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A CALCULATION OF GRAVITATIONAL RADIATION

by

©

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B.Sc., The University of British Columbia, 1977

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

Department of Physics

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

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Date 14 August 1980

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ABSTRACT

Approximate gravitational field equations in an alternative theory of gravity are solved for a class of boundary conditions. The generation of gravitational radiation from spatially bounded sources is analyzed, and it is found that the theory predicts the emission of dipole gravitational radiation. However, the dipole radiation vanishes for slow-motion post-Newtonian sources.

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ACKNOWLEDGEMENTS

I would like to thank my adivsor Professor P. Rastall for suggesting the topic and for providing assistance throughout the research.

I also wish to thank Professor W. Unruh for his helpful comments.

Acknowledgement is also made to the Natural Science and Engineering Research Council Canada for awarding a Postgraduate Scholarship for the period of research.

1. INTRODUCTION

Since Einstein introduced general relativity as a theory of space, time and gravitation, many alternative theories of relativistic gravitation have been proposed. These theories all predict results compatible with the "classical" experimental tests of general relativity - gravitational redshift of the frequencies of electromagnetic signals, relativistic perihelion shifts in the orbits of planets, refraction of electromagnetic waves and time delay of radar signals due to the Sun's gravitational field. Various other experiments may be performed to test general relativity and to possibly eliminate competing theories. For example, gravimeters have been used to search for anomalous Earth tides which would indicate "preferred-frame effects" due to the Earth's velocity relative to the rest frame of the Universe. Laser ranging experiments to corner reflectors left on the Moon have been performed in an attempt to detect different accelerations towards the Sun for the Earth and the Moon, indicating their departure from geodesic motion (i.e. the "Nordtvelt effect"). So far neither preferred-frame nor Nordtvelt effects have been observed. An experiment is planned in which the precession of a gyroscope orbiting the earth will be measured to determine non Newtonian effects [1], [2] ch. 40.

The results of such "solar system" experiments can be conveniently compared with the predictions of metric theories of gravitation within the framework of the parametrized post-Newtonian

(PPN) formalism [1], [2] ch. 39. This formalism summarizes the first (post-Newtonian) order corrections to Newtonian gravitation in a large class of metric theories of gravitation in terms of ten parameters. Solar system experiments then determine limits on the sizes of these post-Newtonian parameters. Thus to see whether a given metric theory of gravitation agrees with experiment, one computes its post-Newtonian parameters and compares them with the experimental limits. On this basis, many theories can be rejected. However, several theories - including the one used in this thesis - have the same, or almost the same post-Newtonian parameters as general relativity. These are as yet compatible with all solar system experiments.

To further restrict the class of viable theories of gravitation, one must consider effects beyond the post-Newtonian. Advances in technology during the next decade may allow measurements of higher-order (post-post-Newtonian) deviations from Newtonian gravitation. Refined solar system experiments would be expected to provide further constraints on the viability of a theory of gravitation. One should also consider cosmological models predicted by each theory and compare them with experiment. For example, the constancy of the Newtonian gravitational "constant" G, and the cosmological acceleration parameter must fall within current experimental limits.

Further selection amongst theories of gravitation may be possible by means of gravitational radiation experiments [3].

Some currently viable theories predict speeds of propagation of gravitational radiation different from that of light. A gravitational detector may be able to detect gravitational radiation bursts from nearly supernovae, and the arrival times of the bursts could then be compared with those for the corresponding electromagnetic radiation. In addition, different metric theories of gravitation predict as many as six polarization modes of gravitational radiation - general relativity predicts only two. Currently feasible detectors may be able to measure all six possible modes and thus eliminate some theories of gravity.

The discovery of binary pulsar PSR 1913+16 in 1975 and its subsequent observation have caused much recent interest in gravitational radiation calculations. The observed rate of decrease of the pulsar's orbital period is attributed to the loss of energy from the system via gravitational radiation. It has been claimed that the period decrease is quantitatively accounted for by the Einstein "quadrupole formula" for gravitational energy loss in general relativity [4]. However, the approximations made in deriving this formula (weak field, slow-motion Newtonian source consisting of two point masses in Keplerian orbits) [5] may not be applicable to the binary pulsar. For example, optical observations indicate that the companion star to the binary pulsar may not be a compact object and so cannot reliably be treated as a point mass [6]. In any case the derivation of the formula is purely formal and the validity of predictions of the rate of period decrease based on the quadrupole formula have been seriously questioned [7], [8]. Attempts are being made to find more mathematically meaningful approximation methods for treating systems such as the binary pulsar [9].

One would like to know whether other currently viable metric theories of gravitation can successfully predict the rate of period decrease of the binary pulsar. Calculations for several theories using weak-field, slow-motion post-Newtonian source approximations have been made, and the theories predict dipole gravitational radiation from gravitionally bound sources as compared to the quadrupole radiation predicted in general relativity. However, the masses of the bodies and the nature of the companion star in PSR 1913+16 are as yet not sufficiently certain to enable one to select conclusively amongst the competing theories [10].

In this thesis, the above-mentioned doubts about the validity of the usual formal manipulations are ignored, and weak-field approximations are used to linearize the field equations of an alternative theory of gravity. Solutions of the linearized equations are found for certain boundary conditions, and the solutions are used to calculate the gravitational energy radiated from a slowly-moving spatially bounded system of sources. It is found that the theory predicts dipole gravitational radiation, but that this vanishes for slow-motion post-Newtonian sources.

2. The Gravitational Field Equations and Approximate Equations

The theory of gravitation used in this thesis is described in detail in [11], and is summarized here. The theory contains a pseudo-Riemannian space-time metric tensor field g, and a pseudo-Riemannian flat metric tensor field $\bar{g}$ (i.e., the Riemann tensor of $\bar{g}$ vanishes). In addition there are a covector field n and a real scalar field ψ. The fields are related by (cf.[11] (4.3a), (4.4a), (4.8), (4.9), and (4.29)).

$$(2.1)\quad \begin{cases} \bar{n}^{\mu} = \bar{g}^{\mu\nu} n_{\nu} , \\ n^{\mu} = g^{\mu\nu} n_{\nu} = e^{-2\psi} \bar{n}^{\mu} , \\ g_{\mu\nu} = e^{-2\psi} (\bar{g}_{\mu\nu} + n_{\mu} n_{\nu}) , \\ g^{\mu\nu} = e^{2\psi} (\bar{g}^{\mu\nu} - n^{\mu} n^{\nu}) , \\ e^{4\psi} - 1 = \bar{g}^{\mu\nu} n_{\mu} n_{\nu} , \\ \Gamma = e^{-2\psi} \bar{\Gamma} , \end{cases}$$

where $\Gamma = (-\det g_{\mu\nu})^{\frac{1}{2}}$, $\bar{\Gamma} = (-\det \bar{g}_{\mu\nu})^{\frac{1}{2}}$, lower-case Greek indices have the range $\{0,1,2,3\}$ and obey the summation convention. Since $\bar{g}$ is a flat metric, there exist charts, called $\bar{g}$ inertial charts, in which

$$(2.2)\quad \begin{cases} \bar{g}_{\mu\nu} = \eta_{\mu\nu} , \\ \bar{\Gamma} = 1 , \\ \Gamma = e^{-2\psi} , \end{cases}$$

where $\eta_{\mu o} = -\delta_{\mu o}$, $\eta_{mn} = \delta_{mn}$, lower-case Latin indices have the range {1,2,3} and also obey the summation convention.

In a $\bar{g}$ inertial chart where the n_m are small (i.e. $|n_m| << |n_o|$), eq. (2.1) implies that $e^{4\psi}-1 = -n_o^2 + n_m n_m$. Thus, in a region where $\psi < 0$ one can choose the n_μ to be real, and the space-time metric will differ little from the "newtonian" metric $g_{\mu o} = -\delta_{\mu o} e^{2\psi}$, $g_{mn} = \delta_{mn} e^{-2\psi}$. In a region where $\psi > 0$, the spacetime metric will be nearly newtonian if the n_μ are imaginary. However, the n_μ cannot be real in one space spacetime region and imaginary in another, so that if ψ takes both positive and negative values in a region of space-time, then the space-time metric cannot be everywhere nearly newtonian (cf. [11] remarks following (2.2)).

The Lagrangian density of the gravitational field is given by (cf. [11] (4.14))

$$\text{(2.3)} \qquad \mathcal{L}_G = \Gamma F(N)\, n^{\mu;\nu} n_{\mu;\nu} ,$$

where $N = n^\mu n_\mu = e^{2\psi} - e^{-2\psi}$, semicolons denote covariant derivatives with respect to the spacetime metric g, and the function F is (cf. [11] (5.8))

$$\text{(2.4)} \qquad F(N) = -N/16\pi k(2+N^2) = e^{2\psi}(1-e^{4\psi})/16\pi k(1+e^{8\psi}) ,$$

where $k = Gc^{-4}$, G is the gravitational constant, and c is the speed of light. The field equations in a $\bar{g}$ inertial chart are (cf. [11](7.7)).

$$(2.5)\qquad \delta\mathcal{L}_G/\delta n_\pi - (\delta\mathcal{L}_G/\delta n_{\pi,\sigma})_{,\sigma} = e^{-4\psi}(\tfrac{1}{2} n^\pi T^{\mu\nu} g_{\mu\nu} - T^{\pi\mu} n_\mu),$$

where commas denote partial derivatives with respect to the space-time variables and the functional derivative δ corresponds to the independent fields n_μ, $n_{\mu,\nu}$, $\bar{g}_{\mu\nu}$ with $\mu \leq \nu$, $\bar{g}_{\mu\nu,\pi}$ with $\mu \leq \nu$, and the nongravitational fields (i.e. sources) whose stress-momentum tensor is T. The components $T^{\mu\nu}$ of T satisfy (cf. [11] (4.15), (4.17)

$$(2.6)\qquad \begin{cases} T^{\mu\nu} = T^{\nu\mu}, \\ T^{\mu\nu}{}_{;\nu} = 0 . \end{cases}$$

To study gravitational radiation in the field equations (2.5) are replaced by approximate equations which are easier to solve. Approximate field equations are presented in detail in [12]. Here the reduction of the field equation (2.5) to approximate equations in the case when $\psi \to \psi_0 \neq 0$ at spatial infinity in a $\bar{g}$ inertial chart is outlined. Gravitational radiation in the case when $\psi \to 0$ is treated in [12].

Calculations are simplified if new variables q_μ are introduced, defined by

$$(2.7)\qquad q_\mu = n_\mu (1 - e^{4\psi})^{-1/2} .$$

The transformation (2.7) is non-singular provided that ψ never vanishes. The q_μ are taken to be real, with $q_o < 0$, whether ψ is negative or positive (cf. [12] remarks following (3.1)). From (2.7) and (2.1) it follows that

$$(2.8) \qquad \bar{g}^{\mu\nu} q_\mu q_\nu = -1 ,$$

which in a $\bar{g}$ inertial chart becomes $q_o{}^2 = 1 + q_m q_m$. Thus ψ and the q_m can be chosen as independent fields, and the Lagrangian density may be expressed in terms of these variables ([12] (3.3)).

Approximations of fields are made on the basis of "order of magnitude" assignments. The expression $0_{(n)}$ denotes terms which are of the order of magnitude $V^n c^{-n}$ of smaller, where V is a typical speed of the sources. The components of the stress-momentum of the sources are assumed to satisfy $kT^{oo} = 0_{(2)}$, $kT^{mo} = 0_{(3)}$, $kT^{mn} = 0_{(4)}$ (e.g. an ideal fluid). One also assumes that in the neighborhood of the sources, time derivatives are an order of magnitude smaller than space derivatives. Thus $kT^{oo}{}_{,m} = 0_{(2)}$, $kT^{oo}{}_{,o} = 0_{(3)}$, etc.

