# GEOMETRICAL ASPECTS OF LOCALIZATION THEORY 

By

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#### Abstract

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#### Abstract

Some recent developments in topological quantum field theory have focused on localization techniques using equivariant cohomology to reduce functional integrals to finitedimensional expressions from which physical and mathematical characteristics are readily deduced. In this thesis we examine the applicability of these localization techniques by analysing in detail the geometric constraints that these methods assume. After an extensive review of the relevant background material, we focus on the applications of equivariant localization techniques to phase space path integrals and classify the 2 -dimensional Hamiltonian systems with simply-connected phase spaces to which these formalisms can be applied using their fundamental geometric constraints. We show that for maximally symmetric phase spaces the localizable Hamiltonian systems all appear in harmonic oscillator forms, while for non-homogeneous spaces the possibilities are more numerous. In the latter cases the Riemannian structures become rather complicated. We show that these systems all share the common property that their quantum dynamics can be described using coherent states, usually associated with coadjoint Lie group orbits, and we evaluate the associated character formulas.

We then show how these results generalize to the case where the phase space is a multiply-connected compact Riemann surface. After discussing how the previous formalisms should be appropriately modified in this case, we show that the partition function for the localizable Hamiltonian systems describes a rich topological field theory which represents the first homology of the phase space. The coherent states in this case are also constructed and it is shown that the Hilbert space is finite-dimensional. The wavefunctions carry a projective representation of the phase space homology group and describe modular invariants of the quantum theory.


Finally, we discuss some geometric methods for analysing corrections to the semiclassical approximation for dynamical systems whose path integrals do not localize. We show that the usual isometric symmetry needed for localization can be replaced by a weaker conformal symmetry requirement. We then introduce an alternative method to the loop expansion for obtaining corrections to the semi-classical approximation which expresses the correction terms as Poincaré dual forms of homology cycles of the phase space.

## Table of Contents

Abstract ..... ii
Table of Contents ..... iv
Acknowledgements ..... vii
1 Introduction ..... 1
2 Equivariant Cohomology and the Localization Principle ..... 14
2.1 DeRham Cohomology ..... 15
2.2 The Cartan Model of Equivariant Cohomology ..... 23
2.3 Equivariant Characteristic Classes ..... 29
2.4 The Equivariant Localization Principle ..... 36
2.5 The Berline-Vergne Theorem ..... 42
3 Finite-dimensional Localization Theory ..... 47
3.1 Symplectic Geometry ..... 48
3.2 Equivariant Cohomology on Symplectic Manifolds ..... 52
3.3 Stationary-phase Approximation and the Duistermaat-Heckman Theorem ..... 57
3.4 Morse Theory and Kirwan's Theorem ..... 63
3.5 The Height Function of a Riemann Surface ..... 66
3.6 Equivariant Localization and Classical Integrability ..... 71
3.7 Degenerate Version of the Duistermaat-Heckman Theorem ..... 76
3.8 The Witten Localization Formula ..... 80
3.9 The Wu Localization Formula ..... 85
4 Quantum Localization Theory ..... 86
4.1 Phase Space Path Integrals ..... 88
4.2 Loop Space Symplectic Geometry and Equivariant Cohomology ..... 95
4.3 Supersymmetry and the Loop Space Localization Principle ..... 99
4.4 The WKB Localization Formula ..... 106
4.5 Degenerate Path Integrals and the Niemi-Tirkkonen Localization Formula ..... 109
4.6 Connections with the Duistermaat-Heckman Integration Formula ..... 117
4.7 Equivariant Localization and Quantum Integrability ..... 120
4.8 Localization for Functionals of Isometry Generators ..... 123
4.9 Topological Quantum Field Theories ..... 128
5 Equivariant Localization on Simply Connected Phase Spaces ..... 135
5.1 Coadjoint Orbit Quantization and Character Formulas ..... 139
5.2 Isometry Groups of Simply Connected Riemannian Spaces ..... 146
5.3 Euclidean Phase Spaces and Holomorphic Quantization ..... 157
5.4 Coherent States on Homogeneous Kähler Manifolds ..... 166
5.5 Spherical Phase Spaces and Quantization of Spin Systems ..... 169
5.6 Hyperbolic Phase Spaces ..... 181
5.7 Quantization on Non-homogeneous Phase Spaces ..... 185
6 Equivariant Localization on Multiply Connected Phase Spaces ..... 198
6.1 Isometry Groups of Multiply Connected Spaces ..... 201
6.2 Equivariant Hamiltonian Systems in Genus One ..... 203
6.3 Homology Representations and Topological Quantum Field Theory ..... 208
6.4 Holomorphic Quantization and Non-symmetric Coadjoint Orbits ..... 214
6.5 Generalization to Hyperbolic Riemann Surfaces ..... 226
7 Geometrical Characteristics of the Semi-classical Expansion ..... 233
7.1 The Loop Expansion and the Duistermaat-Heckman Formula Revisited ..... 235
7.2 Geometry of the Loop Expansion ..... 238
7.3 Conformal Symmetry and Kähler Structures ..... 249
7.4 The Extended Localization Principle ..... 255
7.5 Poincaré Duality and Corrections to the Duistermaat-Heckman Formula . ..... 259
7.6 Examples ..... 269
7.7 Generalizations to Path Integrals ..... 278
Bibliography ..... 282

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## Chapter 1

## Introduction

In this thesis we will discuss some new geometric methods of solving exactly for the spectrum of a quantum system using the functional integral and the relation between these ideas and the properties of integrable and topological quantum field theories. Among other things, these techniques tie in with some other modern ideas in quantum field theory as well as some interesting mathematics, all of which will be extensively discussed in the following.

The idea of path integration was introduced by Feynman [40] in the 1940's as a novel new approach to quantum theory. Symbolically, the fundamental path integral formula is

$$
\begin{equation*}
\mathcal{K}\left(q^{\prime}, q ; T\right)=\sum_{C_{q q^{\prime}}} \mathrm{e}^{i T L\left[C_{q q^{\prime}}\right]} \tag{1.1}
\end{equation*}
$$

where the 'sum' is over all paths $C_{q q}$ between the points $q$ and $q$ ' on the configuration space of a physical system, and $L\left[C_{q q^{\prime}}\right]$ is the length of the path. The quantity on the lefthand side represents the probability amplitude for the system to evolve from a state with configuration $q$ to one with configuration $q^{\prime}$ in a time span $T$. One of the great advantages of the path integral formulation is that it gives a global (integral) solution of the quantum problem in question, in contrast to the standard approach to quantum mechanics based on the Schrödinger equation which gives a local (differential) formulation of the problem. Of utmost significance at the time was Feynman's generalization of the path integral to quantum electrodynamics from which a systematic derivation of the famous Feynman rules, and hence the basis of most perturbative calculations in quantum field theory, can be carried out [63].

The problem of quantum integrability, i.e. the possibility of solving analytically for the spectrum of a quantum Hamiltonian and the corresponding eigenfunctions, is a nontrivial problem. This is even apparent from the point of view of the path integral, which describes the time evolution of wavefunctions. Very few quantum systems have been solved exactly and even fewer have had an exactly solvable path integral. At the time that the functional integration (1.1) was introduced, the only known examples where it could be evaluated exactly were the harmonic oscillator and the free particle. The path integrals for these 2 examples can be evaluated using the formal functional analog of the classical Gaussian integration formula [139]

$$
\begin{equation*}
\int_{-\infty}^{\infty} \prod_{k=1}^{n} d x^{k} \mathrm{e}^{\frac{i}{2} \sum_{i, j} x^{i} M_{i j} x^{j}+i \sum_{i} \lambda_{i} x^{i}}=\frac{\left(2 \pi \mathrm{e}^{i \pi / 2}\right)^{\frac{n}{2}} \mathrm{e}^{\frac{i}{2} \sum_{i, j} \lambda_{i}\left(M^{-1}\right)^{i j} \lambda_{j}}}{\sqrt{\operatorname{det} M}} \tag{1.2}
\end{equation*}
$$

where $M=\left[M_{i j}\right]$ is a non-singular symmetric $n \times n$ matrix. In this way, the Feynman propagator (1.1) can be evaluated formally for any field theory which is at most quadratic in the field variables. If this is not the case, then one can expand the argument of the exponential in (1.1), approximate it by a quadratic form as in (1.2), and then take the formula (1.2) as an approximation for the integral. For a finite-dimensional integral this is the well-known stationary phase (otherwise known as the saddle-point or steepestdescent) approximation [54]. In the framework of path integration, it is usually referred to as the Wentzel-Kramers-Brillouin (or WKB for short) approximation [83, 116]. Since the result (1.2) is determined by substituting into the exponential integrand the global minimum (i.e. classical value) of the quadratic form and multiplying it by a term involving the second variation of that form (i.e. the fluctuation determinant), this approach to functional integration is also called the semi-classical approximation. In this sense, (1.1) interprets quantum mechanics as a sum over paths fluctuating about the classical trajectories of a dynamical system. When the semi-classical approximation is exact, one can think of the Gaussian integration formula (1.2) as a 'localization' of the complicated looking integral on the left-hand side of (1.2) onto the global minimum of the quandratic form there.

For a long time, these were the only examples of exactly solvable systems. In 1968 Schulman [115] found that a path integral describing the precession of a classical spin vector was given exactly by its WKB approximation. This was subsequently generalized by Dowker [32] who proved the exactness of the semi-classical approximation for the path integral describing free geodesic motion on compact group manifolds. It was not until the late• 1970's that more general methods, beyond the restrictive range of the standard WKB method, were developed. In these methods, the Feynman path integral is calculated rigorously in discretized form (i.e. over piecewise-linear paths) by a careful regularization prescription [76], and then exploiting information provided by functional analysis, the theory of special functions, and the theory of differential equations (see [26] and references therein). With these tricks the list of exactly solvable problems has significantly increased over the last 15 years, so that today one is able to essentially evaluate analytically the path integral for any quantum mechanical problem for which the Schrödinger equation can be solved exactly. We refer to [75] and [52] for an overview of these methods and a complete classification of the known examples of exactly solved quantum mechanical path integrals to present date.

The situation is somewhat better in quantum field theory, which represents the real functional integrals of interest from a physical standpoint. There are many non-trivial examples of classically integrable models (i.e. ones whose classical equations of motion are 'exactly solvable'), for example the sine-Gordon model, where the semi-classical approximation describes the exact spectrum of the quantum field theory [139]. Indeed, for any classically integrable dynamical system one can canonically transform the phase space variables so that, using Hamilton-Jacobi theory [48], the path integral can be formally manipulated to yield a result which if taken naively would imply the exactness of the WKB approximation for any classically integrable system [116]. This is not really the case, because the canonical transformations used in the phase space path integral do not respect the ordering prescription used for the properly discretized path integral and
consequently the integration measure is not invariant under these transformations [25]. However, as these problems stem mainly from ordering ambiguities in the discretization of the path integral, in quantum field theory these ordering ambiguities could disappear by a suitable renormalization, for instance by an operator commutator ordering prescription. This has lead to the conjecture that properly interpreted results of semi-classical approximations in integrable field theories reproduce features of the exact quantum spectrum [139]. One of the present motivations for us is to therefore develop a systematic way to implement realizations of this conjecture.

Another class of field theories where the path integral is exactly solvable in most cases is supersymmetric theories and topological quantum field theories (see [17] for a concise review). Topological field theories have lately been of much interest in both the mathematics and physics literature. A field theory is topological if it has only global degrees of freedom. This means, for example, that its classical equations of motion eliminate all field theoretic degrees of freedom from the problem (so that the classical action vanishes). In particular, the theory cannot depend on any metric of the space on which the fields are defined. The observables of these quantum field theories therefore describe geometrical and topological invariants of the space which are computable by conventional techniques of quantum field theory and are of prime interest in mathematics. Physically, topological quantum field theories bear resemblances to many systems of longstanding physical interest and it is hoped that this special class of field theories might serve to provide insight into the structure of more complicated physical systems and a testing ground for new approaches to quantum field theory. There is also a conjecture that topological quantum field theories represent different (topological) phases of their more conventional counterparts (e.g. 4-dimensional Yang-Mills theory). Furthermore, from a mathematical point of view, these field theories provide novel representations of some global invariants whose properties are frequently transparent in the path integral approach.

Topological field theory essentially traces back to the work of Schwarz [117] in 1978 who showed that a particular topological invariant, the Ray-Singer analytic torsion, could be represented as the partition function of a certain quantum field theory. The most important historical work for us, however, is the observation made by Witten [131] in 1982 that the supersymmetry algebra of supersymmetric quantum mechanics describes exactly the DeRham complex of a manifold, where the supersymmetry charge is the exterior derivative. This gave a framework for understanding Morse theory in terms of supersymmetric quantum mechanics in which the quantum partition function computed exactly the Euler characteristic of the underlying manifold, i.e. the index of the DeRham complex.

Witten's partition function computed the so-called Witten index [132], the difference between the number of bosonic and fermionic zero energy states. In order for the supersymmetry to be broken in the ground state of the supersymmetric model, the Witten index must vanish. As supersymmetry, i.e. a boson-fermion symmetry, is not observed in nature, it is necessary to have some criterion for dynamical supersymmetry breaking if supersymmetric theories are to have any physical meaning. Witten's construction was subsequently generalized by Alvarez-Gaumé [4], and Friedan and Windey [42], to give supersymmetric field theory proofs of the Atiyah-Singer index theorem [35]. In this way, the partition function is reduced to an integral over the underlying manifold $\mathcal{M}$. This occurs because the supersymmetry of the action causes only zero modes of the fields, i.e. points on $\mathcal{M}$, to contribute to the path integral, and the integrals over the remaining fluctuation modes are Gaussian. The resulting integral encodes topological information about the manifold $\mathcal{M}$ and represents a huge reduction of the original infinite-dimensional path integration.

This field began to draw more attention around 1988 when Witten introduced topological field theories in a more general setting [134]. A particular supersymmetric nonabelian gauge theory was shown by Witten to describe a theory with only global degrees
of freedom whose observables are the Donaldson invariants, certain differential invariants which classify differentiable structures on 4 -manifolds. Subsequent work then put these ideas into a general framework so that today the formal field theoretic structures of Witten's actions are well-understood [17]. Furthermore, because of their topological nature, these field theories have become the focal point for the description of topological effects in quantum systems using quantum field theory, for instance for the description of holonomy effects in physical systems arising from the adiabatic transports of particles [119] and extended objects such as strings [13] (i.e. Aharonov-Bohm thype effects). In this way the functional integral has in recent years become a very popular tool lying on the interface between string theory, conformal field theory and topological quantum field theory in physics, and between topology and algebraic geometry in mathematics. Because of the consistent reliability of results that path integrals of these theories can produce when handled with care, functional integration has even acquired a certain degree of respectability among mathematicians.

The common feature of topological field theories is that their path integrals are described exactly by the semi-classical approximation. It would be nice to put semiclassically exact features of functional integrals, as well as the features which reduce them to integrals over finite-dimensional manifolds as described above, into some sort of general framework. More generally, we would like to have certain criteria available for when we expect partition functions of quantum theories to reduce to such simple expressions, or 'localize'. This motivates an approach to quantum integrability in which one can systematically study the properties of integrable field theories and their conjectured semi-classical "exactness" that we mentioned before. In this approach we focus on the general features and properties that path integrals appearing in this context have in common. Foremost among these is the existence of a large number of (super-)symmetries
in the underlying dynamical theory, so that these functional integrals reduce to Gaussian ones and essentially represent finite-dimensional integrals ${ }^{1}$. The transition between the functional and finite-dimensional integrals can then be regarded as a rather drastic localization of the original infinite-dimensional integral, thereby putting it into a form that is useable for extracting physical and mathematical information. The mathematical framework for describing these symmetries, which turn out to be of a cohomological nature, is equivariant cohomology and the approach discussed in this paragraph is usually called 'equivariant localization theory'.

Historically, this subject originated in the mathematics literature in 1982 with the Duistermaat-Heckman theorem [33], which established the exactness of the semi-classical approximation for finite-dimensional oscillatory integrals (i.e. finite-dimensional versions of (1.1)) over compact symplectic manifolds in certain instances. The symmetry responsible for the localization here is the existence of a global Hamiltonian torus action on the manifold. Atiyah and Bott [8] showed that the Duistermaat-Heckman localization was a special case of a more general localization property of equivariant cohomology (with respect to the torus group action in the case of the Duistermaat-Heckman theorem). This fact was used by Berline and Vergne [14, 15] at around the same time to derive a quite general integration formula valid for Killing vectors on general compact Riemannian manifolds.

The first infinite-dimensional generalization of the Duistermaat-Heckman theorem is due to Atiyah and Witten [7], in the setting of a supersymmetric path integral for the index (i.e. the dimension of the space of zero modes) of a Dirac operator. They showed that a formal application of the Duistermaat-Heckman theorem on the loop space $L \mathcal{M}$ of a manifold $\mathcal{M}$ to the partition function of $N=\frac{1}{2}$ supersymmetric quantum mechanics (i.e. a supersymmetric spinning particle in a gravitational background) reproduced the

[^0]well-known Atiyah-Singer index theorem correctly. The crucial idea was the interpretation of the fermion bilinear in the supersymmetric action as a loop space symplectic 2 -form. This approach was then generalized by Bismut [18, 19], within a mathematically rigorous framework, to twisted Dirac operators (i.e. the path integral for spinning particles in gauge field backgrounds), and to the computation of the Lefschetz number of a Killing vector field $V$ (a measure of the number of zeroes of $V$ ) acting on the manifold. Another nice infinite-dimensional generalization of the Duistermaat-Heckman theorem was suggested by Picken [110] who formally applied the theorem to the space of loops over a group manifold to localize the path integral for geodesic motion on the group, thus establishing the well-known semi-classical properties of these systems.

It was the beautiful paper by Stone [123] in 1989 that brought the DuistermaatHeckman theorem to the attention of a wider physics audience. Stone presented a supersymmetric derivation of the Weyl character formula for $S U(2)$ using the path integral for spin and interpreted the result as a Duistermaat-Heckman localization. This supersymmetric derivation was extended by Alvarez, Singer and Windey [3] to more general Lie groups using fiber bundle theory, and the supersymmetries in both of these approaches are very closely related to equivariant cohomology. At around this time, other important papers concerning the quantization of spin appeared. Most notably, Nielsen and Rohrlich [94] viewed the path integral for spin from a more geometrical point of view, using as action functional the solid angle swept out by the closed orbit of the spin. This approach was related more closely to geometric quantization and group representation theory by Alekseev, Faddeev and Shatashvili [1, 2], who calculated the coadjoint orbit path integral for unitary and orthogonal groups, and also for cotangent bundles of compact groups, Kac-Moody groups and the Virasoro group. The common feature is always that the path integrals are given exactly by a semi-classical localization formula that resembles the Duistermaat-Heckman formula.

The connections between supersymmetry and equivariant cohomology in the quantum
mechanics of spin were clarified by Blau in [21], who related the Weinstein action invariant [130] to Chern-Simons gauge theory using the Duistermaat-Heckman integration formula. Based on this interpretation, Blau, Keski-Vakkuri and Niemi [25] introduced a general supersymmetric (or equivariant cohomological) framework for investigating DuistermaatHeckman (or WKB) localization formulas for generic (non-supersymmetric) phase space path integrals, leading to the fair amount of activity in this field which is today the foundation of equivariant localization theory. They showed formally that the partition function for the quantum mechanics of circle actions of isometries on symplectic manifolds localizes. Their method of proof involves formal techniques of Becchi-Rouet-Stora-Tyupin (or BRST for short) quantization of constrained systems [11]. Roughly speaking, BRST quantization of gauge theories is carried out by introducing a nilpotent BRST operator $Q, Q^{2}=0$, associated with a local gauge symmetry of a theory and representing the gauge variation of any functional $\mathcal{O}$ of the fields as a graded commutator $\{Q, \mathcal{O}\}$ with the fermionic charge $Q$. The physical (i.e. gauge-invariant) Hilbert space consists of those states $\Psi$ which are annihilated by $Q, Q \Psi=0$, and any state of the form $\Psi^{\prime}=\Psi+Q \chi$ is regarded as equivalent to $\Psi$, for any other state $\chi$. Thus the physical Hilbert space here is the space of BRST-cohomology classes. In the case of the localization formalism for phase space path integrals, the operator $Q$ is identified as the loop space equivariant exterior derivative of the underlying equivariant cohomological structure, and the "Hilbert space" of physical states consists of loop space functionals which are invariant under the flows of some vector field. BRST-cohomology is also the fundamental structure in topological field theories. In particular, a topological action is a Witten-type action if the classical action is BRST-exact, while it is a Schwarz-type action if the gauge-fixed, quantum action is BRST-exact (but not necessarily classically) [17]. This BRST supersymmetry is always the symmetry that is responsible for localization in these models.

Although we shall mention at appropriate instances the extensive applications of the
equivariant localization formalism to supersymmetric quantum mechanics, cohomological field theories and 2-dimensional Yang-Mills theory, in this thesis we shall explore the geometric features of the localization formalism for phase space path integrals. In particular, we shall focus on how these models can be used to extract information about integrable and topological quantum field theories. In this sense, the path integrals we study can be thought of as "toy models" serving as a testing ground for ideas in some more sophisticated field theories. The reasoning behind this is as follows. First of all, because path integral manipulations are frequently known to produce correct (meaning, for instance, topologically or physically correct) results, any definition of, or approach to, the functional integral should be able to reproduce the results which follow and to incorporate these techniques in some way. Secondly, the kinds of theories we shall study here allow one to study kinematical (i.e. geometrical and topological) aspects of the path integral in isolation from their dynamical properties. This stands in contrast to the situation in interacting field theories, which are typically kinematically linear but dynamically highly non-linear. Our theories will be dynamically linear (i.e. free field theories) but kinematically highly non-linear, and the entire non-triviality of the theories resides in this kinematical non-linearity. Finally, in principle at least, the techniques we shall discuss are also applicable to theories with field theoretical degrees of freedom, in the sense that they provide alternative approximation techniques to the usual perturbative expansion. This will be true, in particular, of the generalized WKB approximation techniques based on the Duistermaat-Heckman theorem and the more geometrical localization aspects of our formalism.

There is another important localization technique that has been of interest lately and is quite different from the equivariant localization formalism, although we are not really interested in it from the point of view of what we are trying to accomplish here. This is the Mathai-Quillen formalism [22, 24, 29, 82, 98], a technique that is used to build partition functions for topological field theories that by construction have a localizable
form. This method is based on an extension of the idea of the Witten complex [131] and it constructs some natural representatives for the equivariant cohomology known as Thom classes. Using these equivariant cohomology classes, one then constructs regularized Euler classes of infinite-dimensional vector bundles (e.g. the loop space $L \mathcal{M} \rightarrow \mathcal{M}$ over a manifold $\mathcal{M}$ or the bundle $\mathcal{A} \rightarrow \mathcal{A} / G$ of gauge connections on a principal $G$-bundle) and uses them to build topological partition functions. The localization can then be thought of as the sort of localization which is represented by the classical Gauss-Bonnet-Chern and Poincaré-Hopf theorems, i.e. the representation of the Euler number of a manifold $\mathcal{M}$ as both an integral over $\mathcal{M}$ of a density constructed from the curvature of some connection on $\mathcal{M}$, and in terms of some alternating sum over the isolated zero point set of a vector field $V$ on $\mathcal{M}$. The relations between the BRST picture of equivariant cohomology and Thom classes [67] leads to new insights and techniques for certain topological field theories [103], such as 4-dimensional topological Yang-Mills theory, the prototype of a (Wittentype) cohomological field theory [17]. Although the BRST-symmetric structure which leads to so-called BRST fixed points [24] ties in with some of the framework of equivariant localization, this is more of a 'constructive' localization technique and therefore has no use for our analysis where we shall focus on properties of fairly generic phase space path integrals.

We shall approach the localization formalism for path integrals in the following manner. Focusing on the notion of localizing a quantum partition function by reducing it using the large symmetry of the dynamical system to a sum or finite-dimensional integral in analogy with the classical Gaussian integration formula (1.2), we shall first analyse the symmetries resposible for the localization of finite-dimensional integrals (where the symmetry of the dynamics is represented by an equivariant cohomology). The main focus of this thesis will then be the formal generalizations of these ideas to phase space path integrals, where the symmetry becomes a "hidden" supersymmetry of the dynamics representing the infinite-dimensional analog of equivariant cohomology. A subsequent
generalization, the real hope of localization theory, would be then to extend these notions to both Poincaré supersymmetric quantum field theories (where the symmetry is represented by the supersymmetry of that model) and topological quantum field theories (where the symmetry is represented by a gauge symmetry). The hope is that then these serve as testing grounds for the more sophisticated quantum field theories of real physical interest. This gives a geometric framework for studying quantum integrability, as well as insights into the structure of topological and supersymmetric field theories, and integrable models. In particular, from this analysis we can hope to uncover the reasons why some quantum problems are exactly solvable, and the reasons why others aren't.

Briefly, the structure of this thesis is as follows. In chapters 2 and 3 we go through some of the mathematical background, in particular equivariant cohomology and the Duistermaat-Heckman theorem, which will be the basis for the later chapters. Chapter 4 then goes through the formal supersymmetry and loop space equivariant cohomology arguments establishing the localization of phase space path integrals when there is a Riemannian structure on the phase space which is invariant under the classical dynamics of the system. Depending on the choice of localization scheme, different sets of phase space trajectories are lifted to a preferred status in the integral. Then all contributions to the functional integral come from these preferred paths along with a term taking into account the quantum fluctuations about these selected loops. Chapters 5 and 6 then use the isometry condition to construct examples of localizable path integrals. Here we encounter numerous examples and gain much insight into the range of applicability of the localization formalism in general. We also see here many interesting features of the localizable partition functions when interpreted as topological field theories, and we discuss in detail various other issues (e.g. coherent state quantization and coadjoint orbit character formulas) which are common to all the localizable examples that we find within this setting. Chapter 7 then takes a slightly different approach to analysing localizable systems, this time by some geometric constructs of the full loop-expansion on the phase
space. In particular, we shall see here that the equivariant cohomological symmetry in the conventional localization formalism can be enlarged, and we introduce some geometric methods that deduce geometric properties of the dynamical systems which localize. We also show how to use the isometric condition locally to formulate a geometric approach to obtaining corrections to the standard WKB approximation for non-localizable partition functions. The analysis of this final chapter is the first step towards a systematic, geometric understanding of the reasons why the localization formulas may not apply to a given dynamical system.

Finally, we close this introductory chapter with some comments about the style of this thesis. Although we have attempted to keep things self-contained and at places where topics aren't developed in full detail we have included ample references for further reading, we do assume that the reader has a relatively solid background in many of the mathematical techniques of modern theoretical physics such as topology, differential geometry and group theory. It would be impossible to go through these techniques in any sort of detail here, and so we indicate places where most of the mathematical material can be found. All of the group theory that is used extensively in this thesis can be found in [128] (or see [46] for a more elementary introduction), while most of the material discussing differential geometry, homology and cohomology, and index theorems can be found in the books [51, 27, 92] and the review articles [17, 35]. For a more detailed introduction to algebraic topology, see [81]. The basic reference for quantum field theory is the classic text [63]. Finally, for a discussion of the issues in supersymmetry theory and BRST quantization, see $[11,17,58,95,122]$ and references therein.

## Chapter 2

## Equivariant Cohomology and the Localization Principle

In this preliminary chapter, we shall begin by introducing some of the mathematical notions that will form the basis for this thesis. The central theme will be the description of the topology of a space when there is an action of some Lie group acting on it. This theory is called equivariant cohomology and it is the "right" cohomology theory which takes into proper account of the group action. It is defined in a manner such that if the group action is trivial, the cohomology reduces to the usual cohomological ideas of the classical DeRham theory. We shall develop these ideas starting with a quick review of the DeRham theory and ultimately end up discussing the important localization property of integration in equivariant cohomology. We will see later on that the localization theorems are then fairly immediate consequences of this general formalism. Throughout we shall be working with an abstract topological space (i.e. a set with a collection of open subsets which is closed under unions and finite intersections), and we always regard 2 topological spaces as the same if there is an invertible mapping between the 2 spaces which preserves their open sets, i.e. a bi-continuous function or 'homeomorphism'. To carry out calculus on these spaces, we shall have to introduce some smooth structure on them (i.e. one that is infinitely-continuously differentiable - or $C^{\infty}$ for short). We shall then generalize these constructions to the case where there is a Lie group (a continuous group with a smooth structure whose group multiplication is also smooth) acting on the space. The construction of topological invariants for these spaces (i.e. structures that are the same for homeomorphic spaces) will then be the foundation for the derivation of general integration formulas in the subsequent chapters.

### 2.1 DeRham Cohomology

To introduce some notation and to provide a basis for some of the more abstract concepts that will be used throughout this thesis, we begin with an elementary 'lightning' review of DeRham cohomology theory and how it probes the topological features of a space. Let $\mathcal{M}$ be a $C^{\infty}$ manifold of dimension $n$, i.e. $\mathcal{M}$ is a paracompact Hausdorff topological space which can be covered by open sets $U_{i}, \mathcal{M}=\bigcup_{i \in I} U_{i}$, each of which is homeomorphic to $n$-dimensional Euclidean space $\mathbb{R}^{n}$ and the local homeomorphisms so used induce $C^{\infty}$ coordinate transformations on the overlaps of patches in $\mathbb{R}^{n}$. This means that locally, in a neighbourhood of any point on $\mathcal{M}$, we can treat the manifold as a copy of the more familiar $\mathbb{R}^{n}$, but globally the space $\mathcal{M}$ may be very different from Euclidean space. One way to characterize the global properties of $\mathcal{M}$, i.e. its topology, which make it quite different from $\mathbb{R}^{n}$ is through the theory of homology and its dual theory, cohomology. Of particular importance to us will be the DeRham theory [27]. We shall always assume that $\mathcal{M}$ is orientable and path-connected (i.e. any 2 points in $\mathcal{M}$ can be joined by a continuous path in $\mathcal{M}$ ). We shall usually assume, unless otherwise stated, that $\mathcal{M}$ is compact. In the non-compact case, we shall assume certain regularity conditions at infinity so that results for the compact case hold there as well.

Around each point of the manifold we choose an open set $U$ which is a copy of $\mathbb{R}^{n}$. In $\mathbb{R}^{n}$ we have the natural notion of tangent vectors to a point, and so we can use the locally defined homeomorphisms to define tangent vectors to a point $x \in \mathcal{M}$. Using the local coordinatization provided by the homeomorphism onto $\mathbb{R}^{n}$, a general linear combination of tangent vectors is denoted as

$$
\begin{equation*}
V=V^{\mu}(x) \frac{\partial}{\partial x^{\mu}} \tag{2.1}
\end{equation*}
$$

where throughout we use the Einstein summation convention for repeated upper and lower indices. A linear combination such as (2.1) will be refered to here as a vector field. Its components $V^{\mu}(x)$ are $C^{\infty}$ functions on $\mathcal{M}$ and are specified by the introduction of
local coordinates from $\mathbb{R}^{n}$. The local derivatives $\left\{\frac{\partial}{\partial x^{\mu}}\right\}_{\mu=1}^{n}$ span an $n$-dimensional vector space over $\mathbb{R}$ which is called the tangent space to $\mathcal{M}$ at $x$ and it is denoted by $T_{x} \mathcal{M}$. The disjoint union of all tangent spaces of the manifold,

$$
\begin{equation*}
T \mathcal{M}=\bigsqcup_{x \in \mathcal{M}} T_{x} \mathcal{M} \tag{2.2}
\end{equation*}
$$

is called the tangent bundle of $\mathcal{M}$.
The tangent bundle is an example of a more general geometric entity known as a fiber bundle. This consists of a quadruple ( $E, \mathcal{M}, F, \pi$ ), where $E$ is a topological space called the total space of the fiber bundle, $\mathcal{M}$ is a topological space called the base space of the fiber bundle (usually we take $\mathcal{M}$ to be a manifold), $F$ is a topological space called the fiber, and $\pi: E \rightarrow \mathcal{M}$ is a surjective continuous map with $\pi^{-1}(x)=F, \forall x \in \mathcal{M}$, which is called the projection of the fiber bundle. A fiber bundle is also defined so that locally it is trivial, i.e. locally the bundle is a product $U \times F$ of an open neighbourhood $U \subset \mathcal{M}$ of the base and the fibers, and $\pi: U \times F \rightarrow U$ is the projection onto the first coordinate. In the case of the tangent bundle, the fibers are $F=T_{x} \mathcal{M}=\mathbb{R}^{n}$ and the projection map is defined on $T \mathcal{M} \rightarrow \mathcal{M}$ by $\pi: T_{x} \mathcal{M} \rightarrow x$. In fact, in this case the fibration spaces are vector spaces, so that the tangent bundle is an example of a vector bundle. If the fiber of a bundle is a Lie group $G$, then the fiber bundle is called a principal fiber bundle with structure group $G$. A right-action of $G$ on the total space $E$ and the base $\mathcal{M}$ then gives a local representation of the group in the fibers (see the next section).

Any vector space $W$ has a dual vector space $W^{*}$ which is the space of linear functionals $\operatorname{Hom}_{\mathbf{R}}(W, \mathbb{R})$ on $W \rightarrow \mathbb{R}$. The dual of the tangent space $T_{x} \mathcal{M}$ is called the cotangent space $T_{x}^{*} \mathcal{M}$ and its basis elements $d x^{\mu}$ are defined by

$$
\begin{equation*}
d x^{\mu}\left(\frac{\partial}{\partial x^{\nu}}\right)=\delta_{\nu}^{\mu} \tag{2.3}
\end{equation*}
$$

The disjoint union of all the cotangent spaces of $\mathcal{M}^{1}$,

$$
\begin{equation*}
T^{*} \mathcal{M}=\bigsqcup_{x \in \mathcal{M}} T_{x}^{*} \mathcal{M} \tag{2.4}
\end{equation*}
$$

[^1]is a vector bundle called the cotangent bundle of $\mathcal{M}$. The space $\left(T_{x}^{*} \mathcal{M}\right)^{\otimes k}$ is the space of $n$-multilinear functionals on $T_{x} \mathcal{M} \times \cdots \times T_{x} \mathcal{M}$ whose elements are the linear combinations
\[

$$
\begin{equation*}
T=T_{\mu_{1} \cdots \mu_{k}}(x) d x^{\mu_{1}} \otimes \cdots \otimes d x^{\mu_{k}} \tag{2.5}
\end{equation*}
$$

\]

The object (2.5) is called a rank- $(k, 0)$ tensor and its components are $C^{\infty}$ functions of $x \in \mathcal{M}$. Similarly, the associated dual space $\left(T_{x} \mathcal{M}\right)^{\otimes \ell}$ consists of the linear combinations

$$
\begin{equation*}
\tilde{T}=\tilde{T}^{\mu_{1} \cdots \mu_{\ell}}(x) \frac{\partial}{\partial x^{\mu_{l}}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_{\ell}}} \tag{2.6}
\end{equation*}
$$

which are called $(0, \ell)$ tensors. The elements of $\left(T_{x}^{*} \mathcal{M}\right)^{\otimes k} \otimes\left(T_{x} \mathcal{M}\right)^{\otimes \ell}$ are called $(k, \ell)$ tensors and one can define tensor bundles analogously to the tangent and cotangent bundles above ${ }^{2}$. Under a local $C^{\infty}$ change of coordinates on $\mathcal{M}$ represented by the diffeomorphism $x \rightarrow x^{\prime}(x),(2.5)$ and (2.6) along with the usual chain rules

$$
\begin{equation*}
\frac{\partial}{\partial x^{\mu}}=\frac{\partial x^{\prime \lambda}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\prime \lambda}} \quad, \quad d x^{\mu}=\frac{\partial x^{\mu}}{\partial x^{\prime \lambda}} d x^{\prime \lambda} \tag{2.7}
\end{equation*}
$$

imply that the components of a generic rank $(k, \ell)$ tensor field $T_{\nu_{1} \cdots \nu_{k}}^{\mu_{1} \ldots \mu_{\ell}}(x)$ transform as

$$
\begin{equation*}
T_{\rho_{1} \cdots \rho_{k}}^{\prime \lambda_{1} \cdots \lambda_{l}}\left(x^{\prime}\right)=\frac{\partial x^{\prime \lambda_{1}}}{\partial x^{\mu_{1}}} \cdots \frac{\partial x^{\prime \lambda_{l}}}{\partial x^{\mu_{l}}} \frac{\partial x^{\nu_{1}}}{\partial x^{\prime \rho_{1}}} \cdots \frac{\partial x^{\nu_{k}}}{\partial x^{\prime \rho_{k}}} T_{\nu_{1} \cdots \nu_{k}}^{\mu_{1} \cdots \mu_{\ell}}(x) \tag{2.8}
\end{equation*}
$$

Such local coordinate transformations can be thought of as changes of bases (2.7) on the tangent and cotangent spaces.

We are now ready to define the DeRham complex of a manifold $\mathcal{M}$. Given the tensor product of copies of the cotangent bundle as above, we define a multi-linear antisymmetric multiplication of elements of the cotangent bundle, called the exterior or wedge

[^2]product, by
\[

$$
\begin{equation*}
d x^{\mu_{1}} \wedge \cdots \wedge d x^{\mu_{k}}=\sum_{P \in S_{k}} \operatorname{sgn}(P) d x^{\mu_{P(1)}} \otimes \cdots \otimes d x^{\mu_{P(k)}} \tag{2.9}
\end{equation*}
$$

\]

where the sum is over all permutations $P$ of $1, \ldots, k$ and $\operatorname{sgn}(P)$ is the sign of $P$, defined as $(-1)^{t(P)}$ where $t(P)$ is the number of transpositions in $P$. For example, for 2 cotangent basis vector elements

$$
\begin{equation*}
d x \wedge d y=d x \otimes d y-d y \otimes d x \tag{2.10}
\end{equation*}
$$

The exterior product is antisymmetric and formally the space of all linear combinations of the basis elements (2.9),

$$
\begin{equation*}
\alpha=\frac{1}{k!} \alpha_{\mu_{1} \cdots \mu_{k}}(x) d x^{\mu_{1}} \wedge \cdots \wedge d x^{\mu_{k}} \tag{2.11}
\end{equation*}
$$

is the antisymmetrization $\mathcal{A}\left(T_{x}^{*} \mathcal{M}\right)^{\otimes k}$ of the $k$-th tensor power of the cotangent bundle. The disjoint union, over all $x \in \mathcal{M}$, of these spaces is a vector bundle $\Lambda^{k} \mathcal{M}$ called the $k$-th exterior power of $\mathcal{M}$. Its elements (2.11) are called differential $k$-forms whose components are $C^{\infty}$ functions on $\mathcal{M}$ which are completely antisymmetric in their indices $\mu_{1}, \ldots, \mu_{k}$. Notice that by the antisymmetry of the exterior product, if $\mathcal{M}$ is $n$-dimensional, then $\Lambda^{k} \mathcal{M}=0$ for all $k>n$. Furthermore, $\Lambda^{0} \mathcal{M}=C^{\infty}(\mathcal{M})$, the space of infinitely continuously-differentiable functions on $\mathcal{M} \rightarrow \mathbb{R}$, and $\Lambda^{1} \mathcal{M}=T^{*} \mathcal{M}$ is the cotangent bundle of $\mathcal{M}$.

The exterior product of a $p$-form $\alpha$ and a $q$-form $\beta$ is the $(p+q)$-form $\alpha \wedge \beta=$ $\frac{1}{(p+q)!}(\alpha \wedge \beta)_{\mu_{1} \cdots \mu_{p+q}}(x) d x^{\mu_{1}} \wedge \cdots \wedge d x^{\mu_{p+q}}$ with local components

$$
\begin{equation*}
(\alpha \wedge \beta)_{\mu_{1} \cdots \mu_{p+q}}(x)=\sum_{P \in S_{p+q}} \operatorname{sgn}(P) \alpha_{\mu_{P(1)} \cdots \mu_{P(p)}}(x) \beta_{\mu_{P(p+1)} \cdots \mu_{P(p+q)}}(x) \tag{2.12}
\end{equation*}
$$

The exterior product of differential forms makes the direct sum of the exterior powers

$$
\begin{equation*}
\Lambda \mathcal{M}=\bigoplus_{k=0}^{n} \Lambda^{k} \mathcal{M} \tag{2.13}
\end{equation*}
$$

into a graded-commutative algebra called the exterior algebra of $\mathcal{M}$. In $\Lambda \mathcal{M}$, the exterior product of a $p$-form $\alpha$ and a $q$-form $\beta$ obeys the graded-commutativity property

$$
\begin{equation*}
\alpha \wedge \beta=(-1)^{p q} \beta \wedge \alpha \quad, \quad \alpha \in \Lambda^{p} \mathcal{M}, \beta \in \Lambda^{q} \mathcal{M} \tag{2.14}
\end{equation*}
$$

On the exterior algebra (2.13), we define a linear operator

$$
\begin{equation*}
d: \Lambda^{k} \mathcal{M} \rightarrow \Lambda^{k+1} \mathcal{M} \tag{2.15}
\end{equation*}
$$

on $k$-forms (2.11) by

$$
\begin{equation*}
(d \alpha)_{\mu_{1} \cdots \mu_{k+1}}(x)=\sum_{P \in S_{k+1}} \operatorname{sgn}(P) \partial_{\mu_{P(1)}} \alpha_{\mu_{P(2)} \cdots \mu_{P(k+1)}}(x) \tag{2.16}
\end{equation*}
$$

and $d \alpha=\frac{1}{(k+1)!}(d \alpha)_{\mu_{1} \cdots \mu_{k+1}}(x) d x^{\mu_{1}} \wedge \cdots \wedge d x^{\mu_{k+1}}$. The operator $d$ is called the exterior derivative and it generalizes the notion of the differential of a function

$$
\begin{equation*}
d f=\frac{\partial f(x)}{\partial x^{\mu}} d x^{\mu} \quad, \quad f \in \Lambda^{0} \mathcal{M}=C^{\infty}(\mathcal{M}) \tag{2.17}
\end{equation*}
$$

to generic differential forms. It satisfies the Leibniz property

$$
\begin{equation*}
d(\alpha \wedge \beta)=d \alpha \wedge \beta+(-1)^{p} \alpha \wedge d \beta \quad, \quad \alpha \in \Lambda^{p} \mathcal{M}, \beta \in \Lambda^{q} \mathcal{M} \tag{2.18}
\end{equation*}
$$

and it is nilpotent ${ }^{3}$,

$$
\begin{equation*}
d^{2}=0 \tag{2.19}
\end{equation*}
$$

which follows from the commutativity of multiple partial derivatives of $C^{\infty}$ functions. Thus the exterior derivative allows one to generalize the common notion of vector calculus to more general spaces other than $\mathbb{R}^{n^{4}}$. The collection of vector spaces $\left\{\Lambda^{k} \mathcal{M}\right\}_{k=0}^{n}$ and nilpotent derivations $d$ form what is called the DeRham complex $\Lambda^{*}(\mathcal{M})$ of the manifold $\mathcal{M}$.

There are 2 important subspaces of the exterior algebra (2.13) as far as the map $d$ is concerned. One is the kernel of $d$,

$$
\begin{equation*}
\operatorname{ker} d=\{\alpha \in \Lambda \mathcal{M}: d \alpha=0\} \tag{2.20}
\end{equation*}
$$

[^3]whose elements are called closed forms, and the other is the image of $d$,
\[

$$
\begin{equation*}
\operatorname{im} d=\{\beta \in \Lambda \mathcal{M}: \beta=d \alpha \text { for some } \alpha \in \Lambda \mathcal{M}\} \tag{2.21}
\end{equation*}
$$

\]

whose elements are called exact forms. Since $d$ is nilpotent, we have $\operatorname{im} d \subset$ ker $d$. Thus we can consider the quotient of the kernel of $d$ by its image. The vector space of closed $k$-forms modulo exact $k$-forms is called the $k$-th DeRham cohomology group (or vector space) of $\mathcal{M}$,

$$
\begin{equation*}
H^{k}(\mathcal{M} ; \mathbb{R})=\left.\operatorname{ker} d\right|_{\Lambda^{k} \mathcal{M}} /\left.\operatorname{im} d\right|_{\Lambda^{k-1} \mathcal{M}} \tag{2.22}
\end{equation*}
$$

The elements of the vector space (2.22) are the equivalence classes of differential $k$-forms where 2 differential forms are equivalent if and only if they differ only by an exact form, i.e. if the closed form $\alpha \in \Lambda^{k} \mathcal{M}$ is a representative of the cohomology class $[\alpha] \in H^{k}(\mathcal{M} ; \mathbb{R})$, then so is the closed form $\alpha+d \beta$ for any differential form $\beta \in \Lambda^{k-1} \mathcal{M}$. One important theorem in the context of DeRham cohomology is Poincarés lemma. This states that if $d \omega=0$ in a star-shaped region of the manifold $\mathcal{M}$ (i.e. one in which the points can be connected together by an affine transformation of the coordinates, such as a simplex - see below), then one can write $\omega=d \theta$ in that region for some other differential form $\theta$. Thus each representative of a DeRham cohomology class can be locally written as an exact form, but globally there may be an obstruction to extending the form $\theta$ over the entire manifold in a smooth way depending on whether or not $[\omega] \neq 0$ in the DeRham cohomology group.

The DeRham cohomology groups are related to the topology of the manifold $\mathcal{M}$ as follows. Consider the following $q$-dimensional subspace of $\mathbb{R}^{q+1}$,

$$
\begin{equation*}
\Delta^{q}=\left\{\left(x_{0}, x_{1}, \ldots, x_{q}\right) \in \mathbb{R}^{q+1}: x_{i} \geq 0, \sum_{i=0}^{q} x_{i}=1\right\} \tag{2.23}
\end{equation*}
$$

which is called the standard $q$-simplex. Geometrically, $\Delta^{q}$ is the convex hull generated by the vertices placed at unit distance along the axes of $\mathbb{R}^{q+1}$. We define the geometric
boundary of the standard $q$-simplex as

$$
\begin{equation*}
\partial \Delta^{q}=\sum_{i=0}^{q}(-1)^{i} \hat{\Delta}_{(i)}^{q} \tag{2.24}
\end{equation*}
$$

where $\hat{\Delta}_{(i)}^{q}$ is the $(q-1)$-simplex generated by all the vertices of $\Delta^{q}$ except the $i$-th one, and the sum on the right-hand side is the formal algebraic sum of simplices (where a minus sign signifies a change of orientation). A singular $q$-simplex of the manifold $\mathcal{M}$ is defined to be a continuous map $\sigma: \Delta^{q} \rightarrow \mathcal{M}$. A formal algebraic sum of $q$-simplices with integer coefficients is called a $q$-chain, and the collection of all $q$-chains in a manifold $\mathcal{M}$ is called the $q$-th chain group $C_{q}(\mathcal{M})$ of $\mathcal{M}$. It defines an abelian group under the formal addition. The boundary of a $q$-chain is the ( $q-1$ )-chain

$$
\begin{equation*}
\partial \sigma=\left.\sum_{i=0}^{q}(-1)^{i} \sigma\right|_{\hat{\Delta}_{(i)}^{q}} \tag{2.25}
\end{equation*}
$$

which is easily verified to give a nilpotent homomorphism

$$
\begin{equation*}
\partial: C_{q}(\mathcal{M}) \rightarrow C_{q-1}(\mathcal{M}) \tag{2.26}
\end{equation*}
$$

of abelian groups. The collection of abelian groups $\left\{C_{q}(\mathcal{M})\right\}_{q \in \mathbb{Z}^{+}}$and nilpotent homomorphisms $\partial$ form the singular chain complex $C_{*}(\mathcal{M})$ of the manifold $\mathcal{M}$.

Nilpotency of the boundary map (2.26) means that every $q$-chain in the image of $\left.\partial\right|_{C_{q+1}}$, the elements of which are called the $q$-boundaries of $\mathcal{M}$, lies as well in the kernel of $\left.\partial\right|_{C_{q}}$, whose elements are called the $q$-cycles of $\mathcal{M}$. The abelian group defined as the quotient of the group of all $q$-cycles modulo the group of all $q$-boundaries is called the $q$-th (singular) homology group of $\mathcal{M}$,

$$
\begin{equation*}
H_{q}(\mathcal{M} ; \mathbb{Z})=\left.\operatorname{ker} \partial\right|_{C_{q}} /\left.\operatorname{im} \partial\right|_{C_{q+1}} \tag{2.27}
\end{equation*}
$$

These groups are homotopy invariants of the manifold $\mathcal{M}$ (i.e. invariant under continuous deformations of the space), and in particular they are topological invariants and diffeomorphism invariants (i.e. invariant under $C^{\infty}$ invertible bi-continuous mappings
of $\mathcal{M}$ ). Intuitively, they measure whether or not a manifold has 'holes' in it or not. If $H_{q}(\mathcal{M} ; \mathbb{Z})=0$, then every $q$-cycle (intuitively a closed $q$-dimensional curve or surface) encloses a $q+1$-dimensional chain and $\mathcal{M}$ has no ' $q$-holes'. For instance, if $\mathcal{M}$ is simplyconnected (i,e. every loop in $\mathcal{M}$ can be contracted to a point) then $H_{1}(\mathcal{M} ; \mathbb{Z})=0$.

Given the abelian groups (2.27), we can form their duals using the universal coefficient theorem

$$
\begin{equation*}
H^{q}(\mathcal{M} ; \mathbb{Z}) \simeq \operatorname{Hom}_{\mathbb{Z}}\left(H_{q}(\mathcal{M} ; \mathbb{Z}), \mathbb{Z}\right) \oplus \operatorname{Ext}_{\mathbb{Z}}\left(H_{q-1}(\mathcal{M} ; \mathbb{Z}), \mathbb{Z}\right) \tag{2.28}
\end{equation*}
$$

which is called the $q$-th singular cohomology group of $\mathcal{M}$ with integer coefficients. Here $\operatorname{Hom}_{\mathbb{Z}}\left(H_{q}(\mathcal{M} ; \mathbb{Z}), \mathbb{Z}\right)=H_{q}(\mathcal{M} ; \mathbb{Z})^{*}$ is the free part of the cohomology group, and $\operatorname{Ext}_{\mathbb{Z}}$ is the torsion subgroup of $H^{q}(\mathcal{M} ; \mathbb{Z})$. The DeRham theorem then states that the DeRham cohomology groups are naturally isomorphic to the singular cohomology groups with real coefficients,

$$
\begin{equation*}
H^{q}(\mathcal{M} ; \mathbb{R})=H^{q}(\mathcal{M} ; \mathbb{Z}) \otimes \mathbb{R}=H_{q}(\mathcal{M} ; \mathbb{R})^{*} \tag{2.29}
\end{equation*}
$$

where the tensor product with the reals means that $H^{q}$ is considered as an abelian group with real instead of integer coefficients, i.e. a vector space over $\mathbb{R}$ (this eliminates the torsion subgroup in (2.28)).

The crux of the proof of DeRham's theorem is Stokes' theorem,

$$
\begin{equation*}
\int_{c} d \omega=\oint_{\partial c} \omega \quad, \quad \omega \in \Lambda^{q} \mathcal{M}, \quad c \in C_{q+1}(\mathcal{M}) \tag{2.30}
\end{equation*}
$$

which relates the integral of an exact $(q+1)$-form over a smooth $(q+1)$-chain $c$ in $\mathcal{M}$ to an integral over the closed $q$-dimensional boundary $\partial c$ of $c$. In particular, (2.30) generalizes to the global version of Stokes' theorem,

$$
\begin{equation*}
\int_{\mathcal{M}} d \omega=\oint_{\partial \mathcal{M}} \omega \quad, \quad \omega \in \Lambda^{n-1} \mathcal{M} \tag{2.31}
\end{equation*}
$$

which relates the integral of an exact form over $\mathcal{M}$ to an integral over the closed ( $n-$ 1)-dimensional boundary $\partial \mathcal{M}$ of $\mathcal{M}$. Here integration over a manifold is defined by
partitioning the manifold up into open sets homeomorphic to $\mathbb{R}^{n}$, integrating a top form (i.e. a differential form of highest degree $n$ on $\mathcal{M}$ ) locally over $\mathbb{R}^{n}$ as usual ${ }^{5}$, and then summing up all of these contributions ${ }^{6}$. In this way, we see how the DeRham cohomology of a manifold measures its topological (or global) features in an analytic way suited for the differential calculus of $C^{\infty}$ manifolds. We refer to [81] and [27] for a more complete introduction to this subject.

### 2.2 The Cartan Model of Equivariant Cohomology

Many situations in theoretical physics involve not only a differentiable manifold $\mathcal{M}$, but also the action of some Lie group $G$ acting on $\mathcal{M}$, which we denote symbolically by

$$
\begin{gather*}
G \times \mathcal{M} \rightarrow \mathcal{M} \\
(g, x) \rightarrow g \cdot x \tag{2.32}
\end{gather*}
$$

By a group action we mean that $g \cdot x=x, \forall x \in \mathcal{M}$ if $g$ is the identity element of $G$, and the group action represents the multiplication law of the group, i.e. $g_{1} \cdot\left(g_{2} \cdot x\right)=$ $\left(g_{1} g_{2}\right) \cdot x, \forall g_{1}, g_{2} \in G$. We shall throughout assume that $G$ is connected and that its action on $\mathcal{M}$ is smooth, i.e. for fixed $g \in G$, the function $x \rightarrow g \cdot x$ is a diffeomorphism of $\mathcal{M}$. Usually $G$ is taken to be the symmetry group of the given physical problem. The common (infinite-dimensional) example in topological field theory is where $\mathcal{M}$ is the space of gauge connections of a gauge theory and $G$ is the group of gauge transformations. The space $\mathcal{M}$ modulo this group action is then the moduli space of gauge orbits. Another example is in string theory where $\mathcal{M}$ is the space of metrics on a Riemann surface and $G$ is the semi-direct product of the Weyl and diffeomorphism groups of that 2-surface. Then $\mathcal{M}$ modulo this group action is the moduli space of the Riemann surface. In such instances we are interested in knowing the cohomology of the manifold $\mathcal{M}$ given this

[^4]action of the group $G$. This cohomology is known as the $G$-equivariant cohomology of $\mathcal{M}$. Given the $G$-action on $\mathcal{M}$, the space of orbits $\mathcal{M} / G$ is the set of equivalence classes where $x$ and $x^{\prime}$ are equivalent if and only if $x^{\prime}=g \cdot x$ for some $g \in G$ (the topology of $\mathcal{M} / G$ is the induced topology from $\mathcal{M}$ ). If the $G$-action on $\mathcal{M}$ is free, i.e. $g \cdot x=x$ if and only if $g$ is the identity element of $G, \forall x \in \mathcal{M}$, then the space of orbits $\mathcal{M} / G$ is also a differentiable manifold ${ }^{7}$ of dimension $\operatorname{dim} \mathcal{M}-\operatorname{dim} G$ and the $G$-equivariant cohomology is defined simply as the cohomology of the coset space $\mathcal{M} / G$,
\[

$$
\begin{equation*}
H_{G}^{k}(\mathcal{M})=H^{k}(\mathcal{M} / G) \tag{2.33}
\end{equation*}
$$

\]

However, if the group action is not free and has fixed points on $\mathcal{M}$, the space $\mathcal{M} / G$ can become singular. In a neighbourhood of a fixed point, the dimension of $\mathcal{M} / G$ can be smaller than the $\operatorname{dimension} \operatorname{dim} \mathcal{M}-\operatorname{dim} G$ of other fixed-point free coordinate neighbourhoods (because then the isotropy subgroup $\{g \in G: g \cdot p=p\}$ of that fixed point $p$ is non-trivial), and there is no smooth notion of dimensionality for the coset $\mathcal{M} / G$. A singular quotient space $\mathcal{M} / G$ is called an orbifold. In such instances, one cannot define the equivariant cohomology of $\mathcal{M}$ in a smooth way using (2.33) and more elaborate methods are needed to define this cohomology. There are many approaches to defining the equivariant cohomology of $\mathcal{M}$, but there is only one that will be used extensively in this thesis. This is the Cartan model of equivariant cohomology and it is defined in a manner similar to the analytic DeRham cohomology which was reviewed in the last section. However, the other models of equivariant cohomology are equally as important - the Weil algebra formulation relates the algebraic models to the topological definition of equivariant cohomology using universal bundles of Lie groups [ $8,28,67,82$ ], while the BRST model relates the Cartan and Weil models and moreover is the basis for the superspace formulation of topological Yang-Mills theory in 4-dimensions and other cohomological field theories [29, 67, 103].

[^5]We begin by generalizing the notion of a differential form to the case where there is a group action on $\mathcal{M}$ as above. We say that a $\operatorname{map} f: \mathcal{M}_{1} \rightarrow \mathcal{M}_{2}$ between 2 manifolds with $G$-actions on them is equivariant with respect to the group action if

$$
\begin{equation*}
f(g \cdot x)=g \cdot f(x) \quad \forall x \in \mathcal{M}_{1} \quad, \quad \forall g \in G \tag{2.34}
\end{equation*}
$$

We want to extend this notion of equivariance to differential forms. Consider the symmetric polynomial functions from the Lie algebra $g$ of $G$ into the exterior algebra $\Lambda \mathcal{M}$ of the manifold $\mathcal{M}$. These maps form the algebra $S\left(\mathbf{g}^{*}\right) \otimes \Lambda \mathcal{M}$, where $S\left(\mathbf{g}^{*}\right)$ is called the symmetric algebra over the dual vector space $\mathbf{g}^{*}$ of $\mathbf{g}$ and it corresponds to the algebra of polynomial functions on $g$. The action of $g \in G$ on an element $\alpha \in S\left(\mathbf{g}^{*}\right) \otimes \Lambda \mathcal{M}$ is given by

$$
\begin{equation*}
(g \cdot \alpha)(X)=g \cdot\left(\alpha\left(g^{-1} X g\right)\right) \tag{2.35}
\end{equation*}
$$

where $X \in \mathbf{g}$. Here we have used the natural coadjoint action of $G$ on $\mathbf{g}^{*}$ and the induced $G$-action on $\Lambda \mathcal{M}$ from that on $\mathcal{M}$ as dictated by the tensor transformation law (2.8) with $x^{\prime}(x)=g \cdot x$. From this it follows immediately that the equivariance condition (2.34) is satisfied for the polynomial maps $\alpha: g \rightarrow \Lambda \mathcal{M}$ in the $G$-invariant subalgebra

$$
\begin{equation*}
\Lambda_{G} \mathcal{M}=\left(S\left(\mathbf{g}^{*}\right) \otimes \Lambda \mathcal{M}\right)^{G} \tag{2.36}
\end{equation*}
$$

where the superscript $G$ denotes the (infinitesimal) $G$-invariant part. The elements of (2.36) are called equivariant differential forms [14, 16].

Elements of $G$ are represented in terms of elements of the Lie algebra $g$ through the exponential map,

$$
\begin{equation*}
g=\mathrm{e}^{c^{a} X^{a}} \tag{2.37}
\end{equation*}
$$

where $c^{a}$ are constants and $X^{a}$ are the generators of $g$ obeying the Lie bracket algebra

$$
\begin{equation*}
\left[X^{a}, X^{b}\right]=f^{a b c} X^{c} \tag{2.38}
\end{equation*}
$$

with $f^{a b c}$ the antisymmetric structure constants of $g$. Here and in the following we shall assume an implicit sum over the Lie algebraic indices $a, b, c, \ldots$. In the context of the
relation (2.37), the generators $X^{a}$ span the tangent space to the identity on the group manifold of $G$ and so the Lie algebra $g$ can be regarded as the tangent space to the group manifold of the Lie group $G$.

The smooth $G$-action on $\mathcal{M}$ can be represented locally as the continuous flow

$$
\begin{equation*}
g_{t} \cdot x=x(t) \quad, \quad t \in \mathbb{R}^{+} \tag{2.39}
\end{equation*}
$$

where $g_{t}$ is a path in $G$ starting at the identity $g_{t=0}$. The induced action on differential forms is defined by pullback, i.e. as

$$
\begin{equation*}
\left(g_{t} \cdot \alpha\right)(x)=\alpha(x(t)) \tag{2.40}
\end{equation*}
$$

For example, we can represent the group action on $C^{\infty}$ functions by diffeomorphisms on $\mathcal{M}$ which are connected to the identity, i.e.

$$
\begin{equation*}
\left(g_{t} \cdot f\right)(x)=f(x(t))=\mathrm{e}^{t V(x(t))} f(x) \quad, \quad f \in \Lambda^{0} \mathcal{M} \tag{2.41}
\end{equation*}
$$

The action (2.41) represents the flow of the group on $C^{\infty}$ functions on $\mathcal{M}$, where $V(x)=$ $V^{\mu}(x) \frac{\partial}{\partial x^{\mu}}$ is a vector field on $\mathcal{M}$ representing a Lie algebra element. It is related to the flows (2.39) on the manifold by

$$
\begin{equation*}
\dot{x}^{\mu}(t)=V^{\mu}(x(t)) \tag{2.42}
\end{equation*}
$$

which defines a set of curves in $\mathcal{M}$ which we will refer to as the integral curves of the group action. If $V^{a}$ is the vector field representing the generator $X^{a}$ of $\mathbf{g}$, then the Lie algebra (2.38) is represented on $C^{\infty}$ functions by

$$
\begin{equation*}
\left[V^{a}, V^{b}\right](h)=f^{a b c} V^{c}(h) \quad, \quad \forall h \in \Lambda^{0} \mathcal{M} \tag{2.43}
\end{equation*}
$$

with Lie bracket represented by the commutator bracket. This defines a representation of $G$ by vector fields in the tangent bundle $T \mathcal{M}$. In this setting, the group $G$ is represented as a subgroup of the connected diffeomorphism group of $\mathcal{M}$ whose Lie algebra is generated by all vector fields of $\mathcal{M}$ with the commutator bracket.

The infinitesimal $(t \rightarrow 0)$ action of the group on $\Lambda^{0} \mathcal{M}$ can be expressed as

$$
\begin{equation*}
V(f)=i_{V} d f \tag{2.44}
\end{equation*}
$$

where

$$
\begin{equation*}
i_{V}: \Lambda^{k} \mathcal{M} \rightarrow \Lambda^{k-1} \mathcal{M} \tag{2.45}
\end{equation*}
$$

is the nilpotent contraction operator, or interior multiplication, with respect to $V$ and it is defined locally on $k$-forms (2.11) by

$$
\begin{equation*}
i_{V} \alpha=\frac{1}{(k-1)!} V^{\mu_{1}}(x) \alpha_{\mu_{1} \mu_{2} \cdots \mu_{k}}(x) d x^{\mu_{2}} \wedge \cdots \wedge d x^{\mu_{k}} \tag{2.46}
\end{equation*}
$$

The infinitesimal $G$-action on the higher-degree differential forms is generated by the Lie derivative along $V$

$$
\begin{equation*}
\mathcal{L}_{V}: \Lambda^{k} \mathcal{M} \rightarrow \Lambda^{k} \mathcal{M} \tag{2.47}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{L}_{V}=d i_{V}+i_{V} d \tag{2.48}
\end{equation*}
$$

generates the induced action of $G$ on $\Lambda \mathcal{M}$. This can be verifed by direct computation from expanding (2.40) about $t=0$ using (2.8) and (2.42), and by noting that

$$
\begin{equation*}
\left[\mathcal{L}_{V^{a}}, \mathcal{L}_{V^{b}}\right](\alpha)=f^{a b c} \mathcal{L}_{V^{c}}(\alpha) \quad, \quad \forall \alpha \in \Lambda \mathcal{M} \tag{2.49}
\end{equation*}
$$

Thus the Lie derivative in general defines a representation of $G$ on $\Lambda \mathcal{M}$. The local components of $\mathcal{L}_{V} T$ for a general $(k, \ell)$ tensor field $T$ are found by substituting into the tensor transformation law (2.8) the infinitesimal coordinate change $x^{\prime \mu}(x)=x^{\mu}(t)=$ $x^{\mu}+t V^{\mu}(x)$. Furthermore, the Lie derivative action on contractions is

$$
\begin{equation*}
\left[i_{V^{a}}, \mathcal{L}_{V^{b}}\right](\alpha)=f^{a b c} i_{V^{c}}(\alpha) \tag{2.50}
\end{equation*}
$$

We are now ready to define the Cartan model for the $G$-equivariant cohomology of $\mathcal{M}[16,28,82]$. We assign a $\mathbb{Z}$-grading to the elements of (2.36) by defining the degree of an equivariant differential form to be the sum of its ordinary form degree and twice
the polynomial degree from the $S\left(\mathbf{g}^{*}\right)$ part. Let $\left\{\phi^{a}\right\}_{a=1}^{\operatorname{dim} G}$ be a basis of $\mathbf{g}^{*}$ dual to the basis $\left\{X^{a}\right\}_{a=1}^{\operatorname{dim}^{G}}$ of $g$. With the above grading, the basis elements $\phi^{a}$ have degree 2 . We define a linear map

$$
\begin{equation*}
D_{\mathrm{g}}: \Lambda_{G}^{k} \mathcal{M} \rightarrow \Lambda_{G}^{k+1} \mathcal{M} \tag{2.51}
\end{equation*}
$$

on the algebra (2.36) by

$$
\begin{equation*}
D_{\mathrm{g}} \phi^{a}=0 \quad, \quad D_{\mathrm{g}} \alpha=\left(1 \otimes d-\phi^{a} \otimes i_{V^{a}}\right) \alpha \quad ; \quad \alpha \in \Lambda \mathcal{M} \tag{2.52}
\end{equation*}
$$

The operator $D_{\mathbf{g}}$ is called the equivariant exterior derivative. Its definition (2.52) means that its action on equivariant differential forms $\alpha \in \Lambda_{G} \mathcal{M}$ is

$$
\begin{equation*}
\left(D_{\mathbf{g}} \alpha\right)(X)=\left(d-i_{V}\right)(\alpha(X)) \tag{2.53}
\end{equation*}
$$

where $V=c^{a} V^{a}$ is the vector field on $\mathcal{M}$ representing the Lie algebra element $X=$ $c^{a} X^{a} \in \mathrm{~g} . \quad D_{\mathrm{g}}$ is not nilpotent in general, but its square is given by the Cartan-Weil identity

$$
\begin{equation*}
D_{\mathbf{g}}^{2}=-\phi^{a} \otimes\left(d i_{V^{a}}+i_{V^{a}} d\right)=-\phi^{a} \otimes \mathcal{L}_{V^{a}} \tag{2.54}
\end{equation*}
$$

Thus the operator $D_{\mathbf{g}}$ is nilpotent on the algebra $\Lambda_{G} \mathcal{M}$ of equivariant differential forms. The set of $G$-invariant algebras $\left\{\Lambda_{G}^{k} \mathcal{M}\right\}_{k \in \mathbb{Z}^{+}}$and nilpotent derivations $D_{\mathrm{g}}$ thereon defines the $G$-equivariant complex $\Lambda_{G}^{*}(\mathcal{M})$ of the manifold $\mathcal{M}$. Both the Lie derivative and the equivariant exterior derivative obey a Leibniz rule (2.18) just like the ordinary exterior derivative.

Thus, just as in the last section, we can proceed to define the cohomology of the operator $D_{\mathbf{g}}$. The space of equivariantly closed forms, i.e. $D_{\mathbf{g}} \alpha=0$, modulo the space of equivariantly exact forms, i.e. $\alpha=D_{\mathbf{g}} \beta$, is called the $G$-equivariant cohomology group of $\mathcal{M}$,

$$
\begin{equation*}
H_{G}^{k}(\mathcal{M})=\left.\operatorname{ker} D_{\mathbf{g}}\right|_{\Lambda_{G}^{k} \mathcal{M}} /\left.\operatorname{im} D_{\mathbf{g}}\right|_{\Lambda_{G}^{k-1} \mathcal{M}} \tag{2.55}
\end{equation*}
$$

With this definition, the cohomology of the operator $D_{\mathbf{g}}$ for a fixed-point free $G$-action on $\mathcal{M}$ reduces to the DeRham cohomology of the quotient space $\mathcal{M} / G$, as in (2.33). The definition (2.55) of equivariant cohomology is known as the Cartan model [16, 28, 82].

We close this section with a few remarks concerning the above construction. First of all, it follows from these definitions that $H_{G}^{k}(\mathcal{M})$ coincides with the ordinary DeRham cohomology of $\mathcal{M}$ if $G$ is the trivial group consisting of only the identity element (i.e. $V \equiv 0$ in the above), and that the $G$-equivariant cohomology of a point is the algebra of $G$-invariant polynomials on $\mathrm{g}, H_{G}(\mathrm{pt})=S\left(\mathrm{~g}^{*}\right)^{G}$, of the given degree. Secondly, if a form $\alpha \in \Lambda_{G} \mathcal{M}$ is equivariantly exact, $\alpha=D_{\mathrm{g}} \beta$, then its top-form component $\alpha^{(n)} \in \Lambda^{n} \mathcal{M}$ is exact in the ordinary DeRham sense. This follows because the $i_{V}$ part of $D_{\mathrm{g}}$ lowers the form-degree by 1 so there is no way to produce a top-form by acting with $i_{V}$. Finally, in what follows we shall have occasion to also consider the $C^{\infty}$ extension $\Lambda_{G}^{\infty} \mathcal{M}$ of $\Lambda_{G} \mathcal{M}$ to include $G$-invariant smooth functions from $g$ to $\Lambda \mathcal{M}$. In this extension we lose the $\mathbb{Z}$-grading described above, but we are left with a $\mathbb{Z}_{2}$-grading corresponding to the differential form being of even or odd parity [16].

### 2.3 Equivariant Characteristic Classes

In the ordinary DeRham theory, we have already introduced the notion of a fiber bundle which 'pins' some geometrical or topological object on each point of a manifold $\mathcal{M}$ (e.g. a vector space in the case of a vector bundle). For instance, if $\mathcal{M}=\mathbb{R}^{n}$, then the tangent bundle $T \mathbb{R}^{n}$ associates the vector space $W=\mathbb{R}^{n}$ to each point of $\mathbb{R}^{n}$. In fact in this case, the tangent bundle is globally given by $T \mathbb{R}^{n}=\mathbb{R}^{n} \times W$, the product of its base and fibers. In this case we say that the bundle is trivial, in that the erecting of points into vector spaces is done without any 'twistings' of the fibers. However, a general vector bundle is only locally trivial and globally the fibers can twist in a very complicated fashion. One way to characterize the non-triviality of fiber bundles is through special cohomology classes of the base manifold $\mathcal{M}$ called characteristic classes [85]. A non-trivial characteristic class in this sense signifies the non-triviality of the vector bundle.

It is possible to extend these notions to the case of the equivariant cohomology of a manifold which signifies the non-triviality of an equivariant bundle. First, we define
what we mean by an equivariant bundle [14, 15]. We say that a fiber bundle $E \xrightarrow{\pi} \mathcal{M}$ is a $G$-equivariant bundle if there are $G$-actions on both $E$ and $\mathcal{M}$ which are compatible with each other in the sense that

$$
\begin{equation*}
g \cdot \pi(x)=\pi(g \cdot x) \quad \forall x \in E \quad, \quad \forall g \in G \tag{2.56}
\end{equation*}
$$

This means that the bundle projection $\pi$ is a $G$-equivariant map. The action of the group $G$ on differential forms with values in the bundle is generated by the Lie derivatives $\mathcal{L}_{\text {Va }}$.

Recall that in the ordinary DeRham case, one defines a connection $\Gamma$ as a geometrical object (such as a 1-form) defined over $\mathcal{M}$ with values in $E$ whose action on tensors of the bundle specifies their parallel transport along fibers, as required when there are 'twists' in the given bundle. The parallel transport is generated by the covariant derivative associated with $\Gamma$,

$$
\begin{equation*}
\nabla=d+\Gamma \tag{2.57}
\end{equation*}
$$

For example, if the bundle is a principal fiber bundle, then $\Gamma$ is a connection 1-form $A$, otherwise known as a gauge field. Another case is where the bundle is the tangent bundle $T \mathcal{M}$ equipped with a Riemannian metric $g$. Then $\Gamma$ is the (affine) Levi-Civita-Christoffel connection $\Gamma_{\mu \nu}^{\lambda}(g)$ associated with $g$. When the bundle being considered is an equivariant bundle, we assume that the covariant derivative (2.57) is $G$-invariant,

$$
\begin{equation*}
\left[\nabla, \mathcal{L}_{V^{a}}\right]=0 \tag{2.58}
\end{equation*}
$$

Mimicking the equivariant exterior derivative (2.51), we define the equivariant covariant derivative

$$
\begin{equation*}
\nabla_{\mathbf{g}}=\mathbf{1} \otimes \nabla-\phi^{a} \otimes i_{V^{a}} \tag{2.59}
\end{equation*}
$$

which is considered as an operator on the algebra $\Lambda_{G}(\mathcal{M}, E)$ of equivariant differential forms on $\mathcal{M}$ with values in $E$. In a local trivialization $E=U \times W, U \subset \mathcal{M}$, this algebra looks like

$$
\begin{equation*}
\Lambda_{G}(U, E)=\left(S\left(\mathrm{~g}^{*}\right) \otimes \Lambda U \otimes W\right)^{G} \tag{2.60}
\end{equation*}
$$

Recalling the Cartan-Weil identity (2.54), we define the equivariant curvature of the connection (2.59)

$$
\begin{equation*}
F_{\mathrm{g}}=\left(\nabla_{\mathrm{g}}\right)^{2}+\phi^{a} \otimes \mathcal{L}_{V^{a}} \tag{2.61}
\end{equation*}
$$

which, using (2.58), then satisfies the equivariant Bianchi identity

$$
\begin{equation*}
\left[\nabla_{\mathbf{g}}, F_{\mathbf{g}}\right]=0 \tag{2.62}
\end{equation*}
$$

Notice that if $G$ is the trivial group, these identities reduce to the usual notions of curvature, etc. Expanding out (2.61) explicitly using (2.58) gives

$$
\begin{equation*}
F_{\mathrm{g}}=1 \otimes F+\mu \tag{2.63}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu=\phi^{a} \otimes \mathcal{L}_{V^{a}}-\left[\phi^{a} \otimes i_{V^{a}}, \mathbf{1} \otimes \nabla\right] \tag{2.64}
\end{equation*}
$$

is called the moment map of the $G$-action with respect to the connection $\nabla$. Here $F=\nabla^{2}$ is the ordinary curvature 2 -form of the connection $\nabla$, and from (2.62) and (2.63) we see that the moment map $\mu$ is a $G$-equivariant extension of $F$ from a covariantly-closed 2 -form, $[\nabla, F]=0$, to an equivariant one in the sense of (2.62).

When evaluated on an element $X \in \mathbf{g}$, represented by a vector field $V \in T \mathcal{M} \otimes W$, we write

$$
\begin{equation*}
F_{\mathbf{g}}(X)=F+\mu(X) \equiv F+\mu_{V} \equiv F_{V} \tag{2.65}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu_{V}=\mathcal{L}_{V}-\left[i_{V}, \nabla\right] \tag{2.66}
\end{equation*}
$$

generates the induced $G$-action on the fibers of the bundle. The moment map in this way can be regarded locally as a function $\mu: \Lambda U \otimes W \rightarrow \mathbf{g}^{*}$. Furthermore, using the equivariant Bianchi identity (2.62) we see that it obeys the important property

$$
\begin{equation*}
\nabla \mu_{V}=i_{V} F \tag{2.67}
\end{equation*}
$$

Later on, we shall encounter 2 important instances of equivariant bundles on $\mathcal{M}$, one associated with a Riemannian structure, and the other with a symplectic structure. In the latter case the moment map is associated with the Hamiltonian of a dynamical system.

