising from a point current source (see equation (2.47)). Suppose that the boundary Γ is the plane $z = z_c$ and that the centre of mass of $\tau(x, y, z_c)$ occurs at the point (x_c, y_c, z_c) . Then a possible expansion of the charge density is

$$\tau(x, y, z_c) = \sum_{i=1}^{\infty} \alpha_i \frac{1}{((x - x_c)^2 + (y - y_c)^2 + (z_i - z_c)^2)^{3/2}}$$

where specification of z_i 's determines the basis function and the α_i 's are the strengths. The potential produced from this charge density is

$$\phi_s(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{\infty} \alpha_i \frac{1}{((x - x_c)^2 + (y - y_c)^2 + (|z_i - z_c| + |z - z_c|)^2)^{1/2}}.$$

The difficulty with this method is in choosing the basis functions, i.e., specifying z_i 's, and in evaluating their strengths, α_i . Ideally, the basis functions should form a complete set so that every surface distribution of charge can be represented. It is noted that if the basis functions are continuous and not complete it is impossible to construct an exact representation of a discontinuous charge distribution. Therefore, an image method solution exists only for those structures with continuous charge distributions. It follows that it is not possible to have an image solution for a conductivity structure that violates condition (2.54).

2.7 Numerical Examples of Charge Distribution

For an arbitrarily shaped conductivity structure, the charge density on all interfaces is found by solving the integral equation (2.29). There are many ways to solve this Fredholm integral equation but most methods center around first dividing all interfaces into a set of small areal segments known as elements. A representation for the charge density on each element is then assumed. Different choices are possible: the charge density may be assumed to be concentrated as a point charge located at the centre of gravity of the element; it may be assumed constant or linearly varying over the element; or it may be represented by some higher order polynomial. Substitution of the representation for the charge density into the integral equation yields a system of linear equation to be solved (e.g., Alfano 1959, Harrington 1968, Pratt 1972, Snyder 1976, Eskola 1979, Brebbia and Walker 1980, Okabe 1984, Oppliger 1984, Shulz 1985, Das and Parasnis 1987).

As an example, consider a pure topographic problem in which a single interface Γ overlies a homogeneous medium of conductivity σ_0 . The surface Γ is divided into M elements and a constant τ_j is assigned to the j^{th} element. Substitution into (2.29) yields

$$\frac{\tau_i}{\epsilon_0} = \frac{Ik_i}{2\pi\sigma_0} \frac{\partial G(\vec{r}_i, \vec{r}_s)}{\partial n_i} + \frac{k_i}{2\pi} \sum_{\substack{j=1 \\ j \neq i}}^M \frac{\tau_j}{\epsilon_0} \iint_{\Delta\Gamma_j} \frac{\partial G(\vec{r}_i, \vec{r}')}{\partial n_i} ds, \quad i = 1, \dots, M \quad (2.58)$$

where $\vec{r}_i$ is the center of gravity of the i^{th} element, and $\tau_i = \tau(\vec{r}_i)$ is the charge density of this element. The integral over i^{th} element vanishes after the singularity is removed.

By reordering, the equations in (2.58) may be written in matrix form:

$$A\vec{\tau} = \vec{\tau}^{(0)}. \quad (2.59)$$

where A is an $M \times M$ coefficient matrix with components:

$$A_{ii} = 1, \quad A_{ij} = \frac{k_i}{2\pi} \iint_{\Delta\Gamma_j} \frac{\partial G(\vec{r}_i, \vec{r}')}{\partial n_i} ds, \quad i \neq j \quad (2.60)$$

$\vec{\tau}$ is the $M \times 1$ vector of the unknown charge densities and $\vec{\tau}^{(0)}$ has components:

$$\tau_i^{(0)} = \frac{Ik_i}{2\pi\sigma_0} \frac{\partial G(\vec{r}_i, \vec{r}_s)}{\partial n_i}. \quad (2.61)$$

The linear system of equations can be solved directly by decomposing the matrix A , or it can be solved through iterative techniques.

In (2.59) I have denoted the right hand side by $\vec{\tau}^{(0)}$. The reason for this is that $\vec{\tau}^{(0)}$ is a first order approximation to the charge density that I am seeking. It is the charge density that would exist if the potential at the boundary were equal to the primary potential. This approximation sometimes yields fairly good estimates of potential anomaly and delineates first order characteristics (Keller and Frischknecht, 1966). The computational ease with which $\vec{\tau}^{(0)}$ can be evaluated makes this approximation very desirable.

To illustrate quantitatively the charge accumulation in more complicated geo-electric structures, I consider two examples. In the first, I compute the charge distribution on a surface topography overlying a uniform earth. The topographic model and the charge distribution arising when a current electrode is located in the middle of the valley are shown in Fig. 2.5(a) and Fig. 2.5(b), respectively. A negative charge is observed on the valley walls but everywhere else the charge is positive. The maximum negative charge densities occur on the wall immediately beside the current source. Along the strike, the charge density decays with the distance away from

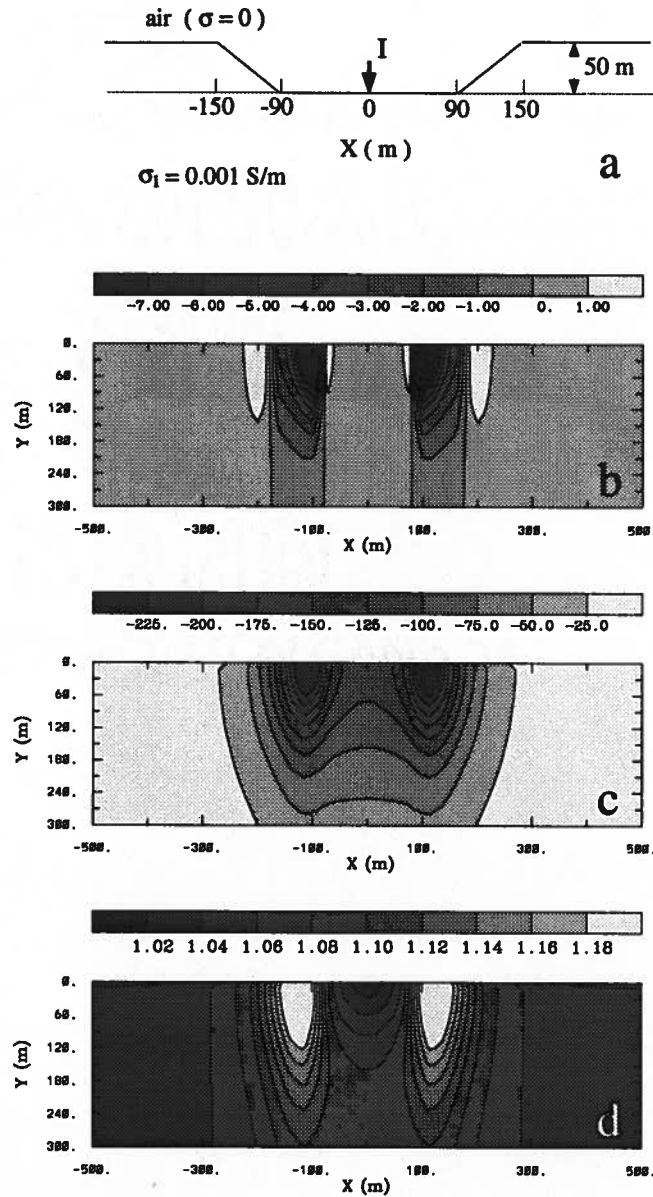


Figure 2.5 The charge density on a 2d topographic surface. The geometry of the cross-section is shown in (a). The charge density on the 2d topographic surface is shown in (b) (coulombs/m² scaled by ϵ_0). The y coordinate delineates offsets in the strike direction. Because of symmetry only charge densities for positive value of y are shown. The surface anomalous potential (mV) is in (c) and the surface apparent conductivity (mS/m) is displayed in (d).

the source. The charge distribution at the bottom of the valley is primarily a result of interaction effects with the negative charge on the valley walls.

This physical charge residing on the earth's surface produces an anomalous potential which is displayed in Fig. 2.5(c). When that potential is added to the primary potential and converted to apparent conductivity using the formula $\sigma_a(r) = I/2\pi r\phi(r)$, I obtain the distorted apparent conductivity map shown in Fig. 2.5(d). In that figure the apparent conductivities are plotted at the locations where the potential is observed. It is noted that the apparent conductivity in and near the topographic depression is greater than the intrinsic half-space conductivity.

This example shows that the physical charge caused by topographic variations gives rise to a perturbation potential that will distort the signal of buried bodies. However, the insight afforded by this example can also be used in reverse. A correction for topographic distortions can be made by first estimating the accumulated surface charge and then subtracting the resultant potential from the field observations. This approach has been used with success by Oppliger (1984).

The second example is selected to illustrate the charge accumulation that exists on a conductive prism buried in a uniform half-space. The geometry of the structure, location of the current source, the charge densities on the prism, the anomalous surface potential, and the surface apparent conductivity are shown in Fig. 2.6. Negative charges exist on the top and left faces of the cube and this is in accordance with current flowing into the prism there. Positive charges are seen on all other faces. Near the upper left corner of the front face (and, by symmetry, on the back face), there is a small area with negative charge indicating that current is flowing into the prism. This is a vivid demonstration of current channelling. In terms of charge accumulation, that negative charge is a consequence of interaction with the strong negative charges on the top and left faces of the cube.

The secondary potential produced by the charge distribution on all faces of the cube is shown in Fig. 2.7(a). It is similar to a dipole field with a negative region centred above the prism edge near the current source and a positive region at the other side of the prism. The apparent conductivity map is shown in Fig. 2.7(b). As in Fig. 2.5, apparent conductivities are plotted at the

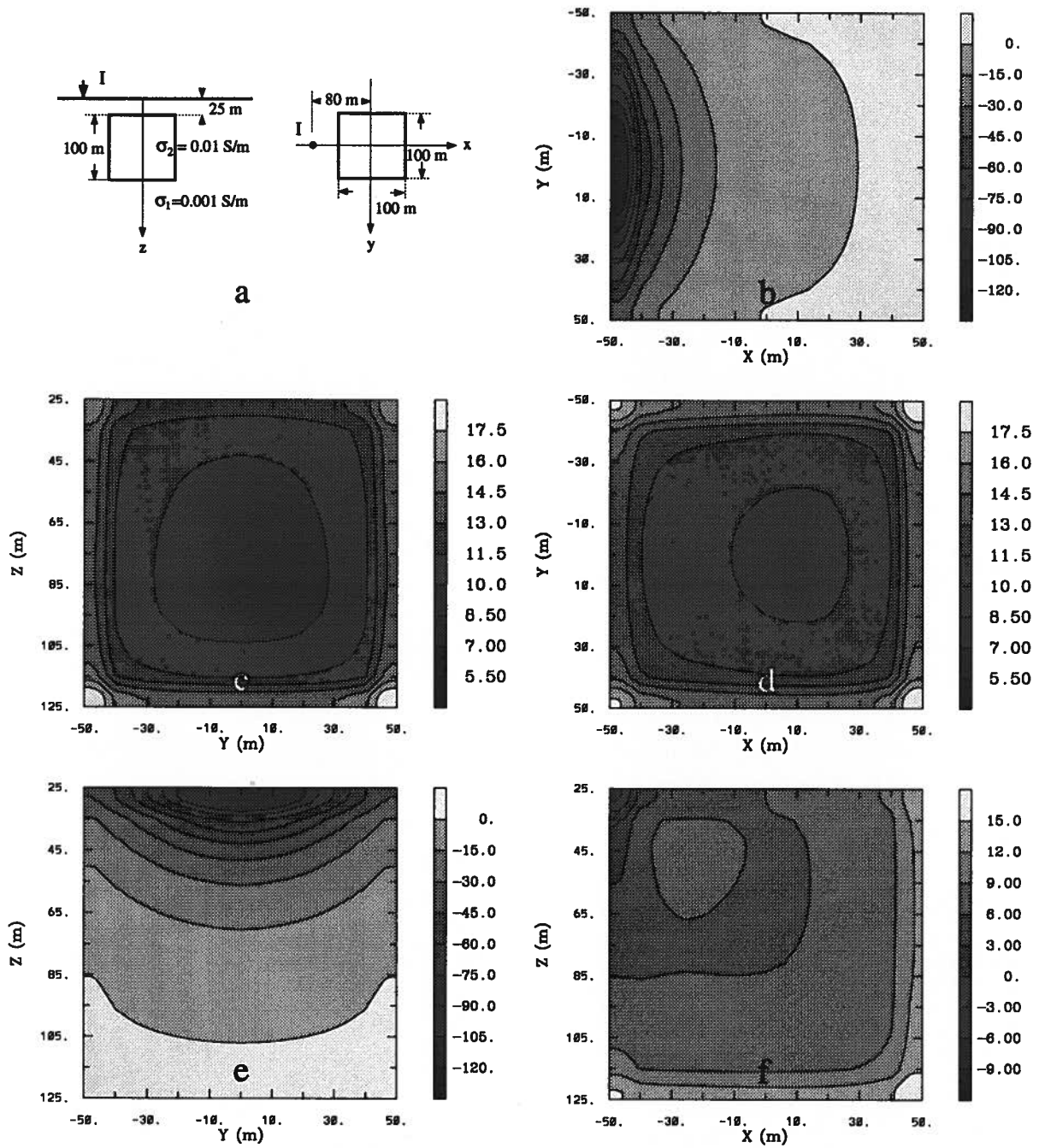


Figure 2.6 The charge density on a buried prism. The geometry and conductivities are given in (a). (b)-(f) respectively shows the charge densities on the top, right, bottom, left, and front faces of the prism. Units are in coulombs/m² scaled by ϵ_0 .

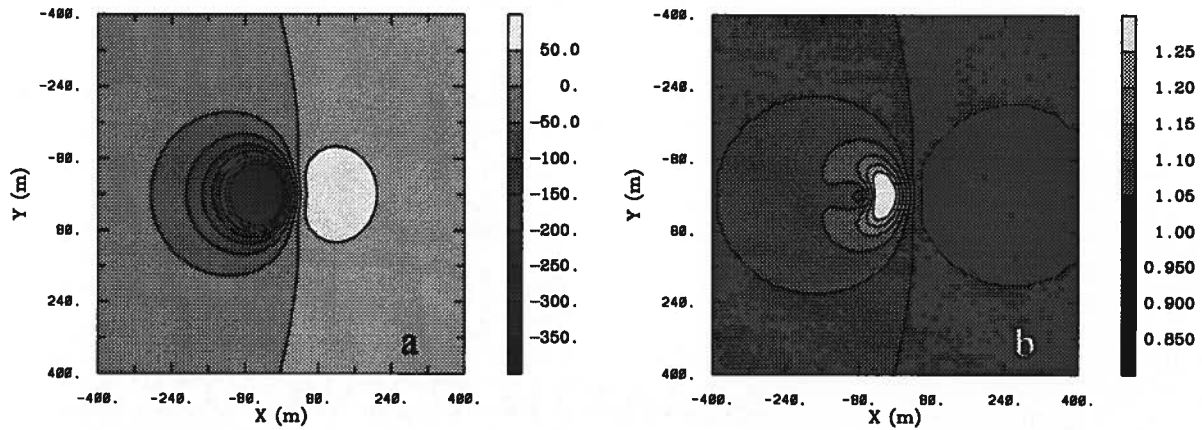


Figure 2.7 (a) shows the secondary potential (mV) at the earth's surface computed from the charge densities shown in Fig. 2.6. (b) is the corresponding apparent conductivity map (mS/m).

locations of the potential electrodes. A conductivity high is centered near the current source but there is a conductivity low at large source-observer distances to the right. It is perhaps contrary to intuition that the conductive prism has produced a resistive anomaly. This is understandable, however, after a somewhat more critical evaluation of the charge distribution. When the potential electrode is to the right of the prism it is close to the positive charge on those faces. Even though the strength of the positive charge is less than the negative charge on the other side, the inverse distance decay in the Coulomb potential allows the secondary potential to be a positive quantity as shown in Fig. 2.7(a). Therefore, a conductive body can produce either a conductive or resistive anomaly depending upon the relative position of potential measurement with respect to charge distribution.

2.8 Born Equation for d.c. Responses

As mentioned in the preceding section, the potential calculated using only the charge accumulation from the primary field may yield a reasonable approximation to the true secondary potential. This is in fact the Born approximation. In this section, I explicitly outline the Born equation for the d.c. potential in a 2d and a 3d environment. This is to establish the basis for the imaging and inversion algorithm which will be presented in the following chapters.

Let $\sigma(\vec{r})$ be the conductivity structure in a lower half-space V whose upper surface is flat. Following sections 2.4 and 2.5, the potential on the surface resulting from a surface point source of strength I is given by

$$\phi(\vec{r}) = \frac{I}{2\pi |\vec{r}_s - \vec{r}| \sigma_s} + \frac{1}{2\pi} \iiint_V \frac{\nabla \sigma(\vec{r}')}{\sigma(\vec{r}')} \cdot \frac{\nabla \phi(\vec{r}')}{|\vec{r}' - \vec{r}|} dv, \quad (2.62)$$

where $\vec{r}_s$ and $\vec{r}$ are the source and observation point respectively, and $\sigma_s = \sigma(\vec{r}_s)$. I choose a Cartesian coordinate system with origin at the surface and z positive down. The integral in (2.62) represents the secondary potential which contains all the information about conductivity anomalies and is denoted by

$$\phi_s(\vec{r}) = \frac{1}{2\pi} \iiint_V \frac{\nabla \sigma(\vec{r}')}{\sigma(\vec{r}')} \cdot \frac{\nabla \phi(\vec{r}')}{|\vec{r}' - \vec{r}|} dv. \quad (2.63)$$

Let the conductivity be represented by $\sigma(\vec{r}) = \sigma_0 \mu(\vec{r})$. Here σ_0 is the conductivity of a uniform background and $\mu(\vec{r})$ is a dimensionless function of spatial position $\vec{r}$. Substituting into (2.63) yields

$$\phi_s(\vec{r}) = \frac{1}{2\pi} \iiint_V \frac{\nabla \ln \mu(\vec{r}') \cdot \nabla \phi(\vec{r}')}{|\vec{r}' - \vec{r}|} dv. \quad (2.64)$$

The potential $\phi(\vec{r})$ is not known, but if the deviation of the conductivity from the background is small over the entire model and if the surface conductivity is equal to σ_0 (thus $\sigma_s = \sigma_0$), I can apply the Born approximation by replacing $\phi(\vec{r})$ in (2.64) with the half-space primary potential

$$\phi_p(\vec{r}) = \frac{I}{2\pi |\vec{r} - \vec{r}_s| \sigma_0},$$

to obtain

$$\phi_s(\vec{r}) = \frac{I}{4\pi^2 \sigma_0} \iiint_V \left(\nabla \ln \mu(\vec{r}') \cdot \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \right) \frac{1}{|\vec{r}' - \vec{r}|} dv. \quad (2.65)$$

This is a Fredholm equation of the first kind with the conductivity perturbation $\ln \mu(\vec{r})$ as the unknown function. Physically, (2.65) approximates the actual secondary potential by considering

only the charge accumulation arising from the primary field. The internal interaction of the accumulated charges is neglected. Because of the linearity between the conductivity perturbation and the secondary potential, (2.65) provides a basis for formulating linear inverse algorithms to image or recover the subsurface conductivity using surface potential data. I shall use it in the *2d* imaging algorithms and *3d* inversion algorithms to be presented in the next two chapters.

In the special case of *2d* structures, the geo-electrical property remains constant in the strike direction. Assume the strike of the *2d* structure is in *y*-direction, then $\mu(\vec{r}) = \mu(x, z)$. Substituting this into (2.64) and recognizing that terms involving $\mu(\vec{r})$ are independent of variable *y*, I can carry out the integration with respect to *y* and obtain

$$\phi_s(\vec{r}) = -\frac{I}{4\pi^2\sigma_0} \int_{-\infty}^{\infty} \int_0^{\infty} \left[\frac{\partial \ln \mu(x', z')}{\partial x'} (x' - x_s) + \frac{\partial \ln \mu(x', z')}{\partial z'} z \right] g_a(\vec{r}'; \vec{r}, \vec{r}_s) dx' dz', \quad (2.66)$$

where $\vec{r} = (x, y, 0)$, $\vec{r}_s = (x_s, y_s, 0)$ and the kernel function g_a is

$$g_a(\vec{r}'; \vec{r}, \vec{r}_s) = \int_{-\infty}^{\infty} \frac{1}{[(x - x')^2 + (y - y')^2 + z^2]^{1/2}} \cdot \frac{1}{[(x_s - x')^2 + (y_s - y')^2 + z^2]^{3/2}} dy'.$$

Performing the integration for the special case $y_s = y$ yields

$$g_a(\vec{r}'; \vec{r}, \vec{r}_s) = \begin{cases} \frac{2}{\eta\eta_s^2(\eta^2 - \eta_s^2)} [\eta^2 E(s) - \eta_s^2 K(s)], & \eta > \eta_s; \\ \frac{2}{\eta_s(\eta^2 - \eta_s^2)} [K(t) - E(t)], & \eta < \eta_s; \\ \frac{\pi}{2\eta^3}, & \eta = \eta_s; \end{cases} \quad (2.67)$$

where

$$\begin{aligned}\eta &= \sqrt{(x' - x)^2 + z^2}, \\ \eta_s &= \sqrt{(x' - x_s)^2 + z^2}, \\ s &= \frac{\sqrt{\eta^2 - \eta_s^2}}{\eta}, \\ t &= \frac{\sqrt{\eta_s^2 - \eta^2}}{\eta_s},\end{aligned}$$

and E and K are the complete elliptic integrals of the first and the second kind, respectively. It should be noted that the integral for a kernel in the general case of $y_s \neq y$ has also been carried out so that off-line potentials can be evaluated.

Since the Born equations are basic to all the algorithms in this thesis work, it is important to examine their validity. I attempt to quantify this for a simple geologic situation in which a conductive or resistive prism, either $2d$ or $3d$ is buried in a half-space. The cross-section of the geologic model and electrode geometry is shown in Fig 2.8(a). The true secondary potentials and those evaluated from (2.65) are given in Fig. 2.8(b) for a $2d$ prism and in Fig. 2.8(c) for the cube. Generally the shape of the Born potential is very similar to that of the true potential but the amplitude can be significantly different. The Born potentials are greater than the true potentials for a resistive prism and smaller than the true potential for a conductive prism. Amplitude discrepancies of 30% are noted. There are also sign differences. For a $2d$ conductive prism the true secondary potential is entirely negative but the Born solution shows a positive side lobe. The true secondary potential requires the interaction between charges; This reduces the positive charge on the far side of the prism and makes the potential negative. Overall, however, the agreement between the true and Born potentials is quite good. In fact, the correspondence for the $3d$ conductive prism is excellent. This provides optimism that analyses carried out using (2.65) or (2.66) will yield useful results.

2.9 Discussion

The basic theory of the charge accumulation in d.c. resistivity experiment is presented. I have clarified that permittivity plays no part in d.c. problems other than supply a portion of

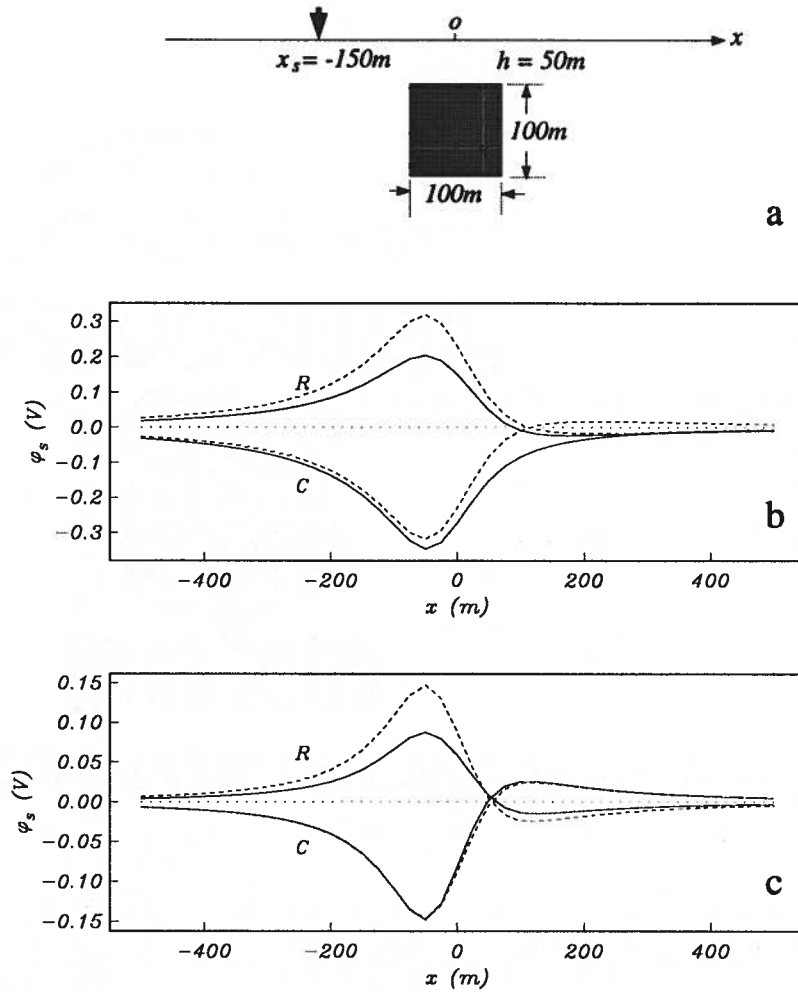


Figure 2.8 The comparison of the true and the Born approximation potentials over $2d$ and $3d$ prisms are shown in panel (b) and (c) respectively. The two prisms have the identical cross-sections, as shown in panel (a), and are embedded in a uniform half-space of 0.01 S/m . The conductivity contrast between the prism and the half-space is 10. The solid lines are the true secondary potentials. The dashed lines represent the potentials from the Born approximation. Curves labeled C and R correspond to conductive and resistive prisms respectively.

the charges accumulated in the regions of conductivity variation. I establish the relationship between the image method of calculating potentials and the effects of true charge accumulation, which states that the essence of the image method is to find a set of fictitious charges which reproduces the potential from the physically accumulated charges. I also show that the charge accumulation is the direct cause for refraction of current at a boundary separating media with

different conductivities. I quantify the charge distributions through the use of integral equations with half-space and whole-space Green's functions, which provides a greater understanding about the d.c. electrical problem.

However, the most important benefit of this investigation of charge accumulations has been in understanding the observed d.c. potentials. When the geologic structures are simple, it is often possible to sketch the magnitude and sign of charges on buried bodies and on the earth's surface that result from an arbitrary specification of the current source location. Thus secondary potentials and apparent conductivity perturbations can be estimated so as to delineate essential features of the conductivity anomaly. This helps to establish an intuitive understanding of the relationship between apparent conductivity anomalies and geo-electrical structures which is immediately useful in making preliminary interpretations of observed potentials or in making first order judgements about the validity of results produced from numerical simulation.

It is also shown that the Born approximation provides a reasonable representation of the true potential. The approximation is especially good in $3d$ cases. This indicates that it may be viable to image or invert for subsurface structures using observed potential data based upon such an approximation.

CHAPTER 3

IMAGING SUBSURFACE STRUCTURES

E-SCAN® data are distinguished from the conventional d.c. data in that the potential field on the earth's surface is recorded at many sites for each location of the current source. The number of data is further compounded by the large number of source locations which are used. This is in sharp contrast with the conventional format of d.c. resistivity experiments where a fixed array configuration is employed and all active electrodes move from one site to another in a group and only the data corresponding to the chosen array configuration are acquired at those sites. Of all the data sets from different surface d.c. resistivity experiments, an E-SCAN® data set provides probably the most complete coverage of the survey area with a great amount of redundancy. At each source location, the input current field energizes the earth differently and hence each set of surface potential measurements associated with a current source provides us with different information about the underground geo-electric structure. Such completeness in the data enables one to extract information by many different means.

For preliminary interpretation, only qualitative information need be extracted directly from the data set. This is a necessary step before inversion methods are applied. The information obtained in this stage can sometimes answer the fundamental questions regarding the existence of prospective anomalies or regional structures of interest. It can also provide a starting point for further quantitative work. Qualitative information is presented as images. The simplest is to plot the data as apparent conductivities. Features seen in the apparent conductivity may be interpretable directly. The next step in sophistication would be to use the potential data to construct a structural image of the actual conductivity. The secondary potential may be inverted to find locations of accumulated charges which are associated with conductivity anomalies. This serves to image boundaries of the anomaly. A crude estimation of the conductivity may also be constructed from the surface potentials. In this chapter, I shall present these imaging methods in the order of increased processing involved.

3.1 Direct Imaging via Apparent Conductivity

The apparent conductivity is a convenient and intuitive format to display and to examine d.c. resistivity data. It is denoted by σ_a and defined by

$$\sigma_a = G \frac{I}{\Delta\phi}, \quad (3.1)$$

where $\Delta\phi$ is the measured total potential difference between the two measuring electrodes, I is the strength of the current source, and G is the geometrical factor determined by the inter-electrode distances of a given array configuration. G is given in its general form by

$$G = \frac{1}{2\pi} \left(\frac{1}{R_{AM}} - \frac{1}{R_{AN}} - \frac{1}{R_{BM}} + \frac{1}{R_{BN}} \right), \quad (3.2)$$

where R_{AM} , R_{AN} , R_{BM} , and R_{BN} are the inter-electrode distances and (A, B) and (M, N) represent the current and potential electrode pairs respectively. When an electrode is placed at infinity, the associated terms vanish. For instance, the geometrical factor for a pole-pole array is given by $G = (2\pi R_{AM})^{-1}$.

The apparent conductivity data are usually presented in the form of plan view maps or pseudo-sections. They are indicative of the subsurface conductivity variation. The multiplicity of the data allows apparent conductivities to be synthesized for arrays with different separations and in different directions. This allows apparent conductivities to be formed in $3d$ space. Such a data volume provides a blurred $3d$ image of the actual conductivity variation and the plan-view and pseudo-section maps are easily extracted.

Of all commonly used configurations, the pole-pole array has the largest signal amplitude and effective depth. Apparent conductivities from all other arrays can be obtained by taking linear combinations of pole-pole apparent conductivities. Therefore, I choose to form the data volume from pole-pole arrays. A pole-pole array is distinctly defined by specifying its separation between the current and potential electrodes, and the orientation in which the two electrodes are aligned. The data for a pole-pole array are commonly registered at the mid-point of the array.

This is justified by the fact that measurements obtained by interchanging the current and potential electrodes are the same due to the reciprocity of the d.c. field.

Since E-SCAN® data are usually acquired over a regular grid, many different pole-pole arrays can be defined. The orientations are constrained by the grid since the electrodes must be on grid points. The electrode separations are multiples of the basic spacing between grid points along the specified direction. For each chosen pole-pole array, many data points can be synthesized throughout the survey grid so that a data map can be formed. It is apparent that for a given electrode separation, there is a number of arrays with different directions.

A set of pole-pole arrays is first chosen and the corresponding apparent conductivity data are gathered. The apparent conductivities of each array are then interpolated to form a data map over the E-SCAN® grid. Since the positions of the array electrodes are limited by the grid, the apparent conductivity is defined only over an smaller region within the E-SCAN® grid. To avoid arbitrary extrapolation near the edge, I assign the mean value of the apparent conductivities gathered from the E-SCAN® data to the edges that have no defined apparent conductivities. The interpolation is then applied to the data consisting of the directly gathered data and the assigned points on the edge. The maps with the same separation but different directions are then averaged to form a directionally averaged map. Each averaged map is assigned at a pseudo-depth proportional to the separation. The composite of all such map forms a 3d prism. Further interpolation in the vertical direction is performed when necessary and this generates the desired apparent conductivity volume.

The two horizontal dimensions of the data volume are consistent with the actual spatial coordinates but the vertical dimension does not correspond to a real depth. Thus the data may indicate the correct horizontal location of an anomaly but provide only a relative depth with respect to other anomalies. With an appropriate choice of the mapping to assign a pseudo-depth Z_e from a given electrode separation, the image from the apparent conductivity data can be indicative of the depth of burial for simple anomalous structures. Such mappings have been studied by a number of authors (e.g., Roy and Apparao 1971, Edwards 1977). Edwards (1977)

provided a rather complete list of the mapping factors for commonly used arrays. The mapping factor for pole-pole arrays is 0.86, i.e., $Z_e = 0.86R_{AM}$.

Based upon the identity derived by Roy (1978) for the surface potential over a half-space V ,

$$\phi(\eta) = \frac{1}{2\pi} \iint_V \nabla \phi(\vec{r}') \cdot \nabla \frac{1}{r'} dv, \quad (3.3)$$

where η is the distance between the current and potential electrodes, Edwards defines a signal contribution function of the depth z

$$f_e(\eta, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \nabla \phi(\vec{r}') \cdot \nabla \frac{1}{r'} dxy. \quad (3.4)$$

so that

$$\phi(\eta) = \int_0^{\infty} f_e(\eta, z) dz.$$

He then defines the pseudo-depth to be the depth above which the conductivity is responsible for half of the potential signal. That is,

$$\frac{\phi(\eta)}{2} = \int_0^{Z_{med}} f(\eta, z) dz. \quad (3.5)$$

This definition is essentially based upon the electric field distribution in the lower half-space.

If the background conductivity deviates greatly from a uniform half-space, the above mapping factor will no longer be valid. I have carried out a study of the depth mapping in the case of layered $1d$ background following the approach of Edwards (1977). I first extend (3.5) to the varying $1d$ background. The signal contribution function $f_e(\eta, z)$ is formed using the numerical solution for the potential assuming the $1d$ earth is composed of a sequence of layers with constant conductivities. The mapping factor given by the ratio of pseudo-depth over the electrode separation defined by Z_{med} , is no longer a constant. Instead, it is a function varying with the separation.

I also show that an identity similar to (3.3) exists, i.e.,

$$\phi(\eta) = \frac{1}{2\pi\sigma_a} \iint_V \sigma(z) \nabla \phi(\vec{r}') \cdot \nabla \frac{1}{r'} dv. \quad (3.6)$$

This is based upon the distribution of electric current field in the lower half-space. A signal contribution function can be defined accordingly as

$$f_c(\eta, z) = \frac{1}{2\pi\sigma_a} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(z) \nabla \phi(\vec{r}') \cdot \nabla \frac{1}{r'} dxy. \quad (3.7)$$

Results from numerical simulations indicate that a signal contribution function formed from weighted sum of $f_e(\eta, z)$ and $f_c(\eta, z)$ is necessary in order for the definition of the mapping factor work for background conductivities with conductive and resistive layers. The criterion is that the apparent conductivity curve plotted using the varying depth mapping should position the anomalies approximately at the depths of the corresponding conductivity layers. One combination is

$$f(\eta, z) = \frac{f_e(\eta, z)}{1 + \mu_a^2} + \frac{\mu_a^2 f_c(\eta, z)}{1 + \mu_a^2}, \quad (3.8)$$

where $\mu_a = \sigma_a(\eta)/\sigma_0$ and σ_0 is the surface conductivity. Simulations suggest that an embedded layer reduces the mapping depth for an array which has strong responses to that layer. Fig. 3.1 shows the mapping depth for a three-layered background. However, the use of such varying depth mapping relies upon the knowledge of the background conductivity. The study of its effectiveness is inconclusive and so is its possible application to field data.

The apparent conductivity data can also be displayed in the form of relative anomalies with respect to the apparent conductivity of the pure $1d$ structure. Numerical experiments indicate that when the relative anomaly data are plotted with the half-space mapping factor, they tend to indicate the depth of burial for simple anomalies.

The data image can be enhanced by applying low-pass and directional filtering at each depth in the data volume. Low-pass filtering is used primarily to accentuate the coherent features in

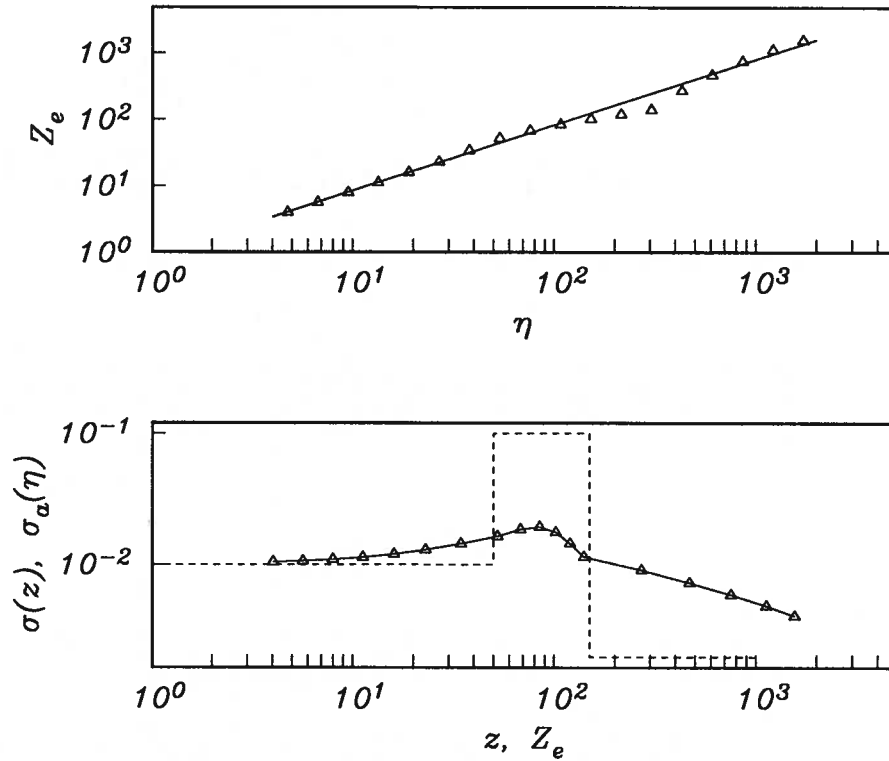


Figure 3.1 Varying depth mapping for a three-layered background. Panel (a) is the pseudo-depth as a function of the pole-pole array separation. The triangles are the $1d$ background mapping and the solid line is the half-space mapping. Panel (b) shows the $1d$ apparent conductivity (solid line) plotted with pseudo-depth in (a). The triangle indicate the actual data points. The dashed line is the true conductivity.

a data map by eliminating the noise components. The coherent features are mainly in the low wavenumber band. There are two major sources of data error. The first is the random noise associated with processes such as potential measurements, electrode positioning, telluric current and cultural noise. The second is the electric signal from shallow or near-surface conductivity anomalies. Small scale conductivity anomalies close to electrodes can distort the electrical measurement even at large current-potential separations. These measurements are assigned to a greater pseudo-depth proportional to the separation. Thus the distortion of the near surface conductivity variations can appear as deep anomalies in the apparent conductivity data. Therefore, the response of an array to near surface anomalies may constitute valid signal at small array separations but it becomes effective noise for the apparent conductivities measured at large

electrode separations. These distortions usually concentrate in the high wavenumber band. When apparent conductivities are displayed as a means of imaging, they should be smoothed to suppress this type of noise. The degree of smoothing to be applied should reflect the decreasing resolution of an array as the electrode separation of the array increases. Low-pass filtering in the wavenumber domain is an effective way to carry out the smoothing. Geological features may exhibit strong linearity in certain directions. Directional filtering is an effective way to either enhance or to suppress such linear features.