In the case when $\psi' \rightarrow \psi_o$ at spatial infinity in a $\bar{g}$ inertial chart x', one assumes that $\psi' - \psi_o = 0_{(2)}$ and $q'_m = 0_{(3)}$. Far from the sources, space and time derivatives may be comparable in size. Thus it is assumed that $\psi'_{,0} = 0_{(2)}$ and $q'_{m,\mu} = 0_{(3)}$ except at points where $T^{\mu\nu} \neq 0$, in which case $\psi'_{,0} = 0_{(3)}$ and $q'_{m,0} = 0_{(4)}$. Similar assumptions are made for the higher partial derivatives. Such assumptions must be checked later for consistency with the solutions of the approximate field equations. The Lagrangian density $\mathcal{L}_G{}'$ in the $\bar{g}$ inertial chart x' is given to seventh order by [12] (5.11).

Define a new chart x by

$$(2.9)\qquad x^m = e^{-\psi_0} x'^m , \qquad x^0 = e^{\psi_0} x'^0 ,$$

so that $\psi \to \psi_0$ and $g_{\mu\nu} \to \eta_{\mu\nu}$ at spatial infinity in x provided that the q_m are negligible. Note that in the $\bar{g}$ inertial chart x', the $g'_{\mu\nu} \not\to \eta_{\mu\nu}$ at spatial infinity even if the q'_m are negligible. If one assumes the constant $\psi_0 \neq 0$ and is not small, all functions of ψ_0 appearing in expressions for the fields or field equations are assigned the order of magnitude $0_{(o)}$. Thus $\psi - \psi_0 = 0_{(2)}$, $q_m = 0_{(3)}$, and so on, in the chart x exactly as assigned in the chart x', since the expressions for the fields in x differ from their counterparts in x' only by factors of order zero.

The field equations to third order in the chart x are given by [12] (5.14) and (5.15):

$$(2.10)\quad (1+e^{8\psi_0})e^{-3\psi_0} q_{m,nn} - 2e^{5\psi_0} q_{m,00} + (1-e^{8\psi_0})e^{-3\psi_0} q_{n,nm} - 2(1+e^{4\psi_0})\psi_{,m0}$$

$$= -16\pi k(1-e^{4\psi_0})^{-1}(1+e^{8\psi_0})T^{m0} + O_{(4)} ,$$

$$(2.11)\quad \psi_{,mm} + 2(1+e^{8\psi_0})^{-1}(-1+e^{4\psi_0}-e^{8\psi_0})\psi_{,00}$$

$$+\tfrac{1}{2}(1+e^{8\psi_0})^{-1}(e^{-4\psi_0}-1-e^{4\psi_0}+e^{8\psi_0})e^{\psi_0}q_{m,m0} = 4\pi k T^{00} + O_{(4)} .$$

Define a scalar function χ by $\chi_{;0} = q_{m,m}$. Then differentiating (2.10) with respect to x^m and summing over m, using $T^{0m}{}_{;m} = -T^{00}{}_{,0} + O_{(5)}$, and integrating the resulting equation with respect to x^0 gives [12] (5.16):

$$(2.12)\quad e^{-3\psi_0}\chi_{,mm} - e^{5\psi_0}\chi_{,00} - (1+e^{4\psi_0})\psi_{,mm}$$

$$= 8\pi k(1-e^{4\psi_0})^{-1}(1+e^{8\psi_0})T^{00} + f + O_{(4)} ,$$

where f is a function of the spatial coordinates only. Eqs. (2.12) and (2.11) with $q_{m,m0} = \chi_{,00}$ are a linear system of partial differential equations of χ and ψ. Its solution is described in the next section.

3. Solutions of Approximate Equations

The approximate gravitational field equations (2.10) and (2.11) are solved in this section. First the system (2.12) and (2.11) with $q_{m,mo} = \chi_{,oo}$ is written in matrix form as

$$(3.1)\qquad \begin{bmatrix} 1 & 0 \\ -(e^{4\psi_0}+1) & e^{-3\psi_0} \end{bmatrix} \begin{bmatrix} \psi \\ \chi \end{bmatrix}_{,mm}$$

$$+ \begin{bmatrix} -2(e^{8\psi_0}+1)^{-1}(e^{8\psi_0}-e^{4\psi_0}+1) & \tfrac{1}{2}e^{-3\psi_0}(e^{8\psi_0}+1)^{-1}(e^{12\psi_0}-e^{8\psi_0}-e^{4\psi_0}+1) \\ 0 & -e^{5\psi_0} \end{bmatrix} \begin{bmatrix} \psi \\ \chi \end{bmatrix}_{,00}$$

$$= -4\pi \begin{bmatrix} -kT^{00} + O(4) \\ 2(e^{4\psi_0}-1)^{-1}(e^{8\psi_0}+1)\,kT^{00} - f/4\pi + O(4) \end{bmatrix} .$$

Multiplying (3.1) on the left by

$$\begin{bmatrix} 1 & 0 \\ e^{3\psi_0}(e^{4\psi_0}+1) & e^{3\psi_0} \end{bmatrix} ,$$

one obtains

$$(3.2)\quad \begin{bmatrix} \psi \\ \\ \chi \end{bmatrix}_{,mm} + \begin{bmatrix} -2(e^{8\psi_0}+1)^{-1}(e^{8\psi_0}-e^{4\psi_0}+1) & \frac{1}{2}e^{-3\psi_0}(e^{8\psi_0}+1)^{-1}(e^{12\psi_0}-e^{8\psi_0}-e^{4\psi_0}+1) \\ \\ -2(e^{8\psi_0}+1)^{-1}(e^{12\psi_0}+1)e^{3\psi_0} & \frac{1}{2}(e^{8\psi_0}+1)^{-1}(-e^{16\psi_0}-4e^{8\psi_0}+1) \end{bmatrix} \begin{bmatrix} \psi \\ \\ \chi \end{bmatrix}_{,00}$$

$$= -4\pi \begin{bmatrix} -kT^{00} + O(4) \\ \\ (e^{4\psi_0}-1)^{-1}(e^{8\psi_0}+3)e^{3\psi_0}kT^{00} + \rho + O(4) \end{bmatrix},$$

where $\rho = -e^{3\psi_0}f/4\pi$. The system of equations (3.2) can be decoupled if the matrix

$$A = \begin{bmatrix} -2(e^{8\psi_0}-e^{4\psi_0}+1) & \frac{1}{2}e^{-3\psi_0}(e^{12\psi_0}-e^{8\psi_0}-e^{4\psi_0}+1) \\ \\ -2(e^{12\psi_0}+1)e^{3\psi_0} & \frac{1}{2}(-e^{16\psi_0}-4e^{8\psi_0}+1) \end{bmatrix}$$

can be diagonalized. This is always possible over the complex numbers, but not necessarily over the reals.

The characteristic polynomial of A is $\det(A-\lambda I)$, where I is the 2 x 2 identity matrix. Thus the eigenvalues of A are the solutions of the quadratic equation

$$(3.3) \qquad 0 = \lambda^2 + \tfrac{1}{2}(e^{16\psi_0} + 8e^{8\psi_0} - 4e^{4\psi_0} + 3)\lambda$$

$$+ 2e^{24\psi_0} - 2e^{20\psi_0} + 4e^{16\psi_0} - 2e^{12\psi_0} + 2e^{8\psi_0}.$$

The solutions $\lambda_\pm$ of (3.3) are given by

$$(3.4) \qquad \lambda_\pm = \tfrac{1}{2}\left[-\tfrac{1}{2}(e^{16\psi_0} + 8e^{8\psi_0} - 4e^{4\psi_0} + 3) \pm \Delta^{1/2}\right],$$

where $\Delta = \frac{1}{4}(e^{4\psi_0}-1)^3(e^{20\psi_0} + 3e^{16\psi_0} - 10e^{12\psi_0} - 14e^{8\psi_0} - 3e^{4\psi_0} - 9)$. One notes that $\Delta = 0$ if $\psi_0 = 0$ or $\psi_0 = \psi_0{*}$, where $\psi_0{*} \approx 0.25$, $\Delta < 0$ if $0 < \psi_0 < \psi_0{*}$, and $\Delta > 0$ for all other ψ_0 (see Appendix). Thus the matrix A is diagonalizable over the reals provided $\psi_0 \leq 0$ or $\psi_0 \geq \psi_0{*}$. In the case when $\psi_0 = 0$, the field equation (2.10) is singular, and in fact the approximations used to obtain eqs. (2.10) and (2.11) do not apply if ψ_0 is small. However, this case is treated separately in [12]. Here only the cases when $\Delta > 0$ and ψ_0 is not small are considered. The eigenvalues $\lambda_\pm$ of A are then real and negative (see Appendix), and can be written as

$$(3.5) \qquad \lambda_\pm = -(e^{8\psi_0} + 1)\,\upsilon_\pm^{-2},$$

where the $v_\pm$ are positive real constants depending on ψ_0. A matrix P which diagonalizes A is found to be

$$P = \begin{bmatrix} a_+\beta & a_-\beta \\ a_+\alpha_+ & a_-\alpha_- \end{bmatrix}, \tag{3.6}$$

where the $a_\pm$ are nonzero but otherwise arbitrary constants, and

$$\alpha_\pm = \tfrac{1}{2} e^{3\psi_0}(e^{12\psi_0}+1)(e^{16\psi_0}+4e^{4\psi_0}-5 \pm \Delta^{1/2}),$$

$$\beta = (e^{4\psi_0}-1)^2(e^{16\psi_0}+e^{12\psi_0}+e^{4\psi_0}+1).$$

The matrix inverse of P is given by

$$P^{-1} = (\det P)^{-1} \begin{bmatrix} a_-\alpha_- & -a_-\beta \\ -a_+\alpha_+ & a_+\beta \end{bmatrix}, \tag{3.7}$$

where $\det P = -a_+ a_- e^{3\psi_0}(e^{4\psi_0} - 1)^2(e^{12\psi_0} + 1)(e^{16\psi_0} + e^{12\psi_0} + e^{4\psi_0} + 1)\Delta^{\frac{1}{2}}$. One checks that

(3.8)
$$P^{-1}AP = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}.$$

Multiplying eq. (3.2) on the left by P^{-1} and using $PP^{-1} = I$ gives

(3.9)
$$P^{-1}\begin{bmatrix} \psi \\ \chi \end{bmatrix}_{,mm} + (e^{8\psi_0}+1)^{-1}P^{-1}APP^{-1}\begin{bmatrix} \psi \\ \chi \end{bmatrix}_{,00} = -4\pi P^{-1}\begin{bmatrix} -kT^{00} + O_{(4)} \\ (e^{4\psi_0}-1)^{-1}(e^{8\psi_0}+3)e^{8\psi_0}kT^{00} + \rho + O_{(4)} \end{bmatrix}.$$

Defining linear combinations ϕ and ω of the fields ψ and χ by

(3.10)
$$\begin{bmatrix} \phi \\ \omega \end{bmatrix} = P^{-1}\begin{bmatrix} \psi \\ \chi \end{bmatrix}$$

and using (3.7) and (3.5), one obtains from (3.9)

(3.11)
$$\begin{bmatrix} \phi \\ \omega \end{bmatrix}_{,mm} - \begin{bmatrix} v_+^{-2} & 0 \\ 0 & v_-^{-2} \end{bmatrix}\begin{bmatrix} \phi \\ \omega \end{bmatrix}_{,00} = -4\pi(\det P)^{-1}\begin{bmatrix} -a_-(\gamma_- kT^{00} + \beta\rho) + O_{(4)} \\ a_+(\gamma_+ kT^{00} + \beta\rho) + O_{(4)} \end{bmatrix},$$

where $\gamma_\pm = \alpha_\pm + (e^{4\psi_0}-1)^{-1}(e^{8\psi_0}+3)\, e^{3\psi_0}\beta$

If the sources are spatially bounded, i.e.