Now we are ready to define the notion of an equivariant characteristic class. First, we recall how to construct conventional characteristic classes [85]. Given a Lie group $H$ with Lie algebra h, we say that a real- or complex-valued function $P$ is an invariant polynomial on $\mathbf{h}$ if it is invariant under the natural adjoint action of $H$ on $\mathbf{h}$,

$$
\begin{equation*}
P\left(h^{-1} Y h\right)=P(Y) \quad \forall h \in H, \forall Y \in \mathbf{h} \tag{2.68}
\end{equation*}
$$

An invariant polynomial $P$ can be used to define characteristic classes on principal fiber bundles with structure group $H$. If we consider the polynomial $P$ in such a setting as a function on $\mathbf{h}$-valued 2 -forms on $\mathcal{M}$, then the $H$-invariance (2.68) of $P$ implies that

$$
\begin{equation*}
d P(\alpha)=r P(\nabla \alpha) \quad, \quad \alpha \in \Lambda^{2} \mathcal{M} \otimes \mathbf{h} \tag{2.69}
\end{equation*}
$$

where $r$ is the degree of $P$. In particular, taking the argument $\alpha$ to be the curvature 2-form $\alpha=F=\nabla^{2}$ on the principal $H$-bundle $E \xrightarrow{\pi} \mathcal{M}$ (which is locally an h-valued 2 -form), we have

$$
\begin{equation*}
d P(F)=0 \tag{2.70}
\end{equation*}
$$

as a consequence of the Bianchi identity for $F$. This means that $P(F)$ defines a (DeRham) cohomology class of $\mathcal{M}$.

What is particularly remarkable about this cohomology class is that it is independent of the particular connection $\nabla$ used to define the curvature $F$. To see this, consider the simplest case where the invariant polynomial is just $P(\alpha)=\operatorname{tr} \alpha^{n}{ }^{8}$, with $\operatorname{tr}$ the invariant Cartan-Killing linear form of the Lie algebra $h$ (usually the ordinary operator trace). Consider a continuous one-parameter family of connections $\nabla_{t}, t \in \mathbb{R}$, with curvatures

[^6]$F_{t}=\nabla_{t}^{2}$. Then
\[

$$
\begin{equation*}
\frac{d}{d t} F_{t}=\left[\nabla_{t}, \frac{d}{d t} \nabla_{t}\right] \tag{2.71}
\end{equation*}
$$

\]

and applying this to the invariant polynomial $\operatorname{tr} F_{t}^{n}$ gives

$$
\begin{equation*}
\frac{d}{d t} \operatorname{tr} F_{t}^{n}=n \operatorname{tr}\left(\frac{d}{d t} F_{t}\right) F_{t}^{n-1}=n \operatorname{tr}\left[\nabla_{t},\left(\frac{d}{d t} \nabla_{t}\right) F_{t}^{n-1}\right]=d \operatorname{tr}\left(\frac{d}{d t} \nabla_{t}\right) F_{t}^{n-1} \tag{2.72}
\end{equation*}
$$

where $d$ is the exterior derivative and in the last equality we have applied (2.69). This means that any continuous deformation of the $2 n$-form tr $F^{n}$ changes it by an exact form, so that the cohomology class determined by it is independent of the choice of connection. In general, the invariant polynomial $P(F) \in \Lambda \mathcal{M}$ is called a characteristic class of the given $H$-bundle.

This notion and construction of characteristic classes can be generalized almost verbatum to the equivariant case [16]. Taking instead the $G$-equivariant curvature (2.61) as the argument of the $G$-invariant polynomial $P,(2.70)$ generalizes to

$$
\begin{equation*}
D_{\mathbf{g}} P\left(F_{\mathbf{g}}\right)=r P\left(\nabla_{\mathbf{g}} F_{\mathbf{g}}\right)=0 \tag{2.73}
\end{equation*}
$$

and now the resulting equivariant characteristic classes $P\left(F_{\mathrm{g}}\right)$ of the given $G$-equivariant bundle are elements of the algebra $\Lambda_{G} \mathcal{M}$. These are denoted by $P_{g}(F)$, or when evaluated on an element $X \in \mathbf{g}$ with associated vector field $V \in T U \otimes W$, we write

$$
\begin{equation*}
P_{\mathbf{g}}(F)(X)=P\left(F_{V}\right) \equiv P_{V}(F) \tag{2.74}
\end{equation*}
$$

The equivariant cohomology class of $P_{\mathbf{g}}(F)$ is independent of the chosen connection on the bundle. Consequently, on a trivial vector bundle $\mathcal{M} \times W$ we can choose a flat connection, $F=0$, and then

$$
\begin{equation*}
P^{\mathcal{M} \times W}\left(F_{\mathbf{g}}\right)(X)=P^{\mathcal{M} \times W}\left(\mu_{V}\right)=P(\rho(X)) \tag{2.75}
\end{equation*}
$$

where $\rho$ is the representation of $G$ defined by the $G$-action on the fibers $W$.
There are 4 equivariant characteristic classes that commonly appear in the localization formalism for topological field theories, all of which are to be understood as elements of
the completion $\Lambda_{G}^{\infty} \mathcal{M}$. These can all be found and are extensively discussed in [16]. The first one is related to the invariant polynomial $\operatorname{tr} \mathrm{e}^{\alpha}$ and is used for $G$-equivariant complex vector bundles (i.e. one in which the fibers are vector spaces over the complex numbers $\mathbb{C}$ ). It is called the $G$-equivariant Chern character

$$
\begin{equation*}
\operatorname{ch}_{\mathrm{g}}(F)=\operatorname{tr} \mathrm{e}^{F_{\mathrm{g}}} \tag{2.76}
\end{equation*}
$$

Note that this is a polynomial on a finite-dimensional manifold because then $\Lambda^{k} \mathcal{M}=0$ for $k>\operatorname{dim} \mathcal{M}$. The other 3 are given by determinants of specific polynomials. On a $G$-equivariant real vector bundle we define the equivariant Dirac $\hat{A}$-genus

$$
\begin{equation*}
\hat{A}_{\mathbf{g}}(F)=\sqrt{\operatorname{det}\left[\frac{\frac{1}{2} F_{\mathbf{g}}}{\sinh \left(\frac{1}{2} F_{\mathbf{g}}\right)}\right]} \tag{2.77}
\end{equation*}
$$

where the inverse of an inhomogeneous polynomial of differential forms is always to be understood in terms of the power series

$$
\begin{equation*}
(1+x)^{-1}=\sum_{k=0}^{\infty}(-1)^{k} x^{k} \tag{2.78}
\end{equation*}
$$

On a complex fiber bundle, the complex version of the equivariant $\hat{A}$-genus is the equivariant Todd class

$$
\begin{equation*}
\operatorname{td}_{\mathbf{g}}(F)=\operatorname{det}\left[\frac{F_{\mathbf{g}}}{\mathrm{e}^{F_{\mathbf{g}}}-1}\right] \tag{2.79}
\end{equation*}
$$

When $G$ is the trivial group, these all reduce to the conventional characteristic classes [85]. Just as for the ordinary $\hat{A}$-genus and Todd classes, their equivariant generalizations inherit the multiplicativity property under Whitney sums of bundles,

$$
\begin{equation*}
\hat{A}_{\mathbf{g}}^{E \oplus F}=\hat{A}_{\mathbf{g}}^{E} \hat{A}_{\mathbf{g}}^{F} \quad, \quad \operatorname{td}_{\mathbf{g}}^{E \oplus F}=\operatorname{td}_{\mathbf{g}}^{E} \operatorname{td}_{\mathbf{g}}^{F} \tag{2.80}
\end{equation*}
$$

Finally, on an orientable real bundle we can define the equivariant generalization of the Euler class,

$$
\begin{equation*}
E_{\mathbf{g}}(F)=\operatorname{Pfaff}\left(F_{\mathbf{g}}\right) \tag{2.81}
\end{equation*}
$$

where the Pfaffian (or Salam-Mathiews determinant) of a $2 N \times 2 N$ antisymmetric matrix $M=\left[M_{i j}\right]$ is defined as

$$
\begin{equation*}
\text { Pfaff } M=\epsilon^{i_{1} \cdots i_{2 N}} M_{i_{1} i_{2}} \cdots M_{i_{2 N-1} i_{2 N}}=\frac{1}{2^{N} N!} \sum_{P \in S_{2 N}} \operatorname{sgn}(P) \prod_{k=1}^{N} M_{P(2 k-1), P(2 k)} \tag{2.82}
\end{equation*}
$$

with the property that

$$
\begin{equation*}
\operatorname{det} M=(\text { Pfaff } M)^{2} \tag{2.83}
\end{equation*}
$$

The sign of the Pfaffian when written as the square root of the determinant as in (2.83) is chosen so that it is the product of the upper skew-diagonal eigenvalues in a skewdiagonalization of the antisymmetric matrix $M$. In (2.82), $\epsilon^{i_{1} \cdots i_{N}}$ is the antisymmetric tensor with the convention $\epsilon^{123 \cdots N}=+1$. Pfaffians arise naturally, as we will see, as fermionic determinants from the integration of fermion bilinears in supersymmetric and topological field theories. Transformations which change the orientation of the bundle change the sign of the Pfaffian. When $F=R$ is the Riemann curvature 2-form associated with the tangent bundle $T \mathcal{M}$ (which can be regarded as a principal $S O(n)$-bundle) of a closed manifold $\mathcal{M}$ of even dimension $2 n$, the integral over $\mathcal{M}$ of the ordinary Euler class is the integer

$$
\begin{equation*}
\chi(\mathcal{M})=\frac{(-1)^{n}}{(4 \pi)^{n} n!} \int_{\mathcal{M}} E(R) \tag{2.84}
\end{equation*}
$$

where

$$
\begin{equation*}
\chi(\mathcal{M})=\sum_{k=0}^{2 n}(-1)^{k} \operatorname{dim}_{\mathbf{R}} H^{k}(\mathcal{M} ; \mathbb{R}) \tag{2.85}
\end{equation*}
$$

is the famous topological invariant called the Euler characteristic of the manifold $\mathcal{M}$. That (2.85) can be written as an integral of a density in (2.84) is a celebrated result of differential topology known as the Gauss-Bonnet-Chern theorem.

Similarly, with $F=F_{A}$ the curvature of a gauge connection $A$ on a principle $H$ bundle over a $2 k$-dimensional manifold $\mathcal{M}$, the integral over $\mathcal{M}$ of the $k$-th term in the expansion of the conventional version of the Chern class (2.76) (which defines the $k$-th Chern class) is the number

$$
\begin{equation*}
c_{k}(\mathcal{M})=\left(-\frac{1}{2 \pi i}\right)^{k} \int_{\mathcal{M}} \operatorname{tr} F_{A}^{k} \tag{2.86}
\end{equation*}
$$

which is a topological invariant of $\mathcal{M}$ called the $k$-th Chern number of $\mathcal{M}$ (or, more precisely, of the complex vector bundle $(E, \mathcal{M}, W, \pi))$. The Chern number is always an integer for closed orientable manifolds. Thus the equivariant characteristic classes defined above lead to interesting equivariant generalizations of some classical topological invariants. We shall see that they appear in most interesting ways within the formalism of topological field theory functional integration.

### 2.4 The Equivariant Localization Principle

We now discuss a very interesting property of equivariant cohomology which is the fundamental feature of all localization theorems. It also introduces the fundamental geometric constraint that will be one of the issues of focus in what follows. In most of our applications we will be concerned with the following situation. Let $\mathcal{M}$ be a compact orientable manifold without boundary and let $V$ be a vector field over $\mathcal{M}$ corresponding to some action of the circle group $G=U(1) \sim S^{1}$ on $\mathcal{M}$. In this case the role of the multiplier $\phi \in S\left(\mathbf{u}(\mathbf{1})^{*}\right)$, which is a linear functional on the 1-dimensional Lie algebra of $U(1)$, will not be important for the discussion that follows. Indeed, we can regard $\phi$ as just some external parameter in this case and 'localize' algebraically by setting $\phi=-1$. As shown in [8], the operations of evaluating $\phi$ on Lie algebra elements and the formation of equivariant cohomology commute for abelian group actions, so that all results below will coincide independently of the interpretation of $\phi$. The corresponding equivariant exterior derivative is then denoted as

$$
\begin{equation*}
D_{\mathbf{u}(\mathbf{1})} \equiv D_{V}=d+i_{V} \tag{2.87}
\end{equation*}
$$

and it is now considered as an operator on the algebra

$$
\begin{equation*}
\Lambda_{V} \mathcal{M}=\left\{\alpha \in \Lambda \mathcal{M}: \mathcal{L}_{V} \alpha=0\right\} \tag{2.88}
\end{equation*}
$$

It was Atiyah and Bott [8] and Berline and Vergne [14, 15] who first noticed that equivariant cohomology is determined by the fixed point locus of the $G$-action. In our
simplified case here, this is the set

$$
\begin{equation*}
\mathcal{M}_{V}=\{x \in \mathcal{M}: V(x)=0\} \tag{2.89}
\end{equation*}
$$

This fact is at the very heart of the localization theorems in both the finite dimensional case and in topological field theory, and it is known as the equivariant localization principle. In this section we shall establish this property in 2 analytic ways. For a more algebraic description of this principle using the Weil algebra and the topological definition of equivariant cohomology, see [8].

Our first argument for localization involves an explicit proof at the level of differential forms. Given an integral $\int_{\mathcal{M}} \alpha$ over $\mathcal{M}$ of an equivariantly closed differential form $\alpha \in$ $\Lambda_{V} \mathcal{M}, D_{V} \alpha=0$, we wish to show that this integral depends only the fixed-point set (2.89) of the $U(1)$-action on $\mathcal{M}$. To show this, we shall explicitly construct a differential form $\lambda$ on $\mathcal{M}-\mathcal{M}_{V}$ satisfying $D_{V} \lambda=\alpha$. This is just the equivariant version of the Poincaré lemma. Thus the form $\alpha$ is equivariantly exact away from the zero locus $\mathcal{M}_{V}$, and we recall that this implies that the top-form component of $\alpha$ is exact. Since integration over $\mathcal{M}$ picks up the top-form component of any differential form, and since $\partial \mathcal{M}=\emptyset$ by hypothesis here, it follows from Stokes' theorem (2.31) that the integral $\int_{\mathcal{M}} \alpha$ only receives contributions from an arbitrarily small neighbourhood of $\mathcal{M}_{V}$ in $\mathcal{M}$.

To construct $\lambda$, we need to impose the following geometric restriction on the manifold $\mathcal{M}$. We assume that $\mathcal{M}$ has a globally-defined $U(1)$-invariant Riemannian structure on it, which means that it admits a globally-defined metric tensor

$$
\begin{equation*}
g=\frac{1}{2} g_{\mu \nu}(x) d x^{\mu} \otimes d x^{\nu} \tag{2.90}
\end{equation*}
$$

which is invariant under the $U(1)$-action generated by $V$, i.e. for which

$$
\begin{equation*}
\mathcal{L}_{V} g=0 \tag{2.91}
\end{equation*}
$$

or in local coordinates on $\mathcal{M}$,

$$
\begin{equation*}
g_{\mu \lambda} \partial_{\nu} V^{\lambda}+g_{\nu \lambda} \partial_{\mu} V^{\lambda}+V^{\lambda} \partial_{\lambda} g_{\mu \nu}=0 \tag{2.92}
\end{equation*}
$$

Alternatively, this Lie derivative constraint can be written as

$$
\begin{equation*}
g_{\nu \lambda} \nabla_{\mu} V^{\lambda}+g_{\mu \lambda} \nabla_{\nu} V^{\lambda}=0 \tag{2.93}
\end{equation*}
$$

where $\nabla$ is the covariant derivative (2.57) constructed from the Levi-Civita-Christoffel connection

$$
\begin{equation*}
\Gamma_{\mu \nu}^{\lambda}=\frac{1}{2} g^{\lambda \rho}\left(\partial_{\mu} g_{\nu \rho}+\partial_{\nu} g_{\mu \rho}-\partial_{\rho} g_{\mu \nu}\right) \tag{2.94}
\end{equation*}
$$

associated with $g$ on the tangent bundle $T \mathcal{M}$. Here $g^{\mu \nu}$ is the matrix inverse of $g_{\mu \nu}$ and the covariant derivative acts on the vector field $V$ in the usual way as

$$
\begin{equation*}
\nabla_{\mu} V^{\nu}=\partial_{\mu} V^{\nu}+\Gamma_{\lambda \mu}^{\nu} V^{\lambda} \tag{2.95}
\end{equation*}
$$

with a plus sign for $(0, k)$-tensors and a minus sign for $(k, 0)$-tensors in front of $\Gamma$, as in (2.95). Notice that the Levi-Civita-Christoffel connection is torsion-free, $\Gamma_{\mu \nu}^{\lambda}=\Gamma_{\nu \mu}^{\lambda}$, and it is compatible with the the metric $g, \nabla_{\lambda} g_{\mu \nu}=0$.

The equivalent equations (2.91)-(2.93) are called the Killing equations and in this case we say that $V$ is a Killing vector field of the metric $g$. Since the map $V \rightarrow \mathcal{L}_{V}$ is linear, the space of Killing vectors of a Riemannian manifold $(\mathcal{M}, g)$ generate the Lie algebra of a Lie group acting on $\mathcal{M}$ by diffeomorphisms which is called the isometry group of $(\mathcal{M}, g)$. We shall describe this group in more detail in chapters 5 and 6 . The Killing equations here are assumed to hold globally over the entire manifold $\mathcal{M}$. If both $\mathcal{M}$ and $G$ are compact, then such a metric can always be obtained from an arbitrary Riemannian metric $h$ on $\mathcal{M}$ by averaging $h$ over the group manifold of $G$ in its Haar measure. However, we shall have occasion to also consider more general vector field flows which aren't necessarily closed or when the manifold $\mathcal{M}$ isn't compact, as are the cases in many physical applications. In such cases the Lie derivative constraint (2.91) is a very stringent one on the manifold. This feature of the localization formalism, that the manifold admit a globally defined metric with the property (2.91) whose components $g_{\mu \nu}(x)=g_{\nu \mu}(x)$ are globally-defined $C^{\infty}$ functions on $\mathcal{M}$, is the crux of all finite- and
infinite-dimensional localization formulas and will be analysed in detail later on in this thesis. For now, we content ourselves with assuming that such a metric tensor has been constructed.

Any metric tensor defines a duality between vector fields and differential 1-forms, i.e. we can consider the metric tensor (2.90) as a map

$$
\begin{equation*}
g: T \mathcal{M} \rightarrow T^{*} \mathcal{M} \tag{2.96}
\end{equation*}
$$

which takes a vector field $V$ into its metric dual 1-form

$$
\begin{equation*}
\beta \equiv g(V, \cdot)=g_{\mu \nu}(x) V^{\nu}(x) d x^{\mu} \tag{2.97}
\end{equation*}
$$

Non-degeneracy, $\operatorname{det} g(x) \neq 0, \forall x \in \mathcal{M}$, of the metric tensor implies that this defines an isomorphism between the tangent and cotangent bundles of $\mathcal{M}$. The 1 -form $\beta$ satisfies

$$
\begin{equation*}
D_{V}^{2} \beta=\mathcal{L}_{V} \beta=0 \tag{2.98}
\end{equation*}
$$

since $\mathcal{L}_{V} V=0$ and $V$ is a Killing vector of $g$. This means that $\beta$ is an equivariant differential 1-form. Furthermore, we have

$$
\begin{equation*}
D_{V} \beta=K_{V}+\Omega_{V} \tag{2.99}
\end{equation*}
$$

where $K_{V}$ is the globally-defined $C^{\infty}$-function

$$
\begin{equation*}
K_{V}=g(V, V)=g_{\mu \nu}(x) V^{\mu}(x) V^{\nu}(x) \tag{2.100}
\end{equation*}
$$

and

$$
\begin{equation*}
\Omega_{V}=d \beta=d g(V, \cdot) \tag{2.101}
\end{equation*}
$$

is the 2-form with local components

$$
\begin{equation*}
\left(\Omega_{V}\right)_{\mu \nu}=g_{\mu \lambda} \nabla_{\nu} V^{\lambda}-g_{\nu \lambda} \nabla_{\mu} V^{\lambda} \tag{2.102}
\end{equation*}
$$

Consequently, away from zero locus $\mathcal{M}_{V}$ of the vector field $V$, the 0 -form part $K_{V}$ of $D_{V} \beta$ is non-zero and hence $D_{V} \beta$ is invertible on $\mathcal{M}-\mathcal{M}_{V}$. Again we understand here
the inverse of an inhomogeneous differential form with non-zero scalar term in analogy with the formula (2.78).

We can now define an inhomogenous differential form by

$$
\begin{equation*}
\xi=\beta\left(D_{V} \beta\right)^{-1} \tag{2.103}
\end{equation*}
$$

on $\mathcal{M}-\mathcal{M}_{V}$, which satisfies $D_{V} \xi=1$ and $\mathcal{L}_{V} \xi=0$ owing to the equivariance (2.98) of $\beta$. Thus we can define an equivariant differential form $\lambda=\xi \alpha$, and since $\alpha$ is equivariantly closed it follows that

$$
\begin{equation*}
\alpha=1 \cdot \alpha=\left(D_{V} \xi\right) \alpha=D_{V}(\xi \alpha) \tag{2.104}
\end{equation*}
$$

Thus, as claimed above, any equivariantly closed form is equivariantly exact away from $\mathcal{M}_{V}$, and in particular the top-form component of an equivariantly closed form is exact away from $\mathcal{M}_{V}$. This establishes the equivariant localization property mentioned above.

The other argument we wish to present here for equivariant localization is less explicit and involves cohomological arguments. First, consider an ordinary closed form $\omega, d \omega=0$. For any other differential form $\lambda$, we have

$$
\begin{equation*}
\int_{\mathcal{M}}(\omega+d \lambda)=\int_{\mathcal{M}} \omega \tag{2.105}
\end{equation*}
$$

by Stokes' theorem (2.31) since $\partial \mathcal{M}=\emptyset$. This means that the integral $\int_{\mathcal{M}} \omega$ of a closed form $\omega$ depends only on the cohomology class defined by $\omega$, not on the particular representative. Since the map $\omega \rightarrow \int_{\mathcal{M}} \omega$ in general defines a linear map on $\Lambda^{k} \mathcal{M} \rightarrow \Lambda^{n-k}(\mathrm{pt})$, it follows that this map descends to a map on $H^{n}(\mathcal{M} ; \mathbb{R}) \rightarrow H^{0}(\mathrm{pt} ; \mathbb{R})=\mathbb{R}$. The same is true for equivariant integration. Since, for a general $G$-action on $\mathcal{M}$, integration of a differential form picks up the top-form component which for an equivariantly exact form is exact, for any equivariantly-closed differential form $\alpha$ we can again invoke Stokes' theorem to deduce

$$
\begin{equation*}
\int_{\mathcal{M}}\left(\alpha+D_{\mathbf{g}} \lambda\right)=\int_{\mathcal{M}} \alpha \tag{2.106}
\end{equation*}
$$

so that the integral of an equivariantly closed form depends only on the equivariant cohomology class defined by it, and not on the particular representative. Note, however,
that equivariant integration for general Lie groups $G$ takes a far richer form. In analogy with the DeRham case above, the integration of equivariant differential forms defines a map on $H_{G}(\mathcal{M}) \rightarrow H_{G}(\mathrm{pt})=S\left(\mathrm{~g}^{*}\right)^{G}$. This we define by

$$
\begin{equation*}
\left(\int_{\mathcal{M}} \alpha\right)(X)=\int_{\mathcal{M}} \alpha(X) \quad, \quad X \in \mathbf{g} \tag{2.107}
\end{equation*}
$$

with integration over the $\Lambda \mathcal{M}$ part of $\alpha$ in the ordinary DeRham sense. Later on, we shall also consider the dual Lie algebra elements $\phi^{a}$ in a more 'dynamical' situation where they are a more integral part of the cohomological description above. We shall see then how this definition of integration should be accordingly modified.

Given that the integral $\int_{\mathcal{M}} \alpha$ depends only on the equivariant cohomology class defined by $\alpha$, we can choose a particular representative of the cohomology class making the localization manifest. Taking the equivariant differential form $\beta$ defined in (2.97), we consider the integral

$$
\begin{equation*}
\mathcal{Z}(s)=\int_{\mathcal{M}} \alpha \mathrm{e}^{-s D_{V} \beta} \tag{2.108}
\end{equation*}
$$

viewed as a function of $s \in \mathbb{R}^{+}$. We assume that (2.108) is a regular function of $s \in \mathbb{R}^{+}$ and that its $s \rightarrow 0$ and $s \rightarrow \infty$ limits exist. Its $s \rightarrow 0$ limit is the integral of interest, $\int_{\mathcal{M}} \alpha$, while from the identities (2.99) and (2.100) we see that the integrand of (2.108) is an increasingly sharply Gaussian peaked form around $\mathcal{M}_{V} \subset \mathcal{M}$ as $s \rightarrow \infty$. The crucial point here is that the equivariant differential form which is the integrand of (2.108) is equivariantly cohomologous to $\alpha$ for all $s \in \mathbb{R}^{+}$. This can be seen by applying Stokes' theorem to get

$$
\begin{align*}
\frac{d}{d s} \mathcal{Z}(s) & =-\int_{\mathcal{M}} \alpha\left(D_{V} \beta\right) \mathrm{e}^{-s D_{V} \beta} \\
& =-\int_{\mathcal{M}}\left\{D_{V}\left(\alpha \beta \mathrm{e}^{-s D_{V} \beta}\right)-\beta D_{V}\left(\alpha \mathrm{e}^{-s D_{V} \beta}\right)\right\}  \tag{2.109}\\
& =s \int_{\mathcal{M}} \alpha \beta\left(\mathcal{L}_{V} \beta\right) \mathrm{e}^{-s D_{V} \beta}=0
\end{align*}
$$

where we have used the fact that $\alpha$ is equivariantly closed and the equivariance property (2.98) of $\beta$. Therefore the integral (2.108) is independent of the parameter $s \in \mathbb{R}^{+}$, and
so its $s \rightarrow 0$ and $s \rightarrow \infty$ limits coincide. Hence, we may evaluate the integral of interest as

$$
\begin{equation*}
\int_{\mathcal{M}} \alpha=\lim _{s \rightarrow \infty} \int_{\mathcal{M}} \alpha \mathrm{e}^{-s D_{V} \beta} \tag{2.110}
\end{equation*}
$$

which establishes the localization of $\int_{\mathcal{M}} \alpha$ to $\mathcal{M}_{V}$.
It should be pointed out though that there is nothing particularly unique about the choice of $\beta$ in (2.110) - indeed the same steps leading to (2.110) can be carried out for an arbitrary equivariant differential form $\beta$, i.e. any one with the property (2.98). In this general case, the localization of $\int_{\mathcal{M}} \alpha$ is onto the subspace of $\mathcal{M}$ which is the support for the non-trivial equivariant cohomology of $\alpha$, i.e. $\int_{\mathcal{M}} \alpha$ localizes to the points where $D_{V} \beta=0$. Different choices of representatives $\beta$ for the equivariant cohomology classes then lead to potentially different localizations other than the one onto $\mathcal{M}_{V}$. This would lead to seemingly different expressions for the integral in (2.110), but of course these must all coincide in some way. In principle this argument for localization could also therefore work without the assumption that $V$ is a Killing vector for some metric on $\mathcal{M}$, but it appears difficult to make general statements in that case. Nonetheless, as everything at the end will be equivariantly closed by our general arguments above, it is possible to reduce the resulting expressions further to $\mathcal{M}_{V}$ by applying the above localization arguments once more, now to the localized expression. The localization formula (2.110) is the basis for the recent applications of equivariant cohomology in physics.

### 2.5 The Berline-Vergne Theorem

The first general localization formula using only the general equivariant cohomological arguments presented in the last section was derived by Berline and Vergne [14, 15]. This formula, as well as some of the arguments leading to the equivariant localization principle, have since been established in many different contexts suitable to other finite dimensional applications and also to path integrals [ $8,9,16,18,19]$. The proof presented here introduces a method that will generalize to functional integrals. For now, we assume
the fixed-point set $\mathcal{M}_{V}$ of the $U(1)$-action on $\mathcal{M}$ consists of discrete isolated points, i.e. $\mathcal{M}_{V}$ is a submanifold of $\mathcal{M}$ of codimension $n=\operatorname{dim} \mathcal{M}^{9}$. We shall discuss the generalization to the case where $\mathcal{M}_{V}$ has non-zero dimension later on.

We wish to evaluate explicitly the right-hand side of the localization formula (2.110). To do this, we introduce an alternative way of evaluating integrals over differential forms. We introduce a set of nilpotent anticommuting variables $\eta^{\mu}, \mu=1, \ldots, n$,

$$
\begin{equation*}
\eta^{\mu} \eta^{\nu}=-\eta^{\nu} \eta^{\mu} \tag{2.111}
\end{equation*}
$$

which generate the exterior algebra $\Lambda \mathcal{M}$. The variables $\eta^{\mu}$ are to be thought of as the basis vectors $d x^{\mu}$ of $\Lambda^{1} \mathcal{M}=T^{*} \mathcal{M}$ with the exterior product of differential forms replaced by the ordinary product of the $\eta^{\mu}$ variables with the algebra (2.111). The $k$-th exterior power $\Lambda^{k} \mathcal{M}$ is then generated by the products $\eta^{\mu_{1}} \cdots \eta^{\mu_{k}}$ and this definition turns $\Lambda \mathcal{M}$ into a graded Grassmann algebra with the generators $\eta^{\mu}$ having grading 1. For instance, suppose the differential form $\alpha$ is the sum

$$
\begin{equation*}
\alpha=\alpha^{(0)}+\alpha^{(1)}+\ldots+\alpha^{(n)} \quad, \quad \alpha^{(k)} \in \Lambda^{k} \mathcal{M} \tag{2.112}
\end{equation*}
$$

with $\alpha^{(k)}$ the $k$-form component of $\alpha$ and $\alpha^{(0)}(x)$ its 0 -form component which is a $C^{\infty}$ function on $\mathcal{M}$. The $k$-form component of $\alpha$ for $k>0$ then has the form

$$
\begin{equation*}
\alpha^{(k)}(x, \eta)=\alpha_{\mu_{1} \cdots \mu_{k}}^{(k)}(x) \eta^{\mu_{1}} \cdots \eta^{\mu_{k}} \quad, \quad k>0 \tag{2.113}
\end{equation*}
$$

and from this point of view differential forms are functions $\alpha(x, \eta)$ on the exterior bundle which is now the $2 n$-dimensional supermanifold $\mathcal{M} \otimes \Lambda \mathcal{M}$ with local coordinates $(x, \eta)$.

The integration of a differential form is now defined by introducing the Berezin rules for integrating Grassman variables [12],

$$
\begin{equation*}
\int d \eta^{\mu} \eta^{\mu}=1 \quad, \quad \int d \eta^{\mu} 1=0 \tag{2.114}
\end{equation*}
$$

[^7]Since the $\eta^{\mu}$ 's are nilpotent, any function of them is a polynomial in $\eta^{\mu}$ and consequently the rules (2.114) unambiguously defines the integral of any function of the anticommuting variables $\eta^{\mu}$. For instance, it is easily verified that with this definition of integration we have

$$
\begin{equation*}
\int d^{n} \eta \mathrm{e}^{\frac{1}{2} \eta^{\mu} M_{\mu \nu} \eta^{\nu}}=\text { Pfaff } M \tag{2.115}
\end{equation*}
$$

where $d^{n} \eta \equiv d \eta^{n} d \eta^{n-1} \cdots d \eta^{1}$. This is the fermionic analog of the Gaussian integration formula (1.2), and the Berezin integral in (2.115) is invariant under similarity transformations.

Given these definitions, we can now alternatively write the integral of any differential form over $\mathcal{M}$ as an integral over the cotangent bundle $\mathcal{M} \otimes \Lambda^{1} \mathcal{M}$. Thus given the localization formula (2.110) with the 1 -form $\beta$ in (2.97) and the identities (2.99)-(2.102), we have

$$
\begin{equation*}
\int_{\mathcal{M}} \alpha=\lim _{s \rightarrow \infty} \int_{\mathcal{M} \otimes \Lambda^{1} \mathcal{M}} d^{n} x d^{n} \eta \alpha(x, \eta) \exp \left(-s g_{\mu \nu}(x) V^{\mu}(x) V^{\nu}(x)-\frac{s}{2}\left(\Omega_{V}\right)_{\mu \nu}(x) \eta^{\mu} \eta^{\nu}\right) \tag{2.116}
\end{equation*}
$$

where the measure $d^{n} x d^{n} \eta$ on $\mathcal{M} \otimes \Lambda^{1} \mathcal{M}$ is coordinate-independent because the measures $d^{n} x \equiv d x^{1} \wedge \cdots \wedge d x^{n}$ and $d^{n} \eta$ transform inversely to each other. To evaluate the large-s limit of (2.116), we use the delta-function representations

$$
\begin{gather*}
\delta(V)=\lim _{s \rightarrow \infty}\left(\frac{s}{\pi}\right)^{n / 2} \sqrt{\operatorname{det} g} \mathrm{e}^{-s g_{\mu \nu} V^{\mu} V^{\nu}}  \tag{2.117}\\
\delta(\eta)=\lim _{s \rightarrow \infty}(-s)^{n / 2} \frac{1}{\text { Pfaff } \Omega_{V}} \mathrm{e}^{-\frac{s}{2}\left(\Omega_{V}\right)_{\mu \nu} \eta^{\mu} \eta^{\nu}} \tag{2.118}
\end{gather*}
$$

as can be seen directly from the respective integrations in local coordinates on $\mathcal{M}$ and $\Lambda^{1} \mathcal{M}$. Notice that from the Killing equations (2.93), the matrix $\left(\Omega_{V}\right)_{\mu \nu}$ is given by

$$
\begin{equation*}
\left(\Omega_{V}\right)_{\mu \nu}=2 g_{\mu \lambda} \nabla_{\nu} V^{\lambda} \tag{2.119}
\end{equation*}
$$

Thus using (2.117) and (2.118) we can write (2.116) as

$$
\begin{equation*}
\int_{\mathcal{M}} \alpha=(-\pi)^{n / 2} \int_{\mathcal{M} \otimes \Lambda^{1} \mathcal{M}} d^{n} x d^{n} \eta \alpha(x, \eta) \frac{\operatorname{Pfaff} \Omega_{V}(x)}{\sqrt{\operatorname{det} g(x)}} \delta(V(x)) \delta(\eta) \tag{2.120}
\end{equation*}
$$

The integration over $\Lambda^{1} \mathcal{M}$ in (2.120) kills off all $k$-form components of the form $\alpha$ except its $C^{\infty}$-function part $\alpha^{(0)}(x) \equiv \alpha(x, 0)$, while the integration over $\mathcal{M}$ localizes it onto a sum over the points in $\mathcal{M}_{V}$. This yields

$$
\begin{equation*}
\int_{\mathcal{M}} \alpha=(-\pi)^{n / 2} \sum_{p \in \mathcal{M}_{V}} \frac{\alpha^{(0)}(p)}{\operatorname{det} d V(p) \mid} \frac{\operatorname{Pfaff} \Omega_{V}(p)}{\sqrt{\operatorname{det} g(p)}} \tag{2.121}
\end{equation*}
$$

where the factor $|\operatorname{det} d V(p)|$ comes from the Jacobian of the coordinate transformation $x \rightarrow V(x)$ used to transform $\delta(V(x))$ to a sum of delta-functions $\sum_{p \in \mathcal{M}_{V}} \delta(x-p)$ localizing onto the zero locus $\mathcal{M}_{V}$. Substituting in the identity (2.119) and noting that at a point $p \in \mathcal{M}_{V}$ we have $\nabla V(p)=d V(p)$, the expression (2.121) reduces to

$$
\begin{equation*}
\int_{\mathcal{M}} \alpha=(-2 \pi)^{n / 2} \sum_{p \in \mathcal{M}_{V}} \frac{\alpha^{(0)}(p)}{\text { Pfaff } d V(p)} \tag{2.122}
\end{equation*}
$$

where we emphasize the manner in which the dependence of orientation in the Pfaffian has been transfered from the numerator to the denominator in going from (2.121) to (2.122) (note that a change of orientation on $\mathcal{M}$ by definition changes the sign of $\operatorname{det} g$ ). This is the (non-degenerate form of) the Berline-Vergne integration formula, and it is our first example of what we shall call a localization formula. It reduces the original integral over the $n$-dimensional space $\mathcal{M}$ to a sum over a discrete set of points in $\mathcal{M}$ and it is valid for any equivariantly-closed differential form $\alpha$ on a manifold with a globally-defined circle action (and Riemannian metric for which the associated diffeomorphism generator is a Killing vector). In general, the localization formulas we shall consider will always at least reduce the dimensionality of the integration of interest. This will be particularly important for path integrals, where we shall see that localization theory can be used to reduce complicated infinite-dimensional integrals to finite sums or finite-dimensional integrals.

We close this chapter by noting the appearence of the operator in the denominator of the expression (2.122). For each $p \in \mathcal{M}_{V}$, it is readily seen that the operator $d V(p)$ appearing in the argument of the Pfaffian in (2.122) is just the invertible linear transformation $L_{V}(p)$ induced by the Lie derivative acting on the tangent spaces $T_{p} \mathcal{M}$, i.e. by
the induced infinitesimal group action on the tangent bundle. Explicitly, this operator is defined on vector fields $W=\left.W^{\mu}(x) \frac{\partial}{\partial x^{\mu}}\right|_{x=p} \in T_{p} \mathcal{M}$ by

$$
\begin{equation*}
L_{V}(p) W=\left.\partial_{\nu} V^{\mu}(p) W^{\nu}(p) \frac{\partial}{\partial x^{\mu}}\right|_{x=p} \tag{2.123}
\end{equation*}
$$

Note however that $d V(p)$ is not covariant in general and so this is only true right on the tangent space $T_{p} \mathcal{M}$ and not in general on the entire tangent bundle $T \mathcal{M}$. A linear transformation on the whole of $T \mathcal{M}$ can only be induced from the Lie derivative by introducing a (metric or non-metric) connection $\Gamma_{\mu \nu}^{\lambda}$ of $T \mathcal{M}$ and inducing an operator from $\nabla V$, as in the matrix (2.119). We shall return to this point later on in a more specific setting.

## Chapter 3

## Finite-dimensional Localization Theory

We shall now proceed to discuss a certain class of integrals that can be considered to be toy models for the functional integrals that we are ultimately interested in. The advantage of these models is that they are finite-dimensional and therefore rigorous theorems concerning their behaviour can be formulated. We shall be interested in certain oscillatory integrals $\int_{\mathcal{M}} d \mu e^{i T H}$ representing the Fourier transform of some smooth measure $d \mu$ on a manifold $\mathcal{M}$ in terms of a smooth function $H$. The common method of evaluating such integrals is the stationary phase approximation which expresses the fact that for large- $T$ the main contributions to the integral come from the critical points of $H$. The main result of this chapter is the Duistermaat-Heckman theorem [33] which provides a criterion for the stationary phase approximation to an oscillatory integral to be exact. Although this theorem was originally discovered within the context of symplectic geometry, it turns out to have its most natural explanation in the setting of equivariant cohomology and equivariant characteristic classes [8],[14]-[16]. The Duistermaat-Heckman theorem, and its various extensions that we shall discuss towards the end of this chapter, are precisely those which originally motivated the localization theory of path integrals.

For physical applications, we shall be primarily interested in a special class of differentiable manifolds known as 'symplectic' manifolds. The application of the equivariant cohomological ideas to these manifolds leads quite nicely to the notion of a Hamiltonian, as well as some standard ideas in the geometrical theory of classical integrability. Furthermore, the configuration space of a topological field theory is typically an (infinitedimensional) symplectic manifold (or phase space) [17] and we shall therefore restrict our
attention for the remainder of this thesis to the localization theory for oscillatory integrals over symplectic manifolds. We shall discuss all of these finite dimensional aspects of equivariant cohomology on symplectic manifolds in this chapter.

### 3.1 Symplectic Geometry

Symplectic geometry is the natural mathematical setting for the geometrical formulation of classical mechanics and the study of classical integrability [5]. It also has applications in other branches of physics, such as geometrical optics [55]. In elementary classical mechanics [48], one is introduced to the Hamiltonian formalism of classical dynamics as follows. For a dynamical system defined on some manifold $\mathcal{M}$ (usually $\mathbb{R}^{n}$ ) with coordinates $\left(q^{1}, \ldots, q^{n}\right)$, we introduce the canonical momenta $p_{\mu}$ conjugate to each variable $q^{\mu}$ from the Lagrangian of the system and then the Hamiltonian $H(p, q)$ is obtained by a Legendre transformation of the Lagrangian. In this way one has a description of the dynamics on the $2 n$-dimensional space of the $(p, q)$ variables which is called the phase space of the dynamical system. With this construction the phase space is the cotangent bundle $\mathcal{M} \otimes \Lambda^{1} \mathcal{M}$ of the configuration manifold $\mathcal{M}$. The equations of motion can be represented through the time evolution of the phase space coordinates by Hamilton's equations. For most elementary dynamical systems, this description is sufficient. However, there are very few examples of mechanical systems whose equations of motion can be solved by quadratures and it is desirable to seek other more general formulations of this elementary situation in the hopes of being able to formulate rigorous theorems about when a classical mechanical system has solvable equations of motion, or is 'integrable'. Furthermore, the above notion of a 'phase space' is very local and is strictly speaking only globally valid when the phase space is $\mathbb{R}^{2 n}$, a rather restrictive class of systems. Motivated by the search for more non-trivial integrable models in both classical and quantum physics, theoretical physicists have turned to the general theory of symplectic geometry which encompasses the above local description in a coordinate-free way suitable to the
methods of modern differential geometry. In this section we shall review the basic ideas of symplectic geometry and how these descriptions tie in with the more familiar ones of elementary classical mechanics.

A symplectic manifold is a differentiable manifold $\mathcal{M}$ of even dimension $2 n$ together with a globally-defined non-degenerate closed 2-form

$$
\begin{equation*}
\omega=\frac{1}{2} \omega_{\mu \nu}(x) d x^{\mu} \wedge d x^{\nu} \tag{3.1}
\end{equation*}
$$

called the symplectic form of $\mathcal{M}$. By closed we mean as usual that

$$
\begin{equation*}
d \omega=0 \tag{3.2}
\end{equation*}
$$

or in local coordinates

$$
\begin{equation*}
\partial_{\mu} \omega_{\nu \lambda}+\partial_{\nu} \omega_{\lambda \mu}+\partial_{\lambda} \omega_{\mu \nu}=0 \tag{3.3}
\end{equation*}
$$

Thus $\omega$ defines a DeRham cohomology class in $H^{2}(\mathcal{M} ; \mathbb{R})$. By non-degenerate we mean that the components $\omega_{\mu \nu}(x)$ of $\omega$ define an invertible $2 n \times 2 n$ antisymmetric matrix globally on the manifold $\mathcal{M}$, i.e.

$$
\begin{equation*}
\operatorname{det} \omega(x) \neq 0 \quad \forall x \in \mathcal{M} \tag{3.4}
\end{equation*}
$$

Thus when considered as a map on $T \mathcal{M} \rightarrow T^{*} \mathcal{M}$, the symplectic 2 -form defines an isomorphism of the tangent and cotangent bundles of $\mathcal{M}$. The manifold $\mathcal{M}$ together with its symplectic form $\omega$ defines the phase space of a dynamical system, as we shall see below.

Since $\omega$ is closed, it follows from the Poincare lemma that locally there exists a 1-form

$$
\begin{equation*}
\theta=\theta_{\mu}(x) d x^{\mu} \tag{3.5}
\end{equation*}
$$

such that

$$
\begin{equation*}
\omega=d \theta \tag{3.6}
\end{equation*}
$$

or in local coordinates

$$
\begin{equation*}
\omega_{\mu \nu}=\partial_{\mu} \theta_{\nu}-\partial_{\nu} \theta_{\mu} \tag{3.7}
\end{equation*}
$$

The locally-defined 1-form $\theta$ is called the symplectic potential or canonical 1-form of $\mathcal{M}$. Diffeomorphisms of $\mathcal{M}$ that leave the symplectic 2 -form invariant are called canonical or symplectic transformations. These are determined by $C^{\infty}$-maps that act on the symplectic potential as

$$
\begin{equation*}
\theta \xrightarrow{F} \theta_{F}=\theta+d F \tag{3.8}
\end{equation*}
$$

or in local coordinates

$$
\begin{equation*}
\theta_{\mu}(x) \xrightarrow{F} \theta_{F, \mu}(x)=\theta_{\mu}(x)+\partial_{\mu} F(x) \tag{3.9}
\end{equation*}
$$

so that by nilpotency of the exterior derivative it follows that $\omega$ is invariant under such transformations,

$$
\begin{equation*}
\omega=d \theta \xrightarrow{F} \omega_{F}=d \theta_{F} \equiv \omega \tag{3.10}
\end{equation*}
$$

The function $F(x)$ is called the generating function of the canonical transformation.
The symplectic 2-form defines a bilinear function $\{\cdot, \cdot\}_{\omega}: \Lambda^{0} \mathcal{M} \otimes \Lambda^{0} \mathcal{M} \rightarrow \Lambda^{0} \mathcal{M}$ called the Poisson bracket. It is defined by

$$
\begin{equation*}
\{f, g\}_{\omega}=\omega^{-1}(d f, d g) \quad, \quad f, g \in \Lambda^{0} \mathcal{M} \tag{3.11}
\end{equation*}
$$

or in local coordinates

$$
\begin{equation*}
\{f, g\}_{\omega}=\omega^{\mu \nu}(x) \partial_{\mu} f(x) \partial_{\nu} g(x) \tag{3.12}
\end{equation*}
$$

where $\omega^{\mu \nu}$ is the matrix inverse of $\omega_{\mu \nu}$. Note that the local coordinate functions themselves have Poisson bracket

$$
\begin{equation*}
\left\{x^{\mu}, x^{\nu}\right\}_{\omega}=\omega^{\mu \nu}(x) \tag{3.13}
\end{equation*}
$$

The Poisson bracket is anti-symmetric,

$$
\begin{equation*}
\{f, g\}_{\omega}=-\{g, f\}_{\omega} \tag{3.14}
\end{equation*}
$$

it obeys the Leibniz property

$$
\begin{equation*}
\{f, g h\}_{\omega}=g\{f, h\}_{\omega}+h\{f, g\}_{\omega} \tag{3.15}
\end{equation*}
$$

and it satisfies the Jacobi identity

$$
\begin{equation*}
\left\{f,\{g, h\}_{\omega}\right\}_{\omega}+\left\{g,\{h, f\}_{\omega}\right\}_{\omega}+\left\{h,\{f, g\}_{\omega}\right\}_{\omega}=0 \tag{3.16}
\end{equation*}
$$

This latter property follows from the fact (3.3) that $\omega$ is closed. These 3 properties of the Poisson bracket mean that it defines a Lie bracket. Thus the Poisson bracket makes the space of $C^{\infty}$-functions on $\mathcal{M}$ into a Lie algebra which we call the Poisson algebra of $(\mathcal{M}, \omega)$.

The connection with the elementary formulation of classical mechanics discussed above is given by a result known as Darboux's theorem [55], which states that this connection is always possible locally. More precisely, Darboux's theorem states that locally there exists a system of coordinates $\left(p_{\mu}, q^{\mu}\right)_{\mu=1}^{n}$ on $\mathcal{M}$ in which the symplectic 2-form looks like

$$
\begin{equation*}
\omega=d p_{\mu} \wedge d q^{\mu} \tag{3.17}
\end{equation*}
$$

so that they have Poisson brackets

$$
\begin{equation*}
\left\{p_{\mu}, p_{\nu}\right\}_{\omega}=\left\{q^{\mu}, q^{\nu}\right\}_{\omega}=0 \quad, \quad\left\{p_{\mu}, q^{\nu}\right\}_{\omega}=\delta_{\mu}^{\nu} \tag{3.18}
\end{equation*}
$$

These coordinates are called canonical or Darboux coordinates on $\mathcal{M}$ and from (3.18) we see that they can be identified with the usual canonical momentum and position variables on the phase space $\mathcal{M}$ [48]. In these coordinates the symplectic potential is

$$
\begin{equation*}
\theta=p_{\mu} d q^{\mu} \tag{3.19}
\end{equation*}
$$

and the transformation (3.8) becomes

$$
\begin{equation*}
\theta=p_{\mu} d q^{\mu} \xrightarrow{F} \theta+d F=\theta_{F}=P_{\mu} d Q^{\mu} \tag{3.20}
\end{equation*}
$$

where $\left(P_{\mu}, Q^{\mu}\right)_{\mu=1}^{n}$ are also canonical coordinates according to (3.10). It follows that

$$
\begin{equation*}
p_{\mu} d q^{\mu}-P_{\mu} d Q^{\mu}=d F \tag{3.21}
\end{equation*}
$$

where both $\left(p_{\mu}, q^{\mu}\right)$ and $\left(P_{\mu}, Q^{\mu}\right)$ are canonical momentum and position variables on $\mathcal{M}$. (3.21) is the usual form of a canonical transformation determined by the generating function $F$ [48].

Smooth real-valued functions $H$ on $\mathcal{M}$ (i.e. elements of $\Lambda^{0} \mathcal{M}$ ) will be called classical observables. Exterior products of $\omega$ with itself determine non-trivial closed $2 k$-forms on $\mathcal{M}$. In particular, the $2 n$-form

$$
\begin{equation*}
d \mu_{L}=\omega^{n} / n!=\sqrt{\operatorname{det} \omega(x)} d^{2 n} x \tag{3.22}
\end{equation*}
$$

defines a natural volume element on $\mathcal{M}$ which is invariant under canonical transformations. It is called the Liouville measure, and in the local Darboux coordinates (3.17) it becomes the familiar phase space measure [48]

$$
\begin{equation*}
(-1)^{n(n-1) / 2} \omega^{n} / n!=d p_{1} \wedge \cdots \wedge d p_{n} \wedge d q^{1} \wedge \cdots \wedge d q^{n} \tag{3.23}
\end{equation*}
$$

### 3.2 Equivariant Cohomology on Symplectic Manifolds

In this section we shall specialize the discussion of chapter 2 to the case where the differentiable manifold $\mathcal{M}$ is a symplectic manifold of dimension $2 n$. Consider the action of some connected Lie group $G$ on $\mathcal{M}$ generated by the vector fields $V^{a}$ with the commutator algebra (2.43). We assume that the action of $G$ on $\mathcal{M}$ is symplectic so that it preserves the symplectic structure,

$$
\begin{equation*}
\mathcal{L}_{V^{a} \omega}=0 \tag{3.24}
\end{equation*}
$$

or in other words $G$ acts on $\mathcal{M}$ by symplectic transformations. Since $\omega$ is closed this means that

$$
\begin{equation*}
d i_{V} a \omega=0 \tag{3.25}
\end{equation*}
$$

Let $L \rightarrow \mathcal{M}$ be a complex line bundle with connection 1-form the symplectic potential $\theta$. If $\theta$ also satisfies

$$
\begin{equation*}
\mathcal{L}_{V^{a}} \theta=0 \tag{3.26}
\end{equation*}
$$

then the associated covariant derivative $\nabla=d+\theta$ is $G$-invariant, and according to the general discussion of section 2.3 this defines a $G$-equivariant bundle.

The associated moment map $H: \mathcal{M} \rightarrow \mathbf{g}^{*}$ evaluated on a Lie algebra element $X \in \mathbf{g}$ with associated vector field $V$ is called the Hamiltonian corresponding to $V$,

$$
\begin{equation*}
H_{V}=\mathcal{L}_{V}-\left[i_{V}, \nabla\right]=i_{V} \theta=V^{\mu} \theta_{\mu} \tag{3.27}
\end{equation*}
$$

From (3.6) and (3.26) it then follows that

$$
\begin{equation*}
d H_{V}=-i_{V} \omega \tag{3.28}
\end{equation*}
$$

or equivalently this follows from the general property (2.67) of the moment map since $\omega$ is the curvature of the connection $\theta$. In local coordinates, this last equation reads

$$
\begin{equation*}
\partial_{\mu} H_{V}(x)=V^{\nu}(x) \omega_{\mu \nu}(x) \tag{3.29}
\end{equation*}
$$

In particular, the components $H^{a}$ of the moment map

$$
\begin{equation*}
H=\phi^{a} \otimes H^{a} \tag{3.30}
\end{equation*}
$$

satisfy

$$
\begin{equation*}
d H^{a}=-i_{V a} \omega \tag{3.31}
\end{equation*}
$$

Comparing with the symplecticity condition (3.25) on the group action, we see that this is equivalent to the statement that the closed 1-forms $i_{V^{a} \omega}$ are exact. If $H^{1}(\mathcal{M} ; \mathbb{R})=0$ this is certainly true, but in the following we will want to consider multiply connected phase spaces as well. We therefore impose this exactness requirement from the onset on the action of $G$ on $\mathcal{M}$, or alternatively the equivariance requirement (3.26) on the symplectic potential $\theta$. When such a Hamiltonian function exists as a globally-defined $C^{\infty}$-map on $\mathcal{M}$, we shall say that the group action is Hamiltonian. A vector field $V$ which satisfies (3.28) is said to be the Hamiltonian vector field associated with $H_{V}$, and we shall call the triple $\left(\mathcal{M}, \omega, H_{V}\right)$, i.e. a symplectic manifold with a Hamiltonian $G$-action on it, a Hamiltonian system or a dynamical system.

The integral curves (2.42) defined by the flows (or time-evolution) of a Hamiltonian vector field $V$ as in (3.29) define the Hamilton equations of motion

$$
\begin{equation*}
\dot{x}^{\mu}(t)=\omega^{\mu \nu}(x(t)) \partial_{\nu} H_{V}(x(t))=\left\{x^{\mu}, H_{V}\right\}_{\omega} \tag{3.32}
\end{equation*}
$$

In particular, in the canonical coordinates defined by (3.17) the equations (3.32) read

$$
\begin{equation*}
\dot{q}^{\mu}=\frac{\partial H}{\partial p_{\mu}} \quad, \quad \dot{p}_{\mu}=-\frac{\partial H}{\partial q^{\mu}} \tag{3.33}
\end{equation*}
$$

which are the usual form of the Hamilton equations of motion encountered in elementary classical mechanics [48]. Thus we see that the above formalisms for symplectic geometry encompass all of the usual ideas of classical Hamiltonian mechanics in a general, coordinate-independent setting.

The equivariant curvature of the above defined equivariant bundle is given by the equivariant extension of the symplectic 2 -form,

$$
\begin{equation*}
\omega_{\mathrm{g}}=1 \otimes \omega+\phi^{a} \otimes H^{a} \tag{3.34}
\end{equation*}
$$

and evaluated on $X \in \mathrm{~g}$ we have

$$
\begin{equation*}
\left(D_{\mathbf{g}} \omega_{\mathbf{g}}\right)(X)=\left(d-i_{V}\right)\left(\omega+H_{V}\right)=0 \tag{3.35}
\end{equation*}
$$

which is equivalent to the definition (3.28) of the Hamiltonian vector field $V$. In fact, the extension (3.34) is the unique equivariant extension of the symplectic 2-form $\omega$ [103], i.e. the unique extension of $\omega$ from a closed 2-form to an equivariantly-closed one. Thus, we see that finding an equivariantly-closed extension of $\omega$ is equivalent to finding a moment map for the $G$-action. If $\omega$ defines actually an integer cohomology class $[\omega] \in H^{2}(\mathcal{M} ; \mathbb{Z})$, then the line bundle $L \rightarrow \mathcal{M}$ introduced above can be thought of as the prequantum line bundle of geometric quantization [136], the natural geometric framework (in terms of symplectic geometry) for the coordinate independent formulation of quantum mechanics. Within this framework, the equivariant curvature 2-form $\omega_{V}=\omega_{\mathrm{g}}(X)$ above is refered
to as the prequantum operator. We shall say more about some of the general ideas of geometric quantization later on.

From (3.31) it follows that the Poisson algebra of the Hamiltonians $H^{a}$ is given by

$$
\begin{equation*}
\left\{H^{a}, H^{b}\right\}_{\omega}=\omega\left(V^{a}, V^{b}\right)=\omega_{\mu \nu} V^{a, \mu} V^{b, \nu}=V^{a, \mu} \partial_{\mu} H^{b}=\mathcal{L}_{V^{a}} H^{b}=-\mathcal{L}_{V^{b}} H^{a} \tag{3.36}
\end{equation*}
$$

From the Jacobi identity (3.16) it follows that the map $H^{a} \rightarrow V^{a}$ is a homomorphism of the Lie algebras $\left(\Lambda^{0} \mathcal{M},\{\cdot, \cdot\}_{\omega}\right) \rightarrow(T \mathcal{M},[\cdot, \cdot])$ since

$$
\begin{equation*}
V^{\left\{H^{a}, H^{b}\right\}_{\omega}}=\left[V^{a}, V^{b}\right] \tag{3.37}
\end{equation*}
$$

However, the inverse of this map does not necessarily define a homomorphism. The Hamiltonian function which corresponds to the commutator of 2 group generators may differ from the Poisson bracket of the pertinent Hamiltonian functions as

$$
\begin{equation*}
\left\{H^{a}, H^{b}\right\}_{\omega}=f^{a b c} H^{c}+c^{a b} \tag{3.38}
\end{equation*}
$$

where $c^{a b} \equiv c\left(X^{a}, X^{b}\right)$ is a 2-cocycle in the Lie algebra cohomology of $G[64]$, i.e.

$$
\begin{equation*}
c\left(\left[X_{1}, X_{2}\right], X_{3}\right)+c\left(\left[X_{2}, X_{3}\right], X_{1}\right)+c\left(\left[X_{3}, X_{1}\right], X_{2}\right)=0 \quad \forall X_{1}, X_{2}, X_{3} \in \mathrm{~g} \tag{3.39}
\end{equation*}
$$

If $H^{2}(G)=0$ then we can set $c^{a b}=0$ and the map $X^{a} \rightarrow H^{a}$ determines a homomorphism between the Lie algebra $\mathbf{g}$ and the Poisson algebra of $C^{\infty}$-functions on $\mathcal{M}$.

The appearence of the 2-cocycle $c^{a b}$ in (3.38) is in fact related to the possible noninvariance of the symplectic potential under $G$ (c.f. eq. (3.26)). From the symplecticity (3.24) of the group action and (3.31) it follows that

$$
\begin{equation*}
\mathcal{L}_{V^{a}} \theta=\left(i_{V^{a}} d+d i_{V^{a}}\right) \theta=d g^{a} \tag{3.40}
\end{equation*}
$$

locally in a neighbourhood $\mathcal{N}$ in $\mathcal{M}$ wherein $\omega=d \theta$ and $V^{a} \neq 0$. Here the locally-defined linear functions $g^{a} \equiv g\left(X^{a}\right)=-H^{a}+i_{V^{a}} \theta$ obey the consistency condition

$$
\begin{equation*}
\left\{H\left(X_{1}\right), g\left(X_{2}\right)\right\}_{\omega}-\left\{H\left(X_{2}\right), g\left(X_{1}\right)\right\}_{\omega}=g\left(\left[X_{1}, X_{2}\right]\right) \quad \forall X_{1}, X_{2} \in \mathbf{g} \tag{3.41}
\end{equation*}
$$

However, if there exists a locally-defined function $f$ such that

$$
\begin{equation*}
g^{a}=\left\{H^{a}, f\right\}_{\omega} \quad, \quad a=1, \ldots, \operatorname{dim} G \tag{3.42}
\end{equation*}
$$

then we can remove the functions $g^{a}$ by the canonical transformation $\theta \rightarrow \theta_{f}=\theta+d f$ so that the symplectic potential $\theta_{f}$ is $G$-invariant. Indeed, the 1 -form $\theta_{f}$ obeys

$$
\begin{equation*}
\mathcal{L}_{V^{a}} \theta_{f}=0 \tag{3.43}
\end{equation*}
$$

which implies that in the neighbourhood $\mathcal{N}$,

$$
\begin{equation*}
i_{V^{a}} \theta_{f}=H^{a}+C \tag{3.44}
\end{equation*}
$$

where $C$ is a constant. This constant is irrelevant here because we can introduce functions $K^{a}$ in $\mathcal{N}$ such that

$$
\begin{equation*}
\left\{H^{a}, K^{a}\right\}_{\omega}=V^{a, \mu} \partial_{\mu} K^{a}=1 \tag{3.45}
\end{equation*}
$$

and defining $F^{a}=f+C K^{a}$ we find

$$
\begin{equation*}
i_{V^{a}} \theta_{F^{a}}=H^{a} \tag{3.46}
\end{equation*}
$$

However, notice that the $G$-invariance (3.46) of the symplectic potential in general holds only locally in $\mathcal{M}$, and furthermore the canonical transformation $\theta \rightarrow \theta_{f}$ above does not remove the functions $g^{a}$ for the entire Lie algebra $g$, but only for a closed subalgebra of $\mathbf{g}$ which depends on the function $f$ and on the phase space $\mathcal{M}$ where $G$ acts $[55,100]$. In this subspace, the symplectic potential is $G$-invariant and the identity (3.27) relating the Hamiltonians to the symplectic potential by $H^{a}=i_{V^{a}} \theta$ holds. In general though, on the entire Lie algebra $\mathbf{g}$, defining $h^{a}=-i_{V^{a}} d F^{a}$ in the above we have

$$
\begin{equation*}
i_{V^{a}} \theta=H^{a}+h^{a} \tag{3.47}
\end{equation*}
$$

and then the Poisson bracket (3.36) implies that the 2-cocycle appearing in (3.38) is given by

$$
\begin{equation*}
c^{a b}=f^{a b c} h^{c}-\mathcal{L}_{V^{a}} h^{b}+\mathcal{L}_{V^{b}} h^{a} \tag{3.48}
\end{equation*}
$$

It is only when $c^{a b}=0$ for all $a, b$ that the $G$-action of the vector fields $V^{a}$ lifts isomorphically to the Poisson action of the corresponding Hamiltonians $H^{a}$ on $\mathcal{M}$. Notice that this is certainly true on the Cartan subalgebra of the Lie algebra $g$ (i.e. the maximal commuting subalgebra of $\mathbf{g}$ ), since $H^{2}(U(1))=H^{2}\left(S^{1}\right)=0$. We shall see in chapter 4 that the dynamical systems for which the equivariance condition (3.27) holds determine a very special class of quantum theories.

### 3.3 Stationary-phase Approximation and the Duistermaat-Heckman Theorem

We now start examining localization formulas for a specific class of phase space integrals which can be thought of as finite-dimensional versions of the functional integrals that we consider later on. It is best to proceed first with a finite-dimensional analysis because there everything is well-defined and rigorous theorems can be formulated. In the infinite-dimensional case, although the techniques used will be standard methods of supersymmetry and topological field theory, a lot of rigor is lost due the ill-definedness of infinite-dimensional manifolds and functional integrals. A lot can therefore be learned by looking closely at some finite-dimensional cases. We shall concentrate for now on the case of an abelian circle action on the manifold $\mathcal{M}$, as we did in section 2.5. We shall also assume that the Hamiltonian $H$ defined as in the last Section is a Morse function. This means that the critical points $p$ of the Hamiltonian, defined by $d H(p)=0$, are isolated and the Hessian matrix of $H$,

$$
\begin{equation*}
\mathcal{H}(x)=\left[\frac{\partial^{2} H(x)}{\partial x^{\mu} \partial x^{\nu}}\right] \tag{3.49}
\end{equation*}
$$

at each critical point $p$ is a non-degenerate matrix, i.e.

$$
\begin{equation*}
\operatorname{det} \mathcal{H}(p) \neq 0 \tag{3.50}
\end{equation*}
$$

The Hamiltonian vector field $V$ is defined by (3.29) and it represents the action of some 1-parameter group on the phase space $\mathcal{M}$. We shall assume here that the orbits (2.42)
of $V$ generate the circle group $U(1) \sim S^{1}$. Later on we shall consider more general cases. Notice that the critical points of $H$ coincide with zero locus $\mathcal{M}_{V}$ of the vector field $V$.

There is an important quantity of physical interest for the statistical mechanics of a classical dynamical system called the partition function. It is constructed as follows. Each point $x$ of the phase space $\mathcal{M}$ represents a classical state of the dynamical system which in canonical coordinates is specified by its configuration $q$ and its momentum $p$. The energy of this state is determined by the Hamiltonian $H$ of the dynamical system which as usual is its energy function. According to the general principles of classical statistical mechanics [114] the partition function is built by attaching to each point $x \in \mathcal{M}$ the Boltzmann weight $e^{i T H(x)}$ and 'summing' them over all states of the system. Here the parameter $i T$ is 'physically' to be identified with $-\beta / k_{B}$ where $k_{B}$ is Boltzmann's constant and $\beta$ is the inverse temperature. However, for mathematical ease in the following, we shall assume that the parameter $T$ is real. In the canonical position and momentum coordinates we would just simply integrate up the Boltzmann weights. However, we would like to obtain a quantity which is invariant under transformations which preserve the (symplectic) volume of the phase space $\mathcal{M}$ (i.e. those which preserve the classical equations of motion (3.32) and hence the density of classical states), and so we integrate using the Liouville measure (3.22) to obtain a canonically invariant quantity. This defines the classical partition function of the dynamical system,

$$
\begin{equation*}
Z(T)=\int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{i T H}=\int_{\mathcal{M}} d^{2 n} x \sqrt{\operatorname{det} \omega(x)} \mathrm{e}^{i T H(x)} \tag{3.51}
\end{equation*}
$$

The partition function determines all the usual thermodynamic quantities of the dynamical system [114], such as its free energies and specific heats, as well as all statistical averages in the canonical ensemble of the classical system.

However, it is very seldom that one can actually obtain an exact closed form for the partition function (3.51) as the integrals involved are usually rather complicated. But there is a method of approximating the integral (3.51), which is very familiar to physicists and mathematicians, called the stationary-phase approximation [55, 61, 136].

This method is often employed when one encounters oscillatory integrals such as (3.51) to obtain an idea of its behaviour, at least for large $T$. It works as follows. Notice that for $T \rightarrow \infty$ the integrand of $Z(T)$ oscillates very rapidly and begins to damp to 0 . The integral therefore has a large- $T$ expansion in powers of $1 / T$. The larger $T$ gets the more the integrand tends to localize around its stationary values wherever the function $H(x)$ has extrema (equivalently where $d H(p)=0)^{1}$. To evaluate these contributions, we expand both $H$ and the Liouville density in (3.51) in a neighbourhood $U_{p}$ about each critical point $p \in \mathcal{M}_{V}$ in a Taylor series, where as usual integration in $U_{p}$ can be thought of as integration in the more familiar $\mathbb{R}^{2 n}$. We expand the exponential of all derivative terms in $H$ of order higher than 2 in the exponential power series, and in this way we are left with an infinite series of Gaussian moment integrals with Gaussian weight determined by the bilinear form defined by the Hessian matrix (3.49) of $H$ at $p$. The lowest order contribution is just the normalization of the Gaussian, while the $k$-th order moments are down by powers of $1 / T^{k}$ compared to the leading term. Carrying out these Gaussian integrations, taking into careful account the signature of the Hessian at each point, and summing over all points $p \in \mathcal{M}_{V}$, in this way we obtain the standard lowest-order stationary-phase approximation to the integral (3.51),

$$
\begin{equation*}
Z(T)=\left(\frac{2 \pi i}{T}\right)^{n} \sum_{p \in \mathcal{M}_{V}}(-i)^{\lambda(p)} e^{i T H(p)} \sqrt{\frac{\operatorname{det} \omega(p)}{\operatorname{det} \mathcal{H}(p)}}+\mathcal{O}\left(1 / T^{n+1}\right) \tag{3.52}
\end{equation*}
$$

where $\lambda(p)$ is the Morse index of the critical point $p$, defined as the number of negative eigenvalues in a diagonalization of the symmetric Hessian matrix of $H$ at $p$. We shall always ignore a possible regular function of $T$ in the large- $T$ expansion (3.52). The higher-order terms in (3.52) are found from the higher-moment Gaussian integrals [106] and they will be discussed in chapter 7. For now, we concern ourselves only with the lowest-order term in the stationary-phase series of (3.51).

[^8]The field of equivariant localization theory was essentially born in 1982 when Duistermaat and Heckman [33] found a general class of Hamiltonian systems for which the leading-order of the stationary-phase approximation gives the exact result for the partition function (3.51) (i.e. for which the $\mathcal{O}\left(1 / T^{n+1}\right)$ correction terms in (3.52) all vanish). Roughly speaking, the Duistermaat-Heckman theorem goes as follows. Let $\mathcal{M}$ be a compact symplectic manifold. Suppose that the vector field $V$ defined by (3.29) generates the global Hamiltonian action of a torus group $\mathcal{T}=\left(S^{1}\right)^{m}$ on $\mathcal{M}$ (where we shall usually assume that $m=1$ for simplicity). Since the critical point set of the Hamiltonian $H$ coincides with the fixed-point set $\mathcal{M}_{V}$ of the $\mathcal{T}$-action on $\mathcal{M}$ we can apply the equivariant Darboux theorem to the Hamiltonian system at hand [55]. This generalization of Darboux's theorem tells us that not only can we find a local canonical system of coordinates in a neighbourhood of each critical point in which the symplectic 2-form looks like (3.17), but these coordinates can further be chosen so that the origin $p_{\mu}=q^{\mu}=0$ of the coordinate neighbourhood represents the fixed point $p$ of the given compact group action on $\mathcal{M}$. This means that in these canonical coordinates the torus action is (locally) linear and has the form [33]

$$
\begin{equation*}
V=\sum_{\mu=1}^{n} \frac{\lambda_{\mu}(p)}{i}\left(p_{\mu} \frac{\partial}{\partial q^{\mu}}-q^{\mu} \frac{\partial}{\partial p_{\mu}}\right) \quad, \quad p \in \mathcal{M}_{V} \tag{3.53}
\end{equation*}
$$

where $\lambda_{\mu}(p)$ are weights that will be specified shortly. From the Hamilton equations (3.29) it follows that the Hamiltonian near each critical point $p$ can be written in the quadratic form

$$
\begin{equation*}
H(x)=H(p)+\sum_{\mu=1}^{n} \frac{i \lambda_{\mu}(p)}{2}\left(p_{\mu}^{2}+q_{\mu}^{2}\right) \tag{3.54}
\end{equation*}
$$

In these coordinates the flows determined by the Hamilton equations of motion (3.33) are the circles $p_{\mu}(t), q^{\mu}(t) \sim \mathrm{e}^{i \lambda_{\mu} t}$ about the critical points, which gives an explicit representation of the Hamiltonian $\mathcal{T}$-action locally on $\mathcal{M}$ and the group action preserves the Darboux coordinate neighbourhood. Thus each neighbourhood integration above is purely Gaussian and so all higher-order terms in the stationary-phase evaluation of
(3.51) vanish and the partition function is given exactly by the leading term in (3.52) of its stationary-phase series. This theorem therefore has the potential of supplying a large class of dynamical systems whose partition function (and hence all thermodynamic and statistical observables) can be evaluated exactly.

Atiyah and Bott [8] pointed out that the basic principle underlying the DuistermaatHeckman theorem is not that of stationary-phase, but rather of the more general localization properties of equivariant cohomology that we discussed in the last chapter. Suppose that the Hamiltonian vector field $V$ generates a global, symplectic circle action on the phase space $\mathcal{M}$. Suppose further that $\mathcal{M}$ admits a globally defined Riemannian structure for which $V$ is Killing vector, as in section 2.4. Recall from the last section that the symplecticity of the circle action implies that $\omega+H$ is the equivariant extension of the symplectic 2 -form $\omega$, i.e. $D_{V}(\omega+H)=0$. Since integration over the $2 n$-dimensional manifold $\mathcal{M}$ picks up the $2 n$-degree component of any differential form, it follows that the partition function (3.51) can be written as

$$
\begin{equation*}
Z(T)=\int_{\mathcal{M}} \alpha \tag{3.55}
\end{equation*}
$$

where $\alpha$ is the inhomogeneous differential form

$$
\begin{equation*}
\alpha=\frac{1}{(i T)^{n}} \mathrm{e}^{i T(H+\omega)}=\frac{1}{(i T)^{n}} \mathrm{e}^{i T H} \sum_{k=0}^{n} \frac{(i T)^{k}}{k!} \omega^{k} \tag{3.56}
\end{equation*}
$$

whose $2 k$-form component is $\alpha^{(2 k)}=\mathrm{e}^{i T H} \omega^{k} /(i T)^{n-k} k$ !. Since $H+\omega$ is equivariantly closed, it follows that $D_{V} \alpha=0$. Thus we can apply the Berline-Vergne localization formula (2.122) to the integral (3.55) to get

$$
\begin{equation*}
Z(T)=\left(\frac{2 \pi i}{T}\right)^{n} \sum_{p \in \mathcal{M}_{V}} \frac{\mathrm{e}^{i T H(p)}}{\text { Pfaff } d V(p)} \tag{3.57}
\end{equation*}
$$

In the case at hand the denominator of (3.57) at a critical point $p$ is found from the Hamilton equations (3.29) which give

$$
\begin{equation*}
d V(p)=\omega^{-1}(p) \mathcal{H}(p) \tag{3.58}
\end{equation*}
$$

and so we see how the determinant factors appear in the formula (3.52). However, we have to remember that the Pfaffian also encodes a specific choice of sign when taking the square root determinant. The sign of the Pfaffian Pfaff $d V(p)$ can be determined by examining it in the equivariant Darboux coordinates above in which the matrix $\omega(p)$ is skew-diagonal with skew-eigenvalues 1 and the Hessian $\mathcal{H}(p)$ which comes from (3.54) is diagonal with eigenvalues $i \lambda_{\mu}(p)$ each of multiplicity 2 . It follows that in these coordinates the matrix $d V(p)$ is skew-diagonal with skew-eigenvalues $i \lambda_{\mu}(p)$. Introducing the etainvariant $\eta(\mathcal{H}(p))$ of $\mathcal{H}(p)$, defined as the difference between the number of positive and negative eigenvalues of the Hessian of $H$ at $p$, i.e. its spectral asymmetry, we find

$$
\begin{equation*}
\eta(\mathcal{H}(p))=2 \sum_{\mu=1}^{n} \operatorname{sgn} i \lambda_{\mu}(p) \tag{3.59}
\end{equation*}
$$

which is related to the Morse index of $H$ at $p$ by

$$
\begin{equation*}
\eta(\mathcal{H}(p))=2 n-2 \lambda(p) \tag{3.60}
\end{equation*}
$$

Using the identity $\pm 1=e^{i \frac{\pi}{2}( \pm 1-1)}$ it follows that

$$
\begin{equation*}
\operatorname{sgn} \text { Pfaff } d V(p)=\prod_{\mu=1}^{n} \operatorname{sgn} i \lambda_{\mu}(p)=\mathrm{e}^{i \frac{\pi}{2}\left(\frac{1}{2} \eta(\mathcal{H}(p))-n\right)}=\mathrm{e}^{-i \frac{\pi}{2} \lambda(p)}=(-i)^{\lambda(p)} \tag{3.61}
\end{equation*}
$$

and so substituting (3.58) and (3.61) into (3.57) we arrive finally at the DuistermaatHeckman integration formula

$$
\begin{equation*}
Z(T)=\left(\frac{2 \pi i}{T}\right)^{n} \sum_{p \in \mathcal{M}_{V}}(-i)^{\lambda(p)} \mathrm{e}^{i T H(p)} \sqrt{\frac{\operatorname{det} \omega(p)}{\operatorname{det} \mathcal{H}(p)}} \tag{3.62}
\end{equation*}
$$

Recall from section 2.5 that $d V(p)$ is associated with the anti-self-adjoint linear operator $L_{V}(p)$ which generates the infinitesimal circle (or torus) action on the tangent space $T_{p} \mathcal{M}$. From the above it then follows that the complex numbers $\lambda_{\mu}(p)$ introduced in (3.53) are just the weights (i.e. eigenvalues of the Cartan generators) of the complex linear representation of the circle (or torus) action in the tangent space at $p$ and the determinant factors from (3.57) appear in terms of them as the products

$$
\begin{equation*}
e(p)=(-1)^{\lambda(p) / 2} \prod_{\mu=1}^{n} \lambda_{\mu}(p) \tag{3.63}
\end{equation*}
$$

as if each unstable mode contributes a factor of $i$ to the integral for $Z(T)$ above. Given the remarkable cohomological derivation of the Duistermaat-Heckman formula above which followed from the quite general principles of equivariant cohomology of the last chapter, one could hope to develop more general types of localization formulas from these general principles in the hopes of being able to generate more general types of integration formulas for the classical partition function. Moreover, given the localization criteria of the last chapter this has the possibility of expanding the set of dynamical systems whose partition functions are exactly solvable. We stress again that the crucial step in this cohomological derivation is the assumption that the Hamiltonian flows of the dynamical system globally generate isometries of a metric $g$ on $\mathcal{M}$, i.e. the Hamiltonian vector field $V$ is a global Killing vector of $g$. This condition and a classification of the dynamical systems for which these localization constraints do hold true will be one of the main topics of this thesis. The extensions of the Duistermaat-Heckman localization formula using the geometric conditions above will be the focus of the remainder of this chapter, as will be the various applications of the formalism of equivariant cohomology for dynamical systems.