The pole-pole array is advantageous for its large effective depth and simple anomaly pattern. However, another good candidate is the square array. It was introduced by Habberjam (1979). A square array involves four active electrodes on the surface. The electrodes **A**, **B**, **N**, **M** are arranged so that each occupies a corner of a square in that order. As the array expands all the inter-electrode distances expand simultaneously. Although acquiring square array data is more difficult in the field, synthesizing them from an E-SCAN® data set can be readily achieved. In addition, since E-SCAN® data are acquired over a surface with planar arrays (in contrast to linear arrays), it is reasonable to display the data using a planar array such as the square array. The pseudo-depth of a square array is given by $Z_e = 0.45a$, where a is the length of one side of the array.

Despite its relatively shallow effective depth compared with a pole-pole array, there are two advantages with the square array. Firstly, the data synthesized for square arrays from E-SCAN® data sets will not be affected by the placement error of the infinite electrodes **B** and **N**. Secondly, a square array measures a very compact anomaly over a body of finite geometry and the response of shallow structures at large off-set is very small. This renders the array insensitive to the near-surface variation in conductivity. Therefore, a square array may represent a good tradeoff in the quest for a large effective depth and reduced sensitivities to near surface variations.

To conclude the discussion, I present a simple example in Fig. 3.2. A conductivity model is formed which has a conductive cube buried beneath a surface layer of random conductivity variations in an otherwise uniform half-space. The cube is 200 metres on side and located

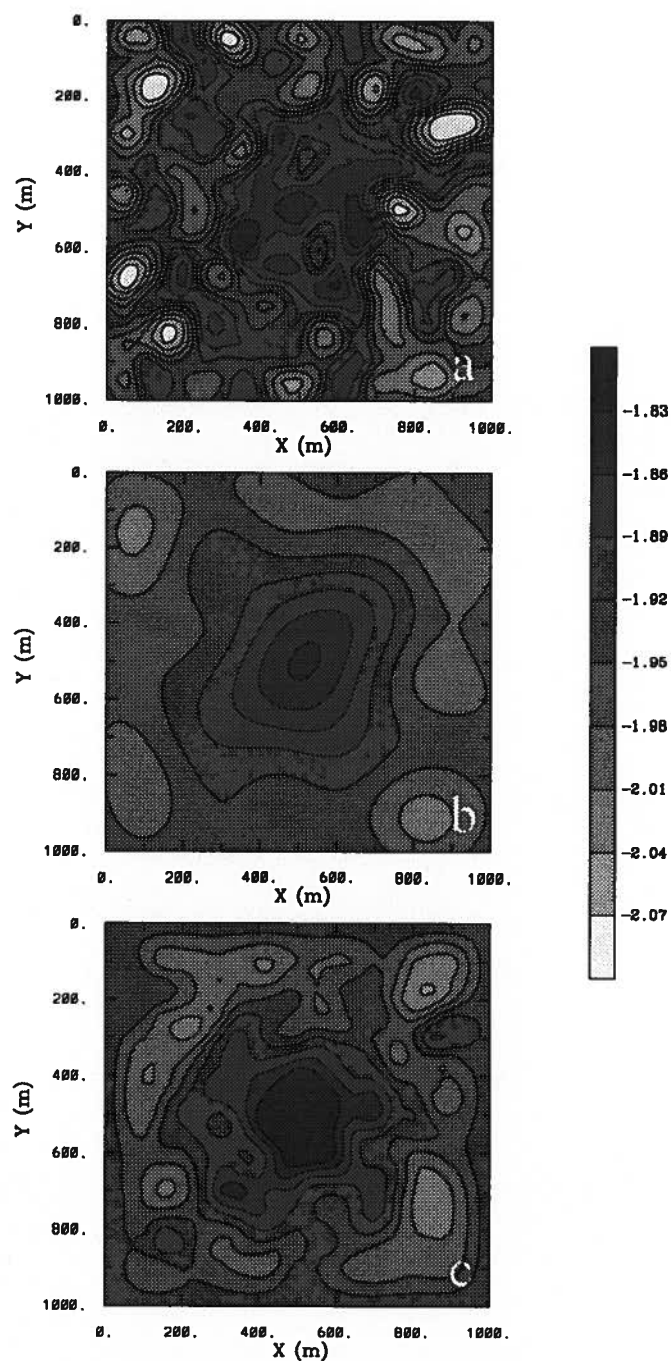


Figure 3.2 Responses of the pole-pole array and the square array to a conductive cube buried beneath a surface layer with correlated random conductivity variations. The apparent conductivity from pole-pole array is shown in (a) for pseudo-depth $Z_e = 100\text{m}$. A low-pass filtered version of (a) is shown in panel (b), which has greatly suppressed the effect of surface variation. The data from the square array at the same pseudo-depth are shown in panel (c). It is clear that the square array is less sensitive to surface variations. The grey scale is in $\log_{10} \sigma_a$.

50 metres below the surface. The surface layer consists of correlated random perturbations to the uniform background with a correlation length of 50 metres. The response to this random layer dominates at all separations in the data from pole-pole arrays. The slice at pseudo-depth $Z_e = 100\text{m}$, as shown in Fig. 3.2(a), is characteristic of this “geological” noise. The same slice from the low-pass filtered version is shown in Fig. 3.2(b); the anomaly caused by the buried conductive cube is clearly displayed. As a comparison, a slice of data at the same pseudo-depth from square arrays is shown in Fig. 3.2(c). No low-pass filtering was applied, yet a concentrated anomaly for the buried cube is well defined over the relatively weak response to the near-surface variations.

3.2 Imaging via Equivalent Source from Secondary Potential

The secondary potential arises from the accumulated charges associated with conductivity anomalies. An image of the charge constructed from the secondary potential, therefore, should provide useful information. In this section, I seek to construct such an image by a sequence of equivalent source layers.

The potential field is related to the subsurface charge density by (see chapter 2)

$$\phi_s(\vec{r}) = \frac{1}{2\pi\epsilon_0} \iiint_V \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} dv, \quad (3.9)$$

where $\vec{r}$ and $\vec{r}'$ are the observation and source points respectively, ρ is the charge density, and V is the lower half-space. Adopting a right handed Cartesian coordinate system with origin at the surface and the z -axis pointing downward, the above equation can be rewritten as

$$\phi_s(x, y) = \frac{1}{2\pi\epsilon_0} \int_0^\infty \rho(x, y, z') \otimes \otimes \frac{1}{\sqrt{x^2 + y^2 + z'^2}} dz', \quad (3.10)$$

where the symbol $\otimes \otimes$ denotes a $2d$ convolution in the x - y domain. It follows from (3.10) that there exists a layer of equivalent source distribution at any depth which will reproduce the given

potential on the surface. Thus, I can proceed to construct a set of equivalent source layers with a unit thickness at depths z_i ($i = 1, \dots, n$) from

$$\phi_s(x, y) = \frac{1}{2\pi\epsilon_0} \rho(x, y, z_i) \otimes \otimes \frac{1}{\sqrt{x^2 + y^2 + z_i^2}}. \quad (3.11)$$

When these source layers are assembled together according to their depths, they form a $3d$ equivalent source image which may indicate the concentration of accumulated charges. A similar technique is proposed by Karous and Hjelt (1983) for interpreting VLF-EM measurements.

The solution of (3.11) is a linear deconvolution problem in a $2d$ domain and can be solved by spectral division. Define the $2d$ Fourier transform pair as (Bracewell 1978)

$$\begin{aligned} \tilde{f}(p, q) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-ipx - iqy} dx dy, \\ f(x, y) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(p, q) e^{ipx + iqy} dp dq. \end{aligned} \quad (3.12)$$

where (p, q) are the wavenumbers in the x - and y -directions respectively. Applying the Fourier transform to (3.11) yields

$$\frac{\tilde{\rho}(p, q, z_i)}{\epsilon_0} = \tilde{\phi}_s(p, q) \sqrt{p^2 + q^2} e^{z_i \sqrt{p^2 + q^2}}, \quad i = 1, \dots, n \quad (3.13)$$

where $\tilde{\phi}_s$ and $\tilde{\rho}$ are the Fourier transform of ϕ_s and ρ respectively. $\tilde{\phi}_s(p, q)$ can be computed by Fourier transforming the secondary potential on the surface. Thus, given a set of depths z_i , $\tilde{\rho}$ can be calculated and the equivalent source layers in the spatial domain are obtained by applying an inverse Fourier transform.

The equivalent source construction is essentially a two-step process. The first step downward continues the surface potential (by the operator $e^{z_i \sqrt{p^2 + q^2}}$) and the second calculates the vertical derivative of the continued potential (by operator $\sqrt{p^2 + q^2}$). The downward continuation is a divergent process. In order to avoid unphysically large amplitudes in the constructed equivalent

source, I apply a low-pass filter to $\tilde{\phi}_s$. The cut-off wavenumber decreases as the depth z_i increases and is determined by

$$k_c = -\frac{\ln \delta}{z_i}, \quad (3.14)$$

where δ is a specified percentage. $1/\delta$ effectively limits the magnification allowed in the downward continuation and, therefore, eliminates higher wavenumber components from the equivalent source at greater depth.

Fig. 3.3 illustrates the imaging method using the secondary potential over a single prism buried in a uniform half-space. The secondary potential map generated by a single point current source over a conductive prism is shown in Fig. 3.3(a). The application of the above imaging method generates a $3d$ image of the equivalent source whose two sections are shown in Fig. 3.3 (b) and (c). The boundary of the buried prism is outlined in each section. It can be seen that the minimum of the equivalent source corresponds to the top portion of the prism. From the viewpoint of imaging, this simple processing of the secondary potential has provided useful information. In general, the extrema in such a $3d$ image serve to indicate the depth of burial of an anomalous conductivity, since the top portion has a greater density of accumulated charges.

With multiple sources, the secondary potential maps can be stacked to simulate the potential from near vertical or horizontal primary fields. The equivalent source obtained using these stacked secondary potential maps provides better images as the effects from the strong asymmetry of the primary field relative to the conductivity anomaly is reduced.

The method is purely based upon the mathematical properties of the potential field. As such, its usefulness is very limited. The image is rudimentary but the approach can be further developed. One can utilize the properties of the actual d.c. field to invert the potential for the charge density, and hence, image the conductivity anomaly. This leads to the next level of the processing, namely, charge density imaging.

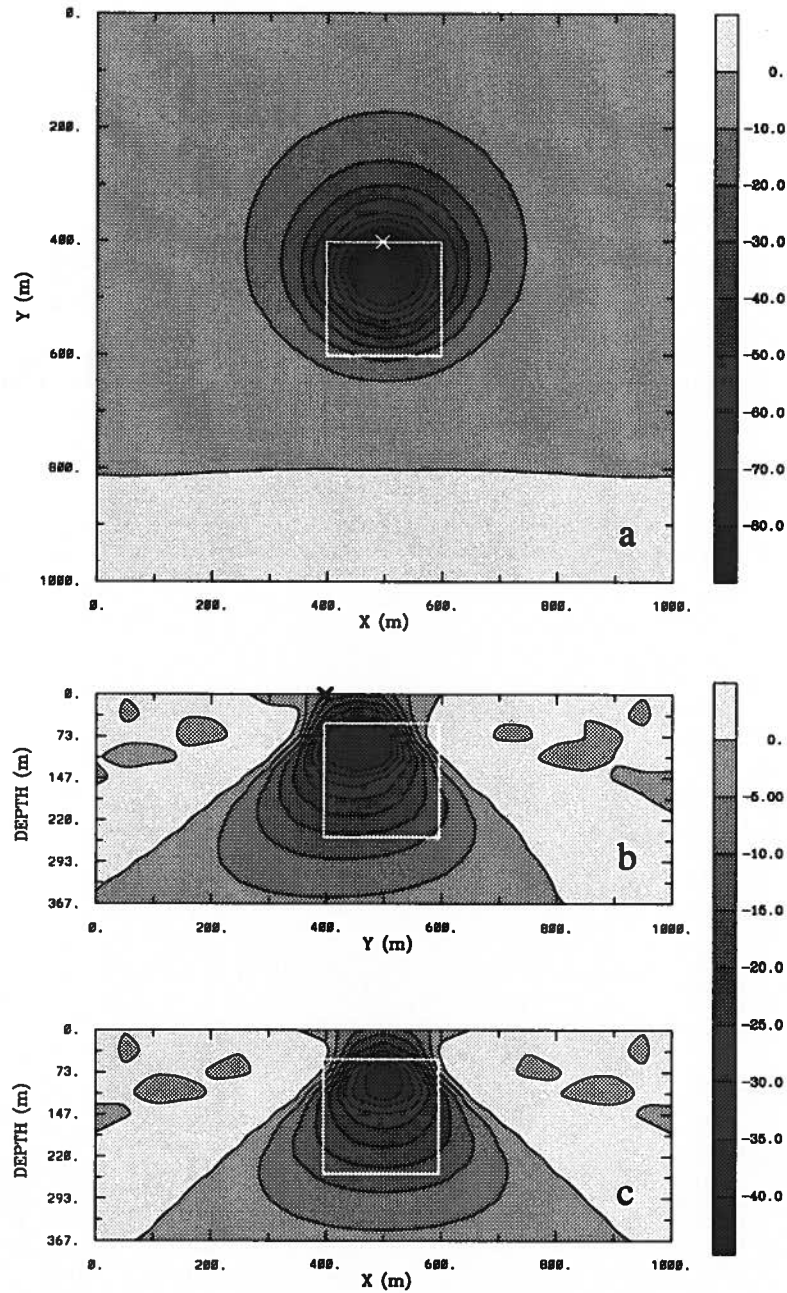


Figure 3.3 Imaging conductivity anomaly using equivalent sources. Panel (a) shows the secondary potential (mV) generated by a point current source (marked by the cross) over a conductive prism in a uniform half-space. Panel (b) is the section at $x = 500\text{m}$ from the constructed $3d$ equivalent source image. The magnitude given by grey scale represents a scaled charge density and only relative strength of the source. Panel (c) is another section at $y = 500\text{m}$. The outline of the prism is shown by the white square in all panels.

3.3 Charge Density Imaging of 2d Structures

In practical applications, if the anomaly due to a geologic structure has a preferred strike and if it is sufficiently elongated, the 2d approximation may be valid. Even though the Born approximation may be somewhat more inaccurate for the 2d models compared to a 3d model, from a computational viewpoint there are benefits to using a 2d formalism if it is possible. For this reason, I develop a 2d imaging algorithm using charge densities based upon (2.66). The term involving $\mu(x, z)$ in (2.66) can be written as

$$c(x, z) = -\frac{I}{4\pi^2\sigma_0} \frac{\nabla \ln \mu(x, z) \cdot \vec{\eta}_s}{\eta_s},$$

where $\vec{\eta}_s = (x - x_s, z)$ is the projection onto the x - z plane of the vector from the source to a given subsurface point. (2.66) can then be written as

$$\phi_s(x) = \int_{-\infty}^{\infty} \int_0^{\infty} c(x, z) g(x', z'; x_s, x) dx' dz', \quad (3.15)$$

where $g(x', z'; x_s, x) = \eta_s g_a$ is the new kernel function. The function $c(x, z)$ represents the total effect of the accumulated charge and behaves like the accumulated charge density: it is non-zero whenever there is a component of $\vec{\eta}_s$ parallel to the conductivity gradient $\nabla \ln \mu(\vec{r})$. It vanishes identically away from any boundary of the conductivity structure.

Equation (3.15) linearly relates the charge density $c(x, z)$ to the secondary potential. As such, secondary potentials recorded at a number of locations x , and arising from a fixed source at x_s , can be inverted to recover an estimate of $c(x, z)$. Given that physical charge accumulates at the boundaries between blocks of different conductivities, it follows that an algorithm which is effective in delineating the locations of the charge, can image the subsurface structure.

This imaging is accomplished by setting up a linear inverse problem. The cross-section perpendicular to the strike is first discretized into $m_x \times m_z$ rectangular cells according to the partitioning $(x_1, \dots, x_j, \dots, x_{m_x})$ and $(z_1, \dots, z_k, \dots, z_{m_z})$. In each cell $c(x, z)$ is assumed to be

constant. Equation (3.15) becomes

$$\phi_s^i = \sum_{j=1}^{m_s} \sum_{k=1}^{m_z} A_{jk}^i c_{jk}, \quad i = 1, \dots, n \quad (3.16)$$

where ϕ_s^i is the datum at location x_i , n is the number of the data available, and A_{jk}^i is the integral of i^{th} kernel function over jk^{th} cell Δ_{jk} ,

$$A_{jk}^i = \iint_{\Delta_{jk}} g(x', z'; x_s, x_i) dx' dz'.$$

The number of data are generally far fewer than the number of cells and hence (3.16) is an underdetermined system. Equivalently, the solution is non-unique. To obtain a particular solution, I seek a "simple" charge density which can explain the observed secondary potential data. The measure of simplicity is defined by the weighted l_1 norm of the charge density. I do not fit the data exactly but rather introduce a misfit variable ϵ_i for each data equation and permit a total misfit bounded by a prescribed value. The linear inverse problem becomes the following minimization problem,

$$\begin{aligned} \text{min.} \quad & \Phi = \sum_{j=1}^{m_s} \sum_{k=1}^{m_z} w_{jk} |c_{jk}|, \\ \text{subject to} \quad & \sum_{j=1}^{m_s} \sum_{k=1}^{m_z} \frac{A_{jk}^i}{\delta_i} c_{jk} + \epsilon_i = \frac{\phi_s^i}{\delta_i}, \quad i = 1, \dots, n \\ & \sum_{i=1}^n |\epsilon_i| \leq \beta n \sqrt{\frac{2}{\pi}}, \end{aligned} \quad (3.17)$$

where δ_i is the estimated standard deviation of the error associated with i^{th} datum, β is a fit parameter which controls the fit to the data. $n\sqrt{2/\pi}$ is the expected value of the total misfit assuming a Gaussian distribution of the noise. w_{jk} is a set of weighting coefficients which ultimately determine the type of model which is obtained. We are free to choose any weighting function. I have specified

$$w_{jk} = (z_{ck} - z_0)^{3/2} \eta_s,$$

where z_{ck} is the central depth of the cell. The first part is introduced to overcome the natural decay of the kernel function with the depth and the second part to compensate the decay of the primary field with the distance.

The above minimization problem is solved using a linear programming technique (e.g., Murty 1983). This choice suits the need for finding a sparse representation of the charge density, since the number of non-zero elements in the LP solution is bounded by the number of constraints. In order to accommodate the absolute value of variables in both the objective function and the misfit constraint, each bipolar variable needs to be expressed as the difference of two positive variables. In this way, when the absolute value is replaced by the sum of the two corresponding positive variables, the solution of the new minimization problem yields the solution for the original problem.

When a current is put into the ground the physical charge density depends upon the source position relative to the conductivity anomaly. The charge density c_{jk} found from the above inversion reflects this relationship. The result only provides a partial image of the boundary. However, the field from a different current source position illuminates the boundary in a different way. A relatively complete image of the boundary can be obtained by inverting for $c(x, z)$ from many different source positions using (3.15).

Taking the advantage of the E-SCAN® data set, I can form many sets of common source potential data. Each set can be inverted to yield a part of the boundary image. When all of the partial images are combined, they give a rather complete picture. Fig. 3.4 shows the result obtained by inverting synthetic data generated from a simple prism buried in a uniform half-space. Seven sets of secondary potential data corresponding to different source locations, shown in Fig. 3.4(a), are used in the inversion. Fig. 3.4(b) is the combined image. The top and the bottom boundary of the prism are correctly imaged. The charge density also has the correct sign. It is negative at the top where the current flows into the conductive body and is positive at the bottom where the current flows out of the prism. Fig. 3.4(c)-(f) show some results from

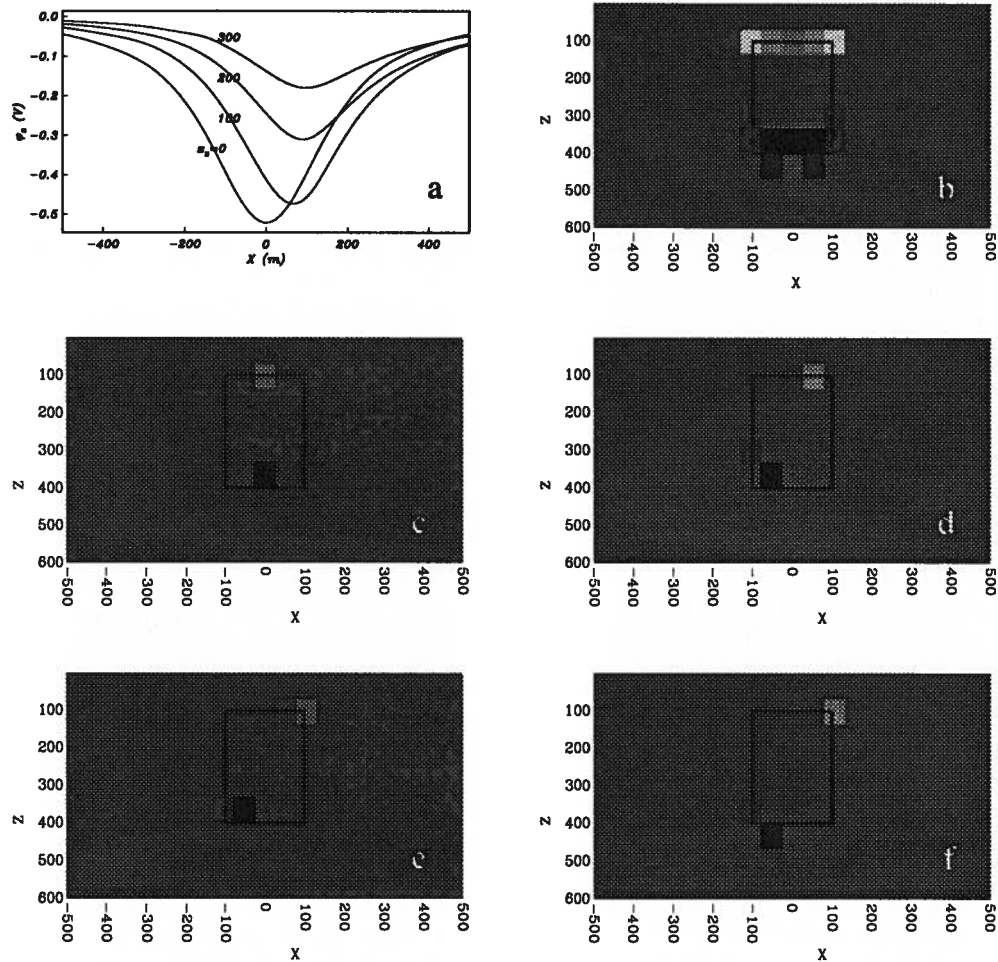


Figure 3.4 Imaging subsurface structure using charge density. Panel (a) shows the secondary potentials corresponding to the labeled source locations over a $2d$ conductive prism buried in a uniform half-space. Panel (b) is the composite of the individually recovered charge density images corresponding to seven sources located from -300 to 300 meters in equal spacing. The light region represents the negative charge density and the dark region the positive charge density. The solid line in the image indicates the boundary of the $2d$ prism. Panels (c)-(f) are images corresponding to sources at 0, 100, 200, and 300 meters, respectively.

different individual inversions. They clearly show the different parts of the boundary imaged by the potentials from the different source locations.

From the viewpoint of physics, this approach is very appealing. The charge accumulation on the boundary is a physical reality in the d.c. experiment and any attempt to recover it is valuable. The above example illustrates that the charge density imaging can work satisfactorily for simple structures. However, for a closed body in a uniform half-space, the sum of the charge accumulated on the boundary is zero. So the secondary potential is the field of dipolar and higher-order sources. The recovery of monopole distributions from such fields is difficult and the solution can be highly non-unique.

Because the subsurface charge density depends upon the source position, there is no single distribution of charges which is compatible with all the data. Therefore, in the recovery of the secondary charge density, each inversion uses a single current field. Results from several such individual inversions form a composite image as the final result. A more stable approach would be to invert for a single model using all the data simultaneously. This leads us to the direct formulation to recover a conductivity image.

3.4 Conductivity Imaging of 2d Structures

Equation (2.66) presents a linear relationship between the gradients of $\ln \mu(\vec{r})$ and the secondary potential. By forming a linear inverse problem to recover $\ln \mu(\vec{r})$ directly, one can simultaneously invert data from many different current locations and thereby greatly reduce the non-uniqueness. I adopt the same type of discretization as in the charge density imaging. The conductivity section is divided into $m_x \times m_z$ cells each of which is specified by a constant value of $\ln \mu$. The model for the inverse problem then becomes

$$m_{jk} = \ln(\mu_{jk}), \quad j = 1, \dots, m_x; \quad k = 1, \dots, m_z$$

Under these assumptions, the gradient becomes discrete. $\frac{\partial \ln \mu}{\partial x}$ is non-zero only on the vertical interfaces between cells and $\frac{\partial \ln \mu}{\partial z}$ is non-zero only on the horizontal interfaces:

$$\begin{aligned} \frac{\partial \ln \mu}{\partial x} &= (m_{jk} - m_{j-1,k}) \delta(x - x_j), \quad (x, z) \in \Gamma_{jk}^x, \quad j=1, \dots, m_x+1, k=1, \dots, m_z \\ \frac{\partial \ln \mu}{\partial z} &= (m_{jk} - m_{j,k-1}) \delta(z - z_k), \quad (x, z) \in \Gamma_{jk}^z, \quad j=1, \dots, m_x, k=1, \dots, m_z+1 \end{aligned} \quad (3.18)$$

Where Γ_{jk}^x and Γ_{jk}^z denote the vertical and horizontal interfaces respectively. When the index j equals 0 or $m_x + 1$, or index k equals 0 or $m_z + 1$, the model parameter m_{jk} is taken to be the background value, which is zero.

Substituting the gradient functions (3.18) into (2.66) yields a discrete data equation of the form

$$\phi_s^i = \sum_{j=1}^{m_x} \sum_{k=1}^{m_z} m_{jk} \gamma_{jk}^i, \quad i = 1, \dots, n \quad (3.19)$$

where

$$\begin{aligned} \gamma_{jk}^i &= \gamma_{jk}^{xi} - \gamma_{j+1,k}^{xi} + \gamma_{jk}^{zi} - \gamma_{j,k+1}^{zi}, \\ \gamma_{jk}^{xi} &= -\frac{I}{4\pi^2 \sigma_0} (x_j - x_{si}) \int_{\Gamma_{jk}^x} g^a(x', z'; x_{si}, x_i) dz', \\ \gamma_{jk}^{zi} &= -\frac{I}{4\pi^2 \sigma_0} z_k \int_{\Gamma_{jk}^z} g^a(x', z'; x_{si}, x_i) dx'. \end{aligned} \quad (3.20)$$

I choose to minimize an objective function composed of a combination of the l_1 norm of the model and the l_1 norm of the model gradient.

$$\begin{aligned} \Phi &= \alpha \sum_{j,k} w_{jk} |m_{jk}| + \\ & (1 - \alpha) \left[\sum_{j,k} w_{jk}^x |m_{jk} - m_{j-1,k}| + \sum_{j,k} w_{jk}^z |m_{jk} - m_{j,k-1}| \right]. \end{aligned}$$

The weighting coefficients w_{jk} , w_{jk}^x and w_{jk}^z are introduced to control the relative importance of a particular cell in the objective function based on the cell size and depth. The parameter α takes on a value in $[0, 1]$ and determines the relative importance between the model and the gradient. Mathematically, the second part is a measure of the variation in the model in the sense of l_1 norm.

When α approaches unity, the inversion produces a smallest model without much restriction on the variation. As α approaches zero, the inversion produces a minimum-variation model (Dosso and Oldenburg 1989).

The minimization problem can be solved using standard methods. Misfit parameters are introduced, as they were in the charge density imaging, so that the final data are fit only to within a global tolerance. Variables which can acquire both positive and negative values are again replaced by the difference of two positive variables and the corresponding absolute value is replaced by the sum of the two new variables. In addition, an upper and lower bound can also be imposed on the model elements as constraints. The kernel γ_{jk}^i are computed numerically. It can be shown that the kernel γ_{jk}^i is symmetric for interchanged source and receiver locations. Therefore, for every pair of electrodes in the E-SCAN® data, only one datum entry is required as the d.c. potential satisfies the reciprocity.

To illustrate the above algorithms, a set of secondary potential data from a synthetic model is inverted. The model consists of a conductive prism and a resistive prism buried in a uniform half-space (see Fig. 3.5(b)). The half-space conductivity is 1mS/m. The conductivities of the two prisms are 4mS/m and 0.25mS/m, respectively. A set of E-SCAN® data are generated over a traverse of 21 grid points. Fig. 3.5(a) shows the corresponding apparent conductivity pseudo-section. 200 secondary potentials corresponding to this pseudo-section are used in the inversion. The model section is divided into 300 cells with thickness increasing with the depth. An upper bound of 10 for conductivity contrast is imposed on the solution. Fig. 3.5(c) shows the results of a minimum variation model with $\alpha = 0.9$. The conductive prism is defined reasonably well. So is the top of the resistive prism. The poorer recovery of the bottom of the resistive prism is due to the fact that the Born approximation differs more from the true potential in the case of resistive prisms. However, for the purpose of imaging the structure, the algorithm has worked satisfactorily.

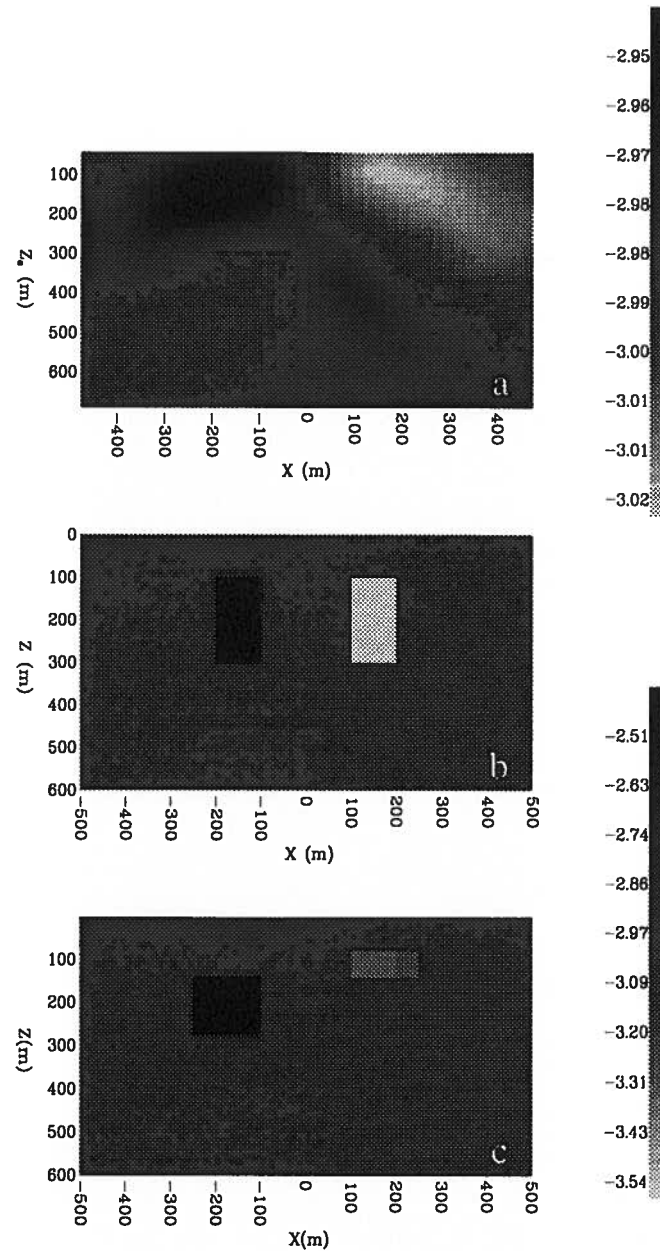


Figure 3.5 Imaging subsurface conductivity. The synthetic model shown in panel (b) consists of a conductive (4mS/m) and a resistive $2d$ prism (0.25mS/m) in a uniform half-space of 1mS/m. 200 secondary potential data on the surface are used in the inversion. Panel (a) is the corresponding apparent conductivity pseudo-section. Panel (c) is the recovered conductivity model. The upper grey scale is for panel (a) and the lower grey scale is for panels (b) and (c). The scale is in $\log_{10} \sigma$.

3.5 Discussion

The display of apparent conductivity for direct imaging is discussed. It is demonstrated that wavenumber filtering can be used to suppress geological noise such as near-surface variations in the conductivity. My study also indicates that a square array is superior in dealing with such noise. An algorithm is also presented to construct a $3d$ equivalent source image from common source secondary potentials on the surface. The image can indicate the depth of burial for simple conductivity anomalies.

Two imaging algorithms are then developed for $2d$ models using the Born equation. The first approach is to image the boundary of subsurface structures using the accumulated charge density. The secondary potential is produced by the accumulated charges on the subsurface boundaries and consequently the recovery of the charge density effectively images these boundaries. Linear programming is employed to invert the secondary potentials for a sparse representation of the charge density due to each of the current sources. The composite of these images yields a relatively complete image for simple subsurface structures. The algorithm has worked satisfactorily for simple models. However, the difficulty lies in combining a final result from several independent inversions. This is generally less stable than producing a final image using all the available data at once.

To overcome the difficulties encountered in the charge density imaging, a direct formulation is developed which inverts approximately for simple conductivity structures. The algorithm also employs linear programming for the inverse solution. An approximate conductivity model is constructed which satisfies all the available data. The method works well for relatively simple structures. For the smallest l_1 model construction, the algorithm is very efficient. As the component of variation increases, the algorithm becomes slower due to the linear programming solution. Because of the approximations involved, the semi-quantitative result mostly serves as an image representation of the model. In general, the approximation is less accurate in the $2d$ cases than it is in $3d$ cases. Consequently, the application of such an inversion can be limited in practice.

CHAPTER 4

APPROXIMATE 3d INVERSION

The d.c. resistivity problem is by its nature a *3d* problem. Firstly, the source in d.c. resistivity experiments is always a point source so the electric field is *3d*. Secondly, geo-electrical structures are generally *3d*. Although the study of *1d* and *2d* problems can provide much insight into theoretical work and data interpretation, in practice their application is rather limited. Only when the *3d* problem is tackled, can we use the d.c. resistivity method to its fullest advantage. In principle, the techniques used in *2d* analyses can be extended to *3d* conductivities. In practice, however, the large numbers of model elements and data result in a large matrix to be inverted and the approach is impractical. I choose to develop a different methodology motivated by the multiplicity in E-SCAN® data.

An integral equation is developed in which the secondary potential anomalies from a pole-pole array is expressed as a depth integral of the logarithmic conductivity anomaly convolved with a kernel function in the horizontal plane. The kernel takes the separation and orientation of specific current and potential electrode pairs as parameters. Fourier transforming the data equation decouples wavenumber components so that the surface response of a given pole-pole configuration at any wavenumber is obtained by carrying out a single integral with respect to depth. This decomposition allows a full *3d* inversion to be carried out by solving a *1d* linear inverse problem at each wavenumber. In the following I shall first develop the integral equation and then proceed to the inversion.

4.1 Data Equation

I begin with the Born equation (2.65),

$$\phi_s(\vec{r}_{obs}) = \frac{I}{4\pi^2\sigma_0} \iiint_V \left(\nabla \ln \mu(\vec{r}') \cdot \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \right) \frac{1}{|\vec{r}' - \vec{r}_{obs}|} dv.$$

Applying the vector identity $\nabla \cdot (\psi \vec{a}) = \psi \nabla \cdot \vec{a} + \nabla \psi \cdot \vec{a}$, for an arbitrary scalar ψ and an arbitrary vector $\vec{a}$, to the integrand in (2.65) yields

$$\begin{aligned} \phi_s(\vec{r}_{obs}) = & -\frac{I}{4\pi^2\sigma_0} \iiint_V \ln \mu(\vec{r}') \nabla \cdot \left(\frac{1}{|\vec{r}' - \vec{r}_{obs}|} \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \right) dv \\ & + \frac{I}{4\pi^2\sigma_0} \iiint_V \nabla \cdot \left(\ln \mu(\vec{r}') \frac{1}{|\vec{r}' - \vec{r}_{obs}|} \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \right) dv. \end{aligned} \quad (4.1)$$

With the aid of Gauss' theorem the last integral in the above equation can be expressed as

$$\beta = \frac{I}{4\pi^2\sigma_0} \iint_S \ln \mu(\vec{r}') \frac{1}{|\vec{r}' - \vec{r}_{obs}|} \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \cdot \hat{n} ds,$$

where S represents the entire boundary of the lower half-space with outward normal vector $\hat{n}$.