$\ell = \sup\{\, |\vec{x}| : T^{\mu\nu}(x^0,\vec{x}) \neq 0 \,\}$ is finite,

then the matrix wave equation (3.11) has a solution corresponding to outgoing disturbances:

$$\begin{bmatrix} \phi(x^0,\vec{x}) \\ \\ \omega(x^0,\vec{x}) \end{bmatrix} = (\det P)^{-1} \begin{bmatrix} -a_-\{\gamma_- k\int R^{-1}[T^{00}]^+_{ret}\, d^3x' + \beta\int R^{-1}\rho(\vec{x}')\, d^3x'\} + O_{(4)} \\ \\ a_+\{\gamma_+ k\int R^{-1}[T^{00}]^-_{ret}\, d^3x' + \beta\int R^{-1}\rho(\vec{x}')\, d^3x'\} + O_{(4)} \end{bmatrix}, \tag{3.12}$$

where $R = |\vec{x}-\vec{x}'|$, $[T^{00}]^\pm_{ret} = T^{00}(x^0 - v_\pm^{-1}R, \vec{x}')$. Inverting eq. (3.10) and imposing boundary conditions $\psi \to \psi_0$ and $\chi \to 0$ at spatial infinity give the solutions to the approximate field equations (2.11) and (2.12):

$$\psi(x^0,\vec{x}) = \psi_0 + k(e^{12\psi_0}+1)^{-1} e^{-3\psi_0}\Delta^{-1/2}\{\gamma_-\int R^{-1}[T^{00}]^+_{ret}\, d^3x' - \gamma_+\int R^{-1}[T^{00}]^-_{ret}\, d^3x'\} + O_{(4)}, \tag{3.13}$$

$$\chi(x^0,\vec{x}) = k(e^{12\psi_0}+1)^{-1} e^{-3\psi_0}\Delta^{-1/2}\{\alpha+\gamma_- -\beta^{-1}\int R^{-1}[T^{00}]^+_{ret}\, d^3x' - \alpha-\gamma_+ +\beta^{-1}\int R^{-1}[T^{00}]^-_{ret}\, d^3x'\} + \int R^{-1}\rho(\vec{x}')\, d^3x' + O_{(4)}. \tag{3.14}$$

The solutions (3.13) and (3.14) may be expressed as

(3.15) $$\psi(x) - \psi_o = \int d^3x' \, R^{-1}\left[\delta_+ T^{00}(x^o - \tfrac{R}{v_+}, \vec{x}') + \delta_- T^{00}(x^o - \tfrac{R}{v_-}, \vec{x}')\right] + O_{(4)} ,$$

(3.16) $$q_{m,m}(x) = \int d^3x' \, R^{-1}\left[\epsilon_+ T^{00}{}_{,o}(x^o - \tfrac{R}{v_+}, \vec{x}') + \epsilon_- T^{00}{}_{,o}(x^o - \tfrac{R}{v_-}, \vec{x}')\right] + O_{(4)} ,$$

where $\delta_\pm = \pm k(e^{12\psi_o}+1)^{-1} e^{-3\psi_o} \Delta^{-\frac{1}{2}} \gamma_\mp$, $\epsilon_\pm = \pm k(e^{12\psi_o}+1)^{-1} e^{-3\psi_o} \Delta^{-\frac{1}{2}} \alpha_\pm \beta^{-1} \gamma_\mp$.

To obtain solutions for the q_m, one multiplies eq. (2.11) by $2e^{3\psi_o}(1-e^{4\psi_o})^{-2}$ and eq. (2.10) by $-\frac{1}{2}(1+e^{8\psi_o})^{-1}$ and gets

(3.17) $$\frac{2e^{3\psi_o}}{(1-e^{4\psi_o})^2} \psi_{,mm} - \frac{4(1-e^{4\psi_o}+e^{8\psi_o})e^{3\psi_o}}{(1-e^{4\psi_o})^2(1+e^{8\psi_o})} \psi_{,oo} + \frac{1+e^{4\psi_o}}{1+e^{8\psi_o}} q_{m,mo} = \frac{8\pi k\, e^{3\psi_o}}{(1-e^{4\psi_o})^2} T^{oo} + O_{(4)} ,$$

(3.18) $$\frac{1+e^{4\psi_o}}{1+e^{8\psi_o}} \psi_{,mo} - \frac{e^{-3\psi_o}}{2} q_{m,nn} + \frac{e^{5\psi_o}}{1+e^{8\psi_o}} q_{m,oo} - \frac{1}{2}\frac{1-e^{8\psi_o}}{1+e^{8\psi_o}} e^{-3\psi_o} q_{n,nm} = \frac{8\pi k}{1-e^{4\psi_o}} T^{mo} + O_{(4)} .$$

Define the Fourier transform $\hat{\psi}$ of $\psi-\psi_o$ by

(3.19) $$\psi(x) - \psi_o = \int d^4k \, \hat{\psi}(k) \, e^{ik_\mu x^\mu}$$

and define $\hat{q}_m$, $\hat{T}^{\mu\nu}$ similarly. Then the Fourier transforms of eqs. (3.17) and (3.18) are

(3.20) $$\left[-\frac{2e^{3\psi_o}}{(1-e^{4\psi_o})^2} k_m k_m + \frac{4(1-e^{4\psi_o}+e^{8\psi_o})e^{3\psi_o}}{(1-e^{4\psi_o})^2(1+e^{8\psi_o})} k_o^2 \right] \hat{\psi} - \frac{1+e^{4\psi_o}}{1+e^{8\psi_o}} k_o k_m \hat{q}_m = \frac{8\pi k\, e^{3\psi_o}}{(1-e^{4\psi_o})^2} \hat{T}^{oo} + O_{(4)} ,$$

(3.21) $$-\frac{1+e^{4\psi_o}}{1+e^{8\psi_o}} k_o k_m \hat{\psi} + \left[\frac{e^{-3\psi_o}}{2} k_n k_n - \frac{e^{5\psi_o}}{1+e^{8\psi_o}} k_o^2 \right] \hat{q}_m + \frac{1}{2}\frac{1-e^{8\psi_o}}{1+e^{8\psi_o}} e^{-3\psi_o} k_m k_n \hat{q}_n = \frac{8\pi k}{1-e^{4\psi_o}} \hat{T}^{mo} + O_{(4)} .$$

Equations (3.20) and (3.21) may be written as a matrix equation:

$$(3.22)\qquad B\Psi = 8\pi k T + O_{(4)},$$

where $B = [B_{\mu\nu}]$ is a 4 x 4 symmetric matrix given by

$$(3.23)\quad \begin{cases} B_{oo} = \left[-\dfrac{2e^{3\phi_0}}{(1-e^{4\phi_0})^2} k_m k_m + \dfrac{4(1-e^{4\phi_0}+e^{8\phi_0})e^{3\phi_0}}{(1-e^{4\phi_0})^2(1+e^{8\phi_0})} k_o^2 \right], \\[2ex] B_{mo} = B_{om} = -\dfrac{1+e^{4\phi_0}}{1+e^{8\phi_0}} k_o k_m, \\[2ex] B_{mn} = \left[\dfrac{e^{-3\phi_0}}{2} k_p k_p - \dfrac{e^{5\phi_0}}{1+e^{8\phi_0}} k_o^2 \right] \delta_{mn} + \dfrac{1}{2} \dfrac{1-e^{8\phi_0}}{1+e^{8\phi_0}} e^{-3\phi_0} k_m k_n, \end{cases}$$

and $\Psi = [\Psi_\mu]$, $T = [T_\mu]$ are four-dimensional column vectors given by

$$(3.24)\qquad \Psi_o = \hat{\phi},\ \Psi_m = \hat{q}_m,\ T_o = \frac{e^{2\phi_0}}{(1-e^{4\phi_0})^2}\hat{T}^{oo},\ T_m = \frac{1}{1-e^{4\phi_0}}\hat{T}^{mo}.$$

Let $|\vec{k}|^{-1}k_m$, $e_m^{(1)}$, $e_m^{(2)}$, define an orthonormal basis for $\mathbb{R}^3$, where $|\vec{k}| = (k_m k_m)^{\frac{1}{2}}$. Then one has the orthonormality relations

$$(3.25)\qquad |\vec{k}|^{-2}k_m k_m = e_m^{(1)}e_m^{(1)} = e_m^{(2)}e_m^{(2)} = 1,\quad k_m e_m^{(1)} = k_m e_m^{(2)} = e_m^{(1)}e_m^{(2)} = 0.$$

The 4x4 matrix $Q = [Q_{\mu\nu}]$ given by

$$(3.26)\qquad Q_{\mu o} = Q_{o\mu} = \delta_{\mu o},\quad Q_{m1} = |\vec{k}|^{-1}k_m,\quad Q_{m2} = e_m^{(1)},\quad Q_{m3} = e_m^{(2)}$$

satisfies $Q^T = Q^{-1}$, where Q^T denotes the matrix transpose of Q.

Defining a new column vector Ψ' by

$$(3.27) \qquad \Psi' = Q^T \Psi ,$$

one has

$$(3.28) \qquad \Psi'_0 = \hat{\psi}, \quad \Psi'_1 \equiv \hat{q}_{\parallel} = |\vec{k}|^{-1} k_m \hat{q}_m , \quad \Psi'_2 \equiv \hat{q}_{\perp}^{(1)} = e_m^{(1)} \hat{q}_m , \quad \Psi'_3 \equiv \hat{q}_{\perp}^{(2)} = e_m^{(2)} \hat{q}_m$$

and Ψ' satisfies the matrix equation

$$(3.29) \qquad B'\Psi' = 8\pi k T' + O_{(4)} ,$$

where

$$(3.30) \quad B' = \begin{pmatrix} B_{00} & -\frac{1+e^{4\phi_0}}{1+e^{8\phi_0}} \frac{k_0}{|\vec{k}|} k_p k_p & 0 & 0 \\ -\frac{1+e^{4\phi_0}}{1+e^{8\phi_0}} \frac{k_0}{|\vec{k}|} k_p k_p & \frac{1}{1+e^{8\phi_0}} \left[e^{-3\phi_0} k_p k_p - e^{5\phi_0} k_0^2 \right] & 0 & 0 \\ 0 & 0 & \frac{e^{-3\phi_0}}{2} k_p k_p - \frac{e^{5\phi_0}}{1+e^{8\phi_0}} k_0^2 & 0 \\ 0 & 0 & 0 & \frac{e^{-3\phi_0}}{2} k_p k_p - \frac{e^{5\phi_0}}{1+e^{8\phi_0}} k_0^2 \end{pmatrix}$$

is found by using the orthonormality relations (3.25), and T' is given by

$$(3.31) \qquad T'_0 = \frac{e^{3\phi_0}}{(1-e^{4\phi_0})^2} \hat{T}^{00} , \quad T'_1 = \frac{1}{1-e^{4\phi_0}} \frac{k_m}{|\vec{k}|} \hat{T}^{m0} , \quad T'_2 = \frac{1}{1-e^{4\phi_0}} e_m^{(1)} \hat{T}^{m0} , \quad T'_3 = \frac{1}{1-e^{4\phi_0}} e_m^{(2)} \hat{T}^{m0} .$$