### 3.4 Morse Theory and Kirwan's Theorem

There is a very interesting and useful connection between the Duistermaat-Heckman theorem and the Morse theory determined by the non-degenerate Hamiltonian $H$. Morse theory relates the structure of the critical points of a Morse function $H$ to the topology of the manifold $\mathcal{M}$ on which it is defined. We very briefly now review some of the basic ideas in Morse theory (see [92] for a comprehensive introduction). Given a Morse function $H$ as above, we define its Morse series

$$
\begin{equation*}
M_{H}(t)=\sum_{p \in \mathcal{M}_{V}} t^{\lambda(p)} \tag{3.64}
\end{equation*}
$$

which is a finite sum because the non-degeneracy of $H$ implies that its critical points are all discrete and the compactness of $\mathcal{M}$ implies that the critical point set $\mathcal{M}_{V}$ is finite. The topology of the manifold $\mathcal{M}$ now enters the problem through the Poincaré series of $\mathcal{M}$, which is defined by

$$
\begin{equation*}
P_{\mathcal{M}}(t ; \mathbb{F})=\sum_{k=0}^{2 n} \operatorname{dim}_{\mathbb{F}} H^{k}(\mathcal{M} ; \mathbb{F}) t^{k} \tag{3.65}
\end{equation*}
$$

where $\mathbb{F}$ is some algebraic field (usually $\mathbb{R}$ or $\mathbb{C}$ ). The fundamental result of Morse theory is the inequality

$$
\begin{equation*}
M_{H}(t) \geq P_{\mathcal{M}}(t ; \mathbb{F}) \tag{3.66}
\end{equation*}
$$

for all fields $\mathbb{F}$. If equality holds in (3.66) for all fields $\mathbb{F}$, then we say that $H$ is a perfect Morse function. The inequality (3.66) leads to many different relations between the critical points of $H$ and the topology of $\mathcal{M}$. These are called the Morse inequalities, and the only feature of them that we shall really need in the following is the fact that the number of critical points of $H$ of a given Morse index $k \geq 0$ is always at least the number $\operatorname{dim}_{\mathbb{R}} H^{k}(\mathcal{M} ; \mathbb{R})$. This puts a severe restriction on the types of non-degenerate functions that can exist as $C^{\infty}$-maps on a manifold of a given topology.

Another interesting relation is obtained when we set $t=-1$ in the Morse and Poincaré series. In the former series we get

$$
\begin{equation*}
M_{H}(-1)=\sum_{p \in \mathcal{M}_{V}} \operatorname{sgn} \operatorname{det} \mathcal{H}(p) \tag{3.67}
\end{equation*}
$$

while (2.85) shows that in the latter series the result is the Euler characteristic $\chi(\mathcal{M})$ of $\mathcal{M}$. That these 2 quantities are equal is known as the Poincaré-Hopf theorem, and employing further the Gauss-Bonnet-Chern theorem (2.84) we find

$$
\begin{equation*}
\sum_{p \in \mathcal{M}_{V}} \operatorname{sgn} \operatorname{det} \mathcal{H}(p)=\frac{(-1)^{n}}{(4 \pi)^{n} n!} \int_{\mathcal{M}} E(R) \tag{3.68}
\end{equation*}
$$

with $E(R)$ the Euler class constructed from a Riemann curvature 2-form $R$ on $\mathcal{M}$. This relation gives a very interesting connection between the structure of the critical point
set of a non-degenerate function and the topology and geometry of the phase space $\mathcal{M}$. We remark that one can also define equivariant versions of the Morse and Poincaré series using the topological definition of equivariant cohomology [92] which is suitable to the equivariant cohomological ideas that we formulated earlier on. These equivariant generalizations which localize topological integrals such as (3.68) onto the zero locus of a vector field is the basis of the Mathai-Quillen formalism and its application to the construction of topological field theories [22, 24, 29, 67, 82, 98].

In regards to the Duistermaat-Heckman theorem, there is a very interesting Morse theoretical result due to Kirwan [72]. Kirwan showed that the only Morse functions for which the stationary phase approximation can be exact are those which have only even Morse indices $\lambda(p)$. This theorem includes the cases where the Duistermaat-Heckman integration formula is exact, and under the assumptions of the Duistermaat-Heckman theorem it is a consequence of the circle action (see the previous section). However, this result is even stronger - it means that when one constructs the full stationaryphase series as described in the last section [106], if that series converges uniformly in $1 / T$ to the exact partition function $Z(T)$, then the Morse index of every critical point of $H$ must be even. From the Morse inequalities mentioned above this furthermore gives a relation between equivariant localization and the topology of the phase space of interest - if the manifold $\mathcal{M}$ has non-trivial cohomology groups of odd dimension, then the stationary phase series diverges for any Morse function defined on $\mathcal{M}$ and in particular the Duistermaat-Heckman localization formula for such phase spaces can never give the exact result for $Z(T)$. In this way, Kirwan's theorem rules out a large number of dynamical systems for which the stationary phase approximation could be exact in terms of the topology of the underlying phase space where the dynamical system lives. Moreover, an application of the Morse lacunary principle [92] shows that, when the stationary-phase approximation is exact so that $H$ has only even Morse indices, $H$ is in fact a perfect Morse function and consequently its Morse inequalities become equalities.

We shall not go into the rather straightforward proof of Kirwan's theorem here, but refer to [72] for the details. In the following we can therefore use Kirwan's theorem as an initial test using the topology of the phase space to determine which dynamical systems will localize in the sense of the Duistermaat-Heckman theorem. In chapter 7 we shall see the direct connection between the higher order terms in the saddle-point series for the partition function and Kirwan's theorem, and more generally the geometry and topology of the manifold $\mathcal{M}$.

### 3.5 The Height Function of a Riemann Surface

It is instructive at this stage to finally present some concrete examples of the equivariant localization formalism presented above. One of the most common examples in both Morse theory and localization theory is the dynamical system whose phase space is a compact Riemann surface $\Sigma^{g}$ of genus $g$ (i.e. a closed surface with $g$ 'handles') and whose Hamiltonian $h_{\Sigma^{g}}$ is the height function on $\Sigma^{g}$ [24, 70, 92, 106, 120]. For instance, consider the Riemann sphere $\Sigma^{0}=S^{2}$ of unit radius viewed in $\mathbb{R}^{3}$ as a sphere standing on end on the $x y$-plane and centered at $z=a$ symmetrically about the $z$-axis. We introduce the usual spherical polar coordinates $x=\sin \theta \cos \phi, y=\sin \theta \sin \phi$ and $z=a-\cos \theta$ for the embedding of the sphere in 3 -space as $S^{2}=\left\{(x, y, z) \in \mathbb{R}^{3}: x^{2}+y^{2}+(z-a)^{2}=1\right\}$, where $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2 \pi$. The symplectic 2-form is the usual volume form on $S^{2}$

$$
\begin{equation*}
\omega_{\Sigma^{0}}=d \cos \theta \wedge d \phi \tag{3.69}
\end{equation*}
$$

induced by the Euclidean metric $g_{\mu \nu}=\delta_{\mu \nu}$ of $\mathbb{R}^{3}$ from the embedding. of $S^{2}$ in 3dimensional space. We wish to consider the $U(1)$-action $\phi \rightarrow \phi+\phi_{0}, \phi_{0} \in[0,2 \pi)$, associated with rigid rotations of $S^{2}$. The vector field generating this compact group action on $S^{2}$ is $V=\frac{\partial}{\partial \phi}$, so that the corresponding moment map is

$$
\begin{equation*}
h_{\Sigma^{0}}(\theta, \phi)=a-\cos \theta \tag{3.70}
\end{equation*}
$$

which is just the height function $z$ restricted to the sphere. Notice that the associated circle action leaves fixed the north pole, at $\theta=\pi$, and the south pole, at $\theta=0$.

The partition function can be evaluated directly and it gives

$$
\begin{align*}
Z_{\Sigma^{0}}(T) & =\int_{\Sigma^{0}} \omega_{\Sigma^{0}} \mathrm{e}^{i T h_{\Sigma^{0}}} \\
& =2 \pi \int_{-1}^{+1} d \cos \theta \mathrm{e}^{i T(a-\cos \theta)}  \tag{3.71}\\
& =\frac{2 \pi i}{T}\left(\mathrm{e}^{-i T(1+a)}-\mathrm{e}^{i T(1-a)}\right)=\frac{4 \pi}{T} \mathrm{e}^{-i T a} \sin T
\end{align*}
$$

The last line in (3.71) shows that the result for the partition function is precisely that anticipated from the Duistermaat-Heckman theorem. It can be expressed as a sum of 2 terms which each correspond to the 2 isolated non-degenerate critical points of the Hamiltonian (3.70) - one from the north pole $\theta=\pi$, which is the maximum of (3.70), and the other from the south pole $\theta=0$, which is its minimum. The relative minus sign in the last line of (3.71) comes from the fact that the Morse index of the maximum $\theta=\pi$ is 2 while that of $\theta=0$ is 0 , i.e. the maximum of $h_{\Sigma^{\circ}}$ is unstable in 2 directions, each of which, heuristically, contributes a factor of $i$. Finally, the factor $2 \pi i / T$ comes from the one-loop determinants from expanding around $\theta=0, \pi$. Note also that the Poincaré series of the 2 -sphere is ${ }^{2}$

$$
\begin{equation*}
P_{S^{2}}(t ; \mathbb{F})=\sum_{k=0}^{2} \operatorname{dim}_{\mathbb{F}} H^{k}\left(S^{2} ; \mathbb{F}\right) t^{k}=1+t^{2} \tag{3.72}
\end{equation*}
$$

which coincides with the Morse series (3.64) for the height function $h_{\Sigma^{0}}$. Thus, consistent with Kirwan's theorem, we see that $h_{\Sigma^{\circ}}$ is a perfect Morse function with even Morse indices.

Notice that the Hamiltonian vector field $V=\frac{\partial}{\partial \phi}$ here generates an isometry of the standard round metric $d \theta \otimes d \theta+\sin ^{2} \theta d \phi \otimes d \phi$ induced by the flat Euclidean metric of $\mathbb{R}^{3}$. The differential form (2.103) with this metric is $\xi=d \phi$, which as expected is

[^9]ill-defined at the 2 poles of $S^{2}$. Now the partition function can be written as
\[

$$
\begin{equation*}
Z_{\Sigma^{0}}(T)=-\frac{1}{i T} \int_{\Sigma^{0}} d\left(\mathrm{e}^{i T h_{\Sigma^{0}}} d \phi\right) \tag{3.73}
\end{equation*}
$$

\]

thus receiving contributions from only the critical points $\theta=0, \pi$, the endpoints of the integration range for $\cos \theta$, in agreement with the explicit evaluation above.

As we shall see later on, the above example for the Riemann sphere is essentially the only Hamiltonian system to which the geometric equivariant localization constraints apply on a simply-connected phase space (i.e. $\left.H_{1}(\mathcal{M} ; \mathbb{Z})=0\right)$. The situation is much different on a multiply-connected phase space, which as we shall see is due to the fact that the non-trivial first homology group of the phase space severely restricts the allowed $U(1)$ group actions on it and hence the Morse functions thereon. For example, consider the case of a genus 1 Riemann surface $[70,106,120]$, i.e. $\Sigma^{1}$ is the 2 -torus $T^{2}=S^{1} \times S^{1}$. The torus can be viewed as a parallelogram in the complex plane with its opposite edges identified. We take as horizontal edge the line segment from 0 to 1 along the real axis and the other slanted edge the line segment from 0 to some complex number $\tau$ in the complex plane. The number $\tau$ is called the modular parameter of the torus and we can take it to lie in the upper complex half-plane

$$
\begin{equation*}
\mathbb{C}^{+}=\{z \in \mathbb{C}: \operatorname{Im} z>0\} \tag{3.74}
\end{equation*}
$$

Geometrically, $\tau$ determines the inner and outer radii of the 2 circles of the torus, and it labels the inequivalent complex structures of $\Sigma^{13}$.

We view the torus embedded in 3 -space as a doughnut standing on end on the $x y$ plane and centered symmetrically about the $z$-axis. If ( $\phi_{1}, \phi_{2}$ ) are the angle coordinates on $S^{1} \times S^{1}$, then the height function on $\Sigma^{1}$ can be written as

$$
\begin{equation*}
h_{\Sigma^{1}}\left(\phi_{1}, \phi_{2}\right)=r_{2}-\left(r_{1}+\operatorname{Im} \tau \cos \phi_{1}\right) \cos \phi_{2} \tag{3.75}
\end{equation*}
$$

[^10]where $r_{1}=|\operatorname{Re} \tau|+\operatorname{Im} \tau$ and $r_{2}=|\operatorname{Re} \tau|+2 \operatorname{Im} \tau$ label the inner and outer radii of the torus. The symplectic volume form on $T^{2}$ is just that induced by the identification of $\Sigma^{1}$ as a parallelogram in the plane with its opposite edges identified, i.e. the Darboux 2 -form
\[

$$
\begin{equation*}
\omega_{D}=d \phi_{1} \wedge d \phi_{2} \tag{3.76}
\end{equation*}
$$

\]

The associated Hamiltonian vector field for this dynamical system has components

$$
\begin{equation*}
V_{\Sigma^{1}}^{1}=-\left(r_{1}+\operatorname{Im} \tau \cos \phi_{1}\right) \sin \phi_{2} \quad, \quad V_{\Sigma^{1}}^{2}=\operatorname{Im} \tau \sin \phi_{1} \cos \phi_{2} \tag{3.77}
\end{equation*}
$$

The Hamiltonian (3.75) has 4 isolated non-degenerate critical points on $S^{1} \times S^{1}$ - a maximum at $\left(\phi_{1}, \phi_{2}\right)=(0, \pi)$ (top of the outer circle), a minimum at $(0,0)$ (bottom of the outer circle), and 2 saddle points at ( $\pi, 0$ ) and ( $\pi, \pi$ ) (corresponding to the bottom and top of the inner circle, respectively). The Morse index of the maximum is 2 , that of the minimum is 0 , and those of the 2 saddle points are both 1 . According to Kirwan's theorem, the appearence of odd Morse indices, or equivalently the fact that

$$
\begin{equation*}
H_{1}\left(\Sigma^{1} ; \mathbb{Z}\right)=\mathbb{Z} \oplus \mathbb{Z} \tag{3.78}
\end{equation*}
$$

with each $\mathbb{Z}$ labelling the windings around the 2 independent non-contractable loops associated with each $S^{1}$-factor, implies that the Duistermaat-Heckman integration formula should fail in this case. Indeed, evaluating the right-hand side of the DuistermatHeckman formula (3.62) gives

$$
\begin{equation*}
\frac{2 \pi i}{T} \sum_{p \in \mathcal{M} v_{V^{1}}} \frac{\mathrm{e}^{i T h_{\Sigma^{1}}(p)}}{e(p)}=\frac{2 \pi i}{T \sqrt{\operatorname{Im} \tau}}\left[r_{2}^{-1 / 2}\left(1+\mathrm{e}^{2 i T r_{2}}\right)+|\operatorname{Re} \tau|^{-1 / 2} \mathrm{e}^{2 i T \operatorname{Im} \tau}\left(1-\mathrm{e}^{2 i T|\operatorname{ReT}|}\right)\right] \tag{3.79}
\end{equation*}
$$

which for the parameter values $i T=1$ and $\tau=1+i$ gives the numerical value

$$
\begin{equation*}
2 \pi \mathrm{e}^{3}\left(\frac{2}{\sqrt{3}} \sinh 3+2 \cosh 1\right) \sim 1849.33 \tag{3.80}
\end{equation*}
$$

On the other hand, an explicit evaluation of the partition function gives

$$
\begin{equation*}
Z_{\Sigma^{1}}(T)=\int_{0}^{2 \pi} \int_{0}^{2 \pi} d \phi_{1} d \phi_{2} \mathrm{e}^{i T h_{\Sigma^{1}}\left(\phi_{1}, \phi_{2}\right)}=2 \pi \mathrm{e}^{r_{2}} \int_{0}^{2 \pi} d \phi_{1} J_{0}\left(i T\left(r_{1}+\operatorname{Im} \tau \cos \phi_{1}\right)\right) \tag{3.81}
\end{equation*}
$$

with $J_{0}$ the regular Bessel function of order 0 [50]. For the parameter values above, a numerical integration in (3.81) gives $Z_{\Sigma^{1}} \sim 2117.13^{4}$, contradicting the result (3.80). Thus even though in this case the Hamiltonian $h_{\Sigma^{1}}$ is a perfect Morse function, it doesn't generate any torus action on the phase space here.

This argument can also be extended to the case where the phase space is a hyperbolic Riemann surface $\Sigma^{g}, g>1$ [120]. For $g>1, \Sigma^{g}=\Sigma^{1} \# \cdots \# \Sigma^{1}$ is the $g$-fold connected sum of 2 -tori and therefore its first homology group is

$$
\begin{equation*}
H_{1}\left(\Sigma^{g} ; \mathbb{Z}\right)=\bigoplus_{i=1}^{2 g} \mathbb{Z} \tag{3.82}
\end{equation*}
$$

It can be viewed in $\mathbb{R}^{3}$ as $g$ doughnuts stuck together on end and standing on the $x y$ plane. The height function on $\Sigma^{g}$ now has $2 g+2$ critical points consisting of 1 maximum, 1 minimum and $2 g$ saddle points. Again the maximum and minimum have Morse indices 2 and 0 , respectively, while those of the $2 g$ saddle points are all 1 . As a consequence the perfect Morse function $h_{\Sigma^{g}}$ generates no torus action on $\Sigma^{g}$.

The above non-exactness of the stationary-phase approximation (and even worse the divergence of the stationary-phase series for (3.75)) is a consequence of the fact that the orbits of the vector field (3.77) do not generate a global, compact group action on $\Sigma^{1}$. Here the orbits of the Hamiltonian vector field bifurcate at the saddle points, and we shall see explicitly in chapter 7 why its flows cannot generate isometries of any metric on $\Sigma^{1}$ and how this makes the stationary phase series diverge. The extension of the equivariant localization principle to non-compact group actions and to non-compact phase spaces are not always immediate [24]. A version of the Duistermaat-Heckman theorem appropriate to both abelian and non-abelian group actions on non-compact manifolds has been presented recently in [111]. The above examples illustrate the strong topological dependence of the dynamical systems to which equivariant localization is applicable. The height function restricted to a compact Riemann surface can only be used for Duistermaat-Heckman

[^11]localization in genus 0 , and the introduction of more complicated topologies restricts even further the class of Hamiltonian systems to which the localization constraints apply. We shall investigate this phenomenon in a more detailed geometric setting later on when we consider quantum localization techniques.

### 3.6 Equivariant Localization and Classical Integrability

In this section we discuss an interesting connection between the equivariant localization formalism and integrable Hamiltonian systems [69, 70]. By an integrable dynamical system we mean this in the sense of the Liouville-Arnold theorem which is a generalized, coordinate independent version of the classical Liouville theorem that dictates when a given Hamiltonian system will have equations of motion whose solutions can be explicitly found by integrating by quadratures [30, 48]. The Liouville-Arnold theorem is essentially a global version of Darboux's theorem and it states that a Hamiltonian is integrable if one can find canonically conjugated action-angle variables $\left(I_{\mu}, \phi^{\mu}\right)_{\mu=1}^{n}$,

$$
\begin{equation*}
\left\{I_{\mu}, \phi^{\nu}\right\}_{\omega}=\delta_{\mu}^{\nu} \tag{3.83}
\end{equation*}
$$

defined almost everywhere on the phase space $\mathcal{M}$, such that the Hamiltonian $H=H(I)$ is a functional of only the action variables [5]. The action variables themselves are supposed to be functionally-independent and in involution,

$$
\begin{equation*}
\left\{I_{\mu}, I_{\nu}\right\}_{\omega}=0 \tag{3.84}
\end{equation*}
$$

and from the Hamilton equations of motion (3.32) it follows that

$$
\begin{equation*}
\dot{I}_{\mu}(t)=\left\{I_{\mu}, H(I)\right\}_{\omega}=0 \tag{3.85}
\end{equation*}
$$

so that the time-evolution of the action variables is constant. Consequently, (3.84) implies that the action variables generate a Cartan subalgebra $\left(S^{1}\right)^{n}$ of the Poisson algebra of the phase space, and the $I_{\mu}$ therefore label a set of canonically invariant tori on the phase
space which are called Liouville tori. The motion of $H(I)$ is constrained to the Liouville tori, and the system is therefore integrable in the sense that we have found $n$ independent degrees of freedom for the classical motion. The symplectic 2-form in the action-angle variables is

$$
\begin{equation*}
\omega=d I_{\mu} \wedge d \phi^{\mu} \tag{3.86}
\end{equation*}
$$

and the corresponding symplectic potential which generates the Hamiltonian as the moment map of a global $U(1)$ group action on $\mathcal{M}$ as in (3.46) is

$$
\begin{equation*}
\theta_{F} \equiv \theta+d F=I_{\mu} d \phi^{\mu} \tag{3.87}
\end{equation*}
$$

The connection between integrability and equivariant localization now becomes rather transparent. The above integrability requirement that $H$ be a functional of some torus action generators is precisely the requirement of the Duistermaat-Heckman theorem. Recall that one of the primary assumptions in the localization framework above was that the phase space admit a Riemannian metric $g$ which is globally invariant under the $U(1)$ action on it. A $U(1)$-invariant metric tensor always exists locally in the regions where $H$ has no critical points. To see this, introduce local equivariant Darboux coordinates $\left(p_{1}, \ldots, p_{n}, q^{1}, \ldots, q^{n}\right)$ in that region in which the Hamiltonian vector field generates translations in $q^{1}$. This means that $H=p_{1}$ is taken as the radius of this equivariant Darboux coordinate system. The $U(1)$-invariant metric tensor can then be taken to be any metric tensor whose components are independent of the coordinate $q^{1}$ (e.g. $g_{\mu \nu}=$ $\delta_{\mu \nu}$ ). However, there may be global obstructions to extending these local metrics to metrics defined globally on the entire phase space in a smooth way. This feature is just equivalent to the well-known fact that any Hamiltonian system is locally integrable. This is easily seen from the local representation (3.53),(3.54) where we can define $p_{\mu}=$ $I_{\mu} \cos \phi^{\mu}, q^{\mu}=I_{\mu} \sin \phi^{\mu}$. Then $H \sim \sum_{\mu=1}^{n} I_{\mu}^{2}$ and $V \sim \sum_{\mu=1}^{n} \frac{\partial}{\partial \phi^{\mu}}$ generates translations in the angle variables $\phi^{\mu}$ (rigid rotations of the local coordinate neighbourhood). Then locally the metric tensor components $g_{\mu \nu}$ should be taken to depend only on the action variables $I_{\mu}$ (i.e. $g$ is radially symmetric in the coordinate neighbourhood).

However, local integrability does not necessarily ensure global integrability. For the latter to follow, it is necessary that the neighbourhoods containing the conserved charges $I_{\mu}$ be patched together in such a way as to yield a complete set of conserved charges defined almost everywhere on the phase space $\mathcal{M}$. Furthermore, global integrability also implements strong requirements on the behaviour of $H$ in the vicinity of its critical points. As we shall see later on, the isometry group of a compact Riemannian manifold is also compact, so that the global existence of an invariant metric tensor in the above for a compact phase space is equivalent to the requirement that $H$ generates the global action of a circle (or more generally a torus). This means that the Hamiltonian vector field $V$ is a Cartan element of the algebra of isometries of the metric $g$ (or equivalently $H$ is a Cartan element of the corresponding Poisson algebra). In other words, $H$ is a globally-defined action variable (or a functional thereof), so that the applicable Hamiltonians within the framework of equivariant localization determine integrable dynamical systems. Thus it is the isometry condition that puts a rather severe restriction on the Hamiltonian functions which generate the circle action through the relation (3.28). These features also appear in the infinite-dimensional generalizations of the localization formalism above and they will be discussed at greater length in chapters 5 and 6.

We note also that for an integrable Hamiltonian $H$ we can construct an explicit representation of the function $F$ which appears in (3.46) and (3.87) above. Indeed, the function $K$ in (3.45) can be constructed locally outside of the critical point set of $H$ by assuming that a given action variable $I_{\mu}$ is such that

$$
\begin{equation*}
\frac{\partial H(I)}{\partial I_{\mu}} \neq 0 \tag{3.88}
\end{equation*}
$$

In this case, the function $K$ can be realized explicitly by

$$
\begin{equation*}
K(I, \phi)=\phi^{\mu} \cdot\left(\frac{\partial H}{\partial I_{\mu}}\right)^{-1} \tag{3.89}
\end{equation*}
$$

and the condition (3.46) becomes

$$
\begin{equation*}
i_{V} \theta_{F}=I_{\mu} \frac{\partial H}{\partial I_{\mu}}+\{H, F\}_{\omega}=H \tag{3.90}
\end{equation*}
$$

which is satisfied by

$$
\begin{equation*}
F=K \cdot\left(H-I_{\mu} \frac{\partial H}{\partial I_{\mu}}\right)+G(I) \tag{3.91}
\end{equation*}
$$

where $G(I)$ is an arbitrary function of the action variables. Consequently, in a neighbourhood where action-angle variables can be introduced and where $H$ does not admit critial points, we get an explicit realization of the function $F$ in (3.46) and thus a locally invariant symplectic potential $\theta_{F}$.

In fact, given the equivariantly closed 2-form $K_{V}+\Omega_{V}$ introduced in (2.99), we note that $\Omega_{V}$ is a closed 2 -form (but not necessarily non-degenerate) and that the function $K_{V}$ satisfies

$$
\begin{equation*}
d K_{V}=-i_{V} \Omega_{V} \tag{3.92}
\end{equation*}
$$

as a consequence of (2.101) and (2.98), respectively. It follows that

$$
\begin{equation*}
V^{\mu}=\Omega_{V}^{\mu \nu} \partial_{\nu} K_{V}=\omega^{\mu \nu} \partial_{\nu} H \tag{3.93}
\end{equation*}
$$

and so the classical equations of motion for the 2 Hamiltonian systems $(\mathcal{M}, \omega, H)$ and $\left(\mathcal{M}, \Omega_{V}, K_{V}\right)$ coincide $^{5}$,

$$
\begin{equation*}
\dot{x}^{\mu}(t)=\left\{x^{\mu}, H\right\}_{\omega}=\left\{x^{\mu}, K_{V}\right\}_{\Omega_{V}} \tag{3.94}
\end{equation*}
$$

This means that these 2 dynamical systems determine a bi-Hamiltonian structure. There are 2 interesting consequences of this structure. The first follows from the fact that if $H=H(I)$ as above is integrable, then these action-angle variables can be chosen so that in addition $K_{V}=K_{V}(I)$ is an integrable Hamiltonian. We can therefore replace $H$ everywhere in (3.88)-(3.91) by the function $K_{V}$ and $\omega$ by $\Omega_{V}$, and after a bit of algebra we find that the 1-form $\theta_{F}^{(V)}$ above which generates $\Omega_{V}$ satisfies

$$
\begin{equation*}
K_{V}+\Omega_{V}=D_{V} \theta_{F}^{(V)} \tag{3.95}
\end{equation*}
$$

[^12]and likewise
\[

$$
\begin{equation*}
H+\omega=D_{V} \theta_{F} \tag{3.96}
\end{equation*}
$$

\]

Since both $H+\omega$ and $K_{V}+\Omega_{V}$ are equivariantly closed, we see that for an integrable bi-Hamiltonian system we can solve explicitly the equivariant version of the Poincaré lemma. The global existence of the 1-forms $\theta_{F}$ and $\theta_{F}^{(V)}$ is therefore connected not only to the non-triviality of the DeRham cohomology of $\mathcal{M}$, but also to the non-triviality of the equivariant cohomology associated with the equivariant exterior derivative $D_{V}$. Note that this derivation could also have been carried out for an arbitrary equivariant differential 1-form $\beta$ with the definition (2.99). This suggests an intimate relationship between the localization formalism, and more generally equivariant cohomology, and the existence of bi-Hamiltonian structures for a given phase space.

Furthermore, it is well-known that the existence alone of a bi-Hamiltonian system is directly connected to integrability [5, 30]. If the symplectic 2 -forms $\omega$ and $\Omega_{V}$ are such that the rank $(1,1)$ tensor

$$
\begin{equation*}
L=\Omega_{V} \cdot \omega^{-1} \tag{3.97}
\end{equation*}
$$

is non-trivial, then one can straightforwardly show [69] that

$$
\begin{equation*}
\dot{L}=V^{\mu} \partial_{\mu} L=[L, d V] \tag{3.98}
\end{equation*}
$$

which is just the Lax equation, so that ( $L, d V$ ) determines a Lax pair [30]. Under a certain additional assumption on the tensor $L$ it can then be shown [69] that the quantities

$$
\begin{equation*}
I_{\mu}=\frac{1}{\mu} \operatorname{tr} L^{\mu} \tag{3.99}
\end{equation*}
$$

give variables which are in involution and which are conserved, i.e. which commute with the Hamiltonian $H$. If these quantities are in addition complete, i.e. the number of functionally independent variables (3.99) is half the phase space dimension, then the Hamiltonian system $(\mathcal{M}, \omega, H)$ is integrable in the sense of the Liouville-Arnold theorem. We refer to [69] for more details of how this construction works. Therefore the equivariant
localization formalism for classical dynamical systems presents an alternative, geometric approach to the problem of integrability.

### 3.7 Degenerate Version of the Duistermaat-Heckman Theorem

In these last 3 sections of this Chapter we shall quickly run through some of the generalizations of the Duistermaat-Heckman theorem which can be applied to more general dynamical systems. The first generalization we consider is to the case where $H$ isn't necessarily non-degenerate and its critical point set $\mathcal{M}_{V}$ is now a submanifold of $\mathcal{M}$ of co-dimension $r=\operatorname{dim} \mathcal{M}-\operatorname{dim} \mathcal{M}_{V}[8,14,16,18,19,33,98]$. In this case some modifications are required in the evaluation of the canonical localization integral (2.116) which was used in the derivation of the Berline-Vergne theorem with the differential form $\alpha$ given in (3.56). The Hessian of $H$ now vanishes everywhere on $\mathcal{M}_{V}$ (because $d H=0$ everywhere on $\mathcal{M}_{V}$ ), but we assume that it is non-vanishing in the directions normal to the critical submanifold $\mathcal{M}_{V}$ [92]. This defines the normal bundle $\mathcal{N}_{V}$ of $\mathcal{M}_{V}$ in $\mathcal{M}$, and the phase space is now locally the disjoint union

$$
\begin{equation*}
\mathcal{M}=\mathcal{M}_{V} \cup \mathcal{N}_{V} \tag{3.100}
\end{equation*}
$$

so that in a neighbourhood near $\mathcal{M}_{V}$ we can decompose the local coordinates on $\mathcal{M}$ as

$$
\begin{equation*}
x^{\mu}=x_{0}^{\mu}+x_{\perp}^{\mu} \tag{3.101}
\end{equation*}
$$

where $x_{0}$ are local coordinates on $\mathcal{M}_{V}$, i.e. $V\left(x_{0}\right)=0$, and $x_{\perp}$ are local coordinates on $\mathcal{N}_{V}$. Similarly, the tangent space at any point $x$ near $\mathcal{M}_{V}$ can be decomposed as

$$
\begin{equation*}
T_{x} \mathcal{M}=T_{x} \mathcal{M}_{V} \oplus T_{x} \mathcal{N}_{V} \tag{3.102}
\end{equation*}
$$

where $T_{x} \mathcal{N}_{V}$ is the space of vectors orthogonal to those in $T_{x} \mathcal{M}_{V}$. We can therefore decompose the Grassmann variables $\eta^{\mu}$ which generate the exterior algebra of $\mathcal{M}$ as

$$
\begin{equation*}
\eta^{\mu}=\eta_{0}^{\mu}+\eta_{\perp}^{\mu} \tag{3.103}
\end{equation*}
$$

where $\eta_{0}^{\mu}$ generate the exterior algebra $\Lambda \mathcal{M}_{V}$ and $\eta_{\perp}^{\mu}$ generate $\Lambda \mathcal{N}_{V}$.
Under the usual assumptions used in deriving the equivariant localization principle, it follows that the tangent bundle, equipped with a Levi-Civita-Christoffel connection $\Gamma$ associated with a $U(1)$-invariant metric tensor $g$ as in (2.94), is an equivariant vector bundle. Recall that the Lie derivative $\mathcal{L}_{V}$ induces a non-trivial action of the group on the fibers of the tangent bundle which is mediated by the matrix $d V$. More precisely, this action is given by

$$
\begin{equation*}
\mathcal{L}_{V}=V^{\mu} \partial_{\mu}+d V^{\mu} i_{V^{\mu}}-d V \tag{3.104}
\end{equation*}
$$

and so the moment map associated with this equivariant bundle is the Riemann moment map [16]

$$
\begin{equation*}
\mu_{V}=\nabla V \tag{3.105}
\end{equation*}
$$

which as always is regarded as a matrix acting on the fiber space. Given the Killing equations for $V$, this moment map is related to the 2 -form $\Omega_{V}$ by

$$
\begin{equation*}
\left(\Omega_{V}\right)_{\mu \nu}=2 g_{\mu \lambda}\left(\mu_{V}\right)_{\nu}^{\lambda} \tag{3.106}
\end{equation*}
$$

and the equivariant curvature of the bundle is

$$
\begin{equation*}
R_{V}=R+\mu_{V} \tag{3.107}
\end{equation*}
$$

where the Riemann curvature 2-form of the tangent bundle is

$$
\begin{equation*}
R_{\nu}^{\mu}=\frac{1}{2} R_{\nu \lambda \rho}^{\mu}(x) \eta^{\lambda} \eta^{\rho} \tag{3.108}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{\sigma \mu \nu}^{\rho}=\partial_{\mu} \Gamma_{\nu \sigma}^{\rho}-\partial_{\nu} \Gamma_{\mu \sigma}^{\rho}+\Gamma_{\mu \lambda}^{\rho} \Gamma_{\nu \sigma}^{\lambda}-\Gamma_{\nu \lambda}^{\rho} \Gamma_{\mu \sigma}^{\lambda} \tag{3.109}
\end{equation*}
$$

are the components of the associated Riemann curvature tensor $R=d \Gamma+\Gamma \wedge \Gamma$. Note that, from the decomposition (3.102), the normal bundle inherits a $U(1)$-invariant connection from $T \mathcal{M}$, and the curvature and moment map on $T \mathcal{N}_{V}$ are just the restrictions of the corresponding objects defined on $T \mathcal{M}$.

Given these features of the 2 -form $\Omega_{V}$, it follows that the generators $\eta_{0}^{\mu}$ of $\Lambda \mathcal{M}_{V}$ satisfy

$$
\begin{equation*}
\left(\Omega_{V}\right)_{\mu \nu}\left(x_{0}\right) \eta_{0}^{\nu}=2\left(g_{\mu \lambda} \partial_{\nu} V^{\lambda}\right)\left(x_{0}\right) \eta_{0}^{\nu}=0 \tag{3.110}
\end{equation*}
$$

since $\eta_{0}^{\mu}$ lie in a direction cotangent to $\mathcal{M}_{V}$. For large $s \in \mathbb{R}^{+}$in (2.116) the integral will localize exponentially to a neighbourhood of $\mathcal{M}_{V}$, and so, in the linearization (3.101) of the coordinates perpendicular to $\mathcal{M}_{V}$ wherein we approximate this neighbourhood with a neighbourhood of the normal bundle $\mathcal{N}_{V}$, we can extend the integration over all values of $x_{\perp}$ there. We now introduce the scaled change of integration variables

$$
\begin{equation*}
x^{\mu}=x_{0}^{\mu}+x_{\perp}^{\mu} \rightarrow x_{0}^{\mu}+x_{\perp}^{\mu} / \sqrt{s} \quad, \quad \eta^{\mu}=\eta_{0}^{\mu}+\eta_{\perp}^{\mu} \rightarrow \eta_{0}^{\mu}+\eta_{\perp}^{\mu} / \sqrt{s} \tag{3.111}
\end{equation*}
$$

and expand the argument of the large-s exponential in (2.116) using the decompositions (3.111). The Jacobian determinants from the anticommuting $\eta_{\perp}^{\mu}$ variables and the commuting $x_{\perp}^{\mu}$ variables cancel each other, and so the integral (2.116) remains unchanged under this coordinate rescaling. A tedious but straightforward calculation using observations such as (3.110) shows that the large-s expansions of the argument of the exponential in (2.116) is given by [98]

$$
\begin{align*}
& \frac{s}{2} \Omega_{V} \xrightarrow{s \rightarrow \infty} \frac{1}{2}\left(\Omega_{V}\right)_{\mu \nu}\left(x_{0}\right) \eta_{\perp}^{\mu} \eta_{\perp}^{\nu}+\frac{1}{2}\left(\Omega_{V}\right)_{\mu \sigma}\left(x_{0}\right) R_{\nu \lambda \rho}^{\sigma}\left(x_{0}\right) x_{\perp}^{\mu} x_{\perp}^{\nu} \eta_{0}^{\lambda} \eta_{0}^{\rho}+\mathcal{O}(1 / \sqrt{s}) \\
& s K_{V} \xrightarrow{s \rightarrow \infty} \frac{1}{2}\left(\mu_{V}\right)_{\mu}^{\rho}\left(x_{0}\right)\left(\Omega_{V}\right)_{\rho \nu}\left(x_{0}\right) x_{\perp}^{\mu} x_{\perp}^{\nu}+\mathcal{O}(1 / \sqrt{s}) \tag{3.112}
\end{align*}
$$

where we have expanded the $C^{\infty}$-functions in (3.112) in their respective Taylor series.
Thus with the coordinate change (3.111), the integration over the normal part of the full integration domain

$$
\begin{equation*}
\mathcal{M} \otimes \Lambda^{1} \mathcal{M}=\left(\mathcal{M}_{V} \otimes \Lambda^{1} \mathcal{M}_{V}\right) \sqcup\left(\mathcal{N}_{V} \otimes \Lambda^{1} \mathcal{N}_{V}\right) \tag{3.113}
\end{equation*}
$$

i.e. over $\left(x_{\perp}, \eta_{\perp}\right)$ in (2.116), is Gaussian and can be carried out explicitly. The result is
an integral over the critical submanifold

$$
\begin{align*}
Z(T) & =\left(\frac{2 \pi i}{T}\right)^{\frac{\Gamma}{2}} \int_{\mathcal{M}_{V} \otimes \Lambda^{1} \mathcal{M}_{V}} d^{r} x_{0} d^{r} \eta_{0} \mathrm{e}^{i T\left(H\left(x_{0}\right)+\omega\left(x_{0}, \eta_{0}\right)\right)} \frac{\text { Pfaff } \Omega_{V}\left(x_{0}\right)}{\sqrt{\operatorname{det} \Omega_{V}\left(x_{0}\right)\left(\mu_{V}\left(x_{0}\right)+R\left(x_{0}, \eta_{0}\right)\right)}} \\
& =\left(\frac{2 \pi i}{T}\right)^{r / 2} \int_{\mathcal{M}_{V}} \frac{\left.\operatorname{ch}_{V}(i T \omega)\right|_{\mathcal{M}_{V}}}{\left.E_{V}(R)\right|_{\mathcal{N}_{V}}} \tag{3.114}
\end{align*}
$$

where we have identified the equivariant Chern and Euler characters (2.76) and (2.81) of the respective fiber bundles. In (3.114) the equivariant Chern and Euler characters are restricted to the critical submanifold $\mathcal{M}_{V}$, and the determinant and Pfaffian there are taken over the normal bundle $\mathcal{N}_{V}$. Note that the above derivation has assumed that the critical submanifold $\mathcal{M}_{V}$ is connected. If $\mathcal{M}_{V}$ consists of several connected components, then the formula (3.114) means a sum over the contributions from each of these components.

There are several comments in order here. First of all, notice that if $\mathcal{M}_{V}$ consists of discrete isolated points, so that $r=2 n$, then, since the curvature of the normal bundle of a point vanishes and so the Riemann moment map $\mu_{V}$ coincides with the usual moment map $d V$ on $T \mathcal{M}$ calculated at that point, the formula (3.114) reduces to the non-degenerate localization formula (3.57) and hence to the Duistermaat-Heckman theorem. Secondly, we recall that the equivariant characterstic classes in (3.114) provide representatives of the equivariantly cohomology of $\mathcal{M}$ and the integration formula (3.114) is formally independent of the chosen metric on $\mathcal{M}$. Thus the localization formulas are topological invariants of $\mathcal{M}$, as they should be, and they represent types of 'index theorems'. This fact will have important implications later on in the formal applications to topological field theory functional integration. However, we shall see later on that naive ambiguities can nevertheless arise from the explicit metric dependence of the localization formulas. Finally, we point out that Kirwan's theorem generalizes to the degenerate case above [72]. In this case, since the Hessian is a non-singular symmetric matrix along the directions normal to $\mathcal{M}_{V}$, we can orthogonally decompose the normal bundle, with the
aid of some locally-defined Riemannian metric on $\mathcal{M}_{V}$, into a direct sum of the positiveand negative-eigenvalue eigenspaces of $\mathcal{H}$. The dimension of the latter subspace is now defined as the index of $\mathcal{M}_{V}$ and Kirwan's theorem now states that the index of every connected component of $\mathcal{M}_{V}$ must be even when the localization formula (3.114) holds. The Morse inequalities for this degenerate case [92] then relate the exactness or failure of (3.114) as before to the homology of the underlying phase space $\mathcal{M}$. One dynamical system to which the formula (3.114) could be applied to is the height function of the torus when the torus is now viewed in 3 -space as a doughnut sitting on a dinner plate (the $x y$-plane). This function has 2 extrema but they are now circles, instead of points, which are parallel to each other and one is a minimum and the other is a maximum. The critical submanifold of $T^{2}$ in this case consists of 2 connected components, $T_{V}^{2}=S^{1} \sqcup S^{1}$.

### 3.8 The Witten Localization Formula

We have thus far only applied the localization formalism to abelian group actions on $\mathcal{M}$. The first generalization of the Duistermaat-Heckman theorem to non-abelian group actions was presented by Guilleman and Prato [53] in the case where the induced action of the Cartan subgroup (or maximal torus) of $G$ has only a finite number of isolated fixed points $p_{i}$ and the stabalizer $\left\{g \in G: g \cdot p_{i}=p_{i}\right\}$ of all these fixed points coincides with the Cartan subgroup. The Guilleman-Prato localization formula reduces the integrals over the dual Lie algebra $\mathbf{g}^{*}$ to integrals over the dual of the Cartan subalgebra using the socalled Weyl integral formula [24]. With this reduction one can apply the standard abelian localization formalism above. This procedure of abelianization thus reduces the problem to the consideration of localization theory for functions of Cartan elements of the Lie group $G$, i.e. integrable Hamiltonian systems. Recently, Witten [135] proposed a more general non-abelian localization formalism and used it to study 2-dimensional Yang-Mills theory. In this section we shall outline the basic features of Witten's localization theory.

Given a Lie group $G$ acting on the phase space $\mathcal{M}$, we wish to evaluate the partition
function with the general equivariant extension (3.34),

$$
\begin{equation*}
Z_{G}=\int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{-\phi^{a} \otimes H^{a}} \tag{3.115}
\end{equation*}
$$

where as usual the Boltzmann weights are given by the symplectic moment map of the $G$-action on $\mathcal{M}$. There are 2 ways to regard the dual algebra functions $\phi^{a}$ in (3.115). We can give the $\phi^{a}$ fixed values, regarding them as the values of elements of $S\left(\mathbf{g}^{*}\right)$ acting on algebra elements, i.e. the $\phi^{a}$ are complex-valued parameters, as is unambiguously the case if $G$ is abelian [8] (in which case we set $\phi=-i T$ in (3.115)). In this case we are integrating with a fixed element of the Lie group $G$, i.e. we are essentially in the abelian case. We shall see that various localization schemes reproduce features of character formulas for the action of the Lie group $G$ on $\mathcal{M}$ at the quantum level. The other possibility is to regard the $\phi^{a}$ as dynamical variables and integrate over them. This case allows a richer intepretation and is the basis of non-abelian localization formulas and the localization formalism in topological field theory.

To employ this latter interpretation for the symmetric algebra elements, we need a definition for equivariant integration. The defintion (2.107) gives a map on $\Lambda_{G} \mathcal{M} \rightarrow$ $S\left(\mathbf{g}^{*}\right)^{G}$, but in analogy with ordinary DeRham integration we wish to obtain a map on $\Lambda_{G} \mathcal{M} \rightarrow \mathbb{C}$. The group $G$ has a natural $G$-invariant measure on it, namely its Haar measure. Since $\mathbf{g}$ is naturally isomorphic to the tangent space of $G$ at the identity, it inherits from the Haar measure a natural translation-invariant measure. Given this measure, the definition we take for equivariant integration is [135]

$$
\begin{equation*}
\oint_{\mathcal{M} \otimes \mathbf{g}^{*}} \alpha=\lim _{s \rightarrow \infty} \frac{1}{\operatorname{vol}(G)} \int_{\mathbf{g}^{*}} \prod_{a=1}^{\operatorname{dim} G} \frac{d \phi^{a}}{2 \pi} \mathrm{e}^{-\frac{1}{2 s}\left(\phi^{a}\right)^{2}} \int_{\mathcal{M}} \alpha \tag{3.116}
\end{equation*}
$$

for $\alpha \in \Lambda_{G} \mathcal{M}$, where $\operatorname{vol}(G)$ is the volume of the group $G$ in its Haar measure. The parameter $s \in \mathbb{R}^{+}$in (3.116) is used to regulate the possible divergence on the completion $\Lambda_{G}^{\infty} \mathcal{M}$. The definition (3.116) indeed gives a map on $\Lambda_{G} \mathcal{M} \rightarrow \mathbb{C}$, and the $\phi^{a}$ 's in it can be regarded as local Euclidean coordinates on $\mathbf{g}^{*}$ such that the measure there coincides with the chosen Haar measure at the identity of $G$. Setting $\alpha=\mathrm{e}^{\omega_{\mathrm{g}}}$ in (3.116), with $\omega_{\mathrm{g}}$ the
equivariant extension (3.34) of the symplectic 2 -form of $\mathcal{M}$, and performing the Gaussian $\phi^{a}$-integrals, we arrive at Witten's localization formula for the partition function (3.115),

$$
\begin{equation*}
Z_{G}=\lim _{s \rightarrow \infty}\left(\frac{s}{2 \pi}\right)^{\operatorname{dim} G / 2} \frac{1}{\operatorname{vol}(G)} \int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{-\frac{s}{2} \sum_{a}\left(H^{a}\right)^{2}} \tag{3.117}
\end{equation*}
$$

The right-hand side of (3.117) localizes onto the extrema of the square of the moment $\operatorname{map} \sum_{a}\left(H^{a}\right)^{2}$. The absolute minima of this function are the solutions to $H=\phi^{a} \otimes H^{a}=$ 0 . The contribution of the absolute minimum to $Z_{G}$ (the dominant contribution for $s \rightarrow \infty$ ) is given by a simple cohomological formula [135]

$$
\begin{equation*}
Z_{G}^{\min }=\left.\lim _{s \rightarrow \infty} \frac{(2 \pi)^{\operatorname{dim} G / 2}}{\operatorname{vol}(G)} \int_{\mathcal{M}_{0}} \mathrm{e}^{\omega+\frac{1}{2 s} \Theta}\right|_{\mathcal{M}_{0}} \tag{3.118}
\end{equation*}
$$

where $\mathcal{M}_{0}=H^{-1}(0) / G$ is the Marsden-Weinstein reduced phase space [80] (or symplectic quotient) and $\Theta$ is a certain element of the cohomology group $H^{4}(\mathcal{M} ; \mathbb{R})$ (we refer to [135] for the details). The localization of the global minima onto $\mathcal{M}_{0}$ is a consequence of the $G$-equivariance of the integration in (3.116). Thus the Witten localization formula can be used to describe the cohomology of the reduced phase space $\mathcal{M}_{0}$ of the given symplectic $G$-action on $\mathcal{M}$.

However, the contributions from the other local extrema of $\sum_{a}\left(H^{a}\right)^{2}$, which correspond to the critical points of $H$ as in the Duistermaat-Heckman integration formula, are in general very complicated functions of the limiting parameter $s \in \mathbb{R}^{+}$. For instance, in the simple abelian example of section 3.5 above where $G=U(1), \mathcal{M}=S^{2}$ and $H=h_{\Sigma^{0}}$ is the height function (3.70) of the sphere, the Witten localization formula (3.117) above becomes

$$
\begin{equation*}
Z_{\Sigma^{0}}=\lim _{s \rightarrow \infty}\left(\frac{s}{2 \pi}\right)^{1 / 2} \int_{-1}^{+1} d \cos \theta \mathrm{e}^{-s(a-\cos \theta)^{2} / 2}=\lim _{s \rightarrow \infty}\left(1-I_{+}(s)-I_{-}(s)\right) \tag{3.119}
\end{equation*}
$$

where we have assumed that $|a|<1$ and $I_{ \pm}(s)$ are the transcendental error functions [50]

$$
\begin{equation*}
I_{ \pm}(s)= \pm\left(\frac{s}{2 \pi}\right)^{1 / 2} \int_{ \pm 1}^{ \pm \infty} d x \mathrm{e}^{-s(a-x)^{2} / 2} \tag{3.120}
\end{equation*}
$$

The 3 final terms in (3.119) are the anticipated contributions from the 3 critical points of $h_{\Sigma^{0}}^{2}=(\cos \theta-a)^{2}$ - the absolute minimum at $\cos \theta=a$ contributes +1 , while the local maxima at $\cos \theta= \pm 1$ contribute negative terms $-I_{ \pm}$to the localization formula. The complicated error functions arise because here the critical point at $\cos \theta=a$ is a degenerate critical point of the canonical localization integral in (2.116). The appearence of these error functions is in marked contrast with the elementary functions that appear as the contributions from the critical points in the usual Duistermat-Heckman formula.

Another interesting application of the Witten localization formalism is that it can be used to derive integration formulas when the argument of the Boltzmann weight in the partition function is instead the square of the moment map. This can be done by reversing the arguments which led to the localization formula (3.117), and further localizing the Duistermaat-Heckman type integral (3.115) using the localization principle of section 2.4. The result (for finite $s$ ) is then a sum of local contributions $\sum_{m} Z_{m}(s)$, where the functions $Z_{m}(s)$ can only be determined explicitly in appropriate instances [65, 135, 137]. Combining these ideas together, we arrive at the localization formula

$$
\begin{align*}
\frac{1}{\operatorname{vol}(G)} & \left(-\frac{i T}{\pi}\right)^{\operatorname{dim} G / 2} \int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{i T \sum_{a}\left(H^{a}\right)^{2}} \\
& =\frac{1}{\operatorname{vol}(G)} \int_{\mathrm{g}^{*}} \prod_{a=1}^{\operatorname{dim} G} \frac{d \phi^{a}}{2 \pi} \mathrm{e}^{-\frac{1}{4 i T}\left(\phi^{a}\right)^{2}} \int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{-\phi^{a} \otimes H^{a}}  \tag{3.121}\\
& =\frac{1}{\operatorname{vol}(G)} \lim _{s \rightarrow \infty} \int_{\mathrm{g}^{*}} \prod_{a=1}^{\operatorname{dim} G} \frac{d \phi^{a}}{2 \pi} \mathrm{e}^{-\frac{1}{4 i T}\left(\phi^{a}\right)^{2}} \int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{-\phi^{a} \otimes H^{a}-s D_{\mathbf{g}} \lambda}
\end{align*}
$$

where $\lambda \in \Lambda_{G}^{1} \mathcal{M}$ and we have applied the localization principle to the DuistermatHeckman type integral over $\mathcal{M}$ on the right-hand side of the second line of (3.121). The localization 1-form $\lambda$ is chosen just as before using a $G$-invariant metric on $\mathcal{M}$ and the Hamiltonian vector field associated with the square of the moment map.

Witten used the formal infinite-dimensional generalization of this last localization formula to evaluate the partition function of 2-dimensional Yang-Mills theory [135]. There $\mathcal{M}$ is the space $\mathcal{A}$ of gauge connections $A$ over a Riemann surface, which has on it a
natural symplectic structure, and the moment map $H(A)=F_{A}$ is the field strength tensor of $A$. Thus the square of the moment map in this case is just the Yang-Mills action and the symplectic quotient $\mathcal{M}_{0}$ is the moduli space of flat gauge connections modulo gauge transformations associated with the gauge group $G$, i.e. the moduli space of solutions to the classical Yang-Mills field equations. In this way, Witten's non-abelian localization formula yields intersection numbers of the moduli space of flat connections on a 2-surface $\Sigma^{g}$ from the solution of Yang-Mills theory on $\Sigma^{g}$. This approach to 2dimensional Yang-Mills theory has been studied in detail for genus 0 recently in [86] and has been generalized to the $G / G$ Wess-Zumino-Witten model in [20, 23]. In this context, the approach above to using a generalization of the Duistermaat-Heckman integration formula to problems with non-abelian symmetries is equivalent to a relation between physical and topological gauge theories [135].

Finally, we point out the work of Jeffrey and Kirwan [65] who rigorously derived, in certain special cases, the contribution to $Z_{G}$ from the reduced phase space $\mathcal{M}_{0}=$ $H^{-1}(0) / G$ in (3.118). Let $H_{C} \subset G$ be the Cartan subgroup of $G$, and assume that the fixed points $p$ of the induced $H_{C}$-action on $\mathcal{M}$ are isolated and non-degenerate. Then for any equivariantly-closed differential form $\alpha$ of degree $\operatorname{dim} \mathcal{M}_{0}$ in $\Lambda_{G} \mathcal{M}$, we have the so-called residue formula [65]

$$
\begin{equation*}
\left.\int_{\mathcal{M}_{0}} \alpha\right|_{\mathcal{M}_{0}}=\sum_{p \in \mathcal{M}_{H_{C}}} \operatorname{Res}\left[\mathrm{e}^{-\phi^{a} \otimes H^{a}(p)} \frac{\left(\Pi_{\beta} \beta\right) \alpha^{(0)}(p)}{e(p)}\right] \tag{3.122}
\end{equation*}
$$

where $\beta$ are the roots associated to $H_{C} \subset G$, and Res is Jeffrey-Kirwan-Kalkman residue, defined as the coefficient of $\frac{1}{\phi}$ where $\phi$ is the element of the symmetric algebra $S\left(\mathrm{~g}^{*}\right)$ representing the induced $H_{C}$-action on $\mathcal{M}$ (see [65] and [68] for its precise definition). This residue, whose explicit form was computed by Kalkman [68], depends on the fixedpoint set $\mathcal{M}_{H_{C}}$ of the $H_{C}$-action on $\mathcal{M}$ and it can be expressed in terms of the weight determinants $e(p)$ in (3.63) of the $H_{C}$-action and the values $H(p)=\phi^{a} \otimes H^{a}(p)$. It is
in forms similar to (3.122) that the first non-abelian generalizations of the DuistermatHeckman theorem due to Guilleman and Prato appeared [53]. The residue formula can explicitly be used to obtain information about the cohomology ring of the reduced phase space $\mathcal{M}_{0}$ above $[65,68]$.

### 3.9 The Wu Localization Formula

The final generalization of the Duistermaat-Heckman theorem that we shall present here is an interesting application, due to Wu [137], of Witten's localization formula in the form (3.121) when applied to a global $U(1)$-action on $\mathcal{M}$. This yields a localization formula for Hamiltonians which are not themselves the associated symplectic moment map, but are functionals of such an observable $H$. This is accomplished via the localization formula

$$
\begin{equation*}
Z_{U(1)}(T)=\int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{i T H^{2}}=\left(-\frac{1}{4 \pi i T}\right)^{1 / 2} \lim _{s \rightarrow \infty} \int_{0}^{2 \pi} d \phi \mathrm{e}^{-\frac{1}{4 i T} \phi^{2}} \int_{\mathcal{M}} \frac{\omega^{n}}{n!} \mathrm{e}^{-\phi \otimes H-s D_{\mathrm{u}(1)} \lambda} \tag{3.123}
\end{equation*}
$$

The final integral on the right-hand side of (3.123) is just that which appears in the canonical localization integral (2.108) used in the derivation of the Duistermaat-Heckman formula. Working this out just as before and performing the resulting Gaussian $\phi$-integral yields Wu's localization formula for circle actions [137]

$$
\begin{equation*}
Z_{U(1)}(T)=\frac{(2 \pi)^{n}}{(n-1)!} \sum_{p \in \mathcal{M}_{V}} \frac{1}{e(p)} \int_{0}^{\infty} d s s^{n-1} \mathrm{e}^{i T(s+|H(p)|)^{2}}+\left.\int_{\mathcal{M}_{0}} \mathrm{e}^{\omega+i F / 4 T}\right|_{\mathcal{M}_{0}} \tag{3.124}
\end{equation*}
$$

where $F=d A$ is the curvature of an abelian gauge connection on the (non-trivial) principal $U(1)$-bundle $H^{-1}(0) \rightarrow \mathcal{M}_{0}$. The formula (3.124) can be used to determine the symplectic volume of the Marsden-Weinstein reduced phase space $\mathcal{M}_{0}$ [137]. This gives an alternative localization for Hamiltonians which themselves do not generate an isometry of some metric $g$ on $\mathcal{M}$, but are quadratic in such isometry generators. As we shall see, the path integral generalizations of Wu's formula are rather important for certain physical problems.

## Chapter 4

## Quantum Localization Theory

In quantum mechanics there are not too many path integrals that can be evaluated explicitly and exactly, while the analog of the stationary phase approximation, i.e. the semiclassical approximation, can usually be obtained quite readily. In this chapter we shall investigate the possibility of obtaining some path integral analogs of the DuistermaatHeckman formula and its generalizations. A large class of examples where one has an underlying equivariant cohomology which could serve as a structure responsible for localization is provided by phase space path integrals, i.e. the direct loop space analogs of (3.51). Of course, as path integrals in general are mathematically awkward objects, the localization formulas that we will obtain in this way are not really definite predictions but rather suggestions for what kind of results to expect. Because of the lack of rigor that goes into deriving these localization formulas it is perhaps surprising then that some of these results are not only conceptually interesting but also physically reasonable.

Besides these there are many other field-theoretic analogies with the functional integral generalization of the Duistermaat-Heckman theorem, the common theme being always some underlying geometrical or topological structure which is ultimately responsible for localization. We have already encountered one of these in the last chapter, namely the Witten localization formula which is in principle the right framework to apply equivariant localization to a cohomological formulation of 2-dimensional quantum Yang-Mills theory. Another large class of quantum models for which the DuistermaatHeckman theorem seems to make sense is $N=\frac{1}{2}$ supersymmetric quantum mechanics [7]. This formal application, due to Atiyah and Witten, was indeed the first encouraging
evidence that such a path integral generalization of the rigorous localization formulas of the last chapter exists. Strictly speaking though, this example really falls into the category of the Berline-Vergne localization of section 2.5 as the free loop space of a configuration manifold is not quite a symplectic manifold in general [16]. More generally, the Duistermaat-Heckman localization can be directly generalized to the infinite-dimensional case within the Lagrangian formalism, if the loop space defined over the configuration space has on it a natural symplectic structure. This is the case, for example, for geodesic motion on a Lie group manifold, where the space of based loops is a Kähler manifold [112] and the stationary phase approximation is well-known to be exact [32, 115]. This formal localization has been carried out by Picken [110].

We will discuss some of these other models later on this chapter, but we are really interested in obtaining some version of the equivariant localization formulas available which can be applied to non-supersymmetric models and when the partition functions cannot be calculated directly by some other means. The Duistermat-Heckman theorem in this context would now express something like the exactness of the one-loop approximation to the path integral. These functional integral formulas, and their connections to the finite-dimensional formulas of chapter 3 , will be discussed in this chapter. The formal techniques we shall employ throughout use ideas from supersymmetric and topological field theories, and indeed we shall see how to interpret an arbitrary phase space path integral quite naturally both as a supersymmetric and as a topological field theory partition function. In the Hamiltonian approach to localization, therefore, topological field theories fit quite naturally into the loop space equivariant localization framework. As we shall see, this has deep connections with the integrability properties of these models. In all of this, the common mechanism will be a fundamental cohomological nature of the model which can be understood in terms of a supersymmetry allowing one to deform the integrand without changing the integral.

### 4.1 Phase Space Path Integrals

We begin this chapter by deriving the quantum mechanical path integral for a bosonic quantum system with no internal degrees of freedom. For simplicity, we shall present the calculation for $n=1$ degree of freedom in Darboux coordinates on $\mathcal{M}$, i.e. we essentially carry out the calculation on the plane $\mathbb{R}^{2}$. The extension to $n>1$ will then be immediate, and then we simply add the appropriate symplectic quantities to obtain a canonically-invariant object on a general symplectic manifold $\mathcal{M}$ to ensure invariance under transformations which preserve the density of states.

To transform the classical theory of the last chapter into a quantum mechanical one (i.e. to 'quantize' it), we replace the phase space coordinates $(p, q)$ with operators $(\hat{p}, \hat{q})$ which obey an operator algebra that is obtained by replacing the Poisson algebra of the Darboux coordinates (3.18) by allowing the commutator bracket of the basis operators $(\hat{p}, \hat{q})$ to be simply equal to the Poisson brackets of the same objects as elements of the Poisson algebra of $C^{\infty}$-functions on the phase space, times an additional factor of $i \hbar$ where $\hbar$ is Planck's constant,

$$
\begin{equation*}
[\hat{p}, \hat{q}]=i \hbar \tag{4.1}
\end{equation*}
$$

The operators $(\hat{p}, \hat{q})$ with the canonical commutation relation (4.1) make the space of $C^{\infty}$-functions on $\mathcal{M}$ into an infinite-dimensional associative algebra called the Heisenberg algebra ${ }^{1}$. This algebra can be represented on the space $L^{2}(q)$ of square integrable functions of the configuration space coordinate $q$ by letting the operator $\hat{q}$ act as multiplication by $q$ and $\hat{p}$ as the derivative

$$
\begin{equation*}
\hat{p}=i \hbar \frac{\partial}{\partial q} \tag{4.2}
\end{equation*}
$$

This representation of the Heisenberg algebra is called the Schrödinger picture and the elements of the Hilbert space $L^{2}(q)$ are called the wavefunctions or physical states of the

[^13]dynamical system ${ }^{2}$.
The eigenstates of the position and momentum operators are denoted by the usual Dirac bra-ket notation
\[

$$
\begin{equation*}
\hat{q}|q\rangle=q|q\rangle \quad, \quad \hat{p}|p\rangle=p|p\rangle \tag{4.3}
\end{equation*}
$$

\]

These states are orthonormal,

$$
\begin{equation*}
\left\langle q \mid q^{\prime}\right\rangle=\delta\left(q-q^{\prime}\right) \quad, \quad\left\langle p \mid p^{\prime}\right\rangle=\delta\left(p-p^{\prime}\right) \tag{4.4}
\end{equation*}
$$

and they obey the momentum and position space completeness relations

$$
\begin{equation*}
\int_{-\infty}^{\infty} d p|p\rangle\langle p|=\int_{-\infty}^{\infty} d q|q\rangle\langle q|=\mathbf{1} \tag{4.5}
\end{equation*}
$$

with 1 the identity operator on the respective space. In the representation (4.2) on $L^{2}$-functions the momentum and configuration space representations are related by the usual Fourier transformation

$$
\begin{equation*}
|q\rangle=\int_{-\infty}^{\infty} \frac{d p}{\sqrt{2 \pi \hbar}} \mathrm{e}^{-i p q / \hbar}|p\rangle \tag{4.6}
\end{equation*}
$$

which identifies the matrix element

$$
\begin{equation*}
\langle p \mid q\rangle=\langle q \mid p\rangle^{*}=\frac{1}{\sqrt{2 \pi \hbar}} \mathrm{e}^{-i p q / \hbar} \tag{4.7}
\end{equation*}
$$

and the basis operators have the matrix elements

$$
\begin{equation*}
\langle p| \hat{q}|q\rangle=q\langle p \mid q\rangle \quad, \quad\langle p| \hat{p}|q\rangle=p\langle p \mid q\rangle=i \hbar \frac{\partial}{\partial q}\langle p \mid q\rangle \tag{4.8}
\end{equation*}
$$

All observables (i.e. real-valued $C^{\infty}$-functions of $\left.(p, q)\right)$ now become Hermitian operators acting on the Hilbert space. In particular, the Hamiltonian of the dynamical system now becomes a Hermitian operator $\hat{H} \equiv H(\hat{p}, \hat{q})$ with the matrix elements

$$
\begin{equation*}
\langle p| \hat{H}|q\rangle=H(p, q)\langle p \mid q\rangle=H(p, q) \frac{\mathrm{e}^{-i p q / \hbar}}{\sqrt{2 \pi \hbar}} \tag{4.9}
\end{equation*}
$$

[^14]and the eigenvalues of this operator determine the energy levels of the physical system.
We shall henceforth assume that the Hamiltonian of the dynamical system does not depend explicitly on time, so that the energy of the system is a constant of the motion. The time evolution of any quantum operator is determined by the quantum mapping above of the Hamilton equations of motion (3.32). In particular, the time evolution of the position operator is determined by
\[

$$
\begin{equation*}
\dot{\hat{q}}(t)=\frac{1}{i \hbar}[\hat{q}, \hat{H}] \tag{4.10}
\end{equation*}
$$

\]

which may be solved formally by

$$
\begin{equation*}
\hat{q}(t)=\mathrm{e}^{i \hat{H} t / \hbar} \hat{q}(0) \mathrm{e}^{-i \hat{H} t / \hbar} \tag{4.11}
\end{equation*}
$$

so that the time evolution is determined by a unitary transformation of the position operator $\hat{q}(0)$. In the Schrödinger representation, we treat the operators as time-independent quantities using the unitary transformation (4.11) and consider the time-evolution of the quantum states. The configuration of the system at a time $t$ is defined using the unitary time-evolution operator in (4.11) acting on an initial configuration $|q\rangle$ at time $t=0$,

$$
\begin{equation*}
|q, t\rangle=\mathrm{e}^{i \hat{H} t / \hbar}|q\rangle \tag{4.12}
\end{equation*}
$$

which is an eigenstate of (4.11) for all $t$.
An important physical quantity is the quantum propagator

$$
\begin{equation*}
\mathcal{K}\left(q^{\prime}, q ; T\right)=\left\langle q^{\prime}, T \mid q, 0\right\rangle=\left\langle q^{\prime}\right| \mathrm{e}^{-i \hat{H} T / \hbar}|q\rangle \tag{4.13}
\end{equation*}
$$

which, according to the fundamental principles of quantum mechanics [83], represents the probability of the system evolving from a state with configuration $q$ to one with configuration $q^{\prime}$ in a time interval $T$. The propagator (4.13) satisfies the Schrödinger wave equation

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial T} \mathcal{K}\left(q^{\prime}, q ; T\right)=\hat{H} \mathcal{K}\left(q^{\prime}, q ; T\right) \tag{4.14}
\end{equation*}
$$

where the momentum operators involved in the Hamiltonian $\hat{H}$ on the right-hand side of (4.14) are represented in the Schrödinger polarization (4.2). The Schrödinger equation is to be solved with the Dirac delta-function initial condition

$$
\begin{equation*}
\mathcal{K}\left(q^{\prime}, q ; T=0\right)=\delta\left(q^{\prime}-q\right) \tag{4.15}
\end{equation*}
$$

Thus the propagator represents the fundamental quantum dynamics of the system and the stationary state solutions to the Schrödinger equation (4.14) determine the energy eigenvalues of the dynamical system.

The phase space path integral provides a functional representation of the quantum propagator in terms of a 'sum' over continuous trajectories on the phase space. It is constructed as follows [116]. Between the initial and final configurations $q$ and $q^{\prime}$ we introduce $N-1$ intermediate configurations $q_{0}, \ldots, q_{N}$ with $q_{0} \equiv q$ and $q_{N} \equiv q^{\prime}$, and each separated by the time interval

$$
\begin{equation*}
\Delta t=T / N \tag{4.16}
\end{equation*}
$$

Introducing intermediate momenta $p_{1}, \ldots, p_{N}$ and inserting the completeness relations

$$
\begin{equation*}
\int_{-\infty}^{\infty} d q_{j-1} d q_{j} d p_{j}\left|q_{j}\right\rangle\left\langle q_{j} \mid p_{j}\right\rangle\left\langle p_{j} \mid q_{j-1}\right\rangle\left\langle q_{j-1}\right|=\mathbf{1} \quad, \quad j=1, \ldots, N \tag{4.17}
\end{equation*}
$$

into the matrix element (4.13) we obtain

$$
\begin{align*}
\mathcal{K}\left(q^{\prime}, q ; T\right)= & \left\langle q^{\prime}\right|\left(\mathrm{e}^{-i \hat{H} \Delta t / \hbar}\right)^{N}|q\rangle \\
= & \int_{-\infty}^{\infty} \prod_{j=1}^{N} d q_{j-1} d q_{j} d p_{j}\left\langle q^{\prime} \mid q_{j}\right\rangle\left\langle q_{j}\right| \mathrm{e}^{-i \hat{H} \Delta t / \hbar}\left|p_{j}\right\rangle\left\langle p_{j} \mid q_{j-1}\right\rangle\left\langle q_{j-1} \mid q\right\rangle  \tag{4.18}\\
= & \int_{-\infty}^{\infty} \prod_{j=1}^{N-1} d q_{j} \prod_{j=1}^{N} \frac{d p_{j}}{2 \pi \hbar} \exp \left\{\frac{i}{\hbar} \sum_{i=1}^{N}\left(p_{i} \frac{q_{i}-q_{i-1}}{\Delta t}-H\left(p_{i}, q_{i}\right)\right) \Delta t\right\} \\
& \quad \times \delta\left(q_{0}-q\right) \delta\left(q_{N}-q^{\prime}\right)
\end{align*}
$$

where we have used the various identities quoted above. In the limit $N \rightarrow \infty$, or equivalently $\Delta t \rightarrow 0$, the discrete points $\left(p_{j}, q_{j}\right)$ describe paths $(p(t), q(t))$ in the phase space
between the configurations $q$ and $q^{\prime}$, and the sum in (4.18) becomes the continuous limit of a Riemann sum representing a discretized time integration. Then (4.18) becomes

$$
\begin{align*}
\mathcal{K}\left(q^{\prime}, q ; T\right)=\lim _{N \rightarrow \infty} \int_{-\infty}^{\infty} & \prod_{j=1}^{N-1} d q_{j} \prod_{j=1}^{N} \frac{d p_{j}}{2 \pi \hbar} \exp \left\{\frac{i}{\hbar} \int_{0}^{T} d t(p(t) \dot{q}(t)-H(p, q))\right\}  \tag{4.19}\\
& \times \delta(q(0)-q) \delta\left(q(T)-q^{\prime}\right)
\end{align*}
$$

Note that the argument of the exponential in (4.19) is just the classical action of the dynamical system, because its integrand is the usual Legendre transformation between the Lagrangian and Hamiltonian descriptions of the classical dynamics [48]. Notice also that, in light of the Heisenberg uncertainty principle $\Delta q \Delta p \sim 2 \pi \hbar$, the normalization factors $2 \pi \hbar$ there can be physically interpreted as the volume of an elementary quantum state in the phase space.

The integration measure in (4.19) formally gives an integral over all phase space paths defined in the time interval $[0, T]$. This measure is denoted by

$$
\begin{equation*}
[d p d q] \equiv \lim _{N \rightarrow \infty} \prod_{j=1}^{N} \frac{d p_{j}}{2 \pi \hbar} \prod_{j=1}^{N-1} d q_{j} \equiv \prod_{t \in[0, T]} \frac{d p(t)}{2 \pi \hbar} d q(t) \tag{4.20}
\end{equation*}
$$

and it is called the Feynman measure. The last equality means that it is to be understood as a 'measure' on the infinite-dimensional functional space of phase space trajectories $(p(t), q(t))$, where for each fixed time slice $t \in[0, T], d p(t) d q(t)$ is ordinary RiemannLebesgue measure. Being an infinite-dimensional quantity, it is not rigorously defined, and some special care must taken to determine the precise meaning of the limit $N \rightarrow \infty$ above. This has been a topic of much dispute over the years and we shall make no attempt in this thesis to discuss the ill-defined ambiguities associated with the Feynman measure. Many rigorous attempts at formulating the path integral have been proposed in constructive quantum field theory. For instance, it is possible to give the limit (4.20) a somewhat precise meaning using the so-called Lipschitz functions of order $\frac{1}{2}$ which assumes that the paths which contribute in (4.19) grow no faster than $\mathcal{O}(\sqrt{t})$ (these
functional integrals are called Wiener integrals) ${ }^{3}$. We shall at least assume that the integration measure (4.20) is supported on $C^{\infty}$ phase space paths and that the quantum mechanical propagator given by (4.19) is a tempered distribution, i.e. it can diverge with at most a polynomial growth. This latter restriction on the path integral is part of the celebrated Wightman axioms for quantum field theory which allows one to at least carry out certain formal rigorous manipulations from the theory of distributions.