The contribution to β from the flat upper surface of the half-space is zero because

$$\nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \cdot \hat{n} = 0$$

there. Along the remaining boundaries,

$$\frac{1}{|\vec{r}' - \vec{r}_{obs}|} \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \cdot \hat{n} \propto \frac{1}{|\vec{r}'|^3} \quad \text{as } |\vec{r}'| \rightarrow \infty.$$

and hence the boundary integral vanishes. Thus β is identically zero.

Applying the same vector identity to the first integral in (4.1) yields

$$\begin{aligned} \phi_s(\vec{r}_{obs}) = & -\frac{I}{4\pi^2\sigma_0} \iiint_V \ln \mu(\vec{r}') \nabla \frac{1}{|\vec{r}' - \vec{r}_{obs}|} \cdot \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} dv \\ & + \frac{I}{4\pi^2\sigma_0} \iiint_V \ln \mu(\vec{r}') \frac{1}{|\vec{r}' - \vec{r}_{obs}|} \nabla^2 \frac{1}{|\vec{r}' - \vec{r}_s|} dv. \end{aligned} \quad (4.2)$$

The second term in (4.2) vanishes identically since

$$\nabla^2 \frac{1}{|\vec{r}' - \vec{r}_s|} = -4\pi \delta(\vec{r}' - \vec{r}_s)$$

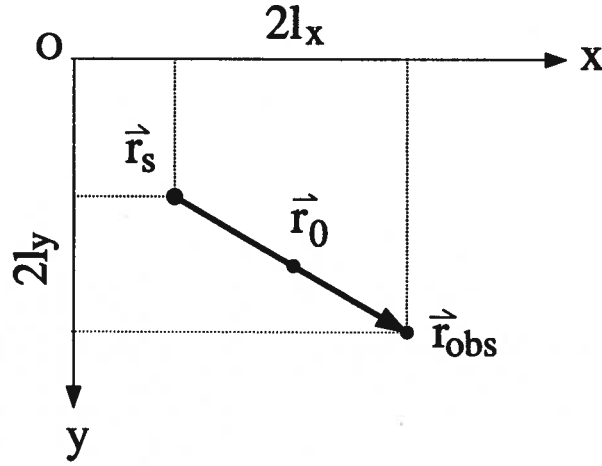


Figure 4.1 Geometry of the surface pole-pole array is specified by a vector $2\vec{l}$ pointing from the source location $\vec{r}_s$ to observation $\vec{r}_{obs}$. $\vec{r}_0$ is the midpoint of the array, where the pole-pole potential is recorded.

and $\ln \mu(\vec{r}_s) = 0$ because it has been assumed that the surface conductivity is σ_0 . Thus I obtain the following expression for the secondary potential on the surface,

$$\phi_s(\vec{r}) = -\frac{I}{4\pi^2\sigma_0} \iiint_V \ln \mu(\vec{r}') \nabla \frac{1}{|\vec{r}' - \vec{r}_s|} \cdot \nabla \frac{1}{|\vec{r}' - \vec{r}_{obs}|} dv. \quad (4.3)$$

Equation (4.3) is a Fredholm equation of the first kind relating the secondary potential to the perturbation of logarithmic conductivity. The equation is similar to that presented by Boerner and West (1989) which was derived using an integral equation for the potential on the boundary of an isolated inhomogeneity with constant conductivity.

Since E-SCAN® data are generally acquired on a regular grid, it is convenient to explicitly consider data generated from pole-pole arrays with fixed separations and orientations. As in Fig. 4.1, let $\vec{r}_0 = (x, y, 0)$ be the midpoint of the array, and let $2\vec{l}$ be the vector pointing from source electrode to potential electrode. Then

$$\begin{aligned} \vec{r}_s &= \vec{r}_0 - \vec{l}, \\ \vec{r}_{obs} &= \vec{r}_0 + \vec{l}. \end{aligned}$$

Substituting into (4.3) and converting into relative anomalies $\delta\phi = \phi_s/\phi_p$ yields

$$\delta\phi(\vec{r}_0; \vec{l}) = -\frac{l}{\pi} \iiint_V \ln \mu(\vec{r}') \nabla \frac{1}{|\vec{r}_0 + \vec{l} - \vec{r}'|} \cdot \nabla \frac{1}{|\vec{r}_0 - \vec{l} - \vec{r}'|} dv, \quad (4.4)$$

where $l = |\vec{l}|$. Since $\vec{r}_0 = (x, y, 0)$ and $\vec{l} = (l_x, l_y, 0)$, it is recognized that (4.4) involves a convolution operation in the x - y domain. Thus the surface pole-pole potential anomaly can be expressed as

$$\delta\phi(\vec{r}_0; \vec{l}) = \int_0^\infty \ln \mu(\vec{r}) \otimes \otimes g(\vec{r}; \vec{l}) dz, \quad (4.5)$$

where $g(\vec{r}; \vec{l})$ is the kernel function

$$\begin{aligned} g(\vec{r}; \vec{l}) &= -\frac{l}{\pi} \nabla \frac{1}{|\vec{r} + \vec{l}|} \cdot \nabla \frac{1}{|\vec{r} - \vec{l}|} \\ &= -\frac{l [(x+l_x)(x-l_x) + (y+l_y)(y-l_y) + z^2]}{\pi [(x+l_x)^2 + (y+l_y)^2 + z^2]^{3/2} [(x-l_x)^2 + (y-l_y)^2 + z^2]^{3/2}}. \end{aligned} \quad (4.6)$$

Equation (4.5) relates the surface pole-pole potential anomaly data from a pole-pole array to the subsurface conductivity structure. It is noticed that the anomaly is a depth integral of the conductivity perturbation convolved with a kernel in the x - and y -directions.

Define the Fourier transform $\mathcal{F}_{xy}$ over the infinite plane and its inverse transform $\mathcal{F}_{xy}^{-1}$ as given by (3.12)

$$\begin{aligned} \tilde{f}(p, q) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-ipx - iqy} dx dy, \\ f(x, y) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(p, q) e^{ipx + iqy} dp dq. \end{aligned}$$

Taking the 2d Fourier transform of (4.5) and applying the convolution theorem yields

$$\tilde{e}_j(p, q) = \int_0^\infty \tilde{m}(p, q, z) \tilde{g}_j(p, q, z) dz, \quad j = 1, \dots, n_l \quad (4.7)$$

where (p, q) are transform variables representing the wavenumber in x - and y -directions and

$$\begin{aligned}\tilde{e}_j(p, q) &= \mathcal{F}_{xy} \left[\delta\phi(\vec{r}_0; \vec{l}_j) \right], \quad j = 1, \dots, n_l, \\ \tilde{m}(p, q, z) &= \mathcal{F}_{xy} \left[\ln \mu(\vec{r}) \right], \\ \tilde{g}_j(p, q, z) &= \mathcal{F}_{xy} \left[g(\vec{r}; \vec{l}_j) \right], \quad j = 1, \dots, n_l.\end{aligned}$$

The index j identifies the j^{th} pole-pole array. In an E-SCAN® data set, many different pole-pole arrays exist; each is specified by a unique separation and orientation of the current and potential electrodes. I denote the number of distinct pole-pole arrays by n_l .

Equation (4.7) is a Fredholm integral equation of the first kind. The data $\tilde{e}_j(p, q)$ and model $\tilde{m}(p, q, z)$ are complex because they are the Fourier transforms of arbitrary real functions. However, the kernels are real. This results because of the symmetry of the kernel function (4.6) about the origin. At each wavenumber, (4.7) shows that there are n_l constraints upon the model $\tilde{m}(p, q, z)$. Linear inverse theory can therefore be used to recover $\tilde{m}(p, q, z)$ as a function of the depth z (Backus and Gilbert 1967, Oldenburg 1984). Completing this inversion at all wavenumbers yields the model $\tilde{m}(p, q, z)$. The 3d spatial distribution of the conductivity is obtained by performing an inverse 2d Fourier transform in the horizontal directions at sampled depths.

The implementation of the above inversion method requires: (1) that a background conductivity be estimated in order to compute the relative potential anomaly; (2) that pole-pole data maps corresponding to different separations and directions are gathered and interpolated to a common grid so they can be Fourier transformed; (3) that kernel functions are evaluated; and (4) that 1d inverse problems are solved. The 1d inverse problem can be formulated to recover either a continuous representation or a discrete form of $\tilde{m}(p, q, z)$ as a function of depth z . Both approaches have merits in different aspects and will be discussed. These aspects are addressed in the following sections.

4.2 Data

The data required for the $1d$ inversions are generated by applying a $2d$ spatial Fourier transform to the relative potential anomaly $\delta\phi$ for each electrode configuration. The choice of the relative anomaly eliminates the effect of electrode separation and the data therefore reflect only the geo-electrical variation. The relative anomaly is also a quantity consistent with the model of logarithmic conductivity perturbation.

Since the E-SCAN® experiment measures the total potentials ϕ_t , the relative potential anomaly $\delta\phi$ is calculated by

$$\delta\phi(\vec{r}; \vec{l}) = \frac{\phi_t(\vec{r}; \vec{l}) - \phi_p(\vec{r}; \vec{l})}{\phi_p(\vec{r}; \vec{l})}.$$

This requires a reliable estimate of the background conductivity σ_0 so that the primary potential ϕ_p can be calculated accurately. The simplest approach to estimate σ_0 is to use the mean value of the apparent conductivity data. That is, I take the best fitting half-space model as the background. Experiments with both synthetic and field data sets show that this works well, especially for structures with very localized inhomogeneities or where the conductivity value is distributed about a regional value.

Alternatively, one can formulate the $1d$ inversion at the zero wavenumber so that the background conductivity σ_0 is solved for as a unknown parameter together with the conductivity perturbation. This requires only the mean value of each potential data map. I shall discuss this approach following the $1d$ inversion.

The process of data preparation also requires that the potential data for each pole-pole array be gathered from initial E-SCAN® field data and be interpolated and extrapolated. The procedure is best explained by using Fig. 4.2. An E-SCAN® survey is generally carried out over a regular grid. A basic grid is shown by the solid lines in Fig. 4.2. Each node in that grid would have been occupied by a current or potential electrode as the survey data are collected. Suppose that a data map is desired for $l_j = (\Delta x/2, \Delta y)$. This pole-pole configuration is given by the arrows in Fig. 4.2. The potential datum ϕ_t for each electrode pair having this orientation can be obtained

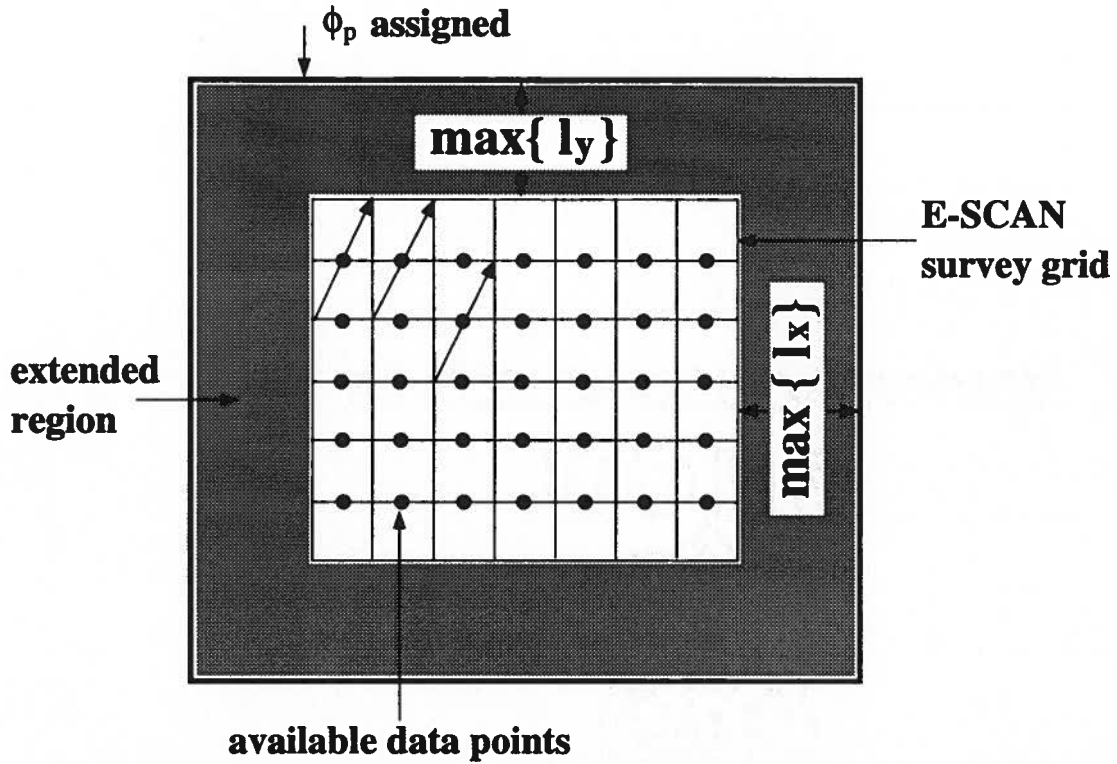


Figure 4.2 Specification of the data map. The intersections of solid lines indicate the E-SCAN® grid points. The potential data are acquired over all grid points but the pole-pole potential is defined only in a smaller area. One possible array configuration is shown by the arrows in the diagram, which point from the source to the potential location. The potential data for this array can be formed only at locations indicated by solid dots. This activated area is smaller than the survey area and decreases as $|\vec{l}|$ increases. To generate a data map suitable for inversion, a mathematical area is defined by extending the original E-SCAN® survey region half of the maximum array separation in x - and y -directions.

and assigned to the midpoint between these two electrodes. The solid dots indicate the locations at which potential data can be gathered for this particular electrode pair. Notice that this area activated by the solid dots is smaller than the initial survey area. The size of the activated area decreases as $|\vec{l}|$ increases. The gathered data in this area need to be interpolated and extrapolated to a common grid over the domain on which the inversion is to be carried out.

I have selected the domain to be a rectangle which is larger than the initial survey area and contains the initial survey at its center. Typically, as illustrated in Fig. 4.2, the distance between

the survey grid and the boundaries of the inversion domain have been set equal to the maximum values of l_x and l_y in the x - and y -directions respectively.

The remaining step is to define a rectangular grid on the surface of the inversion model domain. Because some data maps (for example that shown in Fig. 4.2) yield data at locations midway between the nodes of the survey grid, it is reasonable to select a spatial discretization for the Fourier transform grid which is half of the field survey grid spacing. With this choice of gridding there will be many points on the inversion grid that are not defined directly by values from the survey grid. Both interpolation and extrapolation are required. Interpolation inside an activated area (e.g. within the rectangle generated by the solid dots in Fig. 4.2) is not difficult but extrapolation is always dangerous. In order to avoid the introduction of extraneous artifacts and also ensure that the data to be Fourier transformed are zero around the edges, I specify the potential data around the boundary of the inversion grid to be $\phi_p = I/4\pi l\sigma_0$, the potential corresponding to the background conductivity. The evaluation of potentials at all nodes on the inversion grid can now be obtained by interpolating a data set consisting of the interior activated points and the outside boundary points. I apply an interpolator consisting of a combination of Laplacian and cubic spline components to $\ln(\phi_t/I)$. After interpolation, the relative anomaly $\delta\phi$ is calculated and Fourier transformed to generate the data actually used in the inversion.

4.3 Kernel Functions

Equation (4.6) gives the kernel function in the spatial domain. For illustration, Fig. 4.3 shows three slices through the spatial kernel function, $g(\vec{r}; \vec{l})$, for a given $\vec{l}$. The distance between the electrodes is 50 m and the depth slices are at $z = 1, 20$, and 40 m respectively. At shallow depths the character of the kernel function is dominated by two large dipole-like features which are directly beneath the current and potential electrodes. These features become singularities as $z \rightarrow 0$. At greater depths these peaks become smoother and spread out horizontally. It should be noted, however, that for shallow depths, (e.g. Fig. 4.3(b), (c)) the spatial kernel function has inner regions of opposite sign. These observations are all in agreement with our understanding

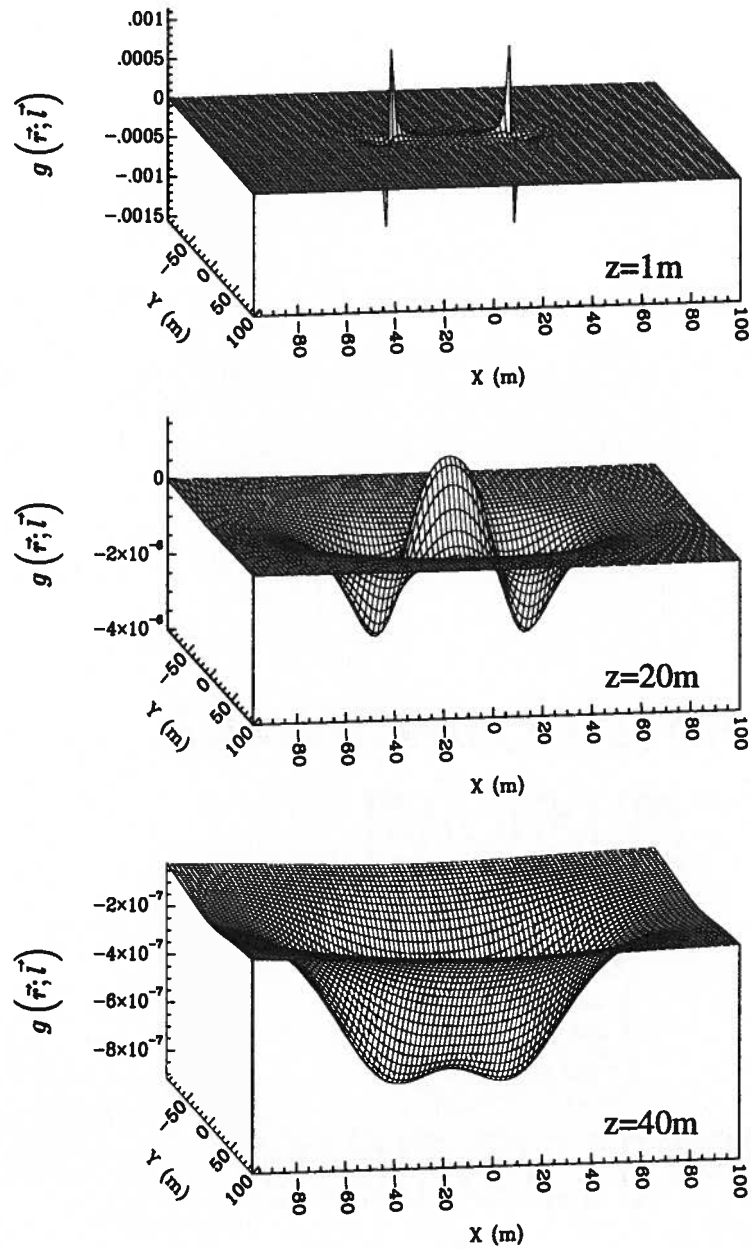


Figure 4.3 Spatial domain kernel function corresponding to $(l_x, l_y) = (25, 0)$ at depths $z = 1, 20, 40$ m respectively.

of the pole-pole dc resistivity experiment. For a given separation, the array is sensitive to smaller changes in the conductivity structure from shallow regions but it responds only to a broader average of conductivity structure at deeper regions. The positive region agrees with the fact that the array has a negative response to relatively shallow anomalies. For instance, it measures a resistive anomaly over a shallow conductor (cf chapter 2).

The kernels $\tilde{g}(p, q, z)$ needed in the 1d inversion require that (4.6) be spatially Fourier transformed. I have not been able to obtain an analytic solution and therefore have calculated them numerically. The approach is to evaluate each 3d kernel at a specified set of depths $(z_1, \dots, z_n)$ and then Fourier transform each depth slice. The results are combined to yield a discretized kernel $\tilde{g}(p, q, z_k)$ ($k = 1, \dots, n$). This discrete representation is interpolated to provide a continuous representation of each kernel, which can be used directly in the continuous form of inversion or be further integrated to form the parametric kernels for the discrete form of inversion.

The computations proceed in the following manner. The set of depths, $(z_1, \dots, z_n)$, for the kernel computation is spaced at intervals increasing with the depth. At each depth, the spatial kernel for a given $\vec{l}_j$ is sampled very finely. The kernel must be sampled frequently enough to avoid aliasing. Ascertaining the appropriate digitization interval is facilitated by the form of the kernel functions in the wavenumber domain. Consider the spatial transform of the kernel in Fig. 4.3(a). The kernel $\tilde{g}$ has a very broad bandwidth whose amplitude decays smoothly as the wavenumber increases. A cut-off wavenumber is chosen so that the contribution from wavenumbers higher than this cutoff point is negligible. It is observed that the envelope of $\tilde{g}$ along the p -axis for $\vec{l}_j = (l_x, 0)$ can be approximated by the function $e^{-\pi p/l_j}$ as shown in Fig. 4.4. For a given depth z , the sampling interval is chosen so that the envelope has decayed to a certain fraction at Nyquist wavenumber compared with zero wavenumber, i.e.,

$$\Delta = -\frac{\pi z}{\ln \eta},$$

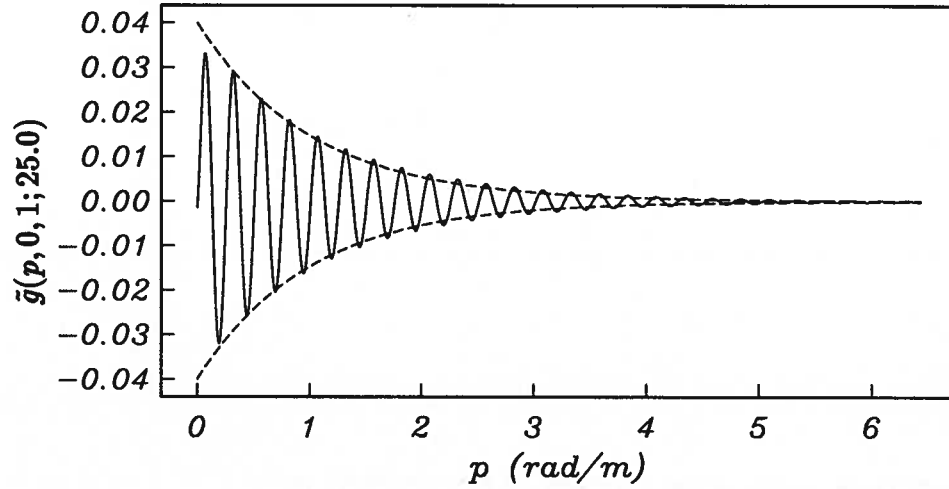


Figure 4.4 Amplitude change of the wavenumber domain kernel function. The solid curve shows the wavenumber kernel along the p axis, which corresponds to array $\vec{l} = (25, 0)m$. The envelope for the kernel along the axis is approximated by the function e^{-zp}/l_x as shown by the dashed curve.

where Δ is the sampling interval and η the specified fraction. η is usually taken to be 0.01. With this criterion, an increasingly larger sampling interval can be used for larger depth values. This ensures adequate sampling at each depth level to avoid the aliasing error and at the same time achieves the optimum computational efficiency.

However, Δ becomes very small for small z and the number of function evaluations becomes large. This difficulty is overcome by specially treating the singularities. The function,

$$g_s(\vec{r}, \vec{l}) = \frac{-l}{\pi (4l_x^2 + 4l_y^2 + z^2)^{3/2}} \left\{ \frac{2l_x(x - l_x) + 2l_y(y - l_y) + z^2}{[(x - l_x)^2 + (y - l_y)^2 + z^2]^{3/2}} + \frac{-2l_x(x - l_x) - 2l_y(y - l_y) + z^2}{[(x - l_x)^2 + (y - l_y)^2 + z^2]^{3/2}} \right\},$$

closely approximates the singularities for small z and has a known analytic solution for its Fourier transform. The subtraction of $g_s(\vec{r}, \vec{l})$ from $g(\vec{r}, \vec{l})$ removes the major variation in the amplitude and hence the high wavenumber components. Therefore, I only sample and perform digital Fourier transform on the residual $(g - g_s)$. The analytic transform of g_s is then added back to the transform of the residual to give the complete Fourier transform of $g(\vec{r}, \vec{l})$.

The spatial Fourier transforms of the slices shown in Fig. 4.3 are given in Fig. 4.5. The Fourier transform at shallow depth is characterized by numerous oscillations and large bandwidth. For greater depths, the transforms die out more rapidly with increasing wavenumber. At any wavenumber, these three plots provide three point evaluations of $\tilde{g}(p, q, z)$. The complete kernels $\tilde{g}_j(p, q, z)$ corresponding to $(p, q) = (0.012, 0)$ are shown in Fig. 4.6. Kernel functions corresponding to small $|l_j|$ have more structure and amplitude toward the surface than do those for larger offsets. Kernel functions for large source-receiver separation decay more slowly with depth. This is consistent with the fact that an array with a larger separation has a greater depth of penetration. It is noted also that the kernels are smooth functions of depth. Since the array separations are increasing linearly, the kernels for large offsets become similar. Effectively the kernel for $l_s = 200$ m provides little new information that is not already in the kernel for $l_s = 175$ m. That is, the kernels associated with the two largest offsets are almost linearly dependent for this particular choice of (p, q) . The smoothness and lack of structure exhibited by the kernels also adds insight into the nature of the inversion results. One would not expect for example that these kernels could provide any vertical resolution about the conductivity below depths of about 200 m.

For a fixed array separation, the dominating portion of the kernel function shifts towards the surface as the wavenumber increases. This can be characterized by the median depth, Z_{med} , of the absolute area $a_j(p, q, z)$ covered by the kernel function. Z_{med} is given by

$$\frac{a_j(p, q, \infty)}{2} = \int_0^{z_{\text{med}}} |\tilde{g}_j(p, q, z)| dz.$$

At zero wavenumber this median depth is in fact the effective depth of pole-pole array defined by Edwards (1977). The decrease of this depth with increasing wavenumber indicates that Edwards' effective depth sets an upper bound on the effective depth. It becomes smaller as the lateral resolution requirement increases. For the same reason, the recovered conductivity model

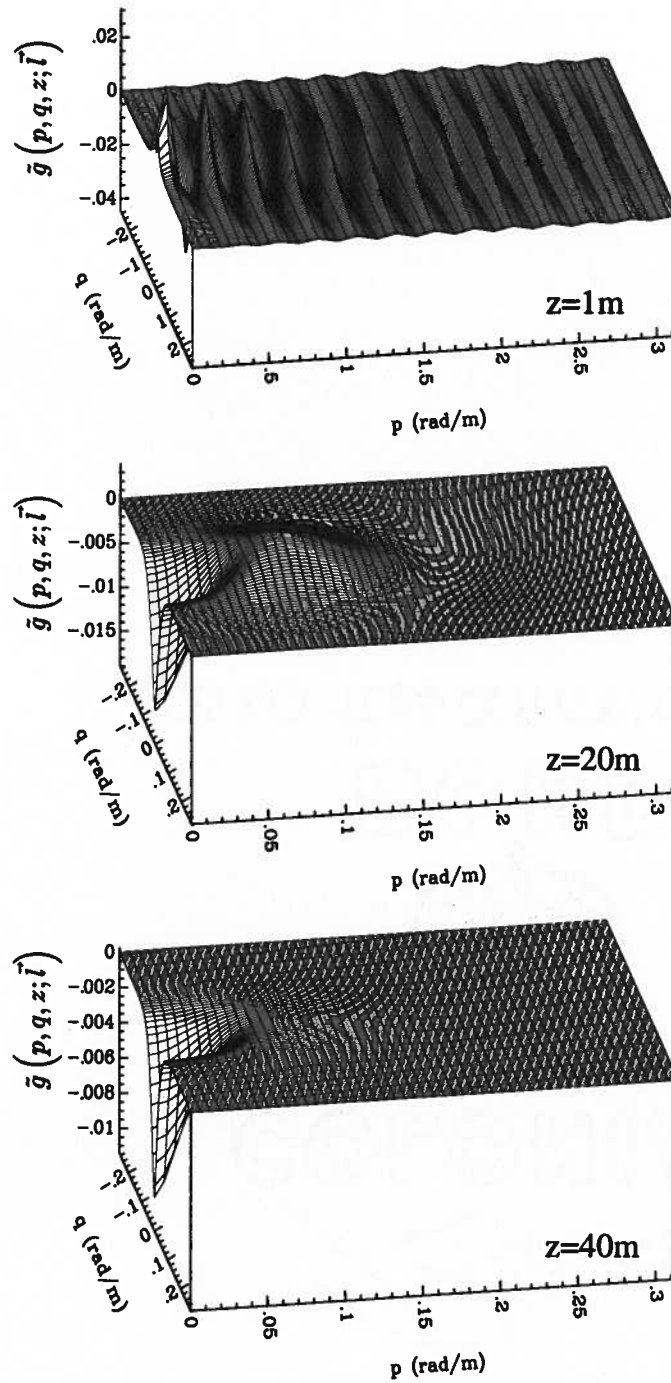


Figure 4.5 Wavenumber domain kernel function corresponding for $\vec{l} = (25, 0)$ at the depth of 1m, 20m, and 40m. These three slices are the 2d Fourier transforms of the spatial kernels in Fig. 4.3. They provide three point evaluations of the kernel corresponding to the given $\vec{l}$ used in the all wavenumbers. The complete kernel is obtained by evaluating all depth slices.

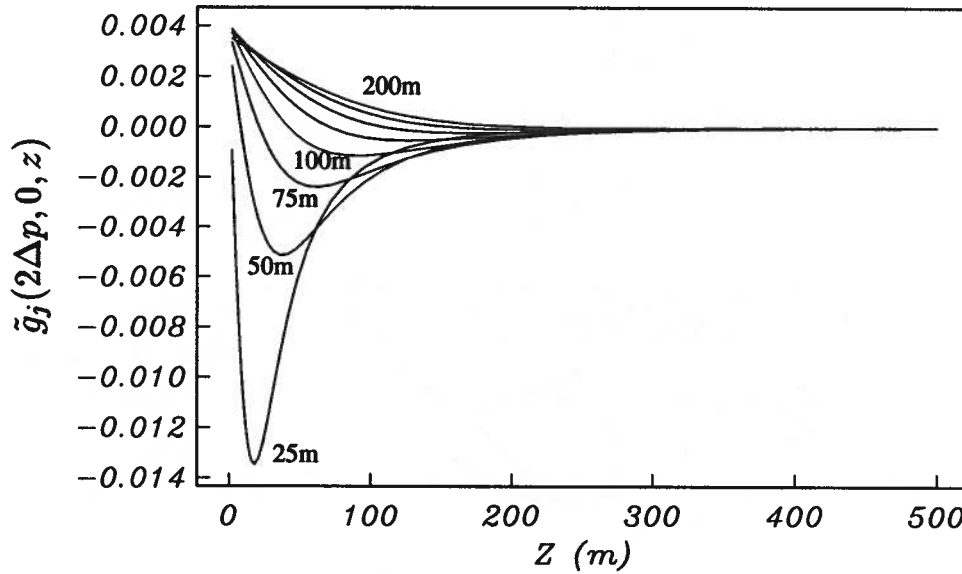


Figure 4.6 A set of kernels at wavenumber $(p, q) = (0.012, 0)$ for arrays in the $\vec{l} = (l_x, 0)$. The curves correspond to arrays with l_x ranging from 25 to 200 m in equal increments of 25 metres.

becomes smoother as depth increases. Fig. 4.7 shows a set of kernels at different wavenumbers and Z_{med} normalized by the value at zero wavenumber.

4.4 Inversion

With the data and kernel functions evaluated in the wavenumber domain, the solution of the inverse problem can now be tackled. According to equation (4.7), at each wavenumber (p, q) a Fredholm equation of the first kind needs to be inverted to recover a complex model $\tilde{m}(p, q, z)$. There exists an infinite number of models which adequately reproduce the n_l complex and inaccurate data e_j ($j = 1, \dots, n_l$). I will formulate the inversion to generate a particular model of “minimum” structure with respect to the background conductivity.

The mathematical development of the approximate 3d inversion made use of the Born approximation. That approximation is valid when the conductivity differs only slightly from the half-space value. For consistency, it is therefore reasonable to compute a conductivity model which deviates as little as possible from this background model. Such a 3d conductivity model

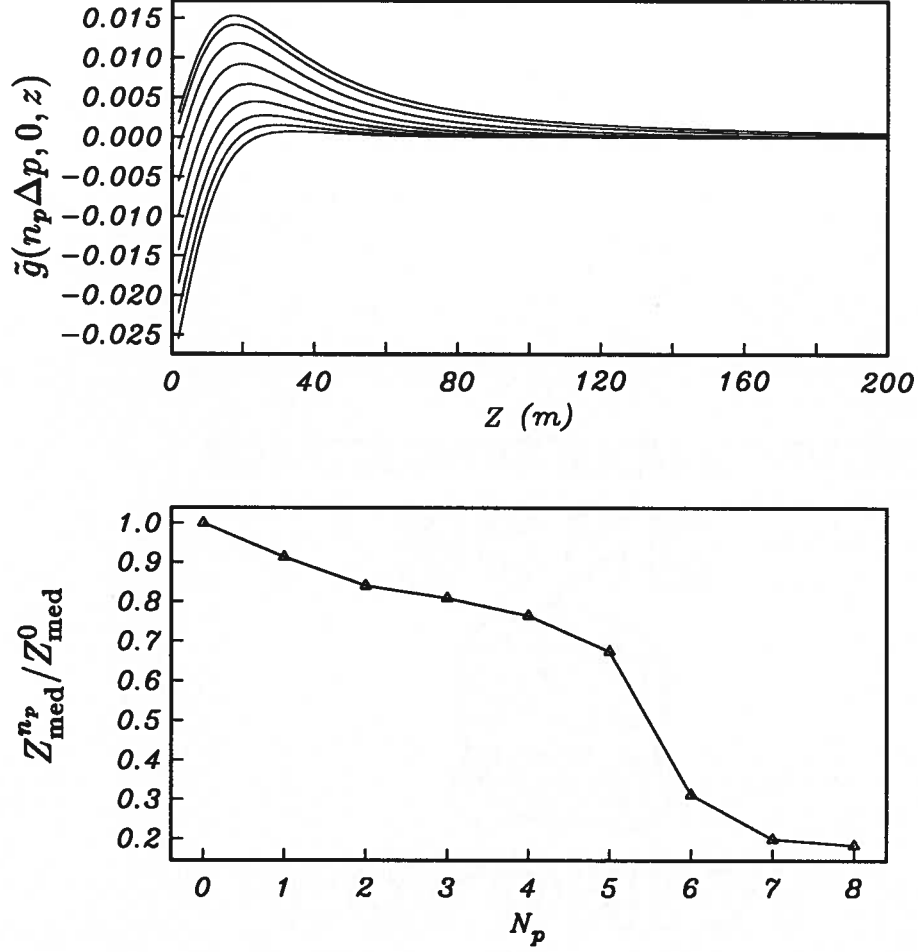


Figure 4.7 Panel (a) displays a set of kernel functions for array $\vec{L}=(25, 0)\text{m}$ at wavenumber $(p, q)=(n_p \Delta p, 0)$. The curves from top to bottom correspond to $n_p=0, \dots, 8$. Panel (b) is the normalized median depth Z_{med} of the absolute area under the kernels.

is constructed by finding perturbations to a base model defined by the estimated half-space conductivity σ_0 , which is used in the data reduction. Therefore a model is sought which minimizes

$$\Phi(\mu) = \int \int \int w(z) [\ln \mu(\vec{r})]^2 dx dy dz, \quad (4.8)$$

where $w(z)$ is an arbitrary positive weighting function. Using Parseval's theorem, (4.8) can be written as

$$\Phi(\mu) = \sum_p \sum_q \int w(z) |\tilde{m}(p, q, z)|^2 dz.$$

In this form, it is observed that the total objective function is a sum of positive quantities. The objective function is effectively decoupled and the total objective function is minimized by minimizing the quantity

$$\Phi(\tilde{m}) = \int_0^{\infty} w(z) |\tilde{m}(p, q, z)|^2 dz \quad (4.9)$$

at each wavenumber.

At each wavenumber the data equations (4.7) have the form

$$\tilde{e}_j = \int_0^{\infty} \tilde{m}(p, q, z) \tilde{g}_j(p, q, z) dz, \quad j = 1, \dots, n_l$$

where $\tilde{e}_j$ and $\tilde{m}(p, q, z)$ are complex and $\tilde{g}_j(p, q, z)$ are real. The 1d inverse problem is then to find a solution for $\tilde{m}$ minimizing $\Phi(\tilde{m})$ in (4.9) subject to fitting data $\tilde{e}_j$ ($j = 1, \dots, n_l$). Either a continuous or blocky $\tilde{m}$ can be sought as a function of depth z . Each form has merit and I shall present both in the following.