Writing out eq. (3.29) explicitly, one gets

$$(3.32)\quad \left[-\frac{2e^{3\psi_0}}{(1-e^{4\psi_0})^2}k_pk_p+\frac{4(1-e^{4\psi_0}+e^{8\psi_0})e^{3\psi_0}}{(1-e^{4\psi_0})^2(1+e^{8\psi_0})}k_0{}^2\right]\hat{\psi}-\frac{1+e^{4\psi_0}}{1+e^{8\psi_0}}k_0k_m\hat{q}_m=\frac{8\pi k e^{3\psi_0}}{(1-e^{4\psi_0})^2}\hat{T}^{oo}+O_{(4)}\,,$$

$$(3.33)\quad -\frac{1+e^{4\psi_0}}{1+e^{8\psi_0}}\frac{k_pk_p}{|\vec{k}|}k_0\hat{\psi}+\frac{1}{1+e^{8\psi_0}}\left[e^{-3\psi_0}k_pk_p-e^{5\psi_0}k_0^2\right]\frac{k_m}{|\vec{k}|}\hat{q}_m=\frac{8\pi k}{1-e^{4\psi_0}}\frac{k_m}{|\vec{k}|}\hat{T}^{mo}+O_{(4)}\,,$$

$$(3.34)\quad \left[\tfrac{1}{2}e^{-3\psi_0}k_pk_p-\frac{e^{5\psi_0}}{1+e^{8\psi_0}}k_0{}^2\right]\hat{q}_\perp^{(j)}=\frac{8\pi k}{1-e^{4\psi_0}}e_m^{(j)}\hat{T}^{mo}+O_{(4)}\quad (j=1,2)\,.$$

Multiplying (3.32) by $\frac{1}{2}\,e^{-3\psi_o}\,(1-e^{4\psi_o})^2$, (3.33) by $-i|\vec{k}|(1+e^{8\psi_o})$ and (3.34) by $-2e^{3\psi_o}$, one obtains

$$(3.35)\quad \left[-k_mk_m+2\frac{1-e^{4\psi_0}+e^{8\psi_0}}{1+e^{8\psi_0}}k_0{}^2\right]\hat{\psi}-\frac{1}{2}\frac{(1-e^{4\psi_0})^2(1+e^{4\psi_0})}{1+e^{8\psi_0}}e^{-3\psi_0}k_0k_m\hat{q}_m=4\pi k\hat{T}^{oo}+O_{(4)}\,,$$

$$(3.36)\quad i(1+e^{4\psi_0})k_pk_pk_0\hat{\psi}-i\left[e^{-3\psi_0}k_pk_p+e^{5\psi_0}k_0{}^2\right]k_m\hat{q}_m=-8\pi k\frac{1+e^{8\psi_0}}{1-e^{4\psi_0}}ik_m\hat{T}^{mo}+O_{(4)}\,,$$

$$(3.37)\quad \left[-k_mk_m+\frac{2e^{8\psi_0}}{1+e^{8\psi_0}}k_0{}^2\right]\hat{q}_\perp^{(j)}=-16\pi k\frac{e^{3\psi_0}}{1-e^{4\psi_0}}e_m^{(j)}\hat{T}^{mo}+O_{(4)}\quad (j=1,2).$$

Equations (3.35) and (3.36) are the Fourier transforms of

$$(3.38)\quad \psi_{,mm}-2\frac{1-e^{4\psi_0}+e^{8\psi_0}}{1+e^{8\psi_0}}\psi_{,oo}+\frac{1}{2}\frac{e^{-4\psi_0}-1-e^{4\psi_0}+e^{8\psi_0}}{1+e^{8\psi_0}}e^{\psi_0}q_{m,mo}=4\pi k\,T^{oo}+O_{(4)}\,,$$

$$(3.39)\quad -(1+e^{4\psi_0})\psi_{,opp}+e^{-3\psi_0}q_{m,mpp}-e^{5\psi_0}q_{m,moo}=8\pi k\frac{1+e^{8\psi_0}}{1-e^{4\psi_0}}T^{oo}{}_{,o}+O_{(4)}\,,$$

where one uses $T^{om}{}_{,m} = -\,T^{oo}{}_{,o} + 0_{(5)}$ ([12](4.5)). Eqs. (3.38) and (3.39) have the solutions (3.15) and (3.16). The Fourier transforms of the solutions are

$$(3.40)\quad \hat{\psi}(k) = 4\pi\left[\delta_+\left(k_m k_m - \tfrac{1}{v_+^2}k_0^2\right)^{-1}\hat{T}^{00}(k) + \delta_-\left(k_m k_m - \tfrac{1}{v_-^2}k_0^2\right)^{-1}\hat{T}^{00}(k)\right] + O_{(4)},$$

$$(3.41)\quad \hat{q}_{\parallel}(k) = 4\pi\left[\epsilon_+\left(k_m k_m - \tfrac{1}{v_+^2}k_0^2\right)^{-1}\frac{k_0}{|\vec{k}|}\hat{T}^{00}(k) + \epsilon_-\left(k_m k_m - \tfrac{1}{v_-^2}k_0^2\right)^{-1}\frac{k_0}{|\vec{k}|}\hat{T}^{00}(k)\right] + O_{(4)}.$$

Using the Fourier transform of [12](4.5), one obtains

$$(3.42)\quad \hat{q}_{\parallel}(k) = -4\pi\left[\epsilon_+\left(k_m k_m - \tfrac{1}{v_+^2}k_0^2\right)^{-1}\frac{k_p}{|\vec{k}|}\hat{T}^{p0}(k) + \epsilon_-\left(k_m k_m - \tfrac{1}{v_-^2}k_0^2\right)^{-1}\frac{k_p}{|\vec{k}|}\hat{T}^{p0}(k)\right] + O_{(4)}.$$

Equation (3.37) has the solution

$$(3.43)\quad \hat{q}_{\perp}^{(j)}(k) = 4\pi\theta\left(k_m k_m - \tfrac{1}{u^2}k_0^2\right)^{-1} e_p^{(j)}\hat{T}^{p0}(k) + O_{(4)} \quad (j=1,2),$$

where $\theta = 4k\frac{e^{3\phi_0}}{1-e^{9\phi_0}}$, $u^2 = \frac{1+e^{8\phi_0}}{2e^{8\phi_0}} > 0$.

Inverting the change of coordinates eq. (3.27), one has

$$(3.44)\quad \hat{q}_m = \frac{k_m}{|\vec{k}|}\hat{q}_{\parallel} + \sum_{j=1}^{2} e_m^{(j)} q_{\perp}^{(j)}$$

Substituting the expressions (3.42) and (3.43) for $\hat{q}_{\parallel}$ and $q_{\perp}^{(j)}$ into eq. (3.44) and using the identity $\sum_{j=1}^{2} e_m^{(j)}e_p^{(j)} = \delta_{mp} - |\vec{k}|^{-2}k_m k_p$, one gets

$$(3.45)\quad \hat{q}_m(k) = -4\pi\left[\epsilon_+\left(k_n k_n - \tfrac{k_0^2}{v_+^2}\right)^{-1} + \epsilon_-\left(k_n k_n - \tfrac{k_0^2}{v_-^2}\right)^{-1} + \theta\left(k_n k_n - \tfrac{k_0^2}{u^2}\right)^{-1}\right]\frac{k_m k_p}{|\vec{k}|^2}\hat{T}^{p0}(k)$$
$$+ 4\pi\theta\left(k_n k_n - \tfrac{k_0^2}{u^2}\right)^{-1}\hat{T}^{m0}(k) + O_{(4)},$$

$$(3.46)\quad q_m(x) = \int d^4k\, \hat{q}_m(k)\, e^{ik_\alpha x^\alpha} + O_{(4)}.$$

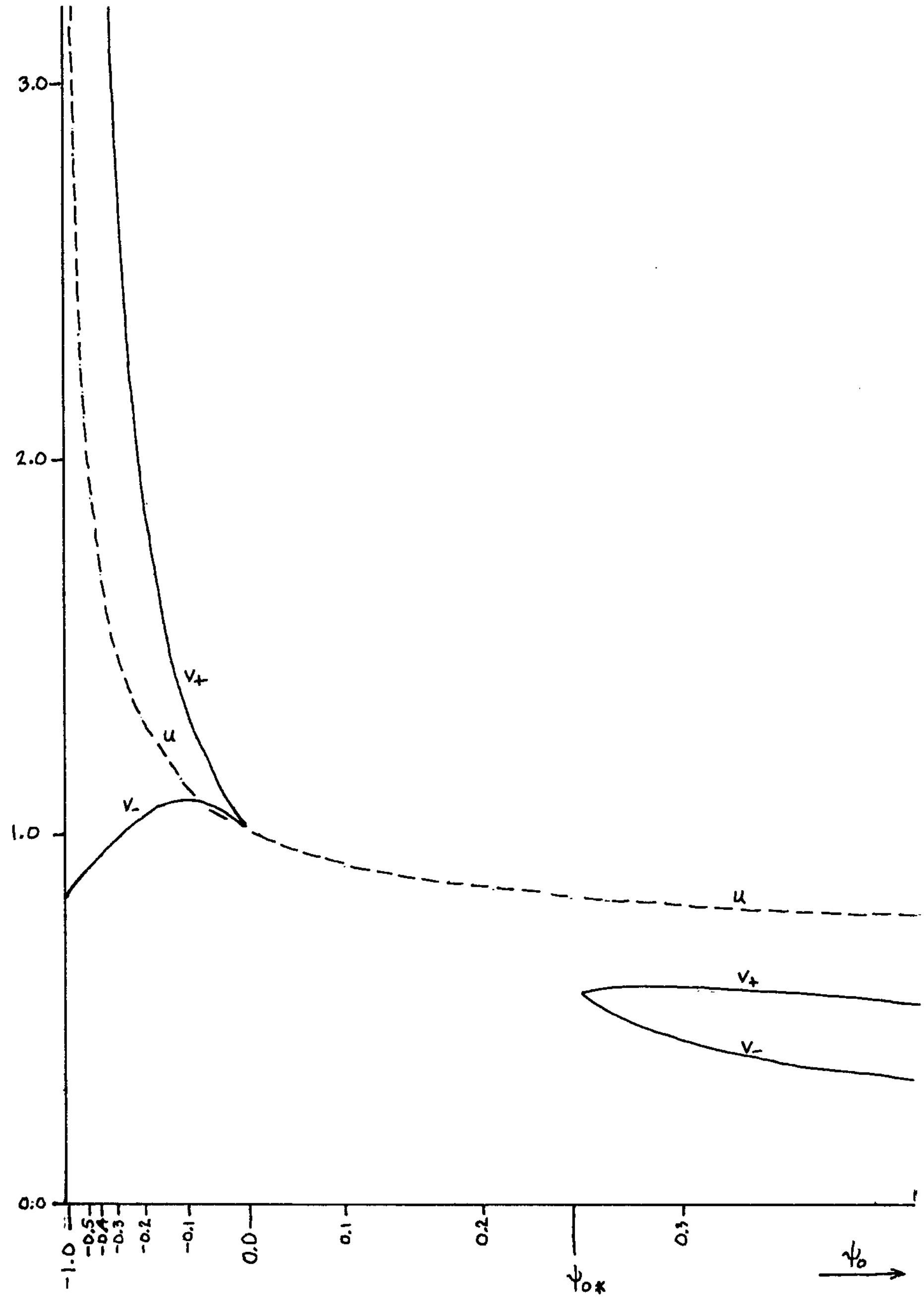
where one imposes the boundary condintion that the $q_m \to 0$ at spatial infinity. Equations (3.15) and (3.46) give solutions to the third-order field equations (2.10) and (2.11). They are consistent with the assumptions $\psi - \psi_o = 0_{(2)}$ and $q_m = 0_{(3)}$.

Three speeds of propagation appear in the solutions to the field equations. Two of the speeds, v_+ and v_-, are associated with ψ and the longitudinal component of q_m, while the transverse components of q_m propagate with speed u. These speeds are distinct from each other and from the speed of light (see Fig. 1), except in certain limiting cases noted in Table I (see also the Appendix).

Table I. Limiting values of speeds of propagation v_+, v_-, u

Limiting case	lim v_+	lim v_-	lim u
$\psi_o \to -\infty$	∞	≈ 0.82	∞
$\psi_o \to 0_+$	1	1	1
$\psi_o \to \psi_{o*+} \approx 0.25$	≈ 0.56	≈ 0.56	≈ 0.82
$\psi_o \to \infty$	$\frac{1}{2}$	0	≈ 0.71

Figure 1. Speeds of propagation v_+, v_-, u



Graphs of speeds of propagation as functions of the value ψ_0 of $\psi(x)$ at spatial infinity. For definitions, see eqs. (3.5) and (3.43).