However, a physicist will typically proceed without worry and succeed in extracting a surprising amount of information from formulas such as (4.19) without the need to investigate in more detail the implications of the limit $N \rightarrow \infty$ above. To actually carry out functional integrations such as (4.19) one uses formal functional analogs of the usual rules of Riemann-Lebesgue integration in the straightforward sense, where all time integrals are treated as continuous sums on the functional space (i.e. the time parameter $t$ is regarded as a continuous index).

If we set $q=q^{\prime}$ and integrate over all $q$, then the left-hand side of (4.19) yields

$$
\begin{equation*}
\int_{-\infty}^{\infty} d q\langle q| \mathrm{e}^{-i \hat{H} T / \hbar}|q\rangle \equiv \operatorname{tr}\left\|\mathrm{e}^{-i \hat{H} T / \hbar}\right\|=£ d E \mathrm{e}^{-i E T / \hbar} \tag{4.21}
\end{equation*}
$$

where $E$ are the energy eigenvalues of $\hat{H}$ and the symbol $\|\cdot\|$ will be used to emphasize that the matrix of interest is considered as an infinite dimensional one over either the Hilbert space of physical states or the functional trajectory space. On the other hand, the right-hand side of (4.19) becomes

$$
\begin{equation*}
Z(T)=\int[d p d q] \exp \left\{\frac{i}{\hbar} \int_{0}^{T} d t(p(t) \dot{q}(t)-H(p, q))\right\} \delta(q(0)-q(T)) \tag{4.22}
\end{equation*}
$$

which is called the quantum partition function. From (4.21) we see that the quantum partition function describes the spectrum of the quantum Hamiltonian of the dynamical system and that the poles of its Fourier transform

$$
\begin{equation*}
G(E)=\int_{0}^{\infty} d T \mathrm{e}^{i E T / \hbar} Z(T) \tag{4.23}
\end{equation*}
$$

[^15]give the bound state spectrum of the system [83]. The quantity (4.23) is none other than the energy Green's function which is associated with the Schrödinger equation (4.14). Thus the quantum partition function is in some sense the fundamental quantity which describes the quantum dynamics (i.e. the energy spectrum) of a Hamiltonian system.

Finally, the generalization to an arbitrary symplectic manifold $(\mathcal{M}, \omega)$ of dimension $2 n$ is immediate. The factor $p \dot{q}$ becomes simply $p_{\mu} \dot{q}^{\mu}$ in higher dimensions, and, in view of (3.19), the canonical form of this is $\theta_{\mu}(x) \dot{x}^{\mu}$ in an arbitrary coordinate system on $\mathcal{M}$. Likewise, the phase space measure $d p \wedge d q$ according to (3.23) should be replaced by the canonically-invariant Liouville measure (3.22). Thus the quantum partition function for a generic dynamical system $(\mathcal{M}, \omega, H)$ is just

$$
\begin{equation*}
Z(T)=\int_{L \mathcal{M}}\left[d \mu_{L}(x)\right] \mathrm{e}^{i S[x]}=\int_{L \mathcal{M}}\left[d^{2 n} x\right] \prod_{t \in[0, T]} \sqrt{\operatorname{det}\|\omega(x(t))\|} \mathrm{e}^{i S[x]} \tag{4.24}
\end{equation*}
$$

where

$$
\begin{equation*}
S[x]=\int_{0}^{T} d t\left(\theta_{\mu}(x) \dot{x}^{\mu}-H(x)\right) \tag{4.25}
\end{equation*}
$$

is the classical action of the Hamiltonian system. Here and in the following we shall set $\hbar \equiv 1$ for simplicity, and the functional integration in (4.24) is taken over the loop space $L \mathcal{M}$ of $\mathcal{M}$, i.e. the infinite-dimensional space of paths $x(t):[0, T] \rightarrow \mathcal{M}$ obeying periodic boundary conditions $x^{\mu}(0)=x^{\mu}(T)$. Although much of the formalism which follows can be applied to path integrals over the larger trajectory space of all paths, we shall find it convenient to deal exclusively with the loop space over the phase space. The partition function (4.24) can be regarded as the formal infinite-dimensional analog of the classical integral (3.51), or, as mentioned before, the prototype of a topological field theory functional integral regarded as a $(0+1)$-dimensional quantum field theory. In the latter application the discrete index sums over $\mu$ contain as well integrals over the manifold on which the fields are defined.

Notice that the symplectic potential $\theta$ appearing in (4.24),(4.25) is only locally defined, and so some care must be taken in defining (4.24) when $\omega$ is not globally exact.

We shall discuss this procedure later on. Note also that the Liouville measure in (4.24), which is defined by the last equality in (4.20), differs from that of (4.22) in that in the latter case there is one extra momentum integration in the phase space Feynman measure (4.20), so that the endpoints are fixed and we integrate over all intermediate momenta. Thus one must carefully define appropriate boundary conditions for the integrations in (4.24) for the Schrödinger path integral measure in order to maintain a formal analogy between the finite and infinite dimensional cases, a point which we shall return to later on. A discussion of this and the proper discretizations and ordering prescriptions that are needed to define the functional integrations that appear above can be found in [116] and [76].

### 4.2 Loop Space Symplectic Geometry and Equivariant Cohomology

Following the lessons we have learned in the finite dimensional cases of the last 2 chapters, we shall now focus on some geometric methods of determining quantum partition functions of dynamical systems. Given the formulation of the path integral above on a general symplectic manifold, we can treat the problem of its exact evaluation within the geometric context of chapter 3. For this, we need a formulation of exterior and symplectic differential geometry on the loop space $L \mathcal{M}$ over the phase space $\mathcal{M}$. This will ultimately lead to a formal, infinite-dimensional generalization of the equivariant localization priniciple for path integrals, and thus formal conditions and methods for evaluating exactly these functional integrations which in general are far more difficult than their classical counterparts. As with the precise definition of the functional integrals above, we shall be rather cavalier here about the details of the geometry of the infinite-dimensional manifold $L \mathcal{M}$.

Given any functional $F[x]$ of closed paths on the loop space $L \mathcal{M}$, we define functional
differentiation, for which functional integration is the anti-derivative thereof, by the rule

$$
\begin{equation*}
\frac{\delta}{\delta x^{\mu}(t)} F\left[x\left(t^{\prime}\right)\right]=\delta\left(t-t^{\prime}\right) F^{\prime}\left[x\left(t^{\prime}\right)\right] \tag{4.26}
\end{equation*}
$$

We define the exterior algebra $L \Lambda \mathcal{M}$ by lifting the Grassmann generators $\eta^{\mu}$ of $\Lambda \mathcal{M}$ to anti-commuting periodic paths $\eta^{\mu}(t)$ which generate $L \Lambda \mathcal{M}$ and which are to be identified as the basis $d x^{\mu}(t)$ of loop space 1-forms. With this, we can define loop space differential $k$-forms

$$
\begin{equation*}
\alpha=\int_{0}^{T} d t_{1} \cdots d t_{k} \frac{1}{k!} \alpha_{\mu_{1} \cdots \mu_{k}}\left[x ; t_{1}, \ldots, t_{k}\right] \eta^{\mu_{1}}\left(t_{1}\right) \cdots \eta^{\mu_{k}}\left(t_{k}\right) \tag{4.27}
\end{equation*}
$$

and the loop space exterior derivative is defined by lifting the exterior derivative of the phase space $\mathcal{M}$,

$$
\begin{equation*}
d_{L}=\int_{0}^{T} d t \eta^{\mu}(t) \frac{\delta}{\delta x^{\mu}(t)} \tag{4.28}
\end{equation*}
$$

The loop space symplectic geometry is determined by a loop space symplectic 2-form

$$
\begin{equation*}
\Omega=\int_{0}^{T} d t d t^{\prime} \frac{1}{2} \Omega_{\mu \nu}\left[x ; t, t^{\prime}\right] \eta^{\mu}(t) \eta^{\nu}\left(t^{\prime}\right) \tag{4.29}
\end{equation*}
$$

which is closed

$$
\begin{equation*}
d_{L} \Omega=0 \tag{4.30}
\end{equation*}
$$

or in local coordinates $x^{\mu}(t)$ on $L \mathcal{M}$,

$$
\begin{equation*}
\frac{\delta}{\delta x^{\mu}(t)} \Omega_{\nu \lambda}\left[x ; t^{\prime}, t^{\prime \prime}\right]+\frac{\delta}{\delta x^{\nu}(t)} \Omega_{\lambda \mu}\left[x ; t^{\prime}, t^{\prime \prime}\right]+\frac{\delta}{\delta x^{\lambda}(t)} \Omega_{\mu \nu}\left[x ; t^{\prime}, t^{\prime \prime}\right]=0 \tag{4.31}
\end{equation*}
$$

Thus we can apply the infinite-dimensional version of Poincaré's lemma to represent $\Omega$ locally in terms of the exterior derivative of a loop space 1-form

$$
\begin{equation*}
\vartheta=\int_{0}^{T} d t \vartheta_{\mu}[x ; t] \eta^{\mu}(t) \tag{4.32}
\end{equation*}
$$

as

$$
\begin{equation*}
\Omega=d_{L} \vartheta \tag{4.33}
\end{equation*}
$$

We further assume that (4.29) is non-degenerate, i.e. the matrix $\Omega_{\mu \nu}\left[x ; t, t^{\prime}\right]$ is invertible on the loop space.

The canonical choice of symplectic structure on $L \mathcal{M}$ which coincides with the loop space Liouville measure introduced in (4.24) is that which is induced from the symplectic structure of the phase space,

$$
\begin{equation*}
\Omega_{\mu \nu}\left[x ; t, t^{\prime}\right]=\omega_{\mu \nu}(x(t)) \delta\left(t-t^{\prime}\right) \tag{4.34}
\end{equation*}
$$

which is diagonal in the loop space indices $t, t^{\prime}$. We shall use similar liftings of other quantities from the phase space to the loop space. In this way, elements $\alpha(x)$ of $L \Lambda_{x} \mathcal{M}$ (or $L T_{x} \mathcal{M}$ ) at a loop $x \in L \mathcal{M}$ are regarded as deformations of the loop, i.e. as elements of $\Lambda \mathcal{M}$ (or $T \mathcal{M}$ ) restricted to the loop $x^{\mu}(t)$ such that $\alpha[x ; t] \in \Lambda_{x(t)} \mathcal{M}$ (or $\left.T_{x(t)} \mathcal{M}\right)$. This means that these vector bundles over $L \mathcal{M}$ are infinite-dimensional spaces of sections of the pull-back of the phase space bundles to $[0, T]$ by the $\operatorname{map} x(t):[0, T] \rightarrow \mathcal{M}$. In particular, we define loop space canonical transformations as loop space changes of variable $F[x(t)]$ that leave $\Omega$ invariant. These are the transformations of the form

$$
\begin{equation*}
\vartheta \xrightarrow{F} \vartheta_{F}=\vartheta+d_{L} F \tag{4.35}
\end{equation*}
$$

Thus in the context of the loop space symplectic geometry determined by (4.34), the quantum partition function is an integral over the infinite-dimensional symplectic manifold ( $L \mathcal{M}, \Omega$ ) with the loop space Liouville measure there determined by the canonicallyinvariant closed form on $L \mathcal{M}$ given by exterior products of $\Omega$ with itself,

$$
\begin{equation*}
\left[d \mu_{L}(x)\right]=\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \tag{4.36}
\end{equation*}
$$

The loop space Hamiltonian vector field associated with the action (4.25) has components

$$
\begin{equation*}
V_{S}^{\mu}[x ; t]=\int_{0}^{T} d t^{\prime} \Omega^{\mu \nu}\left[x ; t, t^{\prime}\right] \frac{\delta S[x]}{\delta x^{\nu}\left(t^{\prime}\right)}=\dot{x}^{\mu}(t)-V^{\mu}(x(t)) \tag{4.37}
\end{equation*}
$$

with $V^{\mu}=\omega^{\mu \nu} \partial_{\nu} H$ as usual the Hamiltonian vector field on $\mathcal{M}$. The zeroes of $V_{S}$

$$
\begin{equation*}
L \mathcal{M}_{S}=\left\{x(t) \in L \mathcal{M}: V_{S}[x(t)]=0\right\} \tag{4.38}
\end{equation*}
$$

are the extrema of the action (4.25) and coincide with the classical trajectories of the dynamical system, i.e. the solutions of the classical Hamilton equations of motion. The loop space contraction operator with respect to a loop space vector field $W^{\mu}[x ; t]$ is given by

$$
\begin{equation*}
i_{W}=\int_{0}^{T} d t W^{\mu}[x ; t] \frac{\delta}{\delta \eta^{\mu}(t)} \tag{4.39}
\end{equation*}
$$

Thus we can define a loop space equivariant exterior derivative

$$
\begin{equation*}
Q_{W}=d_{L}+i_{W} \tag{4.40}
\end{equation*}
$$

whose square is the Lie derivative along the loop space vector field $W$,

$$
\begin{equation*}
Q_{W}^{2}=d_{L} i_{W}+i_{W} d_{L}=\mathcal{L}_{W}=\int_{0}^{T} d t\left(W^{\mu} \frac{\delta}{\delta x^{\mu}}+\partial_{\nu} W^{\mu} \eta^{\nu} \frac{\delta}{\delta \eta^{\mu}}\right) \tag{4.41}
\end{equation*}
$$

When $W=V_{S}$ is the loop space Hamiltonian vector field, we shall for ease denote the corresponding operators above as $i_{V_{S}} \equiv i_{S}$, etc.

The partition function can be written as in the finite-dimensional case using the functional Berezin integration rules to absorb the determinant factor into the exponential in terms of the anti-commuting periodic fields $\eta^{\mu}(t)$,

$$
\begin{align*}
Z(T) & =\int_{L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \exp \left\{i S[x]+\frac{i}{2} \int_{0}^{T} d t \omega_{\mu \nu}(x(t)) \eta^{\mu}(t) \eta^{\nu}(t)\right\}  \tag{4.42}\\
& =\int_{L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}^{[ }}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \mathrm{e}^{i(S[x]+\Omega[x, \eta])}
\end{align*}
$$

so that in this way $Z(T)$ is written in terms of an augmented action $S+\Omega$ on the superloop space $L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}$. From this we can now formally describe the $S^{1}$-equivariant cohomology of the loop space.

The operator $Q_{S}$ is nilpotent on the subspace

$$
\begin{equation*}
L \Lambda_{S} \mathcal{M}=\left\{\alpha \in L \Lambda \mathcal{M}: \mathcal{L}_{S} \alpha=0\right\} \tag{4.43}
\end{equation*}
$$

of equivariant loop space functionals. The loop space observable $S[x]$ defines the loop space Hamiltonian vector field through

$$
\begin{equation*}
d_{L} S=-i_{S} \Omega \tag{4.44}
\end{equation*}
$$

from which it follows that the integrand of the quantum partition function (4.42) is equivariantly closed,

$$
\begin{equation*}
Q_{S}(S+\Omega)=\left(d_{L}+i_{S}\right)(S+\Omega)=0 \tag{4.45}
\end{equation*}
$$

and so the augmented action $S+\Omega$ can be locally represented as the equivariant exterior derivative of a 1 -form $\hat{\boldsymbol{\vartheta}}$,

$$
\begin{equation*}
S+\Omega=Q_{S} \hat{\vartheta}=\int_{0}^{T} d t\left(V_{S}^{\mu} \hat{\vartheta}_{\mu}+\frac{1}{2} \Omega_{\mu \nu} \eta^{\mu} \eta^{\nu}\right) \tag{4.46}
\end{equation*}
$$

From (4.45) we find that

$$
\begin{equation*}
Q_{S}^{2} \hat{\vartheta}=\mathcal{L}_{S} \hat{\vartheta}=0 \tag{4.47}
\end{equation*}
$$

and so $\hat{\vartheta}$ lies in the subspace (4.43). If $\Phi_{S}$ is some globally defined loop space 0 -form with

$$
\begin{equation*}
\mathcal{L}_{S}\left(d_{L} \Phi_{S}\right)=0 \tag{4.48}
\end{equation*}
$$

then we see that $\hat{\vartheta}$ is not unique but the augmented action (4.46) is invariant under the loop space canonical transformation

$$
\begin{equation*}
\hat{\vartheta} \rightarrow \hat{\vartheta}+d_{L} \Phi_{S} \tag{4.49}
\end{equation*}
$$

Thus the partition function (4.42) has a very definite interpretation in terms of the loop space equivariant cohomology $H_{S}(L \mathcal{M})$ determined by the operator $Q_{S}$ on $L \Lambda_{S} \mathcal{M}$.

### 4.3 Supersymmetry and the Loop Space Localization Principle

The fact that the integrand of the partition function above can be interpreted in terms of a loop space equivariant cohomology suggests that we can localize it by choosing an appropriate representative of the loop space equivariant cohomology class determined by the augmented action $S+\Omega$. However, the arguments which showed in the finitedimensional cases that the partition function integral is invariant under such deformations cannot be straightforwardly applied here since there is no direct analog of Stokes' theorem for infinite-dimensional manifolds. Nonetheless, the localization priniciple can be
established by interpreting the equivariant cohomological structure on $L \mathcal{M}$ as a "hidden" supersymmetry of the quantum theory. In this way one has a sort of Stokes' theorem in the form of a Ward identity associated with this supersymmetry, where we interpret the fundamental localization property (2.106) as an infinitesimal change of variables in the integral. The partition function (4.42) can be interpreted as a BRST gauge-fixed path integral [17] with the $\eta^{\mu}(t)$ viewed as fermionic ghost fields and $x^{\mu}(t)$ as the fundamental bosonic fields of the model. The supersymmetry is suggested by the ungraded structure of $Q_{S}$ on $L \Lambda_{S} \mathcal{M}$ which maps even-degree loop space forms (bosons) into odd-degree forms (fermions). Since the fermion fields $\eta^{\mu}(t)$ appear by themselves without a conjugate partner, this determines an $N=\frac{1}{2}$ supersymmetry (the $N$ in general denoting the number of adjoint fermion pairs and corresponding supersymmetry charges $\bar{Q}^{i} Q^{i}$ ). The $N=\frac{1}{2}$ supersymmetry algebra $Q_{S}^{2}=\mathcal{L}_{S}$ implies that $Q_{S}$ is a supersymmetry charge on the subspace $L \Lambda_{S} \mathcal{M}$, and the augmented action is supersymmetric, $Q_{S}(S+\Omega)=0$. Thus here $L \Lambda_{S} \mathcal{M}$ coincides with the BRST complex of physical states, and the BRST transformations of the fundamental bosonic fields $x^{\mu}(t)$ and their superpartners $\eta^{\mu}(t)$ are ${ }^{4}$

$$
\begin{equation*}
Q_{S} x^{\mu}=\eta^{\mu} \quad, \quad Q_{S} \eta^{\mu}=V_{S}^{\mu} \tag{4.50}
\end{equation*}
$$

This formal identification of the equivariant cohomological structure as a hidden supersymmetry allows one to interpret the quantum theory as a supersymmetric or topological field theory. It was Blau, Keski-Vakkuri and Niemi [25] who first realized that a

[^16]quite general localization principle could be formulated for path integrals using rather formal functional techniques introduced in the BRST quantization of first class constrained systems [95]. In these theories a BRST transformation produces a super-Jacobian on the super-loop space $L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}$ whose corrections are related to anomalies and BRST supersymmetry breaking. The arguments below are therefore valid provided that the $Q_{S}$-supersymmetry above is not broken in the quantum theory.

The argument for infinite-dimensional localization proceeds as follows. Consider the 1-parameter family of phase space path integrals

$$
\begin{equation*}
\mathcal{Z}(\lambda)=\int_{L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \mathrm{e}^{i\left(S[x]+\Omega[x, \eta]+\lambda Q_{S} \psi[x, \eta]\right)} \tag{4.51}
\end{equation*}
$$

where $\lambda \in \mathbb{R}$ and $\psi \in L \Lambda_{S}^{1} \mathcal{M}$ is a gauged fermion field which is homotopic to 0 under the supersymmetry transformation generated by $Q_{S}$. As in the finite-dimensional case, we wish to establish the $\lambda$-independence of this path integral. This amounts to a choice of representative of $S+\Omega$ in its loop space equivariant cohomology class and different choices of non-trivial representatives then lead to the desired localization schemes. Consider an infinitesimal variation $\lambda \rightarrow \lambda+\delta \lambda$ of the argument of (4.51). Let $\psi \rightarrow \psi+\delta \psi$ with

$$
\begin{equation*}
\delta \psi=\delta \lambda \cdot \psi \tag{4.52}
\end{equation*}
$$

and consider the infinitesimal supersymmetry transformation on the super-loop space parametrized by the gauge fermion $\delta \psi \in L \Lambda^{1} \mathcal{M}$,

$$
\begin{align*}
& x^{\mu} \rightarrow \bar{x}^{\mu}=x^{\mu}+\delta x^{\mu}=x^{\mu}+\delta \psi \cdot Q_{S} x^{\mu}=x^{\mu}+\delta \psi \cdot \eta^{\mu} \\
& \eta^{\mu} \rightarrow \bar{\eta}^{\mu}=\eta^{\mu}+\delta \eta^{\mu}=\eta^{\mu}+\delta \psi \cdot Q_{S} \eta^{\mu}=\eta^{\mu}+\delta \psi \cdot V_{S}^{\mu} \tag{4.53}
\end{align*}
$$

Since $Q_{S}(S+\Omega)=\mathcal{L}_{S} \psi=0$, the argument of the path integral (4.51) is BRST-invariant.
However, the corresponding super-Jacobian arising in the Feynman measure in (4.51) on $L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}$ is non-trivial and it has precisely the same functional form as that in a standard BRST transformation [95]. The pertinent super-Jacobian here is given by the
super-determinant

$$
\left[d^{2 n} \bar{x}\right]\left[d^{2 n} \bar{\eta}\right]=\operatorname{sdet}\left\|\begin{array}{ll}
\frac{\delta \bar{x}}{\delta x} & \frac{\delta \bar{x}}{\delta \eta}  \tag{4.54}\\
\frac{\delta \bar{\eta}}{\delta x} & \frac{\delta \bar{\eta}}{\delta \eta}
\end{array}\right\|\left[d^{2 n} x\right]\left[d^{2 n} \eta\right]
$$

and the path integral (4.51) is invariant under arbitrary changes of variables. For infinitesimal $\delta \lambda$, the identity

$$
\begin{equation*}
\operatorname{tr} \log \|A\|=\log \operatorname{det}\|A\| \tag{4.55}
\end{equation*}
$$

implies that the super-determinant in (4.54) can be computed in terms of the super-trace, the loop space sum of the diagonal entries in (4.54), as sdet $\|A\|=1+\operatorname{str}\|A\|$. This gives

$$
\begin{align*}
{\left[d^{2 n} \bar{x}\right]\left[d^{2 n} \bar{\eta}\right] } & =\left\{1+\int_{0}^{T} d t\left(\frac{\delta}{\delta x^{\mu}}(\delta \psi) \eta^{\mu}+\frac{\delta}{\delta \eta^{\mu}}(\delta \psi) V_{S}^{\mu}\right)\right\}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \\
& =\left\{1+\int_{0}^{T} d t\left(\eta^{\mu} \frac{\delta}{\delta x^{\mu}}+V_{S}^{\mu} \frac{\delta}{\delta \eta^{\mu}}\right) \delta \psi\right\}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right]  \tag{4.56}\\
& =\left(1+Q_{S} \delta \psi\right)\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \sim \mathrm{e}^{\delta \lambda \cdot Q_{S} \psi}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right]
\end{align*}
$$

Thus substituting the change of variables (4.53) with super-Jacobian (4.56) into the path integral (4.51) we immediately see that

$$
\begin{equation*}
\mathcal{Z}(\lambda)=\mathcal{Z}(\lambda+\delta \lambda) \tag{4.57}
\end{equation*}
$$

which establishes the independence of the path integral (4.51) under homotopically-trivial deformations which live in the subspace (4.43). This proof of the $\lambda$-independence (or the $\psi$-independence more generally) of (4.51) is a specialization of the Fradkin-Vilkovisky theorem [95] to the supersymmetric theory above, which states that local supersymmetric variations of gauge fermions in a supersymmetric BRST gauge-fixed path integral leave it invariant. Indeed, the addition of the BRST-exact term $Q_{S} \psi$ can be regarded as a gauge-fixing term (the reason why $\psi$ is termed here a 'gauge fermion') which renormalizes the theory but leaves it invariant under these perturbative deformations. The addition of this term to the action of the quantum theory above is therefore regarded as
a topological deformation, in that it does not change the value of the original partition function which is the $\lambda \rightarrow 0$ limit of (4.51) above. This is consistent with the general ideas of topological field theory, in which a supersymmetric BRST-exact action is known to have no propagating degrees of freedom and so can only describe topological invariants of the underlying space. We shall discuss these more topological aspects of BRST-exact path integrals, also known as Witten-type topological field theories [17], in due course. In any case, we can now write down the loop space localization principle

$$
\begin{equation*}
Z(T)=\lim _{\lambda \rightarrow \infty} \int_{L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \mathrm{e}^{i\left(S[x]+\Omega[x, \eta]+\lambda Q_{S} \psi[x, \eta]\right)} \tag{4.58}
\end{equation*}
$$

so that the quantum partition function localizes onto the zeroes of the gauge fermion field $\psi$.

Given the localization property (4.58) of the quantum theory, we would now like to pick a suitable representative $\psi$ making the localization manifest. As in the finite dimensional cases, the localizations of interest both physically and mathematically are usually the fixed point locuses of loop space vector fields $W$ on $L \mathcal{M}$. To translate this into a loop space differential form, we introduce a metric tensor $G$ on the loop space and take $\psi$ to be the associated metric-dual form

$$
\begin{equation*}
\psi=\int_{0}^{T} d t d t^{\prime} G_{\mu \nu}\left[x ; t, t^{\prime}\right] W^{\mu}[x ; t] \eta^{\nu}\left(t^{\prime}\right) \tag{4.59}
\end{equation*}
$$

of the loop space vector field $W$. The supersymmetry condition $\mathcal{L}_{S} \psi=0$ is then equivalent to the Killing equation $\mathcal{L}_{S} G=0$ and the additional requirement $\mathcal{L}_{S} W=0$ on $W^{5}$, where

$$
\begin{equation*}
\mathcal{L}_{S} W=\int_{0}^{T} d t\left(\frac{d}{d t}-\mathcal{L}_{V(x(t))}\right) W[x: t] \tag{4.60}
\end{equation*}
$$

There are many useful choices for $W$ obeying such a restriction, but we shall be concerned

[^17]mostly with those which can be summarized in
\[

$$
\begin{equation*}
W^{\mu}[x ; t]=r \dot{x}^{\mu}(t)-s V^{\mu}(x(t)) \tag{4.61}
\end{equation*}
$$

\]

where the parameters $r, s$ are chosen appropriate to the desired localization scheme.
As for the metric in (4.59), there are also many possibilities. However, there only seems to be 1 general class of loop space metric tensors to which general arguments and analyses can be applied. To motivate these, we note first that the equivariant exterior derivative $Q_{S}$ can be written as

$$
\begin{equation*}
Q_{S}=Q_{\dot{x}}-i_{V}=d_{L}+i_{\dot{x}}-i_{V} \tag{4.62}
\end{equation*}
$$

and the square of the operator $Q_{\dot{x}}$ is just the generator of time translations

$$
\begin{equation*}
Q_{\dot{x}}^{2}=\mathcal{L}_{\dot{x}}=d_{L} i_{\dot{x}}+i_{\dot{x}} d_{L}=\int_{0}^{T} d t \frac{d}{d t} \tag{4.63}
\end{equation*}
$$

This operator arises when we assume that the loop space Hamiltonian vector field generates an $S^{1}$-flow on the loop space, parametrized by a parameter $\tau \in[0,1]$ so that the flow is $x^{\mu}(t) \rightarrow x^{\mu}(t ; \tau)$ with $x^{\mu}(t ; 0)=x^{\mu}(t ; 1)$, such that in the selected loop space coordinates $x^{\mu}(t)$ the flow parameter $\tau$ also shifts the loop (time) parameter $t \rightarrow t+\tau$. In this case we have

$$
\begin{equation*}
V_{S}^{\mu}[x ; t]=\left.\frac{\partial x^{\mu}(t ; \tau)}{\partial \tau}\right|_{\tau=0}=\dot{x}^{\mu}(t) \tag{4.64}
\end{equation*}
$$

and the supersymmetry transformation (4.50) becomes

$$
\begin{equation*}
Q_{\dot{x}} x^{\mu}=\eta^{\mu} \quad, \quad Q_{\dot{x}} \eta^{\mu}=\dot{x}^{\mu} \tag{4.65}
\end{equation*}
$$

In particular, the effective action is now (locally) of the functional form

$$
\begin{equation*}
S+\Omega=\int_{0}^{T} d t\left(\theta_{\mu}(x) \dot{x}^{\mu}+\frac{1}{2} \omega_{\mu \nu}(x(t)) \eta^{\mu} \eta^{\nu}\right)=\left(d_{L}+i_{\dot{x}}\right) \vartheta=Q_{\dot{x}} \hat{\vartheta} \tag{4.66}
\end{equation*}
$$

and the topological invariance of the quantum theory, i.e. the invariance of (4.66) under BRST-deformations by elements $\psi$ of the subspace $L \Lambda_{S}^{1} \mathcal{M}$, is according to (4.63) determined by arbitrary globally defined single-valued functionals on $L \mathcal{M}$, i.e. $\psi(0)=\psi(T)$.

This form of the $U(1)$-equivariant cohomology on the loop space is called the modelindependent circle action.

We shall therefore demand that the localization functionals in (4.59) be invariant under the model-independent $S^{1}$-action on $L \mathcal{M}$ (i.e. rigid rotations $x(t) \rightarrow x(t+\tau)$ of the loops). This requires that the loop space metric tensor above obey $\mathcal{L}_{\dot{x}} G=0$, or equivalently that $G_{\mu \nu}\left[x ; t, t^{\prime}\right]=G_{\mu \nu}\left[x ; t-t^{\prime}\right]$ is diagonal in its loop space indices. Since the quantum theory is to describe the dynamics of a given Hamiltonian system for which we know the underlying manifold $\mathcal{M}$, the best way to pick the Riemannian structure on $L \mathcal{M}$ is to lift a metric tensor $g$ from $\mathcal{M}$ so that $G$ takes the ultra-local form

$$
\begin{equation*}
G_{\mu \nu}\left[x ; t, t^{\prime}\right]=g_{\mu \nu}(x(t)) \delta\left(t-t^{\prime}\right) \tag{4.67}
\end{equation*}
$$

and its action on loop space vector fields is given by

$$
\begin{equation*}
G\left(V_{1} ; V_{2}\right)=\int_{0}^{T} d t g_{\mu \nu}(x(t)) V_{1}^{\mu}[x ; t] V_{2}^{\nu}[x ; t] \tag{4.68}
\end{equation*}
$$

Because of the reparametrization invariance of the integral (4.68), the metric tensor $G$ is invariant under the flow generated by $\dot{x}$. The Lie derivative condition on $G$ is then equivalent to the Lie derivative condition (2.91) with respect to the Hamiltonian vector field $V$ on $\mathcal{M}$. Thus infinite dimensional localization requires as well that the phase space $\mathcal{M}$ admit a globally-defined $U(1)$-invariant Riemannian structure on $\mathcal{M}$. As discussed before, the condition that the Hamiltonian $H$ generates an isometry of a metric $g$ on $\mathcal{M}$ (through the induced Poisson structure on $(\mathcal{M}, \omega)$ ) is a very restrictive condition on the Hamiltonian dynamics. Essentially it means that $H$ must be related to the global action (2.32) of a group $G$ on $\mathcal{M}$. We shall analyse this feature of equivariant localization very carefully in the following.

As we mentioned earlier, the infinite dimensional results above, in particular the evaluation of the super-Jacobian in (4.56), are as reliable as the corresponding calculations in standard BRST quantization, provided that the boundary conditions in (4.51) are also
supersymmetric. Provided that the assumptions on the classical properties of the Hamiltonian are satisfied (as for the finite-dimensional cases), the above derivation will stand correct unless the supersymmetry $Q_{S}^{2}=\mathcal{L}_{S}$ is broken in the quantum theory, for instance by a scale anomaly in the rescaling of the metric $G_{\mu \nu} \rightarrow \lambda \cdot G_{\mu \nu}$ above. Recently, Nersessian [93] has naturally incorporated the geometrical objects of the Batalin-Vilkovisky formalism [10, 17, 58, 95] (i.e. a Grassmann-odd degree symplectic structure and a corresponding nilpotent Hamiltonian operator) into the equivariant localization formalism. The presence of supersymmetric bi-Hamiltonian dynamics with even and odd symplectic structures allows novel proofs of the localization principles that eludes many of the geometric restrictions above of the standard equivariant localization constraints. It does, however, require a rather large supersymmetric structure in the quantum theory, but it leads to a derivation of the localization principle via the Batalin-Vilkovisky formalism instead of the standard BRST approach above. For a discussion of the connection this implies between the BRST model of equivariant cohomology and the Batalin-FradkinVilkovisky approach to Hamiltonian BRST quantization of constrained systems, see [103].

### 4.4 The WKB Localization Formula

We shall now begin examining the various types of localization formulas that can be derived from the general principles of the last section. The first infinite-dimensional localization formula that we shall present is the formal generalization of the DuistermaatHeckman integration formula, whose derivation follows the loop space versions of the steps used in sections 2.5 and 3.3. We assume that the action $S$ has isolated and nondegenerate critical trajectories, so that the zero locus (4.38) consists of isolated classical loops in $L \mathcal{M}$. We further assume that the determinant of the associated Jacobi fields arising from a second-order variation of $S$ is non-vanishing on these classical trajectories.

Under these assumptions, we set $r=s=1$ in (4.61), so that

$$
\begin{equation*}
\psi=\int_{0}^{T} d t g_{\mu \nu} V_{S}^{\mu} \eta^{\nu} \quad, \quad Q_{S} \psi=\int_{0}^{T} d t\left[g_{\mu \nu} V_{S}^{\mu} V_{S}^{\nu}+\eta^{\mu}\left(g_{\mu \nu} \partial_{t}-g_{\nu \lambda} \partial_{\mu} V^{\lambda}+V_{S}^{\lambda} \partial_{\mu} g_{\nu \lambda}\right) \eta^{\nu}\right] \tag{4.69}
\end{equation*}
$$

Proceeding just as in the finite-dimensional case, the evaluation of the localization integral (4.58) gives

$$
\begin{align*}
Z(T) & \sim \int_{L \mathcal{M}}\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \sqrt{\operatorname{det}\left\|\delta V_{S}\right\|} \delta\left(V_{S}\right) \mathrm{e}^{i S[x]} \\
& \sim \int_{L \mathcal{M}}\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \sqrt{\operatorname{det}\left\|\delta_{\nu}^{\mu} \partial_{t}-\partial_{\nu}\left(\omega^{\mu \lambda} \partial_{\lambda} H\right)\right\|} \delta\left(\dot{x}^{\mu}-\omega^{\mu \nu} \partial_{\nu} H\right) \mathrm{e}^{i S[x]} \\
& \sim \sum_{x(t) \in L \mathcal{M}_{s}} \frac{\sqrt{\operatorname{det}\|\omega(x(t))\|} \mathrm{e}^{i S[x]}}{\sqrt{\operatorname{det}\left\|\delta_{\nu}^{\mu} \partial_{t}-\partial_{\nu}\left(\omega^{\mu \lambda} \partial_{\lambda} H\right)\right\|}} \tag{4.70}
\end{align*}
$$

where here and in the following the symbol $\sim$ will be used to signify the absorption of infinite prefactors into the determinants which arise from the functional Gaussian integrations. We shall discuss the regularization and evaluation of these infinite factors and functional determinants [84] in (4.70) in the next section.

This is the famous WKB approximation to the partition function [116], except that it is summed over all classical paths and not just those which minimize the action $S$. If we reinstate the factors of $\hbar$, then it is formally the leading term of the stationary phase expansion of the partition function in powers of $\hbar$ as $\hbar \rightarrow 0$. The limit $\hbar \rightarrow 0$ is called the classical limit of the quantum mechanics problem above, since then according to (4.1) the operators $\hat{p}$ and $\hat{q}$ behave as ordinary commuting $c$-numbers as in the classical theory. For $\hbar \rightarrow 0$ we can naturally evaluate the path integral by the stationary-phase method discussed in section 3.3, i.e. we expand the trajectories $x(t)=x_{0}(t)+\delta x(t)$ in the action, with $x_{0}(t) \in L \mathcal{M}_{S}$ and $\delta x(t)$ the fluctuations about the classical paths $x_{0}(t)$ with $\delta x(0)=\delta x(T)=0$, and then carry out the leading Gaussian functional integration over these fluctuations. Indeed, this was the way Feynman originally introduced the path integral to describe quantum mechanics as a sum over trajectories which fluctuate around the classical paths of the system. This presentation of quantum mechanics thus
leads to the dynamical Hamilton action principle of classical mechanics [48], i.e. the classical paths of motion of a dynamical system are those which minimize the action, as a limiting case. If the classical trajectory were unique, then we would only obtain the factor $e^{i S[x] / \hbar}$ above as $\hbar \rightarrow 0$. Quantum mechanics can then be interpreted as implying fluctuations (the one-loop determinant factors in (4.70)) around this classical trajectory. The higher-loop quantum fluctuation terms when (4.70) is not the exact result will be discussed in chapter 7.

We should point out here that the standard WKB formulas are usually given for configuration space path integrals where the determinant $\left(\operatorname{det}\left\|L_{S}(x(t))\right\|\right)^{-1 / 2}$ appearing in (4.70) is the so-called Van Vleck determinant which is essentially the Hessian of $S$ in the configuration space coordinates $q$. Here the determinant is the functional determinant of the Jacobi operator which arises from the usual Legendre transformation to phase space coordinates $(p, q)$. This operator is important in the Hamilton-Jacobi theory of classical mechanics [5, 48], and this determinant can be interpreted as the density of classical trajectories. The result (4.70) and the assumptions that went into deriving it, such as the non-vanishing of the determinant of the Jacobi fields and the existence of an invariant phase space metric, are certainly true for the classic examples in quantum mechanics and field theory where the semi-classical approximation is known to be exact, such as for the propagator of a particle moving on a group manifold [32, 110, 115]. The above localization principle yields sufficient, geometric conditions for when a given path integral is given exactly by its WKB approximation, and it therefore has the possibility of expanding the set of quantum systems for which the Feynman path integral is WKB exact and localizes onto the classical trajectories of the system.

### 4.5 Degenerate Path Integrals and the Niemi-Tirkkonen Localization Formula

There are many instances in which the WKB approximation is unsuitable for a quantum mechanical path integral, such as a dynamical system whose classical phase space trajectories coalesce at some point. It is therefore desirable to seek alternative, more general localization formulas which can be applied to larger classes of quantum systems. Niemi and Palo [98] have investigated the types of degeneracies that can occur for phase space path integrals and have argued that for Hamiltonians which generate circle actions the classical trajectories can be characterized as follows. In general, the critical point set of the action $S$ with non-trivial periodic solutions $x^{\mu}(T)=x^{\mu}(0)=x_{0}^{\mu}$ lie on a compact submanifold $L \mathcal{M}_{S}$ of the phase space $\mathcal{M}$. In this context, $L \mathcal{M}_{S}$ is referred to as the moduli space of $T$-periodic classical solutions and it is in general a non-isolated set for only some discrete values of the propagation time $T$. For generic values of $T$ the periodic solutions with $x^{\mu}(T)=x^{\mu}(0)=x_{0}^{\mu}$ exist only if $x_{0}^{\mu}$ lies on the critical submanifold $\mathcal{M}_{V}$ of the Hamiltonian $H$. Then the classical equations of motion reduce to $\dot{x}^{\mu}=V^{\mu}=\omega^{\mu \nu} \partial_{\nu} H=0$ and so the moduli space $L \mathcal{M}_{S}$ coincides with the critical point set $\mathcal{M}_{V} \subset \mathcal{M}$. We shall see some specific examples of this later on.

With this in mind we can derive a loop space analog of the degenerate DuistermaatHeckman formula of Section 3.7. We decompose $L \mathcal{M}$ and $L \Lambda^{1} \mathcal{M}$ into classical modes and fluctuations about the classical solutions and scale the latter by $1 / \sqrt{\lambda}$,

$$
\begin{equation*}
x^{\mu}(t)=\bar{x}^{\mu}(t)+x_{f}^{\mu}(t) / \sqrt{\lambda} \quad, \quad \eta^{\mu}(t)=\bar{\eta}^{\mu}(t)+\eta_{f}^{\mu}(t) / \sqrt{\lambda} \tag{4.71}
\end{equation*}
$$

where $\bar{x}(t) \in L \mathcal{M}_{S}$ are the solutions of the classical equations of motion, i.e. $V_{S}(\bar{x}(t))=$ $\dot{\bar{x}}^{\mu}-\omega^{\mu \nu}(\bar{x}) \partial_{\nu} H(\bar{x})=0$, and $\bar{\eta}^{\mu}(t) \sim d \bar{x}^{\mu}(t) \in \Lambda^{1} L \mathcal{M}_{S}$ span the kernel of the loop space Riemann moment map,

$$
\begin{equation*}
\left(\Omega_{S}\right)_{\mu \nu}(\bar{x}) \bar{\eta}^{\nu}=0 \tag{4.72}
\end{equation*}
$$

where

$$
\begin{equation*}
\Omega_{S}=d_{L} \psi=\int_{0}^{T} d t \frac{\delta}{\delta x^{\mu}}\left(g_{\nu \lambda} V_{S}^{\lambda}\right) \eta^{\mu} \eta^{\nu} \tag{4.73}
\end{equation*}
$$

with $\psi$ given in (4.69). In particular, this implies that $\bar{\eta}^{\mu}(t)$ are Jacobi fields, i.e. they obey the fluctuation equation

$$
\begin{equation*}
\left(\delta_{\nu}^{\mu} \partial_{t}-\partial_{\nu} V^{\mu}(\bar{x})\right) \bar{\eta}^{\nu}=0 \tag{4.74}
\end{equation*}
$$

The fluctuation modes in (4.71) obey the boundary conditions $x_{f}^{\mu}(0)=x_{f}^{\mu}(T)=0$ and $\eta_{f}^{\mu}(0)=\eta_{f}^{\mu}(T)=0$.

The super-loop space measure with this decomposition is then

$$
\begin{equation*}
\left[d^{2 n} x\right]\left[d^{2 n} \eta\right]=d^{2 n} \bar{x}(t) d^{2 n} \bar{\eta}(t) \prod_{t \in[0, T]} d^{2 n} x_{f}(t) d^{2 n} \eta_{f}(t) \tag{4.75}
\end{equation*}
$$

where as usual the change of variables (4.71) has unit Jacobian because the determinants from the bosonic and fermionic fluctuations cancel (this is the powerful manifestation of the "hidden" supersymmetry in these theories). The calculation now proceeds analogously to that in section 3.7 , so that evaluating the Gaussian integral over the fluctuation modes localizes the path integral to a finite-dimensional integral over the moduli space $L \mathcal{M}_{S}$ of classical solutions,

$$
\begin{equation*}
\left.Z(T) \sim \int_{L \mathcal{M}_{S}} d^{2 n} \bar{x}(t) \frac{\sqrt{\operatorname{det} \omega(\bar{x})} \mathrm{e}^{i S[\bar{x}]}}{\operatorname{Pfaff}\left\|\delta_{\nu}^{\mu} \partial_{t}-\left(\mu_{S}\right)_{\nu}^{\mu}(\bar{x})-R_{\nu}^{\mu}(\bar{x})\right\|}\right|_{\mathcal{N L M}_{S}} \tag{4.76}
\end{equation*}
$$

where $\mu_{S}=g^{-1} \cdot \Omega_{S}$ and $R$ is as usual the Riemann curvature 2-form of the metric $g$ evaluated on $L \mathcal{M}_{S}$. In (4.76) the Pfaffian is taken over the fluctuation modes $x_{f}^{\mu}(t)$ about the classical trajectories $\bar{x}^{\mu}(t) \in L \mathcal{M}_{S}$ (i.e. along the normal bundle $\mathcal{N} L \mathcal{M}_{S}$ in $L \mathcal{M}$ ), and the measure there is an invariant measure over the moduli space of classical solutions which is itself a symplectic manifold. The localization formula (4.76) is the loop space version of the degenerate localization formula (3.114) in which the various factors can be interpreted as loop space extensions of the equivariant characteristic classes. In particular, note that in the limit where the solutions to the classical equations of motion
$V_{S}^{\mu}(x(t))=0$ become isolated and non-degenerate paths the integration formula (4.76) reduces to the standard WKB localization formula (4.70).

However, the degenerate localization formula (4.76) is hard to use in practise because in general the moduli space of classical solutions has a complicated, $T$-dependent structure ${ }^{6}$. We would therefore like to obtain alternative degenerate localization formulas which are applicable independently of the structure of the moduli space $L \mathcal{M}_{S}$ above. Given the form of (4.76), we could then hope to obtain a localization onto some sort of equivariant characteristic classes of the manifold $\mathcal{M}$. The first step in this direction was carried out by Niemi and Tirkkonen in [101]. Their localization formula can be derived by setting $s=0, r=1$ in (4.61) so that

$$
\begin{equation*}
\psi=\int_{0}^{T} d t g_{\mu \nu} \dot{x}^{\mu} \eta^{\nu} \quad, \quad Q_{S} \psi=\int_{0}^{T} d t\left[g_{\mu \nu} \dot{x}^{\mu}\left(\dot{x}^{\nu}-V^{\nu}\right)+\eta^{\mu}\left(g_{\mu \nu} \partial_{t}+\dot{x}^{\lambda} g_{\lambda \rho} \Gamma_{\mu \nu}^{\rho}\right) \eta^{\nu}\right] \tag{4.77}
\end{equation*}
$$

Here the zero locus of the vector field (4.61) consists of the constant loops $\dot{x}^{\mu}=0$, i.e. points on $\mathcal{M}$, so that the canonical localization integral will reduce to an integral over the finite-dimensional manifold $\mathcal{M}$, rather than a sum or integral over the moduli space of classical solutions as above.

To evaluate the right-hand side of (4.58) with (4.77), we use a trick analogous to that used above. We decompose $L \mathcal{M}$ and $L \Lambda^{1} \mathcal{M}$ into constant modes and fluctuation modes and scale the latter by $1 / \sqrt{\lambda}$,

$$
\begin{equation*}
x^{\mu}(t)=x_{0}^{\mu}+\hat{x}^{\mu}(t) / \sqrt{\lambda} \quad, \quad \eta^{\mu}(t)=\eta_{0}^{\mu}+\hat{\eta}^{\mu}(t) / \sqrt{\lambda} \tag{4.78}
\end{equation*}
$$

where

$$
\begin{gather*}
x_{0}^{\mu}=\frac{1}{T} \int_{0}^{T} d t x^{\mu}(t) \quad, \quad \eta_{0}^{\mu}=\frac{1}{T} \int_{0}^{T} d t \eta^{\mu}(t) \\
\partial_{t} x_{0}^{\mu}=\partial_{t} \eta_{0}^{\mu}=0 \quad, \quad \int_{0}^{T} d t \hat{x}^{\mu}(t)=\int_{0}^{T} d t \hat{\eta}^{\mu}(t)=0 \tag{4.79}
\end{gather*}
$$

[^18]The decomposition (4.78) is essentially a Fourier decompostion in terms of some complete sets of states $\left\{x_{k}^{\mu}(t)\right\}_{k \in \mathbb{Z}}$ and $\left\{\eta_{k}^{\mu}(t)\right\}_{k \in \mathbb{Z}}$, so that

$$
\begin{equation*}
\hat{x}^{\mu}(t)=\sum_{k \neq 0} s_{k}^{\mu} x_{k}^{\mu}(t) \quad, \quad \hat{\eta}^{\mu}(t)=\sum_{k \neq 0} \sigma_{k}^{\mu} \eta_{k}^{\mu}(t) \tag{4.80}
\end{equation*}
$$

and the Feynman measure in the path integral is then defined just as before as

$$
\begin{equation*}
\left[d^{2 n} x\right]\left[d^{2 n} \eta\right]=d^{2 n} x_{0} d^{2 n} \eta_{0} \prod_{t \in[0, T]} d^{2 n} \hat{x}(t) d^{2 n} \hat{\eta}(t)=d^{2 n} x_{0} d^{2 n} \eta_{0} \prod_{k \neq 0} d^{2 n} s_{k} d^{2 n} \sigma_{k} \tag{4.81}
\end{equation*}
$$

With the rescaling in (4.78) of the fluctuation modes, the gauge fixing term $Q_{S} \psi$ is

$$
\begin{equation*}
Q_{S} \psi=\int_{0}^{T} d t\left[x^{\mu}\left(\left(\Omega_{V}\right)_{\mu \nu} \partial_{t}-g_{\mu \nu} \partial_{t}^{2}\right) x^{\nu}+\frac{1}{2} R_{\mu \nu} x^{\mu} \dot{x}^{\nu}+\hat{\eta}^{\mu} g_{\mu \nu} \partial_{t} \hat{\eta}^{\nu}\right]+\mathcal{O}(1 / \sqrt{\lambda}) \tag{4.82}
\end{equation*}
$$

where we have integrated by parts over $t$ and used the periodic boundary conditions.
In (4.82) we see the appearence of the equivariant curvature of the Riemannian manifold $(\mathcal{M}, g)$. Since $\Omega_{V}$ and $R$ there act on the fluctuation modes, as usual they can be interpreted as forming the equivariant curvature of the normal bundle of $\mathcal{M}$ in $L \mathcal{M}$. With the above rescaling the fluctuation and zero modes decouple in the localization limit $\lambda \rightarrow \infty$, just as before. The integrations over the fluctuations are as usual Gaussian, and the result of these integrations is

$$
\begin{equation*}
Z(T) \sim \int_{\mathcal{M}} \operatorname{ch} V(-i T \omega) \wedge\left(\operatorname{det}^{\prime}\left\|\delta_{\nu}^{\mu} \partial_{t}-\left(R_{V}\right)_{\nu}^{\mu}\right\|\right)^{-1 / 2} \tag{4.83}
\end{equation*}
$$

where the prime on the determinant means that it is taken over the fluctuation modes with periodic boundary conditions (i.e. the determinant with zero modes excluded). This form of the partition function is completely analogous to the degenerate localization formula of section 3.7, and it is also similar to the formula (4.76), except that now the domain of integration has changed from the moduli space $L \mathcal{M}_{S}$ of classical solutions to the entire manifold $\mathcal{M}$. This makes the formula (4.83) much more appealing, in that there is no further reference to the $T$-dependent submanifold $L \mathcal{M}_{S}$ of $\mathcal{M}$. Note that (4.83) differs from the classical partition function for the dynamical system $(\mathcal{M}, \omega, H)$ by a one-loop
determinant factor which can be thought of as encoding the information due to quantum fluctuations.

Notice also that here the infinite-dimensional Pfaffian arising from the fermionic integration cancels from the result of the infinite-dimensional Gaussian integral over the bosonic fluctuation modes. Thus, just as in the finite-dimensional case, the sign dependence of the Pfaffian gets transferred to the inverse square root of the determinant. The spectral asymmetry associated with the sign of the infinite-dimensional Pfaffian (see (3.61)) has to be regulated and is given by the Atiyah-Patodi-Singer eta-invariant $[35,124]$ of the Dirac operator $\partial_{t}-R_{V}$,

$$
\begin{equation*}
\eta\left(\partial_{t}-R_{V}\right)=\lim _{s \rightarrow 0} f d \lambda \operatorname{sgn}(\lambda)|\lambda|^{-s} \tag{4.84}
\end{equation*}
$$

where the integration (and/or sum) is over all non-zero eigenvalues $\lambda$ of $\partial_{t}-R_{V}$. This eta-function regularization has been discussed in some detail recently in [84].

The localization formula (4.83) can be written in a much nicer form by evaluating the determinant using standard supersymmetry regularizations [4, 42] for first-order differential operators defined on a circle. The most convenient such choice is Riemann zeta-function regularization. The non-constant eigenfunctions of the operator $\partial_{t}$ on the interval $[0, T]$ with periodic boundary conditions are $\mathrm{e}^{2 \pi i k t / T}$, where $k$ are non-zero integers. Since the matrix $R_{V}$ is antisymmetric, it can be skew-diagonalized into $n 2 \times 2$ skew-diagonal blocks $R_{V}^{(j)}$ with skew eigenvalues $\lambda_{j}$, where $j=1, \ldots, n$. For each such block $R_{V}^{(j)}$, we get the formal contribution to the determinant in (4.83),

$$
\begin{equation*}
\operatorname{det}^{\prime}\left\|\partial_{t}-R_{V}^{(j)}\right\|=\prod_{k \neq 0}\left(\frac{2 \pi i k}{T}+\lambda_{j}\right)\left(\frac{2 \pi i k}{T}-\lambda_{j}\right)=g\left(T \lambda_{j} / 2 \pi i\right) g\left(-T \lambda_{j} / 2 \pi i\right) \prod_{k \neq 0}\left(\frac{2 \pi i}{T}\right)^{2} \tag{4.85}
\end{equation*}
$$

where we have defined the function $g(z)$ as the formal product

$$
\begin{equation*}
g(z)=\prod_{k \neq 0}(k+z) \tag{4.86}
\end{equation*}
$$

We can determine the regulated form of the function $g(z)$ by examining its logarithmic derivative $g^{\prime}(z) / g(z)$ [42]. This is, as a function of $z \in \mathbb{C}$, a function with simple poles of
residue 1 at $z=k$ a non-zero integer. Thus we take $g^{\prime}(z) / g(z)=\pi \cot \pi z-1 / z+b$ with $b$ a constant related to the eta-invariant (4.84), and integrating this we get

$$
\begin{equation*}
g(z)=\sin \pi z \mathrm{e}^{b z} / \pi z \tag{4.87}
\end{equation*}
$$

where we have normalized $g(z)$ so that $g(0)=1$.
The infinite prefactor in (4.85) is regularized using the Riemann zeta-function

$$
\begin{equation*}
\zeta(s)=\sum_{k=1}^{\infty} \frac{1}{k^{s}} \tag{4.88}
\end{equation*}
$$

which is finite for $s \geq 0$ with $\zeta(0)=-1 / 2[50]$. We find that

$$
\begin{equation*}
\prod_{k \neq 0}\left(\frac{2 \pi i}{T}\right)^{2}=\prod_{k>0}\left(\frac{2 \pi i}{T}\right)^{4}=\left(\frac{2 \pi i}{T}\right)^{\left.4\left(\sum_{k=1}^{\infty} \frac{1}{k^{s}}\right)\right|_{s=0}}=\left(\frac{2 \pi i}{T}\right)^{4 \zeta(0)}=\left(\frac{2 \pi i}{T}\right)^{-2} \tag{4.89}
\end{equation*}
$$

and thus the block contribution (4.85) to the functional determinant in (4.83) is

$$
\begin{equation*}
\operatorname{det}^{\prime}\left\|\partial_{t}-R_{V}^{(j)}\right\|=\frac{1}{\pi^{2}}\left(\frac{\sin \frac{i T \lambda_{j}}{2}}{\lambda_{j}}\right)^{2}=\left(\frac{T}{2 \pi i}\right)^{2} \operatorname{det}\left[\frac{\sinh \frac{T}{2} R_{V}^{(j)}}{\frac{T}{2} R_{V}^{(j)}}\right] \tag{4.90}
\end{equation*}
$$

Multiplying the blocks together we see that the fluctuation determinant appearing in (4.83) is just given by the equivariant $\hat{A}$-genus (2.77) with respect to the equivariant curvature $R_{V}$, and the localization formula (4.83) becomes

$$
\begin{equation*}
Z(T) \sim \int_{\mathcal{M}} \operatorname{ch}_{V}(-i T \omega) \wedge \hat{A}_{V}(T R) \tag{4.91}
\end{equation*}
$$

The formula (4.91) is the Niemi-Tirkkonen localization formula [101] and it expresses the quantum partition function as an integral over the phase space $\mathcal{M}$ of equivariant characteristic classes in the $U(1)$-equivariant cohomology generated by the Hamiltonian vector field $V$ on $\mathcal{M}$. The huge advantage of this formula over the localization formula of the last section is that no assumptions appear to have gone into its derivation (other than the standard localization constraints). It thus applies not only to the cases covered by the WKB localization theorem, but also to those where the WKB approximation breaks down (e.g. when classical paths coalesce in $L \mathcal{M}$ ). Indeed, being a localization onto
time-independent loops it does not detect degenerate types of phase space trajectories that a dynamical system may possess.

In fact, the localization formula (4.91) can be viewed as an integral over the equivariant generalization of the Atiyah-Singer index density of a Dirac operator with background gravitational and gauge fields, and it therefore represents a sort of equivariant generalization of the Atiyah-Singer index theorem for a twisted spin complex. Indeed, when $H, V \rightarrow 0$ the effective action in the canonical localization integral is

$$
\begin{equation*}
\left.\left(S+\Omega+\lambda Q_{S} \psi\right)\right|_{H=V=0}=\int_{0}^{T} d t\left[\lambda g_{\mu \nu} \dot{x}^{\mu} \dot{x}^{\nu}+\theta_{\mu} \dot{x}^{\mu}+\lambda g_{\mu \nu} \eta^{\mu} \nabla_{t} \eta^{\nu}+\frac{1}{2} \eta^{\mu} \omega_{\mu \nu} \eta^{\nu}\right] \tag{4.92}
\end{equation*}
$$

where $\nabla_{t}$ is the covariant derivative along the loop $x(t)$ induced by the Riemannian connection $\nabla$ on $\mathcal{M}$ (see (4.77)). On the other hand, the left-hand side of the localization formula (4.58) becomes

$$
\begin{equation*}
\left.Z(T)\right|_{H=0}=\left.\operatorname{tr}\left\|\mathrm{e}^{-i \hat{H} T}\right\|\right|_{T=0}=\operatorname{dim} \mathcal{H}_{\mathcal{M}} \tag{4.93}
\end{equation*}
$$

which is an integer representing the dimension of the free Hilbert space associated with $S(H=0)$ and which can therefore only describe the topological characteristics of the manifold $\mathcal{M}$. The action (4.92) is the supersymmetric action for a bosonic field $x^{\mu}(t)$ and its Dirac fermion superpartner field $\eta^{\mu}(t)$ in the background of a gauge field $\theta_{\mu}$ and a gravitational field $g_{\mu \nu}$, i.e. the action of $N=\frac{1}{2}$ Dirac supersymmetric quantum mechanics. Moreover, the integer (4.93) coincides with the $V=0$ limit of (4.91) which is the ordinary Atiyah-Singer index for a twisted spin complex (the 'twisting' here associated with the usual symplectic line bundle $L \rightarrow \mathcal{M}$ ). Thus the localization formalism here reproduces quite beautifully the celebrated Atiyah-Singer index theorem which states that the index of a Dirac operator, representing the dimension of the space of its zero modes (or equivalently its spectral asymmetry (4.84)), is a topological invariant of the background fields [35, 124].

This feature is not that surprising, for the above just reproduces the original infinitedimensional application of the Duistermaat-Heckman theorem due to Atiyah and Witten
[7], and the later generalizations to twisted Dirac operators by Bismut [18, 19] and Jones and Petrack [66]. It is well-known that the Atiyah-Singer index on the twisted spin complex of $\mathcal{M}$ can be evaluated as above from the action (4.92) representing a BRST gauge-fixed path integral in the proper-time gauge [4, 42]. The supersymmetry here is given by (4.65) representing the model independent $S^{1}$-equivariant cohomology. The $N=\frac{1}{2}$ supersymmetry algebra $Q_{\dot{x}}^{2}=\mathcal{L}_{\dot{x}}$ is in this context a canonical realization of the graded constraint representing the zero mode equation for the pertinent elliptic Dirac operator $D \sim Q_{\dot{x}}$ on a compact even-dimensional Riemannian manifold, and the $\lambda \rightarrow \infty$ limit of the canonical localization integral (4.58) is just the path integral representation of the analytical index of this Dirac operator via the Witten index
index $D \equiv \operatorname{dim} \operatorname{ker} D-\operatorname{dim} \operatorname{ker} D^{\dagger}=\lim _{\lambda \rightarrow \infty} \operatorname{tr}\left\|(-1)^{F} \mathrm{e}^{-\lambda \mathcal{H}}\right\|=\lim _{\lambda \rightarrow \infty} \operatorname{tr}\left\|\gamma_{5}\left(\mathrm{e}^{-\lambda D^{\dagger} D}-\mathrm{e}^{-\lambda D D^{\dagger}}\right)\right\|$
for the corresponding supersymmetric model [132]. Here $F$ is the fermion number operator and $\gamma_{5}$ is the Dirac chirality matrix. Moreover, from (4.66) we see that this action is BRST-exact, as anticipated for a bilinear supersymmetric field theory, so that the path integral actually describes a cohomological Witten-type topological field theory [17]. We shall describe this cohomological field theory in more detail later on, but we point out here that this is one of the essential features of the localization formalism which leads to this path integral derivation of the Atiyah-Singer index theorem ${ }^{7}$.

In fact, one could even proceed in the opposite direction to analyse a generic supersymmetric field theory using the canonical loop space symplectic geometry defined in section 4.2 above. This has been argued to be possible for any Poincaré supersymmetric quantum field theory, in addition to the $N=\frac{1}{2}$ model above [60, 88, 89, 104]. If the supersymmetric field theory is bilinear in the fermionic degrees of freedom, then the path

[^19]integral induces a loop space symplectic structure from Berezin integration of the bilinear, the periodic bosonic fields define the loop space coordinates, and the BRST-charge yields the equivariant cohomological structure. This construction has been carried out explicitly for various models such as $N=1$ DeRham supersymmetric quantum mechanics [88, 89, 131], the $N=1$ Wess-Zumino model and $N=1$ supersymmetric Yang-Mills theory [89]. Recently, Palo [104] has generalized this loop space symplectic structure to include the case of general $N=1$ supermultiplets and applied it to the supersymmetric non-linear sigma-model. This underlying symplectic geometry of supersymmetry provides a convenient, conceptual geometric approach to Poincaré supersymmetric quantum field theories. This way of looking at these models provides some additional insights and flexibility in the evaluation of the path integrals. The equivariant cohomological structure of these theories is consistent with the topological nature of supersymmetric models (the basic topological field theories - see section 4.9 below) and they yield certain topological invariants of the underlying manifolds such as the Atiyah-Singer and Callias indices [59] ${ }^{8}$. We shall dispense with further discussion of these topological features of equivariant localization until section 4.9.

### 4.6 Connections with the Duistermaat-Heckman Integration Formula

In this section we shall point out some relations between the path integral localization formulas derived thus far, and, in particular, the relations to the finite-dimensional Duistermaat-Heckman formula. Since the localization formulas are all derived from the same fundamental geometric constraints, one would expect that, in some limits at least, they are all related to each other. In particular, when 2 localization formulas hold for a certain quantum mechanical path integral, they must both coincide somehow. We

[^20]can relate the various localization formulas by noting that the integrand of (4.91) is an equivariantly closed differential form on $\mathcal{M}$ (being an equivariant characteristic class) with respect to the finite-dimensional equivariant cohomology defined by the ordinary Cartan derivative $D_{V}=d+i_{V}$. Thus we can apply the Berline-Vergne theorem (in degenerate form - compare with section 3.7) of section 2.5 to localize the equivariant Atiyah-Singer index onto the critical points of the Hamiltonian $H$ to obtain
\[

$$
\begin{equation*}
\left.Z(T) \sim \int_{\mathcal{M}_{V}} \frac{\operatorname{ch}_{V}(-i T \omega)}{\left.E_{V}(R)\right|_{\mathcal{N}_{V}}} \wedge \hat{A}_{V}(T R)\right|_{\mathcal{M}_{V}} \tag{4.95}
\end{equation*}
$$

\]

Note that this differs from the finite-dimensional localization formula (3.114) only in the appearence of the equivariant $\hat{A}$-genus which arises from the evaluation of the temporal determinants which occur. This factor therefore encodes the quantum fluctuations about the classical values, and its appearence is quite natural according to the general supersymmetry arguments above. Furthermore, the localization formula (4.95) follows from the moduli space formula (4.76) for certain values of the propagation time $T$ (see the discussion at the beginning of the last section).

The connection between the WKB and Niemi-Tirkkonen localization formulas is now immediate if we assume that the critical point set $\mathcal{M}_{V}$ of the Hamiltonian consists of only isolated and non-degenerate points (i.e. the Hamiltonian $H$ is a Morse function). Then in the canonical localization vector field (4.61) we can set $r=0$ and $s=-1$ so that

$$
\begin{equation*}
\psi=i_{V} g=\int_{0}^{T} d t g_{\mu \nu} V^{\mu} \eta^{\nu} \quad, \quad Q_{S} \psi=\int_{0}^{T} d t\left[\frac{1}{2}\left(\Omega_{V}\right)_{\mu \nu} \eta^{\mu} \eta^{\nu}+V^{\mu} g_{\mu \nu}\left(\dot{x}^{\nu}-V^{\nu}\right)\right] \tag{4.96}
\end{equation*}
$$

We use the rescaled decomposition (4.78) again which decouples the zero modes from the fluctuation modes. The Gaussian integration over the fluctuation modes then yields

$$
\begin{equation*}
Z(T) \sim \int_{\mathcal{M}} d^{2 n} x_{0} \mathrm{e}^{-i T H} \sqrt{\frac{\operatorname{det} \Omega_{V}}{\operatorname{det}^{\prime}\left\|\partial_{t}-\Omega_{V}\right\|}} \delta(V) \sim \sum_{p \in \mathcal{M}_{V}} \frac{\mathrm{e}^{-i T H(p)}}{\sqrt{\operatorname{det} \Omega_{V}}} \hat{A}\left(T \Omega_{V}\right) \tag{4.97}
\end{equation*}
$$

where the (ordinary) Dirac $\hat{A}$-genus arises from evaluating the temporal determinant in (4.97) as described above and we recall that $\Omega_{V}(p)=2 d V(p)=2 \omega^{-1}(p) \mathcal{H}(p)$ at a critical
point $p \in \mathcal{M}_{V}$. Thus under these circumstances we can localize the partition function path integral onto the time-independent classical trajectories of the dynamical system, yielding a localization formula that differs from the standard Duistermaat-Heckman formula (3.62) only by the usual quantum fluctuation term.

The localization formula (4.97) of course also follows directly from the degenerate formula (4.95) in the usual way, and it can be shown [70] to also follow from the WKB formula (4.70) using the Weinstein action invariant [21, 130] which probes the first cohomology group of the symplectomorphism group of the symplectic manifold (i.e. the diffeomorphism subgroup of canonical transformations). This latter argument requires that $\mathcal{M}$ is compact, the classical trajectories are non-intersecting and each classical trajectory can be contracted to a critical point of $H$ through a family of classical trajectories (for instance when $H^{1}(\mathcal{M} ; \mathbb{R})=0$ ), and that the period $T$ is such that the boundary condition $x^{\mu}(0)=x^{\mu}(T)$ admits only constant loops as solutions to the classical equations of motion. The localization onto the critical points of the Hamiltonian is not entirely surprising, since as discussed at the beginning of the last section for Hamiltonian circle actions on $\mathcal{M}$ the zero locuses $L \mathcal{M}_{S}$ and $\mathcal{M}_{V}$ in general coincide. Drawing from the analogy of (4.97) with the Duistermaat-Heckman theorem (i.e. that the equivariant Atiyah-Singer index (4.91) is given exactly by its stationary phase approximation), one can, in particular, in this case conclude from Kirwan's theorem that the Hamiltonian $H$ is a perfect Morse function that admits only even Morse indices [70].

We have therefore seen that localization formulas and various Morse theoretic arguments (such as Kirwan's theorem) follow (formally) exactly for path integrals in the same way that they followed for ordinary finite-dimensional phase space integrals. For the remainder of this chapter we shall discuss some more formal features of the localization formalism for path integrals, as well as some extensions of them, in parallel to the last few sections of chapter 3 above.

### 4.7 Equivariant Localization and Quantum Integrability

We have shown in chapter 3 that there is an intimate connection between classical integrability and the localization formalism for dynamical systems. With this in mind, we can use the localization formalism to construct an alternative, geometric formulation of the problem of quantum integrability [41, 97] (in the sense that the quantum partition function can be evaluated exactly) which differs from the usual approaches to this problem [30]. As in section 3.6 we consider a generic integrable Hamiltonian which is a functional $H=H(I)$ of action variables $I^{a}$ which are in involution as in (3.84). From the point of view of the localization constraints above, the condition that $H$ generates a circle action which is an isometry of some Riemannian geometry on $\mathcal{M}$ means that the action variables $I^{a}$ generate the Cartan subalgebra of the associated isometry group of $(\mathcal{M}, g)$ in its Poisson bracket realization on $(\mathcal{M}, \omega)$.

For such a dynamical system, we use a set of generating functionals $J^{a}(t)$ to write the quantum partition function as
$Z(T)=\left.\exp \left(-i \int_{0}^{T} d t H\left[\frac{\delta}{i \delta J(t)}\right]\right) \int_{L \mathcal{M}}\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \exp \left\{i \int_{0}^{T} d t\left(\theta_{\mu} \dot{x}^{\mu}-J^{a} I^{a}\right)\right\}\right|_{J=0}$
To evaluate the path integral in (4.98), we consider an infinitesimal variation of its action

$$
\begin{equation*}
\delta\left(\theta_{\mu} \dot{x}^{\mu}-J^{a} I^{a}\right)=\delta x^{\mu}\left(\omega_{\mu \nu} \dot{x}^{\nu}-J^{a} \partial_{\mu} I^{a}\right) \tag{4.99}
\end{equation*}
$$

with the infinitesimal Poisson bracket variation

$$
\begin{equation*}
\delta x^{\mu}=\epsilon^{a}\left\{I^{a}, x^{\mu}\right\}_{\omega}=-\epsilon^{a} \omega^{\mu \nu} \partial_{\nu} I^{a} \tag{4.100}
\end{equation*}
$$

where $\epsilon^{a}$ are infinitesimal coordinate-independent parameters. The transformation (4.99), (4.100) corresponds to the leading order infinitesimal limit of the canonical transformation

$$
\begin{equation*}
x^{\mu} \rightarrow \mathrm{e}^{-\epsilon^{a} I^{a}} x^{\mu} \mathrm{e}^{\epsilon^{a} I^{a}} \equiv x^{\mu}+\epsilon^{a}\left\{x^{\mu}, I^{a}\right\}_{\omega}+\frac{1}{2} \epsilon^{a} \epsilon^{b}\left\{\left\{x^{\mu}, I^{a}\right\}_{\omega}, I^{b}\right\}_{\omega}+\ldots \tag{4.101}
\end{equation*}
$$

and it gives

$$
\begin{equation*}
\delta\left(\theta_{\mu} \dot{x}^{\mu}-J^{a} I^{a}\right)=-\dot{\epsilon}^{a} I^{a} \tag{4.102}
\end{equation*}
$$

after an integration by parts over time. Since the Liouville measure in (4.98) is invariant under canonical transformations, it follows that the only effect of the variation (4.102) on the loop space coordinates in (4.98) is to shift the external sources as $J^{a} \rightarrow J^{a}+\dot{\epsilon}^{a}$. Note that if we identify $J^{a}(t)$ as the temporal component $A_{0}^{a}$ of a gauge field then this shift has the same functional form as a time-dependent abelian gauge transformation [97]. Thus if for some reason the quantum theory breaks the invariance of the Liouville measure under these coordinate transformations, we would expect to be able to relate the non-trivial Jacobian that arises to conventional gauge anomalies [118].

Thus if we Fourier decompose the fields $J^{a}(t)$ into their zero modes $J_{0}^{a}$ and fluctuation modes $\hat{J}^{a}(t)$ as in (4.78), we can use this canonical transformation to 'gauge' away the time-dependent parts of $J^{a}$ in (4.98) so that the path integral there depends only on the constant modes $J_{0}^{a}$ of the generating functionals and the partition function is given by

$$
\begin{equation*}
Z(T)=\left.\exp \left(-i T H\left[\frac{1}{i} \frac{\partial}{\partial J_{0}}\right]\right) \int_{L \mathcal{M}}\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \exp \left\{i \int_{0}^{T} d t\left(\theta_{\mu} \dot{x}^{\mu}-J_{0}^{a} I^{a}\right)\right\}\right|_{J_{0}=0} \tag{4.103}
\end{equation*}
$$

Since the Hamiltonian $J_{0}^{a} I^{a}$ in the action in (4.103) generates an abelian group action on $\mathcal{M}$, we can localize it using the Niemi-Tirkkonen formula (4.91) to arrive at

$$
\begin{equation*}
\left.Z(T) \sim \exp \left(-i T H\left[\frac{1}{i} \frac{\partial}{\partial J_{0}}\right]\right) \int_{\mathcal{M}} \operatorname{ch}_{J_{0}^{a} I^{a}}(-i T \omega) \wedge \hat{A}_{J_{0}^{a} I^{a}}(T R)\right|_{J_{0}=0} \tag{4.104}
\end{equation*}
$$

and so the path integral now localizes to a derivative expansion of equivariant characteristic classes. The localization formula (4.104) is valid for any integrable Hamiltonian system whose conserved charges $J_{0}^{a} I^{a}$ generate a global isometry on $\mathcal{M}$, and consequently the localization formalism can be used to establish the exact quantum solvability of generic integrable models.