4.4.1 Continuous Formulation

The most general approach to the above-stated inverse problem is to formulate the inversion so that a continuous function $\tilde{m}(p, q, z)$ is recovered. This has the advantage that the model does not depend upon any specific *a priori* parameterization. The model and kernels are assumed to reside in a Hilbert space $\mathcal{M}$ of complex functions defined on the region $[0, z_{\max}]$ where $z_{\max}$ is some maximum depth of interest and is sufficiently great so that all kernel functions have negligible amplitude by that depth. If $\tilde{g}$ and $\tilde{h}$ are any two functions in $\mathcal{M}$, then their inner product is given by

$$(\tilde{g}, \tilde{h}) = \int_0^{z_{\max}} w(z) \tilde{g}(z) \tilde{h}^*(z) dz, \quad (4.10)$$

where the $*$ denotes complex conjugate. The norm on $\mathcal{M}$ is defined by

$$\|\tilde{m}\| = \left(\int_0^{z_{\max}} w(z) |\tilde{m}(z)|^2 dz \right)^{1/2}. \quad (4.11)$$

The data provide only n_l constraints upon the model and hence $\tilde{m}(p, q, z)$ cannot be obtained unambiguously from them. There are, in fact, infinitely many models which fit the data exactly (e.g. Backus and Gilbert 1967). Given that the data are also inaccurate, I choose to find that particular model which minimizes

$$\Phi(\tilde{m}) = \|\tilde{m}\|^2 + \theta^{-1} \sum_j |\tilde{e}_j - (\tilde{m}, \tilde{g}_j^w)|^2, \quad (4.12)$$

where θ is a Lagrange multiplier and $\tilde{g}_j^w(z) = \tilde{g}_j(z)/w(z)$. The minimization is straightforward. Any function $\tilde{m} \in \mathcal{M}$ can be written as $\tilde{m} = \tilde{m}^{\parallel} + \tilde{m}^{\perp}$ where $\tilde{m} \in \mathcal{M}^{\parallel} = \text{asp}\{\tilde{g}_j^w\}$ and $\mathcal{M}^{\perp}$ is the complement of $\mathcal{M}^{\parallel}$. It is clear that $\tilde{m}^{\perp}$ does not contribute to the data misfit but it will contribute to the model norm. As such, a minimization of $\Phi(\tilde{m})$ demands that

$$\tilde{m}(p, q, z) = \sum_{j=1}^n \alpha_j \tilde{g}_j^w(p, q, z) \quad (4.13)$$

where α_j are coefficients to be determined. Substituting into (4.12) and writing the objective function in a matrix/vector notation yields

$$\Phi(\tilde{m}) = \tilde{\alpha}^H \Gamma \tilde{\alpha} + \theta^{-1} (\tilde{e} - \Gamma \tilde{\alpha})^H (\tilde{e} - \Gamma \tilde{\alpha}), \quad (4.14)$$

where H denotes complex conjugate transpose and Γ is a real symmetric positive definite inner product matrix with elements

$$\Gamma_{ij} = \int_0^{z_{\max}} w(z) \tilde{g}_i^w(p, q, z) \tilde{g}_j^w(p, q, z) dz$$

and $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{n_l})^T$, $\tilde{e} = (\tilde{e}_1, \dots, \tilde{e}_{n_l})^T$. Perturbing $\tilde{\alpha}$ in (4.14) and carrying out the minimization using a variational method, subject to the fact that $\Phi(\tilde{m})$ must always be real valued, yields

$$\Gamma \tilde{\alpha} - \theta^{-1} \Gamma^H (\tilde{e} - \Gamma \tilde{\alpha}) = 0.$$

Since Γ is positive definite and symmetric this may be written as

$$(\Gamma + \theta I) \vec{\alpha} = \vec{e}. \quad (4.15)$$

The solution of (4.15) is most easily obtained by writing $\Gamma = R\Lambda R^T$, where Λ is a diagonal matrix containing the eigenvalues of Γ in descending order, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_{n_l})$, and R is a unitary matrix composed of corresponding eigenvectors $R = (\vec{r}_1, \dots, \vec{r}_{n_l})$. The solution of (4.15) is then given by

$$\vec{\alpha} = R(\Lambda + I\theta)^{-1} R^T \vec{e}. \quad (4.19)$$

The major difficulty in carrying out the inversion is specifying the value of θ for each $1d$ inversion. If the data errors on $\tilde{e}_j$ ($j = 1, \dots, n_l$) were Gaussian, independent with zero mean and known standard deviation there would be no problem. The procedure would be first to normalize each data equation by the standard deviation of the datum and then adjust θ until a desired misfit value is reached. A desired misfit might be $\mathcal{E}[\chi^2] \approx n_l$.

The errors in the above inversion do not comply with the simple Gaussian assumption. In addition to measurement errors on the initial potential data, there are several other major sources of errors. Errors are introduced into the data by the extrapolation of the known data to fill the mathematical area in which the inversion is carried out; errors exist in the model representation associated with the $1d$ Fredholm equation which is inexact due to the Born approximation; errors are introduced into both data and model by estimation error of the background conductivity σ_0 . Therefore, the data are biased and non-Gaussian in realistic problems. Nevertheless, I have attempted to incorporate a Gaussian type strategy as much as possible.

The approach has been to first estimate an approximate error for each datum. These will be interpreted either as approximate standard deviations or at least used as a relative weighting for the different data to be inverted. The estimation of the errors are obtained in the following manner. The errors on initial potential data in each map are converted to the variance of the wavenumber domain datum $\tilde{e}_j(p, q)$. Since the initial data errors are assumed Gaussian with zero mean and

uncorrelated and the Fourier transform kernel has a unit amplitude, this variance is equal to the sum of the variance of the initial errors. This error is then scaled by the ratio of the total area of the inversion to activated area. In the estimation of the initial error, an upper and lower bound on the possible error are imposed. The lower bound is especially useful for those data with small magnitude.

The above errors have been used in three ways. In the first these values have been accepted as valid standard deviations, used to normalize the data equations, and θ is chosen so that the chi-squared misfit was equal to n_l . In all the test examples, this method failed to work satisfactorily. The amplitude of the data $\tilde{e}_j(p, q)$ decays rather rapidly with the increasing wavenumber while the estimated “standard deviation” stays the same. This results in the domination of the final model by a very few low wavenumber components and the conductivity model is geophysically unacceptable.

In the second method the estimated standard deviations are used only as relative weightings for the data equations and θ is chosen in accordance with its effect on the model norm. The model norm increases monotonically as θ decreases and is easily evaluated in the spectral domain. Using the decomposition of Γ , we can define a set of rotated basis functions (Parker, 1977),

$$\psi_i(z) = \sum_{j=1}^n r_{ji} \tilde{g}_j^w(p, q, z), \quad (4.17)$$

where r_{ji} is the j^{th} element of eigenvector $\vec{r}_i$. The $\psi_i(z)$'s form an orthogonal set, i.e.,

$$(\psi_i, \psi_j) = \lambda_i \delta_{ij}, \quad (4.18)$$

where δ_{ij} is the Kronecker delta. The regularized solution is given by

$$\tilde{m}(p, q, z) = \sum_{i=1}^n \frac{\hat{e}_i}{\lambda_i + \theta} \psi_i(z), \quad (4.19)$$

where $\hat{e}_i$ is the i^{th} element of the rotated data vector $\hat{e} = R^T \vec{e}$. It follows from the orthogonality of the $\psi_i(z)$'s that

$$||\tilde{m}||^2 = \sum_{i=1}^n \lambda_i \left(\frac{\hat{e}_i}{\lambda_i + \theta} \right)^2. \quad (4.20)$$

The curve $||\tilde{m}||^2(\theta)$ is usually smooth and often characterized by a near vertical slope for small θ and near zero slope for large θ . The value of θ corresponding to the onset of rapid norm increase can serve as estimates for θ used in the inversion. This is implemented by normalizing both θ and $||\tilde{m}||^2$ to interval $[0, 1]$. The value of θ where the tangent to the normalized curve has a given slope k (e.g. $k = -1.0$) yields the desired estimate. This approach has proven very stable and produced models very representative of the true models in synthetic examples. However, without referring to the data misfit, this approach is somewhat arbitrary in the choice of the slope k within this permissible small range. Therefore, when choosing θ based upon a slope criterion, the value of the slope k is best chosen by incorporating a global misfit as done in conjunction with using a single value for θ as presented next.

The third approach is an attempt to find the value of θ according to the misfit to the original data in the spatial domain. A single value of θ is used to regularize all 1d inversions. A given θ will generate a model from which the Born data can be calculated. The total chi-squared misfit, evaluated only for the original field data is given by

$$\chi^2 = \sum_{j=1}^{n_t} \left(\frac{\phi_j^o - \phi_j^p}{\delta_j} \right)^2, \quad (4.21)$$

where ϕ_j^o and ϕ_j^p are observed and predicted total potential values respectively, δ_j is the estimated error of each datum and n_t is the total number of the potential data used in the inversion. The calculation ϕ_j^p is efficiently carried out by first evaluating the Fourier transform of the relative anomaly $\delta\phi$ by (4.7). Applying the inverse Fourier transform yields $\delta\phi$, from which the total potential ϕ^p is obtained. χ^2 is the misfit for the approximate equation rather than a true misfit. Carrying out the inversion for a number of values of θ yields a curve $\chi^2(\theta)$ which can be interrogated to estimate an optimum value or to find that θ which corresponds to an expected χ^2 .

This approach has the advantage that the regularization is controlled by misfit to the original data. It does require that the inversion be carried out a number of times but that poses no computational difficulties. The CPU time needed for all $1d$ inversions is only a small fraction of that used for computing the kernel functions. Once the kernels are computed, they can be stored and additional inversions can be carried out efficiently.

However, the eigenvalues of the inner product matrix Γ decreases with increasing wavenumber since the amplitude of the kernel function decreases. When a single value of θ is used to regularize all $1d$ inversions, it tends to over-damp the solution at higher wavenumbers. As a result, the recovered conductivity model has few details.

Experiments with synthetic examples indicate that a better model is produced when the regularization is applied to each individual inversion based upon the model norm. Therefore, I choose to use second approach of regularization discussed in this section.

4.4.2 Discrete Formulation

It is often desirable to formulate the inversion so that a blocky model is obtained directly. This is especially advantageous when a numerical modelling scheme such as a finite difference algorithm is used for forward modelling. Introduce a depth partitioning $(z_0, \dots, z_n)$ where $z_0 = 0$ and $z_n = z_{\max}$. A set of boxcar functions can be defined according to the partitioning as the basis function for the model $\tilde{m}(p, q, z)$,

$$b_k(z) = \begin{cases} 1, & z \in [z_{k-1}, z_k], \\ 0, & \text{otherwise.} \end{cases} \quad k = 1, \dots, n \quad (4.22)$$

The model can then be expanded as

$$\tilde{m}(p, q, z) = \sum_{k=1}^n \tilde{m}_k b_k(z), \quad (4.23)$$

where the coefficients $\tilde{m}_k$ are now constants to be determined. To be consistent with the model being sought, a weighting function expandable in $\{b_k(z)\}$ would be reasonable:

$$w(z) = \sum_{k=1}^n w_k b_k(z),$$

where w_k are a set of positive real numbers. The basis $\{b_k(z)\}$ spans a portion of the model space $\mathcal{M}$, which can be denoted by $\mathcal{M}^b = \text{asp}\{b_k\}$. Since $b_k(z)$ are not orthogonal to any of kernel functions, $\mathcal{M}^b$ intersects with $\mathcal{M}^{\parallel}$. It is in this intersection where the model $\tilde{m}(p, q, z)$ will be residing.

Substituting (4.23) into (4.12) yields the new objective function

$$\Phi(\tilde{m}) = \tilde{m}^H W \tilde{m} + \theta^{-1} (G\tilde{m} - \vec{e})^H (G\tilde{m} - \vec{e}), \quad (4.24)$$

where $\tilde{m} = (\tilde{m}_1, \dots, \tilde{m}_n)$, $G = (\vec{g}_1, \dots, \vec{g}_{n_l})^T$ and the discrete kernel $\vec{g}_j = (\vec{g}_j^1, \dots, \vec{g}_j^n)^T$ ($j = 1, \dots, n_l$) is given by

$$\vec{g}_j^k(p, q) = \int_{z_{k-1}}^{z_k} \tilde{g}_j(p, q, z) dz.$$

W is the weighting matrix and $W = \text{diag}(w_1(z_1 - z_0), \dots, w_n(z_n - z_{n-1}))$. Minimizing (4.24) with respect to model parameters $\tilde{m}$ in the similar manner as minimizing (4.14) yields

$$(G^T G + \theta W) \tilde{m} = G^T \vec{e}.$$

Since the weighting matrix W is positive definite and diagonal, $W^{-\frac{1}{2}}$ exists and is diagonal. Therefore, the above equation may be written as

$$(G_w^T G_w + \theta I) \tilde{m}_w = G_w^T \vec{e}, \quad (4.25)$$

where $G_w = W^{-\frac{1}{2}} G$ and $\tilde{m}_w = W^{\frac{1}{2}} \tilde{m}$.

Equation (4.25) is solved efficiently using the singular value decomposition (SVD) of the matrix G_w . Let $G_w = U S V^T$, where S is a diagonal matrix of singular values, $S = \text{diag}(s_1, \dots, s_{n_r})$

and n_r is the effective rank of the matrix G_w . U and V are left and right singular vector matrix, respectively. The solution of (4.25) is then given by

$$\vec{m}_w = VS(S^2 + \theta I)^{-1}U^T \vec{e}, \quad (4.26)$$

and the final model is obtained by unweighting $\vec{m}_w$:

$$\vec{m} = W^{-\frac{1}{2}} \vec{m}_w.$$

As in the continuous formulation, the crucial step is to choose a proper value for θ . I have employed the same approaches discussed in the continuous formulation. As can be expected, it does not work to take the estimated errors as the valid standard deviations and to misfit the data accordingly. The second and the third approaches have worked in this discrete formulation as in the continuous formulation. In the second approach the model norm needs to be computed efficiently for a series of θ values. The norm of the model expressed by (4.26) is given by

$$||\vec{m}||^2 = \sum_{i=1}^{n_r} |\hat{e}_i|^2 \left(\frac{s_i}{s_i^2 + \theta} \right)^2, \quad (4.27)$$

where $\hat{e}_i$ are the elements of the rotated data vector $\hat{e} = U^T \vec{e}$. With (4.27), an estimate for θ can be obtained by exactly the same procedure described in the continuous formulation.

4.5 Calculation of Background Conductivity σ_0

When the background conductivity is uncertain, a desirable alternative approach is to estimate it through inversion. Note that the data used in *1d* inversions are the transform of the percentage anomaly $\delta\phi = (\phi - \phi_p)/\phi_p$. Since the primary potential (for unit current strength)

$$\phi_p(\vec{r}) = \frac{1}{4\pi l \sigma_0},$$

is constant for a given array configuration, σ_0 contributes only to the data at zero wavenumber and acts as a scale factor at all non-zero wavenumbers. Substituting $\delta\phi$ into the spatial data equation

(4.5) and rearranging the terms yields

$$\phi(\vec{r}) = \frac{1}{4\pi l \sigma_0} + \frac{1}{4\pi l \sigma_0} \int_0^\infty \ln \mu(\vec{r}') \otimes \otimes g(\vec{r}'; \vec{l}) dz.$$

The zero-wavenumber component after Fourier transforming is, therefore,

$$\bar{\phi}_j = \frac{1}{4\pi l_j} \frac{1}{\sigma_0} + \frac{1}{4\pi l_j} \frac{1}{\sigma_0} \int_0^\infty \bar{m}(z) \bar{g}_j(z) dz, \quad (4.28)$$

where

$$\bar{\phi}_j = \frac{1}{\Delta S} \iint_{\Delta S} \phi_{ij}(x, y) dx dy,$$

$$\bar{m}(z) = \frac{1}{\Delta S} \iint_{\Delta S} \ln(\mu(x, y, z)) dx dy,$$

and

$$\bar{g}_j(z) = \frac{1}{\Delta S} \iint_{\Delta S} g(\vec{r}; \vec{l}_j) dx dy.$$

ΔS denotes the mathematical area defined for the inversion (see Fig. 4.2). Equation (4.28) has both σ_0 and $\bar{m}(z)$ as unknowns and provides the opportunity to recover both simultaneously.

Equation (4.28) can be written

$$\bar{e}_j = \frac{1}{\sigma_0} + \frac{1}{\sigma_0} \int_0^\infty \bar{m}(z) \bar{g}_j(z) dz, \quad (4.29)$$

where $\bar{e}_j = 4\pi l_j \bar{\phi}_j$. Now, (4.29) is a slight variation of the Fredholm equation of the first kind in which σ_0 appears as an extra unknown. As at any other wavenumber, I look for a weighted smallest model $\bar{m}(z)$ subject to (4.29) as constraints. For generality, I shall present the solution in the continuous form in the following. Choose an objective function consistent with the previously used form,

$$\Phi(\bar{m}) = ||\bar{m}||^2 + \theta^{-1} \sum_{j=1}^{n_i} |\sigma_0 \bar{e}_j - 1 - (\bar{m}, \bar{g}_j^w(z))|^2, \quad (4.30)$$

where $\bar{g}_j^w(z) = \bar{g}_j(z)/w(z)$. θ is a Lagrange multiplier. Based upon the similar argument, the formal minimization of Φ with respect to $\bar{m}$ and σ_0 yields

$$\bar{m}(z) = \sum_{j=1}^n \alpha_j \bar{g}_j^w(z), \quad (4.31)$$

$$(\Gamma + \theta_0) \vec{\alpha} = \sigma_0 \vec{e} - \vec{Q}, \quad (4.32)$$

and

$$\vec{e}^T \vec{\alpha} = 0, \quad (4.33)$$

where Γ is again the inner product matrix of kernel functions as defined before, $\vec{e} = (\bar{e}_1, \dots, \bar{e}_{n_t})^T$, and $\vec{Q} = (1, \dots, 1)^T$. It should be noted, however, that $\vec{Q} = (1/\delta_1, \dots, 1/\delta_{n_t})^T$ when the data equations are normalized by their estimated errors δ_j .

Equations (4.31) and (4.32) represent a similar solution as that given by (4.13) and (4.15). But (4.33) imposes a further constraint on the kind of solution permissible. Solving (4.32) and (4.33) for σ_0 yields

$$\sigma_0 = \frac{\vec{e}^T (\Gamma + \theta I)^{-1} \vec{Q}}{\vec{e}^T (\Gamma + \theta I)^{-1} \vec{e}}. \quad (4.34)$$

Since the value of θ is to be determined based upon its effect on either the data misfit or the model norm, (4.32) and (4.34) need to be solved simultaneously. One approach would be to start with an initial θ and solve the two equations iteratively. Once σ_0 is obtained through the 1d inversion at zero-wavenumber, it can be used in the inversions at non-zero wavenumbers as the estimated background conductivity.

The σ_0 obtained through this is the optimum value for the smallest l_2 norm $\bar{m}(0, 0, z)$. This procedure amounts to finding the background conductivity which represents the surface data as much as possible so that the conductivity perturbation is minimized. A similar technique has been used by Aldridge *et al.* (1991) in the construction of the flattest models.

The question remains as how to calculate the mean of each potential data map which is consistent with the mathematical area of the inversion. One can use the mean of the data gathered

from the E-SCAN® data set as the mean for the inversion region. Since the area activated by these data always dominates the inversion region, such extrapolation is not unreasonable. Alternatively, one can take a two-step approach. The mean value of all apparent conductivities gathered from the E-SCAN® data can be used as a first estimate of σ_0 . This first estimate is used to interpolate the data maps by augmenting them with primary potential on the boundary of the mathematical area (see section 4.2). The mean values of these interpolated maps can then be inverted to obtain the final estimate for σ_0 .

4.6 Synthetic Example

The algorithm has been tested on several synthetic data sets. As an illustration, I present the inversion results in the continuous form for one synthetic data set. The data set is generated over a 21×21 grid of 50m spacing for a model consisting of five prisms buried in a uniform half-space. Table 4.1 gives the parameters of the model. The perspective view of the model layout is shown in Fig. 4.8. Prisms **B1** and **B2** are the buried targets. Prisms **S1**, **S2**, and **S3** simulate near surface variations in the conductivity. 8 potential data maps are gathered in both x - and y -directions. The electrode separations are equal to $n\Delta$ ($n = 1, \dots, 8$), where Δ is the grid spacing and equal to 50m. 5 maps are also gathered in both xy - and yx -diagonal directions with separations equal to $n\sqrt{2}\Delta$ ($n = 1, \dots, 5$). These 26 data maps are used in the inversion (i.e., there are 26 complex data for each $1d$ inversion in the wavenumber domain). No *a priori* information except the background conductivity is given. A weighting function $w(z) = 1/(z + z_0)$ is applied, where $z_0 = 5\text{m}$. With such a weighting function, the inversion is effectively carried out on a logarithmic depth $\ln(z)$. The regularization parameter θ for each $1d$ inversion is chosen according to the norm of the model. A slope of $\tan(\pi/3)$ is specified.

Fig. 4.9 is the comparison of the true and recovered model in a section at $x = 450\text{m}$ which cuts through the four major prisms. Fig. 4.10 and Fig. 4.11 show the comparison at depth $z = 30\text{m}$ and $z = 150\text{m}$ respectively. Geometrically, the definition of the surface prisms are excellent. The image becomes less sharp as the depth increases. The magnitude of the recovered anomaly

Table 4.1 Parameters of the 5-prism model.

Prism	x-dimension (m)	y-dimension (m)	z-dimension (m)	Conductivity (mS/m)
S1	225 - 325	225 - 325	0 - 40	10
S2	275 - 675	375 - 475	0 - 40	5
S3	325 - 675	625 - 675	0 - 40	0.5
B1	275 - 625	275 - 375	50 - 250	0.5
B2	375 - 475	475 - 675	75 - 275	10

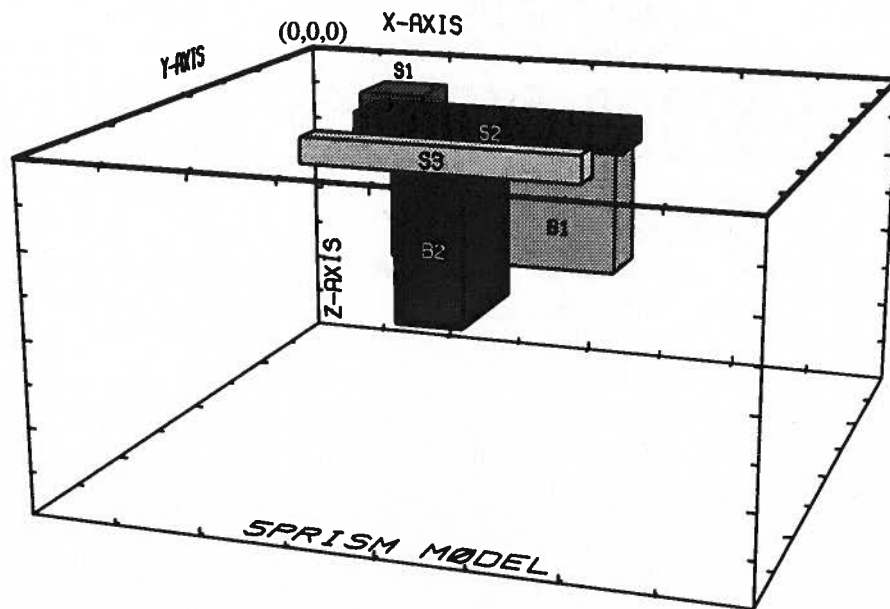


Figure 4.8 The perspective view of the synthetic model consisting of five prisms embedded in a uniform half-space of 1mS/m. The surface prisms S1 (5mS/m), S2 (10mS/m), and S3 (0.5mS/m) simulate the near surface variations in the conductivity. The B1 (0.5mS/m) and B2 (10mS/m) are the buried targets.

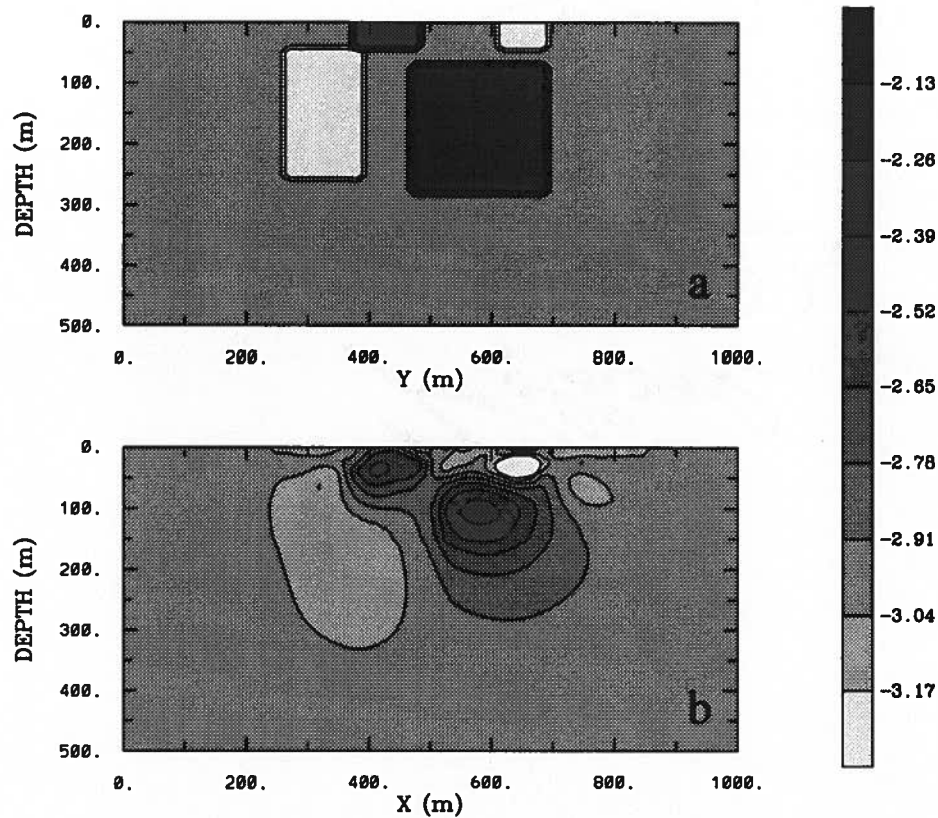


Figure 4.9 The comparison of the true model (a) and the recovered model (b) in the section $x=450$ m (scale in $\log_{10} \sigma$).

decreases with the depth. In general, the conductivity of the buried blocks is under-estimated. Because of the loss of resolution and magnitude at the depth, the bottoms of the buried prisms are poorly defined. Instead, there is a smooth transition from the anomaly to the background. The loss of resolution is due to the limited bandwidth of the data and the fact that the kernel function in the wavenumber domain decays more rapidly with depth at higher wavenumbers. The loss of amplitude is due to the nature of smallest model construction, where a ridge regression parameter is used to regularize the solution. Overall the recovered model appears as a depth varying filtered version of the true model. Nevertheless, all anomalies are well resolved and despite the influence of the surface conductivity variations the buried prisms are clearly defined.

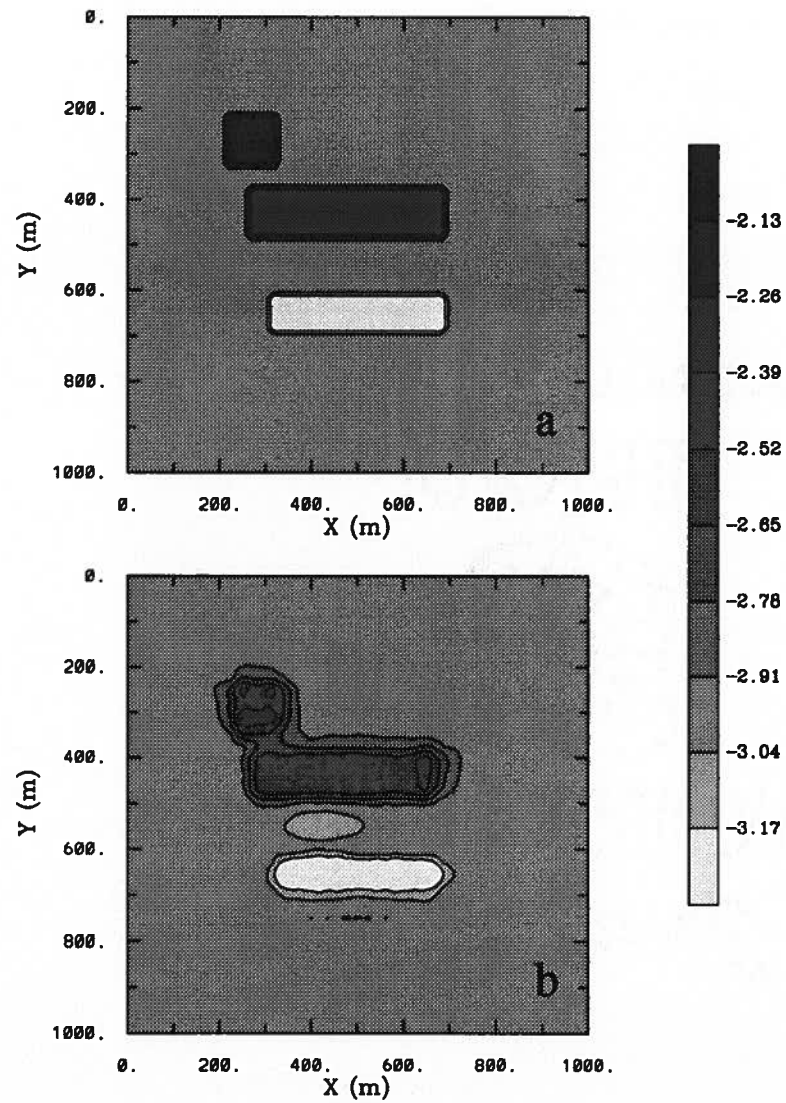


Figure 4.10 The comparison of the true model (a) and the recovered model (b) at depth of 30 m (scale in $\log_{10} \sigma$).

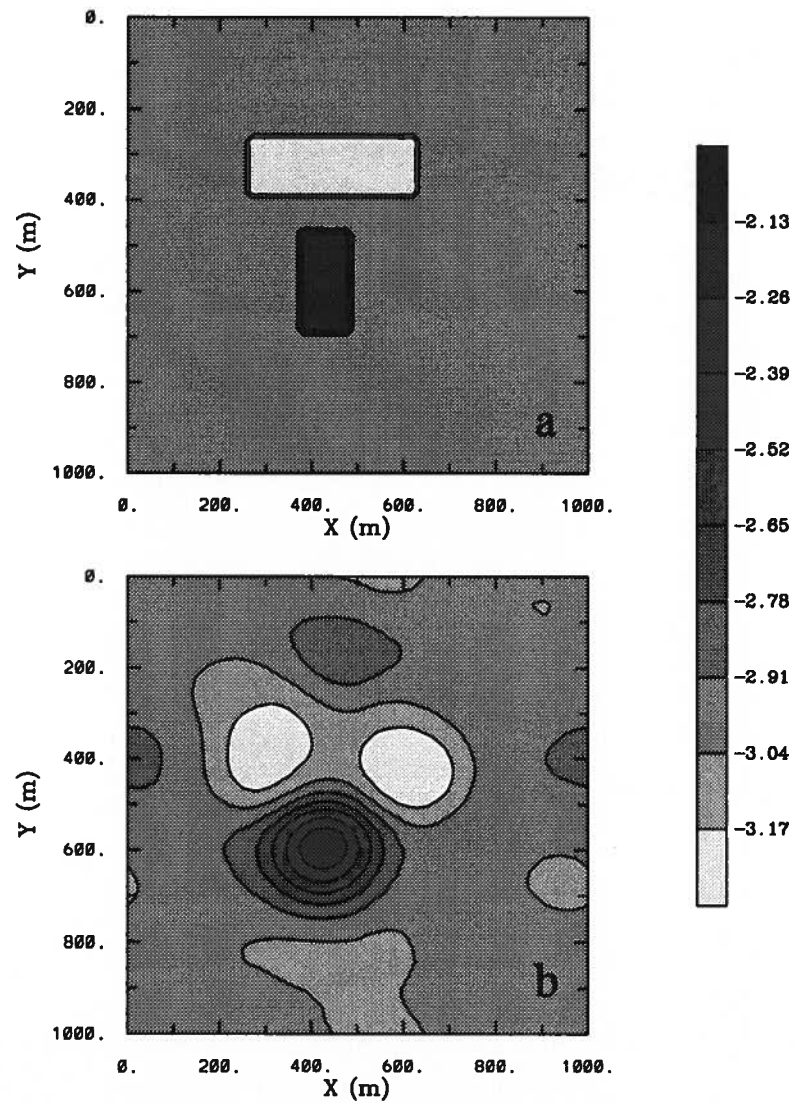


Figure 4.11 The comparison of the true model (a) and the recovered model (b) at depth of 150 m (scale in $\log_{10} \sigma$).

It is noteworthy that such good results have been obtained even though three conductivity blocks with high contrasts are placed at the surface. This is apparently in violation of the assumption that the surface conductivity be constant and the conductivity deviation is small. This shows that the algorithm is quite robust.

4.7 Discussion

A 3d approximate inversion algorithm has been developed based upon the integral equation for surface pole-pole potential anomaly. The algorithm is designed to work for general 3d models composed of perturbations to a background conductivity. Synthetic examples show that it can handle fairly large conductivity contrasts and, to a certain degree, surface variations. The approach I have taken has some general implications. The algorithm can be viewed from different aspects. By applying a linear operator (2d Fourier transform) to the basic data equation, the 3d inverse problem is broken into a set of independent 1d inverse problems. Each 1d problem can be solved easily and efficiently. This results in an algorithm which is much more efficient than solving the original problem directly.

From the view point of inversion, this approach also has merits. The Fourier transform separates the large scale features (low wavenumber components) and small scale features (high wavenumber components) in the data. Features of different scales are fit by finding the model components of corresponding scales. The horizontal variability in the conductivity is controlled by the largest wavenumber included in the inversion. Therefore, the method only attempts to generate features in the model which are required by the data. Variations beyond the resolution provided by the data are excluded. This process makes the inversion stable and less prone to producing spurious structures in the model.

The parameter σ_0 only acts as a scaling factor for the data at non-zero wavenumbers, but provides an additive term to the data at zero wavenumber. Thus an inaccurate choice of σ_0 will only scale the model components at non-zero wavenumbers and the structural image is not altered in the horizontal directions. When the error on the estimated σ_0 is large, it will produce an

incorrect mean value (from the inversion at zero wavenumber) of the model at each depth. This will result in a series of false layers in the spatial domain. The final result would be the correct structural image (with scaled amplitude) superimposed on these false layers. Because of this fact, a small variation in the estimate of σ_0 does not significantly alter the resultant conductivity model.

It should also be noted that the application of any transform introduces an extra level of complication in data processing. This can cause difficulties. For example, in order to apply the Fourier transform, the observed potential data must be interpolated and extrapolated to a regular grid. This introduces additional errors whose statistical properties are unknown. Consequently, the statistical properties of the transformed data are uncertain. If the original data are aliased upon acquisition, the transformed data will also be incorrect.

As the algorithm is based upon the Born approximation, its result is semi-quantitative. The recovered conductivity model is not likely an acceptable model since it does not reproduce the observed data via a full forward modelling algorithm. Therefore, the result of the approximate 3d inversion is best regarded as an image. However, the success with the test examples provides a base for optimism for this approximate inversion. The next step will be to input the recovered conductivity model into a full 3d forward modelling algorithm to quantify the misfit between observed and predicted data. The approximate inverse mapping can then be incorporated into an iterative algorithm to yield a quantitative and rigorous solution to the 3d resistivity problem. This leads to the application of the AIM (Approximate Inverse Mapping) inversion to the 3d resistivity problem using the 3d approximate inversion as an approximate inverse mapping.

CHAPTER 5

AIM INVERSION

The interpretation of a data set often requires that an acceptable conductivity model be constructed. The inversion of d.c. data is a non-linear problem. The most common approach for such problems has been to first linearize the problem (by Taylor expansion) and then to construct the model iteratively using Newton's method. This requires that, at each iteration, the sensitivity matrix (or the Fréchet derivatives) be computed so that the change in the data can be mapped to a change in the model. In the cases of a Newton-type method, the matrix needs also to be inverted. These operations involve a great amount of computation and therefore pose difficulties in practical applications.

However, the *3d* approximate inversion is a very efficient method which maps a set of potential data to a conductivity model without resorting to the generation and inversion of a *3d* sensitivity matrix. If an algorithm is constructed to utilize such an inverse mapping, it will have a great computational advantage. The formalism of AIM (Approximate Inverse Mapping) inversion developed by Oldenburg and Ellis (1991) provides the framework for such an approach. In the AIM inversion, an approximate inverse mapping is used in conjunction with an exact forward mapping to update the model successively by finding either a perturbation to the model or a modification for the data to be inverted. The algorithm iterates towards an acceptable model without necessarily computing and inverting a large sensitivity matrix. In this chapter, I shall first briefly review the AIM formalism and then present its application to *3d* d.c. resistivity problem using the approximate *3d* inversion.

5.1 Review of the AIM Formalism

Suppose a geophysical experiment is described by a forward operator $\mathcal{F}$ which maps an element m in model space to an element $\vec{e}$ in data space,

$$\mathcal{F}[m] = \vec{e}. \quad (5.1)$$

For the inverse problem of this experiment, one would ideally like to have an inverse operator $\mathcal{F}^{-1}$ so that it maps the data $\vec{e}$ back to the model m ,

$$\mathcal{F}^{-1}[\vec{e}] = m. \quad (5.2)$$

If the problem is non-linear, it is most commonly tackled by first linearizing it using mathematical tools such as a Taylor series expansion. By such linearization, the Fréchet derivatives (or sensitivity matrix) are used to relate the change in the data to the change in the model. The model is updated by applying the inverse of the matrix to the difference between the observed and the predicted data. Because of the intensive computations involved, such approaches are marginally applicable on present day workstations to large scale problems in the 3d environment. In order to overcome this difficulty, Oldenburg and Ellis (1991) developed the AIM (Approximate Inverse Mapping) inversion as an alternative approach. The AIM inversion utilizes an approximate inverse mapping and an exact forward mapping to update the model successively without resorting to the full sensitivity matrix. Thus it can avoid the need for intensive computations.

The philosophy of the AIM inversion is to break up a non-linear inverse problem according to the different levels of physical processes involved, instead of using mathematical tools. This approach enables the physical understanding of the problem be introduced optimally into the derivation of the solution. It is recognized that most geophysical problems can have their physics divided into primary and secondary (or residual) processes. For example, the charge accumulation due to the primary electric field would constitute the primary physical process (whose field is governed by the Born approximation). The interaction between the accumulated charges would be the secondary process. Once the primary process is identified, an approximate inverse mapping, denoted by $\tilde{\mathcal{F}}^{-1}$, can be defined accordingly. The division between the primary and secondary physics is problem dependent. They can be chosen so that the primary physics has either a lower dimensionality or a reduced non-linearity and its inverse $\tilde{\mathcal{F}}^{-1}$ can be realized with less computational effort. Thus one can use the approximate inverse mapping to obtain that portion

of the model which can be seen “through” the primary process and find the remaining portion from the difference between the exact and approximate mappings.

