OFF-SHELL PION ELECTROMAGNETIC FORM FACTORS IN CHIRAL PERTURBATION THEORY

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Abstract

The electromagnetic form factors of the pion play a role in investigations of various electromagnetic interactions. We calculate the off-shell pion form factors using chiral perturbation theory, which is the effective theory for QCD at low energies. Chiral perturbation theory has previously been used to calculate the on-shell form factors. The formalism is here described, and it is modified to accomodate possible off-shell contributions to Green functions in the pseudoscalar meson sector. We find that in addition to the phenomenological constant 'L_9' associated with the pion charge radius, another phenomenological constant enters the electromagnetic form factor f_π^+. This constant scales a contribution that is linear in $(p^2 - m_\pi^2)$ for either of the pions at the vertex.

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Chapter 1

Introduction

Effective field theory is a useful tool for systematically analyzing the low energy limit of some underlying, more complete theory [1]. Chiral perturbation theory is the effective theory of the Standard Model at the lowest energies; it is used to study several types of low energy QCD processes. Although it is an extension of ideas that have been around since the 1960's, chiral perturbation theory really has its origins in a 1979 paper by S. Weinberg [2], in which he showed how the low energy sector of QCD, consisting of almost-massless Goldstone bosons, can be treated perturbatively. The theory was then systematized by J. Gasser and H. Leutwyler in 1984 and 1985 [3, 4], and has become quite popular in recent years. (For reviews of chiral perturbation theory, see, for example, Meiβner [1], Bijnens [5], Pich [6].)

The most distinguished characteristic of chiral perturbation theory is that it is based only on symmetry. It is not a model; rather it is a calculational tool which exploits observed symmetries of QCD to make a range of phenomenological predictions. The predictions of the theory are valid to the degree to which the symmetries hold in the real world. Symmetries tie the Standard Model together in profound ways, and can give surprising relationships connecting sectors of particle physics that are usually conceptualized separately. A famous example is the Goldberger-Treiman relation [7], which is a simple connection between constants of the weak and the strong interactions. The origin of the relation is chiral symmetry.

In this study, chiral perturbation theory will be used to calculate the pion electromagnetic form factors generalized to off-mass-shell ($p^2 \neq m^2$) pions. This is an extension of previous work which has addressed only the on-shell behaviour of these form factors.

A pion consists of a quark and an antiquark bound by the strong force. In general, particles comprised of quarks are known as hadrons. Any composite particle has dynamic structure, which is manifested in its interactions.

A form factor (also known as a 'structure function') is a function of Lorentz scalars which characterizes the interaction of a composite particle with another particle over a range of energies. It can be thought of as the 'shape' of a particle in momentum space, in analogy with the concept of shape in position space. For example, in a nonrelativistic framework, the electromagnetic form factor can be interpreted as the Fourier transform of the charge distribution. The form factor is an object of common utility — as opposed to a spatial distribution — because scattering theory is formulated in terms of energy-momentum. (For an elementary discussion of form factors, see, for example, Halzen and Martin [8].)

The pion is the lightest hadron. The form factors to be calculated will describe the interaction of this charged bound state with a single photon. Because of the small masses of the particles involved, this interaction is one of the simplest and cleanest in QCD. It is ideally suited for an effective theory calculation. The calculation of the on-shell pion electromagnetic form factors using chiral perturbation theory can, for example, be found in references [3, 9, 10].

It is useful to discuss off-shell form factors, since the form factors characterize a Green function — in the present case, that of the two-pion/one-photon vertex — and this object has completely general applicability in the analysis of scattering processes involving the particles in question. What we will find is a genuine off-shell characteristic of one of the form factors when we consider the vertex in its full generality. A new term appears which is not present on-shell, and this term is parametrized by an undetermined phenomenological constant. One can hope that this virtual effect will be measurable in an experiment which incorporates a half off-shell pion/photon vertex.

Before the result is derived, several chapters must be devoted to introducing the theory behind the calculation. In Chapter 2, a general background is given on chiral perturbation theory, and we describe the change that must be made for the present application. In Chapter 3, the effective Lagrangian is constructed starting from an elementary set of assumptions. In Chapter 4 we renormalize the modified effective Lagrangian theory, and finally, in Chapter 5 the electromagnetic form factors are calculated.

Chapter 2

The Chiral Perturbation Theory Formalism

2.1 Introduction

Chiral perturbation theory is based on an effective, or phenomenological, Lagrangian. Parameters in an effective Lagrangian are fitted to describe the low energy regime of a more complicated theory. An effective theory, in general, breaks down as the energy rises. An energy scale enters the theory, and it is relative to this scale that energies will be considered as 'low'. The phenomenological parameters characterizing the effective theory contain information about the underlying theory. They represent the degrees of freedom 'frozen in' at low energy. In principle, the values of these parameters are calculable from the underlying theory; however, in practice, they are usually determined using experimental information.

The effective theory for the lowest energy hadrons is a reformulation of QCD, whose degrees of freedom are the quarks and gluons. We will be studying in particular the sector of the theory which describes the octet of pseudoscalar mesons. Although it is written down in a different form compared to QCD, the effective theory for these particles possesses the same symmetries, and this is the important connection. The symmetries demanded by QCD are Lorentz, C, P and T symmetries, and, in the limit of vanishing quark masses, chiral symmetry.

In this chapter the meaning of chiral symmetry will be explained, and the formalism built upon that symmetry will be introduced.

2.2 Chiral Symmetry in QCD

The QCD Lagrangian is

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2}\langle G_{\mu\nu}G^{\mu\nu}\rangle + \overline{q}\,(i\gamma^{\mu}D_{\mu} - M)\,q \tag{2.1}$$

where D_μ is the gauge covariant derivative, $G_{\mu\nu}$ is the gluon field strength tensor, and M is the quark mass matrix. The theory possesses a symmetry called chiral symmetry in the limit that $M = 0$.

Chiral symmetry is invariance of the theory under separate $SU(3)$ transformations of

$$q_L = \frac{1}{2}(1-\gamma_5)\,q\,, \qquad q_R = \frac{1}{2}(1+\gamma_5)\,q\,, \tag{2.2}$$

which are the the left- and right-handed helicity components of the quark wavefunction (hence the term 'chiral'). The left- and right-handed division means there are two entirely non-interacting sectors of the theory, each invariant under its own transformations. The symmetry group is therefore written $SU(3)_R \times SU(3)_L$, or referred to as 'chiral $SU(3) \times SU(3)$'.

The masses of the three lightest quarks, the up, down, and strange, are in fact very small relative to typical energy scales of QCD [1], whereas the charm, or c, quark and the other quarks are much heavier. Thus the usual chiral perturbation theory is a theory for just these three flavours:

$$q = \begin{bmatrix} u \\ d \\ s \end{bmatrix}. \tag{2.3}$$

To the extent that the three masses can be taken as zero, $SU(3)_R \times SU(3)_L$ is a good symmetry. The fact that this is only an approximation in the real world is the motivation for chiral perturbation theory. A perturbative expansion around the zero mass limit comprises the theory.

The breaking of the symmetry occurs on two different levels. The $SU(3)_R \times SU(3)_L$ is *spontaneously* broken down to $SU(3)$, which means there is a symmetry of the theory not shared by the vacuum. This, by Goldstone's theorem, leaves a spectrum of eight massless bosons. Secondly, the $SU(3)$ under which these Goldstone bosons transform is actually only an approximate symmetry; it is broken *explicitly* by the nonzero values of the quark masses.

We will not discuss Goldstone's theorem other than to say that first, it dictates the number of massless excitations based on the dimension of the broken symmetry subgroup, and second, it dictates their quantum numbers based on the broken symmetry of the vacuum. The number of Goldstone bosons in this case is eight, and they have spin and parity $J^P = 0^-$, making them pseudoscalars. These are the mesons to be considered in the present form factor calculation. The pion is one example of this type of particle.

The explicit symmetry breaking is introduced into the theory by the mass term M. The quark masses impart mass to the Goldstone bosons in turn, and consequently these particles are known as 'almost-Goldstone' bosons.

2.3 Chiral Symmetry in the Goldstone Boson Theory

We must motivate the transition from the QCD quark theory to an effective theory based instead on the Goldstone bosons.

The $SU(3)$ symmetry obeyed approximately by these Goldstone bosons is an extension of isospin symmetry. Isospin, or 'isotopic' spin, was invented to describe the approximate symmetry evident in the nucleon system, where the proton and the neutron are almost degenerate in mass. In the quark model, this observed symmetry is described by the approximate degeneracy in the u and d quark masses. Generalizing to $SU(3)$, one includes a third flavour of quark, the s, and the fundamental representation becomes

3-dimensional. When necessary, we will refer to this symmetry as 'generalized isospin'.

One can write a new representation for a symmetry group. As an example, consider the $SU(2)$ symmetry obeyed (almost) by the u and d quarks. These quarks form a 2-dimensional, or spinor, representation of this symmetry group, and analogously, the two nucleons p and n form a 2-dimensional representation. However, we find the same underlying symmetry manifested in a different set of particles — the pions π^+, π^-, and π^0 — which are also comprised of u and d quarks. These three particles form a 3-dimensional, or vector, representation of the same $SU(2)$ group.

In the case of $SU(3)$, the effective theory will be based on the 8-dimensional representation of the group, which will be just the collection of the pseudoscalar Goldstone bosons. This representation is written as the octet

$$\phi' = \sum_{i=1}^{8} \lambda_i \pi'_i \tag{2.4}$$

where the λ_i's are the eight Gell-Mann matrices and the π'_i parameters are a set of fields with the quantum numbers $J^P = 0^-$. The matrix thus generated is traceless and hermitean. It is equal to

$$\phi' = \begin{bmatrix} \pi_3 + \frac{1}{\sqrt{3}}\eta_8 & \pi_1 - i\pi_2 & K_4 - iK_5 \\ \pi_1 + i\pi_2 & -\pi_3 + \frac{1}{\sqrt{3}}\eta_8 & K_6 - iK_7 \\ K_4 + iK_5 & K_6 + iK_7 & -\frac{2}{\sqrt{3}}\eta_8 \end{bmatrix}. \tag{2.5}$$

The π'_i fundamental fields are represented in convenient notation; however, we have yet to do a translation to the octet of physical particles. We will refer to the basis in which the particles in ϕ' are written as the Pauli or numerical basis. The i indices are generalized isospin vector indices.

The physical pions that are eigenstates of electric charge are defined as follows in terms of the fundamental fields:

$$\begin{aligned} \pi^+ &= \frac{\pi_1 - i\,\pi_2}{\sqrt{2}} \\ \pi^- &= \frac{\pi_1 + i\,\pi_2}{\sqrt{2}} \\ \pi^0 &= \pi_3. \end{aligned} \tag{2.6}$$

Notice that the charged pions are complex conjugates of each other. With similar definitions for the kaons and the eta, one changes the basis and writes the octet

$$\phi \equiv \begin{bmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\,\overline{K^0} & -\frac{2}{\sqrt{3}}\eta \end{bmatrix}. \tag{2.7}$$

The matrix ϕ will be the basis for the theory we will write down. Reference will also be made, however, to the fundamental fields shown in equation (2.5).

In chiral perturbation theory the representation for the chiral Lagrangian [2, 11] is a nonlinear function of the meson field ϕ. The simplest choice among an infinite number of possibilities is the exponential

$$U(\phi(x)) = e^{\frac{i}{F_0}\phi(x)} \tag{2.8}$$

where F_0 is a constant having the appropriate dimensions, which turns out to be the pion decay constant. The matrix-valued function U is the building block from which an effective Lagrangian is constructed. U transforms linearly under chiral $SU(3) \times SU(3)$; this transformation will be discussed in the next chapter.

2.4 Issues in the Use of the Effective Lagrangian

2.4.1 The Formalism to $O(p^4)$

There exists a unique series expansion of the S-matrix in powers of the energy-momentum p. This is generically referred to as the 'energy expansion'. The justification for a theory based on the nonlinear representation is that the S-matrix expansion generated should represent that of the complete theory, order by order. Therefore the lowest terms in the expansion yield a phenomenological approximation to the complete theory [12].

The Lagrangian that describes the Goldstone boson system at leading order is

$$\mathcal{L}_2 = \frac{F_0^2}{4}\langle D^\mu U^\dagger D_\mu U\rangle + \frac{F_0^2}{4}\langle \chi^\dagger U + \chi U^\dagger\rangle \tag{2.9}$$

where D_μ is the chiral covariant derivative and χ introduces the nonzero quark mass matrix M. This is the unique lowest order Lagrangian that follows from the assumptions of the linearly transforming U and the addition of a mass term χ that transforms chirally in the same way as U. The subscript 2 refers to the energy dimension of the Lagrangian. Each covariant derivative supplies one power of p in the energy expansion, and the mass matrix is assumed to be $O(p^2)$.

To extend the formalism to $O(p^4)$, a Lagrangian $\mathcal{L}_4$ is added which contains four derivatives (or two derivatives and one mass insertion, or two mass insertions). This extension is nontrivial because of the introduction of loops, and therefore divergences, into S-matrix elements. The Gasser and Leutwyler formalism [3, 4] is the general theory based on $\mathcal{L}_2 + \mathcal{L}_4$. Their theory possesses a tractably small number of phenomenological parameters — all of which have been estimated from experimental data [10].

2.4.2 Perturbation Expansion

Some general comments should be made concerning the use of the effective Lagrangian to do perturbation theory.

The Gasser and Leutwyler formalism gives one the material to calculate the energy expansion up to order p^4. In practice, one thinks about a calculation in terms of Feynman diagrams. To calculate these diagrams from the different components of the Lagrangian, one must keep in mind the 'order' of the particular part of the Lagrangian. This translates into the number of derivatives in its derivative-only interaction term — this number directly represents the energy dimension.

A result that neatly summarizes what we need to know is the following expression for a general S-matrix element [2]. The only dimensional quantities that appear in the theory are the energy scale p of the scattering, the fixed renormalization scale μ, and the phenomenological coupling constants, and one will find that all terms in the calculated S-matrix element have this form:

$$S_{fi} = p^D f(\frac{p}{\mu}) \tag{2.10}$$

where

$$D = 2 + \sum_d N_d (d-2) + 2N_L \tag{2.11}$$

characterizes the energy dimension. The parameters d, N_d, and N_L describe the Feynman diagram from which a given term arises: N_d is the number of vertices coming from interaction Lagrangian terms with d derivatives, and N_L is the number of loops.

One can see from this formula that the $\mathcal{L}_2$ Lagrangian, whose interaction term has $d = 2$, will contribute with p^2 at tree-diagram level ($N_L = 0$), and with p^4 at one-loop level. The $\mathcal{L}_4$ Lagrangian has $d = 4$, and at tree-level will contribute at $O(p^4)$. However, if N_L is nonzero, then N_d must be nonzero, so loop diagrams calculated from this Lagrangian will contribute to the S-matrix element at least at $O(p^6)$, and are to be neglected.

2.4.3 The Off-Shell Modification

The highest order Lagrangian included in calculations ($\mathcal{L}_4$ in the present formalism) will always be used at tree-level. This has an important implication. In the description of *physical* processes, this part of the Lagrangian will describe only states that satisfy an equation of motion, which is to say states that are 'on their mass shell'. In such cases it is appropriate to restrict the form of the Lagrangian. One uses the effective equation of motion at next-to-highest order. The Gasser and Leutwyler $\mathcal{L}_2 + \mathcal{L}_4$ formalism does incorporate the $\mathcal{L}_2$ equation of motion as a constraint on $\mathcal{L}_4$ [4].

Interaction diagrams having more than one vertex can be decomposed into tree-level diagrams, each of which is described by a vertex function having one or more legs off-shell. Such a vertex function is characterized by off-shell form factors. If one wishes to discuss these unphysical vertex functions, one cannot restrict the interaction Lagrangian by an equation of motion [13]. One must in general add back into the Lagrangian those terms which vanished through the use of the equation of motion. It has been crucial to recognize this fact, for the following reason. Because of the nature of the effective Lagrangian approach, one does derive a self-consistent result starting with the restricted effective Lagrangian. In the case of the form factor, this self-consistency is manifested in the solution of the Ward-Takahashi identity. We will discover that if one carries out the off-shell form factor calculation with the standard $\mathcal{L}_2 + \mathcal{L}_4$ Lagrangian, one will obtain a result fully consistent with the Ward-Takahashi identity, even though in fact the Lagrangian is *not* the most general for the purposes of that calculation.

Thus, for the present undertaking, the important first step is to examine the derivation of the Gasser and Leutwyler effective Lagrangian. We must make sure to keep the full generality of the highest order component $\mathcal{L}_4$. As will be seen, two new structures appear if the on-shell constraint is not applied. These structures involve the second derivative of a field (to be precise, the chiral covariant second derivative).

The second derivative interaction is not new to the present treatment, but in fact appears in the calculation of the L_9 component within the on-shell form factor; thus, our result seems fully justified by preceding work. Nonetheless, it is worth pointing out that this issue has been under study recently. The use of interactions involving higher than single derivatives is a departure from the usual Hamilton-Lagrange formalism underlying quantum field theories. In fact, in general, theories with higher order derivatives have unsatisfactory properties. These interactions can, however, be used successfully in the context of *effective* field theories [14]. For a recent proof concerning the use of effective Lagrangians at first order in perturbation theory, see Grosse-Knetter [14].

Chapter 3

The Effective Lagrangian

3.1 Introduction

In this chapter we will derive the effective Lagrangian. The goal is to construct the most general possible Lagrangian obeying $SU(3)_R \times SU(3)_L$ symmetry that also incorporates a symmetry-breaking mass term. This Lagrangian must also be invariant under the symmetries C and P. We will see that the $O(p^4)$ part of this most general Lagrangian includes two terms which vanish when the equation of motion of the $O(p^2)$ part is satisfied. These form an addition to the usual $\mathcal{L}_4$ used for on-shell calculations.

3.2 Definitions

The theory is a spontaneously broken gauge theory based on the linearly transforming $U(x)$, and gauge fields denoted $v_\mu(x)$ and $a_\mu(x)$. In this section we present the basic properties of these objects [4], introduce the chiral covariant derivative, and define the explicit symmetry breaking scheme.

Chiral symmetry is invariance under the local transformation

$$\begin{aligned} A(x) &\rightarrow \Omega_R(x)\, A(x)\, \Omega_L^\dagger(x) \\ &\Omega_{R,L} \in SU(3) \end{aligned} \tag{3.1}$$

for matrices $A(x)$ that comprise the Lagrangian. Objects $A(x)$ that transform this way will be referred to as 'chiral matrices'.

The explicit representation used for U has been given in equation (2.8). The transformation for this matrix is

$$U(x) \quad \rightarrow \quad \Omega_R(x)U(x)\Omega_L^\dagger(x). \tag{3.2}$$

For $U^\dagger$, which also enters the nonlinear represention,

$$U^\dagger(x) \quad \rightarrow \quad \Omega_L(x)U^\dagger(x)\Omega_R^\dagger(x) \tag{3.3}$$

which is the hermitean conjugate of (3.2). U is unitary:

$$U^\dagger U \quad = \quad 1 \tag{3.4}$$

and has determinant 1.

Transformation (3.1) is to represent a local gauge symmetry. One constructs a covariant derivative to be used in place of the ordinary derivative, with the defining property that the covariant derivative of a chiral matrix must transform also as a chiral matrix. The definition is

$$D_\mu U \quad = \quad \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu), \tag{3.5}$$

which brings the gauge field coupling into the Lagrangian. We will show at the end of this section that $D_\mu U$ transforms according to equation (3.1).

For reference, we list here

$$D_\mu U^\dagger \quad = \quad \partial_\mu U^\dagger + iU^\dagger(v_\mu + a_\mu) - i(v_\mu - a_\mu)U^\dagger, \tag{3.6}$$

which is simply the hermitean conjugate of (3.5) — subject to the important definition (3.7) below.

Multiple derivatives $D_\mu D_\nu U$, etc., are also allowed, and always have the transformation property (3.1).

We use this notational convention for adjoint quantities:

$$D_\mu U^\dagger \equiv (D_\mu U)^\dagger \tag{3.7}$$

wherever single, or multiple, covariant derivatives are involved. Care must be taken when explicitly writing out the expansion of a daggered multiple derivative.

A further convention is as follows. A derivative operator will be understood to apply only to the field that directly follows it, so we will suppress the parentheses:

$$D_\mu U U^\dagger \equiv (D_\mu U)\, U^\dagger \tag{3.8}$$

or

$$\partial_\mu \phi\phi\phi\phi \equiv (\partial_\mu \phi)\, \phi\phi\phi. \tag{3.9}$$

Where necessary for clarity, parentheses will be used.

The gauge fields (note they are 3×3 matrices) are written as $(v_\mu + a_\mu)$ and $(v_\mu - a_\mu)$ because the two $SU(3)$ symmetries hold independently, and these are the linear combinations of gauge fields which couple to the right- and left-handed components of the meson wavefunction, respectively. We will use the abbreviations

$$\begin{aligned} r_\mu &\equiv v_\mu + a_\mu \\ l_\mu &\equiv v_\mu - a_\mu \end{aligned}$$

for these gauge fields or for what can equivalently be referred to as 'external currents'.

The external currents must transform according to

$$\begin{aligned} r_\mu &\rightarrow \Omega_R (r_\mu + i\partial_\mu) \Omega_R^\dagger \\ l_\mu &\rightarrow \Omega_L (l_\mu + i\partial_\mu) \Omega_L^\dagger \end{aligned} \tag{3.10}$$

to compensate for the transformation (3.2) of U and guarantee the chiral transformation of $D_\mu U$.

An important property satisfied by both v_μ and a_μ is that they are traceless matrices [4].

'Field strength' tensors are defined by

$$\begin{aligned} F_{\mu\nu}U &\equiv i[D_\mu, D_\nu]U \\ &= F^R_{\mu\nu}U - UF^L_{\mu\nu} \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} F^R_{\mu\nu} &= \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu] \\ F^L_{\mu\nu} &= \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \end{aligned} \tag{3.12}$$

Taking the adjoint of (3.11) yields

$$\begin{aligned} (F_{\mu\nu}U)^\dagger &= U^\dagger F^R_{\mu\nu} - F^L_{\mu\nu}U^\dagger \\ &= -i([D_\mu, D_\nu]U)^\dagger \end{aligned} \tag{3.13}$$

which we list here for later use.

It is the components $F^R_{\mu\nu}$ and $F^L_{\mu\nu}$ which will be considered as separately transforming structures, for full generality. These tensors however, as suggested by their categorization under 'R' and 'L' superscripts, are not $SU(3)_R \times SU(3)_L$ building blocks. They transform according to

$$\begin{aligned} F^R_{\mu\nu} &\to \Omega_R(x)\, F^R_{\mu\nu}\, \Omega^\dagger_R(x) \\ F^L_{\mu\nu} &\to \Omega_L(x)\, F^L_{\mu\nu}\, \Omega^\dagger_L(x) \end{aligned} \tag{3.14}$$

as can be shown using transformations (3.10). Notice that the inclusion of the matrix U either on the right or left side of $F^{R,L}_{\mu\nu}$ leaves an $SU(3)_R \times SU(3)_L$ object. For example, one has the transformation

$$\begin{aligned} F^R_{\mu\nu} U &\rightarrow \Omega_R F^R_{\mu\nu} \Omega_R^\dagger \Omega_R U \Omega_L^\dagger \\ &= \Omega_R F^R_{\mu\nu} U \Omega_L^\dagger \end{aligned}$$

using the fact that the transformation matrix Ω_R is unitary: $\Omega_R^\dagger \Omega_R = 1$. A covariant derivative of a tensor must incorporate U (or $U^\dagger$, as appropriate) with the tensor, so that the derivative acts on a chiral matrix.

Note the following useful properties of the field strength tensors:

$$F^{R,L\dagger}_{\mu\nu} = F^{R,L}_{\mu\nu} \tag{3.15}$$

$$F^{R,L}_{\mu\nu} = -F^{R,L}_{\nu\mu}. \tag{3.16}$$

Finally, one introduces the symmetry-breaking component in the form of a general complex matrix, written

$$\chi = 2B_0 (s + ip) \tag{3.17}$$

where s and p are scalar and pseudoscalar densities. One assumes that χ transforms chirally

$$\chi(x) \rightarrow \Omega_R(x) \chi(x) \Omega_L^\dagger(x) \tag{3.18}$$

so as to leave the Lagrangian $SU(3)_R \times SU(3)_L$ invariant. This is just an ansatz used to set up a general Lagrangian whose symmetry can be broken. The assumption is that to apply the theory to the real world, one will break the symmetry in a very specific way, by setting $s = M$ and $p = 0$, where M is the quark mass matrix

$$M = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}. \tag{3.19}$$

One is then expanding in the (very small) quark masses.

The power counting scheme is set up in such a way that χ is proportional to the squared meson masses, but, containing a single power of M, it depends linearly on the quark masses. The phenomenological constant B_0 makes the connection between the two. An interesting discussion of this part of the theory can be found in Stern et al. [15]. These authors have made a generalization of chiral perturbation theory which leads to a different power counting scheme and a different set of Lagrangians than the standard $\mathcal{L}_2$ and $\mathcal{L}_4$ with which we will be dealing.

The important point for present purposes is that χ has $O(p^2)$ energy dimension. The definition of a covariant derivative $D_\mu \chi$ can be made, and is the same as that for the covariant derivative of U, since both U and χ transform as chiral matrices.

The energy dimensions of the other quantities are as follows: U is dimensionless; the covariant derivative is $O(p)$; the field strength tensors, deriving from the second covariant derivative, are $O(p^2)$.

$\langle A \rangle$ will be used to denote the trace of a matrix A in $SU(3)$ space.

3.2.1 Chiral Transformation Property of the Covariant Derivative

Let us verify the transformation property of the covariant derivative of a chiral matrix. In the following proof we make use of

$$\partial_\mu U^\dagger U \;=\; -U^\dagger \partial_\mu U \tag{3.20}$$

which holds for a unitary matrix U. This identity follows directly from definition (3.4).

The covariant derivative must, under chiral transformations of the fields and the external sources, have the invariant form indicated by equation (3.5). Hence, we apply the transformations

$$\begin{aligned} U &\rightarrow \Omega_R U \Omega_L^\dagger \\ r_\mu &\rightarrow \Omega_R (r_\mu + i\partial_\mu)\Omega_R^\dagger \\ l_\mu &\rightarrow \Omega_L (l_\mu + i\partial_\mu)\Omega_L^\dagger \end{aligned} \tag{3.21}$$

to derive

$$\begin{aligned} D_\mu U &\rightarrow \partial_\mu(\Omega_R U \Omega_L^\dagger) - i\Omega_R r_\mu U \Omega_L^\dagger + i\Omega_R U l_\mu \Omega_L^\dagger + \Omega_R \partial_\mu \Omega_R^\dagger \Omega_R U \Omega_L^\dagger - \Omega_R U \partial_\mu \Omega_L^\dagger \\ &= \Omega_R (\partial_\mu U - i r_\mu U + i U l \mu)\, \Omega_L^\dagger \\ &\qquad + \partial_\mu \Omega_R U \Omega_L^\dagger + \Omega_R \partial_\mu \Omega_R^\dagger \Omega_R U \Omega_L^\dagger \\ &= \Omega_R\, D_\mu U\, \Omega_L^\dagger \end{aligned} \tag{3.22}$$

We have used $\Omega_R^\dagger \Omega_R = \Omega_L^\dagger \Omega_L = 1$, and in the last step, the two left-over terms have cancelled using identity (3.20) applied to Ω_R.

This is the chiral transformation (3.1) that we require. Furthermore, since the object $D_\mu U$ transforms in the same manner as U, the proof extends immediately to multiple covariant derivatives.

3.3 Finding all Chiral + C + P Invariants

We now turn to assembling a Lagrangian from the components we have discussed. The building blocks which transform according to equation (3.1) and are at most $O(p^4)$ are

$$U$$

$$D_\mu U$$

$$D_\mu D_\nu U$$

$$
\begin{aligned}
& D_\mu D_\nu D_\lambda U \\
& D_\mu D_\nu D_\lambda D_\rho U \\
& \chi \\
& D_\mu \chi \\
& D_\mu D_\nu \chi .
\end{aligned}
\tag{3.23}
$$

Further building blocks can be formed from the tensors $F^R_{\mu\nu}$ and $F^L_{\mu\nu}$ if, as mentioned, they are combined with objects that do transform chirally. The tensors are $O(p^2)$, so this list of possibilities is

$$
\begin{array}{ll}
F^R_{\mu\nu} U & U F^L_{\mu\nu} \\
F^R_{\mu\nu} D_\lambda U & D_\lambda U F^L_{\mu\nu} \\
F^R_{\mu\nu} D_\lambda D_\rho U & D_\lambda D_\rho U F^L_{\mu\nu} \\
D_\lambda (F^R_{\mu\nu} U) & D_\lambda (U F^L_{\mu\nu}) \\
D_\lambda (F^R_{\mu\nu} D_\rho U) & D_\lambda (D_\rho U F^L_{\mu\nu}) \\
D_\rho D_\lambda (F^R_{\mu\nu} U) & D_\rho D_\lambda (U F^L_{\mu\nu}).
\end{array}
\tag{3.24}
$$

The matrix χ (remember, it is also $O(p^2)$) will not appear together with the tensors at $O(p^4)$, for the simple reason that Lorentz indices must be contracted. We have seen that $F^\mu_\mu = 0$.

3.3.1 Forming Chiral Invariant Structures

The objects that we have collected are 3×3 matrices that transform in a well-defined manner under the chiral group. Chiral invariants are formed by combining the above with their adjoints and taking the trace. This is because the transformation law for a hermitean conjugate quantity is just the hermitean conjugate of equation (3.1). Any structure having the following form satisfies the chiral $SU(3) \times SU(3)$ symmetry:

$$\begin{aligned} \langle A^\dagger B C^\dagger D ...\rangle &\rightarrow \langle \Omega_L A^\dagger \Omega_R^\dagger \Omega_R B \Omega_L^\dagger \Omega_L C^\dagger \Omega_R^\dagger \Omega_R D \Omega_L^\dagger ...\rangle \\ &= \langle A^\dagger B C^\dagger D ...\rangle \end{aligned} \tag{3.25}$$

since the cyclic property of the trace allows the matrix at the end to be brought around to the front. Strictly speaking, an invariant Lagrangian term can be under one overall trace as implied by (3.25), or it can be a product of these trace expressions.

We will assemble an initial list of all possible chiral invariant terms that can comprise the $O(p^2)$ and $O(p^4)$ Lagrangians. The size of this list can then be greatly reduced by considering parity invariance, C invariance, and special constraints.

There are several guidelines to keep in mind in constructing these invariants. First of all, we have the two relations:

$$U^\dagger U = 1 \tag{3.26}$$

$$D_\mu U^\dagger U = -U^\dagger D_\mu U \tag{3.27}$$

the latter of which can be shown quite easily by writing out the covariant derivative on each side. In general, of course, the covariant derivative does not satisfy relations that would be satisfied by the ordinary derivative, because it does not obey the chain rule for derivatives. As we have defined it, the covariant derivative can only act upon matrices which transform chirally according to equation (3.1). Thus it must be used with discretion.

The guidelines are as follows. Identity (3.26) means invariants will contain a 'minimal' number of U or $U^\dagger$ matrices. There are not an infinite number of allowed expressions differing only in number of factors. This makes sense, thinking of our goal of the 'most general' effective Lagrangian for the meson fields ϕ, since U already exponentiates these fields.

Identity (3.27), as well, permits expressions to be simplified by reducing the number of U matrices. $D_\mu U^\dagger U$ factors can be interchanged with $U^\dagger D_\mu U$ factors in order to move U and $U^\dagger$ matrices next to each other. In fact, this identity has a special status in rewriting expressions into a common, consistent form, or to show equivalence of the expressions we will be dealing with.

As an example, here we derive a simple substitution. We insert $U^\dagger U$ pairs as required, apply equation (3.27), and take advantage of the trace to obtain

$$\begin{aligned} \langle D_\mu U^\dagger D_\nu U\rangle &= \langle D_\mu U^\dagger U U^\dagger D_\nu U\rangle \\ &= \langle D_\mu U D_\nu U^\dagger\rangle, \end{aligned} \tag{3.28}$$

a result that can be used to commute Lorentz indices in such a trace expression.

This kind of substitution will be implicitly used in the following, but note that only single covariant derivatives will be involved. Multiple covariant derivative factors cannot be substituted for or moved around in the same way; the key is the simple form of (3.27).

Nonetheless, relations analogous to (3.27) can be found for double and all other multiple covariant derivatives, and for the $F_{\mu\nu}^{R,L}$ building blocks we have listed. As an example, we show just the double derivative relation:

$$D_\mu D_\nu U^\dagger U = -U^\dagger D_\mu D_\nu U - D_\mu U^\dagger D_\nu U - D_\nu U^\dagger D_\mu U. \tag{3.29}$$

This can be proven by expanding the covariant derivatives using equations (3.5) and (3.6). Consider what is implied by this relation. Any structure incorporating $D_\mu D_\nu U^\dagger U$ is not independent of the same structure containing instead $U^\dagger D_\mu D_\nu U$. Therefore, at this stage, we can consider the structures we write down containing $D_\mu D_\nu U$ as representative of all the structures with either $D_\mu D_\nu U$ or its adjoint. We are interested only in creating a list of truly independent structures. The same consideration holds for the triple and quadruple derivatives. We will only worry about adjoints of the derivatives later — the explicit set of expressions in both normal and adjoint quantities will necessarily emerge when parity invariance of the Lagrangian is considered.