Indeed, there are several non-trivial examples of integrable models where the WKB localization formula (4.70) is known to be valid, and this has lead to the conjecture
that for a large class of integrable field theories a "proper" version of the semi-classical approximation should yield a reliable reproduction of the features of the exact quantum theory [139]. The formula (4.104) is one such candidate, in that its right-hand side could be expanded out in powers of $1 / T$ and corrections to the WKB approximation could hence be studied. We shall return to this point in chapter 7. However, one may also hope that the localization principle of section 4.3 above could be used to derive weaker versions of the localization formulas above for some dynamical systems which are not necessarily completely integrable [70] (in the sense that the localization formalism above does not carry through). For this, we consider a Hamiltonian with $r<n$ conserved charges $I^{a}$ which are in involution as in (3.84),(3.85), and which have the classical equations of motion $\dot{I}^{a}=0$. We then set

$$
\begin{equation*}
\psi=\int_{0}^{T} d t I^{a} \partial_{\mu} I^{a} \eta^{\mu} \quad, \quad Q_{S} \psi=\int_{0}^{T} d t\left(\dot{I}^{a}\right)^{2} \tag{4.105}
\end{equation*}
$$

in the canonical localization integral (4.58). The cohomological relation $Q_{S}^{2} \psi=\mathcal{L}_{S} \psi=0$ follows from the involutary property of the charges $I^{a}$. Then the right-hand side of (4.58) yields a localization of the path integral onto the constant values of the conserved charges $I^{a}$,

$$
\begin{equation*}
Z(T)=\int_{L \mathcal{M}}\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \prod_{a=1}^{r} \delta\left(\dot{I}^{a}\right) \mathrm{e}^{i S[x]} \tag{4.106}
\end{equation*}
$$

The formula (4.106) is a weaker version of the above localization formulas which is valid for any non-integrable system that admits conserved charges. It can be viewed as a quantum generalization of the classical reduction theorem [6] which states that conserved charges in involution reduce the dynamics onto the symplectic subspace of the original phase space determined by the constant values of the integrals of motion $I^{a}$. When $H$ is completely integrable this subspace coincides with the invariant Liouville tori discussed in section 3.6. Thus even when there are corrections to the various localization formulas above (e.g. the WKB approximation), the supersymmetry arguments of section 4.3 can be used to derive weaker versions of the localization formulas. Notice that, as
anticipated, the localization formula (4.106) does not presume any isometric structure on the phase space (see the discussion of section 3.6). Equivariant cohomology might therefore provide a natural geometric framework for understanding quantum integrability, and the localization formulas associated with general integrable models represent equivariant characteristic classes of the phase space. For more details about this and other connections between equivariant localization and integrability, see [69] and [70].

### 4.8 Localization for Functionals of Isometry Generators

In the last section we considered a particular class of Hamiltonians which were functionals of action variables and we were able to derive a quite general localization formula for these dynamical systems. It is natural to explore now whether or not localization formulas could be derived for Hamiltonians which are more general types of functionals. We begin with the case where the Hamiltonian of a dynamical system is an a priori arbitrary functional $\mathcal{F}(H)$ of an observable $H$ which generates an abelian isometry through the Hamiltonian equations for $H$ in the usual sense. Thus we want to evaluate the path integral [102]

$$
\begin{equation*}
Z(T \mid \mathcal{F}(H))=\int_{L \mathcal{M}}\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \exp \left\{i \int_{0}^{T} d t\left(\theta_{\mu} \dot{x}^{\mu}-\mathcal{F}(H)\right)\right\} \tag{4.107}
\end{equation*}
$$

We shall see shortly that such path integrals are important for certain physical applications. Note, however, that although such functionals may seem arbitrary, we must at least require that $\mathcal{F}(H)$ be a semi-bounded functional of the observable $H$ [125]. Otherwise, a Wick rotation off of the real time axis to imaginary time may produce a propagator $\operatorname{tr}\left\|\mathrm{e}^{-i T \mathcal{F}(H)}\right\|$ which is not a tempered distribution and thus eliminating any rigorous attempts to make the path integral a well-defined mathematical entity.

The formalism used to treat path integrals such as (4.107) is the auxilliary field formalism for supersymmetric theories [ $60,88,89$ ] which enables one to relate the loop space equivariant cohomology determined by the derivative $Q_{S}$ to the more general modelindependent $S^{1}$ loop space formalism, i.e. that determined by the equivariant exterior
derivative $Q_{\dot{x}}$. We recall from section 4.3 that in this formulation the path integral action is BRST-exact, as required for supersymmetric field theories. Here the auxilliary fields that are introduced turn out to coincide with those used to formulate generic Poincaré supersymmetric theories in terms of the model-independent $S^{1}$ loop space equivariant cohomology which renders their actions BRST-exact.

To start, we assume that there is a function $\phi(\xi)$ such that the quantity $\mathrm{e}^{-i \int_{0}^{T} d t \mathcal{F}(H)}$ is a Gaussian functional integral transformation of it,

$$
\begin{equation*}
\exp \left(-i \int_{0}^{T} d t \mathcal{F}(H)\right)=\int_{L \mathbf{R}}[d \xi] \exp \left\{i \int_{0}^{T} d t\left(\frac{1}{2} \xi^{2}-\phi(\xi) H\right)\right\} \tag{4.108}
\end{equation*}
$$

Locally such a function $\phi(\xi)$ can always be constructed, but there may be obstructions to constructing $\phi(\xi)$ globally on the loop space $L \mathcal{M}$, for the reasons discussed above. The transformation $\xi \rightarrow \phi$ which maps the Gaussian in $\xi$ to a non-linear functional of $\phi$ is just the Nicolai transformation in supersymmetry theory [17], i.e. the change of variables that maps the bosonic part of the supersymmetric action into a Gaussian such that the Jacobian for this change of variables coincides with the determinant obtained by integrating over the bilinear fermionic part of the supersymmetric action. This observation enables one to explicitly construct a localization for the path integral (4.107).

Notice that when $\mathcal{F}(H)$ is either linear or quadratic in the observable $H$, the Nicolai transform $\mathrm{e}^{\frac{i}{2} \int_{0}^{T} d t \xi^{2}}$ is directly related to the functional Fourier transformation of $\mathrm{e}^{-i \int_{0}^{T} d t \mathcal{F}(H)}$,

$$
\begin{equation*}
\exp \left(-i \int_{0}^{T} d t \mathcal{F}(H)\right)=\int_{L \mathbf{R}}[d \phi] \exp \left(-i \int_{0}^{T} d t \hat{F}(\phi)\right) \exp \left(-i \int_{0}^{T} d t \phi H\right) \tag{4.109}
\end{equation*}
$$

However, for more complicated functionals $\mathcal{F}(H)$ this connection is less straightforward. In particular, if we change variables $\xi \rightarrow \phi$ in the Gaussian transformation (4.108), we find

$$
\begin{equation*}
\exp \left(-i \int_{0}^{T} d t \mathcal{F}(H)\right)=\int_{L \mathrm{R}}[d \phi] \prod_{t \in[0, T]} \xi^{\prime}(\phi) \exp \left\{i \int_{0}^{T} d t\left(\frac{1}{2} \xi^{2}(\phi)-\phi H\right)\right\} \tag{4.110}
\end{equation*}
$$

so that the effect of this transformation is to isolate the isometry generator $H$ and make it contribute linearly to the effective action in (4.107). This allows one to localize (4.107) using the general prescriptions of section 4.3 above.

Substituting (4.110) into (4.107), we then carry out the same steps which led to the Niemi-Tirkkonen localization formula (4.91). However, now there is an auxilliary field $\phi$ which appears in the path integral action which must be incorporated into the localization procedure. These fields appear in the terms $\phi H$ above and are therefore interpreted as the dynamical generators of $S\left(\mathbf{u}(\mathbf{1})^{*}\right)$. We introduce a superpartner $\eta$ for the auxilliary field $\phi$ whose Berezin integration absorbs the Jacobian factor in (4.110). The path integral (4.107) thus becomes a functional integral over an extended superloop space. One can introduce an extended BRST-operator incorporating the super-multiplet $(\phi, \eta)$ such that the partition function is evaluated with a BRST-exact action whose argument lies in the BRST-complex of physical states and the Niemi-Tirkkonen localization of section 4.5 above becomes manifest. We remark that this extended BRST-operator is the so-called Weil differential whose cohomology defines the BRST model for the $U(1)$-equivariant cohomology [82, 103]. This more sophisticated technique is required whenever the basis elements $\phi^{a}$ of the symmetric algebra $S\left(\mathbf{g}^{*}\right)$ are made dynamical, as is the case here. This extended superspace formalism is also the building block for the Mathai-Quillen construction of topological field theories [22, 82, 98].

We shall not enter into the cumbersome details of this extended superspace evaluation of (4.107), but merely refer to [102] for the details. The final result is the integration formula

$$
\begin{equation*}
Z(T \mid \mathcal{F}(H)) \sim \int_{-\infty}^{\infty} d \phi_{0} \xi_{0}^{\prime}\left(\phi_{0}\right) \mathrm{e}^{i T \xi_{0}^{2}\left(\phi_{0}\right) / 2} \int_{\mathcal{M}} \operatorname{ch}_{\phi_{0} V}(-i T \omega) \wedge \hat{A}_{\phi_{0} V}(T R) \tag{4.111}
\end{equation*}
$$

where $\phi_{0}$ are the zero modes of the auxilliary field $\phi$. (4.111) is valid (formally) for any semi-bounded functional $\mathcal{F}(H)$ of an isometry generator $H$ on $\mathcal{M}$. Thus even for functionals of Hamiltonian isometry generators the localization formula is a relatively simple expression in terms of equivariant characteristic classes. The only computational
complication in these formulas is the identification of the function $\xi(\phi)$ (or the functional Fourier transform $\hat{F}(\phi)$ ). We note that when $\mathcal{F}(H)=H$, we have $\phi(\xi)=1$ and (4.111) reduces consistently to the Niemi-Tirkkonen localization formula (4.91). In the important special case $\mathcal{F}(H)=H^{2}$, we find $\phi(\xi)=\xi$ (i.e. $\hat{F}(\phi)=\phi^{2}$ ) and the localization formula (4.111) becomes

$$
\begin{equation*}
Z\left(T \mid H^{2}\right) \sim \int_{-\infty}^{\infty} d \phi_{0} \mathrm{e}^{i T \phi_{0}^{2} / 2} \int_{\mathcal{M}} \operatorname{ch}_{\phi_{0} V}(-i T \omega) \wedge \hat{A}_{\phi_{0} V}(T R) \tag{4.112}
\end{equation*}
$$

which is the formal path integral generalization of the Wu localization formula (3.124).
In fact, the above dynamical treatment of the multipliers $\phi$ suggest a possible nonabelian generalization of the localization formulas and hence a path integral generalization of the Witten localization formula of section 3.8 [127]. At the same time we generalize the localization formalism of section 4.7 above to the case where the Hamiltonian is a functional of the generators of the full isometry group of $(\mathcal{M}, g)$, and not just simply the Cartan subgroup thereof. We consider a general non-abelian Hamiltonian moment map (3.30) where the component functions $H^{a}$ are assumed to generate a Poisson algebra realization of the isometry group $G$ of some Riemannian metric $g$ on $\mathcal{M}$. As mentioned in section 3.8, when the $\phi^{a}$ are fixed we are essentially in the abelian situation above and this case will be discussed in more detail in the subsequent sections. Here we assume that the multipliers $\phi^{a}$ are time-dependent and we integrate over them in the path integral following the same prescription for equivariant integration introduced in section 3.8. This corresponds to modelling the $G$-equivariant cohomology of $\mathcal{M}$ in the Weil algebra using the BRST formalism [103, 127]. When the $\phi^{a}$ are fixed parameters, the action (4.25) generates the action of $S^{1}$ on $L \mathcal{M}$ in the model independent circle action described in section 4.3 above. However, when the $\phi^{a}$ are dynamical quantities, $S$ generates the action of the semi-direct product $L G \times S^{1}$, where the action of $S^{1}$ corresponds to translations of the loop parameter $t$ and $L G=C^{\infty}\left(S^{1}, G\right)$ is the loop group of the isometry group $G$.

These actions are generated, respectively, by the loop space vector fields

$$
\begin{gather*}
V_{S^{1}}=\int_{0}^{T} d t \dot{x}^{\mu}(t) \frac{\delta}{\delta x^{\mu}(t)} \\
V_{L G}=\int_{0}^{T} d t \phi^{a}(t) \omega^{\mu \nu}(x(t))\left(\frac{\delta}{\delta x^{\nu}(t)} H^{a}\right) \frac{\delta}{\delta x^{\mu}(t)}=\int_{0}^{T} d t \phi^{a}(t) V^{a}(t) \tag{4.113}
\end{gather*}
$$

The commutator algebra of the vector fields (4.113) is that of $L G \nVdash S^{1}$ on $L \mathcal{M}$,

$$
\begin{equation*}
\left[V_{S^{1}}, V_{L G}\right]=\int_{0}^{T} d t \dot{\phi}^{a} H^{a} \quad, \quad\left[V^{a}(t), V^{b}\left(t^{\prime}\right)\right]=f^{a b c} V^{c}(t) \delta\left(t-t^{\prime}\right) \tag{4.114}
\end{equation*}
$$

The equivariant extension of the symplectic 2 -form $\Omega$ on $L \mathcal{M}$ is therefore $S+\Omega$.
If the multipliers $\phi^{a}$ (now regarded as local coordinates on $L \mathbf{g}^{*}$ ) are integrated over directly, then the isometry functions $H^{a}$ become constraints because the $\phi^{a}$ appear linearly in the action and so act as Lagrange multipliers. In this case we are left with a topological quantum theory (i.e. there are no classical degrees of freedom) with vanishing classical action, in parallel to the finite-dimensional case of Section 3.8. Alternatively, we can add a functional $F=F\left(\phi^{a}\right)$ to the argument of the exponential term in the partition function such that the quantity $S+\Omega+F$ is equivariantly closed. We then introduce a non-abelian generalization of the procedure outlined above [127]. As remarked there, from supersymmetric manipulations of the path integral one derives an extended equivariant BRST operator $Q_{T}$, which is here the non-abelian version of that above, for the semi-direct product action of $L \dot{G} \not \ngtr S^{1}$ on $L \mathcal{M} . Q_{T}$ is then the sum of the BRST operator for the equivariant cohomology of $L \mathcal{M}$ and the model-independent $S^{1}$ BRST charge $Q_{\dot{x}}$. It turns out that $S+\Omega+F$ is equivariantly closed with respect to $Q_{T}$ only for either $F=0$ or $F=\frac{1}{2}\left(\phi^{a}\right)^{2}$, where the latter is the invariant polynomial corresponding to the quadratic Casimir element of $G$. Note that this is precisely the choice that was made in our definition of equivariant integration in Section 3.8. With these modifications the total extended superloop space action $S_{T}+\Omega+\frac{1}{2}\left(\phi^{a}\right)^{2}$ is BRST-closed with respect to $Q_{T}$, so that $S_{T}$ is the moment map for the action of $L G \ngtr S^{1}$ on $L \mathcal{M}$. Furthermore, $\mathcal{L}_{V}(H+\omega)=0$, so that the action of $L G \nless S^{1}$ on $L \mathcal{M}$ is symplectic. Thus within this
framework we can reproduce loop space generalizations of the cohomological formulation of Section 3.2 for the Hamiltonian dynamics. We refer to [103] and [127] for the details of this approach.

The rest of the localization procedure now carries through parallel to that above and in the Niemi-Tirkkonen localization, and it yields the localization formula [127]

$$
\begin{equation*}
Z(T) \sim \int_{\mathrm{g}^{*}} \prod_{a=1}^{\operatorname{dim} G} d \phi_{0}^{a} \mathrm{e}^{i T\left(\phi_{0}^{a}\right)^{2} / 2} \int_{\mathcal{M}} \operatorname{ch}_{\phi_{0}^{a} V^{a}}(-i T \omega) \wedge \hat{A}_{\phi_{0}^{a} V^{a}}(T R) \tag{4.115}
\end{equation*}
$$

which is a non-abelian version of the quadratic localization formula (4.112) and is the path integral generalization of the Witten localization formula presented in Section 3.8. As such, it can be applied to problems such as geodesic motion on group manifolds, and in particular it reproduces the results of Picken [110] in the Hamiltonian framework [127]. In these cases there is the natural $G$-invariant metric $g=\operatorname{tr}\left(\lambda \otimes \lambda^{\dagger}\right)$ defined on the group manifold of $G$, where $\lambda=h^{-1} d h$ is the adjoint representation of the Cartan-Maurer 1form which takes values in the Lie algebra $g$ of $G$. It therefore also applies to the basic integrable models such as 2-dimensional Yang-Mills theory, supersymmetric quantum mechanics and Calegero-Moser type theories. These describe the quantum mechanics of integrable models related to Hamiltonian reduction of free field theories [38, 49]. Notice that, however, the primary difference between this non-abelian localization and its abelian counterpart is that in the latter the functional $F(\phi)$ is a priori arbitrary.

### 4.9 Topological Quantum Field Theories

In this last Section of this Chapter, we return to the case where the dual basis elements of $S\left(\mathbf{g}^{*}\right)$ are fixed numbers. We wish to study the properties of the quantum theory when the effective action is BRST-exact as in (4.46) locally on the loop space [70, 100]. In this case the quantum theory is said to be topological, in that there are no physical degrees of freedom and the remaining partition function can only describe topological invariants of the space on which it is defined [17]. We shall see this explicitly below,
and indeed we have already seen hints of this in the expressions for the path integral in terms of equivariant characteristic classes above. To get a flavour for this, we first consider a quantum theory that admits a model independent circle action globally on the loop space, i.e. whose loop space Hamiltonian vector field generates a global constant velocity $U(1)$ action on $L \mathcal{M}$, so that its action is given locally by (4.66). In this case, the determinant that appears in the denominator of the WKB localization formula (4.70) is

$$
\begin{equation*}
\left.\operatorname{det}\left\|\delta^{2} S\right\|\right|_{\dot{x}=0}=\left.\operatorname{det}\|\delta(\Omega \cdot \dot{x})\|\right|_{x=x_{0}}=\left.\operatorname{det}\left\|\Omega \partial_{t}\right\|\right|_{x=x_{0}} \tag{4.116}
\end{equation*}
$$

where the localization is now onto the constant loops $x_{0} \in \mathcal{M}$. Since the determinants on the right-hand side of (4.70) now cancel modulo the factor $\operatorname{det}\left\|\partial_{t}\right\|$, only the zero modes of $\partial_{t}$ can contribute. Thus the (degenerate) WKB localization formula in this case becomes

$$
\begin{equation*}
\left.Z(T) \sim \int_{\mathcal{M}} d^{2 n} x_{0} \sqrt{\operatorname{det}\left\|\partial_{t}\right\|} \sqrt{\operatorname{det}\left\|\left.\Omega\right|_{\partial_{t}=0}\right\|}\right|_{x=x_{0}} \tag{4.117}
\end{equation*}
$$

and only the zero modes of the symplectic 2 -form contribute. Since, as discussed in Section 4.5 above, this path integral yields the topological Witten index of the corresponding supersymmetric theory [132], the localization formula identifies the loop space characteristic class which corresponds to the Witten index of which the ensuing Atiyah-Singer index counts the zero modes of the associated Dirac operator. This is one of the new insights gained into supersymmetric theories from the equivariant localization formalism. In Section 4.5 we argued that this was a purely cohomological representative of the manifold $\mathcal{M}$ which contained no physical information.

We now consider the general case of an equivariantly-exact action (4.46). Note that this is precisely the solution to the problem of solving the loop space equivariant Poincaré lemma for $S+\Omega$. If we assume that the symplectic potential is invariant under the global $U(1)$-action on $\mathcal{M}$, as in (3.26), then the Hamiltonian is given by $H=i_{V} \theta$ and the loop space 1 -form $\hat{\vartheta}$ in (4.46) is given by

$$
\begin{equation*}
\hat{\vartheta}=\int_{0}^{T} d t \theta_{\mu}(x(t)) \eta^{\mu}(t) \tag{4.118}
\end{equation*}
$$

The loop space localization principle naively implies that the resulting path integral should be trivial. Indeed, since the 1-form (4.118) lies in the subspace (4.43), the partition function can be written as

$$
\begin{equation*}
Z(T)=\int_{L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \mathrm{e}^{i \lambda Q_{s} \hat{\vartheta}} \tag{4.119}
\end{equation*}
$$

and it is independent of the parameter $\lambda \in \mathbb{R}^{+}$. In particular, it should be independent of the action $S$.

However, the above argument for the triviality of the path integral assumes that $\theta$ is homotopic to 0 in the subspace (4.43) under the supersymmetry generated by $Q_{S}$, i.e. that (4.46) holds globally for all loops. For the remainder of this Chapter we will assume that the manifold $\mathcal{M}$ is simply connected, so that $H^{1}(\mathcal{M} ; \mathbb{R})=0$. Then the above argument presumes that the second DeRham cohomology group $H^{2}(\mathcal{M} ; \mathbb{R})=0$ is trivial. If this is not the case, then one must be careful about arguing the $\lambda$-independence of the path integral (4.119). Consider the family of symplectic 2 -forms

$$
\begin{equation*}
\omega^{(\lambda)}=\lambda d \theta=\lambda \omega \tag{4.120}
\end{equation*}
$$

associated with the action in (4.119). We consider a closed loop $\gamma$ in the phase space $\mathcal{M}$ parametrized by the periodic trajectory $x(t):[0, T] \rightarrow \mathcal{M}$. Since by assumption $\gamma$ is the boundary of a 2-surface $\Sigma_{1}$ in $\mathcal{M}$, Stokes' theorem implies that the kinetic term $\theta^{(\lambda)}=\lambda \theta$ in (4.119) can be written as

$$
\begin{equation*}
\int_{0}^{T} d t \theta_{\mu}^{(\lambda)}(x(t)) \dot{x}^{\mu}(t)=\oint_{\gamma} \theta^{(\lambda)}=\int_{\Sigma_{1}} \omega^{(\lambda)} \tag{4.121}
\end{equation*}
$$

For consistency of the path integral (4.119), which is expressed as a sum over closed loops in $\mathcal{M}$, the phase (4.121) must be independent of the representative surface $\Sigma_{1}$ spanning $\gamma$, owing to the topological invariance of the partition function $Z(T)$ over $L \mathcal{M}$. Thus if we introduce another surface $\Sigma_{2}$ with boundary $\gamma$ and let $\Sigma$ be the closed surface (sphere) which is divided into 2 halves $\Sigma_{1}$ and $\Sigma_{2}$ by $\gamma$, we must have

$$
\begin{equation*}
\mathrm{e}^{i \oint_{\Sigma} \omega^{(\lambda)}}=\mathrm{e}^{i \int_{\Sigma_{1}} \omega(\lambda)} \mathrm{e}^{-i \int_{\Sigma_{2}} \omega(\lambda)} \tag{4.122}
\end{equation*}
$$

and consequently the integral of $\omega^{(\lambda)}$ over any closed orientable surface $\Sigma$ in $\mathcal{M}$ must satisfy a version of the Dirac or Wess-Zumino-Witten quantization condition [133]

$$
\begin{equation*}
\frac{1}{2 \pi} \oint_{\Sigma} \omega^{(\lambda)}=\frac{\lambda}{2 \pi} \oint_{\Sigma} \omega \in \mathbb{Z} \tag{4.123}
\end{equation*}
$$

This means that $\omega^{(\lambda)}$ is an integral element of $H^{2}(\mathcal{M} ; \mathbb{R})$, i.e. it defines an integer cohomology class in $H^{2}(\mathcal{M} ; \mathbb{Z})$, which is possible only for certain discrete values of $\lambda \in$ $\mathbb{R}^{+}$. It follows that a continuous variation $\delta \lambda$ of $\lambda$ cannot leave the path integral (4.119) invariant and it depends non-trivially on the localization 1-form $\psi \equiv \vartheta$ and thus also on the action $S$.

Thus the path integral (4.119) defines a consistent quantum theory only when the symplectic 2 -form (4.120) defines an integral curvature on $\mathcal{M}$. However, if we introduce a variation $\theta \rightarrow \theta+\delta \theta$ of the symplectic potential in (4.119) corresponding to a variation $\omega \rightarrow \omega+\delta \omega$ with $\delta \omega=d \alpha$ a trivial element of $H^{2}(\mathcal{M} ; \mathbb{R})$ in the subspace (4.43), then the localization principle implies that the path integral remains unchanged. Thus the path integral depends only on the cohomology class of $\omega$ in $H^{2}(\mathcal{M} ; \mathbb{R})$, not on the particular representative $\omega=d \theta$, which means that the partition function (4.119) determines a cohomological topological quantum field theory on the phase space $\mathcal{M}$.

Furthermore, we note that within the framework of the Niemi-Tirkkonen localization formula, the BRST-exact term $Q_{S}(\lambda \psi+\hat{\vartheta})$, with $\hat{\vartheta}$ given by (4.118) and $\psi$ given in (4.77), gives the effective action in the canonical localization integral (4.51). We showed in Section 4.5 that the $Q_{\dot{x}}$-exact piece of this action could be identified with a Dirac operator $D$ in the background of a $U(1)$ gauge field $\theta_{\mu}$ and a gravitational field. The remaining terms there, given by the $i_{V}$-exact pieces, then coincide with the terms that one expects in a supersymmetric path integral representation of the infinitesimal Lefschetz number (or equivariant $G$-index)

$$
\begin{equation*}
\operatorname{index}_{H}(D ; T)=\lim _{\lambda \rightarrow \infty} \operatorname{tr}\left\|\mathrm{e}^{i T H}\left(\mathrm{e}^{-\lambda D^{\dagger} D}-\mathrm{e}^{-\lambda D D^{\mathrm{t}}}\right)\right\| \tag{4.124}
\end{equation*}
$$

for the Hamiltonian $H[16,18,19,97,101]$. This number is a regulated measure of the
number of zeroes of $H$ (fixed points of $e^{i T H}$ ). Consequently, in the case of Hamiltonian systems for which $\mathcal{L}_{V} g=\mathcal{L}_{V} \theta=0$, the Niemi-Tirkkonen localization formula (4.91) reproduces the Lefschetz fixed point formulas of Bismut [18, 19] and Atiyah, Bott and Singer [35], provided that boundary conditions for the path integral have been properly selected. Thus a purely bosonic theory can be related to the properties of a (functional) Dirac operator defined in the canonical phase space of the bosonic theory, and this analogy leads one to the hope that the above localization prescriptions can be made quite rigorous in a number of interesting infinite-dimensional cases. Note also that the path integral (4.119) has the precise form of a Witten-type or cohomological quantum field theory, which is characterized by a classical action which is BRST-exact with the BRST charge $Q_{S}$ representing gauge and other symmetries of the classical theory. These types of topological field theories are known to have partition functions which are given exactly by their semi-classical approximation - more precisely, they admit Nicolai maps which trivialize the action and restrict to the moduli space of classical solutions [17]. Thus the topological and localization properties of supersymmetric and topological field theories find their natural explanation within the framework of loop space equivariant localization.

Of course, the above results rely heavily on the symplecticity condition (3.26) for the symplectic potential $\theta$. In the general case, we recall from Section 3.2 that we have the relation (3.46) which holds locally in a neighbourhood $\mathcal{N}$ in $\mathcal{M}$ away from the critical points of $H$ and in which $\omega=d \theta$. In this case, (3.46) gives a solution to the equivariant Poincaré lemma and although the action is locally BRST-exact, globally the quantum theory is non-trivial and may not be given exactly by the WKB approximation. Then the path integral (4.51) has the precise form of a gauge-fixed topological field theory, otherwise known as a Schwarz-type or quantum topological field theory [17], with $Q_{S}$ the BRST charge representing the gauge degrees of freedom. With $\hat{\vartheta}$ as in (4.118), the loop
space equivariant symplectic 2-form can be written in the neighbourhood $L \mathcal{N}$ as

$$
\begin{equation*}
S+\Omega=Q_{S}\left(\hat{\vartheta}+d_{L} F\right)-\int_{0}^{T} d F(x(t)) \tag{4.125}
\end{equation*}
$$

and the path integral can be represented locally as

$$
\begin{equation*}
Z(T)=\int_{L \mathcal{M} \otimes L \Lambda^{1} \mathcal{M}}\left[d^{2 n} x\right]\left[d^{2 n} \eta\right] \mathrm{e}^{i\left(Q_{S}\left(\hat{\vartheta}+d_{L} F\right)-i \oint_{\gamma(x)} d F\right)} \tag{4.126}
\end{equation*}
$$

If we assume that $\mathcal{M}$ is simply connected, so that $H^{1}(\mathcal{M} ; \mathbb{R})=0$, then the $d F$ term in (4.126) can be ignored for closed trajectories on the phase space ${ }^{9}$. Since from (4.125) we have

$$
\begin{equation*}
\mathcal{L}_{S}\left(\hat{\vartheta}+d_{L} F\right)=Q_{S}(S+\Omega)=0 \tag{4.127}
\end{equation*}
$$

it follows that $\hat{\vartheta}+d_{L} F \in L \Lambda_{S}^{1} \mathcal{M}$ and the effective classical action $S+\Omega$ is equivariantlyexact in the neighbourhood $L \mathcal{N}$. If we interpret the coefficient in front of the $Q_{S}$-exact term as Planck's constant $\hbar$, then this is just another way of seeing that the semi-classical approximation for these supersymmetry-type models is exact.

The non-triviality of the path integral now depends on the non-triviality that occurs when the local neighbourhoods $\mathcal{N}$ above are patched together. In particular, we can invoke the above argument to conclude that the partition function (4.126) depends only on the cohomology class of $\omega$ in $H^{2}(\mathcal{M} ; \mathbb{R})$, in addition to the critical point set of the action $S$. Thus the partition function in the general case locally determines a cohomological topological quantum field theory. From the discussion of Section 3.6 we see that this is consistent with the fact that the theory is locally integrable outside of the critical point set of $H$. We recall also from that discussion that in a neighbourhood $\mathcal{N}$ where action-angle variables can be introduced and where $H$ does not have any critical points, we can construct an explicit realization of the function $F$ above and hence an explicit realization of the topological quantum theory (4.126). For integrable models

[^21]where action-angle variables can be defined almost everywhere on the phase space $\mathcal{M}$, the ensuing theory is topological, i.e. it can be represented by a topological action of the form (4.125) almost everywhere on the loop space $L \mathcal{M}$. Notice that all of the above arguments stem from the assumption that $H^{1}(\mathcal{M} ; \mathbb{R})=0$. In Chapter 6 we shall encounter a cohomological topological quantum field theory defined on a multiply-connected phase space which obeys all of the equivariant localization criteria. We also remark that in the general case above, when $\omega$ is not globally exact, the Wess-Zumino-Witten prescription above for considering the action (4.25) in terms of surface integrals as in (4.121) makes rigorous the definition of the partition function on a general symplectic manifold, a point which up until now we have ignored for simplicity. In this case the required consistency condition (4.123) means that $\omega$ itself defines an integral curvature, which is consistent with the usual ideas of geometric quantization [136]. We shall see how this prescription works on a multiply-connected phase space in Chapter 6.

## Chapter 5

## Equivariant Localization on Simply Connected Phase Spaces

When the phase space $\mathcal{M}$ of a dynamical system is compact, the condition that the Hamiltonian vector field $V$ generate a global isometry of some Riemannian geometry on $\mathcal{M}$ automatically implies that its orbits must be closed. This feature is absolutely essential for the finite-dimensional localization theorems, but within the loop space localization framework, where the arguments for localization are based on formal supersymmetry arguments on the infinite-dimensional manifold $L \mathcal{M}$, the flows generated by $V$ need not be closed and indeed many of the formal arguments of the last Chapter will still apply to non-compact group actions. For instance, if we wanted to apply the localization formalism to an $n$-dimensional potential problem, i.e. on the non-compact phase space $\mathcal{M}=\mathbb{R}^{2 n}$, then we would certainly be allowed to use a Hamiltonian vector field which generates non-compact global isometries. As we have already emphasized, the underlying feature of equivariant localization is the interpretation of an equivariant cohomological structure of the model as a supersymmetry among the physical, auxilliary or ghost variables. But as shown in Section 4.2, this structure is exhibited quite naturally by arbitrary phase space path integrals, so that, under the seemingly weak conditions outlined there, this formally results in the equivariant localization of these path integrals. This would in turn naively imply the exact computability of any phase space path integral.

Of course, we do not really expect this to be the case, and there is therefore the need to explore the loop space equivariant localization formalism in more detail to see precisely what sort of dynamical systems will localize. In this Chapter we shall explore the range of applicability of the equivariant localization formulas, a problem that was
first tackled in some generality in [34] and in [125]. As we shall see, the global isometry condition on the Hamiltonian dynamics is a very restrictive one, essentially meaning that $H$ is related to a global group action (2.32). The natural examples of such situations are the harmonic oscillator and free particle Hamiltonians on $\mathbb{R}^{2 n}$, and the quantization of spin [94] (i.e. the height function on the sphere), or more generally the quantization of the coadjoint orbits of Lie groups [3, 21, $70,102,123,125]$ and the equivalent KirillovKostant geometric quantization of homogeneous phase space manifolds [1, 2]. Indeed, the exactness of the semi-classical approximation (or the Duistermaat-Heckman formula) for these classes of phase space path integrals was one of the most important inspirations for the development of quantum localization theory and these systems will be extensively studied in this Chapter, along with some generalizations of them. We shall see that the Hamiltonian systems whose phase space path integrals can be equivariantly localized essentially all fall into this general framework, and that the localization formulas always represent deep, group-theoretical invariants called characters, i.e. the traces $\operatorname{tr} g=$ $\operatorname{tr} \mathrm{e}^{c^{a} X^{a}}$ evaluated in an irreducible representation of a group $G$ which are invariant under similarity transformations representing equivalent group representations, and they reproduce, in certain instances, some classical formulas for these characters [71]. In our case the group $G$ will be the group of isometries of a Riemannian structure on $\mathcal{M}$.

As it is essentially the isometry group $G$ that determines the integrable structure of the Hamiltonian system in the equivariant localization framework, we shall study the localization framework from the point of view of what the possible isometries can be for a given phase space manifold. A detailed analysis of this sort will lead to a geometrical characterization of the integrable dynamical systems from the viewpoint of localization and will lead to topological field theoretical interpretations of integrability, as outlined in Section 4.9. It also promises deeper insights into what one may consider to be the geometrical structure of the quantum theory. This latter result is a particularly interesting characterization of the quantum theory because the partition functions considered are
all $a b$ initio independent of any Riemannian geometry on the underlying phase space (as are usually the classical and quantum mechanics). Nonetheless, we shall see that for a given Riemannian geometry, the localizable dynamical systems depend on this geometry in such a way so that they determine Hamiltonian isometry actions.

Most of what is said in this Chapter and the next is only true for a 2-dimensional phase space, as will become clear from the ensuing analysis. The reason for this is two-fold. First of all, the topological and geometrical classifications of Riemann surfaces is a completely solved problem from a mathematical point of view. We may therefore invoke this classification scheme to in turn classify the Hamiltonian systems which fit the localization framework. Such a neat mathematical characterization of higher dimensional manifolds is for the most part an unsolved problem (although much progress has been made over the last 7 years or so in the classification of 3 - and 4 -manifolds), so that a classification scheme such as the one that follows does not generalize to higher-dimensional models. In 2-dimensions, in fact, we shall see that from certain points of view all the localizable Hamiltonians represent "generalized" harmonic oscillators, a sort of feature that is anticipated from the previous integrability arguments and the local forms of Hamiltonians which generate circle actions. These seemingly trivial behaviours are the essence behind the reduction of the complicated functional integrals to Gaussian ones.

Secondly, the restriction to 2-dimensions allows us to carry out functional integrations rather straightforwardly without some of the annoyances that appear from the higher-dimensionality of a problem. Thus we can analyse in full detail the localization formulas of the last Chapter, which will therefore give explicit examples of the cohomological and integrable models that appear quite naturally in loop space equivariant localization theory. This analysis will also provide new integrable quantum systems, as we shall see, which fall into the class of the generalized localization formulas (e.g. the Niemi-Tirkkonen formula (4.91)), but not the more traditional WKB approximation. Such examples represent a major, non-trivial advance of localization theory. We shall
also encounter a naive ambiguity in the localization formulas which in general manifests itself in a coordinate singularity and makes the characteristic class representation of the partition function appear to be explicitly metric-dependent [34, 125].

At the same time we can address some of the issues that arise when dealing with phase space path integrals, which are generally regarded as rather disreputable because of the unusual discretization of momentum and configuration paths that occurs (in contrast to the more conventional configuration space (Langrangian) path integral [116]). For instance, we recall from Section 4.1 that the general identification between the Schrödinger picture path integral and loop space Liouville measures was done rather artificially, basically by drawing an analogy between them. For a generic phase space path integral to represent the actual energy spectrum of the quantum Hamiltonian, one would have to carry out the usual quantization of generic Poisson brackets $\left\{x^{\mu}, x^{\nu}\right\}_{\omega}=\omega^{\mu \nu}(x)$. However, unlike the Heisenberg canonical commutation relations (4.1), the Lie algebra generated by this procedure is not necessarily finite-dimensional and so the representation problem has no straightforward solution when the phase space is not a cotangent bundle $\mathcal{M} \otimes \Lambda^{1} \mathcal{M}$ [79], as is the case for a Euclidean configuration space. This approach is therefore hopelessly complicated and in general hardly consistent. One way around this, as we shall see, is to use instead coherent state path integrals. This enables one to obtain the desired identification above while maintaining the original phase space path integral, and therefore at the same time keeping a formal analogy between the finitedimensional and loop space localization formulas. Furthermore, because of their classical properties, coherent states are particularly well-suited for semi-classical studies of quantum dynamics. We shall see that all the localizable dynamical systems in 2-dimensions have phase space path integrals that can be represented in terms of coherent states, thus giving an explicit evaluation of the quantum propagator and the connection with some of the conventional coadjoint orbit models.

In this Chapter we shall in addition confine our attention to the case of a simplyconnected phase space, leaving the case where $\mathcal{M}$ can have non-contractible loops for the next Chapter. In both cases, however, we shall focus on the construction of localizable Hamiltonian systems starting from a generic phase space metric, which will illustrate explicitly the geometrical dependence of these dynamical systems and will therefore give a further probe into the geometrical nature of (quantum) integrability. In this way, we will get a good general idea of what sort of phase space path integrals will localize and what sort of topological field theories the localization formulas will represent.

### 5.1 Coadjoint Orbit Quantization and Character Formulas

There is a very interesting class of cohomological quantum theories which arise quite naturally within the framework of equivariant localization. These will set the stage for the analysis of this Chapter wherein we shall focus on the generic equivariant Hamiltonian systems with simply connected phase spaces. For a (compact or non-compact) semisimple Lie group $G$ (i.e. one whose Lie algebra $\mathbf{g}$ has no abelian invariant subalgebras), we are interested in the coadjoint action of $G$ on the coset space $\mathcal{M}_{G}=G / H_{C}$, where $H_{C}$ is the Cartan subgroup of $G^{\mathbf{1}}$. The coadjoint orbit

$$
\begin{equation*}
O_{\Lambda^{\prime}}=\left\{\operatorname{Ad}^{*}(g) \Lambda^{\prime}: g \in G\right\} \simeq \mathcal{M}_{G} \quad, \quad \Lambda^{\prime} \in \mathbf{g}^{*} \tag{5.1}
\end{equation*}
$$

is the orbit of maximal dimensionality of $G$. Here $\operatorname{Ad}^{*}(g) \Lambda^{\prime}$ denotes the coadjoint action of $G$ on $\Lambda^{\prime}$, i.e.

$$
\begin{equation*}
\left(\operatorname{Ad}^{*}(g) \Lambda^{\prime}\right)(\gamma)=\Lambda^{\prime}\left(g^{-1} \gamma g\right) \quad, \quad \forall \gamma \in \mathbf{g} \tag{5.2}
\end{equation*}
$$

We assume henceforth that $H^{2}(G)=0$. There is a natural $G$-invariant symplectic structure on the coadjoint orbit (5.1) which is defined by the Kirillov-Kostant 2-form [1, 2]. This 2 -form at the point $\Lambda \in \mathrm{g}^{*}$ is given by

$$
\begin{equation*}
\omega_{\Lambda}=\frac{1}{2} \Lambda([\mathcal{T} \hat{,} \mathcal{T}]) \tag{5.3}
\end{equation*}
$$

[^22]where $\mathcal{T}$ is a 1-form with values in the Lie algebra $\mathbf{g}$ which satisfies the equation
\[

$$
\begin{equation*}
d \Lambda=\operatorname{ad}^{*}(\mathcal{T}) \Lambda \tag{5.4}
\end{equation*}
$$

\]

and $\operatorname{ad}^{*}(\mathcal{T})$ denotes the infinitesimal coadjoint action of the element $\mathcal{T} \in \mathbf{g}$.
The 2-form (5.3) is closed and non-degenerate on the orbit (5.1), and by construction the group $G$ acts on $O_{\Lambda^{\prime}}$ by symplectic (canonical) transformations with respect to the Kirillov-Kostant 2-form. Its main characteristic is that the Poisson algebra with respect to (5.3) isomorphically represents the group $G$,

$$
\begin{equation*}
\left\{X_{1}(\Lambda), X_{2}(\Lambda)\right\}_{\omega_{\Lambda}}=\left[X_{1}, X_{2}\right](\Lambda) \tag{5.5}
\end{equation*}
$$

where $X_{i} \in \mathrm{~g}$ are regarded as linear functionals on the orbit $O_{\Lambda^{\prime}}$ with $X_{i}(\Lambda) \equiv \Lambda\left(X_{i}\right)$. Alekseev, Faddeev and Shatashvili [1, 2] have studied the phase space path integrals for such dynamical systems with Hamiltonians defined on the coadjoint orbit (5.1) (e.g. Cartan generators of $\mathbf{g}$ ) and have shown that, quite generally, the associated quantum mechanical matrix elements correspond to matrix elements of the Hamiltonian generator of $g$ in some irreducible representation of the group $G$. We shall see this feature explicitly later on.

There is a much nicer description of the orbit space (5.1) using its representation as the quotient space $\mathcal{M}_{G}=G / H_{C}$ [57]. The orbit (5.1) has the topological features $H^{1}\left(\mathcal{M}_{G} ; \mathbb{Z}\right)=0$ and $H^{2}\left(\mathcal{M}_{G} ; \mathbb{Z}\right)=\mathbb{Z}^{r}$, where $r=\operatorname{dim} H_{C}$ is the rank of $G$ and $\mathbb{Z}^{r}$ corresponds to the lattice of roots of $H_{C}$ [128]. We can introduce local complex coordinates $\left(z^{\mu}, \bar{z}^{\bar{\mu}}\right)$ on $\mathcal{M}_{G}$ which are generated by a complex structure on $\mathcal{M}_{G}{ }^{2}$. The cohomology classes in $H^{2}\left(\mathcal{M}_{G} ; \mathbb{Z}\right)$ are then represented by the $r$ closed non-degenerate 2-forms

$$
\begin{equation*}
\omega^{(i)}=\frac{i}{2} g_{\mu \bar{\nu}}^{(i)}(z, \bar{z}) d z^{\mu} \wedge d \bar{z}^{\bar{\nu}} \tag{5.6}
\end{equation*}
$$

[^23]The components $g_{\mu \bar{\nu}}$ of (5.6) define Hermitian matrices, $g_{\bar{\nu} \mu}^{*}=g_{\mu \bar{\nu}}$, and the non-degeneracy condition implies that they define metrics on $\mathcal{M}_{G}$ by

$$
\begin{equation*}
g^{(i)}=g_{\mu \bar{\nu}}^{(i)}(z, \bar{z}) d z^{\mu} \otimes d \bar{z}^{\bar{\nu}} \tag{5.7}
\end{equation*}
$$

The closure condition on the 2 -forms (5.6) can be written in terms of the holomorphic and anti-holomorphic components

$$
\begin{equation*}
d=\partial+\bar{\partial} \tag{5.8}
\end{equation*}
$$

of the exterior derivative as

$$
\begin{equation*}
\partial \omega^{(i)}=\bar{\partial} \omega^{(i)}=0 \tag{5.9}
\end{equation*}
$$

The holomorphic exterior derivative $\partial$ is called the Dolbeault operator, and the analogue of the Poincaré lemma for the Dolbeault operator is the Dolbeault-Grothendieck lemma. Since the 2 -forms $\omega^{(i)}$ in the case at hand are closed under both $\partial$ and $\bar{\partial}$, the DolbeaultGrothendieck lemma implies that locally they can be expressed in terms of $C^{\infty}$-functions $F^{(i)}$ on $\mathcal{M}_{G}$ as

$$
\begin{equation*}
\omega^{(i)}=-i \partial \bar{\partial} F^{(i)} \tag{5.10}
\end{equation*}
$$

or in local coordinates

$$
\begin{equation*}
g_{\mu \bar{\nu}}^{(i)}(z, \bar{z})=\frac{\partial^{2} F^{(i)}(z, \bar{z})}{\partial z^{\mu} \partial \bar{z}^{\bar{\nu}}} \tag{5.11}
\end{equation*}
$$

In general, a complex manifold (i.e. one where the $C^{\infty}$ overlap functions on $\mathbb{C}^{n}$ can be taken to be holomorphic) with a symplectic structure such as (5.6) is called a Kähler manifold. The closed 2-forms (5.6) are then referred to as Kähler classes or Kähler 2forms, the associated metrics (5.7) are called Kähler metrics, and the locally-defined functions $F^{(i)}$ in (5.11) are called Kähler potentials. For an elementary, comprehensive introduction to complex manifolds and Kähler structures, we refer to [35] and [51]. In the case at hand here, the above construction yields a $G$-action on $\mathcal{M}_{G}$ by symplectic (canonical) transformations [57], i.e. holomorphic functions $f(z)$ on $\mathcal{M}_{G}$ which act on the Kähler potentials by

$$
\begin{equation*}
F^{(i)}(z, \bar{z}) \xrightarrow{f} \tilde{F}^{(i)}(z, \bar{z})=F^{(i)}(z, \bar{z})+f(z)+\bar{f}(\bar{z}) \tag{5.12}
\end{equation*}
$$

Consequently, the closed 2-forms $\omega^{(i)}$ define $G$-invariant integral symplectic structures on $\mathcal{M}_{G}$. Since $H^{2}(G ; \mathbb{R})=0$, the 2 -cocycles in (3.38) vanish and this $G$-action determines group homomorphisms into the Poisson algebras of $\mathcal{M}$. This also follows directly from the property (5.5) of the Kirillov-Kostant 2-form above.

The generators of $G$ in the Cartan basis have the non-vanishing Lie brackets

$$
\left[H_{i}, E_{\alpha}\right]=\alpha_{i} E_{\alpha} \quad, \quad\left[E_{\alpha}, E_{\beta}\right]= \begin{cases}N_{\alpha \beta} E_{\alpha+\beta} & , \alpha+\beta \neq 0  \tag{5.13}\\ \sum_{i=1}^{r} \alpha_{i} H_{i} & , \beta=-\alpha\end{cases}
$$

where $\alpha, \beta$ are the roots of $\mathrm{g}, H_{i}=H_{i}^{\dagger}, i=1, \ldots, r$, are the generators of the Cartan subalgebra $\mathbf{h} \otimes \mathbb{C}$ of $\mathbf{g} \otimes \mathbb{C}$, and $E_{\alpha}=E_{-\alpha}^{\dagger}$ are the step operators of $\mathbf{g}$ which act as raising operators by $\alpha>0$ (relative to some Weyl chamber) on the representation states which diagonalize the Cartan generators. The unitary irreducible representations of $G$ are characterized by highest weights $\lambda_{i}, i=1, \ldots, r$, which is an eigenvalue of $H_{i}$ whose eigenvector is annihilated by all the $E_{\alpha}$ for $\alpha>0$. Corresponding to each highest weight vector $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ we introduce the $G$-invariant symplectic 2 -form

$$
\begin{equation*}
\omega^{(\lambda)}=\sum_{i=1}^{r} \lambda_{i} \omega^{(i)} \tag{5.14}
\end{equation*}
$$

The symplectic potentials associated with (5.14) are

$$
\begin{equation*}
\theta^{(\lambda)}=\sum_{i=1}^{r} \lambda_{i}\left(\frac{\partial F^{(i)}}{\partial z^{\mu}} d z^{\mu}-\frac{\partial F^{(i)}}{\partial \bar{z}^{\bar{\mu}}} d \bar{z}^{\bar{\mu}}\right)+d F \tag{5.15}
\end{equation*}
$$

To construct a-topological path integral from this symplectic structure, we need to construct a Hamiltonian satisfying (3.27), i.e. a Hamiltonian which is given by generators of the subalgebra of $g$ which leave the symplectic potential (5.15) invariant. These are the canonical choices that give well-defined functions on the coadjoint orbit (5.1). As remarked at the end of Section 3.2, there usually exists a choice of function $F(z, \bar{z})$ in (5.15) for which this subalgebra contains the Cartan subalgebra $\mathbf{h}$ of $\mathbf{g}$. Let $H_{i}^{(\lambda)}$ be the generators of $\mathbf{h}$ in the representation with highest weight $\lambda$. Then the Hamiltonian

$$
\begin{equation*}
H^{(\lambda)}=\sum_{i=1}^{r} h_{i} H_{i}^{(\lambda)} \tag{5.16}
\end{equation*}
$$

satisfies the required conditions and the corresponding path integral will admit the topological form (4.119). Note that this is also consistent with the integrability arguments of the previous Chapters, which showed that the localizable Hamiltonians were those given by the Cartan generators of an isometry group $G$. Thus the path integral for the above dynamical system determines a cohomological topological quantum field theory which depends only on the second cohomology class of the symplectic 2 -form (5.14), i.e. on the representation with highest weight $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$.

To apply the equivariant localization formalism to these dynamical systems, we note that since the Kähler metrics $g^{(i)}$ above are $G$-invariant, the metric

$$
\begin{equation*}
g^{(\lambda)}=\sum_{i=1}^{r} \lambda_{i} g^{(i)} \tag{5.17}
\end{equation*}
$$

obeys the usual localization criteria. We shall soon see that these group theoretic structures are in fact implied by the localization constraints, in that they are the only equivariant Hamiltonian systems associated with homogeneous symplectic manifolds as above. We want to apply the Niemi-Tirkkonen localization formula (4.91) to the dynamical system above. The tangent and normal bundles of $O_{\Lambda^{\prime}}$ in $\mathbf{g}^{*}$ are related by [16]

$$
\begin{equation*}
T_{\Lambda^{\prime}} \mathbf{g}^{*}=T O_{\Lambda^{\prime}} \oplus \mathcal{N} O_{\Lambda^{\prime}}=O_{\Lambda^{\prime}} \times \mathbf{g}^{*} \tag{5.18}
\end{equation*}
$$

From the construction of the coadjoint orbit it follows that the normal bundle $\mathcal{N} O_{\Lambda^{\prime}}$ in $\mathbf{g}^{*}$ is a trivial bundle with trivial $G$-action on the fibers, and product $O_{\Lambda^{\prime}} \times \mathbf{g}^{*}$ is a trivial bundle with the coadjoint action of $G$ in the fibers. Thus using (2.75) and the multiplicativity property (2.80), we can write the $G$-equivariant $\hat{A}$-genus of the orbit $O_{\Lambda^{\prime}}$ as

$$
\begin{equation*}
\hat{A}_{V}=\sqrt{\operatorname{det}\left[\frac{\operatorname{ad} X}{\sinh (\operatorname{ad} X)}\right]} \equiv \frac{1}{\sqrt{j(\operatorname{ad} X)}} \tag{5.19}
\end{equation*}
$$

where ad $X$ is the Cartan element $X \in \mathbf{h}$ in the adjoint representation of $\mathbf{g}$. If we now choose the radius of the orbit to be the Weyl shift of the weight $\lambda$, i.e. $\Lambda^{\prime}=\lambda+\rho$ where

$$
\begin{equation*}
\rho=\frac{1}{2} \sum_{\alpha>0} \alpha \tag{5.20}
\end{equation*}
$$

is the half-sum of positive roots of $G$, then the localization formula (4.91) is none other than the celebrated Kirillov character formula [71]

$$
\begin{equation*}
\operatorname{tr}_{\lambda} \mathrm{e}^{i T X}=\frac{1}{\sqrt{j(\operatorname{ad} T X)}} \int_{O_{\lambda+\rho}} \frac{\left(\omega^{(\lambda)}\right)^{n}}{n!} \mathrm{e}^{i T H^{(\lambda)}} \tag{5.21}
\end{equation*}
$$

where $\operatorname{tr}_{\lambda}$ denotes the trace in the representation with highest weight $\lambda$ and $H^{(\lambda)}$ is the Hamiltonian (5.16) associated with the Cartan element $X \in \mathbf{h}$. If we further apply the finite-dimensional Duistermaat-Heckman theorem to the Fourier transform of the orbit on the right-hand side of (5.21) (i.e. the localization formula (4.97)) we arrive at the famous Harish-Chandra formula $[16,56]$. These applications of the DuistermaatHeckman theorem have been exploited recently in unitary matrix models to prove the Itzykson-Zuber formula and its generalizations [87].

The resulting character formula associated with the Harish-Chandra formula for the Fourier transform of the orbit is the Weyl character formula of $G[1]-[3],[44,71,94$, 113, 123]. Let $W\left(H_{C}\right)=N\left(H_{C}\right) / H_{C}$ be the Weyl group of $H_{C}$, where $N\left(H_{C}\right)$ is the normalizer subgroup of $H_{C}$, i.e. the subgroup of $g \in G$ with $h g H_{C}=g H_{C}, \forall h \in H_{C}$, so that $N\left(H_{C}\right)$ is the subgroup of fixed points of the left action of $H_{C}$ on the orbit $\mathcal{M}_{G}=G / H_{C}$. Given $w=n H_{C} \in W\left(H_{C}\right)$, with $n=\mathrm{e}^{i N} \in N\left(H_{C}\right)$, let $X^{(w)}=n^{-1} N n$ be the respective adjoint representation $\mathrm{e}^{i X^{(w)}}=n^{-1} \mathrm{e}^{i X} n$. The Weyl character formula can then be written as

$$
\begin{equation*}
\operatorname{tr}_{\lambda} \mathrm{e}^{i T X}=\sum_{w \in W\left(H_{C}\right)} \mathrm{e}^{i T(\lambda+\rho)\left(X^{(w)}\right)} \prod_{\alpha>0} \frac{1}{2 i \sin \frac{T}{2} \alpha\left(X^{(w)}\right)} \tag{5.22}
\end{equation*}
$$

where $\alpha\left(X^{(w)}\right)$ are the roots associated to the Cartan elements $X^{(w)}$. We shall see explicitly later on how these character formulas arise from the equivariant localization formulas of the last Chapter, but for now we simply note here the deep group theoretical significance that the localization formulas will represent for the path integral representations of the characters $\operatorname{tr}_{\lambda} e^{i T X}$. Note that the Weyl character formula writes the character of a Cartan group element as a sum of terms, one for each element of the Weyl group, the
group of symmetries of the roots of the Lie algebra g. In the context of the formalism of Chapter 4, the Weyl character formula will follow from the coadjoint orbit path integral over $L O_{\lambda+\rho}$. It was Stone [123] who first related this derivation of the Weyl character formula to the index of a Dirac operator from a supersymmetric path integral and hence to the semi-classical WKB evaluation of the spin partition function, as we did quite generally in Section 3.5 above $^{3}$. The path integral quantization of the coadjoint orbits of semi-simple Lie groups is essential to the quantization of spin systems. One important feature of the above topological field theories is that there is a one-to-one correspondence between the points on the orbits $G / H_{C}$ and the so-called coherent states associated with the Lie group $G$ in the representation with highest weight vector $\lambda$ [108]. The above character formulas can therefore be represented in complex polarizations using coherent state path integrals. We shall discuss these and other aspects of the path integral representations of character formulas later on in this Chapter.

We close our general discussion of these important classes of cohomological quantum field theories with a technical point concerning the above derivation. From the point of view of path integral quantization, the necessity of performing a Weyl shift $\lambda \rightarrow$ $\lambda+\rho$ in the above is rather unsatisfactory. As we shall see, the Weyl character formula follows directly from the WKB formula for the spin partition function [123], and a proper discretization of the trace in (5.21) really does give the path integral over the orbit $O_{\lambda}$ [2, 94]. The Weyl shift is in fact an artifact of the regularization procedure [2, 84, 94, 113, 126] utilized in Section 4.5 in evaluating the fluctuation determinant there which lead to the Niemi-Tirkkonen localization formula (4.91), which leads directly to the Kirillov character formula (5.21). As the $\hat{A}$-genus is inherently related to tangent bundles of real manifolds, the problem here essentially is that the regularization used in Section 4.5 does not respect the complex structure defined on the orbit. We shall see later on how a coherent state analysis avoids this problem, but for now we note that the proper way

[^24]to carry out the regularization procedure here is to attach different signs to the factors of $b$ in Section 3.5 (see (4.87)) corresponding to the holomorphic and antiholomorphic sectors in the regularization (4.85),(4.87) [84]. This restores the holomorphic properties of the path integral wherein the skew-eigenvalues $\left(\lambda_{j},-\lambda_{j}\right)$ of the block $R_{V}^{(j)}$ correspond, respectively, to the holomorphic and anti-holomorphic components of the equivariant curvature 2-form. The correct way to treat the complex tangent bundle here is to then restrict to the holomorphic component of this curvature [ $16,31,84]$. In doing this, the fluctuation determinant in (4.83) is not under a square root in this complex case, because now it arises from Berezin integration over complex Grassmann variables.

Taking the fluctuation regularization factor of Section 4.5 to be $b=\frac{1}{2}$, the evaluation of the fluctuation determinant in (4.83) leads instead to the equivariant Todd class (2.79) of the complex tangent bundle. Then we arrive at a character formula without an explicit Weyl shift,

$$
\begin{equation*}
\operatorname{tr}_{\lambda} \mathrm{e}^{i T X}=\int_{O_{\lambda}} \operatorname{ch}_{V(\lambda)}\left(-i T \omega^{(\lambda)}\right) \wedge \operatorname{td}_{V(\lambda)}\left(T R^{(\lambda)}\right) \tag{5.23}
\end{equation*}
$$

which can be derived as well from the coadjoint orbit path integral over $L O_{\lambda}$ (as opposed to $L O_{\lambda+\rho}$ as in (5.21)). We shall discuss the topological interpretation of (5.23) in section 5.4. The choice of regulating factor $b=\frac{1}{2}$ above is also consistent with a careful evaluation of the effect of the eta-invariant (4.84) associated with the phase of the fluctuation determinant [84].

### 5.2 Isometry Groups of Simply Connected Riemannian Spaces

Given the large class of localizable dynamical systems of the last Section and their novel topological and group theoretical properties, we now turn to an opposite point of view and begin examining what Hamiltonian systems in general fit within the framework of equivariant localization. For this we shall analyse the fundamental isometry condition on the physical theory in a quite general setting, and show that the localizable systems "essentially" all fall into the general framework of the coadjoint orbit quantization of the
last Section. Indeed, this will be consistent with the integrability features implied by the equivariant localization criteria.

We consider a simply-connected, connected and orientable Riemannian manifold $(\mathcal{M}, g)$ of dimension $d$ (not necessarily symplectic for now) and with metric $g$ of Euclidean signature, for definiteness. The isometry group $\mathcal{I}(\mathcal{M}, g)$ is the diffeomorphism subgroup of $C^{\infty}$ coordinate transformations $x \rightarrow x^{\prime}(x)$ which preserve the metric distance on $\mathcal{M}$, i.e. for which $g_{\mu \nu}^{\prime}\left(x^{\prime}\right)=g_{\mu \nu}\left(x^{\prime}\right)$. The generators of the connected component ${ }^{4}$ of $\mathcal{I}(\mathcal{M}, g)$ form the vector field Lie algebra

$$
\begin{equation*}
\mathcal{K}(\mathcal{M}, g)=\left\{V \in T \mathcal{M}: \mathcal{L}_{V} g=0\right\} \tag{5.24}
\end{equation*}
$$

whose generators $V^{a}$ obey the commutation relations (2.43). For a generic simplyconnected space, the Lie group $\mathcal{I}(\mathcal{M}, g)$ is locally compact in the compact-open topology induced by $\mathcal{M}$ [57]. In particular, if $\mathcal{M}$ is compact then so is $\mathcal{I}(\mathcal{M}, g)$.

We shall now quickly run through some of the basic facts concerning isometries of simply-connected Riemannian manifolds, all of whose proofs can be found in $[36,37,57$, 125, 129]. First of all, the number of linearly independent Killing vectors (i.e. generators of (5.24)) is bounded as

$$
\begin{equation*}
\operatorname{dim} \mathcal{K}(\mathcal{M}, g) \leq d(d+1) / 2 \tag{5.25}
\end{equation*}
$$

when $\mathcal{M}$ has dimension $d$, so that the infinitesimal isometries of $(\mathcal{M}, g)$ are therefore characterized by finitely-many linearly independent Killing vectors in $\mathcal{K}(\mathcal{M}, g)$. There are 2 important classes of metric spaces $(\mathcal{M}, g)$ characterized by their possible isometries. We say that a metric space $(\mathcal{M}, g)$ is homogeneous if there exists infinitesimal isometries $V$ that carry any given point $x \in \mathcal{M}$ to any other point in its immediate neighbourhood. ( $\mathcal{M}, g$ ) is said to be isotropic about a point $x \in \mathcal{M}$ if there exists infinitesimal isometries $V$ that leave the point $x$ fixed, and, in particular, if $(\mathcal{M}, g)$ is isotropic about all of its

[^25]points then we say that it is isotropic. The homogeneity condition means that the metric $g$ must admit Killing vectors that at any given point of $\mathcal{M}$ take on all possible values (i.e. any point on $\mathcal{M}$ is geometrically like any other point). The isotropy condition means that an isotropic point $x_{0}$ of $\mathcal{M}$ is always a fixed point of an $\mathcal{I}(\mathcal{M}, g)$-action on $\mathcal{M}$, $V\left(x_{0}\right)=0$ for some $V \in \mathcal{K}(\mathcal{M}, g)$, but whose first derivatives take on all possible values, subject only to the Killing equation $\mathcal{L}_{V} g=0$.

It follows that a homogeneous metric space always admits $d=\operatorname{dim} \mathcal{M}$ linearly independent Killing vectors (intuitively generating translations in the $d$ directions), and a space that is isotropic about some point admits $d(d-1) / 2$ Killing vector fields (intuitively generating rigid rotations about that point). The connection between isotropy and homogeneity of a metric space lies in the fact that any metric space that is isotropic is also homogeneous. The spaces which have the maximal number $d(d+1) / 2$ of linearly independent Killing vectors enjoy some very special properties, as we shall soon see. We shall refer to such spaces as maximally symmetric spaces. The above dimension counting shows that a homogeneous metric space that is isotropic about some point is maximally symmetric, and, in particular, any isotropic space is maximally symmetric. The converse is also true, i.e. a maximally symmetric space is homogeneous and isotropic. We shall therefore also refer to maximally symmetric spaces as simply homogeneous. In these cases, there is only one orbit under the $\mathcal{I}(\mathcal{M}, g)$-action on $\mathcal{M}$, i.e. $\mathcal{M}$ can be represented as the orbit $\mathcal{M}=\mathcal{I}(\mathcal{M}, g) \cdot x$ of any element $x \in \mathcal{M}$, and the space of orbits $\mathcal{M} / \mathcal{I}(\mathcal{M}, g)$ consists of only a single point. In this case we say that the isometry group $\mathcal{I}(\mathcal{M}, g)$ acts transitively on $\mathcal{M}$.

We shall now describe the rich features of maximally symmetric spaces. It turns out that these spaces are uniquely characterized by a special curvature constant $K$. Specifically, $(\mathcal{M}, g)$ is a maximally symmetric Riemannian manifold if and only if there exists a constant $K \in \mathbb{R}$ such that the Riemann curvature tensor of $g$ can be written
locally almost everywhere as

$$
\begin{equation*}
R_{\lambda \rho \sigma \nu} \equiv g_{\lambda \mu} R_{\rho \sigma \nu}^{\mu}=K\left(g_{\sigma \rho} g_{\lambda \nu}-g_{\nu \rho} g_{\lambda \sigma}\right) \tag{5.26}
\end{equation*}
$$

In dimension $d \geq 3$, Schur's lemma [57] states that the existence of such a form for the curvature tensor automatically implies the constancy of $K$. For $d=2$, however, this is not the case, and indeed dimension counting shows that the curvature of a Riemann surface always takes the form (5.26). In this case $K$ is called the Gaussian curvature of $(\mathcal{M}, g)$ and it is in general not constant. The above result implies that the Gaussian curvature $K$ of a maximally symmetric simply connected Riemann surface is constant.

The amazing result here is the isometric correspondence between maximally symmetric spaces. Any 2 maximally symmetric spaces $\left(\mathcal{M}_{1}, g_{1}\right)$ and $\left(\mathcal{M}_{2}, g_{2}\right)$ of the same dimension and with the same curvature constant $K$ are isometric, i.e. there exists a diffeomorphism $f: \mathcal{M}_{1} \rightarrow \mathcal{M}_{2}$ between the 2 manifolds relating their metrics by $g_{1}(x)=g_{2}(f(x))$. Thus given any maximally symmetric space we can map it isometrically onto any other one with the same curvature tensor (5.26). We can therefore model maximally symmetric spaces by some "standard" spaces, which we now proceed to describe. Consider a flat $(d+1)$-dimensional space with coordinates $\left(x^{\mu}, z\right)$ and metric

$$
\begin{equation*}
\eta_{d+1}=\frac{1}{|K|} d x_{\mu} \otimes d x^{\mu}+\frac{1}{K} d z \otimes d z \tag{5.27}
\end{equation*}
$$

where $K$ is a real-valued constant. A d-dimensional space can be embedded into this larger space by restricting the variables $x^{\mu}$ and $z$ to the surface of a (pseudo-) sphere,

$$
\begin{equation*}
\operatorname{sgn}(K) x^{2}+z^{2}=1 \tag{5.28}
\end{equation*}
$$

Using (5.28) to solve for $z(x)$ and substituting this into (5.27), the metric induced on the surface by this embedding is then

$$
g_{K}= \begin{cases}\frac{1}{K}\left(d x_{\mu} \otimes d x^{\mu}+\frac{x_{\mu} x_{\nu}}{1-x^{2}} d x^{\mu} \otimes d x^{\nu}\right) & \text { for } \quad K>0  \tag{5.29}\\ \frac{1}{|K|}\left(d x_{\mu} \otimes d x^{\mu}-\frac{x_{\mu} x_{\nu}}{1-x^{2}} d x^{\mu} \otimes d x^{\nu}\right) & \text { for } \quad K<0 \\ d x_{\mu} \otimes d x^{\mu} \text { for } K=0\end{cases}
$$

These 3 cases represent, respectively, the standard metrics on the $d$-sphere $S^{d}$ of radius $K^{-1 / 2}$, the hyperbolic Lobaschevsky space $\mathcal{H}^{d}$ of constant negative curvature $K$, and Euclidean $d$-space $\mathbb{R}^{d}$ with its usual flat metric $\eta_{E^{d}}$.

From the embedding condition (5.28) and the manifest invariances of the embedding space geometry (5.27) it is straightforward to show that the above spaces all admit a $d(d+1) / 2$-parameter group of isometries. These consist of $d(d-1) / 2$ rigid rotations about the origin and $d$ (quasi-)translations. The first set of isometries always leave some points on the manifold fixed, while the second set translate any point on $\mathcal{M}$ to any other point in its vicinity. The 3 spaces above are there the 3 unique (up to isometric equivalence) maximally symmetric spaces in $d$-dimensions, and any other maximally symmetric space will be isometric to one of these spaces, depending on whether $K=0$, $K>0$ or $K<0$. It is this feature of maximally symmetric spaces that allows the rather complete isometric correspondence that follows. The Killing vector fields that generate the above stated isometries are, respectively,

$$
V_{K}=\left\{\begin{array}{l}
\left(\Omega_{\nu}^{\mu} x^{\nu}+\alpha^{\mu}\left[1-\operatorname{sgn}(K) x^{2}\right]^{1 / 2}\right) \frac{\partial}{\partial x^{\mu}} \text { for } K \neq 0  \tag{5.30}\\
\left(\Omega_{\nu}^{\mu} x^{\nu}+\alpha^{\mu}\right) \frac{\partial}{\partial x^{\mu}} \text { for } K=0
\end{array}\right.
$$

where $\Omega_{\nu}^{\mu}=-\Omega_{\mu}^{\nu}$ and $\alpha^{\mu}$ are real-valued parameters. These Killing vectors generate the respective isometry groups

$$
\begin{equation*}
\mathcal{I}\left(S^{d}\right)=S O(d+1) \quad, \quad \mathcal{I}\left(\mathcal{H}^{d}\right)=S O(d, 1) \quad, \quad \mathcal{I}\left(\mathbb{R}^{d}\right)=E^{d} \tag{5.31}
\end{equation*}
$$

where $E^{d}$ denotes the Euclidean group in $d$-dimensions, i.e. the semi-direct product of the rotation and translation groups in $\mathbb{R}^{d}, S O(d+1)$ is the rotation group of $\mathbb{R}^{d+1}$, and $S O(d, 1)$ is the Lorentz group in $(d+1)$-dimensional Minkowski space. From this we see therefore what sort of group actions should be considered within the localization framework for maximally symmetric spaces. Note that the maximal symmetry of the spaces $S^{d}$ and $\mathcal{H}^{d}$ are actually implied by that of $\mathbb{R}^{d}$, because $S^{d}$ can be regarded as the one-point compactification of $\mathbb{R}^{d}$, i.e. $S^{d}=\mathbb{R}^{d} \cup\{\infty\}$.

Our final general result concerning Killing vectors on generic $d$-dimensional simply connected manifolds is for the cases where the isometry group of $(\mathcal{M}, g)$ has the opposite feature of maximal symmetry, i.e. when $\mathcal{I}(\mathcal{M}, g)$ is 1-dimensional. Consider a 1-parameter group of isometries acting on the metric space $(\mathcal{M}, g)$. Let $V=V^{\mu}(x) \frac{\partial}{\partial x^{\mu}} \in$ $T \mathcal{M}$ be a generator of $\mathcal{I}(\mathcal{M}, g)$, and let $\chi^{\mu}(x)$ be differentiable functions on $\mathcal{M}$ such that the change of variables $x^{\mu \mu}=\chi^{\mu}(x)$ has non-trivial Jacobian

$$
\begin{equation*}
\operatorname{det}\left[\frac{\partial \chi^{\mu}}{\partial x^{\nu}}\right] \neq 0 \tag{5.32}
\end{equation*}
$$

For $\mu=2, \ldots, d$ we can choose the diffeomorphisms $\chi^{\mu}(x)$ to in addition be the $d-1$ linearly independent solutions of the first order linear homogeneous partial differential equation

$$
\begin{equation*}
V\left(\chi^{\mu}\right)=V^{\nu} \partial_{\nu} \chi^{\mu}=0 \quad, \quad \mu=2, \ldots, d \tag{5.33}
\end{equation*}
$$

given by the constant coordinate lines $\chi^{\mu}(x)=$ constant embedded into $\mathcal{M}$ from $\mathbb{R}^{d-1}$. The functions $\chi^{\mu}(x)$ for $\mu=2, \ldots, d$ also have an invertible Jacobian matrix since then

$$
\begin{equation*}
\operatorname{rank}_{2 \leq \mu, \nu \leq d}\left[\frac{\partial \chi^{\mu}}{\partial x^{\nu}}\right]=d-1 \tag{5.34}
\end{equation*}
$$

which owes to the existence of paths under the flow of the isometry group such that

$$
\begin{equation*}
\frac{d x^{1}}{V^{1}}=\frac{d x^{2}}{V^{2}}=\cdots=\frac{d x^{d}}{V^{d}} \tag{5.35}
\end{equation*}
$$

as implied by (5.33) and the flow equation (2.42).
If we now choose the function $\chi^{1}(x)$ so that $\frac{\partial \chi^{1}}{\partial x^{\mu}} \neq 0$ for $\mu=1, \ldots, d$, then the coordinate transformation $x^{\mu} \rightarrow x^{\prime \mu}(x)=\chi^{\mu}(x)$ changes the components of the vector field $V$ to

$$
\begin{equation*}
V^{\prime \mu}=V^{\nu} \frac{\partial}{\partial x^{\nu}} \chi^{\mu} \quad, \quad \mu=1, \ldots, d \tag{5.36}
\end{equation*}
$$

It follows from (5.33) that in these new $x^{\prime}$-coordinates $V$ therefore has components $V^{\prime 1} \neq$ 0 and $V^{\prime \mu}=0$ for $\mu=2, \ldots, d$. Now further change coordinates $x^{\prime} \rightarrow x^{\prime \prime}$ defined by

$$
\begin{equation*}
x^{\prime \prime 1}=\int_{x_{0}^{\prime}}^{x^{\prime}} \frac{d x^{1}}{V^{\prime 1}\left(x^{\prime}\right)} \quad, \quad x^{\prime \prime \mu}=x^{\prime \mu}=\chi^{\mu} \quad \text { for } \quad \mu=2, \ldots, d \tag{5.37}
\end{equation*}
$$

where $x_{0}$ is a fixed basepoint in $\mathcal{M}$. In this way we have shown that, in the case of a 1parameter isometry group action on $(\mathcal{M}, g)$, there exists a local system of $x^{\prime \prime}$-coordinates defined almost everywhere on $\mathcal{M}$ in which the Killing vector of the isometry group has components

$$
\begin{equation*}
V^{\prime \prime 1}=1 \quad, \quad V^{\prime \prime \mu}=0 \quad \text { for } \quad \mu=2, \ldots, d \tag{5.38}
\end{equation*}
$$

Furthermore, an application of the Killing equation (2.92) shows that ( $\mathcal{M}, g$ ) admits a Killing vector if and only if there are local coordinates $x^{\prime \prime}$ on $\mathcal{M}$ in which the metric tensor components $g_{\mu \nu}^{\prime \prime}\left(x^{\prime \prime}\right)$ are independent of the coordinate $x^{\prime \prime 1}$,

$$
\begin{equation*}
\frac{\partial g_{\mu \nu}^{\prime \prime}\left(x^{\prime \prime}\right)}{\partial x^{\prime \prime 1}}=0 \tag{5.39}
\end{equation*}
$$

and then the integral curves of $x^{\prime \prime 1}$ parametrize the paths of the infinitesimal isometry and of the finite total isometry according to (2.42). Moreover, the above derivation also shows that 2 distinct isometries $V_{1}$ and $V_{2}$ of $(\mathcal{M}, g)$ cannot have the same path, since they can be independently chosen to have the single non-vanishing components $V_{1}^{\prime \prime 1}=V_{2}^{\prime \prime 2}=0$. These results mean that locally any isometry of $g$ looks like translations in a single coordinate, and this therefore gives the representation of a 1-parameter isometry as an explicit $\mathbb{R}^{1}$-action on $(\mathcal{M}, g)$ (which is either bounded or is a $U(1)$-action when $\mathcal{M}$ is compact). We shall refer to this system of coordinates as a preferred set of coordinates with respect to a Killing vector field $V$.