There are two ways of utilizing $\tilde{\mathcal{F}}^{-1}$ to update the model. The first seeks a model perturbation at each iteration so that the updated model will reproduce the observed data according to $\mathcal{F}$. The second attempts to find a modification to the data so that, when the approximate inverse mapping is applied to the updated data, the resulting model will reproduce the observed data. Oldenburg and Ellis refer to these two approaches as AIM-MS and AIM-DS, respectively, for the reason that the perturbation is sought in either model space (AIM-MS) or data space (AIM-DS). There are several ways to derive the recursive formulae for the AIM algorithms. In the following, I shall present a very brief derivation. More detailed discussions can be found in Oldenburg and Ellis (1991).

5.1.1 AIM-MS

In the AIM-MS, a sequence of model perturbations is sought to update the model successively. Let $\vec{e}^{obs}$ be the observed data. Let $m^{(n)}$ be the model and $\vec{e}^{(n)}$ be the predicted data at the n^{th} iteration,

$$\mathcal{F}[m^{(n)}] = \vec{e}^{(n)}. \quad (5.3)$$

Application of the approximate inverse mapping $\tilde{\mathcal{F}}^{-1}$ to $\vec{e}^{obs}$ and $\vec{e}^{(n)}$ yields $\tilde{m}^{obs}$ and $\tilde{m}^{(n)}$. That is,

$$\tilde{\mathcal{F}}^{-1}[\vec{e}^{obs}] = \tilde{m}^{obs}, \quad (5.4)$$

$$\tilde{\mathcal{F}}^{-1}[\vec{e}^{(n)}] = \tilde{m}^{(n)}. \quad (5.5)$$

Since $\tilde{\mathcal{F}}^{-1}$ is only approximate, its application to the data $\vec{e}$ produced by a model m according to (5.1) will not recover the original model. The difference is quantified by

$$\Delta[m] = (\mathcal{I}_m - \tilde{\mathcal{F}}^{-1}\mathcal{F})[m], \quad (5.6)$$

where $\mathcal{I}_m$ is an identity mapping on model space. $\Delta[m]$ is called the mapping error on model space. Suppose m is the model which reproduces the observed data $\tilde{e}^{obs}$, then by definition,

$$m = \tilde{m}^{obs} + \Delta[m].$$

Since m is the unknown to be found, the above equation has to be solved iteratively by replacing the m on the two sides of the equation by the models at two successive iterations,

$$m^{(n+1)} = \tilde{m}^{obs} + \Delta[m^{(n)}]. \quad (5.7)$$

Combining (5.6) and (5.7) and substituting in (5.3) and (5.5) yields,

$$m^{(n+1)} = \tilde{m}^{obs} + m^{(n)} - \tilde{m}^{(n)}. \quad (5.8)$$

Equation (5.8) is the final formula for the iterative solution of m . The iteration starts with an initial model $m^{(0)}$, which can be supplied by $\tilde{m}^{obs}$.

The equation (5.8) can be viewed in two ways. Firstly, it can be written as

$$m^{(n+1)} = m^{(n)} + (\tilde{m}^{obs} - \tilde{m}^{(n)}),$$

and shows that the term $(\tilde{m}^{obs} - \tilde{m}^{(n)})$ constitutes a perturbation at $(n + 1)^{th}$ iteration, which is the difference between the models obtained by applying the approximate inverse mapping to the observed data and the data predicted by the current model. Therefore, the model is built up with a series of perturbations. Secondly, as (5.7) shows, the final model is made of two parts. The first part, $\tilde{m}^{obs}$, is the portion derivable directly using the approximate inverse mapping. The second part is a corrective term to account for the incompleteness of the approximate inverse mapping. In fact, the operator $\Delta = \mathcal{I}_m - \tilde{\mathcal{F}}^{-1}\mathcal{F}$ quantifies the portion of the model which is annihilated by the approximate inverse mapping $\tilde{\mathcal{F}}^{-1}$.

It is necessary for the algorithm to converge that the sequence of perturbations $(\tilde{m}^{obs} - \tilde{m}^{(n)}) \rightarrow 0$ as the iteration progresses. Equivalently, the mapping error $\Delta[m^{(n)}]$ should approach a constant,

i.e., $(\Delta[m^{(n+1)}] - \Delta[m^{(n)}]) \rightarrow 0$. This requires that the approximate inverse mapping be stable so that a small change in the data space is mapped to a small change in the model space.

5.1.2 AIM-DS

In the AIM-DS algorithm, a sequence of data perturbations is sought so that the application of the approximate inverse mapping to the updated data will generate a model which reproduces the observed data. For clarity, I present the derivation in parallel to that of AIM-MS. Again denote the observed data by $\vec{e}^{obs}$ and let $m^{(n)}$ be the model and $\tilde{e}^{(n)}$ be the modified data at the n^{th} iteration, thus,

$$m^{(n)} = \tilde{\mathcal{F}}^{-1}[\tilde{e}^{(n)}]. \quad (5.9)$$

Since $\tilde{\mathcal{F}}^{-1}$ is only approximate, a model m obtained by applying $\tilde{\mathcal{F}}^{-1}$ to data element $\tilde{e}$ will not reproduce that data element through the forward mapping. The difference is quantified by

$$\Delta[\tilde{e}] = (\mathcal{I}_e - \mathcal{F}\tilde{\mathcal{F}}^{-1})[\tilde{e}], \quad (5.10)$$

where $\mathcal{I}_e$ is an identity mapping on data space. $\Delta[\tilde{e}]$ is called the mapping error in the data space. Suppose m is a model derived by applying $\tilde{\mathcal{F}}^{-1}$ to the modified data $\tilde{e}$, and m adequately reproduces the observed data $\vec{e}^{obs}$. Then by definition,

$$\tilde{e} = \vec{e}^{obs} + \Delta[\tilde{e}].$$

Since we seek to form the modified data to recover the model m , this equation must be solved iteratively. Replacing $\tilde{e}$ on the two sides with the modified data in two successive iterations yields

$$\tilde{e}^{(n+1)} = \vec{e}^{obs} + \Delta[\tilde{e}^{(n)}], \quad (5.11)$$

which can be further expanded by substituting (5.3), (5.9), and (5.10),

$$\tilde{e}^{(n+1)} = \vec{e}^{obs} + \tilde{e}^{(n)} - \vec{e}^{(n)}. \quad (5.12)$$

This is the final formula for iteratively updating the data in AIM-DS algorithm. Applying the approximate inverse mapping $\tilde{\mathcal{F}}^{-1}$ yields the updated model in each iteration. The inversion again can start with any reasonable initial data $\tilde{e}^{(0)}$. However, the choice $\tilde{e}^{(0)} = \tilde{e}^{obs}$ is appropriate.

There is an alternative derivation for AIM-DS, which relies upon the explicit definition of an approximate forward mapping and offers some intuitive understanding of the algorithm. I formally introduce the approximate mapping $\tilde{\mathcal{F}}$ which describes the primary physical process and satisfies $\tilde{\mathcal{F}}^{-1}\tilde{\mathcal{F}} = \mathcal{I}_e$. Thus

$$\tilde{\mathcal{F}}[m^{(n)}] = \tilde{e}^{(n)}. \quad (5.13)$$

Suppose m is the model solution reproducing the observed data $\tilde{e}^{obs}$, then the data equation can be expressed as

$$\tilde{e}^{obs} = \tilde{\mathcal{F}}[m] + (\mathcal{F} - \tilde{\mathcal{F}})[m]. \quad (5.14)$$

That is, the data consist of the part predicted by the primary physics and a residual from the secondary physics. Rearrange the terms in (5.14) into

$$\tilde{\mathcal{F}}[m] = \tilde{e}^{obs} + \tilde{\mathcal{F}}[m] - \mathcal{F}[m].$$

The unknown m is to be solved iteratively by replacing it with $m^{(n+1)}$ and $m^{(n)}$ on the left and right hand sides respectively. Substituting in (5.13) yields the same equation as (5.12),

$$\tilde{e}^{(n+1)} = \tilde{e}^{(n)} + \tilde{e}^{obs} - \tilde{e}^{(n)}.$$

Thus the AIM-DS approach is equivalent to an iterative inversion where the data are successively updated by the difference between the exact and approximate responses from the current model.

Equation (5.12) can be viewed in either of the following two ways. The modified data are updated by adding to the current data $\tilde{e}^{(n)}$ the perturbation term which is the difference between the observed data and the predicted data, $(\tilde{e}^{obs} - \tilde{e}^{(n)})$. Therefore, the data are built up by a sequence of perturbations. Alternatively, the data are composed of the observed data $\tilde{e}^{obs}$ and a

corrective term given by the mapping error in data space ($\tilde{e}^{(n)} - \tilde{e}'^{(n)}$). In order for the AIM-DS algorithm to converge, it is necessary that the perturbation ($\tilde{e}^{obs} - \tilde{e}'^{(n)}$) $\rightarrow 0$. Therefore it is required that the difference between the data mapping errors at successive iterations approach zero. This means that $\tilde{e}^{obs} + \Delta[\tilde{e}^{(n)}]$ must converge to a fixed point in the data space.

As the iteration progresses, the models from both AIM-MS and AIM-DS will likely to have increased structure. In the AIM-MS algorithm, the model is built up by a sequence of perturbations. The final model cannot be regularized explicitly in general. In the AIM-DS, the data are built up with a sequence of perturbations thus they will exhibit more and more structures. This increased structure will be translated to the resulting model. However, one advantage of the AIM-DS is that the final model is always obtained by a single application of the inverse mapping. Therefore, it is possible that a norm of the model can be minimized so that the structural complexity of the model be controlled to a certain extent. If both the approximate forward and inverse mappings are linear, it can be shown that AIM-MS and AIM-DS degenerate to the same algorithm. This is an uncommon occurrence since the regularization in the inversion renders the mapping non-linear. However, there are situations in which the inverse mapping is nearly linear. Then the results from the two algorithms will not differ significantly.

5.2 AIM Inversion of 3d d.c. Resistivity Data

In this section, I apply the AIM formalism to the 3d d.c. resistivity problem and develop an iterative inversion algorithm to invert E-SCAN® data sets. The two essential components for forming an AIM inversion algorithm are an approximate inverse mapping and an accurate forward mapping. The approximate 3d inversion presented in Chapter 4 is chosen as the approximate inverse mapping $\tilde{\mathcal{F}}^{-1}$. For the forward mapping, $\mathcal{F}$, a 3d finite difference modelling algorithm is used.

The finite difference forward modelling algorithm presented by Dey and Morrison (1979) is chosen. The conductivity model represented by a finite rectangular region is discretized into prismatic cells by a 3d orthogonal mesh. Each cell is assigned a constant conductivity value. The

algorithm can work with general 3d conductivity models but the current source must be placed on a nodal point. The resulting potential is calculated at all other nodal points. The current sink is assumed to be at infinity.

In order to deal with the finite domain, a mixed boundary condition is adopted on the vertical and the bottom sides of the rectangular region, which simulate the infinitely distant planes of the real model. The potential on those surfaces are assumed to have the general form of $\phi(\vec{r}) = A/r$ where A is a constant, and r is the length of the position vector, assuming that the coordinate origin is at the centre of the grid on the surface. By differentiating the potential with respect to the outward normal of the bounding plane, the mixed boundary condition is derived to be

$$\frac{\partial \phi(\vec{r})}{\partial \hat{n}} + \cos \theta \frac{\phi(\vec{r})}{r} = 0, \quad (5.15)$$

where θ is the angle between the normal vector and the position vector $\vec{r}$.

The discrete equation for potentials is formed by integrating the governing differential equation (2.12),

$$\nabla \cdot (\sigma \nabla) = -I\delta(\vec{r} - \vec{r}_s),$$

in a volume around each node. The volume extends halfway into the eight cells adjacent to the node. The resultant coefficient matrix is banded and is solved using a conjugate gradient method. Several tests have been made to compare this finite difference code with other forward modelling codes. For some models and some data I found discrepancies of 10%, but usually discrepancies were less than 5%.

The approximate 3d inversion as discussed in Chapter 4 can take either a continuous or discrete form in the vertical direction. In theory, they can both be used in the AIM application. A discrete form is chosen here because a single vertical grid can then be used for both the approximate inverse mapping and the forward mapping. This avoids rediscrization of the results from inverse mapping for input of the forward modelling.

Two different meshes are used for the forward mapping and for the approximate inverse mapping. For the forward problem, the mesh consists of a core portion and an extension region. The E-SCAN survey grid is used as the horizontal grid of the mesh in the core. A vertical partitioning is then chosen. It extends to a maximum depth so that the kernel functions for the approximate 3d inversion all have negligible amplitude at that depth. Surrounding this core mesh, an extension mesh is added so that the boundary condition in the forward modelling algorithm can be handled. This is usually a three-cell extension with increasing intervals towards the boundary.

The mesh for the inverse mapping is chosen with the horizontal grid conforming to that described in Chapter 4. That is, the horizontal area consists of the original survey grid at the centre and is extended by one half of the maximum array separation in each direction (Fig. 4.2). The grid spacing is chosen to be half of the original survey grid spacing. A vertical partitioning is chosen to coincide with that of the forward modelling mesh.

The resultant conductivity model of inverse mapping attains a discrete point representation in the horizontal directions by the nature of the fast Fourier transform (FFT). Beneath each horizontal point, the model possesses a constant value along the depth within each vertical interval of the mesh. The logarithmic conductivity $\ln(\sigma)$ is assumed to be represented by piecewise bilinear interpolations in the horizontal directions. A conductivity value is then derived from the integrated average of $\ln(\sigma)$ within each cell of the forward mesh. This value is assigned to the corresponding cell as the recovered conductivity value.

With the forward and inverse mapping and their respective meshes being defined, the next step is to define the data and the model of the AIM inversion. Although $\tilde{\mathcal{F}}^{-1}$ takes the relative potential anomaly as data and produces a conductivity model to which the forward mapping $\mathcal{F}$ is applied, the data and the model for the entire process of AIM inversion can be defined differently. For AIM-MS, it is better to define the model as the logarithm of the conductivity values. Therefore, the model for AIM-MS inversion is composed of the set of the logarithmic conductivity values $\ln(\sigma_{ijk})$ ($i = 1, \dots, M_x$, $j = 1, \dots, M_y$, $k = 1, \dots, M_z$), where M_x , M_y , M_z are respectively the number of cells in the x -, y -, and z -direction. Working with logarithmic

conductivities is advantageous because the variation in the earth's conductivity commonly ranges over several orders of magnitude and also it ensures the positivity of the conductivity model. In addition, it is consistent with the logarithmic conductivity perturbation sought in the approximate inverse mapping. Thus, the iterative formula (5.8) takes the form

$$\ln(\sigma^{(n+1)}) = \ln(\tilde{\sigma}^{obs}) + \ln(\sigma^{(n)}) - \ln(\tilde{\sigma}^{(n)}), \quad (5.16)$$

Given the total potential $\phi_j(x, y)$ associated with n_l pole-pole array configurations, the data for the AIM-DS inversion is defined as the logarithm of the potential values, $\ln(\phi_j(x, y))$ ($j = 1, \dots, n_l$). The formula (5.13) becomes

$$\ln(\tilde{\phi}^{(n+1)}) = \ln(\tilde{\phi}^{(n)}) + \ln(\phi^{obs}) - \ln(\phi^{(n)}), \quad (5.17)$$

Adopting the logarithmic potential as the data eases the task of data interpolation and ensures that the updated potential is always positive.

The AIM inversion of 3d d.c. resistivity data proceeds by the following steps:

AIM-MS:

1. Apply the approximate 3d inversion ($\tilde{\mathcal{F}}^{-1}$) to the observed data ϕ_j^{obs} to obtain $\tilde{\sigma}^{obs}$. Let $\sigma^{(1)} = \sigma^{obs}$ and generate the predicted data $\phi_j^{(1)}$ by applying the forward modelling.
2. Apply $\tilde{\mathcal{F}}^{-1}$ to $\phi_j^{(n-1)}$ to obtain model $\tilde{\sigma}^{(n-1)}$.
3. Update the current model by (5.16), $\ln(\sigma^{(n+1)}) = \ln(\tilde{\sigma}^{obs}) + \ln(\sigma^{(n)}) - \ln(\tilde{\sigma}^{(n)})$, to obtain a new model $\sigma^{(n)}$.
4. Apply $\mathcal{F}$ to compute the predicted data $\phi_j^{(n)}$.
5. Compute the misfit between the observed data ϕ_j^{obs} and the predicted data $\phi_j^{(n)}$. Exit the algorithm if the misfit is acceptable. Otherwise, go to step-2 and begin a new iteration.

AIM-DS:

1. Take the observed data ϕ_j^{obs} as the initial data $\tilde{\phi}_j^{(1)}$ and apply the approximate inverse mapping to produce model $\sigma^{(1)}$. Compute the predicted data $\phi_j^{(1)}$ by applying forward modelling to $\sigma^{(1)}$.

2. Update the data by (5.17), $\ln(\tilde{\phi}^{(n+1)}) = \ln(\tilde{\phi}^{(n)}) + \ln(\phi^{obs}) - \ln(\phi^{(n)})$, to obtain new data $\tilde{\phi}_j^{(n)}$.
3. Compute the new model $\sigma^{(n)}$ by applying the inverse mapping to $\tilde{\phi}_j^{(n)}$.
4. Compute the predicted data $\phi_j^{(n)}$ from the new model.
5. Calculate the data misfit. Exit the algorithm if the misfit is acceptable. Otherwise, go to step-2 and begin a new iteration.

5.3 Examples of the AIM Inversion

To illustrate the inversion of the E-SCAN d.c. resistivity data using the AIM formalism, both AIM algorithms are applied to two sets of synthetic data. These examples demonstrate the convergence of the algorithm, the resolution of the algorithm given the grid spacing, and the depth of investigation given the grid spacing and the maximum array separation.

The synthetic data are computed from two cellularized conductivity models using the finite difference forward modelling algorithm. The distribution of the computed data simulates that of the field data over a 21×21 grid with a 50m spacing in both the x - and y -directions. The first model is a 5-prism model similar to that used in Chapter 4 (Fig. 4.2). The locations of the five conductivity anomalies are changed slightly so that their boundaries are aligned with the cell boundaries used in the finite difference program.

The second model consists of correlated random perturbations to a uniform half-space. Two sets of random numbers with different spatial correlation lengths in the $3d$ space are first generated. A weighted sum of the two sets are used as the conductivity perturbation $\ln(\mu)$ to form the model. The component with longer correlation length is tapered to zero near the surface so that the model simulates the increased spatial variation in the conductivity near the surface. The resultant conductivity model is then converted into prismatic cells by integrations. The cellularized conductivity constitutes the "true model" in this example and will be referred as the geo-statistical model.

5.3.1 The 5-prism model

I now proceed to the results of AIM inversion applied to these examples. For the 5-prism model, I choose a vertical grid of 21 nodes in the core portion with intervals increasing with the depth. The horizontal grid is the same as the data grid. A three-cell extension zone is appended to the core mesh. The AIM-MS algorithm is first applied to the data set and twelve iterations are performed. The regularization of the $1d$ inversions in the wavenumber domain is based upon the norm of the model in each individual inversion. (This method of choosing regularization parameter is used for all the examples in this chapter.) Throughout the inversion, the half-space conductivity has been held at the correct value of 1mS/m. Fig. 5.1 shows the comparison of models at the section $x = 460\text{m}$, which cuts through the four major conductivity anomalies in the model. The gray scales in this and all the following figures are in $\log_{10}(\sigma)$. The top panel is the true model. The middle panel is the model after first iteration (i.e., after the simple application of the approximate inverse mapping), and the bottom panel shows the recovered model from the 12th iteration. Fig. 5.2 shows the similar comparisons at two depth slices at $z = 20\text{m}$ and $z = 150\text{m}$ respectively. The first iteration recovers an image of the conductivity model. The surface anomalies and the buried conductive anomaly are clearly defined but the buried resistive anomaly is not defined very well. The amplitudes of the anomalies are far less than that of the true model. However, further updates by the AIM algorithm greatly improves the result. The final model as shown in the bottom panels of the two figures recovers all the conductivity anomalies and has a dynamic range comparable with that in the true model. In terms of the model recovery, the inversion is quite successful.

Fig. 5.3 attempts to illustrate the convergence property of this particular inversion. The top panel is the RMS misfit of the total potentials at each iteration. The RMS misfit is adopted because the data are accurate. The half-space model (0th iteration) and the model from the approximate inverse mapping have a misfit of 12% and 4.2%, respectively. As the iteration progresses, the misfit is steadily reduced. By the end of the 12th iteration, an RMS misfit of 0.5% is achieved. It is noted that the misfit decreases monotonically with the iteration. The middle panel of Fig. 5.3

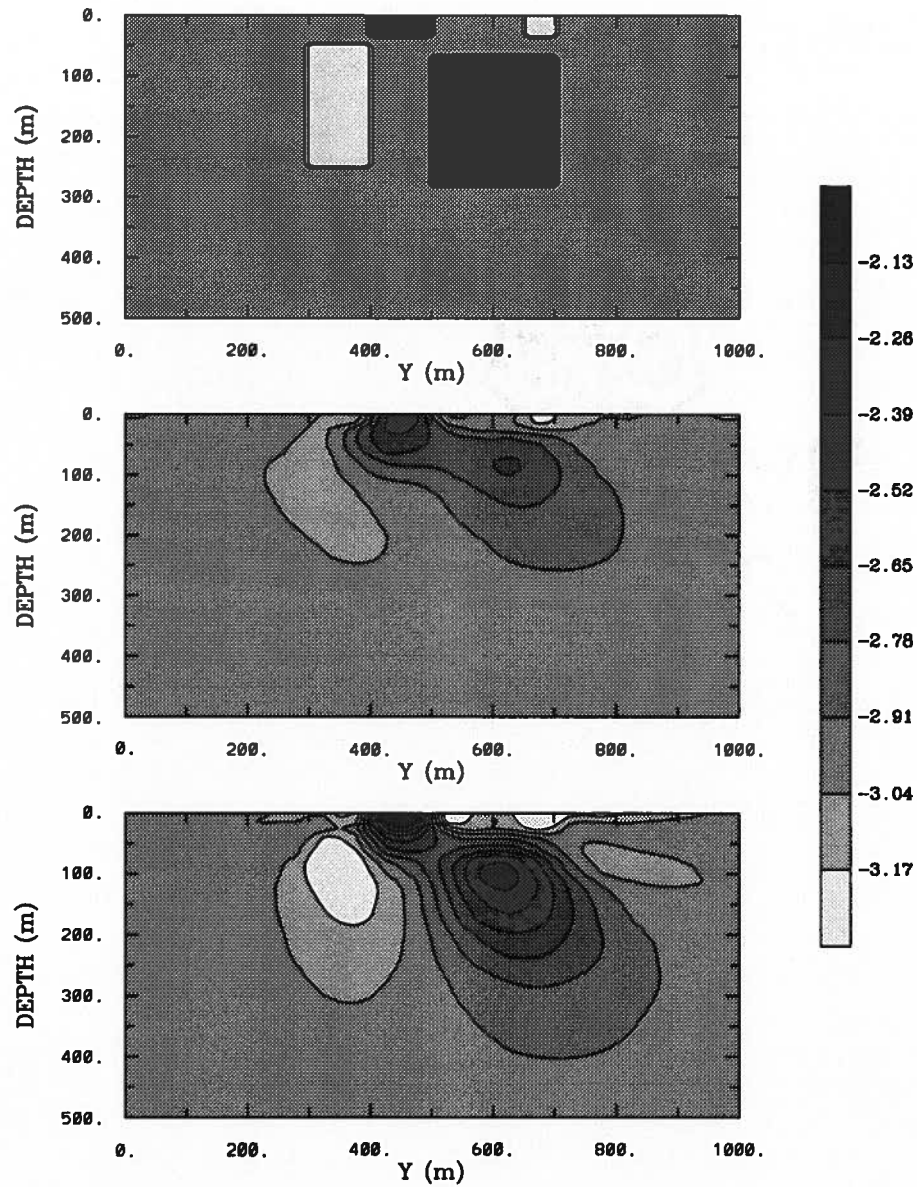


Figure 5.1 Comparison of the 5-prism model and the AIM-MS inversion result in section $x=460\text{m}$. The upper panel is the true model, the middle panel is the model from the first iteration, and the lower panel is the final model.

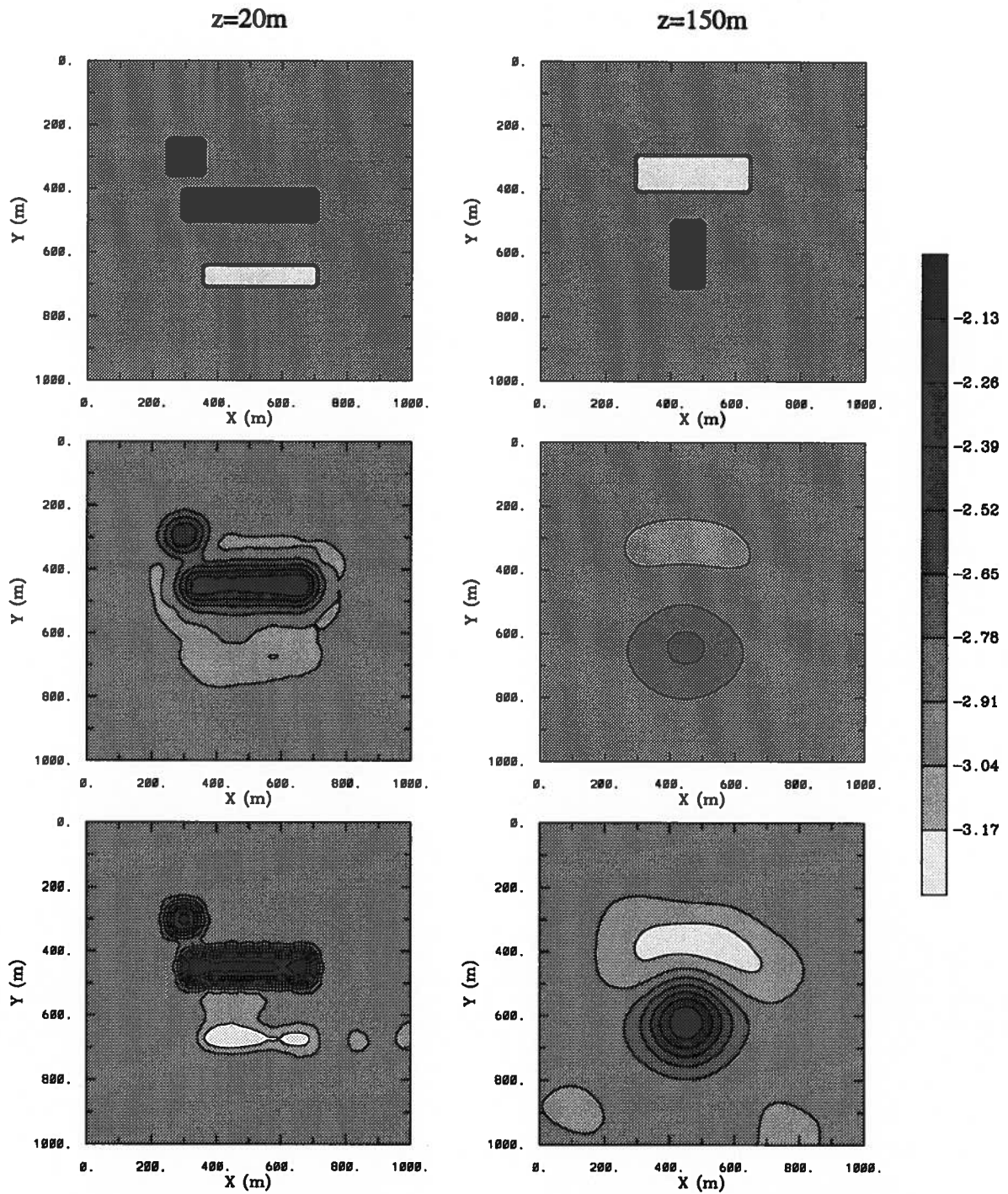


Figure 5.2 Comparison of the 5-prism model and the AIM-MS inversion result at depths $z = 20$ and 150m . The true model and the models from the first and last iteration are shown from top to bottom.

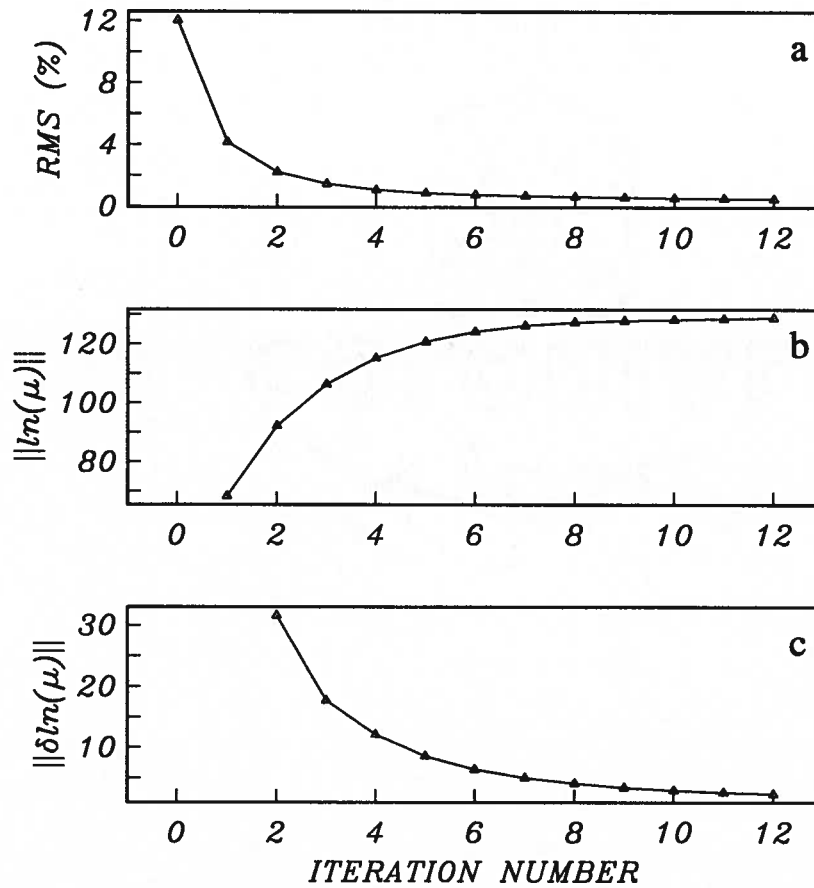


Figure 5.3 The change of the RMS data misfit (a), the model norm (b) and the norm of the model perturbation (c) with the iteration in the AIM-MS inversion for 5-prism model.

shows the norm of the model $\| \ln(\mu(\vec{r})) \|$, where $\mu(\vec{r}) = \sigma(\vec{r})/\sigma_0$. The bottom panel shows the norm of the model perturbation at each iteration. The model norm increases monotonically and approaches an asymptotic value. The norm of the model perturbation decreases in a similar manner and approaches a negligible value compared with that of the model norm (The ratio of the two quantities is 0.012 at the last iteration). Thus the necessary condition outlined in the derivation of AIM-MS algorithm that the model perturbation approach zero is met. Together with the decrease of the misfit, this tends to suggest that this particular application of the AIM-MS inversion is stable and on a path towards convergence.

Next, the AIM-DS inversion is applied. The mesh is the same as that used in the AIM-MS inversion. Again, the inversion has been run for twelve iterations. Fig. 5.4 is the final AIM-DS model at slices $x=460\text{m}$, $z=20\text{m}$, and $z=150\text{m}$. The recovery is good in terms of both locations and amplitudes of the anomaly. It is noticed that this conductivity model is very similar to that obtained by AIM-MS inversion. A detailed comparison finds that the difference between the recovered conductivity values from AIM-MS and AIM-DS can differ by up to 20%. The norm of the AIM-DS model is also slightly smaller than that of the AIM-MS model.

Fig. 5.5 is again to illustrate the convergence properties of the inversion. The top panel in the figure shows the RMS misfit of the total potential. Similar to the result in the AIM-MS inversion, the reduction of the misfit is along a monotonic curve. The final misfit achieved is below 0.5%. It is worth noticing that the major reduction of the misfit is achieved with first few iterations in both the AIM-MS and the AIM-DS inversions. Once the misfit is reduced below a level ($\sim 1.0\%$ in this case), further iteration does not improve it significantly.

The middle and the bottom panel of Fig. 5.5 display respectively the norm of the modified data $\tilde{\epsilon}$ and the norm of the data perturbation $\delta\tilde{\epsilon}$ at each iteration. Here the data are the relative anomaly, $(\phi - \phi_0)/\phi_0$, since that is the quantity used in the inverse mapping. The norm of the modified data increases monotonically towards an asymptotic value while the norm of the perturbation decreases monotonically and levels off at a negligible fraction of the data norm (2.8%). This is very similar to the behavior of model norm and the norm of the model perturbation in AIM-MS. For this ideal case of accurate data, the result conforms with the condition outlined in the derivation of the AIM-DS algorithm that the perturbation of the modified data approach zero.

The above tests have used only the accurate data. A more realistic test, however, is to use data that are contaminated with noise. For the d.c. resistivity experiment there are two major types of noise, namely the geological noise and experimental errors. The former is mostly caused by features such as near surface variations in the conductivity. This type of noise mostly concentrates in the high wavenumber band and can be suppressed in the inversion by limiting the highest wavenumbers. The second type of noise is the error associated with all stochastic

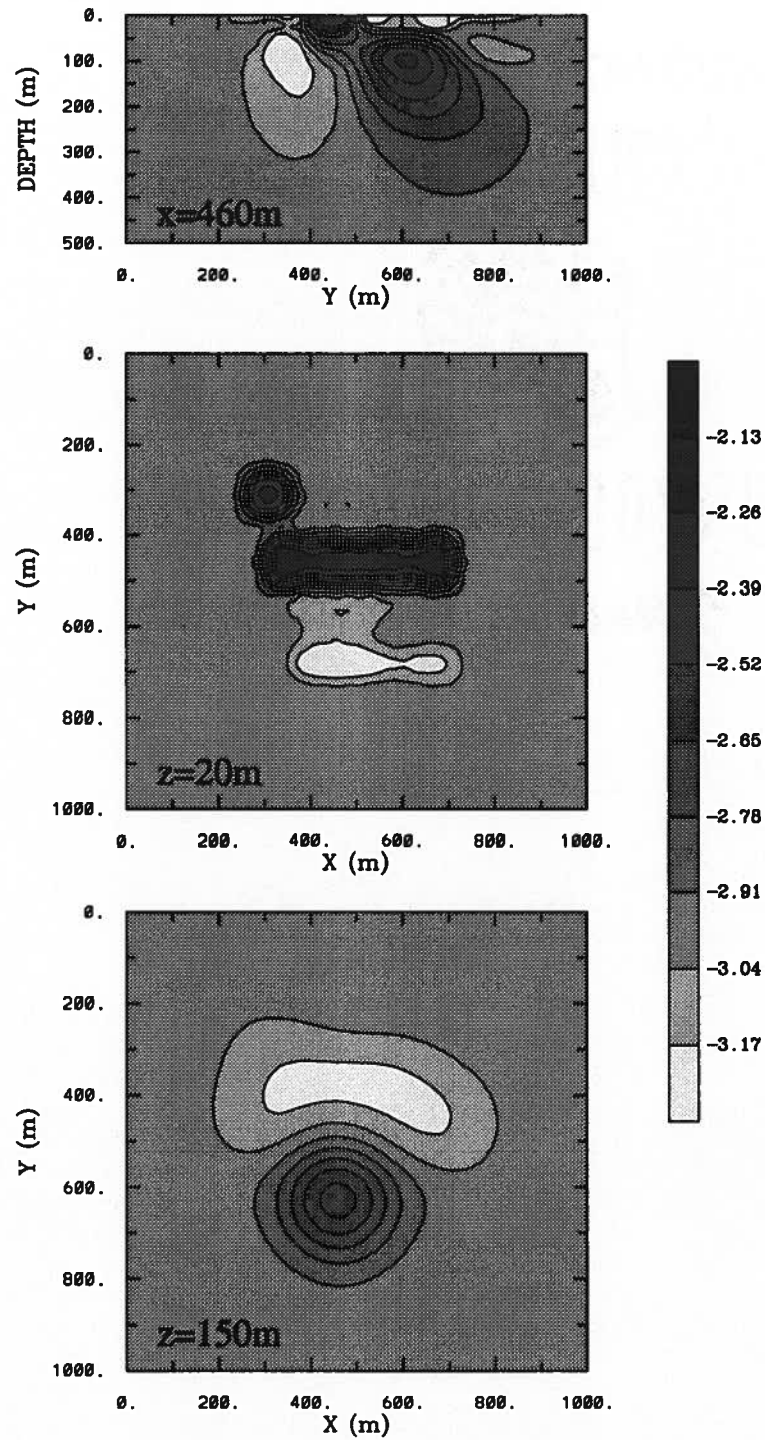


Figure 5.4 AIM-DS inversion result for the 5-prism model. The panels from top to bottom are respectively the slices at $x = 460\text{m}$, $z = 20\text{m}$, and $z = 150\text{m}$.