Because χ is not unitary, no such $\chi \leftrightarrow \chi^\dagger$ relationships exist. Wherever this matrix can appear in the following list of expressions, both χ and $\chi^\dagger$ must be represented, for full generality.

Two other significant relations are

$$\langle D_\mu U^\dagger U \rangle = 0 \tag{3.30}$$

$$\langle F^{R,L}_{\mu\nu} \rangle = 0. \tag{3.31}$$

The second of these is easy to see. Refer to definition (3.12). The external sources v_μ and a_μ are traceless, as we pointed out; thus, their derivatives are traceless. Furthermore, the trace of any commutator is zero. This means all three terms of the tensor are manifestly traceless.

For the identity (3.30), we again have two statements of tracelessness to prove. Refer to the covariant derivative definition (3.5). Again we invoke the tracelessness of the external sources. By the unitarity of U, we have immediately that the external source terms of $D_\mu U^\dagger U$ are traceless, since they are the product of a traceless matrix times the identity.

To prove the derivative part, $\partial_\mu U^\dagger$ and U are expanded as power series. When the series are multiplied together under the trace, cyclic permutation effectively allows the $\partial_\mu \phi$ to be commuted freely in each term; for example, we could write the expansion with all the derivative factors at the front:

$$\langle \partial_\mu U^\dagger U \rangle = \langle -i\partial_\mu \phi - \frac{1}{2!}(2 \times \partial_\mu \phi \phi) + \partial_\mu \phi \phi + \cdots \rangle.$$

The series can be rewritten and separated again as the two series $U^\dagger$ and $\partial_\mu U$ — but the sign does not change. Applying this same-sign relationship, and following this by the application of the ordinary derivative identity (3.20), we conclude

$$\begin{aligned}\langle \partial_\mu U^\dagger U \rangle &= \langle U^\dagger \partial_\mu U \rangle \\ &= -\langle \partial_\mu U^\dagger U \rangle \\ &= 0.\end{aligned}$$

This completes the proof.

Relation (3.30), in particular, eliminates a large number of candidates from our list. It implies that a trace must be over at least an $O(p^2)$ expression — with the consequence that there will be *only* $O(p^2)$ and $O(p^4)$ traces. (Note also that the only trace expression of $O(p^0)$ is $\langle U^\dagger U \rangle =$ constant, which is irrelevant in our constructions.)

The last consideration to be mentioned is contraction of Lorentz indices. We form Lorentz scalars, which means indices will always appear in contracted pairs.

3.3.2 Chiral Invariant List

The possible Lorentz and chiral invariants are compiled in this first list. We have made use of equations (3.15), (3.16), (3.26), (3.27), (3.28), (3.30), and (3.31) in selecting structures. The reason we have not necessarily included the hermitean conjugates of derivatives of U's is that they are not independent, by equations (3.27) and (3.29) and analogous equations for the triple and quadruple derivatives.

Here are the invariants. The first four are $O(p^2)$. The rest are $O(p^4)$.

A–1	$\langle D^\mu U^\dagger D_\mu U \rangle$
A–2	$\langle D^\mu D_\mu U^\dagger U \rangle$
A–3	$\langle \chi^\dagger U \rangle$
A–4	$\langle U^\dagger \chi \rangle$

A–5	$\langle D^\mu U^\dagger D_\mu U\rangle\langle D^\nu U^\dagger D_\nu U\rangle$
A–6	$\langle D^\mu U^\dagger D^\nu U\rangle\langle D_\mu U^\dagger D_\nu U\rangle$
A–7	$\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle$
A–8	$\langle D^\mu U^\dagger D^\nu U D_\mu U^\dagger D_\nu U\rangle$
A–9	$\langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U\rangle$
A–10	$\langle D^\mu U^\dagger D_\mu U\rangle\langle U^\dagger \chi\rangle$
A–11	$\langle D^\mu U^\dagger D_\mu U \chi^\dagger U\rangle$
A–12	$\langle D^\mu U^\dagger D_\mu U U^\dagger \chi\rangle$
A–13	$\langle \chi^\dagger U\rangle\langle \chi^\dagger U\rangle$
A–14	$\langle \chi^\dagger U\rangle\langle U^\dagger \chi\rangle$
A–15	$\langle U^\dagger \chi\rangle\langle U^\dagger \chi\rangle$
A–16	$\langle \chi^\dagger U \chi^\dagger U\rangle$
A–17	$\langle U^\dagger \chi U^\dagger \chi\rangle$
A–18	$\langle \chi^\dagger \chi\rangle$
A–19	$\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger\rangle$ + perm of indices
A–20	$\langle F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle$ + perm of indices
A–21	$\langle F^{R\mu\nu} U F^L_{\mu\nu} U^\dagger\rangle$ + perm of indices
A–22	$\langle F^{R\mu\nu} F^R_{\mu\nu}\rangle$ + perm of indices
A–23	$\langle F^{L\mu\nu} F^L_{\mu\nu}\rangle$ + perm of indices
A–24	$\langle D^\mu D_\mu U^\dagger U\rangle\langle D^\nu U^\dagger D_\nu U\rangle$
A–25	$\langle D^\mu D^\nu U^\dagger U\rangle\langle D_\mu U^\dagger D_\nu U\rangle$

A–26	$\langle D^\mu D_\mu U^\dagger U D^\nu U^\dagger D_\nu U\rangle$
A–27	$\langle D^\mu D^\nu U^\dagger U D_\mu U^\dagger D_\nu U\rangle$ + perm of indices
A–28	$\langle D^\mu D_\mu U^\dagger U\rangle\langle D^\nu D_\nu U^\dagger U\rangle$
A–29	$\langle D^\mu D^\nu U^\dagger U\rangle\langle D_\mu D_\nu U^\dagger U\rangle$ + perm of indices
A–30	$\langle D^\mu D_\mu U^\dagger D^\nu D_\nu U\rangle$
A–31	$\langle D^\mu D^\nu U^\dagger D_\mu D_\nu U\rangle$ + perm of indices
A–32	$\langle D^\mu D_\mu U^\dagger U\rangle\langle \chi^\dagger U\rangle$
A–33	$\langle D^\mu D_\mu U^\dagger U\rangle\langle U^\dagger \chi\rangle$
A–34	$\langle \chi^\dagger D^\mu D_\mu U\rangle$
A–35	$\langle D^\mu D_\mu U^\dagger \chi\rangle$
A–36	$\langle F^R_{\mu\nu} D^\mu D^\nu U U^\dagger\rangle$ + perm of indices
A–37	$\langle F^L_{\mu\nu} D^\mu D^\nu U^\dagger U\rangle$ + perm of indices
A–38	$\langle D^\mu D_\mu D_\nu U^\dagger D^\nu U\rangle$
A–39	$\langle D^\mu D^\nu D_\nu U^\dagger D_\mu U\rangle$
A–40	$\langle D^\mu D^\nu D_\mu U^\dagger D_\nu U\rangle$
A–41	$\langle D^\mu D_\mu D^\nu D_\nu U^\dagger U\rangle$
A–42	$\langle D^\mu D^\nu D_\mu D_\nu U^\dagger U\rangle$
A–43	$\langle D^\mu D^\nu D_\nu D_\mu U^\dagger U\rangle$
A–44	$\langle D^\mu \chi^\dagger D_\mu U\rangle$
A–45	$\langle D^\mu U^\dagger D_\mu \chi\rangle$
A–46	$\langle D^\mu D_\mu \chi^\dagger U\rangle$

A–47	$\langle U^\dagger D^\mu D_\mu \chi \rangle$
A–48	$\langle D^\mu (F^R_{\mu\nu} U)^\dagger D^\nu U \rangle$ + perm of indices
A–49	$\langle D^\mu (U F^L_{\mu\nu})^\dagger D^\nu U \rangle$ + perm of indices
A–50	$\langle D^\mu (F^R_{\mu\nu} D^\nu U)^\dagger U \rangle$ + perm of indices
A–51	$\langle D^\mu (D^\nu U F^L_{\mu\nu})^\dagger U \rangle$ + perm of indices
A–52	$\langle D^\mu D^\nu (F^R_{\mu\nu} U)^\dagger U \rangle$ + perm of indices
A–53	$\langle D^\mu D^\nu (U F^L_{\mu\nu})^\dagger U \rangle$ + perm of indices

Given chirally symmetric components, the goal now is to formulate a Lagrangian that is possessed of all possible structure but that has no redundancy. This then will be the unique basis for the theory, supplying the minimal required set of (undetermined) phenomenological parameters.

3.3.3 First Reductions

This section is devoted to reducing the size of the above list by dealing with permutations of Lorentz indices.

For the $F_{\mu\nu}$ terms that have been listed above, the permutation of indices can in all cases be removed from consideration. Equation (3.16) indicates the dependence $F^{R,L}_{\mu\nu} = -F^{R,L}_{\nu\mu}$. Consider term A–19, for example:

$$\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger \rangle.$$

We consider whether the term

$$\langle F^R_{\mu\nu} D^\nu U D^\mu U^\dagger \rangle$$

— having the same form but permuted Lorentz indices— is independent. No identities can be used to commute the $D_\nu U$ and the $D_\mu U^\dagger$ back to the original form. (In particular, trace identity (3.28) does not apply.) However, switching $\mu \leftrightarrow \nu$ in this new term and then using (3.16) does reveal that there is only one independent expression. Remember that constants do not matter at this stage — only the structural form. Thus the list of terms with the field strength tensor always in the form $F_{\mu\nu}$ is the complete list, without permutation of the indices.

We can use the commutator

$$[D_\mu, D_\nu]U \tag{3.32}$$

that defines the field strength tensor as a constraint for expressions having multiple derivatives. Equation (3.11) yields directly

$$D_\nu D_\mu U \quad = \quad D_\mu D_\nu U + iF^R_{\mu\nu}U - iUF^L_{\mu\nu}. \tag{3.33}$$

This means all double derivatives of a matrix U can be written $D_\mu D_\nu U$ — if we have accounted for all $F^{R,L}_{\mu\nu}$ expressions possible. Equation (3.33) eliminates from consideration the permutations of indices in expressions A–27, A–29, and A–31. It can be seen that these three terms are the only ones for which the dependence of $D_\nu D_\mu U$ on $D_\mu D_\nu U$ must be used to show the uniqueness of the expression given. In other cases where switching one of the pairs $\mu \leftrightarrow \nu$ changes the form of the expression — for example, A–6 and A–8 — the new structure can either be restored to the original form through (3.27) or is identical to an already listed structure.

This leads next to the question of permutations of triple or quadruple derivative indices. Identity (3.33) has one further application in reducing the number of triple derivative forms. (Quadruple derivatives will be discussed later, and are to be eliminated entirely.) Equation (3.33) is applied to one of the triple derivatives to swap the first two indices. Remember that the covariant derivative is simply an operator acting on its chiral matrix argument, so what we are doing here amounts to substituting $D^\nu U$ for U

in equation (3.33). The result of this is

$$D_\nu D_\mu D^\nu U \;=\; D_\mu D_\nu D^\nu U + iF^R_{\mu\nu} D^\nu U - iD^\nu U F^L_{\mu\nu}. \tag{3.34}$$

This eliminates invariant A–40 and leaves only two triple derivative forms that are independent of each other or of other structures.

It can be seen that swapping the *second* two indices of a triple derivative by applying equation (3.33) will yield terms of the form

$$D^\nu (F^R_{\mu\nu} U). \tag{3.35}$$

However, these terms and the two triple derivatives cannot both be eliminated. Later, all terms having the form of derivatives of tensors $F^{R,L}_{\mu\nu}$ will be eliminated in favour of the triple derivatives and other forms.

Finally, a special consideration: The A–8 term

$$\langle D^\mu U^\dagger D^\nu U D_\mu U^\dagger D_\nu U \rangle$$

is not independent of other structures. There is an $SU(3)$ identity [16]

$$\sum_{6\ \text{perm.'s}} \langle A_1 A_2 A_3 A_4 \rangle \;=\; \sum_{3\ \text{perm.'s}} \langle A_1 A_2 \rangle \langle A_3 A_4 \rangle \tag{3.36}$$

which holds for any 3×3 traceless matrices A_i. Since we are working in $SU(3)$, we may use this. (In the $SU(2)$ case, there are even more reductions that can be made [4].)

$D^\mu U^\dagger U$ is traceless, as we have shown, so we substitute it and its adjoint $U^\dagger D^\mu U$ as the matrices in the above identity — taking permutations of Lorentz indices. This yields

$$\begin{aligned} &4\,\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U \rangle + 2\,\langle D^\mu U^\dagger D^\nu U D_\mu U^\dagger D_\nu U \rangle \\ &= \; 2\,\langle D^\mu U^\dagger D^\nu U \rangle \langle D_\mu U^\dagger D_\nu U \rangle + \langle D^\mu U^\dagger D_\mu U \rangle \langle D^\nu U^\dagger D_\nu U \rangle. \end{aligned} \tag{3.37}$$

A–8, then, can be dropped in favour of A–5, A–6 and A–7.

3.3.4 Parity

We group the terms now to create parity invariants, because our final Lagrangian must satisfy this symmetry.

The parity transformation P acts as follows on the chiral matrix U and on the external currents and densities [17, 18]:

$$
\begin{aligned}
U(t,\boldsymbol{x}) &\rightarrow U^{\dagger}(t,-\boldsymbol{x}) \\
\partial_{\mu}U(t,\boldsymbol{x}) &\rightarrow \partial^{\mu}U^{\dagger}(t,-\boldsymbol{x}) \\
v_{\mu}(t,\boldsymbol{x}) &\rightarrow v^{\mu}(t,-\boldsymbol{x}) \\
a_{\mu}(t,\boldsymbol{x}) &\rightarrow -a^{\mu}(t,-\boldsymbol{x}) \\
s(t,\boldsymbol{x}) &\rightarrow s(t,-\boldsymbol{x}) \\
p(t,\boldsymbol{x}) &\rightarrow -p(t,-\boldsymbol{x}).
\end{aligned}
\tag{3.38}
$$

Using these, one establishes that

$$
\begin{aligned}
\chi &\rightarrow \chi^{\dagger} \\
D_{\mu}U &\rightarrow \partial^{\mu}U^{\dagger} - i(v^{\mu}-a^{\mu})U^{\dagger} + iU^{\dagger}(v^{\mu}+a^{\mu}) \\
&= D^{\mu}U^{\dagger} \\
F^{R}_{\mu\nu} &\rightarrow \partial^{\mu}(v^{\nu}-a^{\nu}) - \partial^{\nu}(v^{\mu}-a^{\mu}) - i[(v^{\mu}-a^{\mu}),(v^{\nu}-a^{\nu})] \\
&= F^{L\mu\nu}
\end{aligned}
\tag{3.39}
$$

where the double covariant derivative transformation is the same as that for the single covariant derivative, and likewise, the transformation of $D_{\mu}\chi$ is the same as that of $D_{\mu}U$. It is understood that the space coordinates on the right hand side are reversed in sign compared to those on the left hand side.

Taking transformations (3.38) and (3.39) into account, it is easy to construct structures which remain invariant under P and yield the transformation $\mathcal{L}(t, \boldsymbol{x}) \rightarrow \mathcal{L}(t, -\boldsymbol{x})$. Consider, for example, applying the parity operation to the term A–6. By the relation (3.28) that we derived, each trace expression individually obeys

$$\begin{aligned}\langle D_\mu U^\dagger D_\nu U\rangle &\rightarrow \langle D^\mu U D^\nu U^\dagger\rangle \\ &= \langle D^\mu U^\dagger D^\nu U\rangle,\end{aligned}$$

so the two expressions together, with their Lorentz indices contracted, form an invariant under P. Term A–7 is an invariant by similar reasoning. We see that term A–1 and terms A–5 through A–8 are already parity invariants as written.

Double derivative expressions having well-defined parity must not be elementary like the above. We have

$$\langle D_\mu D_\nu U^\dagger U\rangle \rightarrow \langle D^\mu D^\nu U U^\dagger\rangle,$$

the two sides of which are not equal but are related by equation (3.29). The simplest way to form a parity invariant is just to take the sum or difference of the two terms. The expression

$$\langle D_\mu D_\nu U^\dagger U + D_\mu D_\nu U U^\dagger\rangle \tag{3.40}$$

has $P = +1$, while

$$\langle D_\mu D_\nu U^\dagger U - D_\mu D_\nu U U^\dagger\rangle \tag{3.41}$$

has $P = -1$. The square of either of these has the required even parity and can appear as a term in the $O(p^4)$ Lagrangian.

Notice how the expression A–31, by contrast, is a parity invariant as written. Only the cyclic property of the trace is invoked to show that

$$\langle D^\mu D^\nu U^\dagger D_\mu D_\nu U\rangle \rightarrow \langle D^\mu D^\nu U^\dagger D_\mu D_\nu U\rangle. \tag{3.42}$$

Let us group the remaining List A expressions into $P = 1$ parity invariants.

It is straightforward to form all of the possible even parity expressions. For example, in the triple or quadruple derivative cases, the expression is the sum of two hermitean conjugate terms as in (3.40).

As was the case for expression (3.41), a $P = -1$ expression can be of use to us, if it is multiplied by another $P = -1$ expression. In practice, this means that only the few $O(p^2)$ expressions having odd parity need be considered. No other combinations would be possible, as we know, since there are no $O(p)$ chiral invariants to work with. The only chiral invariants having $P = -1$ that need be considered are expression (3.41), which because of its Lorentz indices can only appear multiplied by itself, and these two expressions:

$$\langle \chi^\dagger U - \chi U^\dagger \rangle \tag{3.43}$$

$$\langle D^\mu D_\mu U^\dagger U - D^\mu D_\mu U U^\dagger \rangle \tag{3.44}$$

which can be taken in their various combinations. Four possibilities are thus generated.

Aside from the above, there is another way to form even parity structures using odd parity components. This is done using the fully antisymmetric $\epsilon^{\mu\nu\lambda\rho}$ tensor, which acts like an object with intrinsic negative parity. Several structures can be created which involve a relative minus sign but do not vanish, and are indexed by four different Lorentz indices, unlike the structures shown in List A. Incorporating the tensor, these transform with $P = +1$.

3.3.5 Parity Invariant List

Having applied the reductions of section 3.3.3, and then having used the transformation properties (3.38) and (3.39) to group the remaining components appropriately, we compile the following list. The structures are invariant under Lorentz transformations,

$SU(3)_R \times SU(3)_L$ transformations, and parity. The first three are the $O(p^2)$ terms.

B–1	$\langle D^\mu U^\dagger D_\mu U\rangle$
B–2	$\langle D^\mu D_\mu U^\dagger U + D^\mu D_\mu U U^\dagger\rangle$
B–3	$\langle \chi^\dagger U + \chi U^\dagger\rangle$
B–4	$\langle D^\mu U^\dagger D_\mu U\rangle\langle D^\nu U^\dagger D_\nu U\rangle$
B–5	$\langle D^\mu U^\dagger D^\nu U\rangle\langle D_\mu U^\dagger D_\nu U\rangle$
B–6	$\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle$
B–7	$\langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle$
B–8	$\langle D^\mu U^\dagger D_\mu U\,(\chi^\dagger U + U^\dagger \chi)\rangle$
B–9	$\langle \chi^\dagger U + \chi U^\dagger\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle$
B–10	$\langle \chi^\dagger U - \chi U^\dagger\rangle\langle \chi^\dagger U - \chi U^\dagger\rangle$
B–11	$\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle$
B–12	$\langle \chi^\dagger \chi\rangle$
B–13	$i\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle$
B–14	$\langle F^{R\mu\nu} U F^L_{\mu\nu} U^\dagger\rangle$
B–15	$\langle F^{R\mu\nu} F^R_{\mu\nu} + F^{L\mu\nu} F^L_{\mu\nu}\rangle$
B–16	$\langle D^\mu D_\mu U^\dagger U + D^\mu D_\mu U U^\dagger\rangle\langle D^\nu U^\dagger D_\nu U\rangle$
B–17	$\langle D^\mu D^\nu U^\dagger U + D^\mu D^\nu U U^\dagger\rangle\langle D_\mu U^\dagger D_\nu U\rangle$
B–18	$\langle (D^\mu D_\mu U^\dagger U + U^\dagger D^\mu D_\mu U)\, D^\nu U^\dagger D_\nu U\rangle$
B–19	$\langle (D^\mu D^\nu U^\dagger U + U^\dagger D^\mu D^\nu U)\, D_\mu U^\dagger D_\nu U\rangle$
B–20	$\langle D^\mu D_\mu U^\dagger U + D^\mu D_\mu U U^\dagger\rangle\langle D^\nu D_\nu U^\dagger U + D^\nu D_\nu U U^\dagger\rangle$

B–21	$\langle D^\mu D_\mu U^\dagger U - D^\mu D_\mu U U^\dagger \rangle \langle D^\nu D_\nu U^\dagger U - D^\nu D_\nu U U^\dagger \rangle$
B–22	$\langle D^\mu D^\nu U^\dagger U + D^\mu D^\nu U U^\dagger \rangle \langle D_\mu D_\nu U^\dagger U + D_\mu D_\nu U U^\dagger \rangle$
B–23	$\langle D^\mu D^\nu U^\dagger U - D^\mu D^\nu U U^\dagger \rangle \langle D_\mu D_\nu U^\dagger U - D_\mu D_\nu U U^\dagger \rangle$
B–24	$\langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle$
B–25	$\langle D^\mu D^\nu U^\dagger D_\mu D_\nu U \rangle$
B–26	$\langle D^\mu D_\mu U^\dagger U + D^\mu D_\mu U U^\dagger \rangle \langle \chi^\dagger U + \chi U^\dagger \rangle$
B–27	$\langle D^\mu D_\mu U^\dagger U - D^\mu D_\mu U U^\dagger \rangle \langle \chi^\dagger U - \chi U^\dagger \rangle$
B–28	$\langle D^\mu D_\mu U^\dagger \chi + D^\mu D_\mu U \chi^\dagger \rangle$
B–29	$i\langle F^R_{\mu\nu} D^\mu D^\nu U U^\dagger + F^L_{\mu\nu} D^\mu D^\nu U^\dagger U \rangle$
B–30	$\langle D^\mu D_\mu D_\nu U^\dagger D^\nu U + D^\mu D_\mu D_\nu U D^\nu U^\dagger \rangle$
B–31	$\langle D^\mu D^\nu D_\nu U^\dagger D_\mu U + D^\mu D^\nu D_\nu U D_\mu U^\dagger \rangle$
B–32	$\langle D^\mu D_\mu D^\nu D_\nu U^\dagger U + D^\mu D_\mu D^\nu D_\nu U U^\dagger \rangle$
B–33	$\langle D^\mu D^\nu D_\mu D_\nu U^\dagger U + D^\mu D^\nu D_\mu D_\nu U U^\dagger \rangle$
B–34	$\langle D^\mu D^\nu D_\nu D_\mu U^\dagger U + D^\mu D^\nu D_\nu D_\mu U U^\dagger \rangle$
B–35	$\langle D^\mu \chi^\dagger D_\mu U + D^\mu \chi D_\mu U^\dagger \rangle$
B–36	$\langle D^\mu D_\mu \chi^\dagger U + D^\mu D_\mu \chi U^\dagger \rangle$
B–37	$i\langle D^\mu (F^R_{\mu\nu} U)^\dagger D^\nu U + D^\mu (U F^L_{\mu\nu}) D^\nu U^\dagger \rangle$
B–38	$i\langle D^\mu (F^R_{\mu\nu} U) D^\nu U^\dagger + D^\mu (U F^L_{\mu\nu})^\dagger D^\nu U \rangle$
B–39	$i\langle D^\mu (F^R_{\mu\nu} D^\nu U)^\dagger U + D^\mu (D^\nu U F^L_{\mu\nu}) U^\dagger \rangle$
B–40	$i\langle D^\mu (F^R_{\mu\nu} D^\nu U) U^\dagger + D^\mu (D^\nu U F^L_{\mu\nu})^\dagger U \rangle$
B–41	$i\langle D^\mu D^\nu (F^R_{\mu\nu} U)^\dagger U + D^\mu D^\nu (U F^L_{\mu\nu}) U^\dagger \rangle$

B–42	$i\langle D^\mu D^\nu (F^R_{\mu\nu} U) U^\dagger + D^\mu D^\nu (U F^L_{\mu\nu})^\dagger U\rangle$
B–43	$i\epsilon^{\mu\nu\lambda\rho}\langle F^R_{\mu\nu} D_\lambda U D_\rho U^\dagger - F^L_{\mu\nu} D_\lambda U^\dagger D_\rho U\rangle$
B–44	$\epsilon^{\mu\nu\lambda\rho}\langle F^R_{\mu\nu} F^R_{\lambda\rho} - F^L_{\mu\nu} F^L_{\lambda\rho}\rangle$
B–45	$\epsilon^{\mu\nu\lambda\rho}\langle (D_\mu D_\nu U^\dagger U - U^\dagger D_\mu D_\nu U)\, D_\lambda U^\dagger D_\rho U\rangle$
B–46	$i\epsilon^{\mu\nu\lambda\rho}\langle F^R_{\mu\nu} D_\lambda D_\rho U U^\dagger - F^L_{\mu\nu} D_\lambda D_\rho U^\dagger U\rangle$
B–47	$i\epsilon^{\mu\nu\lambda\rho}\langle D_\mu (F^R_{\lambda\rho} U)^\dagger D_\nu U - D_\mu (U F^L_{\lambda\rho}) D_\nu U^\dagger\rangle$
B–48	$i\epsilon^{\mu\nu\lambda\rho}\langle D_\mu (F^R_{\lambda\rho} U) D_\nu U^\dagger - D_\mu (U F^L_{\lambda\rho})^\dagger D_\nu U\rangle$
B–49	$i\epsilon^{\mu\nu\lambda\rho}\langle D_\mu (F^R_{\lambda\rho} D_\nu U)^\dagger U - D_\mu (D_\nu U F^L_{\lambda\rho}) U^\dagger\rangle$
B–50	$i\epsilon^{\mu\nu\lambda\rho}\langle D_\mu (F^R_{\lambda\rho} D_\nu U) U^\dagger - D_\mu (D_\nu U F^L_{\lambda\rho})^\dagger U\rangle$
B–51	$i\epsilon^{\mu\nu\lambda\rho}\langle D_\mu D_\nu (F^R_{\lambda\rho} U)^\dagger U - D_\mu D_\nu (U F^L_{\lambda\rho}) U^\dagger\rangle$
B–52	$i\epsilon^{\mu\nu\lambda\rho}\langle D_\mu D_\nu (F^R_{\lambda\rho} U) U^\dagger - D_\mu D_\nu (U F^L_{\lambda\rho})^\dagger U\rangle$

Note that B–9 and B–10 include the three chiral invariants A–13, A–14 and A–15. (While the cross term $2\langle\chi^\dagger U\rangle\langle U^\dagger \chi\rangle$ would appear to cancel, it must be remembered that in the actual Lagrangian the two terms B–9 and B–10 will be separate, each appearing with a different coefficient. The cross-term structure will be expressed as a function of the two coefficients.)

The i is included in the $F^{R,L}_{\mu\nu}$ terms not for reasons of parity invariance but simply because of the tensors' relationship to the other objects, which involves an i (see definition (3.11)).

3.3.6 Charge Conjugation

The symmetry C must yet be verified for the members of List B. C invariance is invariance under the transformations [17, 19]

$$\begin{aligned}
U &\rightarrow U^T \\
\partial_\mu U &\rightarrow \partial_\mu U^T \\
v_\mu &\rightarrow -v_\mu^T \\
a_\mu &\rightarrow a_\mu^T \\
s &\rightarrow s^T \\
p &\rightarrow p^T. \qquad (3.45)
\end{aligned}$$

The above transformation properties lead to

$$\begin{aligned}
\chi &\rightarrow \chi^T \\
D_\mu U &\rightarrow \partial_\mu U^T + i(v_\mu - a_\mu)^T U^T - iU^T(v_\mu + a_\mu)^T \\
&= D_\mu U^T \\
F_{\mu\nu}^R &\rightarrow -\partial_\mu(v_\nu - a_\nu)^T + \partial_\nu(v_\mu - a_\mu)^T - i[(v_\mu - a_\mu)^T, (v_\nu - a_\nu)^T] \\
&= -F_{\mu\nu}^{L\,T}. \qquad (3.46)
\end{aligned}$$

Again, the transformation shown for the covariant derivative carries over to multiple covariant derivatives and derivatives of χ.

All objects upon which C acts appear within a trace. Clearly, $\langle A^T \rangle = \langle A \rangle$, so one can see why C invariance might hold. In most cases in the above list, the structures within the trace are bilinears, for which

$$\langle A^T B^T \rangle = \langle (BA)^T \rangle = \langle BA \rangle = \langle AB \rangle,$$

and the invariance under transposing each matrix is manifest; however, in the other cases, writing the result as one overall transpose will reverse the order of the component matrices, and one must verify that the structure can be restored to its original form. This relationship may not be transparent if there are μ and ν indices that need to be swapped, but one can apply one of the identities that have been given — for example, equation (3.16) or equation (3.33). Consider term B–19, for example, under charge conjugation:

$$\begin{aligned}\langle (D^\mu D^\nu U^\dagger U + U^\dagger D^\mu D^\nu U)\, D_\mu U^\dagger D_\nu U\rangle &\rightarrow \langle (U D^\mu D^\nu U^\dagger + D^\mu D^\nu U U^\dagger)\, D_\nu U D_\mu U^\dagger\rangle \\ &= \langle (D^\mu D^\nu U^\dagger U + U^\dagger D^\mu D^\nu U)\, D_\nu U^\dagger D_\mu U\rangle\end{aligned}$$

where the reversal of order of the terms has come from taking the transpose of the whole expression, then cyclically permuting. It is not immediately apparent that the expressions on the two sides have the same form. Relation (3.33) must be applied to swap the indices within the double derivatives, accompanied by the change $\mu \leftrightarrow \nu$. Extra terms are brought in by that relation, but it happens that these precisely cancel, leaving the right hand side equal to the left hand side.

One easily verifies that all terms listed are invariant under C except for terms B–43 to B–52 containing the $\epsilon^{\mu\nu\lambda\rho}$ tensor. All of these terms do not satisfy the symmetry, so they must be dropped.

The chiral Lagrangian formalism we derive incorporates the C and P symmetries separately, as required.

(With regard to these symmetries, a point should be noted. The formalism with which we are dealing actually possesses *two* discrete symmetries: the 'naive parity' $(t, \boldsymbol{x}) \leftrightarrow (t, -\boldsymbol{x})$ and $U \leftrightarrow U^\dagger$, each individually. QCD itself possesses only the 'true parity' P that we have defined, which is the composition of the two. Thus, to match QCD the effective theory must break these symmetries in such a way that the two do not hold separately, but only in the form of P. This is accomplished through the inclusion of a so-called anomalous term in the Lagrangian, the Wess-Zumino-Witten anomaly [18]. The addition comprises a separate sector of the theory, which will not be relevant for present purposes.)

List B still contains many redundant terms, which we uncover in the following three sections.

3.3.7 Derivative Term Constraints

Two identities which are most instrumental in reducing the size of our list are

$$D_\mu A^\dagger B + A^\dagger D_\mu B \;=\; \partial_\mu(A^\dagger B) - i[l_\mu, A^\dagger B] \tag{3.47}$$

and its hermitean conjugate

$$D_\mu B^\dagger A + B^\dagger D_\mu A \;=\; \partial_\mu(B^\dagger A) - i[l_\mu, B^\dagger A] \tag{3.48}$$

which hold for general chiral matrices A and B, and are derived using the definitions of the covariant derivatives (3.5) and (3.6).

If we choose the matrices

$$A \;=\; D_\nu U \qquad\qquad B \;=\; U$$

then we can take advantage of equation (3.27):

$$D_\nu U^\dagger U \;=\; -U^\dagger D_\nu U$$

to rewrite $A^\dagger B$ as $-B^\dagger A$, and this will allow us to eliminate the total divergence and the commutator parts of equations (3.47) and (3.48) by adding the two equations. The result of adding the equations is this identity relating double derivatives to single derivatives:

$$D_\mu D_\nu U^\dagger U + U^\dagger D_\mu D_\nu U \;=\; -D_\mu U^\dagger D_\nu U - D_\nu U^\dagger D_\mu U. \tag{3.49}$$

This is just equation (3.29). The identity is a relation between parity invariants, and can be seen to immediately eliminate many of the double derivative terms in the B list in favour of single derivative terms.

Subtracting equations (3.47) and (3.48) after having substituted the same matrices A and B yields

$$\begin{aligned} D_\mu D_\nu U^\dagger U - U^\dagger D_\mu D_\nu U &= D_\mu U^\dagger D_\nu U - D_\nu U^\dagger D_\mu U \\ &\quad + 2\partial_\mu (D_\nu U^\dagger U) - 2i[l_\mu, D_\nu U^\dagger U]. \end{aligned} \tag{3.50}$$

This can be turned into a useful identity if we resort to taking its trace. First, by equation (3.30) (trace of a single covariant derivative expression), the total divergence part vanishes. The commutator also vanishes under the trace. The identity resulting from these considerations is

$$\begin{aligned} \langle D_\mu D_\nu U^\dagger U - U^\dagger D_\mu D_\nu U \rangle &= \langle D_\mu U^\dagger D_\nu U - D_\nu U^\dagger D_\mu U \rangle \\ &= \langle D_\mu U^\dagger D_\nu U - D_\nu U D_\mu U^\dagger \rangle \\ &= 0. \end{aligned} \tag{3.51}$$

Whereas equation (3.49) is parity even, this is parity odd. The two identities allow the elimination of terms B–2, B–16 through B–23, B–26 and B–27.