To calculate higher-order contributions to the fields, one keeps terms to seventh order in the Lagrangian density and obtains approximate field equations [12](5.12) and (5.13):

$$(3.47)\quad (1+e^{8\phi_0})e^{-3\phi_0}q_{m,pp} - 2e^{5\phi_0}q_{m,00} + (1-e^{8\phi_0})e^{-3\phi_0}q_{p,pm} - 2(1+e^{4\phi_0})\phi_{,m0}$$

$$+8\,\frac{1+e^{4\phi_0}+5e^{8\phi_0}+e^{12\phi_0}}{(1-e^{4\phi_0})(1+e^{8\phi_0})}(\phi-\phi_0)\phi_{,m0} + 4\,\frac{3e^{4\phi_0}+7e^{8\phi_0}+3e^{12\phi_0}-e^{16\phi_0}}{(1-e^{4\phi_0})(1+e^{8\phi_0})}\phi_{,0}\phi_{,m}$$

$$= -16\pi k\,\frac{1+e^{8\phi_0}}{1-e^{4\phi_0}}\,T^{m0} + O_{(5)}\,,$$

$$(3.48)\quad \phi_{,mm} - \left[2 - \frac{2e^{4\phi_0}}{1+e^{8\phi_0}} - \left(8 - \frac{16e^{12\phi_0}}{1+e^{8\phi_0}}\right)(\phi-\phi_0)\right]\phi_{,00} + \left[4 - \frac{8e^{12\phi_0}}{(1+e^{8\phi_0})^2}\right]\phi_{,0}^{\,2}$$

$$+\frac{1}{2}\,\frac{e^{-4\phi_0}-1-e^{4\phi_0}+e^{8\phi_0}}{1+e^{8\phi_0}}\,e^{\phi_0}q_{m,m0} - \frac{2e^{-4\phi_0}-3+2e^{4\phi_0}-e^{8\phi_0}}{1+e^{8\phi_0}}\,e^{\phi_0}\left[(\phi_{,0}q_m)_{,m} + (\phi_{,m}q_m)_{,0}\right]$$

$$-2\,\frac{e^{-4\phi_0}+4e^{4\phi_0}-4e^{8\phi_0}-e^{12\phi_0}}{1+e^{8\phi_0}}\,e^{\phi_0}(\phi-\phi_0)q_{m,m0} = 4\pi k(T^{00}+T^{mm}) + O_{(6)}\,.$$

If the third-order solutions (3.15) and (3.46) are substituted into the quadratic terms of eqs. (3.47) and (3.48), one has

$$(3.49)\quad (1+e^{8\phi_0})e^{-3\phi_0}q_{m,pp} - 2e^{5\phi_0}q_{m,00} + (1-e^{8\phi_0})e^{-3\phi_0}q_{p,pm} - 2(1+e^{4\phi_0})\phi_{,m0}$$

$$= -16\pi k\,\frac{1+e^{8\phi_0}}{1-e^{4\phi_0}}\,T^{m0} + O_{(5)}\,,$$

$$(3.50)\quad \phi_{,mm} - 2\,\frac{1-e^{4\phi_0}+e^{8\phi_0}}{1+e^{8\phi_0}}\,\phi_{,00} + \frac{1}{2}\,\frac{e^{-4\phi_0}-1-e^{4\phi_0}+e^{8\phi_0}}{1+e^{8\phi_0}}\,e^{\phi_0}q_{m,m0} = 4\pi k(T^{00}+T^{mm}) + O_{(6)}\,.$$

The system of equations (3.49) and (3.50) differs from (2.10) and (2.11) only by the orders of magnitude of the neglected terms and the additional $4\pi kT^{mm}$ term in the sources. The solution methods of this section give the fourth-order solution

$$(3.51)\quad \phi(x)-\phi_0=\int\frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\Big[\delta_+T^{00}\big(x^0-\frac{|\vec{x}-\vec{\xi}|}{v_+},\vec{\xi}\big)+\eta_+T^{mm}\big(x^0-\frac{|\vec{x}-\vec{\xi}|}{v_+},\vec{\xi}\big)$$

$$+\delta_-T^{00}\big(x^0-\frac{|\vec{x}-\vec{\xi}|}{v_-},\vec{\xi}\big)+\eta_-T^{mm}\big(x^0-\frac{|\vec{x}-\vec{\xi}|}{v_-},\vec{\xi}\big)\Big]\;+\;O_{(5)}\,,$$

where $\eta_\pm=\pm k\,(1+e^{12\phi_0})^{-1}e^{-3\phi_0}\Delta^{-1/2}\left[\alpha_\pm+e^{3\phi_0}(1+e^{4\phi_0})\beta\right].$

The expression (3.16) for $q_{m,m}$ is good to fourth order, as are the solutions for the transverse parts of q_m since eq. (3.50) is unchanged from eq. (2.10). Thus one has

$$(3.52)\quad q_m(x)=-\int\frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\int d^4k\,\frac{k_mk_n}{|\vec{k}|^2}\hat{T}^{n0}(k)\Big[\epsilon_+e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{v_+}}+\epsilon_-e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{v_-}}+\theta e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{u}}\Big]e^{ik_0x^0}e^{ik_\rho\xi^\rho}$$

$$+\int\frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\theta\,T^{m0}\big(x^0-\frac{|\vec{x}-\vec{\xi}|}{u},\vec{\xi}\big)\;+\;O_{(5)}\,.$$

4. Radiation of Gravitational Energy

Once solutions to the gravitational field equations are known, it is possible to calculate the rate of gravitational energy radiated away from spatially bounded sources. The components of the canonical gravitational stress-momentum $\Xi_G{}'$ in a $\bar{g}$ inertial chart x' are defined by [12] (2.7) to be

$$(4.1) \qquad \Xi_G{}'{}_\pi{}^\sigma = \delta_{\pi\sigma} \mathcal{L}_G{}' - (\delta \mathcal{L}_G{}' / \delta n'_{\mu,\sigma})\, n'_{\mu,\pi} \; .$$

If the sources are spatially bounded in x', then the rate of gravitational energy radiated out of the spherical surface $(x'^m x'^m)^{\frac{1}{2}} = a$, where $a \geq \ell$, is given by [12] (3.27):

$$(4.2) \qquad \mathcal{E}_G(a) = -c \int_0^{2\pi} d\alpha' \int_0^{\pi} d\theta' \, \Xi_G{}'{}_0{}^s(x'^0, \vec{x}')\, x'^s r' \sin\theta' \, ,$$

where $r' = (x'^m x'^m)^{1/2}$, $x'^1 = r'\cos\theta'$, $x'^2 = r'\sin\theta'\cos\alpha'$, $x'^3 = r'\sin\theta'\sin\alpha'$, $r' = a$. The rate of radiation of gravitational energy is defined by [12] (3.28) as

$$(4.3) \qquad \mathcal{E} = \lim_{a \to \infty} \mathcal{E}_G(a) \; .$$

As noted in [12], terms in $\Xi_G{}'{}_\pi{}^\sigma$ which are $O(r'^{-3})$ as $r' \to \infty$ do not contribute to $\mathcal{E}$:

In the case of small q_m, one uses [12](3.10) and (3.11) to get

$$(4.4)\quad 16\pi k\,\Xi_{4}{}'_{o}{}^{s} = 4\psi'_{,s}\psi'_{,o} - \frac{1-e^{4\psi'}}{1+e^{8\psi'}}\Big[(e^{-4\psi'}-1+e^{4\psi'}-e^{8\psi'})\,q'_{m,s}q'_{m,o}$$

$$+(e^{-4\psi'}-1-e^{4\psi'}+e^{8\psi'})\,q'_{s,m}q'_{m,o} + (8e^{-4\psi'}-4+4e^{4\psi'})(\psi'_{,o}+q'_m\psi'_{,m})\psi'_{,o}q'_s$$

$$+(-2e^{-4\psi'}+2)\,q'_{m,m}q'_s\psi'_{,o} + (-2+2e^{4\psi'})(q'_{s,o}+q'_m q'_{s,m})\psi'_{,o}$$

$$+(-2+2e^{4\psi'})\,q'_s q'_{m,o}\psi'_{,m} + (-2e^{-4\psi'}+2)\,q'_{s,o}(\psi'_{,o}+q'_m\psi'_{,m})\Big] + O(q'^3)\,.$$

The results of section 3 imply $\psi'-\psi_o = 0_{(2)}$, $\psi'_{,o} = 0_{(3)}$, $q'_{m,s} = 0_{(3)}$, $q'_{m,o} = 0_{(4)}$. Thus to eighth order, the energy flux is given by

$$(4.5)\quad 16\pi k\,\Xi_{4}{}'_{o}{}^{s} = 4\psi'_{,s}\psi'_{,o} - \frac{1-e^{4\psi_o}}{1+e^{8\psi_o}}\Big[(e^{-4\psi_o}-1+e^{4\psi_o}-e^{8\psi_o})\,q'_{m,s}q'_{m,o}$$

$$+(e^{-4\psi_o}-1-e^{4\psi_o}+e^{8\psi_o})\,q'_{s,m}q'_{m,o} - 2(e^{-4\psi_o}-e^{4\psi_o})\,q'_{s,o}\psi'_{,o} + O_{(9)}\,.$$

The coordinate transformation (2.9) and tensor transformation rules give

$$(4.6)\qquad \int \frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\, T^{00}\!\left(x^0-\frac{|\vec{x}-\vec{\xi}|}{v},\vec{\xi}\right) = \int \frac{d^3\xi'}{|\vec{x}'-\vec{\xi}'|}\, T'^{00}\!\left(x'^0-e^{-2\psi_0}\frac{|\vec{x}'-\vec{\xi}'|}{v},\vec{\xi}'\right),$$

$$\int \frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\, T^{m0}\!\left(x^0-\frac{|\vec{x}-\vec{\xi}|}{v},\vec{\xi}\right) = e^{-2\psi_0}\int \frac{d^3\xi'}{|\vec{x}'-\vec{\xi}'|}\, T'^{m0}\!\left(x'^0-e^{-2\psi_0}\frac{|\vec{x}'-\vec{\xi}'|}{v},\vec{\xi}'\right),$$

$$\hat{T}^{m0}(k) = e^{-2\psi_0}\,\hat{T}'^{m0}(k'),$$

where k' is given by

$$(4.7)\qquad k'_0 = e^{\psi_0}k_0\,, \qquad k'_m = e^{-\psi_0}k_m$$

The solutions (3.51) and (3.52) in x transform to

$$(4.8)\qquad \psi'(x')-\psi_0 = \int \frac{d^3\xi'}{|\vec{x}'-\vec{\xi}'|}\Big[\delta_+ T'^{00}\!\left(x'^0-\frac{|\vec{x}'-\vec{\xi}'|}{v'_+},\vec{\xi}'\right) + e^{-4\psi_0}\eta_+ T'^{mm}\!\left(x'^0-\frac{|\vec{x}'-\vec{\xi}'|}{v'_+},\vec{\xi}'\right)$$

$$+\delta_- T'^{00}\!\left(x'^0-\frac{|\vec{x}'-\vec{\xi}'|}{v'_-},\vec{\xi}'\right) + e^{-4\psi_0}\eta_- T'^{mm}\!\left(x'^0-\frac{|\vec{x}'-\vec{\xi}'|}{v'_-},\vec{\xi}'\right)\Big] + O_{(5)},$$

$$(4.9)\qquad q'_m(x') = -e^{-3\psi_0}\int\frac{d^3\xi'}{|\vec{x}'-\vec{\xi}'|}\int d^4k'\,\frac{k'_m k'_n}{|\vec{k}|^2}\,\hat{T}'^{n0}(k')\Big[\epsilon_+ e^{-ik'_0\frac{|\vec{x}'-\vec{\xi}'|}{v'_+}} + \epsilon_- e^{-ik'_0\frac{|\vec{x}'-\vec{\xi}'|}{v'_-}}$$

$$+\theta e^{-ik'_0\frac{|\vec{x}'-\vec{\xi}'|}{u'}}\Big]e^{ik'_0x'^0}e^{ik'_p\xi'^p} + e^{-3\psi_0}\int\frac{d^3\xi'}{|\vec{x}'-\vec{\xi}'|}\,\theta\, T'^{m0}\!\left(x'^0-\frac{|\vec{x}'-\vec{\xi}'|}{u'},\vec{\xi}'\right) + O_{(5)},$$

where $v'_\pm = e^{2\psi_0}v_\pm$, $u' = e^{2\psi_0}u$. At this point it is convenient to drop all primes, since the chart x of the last section is no longer needed. All subsequent expressions are to be understood to be expressed in a $\bar{g}$ inertial chart x.