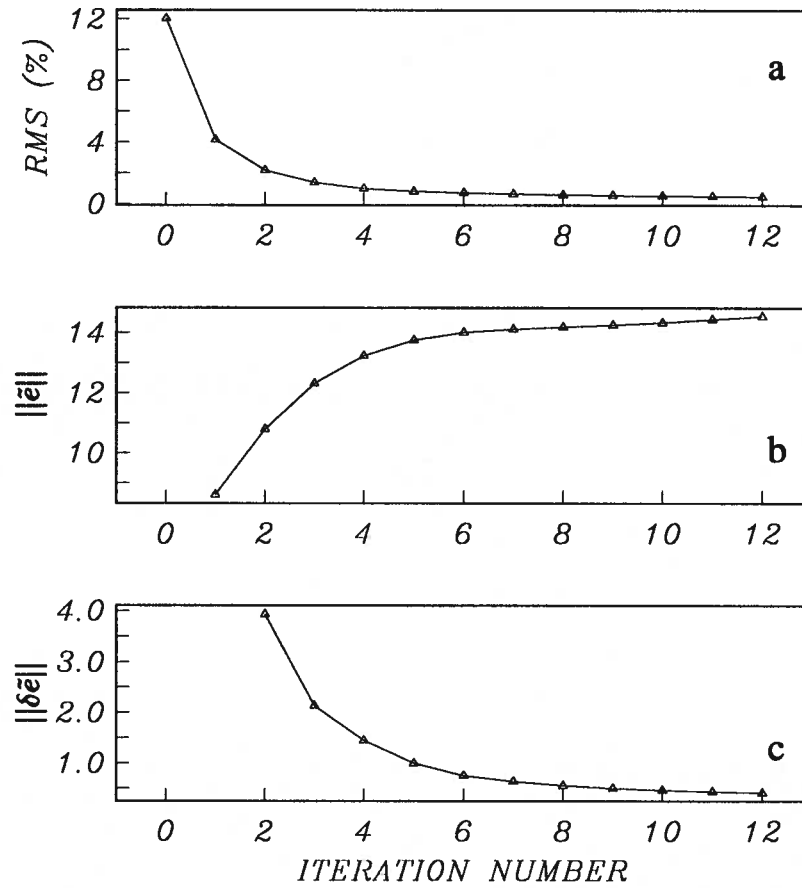


Figure 5.5 The change of the RMS data misfit (a), norm of the modified data (b) and the data perturbation (c) with iteration in the AIM-DS inversion for the 5-prism model.

processes involved in the data collection. Such noise is often uncorrelated and distributed over the entire wavenumber band. It is often simulated by adding to the data independent random numbers with prescribed standard deviation and mean.

For the data from the 5-prism model, I added independent Gaussian noise with zero mean and standard deviations which amount to 5% of the corresponding total potential. A more realistic simulation requires that there be a minimum for the standard deviation of the noise so that the errors on the data from large array separations can be modelled. However, the separations of the array in this numerical experiment only span the short range of 50 to 400 meters. Consequently

the measured potential varies with the separation only in a small range and adding a percentage is reasonable.

The synthesized inaccurate data are input into the AIM-DS algorithm and eight iterations are run. The χ^2 misfit of the total potential is reduced below the expected value after 4 iterations. Since the data now have genuine errors on them, the χ^2 misfit can be computed. Fig. 5.6 shows the model at the 4th iteration. It is immediately clear from the figure that the noise in the data has introduced some spurious structures into the resultant conductivity model. Such distortion is especially strong near the surface. At depth, the shape of buried resistive prism, which is a very weak anomaly, has been altered significantly. However, all anomalies in this particular model are recovered and their definitions are clear.

Because only the norm of the model is minimized in the inversion, there is no explicit control over the structural complexity of the model. Consequently, the spurious structures introduced by the noise cannot be suppressed directly through the regularization in the inversion. However, these structures seem to be uncorrelated and become progressively weak as the depth increases. This is judged to be a direct result of the approximate inverse mapping. From Chapter 4 we know that the kernel functions in the wavenumber domain decays more rapidly with depth at higher wavenumbers. Meanwhile, the energy of the data is concentrated in the lower wavenumber band whereas the energy of the noise is spread throughout the entire band. Thus at lower wavenumber, the signal-to-noise ratio is much higher and the recovered model components have less spurious features. Structures at depth are dominated by these low wavenumber components. On the other hand, the noise component in the data becomes stronger as the wavenumber increases. The corresponding model component becomes more and more affected by the noise. Since the kernel functions are sensitive to shallower regions at higher wavenumbers, the constructed model components are non-zero in the progressively shallower region as the wavenumber increases and the recovered conductivity model in the spatial domain has more noisy features near the surface. This also suggests that these undesirable features can be treated by applying a low-pass filtering to the model with the cut-off wavenumber decreasing with the depth. The type of filter and the

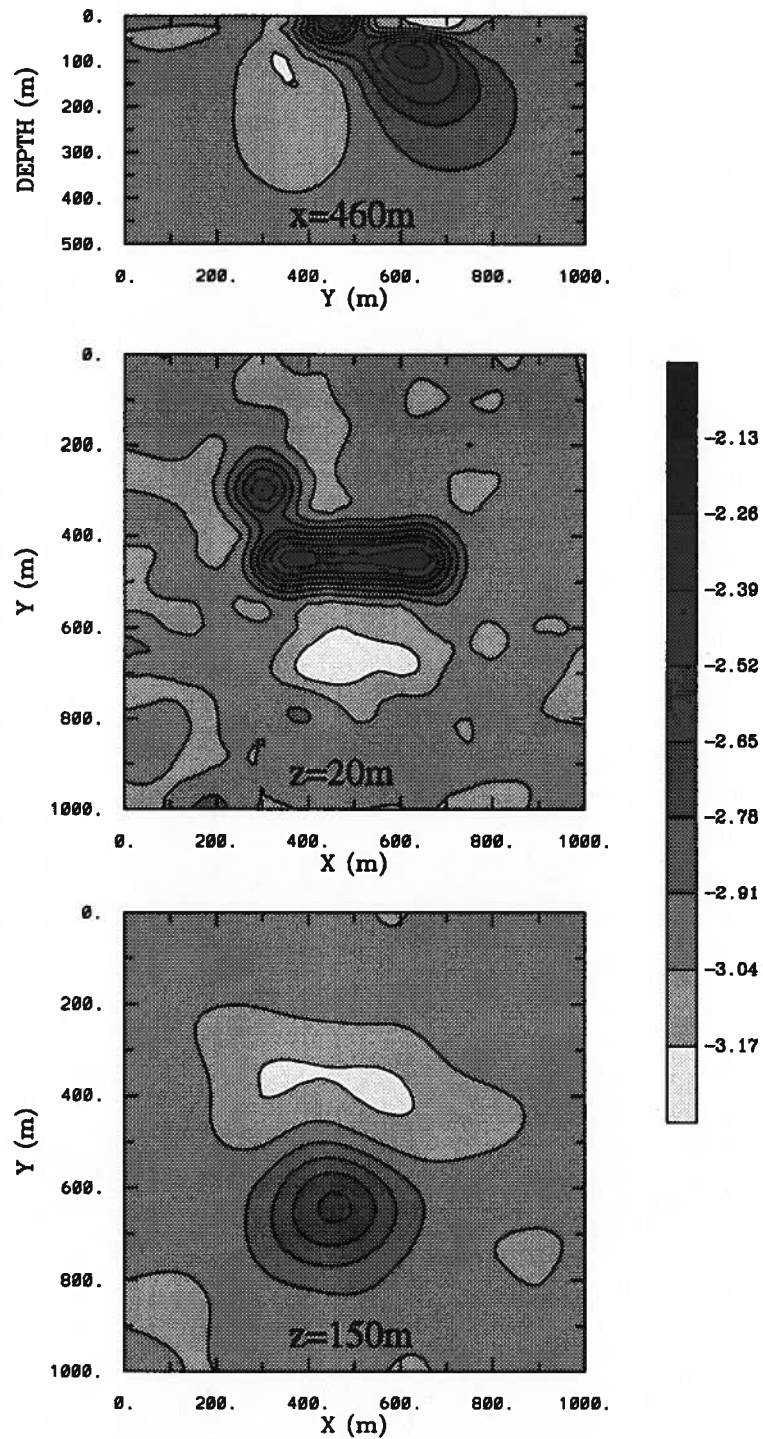


Figure 5.6 AIM-DS inversion result of the noisy data from 5-prism model. The panels from top to bottom are respectively the slices at $x=460\text{m}$, $z=20\text{m}$, and $z=150\text{m}$.

cut-off wavenumber will no doubt depend upon the spectrum of data and the noise and, hence, will be problem dependent.

Fig. 5.7 gives some details on the progress of the inversion as a function of iteration. The first panel shows the χ^2 misfit at each iteration. The dashed line indicates the expected value of χ^2 , which is equal to 8804. Again we observe a steady reduction of the misfit. After four iterations, the χ^2 misfit has dropped below the expected value. Further iterations do not result in significant improvement. In fact from the 6th iteration there is a slight increase in χ^2 . This is to be expected. It has been shown (Bertero *et al.*, 1988) that the degree of regularization is inversely proportional to the number of iterations for an iterative inversion without explicit regularization. Such is the case with the AIM inversion as applied here.

The norm of the modified data (panel b) linearly increases with iteration number. The norm of the data perturbation (panel c) is first on a decreasing curve and then starts to increase from the 6th iteration. This turning point seems to coincide with that of the χ^2 misfit curve and indicates that the inversion has begun to diverge. This suggests that, in practical applications, the inversion should be terminated once the expected misfit is achieved. If the expected misfit cannot be reached, the onset of the increase in data perturbation should serve as the criterion for termination. Since this specific implementation of the AIM inversion does not explicitly minimize the model norm subject to a prescribed data misfit, the prolonged iterations increase the structural complexity of the model but do not necessarily improve the data misfit.

It is interesting that the norm of the model increases smoothly with decreasing rate up to the last iteration in spite of the near-divergent behavior of the modified data. Careful examination of the modified data reveals that the large increase of the data norm is mostly due to the outliers in the data where the added noise is strong. These outliers seem to be enhanced by data updating. However, their energy is spread over the wavenumber band and the regularization based on the model norm tends to suppress their effect so that the recovered model is relatively stable.

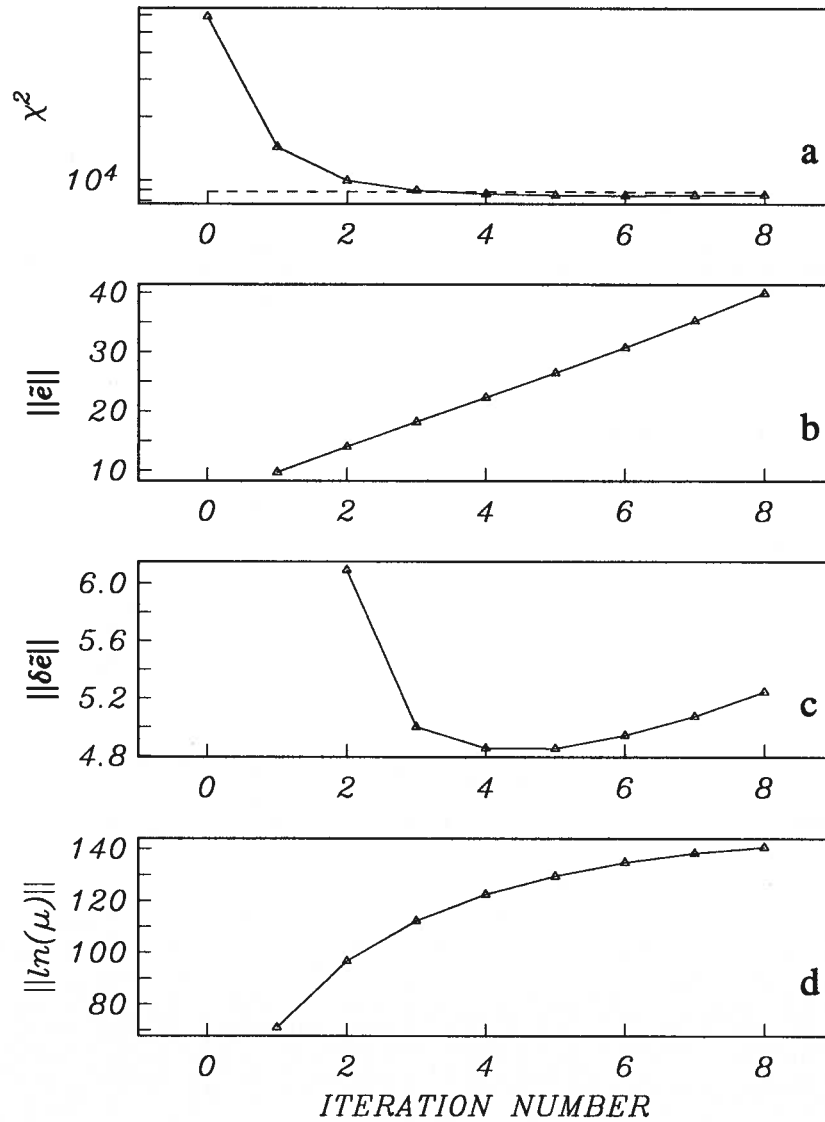


Figure 5.7 Progress of the AIM-DS inversion applied to noisy data from 5-prism model. Panel (a) shows the change of the misfit with the iteration. The dashed line indicates the expected χ^2 misfit. Panels (b) and (c) are the norm of the modified data and the norm of the data perturbation respectively. Panel (d) shows the change of the model norm.

5.3.2 The geo-statistical model

The 5-prism model clearly illustrates both methods of the AIM inversion and demonstrates their ability to derive a conductivity model which fits the data within the error tolerance. It also shows the performance and the complications in the presence of the noise in the data. However the model is simple and the conductivity variations are limited in a relatively shallow region. It is hoped that the geo-statistical model represents a geologically more realistic test which can provide some insight into the performance in more complicated situation. For this model, I choose an inversion mesh similar to the one used for the 5-prism model, but with the core portion being partitioned uniformly with depth at 25 m intervals.

The AIM-MS inversion is first applied. The inversion is terminated after twenty-two iterations. The recovered model after the 16th iteration is shown in Fig. 5.8 and Fig. 5.9 in comparisons with the true model in four slices at different depths. The three panels in each column of the figures display, respectively from top to bottom, the true model, the models at the first and 16th iteration. The first column in Fig. 5.8 is the conductivity in the first layer (between depths 0 to 25m). The second column is the third layer (between 50 to 75m). The conductivity in these layers is recovered with great fidelity, although fewer details are recovered in the third layer. The dynamic range of the recovered conductivity is very close to that of the true model at these depths. The model from the simple application of the inverse mapping is plotted to demonstrate the improvement achieved by the iterative process.

The two columns in Fig. 5.9 show respectively the comparisons in layer-6 (between depths 125 to 150m) and layer-10 (225 to 250m). It is clear that both the detail and the amplitude of the recovered conductivity have decreased considerably. As depth increases, we only see larger scale features with smaller dynamic range present in the recovered model. However, strike direction of the major features are still evident and this could be important in practical applications.

In general it is again observed from the four slices, that the resolution of the model decreases rapidly with the depth. The resultant model appears to be a smoothed version of the true model and the smoothing is increased with the depth. A question of both theoretical and practical

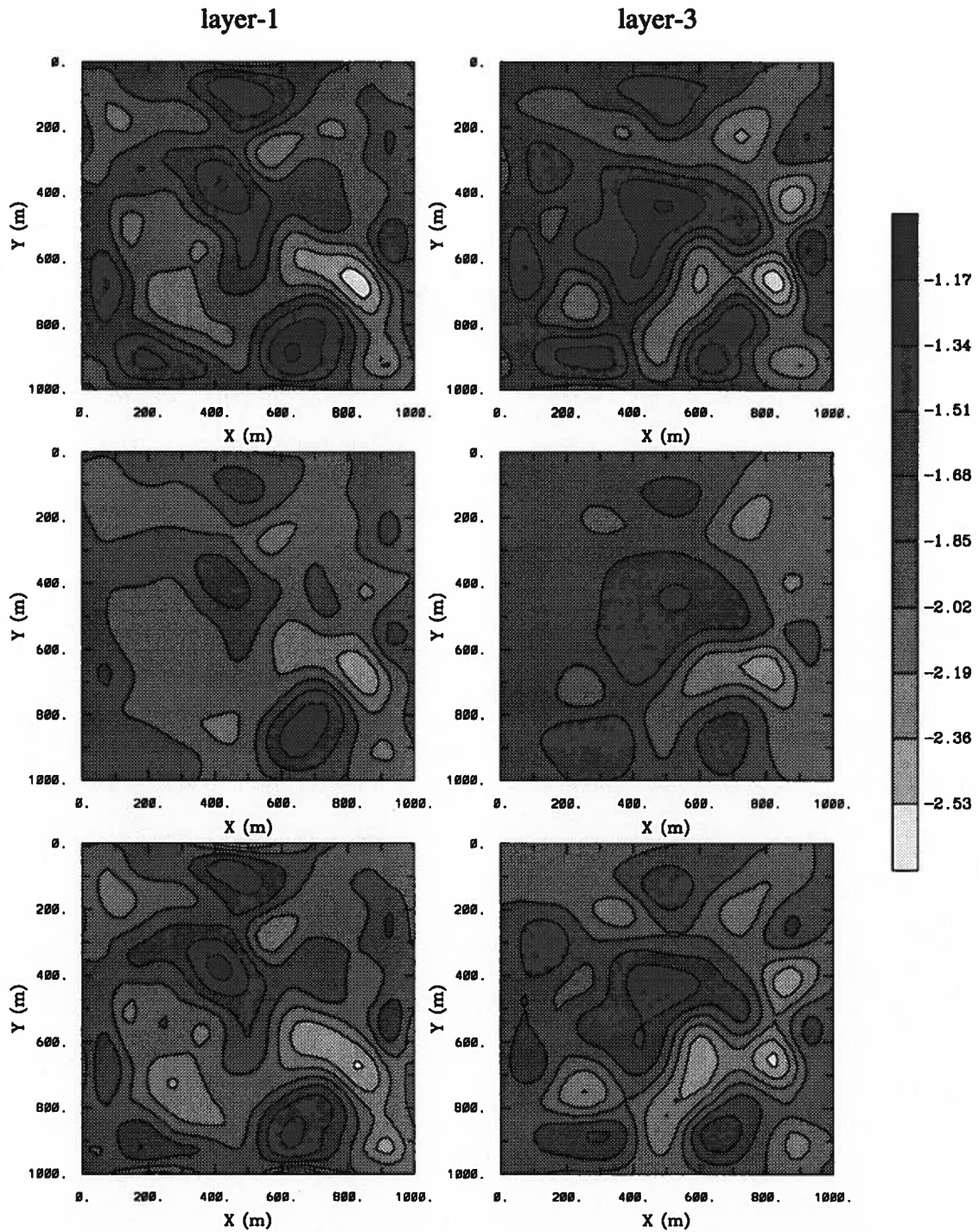


Figure 5.8 Comparison of the geo-statistical model and the AIM-MS inversion results in layer-1 ($z=(0, 25)\text{m}$) and layer-3 ($z=(50, 75)\text{m}$). The panels from top to bottom in each column show respectively the true model and the models at the first and 16th iteration.

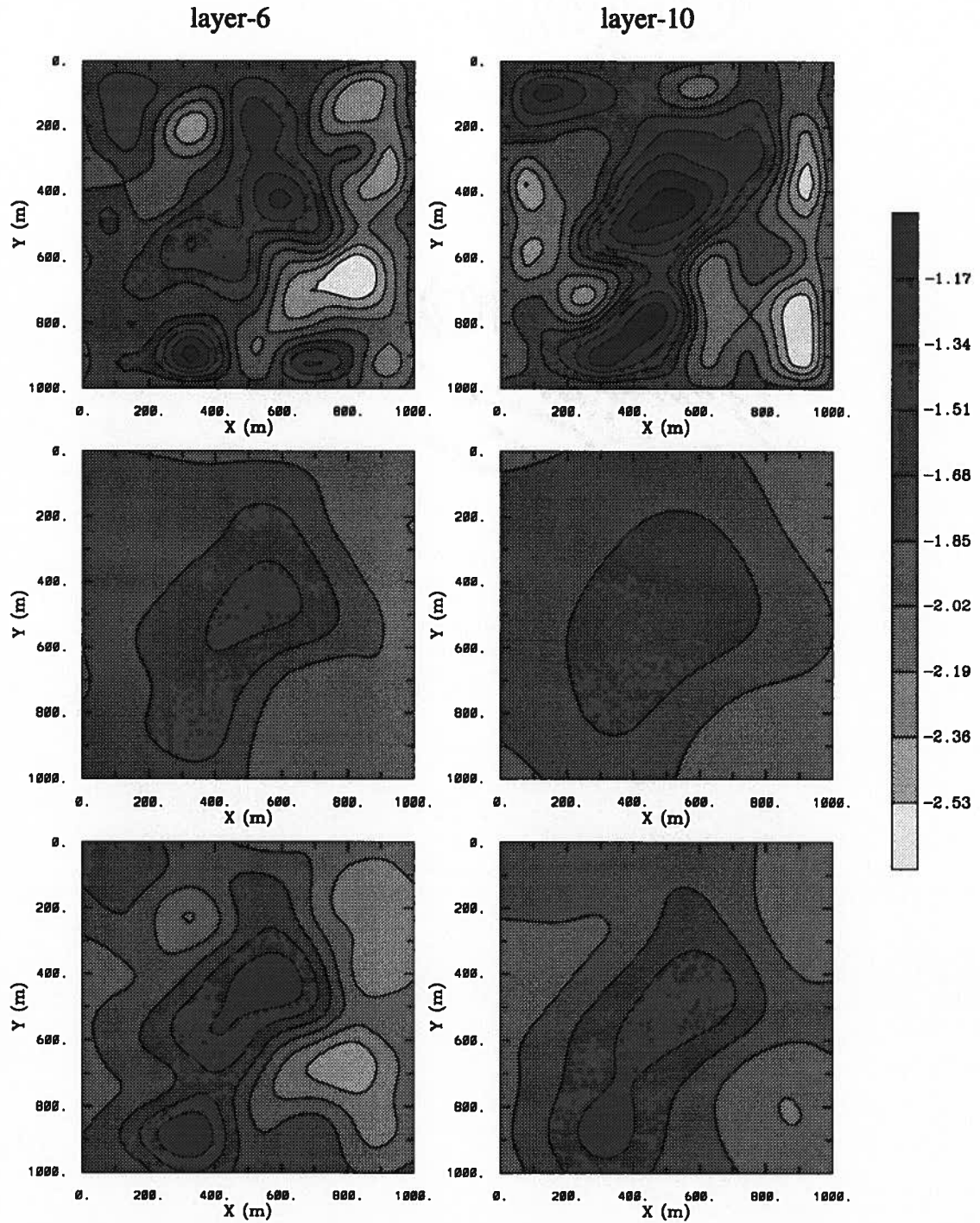


Figure 5.9 Comparison of the geo-statistical model and the AIM-MS inversion results in layer-6 ($z=(125, 150)\text{m}$) and layer-10 ($z=(225, 250)\text{m}$). The panels from top to bottom in each column show respectively the true model and the models at the first and 16th iteration.

importance is: “What is the maximum depth above which the algorithm can recover some conductivity variations from the given data set?” This question is closely related to the question about the depth penetration of the surface d.c. resistivity experiment and its answer is far more involved. Parker (1984) demonstrates that the depth penetration of *1d* d.c. resistivity data is infinitesimal without additional constraints. This has direct bearings on the *3d* experiment. However, if the value of the conductivity is limited to a finite range and arbitrarily rapid variations are restricted, the experiment should still have a finite depth of penetration. The limited Fourier components and the finite partitioning in the vertical direction used in the AIM inversion impose such constraints implicitly. Therefore, I attempt to provide a qualitative evaluation.

Fig. 5.10 and Fig. 5.11 display the comparison in a cross-section of the true and recovered model. The cross-section is taken at $x=420\text{m}$ and is plotted at the scale of the true model. Over the depth extension of the model, the inversion result seems to be a very poor representation of the true model. However, close examination shows that the recovery above the depth of approximately 200m is actually rather good. Below this depth, only the gross features caused by conductive anomalies are present. The recovered model exhibits few features below 300 m. The Fig. 5.11 is a detailed comparison in the region above 200m. It is apparent that the recovered model and the true model agree well. Thus, the maximum depth of the model recovery is estimated at approximately 200 m, one half of the maximum array separation. This result is in agreement with the observation at the end of Section 4.3 that the kernel function does not exhibit significant structure below 200m for an array with 400m separations. To further verify the above maximum depth, the conductivity below 200 m in the true model is muted to the background value and the new responses are computed by the same forward modelling program. The RMS relative difference between the new and original data is 4.5%. The contribution to the difference increases with the array separation. This suggests that the inability to recover conductivity variations below the maximum depth is partly inherent to the data set.

Fig. 5.12 shows the change of misfit, norm of the model and model perturbation. The RMS misfit exhibits the familiar steady reduction up to iteration-16 (this is the iteration whose model is

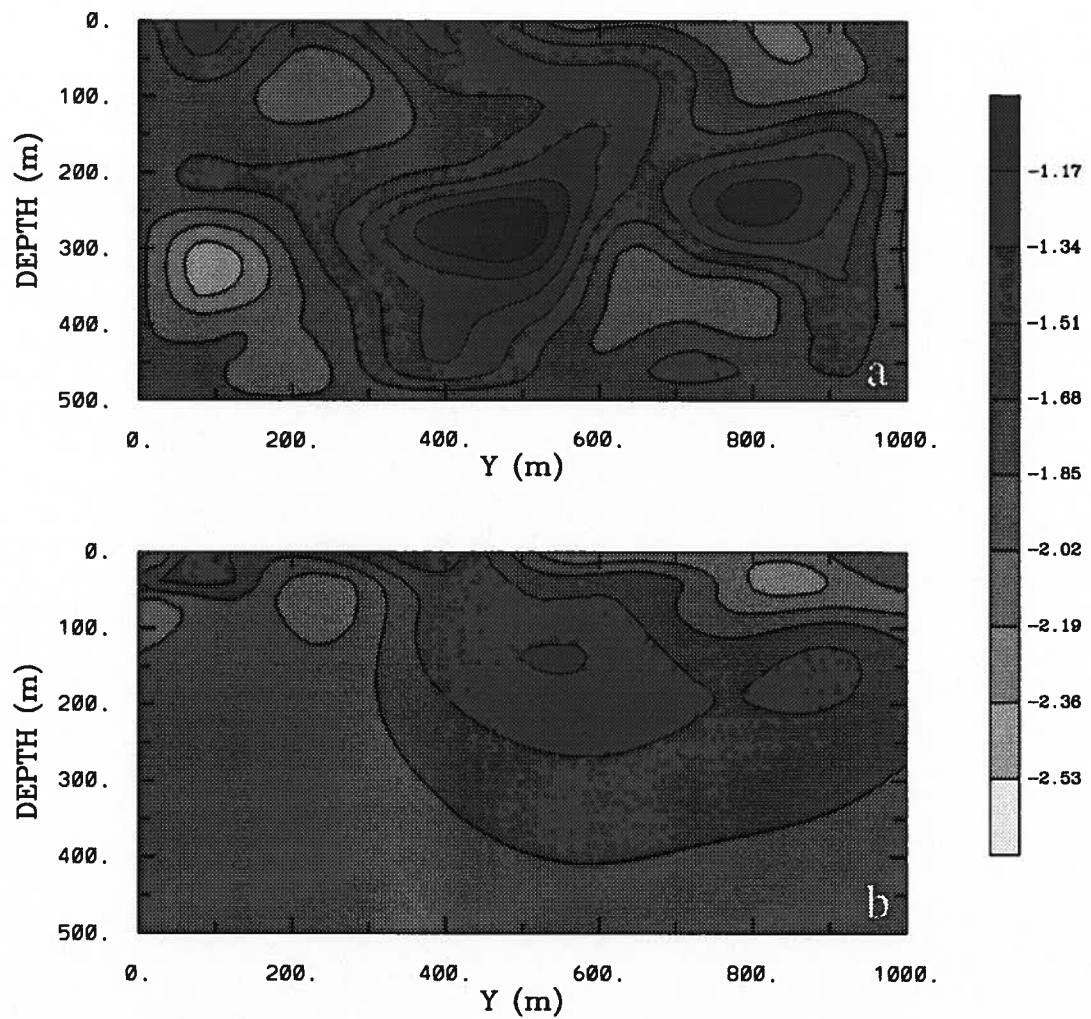


Figure 5.10 Comparison of the geo-statistical model (a) and the AIM-MS inversion result (b) in section $x=420\text{m}$. The recovered model only represent the true model above depth of 200m.

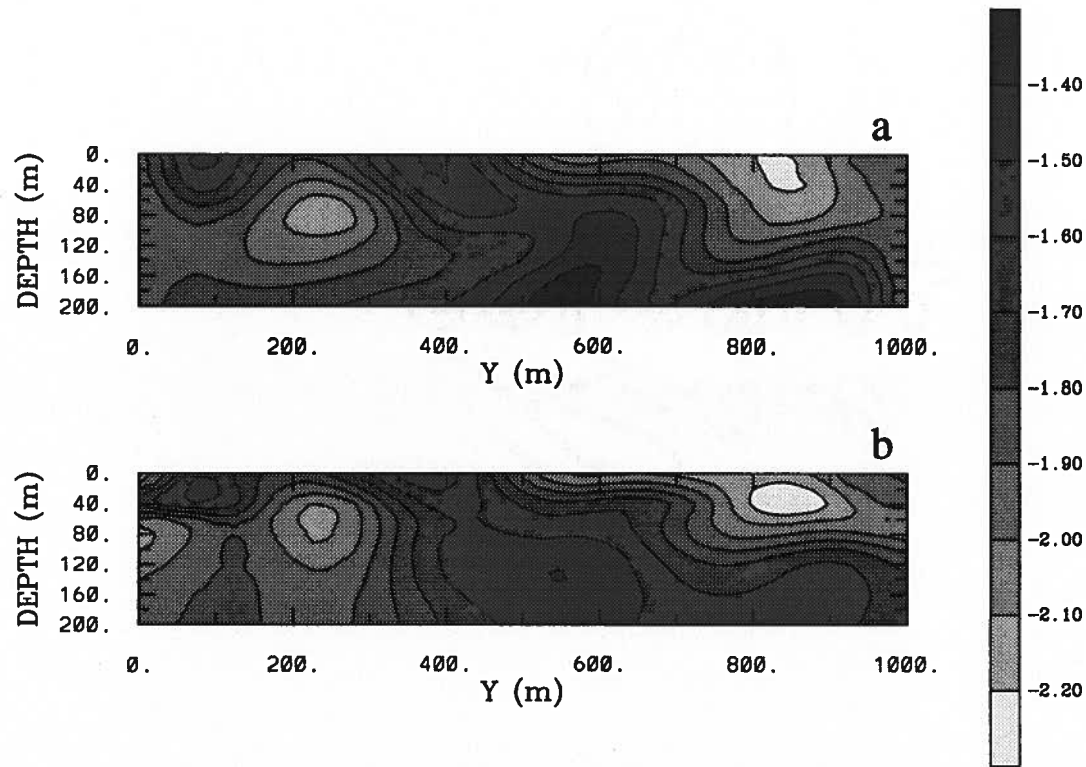


Figure 5.11 Detailed comparison of the geo-statistical model (a) and the AIM-MS inversion result (b) at $x=420\text{m}$ above $z=200\text{m}$

plotted in the preceding figures). An RMS misfit of 0.5% is achieved. However, the misfit starts to increase slightly beginning from iteration-17. So does the norm of the model perturbation. The norm of the model increase approximately linearly with the number of iteration after the initial rapid increase. This indicates that the inversion is starting to diverge.

Next the AIM-DS inversion is applied. Twenty-two iterations are performed. Unlike the AIM-MS results, both the misfit and the norm of the data perturbation are still decreasing. For a consistent comparison, I again plot the model after 16 iterations. Four slices at the same depths as before are shown in Fig. 5.13. They are to be compared with the true and final recovered model in Fig. 5.8 and Fig. 5.9. The AIM-DS model is very similar to the AIM-MS model; differences occur only in the fine details. Calculation shows that the AIM-DS model has a slightly smaller norm. Fig. 5.14 shows the misfit reduction and the change of the norm of the data and the data

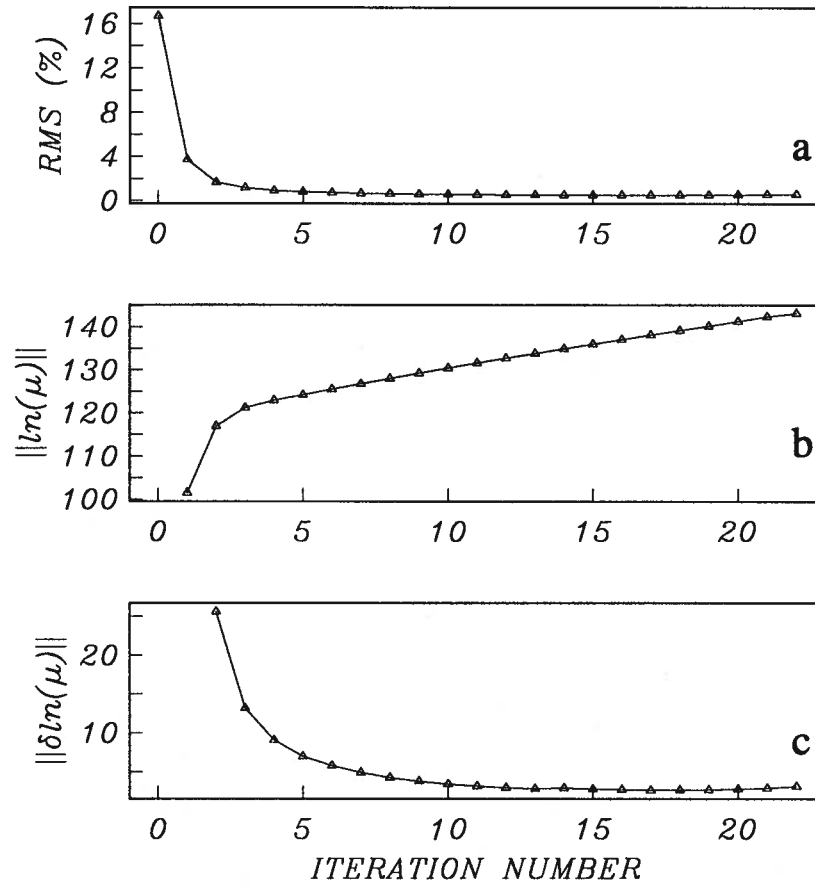


Figure 5.12 The change of the data misfit (a), the model norm (b) and the norm of the model perturbation (c) with the iteration in the AIM-MS inversion for the geo-statistical model.

perturbation. It is noticed that, at the later iterations, the norm of the modified data again increases linearly as does the norm of the model in the AIM-MS inversion.

This behavior of the norm of model and data is different from that in the case of the 5-prism model, where we saw convergence towards an asymptotic value in both model and data. This is due to the fact that the current model has more anomalous structure and the approximate inverse mapping based upon the Born approximation deteriorates. However, this may not be a concern in practical applications, since the misfit is reduced to below 1.0% by the fourth iteration in both cases. Further iteration has not reduced the misfit significantly but only increased the complexity of the model.

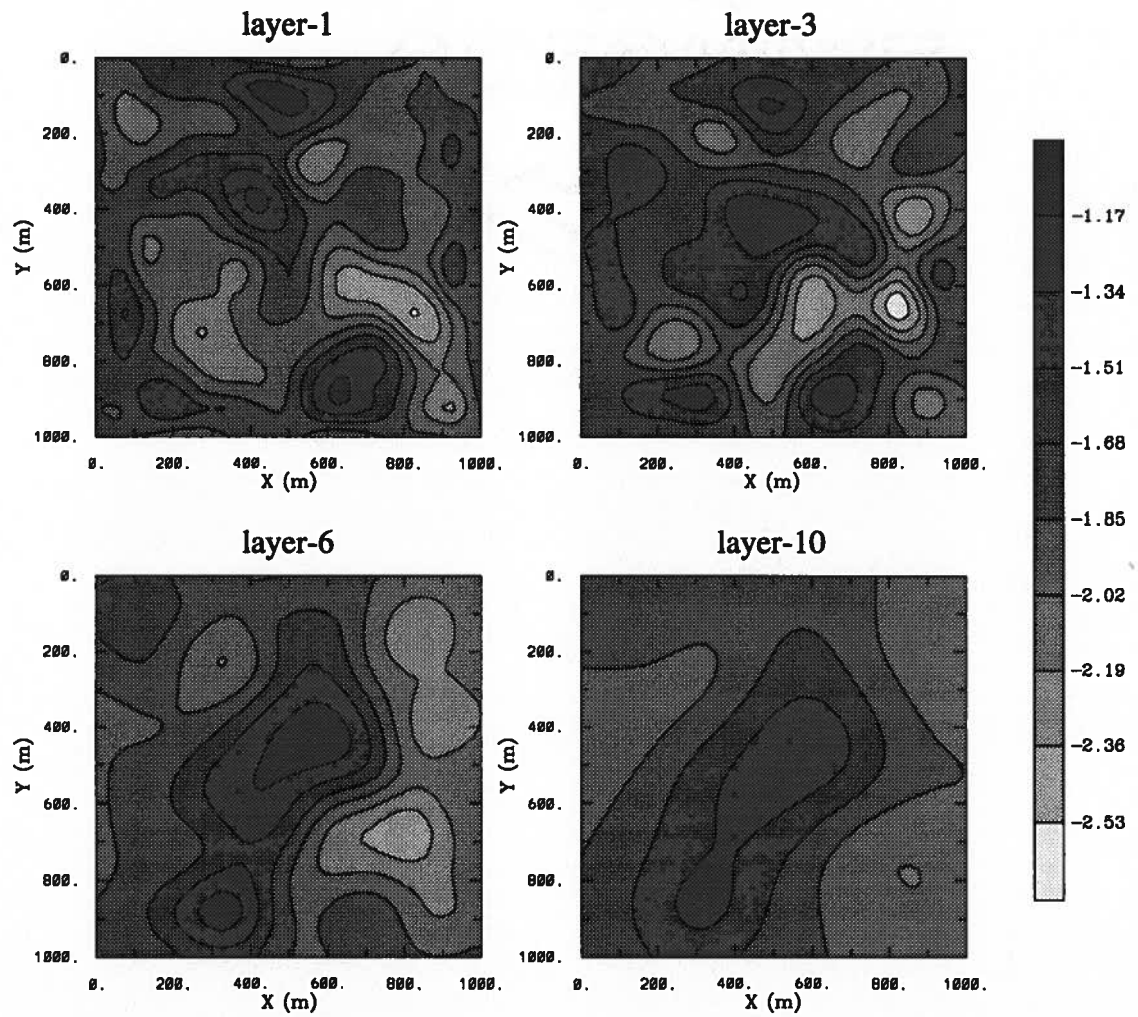


Figure 5.13 AIM-DS inversion result for the geo-statistical model.