Extending the use of the fundamental equations (3.47) and (3.48), we will be able to eliminate all quadruple derivative terms by showing their dependence on triple and double derivative terms. Taking the trace to eliminate the commutators, the sum or difference of these two equations is

$$\langle D_\mu A^\dagger B \pm D_\mu A B^\dagger \rangle + \langle A^\dagger D_\mu B \pm A D_\mu B^\dagger \rangle = \partial_\mu \langle A^\dagger B \pm B^\dagger A \rangle. \tag{3.52}$$

The difference terms are parity odd, and it turns out that they will not lead to any new constraint that we need.

The relation (3.52) will be used in its even parity form to generate constraints among the following list of parity invariants:

$$
\begin{aligned}
I^1_{\mu\nu} &\equiv D_\mu U^\dagger D_\nu U + D_\mu U D_\nu U^\dagger \\
I^2_{\mu\nu} &\equiv D_\mu D_\nu U^\dagger U + D_\mu D_\nu U U^\dagger \\
J^2_{\mu\nu\lambda} &\equiv D_\mu D_\nu U^\dagger D_\lambda U + D_\mu D_\nu U D_\lambda U^\dagger \\
J^3_{\mu\nu\lambda} &\equiv D_\mu D_\nu D_\lambda U^\dagger U + D_\mu D_\nu D_\lambda U U^\dagger \\
K^2_{\mu\nu\lambda\rho} &\equiv D_\mu D_\nu U^\dagger D_\lambda D_\rho U + D_\mu D_\nu U D_\lambda D_\rho U^\dagger \\
K^3_{\mu\nu\lambda\rho} &\equiv D_\mu D_\nu D_\lambda U^\dagger D_\rho U + D_\mu D_\nu D_\lambda U D_\rho U^\dagger \\
K^4_{\mu\nu\lambda\rho} &\equiv D_\mu D_\nu D_\lambda D_\rho U^\dagger U + D_\mu D_\nu D_\lambda D_\rho U U^\dagger .
\end{aligned}
\tag{3.53}
$$

Taking all possible combinations of single, double and triple derivatives of U for A and B and plugging them into (3.52), one finds

$$\langle I^2_{\mu\nu}\rangle + \langle I^1_{\mu\nu}\rangle = 0 \tag{3.54}$$

$$\langle J^2_{\mu\nu\lambda}\rangle + \langle J^2_{\mu\lambda\nu}\rangle = \partial_\mu \langle I^1_{\nu\lambda}\rangle \tag{3.55}$$

$$\langle J^3_{\mu\nu\lambda}\rangle + \langle J^2_{\nu\lambda\mu}\rangle = \partial_\mu \langle I^2_{\nu\lambda}\rangle \tag{3.56}$$

$$\langle K^3_{\mu\nu\lambda\rho}\rangle + \langle K^2_{\mu\rho\nu\lambda}\rangle = \partial_\mu \langle J^2_{\nu\lambda\rho}\rangle \tag{3.57}$$

$$\langle K^4_{\mu\nu\lambda\rho}\rangle + \langle K^3_{\nu\lambda\rho\mu}\rangle = \partial_\mu \langle J^3_{\nu\lambda\rho}\rangle . \tag{3.58}$$

We can use this system of equations to show that $\langle K^4_{\mu\nu\lambda\rho}\rangle$, a general quadruple derivative expression, is not independent of other quantities that appear in the Lagrangian. To do this, we use (3.54), (3.55) and (3.56) to express $\langle J^3\rangle$ in terms of $\langle J^2\rangle$ (i.e. a sum of three terms with permuted indices, with no total divergence). The derivative of this expression then relates (3.57) and (3.58), leaving the $\langle K^4\rangle$ structure expressed only in terms of $\langle K^3\rangle$

and $\langle K^2 \rangle$ structures:

$$\langle K^4_{\mu\nu\lambda\rho} \rangle = -\langle K^3_{\mu\nu\lambda\rho} + K^3_{\mu\nu\rho\lambda} + K^3_{\mu\lambda\rho\nu} + K^3_{\nu\lambda\rho\mu} \rangle - \langle K^2_{\mu\nu\lambda\rho} + K^2_{\mu\lambda\nu\rho} + K^2_{\mu\rho\nu\lambda} \rangle. \quad (3.59)$$

Since we have accounted for all possible triple and double derivative expressions, we will include no quadruple derivatives in the Lagrangian.

3.3.8 $F_{\mu\nu}$ Constraints

Next, the terms involving derivatives of the field strength tensor are addressed.

Consider first the difference of terms B–37 and B–38. Using the definitions of the field strength tensors (3.11) and (3.13), and making use of the notation introduced above, we can show that the difference consists of only triple derivative terms:

$$\begin{aligned}
& i\langle D^\mu (F^R_{\mu\nu} U)^\dagger D^\nu U + D^\mu (U F^L_{\mu\nu}) D^\nu U^\dagger \rangle - i\langle D^\mu (F^R_{\mu\nu} U) D^\nu U^\dagger + D^\mu (U F^L_{\mu\nu})^\dagger D^\nu U \rangle \\
&= i\langle D^\mu (F_{\mu\nu} U)^\dagger D^\nu U - D^\mu (F_{\mu\nu} U) D^\nu U^\dagger \rangle \\
&= \langle (D^\mu D_\mu D_\nu U - D^\mu D_\nu D_\mu U) D^\nu U^\dagger + (D^\mu D_\mu D_\nu U^\dagger - D^\mu D_\nu D_\mu U^\dagger) D^\nu U \rangle \\
&= \langle K^{3\mu}{}_{\mu\nu}{}^{\nu} \rangle - \langle K^{3\mu}{}_{\nu\mu}{}^{\nu} \rangle. \qquad (3.60)
\end{aligned}$$

The sum of terms B–37 and B–38 is a second, independent constraint. First, we have

$$i\langle D^\mu (F^R_{\mu\nu} U)^\dagger D^\nu U \rangle = -i\langle D^\mu (F^R_{\mu\nu} U) D^\nu U^\dagger \rangle \qquad (3.61)$$

(and the same for the expression with $F^L_{\mu\nu}$). These relations can be proven by explicitly expanding out and comparing the covariant derivatives of the tensors using equations (3.5) and (3.6) and definitions (3.12). Because of the identities, the sum of B–37 and B–38 is

$$\begin{aligned}
& i\langle D^\mu (F^R_{\mu\nu} U)^\dagger D^\nu U + D^\mu (U F^L_{\mu\nu}) D^\nu U^\dagger \rangle + i\langle D^\mu (F^R_{\mu\nu} U) D^\nu U^\dagger + D^\mu (U F^L_{\mu\nu})^\dagger D^\nu U \rangle \\
&= 0. \qquad (3.62)
\end{aligned}$$

The two constraints (3.60) and (3.62) eliminate B–37 and B–38 in favour of other included terms.

Taking the sum and difference of terms B–39 and B–40 yields an analogous result, as it does also for terms B–41 and B–42. The sum of the terms again vanishes in both these cases. The difference relation in the first instance is as follows. One can just substitute $D^\nu U$ in the definition (3.11) of the field strength tensor and carry this quantity through as in (3.60). The result is

$$\begin{aligned} i\langle D^\mu(F^R_{\mu\nu}D^\nu U)^\dagger U + D^\mu(D^\nu U F^L_{\mu\nu})U^\dagger\rangle - i\langle D^\mu(F^R_{\mu\nu}D^\nu U)U^\dagger + D^\mu(D^\nu U F^L_{\mu\nu})^\dagger U\rangle \\ = \langle D^\mu[D_\mu, D_\nu]D^\nu U^\dagger U + D^\mu[D_\mu, D_\nu]D^\nu U U^\dagger\rangle \\ = \langle K^{4\mu}{}_\mu{}^\nu{}_\nu\rangle - \langle K^{4\mu\nu}{}_{\mu\nu}\rangle. \end{aligned} \tag{3.63}$$

Similarly, for the double derivative terms B–41 and B–42 one obtains

$$\begin{aligned} i\langle D^\mu D^\nu(F^R_{\mu\nu}U)^\dagger U + D^\mu D^\nu(U F^L_{\mu\nu})U^\dagger\rangle - i\langle D^\mu D^\nu(F^R_{\mu\nu}U)U^\dagger + D^\mu D^\nu(U F^L_{\mu\nu})^\dagger U\rangle \\ = \langle K^{4\mu\nu}{}_{\mu\nu}\rangle - \langle K^{4\mu\nu}{}_{\nu\mu}\rangle. \end{aligned} \tag{3.64}$$

Because of the above constraints, no derivatives of field strength tensors will appear in the Lagrangian.

Finally, we show that the term B–29 can be related to other terms. Observe, by the definitions of the field strength tensors, that the expression

$$\langle [D^\mu, D^\nu]U^\dagger[D_\mu, D_\nu]U\rangle$$

is equivalent to terms with only the tensors, and it is also related to terms mixing the tensors with double derivatives. Expanding it to show this latter first, we obtain

$$\begin{aligned} \langle [D^\mu, D^\nu]U^\dagger[D_\mu, D_\nu]U\rangle &= 2\langle D^\mu D^\nu U^\dagger D_\mu D_\nu U - D^\mu D^\nu U^\dagger D_\nu D_\mu U\rangle \\ &= 2\langle D^\mu D^\nu U^\dagger[D_\mu, D_\nu]U\rangle \end{aligned}$$

$$
\begin{aligned}
&= 2i\langle D^\mu D^\nu U^\dagger(-F^R_{\mu\nu}U + UF^L_{\mu\nu})\rangle \\
&= 2i\langle F^R_{\mu\nu}D^\mu D^\nu UU^\dagger + F^L_{\mu\nu}D^\mu D^\nu U^\dagger U\rangle \\
&\quad + 2i\langle F^R_{\mu\nu}D^\mu U D^\nu U^\dagger + F^R_{\mu\nu}D^\nu U D^\mu U^\dagger\rangle \\
&= 2i\langle F^R_{\mu\nu}D^\mu D^\nu UU^\dagger + F^L_{\mu\nu}D^\mu D^\nu U^\dagger U\rangle,
\end{aligned}
\tag{3.65}
$$

having used (3.49) and then property (3.16), the antisymmetry of $F^R_{\mu\nu}$. Alternatively, we have by direct substitution (equations (3.11) and (3.13)):

$$
\begin{aligned}
\langle [D^\mu, D^\nu]U^\dagger[D_\mu, D_\nu]U\rangle &= \langle (U^\dagger F^{R\mu\nu} - F^{L\mu\nu}U^\dagger)(F^R_{\mu\nu}U - UF^L_{\mu\nu})\rangle \\
&= \langle F^{R\mu\nu}F^R_{\mu\nu}\rangle + \langle F^{L\mu\nu}F^L_{\mu\nu}\rangle - 2\langle F^{R\mu\nu}UF^L_{\mu\nu}U^\dagger\rangle.
\end{aligned}
\tag{3.66}
$$

This shows that B–29 is dependent on other invariants and can be eliminated.

3.3.9 Elimination of a Total Divergence

In this section we discuss the terms that are related to other included terms to within a total divergence $\partial_\mu F^\mu(x)$. Dropping the total divergence eliminates these terms as independent structures.

In the Hamilton-Lagrange formalism, it is a well-known fact that the equation of motion derived from a Lagrangian is unchanged by the inclusion of any term that is a total divergence — because this term automatically vanishes under the partial integration performed when minimizing the action. This rule carries over from classical mechanics to quantum field theory.

For the purposes of the present calculation, however, we cannot depend on the invariance of the equation of motion, since we have specifically eliminated reference to the equation of motion. More generally, we must be sure that a calculated S-matrix element is unchanged when the underlying Lagrangian is changed by a term that is a total divergence. Here we outline a proof of this. Consider the general operator

$$\partial_\mu F^\mu(x)$$

appearing in the Lagrangian and contributing to the S-matrix. We invoke the property of translational invariance of any operator, where a translation is represented by

$$F^\mu(x) \;=\; e^{iP\cdot x}\, F^\mu(0)\, e^{-iP\cdot x}. \tag{3.67}$$

In the derivation of the Feynman rule for the vertex, one calculates the S-matrix element for $\partial_\mu F^\mu(x)$, translating from the origin. One will find

$$\begin{aligned} \langle f|\; i\!\int d^4x\;\partial_\mu\left(e^{iP\cdot x}\,F^\mu(0)\,e^{-iP\cdot x}\right)\,|i\rangle \;&=\; i\!\int d^4x\;e^{i(p_f-p_i)\cdot x}\,(p_f\,-\,p_i)_\mu\,\langle f|\,F^\mu(0)\,|i\rangle \\ &=\; i\,(2\pi)^4\,\delta^4(p_f-p_i)\;(p_f\,-\,p_i)_\mu\,\langle f|\,F^\mu(0)\,|i\rangle \\ &=\; 0. \end{aligned} \tag{3.68}$$

The action of the total momentum operators has left the integral of an exponential, which is equal to the delta function shown. Thus, the Feynman rule for a vertex arising from $\partial_\mu F^\mu(x)$ incorporates the difference between the total initial and final state momenta — and this difference vanishes, because of momentum conservation at any vertex. Thus, even if the equation of motion is not satisfied by the external states involved, still 4-momentum conservation at a vertex holds, and the Feynman rules are unaffected by a total divergence. Thus, we can invoke the familiar argument for the elimination of any total divergence from the Lagrangian.

Let us return to equation (3.52) relating chiral/parity invariants to total divergences. This equation was used to derive (3.57), which is of interest here. Equation (3.57) yields these relations for the triple derivative terms B–30 and B–31:

$$\langle K^{3\mu\nu}{}_{\nu\mu}\rangle \;=\; -\langle K^{2\mu}{}_{\mu}{}^{\nu}{}_{\nu}\rangle + \partial_\mu\,\langle J^{2}_{\nu}{}^{\nu\mu}\rangle \tag{3.69}$$

$$\langle K^{3\mu}{}_{\mu}{}^{\nu}{}_{\nu}\rangle \;=\; -\langle K^{2\mu\nu}{}_{\mu\nu}\rangle + \partial_\mu\,\langle J^{2\mu\nu}{}_{\nu}\rangle. \tag{3.70}$$

Dropping the total divergence, we relate the two triple derivative terms to double derivatives.

Equation (3.52) can be used just as well for the matrix χ. Substituting $A = D^\mu\chi$ and $B = U$, then $A = \chi$ and $B = D^\mu U$, we obtain, respectively,

$$\begin{aligned}
\langle D^\mu D_\mu \chi^\dagger U + D^\mu D_\mu \chi U^\dagger \rangle &= -\langle D^\mu \chi^\dagger D_\mu U + D^\mu \chi D_\mu U^\dagger \rangle \\
&\quad + \partial_\mu \langle D^\mu \chi^\dagger U + D^\mu \chi U^\dagger \rangle \\
\langle D^\mu \chi^\dagger D_\mu U + D^\mu \chi D_\mu U^\dagger \rangle &= -\langle D^\mu D_\mu U^\dagger \chi + D^\mu D_\mu U \chi^\dagger \rangle \\
&\quad + \partial_\mu \langle \chi^\dagger D^\mu U + \chi D^\mu U^\dagger \rangle
\end{aligned} \tag{3.71}$$

which relate the $D^\mu D_\mu \chi$ term to the $D^\mu\chi$ term, and this to the χ term. Only the χ term — that is, B–28 — will be left in the Lagrangian.

Finally, we eliminate the B–25 term, again using equation (3.57). We start by selecting two $\langle K^2\rangle$ structures and translating them to $\langle K^3\rangle$ structures by dropping the total derivative:

$$\begin{aligned}
\langle D^\mu D^\nu U^\dagger D_\nu D_\mu U\rangle &= \frac{1}{2}\langle D^\mu D^\nu U^\dagger D_\nu D_\mu U \,+\, D^\mu D^\nu U D_\nu D_\mu U^\dagger\rangle \\
&\rightarrow -\frac{1}{2}\langle D^\mu D^\nu D_\mu U^\dagger D_\nu U \,+\, D^\mu D^\nu D_\mu U D_\nu U^\dagger\rangle
\end{aligned} \tag{3.72}$$

$$\begin{aligned}
\langle D^\mu D_\mu U^\dagger D^\nu D_\nu U\rangle &= \frac{1}{2}\langle D^\nu D_\nu U^\dagger D^\mu D_\mu U \,+\, D^\nu D_\nu U D^\mu D_\mu U^\dagger\rangle \\
&\rightarrow -\frac{1}{2}\langle D^\nu D^\mu D_\mu U^\dagger D_\nu U \,+\, D^\nu D^\mu D_\mu U D_\nu U^\dagger\rangle
\end{aligned} \tag{3.73}$$

having swapped dummy indices to yield the desired forms that will be useful. Now if we write the two as equalities

$$\begin{aligned}
-\frac{1}{2}\langle D^\mu D^\nu D_\mu U^\dagger D_\nu U \,+\, D^\mu D^\nu D_\mu U D_\nu U^\dagger\rangle &= \langle D^\mu D^\nu U^\dagger D_\nu D_\mu U\rangle \\
-\frac{1}{2}\langle D^\nu D^\mu D_\mu U^\dagger D_\nu U \,+\, D^\nu D^\mu D_\mu U D_\nu U^\dagger\rangle &= \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U\rangle
\end{aligned}$$

and subtract them, the first two derivatives in the triple derivative expressions will form a commutator, which is of course the field strength tensor. We obtain

$$\begin{aligned}
&\frac{1}{2}\langle [D^\mu, D^\nu] D_\mu U^\dagger D_\nu U \;+\; [D^\mu, D^\nu] D_\mu U D_\nu U^\dagger \rangle \\
&= \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \;-\; \langle D^\mu D^\nu U^\dagger D_\nu D_\mu U \rangle \\
&= \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \;-\; \langle D^\mu D^\nu U^\dagger \left(D_\mu D_\nu U + i F^R_{\mu\nu} U - i U F^L_{\mu\nu} \right) \rangle
\end{aligned}$$

having applied identity (3.33), or finally,

$$\begin{aligned}
\langle D^\mu D^\nu U^\dagger D_\mu D_\nu U \rangle \;=\;& \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \\
&+ i \langle F^R_{\mu\nu} D^\mu D^\nu U U^\dagger + F^L_{\mu\nu} D^\mu D^\nu U^\dagger U \rangle \\
&+ i \langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U \rangle . \qquad (3.74)
\end{aligned}$$

(The second term on the right hand side we have related to other terms by equations (3.65) and (3.66).) This shows that expressions B–24 and B–25 are not both independent.

Having done all the other possible reductions first, we were left with five terms that could only be related to other terms to within a total divergence. This completes the reduction of our list to the set of truly independent terms.

3.4 The Lowest Order Lagrangian

As we have seen, there are only two independent terms left at $O(p^2)$. There are consequently only two undetermined parameters in leading order chiral perturbation theory, which are written as the low energy constants F_0 and B_0. The lowest order Lagrangian is written

$$\mathcal{L}_2 \;=\; \frac{F_0^2}{4} \langle D^\mu U^\dagger D_\mu U \rangle + \frac{F_0^2}{4} \langle \chi^\dagger U + \chi U^\dagger \rangle . \qquad (3.75)$$

where $\chi = 2B_0 M$ defines B_0 by relating it to the Goldstone boson masses, and F_0 is the pion decay constant. The two simply represent known observables.

3.5 The $O(p^4)$ Lagrangian

Fourteen independent invariants of $O(p^4)$ emerge from our original list. We have obtained the Lagrangian originally presented by Gasser and Leutwyler:

$$
\begin{aligned}
\mathcal{L}_4 &= L_1 \langle D^\mu U^\dagger D_\mu U\rangle^2 \\
&+ L_2 \langle D^\mu U^\dagger D^\nu U\rangle\langle D_\mu U^\dagger D_\nu U\rangle \\
&+ L_3 \langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle \\
&+ L_4 \langle D^\mu U^\dagger D_\mu U\rangle\langle \chi^\dagger U + \chi U^\dagger\rangle \\
&+ L_5 \langle D^\mu U^\dagger D_\mu U\,(\chi^\dagger U + U^\dagger \chi)\rangle \\
&+ L_6 \langle \chi^\dagger U + \chi U^\dagger\rangle^2 \\
&+ L_7 \langle \chi^\dagger U - \chi U^\dagger\rangle^2 \\
&+ L_8 \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger\rangle \\
&- i\,L_9 \langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle \\
&+ L_{10} \langle F^{R\mu\nu} U F^L_{\mu\nu} U^\dagger\rangle \\
&+ H_1 \langle F^{R\mu\nu} F^R_{\mu\nu} + F^{L\mu\nu} F^L_{\mu\nu}\rangle \\
&+ H_2 \langle \chi^\dagger \chi\rangle
\end{aligned}
\tag{3.76}
$$

and the additional terms:

$$
\begin{aligned}
\mathcal{L}'_{4\,\mathrm{off}} &= P'_1 \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U\rangle \\
&+ P'_2 \langle D^\mu D_\mu U^\dagger \chi + D^\mu D_\mu U \chi^\dagger\rangle.
\end{aligned}
\tag{3.77}
$$

We must now write the off-shell component of the Lagrangian in such a form that the standard theory is without modification in the on-shell limiting case. We can find some linear combination of structures from $\mathcal{L}_4$ and $\mathcal{L}'_{4\,\mathrm{off}}$ which vanishes when the equation of motion is satisfied.

The equation of motion derived from the lowest order Lagrangian $\mathcal{L}_2$ is [4]

$$D^\mu D_\mu U^\dagger U \;-\; U^\dagger D^\mu D_\mu U \;-\; \chi^\dagger U \;+\; U^\dagger \chi \;+\; \frac{1}{3}\langle \chi^\dagger U \;-\; U^\dagger \chi\rangle \;=\; 0. \qquad (3.78)$$

This is written in matrix form to represent the eight equations of motion for the eight pseudoscalar bosons. Notice it involves a trace, the origin of which is the constraint on the U matrix, $\det U = 1$.

To rewrite the P'_1 structure, we may use the above equation of motion in its two forms

$$\begin{aligned} D^\mu D_\mu U^\dagger U \;+\; D^\mu U^\dagger D_\mu U \;-\; \frac{1}{2}(\chi^\dagger U \;-\; U^\dagger \chi) \;+\; \frac{1}{6}\langle \chi^\dagger U \;-\; U^\dagger \chi\rangle \;&=\; 0 \\ U^\dagger D^\mu D_\mu U \;+\; D^\mu U^\dagger D_\mu U \;+\; \frac{1}{2}(\chi^\dagger U \;-\; U^\dagger \chi) \;-\; \frac{1}{6}\langle \chi^\dagger U \;-\; U^\dagger \chi\rangle \;&=\; 0 \qquad (3.79) \end{aligned}$$

which are derived by applying equation (3.29) to eliminate the respective double derivative terms. The P'_1 structure is rewritten by substituting, in place of its original derivative factors, these equations of motion. The new term, when we expand it out and write it in terms of already defined structures, becomes

$$\begin{aligned} & P_1 \langle [D^\mu D_\mu U^\dagger U \;+\; D^\mu U^\dagger D_\mu U \;-\; \frac{1}{2}(\chi^\dagger U \;-\; U^\dagger \chi) \;+\; \frac{1}{6}\langle \chi^\dagger U \;-\; U^\dagger \chi\rangle] \\ & \quad [U^\dagger D^\mu D_\mu U \;+\; D^\mu U^\dagger D_\mu U \;+\; \frac{1}{2}(\chi^\dagger U \;-\; U^\dagger \chi) \;-\; \frac{1}{6}\langle \chi^\dagger U \;-\; U^\dagger \chi\rangle]\rangle \\ & \qquad = \; P_1 \langle\, D^\mu D_\mu U^\dagger D^\nu D_\nu U \\ & \qquad\qquad + (D^\mu D_\mu U^\dagger U \;+\; U^\dagger D^\mu D_\mu U) D^\nu U^\dagger D_\nu U \\ & \qquad\qquad + D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U \\ & \qquad\qquad + \frac{1}{2}(D^\mu D_\mu U^\dagger U \;-\; U^\dagger D^\mu D_\mu U)(\chi^\dagger U \;-\; U^\dagger \chi) \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{6}(D^\mu D_\mu U^\dagger U \, - \, U^\dagger D^\mu D_\mu U) \langle \chi^\dagger U \, - \, U^\dagger \chi \rangle \\
&- \frac{1}{4}(\chi^\dagger U \, - \, U^\dagger \chi)(\chi^\dagger U \, - \, U^\dagger \chi) \\
&+ \frac{1}{6}(\chi^\dagger U \, - \, U^\dagger \chi) \langle \chi^\dagger U \, - \, U^\dagger \chi \rangle \\
&- \frac{1}{36}\langle \chi^\dagger U \, - \, U^\dagger \chi \rangle \langle \chi^\dagger U \, - \, U^\dagger \chi \rangle \; \rangle \\
= \; & P_1 \, [\, \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \\
&- \langle D^\mu D_\mu U^\dagger \chi \, + \, D^\mu D_\mu U \chi^\dagger \rangle \\
&- \langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U \rangle \\
&- \langle D^\mu U^\dagger D_\mu U \, (\chi^\dagger U + U^\dagger \chi) \, \rangle \\
&- \frac{1}{4} \, \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
&+ \frac{1}{2} \, \langle \chi^\dagger \chi \rangle \\
&+ \frac{1}{12} \, \langle \chi^\dagger U \, - \, U^\dagger \chi \rangle^2 \,]
\end{aligned}
\tag{3.80}
$$

where the particular term

$$
\langle \, - \frac{1}{36}\langle \chi^\dagger U \, - \, U^\dagger \chi \rangle \langle \chi^\dagger U \, - \, U^\dagger \chi \rangle \; \rangle
$$

consisting of only trace expressions inside the overall trace was multiplied by 3 when the trace of the identity was taken. It and the term preceding it have contributed the coefficient

$$
+\frac{1}{6} \, - \, \frac{1}{12} \;\; = \;\; \frac{1}{12}
$$

to the last term shown above. The term

$$
-\frac{1}{2} \, \langle (D^\mu D_\mu U^\dagger U \, - \, U^\dagger D^\mu D_\mu U) \, (\chi^\dagger U - U^\dagger \chi) \rangle
$$

was rewritten using this identity:

$$
\begin{aligned}
-\frac{1}{2}\langle (D^\mu D_\mu U^\dagger U - U^\dagger D^\mu D_\mu U)(\chi^\dagger U - U^\dagger \chi)\rangle &= \langle D^\mu D_\mu U^\dagger \chi + D^\mu D_\mu U \chi^\dagger \rangle \\
&+ \langle D^\mu U^\dagger D_\mu U (\chi^\dagger U + U^\dagger \chi)\rangle \quad (3.81)
\end{aligned}
$$

while the term

$$
-\frac{1}{6}\langle D^\mu D_\mu U^\dagger U - U^\dagger D^\mu D_\mu U\rangle \langle \chi^\dagger U - U^\dagger \chi\rangle
$$

vanished, according to the double-derivative identity (3.51).

The P_1 structure is designed to vanish within the standard formalism, where the equation of motion constraint is applied. We will see that quantities calculated from the structure will have the form $(p^2 - m^2)$.

Now we make the other term in $\mathcal{L}'_{4\,\mathrm{off}}$ conform in the same way. The P'_2 structure can be rewritten using identity (3.81) so that the equation of motion (3.78) can be used directly. We choose the left hand side of the identity, the term

$$
-\frac{1}{2}P''_2\langle (D^\mu D_\mu U^\dagger U - U^\dagger D^\mu D_\mu U)(\chi^\dagger U - U^\dagger \chi)\rangle \tag{3.82}
$$

as our starting point, instead of the original P'_2 structure which appears on the right hand side — the independent structure is arbitrary, so we can make this redefinition. We substitute the equation of motion (3.78) in the term (3.82) and obtain

$$
\begin{aligned}
&-\frac{1}{2}P_2\langle [D^\mu D_\mu U^\dagger U - U^\dagger D^\mu D_\mu U + U^\dagger \chi - \chi^\dagger U + \frac{1}{3}\langle \chi^\dagger U - U^\dagger \chi\rangle](\chi^\dagger U - U^\dagger \chi)\rangle \\
&\quad = -\frac{1}{2}P_2[\,\langle (D^\mu D_\mu U^\dagger U - U^\dagger D^\mu D_\mu U)(\chi^\dagger U - U^\dagger \chi)\rangle \\
&\qquad - \langle (\chi^\dagger U - U^\dagger \chi)(\chi^\dagger U - U^\dagger \chi)\rangle \\
&\qquad + \frac{1}{3}\langle \chi^\dagger U - U^\dagger \chi\rangle^2\,]
\end{aligned}
$$

$$
\begin{aligned}
= \; & P_2 [\, \langle D^\mu D_\mu U^\dagger \chi \; + \; D^\mu D_\mu U \chi^\dagger \rangle \\
& + \langle D^\mu U^\dagger D_\mu U \, (\chi^\dagger U + U^\dagger \chi) \, \rangle \\
& + \frac{1}{2} \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
& - \langle \chi^\dagger \chi \rangle \\
& - \frac{1}{6} \langle \chi^\dagger U \; - \; U^\dagger \chi \rangle^2 \,]
\end{aligned}
\tag{3.83}
$$

having applied identity (3.81) again, to write the structure as a double-derivative leading term modified by other standard terms from the Lagrangian $\mathcal{L}_4$.

Thus we arrive at this new contribution to the effective Lagrangian at $O(p^4)$:

$$
\begin{aligned}
\mathcal{L}_{4\,\mathrm{off}} \; = \; & P_1 [\, \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \\
& \quad - \langle D^\mu D_\mu U^\dagger \chi \; + \; D^\mu D_\mu U \chi^\dagger \rangle \\
& \quad - \langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U \rangle \\
& \quad - \langle D^\mu U^\dagger D_\mu U \, (\chi^\dagger U + U^\dagger \chi) \, \rangle \\
& \quad - \frac{1}{4} \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
& \quad + \frac{1}{2} \langle \chi^\dagger \chi \rangle \\
& \quad + \frac{1}{12} \langle \chi^\dagger U \; - \; U^\dagger \chi \rangle^2 \,] \\
& + \; P_2 [\, \langle D^\mu D_\mu U^\dagger \chi \; + \; D^\mu D_\mu U \chi^\dagger \rangle \\
& \quad + \langle D^\mu U^\dagger D_\mu U \, (\chi^\dagger U + U^\dagger \chi) \, \rangle \\
& \quad + \frac{1}{2} \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
& \quad - \langle \chi^\dagger \chi \rangle \\
& \quad - \frac{1}{6} \langle \chi^\dagger U \; - \; U^\dagger \chi \rangle^2 \,].
\end{aligned}
\tag{3.84}
$$

This extends the Gasser and Leutwyler formalism to include the generalization of S-matrix elements to off-shell 'external' states. P_1 and P_2 are new low energy constants that can only be determined by relating off-shell Green functions to observables.

Chapter 4

Renormalization

4.1 Introduction

An effective theory based on a completely general Lagrangian requires an infinite number of counterterms, and therefore is not renormalizable in the usual sense. The full $\mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots$ Lagrangian of chiral pertubation theory is comprised of interactions having all possible powers of derivatives; the nonrenormalizability result follows from this fact, because the coupling constants associated with all these interactions do not have zero or positive mass dimension. The theory is, however, renormalizable order by order in perturbation theory, because at each order there are a finite number of interaction terms, meaning a finite number of phenomenological parameters. With each higher order, one must include more and more of these free parameters.

If the Lagrangian is indeed the most general one possible, up to a given order, then a particular renormalization program can be set out for that order, and relations between all physical observables will be finite. The infinities arising in the calculation are all absorbed into the definitions of the free parameters. In the present case, in particular, ultraviolet loop divergences from $\mathcal{L}_2$ serve to renormalize the ten L_i constants from $\mathcal{L}_4$.

In the present chapter, two main aspects of renormalization will be discussed; first, the L_i renormalization, and second, wavefunction renormalization, which is required in order to relate the Green function calculated in the effective theory to the physical Green function.

Renormalization of the L_i constants is necessary because loops are included in the calculation. We will first examine how loop divergences enter, and demonstrate the regularization scheme, then we will present the Gasser and Leutwyler renormalization results.

Wavefunction renormalization will be determined in two basic steps. The first step is the involved process of uncovering a 'canonical Lagrangian' that represents the underlying effective theory. This process will really only be outlined, but it should give an introduction to the conceptual aspects of using the effective Lagrangian — i.e. using the Lagrangian at a certain order in the fields, relating the Lagrangian terms to Feynman diagrams, and absorbing loop divergences in the phenomenological constants. The second step is to then modify the standard renormalization to accomodate the new higher derivative interactions from $\mathcal{L}_{4\,\mathrm{off}}$. The basis for this procedure will necessarily differ from that of the first step. However, introducing the more axiomatic ideas to carry this out will allow us to tie the results of the chapter together by showing the renormalization of a Green function — the propagator.

4.2 The Self-Energy Loop

The Feynman diagrams of Figure 4.1 are the renormalization diagrams for this scalar-only theory. Diagram (a) is known as the self-energy loop, and diagram (b) represents a 'counterterm' interaction. This latter is also a self-interaction, since it involves only two external particles, and it can be thought of as a tree diagram. In renormalization, as in other calculations, the tree diagrams are associated with both $\mathcal{L}_2$ and $\mathcal{L}_4$, while the loop diagrams are associated with $\mathcal{L}_2$ only. We will see later how this works.