Define the one-dimensional Fourier transform ψ of $\psi-\psi_o$ by

(4.10) $$\psi(x)-\psi_0 = \int_{-\infty}^{\infty} dk_0\, \tilde{\psi}(k_0,\vec{x})\, e^{ik_0x^0} ,$$

and define $\tilde{q}_m$ and $\tilde{T}^{\mu\nu}$ similarly. From (4.8) and (4.9), one obtains

(4.11) $$\tilde{\psi}(k_0,\vec{x}) = \int\frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\Big[\Big(\delta_+\tilde{T}^{00}(k_0,\vec{\xi}) + e^{-4\psi_0}\eta_+\tilde{T}^{mm}(k_0,\vec{\xi})\Big)e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{v_+}}$$

$$+\Big(\delta_-\tilde{T}^{00}(k_0,\vec{\xi}) + e^{-4\psi_0}\eta_-\tilde{T}^{mm}(k_0,\vec{\xi})\Big)e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{v_-}}\Big] \quad + O_{(5)} ,$$

(4.12) $$\tilde{q}_m(k_0,\vec{x}) = -e^{-3\psi_0}\int\frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\int d^3k\,\frac{k_mk_n}{|\vec{k}|^2}\,\hat{T}^{n0}(k)\Big[\epsilon_+e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{v_+}} + \epsilon_-e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{v_-}} + \theta e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{u}}\Big]e^{ik_p\xi^p}$$

$$+ e^{-3\psi_0}\int\frac{d^3\xi}{|\vec{x}-\vec{\xi}|}\,\tilde{T}^{m0}(k_0,\vec{\xi})\,\theta\, e^{-ik_0\frac{|\vec{x}-\vec{\xi}|}{u}} \qquad + \quad O_{(5)} .$$

At large distances from the sources, i.e. if $|\vec{x}| = r >> |k_0|^{-1} \sim \ell$, one has $|\vec{x}-\vec{\xi}| = r - x^m\xi^m/r + O(r^{-1})$. The expression (4.11) may be written as

(4.13) $$\tilde{\psi}(k_0,\vec{x}) = \frac{1}{r}\Big\{e^{ia_px^p}\int d^3\xi\big[\delta_+\tilde{T}^{00}(k_0,\vec{\xi}) + e^{-4\psi_0}\eta_+\tilde{T}^{mm}(k_0,\vec{\xi})\big]e^{-ia_p\xi^p}$$

$$+ e^{ib_px^p}\int d^3\xi\big[\delta_-\tilde{T}^{00}(k_0,\vec{\xi}) + e^{-4\psi_0}\eta_-\tilde{T}^{mm}(k_0,\vec{\xi})\big]e^{-ib_p\xi^p}\Big\} + O_{(5)} + O(r^{-2}),$$

where $a_p = -k_0x^p/v_+r$, $b_p = -k_0x^p/v_-r$. Identifying the spatial integrals as inverse Fourier transforms, one writes

(4.14) $$\tilde{\psi}(k_0,\vec{x}) = \frac{(2\pi)^3}{r}\Big\{e^{ia_px^p}\big[\delta_+\hat{T}^{00}(k_0,\vec{a}) + e^{-4\psi_0}\eta_+\hat{T}^{mm}(k_0,\vec{a})\big]$$

$$+ e^{ib_px^p}\big[\delta_-\hat{T}^{00}(k_0,\vec{b}) + e^{-4\psi_0}\eta_-\hat{T}^{mm}(k_0,\vec{b})\big]\Big\} \quad + O_{(5)} + O(r^{-2}) .$$

Similarly, one has

$$(4.15)\quad \tilde{q}_m(k_0,\vec{x}) = -e^{-3\psi_0}\frac{(2\pi)^3}{r}\Big\{ e^{ia_p x^p}\epsilon_+ \frac{a_m a_n}{a^2}\hat{T}^{no}(k_0,\vec{a}) + e^{ib_p x^p}\epsilon_- \frac{b_m b_n}{b^2}\hat{T}^{no}(k_0,\vec{b})$$

$$+ e^{id_p x^p}\theta\Big[\frac{d_m d_n}{d^2}\hat{T}^{no}(k_0,\vec{d}) - \hat{T}^{mo}(k_0,\vec{d})\Big]\Big\} + O_{(5)} + O(r^{-2}),$$

where $a = |k_o|/v_+$, $b = |k_o|/v_-$, $d_p = -k_o x^p/ur$, $d = |k_o'|/u$.

If the sources are assumed to move slowly, i.e. if $|k_o| = 0_{(1)}$ and $v_+/\ell \sim v_-/\ell >> |k_o|$, one may expand the Fourier transforms:

$$(4.16)\quad \hat{T}^{\mu\nu}(k_0,\vec{k}) = (2\pi)^{-3}\int d^3\xi\, \tilde{T}^{\mu\nu}(k_0,\vec{\xi})\, e^{-ik_p\xi^p}$$

$$= (2\pi)^{-3}\sum_{j=0}^{\infty}\frac{1}{j!}\int d^3\xi\, \tilde{T}^{\mu\nu}(k_0,\vec{\xi})(-ik_p\xi^p)^j ,$$

where in (4.16) and in subsequent expressions, k_p is one of a_p, b_p, d_p. Keeping terms to fourth order, one has

$$(4.17)\quad \begin{cases} ik_0\hat{T}^{oo}(k_0,\vec{k}) = (2\pi)^{-3}\int d^3\xi\,[ik_0\tilde{T}^{oo}(k_0,\vec{\xi}) + k_0 k_p\tilde{T}^{oo}(k_0,\vec{\xi})\,\xi^p] + O_{(5)}, \\ ik_0\hat{T}^{mo}(k_0,\vec{k}) = (2\pi)^{-3}\int d^3\xi\; ik_0\tilde{T}^{mo}(k_0,\vec{\xi}) \quad + O_{(5)}, \\ ik_0\hat{T}^{mm}(k_0,\vec{k}) = \quad O_{(5)} . \end{cases}$$

The Fourier transform of the conservation law $T^{oo}{}_{,o} = -T^{on}{}_{,n} + 0_{(5)}$ (cf.[12](4.5)) is

$$(4.18) \qquad ik_o \tilde{T}^{oo} = -\tilde{T}^{on}{}_{,n} + O_{(5)} .$$

Integrating eq. (4.18) over all space gives

$$(4.19) \qquad \int d^3\xi \; ik_o \tilde{T}^{oo}(k_o,\vec{\xi}) = O_{(5)} ,$$

while multiplying eq. (4.18) first by ξ^m and then integrating over all space gives

$$(4.20) \qquad \int d^3\xi \; ik_o \tilde{T}^{om}(k_o,\vec{\xi}) = -k_o^2 \int d^3\xi \; \tilde{T}^{oo}(k_o,\vec{\xi})\,\xi^m + O_{(5)} .$$

If one defines the Fourier transform of the dipole moment of the sources as

$$(4.20) \qquad \tilde{D}^m(k_o) = \int d^3\xi \; \tilde{T}^{oo}(k_o,\vec{\xi})\,\xi^m ,$$

then from eqs. (4.17) and (4.19)-(4.21) one gets

$$
(4.22)\quad \begin{cases} ik_o \hat{T}^{oo}(k_o,\vec{k}) = (2\pi)^{-3} k_o k_p \tilde{D}^p(k_o) + O_{(5)}, \\ \\ ik_o \hat{T}^{mo}(k_o,\vec{k}) = -(2\pi)^{-3} k_o^2 \tilde{D}^m(k_o) + O_{(5)}. \end{cases}
$$

From eqs. (4.14), (4.15), (4.17) and (4.22) one obtains

$$
(4.23)\qquad ik_o \tilde{\psi}(k_o,\vec{x}) = -\frac{k_o^2}{r}\left[e^{ia_p x^p}\frac{\delta_+}{V_+}\frac{x^m}{r}\tilde{D}^m(k_o) + e^{ib_p x^p}\frac{\delta_-}{V_-}\frac{x^m}{r}\tilde{D}^m(k_o)\right] + O_{(5)} + O(r^{-2}),
$$

$$
(4.24)\qquad \tilde{\psi}_{,s}(k_o,\vec{x}) = \frac{x^s}{r}\frac{k_o^2}{r}\left[e^{ia_p x^p}\frac{\delta_+}{V_+^2}\frac{x^m}{r}\tilde{D}^m(k_o) + e^{ib_p x^p}\frac{\delta_-}{V_-^2}\frac{x^m}{r}\tilde{D}^m(k_o)\right] + O_{(5)} + O(r^{-2}),
$$

$$
(4.25)\qquad ik_o \tilde{q}_m(k_o,\vec{x}) = e^{-3\psi_o}\frac{k_o^2}{r}\left\{e^{ia_p x^p}\epsilon_+\frac{x^m x^n}{r^2}\tilde{D}^n(k_o) + e^{ib_p x^p}\epsilon_-\frac{x^m x^n}{r^2}\tilde{D}^n(k_o)\right.
$$

$$
\left. + e^{id_p x^p}\,\theta\left[\frac{x^m x^n}{r^2}\tilde{D}^n(k_o) - \tilde{D}^m(k_o)\right]\right\} + O_{(5)} + O(r^{-2}),
$$

$$
(4.26)\qquad \tilde{q}_{m,s}(k_o,\vec{x}) = -e^{-3\psi_o}\frac{x^s}{r}\frac{k_o^2}{r}\left\{e^{ia_p x^p}\frac{\epsilon_+}{V_+}\frac{x^m x^n}{r^2}\tilde{D}^n(k_o) + e^{ib_p x^p}\frac{\epsilon_-}{V_-}\frac{x^m x^n}{r^2}\tilde{D}^n(k_o)\right.
$$

$$
\left. + e^{id_p x^p}\frac{\theta}{u}\left[\frac{x^m x^n}{r^2}\tilde{D}^n(k_o) - \tilde{D}^m(k_o)\right]\right\} + O_{(5)} + O(r^{-2}).
$$

The Fourier transform eq. (4.10) may be written

$$\psi(x) - \psi_0 = \int_{-\infty}^{0} dk_0 \left[\tilde{\psi}(k_0, \vec{x})\, e^{ik_0 x^0} + \tilde{\psi}^*(k_0, \vec{x})\, e^{-ik_0 x^0} \right] . \tag{4.27}$$