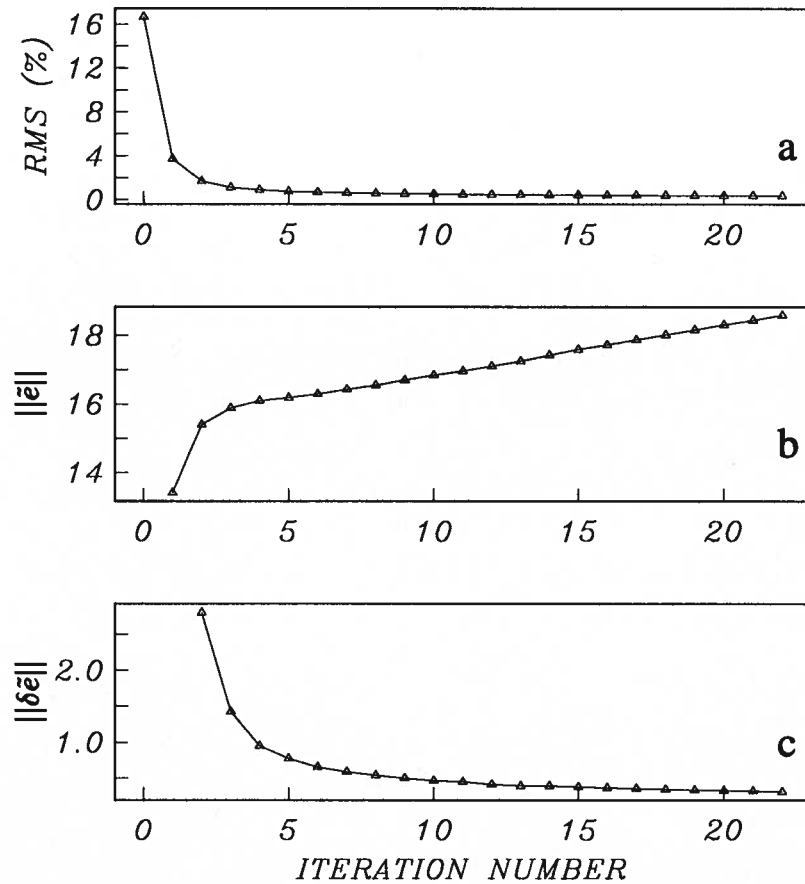


Figure 5.14 The change of the data misfit (a), norm of the modified data (b) and the data perturbation (c) with iteration in the AIM-DS inversion for the geo-statistical model.

To complete the example, I add 5% Gaussian independent noise to the data and apply both AIM-MS and AIM-DS inversion. After four iterations, both algorithms produced models which reduced the χ^2 misfit to the expected value. The two models are very similar, although the AIM-MS model has a slightly larger norm. Fig. 5.15 displays the AIM-DS model in four slices at the same depths as before. It is evident that there are small scale features introduced by the added noise. However, the first layer is not as noisy as that obtained by inverting the noisy data from 5-prism model. This difference is related to the difference in vertical partitioning. The inversion mesh is uniform in the vertical direction for the geo-statistical model whereas it was finely partitioned near the surface for the 5-prism model. The very thin cells near the surface are

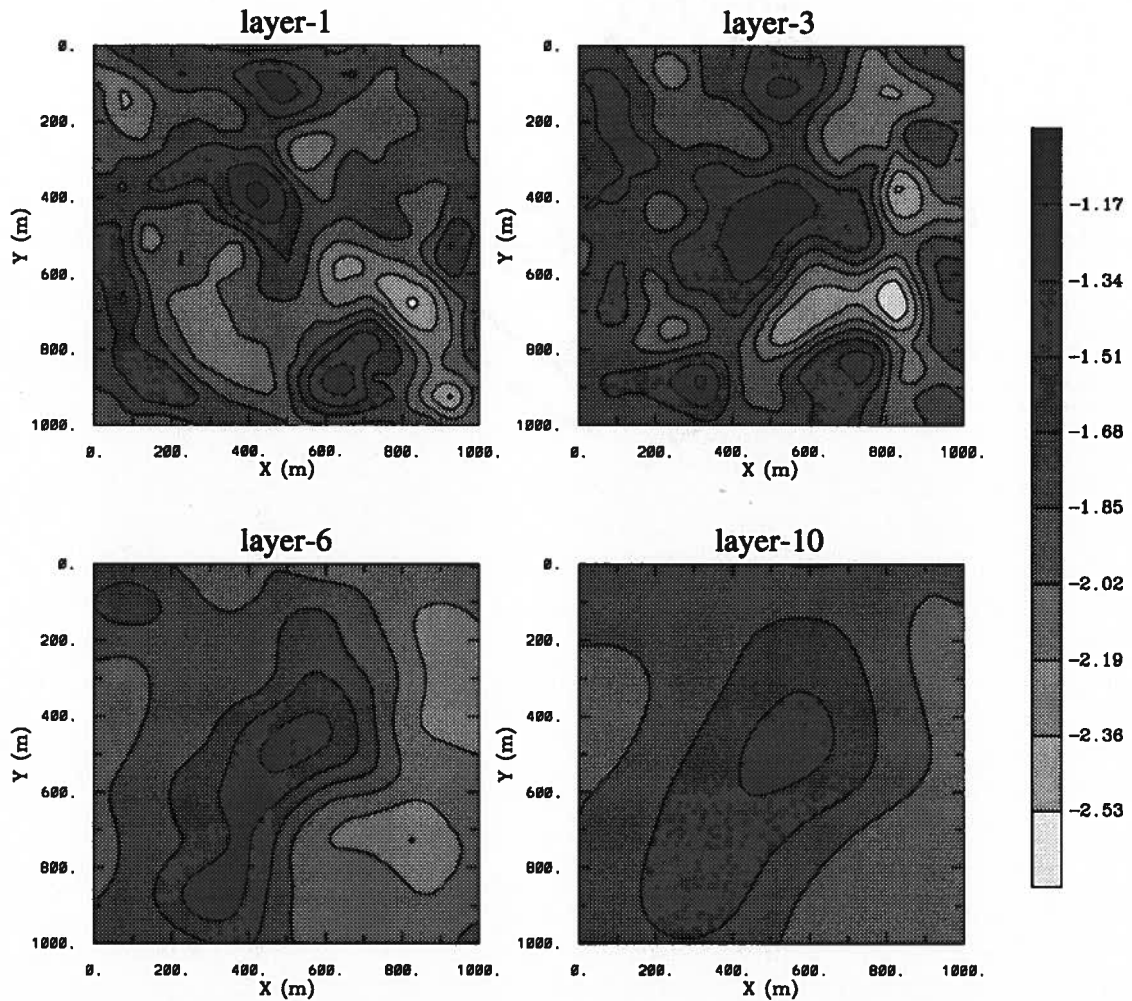


Figure 5.15 The result from AIM-DS inversion of the noisy data from geo-statistical model.

more prone to the effect of the noise in the data, since they allow more structure in the vertical direction. This has direct bearings on the application to field data. Overly fine partitioning should be avoided in the mesh. Fig. 5.16 shows the reduction of the χ^2 misfit and the change of the modified data, its perturbation, and the model in terms of their respective norms. Similar behavior is observed as in the case of 5-prism model. The norm of the data increases linearly but the model norm is more stable.

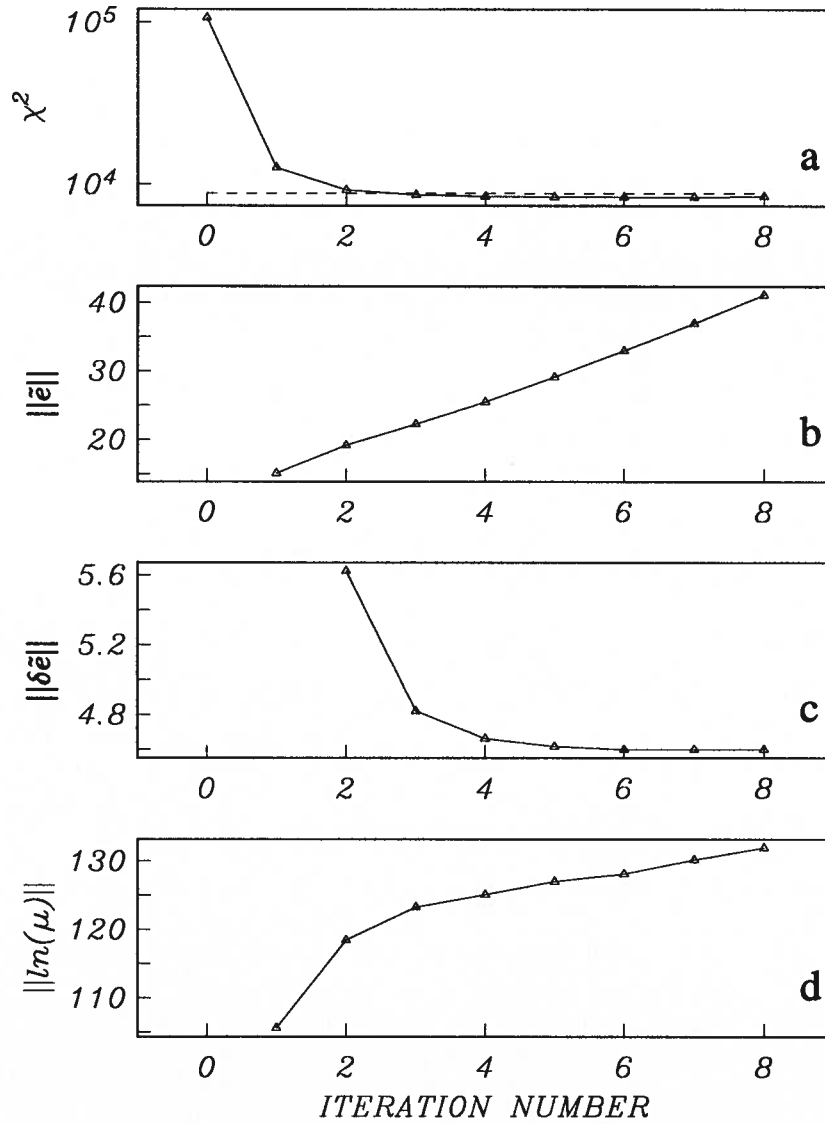


Figure 5.16 Progress of the AIM-DS inversion applied to the noisy data from the geo-statistical model. Panel (a) shows the change of the misfit with the iteration. The dashed line indicates the expected χ^2 misfit. Panels (b) and (c) are the norm of the modified data and the norm of the data perturbation respectively. Panel (d) shows the change of the model norm.

5.4 Discussion

The formalism of the AIM inversion is successfully applied to the $3d$ d.c. resistivity inverse problem and both AIM-MS and AIM-DS iterative inversion algorithms are constructed to invert E-SCAN® data for a $3d$ conductivity model. The model is assumed to have a factorized form of $\sigma(\vec{r}) = \sigma_0\mu(\vec{r})$, where σ_0 is the background conductivity and $\mu(\vec{r})$ is a scale factor. The algorithm employs the approximate $3d$ inversion developed in Chapter 4 based upon the Born approximation, which generates the $3d$ inverse solution by decomposing the problem into a sequence of $1d$ problems in the wavenumber domain. A finite difference forward modelling algorithm is used as the forward mapping. Both accurate and noisy data from two synthetic models are used to test the AIM-MS and AIM-DS inversion algorithms. In all test cases the algorithms have succeeded in producing acceptable models. These models adequately reproduce the data within the tolerance of the associated errors, and represent the true models with reasonable fidelity.

In general, the AIM-DS is advantageous in comparison with AIM-MS inversion. AIM-MS generates the final model by a series of model perturbations and there is no explicit control over the norm of the model. As a result, the model from AIM-MS inversion can acquire excessive structure. The AIM-DS inversion, however, generates the final model from a single application of the approximate inverse mapping. A norm of the model is explicitly minimized to reduce the structural complexity. Thus AIM-DS can potentially produce a better model and is preferable in practical applications. This prompted me to invert the data sets from two synthetic model with both AIM-MS and AIM-DS algorithm. I had expected that the models from the two algorithms would be different. However, the results show that the difference is not large. It is only noted that the norm of the model generated by AIM-DS is slightly smaller than that from AIM-MS and the conductivities of individual cells between the two models can differ up to 20%. Beside these differences, the two algorithms have generated almost identical results. The similarity between the results of the AIM-MS and AIM-DS inversions suggests that the approximate $3d$ inversion may be nearly linear when regularized according to the model norm.

The algorithm has combined the advantages of the approximate $3d$ inversion and the AIM formalism and therefore is very fast. The kernel functions for the approximate $3d$ inversion are computed at the first iteration and stored for subsequent iterations. As a result, the inverse mapping at each iteration is very rapid, since its solution in the wavenumber domain takes only a small fraction of the time needed for computing kernel functions. In addition, the AIM formalism requires one forward mapping at each iteration. This reduces the computation effort at each iteration to a minimum. This is the advantage of the algorithm.

The iterative application of the approximate inverse mapping has been able to construct models which fit the data. The reduction of the misfit to the expected level is achieved rapidly within the first few iterations. The corresponding model is greatly improved compared with that from a simple application of the approximate inverse mapping. However, the algorithm has its limitations. The misfit cannot be further reduced after the initial reduction in the first iterations. It is not guaranteed that these first few iterations will achieved the expected misfit. Prolonged execution of the inversion only increases the structural complexity of the model. Therefore, the method is best used as a means to generate a conductivity model which reduces the data misfit to a low level with minimum computational efforts. The resulting model can be used to draw geologic conclusions or it may be used as an initial model for more rigorous inversion based upon linearization.

CHAPTER 6

ANALYSIS OF FIELD DATA

In this chapter, a set of field data is processed and inverted using the AIM inversion algorithm developed in the preceding chapter. These data are acquired by a survey designed to assist exploration for epithermal type deposits. I first describe the field procedure of data acquisition and the data format pertaining to the E-SCAN® experiments. Next I proceed to processing the data in preparation for the inversion. Then I will present the inversion results and make a comparison with the geology in the area based upon the information available.

6.1 Geologic Objectives

The survey which acquired this data set is a part of an exploration effort for epithermal deposits. They are a class of the hydrothermal mineral deposits. In general, epithermal deposits are considered as having been formed “by hot ascending waters of uncertain origin, but charged with igneous emanations. Deposition and concentration of ore minerals occur at slight depth” (Lindgren, 1933). Such deposits most commonly occur within the upper 600 m in areas with well developed fracture and fault systems. Existing faults or fractures act as feeders of hot fluid which alters the host rock and deposits minerals. Consequently, ore is formed in the alteration zones, which often expand in the upper portion to form cone-like or mushroom-shaped features with deep “roots”. Precious metal deposits are frequently associated with silicification brought about by hydrothermal alteration.

Because of the highly resistive nature of the veins and silicified zones, the ore deposits exhibit localized resistive anomalies as typical signatures in the d.c. resistivity experiment. The E-SCAN® survey is therefore employed to map such silicification zones and possible fault structures. The area is covered with eluvium on the surface. As a result, there are few surface expressions of the geology in this region but it is hoped that the E-SCAN® experiment will provide information about the subsurface conductivity so that ore bodies can be located.

6.2 Data Acquisition

E-SCAN® experiments, especially when applied to mineral exploration, are usually carried out on a regular grid with constant grid spacings in each direction. The spacing can be the same for both directions. Theoretically, each grid point is occupied in turn by a current source and the surface potentials are measured within a prescribed radius around the source location. The measurement of the potential within that radius is over a specified pattern of the grid points. Since the potentials measured with slightly different source-receiver orientations at a large separation do not provide very different information, omitting some of these data points will not reduce the information content of the data set significantly. Such omission is usually for economic reasons. The “infinite” current and potential electrodes **B** and **N** are placed at sufficiently long distances away.

The field data acquisition is controlled by a digital computer. Each electrode at the grid points has an electronic switch box attached to it. Along each line these switch boxes are connected in series and the end of each line is connected via a switch box to a main cable leading to the central control. Each box has a unique address within this network. The central control is thus able to instruct each individual box either to connect to the attached electrode (or the attached line for those on the main cable), or to act as a relay connecting further down the network. With this setup, the controlling computer can measure the potential at any node within the grid. The energizing current is transmitted through a different cable and the current electrode is usually moved to each site manually. The waveform of the source current is the standard periodic square wave with reversing polarities. The total potential is measured just before the source is cut off. The experiment also measures the IP (induced polarization) potential during the decay period after the current is cut off.

Since the measurement radius is usually a fraction of the distance covered by the entire grid, the wire network is laid out only over a group of adjacent lines in a portion of the grid. As the survey progresses, new lines are added in front of the network and lines along which measurements have been taken are removed. To reduce the measurements needed in the field, the

reciprocity of the d.c. field is utilized so that potential measurements are made only in the forward direction. That is, if the current source is moving from left to right on the line sweeping through the grid from top to bottom, the potentials are measured only on the grid points to the right of the source location and below the source line. This process is explained in Fig. 6.1.

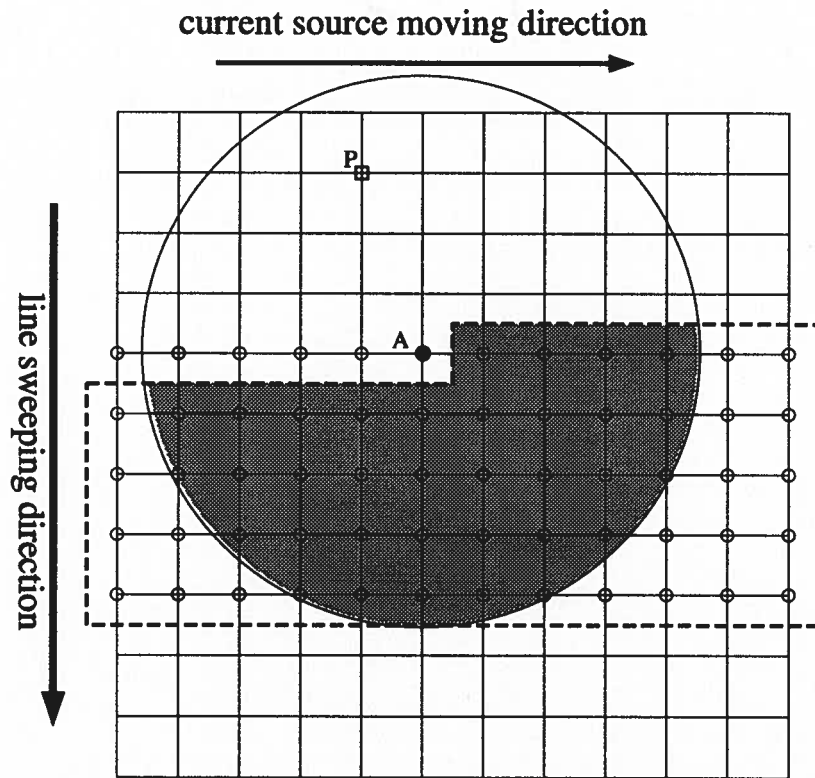


Figure 6.1 Field acquisition procedure of the E-SCAN[®] experiment. The solid grid represents the survey grid. The circles represent the grid points activated by measuring network. For any current source location A, potential data are collected only within a prescribed radius (shown by the large circle) at the nodes in front of the source. These nodes are shown in the shaded area. The potential at the nodes behind the source, for instance at P, are substituted by the reciprocal data, i.e, the potential measured at A when the current is at P. This group of activated nodes sweeps through the grid and the current source moves from one end to another so that all grid points are occupied by the source in turn.

The solid lines in Fig. 6.1 represent the E-SCAN® survey grid. The grid points marked by circles denote activated electrodes (connected into the network). The current source is marked by the solid dot. As the survey continues from the top of the grid to the bottom, the line below this group will be activated and the line on which the current source presently lies will be deactivated when the current source has occupied the last point on this line. So the group moves downward as indicated by the arrow on the left side. The circle centred at the current source marks the boundary in which potential data are collected for this particular source location. The activated grid enclosed by the dashed line consists the points in the “forward” direction. Therefore, the field measurements are made only within the intersection of the circle and the region bound by the dashed line, which is highlighted by the shading. This area of measurements is often extended so that potentials are acquired to the array extremities on the x - and y -axes and 45° diagonal directions.

The particular experiment in consideration was carried out on a 32×35 grid with two small areas missing on the margin. The grid has a spacing of 91.44 metres and covers an area of $2.5\text{km} \times 3\text{km}$. Fig. 6.2 shows the grid layout. The solid lines represent the E-SCAN® grid. The grid points marked by crosses constitute a primary grid with a grid spacing of 182.44 metres. The majority of the data are collected on the primary grid following the procedure described above. Fig. 6.3(a) shows the pattern of points at which the potentials are measured for each current location. The remaining grids are used to acquire supplemental measurements so that near off-set data are collected. Only current sources are placed on these points and their potentials are measured on the primary grid. For these sources, the measurement pattern is specially designed and varies with the type of source locations in the grid. Only a small number of potentials are measured for these sources. The exact patterns are shown in Fig. 6.3 by panels (b), (c), and (d). For these special cases, all the potentials are measured in both the forward and backward directions.

The lines AA' and BB' in Fig. 6.2 mark the positions of two sections whose apparent conductivity pseudo-sections and the conductivity section from the constructed model will be

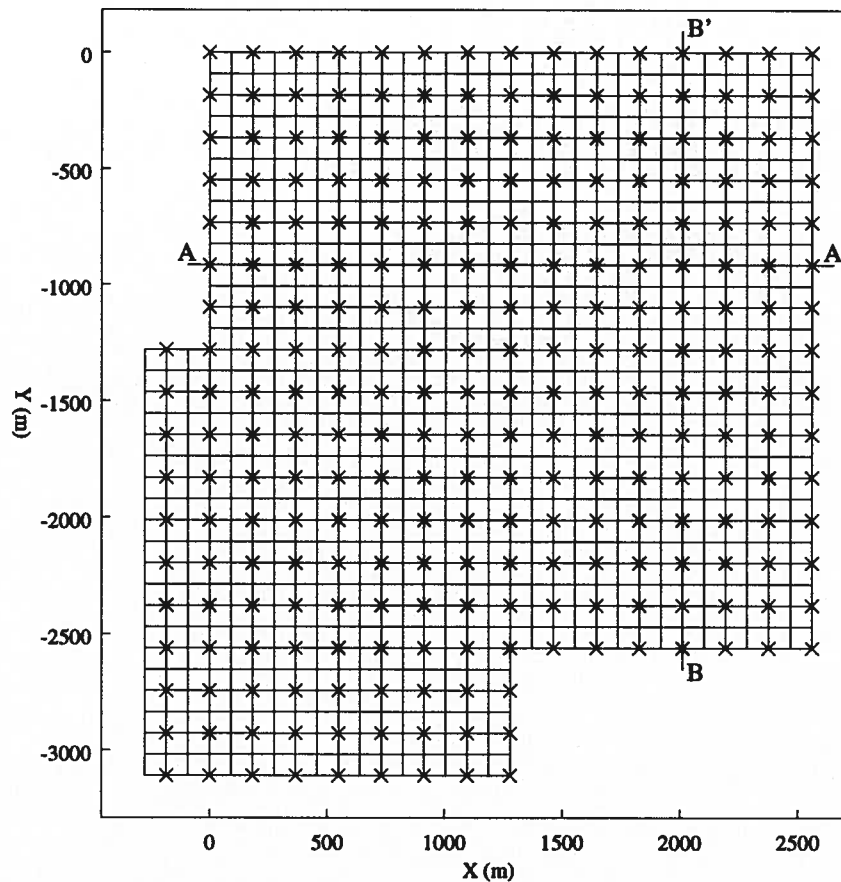
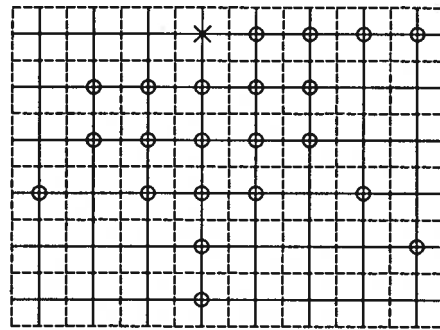


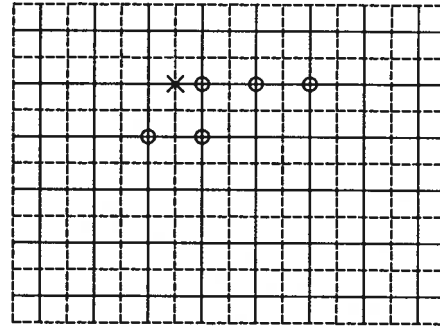
Figure 6.2 Grid layout of the field data set. The solid lines denote the E-SCAN® grid and grid points marked by cross form the primary grid. Potentials are measured only on the primary grid. The coordinates are reproduced from the field positions. The lines AA' and BB' indicate respectively the position of two sections to be discussed. They are to be denoted by section-A and section-B, respectively.

studied. These sections are chosen to coincide with two geological sections compiled from drillhole information. They help to provide some limited verification for the inversion result.

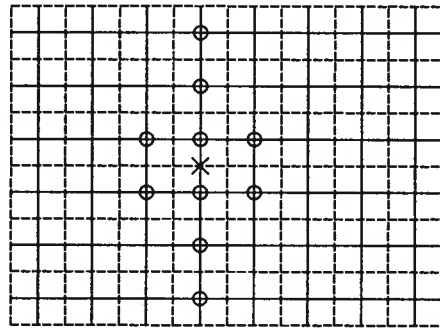
Each datum is a stacked result of measurements taken with both polarities over a number of current supply cycles. Usually the mean value is taken. Ideally, a standard deviation of the associated error should be estimated from the repeated measurements. However, the data set from the field only has a numerical indicator of the noise for each datum. These indicators are qualitative summaries of the measurement repeatability, signal strength, and the background noise such as the strength of cultural interferences and telluric currents. These indicators are useful in



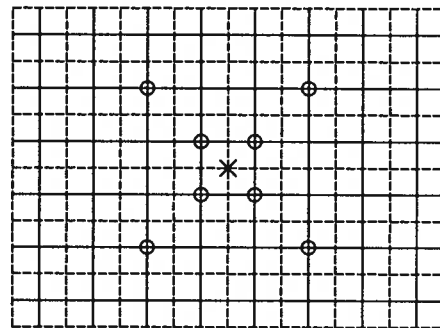
(a) on primary grid



(b) on horizontal line of primary grid



(c) on vertical line of primary grid



(d) not on primary grid

Figure 6.3 Four different patterns of data collection. The solid lines are the primary grid and the dashed lines are the remaining grid lines. The source is marked by the cross and its location is indicated by the labels. The circles are the nodes at which the potentials are measured.

identifying bad data points, but they cannot be used as standard deviations of the data error due to the lack of quantitativeness.

Each datum in an E-SCAN® data set consists of 19 quantities. These include, in the order of the file format:

1. serial number of the datum
2. grid type indicating the direction of the grid axes
- 3-5. (x, y, z) coordinates of the current source **A**
- 6-8. (x, y, z) coordinates of the potential site **M**

- 9-11. (x, y, z) coordinates of the current sink **B** (at “infinity”)
- 12-14. (x, y, z) coordinates of the reference potential site **N** (at “infinity”)
- 15. the transmitted current strength
- 16. measured total potential
- 17. measured IP potential (in percentage of the total potential)
- 18. noise indicator for the total potential
- 19. noise indicator for the IP potential

These 19 quantities provide the complete information about each datum. These data records collectively define the survey grid and they also indicate the sequence in which the experiment is carried out.

6.3 Data Processing

The field data set in the above format needs to be processed before it can be used in the inversion to construct a conductivity model. The first problem is the topographic distortion. Since the inversion algorithm assumes a flat earth surface, any significant topographic relief in the survey area will be translated by the inversion into spurious conductivity structures beneath the surface. Fortunately, the topographic relief in the survey area is small. Any distortion due to such small topography can be effectively treated as geological noise and be suppressed in the process of inversion by limiting the high wavenumber components. Thus, the data set is treated as being collected from plane surface and the inversion is applied directly.

The “infinite” electrodes, **B** and **N**, in this particular experiment are placed over 7km from the centre of the grid. This is sufficiently distant for measurements from short source-receiver separations, but they are relatively close in comparison to large separations and have significant contributions to the measured potential. When the pole-pole data are formed, this contribution tends to reduce the potential and to introduce spurious large scale apparent conductivity highs. These spurious features will lead to false structures in the constructed conductivity model.

Therefore, the data must be corrected first for such biases before they can be used to recover a conductivity model. However, the difficulty is in that the correction relies upon the knowledge of the conductivity structure in the region. Thus, I only attempt to make the first order correction. A best fitting half-space model σ_0 of the region can be derived from the data set. At this stage the data should be correctly treated as measurements from four-electrode configurations (the data set has complete information on the electrode locations). A value of 0.04S/m is estimated for this region. The data set is then corrected for the “infinite” electrodes using this value by subtracting away the contribution of the B and N electrodes over the half-space. The results show that the corrections can be as large as 20% and they are mainly for the large off-set data. Once this correction is done, the data are dealt with as true pole-pole potentials.

For most data processing, it is necessary to have a complete “common current gather” for every current site in the data set. Such a gather consists of all measurements of the potential generated by a common current source. The direct measurement from the field experiment only provides half of the gather as a result of the acquisition procedure. Therefore, the reciprocity of the d.c. potential field is invoked to complete the gather. As shown in Fig. 6.1, suppose A is the source location and P is a point without a direct measurement. This datum is substituted by the potential measured at A with a source at P. Such measurements are called indirect measurements. Fig. 6.4 illustrates the composition for one current site, where the circles and squares represent the direct and indirect measurements, respectively.

With the completed common current gathers, the data can be easily sorted into different formats for subsequent use. For example, data maps for pole-pole arrays can be synthesized and the apparent conductivity volume can be formed. The pole-pole data maps also constitute the data for 3d inversion. 26 data maps with good areal distributions are gathered in the horizontal, vertical and 45° diagonal directions. The corresponding pole-pole separations range from 1 to 10 basic grid spacings, which are sufficiently large for the interested depth. These 26 maps are to be used in the AIM inversion.

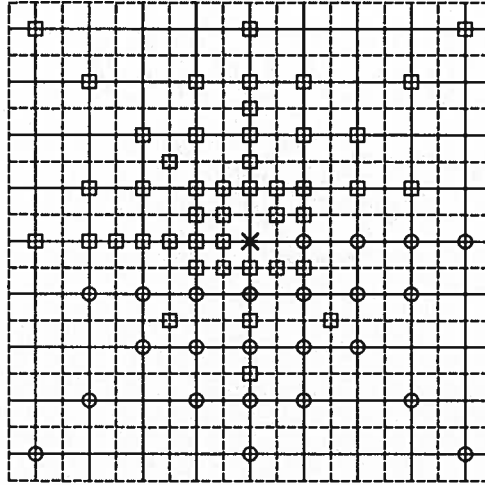


Figure 6.4 Composition of potential measurements in a common current gather. The source location is marked by the cross. The direct and reciprocal measurements are indicated by circles and squares respectively.

There is a highly resistive anomaly in the northwest corner of the grid. This is caused by resistive volcanic tuff, but it has no bearing on the structures of interest in the rest of the region. Because its magnitude is much higher than that of other anomalies, it could have an adverse effect on the inversion by masking the signal from those anomalies. Therefore, the potential at the points associated with this anomaly have been replaced by the mean value from the rest of data points in each map. This replacement should not cause problems as long as no significance is attached to the conductivity in this region in interpreting the conductivity model from the inversion.

It is also clear that some data maps are very noisy. In the extreme cases, the data maps have many isolated spikes. As mentioned earlier, each datum does not have an estimated error so erroneous data cannot be isolated on that basis. The potential data are to be Fourier transformed and any erroneous data points in the spatial domain will affect all data in the wavenumber domain. To improve the quality of the data, I smooth each data map (with the resistive anomaly replaced) using spline interpolation. The spline algorithm constructs a smooth representation, $f(x, y)$, of

the data, which minimizes the functional

$$\phi(\lambda) = \frac{1}{n} \sum_{i=1}^n (f(x_i, y_i) - e_i)^2 + \lambda \iint \left(\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 \right) dx dy,$$

where e_i is the quantity to be smoothed, which is usually the logarithm of the total potential. λ is the tradeoff parameter determining the degree of the smoothing. Since the data error is uncertain, the generalized cross-validation (GCV) technique (Craven and Wahba 1979) is used to estimate the optimum value of λ . The smoothed data are then used in the inversion as the observed data.

Fig. 6.5 shows two apparent conductivity maps formed directly from the raw data. The corresponding arrays are in x -direction and the electrode separations are 1 and 4 grid spacings. Fig. 6.6 displays the same two maps after the aforementioned processing steps. The resistive region at the upper left corner of the grid in Fig. 6.5 is the anomaly which has been replaced. It is evident that the GCV smoothing has improved the data quality by suppressing the outliers. Fig. 6.7 shows two apparent conductivity pseudo-sections in section-A and section-B, respectively. The pseudo-sections are formed from the arrays aligned with the orientation of the section and the processed data are used.

6.4 Inversion

The AIM-DS algorithm is applied to the processed data maps to construct a 3d conductivity model. Again, a dual mesh system is used. The mesh for the inversion is designed as described in Chapter 5. The area of the inversion is obtained by expanding the survey area by half maximum array separation (457.2 m) in horizontal directions. The horizontal mesh over this area has a spacing equal to half of the survey grid spacing. The vertical mesh has a uniform partitioning with a spacing of 30 metres in the interval [0, 600]m. Three cells with increasing spacing are then appended to extend the mesh down to 900 metres.

For the forward modelling mesh, the horizontal grid has, in the core portion, the field survey grid of 32×35 with 91.44m spacing (the two missing areas are filled in). Three cells with increasing spacings are appended to four sides to complete the grid. The vertical grid coincides

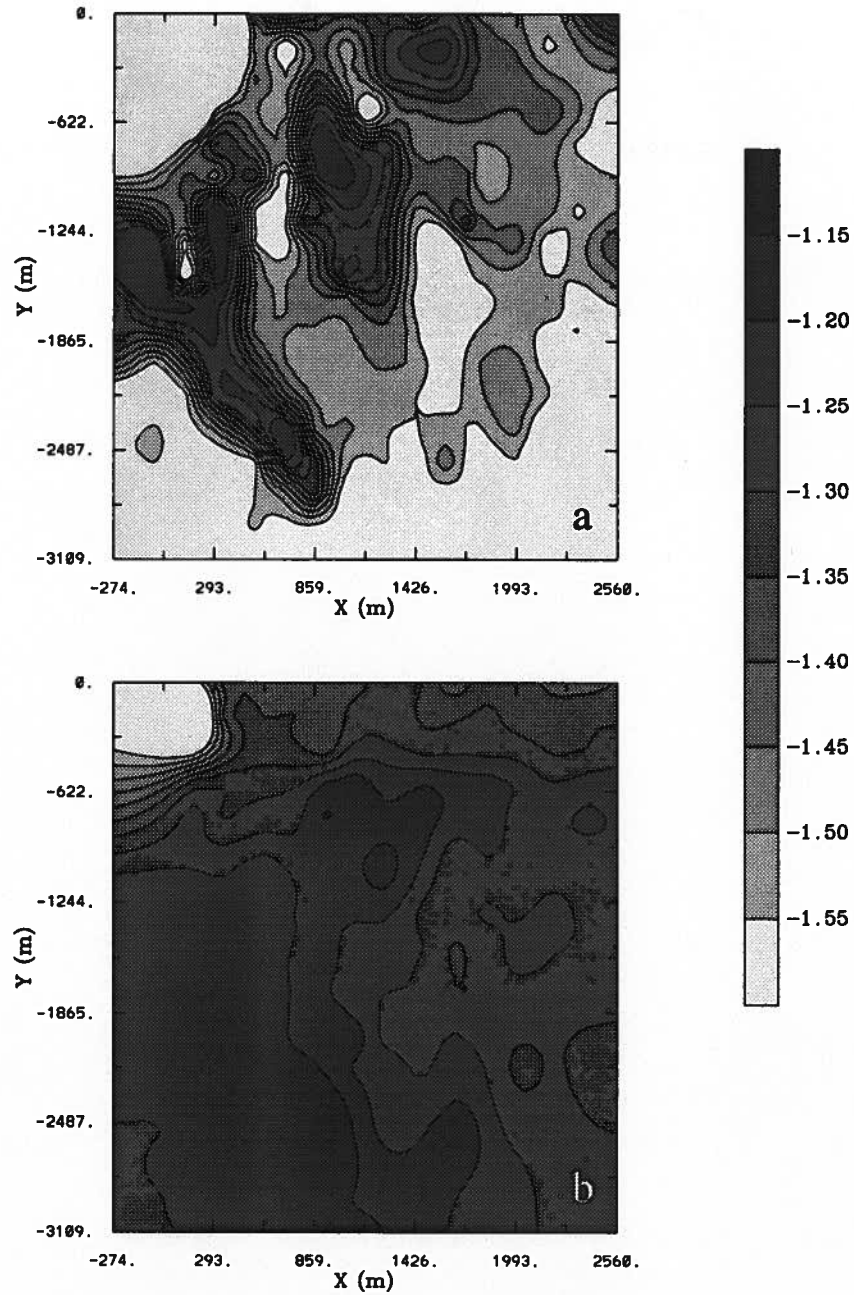


Figure 6.5 Two apparent conductivity data maps from the raw field data. Panels (a) and (b) correspond to arrays aligned in the x -direction and with separations equal to 91.44m and 365.76m respectively.