The aim in this section is to write the expression for the self-energy loop in an analytic

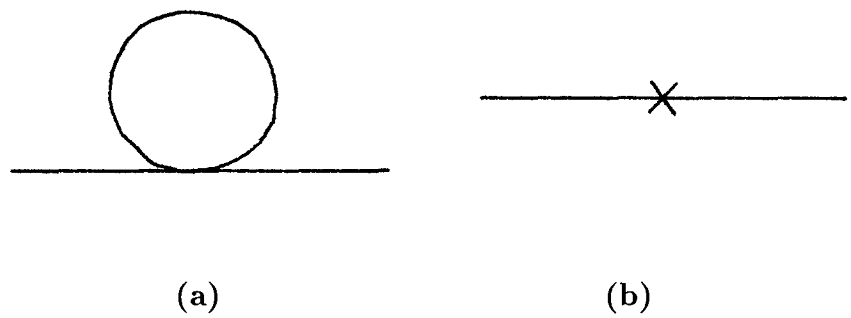

Figure 4.1: Self-Energy Diagrams

form, to establish a system for handling ultraviolet divergences. To do loop-level calculations, one must define a prescription for consistently keeping track of the ultraviolet-divergent 4-momenta from integrals. This prescription is known as regularization. In the established Gasser and Leutwyler formalism, dimensional regularization is used.

The self-energy loop diagram contributes to the full propagator. We begin by introducing the free, or 'bare' propagator. This is defined as a contraction of two free fields, where a contraction is the vacuum expectation value of the time-ordered product of two fields [17]:

$$i\,\triangle_{m^2}(x-y) \;\equiv\; \langle 0|\,T\,(\,\pi^*(x)\pi(y)\,)|0\rangle. \tag{4.1}$$

(Throughout this section we use π to represent any one of the members of the pseudoscalar meson octet.)

In general, this is equal to

$$i\,\triangle_{m^2}(x-y) \;=\; \int \frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon} \tag{4.2}$$

which is the Fourier Transform of the familiar Feynman propagator in momentum space.

When the contraction is between two fields evaluated at the same space-time point x, it will contribute to the self-energy loop represented by diagram (a). The propagator connects a single vertex to itself. However, the integral of equation (4.2) is divergent for $x = y$. One defines the self-energy integral from integral (4.2) as [10]

$$I(m^2) \;\equiv\; i\,\triangle_{m^2}(0) \;=\; \mu^{4-d}\int \frac{d^d k}{(2\pi)^d}\,\frac{i}{k^2 - m^2} \tag{4.3}$$

where d is in general complex, and μ is a constant having the dimensions of energy. This integral converges for a suitable choice of d. (We suppress the $i\epsilon$ from the denominator of the propagator henceforth, since it is not needed in this regularization scheme.)

Parametrizing the dimension of an integral is known as dimensional regularization. This method relies on the principle of analytic continuation in the number of space-time dimensions [20], which guarantees that one can return to dimension $d = 4$ after having begun with the above generalized integral and calculated it as some function of d. One performs manipulations such as symmetric integration and shift of integration variable away from $d = 4$, when the integral $I(d = 4)$ would not be well-defined. The limit $d \longrightarrow 4$ is taken only at the end of the calculation, and the field theory renormalization scheme guarantees that any calculated result will be finite.

The constant μ has been incorporated in the integral to maintain a consistent overall dimension for $I(m^2)$. This follows the technique used by Donoghue et al. [10]. It should be noted that the prescription used here for the introduction of μ differs from that of Gasser and Leutwyler, but that nevertheless, this constant has a well-defined meaning: It is the cutoff determining the energy scale of the theory. We will presently see how this interpretation arises.

To proceed with the evaluation of (4.3), we will use the standard formula for a dimensionally regularized integral [21]:

$$\int \frac{d^{2\omega}k\;(k^2)^t}{[k^2\,-\,m^2]^q} \;=\; \frac{i\,\pi^{\omega}\,\Gamma(\omega + t)\,\Gamma(q - \omega - t)}{\Gamma(\omega)\,\Gamma(q)\,(-m^2)^{q-\omega-t}} \tag{4.4}$$

where $\Re e\,(q-\omega-t)\neq 0$.

Note that the integral is defined over Minkowski space — no Wick rotation is to be performed when applying the formula. One defines the metric

$$g_{\mu\nu} \;=\; \mathrm{diag}\,(1,-1,\ldots-1)$$

generalized to 2ω dimensions [20].

Using (4.4) with $2\omega=d$, $t=0$, and $q=1$, the integral $I(m^2)$ is equal to

$$I(m^2) \;=\; \frac{\mu^{4-d}}{(4\pi)^{\frac{d}{2}}}\,(m^2)^{\frac{d}{2}-1}\,\Gamma(1-\frac{d}{2}). \tag{4.5}$$

The next step in the procedure is to expand in a Taylor series about $d=4$. (We have actually taken the limit $d\to 4$ where d appeared in the factor $(-1)^{1-\frac{d}{2}}$ yielding the overall sign. This is legitimate because it will not affect the expansion to be made.)

The factors of d in the expression are manipulated as follows. The recurrence relation

$$\Gamma(z+1) \;=\; z\Gamma(z) \tag{4.6}$$

is used to change the argument of the gamma function to $(2-\frac{d}{2})$. This will allow the function to be rewritten in terms of the expansion

$$\Gamma(z) \;=\; \frac{1}{z}-\gamma+O(z) \tag{4.7}$$

where γ is Euler's constant. One also collects the other factors under the common exponent $(2-\frac{d}{2})$, to arrive at the following expression:

$$I(m^2) \;=\; \frac{m^2}{(4\pi)^2}\left(\frac{4\pi\mu^2}{m^2}\right)^{2-\frac{d}{2}}\frac{1}{1-\frac{d}{2}}\,\Gamma(2-\frac{d}{2}). \tag{4.8}$$

Now let

$$\epsilon \;=\; 2-\frac{d}{2}.$$

By expanding around $\epsilon = 0$ and throwing away terms of $O(\epsilon)$ or higher, one derives the following approximation:

$$\begin{aligned}
\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon} \frac{1}{\epsilon-1}\Gamma(\epsilon) &= e^{\epsilon \ln\left(\frac{4\pi\mu^2}{m^2}\right)} \frac{1}{\epsilon-1}\Gamma(\epsilon) \\
&= -\left[1 + \epsilon \ln\left(\frac{4\pi\mu^2}{m^2}\right) + O(\epsilon^2)\right]\left[1 + \epsilon + O(\epsilon^2)\right]\left[\frac{1}{\epsilon} - \gamma + O(\epsilon)\right] \\
&= -\frac{1}{\epsilon} - \ln 4\pi + \ln\frac{m^2}{\mu^2} + \gamma - 1 + O(\epsilon). \qquad (4.9)
\end{aligned}$$

For the self-energy integral, then, we have

$$\begin{aligned}
I(m^2) &= \frac{m^2}{16\pi^2}\left[\frac{2}{d-4} - \ln 4\pi + \gamma - 1 + \ln\frac{m^2}{\mu^2}\right] \\
&= \frac{m^2}{16\pi^2}\left[R + \ln\frac{m^2}{\mu^2}\right] \qquad (4.10)
\end{aligned}$$

with the definition

$$R = \frac{2}{d-4} - \ln 4\pi + \gamma - 1. \qquad (4.11)$$

The constant R contains the divergence of the unrenormalized Green functions, which occurs at the $d = 4$ pole. We will use R throughout our calculations.

Result (4.10) demonstrates that an energy scale enters the theory. This scale, μ, is often taken to be the eta mass [10] when it is necessary to ascribe numerical values to scale-dependent quantities. The mass ratio term gives a non-power law behaviour to the low energy expansion. The logarithm, whose argument is by assumption small, cannot be expanded in a power series around zero.

Another result that will be of use is [10]

$$\partial_x^\mu i \triangle_{m^2}(x-y)|_{x=y} = 0. \qquad (4.12)$$

This identity holds by symmetric integration:

$$\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{i\,k^\mu}{k^2-m^2} \;=\; 0. \tag{4.13}$$

The argument applies because the integral is from $k=-\infty$ to $+\infty$, and the integrand is antisymmetric under $k\to -k$. The symmetric integration argument is only valid, however, for d chosen so that the integral converges. After integrating, one analytically continues to other dimensions.

Equations (4.10) and (4.12) are the basis for the loop calculations we will be carrying out in the following chapter since, by Wick's Theorem, the S-matrix expansion is reduced to contractions of pairs of fields [17].

Furthermore, we have presented the background required for dealing with the L_i constants that arise in the calculation. The original prescription laid out by Gasser and Leutwyler [4] specifies how the divergences represented by the constant R are absorbed into the set of bare $\mathcal{L}_4$ parameters. The following are their results:

$$L_i^r \;=\; L_i - \frac{\gamma_i}{32\pi^2}R$$

where

$$\begin{array}{ll} \gamma_1=\frac{3}{32} & \gamma_6=\frac{11}{144} \\ \gamma_2=\frac{3}{16} & \gamma_7=0 \\ \gamma_3=0 & \gamma_8=\frac{5}{48} \\ \gamma_4=\frac{1}{8} & \gamma_9=\frac{1}{4} \\ \gamma_5=\frac{3}{8} & \gamma_{10}=-\frac{1}{4} \end{array} \tag{4.14}$$

and R has been defined in equation (4.11). The form of R and the procedure outlined from equations (4.5) — (4.10) dictate the finite constants that are absorbed into the L_i parameters, since this choice is ambiguous.

In addition to the L_i constants, there are three other renormalized quantities in the theory: the meson wavefunction π, the bare mass $\mathring{m}_\pi{}^2$ and the pion decay constant F_0 which come from the effective Lagrangian. We now turn to a discussion of the wavefunction renormalization, which will have direct application in calculating the form factor.

4.3 Renormalization Constants

In this section we will examine the Z_π and J_π wavefunction and mass renormalization constants for the mesons in our theory. The constants are defined by

$$\begin{aligned} \pi^{\mathrm{r}} &= Z_\pi^{-\frac{1}{2}}\,\pi \\ m_\pi^2 &= J_\pi\,\mathring{m}_\pi^2 \end{aligned} \tag{4.15}$$

again using the generic notation 'π' to represent a meson field, which could be the π, K, or η. The field π^{r} is the physical meson, and m_π^2, of course, is the physical mass. We denote unrenormalized masses using a circle.

We should begin by remarking that the meson spectrum will be treated as three families, each degenerate in mass. The spectrum is comprised of a triplet of pions, two doublets of kaons, and the eta, if one assumes isospin symmetry, or unbroken $SU(2)$. We set

$$m_u = m_d = \hat{m} \tag{4.16}$$

and invoke this approximation when expanding Lagrangian terms containing χ. This is a very good approximation, and in fact, it is fully justified in the context of the present calculation, because the differentiation between masses in an isospin multiplet, for example $m_{\pi^\pm}^2 \leftrightarrow m_{\pi^0}^2$, occurs at higher order in the energy expansion than we will be working.

The constants of equation (4.15) are intimately associated with the effective Lagrangian. In sections 4.3.1 — 4.3.5, we take the effective Lagrangian consisting of $\mathcal{L}_2 + \mathcal{L}_4$ (i.e. not including $\mathcal{L}_{4\,\mathrm{off}}$) and formulate it in terms of a 'canonical' effective Lagrangian from which Z_π and J_π are directly determined.

4.3.1 Canonical Form of the Lagrangian

The effective Lagrangian without external currents ($D^\mu \longrightarrow \partial^\mu$) is to be cast into the canonical form

$$\mathcal{L}_{\mathrm{eff}} \;=\; \frac{1}{2}\,\partial^\mu \boldsymbol{\pi}^{\mathrm{r}} \cdot \partial_\mu \boldsymbol{\pi}^{\mathrm{r}} \;-\; \frac{1}{2}\,m_\pi^2\,\boldsymbol{\pi}^{\mathrm{r}} \cdot \boldsymbol{\pi}^{\mathrm{r}} \tag{4.17}$$

which describes the physical fields, and whose mass parameter m_π^2 is the renormalized, or fully dressed, mass of the particle.

There is an effective Lagrangian of this form for each family of mesons — that is, one for the pions, one for the kaons, and one for the eta. The dot product indicates that the expression is written using isospin vectors — for the pion Lagrangian one has

$$\boldsymbol{\pi} \cdot \boldsymbol{\pi} \;=\; \pi_1\,\pi_1 \,+\, \pi_2\,\pi_2 \,+\, \pi_3\,\pi_3$$

and, using the notation introduced in equation (2.5), one has

$$\boldsymbol{K} \cdot \boldsymbol{K} \;=\; K_4\,K_4 \,+\, K_5\,K_5 \,+\, K_6\,K_6 \,+\, K_7\,K_7$$

for the kaon Lagrangian. The Pauli basis form (numerical indices) is convenient for deriving the renormalization constants Z_π and J_π; the same constants will then apply for a calculation in the physical basis, where the particles are the π^+, π^-, π^0, etc.

How does one make the connection between the effective Lagrangian $\mathcal{L}_2 + \mathcal{L}_4$ and the canonical Lagrangian (4.17)? The canonical Lagrangian is a bilinear operator expression whose matrix elements will give us 1-particle irreducible Green functions. This simply means that the Lagrangian will already account for self-energy loops and other renormalization associated with external legs, so that the explicit diagrams need not be included in calculations.

We should recall the comments in section 2.4.2 concerning the use of the effective Lagrangian in perturbation theory. Diagrammatically, the renormalization we are undertaking involves tree and one-loop graphs. We expand both the Lagrangians $\mathcal{L}_2$ and $\mathcal{L}_4$ to $O(\phi^2)$ to find the former, and just $\mathcal{L}_2$ to $O(\phi^4)$ to find the latter. The notation $O(\phi^n)$ refers to an order in the expansion of the exponential $U(\phi)$ defined in equation (2.8). In this order notation it should be noted that ϕ may represent also derivatives of the field, $\partial^\mu\phi$.

Determining the self-energy loops having the form of diagram (a) proceeds by making all possible contractions in the $O(\phi^4)$ terms of $\mathcal{L}_2$. To evaluate a self-energy loop one will be taking a matrix element; the objective here is to find the single contractions in

$$\langle 0|\, T\,(\,\mathcal{L}_{(4\pi)}(x)\,)|0\rangle \tag{4.18}$$

(compare equation (4.1)). The π represents one of the meson fields. Taking only one contraction in this $O(\phi^4)$ expression will leave a bilinear operator; this operator will form a component of the canonical Lagrangian, which is to be evaluated between single-particle external states.

Since all fields in a Lagrangian are evaluated at the same space-time point x, we will obtain from a contraction in expression (4.18) the Feynman propagator $i\triangle_{m_\pi^2}(x-x) = i\triangle_{m_\pi^2}(0)$, which has been evaluated in section 4.2.

From these loops, and from the $O(\phi^2)$ part of $\mathcal{L}_2$ and the $O(\phi^2)$ part of $\mathcal{L}_4$, we will assemble a Lagrangian having the canonical $\boldsymbol{\pi}\cdot\boldsymbol{\pi}$ form of equation (4.17).

4.3.2 Expanding the Lowest Order Lagrangian

The renormalization diagrams of Figure 4.1 represent the strong interaction alone, with no effects from external sources. The Lagrangian is therefore expanded and analyzed here with $D^\mu = \partial^\mu$. In this chapter, we will not go into the details of this expansion, or of the evaluation of traces which leads to the extraction of the fields. The intent here is just to obtain the required renormalization result. The steps taken are all analogous to steps which will be outlined in detail in Chapter 5, when we calculate the form factors.

First, we expand the exponentials $U(\phi)$ and $U^\dagger(\phi)$ in order to write the Lagrangian expressed in terms of these objects as a Lagrangian for the actual meson fields π, K, and η. Expanding $U(\phi)$ and $U^\dagger(\phi)$ and keeping only terms up to $O(\phi^4)$, the Lagrangian $\mathcal{L}_2$ in the absence of external sources is

$$
\begin{aligned}
\mathcal{L}_2 &= \mathcal{L}_2^0 + \mathcal{L}_2^{\text{int}} \\
\mathcal{L}_2^0 &= \frac{1}{4}\langle\, \partial^\mu\phi\partial_\mu\phi \;-\; 2B_0 M\phi\phi \,\rangle \\
\mathcal{L}_2^{\text{int}} &= \frac{1}{24F_0^2}\langle\, \partial^\mu\phi\phi\partial_\mu\phi\phi \;-\; \partial^\mu\phi\partial_\mu\phi\phi\phi \;+\; B_0 M\phi\phi\phi\phi \,\rangle
\end{aligned}
\tag{4.19}
$$

where we have dropped an irrelevant constant term and represented the higher order term as an interaction piece.

Now we reduce from the ϕ matrix form to the actual field operators. (See equation (2.5) for the definition of ϕ.) This is done by multiplying the matrices together and taking the trace. (See section 5.3.1 for an example showing this type of procedure.) The $O(\phi^2)$ part of Lagrangian (4.19) becomes

$$
\begin{aligned}
\mathcal{L}_2^0 \;=\; & \frac{1}{2}\,\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + \frac{1}{2}\,\partial^\mu\boldsymbol{K}\cdot\partial_\mu\boldsymbol{K} + \frac{1}{2}\,\partial^\mu\boldsymbol{\eta}\cdot\partial_\mu\boldsymbol{\eta} \\
& - \frac{1}{2}\,\mathring{m}_\pi^2\,\boldsymbol{\pi}\cdot\boldsymbol{\pi} - \frac{1}{2}\,\mathring{m}_K^2\,\boldsymbol{K}\cdot\boldsymbol{K} - \frac{1}{2}\,\mathring{m}_\eta^2\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}
\end{aligned}
\tag{4.20}
$$

with

$$B_0\hat{m} = \frac{1}{2}\overset{\circ}{m}_\pi^{\,2} \tag{4.21}$$

$$B_0(\hat{m}+m_s) = \overset{\circ}{m}_K^{\,2} \tag{4.22}$$

$$B_0(\hat{m}+2m_s) = \frac{3}{2}\overset{\circ}{m}_\eta^{\,2} \tag{4.23}$$

and, derived from these,

$$B_0 m_s = -\overset{\circ}{m}_K^{\,2} + \frac{3}{2}\overset{\circ}{m}_\eta^{\,2}\,. \tag{4.24}$$

These important relationships between quark masses and meson masses are dictated by the fact that $\overset{\circ}{m}_\pi^{\,2}$, $\overset{\circ}{m}_K^{\,2}$, and $\overset{\circ}{m}_\eta^{\,2}$ are the constants that must appear in front of the $\boldsymbol{\pi}\cdot\boldsymbol{\pi}$ terms of the canonical scalar Lagrangian, when this Lagrangian is compared to the effective Lagrangian of the theory taken at lowest order. The relations are valid only to leading order in the energy expansion. Also, they are already written as approximations because we have assumed $m_u = m_d = \hat{m}$. We will refer to these relations later when rewriting B_0 terms.

Equation (4.20), from the $O(\phi^2)$ part of equation (4.19), is the leading order piece of the canonical Lagrangian. The higher order corrections to this are derived in the following two sections.

4.3.3 Rewriting the $O(\phi^4)$ Part

The $O(\phi^4)$ part of equation (4.19) expands into an expression too long to show here. It has terms consisting of four fields multiplied by $B_0\hat{m}$, a similar set of terms multiplied by $B_0 m_s$, and another set of terms containing derivatives of two of the fields.

Let us consider just the $B_0\hat{m}$ terms to illustrate. These are:

$$\begin{array}{rcllll}\mathcal{L}_2^{\rm int}(B_0\hat{m}) & = & \frac{B_0\hat{m}}{12F_0^2} & & & \\ & & [\quad \pi^0\pi^0\pi^0\pi^0 & + \ 4\pi^+\pi^-\pi^0\pi^0 & + \ 4\pi^+\pi^-\pi^+\pi^- & + \ 3K^0\overline{K^0}\pi^0\pi^0 \\ & & + \ 3K^+K^-\pi^0\pi^0 & + \ 6K^0\overline{K^0}\pi^+\pi^- & + \ 6K^+K^-\pi^+\pi^- & + \ 2K^0\overline{K^0}K^0\overline{K^0} \\ & & + \ 4K^+K^-K^0\overline{K^0} & + \ 2K^+K^-K^+K^- & + \ 2\eta\eta\pi^0\pi^0 & + \ 4\eta\eta\pi^+\pi^- \\ & & - \ \frac{4}{\sqrt{3}}K^0\overline{K^0}\eta\pi^0 & + \ \frac{4}{\sqrt{3}}K^+K^-\eta\pi^0 & + \ 2\sqrt{\frac{2}{3}}K^+\overline{K^0}\eta\pi^- & + \ 2\sqrt{\frac{2}{3}}K^-K^0\eta\pi^+ \\ & & + \ \eta\eta K^0\overline{K^0} & + \ \eta\eta K^+K^- & + \ \frac{1}{9}\eta\eta\eta\eta \quad]. & \end{array} \quad (4.25)$$

We take all possible contractions to find the self-energy loops. The resultant operator expression will be denoted '$\mathcal{L}_{2\,\rm loop}$', and will form a component of the canonical Lagrangian.

The above expansion has been done in the physical basis to make the process of forming the contractions more transparent. Nonzero contractions are performed between complex conjugate pairs of particles, as indicated by equation (4.1). Note that the K^0 and $\overline{K^0}$ are a complex conjugate pair, and the K^+ and K^- are another pair. The π^0 and the η are, of course, different particles, and cannot be contracted together. In a term like

$$\pi^0\pi^0\pi^0\pi^0 \quad (4.26)$$

consisting of identical particles, six distinct contractions are possible. In a term like

$$4\pi^+\pi^-\pi^0\pi^0 \quad (4.27)$$

there are only two contractions possible, while in a term like

$$2\sqrt{\frac{2}{3}}K^+\overline{K^0}\eta\pi^- \quad (4.28)$$

none will appear.

Due to our assumed isospin symmetry, two different contractions of particles within the same family will form the same Feynman propagator. For the pion multiplet we have

$$\langle 0|T\,(\,\pi^+\pi^-\,)|0\rangle \;=\; \langle 0|T\,(\,\pi^0\pi^0\,)|0\rangle \;=\; i\triangle_{\overset{\circ}{m}{}_\pi^2}(0). \quad (4.29)$$

(From here on, 'π' refers specifically to the pion.)

To continue with the example, consider the terms that will lead to the pion loop of equation (4.29), and that will have two π^0's left over as the external fields. These will be only the terms (4.26) and (4.27) shown above. From them, we find six π^0 loop contributions, as mentioned, and we find one $\pi^\pm$ loop contribution with coefficient 4. Taking the contractions, this particular result is then the term

$$10\left(\frac{B_0\hat{m}}{12F_0^2}\right)\; i\triangle_{\overset{\circ}{m}{}_\pi^2}(0)\;\pi^0\pi^0.$$

Going on to form all of the contractions in expression (4.25), we find the following canonical terms multiplying the three propagators:

$$\begin{aligned}\mathcal{L}_{2\,\mathrm{loop}}(B_0\hat{m}) \;=\; \frac{B_0\hat{m}}{24F_0^2}\Big[\; & i\triangle_{\overset{\circ}{m}{}_\pi^2}(0)\,(\,20\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,+\,9\boldsymbol{K}\cdot\boldsymbol{K}\,+\,12\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,) \\ & + i\triangle_{\overset{\circ}{m}{}_K^2}(0)\,(\,12\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,+\,12\boldsymbol{K}\cdot\boldsymbol{K}\,+\,4\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,) \\ & + i\triangle_{\overset{\circ}{m}{}_\eta^2}(0)\,(\,4\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,+\,\boldsymbol{K}\cdot\boldsymbol{K}\,+\,\frac{4}{3}\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,)\;\Big]. \end{aligned} \tag{4.30}$$

We have changed from the physical basis to the Pauli basis in order to write the expression in this form (see equation (2.6)). Equation (4.30) indicates the emergence of the corrections that will occur to the leading order canonical Lagrangian (4.20). The propagator is just the expression $I(m^2)$ that has been given in equation (4.10), involving a constant and the mass m^2; $B_0\hat{m}$ of course is proportional to m^2, by equations (4.21) — (4.23). Thus we see that the expression has energy dimension $O(p^4)$.

The above terms, because of their $B_0\hat{m}$ coefficient, ultimately form a component of the

$$-\frac{1}{2}\,\overset{\circ}{m}{}_\pi^2\;\boldsymbol{\pi}\cdot\boldsymbol{\pi}$$

term in the canonical Lagrangian shown below, by equation (4.21). Checking the coefficient of the $\pi^0\pi^0$ contribution against (4.31), however, reveals that this is not the whole story. There is an interplay between the $B_0\hat{m}$ and the $B_0 m_s$ parts of the full expression, as can be understood by comparing equations (4.21), (4.22), and (4.23) for the meson masses. The simple structures shown below arise in a straightforward way from the application of these equations.

For the full $O(\phi^4)$ part of equation (4.19), the end result of forming the contractions is this expression:

$$\begin{aligned}
\mathcal{L}_{2\,\text{loop}} &= \frac{1}{2}\,\partial^\mu \boldsymbol{\pi}\cdot\partial_\mu \boldsymbol{\pi}\,\left[-\frac{2}{3}\,I(\mathring{m}_\pi^2) - \frac{1}{3}\,I(\mathring{m}_K^2)\right] \\
&+ \frac{1}{2}\,\partial^\mu \boldsymbol{K}\cdot\partial_\mu \boldsymbol{K}\left[-\frac{1}{4}\,I(\mathring{m}_\pi^2) - \frac{1}{2}\,I(\mathring{m}_K^2) - \frac{1}{4}\,I(\mathring{m}_\eta^2)\right] \\
&+ \frac{1}{2}\,\partial^\mu \boldsymbol{\eta}\cdot\partial_\mu \boldsymbol{\eta}\,\left[-I(\mathring{m}_K^2)\right] \\
&- \frac{1}{2}\,\mathring{m}_\pi^2\,\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\left[-\frac{1}{6}\,I(\mathring{m}_\pi^2) - \frac{1}{3}\,I(\mathring{m}_K^2) - \frac{1}{6}\,I(\mathring{m}_\eta^2)\right] \\
&- \frac{1}{2}\,\mathring{m}_K^2\,\boldsymbol{K}\cdot\boldsymbol{K}\left[-\frac{1}{4}\,I(\mathring{m}_\pi^2) - \frac{1}{2}\,I(\mathring{m}_K^2) + \frac{1}{12}\,I(\mathring{m}_\eta^2)\right] \\
&- \frac{1}{2}\,\mathring{m}_\eta^2\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,\left[-\frac{1}{2}\,I(\mathring{m}_\pi^2) + \frac{1}{3}\,\frac{\mathring{m}_\pi^2}{\mathring{m}_K^2}\,I(\mathring{m}_K^2) + \frac{1}{6}\left(\frac{\mathring{m}_\pi^2}{\mathring{m}_\eta^2} - 4\right)\,I(\mathring{m}_\eta^2)\right].
\end{aligned} \tag{4.31}$$

The divergences contained in this will all be absorbed by the counterterms of the $\mathcal{L}_4$ piece of the Lagrangian.

As we pointed out, the terms of equation (4.31) are $O(p^4)$. We can actually write $\mathring{m}^2$ as m^2 wherever it appears in a correction to the lowest order (that is, $O(p^2)$) Lagrangian, since the difference between $\mathring{m}^2$ and m^2 is itself $O(p^2)$ and will not be relevant. We do this in the following section, where again the Lagrangian we are deriving is $O(p^4)$.

4.3.4 Contributions From $\mathcal{L}_4$

As opposed to self-energy loops, the $\mathcal{L}_4$ part supplies counterterms represented by diagram (b) of Figure 4.1. We expand $\mathcal{L}_4$ to $O(\phi^2)$, and collect the common expressions together as we have done in the preceding section. Again, ∂^μ is substituted for D^μ.

The following are the structures of $\mathcal{L}_4$ expanded to $O(\phi^2)$, excluding the H_1 and H_2 terms, which do not contain meson fields. Notice that several of these terms vanish at $O(\phi^2)$. In the case of the first three terms, this is because $\partial^\mu U$ has a leading order term of $O(\phi)$; for expressions C–9 and C–10, the explanation is that $F^R_{\mu\nu} = F^L_{\mu\nu} = 0$ in the absence of the external fields.

C–1

$$\langle D^\mu U^\dagger D_\mu U\rangle^2 \;=\; 0$$

C–2

$$\langle D^\mu U^\dagger D^\nu U\rangle\langle D_\mu U^\dagger D_\nu U\rangle \;=\; 0$$

C–3

$$\langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U\rangle \;=\; 0$$

C–4

$$\begin{aligned}
&\langle D^\mu U^\dagger D_\mu U\rangle\langle\chi^\dagger U + \chi U^\dagger\rangle \\
&\quad= \frac{4B_0}{F_0^2}\langle\partial^\mu\phi\partial_\mu\phi\rangle\langle M\rangle \\
&\quad= \frac{8B_0}{F_0^2}(m_u + m_d + m_s)\,[\,\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + \partial^\mu\boldsymbol{K}\cdot\partial_\mu\boldsymbol{K} + \partial^\mu\boldsymbol{\eta}\cdot\partial_\mu\boldsymbol{\eta}\,] \\
&\quad= \frac{4(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,[\,\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} + \partial^\mu\boldsymbol{K}\cdot\partial_\mu\boldsymbol{K} + \partial^\mu\boldsymbol{\eta}\cdot\partial_\mu\boldsymbol{\eta}\,]
\end{aligned}$$

C–5

$$
\begin{aligned}
&\langle D^\mu U^\dagger D_\mu U\,(\chi^\dagger U + U^\dagger \chi)\rangle \\
&= \frac{4B_0}{F_0^2}\,\langle M\partial^\mu\phi\partial_\mu\phi\rangle \\
&= \frac{4B_0}{F_0^2}\,[\,(m_u+m_d)\,\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} \;+\; (\hat{m}+m_s)\,\partial^\mu\boldsymbol{K}\cdot\partial_\mu\boldsymbol{K} \;+\; \frac{2}{3}(\hat{m}+2m_s)\,\partial^\mu\boldsymbol{\eta}\cdot\partial_\mu\boldsymbol{\eta}\,] \\
&= \frac{4}{F_0^2}\,[\,m_\pi^2\,\partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi} \;+\; m_K^2\,\partial^\mu\boldsymbol{K}\cdot\partial_\mu\boldsymbol{K} \;+\; m_\eta^2\,\partial^\mu\boldsymbol{\eta}\cdot\partial_\mu\boldsymbol{\eta}\,]
\end{aligned}
$$

C–6

$$
\begin{aligned}
&\langle\chi^\dagger U + \chi U^\dagger\rangle^2 \\
&= 4B_0^2\,\langle 2M \;-\; \frac{1}{F_0^2}\,M\phi\phi\rangle^2 \\
&= -\frac{16B_0^2}{F_0^2}\,\langle M\rangle\langle M\phi\phi\rangle \\
&= -\frac{16B_0^2}{F_0^2}\,(2\hat{m}+m_s)\,[\,2\hat{m}\,\boldsymbol{\pi}\cdot\boldsymbol{\pi} \;+\; (\hat{m}+m_s)\,\boldsymbol{K}\cdot\boldsymbol{K} \;+\; \frac{2}{3}(\hat{m}+2m_s)\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,] \\
&= -\frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,[\,m_\pi^2\,\boldsymbol{\pi}\cdot\boldsymbol{\pi} \;+\; m_K^2\,\boldsymbol{K}\cdot\boldsymbol{K} \;+\; m_\eta^2\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,]
\end{aligned}
$$

C–7

$$
\begin{aligned}
&\langle\chi^\dagger U - \chi U^\dagger\rangle^2 \\
&= -\frac{16B_0^2}{F_0^2}\,\langle M\phi\rangle^2 \\
&= -\frac{64B_0^2}{3F_0^2}\,(\hat{m}-m_s)^2\,\boldsymbol{\eta}\cdot\boldsymbol{\eta} \\
&= -\frac{16}{3F_0^2}\,(\,m_\pi^4 \;+\; 4m_\pi^2 m_K^2 \;-\; 6m_\pi^2 m_\eta^2 \;+\; 4m_K^4 \;-\; 12m_K^2 m_\eta^2 \;+\; 9m_\eta^4\,)\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}
\end{aligned}
$$

C–8

$$
\begin{aligned}
&\langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
&\quad = -\frac{8B_0^2}{F_0^2} \langle M\phi M\phi + MM\phi\phi \rangle \\
&\quad = -\frac{8B_0^2}{F_0^2} [4\hat{m}^2 \boldsymbol{\pi}\cdot\boldsymbol{\pi} + (\hat{m}^2 + 2\hat{m}m_s + m_s^2)\boldsymbol{K}\cdot\boldsymbol{K} + \frac{4}{3}(\hat{m}^2 + 2m_s^2)\boldsymbol{\eta}\cdot\boldsymbol{\eta}] \\
&\quad = -\frac{8}{F_0^2} [m_\pi^4 \boldsymbol{\pi}\cdot\boldsymbol{\pi} + m_K^4 \boldsymbol{K}\cdot\boldsymbol{K} + (2m_\pi^2 m_K^2 - 3m_\pi^2 m_\eta^2 + 2m_K^2 m_\eta^2)\boldsymbol{\eta}\cdot\boldsymbol{\eta}]
\end{aligned}
$$

C–9

$$\langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U \rangle = 0$$

C–10

$$\langle F^{R\mu\nu} U F^L_{\mu\nu} U^\dagger \rangle = 0.$$

In writing this list, we had to choose how to convert the quark mass expressions into combinations of meson masses by equations (4.21) — (4.24), since these relationships are not uniquely determined. For example, one could write equivalently

$$2B_0(2\hat{m} + m_s) = m_\pi^2 + 2m_K^2$$

or

$$2B_0(2\hat{m} + m_s) = 2m_\pi^2 - 2m_K^2 + 3m_\eta^2.$$

The choice is made on the basis of how the expression reduces to $SU(2)$. It is convenient if one can just drop the m_K^2 and m_η^2 masses from the expression and recover the result derived from a strictly $SU(2)$ treatment. (See section 5.3.1 for a discussion on the point of using $SU(2)$ instead of $SU(3)$ matrices.)