If the sources consist of a single Fourier component, ie, if

$$T^{\mu\nu}(x) = \tilde{T}^{\mu\nu}(k_0, \vec{x})\, e^{ik_0 x^0} + \tilde{T}^{\mu\nu *}(k_0, \vec{x})\, e^{-ik_0 x^0} , \tag{4.28}$$

where $k_0 < 0$, then so do the fields $\psi - \psi_0$ and q_m. For such cases one has

$$\psi_{,s}\psi_{,0} = 2\,\mathrm{Re}\left\{ \tilde{\psi}_{,s}(ik_0\tilde{\psi})^* + \tilde{\psi}_{,s}(ik_0\tilde{\psi})\, e^{i2k_0 x^0} \right\} , \tag{4.29}$$

where Re{z} denotes the real part of a complex number z. Substituting the expressions (4.23) and (4.24) into (4.29), one obtains

$$\begin{aligned} \psi_{,s}\psi_{,0} = {} & -2 \frac{x^s}{r} \frac{k_0^4}{r^2}\, \mathrm{Re}\left\{ \frac{\delta_+^2}{V_+^3} \frac{x^m x^n}{r^2} \tilde{D}^m(k_0) \tilde{D}^{n*}(k_0) + \frac{\delta_-^2}{V_-^3} \frac{x^m x^n}{r^2} \tilde{D}^m(k_0) \tilde{D}^{n*}(k_0) \right. \\ & \left. + \left[\frac{\delta_+\delta_-}{V_+^2 V_-} e^{i(a_p - b_p)x^p} + \frac{\delta_+\delta_-}{V_+ V_-^2} e^{i(b_p - a_p)x^p} \right] \frac{x^m x^n}{r^2} \tilde{D}^m(k_0) \tilde{D}^{n*}(k_0) + \tilde{\psi}_{,s}(ik_0\tilde{\psi}) e^{i2k_0 x^0} \right\} \\ & + O_{(9)} + O(r^{-3}) . \end{aligned} \tag{4.30}$$

Taking the average over a large region of spacetime compared with the wavelengths and frequency, the oscillating terms give no net contribution, and one has

$$\langle \psi_{,s}\psi_{,0} \rangle = -2 \frac{x^s}{r} \frac{k_0^4}{r^2}\, \mathrm{Re}\left\{ \left(\frac{\delta_+^2}{V_+^3} + \frac{\delta_-^2}{V_-^3} \right) \frac{x^m x^n}{r^2} \tilde{D}^m(k_0) \tilde{D}^{n*}(k_0) \right\} + O_{(9)} + O(r^{-3}) , \tag{4.31}$$

where < > denotes the average taken over a large region of spacetime. Similarly,

$$(4.32)\qquad \langle q_{m,s}\, q_{m,0}\rangle = -2e^{-6\phi_0}\,\frac{x^s}{r}\,\frac{k_0^4}{r^2}\,\mathrm{Re}\Big\{\Big(\frac{\epsilon_+^2}{V_+}+\frac{\epsilon_-^2}{V_-}-\frac{\theta^2}{u}\Big)\frac{x^m x^n}{r^2}\,\tilde{D}^m(k_0)\tilde{D}^{n*}(k_0)$$

$$+\frac{\theta^2}{u}\,\tilde{D}^m(k_0)\tilde{D}^{m*}(k_0)\Big\} + O_{(9)} + O(r^{-3}),$$

$$(4.33)\qquad \langle q_{s,m}\, q_{m,0}\rangle = -2e^{-6\phi_0}\,\frac{x^s}{r}\,\frac{k_0^4}{r^2}\,\mathrm{Re}\Big\{\Big(\frac{\epsilon_+^2}{V_+}+\frac{\epsilon_-^2}{V_-}\Big)\frac{x^m x^n}{r^2}\,\tilde{D}^m(k_0)\tilde{D}^{n*}(k_0)\Big\} + O_{(9)} + O(r^{-3}),$$

$$(4.34)\qquad \langle q_{s,0}\, \phi_{,0}\rangle = -2e^{-3\phi_0}\,\frac{x^s}{r}\,\frac{k_0^4}{r^2}\,\mathrm{Re}\Big\{\Big(\frac{\delta_+\epsilon_+}{V_+}+\frac{\delta_-\epsilon_-}{V_-}\Big)\frac{x^m x^n}{r^2}\,\tilde{D}^m(k_0)\tilde{D}^{n*}(k_0)\Big\} + O_{(9)} + O(r^{-3}).$$

If one contracts eqs. (4.31)-(4.34) with the unit vector x^s/r the angular variables one obtains

$$(4.35)\qquad \int d\Omega\,\frac{x^s}{r}\langle \phi_{,s}\phi_{,0}\rangle = -\frac{8\pi}{3}\,\frac{k_0^4}{r^2}\Big(\frac{\delta_+^2}{V_+^3}+\frac{\delta_-^2}{V_-^3}\Big)\tilde{D}^m(k_0)\tilde{D}^{m*}(k_0) + O_{(9)} + O(r^{-3}),$$

$$(4.36)\qquad \int d\Omega\,\frac{x^s}{r}\langle q_{m,s}q_{m,0}\rangle = -\frac{8\pi}{3}\,e^{-6\phi_0}\,\frac{k_0^4}{r^2}\Big(\frac{\epsilon_+^2}{V_+}+\frac{\epsilon_-^2}{V_-}+2\frac{\theta^2}{u}\Big)\tilde{D}^m(k_0)\tilde{D}^{m*}(k_0) + O_{(9)} + O(r^{-3}),$$

$$(4.37)\qquad \int d\Omega\,\frac{x^s}{r}\langle q_{s,m}q_{m,0}\rangle = -\frac{8\pi}{3}\,e^{-6\phi_0}\,\frac{k_0^4}{r^2}\Big(\frac{\epsilon_+^2}{V_+}+\frac{\epsilon_-^2}{V_-}\Big)\tilde{D}^m(k_0)\tilde{D}^{m*}(k_0) + O_{(9)} + O(r^{-3}),$$

$$(4.38)\qquad \int d\Omega\,\frac{x^s}{r}\langle q_{s,0}\phi_{,0}\rangle = -\frac{8\pi}{3}\,e^{-3\phi_0}\,\frac{k_0^4}{r^2}\Big(\frac{\delta_+\epsilon_+}{V_+}+\frac{\delta_-\epsilon_-}{V_-}\Big)\tilde{D}^m(k_0)\tilde{D}^{m*}(k_0) + O_{(9)} + O(r^{-3}),$$

where $\int d\Omega = \int_0^{2\pi} d\alpha \int_0^{\pi} d\theta\ \sin\theta$ (cf. (4.2)), $\int d\Omega\ x^m x^n/r^2 = (4\pi/3)\ \delta_{mn}$.

Substituting eqs. (4.35)-(4.38) into eq. (4.5), one obtains

$$(4.39)\quad k\int r^2 d\Omega\, \frac{x^s}{r}\langle \Xi_{40}{}^s\rangle = -k_0^4\,\tilde{D}^m(k_0)\tilde{D}^{m*}(k_0)\Big\{\tfrac{2}{3}\Big(\frac{\delta_+^2}{V_+^3}+\frac{\delta_-^2}{V_-^3}\Big) - \tfrac{1}{3}e^{-10\psi_0}(1-e^{4\psi_0})^2$$

$$\times\Big[\frac{1}{1+e^{8\psi_0}}\Big(\frac{\epsilon_+^2}{V_+}+\frac{\epsilon_-^2}{V_-}\Big) + \frac{\theta^2}{u} - e^{3\psi_0}\frac{1+e^{4\psi_0}}{1+e^{8\psi_0}}\Big(\frac{\delta_+\epsilon_+}{V_+}+\frac{\delta_-\epsilon_-}{V_-}\Big)\Big]\Big\} + O_{(9)} + O(r^{-1})$$

By (4.2) and (4.3), the power lost from sources consisting of a single Fourier component is

$$(4.40)\quad \mathcal{E} = \frac{c}{3k}\,k_0^4\,\tilde{D}^m(k_0)\tilde{D}^{m*}(k_0)\Big\{2\Big(\frac{\delta_+^2}{V_+^3}+\frac{\delta_-^2}{V_-^3}\Big) - e^{-10\psi_0}(1-e^{4\psi_0})^2$$

$$\times\Big[\frac{1}{1+e^{8\psi_0}}\Big(\frac{\epsilon_+^2}{V_+}+\frac{\epsilon_-^2}{V_-}\Big) + \frac{\theta^2}{u} - e^{3\psi_0}\frac{1+e^{4\psi_0}}{1+e^{8\psi_0}}\Big(\frac{\delta_+\epsilon_+}{V_+}+\frac{\delta_-\epsilon_-}{V_-}\Big)\Big]\Big\} + O_{(9)}\ .$$

For small values of $\psi_0 < 0$, one has

$$(4.41)\quad \begin{cases} \delta_\pm = \pm k\,2^{-\frac{1}{2}}\,6\,(1-e^{4\psi_0})^{-\frac{1}{2}} + O(1)\,, \quad \epsilon_\pm = \pm k\,2^{-\frac{1}{2}}\,3\,(1-e^{4\psi_0})^{-\frac{1}{2}} + O(1) \\ \theta = 4k\,(1-e^{4\psi_0})^{-1} + O(1) \\ V_\pm = 1 + O((1-e^{4\psi_0}))\,, \quad u = 1 + O((1-e^{4\psi_0}))\,, \end{cases}$$

and hence

$$(4.42) \qquad \mathcal{E} = 12\,ck\,k_0^{4}\,\tilde{D}^m(k_0)\,\tilde{D}^{m*}(k_0)\,(1-e^{4\psi_0})^{-1}\left[1+O((1-e^{4\psi_0}))\right] + O_{(9)}$$

as $\psi_0 \to 0_-$. Thus the power loss has a physically reasonable sign as $\psi_0 \to 0_-$, although it becomes infinitely large. One recalls that the approximations used to obtain the field equations (2.10) and (2.11) assumed that ψ_0 is not small (cf. [12] remarks following (4.21), remarks following (5.2) and (5.10) concerning analogous post-Newtonian equations) and that the approximations are not valid for small ψ_0. In particular, one cannot assume that $q_m = 0_{(3)}$. However, eq. (4.42) gives some hope that energy is lost rather than gained by gravitational radiation for at least some ψ_0. Further work (for example, numerical calculations) would be needed to verify that the energy loss $\mathcal{E}$ remains positive for all admissible values of ψ_0.

Inverting the Fourier transforms (4.23)-(4.26) give

$$(4.43) \qquad \psi_{,0}(x) = \frac{1}{r}\left[\frac{\delta_+}{V_+}\frac{x^m}{r}\ddot{D}^m\left(x^0-\frac{r}{V_+}\right) + \frac{\delta_-}{V_-}\frac{x^m}{r}\ddot{D}^m\left(x^0-\frac{r}{V_-}\right)\right] + O_{(5)} + O(r^{-2}) ,$$

$$(4.44) \qquad \psi_{,s}(x) = -\frac{x^s}{r^2}\left[\frac{\delta_+}{V_+^2}\frac{x^m}{r}\ddot{D}^m\left(x^0-\frac{r}{V_+}\right) + \frac{\delta_-}{V_-^2}\frac{x^m}{r}\ddot{D}^m\left(x^0-\frac{r}{V_-}\right)\right] + O_{(5)} + O(r^{-2}),$$

$$(4.45) \qquad q_{m,0}(x) = -\frac{e^{-3\psi_0}}{r}\left[\epsilon_+\frac{x^m x^n}{r^2}\ddot{D}^n\left(x^0-\frac{r}{V_+}\right) + \epsilon_-\frac{x^m x^n}{r^2}\ddot{D}^n\left(x^0-\frac{r}{V_-}\right) + \theta\frac{x^m x^n}{r^2}\ddot{D}^n\left(x^0-\frac{r}{u}\right)\right.$$

$$\left. - \theta\ddot{D}^m\left(x^0-\frac{r}{u}\right)\right] + O_{(5)} + O(r^{-2}),$$

$$(4.46) \qquad q_{m,s}(x) = e^{-3\psi_0} \frac{x^s}{r^2} \Big[\frac{\epsilon_+}{v_+} \frac{x^m x^n}{r^2} \ddot{D}^n(x^0 - \tfrac{r}{v_+}) + \frac{\epsilon_-}{v_-} \frac{x^m x^n}{r^2} \ddot{D}^n(x^0 - \tfrac{r}{v_-}) + \frac{\theta}{u} \frac{x^m x^n}{r^2} \ddot{D}^n(x^0 - \tfrac{r}{u})$$

$$- \frac{\theta}{u} \ddot{D}^m(x^0 - \tfrac{r}{u}) \Big] + O_{(5)} + O(r^{-2}),$$

where the dot denote the derivative. One may use (4.43)-(4.46), (4.5), (4.2) and (4.3) to obtain an expression for the energy loss $\mathcal{E}$ from sources consisting of a smooth distribution of frequencies. One notes that cross-terms such as $\ddot{D}^m(x^0-r/v_+)\ \ddot{D}^m(x^0-r/v_-)$ survive in general, while for periodic sources the cross-terms correspond to best periods in the expression for $\mathcal{E}$ which average zero over a long time interval.