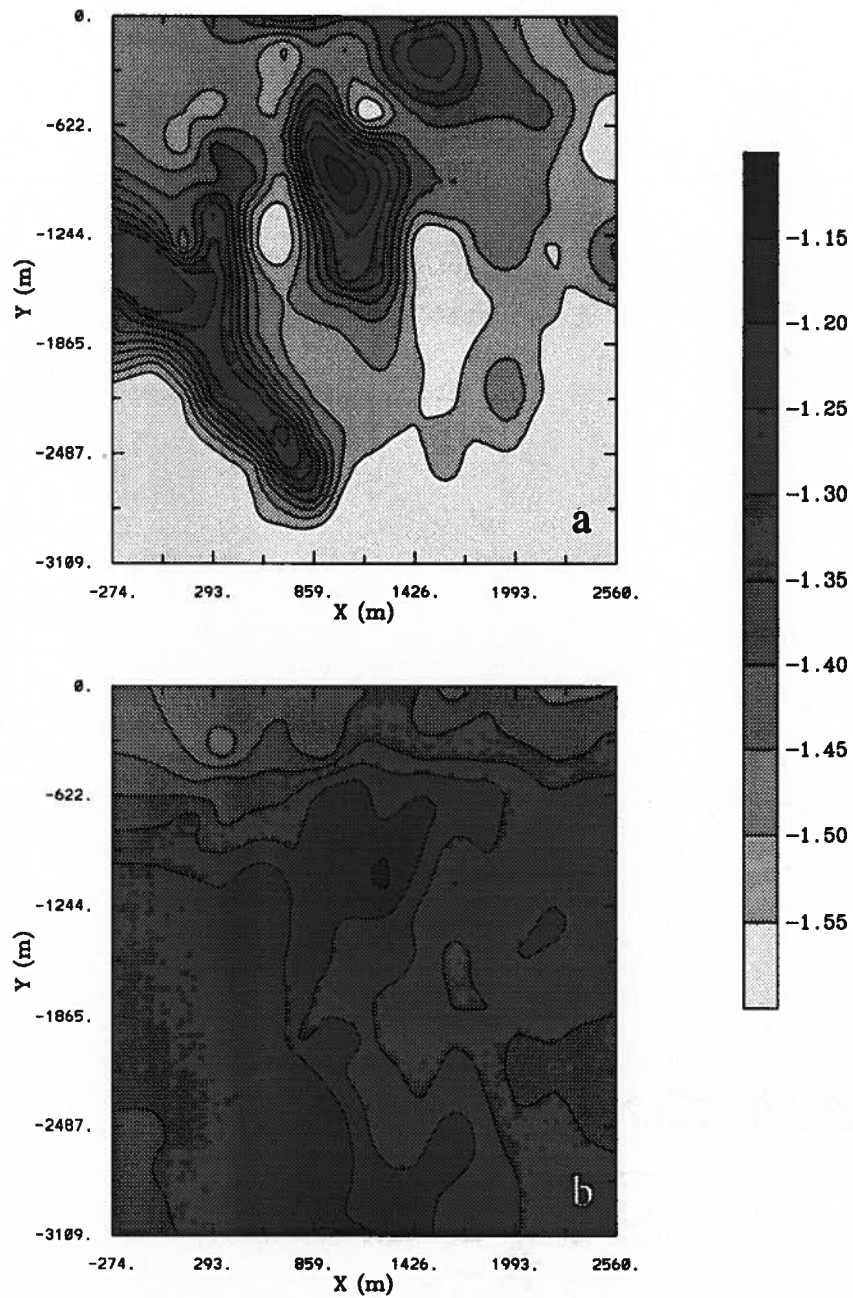


Figure 6.6 Two processed apparent conductivity data maps from the field data set. Panels (a) and (b) correspond to arrays aligned in the x -direction and with separations equal to 91.44m and 365.76m respectively. These two maps are to be compared with those in Fig. 6.5.

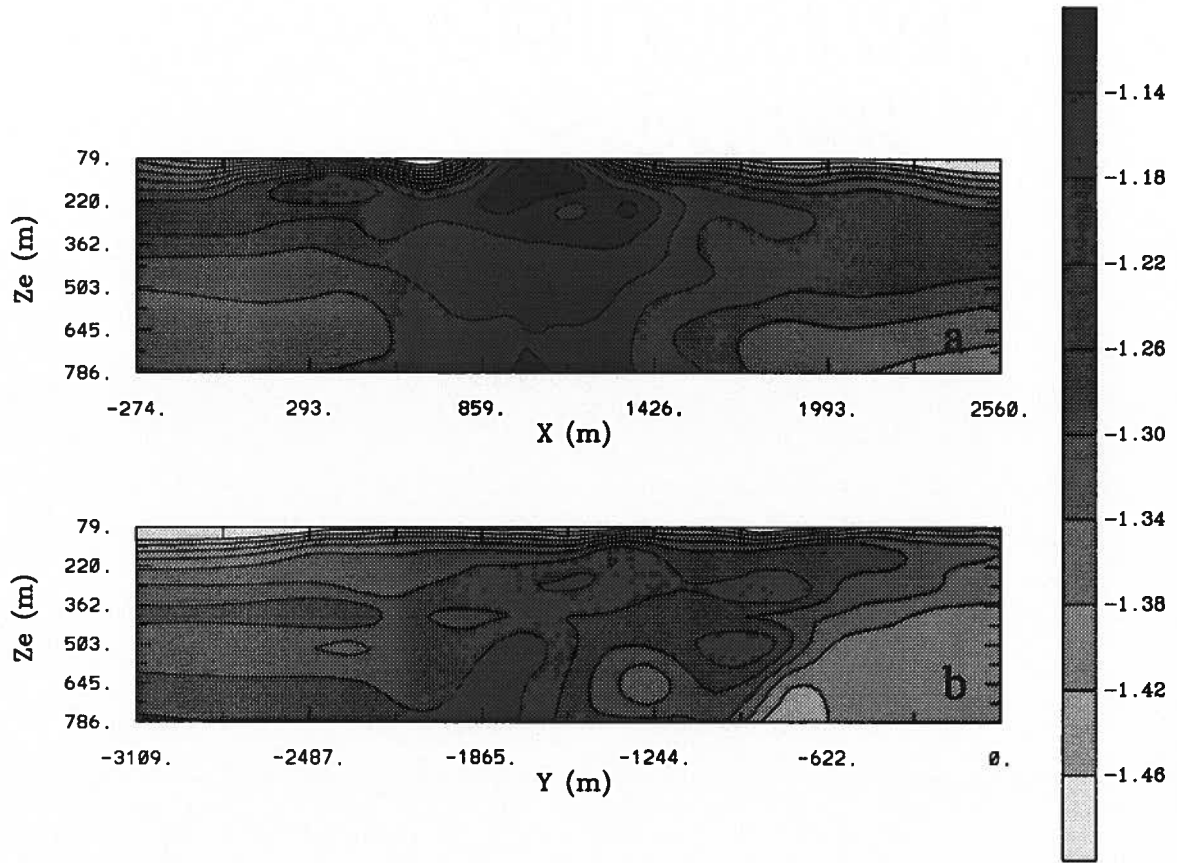


Figure 6.7 Two apparent conductivity pseudo-sections from the processed field data. Panels (a) and (b) show respectively the section-A and section-B. Each pseudo-section is formed from pole-pole arrays aligned with the section.

with the partitioning in the inversion mesh. This results in a mesh with $38 \times 41 \times 24$ nodal points. Thus the AIM inversion has a total of 34,040 unknowns.

As described in section 6.2, in the original survey the current source is placed at every grid point in turn but the potential is measured only over the primary grid. Because of the reciprocity in the d.c. field, this data set is equivalent to one in which the potential is measured over the entire grid while the current source is placed only at the primary grid locations. Thus I treat the original potential sites as the source locations and compute the potentials at the original source sites. This reduces the computation required for the forward modelling by half, since the operation is proportional to the number of the source locations.

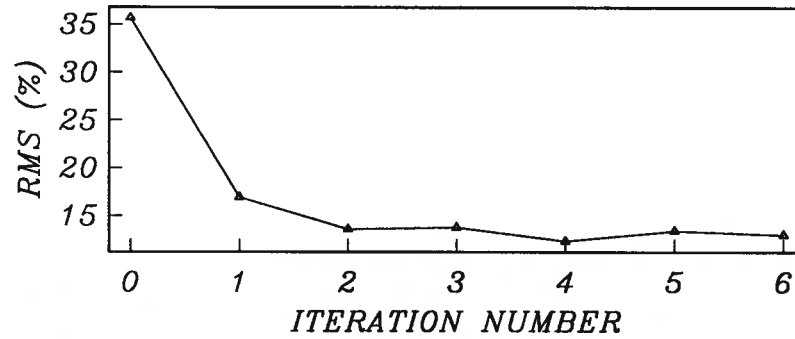


Figure 6.8 Reduction of the data misfit in the AIM-DS inversion of the field data set.

An AIM-DS inversion is carried out for 6 iterations. A weighting function $w(z) = 1/(z + z_0)$ is applied, where $z_0 = 20\text{m}$. The half-space conductivity is fixed at 0.0526S/m , which is the best fitting half-space model from the 26 data maps. The half-space model has a RMS misfit of 36%. This initial inversion achieved a lowest misfit of 12.3%. However, the misfit curve exhibits strong oscillations as shown in the Fig. 6.8. This indicates that the modified data $\tilde{e}$ are likely over-corrected. A second attempt is made in which a relaxation parameter is introduced to limit the stepsize of the data update.

The quantity $(\tilde{e}^{obs} - \tilde{e}^{(n)})$ in equation (5.13),

$$\tilde{e}^{(n+1)} = \tilde{e}^{(n)} + (\tilde{e}^{obs} - \tilde{e}^{(n)}),$$

can be considered to provide both the direction and the stepsize for the perturbation. We can perturb the data in this direction but control the stepsize. That is, the data can be updated by,

$$\tilde{e}^{(n+1)} = \tilde{e}^{(n)} + \omega(\tilde{e}^{obs} - \tilde{e}^{(n)}),$$

where ω is the relaxation parameter. A choice of ω less than unity should prevent an over correction of the data to a certain extent and make the iterative process more stable. A similar relaxation parameter can be applied to the model perturbation in the AIM-MS algorithm.

In the second attempt, a relaxation with $\omega = 0.6$ is applied to the data update. The stability is apparently improved. This is indicated by the smooth decrease of the misfit. Fig. 6.9 shows

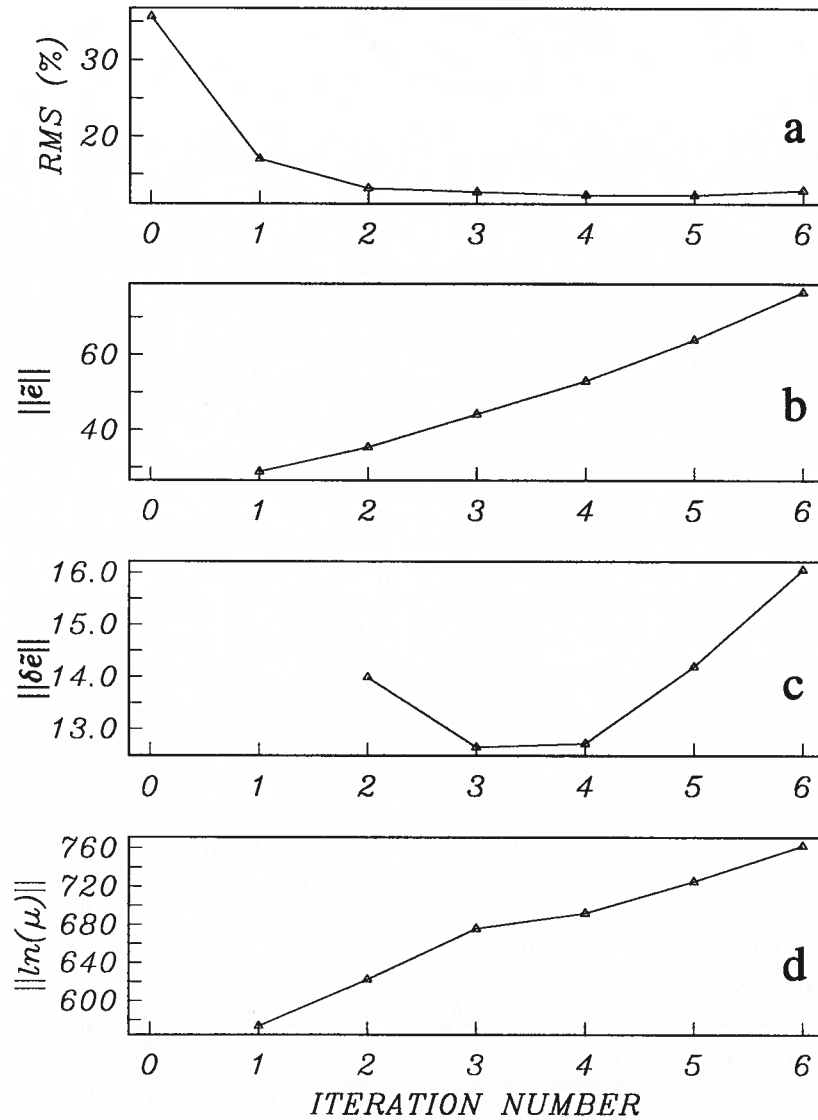


Figure 6.9 Reduction of the data misfit in the AIM-DS inversion with a relaxation parameter. (a) is the RMS misfit of the predicted data. (b) and (c) are respectively the norms of the modified data and the data perturbation. (d) is the norm of the model. Note that both data misfit and data perturbation start to increase after 4 iterations.

the reduction of the misfit, and the change of the model, data and data perturbation measured in their respective norms as the inversion progresses. It should be noted that the achieved best fit to the observed data is about the same for both inversions, although the stepsize of the data update is different. It is also noted that the major reduction of the misfit is achieved in the first two iterations. The remaining iterations have little effect on the misfit, but the model norm does increase considerably.

The inversion is apparently on a path towards divergence. This is indicated by the increase of the data misfit at the last iteration as well as the commencement of the rapid increase in data perturbation after four iterations. Since after four iterations the data misfit does not improve, the inversion should be terminated at this point. If the desired misfit is not achieved or is unknown, the onset of the increase in data misfit, data perturbation (for AIM-DS inversion) or model perturbation (for AIM-MS inversion) can serve as the criterion for termination. The prolonged execution of the inversion here is to examine the convergence properties of the algorithm. In keeping with this argument, I take the model from the 4th iteration as the final conductivity model from this inversion.

Fig. 6.10 shows two apparent conductivity maps predicted by this model. Fig. 6.11 shows two pseudo-sections of the predicted data. These are to be compared with the observed data shown in Fig. 6.6 and Fig. 6.7. It is seen that the large scale features in the field data are reproduced quite well. However, some fine details are different. Noticeably, the predicted data have smoother contour lines and fewer isolated small peaks. These isolated points may be contributing a great deal to the data misfit. Thus the structural fit to the data might be better than that indicated by the RMS misfit.

Fig. 6.12 shows the constructed conductivity model at section-A and section-B. The recovered conductivity model has many structural details at shallow depth but becomes increasingly smooth at greater depth. This is similar to the results from synthetic examples presented in Chapter 5. These conductivity sections can be compared with the apparent conductivity pseudo-sections displayed in Fig. 6.7. The benefits of carrying out the inversion can now be seen. The

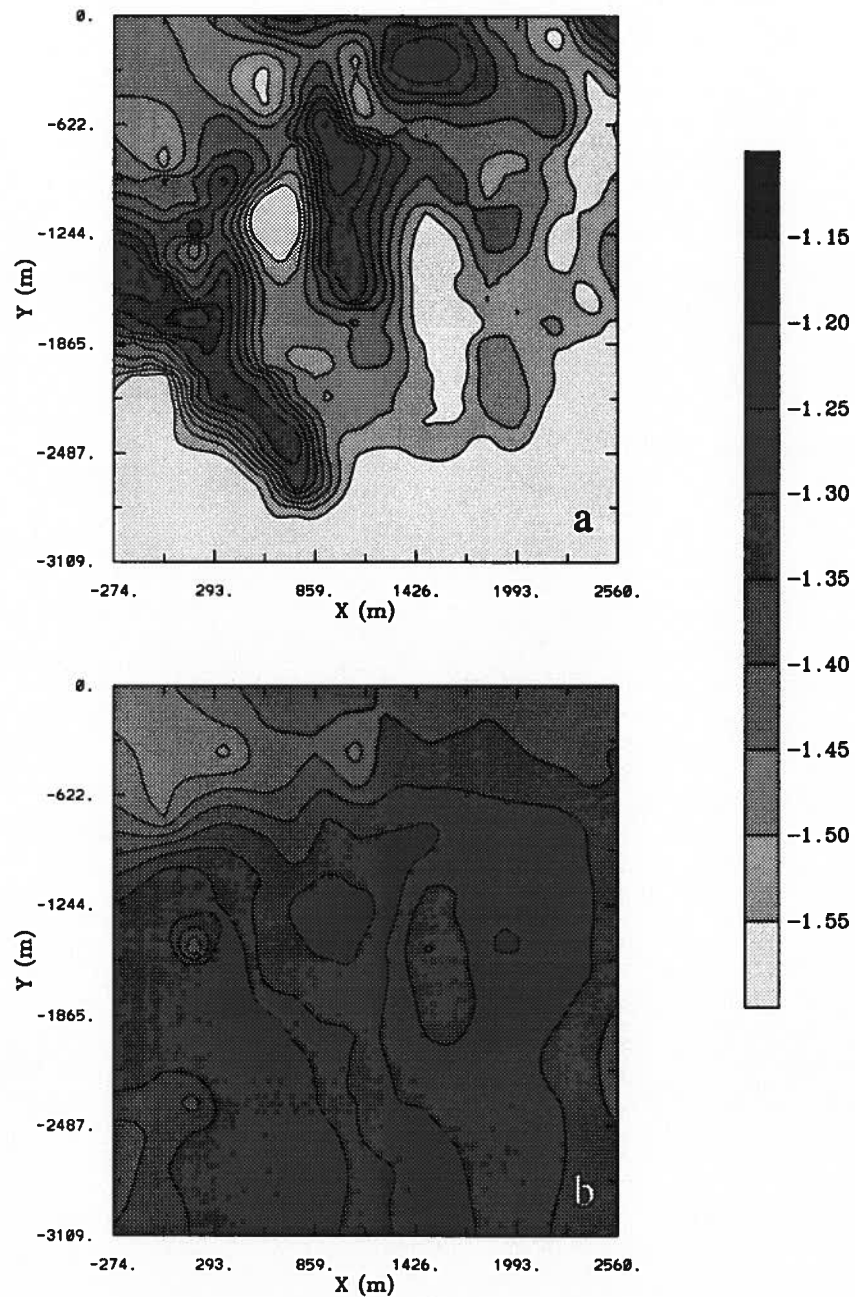


Figure 6.10 Two apparent conductivity maps from the data predicted by the constructed conductivity model. To be compared with the maps in Fig. 6.6.

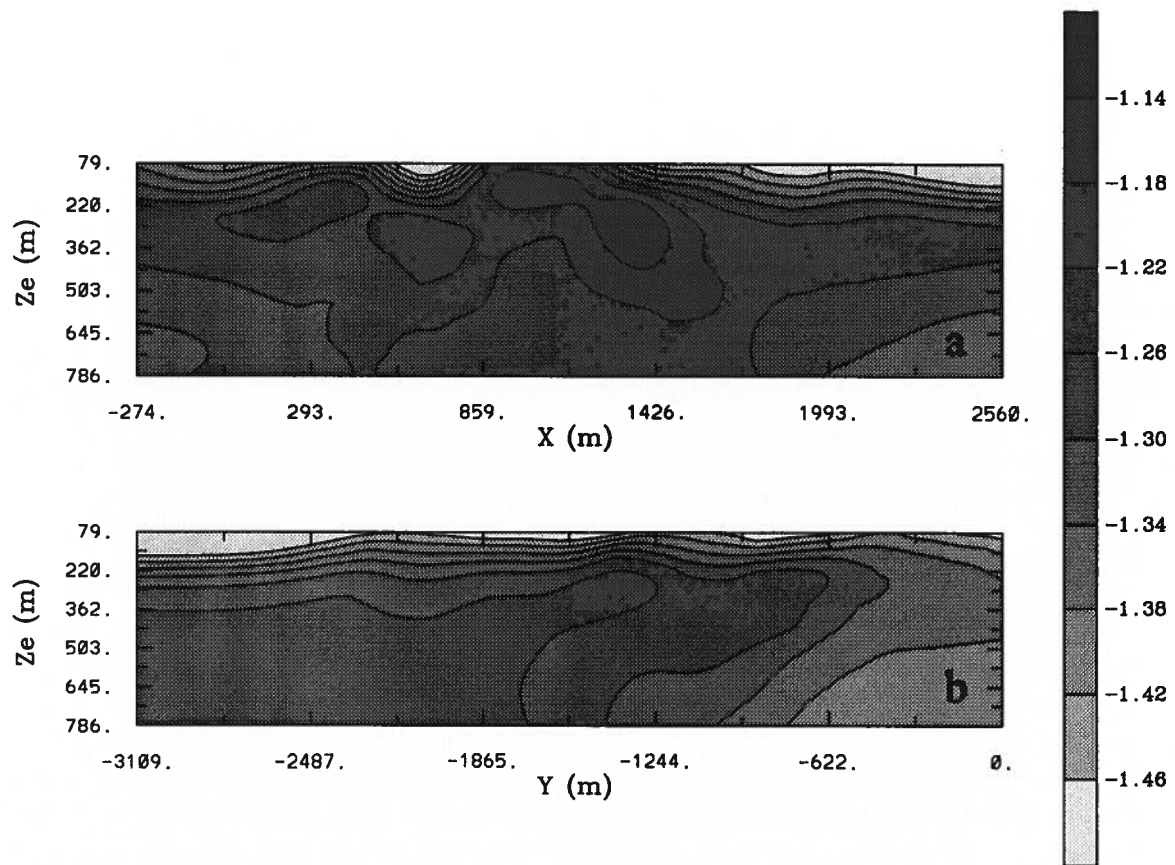


Figure 6.11 Two apparent conductivity pseudo-sections from the data predicted by the constructed conductivity model. To be compared with the sections in Fig. 6.7.

model obtained from the inversion displays many structures not visible or clear in the apparent conductivity pseudo-sections.

Known geology in this area seems to support some features observed in the conductivity model. The layer of eluvium covering the area is resistive. This may have been correctly represented in the model by the resistive layering near the surface. Borehole information in the section-A indicates that there is a fault between $x = 1000\text{m}$ and $x = 1200\text{m}$. The sharp conductivity contrast at the similar location (with the steep contour lines) and the structural change across this position may reflect the existence of the fault. In the section-B, the sharp conductivity contrast near $y = -400\text{m}$ seems to indicate the position of a fault known from borehole information.

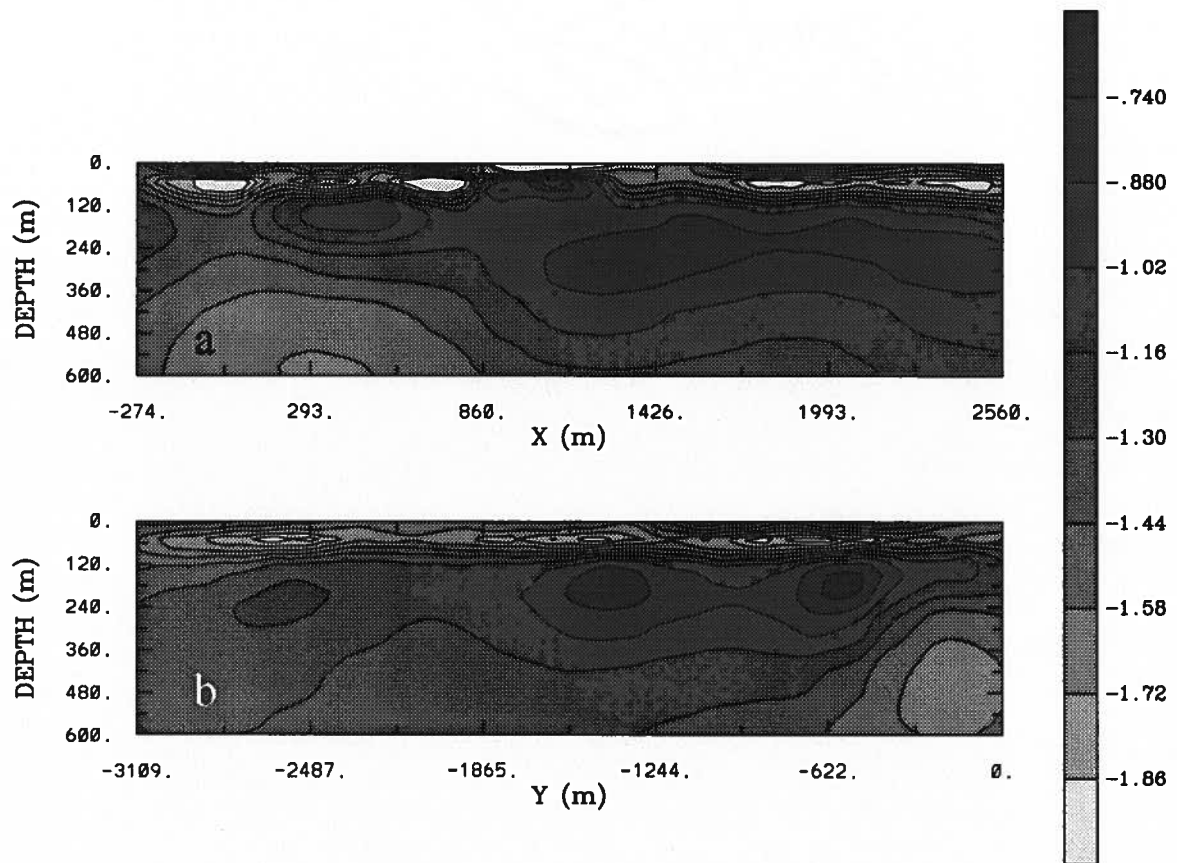


Figure 6.12 Two sections from the constructed conductivity model. Panels (a) and (b) show the conductivity in the section-A and section-B, respectively. These sections are chosen to coincide with the apparent conductivity pseudo-sections displayed in Fig. 6.7. Some geological information is available at these locations.

The model exhibits clear definitions of various conductive and resistive units. Ideally, these definitions should be related to different lithologic units using the information obtained from drilling. Although five major lithologic units have been identified in this region, their geo-electric properties are still uncertain. This, together with the limited availability of drilling information, makes such verification difficult at this stage.

From the viewpoint of the algorithm performance, the model may also prove to have its validity. The inversion starts from a best fitting half-space model with a misfit of 36%, and constructed a sequence of minimum norm models which steadily reduce the misfit to 12%. Such a fit is not overly loose for a field data set. Due to the nature of the algorithm, there is little chance

that large scale spurious structures are introduced. Therefore, features appearing in the model are most likely required by the observed data and may reflect true large scale conductivity variations.

6.5 Discussion

A set of field E-SCAN® data is analyzed using the techniques developed in this thesis. The data set was acquired as a part of an exploration effort for epithermal deposits. The field procedure of data acquisition is first reviewed. I then apply a first order correction for the placement error of the “infinite” electrode using an estimated value for the regional background conductivity. The maximum correction is about 20%, which indicates that the correction is necessary for field data sets. The common source gathers of the potential measurements are then formed from the corrected data by invoking the reciprocity of the d.c. potential field. Pole-pole data maps are formed and smoothed using the generalized cross-validation (GCV) smoothing technique.

The AIM-DS iterative inversion is then applied to 26 data maps with a total of 6825 potential data. The constructed conductivity model is represented by 34,040 parameters and achieved a RMS relative misfit of 12%. The model exhibits various structures and geo-electrical units which are not visible in the apparent conductivity images. However, due to the lack of the information about the true geology in the area, the conductivity model is difficult to verify, but it seems that a few known structures are reflected in the model.

Of course, the interpretation of a data set has at least two steps. The first is to construct a conductivity model which satisfies the data and reflects the true conductivity structures beneath the surface. The second is then to correlate the recovered conductivity structure with various lithologic units and to extract information about unknown geological structures. The first stage falls in the realm of the inversion. Although one always like to see the verification of the model by the ground truth in the second stage, such a procedure is beyond the scope of an inversion algorithm.

For the purpose of finding an acceptable conductivity model, the AIM algorithm has worked well. It constructs a model which fits the field data reasonably. The reduction of the misfit is

achieved within 4 iterations. Each iteration takes less than 300 minutes CPU time on a 4/330 SUN workstation with 32 MB memory. This shows that the application of the algorithm to field data is practical and efficient.

CHAPTER 7

CONCLUSIONS

The goal of the research in this thesis is to develop techniques for interpreting *3d* d.c. resistivity data to extract information about subsurface conductivity structures. Several imaging and inversion algorithms have been developed based upon the understanding of charge accumulation in d.c. resistivity experiments and the Born approximation of the surface electrical potential. All work on imaging and inversion has concentrated on aspects of model construction.

Various aspects of charge accumulation in the d.c. resistivity experiment are studied in Chapter 2. When electric current is introduced into a conductive medium, charges accumulate in the region where there is a non-zero component of electric field parallel to the conductivity gradient. It is these accumulated charges which give rise to the observed potentials. The behavior of a d.c. field can be explained with better understanding by examining the distribution of these charges. It is demonstrated that the refraction of current across a boundary separating two media with differing conductivity is a direct effect of the charges accumulated on the boundary. For simple geologic structures, it is often possible to sketch out the approximate potential response based upon the understanding of how the accumulated charges will distribute. This is useful in making preliminary interpretations of observed data, in understanding the result generated by an inversion algorithm, or in making first order judgements about the validity of the result from a numerical forward simulation. The traditional image method of solving potential problems is shown to be finding the set of fictitious point charges which produces the same potential as does the true accumulated charge. For systems consisting of isolated bodies of constant conductivity in a uniform background, the integral equation for the accumulated charges is generalized to include an undulating surface of the earth. This allows the solution of problems with surface topography. Both analytic and numerical examples are presented in which the charge accumulation is quantified using an integral equation.

In Chapter 3, different imaging techniques are developed for extracting first order information directly from the data. The display of apparent conductivity data volume with a pseudo-depth as a means of imaging is examined. It is demonstrated that filtering techniques can be used to enhance the data image. However, such data images only provide an indication of the lateral variation of the conductivity and possible relative depth of different anomalies. It provides neither the absolute depth nor the actual structural information. A second method is developed to image the depth of burial of simple structures by equivalent source images. The secondary electric potential is produced by subsurface charges. Potential field theory allows calculation of an equivalent source layer at any depth. Therefore, a $3d$ image can be obtained by constructing such equivalent sources at a sequence of depths. Numerical results show that the depth of burial for simple anomalies can be estimated from such images.

In the special case of $2d$ structures, the Born approximation of the potential field is used to develop two algorithms for imaging the conductivity structures. Under the approximation, the secondary potential on the surface is linearly related to a “pseudo-charge” which is a function of the conductivity gradient and the position of the current source. This function represents the integrated effect of accumulated charges distributed along the strike direction, hence the name pseudo-charge. Since the charge only accumulates in the region where the conductivity changes, the recovery of this function can yield a structural image for the conductivity. This is the basis of the first algorithm. A sparse representation of the pseudo-charge for each source location is constructed by inverting the corresponding secondary potentials. Linear programming is used to solve the inverse problem. The composite of these pseudo-charges from multiple source locations forms the desired structural image. The drawbacks of the method are the severe non-uniqueness in recovering the charge distribution from its potentials, and the instability in combining several individual inversions.

In the second approach, the same Born equation is used to formulate the inversion so that an image of the conductivity is recovered from the secondary potentials generated by all current sources. The pseudo-charge density can be expressed explicitly as a function of the conductivity.

In this way, the surface secondary potential becomes a linear functional of the conductivity. Unlike estimation of the charge location, the inversion now has a single function to recover. Consequently, all surface data can be inverted at once. The algorithm constructs a piecewise constant conductivity model which minimizes a combination of the l_1 norm of the model and the l_1 norm of the model gradient in a section parametrized into rectangular cells. Different proportions of the two model components results in models having different flatness. Linear programming is again used to solve the problem. Due to the approximation involved, the solution is to be interpreted as an image representation of the true conductivity.

An efficient $3d$ inversion algorithm is developed in Chapter 4 based upon the Born approximation. The algorithm can be used to construct general $3d$ conductivity models from pole-pole data maps. The model is assumed to be composed of perturbations over a given uniform half-space, i.e., $\sigma(\vec{r}) = \sigma_0\mu(\vec{r})$, where σ_0 is the background conductivity and $\mu(\vec{r})$ is the dimensionless function representing the perturbation. With this model assumption and the Born approximation, an integral equation is derived for the potential anomalies of the surface pole-pole array configuration. The equation expresses the pole-pole potential anomaly as a depth integral of the $2d$ convolution between the conductivity perturbation, $\ln(\mu)$, and a kernel function in the horizontal plane. This convolutional representation then enables the data equation to be decomposed into a sequence of independent components in the wavenumber domain by the Fourier transform. That is, each component of the potential data from any pole-pole array is related to the same component of model perturbation by a single depth integral with a wavenumber domain kernel. It follows that the $3d$ inverse problem of recovering a multiplicative model perturbation is solved by a sequence of $1d$ inversions in the wavenumber domain. Each available pole-pole data map in the spatial domain provides one (complex) datum at every wavenumber via a $2d$ Fourier transform. At each wavenumber, the $1d$ inversion is solved using linear inverse theory and a smallest model as a function of depth is constructed for the conductivity perturbation. By Parseval's theorem, this gives a smallest model for the perturbation in the spatial domain. Thus the conductivity model is essentially a smallest deviatoric model with a uniform base model $\ln(\sigma_0)$.

This formulation of the $3d$ inverse problem has several advantages. It is computationally efficient. With the convolutional representation of the surface data, the application of a Fourier transform decomposes the $3d$ problem into a set of independent $1d$ problems. Each $1d$ problem is solved with minimal computational effort. The solution of the $1d$ problems collectively yield the solution for the original $3d$ problem through an inverse Fourier transform. This approach is much more rapid than solving the $3d$ problem directly. For a problem consisting of $20,490$ ($32 \times 32 \times 20$) cells and $8,800$ potential data, the CPU time needed is about 100 minutes on a SUN 4/330 workstation with 32 MB memory. Most of the CPU time is used in computing the kernel functions.

The algorithm is capable of solving large problems. Since the problem is solved at each wavenumber through $1d$ inversions, the computing resources and time required for the inverse solution is greatly reduced. Therefore, it can cope with large numbers of model parameters and large numbers of data. This is an important step toward practical applications.

The algorithm is stable and less likely to produce large scale spurious structures. From an alternative view, the Fourier transform decomposes a single large inverse problem into many smaller problems each of which is sensitive to a different spatial scale length. The horizontal variability is limited by the smallest scale (highest wavenumber) of component included. In addition, a model of smallest norm is sought at each component. The kernel function used for model construction decays more rapidly with depth for higher wavenumbers. Therefore, the solution eliminates from the model those features which cannot be resolved by the data. At the same time, the algorithm naturally reflects in the model the decrease of resolution with the depth.

In Chapter 5, the $3d$ approximate inversion and AIM formalism is used to formulate an iterative algorithm which constructs a $3d$ conductivity model reproducing the observed data. The $3d$ approximate inversion is used as an approximate inverse mapping under the AIM formalism. Together with a finite difference forward modelling algorithm, it updates successively a initial model to produce a conductivity model which predicts the observed data within the tolerance of the data errors. The model obtained by simple application of the approximate $3d$ inversion

has been used as an initial model in all test cases, however, any appropriate model can serve the purpose. Both AIM-MS and AIM-DS algorithms are constructed and tested with several synthetic data sets.

The performance of the AIM algorithm depends to a great extent upon that of the approximate inverse mapping. Thus the constructed iterative inversion algorithm for $3d$ d.c. resistivity inherits the advantages of the approximate $3d$ inversion. It is rapid, stable, and capable of solving large problems. The kernel functions in the wavenumber domain are computed at the first iteration and stored for use in subsequent iterations. The computational effort for the solution of $1d$ systems is a very small fraction of that needed to calculate kernels. Thus the CPU time needed for applying inverse mapping is almost negligible in each iteration. In addition, the forward mapping needs to be performed only once at each iteration. Therefore, the total computation is minimal – one forward modelling is required in order to compute the data misfit in any iterative inversion. In all the synthetic test cases, the algorithm has steadily reduced the data misfit to the expected level for both accurate and noisy data sets. The reduction of the misfit is achieved within the first few iterations. The original conductivity models are reproduced with great fidelity in the depth range to which the given data set has sufficient responses.

The AIM inversion algorithm is then applied in Chapter 6 to a field E-SCAN® data set to construct a $3d$ conductivity model. The data set is acquired in an epithermal deposit area. The purpose of the E-SCAN® survey is to locate the fault structures and silicified zones with which ore deposits may be associated. The inversion constructs a $3d$ model which fits the observed data. The model defines various conductive and resistive units. Due to the lack of available geological information, the validity of the model is difficult to verify. However, limited geological sections seem to support several features observed in the constructed model.

The iterative inversion algorithm developed in this thesis offers a rapid approach to constructing $3d$ conductivity models by inverting large d.c. resistivity data sets from E-SCAN® experiments, or any surface experiment which can provide multiple set of pole-pole data maps. The implementation is stable and efficient. The resultant conductivity model may fit the observed data

adequately and suffice to provide the qualitative and quantitative information needed in the data interpretation. However, it is observed that the algorithm is likely to diverge after the initial reduction of the data misfit. The algorithm does not explicitly minimize the norm of the model subject to a desired misfit. As the number of iterations becomes large, the structural complexity of the model will increase and this will lead to the increase in the data misfit. As a result, the algorithm does not necessarily construct a model which fits the data to an expected accuracy. In such cases, the resultant model from this algorithm can be used as a starting model for other linearized inversions. Such a scheme would be able to produce a model which approximately fits the data with minimal effort and then to carry out detailed analysis with a more rigorous inversion if necessary.

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