Using the above list (each list item C–i carrying the coefficient L_i), we collect all the $\mathcal{L}_4$ structures into the canonical form

$$\begin{aligned}
\mathcal{L}_{4\,\text{tree}} &= \frac{1}{2}\,\partial^\mu \boldsymbol{\pi}\cdot\partial_\mu \boldsymbol{\pi}\,\left[\frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,L_4 + \frac{8m_\pi^2}{F_0^2}\,L_5\right] \\
&+ \frac{1}{2}\,\partial^\mu \boldsymbol{K}\cdot\partial_\mu \boldsymbol{K}\left[\frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,L_4 + \frac{8m_K^2}{F_0^2}\,L_5\right] \\
&+ \frac{1}{2}\,\partial^\mu \boldsymbol{\eta}\cdot\partial_\mu \boldsymbol{\eta}\,\left[\frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,L_4 + \frac{8m_\eta^2}{F_0^2}\,L_5\right] \\
&- \frac{1}{2}\,\mathring{m}_\pi^2\,\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\left[\frac{16(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,L_6 + \frac{16m_\pi^2}{F_0^2}\,L_8\right] \\
&- \frac{1}{2}\,\mathring{m}_K^2\,\boldsymbol{K}\cdot\boldsymbol{K}\left[\frac{16(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,L_6 + \frac{16m_K^2}{F_0^2}\,L_8\right] \\
&- \frac{1}{2}\,\mathring{m}_\eta^2\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,\left[\frac{16(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\,L_6 + \frac{128}{3F_0^2}\,\frac{B_0^2(\hat{m} - m_s)^2}{m_\eta^2}\,L_7\right. \\
&\qquad\qquad\qquad\qquad\left. + \frac{64}{3F_0^2}\,\frac{B_0^2(\hat{m}^2 + 2m_s^2)}{m_\eta^2}\,L_8\right]. \qquad (4.32)
\end{aligned}$$

Notice that for the corrections (i.e. the terms inside the square brackets which multiply the $\mathring{m}^2$) we have not bothered with the distinction between $\mathring{m}^2$ and m^2.

The contributions from $\mathcal{L}_{4\,\text{off}}$ are not in the form of corrections to the canonical Lagrangian. These will be dealt with in section 4.4.

4.3.5 Collecting the Terms of the Canonical Lagrangian

The sum of equations (4.20), (4.31), and (4.32) forms this renormalized effective Lagrangian:

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \frac{1}{2}\,\partial^\mu \boldsymbol{\pi}\cdot\partial_\mu \boldsymbol{\pi}\,\frac{1}{Z_\pi} + \frac{1}{2}\,\partial^\mu \boldsymbol{K}\cdot\partial_\mu \boldsymbol{K}\,\frac{1}{Z_K} + \frac{1}{2}\,\partial^\mu \boldsymbol{\eta}\cdot\partial_\mu \boldsymbol{\eta}\,\frac{1}{Z_\eta} \\
&- \frac{1}{2}\,\mathring{m}_\pi^2\,\boldsymbol{\pi}\cdot\boldsymbol{\pi}\,\frac{J_\pi}{Z_\pi} - \frac{1}{2}\,\mathring{m}_K^2\,\boldsymbol{K}\cdot\boldsymbol{K}\,\frac{J_K}{Z_K} - \frac{1}{2}\,\mathring{m}_\eta^2\,\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,\frac{J_\eta}{Z_\eta}
\end{aligned}$$

with

$$
\begin{aligned}
Z_\pi &= 1 - \frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2} L_4 - \frac{8m_\pi^2}{F_0^2} L_5 \\
&\quad + \frac{m_\pi^2}{24\pi^2 F_0^2} (R + \ln\frac{m_\pi^2}{\mu^2}) + \frac{m_K^2}{48\pi^2 F_0^2} (R + \ln\frac{m_K^2}{\mu^2}) \\
Z_K &= 1 - \frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2} L_4 - \frac{8m_K^2}{F_0^2} L_5 \\
&\quad + \frac{m_\pi^2}{64\pi^2 F_0^2} (R + \ln\frac{m_\pi^2}{\mu^2}) + \frac{m_K^2}{32\pi^2 F_0^2} (R + \ln\frac{m_K^2}{\mu^2}) + \frac{m_\eta^2}{64\pi^2 F_0^2} (R + \ln\frac{m_\eta^2}{\mu^2}) \\
Z_\eta &= 1 - \frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2} L_4 - \frac{8m_\eta^2}{F_0^2} L_5 + \frac{m_K^2}{16\pi^2 F_0^2} (R + \ln\frac{m_K^2}{\mu^2}) \\
\frac{J_\pi}{Z_\pi} &= 1 + \frac{16(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2} L_6 + \frac{16m_\pi^2}{F_0^2} L_8 \\
&\quad - \frac{m_\pi^2}{96\pi^2 F_0^2} (R + \ln\frac{m_\pi^2}{\mu^2}) - \frac{m_K^2}{48\pi^2 F_0^2} (R + \ln\frac{m_K^2}{\mu^2}) - \frac{m_\eta^2}{96\pi^2 F_0^2} (R + \ln\frac{m_\eta^2}{\mu^2}) \\
\frac{J_K}{Z_K} &= 1 + \frac{16(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2} L_6 + \frac{16m_K^2}{F_0^2} L_8 \\
&\quad - \frac{m_\pi^2}{64\pi^2 F_0^2} (R + \ln\frac{m_\pi^2}{\mu^2}) - \frac{m_K^2}{32\pi^2 F_0^2} (R + \ln\frac{m_K^2}{\mu^2}) + \frac{m_\eta^2}{128\pi^2 F_0^2} (R + \ln\frac{m_\eta^2}{\mu^2}) \\
\frac{J_\eta}{Z_\eta} &= 1 + \frac{16(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2} L_6 + \frac{128}{3F_0^2} \frac{B_0^2(\hat{m} - m_s)^2}{m_\eta^2} L_7 \\
&\quad + \frac{64}{3F_0^2} \frac{B_0^2(\hat{m}^2 + 2m_s^2)}{m_\eta^2} L_8 - \frac{m_\pi^2}{32\pi^2 F_0^2} (R + \ln\frac{m_\pi^2}{\mu^2}) \\
&\quad + \frac{m_\pi^2}{48\pi^2 F_0^2} (R + \ln\frac{m_K^2}{\mu^2}) + \frac{(m_\pi^2 - 4m_\eta^2)}{96\pi^2 F_0^2} (R + \ln\frac{m_\eta^2}{\mu^2}).
\end{aligned}
\tag{4.33}
$$

With the definitions of renormalized fields and masses

$$\begin{aligned} \pi^{\mathrm{r}} &= Z_\pi^{-\frac{1}{2}}\,\pi & m_\pi^2 &= J_\pi\,\overset{\circ}{m}{}_\pi^2 \\ K^{\mathrm{r}} &= Z_K^{-\frac{1}{2}}\,K & m_K^2 &= J_K\,\overset{\circ}{m}{}_K^2 \\ \eta^{\mathrm{r}} &= Z_\eta^{-\frac{1}{2}}\,\eta & m_\eta^2 &= J_\eta\,\overset{\circ}{m}{}_\eta^2 \end{aligned} \tag{4.34}$$

the Lagrangian $\mathcal{L}_{\mathrm{eff}}$ of equation (4.33) takes the form of equation (4.17), and we have met the goal set out at the beginning of this section. The relation

$$\begin{aligned} \frac{1}{1-x} &= 1 + x + O(x^2) \\ &\approx 1 + x \end{aligned} \tag{4.35}$$

has been used to invert the Z constants, because they will only be required to $O(p^2)$ in the energy expansion. Equation (4.35) is an approximation that will also be useful later.

We have established, in particular, the wavefunction renormalization Z_π for the standard theory.

4.3.6 Mass Renormalization

From the above, one determines the mass renormalizations. Consider J_π. Multiplying J_π/Z_π by Z_π and keeping only the $O(p^2)$ terms, one finds that the low energy constants in the expression are renormalized according to equation (4.14), and the result for the $O(p^4)$ mass correction involves only phenomenologically determined quantities:

$$\begin{aligned} m_\pi^2 = \overset{\circ}{m}{}_\pi^2 \Bigg[1 &+ \frac{1}{96\pi^2 F_0^2}\left(3m_\pi^2 \ln\frac{m_\pi^2}{\mu^2} - m_\eta^2 \ln\frac{m_\eta^2}{\mu^2}\right) \\ &+ \frac{8}{F_0^2}\Big((2m_\pi^2 - 2m_K^2 + 3m_\eta^2)(2L_6^r - L_4^r) + m_\pi^2(2L_8^r - L_5^r)\Big)\Bigg]. \end{aligned} \tag{4.36}$$

Each L_i^r constant is scale dependent, but this scale dependence cancels out of the full expression. This is easily checked by using the results of equation (4.14) and differentiating the expression with respect to μ. Equation (4.36) reproduces the accepted result [4].

4.3.7 Relation of the Constants to the Propagator

Here we indicate how the propagator is renormalized using the same constants that renormalize the bare Lagrangian.

The full propagator, known as the 2-point Green function, is derived by computing self-energy diagrams to the order desired, and finding that they comprise a modification to the free propagator having this form:

$$i\,\triangle(p) \;=\; \frac{i}{p^2 - \mathring{m}^2 - \Sigma(p^2)}. \tag{4.37}$$

The self-energy $\Sigma(p^2)$ renormalizes the bare mass. However, the propagator must itself be renormalized again through multiplication by the wavefunction renormalization constant. The reason for this is that the physical propagator is defined to have a pole of residue 1 at the physical mass m^2. This mass is defined by the general relation [22]

$$m^2 \;=\; \mathring{m}^2 + \Sigma(m^2). \tag{4.38}$$

In our particular case the self-energy contains no terms higher than order p^2, so it has the form

$$\Sigma(p^2) \;=\; A + Bp^2. \tag{4.39}$$

The coefficients A and B have the appropriate energy dimensions (being ratios of dimensional quantities), and are associated with a certain *order* in the low-energy expansion. A is of order 4 and B is of order 2.

In general, the wavefunction renormalization constant is given by [22]

$$Z \;=\; \frac{1}{1 - \frac{d}{dp^2}\Sigma(p^2)|_{m^2}}. \tag{4.40}$$

For our particular calculation this means

$$Z \;=\; \frac{1}{1-B} \;\approx\; 1 + B \tag{4.41}$$

which is correct to order p^2 in the energy expansion. Any higher terms will not modify the Green function at the order to which we are calculating it, as will be seen later.

These definitions yield the renormalized propagator from the free propagator as follows:

$$\begin{aligned} i\,\Delta^{\mathrm{r}}(p) &= \frac{1}{Z}\,\frac{i}{p^2 - \mathring{m}^2 - \Sigma(p^2)} \\ &= (1-B)\,\frac{i}{p^2(1-B) - \mathring{m}^2 - A} \\ &= \frac{i}{p^2 - \frac{1}{(1-B)}(\mathring{m}^2 + A)} \\ &= \frac{i}{p^2 - m^2} \end{aligned} \tag{4.42}$$

with, to $O(p^4)$,

$$m^2 = \frac{(\mathring{m}^2 + A)}{(1-B)} \approx \mathring{m}^2\,(1+B) + A. \tag{4.43}$$

Writing relation (4.43) in terms of a mass renormalization constant, following equation (4.34), shows that

$$\begin{aligned} J &\approx 1 + \frac{A}{\mathring{m}^2} + B \\ &\approx 1 + \frac{\Sigma(m^2)}{\mathring{m}^2} \end{aligned} \tag{4.44}$$

so that we have both Z and J defined as functions of the self-energy.

4.4 Modifications Due to $\mathcal{L}_{4\,\mathrm{off}}$

We will concentrate only on the pion part of the theory henceforth. The wavefunction renormalization Z_π has been established for the standard theory by examining the form of the $\boldsymbol{\pi}$-field effective Lagrangian. This renormalization must be modified to account for the new additions to the Lagrangian. We approach the modification by calculating the self-energies $\Sigma_{P_1}(p^2)$ and $\Sigma_{P_2}(p^2)$ attributable to $\mathcal{L}_{4\,\mathrm{off}}$, and seeing how they affect Z_π.

First, here are the two new second derivative structures found in $\mathcal{L}_{4\,\mathrm{off}}$. These are expanded to $O(\phi^2)$, as were the terms in list C:

$$
\begin{aligned}
&\langle D^\mu D_\mu U^\dagger D^\nu D_\nu U\rangle \\
&\quad = \frac{1}{F_0^2}\langle \partial^\mu\partial_\mu\phi\partial^\nu\partial_\nu\phi\rangle \\
&\quad = \frac{2}{F_0^2}(\partial^2\boldsymbol{\pi}\cdot\partial^2\boldsymbol{\pi} + \partial^2\boldsymbol{K}\cdot\partial^2\boldsymbol{K} + \partial^2\boldsymbol{\eta}\cdot\partial^2\boldsymbol{\eta})
\end{aligned} \tag{4.45}
$$

$$
\begin{aligned}
&\langle D^\mu D_\mu U^\dagger \chi + D^\mu D_\mu U\chi^\dagger\rangle \\
&\quad = -\frac{2B_0}{F_0^2}\langle M(\partial^\mu\partial_\mu\phi\phi + 2\partial^\mu\phi\partial_\mu\phi + \phi\partial^\mu\partial_\mu\phi)\rangle \\
&\quad = -\frac{4}{F_0^2}[\overset{\circ}{m}{}_\pi^2(\partial^2\boldsymbol{\pi}\cdot\boldsymbol{\pi} + \partial^\mu\boldsymbol{\pi}\cdot\partial_\mu\boldsymbol{\pi}) + \overset{\circ}{m}{}_K^2(\partial^2\boldsymbol{K}\cdot\boldsymbol{K} + \partial^\mu\boldsymbol{K}\cdot\partial_\mu\boldsymbol{K}) \\
&\qquad\qquad + \overset{\circ}{m}{}_\eta^2(\partial^2\boldsymbol{\eta}\cdot\boldsymbol{\eta} + \partial^\mu\boldsymbol{\eta}\cdot\partial_\mu\boldsymbol{\eta})].
\end{aligned} \tag{4.46}
$$

Note that these expressions do not have the form of a canonical Lagrangian. This is why for $\mathcal{L}_{4\,\mathrm{off}}$ we must calculate the counterterms explicitly.

4.4.1 The Counterterms

One computes the additional counterterms by taking the matrix elements of the $\mathcal{L}_{4\,\mathrm{off}}$ interactions between states. These amplitudes, represented by diagram (b) of Figure 4.1, are self-energies. They are written, for the pion case [22],

$$
-i\Sigma(p^2) = \langle\pi^+(\boldsymbol{p})|\, i\int d^4x\, \mathcal{L}_{4\,\mathrm{off}}\, |\pi^+(\boldsymbol{p})\rangle \tag{4.47}
$$

where the $O(\phi^2)$ component of $\mathcal{L}_{4\,\mathrm{off}}$ contributes.

Refer to equation (3.84), which gives the form of the $\mathcal{L}_{4\,\mathrm{off}}$ terms. The following are the two terms expanded to $O(\phi^2)$, using the results C–5 and C–8 and equations (4.45) and (4.46) (the rest of the structures in these terms do not contribute at this order):

$$
\begin{aligned}
&P_1 [\, \langle \, D^\mu D_\mu U^\dagger D^\nu D_\nu U \; - \; (D^\mu D_\mu U^\dagger \chi \, + \, D^\mu D_\mu U \chi^\dagger) \; - \; D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U \\
&\qquad - \; D^\mu U^\dagger D_\mu U \, (\chi^\dagger U + U^\dagger \chi) \; - \; \frac{1}{4} (\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger) \; + \; \frac{1}{2} \chi^\dagger \chi \, \rangle \\
&\qquad + \; \frac{1}{12} \langle \chi^\dagger U \; - \; U^\dagger \chi \rangle^2 \,] \\
= \; & \frac{2P_1}{F_0^2} [\, \partial^2 \boldsymbol{\pi} \cdot \partial^2 \boldsymbol{\pi} \, + \, \partial^2 \boldsymbol{K} \cdot \partial^2 \boldsymbol{K} \, + \, \partial^2 \boldsymbol{\eta} \cdot \partial^2 \boldsymbol{\eta} \\
&\qquad + \, 2 \, \overset{\circ}{m}{}_\pi^2 \; \partial^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi} \, + \, 2 \, \overset{\circ}{m}{}_K^2 \; \partial^2 \boldsymbol{K} \cdot \boldsymbol{K} \, + \, 2 \, \overset{\circ}{m}{}_\eta^2 \; \partial^2 \boldsymbol{\eta} \cdot \boldsymbol{\eta} \\
&\qquad + \, \overset{\circ}{m}{}_\pi^4 \; \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \, \overset{\circ}{m}{}_K^4 \; \boldsymbol{K} \cdot \boldsymbol{K} + \, \overset{\circ}{m}{}_\eta^4 \; \boldsymbol{\eta} \cdot \boldsymbol{\eta} \,]
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
&P_2 [\, \langle \, (D^\mu D_\mu U^\dagger \chi \, + \, D^\mu D_\mu U \chi^\dagger) \, + \, D^\mu U^\dagger D_\mu U \, (\chi^\dagger U + U^\dagger \chi) \\
&\qquad + \, \frac{1}{2} (\chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger) \; - \; \chi^\dagger \chi \, \rangle \; - \; \frac{1}{6} \langle \chi^\dagger U \; - \; U^\dagger \chi \rangle^2 \,] \\
= \; & - \frac{4P_2}{F_0^2} [\, \overset{\circ}{m}{}_\pi^2 \; \partial^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \, \overset{\circ}{m}{}_K^2 \; \partial^2 \boldsymbol{K} \cdot \boldsymbol{K} + \, \overset{\circ}{m}{}_\eta^2 \; \partial^2 \boldsymbol{\eta} \cdot \boldsymbol{\eta} \\
&\qquad + \, \overset{\circ}{m}{}_\pi^4 \; \boldsymbol{\pi} \cdot \boldsymbol{\pi} + \, \overset{\circ}{m}{}_K^4 \; \boldsymbol{K} \cdot \boldsymbol{K} + \, \overset{\circ}{m}{}_\eta^4 \; \boldsymbol{\eta} \cdot \boldsymbol{\eta} \,].
\end{aligned}
\tag{4.49}
$$

We need to consider only the pion parts of these expressions. Performing a simple calculation of the type demonstrated in section 5.6 of the next chapter, we obtain the amplitudes of these interactions between pion states. That for the first term is, using (4.47),

$$
\Sigma_{P_1}(p^2) \;\; = \;\; -\frac{4P_1}{F_0^2} \, (p^2 - \, \overset{\circ}{m}{}_\pi^2)^2 \tag{4.50}
$$

and that for the second term is

$$
\Sigma_{P_2}(p^2) \;\; = \;\; -\frac{8P_2}{F_0^2} \; \overset{\circ}{m}{}_\pi^2 \; (p^2 - \, \overset{\circ}{m}{}_\pi^2). \tag{4.51}
$$

The first thing to notice is that these counterterms do not modify the physical mass expression that we found, equation (4.36). This follows from equation (4.44), because both Σ_{P_1} and Σ_{P_2} vanish when evaluated at the physical mass m_π^2, and taken to $O(p^4)$. These self-energy counterterms are already $O(p^4)$, so the difference between m_π^2 and $\mathring{m}_\pi^2$ will not form a contribution.

4.4.2 Pion Wavefunction Renormalization Constant

Self-energy contributions $\Sigma(p^2)$ modify the wavefunction renormalization according to equations (4.40) and (4.41):

$$Z \approx 1 + \frac{d}{dp^2}\Sigma(p^2)|_{m^2}. \tag{4.52}$$

Therefore, to find the modified Z_π, we simply add the first derivatives of expressions (4.50) and (4.51), evaluated at m_π^2:

$$Z_\pi \rightarrow Z_\pi + \frac{d}{dp^2}\Sigma_{P_1}(p^2)|_{m_\pi^2} + \frac{d}{dp^2}\Sigma_{P_2}(p^2)|_{m_\pi^2} \tag{4.53}$$

The P_1 term vanishes, but the second term is

$$\frac{d}{dp^2}\Sigma_{P_2}(p^2)|_{m_\pi^2} = -\frac{8P_2}{F_0^2}\, m_\pi^2. \tag{4.54}$$

Finally, then, we quote the basic result derived in this chapter, the renormalization constant

$$\begin{aligned} Z_\pi = {} & 1 - \frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_0^2}\, L_4 - \frac{8m_\pi^2}{F_0^2}\, L_5 - \frac{8m_\pi^2}{F_0^2}\, P_2 \\ & + \frac{m_\pi^2}{24\pi^2 F_0^2}\,(\, R + \ln\frac{m_\pi^2}{\mu^2}\,) + \frac{m_K^2}{48\pi^2 F_0^2}\,(\, R + \ln\frac{m_K^2}{\mu^2}\,). \end{aligned} \tag{4.55}$$

This constant will be required in the calculation of the pion form factor.

4.4.3 Renormalized Propagator

We showed the renormalized propagator in equation (4.42). For the pion, of physical mass m_π, this is

$$i\,\triangle^{\mathrm{r}}_{m_\pi^2}(p) = \frac{i}{p^2 - m_\pi^2}. \tag{4.56}$$

This is derived for the standard theory without off-shell counterterms. It incorporates Z_π from equation (4.33), and the self-energy implicit in that equation which leads to the physical mass m_π^2 shown in equation (4.36). The simple form of (4.56), having a p^2 term but no p^4 term, is a consequence of the form of the self-energy, equation (4.39), in the standard approach. Note that the propagator does, however, contain corrections at $O(p^4)$ in the energy expansion (i.e. m_π^2 is $O(p^4)$).

We now have finite modifications to both the self-energy and the wavefunction renormalization, and it is easy to apply these modifications to the propagator. Consider the origin of the renormalized propagator, from equation (4.42):

$$i\,\Delta^{\mathrm{r}}_{m_\pi^2}(p) \;=\; \frac{1}{Z_\pi}\,\frac{i}{p^2 - \mathring{m}_\pi^2 - \Sigma(p^2)}. \tag{4.57}$$

From the denominator we must simply subtract the additional self-energy counterterms (4.50) and (4.51)

$$\Sigma(p^2) \;\rightarrow\; \Sigma(p^2) - \frac{4P_1}{F_0^2}(p^2 - \mathring{m}_\pi^{\,2})^2 - \frac{8P_2}{F_0^2}\,\mathring{m}_\pi^{\,2}\,(p^2 - \mathring{m}_\pi^{\,2}) \tag{4.58}$$

and we must multiply the expression by the new wavefunction renormalization

$$Z_\pi \;\rightarrow\; Z_\pi - \frac{8P_2}{F_0^2}\,m_\pi^2 \tag{4.59}$$

that we have found from equation (4.53). Because Z_π is equal to 1 plus a small constant, we can obtain the same effect by forming a multiplicative renormalization

$$\frac{1}{Z_{P_2}} \;=\; \frac{1}{1 - \frac{8P_2}{F_0^2}\,m_\pi^2}, \tag{4.60}$$

and multiplying the original renormalized propagator from equation (4.56).

The modified renormalized propagator (4.56) is

$$\begin{aligned} i\,\Delta^{\mathrm{r}}_{m_\pi^2}(p) &= \left(\frac{1}{1 - \frac{8P_2}{F_0^2}\,m_\pi^2}\right)\frac{i}{p^2 - m_\pi^2 + \frac{4P_1}{F_0^2}(p^2 - m_\pi^2)^2 + \frac{8P_2}{F_0^2}\,m_\pi^2\,(p^2 - m_\pi^2)} \\ &= \frac{i}{p^2 - m_\pi^2 + \frac{4P_1}{F_0^2}(p^2 - m_\pi^2)^2} \end{aligned} \tag{4.61}$$

having kept only $O(p^4)$ terms. This result will be used at the end of the form factor calculation, to verify the Ward identity.

Chapter 5

The Form Factor Calculation

5.1 Introduction

The chiral perturbation theory formalism laid out in the preceding two chapters will now be used to examine the pion/photon interaction. Several definitions must first be made.

5.2 Definitions

The effective Lagrangian incorporates couplings to eight vector and eight axial currents through the matrices v_μ and a_μ in the covariant derivative. We can study a given current interacting with the almost-Goldstone bosons by considering it as an external probe introduced into the otherwise purely QCD effective Lagrangian.

The electromagnetic current transforms as a vector. It is defined as follows in terms of the gauge coupling matrices:

$$a_\mu = 0 \tag{5.1}$$

$$v_\mu = -eA_\mu Q \tag{5.2}$$

where $e > 0$ is the electromagnetic coupling constant and A_μ is the photon field. The current involves the diagonal charge matrix

$$Q = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}. \tag{5.3}$$

This matrix describes the relative strengths of the electromagnetic couplings of the up, down and strange quarks. With these definitions, the covariant derivative originally defined in equation (3.5) becomes

$$D_\mu U \;=\; \partial_\mu U + ieA_\mu\,[Q, U] \tag{5.4}$$

and we have for the field strength tensors of equations (3.12)

$$F^R_{\mu\nu} \;=\; F^L_{\mu\nu} \;=\; -e\,(\partial_\mu A_\nu \,-\, \partial_\nu A_\mu)\,Q. \tag{5.5}$$

Finally, we set the scalar and pseudoscalar densities to

$$\begin{aligned} s &= M \\ p &= 0 \end{aligned} \tag{5.6}$$

where M is the quark mass matrix. These define χ by equation (3.17):

$$\chi \;=\; 2B_0 M. \tag{5.7}$$

The pion electromagnetic form factors f^+_π and f^-_π come from the defining equation [23]

$$\begin{aligned} \Gamma^\mu_{\mathrm{r}}(p_i, p_f) \;=\;& (p_i + p_f)^\mu \;\; f^+(\,p_i^2,\, p_f^2,\, (p_f - p_i)^2\,) \\ +\;& (p_f - p_i)^\mu \;\; f^-(\,p_i^2,\, p_f^2,\, (p_f - p_i)^2\,) \end{aligned} \tag{5.8}$$

where Γ^μ_{r} is the renormalized 3-point Green function for the electromagnetic interaction. Γ^μ_{r} depends only on the initial and final meson momenta because of 4-momentum conservation at a vertex. The right hand side is the most general possible parametrization of this function. Thus, it consists of two independent linear combinations of the available Lorentz vectors, multiplied by functions of the three scalars shown.

We will refer to Γ^μ (whether renormalized or not) as the 'vertex function' since it plays this role in the language of the Feynman rules. One finds that the vertex function is related to the S-matrix by the following equation and definition:

$$-\, ie\, \epsilon_\mu(p_f - p_i)\; \Gamma^\mu(p_i, p_f) \;=\; i\, A_{fi}(p_i, p_f) \tag{5.9}$$

$$(2\pi)^4\, \delta^4(p_f - p_i - q)\; i\, A_{fi}(p_i, p_f) \;=\; S_{fi}(p_i, p_f, q) \tag{5.10}$$

where A_{fi} is known as the invariant amplitude. The delta function enforces 4-momentum conservation at the vertex.

The S-matrix, or scattering matrix, describes the transition from initial to final asymptotically free states through the interaction term in a Hamiltonian. The Dyson expansion of the S-matrix is a fundamental result [17]:

$$\begin{aligned} S &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n\; T\left(\mathcal{H}_I(x_1) \ldots \mathcal{H}_I(x_n)\right) \\ &= T\left(e^{-i \int d^4x\, \mathcal{H}_I(x)}\right) \end{aligned} \tag{5.11}$$

where the integral is over all space.

This definition is in terms of the Hamiltonian. We are working with the Lagrangian, and will use

$$S \;=\; \hat{T}\left(e^{\, i \int d^4x\, \mathcal{L}_I(x)}\right) \tag{5.12}$$

where $\hat{T}()$ is defined as the covariant time-ordered product [24]. In chiral perturbation theory the Lagrangian involves time derivatives of the fields, so one cannot trivially replace $\mathcal{H}$ by $-\mathcal{L}$. The difference between the two is, however, compensated by an appropriate definition of the time-ordering operator, leaving the same Feynman rule formalism in place. This framework is used in scalar electrodynamics in general. Thus the S-matrix can be computed conveniently from the Lagrangian using (5.12).

The S-matrix is a Hilbert space operator — an expression in quantum field operators. An S-matrix *element* is a specific element in this infinite-dimensional matrix. The S-matrix element (and by equations (5.9) and (5.10) the Γ^μ function), is a momentum-space function computed by evaluating the field operator expression between particle states:

$$S_{fi} \;=\; \langle f|\, S\, |i\rangle \tag{5.13}$$

where the generic f and i are the quantum numbers characterizing the states. In the case of the pion, since it is a scalar, these are just the energy-momentum.

We will be computing amplitudes using physical particle states, but these amplitudes will represent individual Feynman diagrams based on 'bare' Lagrangian field operators. Mass and wavefunction renormalization relate the bare Lagrangian to the effective Lagrangian, so in the same way, they relate our perturbation theory result to the real physical quantity. For the Green function Γ^μ, the basic result that will be required is

$$\Gamma^\mu_{\mathrm{r}} \;=\; Z_\pi \Gamma^\mu. \tag{5.14}$$

There is a further multiplicative renormalization — that of the coupling constant [22] — however, it can be shown to be 1 in the case under consideration, and we disregard it. Secondly, since we will be calculating a 1-particle irreducible Green function, the mass renormalization we have already carried out becomes an implicit part of the result, if we just substitute the physical mass in the Green function.

5.3 The Electromagnetic Current to $O(p^4)$

The electromagnetic current, which we denote J^μ, will be simply the field operator expression whose matrix element yields Γ^μ. It is the current J^μ which we are interested in finding first. A general electromagnetic interaction is written

$$\mathcal{L}_I \;=\; -eA_\mu J^\mu \tag{5.15}$$

where A_μ is the photon. The interaction Lagrangian has the form of a gauge field times a current. Our effective Lagrangian, although it contains many other interactions as well

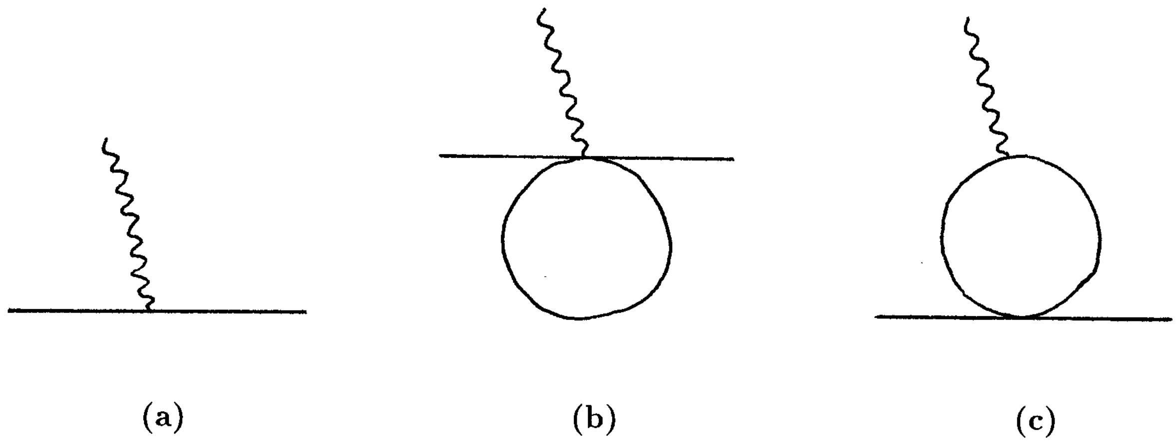

Figure 5.1: Electromagnetic Form Factor Diagrams

(in fact infinitely many), contains this interaction. One has only to isolate the terms in the Lagrangian containing the single-photon coupling eA_μ, and one has found the current, by

$$J^\mu = -\frac{\partial}{\partial(eA_\mu)}\mathcal{L}_I. \tag{5.16}$$

Contributions to the electromagnetic current derive from the three types of Feynman diagrams shown in Figure 5.1. To one-loop order, these are the only distinct graphs. In the present calculation, the external legs represent charged pions. The loops, however, represent any of the mesons in the theory.