For simplicity, we shall now concentrate on the cases where $\mathcal{M}$ is a simply connected 2-dimensional symplectic manifold with metric $g$. The advantage of this insofar as the localization formalism is concerned is that the Riemann uniformization theorem [62, 92, 121] tells us that $g_{\mu \nu}(x)$ has globally only 1 independent component. This situation is therefore amenable to a detailed analysis of the equivariant localization constraints in terms of the single degree of freedom of the metric $g$. Defining complex coordinates $z, \bar{z}=x^{1} \pm i x^{2}$, we can represent the metric as

$$
\begin{equation*}
g=\lambda(d z+\mu d \bar{z}) \otimes(d \bar{z}+\bar{\mu} d z) \tag{5.40}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda=(\operatorname{tr} g+2 \sqrt{\operatorname{det} g}) / 4 \quad, \quad \mu=\left(g_{11}-g_{22}+2 i g_{12}\right) / 4 \lambda \tag{5.41}
\end{equation*}
$$

Orientation-preserving diffeomorphisms of $\mathcal{M}$ which only change the function $\lambda>0$ above are called conformal transformations. The function $\mu$ determines the complex structures of $\mathcal{M}$, and therefore the set of inequivalent complex structures of $\mathcal{M}$ is in a one-to-one correspondence with the space of conformal equivalence classes of metrics on $\mathcal{M}$. A complex coordinate $w$ is said to be an isothermal coordinate for $g$ if $g=\rho d w \otimes d \bar{w}$ for some function $\rho>0$. Using the tensor transfromation law for $g$, it follows from (5.40) that an isothermal coordinate $w$ for $g$ exists if and only if the Beltrami partial differential equation

$$
\begin{equation*}
\frac{\partial w}{\partial \bar{z}}=\mu \frac{\partial w}{\partial z} \tag{5.42}
\end{equation*}
$$

has a $C^{\infty}$-solution $w(z, \bar{z})$. Such a solution always exists provided that the function $\mu(z, \bar{z})$ is uniformly bounded as $\|\mu\|_{\infty}<1$. A complex structure on $\mathcal{M}$ can therefore be identified with the conformal structure represented by the Riemannian metric $g$.

The simple-connectivity of a 2-manifold $\mathcal{M}$ implies that via a diffeomorphism and Weyl rescaling $g \rightarrow \mathrm{e}^{\varphi} g$ of the coordinates the metric can be put globally ${ }^{5}$ into the isothermal form

$$
\begin{equation*}
g_{\mu \nu}(x)=\mathrm{e}^{\varphi(x)} \delta_{\mu \nu} \quad \text { or } \quad g=\mathrm{e}^{\varphi(z, \bar{z})} d z \otimes d \bar{z} \tag{5.43}
\end{equation*}
$$

where $\varphi(x)$ is a globally-defined real-valued function on $\mathcal{M}$ which we shall refer to as the conformal factor of the metric. This means that there is a unique complex structure on the Riemann surface $\mathcal{M}$ which we can define by the standard local complex coordinates $z, \bar{z}=x^{1} \pm i x^{2}$. Notice that these remarks are not true if $H^{1}(\mathcal{M} ; \mathbb{Z}) \neq 0$, because then the metric has additional degrees of freedom from moduli parameters (see the torus example in Section 3.5), i.e. (5.43) should be replaced by

$$
\begin{equation*}
g=\mathrm{e}^{\varphi} \hat{g}(\tau) \tag{5.44}
\end{equation*}
$$

[^26]where $\tau$ labels the additional modular degress of freedom of the metric. We shall discuss the case of multiply-connected phase spaces in the next Chapter.

With this complex structure we define $V^{z, \bar{z}}=V^{1} \pm i V^{2}$ for any vector field $V$, and we set $\partial, \bar{\partial}=\frac{1}{2}\left(\frac{\partial}{\partial x^{1}} \mp i \frac{\partial}{\partial x^{2}}\right)$. The Killing equations (2.92) in these complex coordinates can be written as

$$
\begin{equation*}
\bar{\partial} V^{z}=\partial V^{\bar{z}}=0 \quad, \quad \partial V^{z}+\bar{\partial} V^{\bar{z}}+V^{z} \partial \varphi+V^{\bar{z}} \bar{\partial} \varphi=0 \tag{5.45}
\end{equation*}
$$

The first set of equations in (5.45) are the Cauchy-Riemann equations and they imply that in these local coordinates the Killing vector field $V^{z}$ is a holomorphic function on $\mathcal{M}$. The other equation is a source equation for $V^{z}$ and $V^{\bar{z}}$ that explicitly determines the Killing fields in terms of the single degree of freedom of the metric $g$ (i.e. the conformal factor $\varphi$ ).

The Gaussian curvature scalar $K(x)$ of $(\mathcal{M}, g)$, which is always defined by (5.26) in 2 dimensions, can be written in these isothermal coordinates as

$$
\begin{equation*}
K(x)=-\frac{1}{2} \mathrm{e}^{-\varphi(x)} \nabla^{2} \varphi(x) \tag{5.46}
\end{equation*}
$$

where $\nabla^{2}=\partial \bar{\partial}$ is the 2-dimensional scalar Laplacian on $\mathcal{M}$ associated with the metric (5.43). This follows from noting that the only non-vanishing connection coefficients of the metric (5.43) are

$$
\begin{equation*}
\Gamma_{z z}^{z}=\partial \varphi \quad, \quad \Gamma_{\bar{z} \bar{z}}^{\bar{z}}=\bar{\partial} \varphi \tag{5.47}
\end{equation*}
$$

The Gaussian curvature of $(\mathcal{M}, g)$ then uniquely characterizes the isometry group acting on the phase space. If $K$ is constant, then $(\mathcal{M}, g)$ is maximally symmetric with 3 linearly independent Killing vectors. Moreover, in this case $(\mathcal{M}, g)$ is isometric to either the 2sphere $S^{2}$, the Lobaschevsky plane $\mathcal{H}^{2}$ or the Euclidean plane $\mathbb{R}^{2}$. We shall soon examine these 3 distinct Riemannian spaces in detail. Notice, however, that if $\mathcal{M}=\Sigma^{h}$ is a compact Riemann surface of genus $h$, then the Gauss-Bonnet-Chern theorem (2.84) in the case at hand reads

$$
\begin{equation*}
\int_{\Sigma^{h}} d \operatorname{vol}(g(x)) K(x)=4 \pi(1-h) \tag{5.48}
\end{equation*}
$$

where $d \operatorname{vol}(g(x))=d^{2} x \sqrt{\operatorname{det} g(x)}$ is the metric volume form of $(\mathcal{M}, g)$. Thus a maximally symmetric compact Riemann surface of constant negative curvature must have genus $h \geq 2$. It follows, under the simple-connectivity assumption of this Chapter, that when $K=0$ or $K>0$ the phase space $\mathcal{M}$ can be either compact or non-compact, but when $K<0$ it is necessarily non-compact.

The other extremal case is where $(\mathcal{M}, g)$ admits only a 1-parameter group of isometries. From the above general discussion it follows that in this case there exist 2 differentiable functions $\chi^{1}$ and $\chi^{2}$ on $\mathcal{M}$ and local coordinates $x^{\prime}$ on $\mathcal{M}$ such that

$$
\begin{equation*}
V^{\mu} \frac{\partial}{\partial x^{\mu}} \chi^{2}\left(x^{1}, x^{2}\right)=0 \quad, \quad x^{2}=\chi^{2}\left(x^{1}, x^{2}\right) \tag{5.49}
\end{equation*}
$$

and in these coordinates the Killing vector field has components $V^{\prime 1}=1, V^{\prime 2}=0$. Moreover, the characteristic curves of the coordinate $x^{\prime 2}=\chi^{2}$, defined by the initial data surfaces of the partial differential equation in (5.49), can be chosen to be orthogonal to the paths defined by the isometry generator $V$, i.e. we can choose the initial data for the solutions of (5.49) to lie on a non-characteristic surface. This means that in these new coordinates $g_{12}^{\prime}\left(x^{\prime}\right)=0$. Thus in this case the metric can be written locally as

$$
\begin{equation*}
g=g_{11}^{\prime} d x^{\prime 1} \otimes d x^{\prime 1}+g_{22}^{\prime} d x^{2} \otimes d x^{\prime 2} \tag{5.50}
\end{equation*}
$$

and from (5.39) it follows that $g_{11}^{\prime}$ and $g_{22}^{\prime}$ are functions only of $x^{\prime 2}$. The phase space therefore describes a surface of revolution, for example a cylinder or the 'cigar-shaped' geometries that are described in typical black hole theories [129].

The only other case left to consider here is when $(\mathcal{M}, g)$ has a 2-dimensional isometry group. In this case we have 2 independent vector fields $V_{1}=V_{1}^{\mu} \frac{\partial}{\partial x^{\mu}}$ and $V_{2}=V_{2}^{\mu} \frac{\partial}{\partial x^{\mu}}$ which obey the Lie algebra (2.43) with $a, b, c=1,2$. There are 2 possibilities for this Lie algebra - either the isometry group is abelian, $f^{a b c}=0$, or it is non-abelian, $f^{a b c} \neq 0$ for some $a, b, c$. Since $V_{1}$ and $V_{2}$ cannot have the same path in $\mathcal{M}$, we can choose paths for the constant coordinate lines so that $V_{1}^{2}=V_{2}^{1}=0$. In the abelian case, the commutativity
of $V_{1}$ and $V_{2}$

$$
\begin{equation*}
\left[V_{1}, V_{2}\right]=0 \tag{5.51}
\end{equation*}
$$

implies that $V_{1}^{1}$ is a function of $x^{1}$ alone and $V_{2}^{2}$ is a function only of $x^{2}$. As above, we can choose local coordinates almost everywhere on $\mathcal{M}$ in which $V_{1}^{1}=V_{2}^{2}=1$. In these coordinates, the Killing equations imply that the metric components $g_{\mu \nu}(x)$ are all constant. Thus in this case $(\mathcal{M}, g)$ is isometric to flat Euclidean space, which contradicts the standard maximal symmetry arguments above.

In the non-abelian case, we can choose linear combinations of the isometry generators $V_{1}$ and $V_{2}$ so that their Lie algebra is

$$
\begin{equation*}
\left[V_{1}, V_{2}\right]=V_{1} \tag{5.52}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\partial_{1} V_{2}^{2}=0 \quad, \quad \partial_{2} \log V_{1}^{1}=-1 / V_{2}^{2} \tag{5.53}
\end{equation*}
$$

and so we can choose local coordinates almost everywhere on $\mathcal{M}$ in which $V_{2}^{2}=1$ and $V_{1}^{1}=\mathrm{e}^{-x^{2}}$. The Killing equations then become

$$
\begin{equation*}
\partial_{2} g_{\mu \nu}=\partial_{1} g_{11}=0 \quad, \quad \partial_{1} g_{12}=g_{11} \quad, \quad \partial_{1} g_{22}=2 g_{12} \tag{5.54}
\end{equation*}
$$

which have solutions

$$
\begin{equation*}
g_{11}=\alpha \cdot, \quad g_{12}=\alpha x^{1}+\beta \quad, \quad g_{22}=\alpha\left(x^{1}\right)^{2}+2 \beta x^{1}+\gamma \tag{5.55}
\end{equation*}
$$

where $\alpha, \beta$ and $\gamma$ are real-valued constants. It is then straightforward to compute the Gaussian curvature of $g$ from the identity

$$
\begin{equation*}
K(x)=-R_{1212}(x) / \operatorname{det} g(x) \tag{5.56}
\end{equation*}
$$

which gives $K(x)$ as the constant $K=\alpha /\left(\beta^{2}-\alpha \gamma\right)$, again contradicting the maximal symmetry theorems quoted above.

Thus a 2-dimensional phase space is either maximally symmetric with a 3-dimensional isometry group, or it admits a 1-parameter group of isometries (or, equivalently, has a single 1-dimensional maximally symmetric subspace), because the above arguments show that it clearly cannot have a 2-dimensional isometry group. The fact that there are only 2 distinct classes of isometries in 2 dimensions is another very appealing feature of these cases for the analysis which follows. For the remainder of this Chapter we shall analyse the equivariant Hamiltonian systems which can be studied on each of the 4 possible isometric types of spaces above and discuss the features of the integrable quantum models that arise from the localization formalism. This will provide a large set of explicit examples of the formalism developed thus far, and at the same time clarify some other issues that arise in the formalism of path integral quantization.

### 5.3 Euclidean Phase Spaces and Holomorphic Quantization

We begin our study of general localizable Hamiltonian systems with the case where the phase space $\mathcal{M}$ is locally flat, i.e. $K=0$. The conformal factor $\varphi$ in (5.43) and (5.46) then satisfies the 2-dimensional Laplace equation

$$
\begin{equation*}
\nabla^{2} \varphi(z, \bar{z})=\partial \bar{\partial} \varphi(z, \bar{z})=0 \tag{5.57}
\end{equation*}
$$

whose general solutions are

$$
\begin{equation*}
\varphi(z, \bar{z})=f(z)+\bar{f}(\bar{z}) \tag{5.58}
\end{equation*}
$$

where $f(z)$ is any holomorphic function on $\mathcal{M}$. The Riemannian manifold $(\mathcal{M}, g)$ is isometric to the flat Euclidean space $\left(\mathbb{R}^{2}, \eta_{E^{2}}\right)$ and from the metric tensor transformation law it follows that this coordinate change $z \rightarrow w$ taking the metric (5.43) to $d w \otimes d \bar{w}$ satisfies

$$
\begin{equation*}
\frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}}+\frac{\partial \bar{w}}{\partial z} \frac{\partial w}{\partial \bar{z}}=\mathrm{e}^{\varphi(z, \bar{z})}=\mathrm{e}^{f(z)} \mathrm{e}^{\bar{f}(\bar{z})} \quad, \quad \frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial z}=\frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial \bar{z}}=0 \tag{5.59}
\end{equation*}
$$

It follows from (5.59) that this isometric transformation is the 2-dimensional conformal transformation $z \rightarrow w_{f}(z)$ (i.e. an analytic rescaling of the standard flat Euclidean metric
of the plane) where

$$
\begin{equation*}
w_{f}(z)=\int_{C_{z}} d \xi \mathrm{e}^{f(\xi)} \tag{5.60}
\end{equation*}
$$

and $C_{z} \subset \mathcal{M}$ is a simple curve from some fixed basepoint in $\mathcal{M}$ to $z$. From the last Section (eq. (5.30)) we know that the Killing vectors of $\left(\mathbb{R}^{2}, \eta_{E^{2}}\right)$ in the complex coordinates ( $w, \bar{w}$ ) take on the general form

$$
\begin{equation*}
V_{\mathbf{R}^{2}}^{w}=-i \Omega w+\alpha \quad, \quad V_{\mathbf{R}^{2}}^{\bar{w}}=i \Omega \bar{w}+\bar{\alpha} \tag{5.61}
\end{equation*}
$$

where $\Omega \in \mathbb{R}$ and $\alpha \in \mathbb{C}$ are constants. The Killing vectors (5.61) follow directly from (5.45) with $\varphi=0$ there, and they generate the groups of 2-dimensional rotations $w \rightarrow \mathrm{e}^{i \Omega} w$ and translations $w \rightarrow w+\alpha$ whose semi-direct product forms the Euclidean group $E^{2}$ of the plane.

In these local complex coordinates on $\mathbb{R}^{2}$ the Hamiltonian equations $d H=-i_{V} \omega$ take the form

$$
\begin{equation*}
\partial H=\frac{i}{\overline{2}} \omega(w, \bar{w}) V^{\bar{w}} \quad, \quad \bar{\partial} H=-\frac{i}{2} \omega(w, \bar{w}) V^{w} \tag{5.62}
\end{equation*}
$$

where

$$
\begin{equation*}
\omega=\frac{i}{2} \omega(w, \bar{w}) d w \wedge d \bar{w} \tag{5.63}
\end{equation*}
$$

The symplectic 2-form (5.63) can be explicitly determined here by recalling that the Hamiltonian group action on the phase space is symplectic so that $\mathcal{L}_{V} \omega=0$. In local coordinates this means that

$$
\begin{equation*}
\partial_{\mu}\left(V^{\lambda} \omega_{\nu \lambda}\right)-\partial_{\nu}\left(V^{\lambda} \omega_{\mu \lambda}\right)=0 \tag{5.64}
\end{equation*}
$$

for each $\mu$ and $\nu$. Requiring this symplecticity condition for the full isometry group action of $E^{2}$ on $\mathbb{R}^{2}$, we substitute into (5.64) each of the 3 linearly independent Killing vectors represented by (5.61) (corresponding to $\Omega=0, \alpha^{1}=0$ and $\alpha^{2}=0$ there). The differential equations (5.64) for the function $\omega(w, \bar{w})$ now easily imply that it is constant on $\mathbb{R}^{2}$ with these substitutions. Thus $\omega(w, \bar{w})$ is the Riemannian volume (and in this case the Darboux) 2-form globally on $\mathbb{R}^{2}$. Substituting the Darboux value $\omega(w, \bar{w})=1$ and
the Killing vectors (5.61) into the Hamiltonian equations above and integrating them up to get $H(w, \bar{w})$, we see that the most general equivariant Hamiltonian on a planar phase space $\mathcal{M}$ is

$$
\begin{equation*}
H_{0}(z, \bar{z})=\Omega w_{f}(z) \bar{w}_{f}(\bar{z})+\bar{\alpha} w_{f}(z)+\alpha \bar{w}_{f}(\bar{z})+C_{0} \tag{5.65}
\end{equation*}
$$

where $C_{0} \in \mathbb{R}$ is a constant of integration and $w_{f}(z)$ is the conformal transformation (5.60) from the flat Euclidean space back onto the original phase space.

The fact that the symplectic 2 -form here is uniquely determined to be the volume form associated with the phase space geometry is a general feature of any homogeneous symplectic manifold. Indeed, when a Lie group $G$ acts transitively on a symplectic manifold there is a unique $G$-invariant measure [57], i.e. a unique solution for the $d(d-$ 1)/2 functions $\omega_{\mu \nu}$ from the $d(d-1) \cdot d(d+1) / 4$ differential equations (5.64). Thus $\omega^{n} / n$ ! is necessarily the maximally symmetric volume form of $(\mathcal{M}, g)$ and the phase space is naturally a Kähler manifold, as in Section 5.1. We shall soon see the precise connection between maximally symmetric phase spaces and the coadjoint orbit models of Section 5.1. In the present context, this is one of the underlying distinguishing features between the maximally symmetric and inhomogeneous cases. In the latter case $\omega$ is not uniquely determined form the requirement of symplecticity of the isometry group action on $\mathcal{M}$, leading to numerous possibilities for the equivariant Hamiltonian systems. In the case at hand here, the Darboux 2-form on $\mathbb{R}^{2}$ is the unique 2-form which is invariant under the full Euclidean group, i.e. invariant under rotations and translations in the plane, and on $\mathcal{M}$ it is the Kähler form associated with the Kähler metric (5.43) and (5.58).

The form (5.65) for the planar equivariant Hamiltonian systems illustrates how the integrable dynamical systems which obey the localization criteria depend on the phase space geometry which needs to be introduced in this formalism. These systems are all, however, holomorphic copies of the same initial dynamical system on $\mathbb{R}^{2}$ defined by the Darboux Hamiltonian

$$
\begin{equation*}
H_{0}^{D}(z, \bar{z})=\Omega z \bar{z}+\bar{\alpha} z+\alpha \bar{z}+C_{0} \quad ; \quad z \in \mathbb{C} \tag{5.66}
\end{equation*}
$$

or identifying $z, \bar{z}=p \pm i q$ with $(p, q)$ canonical momentum and position variables, these dynamical Hamiltonians are of the form

$$
\begin{equation*}
H_{0}^{D}(p, q)=\Omega\left(p^{2}+q^{2}\right)+\alpha_{1} p+\alpha_{2} q+C_{0} \tag{5.67}
\end{equation*}
$$

Thus the dependence on the phase space Riemannian geometry is trivial in the sense that these systems all lift to families of holomorphic copies of the planar dynamical systems (5.66). This sort of trivial dependence is to be expected since the (classical or quantum) dynamical problem is initially independent of any Riemannian geometry of the phase space. It is also anticipated from the general topological field theory arguments that we presented earlier. Nonetheless, the general functions $H_{0}(z, \bar{z})$ in (5.65) illustrate how the geometry required for equivariant localization is determined by the different dynamical systems, and vice versa, i.e. the geometries that make these dynamical systems integrable. This probes into what one may consider to be the geometry of the classical or quantum dynamical system.

Thus essentially the only equivariant Hamiltonian system on a planar symplectic manifold is the displaced harmonic oscillator Hamiltonian

$$
\begin{equation*}
H_{0}^{D}=\Omega(z+a)(\bar{z}+\bar{a})=\Omega\left\{\left(p+a_{1}\right)^{2}+\left(q+a_{2}\right)^{2}\right\}+C_{0} \tag{5.68}
\end{equation*}
$$

and in this case we can replace the requirement that $H$ generate a circle action with the requirement that it generate a semi-bounded group action. To compare the localization formulas with some known results from elementary quantum mechanics, we note that the Hamiltonian (5.67) can only describe 2 distinct 1-dimensional quantum mechanical models. These are the harmonic oscillator $\frac{1}{2} z \bar{z}=\frac{1}{2}\left(p^{2}+q^{2}\right)$ wherein we take $\Omega=\frac{1}{2}$ and $\alpha=0$ in (5.66) and apply either the WKB or the Niemi-Tirkkonen localization formulas of the last Chapter, and the free particle $\frac{1}{2} p^{2}$ where we take $\Omega=0$ and $\alpha=1 / 2 \sqrt{2}$ in (5.66) and apply the quadratic localization formula (4.112) (or equivalently (4.104)). In fact, these are the original classic examples, which were for a long time the only known examples, where the Feynman path integral can be evaluated exactly because then their
functional integrals are Gaussian. For the same reasons, these are also the basic examples where the WKB approximation is known to be exact [116].

It is straightforward to verify the Niemi-Tirkkonen localization formula (4.91) for the harmonic oscillator. In polar coordinates $z=r \mathrm{e}^{i \theta}$ with $r \in \mathbb{R}^{+}$and $\theta \in[0,2 \pi]$, we have $\omega_{r \theta}=r,\left(\Omega_{V}\right)_{\theta r}=-2 r$ and $R=0$ on flat $\mathbb{R}^{2}$, and so the integral in (4.91) gives

$$
\begin{equation*}
Z_{\mathrm{harm}}(T) \sim \int_{0}^{\infty} d r \frac{T}{2 \sin \frac{T}{2}} \mathrm{e}^{-i T r^{2} / 2}=\frac{1}{2 i \sin \frac{T}{2}} \tag{5.69}
\end{equation*}
$$

That this is the correct result can be seen by noting that the energy spectrum determined by the Schrödinger equation for the harmonic oscillator is $E_{k}=k+\frac{1}{2}, k \in \mathbb{Z}^{+}$[83], so that

$$
\begin{equation*}
\operatorname{tr}\left\|\mathrm{e}^{-i T\left(\hat{p}^{2}+\hat{q}^{2}\right) / 2}\right\|=\sum_{k} \mathrm{e}^{-i T E_{k}}=\sum_{k=0}^{\infty} \mathrm{e}^{-i T\left(k+\frac{1}{2}\right)}=\frac{1}{2 i \sin \frac{T}{2}} \tag{5.70}
\end{equation*}
$$

This result also follows from the WKB formula (4.70) after working out the regularized fluctuation determinant in a manner similar to that described in Section 4.5. Here the classical trajectories determined by the flows of the vector field $V^{z}=i z / 2$ are the circular orbits $z(t)=z(0) \mathrm{e}^{i t / 2}$. Note that the only way these orbits can be defined on the loop space $L \mathbb{C}$ is to regard $z(t)=z(0) \mathrm{e}^{i t / 2}$ and $\bar{z}(t)=\bar{z}(T) \mathrm{e}^{i(T-t) / 2}$ as independent complex variables. This means that the functional integral should be evaluated in a holomorphic polarization. We shall return to this point shortly. Alternatively, we note that for $T \neq 4 \pi n$ the only $T$-periodic critical trajectoies of this dynamical system are the critical points $z, \bar{z}=0$ of the harmonic oscillator Hamiltonian $z \bar{z}$ and one can easily derive (5.70) from (4.97). For the discretized values $T=4 \pi n$ any initial condition $z(0) \in \mathbb{C}$ leads to $T$ periodic orbits, and the moduli space of critical trajectories is non-isolated and coincides with the entire phase space $\mathcal{M}=\mathbb{R}^{2}$. In that case the degenerate path integral formula (4.76) yields the correct result (5.70). These results therefore all agree with the general assertions made at the beginning of Section 4.5 concerning the structure of the moduli space $T$-periodic classical trajectories for a Hamiltonian circle action on the phase space.

For the free particle partition function, we have $R=\Omega_{V}=0$, and so the $\hat{A}$-genus term in the localization formula (4.112) contributes 1 . The $\phi_{0}$-integral in (4.112) is thus
a trivial Gaussian one and we find

$$
\begin{equation*}
Z_{\mathrm{free}}(T) \sim \int_{-\infty}^{\infty} d p d q \int_{-\infty}^{\infty} d \phi_{0} \mathrm{e}^{i T \phi_{0}^{2}-i T \phi_{0} p / 2 \sqrt{2}} \sim \int_{-\infty}^{\infty} d p d q \mathrm{e}^{-i T p^{2} / 2} \tag{5.71}
\end{equation*}
$$

which also coincides with the exact propagator $\operatorname{tr}\left\|\mathrm{e}^{-i T \hat{p}^{2} / 2}\right\|$ in the phase space representation. In this case the Hamiltonian $\frac{1}{2} p^{2}$ is degenerate on $\mathbb{R}^{2}$, so that the WKB localization formula is unsuitable for this dynamical system and the result (5.71) follows from the degenerate formula (4.76) by noting that $L \mathcal{M}_{S}=\mathbb{R}^{2}$ in this case.

There is another way to look at the path integral quantization of the Darboux Hamiltonian system (5.66) which ties in with some of the general ideas of Section 5.1 above. The Heisenberg-Weyl algebra $\mathrm{g}_{H W}[83]$ is the algebra generated by the usual harmonic oscillator raising and lowering operators

$$
\begin{equation*}
\hat{a}, \hat{a}^{\dagger}=\frac{1}{\sqrt{2}}(\hat{p} \pm i \hat{q}) \tag{5.72}
\end{equation*}
$$

in the canonical quantum theory associated with the phase space $\mathbb{R}^{2}$ and the operator algebra (4.1). The Lie algebra $\mathbf{g}_{H W}$ is generated by the operators $\hat{a}^{\dagger}, \hat{a}$ and $\hat{N} \equiv \hat{a}^{\dagger} \hat{a}=$ $\frac{1}{2}\left(\hat{p}^{2}+\hat{q}^{2}-1\right)$ with the commutation relations

$$
\begin{equation*}
\left[\hat{a}^{\dagger}, \hat{a}\right]=1 \tag{5.73}
\end{equation*}
$$

The (infinite-dimensional) Hilbert space which defines an representation of these operators is spanned by the bosonic number basis $|n\rangle, n \in \mathbb{Z}^{+}$, which form the complete orthonormal system of eigenstates of the number operator $\hat{N}$ with eigenvalue $n$,

$$
\begin{equation*}
\hat{N}|n\rangle=\hat{a}^{\dagger} \hat{a}|n\rangle=n|n\rangle \tag{5.74}
\end{equation*}
$$

and on which $\hat{a}^{\dagger}$ and $\hat{a}$ act as raising and lowering operators, respectively,

$$
\begin{equation*}
\hat{a}^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle \quad, \quad \hat{a}|n\rangle=\sqrt{n}|n-1\rangle \tag{5.75}
\end{equation*}
$$

We now define the canonical coherent states [39, 83, 108] associated with this representation of the Heisenberg-Weyl group $G_{H W}$ as

$$
\begin{equation*}
\mid z) \equiv \mathrm{e}^{z^{\hat{a}}}|0\rangle=\sum_{n=0}^{\infty} \frac{z^{n}}{\sqrt{n!}}|n\rangle \quad ; \quad z \in \mathbb{C} \tag{5.76}
\end{equation*}
$$

These states are normalized as

$$
\begin{equation*}
(z \mid z)=\mathrm{e}^{z \bar{z}} \tag{5.77}
\end{equation*}
$$

with $(z|\equiv| z)^{\dagger}$, and they obey the completeness relation

$$
\begin{align*}
\int \frac{d^{2} z}{2 \pi} \frac{\mid z)(z \mid}{(z \mid z)} & =\int \frac{d^{2} z}{2 \pi} \mathrm{e}^{-z \bar{z}} \sum_{n, m} \frac{z^{n} \bar{z}^{m}}{\sqrt{n!m!}}|n\rangle\langle m| \\
& =\frac{1}{2 \pi} \int_{0}^{\infty} d r r \mathrm{e}^{-r^{2}} \sum_{n, m} \frac{r^{n+m}}{\sqrt{n!m!}} \int_{0}^{2 \pi} d \theta \mathrm{e}^{i(n-m) \theta}|n\rangle\langle m|  \tag{5.78}\\
& =\frac{1}{2 \pi} \int_{0}^{\infty} d r r \mathrm{e}^{-r^{2}} \sum_{n, m} \frac{r^{n+m}}{\sqrt{n!m!}} \cdot 2 \pi \delta_{n m}|n\rangle\langle m|=\sum_{n=0}^{\infty}|n\rangle\langle n|=1
\end{align*}
$$

where we have as usual written $z=r \mathrm{e}^{i \theta}$. The normalized matrix elements of the algebra generators in these states are

$$
\begin{equation*}
\frac{\left(z\left|\hat{a}^{\dagger} \hat{a}\right| z\right)}{(z \mid z)}=z \bar{z} \quad, \quad \frac{(z|\hat{a}| z)}{(z \mid z)}=\bar{z} \quad, \quad \frac{\left(z\left|\hat{a}^{\dagger}\right| z\right)}{(z \mid z)}=z \tag{5.79}
\end{equation*}
$$

Thus the 3 independent terms in the Darboux Hamiltonian (5.66) are none other than the normalized canonical coherent state matrix elements of the Heisenberg-Weyl group generators. These 3 observables represent the Poisson Lie group action of the Euclidean group $E^{2}$ on the coadjoint orbit $G_{H W} / H_{C}=G_{H W} / U(1)=\mathbb{C}^{1}$ with the Darboux Poisson bracket

$$
\begin{equation*}
\{z, \bar{z}\}_{\omega_{D}}=1 \tag{5.80}
\end{equation*}
$$

which is the Poisson algebra representation of the Heisenberg-Weyl algebra (5.73). This correspondence with the coset space $G / H_{C}$ and the general framework of Section 5.1 is not entirely surprising, since it is well-known that homogeneous symplectic manifolds are in general essentially coadjoint orbits of Lie groups [57], as exemplified by this system and the others that we shall encounter in this Chapter. The integrable Hamiltonian systems in this case are functionals of Cartan elements of $\mathbf{g}_{H W}$ (e.g. the harmonic oscillator $\hat{a}^{\dagger} \hat{a}$ or the free particle $\left.\left(\hat{a}+\hat{a}^{\dagger}\right)^{2}\right)$.

The canonical coherent states (5.76) are those quantum states which minimize the

Heisenberg uncertainty principle $\Delta q \cdot \Delta p \geq \frac{1}{2}$ [83], because they diagonalize the annihilation operator $\hat{a}, \hat{a} \mid z)=z \mid z$, and they can be generalized to arbitrary Lie groups [108], as we shall soon see. The Darboux 2-form

$$
\begin{equation*}
\omega_{D}=\frac{i}{2} d z \wedge d \bar{z} \tag{5.81}
\end{equation*}
$$

is defined globally on $\mathbb{C}$ and, since $\mathbb{R}^{2}$ is contractable and hence $H^{2}\left(\mathbb{R}^{2} ; \mathbb{Z}\right)=0$, it can be be generated globally by the symplectic potential

$$
\begin{equation*}
\theta_{D}=-\frac{i}{2}(\bar{z} d z-z d \bar{z}) \tag{5.82}
\end{equation*}
$$

The canonical 1-form (5.82) and the flat Kähler metric associated with (5.81) on $\mathbb{R}^{2}$ can be written in terms of the coherent states (5.76) as

$$
\begin{gather*}
\theta_{D}=\frac{i}{2} \frac{(z|d| z)}{(z \mid z)}  \tag{5.83}\\
g_{D}=d z \otimes d \bar{z}=\frac{\| d \mid z) \|}{\sqrt{(z \mid z)}} \otimes \frac{\| d \mid z) \|}{\sqrt{(z \mid z)}}-\frac{(z|d| z)}{(z \mid z)} \otimes \frac{(z|d| z)^{*}}{(z \mid z)} \tag{5.84}
\end{gather*}
$$

and the Kähler potential associated with (5.82) is

$$
\begin{equation*}
F_{\mathbf{R}^{2}}(z, \bar{z})=z \bar{z} \tag{5.85}
\end{equation*}
$$

The path integral here then coincides with the standard coherent state path integral $\operatorname{tr} \mathrm{e}^{-i T \hat{\mathcal{H}}}=\int \frac{d^{2} z}{2 \pi} \frac{\left(z\left|\mathrm{e}^{-i T \hat{\mathcal{H}}}\right| z\right)}{(z \mid z)}=\int_{L \mathbf{R}^{2}} \prod_{t \in[0, T]} \frac{d z(t) d \bar{z}(t)}{2 \pi} \exp \left\{i \int_{0}^{T} d t(z \dot{\bar{z}}-\bar{z} \dot{z}-H(z, \bar{z}))\right\}$
where

$$
\begin{equation*}
H(z, \bar{z})=(z|\hat{\mathcal{H}}| z) /(z \mid z) \tag{5.87}
\end{equation*}
$$

is the coherent state matrix element of some operator $\hat{\mathcal{H}}=\hat{\mathcal{H}}\left(\hat{a}, \hat{a}^{\dagger}\right)$ on the underlying representation space of the Heisenberg-Weyl algebra. The derivation of (5.86) is identical to that in Section 4.1 except that now we use the modified completeness relation (5.78)
for the coherent state representation. This manner of describing the quantum dynamics goes under the equivalent names of holomorphic, coherent state or Kähler polarization. One of its nice features in general is that it provides a natural identification of the path integral and loop space Liouville measures. We recall from (4.20) that in the former measure there is one unpaired momentum in general and, besides the periodic boundary conditions, there is a formal analog between the measures in (4.20) and (4.24) only if the initial configuration of the propagator depends on the position variables $q$ and the final configuration on the momentum variables $p$, or vice versa. In the holomorphic polarization above, however, the initial configuration depends on the $z$ variables, the final one on the $\bar{z}$ variables, and the path integral measure is the formal $N \rightarrow \infty$ limit of $\prod_{i=1}^{N} d z_{i} d \bar{z}_{i} / 2 \pi$. Since the number of $z$ and $\bar{z}$ integrations are the same, we obtain the desired formal identifications. Besides providing one with a formal analog between the path integral localization formulas and the Duistermaat-Heckman theorem and its generalizations, this enables one to also ensure that the loop space supersymmetry encountered in Section 4.3, which is intimately connected with the definition of the path integral measure (as are the boundary conditions for the propagator), is consistent with the imposed boundary conditions.

Thus on a planar phase space essentially the only equivariant Hamiltonian systems are harmonic oscillators ${ }^{6}$, generalized as in (5.65) to the inclusion of a generic flat geometry so that the remaining Hamiltonian systems are merely holomorphic copies of these displaced oscillators defined by the analytic coordinate transformation (5.60). These systems generate a topological quantum theory of the sort discussed in Sections 4.8 and 5.1, with the Darboux Hamiltonian (5.66) related to the symplectic potential (5.82) by the usual topological condition $H_{0}^{D}=i_{V_{\mathbf{R}^{2}}} \theta_{D}$ reflecting the fact that (5.82) is invariant under the action of the rotation group of the plane. It is not, however, invariant under the

[^27]translation group action, so that the translation generators do not determine a Wittentype topological field theory like the harmonic oscillator Hamiltonians do. This means that there are no $E^{2}$-invariant symplectic potentials on the plane, i.e. it is impossible to find a function $F$ in (5.15) that gives an invariant potential simultaneously for all 3 of the independent generators in (5.66). In any case, the harmonic oscillator nature of these systems is consistent with their global integrability properties. The holomorphic polarization of the quantum theory associates the canonical quantum theory above with the topological coadjoint orbit quantum theory of Section 5.1 and the coherent state path integral (5.86) yields character formulas for the isometry group of the phase space. This will be the general characteristic feature of the localizable systems we shall find. In the case at hand, the character formulas are associated with the Cartan elements of the Heisenberg-Weyl group.

### 5.4 Coherent States on Homogeneous Kähler Manifolds

Before carrying on with our geometric determination of the localizable dynamical systems and their path integral representations, we pause to briefly discuss how the holomorphic quantization introduced above on the coadjoint orbit $\mathbb{R}^{2}$ can be generalized to the action of an arbitrary semi-simple Lie group $G[12,74,108]$. This representation of the quantum dynamics proves to be the most fruitful on homogeneous spaces $G / H_{C}$, and later on we shall even generalize this construction to apply to non-homogeneous phase spaces and even non-symmetric multiply connected phase spaces. As the coherent states are those which are closest to "classical" states, in that they are the most tightly peaked ones about their locations, they are the best quantum states in which to study the semiclassical localizations for quantum systems. In the next Chapter we shall see that they are also related to the geometric quantization of dynamical systems [136].

Given any irreducible unitary representation $D(G)$ of the group $G$ and some normalized state $|0\rangle$ in the representation space, we define the (normalized) state $|g\rangle$ by

$$
\begin{equation*}
|g\rangle=D(g)|0\rangle \tag{5.88}
\end{equation*}
$$

If $d g$ denotes Haar measure of $G$, then Schur's lemma [128] and the completeness of the representation $D(G)$ implies the completeness relation

$$
\begin{equation*}
\frac{\operatorname{dim} D(G)}{\operatorname{vol}(G)} \int_{G} d g|g\rangle\langle g|=\mathbf{1} \tag{5.89}
\end{equation*}
$$

Following the derivation of Section 4.1, it follows that the partition function associated with an operator $\hat{\mathcal{H}}$ acting on the representation space of $D(G)$ can be represented by the path integral

$$
\begin{align*}
\operatorname{tr}_{D(G)} \mathrm{e}^{-i T \hat{\mathcal{H}}} & =\int_{G} d g\langle g| \mathrm{e}^{-i T \hat{\mathcal{H}}}|g\rangle \\
& =\int_{L G} \prod_{t \in[0, T]} \frac{\operatorname{dim} D(G)}{\operatorname{vol}(G)} d g(t) \exp \left\{i \oint_{\gamma(g)}\langle g| d|g\rangle-i \int_{0}^{T} d t\langle g| \hat{\mathcal{H}}|g\rangle\right\} \tag{5.90}
\end{align*}
$$

However, if we take $|0\rangle$ to be a simultaneous eigenstate of the generators of $H_{C} \subset G$ (i.e. a weight state), then the 'coherent' states $|g\rangle$ associated with any one coset of $G / H_{C}$ are all phase multiples of one another. Thus the set of coherent states form a principal $H_{C}$-bundle over $G / H_{C}$ and the coherent state path integral (5.90) is in fact taken over paths in the homogeneous space $G / H_{C}$. This geometrical method for constructing irreducible representations of semi-simple Lie groups as sections of the principal fiber bundle $G \rightarrow G / H_{C}$ is known as the Borel-Weil-Bott method [112].

What is most interesting about the character representation (5.90) is that it is closely related to the Kähler geometry of the homogeneous space $G / H_{C}$. To see this, we first define the Borel subgroups $B_{ \pm}$of $G$ which are the exponentiations of the subalgebras $\mathcal{B}_{ \pm}$ spanned by $H_{i} \in \mathbf{h} \otimes \mathbb{C}$ and $E_{\alpha}$ for $\alpha>0$ and $\alpha<0$, respectively. The complexification of the coadjoint orbit $\mathcal{M}_{G}$ is then provided by the isomorphism $G / H_{C} \simeq G^{c} / B_{ \pm}$, where
$G^{c}$ is the complexification of $G$ [128]. Almost any $g \in G$ can be factored as a Gaussian decomposition

$$
\begin{equation*}
g=\zeta_{+} h \zeta_{-} \tag{5.91}
\end{equation*}
$$

where $h \in H_{C}^{c}$ and

$$
\begin{equation*}
\zeta_{+}=\mathrm{e}^{\sum_{\alpha>0} z^{\alpha} E_{\alpha}} \quad, \quad \zeta_{-}=\mathrm{e}^{\sum_{\alpha<0} z^{\alpha} E_{\alpha}} \tag{5.92}
\end{equation*}
$$

Here $z^{\alpha} \in \mathbb{C}$, and if we now apply the representation operator $D(g)$ to a lowest weight state, then $\zeta_{-}$acts as the identity and the set of physically distinct states are in a one-to-one correspondence with the coset space $G^{c} / B_{+}$. Since $\mathcal{B}_{+}$is a closed subalgebra of $\mathbf{g} \otimes \mathbb{C}$, the parameters $z^{\alpha} \in \mathbb{C}$ define a complex structure on $\mathcal{M}_{G}$. In this way, we can now write down coherent state path integral representations of the character formulas of Section 5.1 above. The choice of $|0\rangle$ above as a lowest weight state ensures that the coherent states $\mid z) \equiv \zeta_{+}|0\rangle$ are holomorphic. Note that their coherency follows from the fact that $\left.\left.E_{-\alpha} \mid z\right)=z^{\alpha} \mid z\right)$ for $\alpha>0$.

The Kähler potentials are now given by the normalizations

$$
\begin{equation*}
\mathrm{e}^{F^{(h)}(z, \bar{z})}=(z \mid z) \equiv\langle 0| g|0\rangle=\langle 0| h|0\rangle \tag{5.93}
\end{equation*}
$$

with the potentials $F^{(i)}(z, \bar{z}), i=1, \ldots, r$, each associated with the Cartan generator $H_{i}$ in (5.93) (compare eqs. (5.77) and (5.85)). From this it follows that the associated Kähler metrics $g^{(i)}$ and symplectic potentials $\theta^{(i)}$ can be represented as coherent state matrix elements as in (5.83) and (5.84). In this way the kinetic term in the coherent state path integral (5.90) coincides with the usual one induced by the symplectic Kähler structures of the homogeneous space $\mathcal{M}_{G}$ and the path integral measure becomes the loop space Liouville measure. Moreover, the symplectic line bundle $L^{(\lambda)} \rightarrow \mathcal{M}_{G}$ associated with the principal $G$-bundle $G \rightarrow \mathcal{M}_{G}$ with connection 1-form $\theta^{(\lambda)}$ defines a twisted (covariant) Dolbeault derivative $\bar{\partial}_{\lambda}=\bar{\partial}+\theta_{\bar{z}}^{(\lambda)}$ which annihilates the normalized coherent states $\mid z) /(z \mid z)$. Then the equivariant index of $\bar{\partial}_{\lambda}$ is the Lefschetz number

$$
\begin{equation*}
\operatorname{index}_{H^{(\lambda)}}\left(\bar{\partial}_{\lambda} ; T\right)=\lim _{\beta \rightarrow \infty} \operatorname{tr}\left\|\mathrm{e}^{-i T H^{(\lambda)}}\left(\mathrm{e}^{-\beta \bar{\partial}_{\lambda}^{\dagger} \bar{\partial}_{\lambda}}-\mathrm{e}^{-\beta \bar{\partial}_{\lambda} \bar{\partial}_{\lambda}^{\dagger}}\right)\right\| \tag{5.94}
\end{equation*}
$$

which is equal to the character (5.23) represented by the equivariant Riemann-RochHirzebruch index ${ }^{7}$. We recall from section 4.9 that the Atiyah-Singer index contribution to (5.21) evaluates the spectral asymmetry of the zero mode representation of $\mathbf{g}$ determined by the pertinent Dirac operator, while the Lefschetz number coincides with the character of that representation of the spin complex. From the formulas (5.23),(5.94), however, we see that the character of a Lie group $G$ is a Lefschetz number related to the $G$-index theorem of the holomorphic Dolbeault complex [16, 84, 123], rather than to the Atiyah-Singer index theorem of the spin complex.

### 5.5 Spherical Phase Spaces and Quantization of Spin Systems

We are now ready to continue with our general isometric classification. The next case we consider is when the phase space $\mathcal{M}$ has a positive constant Gaussian curvature $K>0$. In this case the conformal factor solves the Liouville field equation

$$
\begin{equation*}
\nabla^{2} \varphi(z, \bar{z})=-2 K \mathrm{e}^{\varphi(z, \bar{z})} \tag{5.95}
\end{equation*}
$$

which is a completely integrable system [30] whose general solutions are

$$
\begin{equation*}
\varphi(z, \bar{z})=\log \left[\frac{\partial f(z) \bar{\partial} \bar{f}(\bar{z})}{\left(\frac{K}{4}+f(z) \bar{f}(\bar{z})\right)^{2}}\right] \tag{5.96}
\end{equation*}
$$

By the essential uniqueness of maximally symmetric spaces, we know that in this case $(\mathcal{M}, g)$ is isometric to the sphere $S^{2}$ of radius $K^{-1 / 2}$ with its standard round metric given in (5.29). From the transformation law of the metric tensor $g$ and (5.96) it is straightforward to work out the explicit diffeomorphism $(z, \bar{z}) \rightarrow(w(z, \bar{z}), \bar{w}(z, \bar{z}))$ which accomplishes this isometric correspondence.

[^28]First of all, we rewrite the spherical metric in (5.29) in complex coordinates $w, \bar{w}=$ $x^{1} \pm i x^{2}$, with $x^{\mu}$ the spherical coordinates defined in (5.29), to get

$$
\begin{equation*}
g_{S^{2}}=\frac{1}{4 K}\left[\frac{\bar{w}^{2}}{1-w \bar{w}} d w \otimes d w+\frac{w^{2}}{1-w \bar{w}} d \bar{w} \otimes d \bar{w}+2\left(2+\frac{w \bar{w}}{1-w \bar{w}}\right) d w \otimes d \bar{w}\right] \tag{5.97}
\end{equation*}
$$

where $w \bar{w} \leq 1$. If we view the unit sphere as centered in the $x^{\prime} y^{\prime}$-plane in $\mathbb{R}^{2}$ and symmetrically about the $z^{\prime}$-axis, then we can map $S^{2}$ onto the complex plane via the standard stereographic projection from the south pole $z^{\prime}=-1$,

$$
\begin{equation*}
w=\frac{2 w^{\prime}}{1+w^{\prime} \bar{w}^{\prime}} \quad, \quad z^{\prime}=\sqrt{1-w \bar{w}}=\frac{1-w^{\prime} \bar{w}^{\prime}}{1+w^{\prime} \bar{w}^{\prime}} \tag{5.98}
\end{equation*}
$$

This gives a diffeomorphism of $S^{2}$ with the compactified plane $\mathbb{C} \cup\{\infty\}$. From (5.97), the metric tensor transformation law and (5.96) we find after some algebra that the coordinate transformation above must satisfy

$$
\begin{equation*}
\frac{1}{\left(1+w^{\prime} \bar{w}^{\prime}\right)^{2}}\left(\frac{\partial w^{\prime}}{\partial z} \frac{\partial \bar{w}^{\prime}}{\partial \bar{z}}+\frac{\partial w^{\prime}}{\partial \bar{z}} \frac{\partial \bar{w}^{\prime}}{\partial z}\right)=K \mathrm{e}^{\varphi(z, \bar{z})}=K \frac{\partial f(z) \bar{\partial} \bar{f}(\bar{z})}{\left(\frac{K}{4}+f(z) \bar{f}(\bar{z})\right)^{2}} \tag{5.99}
\end{equation*}
$$

From (5.99) and (5.98) it then follows that the desired coordinate transformation from $(\mathcal{M}, g)$ to $S^{2}$ with the standard round metric (5.97) is given by

$$
\begin{equation*}
w(z, \bar{z})=\frac{4 K^{-1 / 2} f(z)}{1+4 K^{-1} f(z) \bar{f}(\bar{z})} \tag{5.100}
\end{equation*}
$$

The mapping (5.100) is just a generalized stereographic projection from the south pole of $S^{2}$ where $f(z)$ maps $(\mathcal{M}, g)$ onto the entire complex plane with the usual Kähler geometry of $S^{2}$ defined by the coordinates in (5.98),

$$
\begin{align*}
& g_{S^{2}}=4 \partial \bar{\partial} F_{S^{2}}(z, \bar{z}) d z \otimes d \bar{z}=\frac{4}{(1+z \bar{z})^{2}} d z \otimes d \bar{z} \\
& \omega_{S^{2}}=2 i \partial \bar{\partial} F_{S^{2}}(z, \bar{z}) d z \wedge d \bar{z}=\frac{2 i}{(1+z \bar{z})^{2}} d z \wedge d \bar{z} \tag{5.101}
\end{align*}
$$

where the associated Kähler potential is

$$
\begin{equation*}
F_{S^{2}}(z, \bar{z})=\log (1+z \bar{z}) \tag{5.102}
\end{equation*}
$$

Notice that the diffeomorphism (5.100) obeys $w \bar{w} \leq 1$, as required for $(w, \bar{w}) \in S^{2}$, and that the Kähler metric $g_{S^{2}}$ in (5.101) coincides with the original phase space geometry (5.43) when $f(z)=\frac{1}{2} K^{1 / 2} z$ in (5.96) above.

From our general considerations of Section 5.2 above we know that the Killing vectors of the metric (5.97) are

$$
\begin{equation*}
V_{S^{2}}^{w}=-i \Omega w+\alpha(1-w \bar{w})^{1 / 2} \quad, \quad V_{S^{2}}^{\bar{w}}=i \Omega \bar{w}+\bar{\alpha}(1-w \bar{w})^{1 / 2} \tag{5.103}
\end{equation*}
$$

The Killing vectors (5.103) generate the rigid rotations $w \rightarrow \mathrm{e}^{i \Omega} w$ of the sphere and the quasi-translations $w \rightarrow w+\alpha(1-w \bar{w})^{1 / 2}$ (i.e. translations along the geodesical great circles of $S^{2}$ ), and they together generate the Lie group $S O(3)$. Requiring the symplecticity condition (5.64) again under the full $S O(3)$ group action generated by (5.103) on the symplectic 2-form (5.63), we find after some algebra that the equations (5.64) are uniquely solved by

$$
\begin{equation*}
\omega_{S^{2}}(w, \bar{w})=1 / K(1-w \bar{w})^{1 / 2} \tag{5.104}
\end{equation*}
$$

This symplectic 2-form is again the volume form associated with (5.97). It is a nontrivial element of $H^{2}\left(S^{2} ; \mathbb{Z}\right)=\mathbb{Z}$ and it coincides with the Kähler classes in (5.101) in the stereographic coordinates (5.98). We now substitute (5.103) and (5.104) into the Hamiltonian equations (5.62), which are easily solved on $S^{2}$ in the $w$-coordinates above, and then apply the generalized stereographic projection (5.100) to get the most general equivariant Hamiltonian on a spherical phase space as

$$
\begin{equation*}
H_{+}(z, \bar{z})=\frac{\Omega\left(\frac{K}{4}-f(z) \bar{f}(\bar{z})\right)}{\frac{K}{4}+f(z) \bar{f}(\bar{z})}+\frac{\alpha \bar{f}(\bar{z})+\bar{\alpha} f(z)}{\frac{K}{4}+f(z) \bar{f}(\bar{z})}+C_{0} \tag{5.105}
\end{equation*}
$$

Thus, again the Riemannian geometry of the phase space $\mathcal{M}$ is realized (or even determined) by the equivariant Hamiltonian systems which can be defined on $\mathcal{M}$. The transformation to Darboux coordinates on $\mathcal{M}$, defined as usual as those coordinates $(v, \bar{v})$ in which the symplectic 2-form is locally $\omega_{S^{2}}=\frac{i}{2} d v \wedge d \bar{v}$, can be found from
the fact that $\omega_{S^{2}}$ is the (Kähler) volume form associated with (5.96) and applying the tensor transformation law (2.8) for $\omega$. After some algebra we find that the local Darboux coordinates on $\mathcal{M}$ are defined by the diffeomorphism $(z, \bar{z}) \rightarrow(v(z, \bar{z}), \bar{v}(z, \bar{z}))$, where the function

$$
\begin{equation*}
v(z, \bar{z})=\frac{f(z)}{\left(\frac{K}{4}+f(z) \bar{f}(\bar{z})\right)^{1 / 2}} \tag{5.106}
\end{equation*}
$$

maps $\mathcal{M}$ onto the unit disc

$$
\begin{equation*}
D^{2}=\{z \in \mathbb{C}: z \bar{z} \leq 1\} \tag{5.107}
\end{equation*}
$$

which is the Darboux phase space associated with a general spherical phase space geometry. Thus, applying the transformation (5.106) to (5.105), we see that the general Darboux Hamiltonians in the present case are

$$
\begin{equation*}
H_{+}^{D}(z, \bar{z})=\Omega z \bar{z}+(\bar{\alpha} z+\alpha \bar{z})(1-z \bar{z})^{1 / 2}+C_{0} \quad ; \quad z \in D^{2} \tag{5.108}
\end{equation*}
$$

which correspond to the quasi-displaced harmonic oscillators

$$
\begin{equation*}
H_{+}^{D}(z, \bar{z})=\Omega\left[z+a(1-z \bar{z})^{1 / 2}\right]\left[\bar{z}+\bar{a}(1-z \bar{z})^{1 / 2}\right] \tag{5.109}
\end{equation*}
$$

with compactified position and momentum ranges. Thus here the criterion of a (compact) circle action cannot be removed, in contrast to the case of the planar geometries of Section 5.3 where the Darboux phase space was the entire complex plane $\mathbb{C}$. Notice that all translations in the planar case become quasi-translations in the spherical case, which is a measure of the presence of a curved Riemannian geometry on $\mathcal{M}$.

The mapping onto Darboux coordinates above shows that once again all the general spherical Hamiltonians are holomorphic copies of each other, as they all define the same Darboux dynamics. We shall therefore focus our attention to the quantum dynamics defined on the phase space $S^{2}$ (i.e. $f(z)=K^{1 / 2} z / 2$ above), and for simplicity we normalize the coordinates so that now $K=1$, i.e. $S^{2}$ has unit radius. First of all, we write the 3 independent observables appearing in (5.105) above as

$$
\begin{equation*}
J_{3}^{(j)}(z, \bar{z})=-j \frac{1-z \bar{z}}{1+z \bar{z}} \quad, \quad J_{+}^{(j)}(z, \bar{z})=2 j \frac{\bar{z}}{1+z \bar{z}} \quad, \quad J_{-}^{(j)}(z, \bar{z})=2 j \frac{z}{1+z \bar{z}} \tag{5.110}
\end{equation*}
$$

where the parameters $j$ will be specified below. Using (5:101) we define the Kähler 2-form

$$
\begin{equation*}
\omega^{(j)}=j \omega_{S^{2}} \tag{5.111}
\end{equation*}
$$

and working out the associated Poisson algebra of the functions (5.110)

$$
\begin{equation*}
\left\{J_{3}^{(j)}, J_{ \pm}^{(j)}\right\}_{\omega^{(j)}}= \pm J_{ \pm}^{(j)} \quad, \quad\left\{J_{+}^{(j)}, J_{-}^{(j)}\right\}_{\omega^{(j)}}=2 J_{3}^{(j)} \tag{5.112}
\end{equation*}
$$

shows that they realize the $S U(2)$ (angular momentum) Lie algebra [128]. The functions (5.110) therefore generate the Poisson-Lie group action of the $S^{2}$ isometry group $S O(3)$ on the coadjoint orbit

$$
\begin{equation*}
G / H_{C}=S U(2) / U(1) \simeq S^{3} / S^{1}=S^{2} \tag{5.113}
\end{equation*}
$$

and we obtain the usual coadjoint orbit topological quantum theory by choosing the Hamiltonian to be an element of the Cartan subalgebra $\mathbf{u}(\mathbf{1})$ of $\mathbf{s u}(2)$. Notice that, comparing (5.110) with the stereographic coordinates (5.98), we see that these observables just describe the precession of a classical spin vector of unit length $J= \pm 1$. The coadjoint orbit path integral associated with the observables (5.110) will therefore describe the quantum dynamics of a classical spin system, e.g. the system with Hamiltonian $H=J_{3}$ describes the Pauli interaction between a spin $\vec{J}$ and a uniform magnetic field directed along the $z$-axis. Thus in this case $S^{2}$ is actually naturally the configuration space for a spin system, which has on it a natural symplectic structure and so the corresponding path integral can be regarded as one for the Lagrangian formulation of the theory, rather than the Hamiltonian one [110]. This is also immediate from noting that the stereographic complex coordinates above can be written as

$$
\begin{equation*}
z=\mathrm{e}^{-i \phi} \tan (\theta / 2) \tag{5.114}
\end{equation*}
$$

in terms of the usual spherical polar coordinates $(\theta, \phi)$, so that the observable $J_{3}$ in (5.110) coincides with the height function (3.70) of $S^{2}$ with $a=1$ (up to an additive constant), and the Kähler geometry above becomes the standard round geometry of $S^{2}$.

To construct a topological Hamiltonian along the lines of the theory of Section 5.1, we consider an irreducible spin- $j$ representation of $S U(2)$, where $j=\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots[128]$. The state space for this representation with heighest weight $j$ is spanned by the complete set of orthonormal basis states $|j, m\rangle$, where $m$ are the magnetic quantum numbers with the range $m=-j,-j+1, \ldots, j-1, j$. The $S U(2)$ generators act on these states as

$$
\begin{equation*}
\hat{J}_{3}|j, m\rangle=m|j, m\rangle \quad, \quad \hat{J}_{ \pm}|j, m\rangle=\sqrt{(j \mp m)(j \pm m+1)}|j, m \pm 1\rangle \tag{5.115}
\end{equation*}
$$

Following the last Section, we define the $S U(2)$ coherent states by successive applications of the raising operator $\hat{J}_{+}$to the lowest weight (vacuum) state $|j,-j\rangle[39,108]$,

$$
\begin{equation*}
\mid z)=\mathrm{e}^{-i j \rho} \mathrm{e}^{z \hat{J}_{+}}|j,-j\rangle=\mathrm{e}^{-i j \rho} \sum_{m=-j}^{j}\binom{2 j}{j+m}^{1 / 2} z^{j+m}|j, m\rangle \quad ; \quad z \in \mathbb{C} \tag{5.116}
\end{equation*}
$$

where for $n, m \in \mathbb{Z}^{+}$with $n \geq m$ the binomial coefficient is defined by

$$
\begin{equation*}
\binom{n}{m}=\frac{n!}{m!(n-m)!} \tag{5.117}
\end{equation*}
$$

and where the function $\rho(z, \bar{z})$ is an arbitrary phase which as we shall see is related to the function $F(z, \bar{z})$ in (5.15). It is easily verified that then the $S U(2)$ generators (5.110) are the normalized matrix elements of the operators $\hat{J}_{3}, \hat{J}_{ \pm}$in the coherent states $(5.116)$, respectively.

The coherent states (5.116) are normalized as

$$
\begin{equation*}
\left(z_{2} \mid z_{1}\right)=\left(1+z_{1} \bar{z}_{2}\right)^{2 j} \mathrm{e}^{i j\left[\rho\left(z_{2}, \bar{z}_{2}\right)-\rho\left(z_{1}, \bar{z}_{1}\right)\right]} \tag{5.118}
\end{equation*}
$$

where we have used the binomial theorem

$$
\begin{equation*}
(x+y)^{n}=\sum_{k=0}^{n}\binom{n}{k} x^{k} y^{n-k} \tag{5.119}
\end{equation*}
$$

and they obey the completeness relation

$$
\begin{equation*}
\int d \mu^{(j)}(z, \bar{z}) \frac{\mid z)(z \mid}{(z \mid z)}=1^{(j)} \tag{5.120}
\end{equation*}
$$

where $\mathbf{1}^{(j)}$ is the identity operator in the spin- $j$ representation of $S U(2)$ and the coherent state measure is

$$
\begin{equation*}
d \mu^{(j)}(z, \bar{z})=\frac{i}{2 \pi} \frac{2 j+1}{(1+z \bar{z})^{2}} d z \wedge d \bar{z} \tag{5.121}
\end{equation*}
$$

which coincides with the symplectic 2-form of the spin system above. The identity (5.120) follows from a calculation analogous to that in (5.78). Note that, as explained in the last Section, the Kähler structure is generated through the identity $(z \mid z)=\mathrm{e}^{2 j F_{S^{2}}(z, \bar{z})}$.

We want to evaluate the propagator

$$
\begin{equation*}
\mathcal{K}\left(z_{2}, z_{1} ; T\right)=\left(z_{2}\left|\mathrm{e}^{-i T \hat{\mathcal{H}}}\right| z_{1}\right) / \sqrt{\left(z_{2} \mid z_{2}\right)\left(z_{1} \mid z_{1}\right)} \tag{5.122}
\end{equation*}
$$

for some $S U(2)$ operator $\hat{\mathcal{H}}$ given the one-to-one correspondence between the points on the coadjoint orbit $S U(2) / U(1)=S^{2} \simeq \mathbb{C} \cup\{\infty\}$ and the $S U(2)$ coherent states (5.116). Dividing the time interval in (5.122) up into $N$ segments and letting $N \rightarrow \infty$, following the analogous steps as in Section 4.1 using the completeness relation (5.120) we arrive at the coherent state path integral

$$
\begin{align*}
\mathcal{K}\left(z_{2}, z_{1} ; T\right)= & \mathcal{N} \int_{L \mathbf{R}^{2}} \prod_{t \in[0, T]} d z(t) d \bar{z}(t) \sqrt{\operatorname{det}\left\|\Omega^{(j)}\right\|} \exp \left\{j \log \left(1+z_{2} \bar{z}_{2}\right)+j \log \left(1+z_{1} \bar{z}_{1}\right)\right. \\
& \left.+i \int_{0}^{T} d t\left[\frac{i j}{1+z \bar{z}}(\bar{z} \dot{z}-z \dot{\bar{z}})-i j\left(\frac{\partial \rho}{\partial z} \dot{z}+\frac{\partial \rho}{\partial \bar{z}} \dot{\bar{z}}\right)-H(z, \bar{z})\right]\right\} \tag{5.123}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{N}=\lim _{N \rightarrow \infty} \prod_{k=1}^{N-1} \frac{2 j+1}{2 j \pi} \tag{5.124}
\end{equation*}
$$

is a normalization constant and $H(z, \bar{z})$ denotes the matrix elements (5.87) in the coherent states (5.116). Here we see once again the formal equivalence of the path integral and Liouville measures defined by the Kähler polarization above. In particular, the local symplectic potential generating the Kähler structures (5.111) are

$$
\begin{equation*}
\theta^{(j)}=\frac{i j}{1+z \bar{z}}(\bar{z} d z-z d \bar{z})-i j d \rho \tag{5.125}
\end{equation*}
$$

and they coincide with the standard coherent state canonical 1-forms (5.83). Similarly, the Kähler structure (5.101) can be represented in the standard coherent state form (5.84).

The Wess-Zumino-Witten quantization condition (4.123) applied to $\omega^{(j)}$ implies that $j$ must be a half-integer, since $\int_{S^{2}} \omega_{S^{2}}=4 \pi$, corresponding to the unitary irreducible representations of $G=S U(2)$ [128]. To construct a topological quantum theory (or equivalently an integrable quantum system) as described in Sections 4.8 and 5.1, we need to choose the phase function $\rho(z, \bar{z})$ in the above so that $i_{V} \theta^{(j)}=H$. This problem was analysed in detail by Niemi and Pasanen [100] who showed that it is impossible to satisfy this integrability requirement simultaneously for all 3 of the generators in (5.110). Again, this means that there are no $S U(2)$-invariant symplectic potentials on the sphere $S^{2}$. However, such 1 -forms do exist on the cylindrical representation of $S U(2)$ [100], i.e. the complex plane with the origin removed, which is conformally equivalent to the Kähler representation of $S^{2}$ above under the transformation $z=\mathrm{e}^{s^{1}+i s^{2}}$ which maps $\left(s^{1}, s^{2}\right) \in \mathbb{R} \times S^{1}$ to $z \in \mathbb{C}-\{0\}$. In this latter representation, the Hamiltonian in (5.123) can be taken to be an arbitrary linear combination of the $S U(2)$ generators, and the coherent state path integral (5.123) determines a topological quantum field theory with $\rho=0$ in (5.125). This is not true, however, in the Kähler representation above, but we do find, for example, that the symplectic invariance condition can be fulfilled by choosing the basis $H(z, \bar{z})=J_{3}^{(j)}(z, \bar{z})$ of the Cartan subalgebra $\mathbf{u}(1)$ and $\rho(z, \bar{z})=\frac{1}{2} \log (z / \bar{z})$. The ensuing topological path integral (5.123) then describes the quantization of spin.

To evaluate this spin partition function, we set $\rho=0$ above. Although the ensuing quantum theory now does not have the topological form in terms of a BRST-exact action, it still maintains the Schwarz-type topological form described in Section 4.8, since the Hamiltonian then satisfies (3.44) with $C=j$ and the function $K$ in (3.45) is

$$
\begin{equation*}
K(z, \bar{z})=\frac{i}{2} \log \left(\frac{z}{\bar{z}}\right) \tag{5.126}
\end{equation*}
$$

so that (5.123) is a topological path integral of the form (4.126), i.e. the quantum theory
determines a Schwarz-type topological field theory, as opposed to a Witten-type one as above. We first analyse the WKB localization formula (4.70) for the coadjoint orbit path integral (5.123). We note first of all that the boundary conditions in (5.123) are $z(0)=z_{1}$ and $\bar{z}(T)=\bar{z}_{2}$. In particular, the final value $z(T)$ and the initial value $\bar{z}(0)$ are not specified, and the boundary terms in (5.123) ensure that with these boundary conditions there is no boundary contribution to the pertinent classical equations of motion

$$
\begin{equation*}
\dot{z}+i z=0 \quad, \quad \dot{\bar{z}}-i \bar{z}=0 \tag{5.127}
\end{equation*}
$$

In general, if $z(t)$ and $\bar{z}(t)$ are complex conjugates of each other, then there are no classical trajectories that connect $z(0)=z_{1}$ with $\bar{z}(T)=\bar{z}_{2}$ on the sphere $S^{2}$. But if we view the path integral (5.123) instead as a matrix element between 2 configurations in different polarizations, then there is always the following solution to the equations of motion (5.127) with the required boundary conditions for arbitrary $z_{1}$ and $\bar{z}_{2}$,

$$
\begin{equation*}
z(t)=z_{1} \mathrm{e}^{-i t} \quad, \quad \bar{z}(t)=\bar{z}_{2} \mathrm{e}^{-i(T-t)} \tag{5.128}
\end{equation*}
$$

The solution (5.128) is complex, and hence $z(t)$ and $\bar{z}(t)$ must be regarded as independent variables. This is one of the characteristic features behind the holomorphic quantization formalism that makes it suitable to describe topological field theories. The trajectories (5.128) are therefore regarded as describing a complex saddle-point of the path integral [38, 70, 113]. We shall see other forms of this later on.

Substituting the solutions (5.128) into the WKB formula (4.70) we find the propagator

$$
\begin{equation*}
\mathcal{K}\left(z_{2}, z_{1} ; T\right)=\frac{\left(1+z_{1} \bar{z}_{2} \mathrm{e}^{-i T}\right)^{2 j} \mathrm{e}^{-i j T}}{\left(1+z_{1} \bar{z}_{1}\right)^{j}\left(1+z_{2} \bar{z}_{2}\right)^{j}} \tag{5.129}
\end{equation*}
$$

The exact propagator from a direct calculation is

$$
\begin{equation*}
\frac{\left(z_{2}\left|\mathrm{e}^{-i T \hat{J}_{3}}\right| z_{1}\right)}{\sqrt{\left(z_{1} \mid z_{1}\right)\left(z_{2} \mid z_{2}\right)}}=\frac{1}{\left(1+z_{1} \bar{z}_{1}\right)^{j}\left(1+z_{2} \bar{z}_{2}\right)^{j}} \sum_{m=-j}^{j}\binom{2 j}{j+m}\left(z_{1} \bar{z}_{2} \mathrm{e}^{-i T}\right)^{j+m} \mathrm{e}^{i j T} \tag{5.130}
\end{equation*}
$$

which coincides with (5.129) upon application of the binomial theorem (5.119). In particular, setting $z_{1}=z_{2}=z$ and integrating over $z \in \mathbb{C}$ using the coherent state measure
(5.121), we find the partition function

$$
\begin{align*}
Z_{S U(2)}(T) & =\int d \mu^{(j)}(z, \bar{z}) \frac{\left(z\left|\mathrm{e}^{-i T \hat{J}_{3}}\right| z\right)}{(z \mid z)} \\
& =\int_{0}^{\infty} d r r \frac{(2 j+1)\left(1+r^{2} \mathrm{e}^{-i T}\right)^{2 j} \mathrm{e}^{-i j T}}{\left(1+r^{2}\right)^{2 j}}=\frac{\sin \left(\frac{T}{2}(2 j+1)\right)}{\sin \frac{T}{2}} \tag{5.131}
\end{align*}
$$

which also coincides with the exact result

$$
\begin{equation*}
\operatorname{tr}_{j} \mathrm{e}^{-i T \hat{J}_{3}}=\sum_{m=-j}^{j} \mathrm{e}^{-i T m}=\frac{\mathrm{e}^{i T j}}{1-\mathrm{e}^{-i T j}}+\frac{\mathrm{e}^{-i T j}}{1-\mathrm{e}^{i T j}} \tag{5.132}
\end{equation*}
$$

Note that the right-hand side of (5.132) is precisely what one anticipates from the Weyl character formula (5.22). The roots of $S U(2)$ are $\alpha= \pm 1$ [128], and the Cartan subalgebra is $\mathbf{u}(\mathbf{1})$ consisting of the single element $\hat{J}_{3}$. The Weyl group is $W=\mathbb{Z}_{2}$ and it has 2 elements, the identity map and the reflection map $\hat{J}_{3} \rightarrow-\hat{J}_{3}$. Thus the formula (5.132) is simply the Weyl character formula (5.22) for the spin- $j$ representation of $S U(2)$.

In the framework of the Duistermaat-Heckman theorem, the terms summed in (5.132) are each associated with one of the poles of the sphere $S^{2}$, i.e. with the critical points of the height function on $S^{2}$. Indeed, since this Hamiltonian is a perfect Morse function with even Morse indices, we expect that the Weyl character formula above coincides with the pertinent stronger version (4.97) of the localization formulas. Because of the Kähler structure (5.101) on $S^{2}$ (see (5.47)), the Riemann moment map has the non-vanishing components

$$
\begin{equation*}
\left(\mu_{V^{(j)}}\right)_{z}^{z}=-\left(\mu_{V^{(j)}}\right)_{\bar{z}}^{\bar{z}}=i J_{3}^{(j)}(z, \bar{z}) / j \tag{5.133}
\end{equation*}
$$

and consequently the $\operatorname{Dirac} \hat{A}$-genus is

$$
\begin{equation*}
\hat{A}\left(T \Omega_{V^{(j)}}\right)=\frac{T}{2 j} \frac{J_{3}^{(j)}}{\sin \left(\frac{T}{2 j} J_{3}^{(j)}\right)} \tag{5.134}
\end{equation*}
$$

Substituting these into the localization formula (4.97) yields precisely the Weyl character formula (5.132). This localization onto the critical points of the Hamiltonian, as for the harmonic oscillator example of Section 5.3, agrees with the general arguments at the
beginning of Section 4.5. Substituting the stereographic projection map (5.114) into the classical equations of motion (5.127) gives

$$
\begin{equation*}
\dot{\theta} \sin \theta=0 \quad, \quad \dot{\phi}+1=0 \tag{5.135}
\end{equation*}
$$

For $T \neq 2 \pi n, n \in \mathbb{Z}$, the only $T$-periodic critical trajectories of coincide with the critical points of the Hamiltonian $j(1-\cos \theta)$, i.e. $\theta=0, \pi$, and in this case the critical point set of the action is isolated and non-degenerate. However, for $T=2 \pi n, n \in \mathbb{Z}$, we find $T$-periodic classical solutions for any initial value of $\theta$ and $\phi$ in (5.135) and the critical point set of the classical action coincides with the original phase space $S^{2}$. Thus the moduli space of classical solutions in this case is $L \mathcal{M}_{S}=S^{2}$, and the localization onto this moduli space is now easily verified from (4.76) to give the correct anticipated result above. From the discussion of Section 4.8, it also follows that the sum of the terms in (5.132) describes exactly the properly normalized period group of the symplectic 2 -form $\omega^{(j)}$ on the sphere [70], i.e. the integer-valued surface integrals of $\omega^{(j)}$ as in (4.123). We shall see in the next Chapter that quantizations of the propagation time $T$ as above lead to interesting quantum theories in certain other instances of the localization framework.

It is an interesting exercise to work out the Niemi-Tirkkonen localization formula (4.91) for the above dynamical system. For this we note that, again because of the Kähler geometry of $S^{2}$, the Riemann curvature 2-form has the non-vanishing components

$$
\begin{equation*}
R_{z}^{z}=-R_{\bar{z}}^{\bar{z}}=-i \omega^{(j)} / j \tag{5.136}
\end{equation*}
$$

and so combined with (5.133) we see that the equivariant $\hat{A}$-genus here is

$$
\begin{equation*}
\hat{A}_{V^{(j)}}(T R)=\frac{T}{2 j} \frac{J_{3}^{(j)}-\omega^{(j)}}{\sin \left(\frac{T}{2 j}\left(J_{3}^{(j)}-\omega^{(j)}\right)\right)} \tag{5.137}
\end{equation*}
$$

The equivariant extension of $\omega^{(j)}$ is

$$
\begin{equation*}
J_{3}^{(j)}-\omega^{(j)}=j\left(\frac{1-z \bar{z}}{1+z \bar{z}}-\frac{2 i}{(1+z \bar{z})^{2}} \eta \bar{\eta}\right)=j\left(\frac{1-z \bar{z}-\eta \bar{\eta}}{1+z \bar{z}+\eta \bar{\eta}}\right) \tag{5.138}
\end{equation*}
$$

where we have redefined the Grassmann variables $\eta^{\mu} \rightarrow \sqrt{i} \cdot \eta^{\mu}$. The Niemi-Tirkkonen localization formula (4.91) can then be written as

$$
\begin{equation*}
Z_{S U(2)}(T) \sim \frac{i}{\pi T} \int_{\mathbf{R}^{2} \otimes \Lambda^{1} \mathbf{R}^{2}} d z d \bar{z} d \eta d \bar{\eta} L(z \bar{z}+\eta \bar{\eta}) \tag{5.139}
\end{equation*}
$$

where

$$
\begin{equation*}
L(y)=\frac{\frac{T}{2} \frac{1-y}{1+y}}{\sin \left(\frac{T}{2}\left(\frac{1-y}{1+y}\right)\right)} \exp \left[-i j T\left(\frac{1-y}{1+y}\right)\right] \tag{5.140}
\end{equation*}
$$

Using the Parisi-Sourlas integration formula [107]

$$
\begin{equation*}
\frac{1}{\pi} \int_{\mathbf{R}^{2} \otimes \Lambda^{1} \mathbf{R}^{2}} d^{2} x d \eta d \bar{\eta} L\left(x^{2}+\eta \bar{\eta}\right)=\int_{0}^{\infty} d u \frac{d L(u)}{d u}=L(\infty)-L(0) \tag{5.141}
\end{equation*}
$$

we obtain from (5.139) the partition function

$$
\begin{equation*}
Z(T) \sim \sin (T j) / \sin (T / 2) \tag{5.142}
\end{equation*}
$$

Introducing the Weyl shift $j \rightarrow j+\frac{1}{2}$ in (5.142) then yields the correct Weyl character formula (5.131) for $S U(2)^{8}$. Note that (5.141) shows explicitly how the localization in (5.139) comes directly from the extrema of the height function at $z=\infty$ and $z=0$.