Integrating the equation

$$(4.47) \qquad T^{p0}{}_{,0} = -T^{pm}{}_{,m} - T^{00}\psi_{,p} + O_{(6)}$$

(cf. [12](4.5)) over all space, one has

$$(4.48) \qquad \ddot{D}^p(x^0) = -\int d^3x\, T^{00}(x)\, \psi_{,p}(x) \quad + O_{(5)}$$

Substituting the solution (3.15) for $\psi-\psi_0$ into eq. (4.48), one gets

$$(4.49) \qquad \ddot{D}^p(x^0) = (\delta_+ + \delta_-) \iint d^3x\, d^3x'\, T^{00}(x^0,\vec{x}) \frac{x^p - x'^p}{|\vec{x}-\vec{x}'|^3} T^{00}(x^0,\vec{x}') \quad + O_{(5)}$$

for slowly moving sources. Under the interchange of $\vec{x}$ and $\vec{x}'$, the integrand in (4.49) is odd, hence the integral vanishes and one has

(4.50) $$\ddot{D}^{p}(x^{0}) = O_{(5)} \; .$$

Thus to fourth order, the $\ddot{D}^{m}$ vanish and

(4.51) $$\mathcal{E} = O_{(9)} \; .$$

No energy is lost by gravitational radiation if terms of ninth order are negligible.

5. Conclusions

It has been shown that the weak-field approximations to the field equations have wavelike solutions that correspond to three speeds of propagation. These solutions satisfy $\psi \to \psi_o$ and $q_m \to 0$ at spatial infinity, provided that ψ_o is not too small, and that $\Delta > 0$ (see remarks following eq. (3.4)). In the slow-motion approximation, the leading term in the gravitational energy loss is due to dipole radiation. However, this term vanishes for post-Newtonian sources.

The results of this thesis are not incompatible with the Einstein quadrupole formula or the measured rate of decrease of the period of the binary pulsar. Higher-order contributions to the energy loss need to be computed to give more conclusive results.

References

[1] C.M. Will, in Experimental Gravitation, ed. B. Bertotti (Academic Press, New York, N.Y., 1974).

[2] C.W. Misner, K.S. Thorne and J.A. Wheeler, Gravitation (W.H. Freeman, San Francisco, Calif.,1973).

[3] D.M. Eardley, D.L. Lee, A.P. Lightman, R.V. Wagoner and C.M. Will, Phys. Rev. Lett, 30, 884 (1973).

[4] J.H. Taylor, L.A. Fowler and P.M. McCulloch, Nature 277, 437 (1979).

[5] P.C. Peters and J. Mathews, Phys. Rev. 131, 435 (1963).

[6] P. Crane, J.E. Nelson and J.A. Tyson, Nature 280, 367 (1979).

[7] J. Ehlers, A. Rosenblum, J.N. Goldberg and P. Havas, Astrophys. J. Lett. 208, L77 (1976).

[8] M. MacCullum, Nature 280, 449 (1979).

[9] J. Ehlers, ed., Isolated Gravitating Systems in General Relativity (North-Holland, Amsterdam, 1979).

[10] C.M. Will, Astrophys. J. 214, 826 (1977).

[11] P. Rastall, Can. J. Phys. 57, 944 (1979).

[12] P. Rastall, "Approximate gravitational field equations," to be published in Can. J. Phys. (1980).

BIBLOGRAPHY

Crane, P., J.E. Nelson and J.A. Tyson. Nature 280, 367 (1979).

Eardley, D.M., D.L. Lee, A.P. Lightman, R.V. Wagoner and C.M. Will. Phys. Rev. Lett. 30, 884 (1973).

Ehlers, J., ed. Isolated Gravitating Systems in General Relativity. North-Holland, Amsterdam, 1979.

Ehlers, J., A. Rosenblum, J.N. Goldberg and P. Havas. Astrophys. J. Lett. 208, L77 (1976).

MacCullum, M. Nature 280, 449 (1979).

Misner, C.W., K.S. Thorne and J.A. Wheeler, Gravitation. W.H. Freeman, San Fransico, Calif., 1973.

Peters, P.C. and J. Mathews. Phys. Rev. 131, 435 (1963).

Rastall, P. Can. J. Phys. 57, 944 (1979).

Rastall, P. "Approximate gravitational field equations." To be published.

Taylor, J.H., L.A. Fowler and P.M. McCulloch. Nature 277, 437 (1979).

Will, C.M. In Experimental Gravitation. Ed. B. Bertotti. Academic Press, New York, N.Y., 1974.

Will, C.M. Astrophys. J. 214, 826 (1977).

Appendix: Expressions depending on ψ_o

In sections 3 and 5, several expressions appear which depend on the boundary value ψ_o. Here the values of ψ_o for which the expressions are valid are determined. To simplify notation, define

(A.1) $$\zeta = e^{4\psi_o}$$

so that $\zeta > 0$ for any real ψ_o .

The discriminant Δ appearing in eq. (3.4) is given by

(A.2) $$\Delta = \tfrac{1}{4}(\zeta-1)^3\Theta(\zeta)$$

where $\Theta(\zeta) = \zeta^5 + 3\zeta^4 - 10\zeta^3 - 14\zeta^2 - 3\zeta - 9$.

Several cases are considered:

(i) If $0 < \zeta < 1$, then $(\zeta - 1)^3 < 0$, hence $\Delta > 0$ (to see that $\theta(\zeta) < 0$, one notes that $\zeta^5 < \zeta^3$ and $3\zeta^4 < 3\zeta^3$ together imply that

$\Theta(\zeta) < -6\zeta^3 - 14\zeta^2 - 3\zeta - 9 < 0$).

(ii) If $\zeta = 1$, then $\Delta = 0$

(iii) For $\zeta > 1$, one notes that $(\zeta - 1)^3 > 0$. $\Theta(1) = -32$ and $\lim_{\zeta \to \infty} \Theta(\zeta) = \infty$, the continuous function $\Theta(\zeta)$ must have at least one positive zero. Resorting to numerical computation, one finds the smallest positive zero $\zeta_* \simeq 2.79$. If $1 < \zeta < \zeta_*$, then $\Delta < 0$.

(iv) If $\zeta = \zeta_*$, then $\Delta = 0$.

(v) Numerical computation gives $\Theta(\zeta) > 0$ for $\zeta_* < \zeta < 3$. For $\zeta \geq 3$, one checks that $-10\zeta^3 - 14\zeta^2 - 3\zeta - 9 > -16\zeta^3$, thus $\Theta(\zeta) > \zeta^5 + 3\zeta^4 - 16\zeta^3 = \zeta^3(\zeta^2 + 3\zeta - 16)$.
One shows that $\zeta^2 + 3\zeta - 16 > 0$ for $\zeta \geq 3$, hence $\Theta(\zeta) > 0$.
If $\zeta > \zeta_*$, then $\Delta > 0$.

From cases (i) - (v) and eq. (A.1), one concludes that $\Delta > 0$ if, and only if, $\psi_0 \in D = \{\psi_0 : \psi_0 < 0 \text{ or } \psi_0 > \psi_{0*} = \frac{1}{4} \ln \zeta_* \simeq 0.25\}$.

The eigenvalues $\lambda_\pm$ of the matrix A are given by

$$\lambda_\pm = \tfrac{1}{2}(-\Lambda \pm \Delta^{1/2}), \tag{A.3}$$

where $\Lambda(\zeta) = \frac{1}{2}(\zeta^4 + 8\zeta^2 - 4\zeta + 3)$ (cf. (3.4)). One finds that

$$\Lambda^2 - \Delta = 8\zeta^2(\zeta^2+1)(\zeta^2 - \zeta + 1) > 0 \tag{A.4}$$

for all $\zeta > 0$, so in particular $\Lambda > \Delta^{\frac{1}{2}}$ and hence $\lambda_+ < 0$ for $\psi_0 \in D$. Λ has an absolute minimum at $\zeta \simeq 0.84$, for which $\Lambda \simeq 4.17$. Thus $\Lambda > 0$ and $\lambda_- < 0$ for $\psi_0 \in D$. One considers the boundaries of D:

(i) As $\psi_0 \to -\infty$ $(\zeta \to 0_+)$, one has, by (A.4), $\Lambda^2 - \Delta \to 0$, hence $\lambda_+ \to 0$ and $\lambda_- \to -\Lambda(0) = -\frac{3}{2}$.

(ii) As $\psi_0 \to 0_-$ $(\zeta \to 1_-)$, one has $\Delta \to 0$, hence $\lambda_\pm \to -\frac{1}{2}\Lambda(1) = -2$.

(iii) As $\psi_0 \to \psi_{0*+}$ $(\zeta \to \zeta_{*+})$, one has $\Delta \to 0$, hence $\lambda_\pm \to -\frac{1}{2}\Lambda(\zeta_*)$.

(iv) As $\psi_0 \to \infty$ $(\zeta \to \infty)$, one has $\Lambda = \frac{1}{2}\zeta^4[1 + 8\zeta^{-2} + O(\zeta^{-3})]$, $\Delta^{1/2} = \frac{1}{2}\zeta^4[1 - 8\zeta^{-2} + O(\zeta^{-3})]$, hence $\lambda_+ = -4\zeta^2[1 + O(\zeta^{-1})]$, $\lambda_- = -\frac{1}{2}\zeta^4[1 + O(\zeta^{-2})]$.

The propagation speeds $v_\pm$ are given by (cf. (3.5))

$$(A.5) \qquad v_\pm = (\zeta^2 + 1)^{1/2}(-\lambda_\pm)^{-1/2} .$$

Since $\zeta^2 + 1 > 0$ for all ζ, and $\lambda_\pm < 0$ for $\psi_0 \in D$, one can take the $v_\pm$ real and positive for $\psi_0 \in D$. The limiting values of $v_\pm$ corresponding to the boundaries of D are:

(i) As $\psi_0 \to -\infty$ $(\zeta \to 0_+)$, one has $\zeta^2 + 1 \to 1$, hence $v_\pm \to \lim_{\zeta \to 0_+} (-\lambda_\pm)^{-1/2}$, i.e., $v_+ \to \infty$, $v_- \to (\frac{3}{2})^{-1/2} \simeq 0.82$.

(ii) As $\psi_0 \to 0_-$ $(\zeta \to 1_-)$, one has $\zeta^2 + 1 \to 2$, hence $v_\pm \to 1$.

(iii) As $\psi_0 \to \psi_{0*+}$ $(\zeta \to \zeta_{*+})$, $v_\pm \to [2(\zeta_*^2 + 1)/\Lambda(\zeta_*)]^{1/2} \simeq 0.56$.

(iv) As $\psi_0 \to \infty$ $(\zeta \to \infty)$, $v_+ = \frac{1}{2} + O(\zeta^{-1})$, $v_- = 2^{1/2}\zeta^{-1} + O(\zeta^{-3})$.

The constants $\alpha_{\pm}$ and β in the P in eq. (3.6), and the coefficients $\gamma_{\pm}$ and $\delta_{\pm}$ appearing in eqs. (3.11) and (5.3) are all well-defined for $\psi_0 \in D$. Note that since $0 \notin D$, one has $\beta \neq 0$ and $\det P \neq 0$ for $\psi_0 \in D$.