It is important to understand the relationship between the diagrams and the formalism with which we are working, so we repeat some of the comments made in Chapter 2. In effective field theory, a given Feynman diagram consists of contributions from *all* orders of the Lagrangian. The Lagrangian denoted $\mathcal{L}_2$ contributes at $O(p^2)$, the leading order in the energy expansion, so it is naturally associated with the tree graph, diagram (a). However, any higher order Lagrangian $\mathcal{L}_n$ plays the role of supplying a correction to that same graph at $O(p^n)$. Using the modified Gasser and Leutwyler formalism, we have two orders, $\mathcal{L}_2$ and $\mathcal{L}_4 + \mathcal{L}_{4\,\mathrm{off}}$, of the full effective Lagrangian. We calculate the tree diagram from each of these Lagrangians. Then in calculating the loop diagrams (b) and (c), only $\mathcal{L}_2$ applies — the reason being that these diagrams calculated using $\mathcal{L}_4 + \mathcal{L}_{4\,\mathrm{off}}$ would supply corrections at $O(p^6)$ or $O(p^8)$ in the energy expansion, according to equation (2.11).

5.3.1 Tree-Level Contribution From $\mathcal{L}_2$

The Feynman diagram (a) of Figure 5.1 represents the collection of all terms from the Lagrangian having two meson fields ϕ and one photon field A_μ. It is straightforward to extract these terms. In the notation $O(\phi^n)$ used in the following, or when 'ϕ' is used in a generic sense, the field may in general represent ϕ or $\partial^\mu \phi$.

We expand the Lagrangian by expanding $U(\phi)$, and we retain only the terms of $O(\phi^2)$ which are also $O(eA_\mu)$. As will be seen in the following manipulations, the power of the matrix ϕ in a trace expression is equivalent to the power of the fundamental field π_i (or K_i or η) emerging from taking the trace. Thus we simply count powers of ϕ. For the purposes of expanding U and $U^\dagger$, we make use of the fact that ϕ is hermitean ($\phi^\dagger = \phi$). This means $U^\dagger$ is obtained from U simply by reversing the sign of the i in equation (2.8):

$$U^\dagger = e^{-\frac{i}{F_0}\phi}. \tag{5.17}$$

What does the Lagrangian $\mathcal{L}_2$ contribute to the current J^μ at tree-level? The field A_μ that we are interested in will only come from the covariant derivative-containing term of the Lagrangian:

$$\frac{F_0^2}{4}\langle D^\mu U^\dagger D_\mu U \rangle.$$

(Refer to equation (3.75).)

We illustrate the two basic steps in extracting the current. First one expands the Lagrangian in ϕ and selects the terms appropriate to the topology of the Feynman diagram in question. Secondly, one multiplies out the ϕ's and other matrices, takes the trace in flavour $SU(3)$ space, and selects again — this time for the two mesons corresponding to the external legs of the Feynman diagram.

Here is what happens when the above term is expanded to second order in ϕ to make sure all possible $O(\phi^2)$ terms are uncovered:

$$\frac{F_0^2}{4}\langle D^\mu U^\dagger D_\mu U\rangle$$

$$\begin{aligned}
&= \frac{F_0^2}{4}\langle [\ \partial^\mu(-\frac{i}{F_0}\phi - \frac{1}{2F_0^2}\phi\phi) - ieA^\mu(-\frac{i}{F_0}\phi Q + \frac{i}{F_0}Q\phi - \frac{1}{2F_0^2}\phi\phi Q + \frac{1}{2F_0^2}Q\phi\phi)\ ] \\
&\qquad [\ \partial_\mu(+\frac{i}{F_0}\phi - \frac{1}{2F_0^2}\phi\phi) + ieA_\mu(+\frac{i}{F_0}Q\phi - \frac{i}{F_0}\phi Q - \frac{1}{2F_0^2}Q\phi\phi + \frac{1}{2F_0^2}\phi\phi Q)\ ]\rangle \\
&= \frac{F_0^2}{4}\langle\ -\frac{eA_\mu}{F_0}Q\phi(-\frac{i}{F_0}\partial^\mu\phi)\ +\ \frac{eA_\mu}{F_0}\phi Q(-\frac{i}{F_0}\partial^\mu\phi) \\
&\qquad +\frac{eA^\mu}{F_0}(\frac{i}{F_0}\partial_\mu\phi)Q\phi\ -\ \frac{eA^\mu}{F_0}(\frac{i}{F_0}\partial_\mu\phi)\phi Q\ \rangle \\
&= \frac{ieA_\mu}{2}\langle\ Q(\phi\partial^\mu\phi\ -\ \partial^\mu\phi\phi)\ \rangle.
\end{aligned} \tag{5.18}$$

Remember that $D^\mu U^\dagger D_\mu U$ has the meaning $(D^\mu U^\dagger)D_\mu U$, and similarly, $\partial^\mu\phi\phi = (\partial^\mu\phi)\,\phi$. At the second step we retained the $O(\phi^2)$ terms, and simply dropped everything else. We will not indicate the implicit presence of extra terms of different order in these types of expansions. (This will not be a problem, since each expansion is carried out in a specific context with a specific extraction in mind.)

First we noted that an eA_μ term from the first factor must multiply a term not containing eA_μ from the second factor, and vice versa, and then we concerned ourselves with the powers of ϕ occurring at leading order in the two factors. The first term of the exponential expansion is the identity. Since $\partial_\mu 1 = 0$, and furthermore, $Q1 - 1Q = 0$, there is already one power of ϕ at leading order throughout the expression; none but the leading order terms contributed in this case.

We will describe in more detail the process of expanding in ϕ and selecting terms. To derive equation (5.18), one must understand the roles of the fields ϕ or π_i, the field A_μ, the charge matrix Q, and the action of the trace and of the derivative ∂_μ.

First of all, it must be kept in mind that the exponential U forms an expansion in matrices, which don't commute, so one must maintain the ordering in this expansion. To emphasize that ϕ is a matrix, we have written out the expansion terms explicitly, without using the abbreviation ϕ^2, for example. The important point to keep in mind is just that $\partial^\mu\phi\phi \neq \phi\partial^\mu\phi$ and $Q\phi \neq \phi Q$.

The cyclic property of the trace is used to write the matrix expressions in a convenient form; for example, to move Q always to the front. This does not constitute any actual re-ordering, of course.

Taking the trace of a matrix expression involving ϕ's reduces it to an expression in the field operators π_i, K_i and η. These are not matrices, but simply operators in Hilbert space. The meson field operators and their derivatives freely commute, so that after the trace has been taken, one may write equivalently $\partial^\mu\pi^+\pi^-$ or $\pi^-\partial^\mu\pi^+$, etc. These expressions are simply sequences of creation and annihilation operators (see equation (5.68), for example, which defines the π^+). On this point, it is important to bear in mind that the Lagrangian is defined not to be normal ordered.

The ordinary derivative $\partial_\mu = \partial/\partial x^\mu$ acts on the fields $\phi(x)$ and $A_\mu(x)$. (Derivatives of the latter will be of concern when we are dealing with double covariant derivatives or field strength tensors.) The charge matrix Q is just a constant, as is the mass matrix M.

The next step is to explicitly multiply out the matrices in equation (5.18) and take the trace. In the present derivation we can save a lot of unnecessary labour by taking advantage of the fact that $SU(2)$ is a subgroup of $SU(3)$. If we neglect all of the fields in the octet except for the pions, we are working with the matrix

$$\phi = \begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ & 0 \\ \sqrt{2}\pi^- & -\pi^0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.19}$$

which contains the $SU(2)$ triplet of pions as a sub-matrix. In the resulting expression, we will obtain all possible pion terms, and it is only these terms that we want.

At this stage also we apply equation (5.16). We drop the eA_μ from the expression and reverse the sign to recover the following electromagnetic current:

$$\begin{aligned} J^\mu_{2\,\text{tree}} &= -\frac{i}{2}\,\text{trace}\left\{ \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix} \begin{bmatrix} \partial^\mu\pi^0 & \sqrt{2}\partial^\mu\pi^+ \\ \sqrt{2}\partial^\mu\pi^- & -\partial^\mu\pi^0 \end{bmatrix} \right. \\ &\qquad \left. - \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \partial^\mu\pi^0 & \sqrt{2}\partial^\mu\pi^+ \\ \sqrt{2}\partial^\mu\pi^- & -\partial^\mu\pi^0 \end{bmatrix} \begin{bmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{bmatrix} \right\} \\ &= -\frac{i}{2}\Big[\ \frac{2}{3}(\pi^0\partial^\mu\pi^0 + 2\pi^+\partial^\mu\pi^-) - \frac{1}{3}(2\pi^-\partial^\mu\pi^+ + \pi^0\partial^\mu\pi^0) \\ &\qquad - \frac{2}{3}(\partial^\mu\pi^0\pi^0 + 2\partial^\mu\pi^+\pi^-) + \frac{1}{3}(2\partial^\mu\pi^-\pi^+ + \partial^\mu\pi^0\pi^0)\ \Big] \\ &= -\frac{i}{6}[\,4\pi^+\partial^\mu\pi^- - 2\pi^-\partial^\mu\pi^+ - 4\pi^-\partial^\mu\pi^+ + 2\pi^+\partial^\mu\pi^-\,] \\ &= -\,i\,[\,\pi^+\partial^\mu\pi^- - \pi^-\partial^\mu\pi^+\,]. \end{aligned} \tag{5.20}$$

Note that all neutral fields disappeared, as they must in a minimal coupling to the photon.

Let us transform this current to the Pauli basis, writing it in terms of the fundamental $SU(2)$ fields indexed by a number (which represents the isospin component). The transformation

$$\begin{aligned}\pi^+ &= \frac{\pi_1 - i\,\pi_2}{\sqrt{2}} \\ \pi^- &= \frac{\pi_1 + i\,\pi_2}{\sqrt{2}}\end{aligned} \tag{5.21}$$

yields the following identity:

$$\begin{aligned} i\,(\pi^+\partial^\mu\pi^- \;-\; \pi^-\partial^\mu\pi^+) &= -\pi_1\partial^\mu\pi_2 \;+\; \pi_2\partial^\mu\pi_1 \\ &= -(\boldsymbol{\pi} \times \partial^\mu\boldsymbol{\pi})_3 \end{aligned} \tag{5.22}$$

which in the present case gives

$$J^\mu_{2\,\mathrm{tree}} = (\boldsymbol{\pi} \times \partial^\mu\boldsymbol{\pi})_3. \tag{5.23}$$

The electromagnetic current of the pion always has the form of the third component of a vector in isospin space.

5.3.2 Contribution From $\mathcal{L}_4$

The tree diagram (a) is comprised of all the first order interactions that we can find in $\mathcal{L}_4$ and $\mathcal{L}_{4\,\mathrm{off}}$, in addition to the one found in the previous section. As we will see, contributions from these Lagrangians will correct the leading order constant term of the form factor (i.e. 1) with terms that are $O(p^2)$.

Expanding the $\mathcal{L}_4$ part, equation (3.76), we find that only the terms with the coefficients L_4, L_5, and L_9 have an $O(\phi^2)$ component also containing eA_μ.

The term with coefficient L_4 yields the following $O(eA_\mu \phi^2)$ components:

$$
\begin{aligned}
& L_4 \langle D^\mu U^\dagger D_\mu U\rangle\langle\chi^\dagger U + \chi U^\dagger\rangle \\
& \qquad = L_4 \frac{4ieA_\mu}{F_0^2} \langle Q\,(\phi\partial^\mu\phi - \partial^\mu\phi\phi)\,\rangle\,\langle 2B_0 M\rangle. \qquad (5.24)
\end{aligned}
$$

In taking these traces, there is a subtle point to note. Even though in evaluating (5.20) we were able to make the replacement of the $SU(3)$ matrix by the $SU(2)$ sub-matrix, we must use the 3×3 matrices in general, because we have renormalized for $SU(3)$. (One can renormalize the theory for $SU(2)$ and work with only pions from the outset, as illustrated in [10].) Thus, for the present reduction of (5.24), it will be found that the $SU(2)$ substitution can be made in the first trace, because we are simply picking out the pion fields; however, the $SU(3)$ M matrix must be substituted in the second trace.

The factor from the first trace will be the same as that derived previously for the $\mathcal{L}_2$ current.

The mass matrix part is

$$
\begin{aligned}
\langle 2B_0 M\rangle &= 2B_0\,(m_u + m_d + m_s) \\
&= 4B_0\hat{m} \,+\, 2B_0 m_s \\
&= 2m_\pi^2 - 2m_K^2 + 3m_\eta^2 \qquad (5.25)
\end{aligned}
$$

using the isospin approximation (4.16), $m_u = m_d = \hat{m}$, and applying equations (4.21) and (4.24).

The resulting electromagnetic current is

$$
J^\mu_{L_4} \;=\; \frac{8\,(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_\pi^2}\,L_4\,(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3 \qquad (5.26)
$$

after transforming to the Pauli basis. Again, note the structure of the current.

Turning to the coefficient-L_5 term, the derivation differs from the above because the mass matrix multiplies the meson fields *before* the trace is taken. The Lagrangian term looks like this:

$$L_5 \langle D^\mu U^\dagger D_\mu U (\chi^\dagger U + U^\dagger \chi)\rangle$$

$$= L_5 \frac{4ieA_\mu}{F_0^2} \langle 2B_0 MQ (\phi\partial^\mu\phi - \partial^\mu\phi\phi) \rangle. \quad (5.27)$$

A particular quark mass from the diagonal of M will go with each field operator term.

We do the matrix multiplication, take the trace, and write the electromagnetic current. The neutral pion terms still cancel, and the remaining terms are

$$J^\mu_{L_5} = -\frac{8i}{F_\pi^2} L_5 [\, 2B_0(\frac{2}{3}m_u + \frac{1}{3}m_d)\, \pi^+\partial^\mu\pi^- - 2B_0(\frac{2}{3}m_u + \frac{1}{3}m_d)\, \pi^-\partial^\mu\pi^+ \,]. \quad (5.28)$$

Invoking the isospin approximation allows us to use equation (4.21), which puts an m_π^2 in front of each of the above terms. We again use the identity (5.22) to change the pion fields to the Pauli basis, and the expression becomes simply

$$J^\mu_{L_5} = \frac{8\, m_\pi^2}{F_\pi^2} L_5\, (\boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi})_3. \quad (5.29)$$

Finally, we find the current associated with the L_9 term. We insert expression (5.5) for $F^R_{\mu\nu}$ and $F^L_{\mu\nu}$, which we have seen are equal:

$$-\, i\, L_9 \langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle$$

$$= -i\, L_9 \langle\, e\, (\partial_\nu A_\mu - \partial_\mu A_\nu)\, Q\, (D^\mu U D^\nu U^\dagger + D^\mu U^\dagger D^\nu U) \rangle. \quad (5.30)$$

The field strength tensors contain only derivatives $\partial_\nu A_\mu$, so it is not obvious that the A_μ structure will be recovered from the L_9 expansion. One must consider all *possible* contributions from the effective Lagrangian, which in this case means we must uncover the A_μ structure by dropping a total divergence from the Lagrangian. In section 3.3.9 it was explained that any total divergence can be neglected; in this context, we have

$$\partial_\nu(A_\mu J(\phi)) \;=\; \partial_\nu A_\mu J(\phi) + A_\mu \partial_\nu J(\phi) \tag{5.31}$$

so evidently we can make the replacement

$$\partial_\nu A_\mu J(\phi) \;\rightarrow\; - A_\mu \partial_\nu J(\phi) \tag{5.32}$$

and the L_9 field strength term actually gives an electromagnetic vertex contribution.

The current is derived as follows. First we expand the Lagrangian expression (5.30), and as usual keep the terms of $O(\phi^2)$. Because of the $e\partial_\mu A_\nu$ terms, the entire expression is of $O(eA_\mu)$. We apply relation (5.32), and in the last step we swap the dummy indices $\mu \leftrightarrow \nu$ in one of the terms:

$$\begin{aligned}
&- i\, L_9 \langle F^R_{\mu\nu} D^\mu U D^\nu U^\dagger + F^L_{\mu\nu} D^\mu U^\dagger D^\nu U \rangle \\
&\qquad = \; - L_9 \frac{2ie}{F_0^2} \langle Q\, [\, \partial_\nu A_\mu (\partial^\mu\phi\partial^\nu\phi) \;-\; \partial_\mu A_\nu (\partial^\mu\phi\partial^\nu\phi)\,]\,\rangle \\
&\qquad \rightarrow \; L_9 \frac{2ie}{F_0^2} \langle Q\, [\, A_\mu \partial_\nu (\partial^\mu\phi\partial^\nu\phi) \;-\; A_\nu \partial_\mu (\partial^\mu\phi\partial^\nu\phi)\,]\,\rangle \\
&\qquad = \; L_9 \frac{2ieA_\mu}{F_0^2} \langle Q\, \partial_\nu (\partial^\mu\phi\partial^\nu\phi \;-\; \partial^\nu\phi\partial^\mu\phi)\,\rangle .
\end{aligned} \tag{5.33}$$

From this, the current is

$$J^\mu_{L_9} \;=\; \frac{4}{F_\pi^2} L_9\, \partial_\nu (\partial^\mu \boldsymbol{\pi} \times \partial^\nu \boldsymbol{\pi})_3 . \tag{5.34}$$

Note that the effective interaction underlying this result involves the second derivative of a field.

This is the third and last component of the current contributed by $\mathcal{L}_4$.

5.3.3 Contribution From $\mathcal{L}_{4\,\mathrm{off}}$

We consider the P_1 part of equation (3.84):

$$
\begin{aligned}
P_1 [\; & \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \\
& - \langle D^\mu D_\mu U^\dagger \chi + D^\mu D_\mu U \chi^\dagger \rangle \\
& - \langle D^\mu U^\dagger D_\mu U D^\nu U^\dagger D_\nu U \rangle \\
& - \langle D^\mu U^\dagger D_\mu U (\chi^\dagger U + U^\dagger \chi) \rangle \\
& - \frac{1}{4} \langle \chi^\dagger U \chi^\dagger U + \chi U^\dagger \chi U^\dagger \rangle \\
& + \frac{1}{2} \langle \chi^\dagger \chi \rangle \\
& + \frac{1}{12} \langle \chi^\dagger U - U^\dagger \chi \rangle^2 \;].
\end{aligned}
$$

At $O(eA_\mu \phi^2)$, there are nonzero contributions only from the first and fourth terms. The first term expands as

$$
\begin{aligned}
P_1 \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \;=\; & P_1 \frac{ie}{F_0^2} \langle \, 2 \partial^\mu A_\mu Q \, (\phi \partial^\nu \partial_\nu \phi - \partial^\nu \partial_\nu \phi \phi) \\
& + 4 \, A_\mu Q \, (\partial^\mu \phi \partial^\nu \partial_\nu \phi - \partial^\nu \partial_\nu \phi \partial^\mu \phi) \, \rangle
\end{aligned}
\tag{5.35}
$$

where, dropping a total divergence as we did previously, we are able to write this in terms of the photon field without derivative:

$$
\begin{aligned}
P_1 \langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle \;\to\; & P_1 \frac{2ieA_\mu}{F_0^2} \langle Q \, (\partial^\mu \partial^2 \phi \phi - \phi \partial^\mu \partial^2 \phi \\
& + \partial^\mu \phi \partial^2 \phi - \partial^2 \phi \partial^\mu \phi) \, \rangle.
\end{aligned}
\tag{5.36}
$$

The fourth term, following equation (5.27), is

$$
\begin{aligned}
& - P_1 \langle D^\mu U^\dagger D_\mu U \, (\chi^\dagger U + U^\dagger \chi) \rangle \\
& \qquad = \; - P_1 \frac{4ieA_\mu}{F_0^2} \langle \, 2 B_0 M Q \, (\phi \partial^\mu \phi - \partial^\mu \phi \phi) \, \rangle.
\end{aligned}
\tag{5.37}
$$

We substitute the matrices and reduce equations (5.36) and (5.37) to the explicit π_i operator form to be used in the calculation. They yield the electromagnetic current

$$J^{\mu}_{P_1} = \frac{4}{F_0^2} P_1 \left[(\partial^{\mu}\partial^2 \boldsymbol{\pi} \times \boldsymbol{\pi})_3 + (\partial^{\mu}\boldsymbol{\pi} \times \partial^2 \boldsymbol{\pi})_3 - 2m_{\pi}^2 (\boldsymbol{\pi} \times \partial^{\mu}\boldsymbol{\pi})_3 \right]. \quad (5.38)$$

From the P_2 part of the off-shell Lagrangian:

$$\begin{aligned} P_2 [& \langle D^{\mu} D_{\mu} U^{\dagger} \chi + D^{\mu} D_{\mu} U \chi^{\dagger} \rangle \\ & + \langle D^{\mu} U^{\dagger} D_{\mu} U (\chi^{\dagger} U + U^{\dagger} \chi) \rangle \\ & + \frac{1}{2} \langle \chi^{\dagger} U \chi^{\dagger} U + \chi U^{\dagger} \chi U^{\dagger} \rangle \\ & - \langle \chi^{\dagger} \chi \rangle \\ & - \frac{1}{6} \langle \chi^{\dagger} U - U^{\dagger} \chi \rangle^2] \end{aligned}$$

only the second term contributes to the current, and this is the same contribution that we have already seen. Thus

$$J^{\mu}_{P_2} = \frac{8m_{\pi}^2}{F_0^2} P_2 \, (\boldsymbol{\pi} \times \partial^{\mu}\boldsymbol{\pi})_3. \quad (5.39)$$

5.3.4 The Tadpole Diagram

The tadpole diagram is diagram (b) of Figure 5.1. In our calculation there are actually four Feynman diagrams of this form — the loop can be a pion, either charged or neutral, or a kaon, either charged or neutral.

We begin by considering the structure of the diagram and how it is derived from our Lagrangian. It consists of a 4-meson/one photon interaction at a single space-time point x. There exists an interaction term of this form in the effective Lagrangian. It is the $O(eA_{\mu}\phi^4)$ term. The origin of the loop, then, is in the Wick's theorem contractions that occur in the evaluation of the S-matrix element for two 'external' pions from this originating interaction. (Strictly speaking, at the end of the calculation we will be considering the 'external' pions to be off-shell. This will be discussed further in section 5.6.) Diagram (b) is simply all single contractions of $O(eA_\mu\phi^4)$ terms in $\mathcal{L}_2$. It should be recalled that we will not be including contractions from $\mathcal{L}_4$ and $\mathcal{L}_{4\,\mathrm{off}}$ at all.

Our starting point therefore is the first term in the Dyson-Wick expansion of equation (5.12):

$$S_{(\mathrm{b})} = \hat{T}\left(+i\int d^4x\,\mathcal{L}_{(\mathrm{b})}(x)\right)$$

This term in the expansion contains only one space-time variable x, which is what we require.

We proceed to extract the $\mathcal{L}_{(\mathrm{b})}$ interaction and derive the current from it. For illustration, all the steps — expanding and collecting the terms, taking the trace, and then forming the possible contractions — will be shown.

The interaction derives from the covariant derivative-containing part of $\mathcal{L}_2$. As we have seen in the expansion of $\mathcal{L}_2$ leading to equation (5.18), the leading order in the covariant derivative is at $O(\phi)$. This implies that here we only need to keep terms up to $O(\phi^3)$ in each covariant derivative expansion, since we are interested in an $O(\phi^4)$ expression overall. The following terms are collected (neglecting the terms without eA_μ):

$$\frac{F_0^2}{4}\langle D^\mu U^\dagger D_\mu U\rangle$$

$$\begin{aligned}= \frac{F_0^2}{4}\langle[\; &-\frac{i}{F_0}\partial^\mu\phi - \frac{1}{2F_0^2}\partial^\mu\phi\phi - \frac{1}{2F_0^2}\phi\partial^\mu\phi + \frac{i}{6F_0^3}\partial^\mu\phi\phi\phi + \frac{i}{6F_0^3}\phi\partial^\mu\phi\phi + \frac{i}{6F_0^3}\phi\phi\partial^\mu\phi \\ &+ ieA^\mu(-\frac{i}{F_0}Q\phi + \frac{i}{F_0}\phi Q - \frac{1}{2F_0^2}Q\phi\phi + \frac{1}{2F_0^2}\phi\phi Q + \frac{i}{6F_0^3}Q\phi\phi\phi - \frac{i}{6F_0^3}\phi\phi\phi Q)\;]\end{aligned}$$

$$\times [\ +\frac{i}{F_0}\partial_\mu\phi - \frac{1}{2F_0^2}\partial_\mu\phi\phi - \frac{1}{2F_0^2}\phi\partial_\mu\phi - \frac{i}{6F_0^3}\partial_\mu\phi\phi\phi - \frac{i}{6F_0^3}\phi\partial_\mu\phi\phi - \frac{i}{6F_0^3}\phi\phi\partial_\mu\phi$$

$$+ ieA_\mu(+\frac{i}{F_0}Q\phi - \frac{i}{F_0}\phi Q - \frac{1}{2F_0^2}Q\phi\phi + \frac{1}{2F_0^2}\phi\phi Q - \frac{i}{6F_0^3}Q\phi\phi\phi + \frac{i}{6F_0^3}\phi\phi\phi Q)\ ]\rangle$$

$$\begin{aligned} = \ & \frac{ieA_\mu}{4F_0^2}\langle -\frac{1}{3}Q\,\phi\,(\partial^\mu\phi\phi\phi + \phi\partial^\mu\phi\phi + \phi\phi\partial^\mu\phi) \\ & + \frac{1}{3}Q\,(\partial^\mu\phi\phi\phi + \phi\partial^\mu\phi\phi + \phi\phi\partial^\mu\phi)\,\phi \\ & + \frac{1}{2}Q\,(\phi\phi\partial^\mu\phi\phi + \phi\phi\phi\partial^\mu\phi) \\ & - \frac{1}{2}Q\,(\partial^\mu\phi\phi\phi\phi + \phi\partial^\mu\phi\phi\phi) \\ & - \frac{1}{3}Q\,(\phi\phi\phi\partial^\mu\phi - \partial^\mu\phi\phi\phi\phi)\ \rangle \end{aligned}$$

$$= \frac{ieA_\mu}{4F_0^2}\langle \frac{1}{6}Q\,\partial^\mu\phi\phi\phi\phi - \frac{1}{2}Q\,\phi\partial^\mu\phi\phi\phi + \frac{1}{2}Q\,\phi\phi\partial^\mu\phi\phi - \frac{1}{6}Q\,\phi\phi\phi\partial^\mu\phi\ \rangle. \tag{5.40}$$

Now the trace is taken. The use of $SU(3)$ matrices takes account of all possible loops, whether pion, kaon or eta. Consider the following three examples of field operator expressions that might emerge from taking the trace:

$$\partial^\mu\pi^+\pi^-\pi^+\pi^-$$

$$\partial^\mu\pi^+\pi^-K^+K^-$$

$$\partial^\mu K^+K^-K^+K^-$$

A contraction within the first will yield a pion loop for the pion form factor. From the second, one will obtain a kaon loop. The third, of course, will not apply in this calculation, since two external pions are required. The second and third types will be of interest in calculating the kaon form factor.

For the present illustrative example, we substitute only the $SU(2)$ matrices and pick out the pion loop component. However, the full expansion must be made, from which the other components may be picked out. It turns out that only charged and neutral kaon fields are generated. There is no eta loop in the case of the pion current.

The following, then, is the electromagnetic current with four external pion fields:

$$
\begin{aligned}
J^{\mu}_{2\,(4\pi)} & \\
= \frac{i}{4F_0^2}\Big[\ & \frac{1}{18}\left(-4\partial^\mu\pi^+\pi^-\pi^0\pi^0 - 8\partial^\mu\pi^+\pi^-\pi^+\pi^- + 2\pi^+\partial^\mu\pi^-\pi^0\pi^0 + 4\pi^+\partial^\mu\pi^-\pi^+\pi^-\right) \\
& -\frac{1}{6}\left(+2\partial^\mu\pi^+\pi^-\pi^0\pi^0 + 4\partial^\mu\pi^+\pi^-\pi^+\pi^- - 4\pi^+\partial^\mu\pi^-\pi^0\pi^0 - 8\pi^+\partial^\mu\pi^-\pi^+\pi^-\right) \\
& +\frac{1}{6}\left(-4\partial^\mu\pi^+\pi^-\pi^0\pi^0 - 8\partial^\mu\pi^+\pi^-\pi^+\pi^- + 2\pi^+\partial^\mu\pi^-\pi^0\pi^0 + 4\pi^+\partial^\mu\pi^-\pi^+\pi^-\right) \\
& -\frac{1}{18}\left(+2\partial^\mu\pi^+\pi^-\pi^0\pi^0 + 4\partial^\mu\pi^+\pi^-\pi^+\pi^- - 4\pi^+\partial^\mu\pi^-\pi^0\pi^0 - 8\pi^+\partial^\mu\pi^-\pi^+\pi^-\right)\Big] \\
= \frac{i}{3F_0^2}\Big[\ & \pi^+\partial^\mu\pi^-\pi^0\pi^0 - \partial^\mu\pi^+\pi^-\pi^0\pi^0 + 2\,\pi^+\partial^\mu\pi^-\pi^+\pi^- - 2\,\partial^\mu\pi^+\pi^-\pi^+\pi^-\ \Big]. \qquad (5.41)
\end{aligned}
$$

It remains to perform all possible contractions. A contraction between two of the fields is equal to the Feynman propagator $i\,\triangle_{m_\pi^2}(x-x) = i\,\triangle_{m_\pi^2}(0)$. As we discovered in evaluating these expressions, a propagator incorporating a single derivative vanishes (see equation (4.12)); therefore, we only form contractions among the non-derivative fields. The following are all the non-vanishing contractions:

$$
\begin{aligned}
J^{\mu}_{(\mathrm{b})\,(\pi)} = \frac{i}{3F_0^2}\Big[\ & \pi^+\partial^\mu\pi^-\ \underbrace{\pi^0\pi^0} - \partial^\mu\pi^+\pi^-\ \underbrace{\pi^0\pi^0} \\
& + 2\,\pi^+\partial^\mu\pi^-\ \underbrace{\pi^+\pi^-} - 2\,\partial^\mu\pi^+\pi^-\ \underbrace{\pi^+\pi^-} \\
& + 2\,\underbrace{\pi^+\partial^\mu\pi^-\pi^+\pi^-} - 2\,\partial^\mu\pi^+\underbrace{\pi^-\pi^+}\pi^-\ \Big]. \qquad (5.42)
\end{aligned}
$$

Remember that, as shown in equation (4.29), the exact isospin approximation means there is actually only one propagator, with mass m_π^2. Consequently, the current takes on the same isospin vector form that we have already seen:

$$\begin{aligned} J^\mu_{(\mathrm{b})\ (\pi)} &= \frac{5i}{3F_0^2}\,[\ \pi^+\partial^\mu\pi^- \ - \ \pi^-\partial^\mu\pi^+\ ]\ i\triangle_{m_\pi^2}(0) \\ &= -\frac{5}{3F_0^2}\,(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3\ i\triangle_{m_\pi^2}(0) \\ &= -\frac{5}{3F_0^2}\,(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3\ I(m_\pi^2) \end{aligned} \tag{5.43}$$

where we have substituted the properly regularized integral $I(m_\pi^2)$ defined in equations (4.3) — (4.10).

Expanding equation (5.40) in $SU(3)$ reveals a kaon loop contribution of the same form as the above. We do not show the details; the only difference is that when one forms contractions between the kaon fields, one obtains $I(m_K^2)$. Furthermore, there is a difference of an overall factor of 2. The contribution is

$$J^\mu_{(\mathrm{b})\ (\mathrm{K})} \ = \ -\frac{5}{6F_0^2}\,(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3\ I(m_K^2). \tag{5.44}$$

5.4 The 2-Vertex Loop Diagram

Diagram (c) of Figure 5.1 represents two separate interactions. The derivation of the vertex function for this diagram is in principle the same as that for the tree and (b) diagrams — one picks out the appropriate terms from the Lagrangian and evaluates their matrix element according to equation (5.13).

The difference is that here the process being examined is second order in the Dyson-Wick expansion. Specifically, consider this term in equation (5.12):

$$S_{(\mathrm{c})} \ = \ \hat{T}\left(-\frac{2}{2!}\int d^4x\,d^4y\ \mathcal{L}_{2\,(\mathrm{a})}(x)\ \mathcal{L}_{2\,(4\phi)}(y)\right). \tag{5.45}$$

Here the subscript (a) refers to Feynman diagram (a), the tree diagram, and the subscript (4ϕ) denotes a 4-meson interaction. The 4-meson interaction is like that extracted in the previous section for the tadpole diagram, except without the photon. We have actually expanded to find $\mathcal{L}_{2\,(4\phi)}$ already in the context of renormalization (see equation (4.19)).

This S-matrix term contains the two interaction vertices of diagram (c); we will find the diagram itself if we select out the terms containing two contractions, which thus have two external fields left over. The contractions must each be between the interaction at x and the interaction at y. This yields the form of two propagators connecting the two vertices to comprise the required loop. Alternative contractions lead to disconnected diagrams. Note that there are actually two expressions constituting (5.45), each with the order of the interactions reversed. This is the reason for the factor of 2 cancelling the 2! from the Taylor series expansion.

Because the two Lagrangians are evaluated at separate space-time points, we cannot put equation (5.45) in the form of a current J^μ as we have done in the other cases. We must evaluate the S-matrix itself. In the rest of this section we will discuss how this is accomplished.