As a final application for the above dynamical system, we examine the quadratic localization formula (4.112). Now the (degenerate) Hamiltonian is

$$
\begin{equation*}
\mathcal{F}\left(J_{3}^{(j)}\right)=\left(J_{3}^{(j)}\right)^{2}=j^{2}\left(\frac{1-z \bar{z}}{1+z \bar{z}}\right)^{2} \tag{5.143}
\end{equation*}
$$

Following the same steps as above, the localization formula (4.112) can be written as

$$
\begin{equation*}
Z_{S U(2)}\left(T \mid\left(J_{3}^{(j)}\right)^{2}\right) \sim \frac{i}{\sqrt{4 \pi i T}} \int_{-\infty}^{\infty} \frac{d \phi_{0}}{\phi_{0}} \int_{\mathbf{R}^{2} \otimes \Lambda^{1} \mathbf{R}^{2}} d z d \bar{z} d \eta d \bar{\eta} L\left(\phi_{0}, z \bar{z}+\eta \bar{\eta}\right) \tag{5.144}
\end{equation*}
$$

where

$$
\begin{equation*}
L\left(\phi_{0}, y\right)=\frac{\frac{T \phi_{0}}{2} \frac{1-y}{1+y}}{\sin \left(\frac{T \phi_{0}}{2}\left(\frac{1-y}{1+y}\right)\right)} \exp \left[\frac{i T}{4} \phi_{0}^{2}-i j T \phi_{0}\left(\frac{1-y}{1+y}\right)\right] \tag{5.145}
\end{equation*}
$$

[^29]and we have redefined $\eta^{\mu} \rightarrow \sqrt{i / \phi_{0}} \cdot \eta^{\mu}$. Using the Parisi-Sourlas integration formula (5.141) again and introducing the Weyl shift $j \rightarrow j+\frac{1}{2}$, we find
\[

$$
\begin{align*}
Z_{S U(2)}\left(T \mid\left(J_{3}^{(j)}\right)^{2}\right) & \sim \sqrt{\frac{T}{4 \pi i}} \int_{-\infty}^{\infty} d \phi_{0} \mathrm{e}^{i T \phi_{0}^{2} / 4} \frac{\sin \left[\left(j+\frac{1}{2}\right) T \phi_{0}\right]}{\sin \left(T \phi_{0} / 2\right)} \\
& =\sum_{m=-j}^{j} \sqrt{\frac{T}{4 \pi i}} \int_{-\infty}^{\infty} d \phi_{0} \mathrm{e}^{-i T m \phi_{0}} \mathrm{e}^{i T \phi_{0}^{2} / 4}=\sum_{m=-j}^{j} \mathrm{e}^{-i T m^{2}} \tag{5.146}
\end{align*}
$$
\]

which is again the correct character $\operatorname{tr}_{j} \mathrm{e}^{-i T \hat{J}_{3}^{2}}$.
Thus on a spherical phase space geometry the equivariant Hamiltonian systems provide a rich example of the topological quantum field theories discussed in Section 4.8, and they are the natural framework for the study of the quantum properties of classical spin systems. The character formula path integrals above describe the quantization of the harmonic oscillator on the sphere, and therefore the only integrable quantum system, up to holomorphic equivalence (i.e. modification by the general geometry of the phase space), that exists within the equivariant localization framework on a general spherical geometry is the harmonic oscillator defined on the reduced compact phase space $D^{2}$.

### 5.6 Hyperbolic Phase Spaces

The situation for the case where the phase space is endowed with a Riemannian geometry of constant negative Gaussian curvature $K<0$ parallels that of the last Section, and we only therefore briefly discuss the essential differences [125]. The phase space $\mathcal{M}$ is now necessarily a non-compact manifold, and we can map it onto the maximally symmetric space $\mathcal{H}^{2}$, the Lobaschevsky plane (or pseudo-sphere) of constant negative curvature, with its standard curved hyperbolic metric $g_{\mathcal{H}^{2}}[36,37,57]$. The Killing vectors of this metric have the general form

$$
\begin{equation*}
V_{\mathcal{H}^{2}}^{w}=-i \Omega w+\alpha(1+w \bar{w})^{1 / 2} \quad, \quad V_{\mathcal{H}^{2}}^{\bar{w}}=i \Omega \bar{w}+\bar{\alpha}(1+w \bar{w})^{1 / 2} \tag{5.147}
\end{equation*}
$$

and they generate the isometry group $S O(2,1)$. The rest of the analysis at the beginning of the last Section now carries through analogously to the case at hand here, where we
replace the $K$ factors everywhere by $-|K|$ and the $K^{1 / 2}$ factors by $|K|^{1 / 2}$.
In particular, with these changes, the generalized stereographic coordinate transformation (5.100) is the same except that now the holomorphic function $f(z)$ there maps the phase space onto the Poincaré disk of radius $\frac{1}{2}|K|^{1 / 2}$, i.e. the disk $D^{2}$ with the Poincaré metric

$$
\begin{equation*}
g_{\mathcal{H}^{2}}=\frac{4}{(1-z \bar{z})^{2}} d z \otimes d \bar{z} \tag{5.148}
\end{equation*}
$$

which defines a Kähler geometry on the disk for which the associated symplectic 2-form is the unique invariant volume form under the transitive $S O(2,1)$-action. The Poincaré disk is the stereographic projection image for the Lobaschevsky plane when we regard it through its embedding in $\mathbb{R}^{3}$ as the pseudo-sphere, so that we can represent it by pseudo-spherical coordinates as $(\tau, \phi) \in \mathbb{R} \times[0,2 \pi]$ by $x^{1}=\sinh \tau \cos \phi, x^{2}=\sinh \tau \sin \phi$ and $z=\cosh \tau$. The stereographic projection is again taken from the projection center $z^{\prime}=-1$, and the boundary of the Poincare disc corresponds to points at infinity of the hyperboloid $\mathcal{H}^{2}$. The pseudo-sphere itself is represented by the interior of the disc. The explicit transformation in terms of pseudo-spherical coordinates is

$$
\begin{equation*}
z=\frac{w^{\prime}}{1+z^{\prime}}=\mathrm{e}^{-i \phi} \tanh (\tau / 2) \tag{5.149}
\end{equation*}
$$

We also note here that the Poincaré disc is conformally equivalent to the upper half plane $\mathbb{C}^{+}$via the Cayley transform $\xi \rightarrow z=(\xi-i) /(\xi+i)$ which takes $\xi \in \mathbb{C}^{+}$onto the Poincaré disk, and the Poincaré metric (5.148) on the (Poincaré) upper-half plane is

$$
\begin{equation*}
g_{\mathcal{H}^{2}}=\operatorname{Im}(\xi)^{-2} d \xi \otimes d \bar{\xi} \tag{5.150}
\end{equation*}
$$

The path integral over such hyperbolic geometries arises in string theory and studies of quantum chaos [26].

The most general localizable Hamiltonian in a hyperbolic phase space geometry is therefore

$$
\begin{equation*}
H_{-}(z, \bar{z})=\frac{\Omega\left(\frac{|K|}{4}+f(z) \bar{f}(\bar{z})\right)}{\frac{|K|}{4}-f(z) \bar{f}(\bar{z})}+\frac{\alpha \bar{f}(\bar{z})+\bar{\alpha} f(z)}{\frac{|K|}{4}-f(z) \bar{f}(\bar{z})}+C_{0} \tag{5.151}
\end{equation*}
$$

The transformation to Darboux coordinates on $\mathcal{M}$ is now accomplished by the diffeomorphism

$$
\begin{equation*}
v(z, \bar{z})=\frac{f(z)}{\left(\frac{\lfloor K\rfloor}{4}-f(z) \bar{f}(\bar{z})\right)^{1 / 2}} \tag{5.152}
\end{equation*}
$$

which maps $\mathcal{M}$ onto the complement of the unit disc $\mathbb{C}-\operatorname{int}\left(D^{2}\right)$ in $\mathbb{R}^{2}$. The general Darboux Hamiltonians are therefore

$$
\begin{equation*}
H_{-}^{D}(z, \bar{z})=\Omega z \bar{z}+(\bar{\alpha} z+\alpha \bar{z})(1+z \bar{z})^{1 / 2} \quad ; \quad z \in \mathbb{C}-\operatorname{int}\left(D^{2}\right) \tag{5.153}
\end{equation*}
$$

We note that here there are 2 inequivalent Hamiltonians, corresponding to a choice of "spacelike" and "timelike" Killing vectors, but the generic hyperbolic Hamiltonians are again all holomorphic copies of one another, again reducing to a quasi-displaced harmonic oscillator. However, given that the Darboux phase space is now non-compact, we can again weaken the requirement of a global circle action on the phase space to a semibounded group action.

Considering therefore the quantum problem defined on the Poincare disc of unit radius, we write the 3 independent observables in (5.151) as

$$
\begin{equation*}
S_{3}^{(k)}(z, \bar{z})=k \frac{1+z \bar{z}}{1-z \bar{z}} \quad, \quad S_{+}^{(k)}(z, \bar{z})=2 k \frac{\bar{z}}{1-z \bar{z}} \quad, \quad S_{-}^{(k)}(z, \bar{z})=2 k \frac{z}{1-z \bar{z}} \tag{5.154}
\end{equation*}
$$

Defining the Kähler 2-form $\omega^{(k)}=k \omega_{\mathcal{H}^{2}}$, we see that the associated Poisson algebra of these observables is just the $S U(1,1)$ Lie algebra

$$
\begin{equation*}
\left\{S_{3}^{(k)}, S_{ \pm}^{(k)}\right\}_{\omega^{(k)}}= \pm S_{ \pm}^{(k)} \quad, \quad\left\{S_{+}^{(k)}, S_{-}^{(k)}\right\}_{\omega^{(k)}}=-2 S_{3}^{(k)} \tag{5.155}
\end{equation*}
$$

The Hamiltonians in (5.151) are therefore functions on the coadjoint orbit

$$
\begin{equation*}
S U(1,1) / U(1) \simeq \mathcal{H}^{2} \tag{5.156}
\end{equation*}
$$

of the non-compact Lie group $S U(1,1)$, and the generators (5.154) are the normalized matrix elements of the $S U(1,1)$ generators in the $S U(1,1)$ coherent states

$$
\begin{equation*}
\mid z)=\mathrm{e}^{z \hat{S}_{+}}|k, 0\rangle=\sum_{n=0}^{\infty}\binom{2 k+n+1}{n}^{1 / 2} z^{n}|k, n\rangle \quad ; \quad z \in \operatorname{int}\left(D^{2}\right) \tag{5.157}
\end{equation*}
$$

for the discrete irreducible representation of $S U(1,1)$ characterized by $k=1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ [108]. The (infinite-dimensional) representation states $|k, n\rangle$ defined here are the eigenstates of the generator $\hat{S}_{3}$ with eigenvalues

$$
\begin{equation*}
\hat{S}_{3}|k, n\rangle=(k+n)|k, n\rangle \tag{5.158}
\end{equation*}
$$

and the coherent states (5.157) have the normalization

$$
\begin{equation*}
\left(z_{2} \mid z_{1}\right)=\left(1-z_{1} \bar{z}_{2}\right)^{-2 k} \tag{5.159}
\end{equation*}
$$

where we have used the binomial series expansion

$$
\begin{equation*}
\frac{1}{(1-x)^{n}}=\sum_{m=0}^{\infty}\binom{m+n-1}{m} x^{m} \tag{5.160}
\end{equation*}
$$

which is valid for $n \in \mathbb{Z}^{+}$and $|x|<1$.
Again, the integrable Hamiltonian systems are obtained by taking $H=S_{3}^{(k)}$, which is the height function on $\mathcal{H}^{2}$, and the corresponding coherent state path integral yields the quantization of the harmonic oscillator on the open infinite space $\mathcal{H}^{2}$ (and up to holomorphic equivalence these are the only integrable systems on a general hyperbolic phase space). It is straightforward to analyse the localization formulas for the coherent state path integral just as in the last Section. For instance, the WKB localization formula for the coadjoint orbit path integral

$$
\begin{equation*}
Z_{S U(1,1)}(T)=\int_{L \mathcal{H}^{2}}[d \cosh \tau][d \phi] \exp \left\{i \int_{0}^{T} d t(k \cosh \tau \dot{\phi}-k(1+\cosh \tau))\right\} \tag{5.161}
\end{equation*}
$$

can be shown to coincide with the exact Weyl character formula for $S U(1,1)[44,113]$

$$
\begin{equation*}
Z_{S U(1,1)}(T)=\operatorname{tr}_{k} \mathrm{e}^{-i T \hat{S}_{3}}=\sum_{n=0}^{\infty} \mathrm{e}^{-i T(k+n)}=2 i \frac{\mathrm{e}^{-i T\left(k-\frac{1}{2}\right)}}{\sin \frac{T}{2}} \tag{5.162}
\end{equation*}
$$

Some higher-dimensional examples of these coadjoint orbit models and an explicit verification of the Duistermaat-Heckman localization formula have been worked out recently in [45] for $S U(N)$ coherent states on complex $N$-dimensional projective space $\mathbb{C} P^{N} \simeq S^{2 N+1} / S^{N}$ (i.e. the set of complex lines through the origin in $\mathbb{C}^{N+1}$ ), and their non-compact hyperbolic counterparts (i.e. $S U(N-1,1)$ coherent states), and in [43] for $U(N)$ coherent states on Grassmann manifolds $U(N) /(U(n) \times U(N-n))$.

### 5.7 Quantization on Non-homogeneous Phase Spaces

In this final Section of this Chapter we consider the final remaining possible class of Riemannian geometries on the phase space $\mathcal{M}$, i.e. those with a Gaussian curvature $K(x)$ which is a non-constant function of the coordinates on $\mathcal{M}$, so that $\operatorname{dim} \mathcal{K}(\mathcal{M}, g)=1$. The 2-dimensional geometries which admit only a single Killing vector are far more numerous than the maximally symmetric ones and it is here that one could hope to obtain more non-trivial applications of the localization formulas. Another nice feature of these spaces is that the corresponding Hamiltonian Poisson algebra will be abelian, so that the Hamiltonians so obtained will automatically be Cartan elements, in contrast to the previous cases where the 3 -dimensional Lie algebra $\mathcal{K}(\mathcal{M}, g)$ was non-abelian. Thus the abelian localization formulas of the last Chapter can be applied straightforwardly, and the resulting propagators will yield character formulas for the isometry group elements defined in terms of a topological field theory type path integral. It is possible to study non-abelian localization formulas using the formalisms developed in Sections 3.8 [135] and 4.8 [127], but here we wish to focus on the properties of integrable quantum systems corresponding to Cartan element Hamiltonians so that we can study the appropriate classical character formulas.

Given a 1-parameter isometry group acting on $(\mathcal{M}, g)$, we begin by introducing a set of preferred coordinates ( $x^{11}, x^{2}$ ) defined in terms of 2 differentiable functions $\chi^{1}$ and $\chi^{2}$ as described in Section 5.2, so that in these coordinates the Killing vector $V$ has components $V^{1}=1, V^{\prime 2}=0$. For now, the function $\chi^{1}$ is any non-constant function on $\mathcal{M}$, but we shall soon see how, once a given isometry of the dynamical system is identified, it can be fixed to suit the given problem. For a Hamiltonian system $(\mathcal{M}, \omega, H)$ which generates the flows of the given isometry in the usual way via Hamilton's equations, the defining condition (5.49) for the coordinate function $\chi^{2}$ now reads

$$
\begin{equation*}
\left\{\chi^{2}, H\right\}_{\omega}=0 \tag{5.163}
\end{equation*}
$$

which is assumed to hold away from the critical point set of $H$ (i.e. the zeroes of $V$ ) almost everywhere on $\mathcal{M}$. This means that $\chi^{2}$ is a conserved charge of the given dynamical system, i.e. a function of action variables. In higher dimensions there would be many such possibilities for the conserved charges depending on the integrability properties of the system. However, in 2-dimensions this requirement fixes the action variable to be simply a functional of the Hamiltonian $H$,

$$
\begin{equation*}
\chi^{2}=\mathcal{F}(H) \tag{5.164}
\end{equation*}
$$

and so even in the non-maximally symmetric cases we see the intimate connection here between the equivariant localization formalism and the integrability of a (classical or quantum) dynamical system. We note that this only fixes the requirement (5.163) that the coordinate transformation function be constant along the integral curves of the Killing vector field $V$. The isometry condition (5.64) on the symplectic 2-form now only implies that, in the new $x^{\prime}$-coordinates, $\omega_{\mu \nu}\left(x^{\prime}\right)$ is independent of $x^{11}$ (just as for the metric). The Hamiltonian equations with $V^{\prime 1}=1, V^{\prime 2}=0$ must be solved consistently now using (5.164) and an associated symplectic structure. Notice that this construction is explicitly independent of the other coordinate transformation function $\chi^{1}$ used in the construction of the preferred coordinates for $V$ (c.f. Section 5.2).

Thus for a general metric (5.43) that admits a sole isometry, the general "admissible" Hamiltonians within the framework of equivariant localization are given by the functionals in (5.164) determined by the transformation $x \rightarrow x^{\prime}$ to coordinates in which the (circle or translation) action of the corresponding Killing vector is explicit. The rich structure now arises because the integrability condition $\mathcal{L}_{V} \omega=0$ for the Hamiltonian equations does not uniquely determine the symplectic 2 -form $\omega$, as it did in the case of a maximally symmetric geometry. The above construction could therefore be started with any given symplectic 2-form obeying this requirement, with the hope of being able to analyse quite general classes of Hamiltonian systems. This has the possibility of largely expanding the
known examples of quantum systems where the Feynman path integral could be evaluated exactly, in contrast to the maximally symmetric cases where we saw that there was only a small number of few-parameter Hamiltonians which fit the localization framework. However, it has been argued that the set of Hamiltonian systems in general for which the localization criteria apply is still rather small [34, 125]. For instance, we could from the onset take $\omega$ to be the Darboux 2-form on $\mathcal{M}=\mathbb{R}^{2}$ and hope to obtain localizable examples of 1-dimensional quantum mechanical problems with static potentials. These are defined by the Darboux Hamiltonians

$$
\begin{equation*}
H_{Q M}(p, q)=\frac{1}{2} p^{2}+U(q) \tag{5.165}
\end{equation*}
$$

where $U(q)$ is some potential which is a $C^{\infty}$-function of the position $q \in \mathbb{R}^{1}$. It was Dykstra, Lykken and Raiten [34] who first pointed out that the formalism outlined in Chapter 4 above, which naively seems like it would imply the exact solvability of any phase space path integral, does not work for arbitrary potentials $U(q)$.

To see this, we consider a generic potential $U(q)$ which is bounded from below. By adding an irrelevant constant to the Hamiltonian (5.165) if necessary, we can assume that $U(q) \geq 0$ without loss of generality. We introduce a "harmonic" coordinate $y \in \mathbb{R}$ and polar coordinates $(r, \theta) \in \mathbb{R} \times S^{1}$ by

$$
\begin{equation*}
p=r \sin \theta \quad, \quad U(q)=\frac{1}{2} y^{2}=\frac{1}{2} r^{2} \cos ^{2} \theta \tag{5.166}
\end{equation*}
$$

In these coordinates the Hamiltonian (5.165) takes the usual integrable harmonic oscillator form $H=\frac{1}{2} r^{2}$, so that the function $\chi^{2}$ above defines the radial coordinate $r$ in (5.166) and $\mathcal{F}(H)=\sqrt{2 H}$ in (5.164). The Hamiltonian vector field in these polar coordinates has the single non-vanishing component

$$
\begin{equation*}
V^{\theta}=-\frac{d y}{d q} \tag{5.167}
\end{equation*}
$$

The metric tensor (5.43) will have in general have 3 components $g_{r r}, g_{\theta \theta}$ and $g_{\theta r}$ under the coordinate transformation (5.166), and the Killing equations (2.92) become

$$
\begin{equation*}
V^{\theta} \partial_{\theta} g_{\theta \theta}+2 g_{\theta \theta} \partial_{\theta} V^{\theta}=0 \quad, \quad \partial_{\theta}\left(g_{r \theta} V^{\theta}\right)+g_{\theta \theta} \partial_{r} V^{\theta}=0 \quad, \quad V^{\theta} \partial_{\theta} g_{r r}+2 g_{r \theta} \partial_{r} V^{\theta}=0 \tag{5.168}
\end{equation*}
$$

The 3 equations in (5.168) can be solved in succession by integrating them and the general solution has the form

$$
\begin{equation*}
g_{\theta \theta}=\frac{f(r)}{\left(V^{\theta}\right)^{2}} \quad, \quad g_{r \theta}=\frac{f(r)}{V^{\theta}} \int_{\theta_{0}}^{\theta} d \theta^{\prime} \partial_{r}\left(\frac{1}{V^{\theta^{\prime}}}\right)+\frac{h(r)}{V^{\theta}} \quad, \quad g_{r r}=\frac{\left(V^{\theta}\right)^{2}}{f(r)} g_{r \theta}^{2}+k(r) \tag{5.169}
\end{equation*}
$$

where $f(r), h(r)$ and $k(r)$ are arbitrary $C^{\infty}$-functions that are independent of the angular coordinate $\theta$.

Note that, as expected, there is no unique solution for the conformal factor $\varphi$ in (5.43), only the requirement that it be radially symmetric (i.e. independent of $\theta$ ). However, the equations (5.169) impose a much stronger requirement, this time on the actual coordinate transformation (5.166). If we impose the required single-valuedness property on the metric components above, then the requirement that $g_{r \theta}(r, \theta)=g_{r \theta}(r, \theta+2 \pi)$ is equivalent to the condition

$$
\begin{equation*}
\frac{\partial}{\partial r} \int_{0}^{2 \pi} \frac{d \theta}{V^{\theta}}=0 \tag{5.170}
\end{equation*}
$$

or equivalently that

$$
\begin{equation*}
\int_{0}^{2 \pi} d \theta \frac{d q}{d y}=\text { constant } \tag{5.171}
\end{equation*}
$$

However, the only solution to (5.171) is when the function $\frac{d q}{d y}$ is independent of the radial coordinate $r$, which from (5.166) is possible only when $y=-q$, so that $U(q)=\frac{1}{2} q^{2}$ and $H_{Q M}$ is the harmonic oscillator Hamiltonian. Thus, with the exception of the harmonic oscillator, equivariant localization fails for all 1-dimensional quantum mechanical Hamiltonians with static potentials which are bounded below, due to the non-existence of a single-valued metric satisfying the Lie derivative constraint in this case.

Even for the harmonic oscillator, which is considered trivial from the point of view of localization theory, there are some ambiguities that arise in the above formalism due to the fact that there is a large degree of freedom remaining in the metric tensor which is not determined by the equivariant localization constraints. To see this, we note first that the Hamiltonian vector field (5.167) in this case is $V^{\theta}=1$ which generates a global $S^{1}$-action on $\mathcal{M}=\mathbb{R}^{2}$ given by translations of the angle coordinate $\theta$. Thus one would
expect the localization formulas to be exact for the harmonic oscillator using any radially symmetric geometry (5.43) to make manifest the localization principle. This is certainly true of the WKB formula (4.70) which does not involve the metric tensor at all, but the more general localization formulas, such as the Niemi-Tirkkonen formula (4.91), are explicitly metric dependent through, e.g. the $\hat{A}$-genus terms, although not manifestly so. Explicitly, the non-vanishing components of the metric tensor (5.43) under the coordinate transformation (5.166) in the case at hand are

$$
\begin{equation*}
g_{r r}=\mathrm{e}^{\varphi(r)} \quad, \quad g_{\theta \theta}=r^{2} \mathrm{e}^{\varphi(r)} \tag{5.172}
\end{equation*}
$$

and it is straightforward to work out the Riemann moment map and curvature tensor which with $V^{\theta}=1$ lead to the non-vanishing components

$$
\begin{equation*}
\left(\Omega_{V}\right)_{\theta r}=-\left(\Omega_{V}\right)_{r \theta}=\frac{r}{2} \mathrm{e}^{\varphi(r)}\left(2+r \frac{d \varphi(r)}{d r}\right) \quad, \quad R_{\theta r \theta r}=-\frac{1}{2}\left(\Omega_{V}\right)_{\theta r} \frac{d}{d r} \log \lambda(r) \tag{5.173}
\end{equation*}
$$

where we have introduced the function

$$
\begin{equation*}
\lambda(r)=\mathrm{e}^{-\varphi(r)}\left(\Omega_{V}\right)_{\theta r} / 2 r \tag{5.174}
\end{equation*}
$$

Substituting the above quantities into the Niemi-Tirkkonen formula (4.91) with $\omega_{r \theta}=$ $r$ and working out the Grassmann and $\theta$ integrals there, after some algebra we find the following expression for the harmonic oscillator partition function,

$$
\begin{equation*}
Z_{\mathrm{harm}}(T) \sim \frac{1}{i} \int_{0}^{\infty} d r \frac{d}{d r}\left(\frac{\lambda(r)}{\sin T \lambda(r)} \mathrm{e}^{-i T r^{2} / 2}\right)=\frac{1}{i} \lim _{r \rightarrow 0} \frac{\lambda(r)}{\sin T \lambda(r)} \tag{5.175}
\end{equation*}
$$

Comparing with (5.69), we see that this result coincides with the exact result for the harmonic oscillator partition function only if the function (5.174) behaves at the origin $r=0$ as

$$
\begin{equation*}
\lim _{r \rightarrow 0} \lambda(r)=\frac{1}{2} \tag{5.176}
\end{equation*}
$$

which using (5.173) and (5.174) means that the phase space metric must satisfy, in addition to the radial symmetry constraint, the additional constraint

$$
\begin{equation*}
\lim _{r \rightarrow 0} r \frac{d}{d r} \varphi(r)=0 \tag{5.177}
\end{equation*}
$$

The requirement (5.177) means that the conformal factor $\varphi(r)$ of the Riemannian geometry must be an analytic function of $r$ about $r=0$, and this restriction on the general form of the metric (5.43) (i.e. on the functional properties of the conformal factor $\varphi$ ) ensures that the partition function is independent of this phase space metric, as it should be.

This analyticity requirement, however, simply means that the metric should be chosen so as to eliminate the singularity at the origin of the coordinate transformation to polar coordinates $(r, \theta)$ on the plane. That this transformation is singular at $p=q=0$ is easily seen by computing the Jacobian for the change of variables (5.166) with the harmonic oscillator potential (or by noting that $\omega$ and $g$ are degenerate at $r=0$ in these coordinates). Since the equivariant Atiyah-Singer index which appears as the Niemi-Tirkkonen formula for the quantum mechanical path integral is an integral over characteristic classes, it is manifestly invariant under $C^{\infty}$ deformations of the metric on $\mathcal{M}$. The transformation to polar coordinates is a diffeomorphism only on the punctured plane $\mathbb{R}^{2}-\{0\}$, which destroys the manifest topological invariance of the partition function (at $r=0$ anyway). To obtain a properly defined metric-independent quantity, one should instead consider the quantum theory as defined on the punctured plane, but this is a multiply-connected phase space, since the loops which encircle the origin are non-contractable, so that some of the localization formalism described above must be suitably modified (see the next Chapter). We shall return to some of these points in Chapter 7. As discussed in [34] and [125], this appears to be a general feature of the generalized localization formulas, and one must essentially know the quantum theory $a b$ initio in order to resolve the ambiguities associated with the arbitrariness of the metric (5.43). Indeed, in the set of preferred coordinates for $V$ it has no zeroes and so the critical points are "absorbed" into the symplectic 2 -form $\omega$ and in general also the metric $g$. Thus the preferred coordinate transformation for $V$ is a diffeomorphism only on $\mathcal{M}-\mathcal{M}_{V}$ in general. Nonetheless, this simple example illustrates that quite general, non-homogeneous geometries can still be
used to carry out the equivariant localization framework for path integrals and describe the equivariant Hamiltonian systems which lead to topological quantum theories in terms of the generic phase space geometry.

Although the above arguments appear to have eliminated a large number of interesting physical problems, owing to the fact that their Hamiltonian vector fields do not generate well-defined orbits on the $\theta$-circle, it is still possible that quantum mechanical Hamiltonians with unbounded static potentials could fit the localization framework. Such dynamical systems indeed do represent a rather large class of physically interesting quantum systems. The first such attempt was carried out by Dykstra, Lykken and Raiten [34] who considered the equivariant localization formalism applied to the 1-dimensional hydrogen atom Hamiltonian [78]

$$
\begin{equation*}
H_{h}(p, q)=\frac{1}{2} p^{2}-\frac{1}{|q|} \tag{5.178}
\end{equation*}
$$

The eigenvalues of the associated quantum Hamiltonian form a discrete spectrum with energies

$$
\begin{equation*}
E_{n}=-1 / 2 n^{2} \quad, \quad n=1,2, \ldots \tag{5.179}
\end{equation*}
$$

which resembles the bound state spectrum of the more familiar 3-dimensional hydrogen atom [83]. What is even more interesting about this dynamical system is that the classical bound state orbits all coalesce at the phase space points $q=0, p= \pm \infty$ on $\mathbb{R}^{2}$, so that a localization onto classical trajectories (like the WKB formula) is highly unsuitable for this quantum mechanical problem. This problem could therefore provide an example wherein although the standard WKB approximation cannot be employed, the more general localization formulas, like the Niemi-Tirkkonen formula, which seem to have no constraints on them other than the usual isometry restrictions on the phase space $\mathcal{M}$, could prove of use in describing the exact quantum theory of the dynamical system.

The key to evaluating the localization formulas for the Darboux Hamiltonian (5.178)
is the transformation to the hyperbolic coordinates $(r, \tau)$ with $-\infty \leq r, \tau \leq \infty$,

$$
\begin{equation*}
p=|r| \sinh \tau \quad, \quad q=2 / r|r| \cosh ^{2} \tau \tag{5.180}
\end{equation*}
$$

so that the Hamiltonian is again $H_{h}=-\frac{1}{2} r^{2}$ and the Hamiltonian vector field has the single non-vanishing component

$$
\begin{equation*}
V^{\tau}=-\frac{1}{4} r^{3} \cosh ^{3} \tau \tag{5.181}
\end{equation*}
$$

Now the Killing equations have precisely the same form as in (5.168), with ( $r, \theta$ ) replaced by $(r, \tau)$ there, and thus the general solutions for the metric tensor have precisely the same form as in (5.169). However, because of the non-compact range of the hyperbolic coordinate $\tau$ in the case at hand, we do not encounter a single-valuedness problem in defining the components $g_{r r}$ as $C^{\infty}$ functions on $\mathbb{R}^{2}$ and from (5.169) and (5.181) we find that it is given explicitly by the perfectly well-defined function

$$
\begin{equation*}
g_{r \tau}=\frac{12 f(r)}{r^{4} V^{\tau}}\left(\frac{\sinh \tau}{2 \cosh ^{2} \tau}+\frac{1}{2} \arctan (\sinh \tau)\right)+\frac{h(r)}{V^{\tau}} \tag{5.182}
\end{equation*}
$$

In the context of our isometry analysis above, we again choose the coordinate transformation function $\chi^{2}$ above so that $\mathcal{F}(H)=\sqrt{-2 H}$ in (5.164). The other coordinate function $\chi^{1} \equiv x^{\prime \prime 1}$ is determined by noting that the above $(r, \tau)$ coordinates are the $x^{\prime}$ coordinates in (5.37) from which we wish to define the preferred set of $x^{\prime \prime}$-coordinates for the Hamiltonian vector field $V$. There we identify $\left(x^{11}, x^{\prime 2}\right)=(\tau, r)$ according to that prescription. Carrying out the explicit integration over $x^{\prime 1}=\tau$ using (5.181), and then substituting in the transformation (5.180) back to the original Darboux coordinates, after some algebra we find

$$
\begin{equation*}
\chi^{1}(p, q)=-\left|\frac{2}{|q|}-p^{2}\right|^{-3 / 2}\left[p|q|\left|\frac{2}{|q|}-p^{2}\right|^{1 / 2}+2 \arctan \left(\frac{p}{\left|\frac{2}{|q|}-p^{2}\right|^{1 / 2}}\right)\right] \tag{5.183}
\end{equation*}
$$

Thus the Hamiltonian (5.178) is associated with the phase space metric tensor (5.43) which is invariant under the translations $\chi^{1} \rightarrow \chi^{1}+a_{0}$ of the coordinate (5.183). The
discussion above shows explicitly that the phase space indeed does admit a globally welldefined metric which is translation invariant in the variable (5.183). It is also possible to evaluate the Niemi-Tirkkonen localization formula for this quantum problem in a similar fashion as the harmonic oscillator example above. We shall not go into this computation here, but refer to [34] for the technical details. The only other point we wish to make here is that one encounters in the same way as above a metric ambiguity such as (5.176), which imposes again certain regularity requirements on the conformal factor of the metric (5.43). These conditions are far more complicated than above because of the more complicated form of the translation function (5.183), but they are again associated with the cancelling of the coordinate singularities in (5.180) which make the equivariant Atiyah-Singer index in (4.91) an explicitly metric dependent quantity. With these appropriate geometric restrictions it is enough to argue that the quantum partition function for the Darboux Hamiltonian (5.178) has the form [34]

$$
\begin{equation*}
Z_{h}(T) \sim \sum_{n=1}^{\infty} \mathrm{e}^{i T / 2 n^{2}} \tag{5.184}
\end{equation*}
$$

which from (5.179) we see is indeed the exact spectral propagator for the 1-dimensional hydrogen atom [78].

This example shows that more complicated quantum systems can be studied within the equivariant localization framework on a simply connected phase space, but only for those phase spaces which admit Riemannian geometries which have complicated and unusual symmetries, such as translations in the coordinate (5.183) above. Thus aside from the above noted problem of having to resolve the explicit metric ambiguity in the localization formulas, there is the further general problem as to whether or not a geometry can in fact possess the required symmetry (e.g. for Hamiltonians associated with bounded potentials, there is no such geometry). It is not expected, of course, that any Hamiltonian will have an exactly solvable path integral, and from the point of view of this Chapter the cases where the Feynman path integral fails to be effectively computable within the framework of equivariant localization will be the cases where a required symmetry of the
phase space geometry does not lead to a globally well-defined metric tensor. Nonetheless, the analysis in [34] for the 1-dimensional hydrogen atom is a highly non-trivial success of the equivariant localization formulas for path integrals which goes beyond the range of the standard WKB method.

We conclude this Chapter by showing that it is possible to relate the path integrals for generic dynamical systems on non-homogenous phase spaces which fall into the framework of loop space equivariant localization to character formulas for the associated 1-parameter isometry groups [125]. For this, we need to introduce a formalism for constructing coherent states associated with non-transitive group actions on manifolds [73, 125]. We consider the isothermal metric (5.43) in the preferred $x^{\prime}$-coordinates for a Hamiltonian vector field $V$ on $\mathcal{M}$. Using these coordinates, we define the complex coordinates $z=x^{2} \mathrm{e}^{i x^{\prime 1}}$, in analogy with the case where $V$ defines a rotationally symmetric geometry (as for the harmonic oscillator). Let $f(z \bar{z})$ be an invariant analytic solution of the ordinary differential equation

$$
\begin{equation*}
\frac{d}{d(z \bar{z})} z \bar{z} \frac{d}{d(z \bar{z})} \log f(z \bar{z})=\frac{1}{2} \mathrm{e}^{\varphi(z \bar{z})} \tag{5.185}
\end{equation*}
$$

For the symplectic 2 -form of the phase space, we take the invariant volume form associated with $(\mathcal{M}, g)$,

$$
\begin{equation*}
\omega^{(\varphi)}=i \frac{d}{d(z \bar{z})} z \bar{z} \frac{d}{d(z \bar{z})} \log f(z \bar{z}) d z \wedge d \bar{z} \tag{5.186}
\end{equation*}
$$

whose associated symplectic potential is

$$
\begin{equation*}
\theta^{(\varphi)}=\frac{i}{2} \frac{d}{d(z \bar{z})} \log f(z \bar{z})(\bar{z} d z-z d \bar{z}) \tag{5.187}
\end{equation*}
$$

This definition turns the phase space into a non-homogeneous Kähler manifold with Kähler potential

$$
\begin{equation*}
F^{(\varphi)}(z, \bar{z})=\log f(z \bar{z}) \tag{5.188}
\end{equation*}
$$

Let $N_{\varphi}, 0<N_{\varphi} \leq \infty$, be the integer such that the function $f(z \bar{z})$ admits the Taylor
series expansion

$$
\begin{equation*}
f(z \bar{z})=\sum_{n=0}^{N_{\varphi}}(z \bar{z})^{n} f_{n} \tag{5.189}
\end{equation*}
$$

and let $\rho(z \bar{z})$ be an invariant integrable function whose moments are

$$
\begin{equation*}
\int_{0}^{P} d(z \bar{z})(z \bar{z})^{n} \rho(z \bar{z})=\frac{1}{f_{n}} \quad, \quad 0 \leq n \leq N_{\varphi} \tag{5.190}
\end{equation*}
$$

where $P$ is a real number with $0<P \leq \infty$. Let $\hat{a}^{\dagger}$ and $\hat{a}$ be bosonic creation and annihilation operators on some representation space of the isometry group (as in Section 5.3 above), and let $|n\rangle, n \in \mathbb{Z}^{+}$, be the complete system of eigenstates of the corresponding number operator, $\hat{a}^{\dagger} \hat{a}|n\rangle=n|n\rangle$. The desired coherent states are then defined as

$$
\begin{equation*}
|z\rangle=\sum_{n=0}^{N_{\varphi}} \sqrt{f_{n}} z^{n}|n\rangle \tag{5.191}
\end{equation*}
$$

The states (5.191) have the normalization

$$
\begin{equation*}
(z \mid z)=f(z \bar{z})=\mathrm{e}^{F^{(\varphi)}(z, \bar{z})} \tag{5.192}
\end{equation*}
$$

and they obey a completeness relation analogous to (5.120) in the isometry invariant measure

$$
\begin{equation*}
d \mu^{(\varphi)}(z, \bar{z})=\frac{i}{2 \pi} f(z \bar{z}) \rho(z \bar{z}) \Theta(P-z \bar{z}) d z \wedge d \bar{z} \tag{5.193}
\end{equation*}
$$

where $\Theta(x)$ denotes the step function for $x \in \mathbb{R}$. The completeness of the coherent states (5.191) follows from a calculation analogous to that in (5.78) using the definitions (5.189)-(5.191) above.

Notice that for the functional values $f(z \bar{z})=\mathrm{e}^{z \bar{z}},(1+z \bar{z})^{2 j}$ and $(1-z \bar{z})^{-2 k}(5.191)$ reduces to, respectively, the Heisenberg-Weyl group, spin- $j S U(2)$ and level- $k S(1,1)$ coherent states that we described earlier. Moreover, in that case we consistently find, respectively, the weight functions $\rho(z \bar{z})=\mathrm{e}^{-z \bar{z}}$ with $P=\infty, \rho(z \bar{z})=(2 j+1)(1+z \bar{z})^{-2(j+1)}$ with $P=\infty$, and $\rho(z \bar{z})=(2 k-1)(1-z \bar{z})^{2(k-1)}$ with $P=1$. This is anticipated from (5.185), as then the isothermal metrics in (5.43) correspond to the standard maximally symmetric Kähler geometries described earlier. Here the isometry group acts on the
states (5.191) as $\left.\left.h^{(\tau)} \mid z\right)=\mid \mathrm{e}^{i \tau} z\right), h^{(\tau)} \in \mathcal{I}(\mathcal{M}, g), \tau \in \mathbb{R}^{1}$, which ensures that a Hamiltonian exists (as we shall see explicitly below) such that a time-evolved coherent state remains coherent in this sense, regardless of the choice of $\rho$ [73]. The (holomorphic) dependence of the non-normalized coherent state vectors $\mid z$ ) on only the single complex variable $z$ is, as usual, what makes them amenable to the study of the isometry situation at hand. Notice also that the metric tensor (5.43) and canonical 1 -form (5.187) can as usual be represented in the standard coherent state forms (5.84) and (5.83), respectively.

Considering as usual the coherent state matrix elements (5.87) with respect to (5.191), using (5.187) and (5.193) we can construct the usual coherent state path integral

$$
\begin{align*}
Z^{(\varphi)}(T \mid \mathcal{F}(H))= & \int_{L \mathcal{M}} \prod_{t \in[0, T]} d \mu^{(\varphi)}(z(t), \bar{z}(t)) \\
& \times \exp \left\{i \int_{0}^{T} d t\left[\frac{1}{2} \frac{d}{d(z \bar{z})} \log f(z \bar{z})(z \dot{\bar{z}}-\bar{z} \dot{z})-\mathcal{F}(H)\right]\right\} \tag{5.194}
\end{align*}
$$

where we have again allowed for a possible functional $\mathcal{F}(H)$ of the isometry generator $H$. The observable $H(z, \bar{z})$ in (5.194) can be found by substituting (5.186), written back in the $x^{\prime}$-coordinates using the standard radial form for $z=x^{\prime 2} \mathrm{e}^{i x^{\prime 1}}$ given in (5.172), and $V^{\prime 1}=a_{0}, V^{\prime 2}=0$ into the Hamiltonian equations. Thus the equivariant localization constraints in these cases determine $H$ in terms of the phase space metric as

$$
\begin{equation*}
H^{(\varphi)}(z, \bar{z})=a_{0} z \bar{z} \frac{d}{d(z \bar{z})} \log f(z \bar{z})+C_{0}=a_{0} \frac{\left(z\left|\hat{a}^{\dagger} \hat{a}\right| z\right)}{(z \mid z)}+C_{0} \tag{5.195}
\end{equation*}
$$

where the function $f(z \bar{z})$ is related to the metric (5.43) by (5.185). Notice that (5.195) reduces to the usual harmonic oscillator height functions in the maximally symmetric cases of Sections 5.3, 5.5 and 5.6 above. Thus (5.195) can be considered as the general localizable Hamiltonian valid for any phase space Riemannian geometry, be it maximally symmetric or otherwise (the same is true, of course, for the coherent state path integral (5.194)). This is to be expected, because the localizable Hamiltonian functions in the case of maximal symmetry are simply displaced harmonic oscillators, and these oscillator Hamiltonians correspond to the rotation generators of the isometry groups, i.e.
translations in $\arg (z)=x^{11}$ (this also agrees with the usual integrability arguments). In fact, (5.195) shows explicitly that the function $H$ is essentially just a harmonic oscillator Hamiltonian written in terms of some generalized phase space geometry.

The main difference in the present context between the maximally symmetric and non-homogeneous cases lies in the path integral (5.194) itself. In the former case the coherent state measure $d \mu^{(\varphi)}(z, \bar{z})$ which must be used in the Feynman measure in (5.194) coincides with the the volume form (5.186), because as mentioned earlier if the isometry group acts transitively on the Riemannian manifold $(\mathcal{M}, g)$ then there is a unique leftinvariant measure (i.e. a unique solution to (5.64)) and so $d \mu^{(\varphi)}=\omega^{(\varphi)}$ yields the standard Liouville measure on the loop space $L \mathcal{M}$. In the latter case $d \mu^{(\varphi)} \neq \omega^{(\varphi)}$, and (5.194) is not in the canonical form (4.107) for the quantum partition function associated with the loop space symplectic geometry. Nonetheless, by a suitable modification of the loop space supersymmetry associated with the dynamical system by noting that the coherent state measure in (5.193) is invariant under the action of the isometry group on $\mathcal{M}$, it is still possible to derive appropriate versions of the standard localization formulas with the obvious replacements corresponding to this change of integration measure. Of course, we can alternatively follow the analysis of the former part of this Section and use the standard Liouville path integral measure, but then we lose the formal analogies with the Duistermaat-Heckman theorem and its generalizations. It is essentially this nonuniqueness of an invariant symplectic 2 -form in the case of non-transitive isometry group actions which leads to numerous possibilities for the localizable Hamiltonian systems defined on such geometries, in marked contrast to the maximally symmetric cases where everything was uniquely fixed. If one consistently makes the "natural" choice for $\omega$ as the Kähler 2-form (5.186), then indeed the only admissible Hamiltonian functions $H$ are generalized harmonic oscillators.

## Chapter 6

## Equivariant Localization on Multiply Connected Phase Spaces

In the last Chapter we deduced the general features of the localization formalism on a simply-connected 2-dimensional symplectic manifold. We found general forms for the Hamiltonian functions in terms of the underlying phase space Riemannian geometry which is required for their Feynman path integrals to manifestly localize. This feature is quite interesting from the point of view that, as the quantum theory is always $a b$ initio metric-independent, this analysis probes the role that the geometry and topology plays towards the understanding of quantum integrability. For instance, we saw that the classical trajectories of a harmonic oscillator must be embedded into a rotationallyinvariant geometry on $\mathbb{R}^{2}$ and that as such its orbits were always circular trajectories on the plane. For more complicated systems these quantum geometries are less familiar and endow the phase space with unusual Riemannian structures. In any case, all the localizable Hamiltonians were essentially harmonic oscillators (e.g. the height function for a spherical phase space geometry) and their quantum partition functions could be represented naturally using coherent state formalisms associated with the Poisson-Lie group actions of the isometry groups of the phase space. In the non-homogeneous cases we saw, in particular, that to investigate equivariant localization in general one needs to determine if a Riemannian geometry can possess certain symmetries imposed by some rather ad-hoc restrictions from the dynamical system. In practice, the introduction of such a definite geometry into the problem is highly non-trivial, although we saw that it was possible in some non-trivial examples. These results also impose restrictions on the classes of topological quantum field theories and supersymmetric models which fall into
the framework of these geometric localization principles.
In this Chapter we shall extend the analysis of Chapter 5 to the case when the phase space $\mathcal{M}$ is multiply-connected. The first such extension of the loop space equivariant localization formalism with a detailed analysis as in the last Chapter was carried out in [120]. In particular, we shall consider a compact Riemann surface of genus $h \geq 1$, again because of the wealth of mathematical characterizations that are available for such spaces. We shall explore how the localization formalism differs from that on a simply-connected manifold. Recall that much of the formalism developed in Chapter 4, in particular that of Section 4.9, relied quite heavily on this topological restriction. We shall see that now the topological quantum field theories that appear also describe the non-trivial first homology group of the Riemann surface, and that it is completely independent of the geometrical structures that are used to carry out the equivariant localization on $\mathcal{M}$, such as the conformal factors and the modular parameters. This is typically what a topological field theory should do (i.e. have only global features), and therefore the equivariant Hamiltonian systems that one obtains in these cases are nice examples of how the localization formalism is especially suited to describe the characteristics of topological quantum field theories on spaces with much larger topological degrees of freedom. Again the common feature will be the description of the quantum dynamics using a coherent state formalism, this time associated with some non-symmetric spin system and some of the ideas from geometric quantization $[17,136]$. We shall in addition see that the coherent states span a multi- but finite-dimensional Hilbert space in which the wavefunctions carry a non-trivial representation of the discrete first homology group of the phase space. We shall verify the localization formulas of Chapter 4 in a slightly modified setting, pointing out the important subtleties that arise in trying to apply them directly on a multiplyconnected phase space.

Although we shall attempt to give a quite general argument for what the localizable dynamics are on these spaces, most of our arguments will only be carried explicitly for
genus 1, i.e. on the 2-torus $T^{2}=S^{1} \times S^{1}$, as will become apparent as we go along. In particular, we shall view the torus in a way best suited to describe its complex algebraic geometry, i.e. in the parallelogram representation of Section 3.5, so that we can examine the topological properties of the quantum theory we find and get a good idea of the features of the localization formalism on multiply-connected spaces in general. Another more explicit way to view the torus is by embedding it in $\mathbb{R}^{3}$ by revolving the circle $(y-a)^{2}+x^{2}=b^{2}$ on the $x y$-plane around the $x$-axis, where $0<b<a$, i.e. embedding $T^{2}$ in 3-space by $x=b \sin \phi_{1}, y=\left(a+b \cos \phi_{1}\right) \sin \phi_{2}$ and $z=\left(a+b \cos \phi_{1}\right) \cos \phi_{2}$. The induced metric on the surface from the flat Euclidean metric of $\mathbb{R}^{3}$ is then $b^{2} d \phi_{1} \otimes$ $d \phi_{1}+\left(a+b \cos \phi_{1}\right)^{2} d \phi_{2} \otimes d \phi_{2}$, and the modular parameter $\tau \in \mathbb{C}^{+}$of the parallelogram representation of $T^{2}$ is (c.f. Section 3.5)

$$
\begin{equation*}
\tau=i b / \sqrt{a^{2}-b^{2}} \tag{6.1}
\end{equation*}
$$

If we now introduce the coordinate

$$
\begin{equation*}
\theta=\theta\left(\phi_{1}\right)=\int_{0}^{\phi_{1}} d \phi_{1}^{\prime} \frac{b}{a+b \cos \phi_{1}^{\prime}} \tag{6.2}
\end{equation*}
$$

then it is straightforward to verify that $w=\phi_{2}+i \theta$ is an isothermal coordinate for the induced metric on $T^{2}$ for which its isothermal form is $\rho(\theta)\left(d \phi_{2} \otimes d \phi_{2}+d \theta \otimes d \theta\right)$. This defines a complex structure on $T^{2}$. Since this metric is invariant under translations in $\phi_{2}$, we could heuristically follow the analysis of Section 5.7 to deduce that one class of localizable Hamiltonians are those which are functions only of $\phi_{1}$. In order that these Hamiltonians be well-defined on $T^{2}=S^{1} \times S^{1}$, we require in addition that these be periodic functions of $\phi_{1}$. As we shall soon see, this is consistent with the general localizable dynamical systems we shall find. Topological invariance of the associated quantum theory in this context would say something like the invariance of it under certain rescalings of the modular parameter (6.1), i.e. under rescalings of the radius parameters $a$ or $b$ corresponding to a 'shift' in the local geometry of $T^{2}$. A topological quantum theory shouldn't detect such shifts which aren't considered as ones modifying the topological properties of the torus.

In other words, the topological quantum theory should be independent of the phase space complex structure. We shall see this in a more algebraic form later on in this Chapter.

### 6.1 Isometry Groups of Multiply Connected Spaces

To describe the isometries of a generic path connected, multiply-connected Riemannian manifold ( $\mathcal{M}, g$ ), we lift these isometries up into what is known as the universal covering space of the manifold. The multiple-connectivity of $\mathcal{M}$ means that it has loops in it which cannot be contracted to a point (i.e. $\mathcal{M}$ has 'holes' in it). This is measured algebraically by what is called the fundamental homotopy $\operatorname{group} \pi_{1}(\mathcal{M})$ of $\mathcal{M}$, a similar but rather different mathematical entity as the first homology group $H_{1}(\mathcal{M} ; \mathbb{Z})$. Roughly speaking, this group is defined as follows. We fix a basepoint $x_{0} \in \mathcal{M}$ and consider the loop space of periodic maps $\sigma:[0,1] \rightarrow \mathcal{M}$ with $\sigma(0)=\sigma(1)=x_{0}$. For any 2 loops $\sigma$ and $\tau$ based at $x_{0}$ in this way, the product loop $\sigma \cdot \tau$ is defined to be the loop obtained by first going around $\sigma$, and then going around $\tau$. The set $\pi_{1}(\mathcal{M})$ is the space of all equivalence classes [ $\sigma$ ] of loops, where 2 loops are equivalent if and only if they are homotopic to each other, i.e. there exists a continuous deformation between the loops. It can be shown that the above multiplication of loops then gives a well defined multiplication in $\pi_{1}(\mathcal{M})$ and turns it into a group with identity the homotopy class of the trivial loop $[0,1] \rightarrow x_{0}$ and with inverse defined by reversing the orientation of a loop. In general, this group is non-abelian and discrete, and it is related to the first homology group $H_{1}(\mathcal{M} ; \mathbb{Z})$ as follows. Let $[G, G]$ denote the commutator subgroup of any group $G$, i.e. $[G, G]$ is the normal subgroup of $G$ generated by the products $g h g^{-1} h^{-1}, g, h \in G$. The homology group $H_{1}(\mathcal{M} ; \mathbb{Z})$ is then the abelianization of the fundamental group,

$$
\begin{equation*}
H_{1}(\mathcal{M} ; \mathbb{Z})=\pi_{1}(\mathcal{M})_{\mathrm{ab}} \equiv \pi_{1}(\mathcal{M}) /\left[\pi_{1}(\mathcal{M}), \pi_{1}(\mathcal{M})\right] \tag{6.3}
\end{equation*}
$$

If $\pi_{1}(\mathcal{M})$ is itself abelian, then the homology and homotopy of $\mathcal{M}$ coincide. We refer to [81] for a more complete exposition of homotopy theory and how homology, in the sense
of (6.3), is the natural approximation of homotopy.
The universal covering space of $\mathcal{M}$ is now defined as the smallest simply connected manifold $\tilde{\mathcal{M}}$ covering $\mathcal{M}$. By a covering space we mean that there is a surjective continuous $\operatorname{map} \pi: \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ such that its restriction to any neighbourhood of $\tilde{\mathcal{M}}$ defines a local diffeomorphism. This means that locally on $\mathcal{M}$ we can lift any quantity defined on it to its universal cover and study it on the simply connected space $\tilde{\mathcal{M}}$. The manifold $\mathcal{M}$ and its universal covering space $\tilde{\mathcal{M}}$ are related by the homeomorphism

$$
\begin{equation*}
\mathcal{M} \simeq \tilde{\mathcal{M}} / \pi_{1}(\mathcal{M}) \tag{6.4}
\end{equation*}
$$

where the fundamental group acts freely on $\tilde{\mathcal{M}}$ through what are known as deck or covering transformations [81], i.e. the diffeomorphisms $\sigma: \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}$ such that $\pi(\sigma(x))=$ $\pi(x), \forall x \in \tilde{\mathcal{M}}$. Thus in this setting, the universal covering space is a principal fiber bundle where the total space $\tilde{\mathcal{M}}$ is locally regarded as the space of all pairs ( $x,\left[C_{x}\right]$ ), where $C_{x}$ is a curve in $\mathcal{M}$ from $x_{0}$ to $x$ and $\left[C_{x}\right]$ is its homotopy class ${ }^{1}$. The structure group of the bundle is $\pi_{1}(\mathcal{M})$ and the bundle projection $\tilde{\mathcal{M}} \xrightarrow{\pi} \mathcal{M}$ takes a homotopy class of curves to their endpoint, $\pi:\left[C_{x}\right] \rightarrow x$. Clearly, $\mathcal{M}$ is its own universal cover if it is simply connected, i.e. $\pi_{1}(\mathcal{M})=0$. We shall see some examples in due course.

Consider now a Riemannian metric $g$ defined on $\mathcal{M}$, and let $\pi^{*} g$ be its inverse image under the canonical bundle projection of $\tilde{\mathcal{M}}$ onto $\mathcal{M}$. Then ( $\tilde{\mathcal{M}}, \pi^{*} g$ ) is a simplyconnected Riemannian manifold, and from the discussion of the last Chapter we are well acquainted with the structure of its isometry groups. It is possible to show [77], from the principal fiber bundle interpretation (6.4) above, that to every isometry $h \in \mathcal{I}(\mathcal{M}, g)$ one can associate an isometry $\tilde{h} \in \mathcal{I}\left(\tilde{\mathcal{M}}, \pi^{*} g\right)$ which is compatible with the universal covering projection in the sense that

$$
\begin{equation*}
\pi \circ \tilde{h}=h \circ \pi \tag{6.5}
\end{equation*}
$$

[^30]To prove this one needs to show that the lifting $\tilde{h} \equiv \pi^{*} h$ gives a diffeomorphism of $\tilde{\mathcal{M}}$ which is a well-defined function on the homotopy classes of curves used for the definition of $\tilde{\mathcal{M}}$ [77]. Thus the isometries of the Riemannian manifold $(\mathcal{M}, g)$ lift to isometries of the simply connected space $\left(\tilde{\mathcal{M}}, \pi^{*} g\right)$ of which we have a complete description from the last Chapter. It should be kept in mind though that there may global obstructions from the homotopy of $\mathcal{M}$ to extending an isometry of $\tilde{\mathcal{M}}$ projected locally down onto $\mathcal{M}$ by the bundle projection $\pi$. We shall see how this works in the next Section.

### 6.2 Equivariant Hamiltonian Systems in Genus One

Our prototypical model for a multiply-connected symplectic manifold will be the 2-torus $T^{2}=S^{1} \times S^{1}$ which we first studied in Section 3.5. Notice that the circle is multiplyconnected with $\pi_{1}\left(S^{1}\right)=\mathbb{Z}$ with the integers labelling the number of times that a $\operatorname{map} \sigma: S^{1} \rightarrow S^{1}$ 'winds' around the circle, i.e. to each homotopy class $[\sigma] \in \pi_{1}\left(S^{1}\right)$ we can associated an integer which we call the winding number of the loop $\sigma$ (where a change of sign signifies a change in the direction of traversing the loop). We can describe the homotopy of the torus by introducing 2 loops $a$ and $b$, both fixed at the same basepoint on $S^{1} \times S^{1}$, with $a$ looping once around the inner circle of the torus (i.e. $\left.a: S^{1} \rightarrow\left(\phi_{1}, 0\right) \in S^{1} \times S^{1}\right)$ and $b$ looping once around the outer circle (i.e. $\left.b: S^{1} \rightarrow\left(0, \phi_{2}\right) \in S^{1} \times S^{1}\right)$. Since clearly any other loop in $T^{2}$ is homotopic to some combination of the loops $a$ and $b$, it follows that they generate the fundamental group $\pi_{1}\left(T^{2}\right)$ of the torus, and furthermore they obey the relation

$$
\begin{equation*}
a b a^{-1} b^{-1}=1 \tag{6.6}
\end{equation*}
$$

which is easily seen by simply tracing the loop product in (6.6) around $S^{1} \times S^{1}$. (6.6) means that $\pi_{1}\left(T^{2}\right)$ is abelian and therefore coincides with the first homology group (3.78). Thus the loops $a$ and $b$ defined above are also generators of the first homology group $H_{1}\left(\Sigma^{1} ; \mathbb{Z}\right)$, and they will henceforth be referred to as the canonical homology cycles
of the torus. Note that any homology cycle in $\Sigma^{1}$ which defines the homology class $a$ (respectively $b$ ) can be labelled by the $\phi_{1}$ angle coordinates (respectively $\phi_{2}$ ). Thus any homology class of a genus 1 compact Riemann surface is labelled by a pair of integers ( $n, m$ ) which represents the winding numbers around the canonical homology cycles $a$ and $b$.

Recall from Section 3.5 the description of the torus as a parallelogram with its opposite edges identified in the plane, and with modular parameter $\tau \in \mathbb{C}^{+}$which labels the inequivalent complex analytic structures on the torus (or equivalently the conformal equivalence classes of metrics on $T^{2}$ ) $[92,121]$. This means that it can be represented as the quotient space

$$
\begin{equation*}
\Sigma^{1}=\mathbb{C} /(\mathbb{Z} \oplus \tau \mathbb{Z}) \tag{6.7}
\end{equation*}
$$

where the quotient is by the free bi-holomorphic action of the lattice group $\mathbb{Z} \oplus \tau \mathbb{Z}$ on the simply-connected complex plane $\mathbb{C}$. In other words, the lattice group is the discrete automorphism group of the complex plane and it acts on $\mathbb{C}$ by the translations ${ }^{2}$

$$
\begin{equation*}
z \rightarrow z+2 \pi(n+\tau m) \quad, \quad \bar{z} \rightarrow \bar{z}+2 \pi(n+\bar{\tau} m) \quad ; \quad n, m \in \mathbb{Z} \tag{6.8}
\end{equation*}
$$

under which the canonical bundle projection $\mathbb{C} \xrightarrow{\pi} \Sigma^{1}$ is invariant. That the plane is the universal cover of the torus is easily seen by observing that the real line $\mathbb{R}^{1}$ is the universal cover of the circle $S^{1}$ with the bundle projection $\pi(x)=\mathrm{e}^{2 \pi i x}$ for $x \in \mathbb{R}^{1}$.

With the identification (6.7), we can now consider the most general Euclidean signature metric on $\Sigma^{1}$. From our discussion in Section 5.2, we know that the most general metric on $\mathbb{C}$ can be written in the global isothermal form (5.43). The covering projection in (6.7) in this way induces the most general metric on the torus, which can therefore be written in terms of a flat Kähler metric as

$$
\begin{equation*}
g_{\tau}=\frac{\mathrm{e}^{\varphi(z, \bar{z})}}{\operatorname{Im} \tau} d z \otimes d \bar{z} \tag{6.9}
\end{equation*}
$$

[^31]or in terms of the angle coordinates $\left(\phi_{1}, \phi_{2}\right) \in S^{1} \times S^{1}$
\[

\left[g_{\phi_{\mu} \phi_{\nu}}\right]=\frac{\mathrm{e}^{\varphi\left(\phi_{1}, \phi_{2}\right)}}{\operatorname{Im} \tau}\left($$
\begin{array}{cc}
1 & \operatorname{Re} \tau  \tag{6.10}\\
\operatorname{Re} \tau & |\tau|^{2}
\end{array}
$$\right)
\]

The complex structure on $\Sigma^{1}$ is now defined by the complex coordinates $z=\phi_{1}+\tau \phi_{2}, \bar{z}=$ $\phi_{1}+\bar{\tau} \phi_{2}$ which are therefore considered invariant under the transformations (6.8). The conformal factor $\varphi(z, \bar{z})$ is now a globally defined real-valued function on $\Sigma^{1}$ (i.e. invariant under the translations (6.8)), and the normalization in (6.9) is chosen for simplicity so that the associated metric volume of the torus

$$
\begin{equation*}
\operatorname{vol}_{g_{\tau}}\left(\Sigma^{1}\right)=\int_{\Sigma^{1}} d^{2} \phi \sqrt{\operatorname{det} g_{\tau}}=\int_{\Sigma^{1}} d^{2} \phi \mathrm{e}^{\varphi\left(\phi_{1}, \phi_{2}\right)} \equiv(2 \pi)^{2} v \tag{6.11}
\end{equation*}
$$

is finite and independent of the complex structure of $\Sigma^{1}$ with $v \in \mathbb{R}$ a fixed volume parameter of the torus. The metric (6.9) is further constrained by its Gaussian curvature scalar

$$
\begin{equation*}
K\left(g_{\tau}\right)=-\frac{1}{2} \operatorname{Im}(\tau) \mathrm{e}^{-\varphi} \nabla_{\tau}^{2} \varphi \tag{6.12}
\end{equation*}
$$

which by the Gauss-Bonnet-Chern theorem (5.48) for genus $h=1$ must obey

$$
\begin{equation*}
\int_{\Sigma^{1}} d^{2} \phi \nabla_{\tau}^{2} \varphi\left(\phi_{1}, \phi_{2}\right)=0 \tag{6.13}
\end{equation*}
$$

where $\nabla_{\tau}^{2}=\partial \bar{\partial}$ is the scalar Laplacian

$$
\begin{equation*}
\nabla_{\tau}^{2}=\partial_{\phi_{1}}^{2}+|\tau|^{-2} \partial_{\phi_{2}}^{2}+2 \operatorname{Re}(\tau)|\tau|^{-2} \partial_{\phi_{1}} \partial_{\phi_{2}} \tag{6.14}
\end{equation*}
$$

associated with the Kähler structure in (6.9).
Given this general geometric structure of the 2-torus, following the analysis of the last Chapter we would like to find the most general localizable Hamiltonian system on it which obeys the localization criteria. First of all, the condition that the Hamiltonian $H$ generates a globally integrable isometry of the metric (6.9) implies that the associated Hamiltonian vector fields $V^{\mu}(x)$ must be single-valued functions under the windings (6.8)
around the non-trivial homology cycles of $\Sigma^{1}$. This means that these functions must admit the convergent 2-dimensional harmonic mode expansions

$$
\begin{equation*}
V^{\mu}\left(\phi_{1}, \phi_{2}\right)=\sum_{n, m=-\infty}^{\infty} V_{n, m}^{\mu} \mathrm{e}^{i\left(n \phi_{1}+m \phi_{2}\right)} \tag{6.15}
\end{equation*}
$$

In other words, the components of $V$ must be $C^{\infty}$-functions which admit a 2-dimensional Fourier series plane wave expansion (6.15) appropriate to globally-defined periodic functions on $S^{1} \times S^{1}$. As we shall now demonstrate, these topological restrictions from the underlying phase space severely limit the possible Hamiltonian systems to which the equivariant localization constraints apply.

From (2.92) it follows that the Killing equations for the metric (6.10) are

$$
\begin{gather*}
2 \partial_{\phi_{1}} V^{1}+2 \operatorname{Re}(\tau) \partial_{\phi_{1}} V^{2}+V^{\mu} \partial_{\phi_{\mu}} \varphi=0 \\
2 \operatorname{Re}(\tau) \partial_{\phi_{2}} V^{1}+2|\tau|^{2} \partial_{\phi_{2}} V^{2}+|\tau|^{2} V^{\mu} \partial_{\phi_{\mu}} \varphi=0  \tag{6.16}\\
\partial_{\phi_{2}} V^{1}+\operatorname{Re}(\tau)\left(\partial_{\phi_{2}} V^{2}+\partial_{\phi_{1}} V^{1}\right)+|\tau|^{2} \partial_{\phi_{1}} V^{2}+\operatorname{Re}(\tau) V^{\mu} \partial_{\phi_{\mu}} \varphi=0
\end{gather*}
$$

Substituting in the harmonic expansions (6.15) and using the completeness of the plane waves there to equate the various components of the expansions in (6.16), we find after some algebra that (6.16) generates 2 coupled equations for the Fourier components of the Hamiltonian vector field,

$$
\begin{gather*}
\left(|\tau|^{2} n-\operatorname{Re}(\tau) m\right) V_{n, m}^{1}=|\tau|^{2}(m-\operatorname{Re}(\tau) n) V_{n, m}^{2} \\
(m-\operatorname{Re}(\tau) n) V_{n, m}^{1}=\left[\left(\operatorname{Re}(\tau)^{2}-\operatorname{Im}(\tau)^{2}\right) n-\operatorname{Re}(\tau) m\right] V_{n, m}^{2} \tag{6.17}
\end{gather*}
$$

which hold for all integers $n$ and $m$. It is straightforward to show from the coupled equations (6.17) that for $\tau \in \mathbb{C}^{+}, V_{n, m}^{1}=V_{n, m}^{2}=0$ unless $n=m=0$. Thus the only non-vanishing components of the harmonic expansions (6.15) are the constant modes,

$$
\begin{equation*}
V_{\Sigma^{1}}^{\mu}(x)=V_{0}^{\mu} \tag{6.18}
\end{equation*}
$$

and the only Killing vectors of the metric (6.9) are the generators of translations (by $V_{0}^{\mu} \in \mathbb{R}$ ) along the 2 independent homology cycles of $\Sigma^{1}$. Notice that this result is
completely independent of the structure of the conformal factor $\varphi$ in (6.9), and it simply means that although the torus inherits locally 3 isometries from the maximally symmetric plane, i.e. local rotations and translations, only the 2 associated translations on $\Sigma^{1}$ are global isometries. The independence of this result on the conformal factor is not too surprising, since this just reflects the fact that given any metric on a compact phase space we can make it invariant under a compact group action by averaging it over the group in its Haar measure. The above derivation gives an explicit geometric view of how the non-trivial topology of $\Sigma^{1}$ restricts the allowed global circle actions on the phase space, and we see therefore that the isometry group of any globally-defined Riemannian geometry on the torus is $U(1) \times U(1)$.

The invariance condition (5.64) for the symplectic structure can be solved by imposing the requirement of invariance of $\omega_{\Sigma^{1}}$ independently under the 2 Killing vectors (6.18). This implies that the components $\omega_{\phi_{\mu} \phi_{\nu}}$ must be constant functions, i.e. that $\omega$ must be proportional to the Darboux 2 -form $\omega_{D}$, and thus we take

$$
\begin{equation*}
\omega_{\Sigma^{1}}=v d \phi_{1} \wedge d \phi_{2} \tag{6.19}
\end{equation*}
$$

to be an associated metric-volume form on $\Sigma^{1}$ for the present Riemannian geometry (c.f. (6.11)). It is straightforward to now integrate up the Hamiltonian equations with (6.18) and (6.19), and we find that the Hamiltonian $H_{\Sigma^{1}}$ is given by displacements along the homology cycles of $\Sigma^{1}$,

$$
\begin{equation*}
H_{\Sigma^{1}}\left(\phi_{1}, \phi_{2}\right)=h^{1} \phi_{1}+h^{2} \phi_{2} \tag{6.20}
\end{equation*}
$$

where

$$
\begin{equation*}
h^{1}=v V_{0}^{2} \quad, \quad h^{2}=-v V_{0}^{1} \tag{6.21}
\end{equation*}
$$

are real-valued constants. Note that, as anticipated from (6.7), the invariant symplectic structure here is uniquely determined just as for maximally symmetric phase spaces which have 3 (as opposed to just 2 as above) linearly independent Killing vectors. Thus we see here that the localizable Hamiltonian systems in genus 1 are even more severely restricted
by the equivariant localization constraints as compared to the simply-connected cases. Note that the Hamiltonian (6.20) does not give a globally-defined single-valued function on $\Sigma^{1}$, a point which we shall return to shortly.

### 6.3 Homology Representations and Topological Quantum Field Theory

The Hamiltonian (6.20) defines a rather odd dynamical system on the torus, but besides this feature we see that the allotted Hamiltonians as determined from the geometric localization constraints are in effect completely independent of the explicit form of the phase space geometry and depend only on the topological properties of the manifold $\Sigma^{1}$, i.e. (6.20) is explicitly independent of both the complex structure $\tau$ and the conformal factor $\varphi$ appearing in (6.9). From the analysis of the last Chapter, we see that this is in marked contrast to what occurs in the case of a simply connected phase space, where the conformal factor of the metric entered into the final expression for the observable $H$ and the equivariant Hamiltonian systems so obtained depended on the phase space geometry explicitly (and for non-homogeneous phase spaces this dependence occured in a nontrivial way). In the present case the partition function with the Hamiltonian (6.20) and symplectic 2-form (6.19) obtained as the unique solutions of the equivariant localization constraints can be thought of in this way as defining a topological quantum theory on the torus which is completely independent of any Riemannian geometry on $\Sigma^{1}$. Furthermore, the symplectic potential associated with (6.19) is

$$
\begin{equation*}
\theta_{\Sigma^{1}}=\frac{v}{2}\left(\phi_{1} d \phi_{2}-\phi_{2} d \phi_{1}\right) \tag{6.22}
\end{equation*}
$$

which we note is only locally defined because it involves multi-valued functions in this local form, so that $\omega_{\Sigma^{1}}$ is a non-trivial element of $H^{2}\left(\Sigma^{1} ; \mathbb{Z}\right)=\mathbb{Z}$. The Hamiltonian (6.20) thus admits the local topological form $H_{\Sigma^{1}}=i_{V_{\Sigma^{1}}} \theta_{\Sigma^{1}}$, so that the corresponding partition function defines a cohomological field theory and it will be a topological invariant of the manifold $\Sigma^{1}$.

To explore some of the features of this topological quantum field theory, we note that (6.20) is not defined as a global $C^{\infty}$-function on $\Sigma^{1}$. However, this is not a problem from the point of view of localization theory. Although for the classical dynamics the Hamiltonian can be a multi-valued function on $\Sigma^{1}$, to obtain a well-defined quantum theory we require single-valuedness, under the windings (6.8) around the homology cycles of $\Sigma^{1}$, of the time evolution operator $\mathrm{e}^{-i T \hat{H}_{\Sigma^{1}}}$ which defines the quantum propagator (and also of the Boltzmann weight $e^{i T H_{\Sigma^{1}}}$ if we wish to have a well-defined classical statistical mechanics). This implies that the constants $h^{\mu}$ in (6.20) must be quantized, i.e. $h^{\mu} \in h \mathbb{Z}$ for some $h \in \mathbb{R}$, and then time propagation in this quantum system can only be defined in discretized intervals of the base time $h^{-1}$, i.e. $T=N_{T} h^{-1}$ where $N_{T} \in \mathbb{Z}^{+}$. Such quantizations of coupling parameters in topological gauge theories is a rather common occurence to ensure the invariance of a quantum theory under 'large gauge transformations' when the underlying space has non-trivial topology [17].

In the quantum theory, the Hamiltonian (6.20) therefore represents the winding numbers around the homology cycles of the torus, and therefore to each homology class of $\Sigma^{1}$ we can associate a corresponding Hamiltonian system obeying the equivariant localization constraints. The partition function is now denoted as

$$
\begin{equation*}
Z_{\Sigma^{1}}^{v}\left(k, \ell ; N_{T}\right) \sim \int_{L \Sigma^{1}}\left[d^{2} \phi\right] \exp \left\{i \int_{0}^{N_{T} h^{-1}} d t\left(v \phi_{2} \dot{\phi}_{1}+h\left(k \phi_{1}+\ell \phi_{2}\right)\right)\right\} \tag{6.23}
\end{equation*}
$$

where $k$ and $\ell$ are integers. This path integral can be evaluated directly by first integrating over the loops $\phi_{2}(t)$, which gives

$$
\begin{equation*}
Z_{\Sigma^{1}}^{v}\left(k, \ell ; N_{T}\right) \sim \int_{L S^{1}}\left[d \phi_{1}\right] \delta\left(v \dot{\phi}_{1}+h \ell\right) \exp \left\{i \int_{0}^{N_{T} h^{-1}} d t h k \phi_{1}(t)\right\} \sim \mathrm{e}^{-i k \ell N_{T}^{2} / 2 v} \tag{6.24}
\end{equation*}
$$

Thus the partition function of this quantum system represents the the non-trivial homology classes of the torus, through the winding numbers $k$ and $\ell$ and the time evolution integer $N_{T}$. In fact, (6.24) defines a family of 1-dimensional unitary irreducible representations of the first homology group of $\Sigma^{1}$ through the family of homomorphisms

$$
\begin{equation*}
Z_{\Sigma^{1}}^{v}\left(\cdot, \cdot ; N_{T}\right): H_{1}\left(\Sigma^{1} ; \mathbb{Z}\right) \rightarrow U(1) \otimes U(1) \tag{6.25}
\end{equation*}
$$

from the additive first homology group (3.78) into a multiplicative circle group. Notice that the associated homologically-invariant quantum theory is trivial, in that the sum over all winding numbers of the partition function (6.24) vanishes,

$$
\begin{equation*}
\sum_{k, \ell=-\infty}^{\infty} Z_{\Sigma^{1}}^{v}\left(k, \ell ; N_{T}\right)=\sum_{k=-\infty}^{\infty}\left(\frac{1}{1-\mathrm{e}^{-i k N_{T}^{2} / 2 v}}+\frac{1}{1-\mathrm{e}^{i k N_{T}^{2} / 2 v}}-1\right)=0 \tag{6.26}
\end{equation*}
$$

This sum over all winding numbers is analogous to what one would do in 4-dimensional Yang-Mills theory to include all instanton sectors into the quantum theory [118].