We could expand equation (5.45) using the results we already have, equations (4.19) and (5.18):

$$\begin{aligned} S_{(c)} &= \hat{T}\Big(-\int d^4x\, d^4y\; \mathcal{L}_{2\,(\mathrm{a})}(x)\; \mathcal{L}_{2\,(4\phi)}(y)\Big) \\ &= \hat{T}\Bigg(-\int d^4x\, d^4y\; \frac{ieA_\mu(x)}{2} \langle\, Q\,(\,\phi\partial^\mu\phi \,-\, \partial^\mu\phi\phi\,)\,\rangle\,(x) \\ &\qquad\qquad \times \frac{1}{24F_0^2} \langle\, \partial^\nu\phi\phi\partial_\nu\phi\phi \,-\, \partial^\nu\phi\partial_\nu\phi\phi\phi \,+\, B_0M\phi\phi\phi\phi\,\rangle\,(y)\Bigg)\,. \end{aligned} \tag{5.46}$$

and then form two contractions in all possible ways between the x and y fields, leaving only two external fields in the expression. Doing this, contractions of the following form would appear:

$$\partial_\mu i \triangle_{m_\phi^2}(x-y) = \partial_\mu \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{k^2 - m_\phi^2 + i\epsilon}. \tag{5.47}$$

Because of the presence of the exponential, this expression does not vanish as did the single-derivative propagator of equation (4.12). The calculation becomes somewhat involved owing to this.

However, there is no need to perform the computation from first principles. One finds that this procedure leads to an expression involving two Feynman propagators and two vertex factors — which is nothing but the expression one would write down using the Feynman rules. All we really need to do is to derive the Feynman rules, a procedure which is much simpler than the above. We compute the vertex factor for the photon/meson interaction, and that for the 4-meson interaction. The former will turn out to be the standard result for the photon/scalar interaction [25]

$$-ie\Gamma^\mu_{(\mathrm{a})} = -ie\,(p_i^\mu + p_f^\mu)$$

where $\Gamma^\mu_{(\mathrm{a})}$ is the tree diagram vertex function, which we will actually be calculating in section 5.6.3. The latter, $\Gamma_{(4\phi)}$, we can find as well, after we have discussed how to calculate vertex amplitudes.

For now, we write down the expression for diagram (c). The Feynman rules tell us to construct this diagram as follows. Include

- Two momentum-space meson propagators for the internal lines
- A 4-momentum integral for the unfixed internal momentum
- An ϵ_μ vector for the external photon line
- The photon/meson vertex factor $-ie\Gamma^\mu_{(\mathrm{a})}$

- The 4-meson vertex factor $i\Gamma_{(4\phi)}$.

The calculation is entirely in momentum space. We avoid the x and y integrations, and the delta function giving $q = p_f - p_i$ is implicit. Thus instead of the S-matrix element, the above rules yield the invariant amplitude:

$$iA_{(c)\ (\phi)} \;=\; \int \frac{d^4k}{(2\pi)^4}\, i\Gamma_{(4\phi)}(k,p_i,p_f)\; i\triangle_{m^2_\phi}\;\; i\triangle_{m^2_\phi}\; \left(-ie\Gamma^\mu_{(a)}(k,p_i,p_f)\right)\epsilon_\mu(q). \tag{5.48}$$

By equation (5.9), the vertex function itself is simply this expression without the factor $-e\epsilon(q)$:

$$\Gamma^\mu_{(c)\ (\phi)} \;=\; i\int \frac{d^4k}{(2\pi)^4}\,\Gamma_{(4\phi)}(k,p_i,p_f)\; i\triangle_{m^2_\phi}\;\; i\triangle_{m^2_\phi}\;\;\Gamma^\mu_{(a)}(k,p_i,p_f). \tag{5.49}$$

There will be one expression of this form comprising a charged pion loop, and another comprising a charged kaon loop. The internal lines must be charged particles because of the electromagnetic interaction occurring in the loop; both the pion and the kaon are allowed. The discussion of the two loops will be continued in section 5.6.6.

5.5 Summary of the Contributions

Here we collect all the contributions to the electromagnetic vertex function. We have found the following in the form of electromagnetic current operators:

$$J^\mu_{2\,\mathrm{tree}} \;=\; (\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3 \tag{5.50}$$

$$J^\mu_{L_4} \;=\; \frac{8\,(2m^2_\pi - 2m^2_K + 3m^2_\eta)}{F_0^2}\, L_4\;(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3 \tag{5.51}$$

$$J^\mu_{L_5} \;=\; \frac{8\,m^2_\pi}{F_0^2}\, L_5\;(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3 \tag{5.52}$$

$$J^\mu_{L_9} \;=\; \frac{4}{F_0^2}\, L_9\;\partial_\nu(\partial^\mu\boldsymbol{\pi}\times\partial^\nu\boldsymbol{\pi})_3 \tag{5.53}$$

$$J^\mu_{P_1} \;=\; \frac{4}{F_0^2}\, P_1\;\left[\;(\partial^\mu\partial^2\boldsymbol{\pi}\times\boldsymbol{\pi})_3 + (\partial^\mu\boldsymbol{\pi}\times\partial^2\boldsymbol{\pi})_3 - 2m^2_\pi(\boldsymbol{\pi}\times\partial^\mu\boldsymbol{\pi})_3\;\right] \tag{5.54}$$

$$J^\mu_{P_2} = \frac{8m_\pi^2}{F_0^2} P_2 \, (\boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi})_3 \tag{5.55}$$

$$J^\mu_{(\mathrm{b})\ (\pi)} = -\frac{5}{3F_0^2} \, I(m_\pi^2) \, (\boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi})_3 \tag{5.56}$$

$$J^\mu_{(\mathrm{b})\ (\mathrm{K})} = -\frac{5}{6F_0^2} \, I(m_K^2) \, (\boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi})_3 \tag{5.57}$$

and for the (c) diagram we have found the vertex function expressions

$$\Gamma^\mu_{(\mathrm{c})\ (\pi)} = i\int \frac{d^4k}{(2\pi)^4} \, \Gamma_{(4\pi)}(k, p_i, p_f) \, i\triangle_{m_\pi^2} \;\; i\triangle_{m_\pi^2} \;\; \Gamma^\mu_{(\mathrm{a})(\pi)}(k, p_i, p_f) \tag{5.58}$$

$$\Gamma^\mu_{(\mathrm{c})\ (K)} = i\int \frac{d^4k}{(2\pi)^4} \, \Gamma_{(2K)(2\pi)}(k, p_i, p_f) \, i\triangle_{m_K^2} \;\; i\triangle_{m_K^2} \;\; \Gamma^\mu_{(\mathrm{a})(\mathrm{K})}(k, p_i, p_f). \tag{5.59}$$

The next stage is to calculate the bare, or unrenormalized, amplitude Γ^μ from each of these contributions.

5.6 Calculation of the Electromagnetic Vertex Function

The aim in this section is to calculate the 3-point Green function $\Gamma^\mu(p_i, p_f)$ introduced in equation (5.8). First we introduce the machinery needed to carry this out, then we show the detailed steps of an example calculation whose result can be used in several places. Finally, the various components of Γ^μ deriving from the list in section 5.5 are calculated.

5.6.1 Definitions

By the LSZ reduction procedure [17], a Green function is related to an S-matrix element for asymptotically free states. We will start by discussing the relation we will use between Γ^μ and the current J^μ in place of that between Γ^μ and the full S-matrix.

The relationship between the vertex function Γ^μ and the invariant amplitude A_{fi} is given by equation (5.9):

$$-e\,\epsilon_\mu \Gamma^\mu = A_{fi}.$$

This is analogous to the relationship (5.16) we have used between J^μ and the Lagrangian, except for the fact that the polarization vector is involved in one case, and the photon field itself in the other case. The photon field is written as the polarization vector and a plane wave:

$$A_\mu(x) \;=\; \epsilon_\mu(q)\, e^{-iq\cdot x}. \tag{5.60}$$

Therefore, instead of computing the full S-matrix element

$$S_{fi} \;=\; \langle \pi^+(\boldsymbol{p}_f)|\, i \int d^4x\, \left(-e\, \epsilon_\mu(q)\, e^{-iq\cdot x}\right) J^\mu(x)\, |\pi^+(\boldsymbol{p}_i)\rangle \tag{5.61}$$

– where we have indicated the explicit structure of the Lagrangian, after equation (5.15) – we obtain the Γ^μ function directly, by computing the matrix element

$$(2\pi)^4\, \delta^4(p_f - p_i - q)\, \Gamma^\mu(p_i, p_f, q) \;=\; \langle \pi^+(\boldsymbol{p}_f)| \int d^4x\; e^{-iq\cdot x}\, J^\mu(x)\, |\pi^+(\boldsymbol{p}_i)\rangle. \tag{5.62}$$

This definition can be seen to follow from equations (5.9) — (5.12) relating Γ^μ, the S-matrix and the Lagrangian, equation (5.16) showing the derivation of the current J^μ from the Lagrangian, and equation (5.13) defining the S-matrix element. For the states $\langle f|$ and $|i\rangle$ of equation (5.13), we have used the properly normalized momentum eigenstates that will be defined in equations (5.73) — (5.75) below. The issue of taking the matrix element between these states requires some discussion.

We are interested in allowing the external legs of Γ^μ to go off-shell, and for this purpose we must consider how an off-shell Green function is defined.

Consider the relationship

$$\int d^4y\ d^4z\ e^{ip_f\cdot y}\ \overrightarrow{(\partial_y^2+m_\pi^2)}\ \langle 0|\, T\,(\,\pi^+(y)\,J^\mu(x)\,\pi^-(z)\,)|0\rangle\ \overleftarrow{(\partial_z^2+m_\pi^2)}\ e^{-ip_i\cdot z}$$

$$= \ N\,\langle \pi^+(\boldsymbol{p}_f);\mathrm{out}|\ J^\mu(x)\ |\pi^+(\boldsymbol{p}_i);\mathrm{in}\rangle \tag{5.63}$$

which arises from the LSZ reduction procedure [17]. Here,

$$\Gamma^\mu(x,y,z) \ = \ \langle 0|\, T\,(\,\pi^+(y)\,J^\mu(x)\,\pi^-(z)\,)|0\rangle \tag{5.64}$$

is the Green function in coordinate space, and N is a normalization constant. Even though the right hand side of equation (5.63) is defined for on-shell momenta $p_i^2 = p_f^2 = m_\pi^2$, one can consider the analytic continuation of the expression on the left hand side; nowhere in the evaluation procedure for this expression are the external momenta explicitly required to satisfy the Einstein condition

$$E_{\boldsymbol{k}}^2 \ = \ \boldsymbol{k}^2 + m^2. \tag{5.65}$$

In evaluating the vacuum expectation values on the left hand side, contractions are performed according to Wick's theorem, and these contractions are cancelled through the action of the Klein-Gordon operators, for general p_i and p_f. Off-shell Green functions can be evaluated in this way. Thus, knowing that this axiomatic procedure exists and validates our result, we will save trouble by calculating from the point of view of the right hand side: A normal matrix element calculation between states will be performed with the understanding that at the end we can replace 4-momenta that satisfy the on-shell condition (5.65) with ones that do not.

First we define the pion field operators in terms of creation/annihilation operators [17]. Real scalar fields forming the $SU(3)$ octet ϕ' were introduced in equation (2.4). The triplet of pion fields has the representation

$$\begin{aligned}
\pi_1(x) &= \int d\tilde{k}\,(a_1^\dagger(k)e^{ik\cdot x} + a_1(k)e^{-ik\cdot x}) \\
\pi_2(x) &= \int d\tilde{k}\,(a_2^\dagger(k)e^{ik\cdot x} + a_2(k)e^{-ik\cdot x}) \\
\pi_3(x) &= \int d\tilde{k}\,(a_3^\dagger(k)e^{ik\cdot x} + a_3(k)e^{-ik\cdot x}) \qquad (5.66)
\end{aligned}$$

in terms of creation/annihilation operators and plane waves. These are just three general independent solutions of the free Klein-Gordon equation. We use the notation

$$\int d\tilde{k} \equiv \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\boldsymbol{k}}}} \qquad (5.67)$$

for the integration measure. Since the pion electromagnetic current J^μ has been written in the numerical basis, the above operators will apply. The physical states between which we take matrix elements, however, are the pions π^+, π^-, and π^0 from the octet ϕ. These are to be defined from the above operators.

The charged pion is a complex scalar; it is defined in terms of two fundamental scalar fields in such a way that it is an eigenstate of electric charge. The pion with positive electric charge is written as

$$\pi^+(x) \equiv \int d\tilde{k}\left[a_{(+)}^\dagger(k)e^{ik\cdot x} + a_{(-)}(k)e^{-ik\cdot x}\right] \qquad (5.68)$$

where

$$a_{(+)}^\dagger(k) = \frac{1}{\sqrt{2}}(a_1^\dagger(k) - ia_2^\dagger(k)) \qquad (5.69)$$

creates a quantum of charge +1, and

$$a_{(-)}(k) = \frac{1}{\sqrt{2}}(a_1(k) - ia_2(k)) \qquad (5.70)$$

annihilates a quantum of charge −1. The operators a_1 and a_2 are the creation/annihilation operators for the fundamental fields π_1 and π_2. The pion with negative electric charge is the adjoint of the above:

$$\pi^-(x) \equiv \int d\tilde{k} \left[a^\dagger_{(-)}(k)e^{ik\cdot x} + a_{(+)}(k)e^{-ik\cdot x}\right] \tag{5.71}$$

with

$$\begin{aligned} a^\dagger_{(-)}(k) &= \frac{1}{\sqrt{2}}(a_1^\dagger(k) + ia_2^\dagger(k)) \\ a_{(+)}(k) &= \frac{1}{\sqrt{2}}(a_1(k) + ia_2(k)). \end{aligned} \tag{5.72}$$

The π^0 is just π_3.

We will concentrate only on the π^+ henceforth. The π^+ state of 4-momentum k is created when the creation operator acts on the vacuum:

$$\begin{aligned} |\pi^+(k)\rangle &= a^\dagger_{(+)}(k)\,|0\rangle \\ &= \frac{1}{\sqrt{2}}(a_1^\dagger(k) - ia_2^\dagger(k))\,|0\rangle \end{aligned} \tag{5.73}$$

and the 'out-state' conjugate to this 'in-state' is

$$\begin{aligned} \langle\pi^+(k)| &= \langle 0|\,a_{(+)}(k) \\ &= \langle 0|\,\frac{1}{\sqrt{2}}\,(a_1(k) + ia_2(k)). \end{aligned} \tag{5.74}$$

The creation/annihilation operators connect the vacuum to 4-momentum states, whereas we will use in the calculation the normalized 3-momentum eigenstate

$$|\pi^+(\boldsymbol{k})\rangle = (2\pi)^{\frac{3}{2}}\sqrt{2E_{\boldsymbol{k}}}\;|\pi^+(k)\rangle \tag{5.75}$$

which is defined with $E_{\boldsymbol{k}}$ satisfying (5.65). The relationship between any matrix elements computed using the two different states is given by

$$\begin{aligned} \langle\pi^+(\boldsymbol{p}_f)|\pi^+(\boldsymbol{p}_i)\rangle &= N\,\langle\pi^+(p_f)|\pi^+(p_i)\rangle \\ N &= (2\pi)^3\sqrt{2E_{\boldsymbol{p}_f}}\sqrt{2E_{\boldsymbol{p}_i}} \end{aligned} \tag{5.76}$$

so we must include the normalization factor N when evaluating the matrix element (5.62).

We are thus equipped to find the matrix element between states $|\pi^+(\boldsymbol{k})\rangle$; everything has been written in terms of a_1 and a_2 creation/annihilation operators.

5.6.2 Example Calculation

Consider this definition of a generalized 'current' with three derivatives:

$$J^{\mu\nu\lambda} = (\partial^\mu \partial^\nu \boldsymbol{\pi} \times \partial^\lambda \boldsymbol{\pi})_3. \tag{5.77}$$

We will use $J^{\mu\nu\lambda}$ as an example, and derive the amplitude

$$\Gamma^{\mu\nu\lambda} = [\, p_i^\mu p_i^\nu p_f^\lambda + p_f^\mu p_f^\nu p_i^\lambda \,]. \tag{5.78}$$

Here are the detailed steps of this evaluation. We write down the right hand side of equation (5.62), rewrite the states according to equation (5.76), and show the states and the current in explicit form in terms of creation/annihilation operators so that the vacuum expectation values can be taken:

$$\langle \pi^+(\boldsymbol{p}_f)| \int d^4x \; e^{-iq\cdot x} \, (\partial^\mu \partial^\nu \boldsymbol{\pi} \times \partial^\lambda \boldsymbol{\pi})_3 \, (x) \; |\pi^+(\boldsymbol{p}_i)\rangle$$

$$= N \, \langle \pi^+(p_f)| \int d^4x \; e^{-iq\cdot x} (\partial^\mu \partial^\nu \boldsymbol{\pi} \times \partial^\lambda \boldsymbol{\pi})_3 \, (x) \; |\pi^+(p_i)\rangle$$

$$= N \, \langle \pi^+(p_f)| \int d^4x \; e^{-iq\cdot x} (\partial^\mu \partial^\nu \pi_1 \partial^\lambda \pi_2 - \partial^\lambda \pi_1 \partial^\mu \partial^\nu \pi_2) \, (x) \; |\pi^+(p_i)\rangle$$

$$
\begin{aligned}
= \quad & N \, \langle 0| \, \frac{1}{\sqrt{2}}(a_1(p_f) + i a_2(p_f)) \, \int d^4x \; e^{-iq\cdot x} \\
& \qquad [\, \partial^\mu \partial^\nu \int d\tilde{q}_1 \, (a_1^\dagger(q_1) e^{iq_1\cdot x} + a_1(q_1) e^{-iq_1\cdot x}) \\
& \qquad\qquad \partial^\lambda \int d\tilde{q}_2 \, (a_2^\dagger(q_2) e^{iq_2\cdot x} + a_2(q_2) e^{-iq_2\cdot x}) \\
& \qquad - \partial^\lambda \int d\tilde{q}_1 \, (a_1^\dagger(q_1) e^{iq_1\cdot x} + a_1(q_1) e^{-iq_1\cdot x}) \\
& \qquad\qquad \partial^\mu \partial^\nu \int d\tilde{q}_2 \, (a_2^\dagger(q_2) e^{iq_2\cdot x} + a_2(q_2) e^{-iq_2\cdot x}) \,] \\
& \qquad \frac{1}{\sqrt{2}}(a_1^\dagger(p_i) - i a_2^\dagger(p_i)) \, |0\rangle \\
= \quad & \frac{N}{2} \langle 0| \, (a_1(p_f) + i a_2(p_f)) \, \int d^4x \; e^{-iq\cdot x} \\
& \qquad [\int d\tilde{q}_1 \, (-q_1^\mu q_1^\nu \, a_1^\dagger(q_1) e^{iq_1\cdot x} - q_1^\mu q_1^\nu \, a_1(q_1) e^{-iq_1\cdot x}) \\
& \qquad\qquad \int d\tilde{q}_2 \, (i q_2^\lambda \, a_2^\dagger(q_2) e^{iq_2\cdot x} - i q_2^\lambda \, a_2(q_2) e^{-iq_2\cdot x}) \\
& \qquad - \int d\tilde{q}_1 \, (i q_1^\lambda \, a_1^\dagger(q_1) e^{iq_1\cdot x} - i q_1^\lambda \, a_1(q_1) e^{-iq_1\cdot x}) \\
& \qquad\qquad \int d\tilde{q}_2 \, (-q_2^\mu q_2^\nu \, a_2^\dagger(q_2) e^{iq_2\cdot x} - q_2^\mu q_2^\nu \, a_2(q_2) e^{-iq_2\cdot x}) \,] \\
& \qquad (a_1^\dagger(p_i) - i a_2^\dagger(p_i)) \, |0\rangle
\end{aligned}
$$

$$= \frac{1}{2}(2\pi)^3 \sqrt{2E_{\boldsymbol{p}_f}}\sqrt{2E_{\boldsymbol{p}_i}} \int d^4x \; \frac{d^3q_1}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\boldsymbol{q}_1}}} \; \frac{d^3q_2}{(2\pi)^{\frac{3}{2}}\sqrt{2E_{\boldsymbol{q}_2}}} \; e^{-iq\cdot x}$$

$$\langle 0|\, a_1(p_f) \quad [\,(-q_1^\mu q_1^\nu e^{iq_1\cdot x})\; a_1^\dagger(q_1)\; (-iq_2^\lambda e^{-iq_2\cdot x})\; a_2(q_2)$$
$$-\,(\,iq_1^\lambda e^{iq_1\cdot x})\; a_1^\dagger(q_1)\; (-q_2^\mu q_2^\nu e^{-iq_2\cdot x})\; a_2(q_2)\,] \quad (-i)\; a_2^\dagger(p_i)\; |0\rangle$$

$$+\cdots \tag{5.79}$$

where we write all the possible terms which have a non-vanishing vacuum expectation value. In this case, there will be only one other term (that is, having the same form as the above). It will contain the a_2 operator from the out-state and the $a_1^\dagger$ operator from the in-state.

Clearly, choosing an operator from a $\pi_i(x)$ expression in the matrix element dictates the choice of the other operator that will give a non-vanishing VEV. For example, if the $a_1^\dagger$ is matched up with the out-state, the operator that will correspondingly match up with the in-state is the a_2 (no dagger). Two particles are created from the vacuum and then annihilated. A useful way to think about this matching process is to consider that the field operators from the Lagrangian act once to the left and once to the right on the states.

The VEV's are equal to [22]

$$\langle 0|\, a(p_f)a^\dagger(q_1)a(q_2)a^\dagger(p_i)\,|0\rangle \;=\; \delta^3(\boldsymbol{p}_f-\boldsymbol{q}_1)\;\delta^3(\boldsymbol{p}_i-\boldsymbol{q}_2). \tag{5.80}$$

After taking the VEV's and obtaining these delta functions, the next step is to integrate over the momenta. This saturates the delta functions, the effect of which is to leave $\boldsymbol{p}_f$ and $\boldsymbol{p}_i$ in place of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, respectively, in the remaining expression. When these replacements are made, the exponential part of the expression turns out to be the same in the two contributing terms we have, and indeed in any non-vanishing terms in general. It has the form

$$e^{\,i(p_f - p_i - q)\cdot x},$$

and since the exponential is the same in each term, it can be brought out front.

Another effect of taking the $\boldsymbol{q}$ integration momenta to $\boldsymbol{p}$ external momenta, clearly, is the cancellation of the integration measure factors with the normalization factor out front.

Performing the momentum integrations amounts to systematically replacing all $q_{1,2}$ variables (4-momentum notation) with the appropriate p_f or p_i, depending on which state, initial or final, each collected operator pairs up with.

Having taken the VEV's and done the momentum integrations, we are left with

$$\begin{aligned}
&\langle \pi^+(\boldsymbol{p}_f)|\int d^4x\ e^{-iq\cdot x}\,(\partial^\mu\partial^\nu\boldsymbol{\pi}\times\partial^\lambda\boldsymbol{\pi})_3\,(x)\ |\pi^+(\boldsymbol{p}_i)\rangle \\
&\quad= \frac{1}{2}\int d^4x\ e^{\,i(p_f-p_i-q)\cdot x}\quad [\,(-p_f^\mu p_f^\nu)\,(-ip_i^\lambda)\,(-i)\ -\ (\,ip_f^\lambda)\,(-p_i^\mu p_i^\nu)\,(-i) \\
&\qquad\qquad + \ i\,(-p_i^\mu p_i^\nu)\,(\,ip_f^\lambda)\ -\ i\,(-ip_i^\lambda)\,(-p_f^\mu p_f^\nu)\,] \\
&\quad= \int d^4x\ e^{\,i(p_f-p_i-q)\cdot x}\quad [\,p_i^\mu p_i^\nu p_f^\lambda\ +\ p_f^\mu p_f^\nu p_i^\lambda\,] \\
&\quad= (2\pi)^4\,\delta^4(p_f-p_i-q)\ \ [\,p_i^\mu p_i^\nu p_f^\lambda\ +\ p_f^\mu p_f^\nu p_i^\lambda\,]
\end{aligned} \tag{5.81}$$

where finally the x integral is just the definition of the 4-dimensional delta function. This is result (5.78).

The 4-momentum which we introduced, q, is just the photon momentum, since the delta function defines it to be the difference between the external meson momenta:

$$q \;=\; p_f - p_i. \tag{5.82}$$

This definition of q will subsequently be used.

Having completed the evaluation of the function $\Gamma^\mu(p_i, p_f)$, we can generalize to assume that the momenta p_i and p_f of the 'external' pions do not necessarily satisfy the on-shell condition.

We list here another result similar to the above:

$$\langle \pi^+(\boldsymbol{p}_f)| \int d^4x \; e^{-iq\cdot x} \, (\partial^\mu \partial^\nu \partial^\lambda \boldsymbol{\pi} \times \boldsymbol{\pi})_3(x) \; |\pi^+(\boldsymbol{p}_i)\rangle$$

$$= \; -(2\pi)^4 \, \delta^4(p_f - p_i - q) \, [\, p_i^\mu p_i^\nu p_i^\lambda \; + \; p_f^\mu p_f^\nu p_f^\lambda \,]. \qquad (5.83)$$

Now the amplitudes of all the components of the current can be computed using the techniques illustrated.

5.6.3 Tree-Level Contributions from $\mathcal{L}_2$ and $\mathcal{L}_4$

First consider the tree-level contributions to J^μ given by equations (5.50), (5.51), and (5.52). These are all of the same form, containing a single derivative, so we treat them together.

Carrying out the steps shown in the previous section leads to this simple result for the one-derivative current:

$$\langle \pi^+(\boldsymbol{p}_f)| \int d^4x \; e^{-iq\cdot x} \, (\boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi})_3 \; |\pi^+(\boldsymbol{p}_i)\rangle \;=\; (2\pi)^4 \, \delta^4(p_f - p_i - q) \, [p_i^\mu + p_f^\mu]. \qquad (5.84)$$

The three tree-level components form the first contribution to the overall amplitude, by equation (5.84) — and referring to equation (5.62) to write the result without the delta function pre-factor:

$$\Gamma^\mu_{\text{tree}} \;=\; (p_i + p_f)^\mu \; [\; 1 \; + \; \frac{8\,(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_\pi^2} \, L_4 \; + \; \frac{8\, m_\pi^2}{F_\pi^2} \, L_5 \;]. \qquad (5.85)$$

This and the other components of Γ^μ from the following sections will be collected together in section 5.7.

Secondly, we look at the L_9 tree-level contribution — current (5.53). This consists of two operator expressions of the type computed in the previous section, namely

$$\partial_\nu(\partial^\mu \boldsymbol{\pi} \times \partial^\nu \boldsymbol{\pi})_3 \;=\; (\partial^\mu\partial^\nu \boldsymbol{\pi} \times \partial_\nu \boldsymbol{\pi})_3 \;-\; (\partial^2 \boldsymbol{\pi} \times \partial^\mu \boldsymbol{\pi})_3 \tag{5.86}$$

if we distribute the derivative ∂_ν and then use the antisymmetry of the cross product in rewriting the second term. To find the matrix elements, we simply substitute these into equation (5.78). Here is the combined result for the L_9 current:

$$\begin{aligned}\langle \pi^+(\boldsymbol{p}_f)| & \int d^4x\; e^{-iq\cdot x}\, J^\mu_{L_9}\; |\pi^+(\boldsymbol{p}_i)\rangle \\ &= (2\pi)^4\, \delta^4(p_f - p_i - q)\, \frac{4L_9}{F_0^2}\; [\; -(p_i + p_f)^\mu\, p_i\!\cdot\! p_f \;+\; p_i^\mu\, p_f^2 \;+\; p_f^\mu\, p_i^2 \;].\end{aligned} \tag{5.87}$$

Note here that the extra p_i^μ and p_f^μ factors do not multiply the same quantity, so we are confronted with an unusual structure not consistent with the $(p_i + p_f)^\mu$ form factor.

If one is considering on-shell external momenta

$$p_i^2 \;=\; p_f^2 \;=\; m_\pi^2 \tag{5.88}$$

then the expression does simplify, taking just the form

$$(p_i + p_f)^\mu\, f_\pi^+(q^2). \tag{5.89}$$

If one is considering general off-shell external momenta, then one must introduce the second form factor f_π^- and the structure $(p_f - p_i)^\mu$. The expression (5.87) can be rewritten in two terms which conform to the two required structures. It is not difficult to show that

$$\begin{aligned}(p_i + p_f)^\mu\, p_i\!\cdot\! p_f \;&-\; p_i^\mu\, p_f^2 \;-\; p_f^\mu\, p_i^2 \\ &= \; -\frac{1}{2}\,(p_i + p_f)^\mu\; (\,p_f \;-\; p_i\,)^2 \;+\; \frac{1}{2}\,(p_f - p_i)^\mu\; (\,p_f^2 \;-\; p_i^2\,)\end{aligned} \tag{5.90}$$

so that our general off-shell amplitude is (again writing equation (5.87) in the form of Γ^μ without the delta function):

$$\Gamma^\mu_{L_9} \;=\; \frac{2L_9}{F_0^2}\,[\,(p_i+p_f)^\mu\, q^2 \;+\; (p_f-p_i)^\mu\,(\,p_i^2\,-\,p_f^2\,)\,]. \tag{5.91}$$

5.6.4 Contribution From $\mathcal{L}_{4\,\mathrm{off}}$

For the off-shell part, we have the P_1 contribution from current (5.54) and the P_2 contribution from current (5.55). To calculate the former, we make use of equations (5.78), (5.83), and (5.84), respectively, for the three terms of the current. We obtain

$$\begin{aligned}\Gamma^\mu_{P_1} &= \frac{4P_1}{F_0^2}\;[\,(\,p_i^\mu\, p_i^2\,+\,p_f^\mu\, p_f^2\,) \\ &\qquad +\,(\,p_i^\mu\, p_f^2\,+\,p_f^\mu\, p_i^2\,) \\ &\qquad -\,2m_\pi^2\,(\,p_i^\mu\,+\,p_f^\mu\,)\,] \\ &= \frac{4P_1}{F_0^2}\,(p_i+p_f)^\mu\;[\,p_i^2\,+\,p_f^2\,-\,2m_\pi^2\,].\end{aligned} \tag{5.92}$$

The P_2 current evaluates to

$$\Gamma^\mu_{P_2} \;=\; \frac{8m_\pi^2}{F_0^2}\,P_2\,(p_i+p_f)^\mu, \tag{5.93}$$

and evidently both components contribute to the f_π^+ form factor only.

5.6.5 Contribution From the Tadpole Diagrams

The calculation of the (b)-type amplitude is exactly the same as that already done for the tree-level amplitude. We use equation (5.84). The result, however, contains the divergent loop integral $I(m^2)$. This divergence will ultimately cancel with the same divergence supplied by the (c) loop diagram.

Evaluation of currents (5.56) and (5.57) yields the expression

$$\begin{aligned}\Gamma^{\mu}_{(\mathrm{b})} &= (p_i + p_f)^{\mu} \, [\, -\frac{5}{3F_0^2} \, I(m_{\pi}^2) \, - \, \frac{5}{6F_0^2} \, I(m_K^2) \,] \\ &= (p_i + p_f)^{\mu} \, [\, -\frac{5m_{\pi}^2}{48\pi^2 F_0^2} \, (R \, + \, \ln\frac{m_{\pi}^2}{\mu^2}) \, - \, \frac{5m_K^2}{96\pi^2 F_0^2} \, (R \, + \, \ln\frac{m_K^2}{\mu^2}) \,]. \end{aligned} \quad (5.94)$$

Notice that the tadpole diagrams do not contribute anything to the $(p_f - p_i)^{\mu}$ part of the form factor. This is unlike the (c)-type diagrams, whose structure is more complicated. We compute these diagrams next.

5.6.6 Contribution From the 2-Vertex Loop Diagrams

Let us start with the pion loop diagram. From equation (5.58), the expression we have for the Green function Γ^{μ} for this diagram is

$$\Gamma^{\mu}_{(\mathrm{c})(\pi)} \; = \; i\int \frac{d^4k}{(2\pi)^4} \, \Gamma_{(4\pi)}(k, p_i, p_f) \, i\triangle_{m_{\pi}^2}(k, p_i, p_f) \, i\triangle_{m_{\pi}^2}(k, p_i, p_f) \, \Gamma^{\mu}_{(\mathrm{a})(\pi)}(k, p_i, p_f). \quad (5.95)$$

We have found in section 5.6.3 that the tree diagram vertex function is

$$\Gamma^{\mu}_{(\mathrm{a})(\pi)} \; = \; (p_i + p_f)^{\mu}. \quad (5.96)$$

Now we must derive the Feynman rule for the 4-pion vertex. We know that the internal lines of diagram (c) must be charged particles to interact with the photon; hence we are interested in the $\pi^+\pi^-\pi^+\pi^-$ vertex. We take the matrix element of the interaction $\mathcal{L}_{2\,(4\pi)}$:

$$\langle \pi^+(\boldsymbol{k}_1)\pi^-(\boldsymbol{k}_2)| \; i\int d^4x \, \mathcal{L}_{2\,(4\pi)}(x) \; |\pi^+(\boldsymbol{p}_1)\pi^-(\boldsymbol{p}_2)\rangle, \quad (5.97)$$

which will be the S-matrix element for pion elastic scattering, to first order. In the expansion of $\mathcal{L}_{2\,(4\pi)}$ (from equation (4.19)) we can neglect all but the charged pions.

Doing this, we obtain

$$\begin{aligned}
\mathcal{L}_{2\,(4\pi)} &= \frac{1}{24F_0^2}\langle\, \partial^\nu\phi\phi\partial_\nu\phi\phi \,-\, \partial^\nu\phi\partial_\nu\phi\phi\phi \,+\, B_0 M\phi\phi\phi\phi\,\rangle \\
&= \frac{1}{6F_0^2}\,[\,\partial^\mu\pi^+\pi^-\partial_\mu\pi^+\pi^- \,+\, \pi^+\partial^\mu\pi^-\pi^+\partial_\mu\pi^- \\
&\qquad -\, 2\partial^\mu\pi^+\partial_\mu\pi^-\pi^+\pi^- \,+\, m_\pi^2\pi^+\pi^-\pi^+\pi^-\,].
\end{aligned} \tag{5.98}$$

Now using the steps shown in section 5.6.2, it is not difficult to evaluate the amplitude shown in equation (5.97). One uses the rule that each operator acts twice on the initial and final state operators — once to the right with its positive-frequency component, and once to the left with its negative-frequency component. The fact that there are two pions in the initial and final states means there will be twice as many matches of operators as, for example, we found in section 5.6.2. The result is the vertex amplitude

$$\begin{aligned}
&\langle\pi^+(\boldsymbol{k}_1)\pi^-(\boldsymbol{k}_2)|\; i\int d^4x\,\mathcal{L}_{2\,(4\pi)}\;|\pi^+(\boldsymbol{p}_1)\pi^-(\boldsymbol{p}_2)\rangle \\
&= (2\pi)^4\,\delta^4(k_1+k_2-p_1-p_2)\,\frac{i}{3F_0^2}[\,2m_\pi^2 \,+\, p_1^2 \,+\, p_2^2 \,+\, k_1^2 \,+\, k_2^2 \,-\, 3(p_1-k_2)^2\,]
\end{aligned} \tag{5.99}$$

after some manipulations have been done to reduce the set of 4-vector dot products to just squares plus the quantity $p_1\!\cdot\!k_2$.

We have thus found the Feynman rule for the vertex:

$$\Gamma_{(4\pi)} \;=\; \frac{1}{3F_0^2}\,[\,2m_\pi^2 \,+\, p_1^2 \,+\, p_2^2 \,+\, k_1^2 \,+\, k_2^2 \,-\, 3(p_1-k_2)^2\,]. \tag{5.100}$$

Equations (5.96) and (5.100) are to be substituted into equation (5.95), along with the momentum space expressions for the propagators — these latter are not, of course, the evaluated contraction $I(m^2)$, but just the free propagators:

$$i\,\triangle_{m_\pi^2}(p) \;=\; \frac{i}{p^2 \,-\, m_\pi^2 \,+\, i\epsilon}.$$

We assign the momenta according to Figure 5.2. This choice, because of its symmetry,

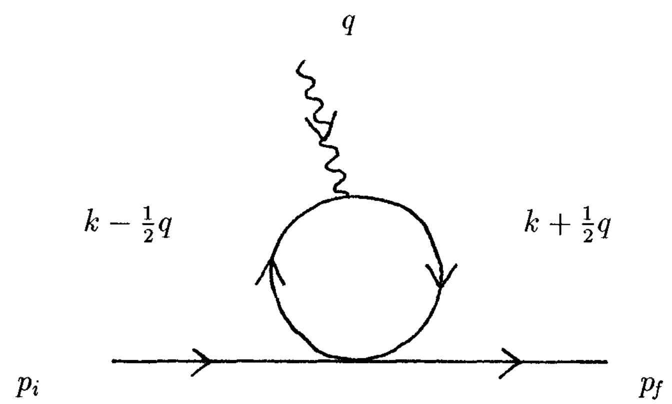

Figure 5.2: 2-Vertex Diagram Momentum Assignments

is most convenient for evaluating equation (5.95), which becomes:

$$\Gamma^{\mu}_{(c)\ (\pi)}$$
$$= -\frac{i}{3F_0^2}\int\frac{d^4k}{(2\pi)^4}\,\frac{2k^{\mu}\,[2m_{\pi}^2 + p_i^2 + p_f^2 + (k+\frac{1}{2}q)^2 + (k-\frac{1}{2}q)^2 - 3(k+\frac{p_i+p_f}{2})^2]}{[(k+\frac{1}{2}q)^2 - m_{\pi}^2 + i\epsilon]\,[(k-\frac{1}{2}q)^2 - m_{\pi}^2 + i\epsilon]}.$$

We introduce dimensional regularization as defined in equation (4.3) to evaluate this. When the dimension has been generalized to d, we can assume the integral converges, and can use the symmetric integration argument. Most of the terms in the integrand are odd in k, and will not survive symmetric integration. Thus we are left with the integral

$$\Gamma^{\mu}_{(c)\ (\pi)} = \frac{2i\mu^{4-d}}{F_0^2}\int\frac{d^dk}{(2\pi)^d}\,\frac{k^{\mu}\,k\cdot(p_i+p_f)}{[(k+\frac{1}{2}q)^2 - m_{\pi}^2][(k-\frac{1}{2}q)^2 - m_{\pi}^2]}. \tag{5.101}$$

This can be evaluated by introducing a Feynman parameter to change the product of factors in the denominator to only one factor; then one can reverse the order of integration, and apply the dimensional regularization formula first, then finally integrate over the Feynman parameter. The result will form two expressions, one contributing to the form factor f_{π}^{+}, the other contributing to f_{π}^{-}.

Before proceeding to do this evaluation, we show the analogous integral for the charged kaon loop. Recall that we expanded $\mathcal{L}_{2\,(4\phi)}$ to extract the $\pi^+\pi^-\pi^+\pi^-$ interaction. Doing the same thing to extract the $K^+K^-\pi^+\pi^-$ interaction, we find

$$\begin{aligned}
\mathcal{L}_{2\,(2K)(2\pi)} \;=\; \frac{1}{6F_0^2}[\, & 2\partial^\mu K^+ K^- \partial_\mu \pi^+ \pi^- \;+\; 2K^+ \partial^\mu K^- \pi^+ \partial_\mu \pi^- \\
& -\; \partial^\mu K^+ K^- \pi^+ \partial_\mu \pi^- \;-\; K^+ \partial^\mu K^- \partial_\mu \pi^+ \pi^- \\
& -\; \partial^\mu K^+ \partial_\mu K^- \pi^+ \pi^- \;-\; K^+ K^- \partial^\mu \pi^+ \partial_\mu \pi^- \\
& +\; (m_\pi^2 + m_K^2) K^+ K^- \pi^+ \pi^- \,]. \qquad (5.102)
\end{aligned}$$

The evaluation of the scattering amplitude

$$\langle \pi^+(\boldsymbol{k}_1) K^-(\boldsymbol{k}_2)|\; i \int d^4x\, \mathcal{L}_{2\,(2K)(2\pi)}\; |\pi^+(\boldsymbol{p}_1) K^-(\boldsymbol{p}_2)\rangle \qquad (5.103)$$

proceeds in the same way as that leading to equation (5.100), and the end result has the same form as that equation:

$$\Gamma_{(2K)(2\pi)} \;=\; \frac{1}{6F_0^2}[\, m_\pi^2 \,+\, m_K^2 \,+\, p_1^2 \,+\, p_2^2 \,+\, k_1^2 \,+\, k_2^2 \,-\, 3(p_1 - k_2)^2\,]. \qquad (5.104)$$

Clearly, substituting this vertex factor into equation (5.59) and again using the momentum assignments shown in Figure 5.2, we end up with an integral that closely matches our preceding result:

$$\Gamma^\mu_{\text{(c) (K)}} \;=\; \frac{i\mu^{4-d}}{F_0^2} \int \frac{d^dk}{(2\pi)^d}\, \frac{k^\mu\, k\cdot(p_i + p_f)}{[\,(k + \frac{1}{2}q)^2 \,-\, m_K^2\,]\,[\,(k - \frac{1}{2}q)^2 \,-\, m_K^2\,]}. \qquad (5.105)$$

The differences are only the particle mass and a factor of 2. (Compare equations (5.100) and (5.104)).

Let us therefore evaluate equation (5.101), the pion loop integral. Introducing a Feynman parameter in the standard way yields

$$\Gamma^\mu_{\text{(c)}\;(\pi)} \;=\; \frac{2i\mu^{4-d}}{F_0^2} \int \frac{d^dk}{(2\pi)^d}\, (2-1)! \int_0^1 dx\, \frac{k^\mu\, k\cdot(p_i + p_f)}{[\,(k + \frac{1}{2}q)^2 \,-\, 2xk\cdot q \,-\, m_\pi^2\,]^2}. \qquad (5.106)$$

We must perform a shift of integration origin to eliminate the dot product in the denominator. In the present case, a variable shift would not be allowed in four dimensions, because the integral is quadratically divergent [26]. However, in dimensional regularization, one is allowed to shift the variable of integration regardless of the degree of divergence of the integral in four space-time dimensions [20]. The shift and the formal evaluation of the integral can be carried out freely after one has regularized the integral, and only at the end does one move back to 4 dimensions.

We shift k as follows:

$$k \;\rightarrow\; k + (x - \frac{1}{2})\, q \tag{5.107}$$

to arrive at the integral

$$\begin{aligned}\Gamma^{\mu}_{(\mathrm{c})\ (\pi)} &= \frac{2i\mu^{4-d}}{F_0^2}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\,\frac{k^\mu\, k\cdot(p_i+p_f) \;+\; (x-\frac{1}{2})^2 q^\mu\, q\cdot(p_i+p_f)}{[k^2+x(1-x)q^2-m_\pi^2]^2} \\ &= \Gamma^{\mu}_{(\mathrm{c-1})\ (\pi)} + \Gamma^{\mu}_{(\mathrm{c-2})\ (\pi)}\end{aligned} \tag{5.108}$$

where, again, we have dropped terms odd in k. The integral is comprised of two terms, to which we refer using the subscripts (c-1) and (c-2).

The momentum integrals in equation (5.108) can be evaluated easily by applying the dimensional regularization formula (4.4). We show the results separately for the two terms.

For the first term, we take advantage of Lorentz covariance to write [21]

$$\int d^dk\, k_\mu k_\nu f(k^2) \;=\; \int d^dk\,\frac{g_{\mu\nu}}{d}\,k^2 f(k^2). \tag{5.109}$$

This allows us to use formula (4.4). The integration yields

$$\begin{aligned}
\Gamma^{\mu}_{(c-1)\ (\pi)} &= \frac{2i\mu^{4-d}}{F_0^2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\,\frac{k^2\,(p_i+p_f)^{\mu}}{d\,[k^2+x(1-x)q^2-m_\pi^2]^2} \\
&= -\frac{2\mu^{4-d}}{F_0^2(2\pi)^d}\,(p_i+p_f)^{\mu}\int_0^1 dx\,\frac{\pi^{\frac{d}{2}}\,\Gamma(\frac{d}{2}+1)\Gamma(1-\frac{d}{2})}{d\,\Gamma(\frac{d}{2})\Gamma(2)\,[x(1-x)q^2-m_\pi^2]^{1-\frac{d}{2}}} \\
&= \frac{\mu^{4-d}}{F_0^2(4\pi)^{\frac{d}{2}}}\,(p_i+p_f)^{\mu}\int_0^1 dx\,\frac{\Gamma(2-\frac{d}{2})}{(1-\frac{d}{2})\,[m_\pi^2-x(1-x)q^2]^{1-\frac{d}{2}}}.
\end{aligned}$$

We took the limit $d \to 4$ where d appeared in the factor $(-1)^{1-\frac{d}{2}}$ yielding the overall sign. The gamma functions have been reduced using formula (4.6) to the single gamma function with argument $(2-\frac{d}{2})$, as we require in order to follow precisely the same procedure shown in equations (4.8) — (4.9) for the integral $I(m^2)$. We group the d's to write

$$\Gamma^{\mu}_{(c-1)\ (\pi)} = \frac{(p_i+p_f)^{\mu}}{16\pi^2F_0^2}\int_0^1 dx\,[m_\pi^2-x(1-x)q^2]\left(\frac{4\pi\mu^2}{m_\pi^2-x(1-x)q^2}\right)^{2-\frac{d}{2}}\frac{\Gamma(2-\frac{d}{2})}{(1-\frac{d}{2})}$$

so that we can carry out the expansion around $d=4$. Just as in the case demonstrated previously, the form is

$$\lim_{\epsilon\to 0}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\frac{1}{\epsilon-1}\,\Gamma(\epsilon).$$

Consequently we obtain

$$\Gamma^{\mu}_{(c-1)\ (\pi)} = \frac{(p_i+p_f)^{\mu}}{16\pi^2F_0^2}\int_0^1 dx\,[m_\pi^2-x(1-x)q^2]\left[R+\ln\left(\frac{m_\pi^2-x(1-x)q^2}{\mu^2}\right)\right] \tag{5.110}$$

with the same divergent constant R.

Finally we integrate over x. This integration yields

$$\begin{aligned}
\Gamma^{\mu}_{(c-1)\ (\pi)} &= (p_i+p_f)^{\mu} \\
&\times\frac{1}{96\pi^2F_0^2}\left[\,(6m_\pi^2-q^2)(R+\ln\frac{m_\pi^2}{\mu^2})+(4m_\pi^2-q^2)\,H(\frac{q^2}{m_\pi^2})-\frac{1}{3}q^2\,\right]
\end{aligned} \tag{5.111}$$

with

$$
\begin{aligned}
H(a) &\equiv \int_0^1 dx \, \ln\left(1 - ax(1-x)\right) \\
&= \left\{ \begin{array}{ll} -2 + 2\sqrt{\frac{4}{a} - 1}\, \cot^{-1} \sqrt{\frac{4}{a} - 1} & (0 < a < 4) \\ -2 + \sqrt{1 - \frac{4}{a}} \left[\ln \frac{\sqrt{1-\frac{4}{a}} - 1}{\sqrt{1-\frac{4}{a}} + 1} + i\pi\theta(a-4) \right] & (\text{otherwise}). \end{array} \right.
\end{aligned} \tag{5.112}
$$

Loop (c) has introduced an imaginary component into the energy expansion, which is necessary to make the S-matrix unitary, but is absent from tree-level calculations [10].

We now address the second term of (5.108). This part gives the f_π^- form factor contribution — notice that

$$
q^\mu \, q \cdot (p_i + p_f) \;=\; (p_f - p_i)^\mu \, (p_f^2 - p_i^2).
$$

We may use formula (4.4), even though we have in this case

$$
\lim_{d \to 4} (q - \omega - t) \;=\; \lim_{d \to 4} (2 - \frac{d}{2}) \;=\; 0.
$$

This is because although the formula gives a divergent result for $d \to 4$, we have a prescription for handling the divergence in this limit. It will be seen below that the final analytic expression we derive is of the familiar form containing R.

In d-dimensional space-time, we integrate to obtain

$$
\begin{aligned}
\Gamma^\mu_{(c-2)\ (\pi)} &= -\frac{2i\mu^{4-d}}{F_0^2} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \, \frac{(x - \frac{1}{2})^2 \, (p_f - p_i)^\mu \, (p_i^2 - p_f^2)}{[k^2 + x(1-x)q^2 - m_\pi^2]^2} \\
&= \frac{2\mu^{4-d}}{F_0^2 (4\pi)^{\frac{d}{2}}} \, (p_f - p_i)^\mu \, (p_i^2 - p_f^2) \int_0^1 dx \, (x - \frac{1}{2})^2 \frac{\Gamma(2 - \frac{d}{2})}{[m_\pi^2 - x(1-x)q^2]^{2-\frac{d}{2}}} \\
&= \frac{(p_f - p_i)^\mu (p_i^2 - p_f^2)}{8\pi^2 F_0^2} \int_0^1 dx \, (x - \frac{1}{2})^2 \left(\frac{4\pi\mu^2}{m_\pi^2 - x(1-x)q^2} \right)^{2-\frac{d}{2}} \Gamma(2 - \frac{d}{2}).
\end{aligned} \tag{5.113}
$$

Again, we expand this around $d = 4$. The expansion proceeds along the same lines as in equations (4.8) — (4.9), giving the same leading order divergence. We have for small ϵ

$$\begin{aligned}\left(\frac{4\pi\mu^2}{M^2}\right)^{\epsilon}\Gamma(\epsilon) &= e^{\epsilon\ln\left(\frac{4\pi\mu^2}{M^2}\right)}\Gamma(\epsilon)\\ &= \left[1+\epsilon\ln\left(\frac{4\pi\mu^2}{M^2}\right)+O(\epsilon^2)\right]\left[\frac{1}{\epsilon}-\gamma+O(\epsilon)\right]\\ &= \frac{1}{\epsilon}+\ln 4\pi-\ln\frac{M^2}{\mu^2}-\gamma+O(\epsilon)\\ &= -\left[R+1+\ln\frac{M^2}{\mu^2}\right].\end{aligned}\tag{5.114}$$

Hence, performing the x integration,

$$\begin{aligned}\Gamma^{\mu}_{(c-2)\ (\pi)} &= (p_f-p_i)^{\mu}\frac{(p_f^2-p_i^2)}{8\pi^2F_0^2}\int_0^1 dx\,(x-\frac{1}{2})^2\left[R+1+\ln\left(\frac{m_\pi^2-x(1-x)q^2}{\mu^2}\right)\right]\\ &= (p_f-p_i)^{\mu}\frac{(p_f^2-p_i^2)}{96\pi^2F_0^2}\left[R+\frac{1}{3}+\ln\frac{m_\pi^2}{\mu^2}-(4\frac{m_\pi^2}{q^2}-1)\,H(\frac{q^2}{m_\pi^2})\right].\end{aligned}\tag{5.115}$$

This completes the evaluation, from equation (5.95), of the pion loop described by Feynman diagram (c).

In addition, there is the kaon loop coming from result (5.59). The two contributions from the kaon loop analogous to results (5.111) and (5.115) simply have m_K^2 substituted for m_π^2, and differ by an overall factor of 2.

Combining the pion and kaon loop components, we obtain the final result for the (c) diagram contribution:

$$\begin{aligned}\Gamma^{\mu}_{(c)} &= (p_i+p_f)^{\mu}\\ &\times\frac{1}{192\pi^2F_0^2}\left[2(6m_\pi^2-q^2)(R+\ln\frac{m_\pi^2}{\mu^2})+2(4m_\pi^2-q^2)\,H(\frac{q^2}{m_\pi^2})-\frac{2}{3}q^2\right.\\ &\qquad\left.+(6m_K^2-q^2)(R+\ln\frac{m_K^2}{\mu^2})+(4m_K^2-q^2)\,H(\frac{q^2}{m_K^2})-\frac{1}{3}q^2\right]\end{aligned}$$

$$
\begin{aligned}
+ \quad & (p_f - p_i)^\mu \, \frac{(p_f^2 - p_i^2)}{q^2} \\
& \times \frac{1}{192\pi^2 F_0^2} \left[2q^2 (R + 1 + \ln\frac{m_\pi^2}{\mu^2}) + 2(q^2 - 4m_\pi^2)\, H(\frac{q^2}{m_\pi^2}) - \frac{4}{3} q^2 \right. \\
& \qquad \left. + \, q^2 (R + 1 + \ln\frac{m_K^2}{\mu^2}) + (q^2 - 4m_K^2)\, H(\frac{q^2}{m_K^2}) - \frac{2}{3} q^2 \right] \\
= \quad & (p_i + p_f)^\mu \\
& \left[\frac{1}{192\pi^2 F_0^2} \left(12 m_\pi^2 (R + \ln\frac{m_\pi^2}{\mu^2}) + 6 m_K^2 (R + \ln\frac{m_K^2}{\mu^2}) - 3q^2 R \right) \right. \\
& \qquad \left. - \frac{q^2}{F_0^2} \left(2F(\frac{m_\pi^2}{q^2}, \frac{m_\pi^2}{\mu^2}) + F(\frac{m_K^2}{q^2}, \frac{m_K^2}{\mu^2}) \right) \right] \\
+ \quad & (p_f - p_i)^\mu \, (p_i^2 - p_f^2) \\
& \left[\frac{1}{192\pi^2 F_0^2} \, (-3R) \; - \frac{1}{F_0^2} \left(2F(\frac{m_\pi^2}{q^2}, \frac{m_\pi^2}{\mu^2}) + F(\frac{m_K^2}{q^2}, \frac{m_K^2}{\mu^2}) \right) \right]
\end{aligned}
\tag{5.116}
$$

where it is convenient to introduce the function

$$
F(\frac{m^2}{q^2}, \frac{m^2}{\mu^2}) \equiv \frac{1}{192\pi^2} \left[\frac{1}{3} + (1 - 4\frac{m^2}{q^2})\, H(\frac{q^2}{m^2}) + \ln\frac{m^2}{\mu^2} \right]. \tag{5.117}
$$

In the next section we will see that the form factors are written in terms of only the function $F()$ and the phenomenological constants.

5.7 The Form Factors

5.7.1 Vertex Function Result

To obtain the physical Green function we must apply the wavefunction renormalization

$$\Gamma^{\mu}_{\mathrm{r}} \;=\; Z_\pi \Gamma^{\mu}.$$

We also substitute F_π for F_0. Since the difference between these is of $O(p^2)$, this change will not affect the energy expansion at the order to which we are working. From equation (4.55), the renormalization constant is

$$\begin{aligned} Z_\pi \;=\; & 1 - \frac{8(2m_\pi^2 - 2m_K^2 + 3m_\eta^2)}{F_\pi^2} L_4 - \frac{8m_\pi^2}{F_\pi^2} L_5 - \frac{8m_\pi^2}{F_\pi^2} P_2 \\ & + \frac{m_\pi^2}{24\pi^2 F_\pi^2} (R + \ln\frac{m_\pi^2}{\mu^2}) + \frac{m_K^2}{48\pi^2 F_\pi^2} (R + \ln\frac{m_K^2}{\mu^2}). \end{aligned}$$

All of the form factors from the Γ^μ components we have calculated consist of $O(p^2)$ terms except for that from $\Gamma^\mu_{\mathrm{tree}}$, which also contains the leading term 1. Thus, to the order that our result is valid, only $\Gamma^\mu_{\mathrm{tree}}$ is actually modified by this renormalization. The corrected amplitude (5.85) is

$$\begin{aligned} \Gamma^{\mu}_{\mathrm{r\,tree}} \;=\; & (p_i + p_f)^\mu \\ & [\, 1 + \frac{m_\pi^2}{24\pi^2 F_\pi^2} (R + \ln\frac{m_\pi^2}{\mu^2}) + \frac{m_K^2}{48\pi^2 F_\pi^2} (R + \ln\frac{m_K^2}{\mu^2}) - \frac{8m_\pi^2}{F_\pi^2} P_2 \,]. \end{aligned} \tag{5.118}$$

In adding all the contributions we have collected, we observe that the R terms all cancel, except for one that renormalizes the constant L_9. In fact, considering first the $(p_i + p_f)^\mu$ part, all the terms that do not contain q^2 vanish. We collect the following terms from equations (5.94), (5.116) and (5.118):

$$\begin{aligned} & \frac{1}{48\pi^2 F_\pi^2} [-5 + 3 + 2]\, m_\pi^2 (R + \ln\frac{m_\pi^2}{\mu^2}) \\ & + \frac{1}{96\pi^2 F_\pi^2} [-5 + 3 + 2]\, m_K^2 (R + \ln\frac{m_K^2}{\mu^2}) \;=\; 0. \end{aligned}$$

The only term left containing R is

$$-\frac{1}{64\pi^2 F_\pi^2}\; R\, q^2$$

which is contributed by the (c) loop diagram, equation (5.116). Similarly, in the $(p_f - p_i)^\mu$ part there is the R term

$$-\frac{1}{64\pi^2 F_\pi^2}\; R\,(p_i^2 - p_f^2)$$

contributed by the (c) loop. These terms serve to renormalize the L_9 constant according to the result from equation (4.14):

$$L_9^{\mathrm{r}} \;=\; L_9 - \frac{1}{128\pi^2}\,R.$$

The bare L_9 is brought into both f_π^+ and f_π^- parts of the vertex function by contribution (5.91).

Summing all the contributions, from equation (5.118) above and from equations (5.91), (5.92), (5.93), (5.94) and (5.116), we find that the P_2 dependence cancels out of our final expression. The total vertex function is

$$\begin{aligned}
\Gamma_{\mathrm{r}}^\mu \;=\; & (p_i + p_f)^\mu \left[1 + \frac{q^2}{F_\pi^2}\left(2L_9^{\mathrm{r}} - 2F(\frac{m_\pi^2}{q^2}, \frac{m_\pi^2}{\mu^2}) - F(\frac{m_K^2}{q^2}, \frac{m_K^2}{\mu^2}) \right) \right. \\
& \left. \qquad + \frac{4P_1}{F_\pi^2}\,(\, p_i^2 + p_f^2 - 2m_\pi^2 \,) \right] \\
& + (p_f - p_i)^\mu\, \frac{(p_i^2 - p_f^2)}{F_\pi^2} \left[2L_9^{\mathrm{r}} - 2F(\frac{m_\pi^2}{q^2}, \frac{m_\pi^2}{\mu^2}) - F(\frac{m_K^2}{q^2}, \frac{m_K^2}{\mu^2}) \right] \qquad (5.119)
\end{aligned}$$

giving

$$f_\pi^+ = 1 + \frac{q^2}{F_\pi^2}\left[2L_9^r - 2F(\frac{m_\pi^2}{q^2}, \frac{m_\pi^2}{\mu^2}) - F(\frac{m_K^2}{q^2}, \frac{m_K^2}{\mu^2}) \right] + \frac{4P_1}{F_\pi^2}(p_i^2 + p_f^2 - 2m_\pi^2)$$

$$f_\pi^- = \frac{(p_i^2 - p_f^2)}{F_\pi^2}\left[2L_9^r - 2F(\frac{m_\pi^2}{q^2}, \frac{m_\pi^2}{\mu^2}) - F(\frac{m_K^2}{q^2}, \frac{m_K^2}{\mu^2}) \right]. \tag{5.120}$$

The f_π^- form factor contains the same functional form as f_π^+. This is a result that must hold because of the Ward identity, as we will see in the following section.

5.7.2 Discussion

The constant L_9^r characterizes what is known as the pion charge radius; measurement of the charge radius pins down the numerical value to be used in the form factors.

We discover that the constant P_1 is not renormalized by loop divergences. In the language of the renormalization program shown in equation (4.14), the gamma constant associated with P_1 vanishes:

$$\gamma_{P_1} = 0. \tag{5.121}$$

The same does not necessarily hold for the constant P_2. One cannot exclude the possibility that the P_2 counterterm might be required in the renormalization of a different n-point Green function taken off shell. In fact, renormalization of new parameters has been reported in the off-shell generalization of chiral perturbation theory in the nucleon sector [13].

On shell, with $p_i^2 = p_f^2 = m_\pi^2$, equation (5.119) reproduces the established result [4, 9]:

$$f_\pi^+ = 1 + \frac{q^2}{F_\pi^2}\left[2L_9^r - 2F(\frac{m_\pi^2}{q^2}, \frac{m_\pi^2}{\mu^2}) - F(\frac{m_K^2}{q^2}, \frac{m_K^2}{\mu^2}) \right] \tag{5.122}$$

Off shell, we have generated a new contribution to the form factor parametrized by P_1. The modified Green function still satisfies the Ward identity.

We can make the observation also that f_π^+ and f_π^- are scale-independent; differentiating with respect to μ yields zero. This is unaffected by the presence of P_1, since the constant is unrenormalized.

The structure

$$\langle D^\mu D_\mu U^\dagger D^\nu D_\nu U \rangle$$

that we parametrized by P_1' in equation (3.77) has appeared previously in simple models of QCD [27, 28, 29]. From the models, numerical values have been estimated for all of the low energy constants. The number associated with the above structure can be translated into the P_1 we have defined [30], and for completeness, we quote the estimate:

$$P_1 \;=\; \frac{N_c}{96\,\pi^2} \;=\; 3.2 \times 10^{-3} \tag{5.123}$$

where $N_c = 3$ is the number of colours in QCD. This value is of the same order of magnitude as the other low energy constants; for example [10], $L_9^r\,(\mu = m_\eta) = 7.1 \times 10^{-3}$.

5.8 Verification of the Ward-Takahashi Identity

The Ward-Takahashi identity connecting the 2- and 3-point Green functions is [17]

$$q_\mu\, \Gamma_{\mathrm{r}}^\mu\,(p_i, p_f) \;=\; \triangle_{\mathrm{r}}^{-1}(p_f) \;-\; \triangle_{\mathrm{r}}^{-1}(p_i) \;. \tag{5.124}$$

where $q = p_f - p_i$ is the photon momentum. This identity is a consequence of electromagnetic current conservation, $\partial_\mu J^\mu = 0$.

To check the identity we substitute result (5.119) and equation (4.61) giving the propagator. First, since

$$\begin{aligned} q_\mu(p_i + p_f)^\mu &= -\,(p_i^2 - p_f^2) \\ q_\mu(p_f - p_i)^\mu &= q^2 \end{aligned} \tag{5.125}$$

we see that the normal on-shell component common to the two form factors

$$\left[2L_9^r \;-\; 2F(\frac{m_\pi^2}{q^2},\frac{m_\pi^2}{\mu^2}) \;-\; F(\frac{m_K^2}{q^2},\frac{m_K^2}{\mu^2}) \right]$$

cancels from the left hand side of the equation. This leaves the left hand side

$$q_\mu\, \Gamma_r^\mu(p_i,p_f) \;=\; (p_f^2 - p_i^2)\left[1 \;+\; \frac{4P_1}{F_\pi^2}\,(\, p_i^2 \;+\; p_f^2 \;-\; 2m_\pi^2\,)\right]. \tag{5.126}$$

The right hand side, then, is

$$\begin{aligned}
\triangle_r^{-1}(p_f) \;-\; \triangle_r^{-1}(p_i) \;&=\; p_f^2 \;-\; m_\pi^2 \;+\; \frac{4P_1}{F_\pi^2}\,(p_f^2 \;-\; m_\pi^2)^2 \\
&-\; p_i^2 \;+\; m_\pi^2 \;-\; \frac{4P_1}{F_\pi^2}\,(p_i^2 \;-\; m_\pi^2)^2 \\
&=\; (p_f^2 - p_i^2) \;+\; \frac{4P_1}{F_\pi^2}\,(p_f^4 \;-\; 2p_f^2 m_\pi^2 \;-\; p_i^4 \;+\; 2p_i^2 m_\pi^2) \\
&=\; (p_f^2 - p_i^2)\left[1 \;+\; \frac{4P_1}{F_\pi^2}\,(\, p_i^2 \;+\; p_f^2 \;-\; 2m_\pi^2\,)\right].
\end{aligned} \tag{5.127}$$

Our result satisfies the identity. The solution of the identity requires that P_1 terms enter both the propagator and the 3-point function.

One can see that in the absence of the P_1 and P_2 terms considered in the present treatment, we would have obtained result (5.122) for f_π^+, the unchanged f_π^- from (5.120), and the propagator (4.56). Together, these satisfy the identity by leaving the left and right hand sides equal to $(p_f^2 - p_i^2)$.

Chapter 6

Summary and Conclusions

We have addressed the question of how chiral perturbation theory can be used to calculate a general Green function with one or more legs off shell. This has been done using the theory for the pseudoscalar meson octet including electromagnetism, and the result has been a successful characterization of off-shell behaviour within this theory.

The electromagnetic vertex function for the pion has been computed, assuming an off-mass-shell interaction. The calculation has been based upon the full effective chiral Lagrangian to $O(p^4)$ unrestricted by the $O(p^2)$ equation of motion. This Lagrangian we have had to derive by invoking the usual symmetry arguments and examining all possible structures, including those built upon a symmetry-breaking mass term. After making sure to eliminate all non-independent structures, we have found that there are two extra terms usually omitted from the $\mathcal{L}_2 + \mathcal{L}_4$ Lagrangian. With the inclusion of these terms, renormalization of the theory has had to be modified. We have evaluated the new wavefunction renormalization constant for the pion, and derived the modified pion propagator.

Our result is presented in equation (5.120), which lists the electromagnetic form factors. There has indeed been a modification to the standard on-shell form factor f_π^+. The low energy constant that we have called P_1 enters the form factor, and describes the off-shell behaviour through a term proportional to

$$p_i^2 + p_f^2 - 2m_\pi^2.$$

The effect vanishes when the pions are on shell. The range of validity of this result is of course limited to reasonable values of $(p^2 - m_\pi^2)$, because it is derived from an effective theory.

The Ward-Takahashi identity has been checked and is found to be satisfied. The solution of this identity is an important check, and gives us confidence in our result.

To study the issue further, one can use a reasonable estimate for the constant P_1 to calculate the magnitude of deviation from the on-shell form factor as the interaction goes off shell. Meaningful predictions can in fact be made of this P_1 effect, since a numerical estimate for the parameter exists.

Many experiments have been done and are being done involving electromagnetic interactions of pions; for example, pion photo- and electo-production, Compton scattering, and analysis of meson exchange currents in nuclei. A possible candidate for showing an off-shell effect is pion Compton scattering. The s- and u-channel diagrams for this process include two half off-shell vertices. In other processes as well, there might be an observable contribution from this virtual effect, but the experimental question is whether it can be isolated unambiguously.

On the theoretical side, one must examine other processes to which the constants P_1 or P_2 might contribute, apart from the electromagnetic vertex. A phenomenological constant is particularly useful if it can form a constraint in the comparison of independent processes. It is an interesting question whether or not these new parameters will have any net effect in any observable at $O(p^4)$, the present state of the art for chiral perturbation theory.

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