However, it is possible to modify what we mean by the quantum propagator on a multiply-connected phase space so that we obtain a partition function which is independent of the homology representative class defined by the Hamiltonian using a modification of the definition of the path integral over a multiply connected space [116]. In general, if the phase space $\mathcal{M}$ is multiply connected, i.e. $\pi_{1}(\mathcal{M}) \neq 0$, then the Feynman path integral representation of the quantum propagator can contain parameters $\chi([\sigma])$ which are not present in the classical theory and which weight the homotopy classes $[\sigma]$ of inequivalent time evolutions of the system ${ }^{3}$,

$$
\begin{equation*}
Z_{\mathrm{hom}}(T)=\sum_{[\sigma] \in \pi_{1}(\mathcal{M})} \chi([\sigma]) \int_{x(t) \in[\sigma]}\left[d^{2 n} x\right] \sqrt{\operatorname{det}\|\Omega\|} \mathrm{e}^{i S[x]} \tag{6.27}
\end{equation*}
$$

Unitarity and completeness of the quantum theory (i.e. of the propagator (4.13)) yield, respectively, the constraints that the parameters $\chi([\sigma])$ are phases,

$$
\begin{equation*}
\chi([\sigma])^{*} \chi([\sigma])=1 \tag{6.28}
\end{equation*}
$$

and that they form a 1 -dimensional unitary representation of $\pi_{1}(\mathcal{M})$,

$$
\begin{equation*}
\chi([\sigma]) \chi\left(\left[\sigma^{\prime}\right]\right)=\chi\left(\left[\sigma \cdot \sigma^{\prime}\right]\right) \tag{6.29}
\end{equation*}
$$

Note that the restriction of the path integration to homotopy classes as in (6.27) makes well-defined the representation of the partition function action $S$ with a local symplectic

[^32]potential following the Wess-Zumino-Witten prescription of Section 4.9. In particular, we can invoke the argument there to conclude that over each homotopy class $[\sigma] \in \pi_{1}(\mathcal{M})$, the path integral depends only on the second cohomology class defined by $\omega$.

In the case at hand, the partition function (6.24) is regarded as that obtained by restricting the path integration in (6.23) to loops in the homology class labelled by $(k, \ell) \in \mathbb{Z}^{2}$. In particular, we can add to the sum in (6.26) the phases $\chi(k, \ell)=\mathrm{e}^{i \alpha(k, \ell)}$ for each $(k, \ell) \in \mathbb{Z}^{2}$, which from (6.29) would then have to satisfy

$$
\begin{equation*}
\alpha\left(k+k^{\prime}, \ell+\ell^{\prime}\right)=\alpha(k, \ell)+\alpha\left(k^{\prime}, \ell^{\prime}\right) \tag{6.30}
\end{equation*}
$$

The condition (6.30) means that the phase $\alpha(k, \ell)$ define a $\mathbf{u}(1)$-valued 1 -cocycle of the fundamental (or homology) group $\mathbb{Z} \oplus \mathbb{Z}$ of $\Sigma^{1}$ as required for them to form a representation of it in the circle group $S^{1}$. When they are combined with the character representation (6.24) and the resulting quantity is summed as in (6.26), we can obtain a propagator which is a non-trivial homological invariant of $\Sigma^{1}$ and which yields a character formula for the non-trivial topological groups of the phase space. We shall see how to interpret these character formulas in a group-theoretic setting, as we did in the last Chapter, in the next Section. Notice that, strictly speaking, the volume parameter $v$ in (6.24) should be quantized in terms of $h, k$ and $\ell$ so that the partition function yields a non-zero result when integrated over the space of $T$-periodic trajectories. In this way, (6.24) also represents the cohomology class defined by the symplectic 2 -form (6.19) through the parameter $v$. We recall from Section 4.9 that for a simply-connected phase space, the localizable partition functions depend only on the second cohomology class defined by $\omega$. Here we find that the multiple-connectivity of the phase space makes it depend in addition on the first homology group of the manifold. Thus the partition function of the localizable quantum systems on the torus yield topological invariants of the phase space representing its (co-)homology groups.

The expression (6.24) for the partition function also follows directly from substituting into the Boltzmann weight $\mathrm{e}^{i S[x]}$ the value of the action in (6.23) evaluated on the classical
trajectories $\dot{x}^{\mu}(t)=V_{\Sigma^{1}}^{\mu}$ for the above quantum system, which here are defined by

$$
\begin{equation*}
\dot{\phi}_{1}(t)=V_{0}^{1} \quad, \quad \dot{\phi}_{2}(t)=V_{0}^{2} \tag{6.31}
\end{equation*}
$$

Thus the path integral (6.23) (trivially) localizes onto the classical loops as in the WKB localization formula (4.70), except that now even the 1-loop fluctuation term vanishes and the path integral is given exactly by its tree-level value. This also independently establishes the quantizations of the propagation time $T$ and the volume parameter $v$, in that $T$-periodic solutions to the classical equations of motion with the degenerate structure of the Hamiltonian (6.20) only exist with the discretizations of the parameters $h^{\mu}$ and $T$ above. This is consistent with the discussion at the beginning of Section 4.5 concerning the structure of the moduli space of classical solutions, and again for these discretizations the path integral can be evaluated using the degenerate localization formula (4.76) while for the non-discretized values the critical trajectory set (trivially) coincides with the critical point set $\mathcal{M}_{V}$ of the Hamiltonian. Furthermore, the fact that the conformal factor $\varphi$ is not involved at all in the solutions of the localization constraints just reflects the fact that the torus is locally flat (as is immediate from its parallelogram representation) and any global 'curving' of its geometry represented by $\varphi$ in (6.9) can only be done in a uniform periodic fashion around the canonical homology cycles of $\Sigma^{1}$ (c.f. eq. (6.13)). However, the Niemi-Tirkkonen formula (4.91) does depend explicitly on $\varphi$. It is here that the geometry of the phase space enters explicitly into the quantum theory, as it did in Chapter 5, if we demand that the metric (6.9) making the equivariant localization manifest be chosen so that the localization formula (4.91) coincides with the exact result (6.24), as of course it should.

In the case at hand (4.91) becomes

$$
\begin{align*}
Z_{\Sigma^{1}}^{v}\left(k, \ell ; N_{T}\right) \sim & \int_{\Sigma^{1}} \operatorname{ch}_{V_{\Sigma^{1}}}\left(-i N_{T} \omega_{\Sigma^{1}} / h\right) \wedge \hat{A}_{V_{\Sigma^{1}}}\left(N_{T} R_{\tau} / h\right) \\
= & \int_{\Sigma^{1}} d^{2} \phi \int d^{2} \eta \exp \left[-\frac{i N_{T}}{h}\left(H_{\Sigma^{1}}(k, \ell)-\frac{1}{2}\left(\omega_{\Sigma^{1}}\right)_{\mu \nu} \eta^{\mu} \eta^{\nu}\right)\right]  \tag{6.32}\\
& \times \sqrt{\operatorname{det}\left[\frac{N_{T}\left(2\left(\nabla_{\tau}\right)_{\mu} V_{\Sigma^{1}}^{\nu}+\left(R_{\tau}\right)_{\mu \lambda \rho}^{\nu} \eta^{\lambda} \eta^{\rho}\right) / 4 h}{\sinh \left(N_{T}\left(2\left(\nabla_{\tau}\right)_{\mu} V_{\Sigma^{1}}^{\nu}+\left(R_{\tau}\right)_{\mu \lambda \rho}^{\nu} \eta^{\lambda} \eta^{\rho}\right) / 4 h\right)}\right]}
\end{align*}
$$

Again, because of the Kähler structure of (6.9), the Riemann moment map and curvature 2 -form have the non-vanishing components

$$
\begin{equation*}
\left(\mu_{V_{\Sigma^{1}}} z_{\bar{z}}^{z}=-\left(\mu_{V_{\Sigma^{1}}}\right)_{z}^{\bar{z}}=V_{\Sigma^{1}}^{z} \partial \varphi+V_{\Sigma^{1}}^{\bar{z}} \bar{\partial} \varphi \quad, \quad R_{z}^{z}=-R_{\bar{z}}^{\bar{z}}=\operatorname{Im}(\tau) \mathrm{e}^{-\varphi} \nabla_{\tau}^{2} \varphi \eta \bar{\eta}\right. \tag{6.33}
\end{equation*}
$$

We substitute (6.18)-(6.21) and (6.33) into (6.32) and carry out the Berezin integrations there. Comparing the resulting expression with the exact one (6.24) for the partition function, we arrive after some algebra at a condition on the conformal factor of the metric (6.9),

$$
\begin{equation*}
\int_{\Sigma^{1}} d^{2} \phi \mathrm{e}^{-i N_{T}\left(k \phi_{1}+\ell \phi_{2}\right)} \sqrt{1-\frac{N_{T}^{2}\left(\ell \partial_{\phi_{1}} \varphi-k \partial_{\phi_{2}} \varphi\right)^{2}}{4 v^{2} \sinh ^{2}\left(\frac{N_{T}}{2 v}\left(\ell \partial_{\phi_{1}} \varphi-k \partial_{\phi_{2}} \varphi\right)\right)}}=-\frac{2 i}{N_{T} v} \mathrm{e}^{-i k \ell N_{T}^{2} / 2 v} \tag{6.34}
\end{equation*}
$$

The Fourier series constraint (6.34) on the metric is rather complicated and it represents a similar sort of metric ambiguity that we encountered in Section 5.7 before. It fixes the harmonic modes of the square-root integrand in (6.34) which should have an expansion such as (6.15). Notice, however, that (6.34) is independent of the phase space complex structure $\tau$, and thus it only depends on the representative of the conformal equivalence class of the metric (6.9). This is typical of a topological field theory path integral [17].

The condition (6.34) can be used to check if a given phase space metric really does result in the correct quantum theory (6.24), and this procedure then tells us what (representatives of the conformal equivalence classes of) quantum geometries in this sense are applicable to the equivariant localization of path integrals on the torus. For example, suppose we tried to quantize a flat torus using equivariant localization. Then from (6.12)
the conformal factor would have to solve the Laplace equation $\nabla_{\tau}^{2} \varphi=0$ globally on $\Sigma^{1}$. Since $\varphi$ is assumed to be a globally-defined function on $\Sigma^{1}$, it must admit a harmonic mode expansion over $\Sigma^{1}$ as in (6.15). From (6.14) and this Fourier series for $\varphi$ we see that the Laplace equation implies that all Fourier modes of $\varphi$ except the constant modes vanish, and so the left-hand side of (6.34) is zero. Thus a flat torus cannot be used to localize the quantum mechanical path integral (6.23) onto the equivariant Atiyah-Singer index in (4.91). This means that a flat Kähler metric (6.9) on $\Sigma^{1}$ does lead to a homotopically trivial localization 1-form $\psi=i_{V_{\Sigma^{1}}} g_{\tau}$ on the loop space $L \Sigma^{1}$ within any homotopy class (c.f. Section 4.3). This simple example shows that the condition (6.34), along with the Riemannian restrictions (6.11) and (6.13), give a very strong probe of the quantum geometry of the torus. Moreover, when (6.34) does hold, we can represent the equivariant characteristic classes in (4.91) in terms of the homomorphism (6.24) of the first homology group of $\Sigma^{1}$.

### 6.4 Holomorphic Quantization and Non-symmetric Coadjoint Orbits

In this Section we shall show that it is possible to interpret the topological path integral (6.23) as a character formula associated with the quantization of a coadjoint orbit corresponding to some novel sort of spin system described by $\Sigma^{1}$, as was the situation in all of the simply connected cases of the last Chapter. For this, we examine the canonical quantum theory defined by the symplectic structure (6.19) in the Schrödinger picture representation. We first rewrite the symplectic 2 -form (6.19) in complex coordinates to get the Kähler structure

$$
\begin{equation*}
\omega_{\Sigma^{1}}=\frac{v}{2 i \operatorname{Im} \tau} d z \wedge d \bar{z}=-i \partial \bar{\partial} F_{\Sigma^{1}} \tag{6.35}
\end{equation*}
$$

with corresponding local Kähler potential

$$
\begin{equation*}
F_{\Sigma^{1}}(z, \bar{z})=v z \bar{z} / \operatorname{Im} \tau \tag{6.36}
\end{equation*}
$$

We then map the corresponding Poisson algebra onto the associated Heisenberg algebra by the standard commutator prescription (c.f. beginning of Section 5.1). With this we obtain the quantum commutator

$$
\begin{equation*}
[\hat{z}, \hat{\bar{z}}]=2 \operatorname{Im}(\tau) / v \tag{6.37}
\end{equation*}
$$

We can represent the algebra (6.37) on the space $\operatorname{Hol}\left(\Sigma^{1} ; \tau\right)$ of holomorphic functions $\Psi(z)$ on $\Sigma^{1}$ by letting $\hat{z}$ act as multipication by the complex coordinate $z=\phi_{1}+\tau \phi_{2}$ and $\hat{\bar{z}}$ as the derivative operator

$$
\begin{equation*}
\hat{\bar{z}}=-\frac{2 \operatorname{Im} \tau}{v} \frac{\partial}{\partial z} \tag{6.38}
\end{equation*}
$$

With this holomorphic Schrödinger polarization, the operators $\hat{z}$ and $\hat{\bar{z}}$ with the commutator algebra (6.37) resemble the creation and annihilation operators (5.72) of the Heisenberg-Weyl algebra with the commutation relation (5.73). In analogy with that situation, we can construct the corresponding coherent states

$$
\begin{equation*}
\mid z)=\mathrm{e}^{(-v / 2 \operatorname{Im} \tau) z \hat{z}}|0\rangle \quad ; \quad z \in \Sigma^{1} \tag{6.39}
\end{equation*}
$$

which are normalized as

$$
\begin{equation*}
(z \mid z)=\mathrm{e}^{-(v / 2 \operatorname{Im} \tau) z \bar{z}}=\mathrm{e}^{-F_{\Sigma^{1}}(z, \bar{z}) / 2} \tag{6.40}
\end{equation*}
$$

and obey the completeness relation

$$
\begin{equation*}
\int_{\Sigma^{1}} \frac{d^{2} z}{(2 \pi)^{2}} \frac{\mid z)(z \mid}{(z \mid z)}=\mathbf{1} \tag{6.41}
\end{equation*}
$$

These coherent states are associated with the quantization of the coadjoint orbit $U(1) \times$ $U(1)=S^{1} \times S^{1}$. However, since $\Sigma^{1}$ is a non-symmetric space, it cannot be considered as a Kähler manifold associated with the coadjoint orbit of a semi-simple Lie group, as was the case in the last Chapter. The orbits above are, however, associated with the action of the isometry group $U(1) \times U(1)$ on $\Sigma^{1}$, which has an interesting Lie algebraic structure that we shall discuss below. In the Schrödinger representation (6.38), we consistently find the
action of the operator $\hat{\bar{z}}$ on the states (6.39) as $\hat{\bar{z}} \mid z$ ). The holomorphic representation space $\operatorname{Hol}\left(\Sigma^{1} ; \tau\right)$ in this context is then regarded as the space of entire functions $\Psi(z)=(z \mid \Psi)$ for each state $\mid \Psi)$ in the span of the coherent states (6.39). An inner product on $\operatorname{Hol}\left(\Sigma^{1} ; \tau\right)$ is then determined from the completeness relation (6.41) and the normalization (6.40) as

$$
\begin{equation*}
\left(\Psi_{1} \mid \Psi_{2}\right)=\int_{\Sigma^{1}} \frac{d^{2} z}{(2 \pi)^{2}} \frac{\left(\Psi_{1} \mid z\right)\left(z \mid \Psi_{2}\right)}{(z \mid z)}=\int_{\Sigma^{1}} \frac{d^{2} z}{(2 \pi)^{2}} \mathrm{e}^{(v / 2 \operatorname{Im} \tau) z \bar{z}} \Psi_{1}^{\dagger}(\bar{z}) \Psi_{2}(z) \tag{6.42}
\end{equation*}
$$

With the inner product (6.42), we find that the operator $\hat{\bar{z}}=\hat{z}^{\dagger}$ is the adjoint of $\hat{z}$, as it consistently should be. An operator $\hat{\mathcal{H}}$ acting on the space of coherent states (6.39) can now be represented on $\operatorname{Hol}\left(\Sigma^{1} ; \tau\right)$ as usual by an integral kernel as in (5.87) with the identification of $\bar{z}$ with the derivative operator (6.38).

The advantage of working with the holomorphic representation space $\operatorname{Hol}\left(\Sigma^{1} ; \tau\right)$ is that we shall want to discuss the explicit structure of the Hilbert space associated with the localizable quantum systems we found above. With the Kähler structure defined by the symplectic 2 -form $\omega_{\Sigma^{1}}$ above, the Hilbert space of the quantum theory is then the space of holomorphic sections of a complex line bundle $L \rightarrow \Sigma^{1}$ called the prequantum line bundle over $\Sigma^{1}$. As such, $\omega_{\Sigma^{1}}$ represents the first Chern characteristic class of $L$, and so such a bundle exists only if $\omega_{\Sigma^{1}}$ is an integral 2 -form on $\Sigma^{1}$. This method of quantizing the Hamiltonian dynamics in terms of the geometry of fiber bundles is called geometric quantization [136]. In light of the requirement of single-valuedness of the quantum propagator that we discussed in the last Section, we require, from the point of view of equivariant localization, that the wavefunctions $\Psi(z)$ change only by a unitary transformation under the winding transformations (6.8) on $\Sigma^{1}$, so that all physical quantities, such as the probability density $\Psi^{\dagger} \Psi$, are well-defined $C^{\infty}$-functions on the phase space $\Sigma^{1}$ and respect the symmetries of the quantum theory as defined by the quantum Hamiltonian, i.e. by the supersymmetry making the dynamical system a localizable one. In this setting, the multivalued wavefunctions, regarded as sections of the associated line bundle $L \rightarrow \Sigma^{1}$ where the structure group $\pi_{1}\left(\Sigma^{1}\right)=\mathbb{Z} \oplus \mathbb{Z}$ acts through a unitary representation, are single-valued functions on the universal cover $\mathbb{C}$ of the torus and so they
can be thought of as single-valued functions of homotopy classes $[\sigma]$ of loops on $\Sigma^{1}$. This also ensures that the coherent states (6.39) remain coherent under the time evolution determined by the localizable Hamiltonians of the last Section (i.e. under the action of $\left.\mathcal{I}\left(\Sigma^{1} ; g_{\tau}\right)\right)$ which will lead to a consistent coherent state path integral representation of (6.23).

To explore this in more detail, we need a representation for the discretized equivariant Hamiltonian generators above of the isometry group $\mathcal{I}\left(\Sigma^{1} ; g_{\tau}\right)$ on the space $\operatorname{Hol}\left(\Sigma^{1} ; \tau\right)$ [13, 120]. Note that translations by $a \in \mathbb{C}$ on $z$ are generated on functions of $z$ by the operator $\mathrm{e}^{a \frac{\partial}{\partial z}}$, and likewise on functions of $\bar{z}$ by $\mathrm{e}^{\bar{a} \frac{\partial}{\partial z}}$. On the holomorphic representation space $\operatorname{Hol}\left(\Sigma^{1} ; \tau\right)$, we represent the latter operator using the commutation relation (6.37) as $\mathrm{e}^{(v / 2 \operatorname{Im} \tau) \tilde{a} z}$, in accordance with the coherent state representation above. Thus the generators of large $U(1)$ transformations around the homology cycles of $\Sigma^{1}$ in the holomorphic Schrödinger polarization above are the unitary quantum operators

$$
\begin{equation*}
U(n, m)=\exp \left(2 \pi(n+m \tau) \frac{\partial}{\partial z}+\frac{\pi v}{\operatorname{Im} \tau}(n+m \bar{\tau}) z\right) \quad ; \quad n, m \in \mathbb{Z} \tag{6.43}
\end{equation*}
$$

which generate simultaneously both of the winding transformations in (6.8). By the above arguments, the quantum states should be invariant (up to unitary equivalence) under their action on the Hilbert space. Solving this invariance condition will then give a representation of the equivariant localization constraints (i.e. of the pertinent cohomological supersymmetry) and of the coadjoint orbit system directly in the Hilbert space of the canonical quantum theory.

In contrast with their classical counterparts, the quantum operators (6.43) do not commute among themselves in general and their products differ from the opposite-ordered products by a $\mathbf{u}(1)$-valued 2-cocycle. The Baker-Campbell-Hausdorff formula,

$$
\begin{equation*}
\mathrm{e}^{X+Y}=\mathrm{e}^{-[X, Y] / 2} \mathrm{e}^{X} \mathrm{e}^{Y} \quad \text { when } \quad[X,[X, Y]]=[Y,[X, Y]]=0 \tag{6.44}
\end{equation*}
$$

implies

$$
\begin{equation*}
\mathrm{e}^{X} \mathrm{e}^{Y}=\mathrm{e}^{-[X, Y] / 2} \mathrm{e}^{Y} \mathrm{e}^{X} \tag{6.45}
\end{equation*}
$$

Applying (6.45) to products of the operators (6.43) and using the commutation relation (6.37) with (6.38), we find that they obey what is called a clock algebra,

$$
\begin{equation*}
U\left(n_{1}, m_{1}\right) U\left(n_{2}, m_{2}\right)=\mathrm{e}^{2 \pi i v\left(n_{2} m_{1}-n_{1} m_{2}\right)} U\left(n_{2}, m_{2}\right) U\left(n_{1}, m_{1}\right) \tag{6.46}
\end{equation*}
$$

To determine the action of the operators (6.43) explicitly on the wavefunctions $\Psi(z)$, we apply the Baker-Campbell-Hausdorff formula (6.44) to get

$$
\begin{equation*}
U(n, m)=\exp \left[\frac{\pi v}{\operatorname{Im} \tau}\left(\pi|n+m \tau|^{2}+(n+m \bar{\tau}) z\right)\right] \mathrm{e}^{2 \pi(n+m \tau) \frac{\partial}{\partial z}} \tag{6.47}
\end{equation*}
$$

so that the action of (6.47) on the quantum states of the theory is

$$
\begin{equation*}
U(n, m) \Psi(z)=\exp \left[\frac{\pi v}{\operatorname{Im} \tau}\left(\pi|n+m \tau|^{2}+(n+m \bar{\tau}) z\right)\right] \Psi(z+2 \pi(n+m \tau)) \tag{6.48}
\end{equation*}
$$

If the volume parameter $v=\operatorname{vol}_{g_{\tau}}\left(\Sigma^{1}\right) /(2 \pi)^{2}$ is an irrational number, then it follows from the clock algebra (6.46) that the $U(1)$ generators above act as infinite-dimensional raising operators in (6.48) and so the Hilbert space of quantum states in this case is infinite-dimensional. However, we recall the necessary quantization requirements for the parameters of the Hamiltonian system required for a consistent quantum theory. With this in mind, we instead consider the case where the volume of the torus is quantized so that

$$
\begin{equation*}
v=v_{1} / v_{2} \quad ; \quad v_{1}, v_{2} \in \mathbb{Z}^{+} \tag{6.49}
\end{equation*}
$$

is rational-valued. Alternatively, such a discretization of $v$ is required in order that the symplectic 2 -form $\omega_{\Sigma^{1}}$ define an integer cohomology class, as in (4.123). In this case, the cocycle relation (6.46) shows that the operator $U\left(v_{2} n, v_{2} m\right)$ commutes with all of the other $U(1)$ generators and the time evolution operator, and so they can be simultaneously diagonalized over the same basis of states. This means that their action (6.48) on the wavefunctions must produce a state that lies on the same ray in the Hilbert space as that defined by $\Psi(z)$, i.e.

$$
\begin{equation*}
U\left(v_{2} n, v_{2} m\right) \Psi(z)=\mathrm{e}^{i \eta(n, m)} \Psi(z) \tag{6.50}
\end{equation*}
$$

for some phases $\eta(n, m) \in S^{1}$. The invariance condition (6.50), expressing the symmetry of the wavefunctions under the action of the (non-simple) Lie group $U(1) \times U(1)$, is called a projective representation of the symmetry group. It must obey a particular consistency condition. The composition law for the group operations induces a composition law for the phases in (6.50),

$$
\begin{align*}
& U\left(v_{2}\left(n_{1}+n_{2}\right), v_{2}\left(m_{1}+m_{2}\right)\right) \Psi(z) \\
& =U\left(v_{2} n_{1}, v_{2} m_{1}\right) U\left(v_{2} n_{2}, v_{2} m_{2}\right) \Psi(z)  \tag{6.51}\\
& =\mathrm{e}^{i \eta\left(n_{1}+n_{2}, m_{1}+m_{2}\right)} \Psi(z) \cdot \exp \left[i\left\{\eta\left(n_{1}, m_{1}\right)+\eta\left(n_{2}, m_{2}\right)-\eta\left(n_{1}+n_{2}, m_{1}+m_{2}\right)\right\}\right]
\end{align*}
$$

If the last phase in (6.51) vanishes, as in (6.30), then the projective phase $\eta(n, m)$ is a 1-cocycle of the symmetry group $U(1) \times U(1)$ and the wavefunctions carry a unitary representation of the group, as required [64]. The determination of these 1-cocycles explicitly below will then yield an explicit representation of the homologically-invariant partition function (6.27).

Comparing (6.50) and (6.48), we see that the invariance of the quantum states under the $U(1)$ action on the phase space can be expressed as

$$
\begin{equation*}
\Psi\left(z+2 \pi v_{2}(n+m \tau)\right)=\exp \left[i \eta(n, m)-\frac{\pi v_{1}}{\operatorname{Im} \tau}\left(\pi v_{2}|n+m \tau|^{2}+(n+m \bar{\tau}) z\right)\right] \Psi(z) \tag{6.52}
\end{equation*}
$$

The only functions which obey quasi-periodic conditions like (6.52) are combinations of the Jacobi theta functions [51, 90, 121]

$$
\begin{equation*}
\Theta^{(D)}\binom{c}{d}(z \mid \Pi)=\sum_{\left\{n^{\ell}\right\} \in \mathbb{Z}^{D}} \exp \left[i \pi\left(n^{\ell}+c^{\ell}\right) \Pi_{\ell p}\left(n^{p}+c^{p}\right)+2 \pi i\left(n^{\ell}+c^{\ell}\right)\left(z_{\ell}+d_{\ell}\right)\right] \tag{6.53}
\end{equation*}
$$

where $c^{\ell}, d_{\ell} \in[0,1]$. The functions (6.53) are well-defined holomorphic functions of $\left\{z_{\ell}\right\} \in \mathbb{C}^{D}$ for $D \times D$ complex-valued matrices $\Pi=\left[\Pi_{\ell p}\right]$ in the Siegal upper half-plane (i.e. $\operatorname{Im} \Pi>0$ ). They obey the doubly semi-periodic conditions

$$
\begin{equation*}
\Theta^{(D)}\binom{c}{d}(z+s+\Pi \cdot t \mid \Pi)=\exp \left[2 \pi i c^{\ell} s_{\ell}-i \pi t^{\ell} \Pi_{\ell p} t^{p}-2 \pi i t^{\ell}\left(z_{\ell}+d_{\ell}\right)\right] \Theta^{(D)}\binom{c}{d}(z \mid \Pi) \tag{6.54}
\end{equation*}
$$

where $s=\left\{s_{\ell}\right\}$ and $t=\left\{t^{\ell}\right\}$ are integer-valued vectors, and

$$
\begin{equation*}
\Theta^{(D)}(z+\alpha \Pi \cdot t \mid \Pi)=\exp \left[-i \pi \alpha^{2} t^{\ell} \Pi_{\ell p} t^{p}-2 \pi i \alpha t^{\ell}\left(z_{\ell}+d_{\ell}\right)\right] \Theta^{(D)}\binom{c-\alpha t}{d}(z \mid \Pi) \tag{6.55}
\end{equation*}
$$

for any non-integer constant $\alpha \in \mathbb{R}$. We remark here that the transformations in (6.54) can be applied in many different steps with the same final result, but successive applications of (6.54) and (6.55) do not commute [13]. In the context of the action of the unitary operators $U(n, m)$ above, when these transformations are applied in different orders in (6.52), the final results differ by a phase which forms a representation of the clock algebra (6.46). To avoid this minor ambiguity, we simply define the operators $U(n, m)$ by their action on the states $\Psi(z)$ with the convention that the transformation (6.54) is applied before (6.55).

After some algebra, we find that the algebraic constraints (6.52) are uniquely solved by the $v_{1} v_{2}$ independent holomorphic wavefunctions

$$
\begin{equation*}
\Psi_{p, r}\binom{c}{d}(z)=\mathrm{e}^{-(v / 4 \operatorname{Im} \tau) z^{2}} \Theta^{(1)}\binom{\frac{c+2 \pi v_{1} p+v_{2} r}{2 \pi v_{1} v_{2}}}{d}\left(v_{1} z \mid 2 \pi v_{1} v_{2} \tau\right) \tag{6.56}
\end{equation*}
$$

where $p=1,2, \ldots, v_{2}$ and $r=1,2, \ldots, v_{1}$. The phases in (6.50) are the non-trivial 1-cocycles

$$
\begin{equation*}
\eta(n, m) / 2 \pi=c n-d m+\pi v_{1} v_{2} n m \tag{6.57}
\end{equation*}
$$

of the global $U(1) \times U(1)$ group acting on $\Sigma^{1}$ here. Furthermore, the winding transformations (6.48) can be written as

$$
\begin{equation*}
U(n, m) \Psi_{p, r}\binom{c}{d}(z)=\sum_{p^{\prime}=1}^{v_{2}}[U(n, m)]_{p p^{\prime}} \Psi_{p^{\prime}, r}\binom{c}{d}(z) \tag{6.58}
\end{equation*}
$$

where the finite-dimensional unitary matrices

$$
\begin{equation*}
[U(n, m)]_{p p^{\prime}}=\exp \left[\frac{2 \pi i}{v_{2}}\left(c n-d m+\pi v_{1} n(m+2 p)\right)\right] \delta_{p+m, p^{\prime}} \tag{6.59}
\end{equation*}
$$

form a $v_{2}$-dimensional projective representation, which is cyclic of period $v_{2}$, of the clock algebra (6.46). The projective phase here is the non-trivial $U(1) \times U(1)$ 1-cocycle

$$
\begin{equation*}
\eta^{(p)}(n, m) / 2 \pi=\left(c n-d m+\pi v_{1} n(m+2 p)\right) / v_{2} \tag{6.60}
\end{equation*}
$$

which could also therefore be used to construct an unambiguous partition function as in the last Section by

$$
\begin{equation*}
Z_{\mathrm{hom}}(T)=\sum_{k, \ell=-\infty}^{\infty} \mathrm{e}^{i \eta(k, \ell)} Z_{\Sigma^{1}}^{v}\left(k, \ell ; N_{T}\right) \tag{6.61}
\end{equation*}
$$

Thus the Hilbert space is $v_{1} v_{2}$-dimensional and the quantum states carry a $v_{2}$-dimensional projective representation of the equivariant localization constraints via the clock algebra (6.46) which involves the $\mathbf{u}(1)$-valued 2-cocycle

$$
\begin{equation*}
\xi\left(n_{1}, m_{1} ; n_{2}, m_{2}\right) / 2 \pi=v_{1}\left(n_{2} m_{1}-n_{1} m_{2}\right) / v_{2} \tag{6.62}
\end{equation*}
$$

of the $U(1) \times U(1)$ isometry group of $\Sigma^{1}$. This shows explicitly how the $U(1)$ equivariant localization constraints and the topological toroidal restrictions are realized in the canonical quantum theory, as then these conditions imply that the only invariant operators on the Hilbert space here are essentially combinations of the generators (6.43). In particular, this implies, by construction, that the coherent state wavefunctions (6.56) are complete. This is much different than the situation for the coherent states associated with simply-connected phase spaces where there are no such topological symmetries to be respected for the supersymmetric localization of the path integral and the Hilbert space is 1-dimensional. Intuitively, the finite-dimensionality of the Hilbert space of physical states is expected from the compactness of the phase space $\Sigma^{1}$.

Notice though that the wavefunctions (6.56) contain the 2 free parameters $c$ and $d$. We can eliminate one of them by requiring that the Hamiltonian (6.20) in this basis of states does indeed lead to the correct propagator (6.24), i.e. that (6.24) be equal to the trace of the time evolution operator on the finite dimensional vector space spanned by the coherent states (6.56),

$$
\begin{equation*}
\operatorname{tr} \mathrm{e}^{-i N_{T} \hat{H}_{\Sigma^{1}}(k, \ell) / h}=\sum_{p=1}^{v_{2}} \sum_{r=1}^{v_{1}}\left(\Psi_{p, r}\left|\mathrm{e}^{-i N_{T} \hat{H}_{\Sigma^{1}}(k, \ell) / h}\right| \Psi_{p, r}\right) \tag{6.63}
\end{equation*}
$$

where the coherent state inner product is given by (6.42). With this inner product, it is straightforward to show that the states (6.56) define an orthonormal basis of the Hilbert
space,

$$
\begin{equation*}
\left(\Psi_{p_{1}, r_{1}} \mid \Psi_{p_{2}, r_{2}}\right)=\delta_{p_{1}, p_{2}} \delta_{r_{1}, r_{2}} \tag{6.64}
\end{equation*}
$$

Substituting the identity $\mathrm{e}^{-i N_{T} \hat{H}_{\Sigma^{1}}(k, \ell) / h}=[U(\ell,-k)]^{N_{T} v_{2} / 2 \pi v_{1}}$ into (6.63) and using $(6.58),(6.59)$ and (6.64) we find

$$
\begin{equation*}
\operatorname{tr} \mathrm{e}^{-i N_{T} \hat{H}_{\Sigma^{1}}(k, \ell) / h}=(-1)^{k \ell N_{T}} \mathrm{e}^{i N_{T}(c k+d \ell) / v_{1}} \tag{6.65}
\end{equation*}
$$

Comparing the result (6.65) with the exact one (6.24), we find that the parameter $d$ appearing in the wavefunctions (6.56) can be determined as

$$
\begin{equation*}
d=\left(k \ell N_{T}-2 c k\right) / 2 \ell \tag{6.66}
\end{equation*}
$$

Another way to eliminate the parameters $c$ and $d$ appearing in (6.56) is to regard the quantum theory as a topological field theory. The above construction produces a Hilbert space $\mathcal{H}^{\tau}$ of holomorphic sections of a complex line bundle $L^{\tau} \rightarrow \Sigma^{1}$ for each modular parameter $\tau$. If we smoothly vary the complex structure $\tau$, then this gives a family of finite-dimensional Hilbert spaces which can be regarded as forming in this way a holomorphic vector bundle over the Teichmüller space $\mathbb{C}^{+}$of the torus for which the projective representations above define a canonical projectively-flat connection. This is a typical feature of the Hilbert space for a Schwarz-type topological gauge theory [17]. Equivalent complex structures (i.e. those which generate the same conformal equivalence classes as (6.9)) in the sense of the topological field theory of this Chapter should be regarded as leading to the same quantum theory, and this should be inherent in both the homological partition functions of the last Section and in the canonical quantum theory above. It can be shown [92] that 2 toroidal complex structures $\tau, \tau^{\prime} \in \mathbb{C}^{+}$define conformally equivalent metrics (i.e. $g_{\tau}=\rho g_{\tau^{\prime}}$ for some $\rho>0$ ) if and only if they are related by the projective transformation ${ }^{4}$

$$
\begin{equation*}
\tau^{\prime}=\frac{\alpha \tau+\beta}{\gamma \tau+\delta} \quad \text { with } \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \alpha \delta-\beta \gamma=1 \tag{6.67}
\end{equation*}
$$

[^33]on $\mathbb{C}^{+} \rightarrow \mathbb{C}^{+}$. The transformations (6.67) generate the action of the group $S L(2, \mathbb{Z}) / \mathbb{Z}_{2}$ on $\mathbb{C}^{+}$, which is a discrete subgroup of the Möbius group $S L(2, \mathbb{C}) / \mathbb{Z}_{2}$ of linear fractional transformations of $\mathbb{C}$ wherein we take $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ in (6.67). We call this discrete group the modular or mapping class group $\Gamma_{\Sigma^{1}}$ of the Riemann surface $\Sigma^{1}$ and it consists of the discrete automorphisms of $\Sigma^{1}$ (i.e. the conformal diffeomorphisms of $\Sigma^{1}$ which aren't connected to the identity and so cannot be represented as global flows of vector fields). The Teichmüller space $\mathbb{C}^{+}$modulo this group action, i.e. the space of inequivalent complex structures on $\Sigma^{1}$, is called the moduli space $\mathcal{M}_{\Sigma^{1}} \equiv \mathbb{C}^{+} / \Gamma_{\Sigma^{1}}$ of $\Sigma^{1}$. The topological quantum theory above therefore should also reflect this sort of full topological invariance on the torus, because it is independent of the conformal factor $\varphi$ in (6.9).

Under the modular transformation (6.67), it is possible to show that, up to an overall phase, the 1-dimensional Jacobi theta functions in (6.53) transform as [90]

$$
\begin{equation*}
\Theta^{(1)}\binom{c}{d}(z \mid \tau) \rightarrow \Theta^{(1)}\binom{c^{\prime}}{d^{\prime}}\left(z^{\prime} \mid \tau^{\prime}\right)=\sqrt{\gamma \tau+\delta} \mathrm{e}^{i \pi z^{2} /(\gamma \tau+d)} \Theta^{(1)}\binom{c}{d}(z \mid \tau) \tag{6.68}
\end{equation*}
$$

where

$$
\begin{equation*}
z^{\prime}=\phi_{1}+\tau^{\prime} \phi_{2}=z /(\gamma \tau+\delta) \tag{6.69}
\end{equation*}
$$

is the new (but equivalent) complex structure defined by (6.67) and the new parameters $c^{\prime}$ and $d^{\prime}$ are given by

$$
\begin{equation*}
c^{\prime}=\delta c-\gamma d-\gamma \delta / 2 \quad, \quad d^{\prime}=\alpha d-\beta c-\alpha \beta / 2 \tag{6.70}
\end{equation*}
$$

Using (6.68) we find after some algebra that the wavefunctions (6.56) transform under the modular transformation of isomorphic complex structures as

$$
\begin{equation*}
\Psi_{p, r}\binom{c}{d}(z) \rightarrow \frac{1}{\sqrt{\gamma \tau+\delta}} \Psi_{p, r}\binom{\tilde{c}^{\prime}}{\tilde{d}^{\prime}}\left(z^{\prime}\right) \tag{6.71}
\end{equation*}
$$

with

$$
\begin{equation*}
\tilde{c}^{\prime}=\delta c-\gamma d-\pi v_{1} v_{2} \gamma \delta \quad, \quad \tilde{d}^{\prime}=\alpha d-\beta c-\pi v_{1} v_{2} \alpha \beta \tag{6.72}
\end{equation*}
$$

It follows that a set of modular invariant wavefunctions can exist only when the combination $v_{1} v_{2}$ is an even integer, in which case the invariance condition requires $c=d=0$. For $v_{1} v_{2}$ an odd integer, we can take $c, d \in\left\{0, \frac{1}{2}\right\}$, and then the holomorphic wavefunctions carry a spinor representation of the modular group as defined by (6.71). These choices of $c$ and $d$ correspond to the 4 possible choices of spin structure on the torus [51] (i.e. representations of the 2-dimensional spinor group $U(1)$ in the tangent bundle of $\Sigma^{1}$ ), and it increases the number of basis wavefunctions (6.56) by a factor of 4.

It is in this way that one may adjust the parameters $c$ and $d$ so that the wavefunctions (6.56) are modular invariants, as they should be since the topological quantum theory defined by (6.23) is independent of the phase space complex structure. We note also that these specific choices of the parameters in turn then fix the propagation time integers $N_{T}$ by (6.66), so that these topological requirements completely determine the topological quantum field theory in this case. Thus one can remove all apparent ambiguities here and obtain a situation that parallels the topological quantum theories in the simply connected cases, although now the emerging topological and group theoretical structures are far more complicated. In any case, with these appropriate choices of parameter values, the propagator (6.63) then coincides with the coherent state path integral

$$
\begin{align*}
& Z_{\Sigma^{1}}^{v}\left(k, \ell ; N_{T}\right) \sim \int_{L \Sigma^{1}} \prod_{t \in[0, T]} \frac{d z(t) d \bar{z}(t)}{(2 \pi)^{2}} \\
& \times \exp \left\{\frac{1}{2 i \operatorname{Im} \tau} \int_{0}^{N_{T} h^{-1}} d t\left[\frac{v_{1}}{2 v_{2}}(\bar{z} \dot{z}-z \dot{\bar{z}})+i h((\ell-\tau k) \bar{z}-(\ell-\bar{\tau} k) z)\right]\right\} \tag{6.73}
\end{align*}
$$

The coherent state path integral (6.73) models the quantization of some novel, unusual spin system defined by the Hamiltonians (6.20) which are associated with the quantized, non-symmetric coadjoint Lie group orbit $U(1) \times U(1)=S^{1} \times S^{1}$. Notice that, by adding appropriate constants to the 2 independent Hamiltonian generators $H^{1}, H^{2}$ in (6.20), their Poisson algebra generates the $\mathbf{u}(\mathbf{1}) \oplus \mathbf{u}(\mathbf{1})$ Lie algebra

$$
\begin{equation*}
\left\{H^{1}, H^{2}\right\}_{\omega_{\Sigma 1}}=0 \tag{6.74}
\end{equation*}
$$

of the non-simple torus group $U(1) \times U(1)$. Consequently, this abelian orbit is an unreduced one as it already is its own maximal torus. We can think of this spin system therefore as 2 independent planar spins tracing out circles. The points on this orbit are in one-to-one correspondence with the coherent state representations above of the projective clock algebra (6.46) of the discrete first homology group of the torus. The associated character formula represented by (6.73) gives path integral representations of the homology classes of $\Sigma^{1}$, in accordance with the fact that it defines a topological quantum field theory, and these localizable quantum systems are exactly solvable via both the functional integral and canonical quantization formalisms, as above. In this latter formalism, the Hilbert space of physical states is finite-dimensional and the basis states carry a non-trivial projective representation of the first homology group of the phase space, in addition to the usual representation of $H^{2}(\mathcal{M} ; \mathbb{Z})$.

We close this Section with a brief discussion about the possibilities of using functionals $\mathcal{F}\left(H_{\Sigma^{1}}\right)$ of the isometry generator (6.20) for localization as in Section 4.8. Here the arbitrariness of these functionals is not as great as it was in the simply connected cases of Chapter 5. There we required generally only that $\mathcal{F}$ be bounded from below, while in the case at hand the discussion of Section 6.3 above shows that we need in addition the requirement that $\mathcal{F}$ be formally a periodic functional of the observable (6.20). In general, this will not impose any quantization condition on the time translation $T$, as it did above. For such functionals, however, it is in general rather difficult to determine explicitly the Nicolai transform in (4.110) required for the localization (4.111). Alternatively, one can try to localize the system using (4.104) and the above description of the quantum theory as a topological one to interpret the independent Hamiltonians in (6.20) as conserved charges of some integrable dynamical system with phase space the torus. These remarks imply, for example, that one cannot equivariantly quantize a free particle or harmonic oscillator (with compactified momentum and position ranges) on
the torus, so that the localizable dynamical systems do not represent generalized harmonic oscillators as they did in the simply connected cases. The same is true of the torus height function (3.75), as anticipated. However, in these cases the periodicity of the Hamiltonian function leads to a much better defined propagator in the sense of it being a tempered distribution represented by a functional integral. Notice that this also shows explicitly, in a rather transparent way, how the Hamiltonian functions on $T^{2}$ are restricted by Kirwan's theorem.

### 6.5 Generalization to Hyperbolic Riemann Surfaces

We conclude this Chapter by indicating how the above features of equivariant localization could generalize to the case where the phase space is a hyperbolic Riemann surface [120], although our conclusions are somewhat heuristic and more care needs to be exercised in order to study these examples in more detail. Since for $h>1, \Sigma^{h}$ can be regarded as $h$ tori stuck together, its homotopy can be described by the $2 h$ loops $a_{i}, b_{i}, i=1, \ldots, h$, where each pair $a_{i}, b_{i}$ loop abound the 2 holes of the $i$-th torus in the connected sum representation of $\Sigma^{h}$. The constraint (6.6) on the fundamental homotopy generators now generalizes to

$$
\begin{equation*}
\prod_{i=1}^{h} a_{i} b_{i} a_{i}^{-1} b_{i}^{-1}=1 \tag{6.75}
\end{equation*}
$$

and so the commutator subgroup of $\pi_{1}\left(\Sigma^{h}\right)$ for $h>1$ is non-trivial and the fundamental group of a hyperbolic Riemann surface is non-abelian. It's first homology group is given by (3.82), and, using an abusive notation, we shall denote its generators as well by $a_{i}, b_{i}$, $i=1, \ldots, h$ and call them a canonical basis of homology cycles for $\Sigma^{h}$.

According to the Riemann uniformization theorem [92], there are only 3 (compact or non-compact) simply-connected Riemann surfaces - the 2 -sphere $S^{2}$, the plane $\mathbb{C}$ and the Poincaré upper half-plane $\mathcal{H}^{2}$, each equipped with their standard metrics as discussed in the last Chapter. The sphere is its own universal cover of course (being simply-connected and having a unique complex structure), while $\mathbb{C}$ is the universal cover of the torus. The
hyperbolic plane $\mathcal{H}^{2}$ is always the universal cover of a Riemann surface of genus $h \geq 2$, which is represented as

$$
\begin{equation*}
\Sigma^{h}=\mathcal{H}^{2} / F_{h} \tag{6.76}
\end{equation*}
$$

where $F_{h}=\pi_{1}\left(\Sigma^{h}\right)$ is in this context refered to as a discrete Fuchsian group. The quotient in (6.76) is by the fixed-point free bi-holomorphic action of $F_{h}$ on $\mathcal{H}^{2}$. The group of analytic automorphisms of the upper half-plane $\mathcal{H}^{2}$ is $P S L(2, \mathbb{R})=S L(2, \mathbb{R}) / \mathbb{Z}_{2}$, the group of projective linear fractional transformations as in (6.67) except that now the coefficients $\alpha, \beta, \gamma, \delta$ are taken to be real-valued. Then $\pi_{1}\left(\Sigma^{h}\right)$ is taken as a discrete subgroup of this $P S L(2, \mathbb{R})$-action on $\mathcal{H}^{2}$ and the different isomorphism classes of complex analytic structures of $\Sigma^{h}$ are essentially the different possible classes of discrete subgroups. Note that this generalizes the genus 1 situation above, where the automorphism group of $\mathbb{C}$ was the group $P S L(2, \mathbb{C})$ of global conformal transformations in 2-dimensions and $\pi_{1}\left(\Sigma^{1}\right)$ was taken to be the lattice subgroup. Indeed, it is possible to regard $\Sigma^{h}$ as a $4 h$-gon in the plane with edges identified appropriately to give the 'holes' in $\Sigma^{h}$.

It is difficult to generalize the explicit construction of the last few Sections because of the complicated, abstract fashion in (6.76) that the complex coordinatization of $\Sigma^{h}$ occurs. For the various ways of describing the Teichmüller space and Fuchsian groups of hyperbolic Riemann surfaces without the explicit introduction of local coordinates, see [62]. The Teichmüller space of $\Sigma^{h}$ can be naturally given the geometric structure of a noncompact complex manifold which is homeomorphic to $\mathbb{C}^{3 h-3}$, so that the coordinatization of $\Sigma^{h}$ is far more intricate for $h \geq 2$ because it now involves $3 h-3$ complex parameters, as opposed to just 1 as before. Nonetheless, it is still possible to deduce the unique localizable Hamiltonian system on a hyperbolic Riemann surface and deduce some general features of the ensuing topological quantum field theory just as we did above.

We choose a complex structure on $\Sigma^{h}$ for which the universal bundle projection in (6.76) is holomorphic (as for the torus), and then the metric $g_{\Sigma^{h}}$ induced on $\Sigma^{h}$ by this projection involves a globally-defined conformal factor $\varphi$ as in (6.9) and a constant
negative curvature Kähler metric (the hyperbolic Poincaré metric - c.f. Section 5.6). The condition now that the Killing vectors of this metric be globally-defined on $\Sigma^{h}$ means that they must be single-valued under windings around the canonical homology cycles $\left\{a_{\ell}, b_{\ell}\right\}_{\ell=1}^{h} \in H_{1}\left(\Sigma^{h} ; \mathbb{Z}\right)$, or equivalently that

$$
\begin{equation*}
\oint_{a_{\ell}} d V^{\mu}=\oint_{b_{\ell}} d V^{\mu}=0 \quad, \quad \ell=1, \ldots, h \tag{6.77}
\end{equation*}
$$

Using this single-valued condition and the Killing equations

$$
\begin{equation*}
g_{\Sigma^{h}}(d V, \cdot)=-i_{V} d g_{\Sigma^{h}} \tag{6.78}
\end{equation*}
$$

we can now deduce the general form of the Killing vectors of $\Sigma^{h}$. For this, we apply the Hodge decomposition theorem $[17,27]$ to the metric-dual 1-form $g(V, \cdot) \in \Lambda^{1} \Sigma^{h}$,

$$
\begin{equation*}
g(V, \cdot)=d \chi+\star d \xi+\lambda_{h} \tag{6.79}
\end{equation*}
$$

where $\chi$ and $\xi$ are $C^{\infty}$-functions on $\Sigma^{h}$ and $\lambda_{h}$ is a harmonic 1-form, i.e. a solution of the zero-mode Laplace equation for 1 -forms,

$$
\begin{equation*}
\Delta_{1} \lambda_{h} \equiv(\star d \star d+d \star d \star) \lambda_{h}=0 \tag{6.80}
\end{equation*}
$$

In the above, $\star$ denotes the Hodge duality operator and, on a general $d$-dimensional Riemannian manifold $(\mathcal{M}, g)$, it encodes the Riemannian geometry directly into the DeRham cohomology. It is defined as the map

$$
\begin{equation*}
\star: \Lambda^{k} \mathcal{M} \rightarrow \Lambda^{d-k} \mathcal{M} \tag{6.81}
\end{equation*}
$$

which is given locally by

$$
\begin{equation*}
\star \alpha=\frac{1}{(d-k)!} \sqrt{\operatorname{det} g(x)} g_{j_{1} \lambda_{1}}(x) \cdots g_{j_{d-k} \lambda_{d-k}}(x) \epsilon^{\lambda_{1} \ldots \lambda_{d-k} i_{1} \ldots i_{k}} \alpha_{i_{1} \ldots i_{k}}(x) d x^{j_{1}} \wedge \cdots \wedge d x^{j_{d-k}} \tag{6.82}
\end{equation*}
$$

and satisfies

$$
\begin{equation*}
(\star)^{2}=(-1)^{(d-1) k} \tag{6.83}
\end{equation*}
$$

on $\Lambda^{k} \mathcal{M}$. It defines an inner product $\int_{\mathcal{M}} \alpha \Lambda \star \beta$ on each vector space $\Lambda^{k} \mathcal{M}$. Using this inner product it is possible to show that a differential form $\lambda_{h}$ as above is harmonic if and only if

$$
\begin{equation*}
d \lambda_{h}=d \star \lambda_{h}=0 \tag{6.84}
\end{equation*}
$$

and the Hodge decomposition theorem (6.79) (which can be generalized to arbitrary degree differential forms in the general case) implies that the DeRham cohomology groups of $\mathcal{M}$ are spanned by a basis of harmonic forms.

The Hodge decomposition (6.79) is unique and the components involved there are explicitly given by

$$
\begin{equation*}
\chi=\frac{1}{\nabla_{\Sigma^{h}}^{2}} \star d \star g(V, \cdot) \quad, \quad \xi=\frac{1}{\nabla_{\Sigma^{h}}^{2}} \star d g(V, \cdot) \tag{6.85}
\end{equation*}
$$

where the scalar Laplacians $\nabla_{\Sigma^{h}}^{2} \equiv \star d \star d$ in (6.85) are assumed to have their zero modes removed. The 1 -form $\lambda_{h}$ in (6.79) can be written as a linear combination of basis elements of the DeRham cohomology group $H^{1}\left(\Sigma^{h} ; \mathbb{R}\right)$. According to the Poincaré-Hodge duality theorem [27], we can in particular choose an orthonormal basis of harmonic 1-forms $\left\{\alpha_{\ell}, \beta_{\ell}\right\}_{\ell=1}^{h} \in H^{1}\left(\Sigma^{h} ; \mathbb{R}\right)$ which are Poincaré-dual to the chosen canonical homology basis $\left\{a_{\ell}, b_{\ell}\right\}_{\ell=1}^{h} \in H_{1}(\mathcal{M} ; \mathbb{Z})$ above, i.e.

$$
\begin{equation*}
\oint_{a_{\ell}} \alpha_{\ell^{\prime}}=\oint_{b_{\ell}} \beta_{\ell^{\prime}}=\delta_{\ell^{\prime}} \quad, \quad \oint_{b_{\ell}} \alpha_{\ell^{\prime}}=\oint_{a_{\ell}} \beta_{\ell^{\prime}}=0 \tag{6.86}
\end{equation*}
$$

We remark here that the local parts of the decomposition (6.79) simply form the decomposition of the vector $g_{\Sigma^{h}}(V, \cdot)$ into its curl-free, longitudinal and divergence-free, transverse pieces as $\nabla_{\Sigma^{h}} \chi+\nabla_{\Sigma^{h}} \times \xi$. The harmonic part $\lambda_{h}$ accounts for the fact that this 1-form may sit in a non-trivial DeRham cohomology class of $H^{1}\left(\Sigma^{h} ; \mathbb{R}\right)$.

We can now write the general form of the isometries of $\Sigma^{h}$. As before, $\Sigma^{h}$ inherits 3 local isometries via the bundle projection in (6.76) from the maximally symmetric Poincaré upper half-plane. However, only the 2 quasi-translations on $\mathcal{H}^{2}$ become global isometries of $\Sigma^{h}$, and they can be expressed in terms of the canonical homology basis
using the above relations. This global isometry condition along with (6.77) and (6.78) imply that Hodge decomposition (6.79) of the metric-dual 1-form to the Hamiltonian vector field $V_{\Sigma^{h}}$ is simply given by its harmonic part which can be written as

$$
\begin{equation*}
g_{\Sigma^{h}}\left(V_{\Sigma^{h}}, \cdot\right)=\sum_{\ell=1}^{h}\left(V_{1}^{\ell} \alpha_{\ell}+V_{2}^{\ell} \beta_{\ell}\right) \tag{6.87}
\end{equation*}
$$

The harmonic decomposition (6.87) is the generalization of (6.18). Indeed, on the torus we can identify the canonical harmonic forms above as $\alpha=d \phi_{1} / 2 \pi$ and $\beta=d \phi_{2} / 2 \pi$. The Killing vectors dual to (6.87) generate translations along the homology cycles of $\Sigma^{h}$, and the isometry group of $\Sigma^{h}$ is $\prod_{i=1}^{2 h} U(1)$. The usual equivariance condition $\mathcal{L}_{V_{\Sigma^{h}}} \omega=0$ on the symplectic 2 -form of $\Sigma^{h}$ now becomes

$$
\begin{equation*}
d i V_{\Sigma^{h}} \omega=\sum_{\ell=1}^{h} d\left(\bar{\omega} \star\left\{V_{1}^{\ell} \alpha_{\ell}+V_{2}^{\ell} \beta_{\ell}\right\}\right)=0 \tag{6.88}
\end{equation*}
$$

where $\bar{\omega}(x)$ is the $C^{\infty}$-function on $\Sigma^{h}$ defined by $\omega_{\mu \nu}(x)=\bar{\omega}(x) \epsilon_{\mu \nu}$, and (6.88) implies that it is constant on $\Sigma^{h}$, just as in (6.19).

Integrating up the Hamiltonian equations we see therefore that the unique equivariant (Darboux) Hamiltonians have the form

$$
\begin{equation*}
H_{\Sigma^{h}}(x)=\sum_{\ell=1}^{h} \int_{C_{x}}\left(h_{1}^{\ell} \alpha_{\ell}+h_{2}^{\ell} \beta_{\ell}\right) \tag{6.89}
\end{equation*}
$$

where $h_{\mu}^{\ell}$ are real-valued constants and $C_{x} \subset \Sigma^{h}$ is a simple curve from some fixed basepoint to $x$. The Hamiltonian (6.89) is multi-valued because it depends explicitly on the particular representatives $\alpha_{\ell}, \beta_{\ell}$ of the DeRham cohomology classes in $H^{1}\left(\Sigma^{h} ; \mathbb{R}\right)=$ $\mathbb{R}^{2 h}$. As before, single-valuedness of the time-evolution operator requires that $h_{\mu}^{\ell}=$ $n_{\mu}^{\ell}$, for some $n_{\mu}^{\ell} \in \mathbb{Z}$ and $h \in \mathbb{R}$, and the propagation times are again the discrete intervals $T=N_{T} h^{-1}$. Thus the Hamiltonian (6.89) represents the windings around the non-trivial homology cycles of $\Sigma^{h}$ and the partition function defines a topological quantum field theory which again represents the homology classes of $\Sigma^{h}$ through a family of homomorphisms from $\bigoplus_{i=1}^{2 h} \mathbb{Z}$ into $U(1)^{\otimes 2 h}$. Again, the partition function path integral
should be properly defined in the homologically-invariant form (6.27) to make the usual quantities appearing in the associated action $S$ well-defined by restricting the functional integrations to homotopically equivalent loops. We note that again the general conformal factor involved in the metric $g_{\Sigma^{h}}$ obeys Riemannian restrictions from the Gauss-BonnetChern theorem and a volume constraint similar to those in Section 6.2 above. When the volume parameter is quantized as in (6.49), we expect that the Hilbert space of physical states will be $\left(v_{1} v_{2}\right)^{3 h-3}$ dimensional (one copy of the genus 1 Hilbert spaces for each of the $3 h-3$ modular degrees of freedom in this case) and the coherent state wavefunctions, which can be expressed in terms of $D=3 h-3$ dimensional Jacobi theta functions (6.53), will in addition carry a $\left(v_{2}\right)^{3 h-3}$ dimensional projective representation of the discrete first homology group of $\Sigma^{h}$ (i.e. of the equivariant localization constraint algebra). The explicit proofs of all of the above facts appear to be difficult, because of the lack of complex coordinatization for these manifolds which is required for the definition of coherent states associated with the isometry group action $\prod_{i=1}^{2 h} U(1)=\prod_{i=1}^{2 h} S^{1}$ on the non-symmetric space $\Sigma^{h}=\left(S^{1} \times S^{1}\right)^{\# h}$.

Thus the general feature of abelian equivariant localization of path integrals on multiply connected compact Riemann surfaces is that it leads to a topological quantum theory whose associated topologically invariant partition function represents the non-trivial homology classes of the phase space. The coherent states in the finite-dimensional Hilbert space also carry a multi-dimensional representation of the discrete first homology group, and the localizable Hamiltonians on these phases spaces are rather unusual and even more restricted than in the simply-connected cases. The invariant symplectic 2 -forms in these cases are non-trivial elements of $H^{2}\left(\Sigma^{h} ; \mathbb{Z}\right)=\mathbb{Z}$, as in the maximally-symmetric cases, and it is essentially the global topological features of these multiply-connected phase spaces which leads to these rather severe restrictions. The coherent state quantization of these systems shows that the path integral describes the coadjoint orbit quantization of an unusual spin system described by the Riemann surface. These spin systems are
exactly solvable both from the point of view of path integral quantization on the loop space and of canonical holomorphic quantization in the Schrödinger polarization. The integrable systems one obtains in these cases are rather trivial in appearence and are associated with abelian isometry groups acting on the phase spaces which makes them automatically integrable and topological in the usual sense of this Thesis. However, these quantum theories probe deep geometric and topological features of the phase spaces, such as their complex algebraic geometry and their homology. This is contrast with the topological quantum field theories that we found in the simply-connected cases, where at best the topological path integral could only represent the possible non-trivial cohomology classes in $H^{2}(\mathcal{M} ; \mathbb{Z})$. It is not completely clear though how these path integral representations correspond to analogs of the standard character formulas on homogeneous symplectic manifolds which are associated with semi-simple Lie groups, since, for instance, the usual Kähler structure between the Riemannian and symplectic geometries is absent in these non-symmetric cases.

## Chapter 7

## Geometrical Characteristics of the Semi-classical Expansion

In this final Chapter of this Thesis, we shall examine a different approach to the problem of localization [106]. We return to the general finite-dimensional analysis of Chapter 3 and consider a Hamiltonian system whose Hamiltonian function is a Morse function ${ }^{1}$. From this we will construct the full $\frac{1}{T}$-expansion for the classical partition function, as we outlined briefly in Section 3.3. A proper covariantization of this expansion will then allow us to determine somewhat general geometrical characteristics of dynamical systems whose partition functions localize, which in this context will be the vanishing of all terms in the perturbative loop expansion beyond 1-loop order. The possible advantages of this analysis are numerous. For instance, we can analyse the fundamental isometry condition required for equivariant localization and see more precisely what mechanism or symmetry makes the higher-order terms disappear. This could then expand the set of localizable systems beyond the ones we have already encountered that are predicted from localization theory, and at the same time probe deeper into the geometrical structures of the phase space or the whole dynamical system thus prividing richer examples of topological field theories. Indeed we shall find some noteworthy geometrical significances of when a partition function is given exactly by its semi-classical approximation. This approach to the Duistermaat-Heckman integration formula using the perturbative loopexpansion has been discussed in a different context recently in [138].

In particular, we shall find that the condition that the Hamiltonian be a Killing vector

[^34]of some globally-defined Riemannian geometry on $\mathcal{M}$ can be replaced by the weaker condition that it be only a conformal Killing vector [105], i.e. the localization metric $g$ need only be invariant under the flows of $V$ up to rescaling by some globally defined $C^{\infty}$-function $\Upsilon(x)$ on $\mathcal{M}$, so that the basic isometry condition $\mathcal{L}_{V} g=0$ is replaced by the weaker conformal symmetry requirement $\mathcal{L}_{V} g=\Upsilon g$. We shall discuss this feature in some detail in this Chapter, in particular what it implies for the general localization principle.

Recalling that the isometry condition can always be satisfied at least locally on $\mathcal{M}$, we then develop a novel geometric method for systematically constructing corrections to the Duistermaat-Heckman formula. Given that a particular system does not localize, the idea is that we can "localize" in local neighbourhoods on $\mathcal{M}$ where the Killing equation can be satisfied. The correction terms are then picked up when these open sets are patched back together on the manifold, as then there are non-trivial singular contributions to the usual 1-loop term owing to the fact that the Lie derived metric tensor cannot be defined globally in a smooth way over the entire manifold $\mathcal{M}$. Recalling from Section 3.6 that the properties of such a metric tensor are intimately related to the integrability properties of the dynamical system, we can explore the integrability problem again in a (different) geometric setting now by closely examining these correction terms. In fact, we shall see that, in 2-dimensions at least, they can be interpreted as giving a different, nontrivial representative of the DeRham cohomology class $i_{V} \omega \in H^{1}(\mathcal{M} ; \mathbb{R})$. As we shall see, this is because the correction term can be represented as an intersection number of the phase space and the 1 -form $i_{V} \omega$ has to sit in the same cohomology class as a 1-form which is the Poincaré dual of a certain set of homology 1-cycles in $\mathcal{M}$ in order for the higher-order correction terms to vanish. This provides a highly non-trivial geometric classification of the localizability of a dynamical system which is related to the homology of $\mathcal{M}$, the integrability of the dynamical system, and is moreover completely consistent with Kirwan's theorem. We shall illustrate all of these geometrical characterizations with
several explicit examples.
Unfortunately, the generalization to path integrals is not yet known, but we discuss the situation somewhat heuristically at the end of this Chapter and indicate how our analysis could prove of use. Nevertheless, given the large amount of progress that was made in quantum localization theory from relatively direct generalizations of the finitedimensional localization theorems, this is a non-trivial first step to a full analysis of corrections to path integral localization formulas (e.g. corrections to the WKB approximation), and to uncovering systematically the reasons why these approximations aren't exact. The first step in this direction was carried out in [106].

### 7.1 The Loop Expansion and the Duistermaat-Heckman Formula Revisited

Throughout this Chapter we return to the situation of Section 3.3 where the Hamiltonian $H$ is a Morse function on a (usually compact) symplectic manifold $\mathcal{M}$. For now we assume that $\partial \mathcal{M}=\emptyset$, but later we shall also consider manifolds with boundary. We now explicitly work out the full stationary phase series whose construction we briefly outlined in Section 3.3 [61]. We first expand the $C^{\infty}$-function $H$ in a neighbourhood $U_{p}$ of a given critical point $p \in \mathcal{M}_{V}$ in a Taylor series

$$
\begin{equation*}
H(x)=H(p)+\mathcal{H}(p)_{\mu \nu} x_{p}^{\mu} x_{p}^{\nu} / 2+g(x ; p) \quad, \quad x \in U_{p} \tag{7.1}
\end{equation*}
$$

where $x_{p}=x-p \in U_{p}$ are the fluctuation modes about the extrema of $H$ and $g(x ; p)$ is the Gaussian deviation of $H(x)$ in the neighbourhood $U_{p}$ (i.e. all terms in the Taylor series beyond quadratic order). The determinant of the symplectic 2 -form which appears in (3.51) is similarly expanded in $U_{p}$ as

$$
\begin{equation*}
\sqrt{\operatorname{det} \omega(x)}=\sqrt{\operatorname{det} \omega(p)}+\left.\sum_{k=1}^{\infty} \frac{1}{k!} x_{p}^{\mu_{1}} \cdots x_{p}^{\mu_{k}} \partial_{\mu_{1}} \cdots \partial_{\mu_{k}} \sqrt{\operatorname{det} \omega(x)}\right|_{x=p} \quad, \quad x \in U_{p} \tag{7.2}
\end{equation*}
$$

We substitute (7.1) and (7.2) into (3.51), expand the exponential function there in powers of the Gaussian deviation function, and then integrate by parts within each
of the neighbourhoods $U_{p}$. In this way we arrive at a series expansion of (3.51) for large- $T$ in terms of Gaussian moment integrals over the fluctuations $x_{p}$, with Gaussian weight $\mathrm{e}^{i T \mathcal{H}(p)_{\mu \nu} x_{p}^{\mu} x_{p}^{\nu}}$, associated with each open neighbourhood $U_{p}$ for $p \in \mathcal{M}_{V}$. The Gaussian moments $\left\langle x^{\mu_{1}} \cdots x^{\mu_{k}}\right\rangle$ can be found from the Gaussian integration formula (1.2) in the usual way by applying the operator $\frac{\partial}{\partial \lambda_{\mu_{1}}} \cdots \frac{\partial}{\partial \lambda_{\mu_{k}}}$ to both sides of (1.2) and then setting all the $\lambda$ 's to 0 . The odd-order moments vanish, since these integrands are odd functions, and the $2 k$-th order moment contributes a term of order $\mathcal{O}\left(1 / T^{n+k}\right)$. Rearranging terms carefully, taking into account the signature of the Hessian at each critical point, and noting that for large- $T$ the integral will localize around each of the disjoint neighbourhoods $U_{p}$, we arrive at the standard stationary-phase expansion ${ }^{2}$

$$
\begin{equation*}
Z(T)=\left(\frac{2 \pi i}{T}\right)^{n} \sum_{p \in \mathcal{M}_{V}}(-i)^{\lambda(p)} \mathrm{e}^{i T H(p)} \sum_{\ell=0}^{\infty} \frac{A_{\ell}(p)}{(-2 T)^{\ell}} \tag{7.3}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{\ell}(p)=\left.\frac{1}{\sqrt{\operatorname{det} \mathcal{H}(p)}} \sum_{j=0}^{2 \ell} \frac{(-1)^{j}}{2^{j} j!(\ell+j)!}\left(\mathcal{H}(p)^{\mu \nu} \partial_{\mu} \partial_{\nu}\right)^{\ell+j}\left(g(x ; p)^{j} \sqrt{\operatorname{det} \omega(x)}\right)\right|_{x=p} \tag{7.4}
\end{equation*}
$$

and $\mathcal{H}(x)^{\mu \nu}$ is the matrix inverse of $\mathcal{H}(x)_{\mu \nu}$.
Of course, if the stationary-phase series diverges (e.g. applying Kirwan's theorem in appropriate instances), then (7.3) is to be understood formally order by order in the $\frac{1}{T}$-expansion. Borrowing terminology from quantum field theory, we shall refer to the series (7.3) as the loop-expansion, because each of the $2 \ell+1$ terms in (7.4) can be understood from pairing fluctuation modes $x_{p}^{\mu} x_{p}^{\nu}$ (i.e. a loop) associated with each derivative operator there. Indeed, the expansion (7.3),(7.4) is just the finite-dimensional version of the perturbation expansion (for large- $T$ ) in quantum field theory. We shall call the $\mathcal{O}\left(1 / T^{n+\ell}\right)$ contributions to the series (7.3) the $(\ell+1)$-loop term.

In this Chapter we shall be interested in extracting information from the loopexpansion. In particular, we will want to focus on the $k$-loop contributions for $k>1$.

[^35]However, let us see first how to interpret more geometrically the lowest order (1-loop) contribution $A_{0}(p)$ above [8], i.e. the Duistermaat-Heckman formula (3.62). We recall from Chapter 3 that, under the usual assumptions of the Duistermaat-Heckman theorem, the Pfaffian Pfaff $d V(p)$ which appears in (3.57) was none other than the equivariant Euler characteristic class $E_{V}\left(\mathcal{N}_{p}\right)=$ Pfaff $\left.R_{V}\right|_{\mathcal{N}_{p}}$ of the normal bundle $\mathcal{N}_{p}$ in $\mathcal{M}$ of each critical point $p \in \mathcal{M}_{V}$. This defines an equivariant cohomology class in $H_{U(1)}^{2 n}(\mathcal{M})$. From (3.53) it follows that the induced circle action on $\mathcal{N}_{p}$ is through non-trivial irreducible representations and we can therefore decompose the normal bundle at $p \in \mathcal{M}_{V}$ into a direct (Whitney) sum of 2-plane bundles,

$$
\begin{equation*}
\mathcal{N}_{p}=\bigoplus_{\mu=1}^{n} N_{p}^{(\mu)} \tag{7.5}
\end{equation*}
$$

From (3.53) it then follows that the equivariant Euler class of $N_{p}^{(\mu)}$ is simply $E_{V}\left(N_{p}^{(\mu)}\right)=$ $i \lambda_{\mu}(p) / 2$. Taking into account the proper orientation of $\mathcal{N}_{p}$ induced by the Hamiltonian vector field near $x=p$ and the Liouville measure, and using the multiplicativity of the Euler class under Whitney sums of bundles [16], we find that the equivariant Euler class of the normal bundle at $p$ is

$$
\begin{equation*}
E_{V}\left(\mathcal{N}_{p}\right)=\prod_{\mu=1}^{n} E_{V}\left(N_{p}^{(\mu)}\right) \equiv e(p) \tag{7.6}
\end{equation*}
$$

which is just the weight product (3.63). Thus, for Hamiltonians that generate circle actions, the 1-loop contribution to the series (7.3) (i.e. the Duistermaat-Heckman formula in the form (3.57)) describes the equivariant cohomology of the phase space with respect to the Hamiltonian circle action on $\mathcal{M}$. The particular value of the Duistermaat-Heckman formula depends on the equivariant cohomology group $H_{U(1)}^{2 n}(\mathcal{M})$ of the manifold $\mathcal{M}$.

In the next Section, we shall attempt similar sorts of geometrical and topological characterizations of the loop expansion (7.3) with the hope of being able to interpret the terms there in the context of topological and global geometrical features of the underlying phase space. This will give us a very interesting interpretation of the symmetries responsible for localization as well as some new localization mechanisms.

### 7.2 Geometry of the Loop Expansion

Motivated by the above geometrical and topological interpretation of the lowest order contribution $A_{0}(p)$ to the loop-expansion in terms of equivariant cohomology, we shall now try to do the same for the rest of the terms in the series (7.3). Of course, given our experience now with the Duistermaat-Heckman theorem, we will remove any requirements on the flows of the Hamiltonian vector field and leave these as quite arbitrary for now. When these orbits describe tori, we already have a thorough understanding of the localization in terms of equivariant cohomology, and we shall therefore look at dynamical systems which do not necessarily obey this requirement. Thus any classification that we obtain below that is described solely by the vanishing of higher-loop contributions will for the most part be of a different geometrical nature than the situation that prevails in Duistermaat-Heckman localization. This then has the possibility of expanding the cohomological symmetries usually resposible for localization. We shall see this in a somewhat more general setting in Section 7.4.

The perturbative series (7.3), however, must be appropriately modified before we can put it to use. This is because, although the original partition function (3.51) is coordinateindependent (i.e. manifestly a topological invariant) and has a well-defined $\frac{1}{T}$-expansion, the loop-expansion (7.3) is explicitly coordinate-dependent, a result of having to pick local coordinates on $\mathcal{M}$ to carry out explicitly the Gaussian integrations in $\mathbb{R}^{2 n}$. For each order of the $\frac{1}{T}$-expansion we should have a manifestly coordinate independent quantity, i.e. a scalar. To write the contributions (7.4) in such a fashion so as to be manifestly invariant under local diffeomorphisms of $\mathcal{M}$, we have to introduce a Christoffel connection $\Gamma_{\mu \nu}^{\lambda}$ on the tangent bundle of $\mathcal{M}$ which makes the derivative operators appearing in (7.4) manifestly covariant objects, i.e. write them in terms of covariant derivatives $\nabla=d+\Gamma$. Because $d H(p)=0$ at a critical point $p \in \mathcal{M}_{V}$, the Hessian evaluated at a critical point is automatically covariant, i.e. $\nabla \nabla H(p)=\mathcal{H}(p)$. This process, which we shall call 'covariantization', will then ensure that each term (7.4) is manifestly a scalar. We note
that the Morse index of any critical point is a topological invariant in this sense.
Our first observation is that it is enough for our purposes here to restrict attention to only the 2-loop correction $A_{1}(p)$ in (7.3). To see this, we first cycle out the symplectic factors in (7.4) to get

$$
\begin{equation*}
A_{\ell}(p)=\left.A_{0}(p) \sum_{j=0}^{2 \ell} \frac{(-1)^{j}}{2^{j} j!(\ell+j)!}\left(\mathcal{H}(p)^{\mu \nu} \mathcal{D}_{\mu} \mathcal{D}_{\nu}\right)^{\ell+j} g(x ; p)^{j}\right|_{x=p} \tag{7.7}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{0}(p)=\sqrt{\frac{\operatorname{det} \omega(p)}{\operatorname{det} \mathcal{H}(p)}} \tag{7.8}
\end{equation*}
$$

is the Duistermaat-Heckman (1-loop) contribution to (7.3), and

$$
\begin{equation*}
\mathcal{D}=d+\gamma \tag{7.9}
\end{equation*}
$$

where we have introduced the one-component connection

$$
\begin{equation*}
\gamma=h_{L}^{-1} d h_{L} \tag{7.10}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{L}(x)=\sqrt{\operatorname{det} \omega(x)} \tag{7.11}
\end{equation*}
$$

is the Liouville volume density. The derivative operator $\mathcal{D}$ transforms like an abelian gauge connection under local diffeomorphisms $x \rightarrow x^{\prime}(x)$ of $\mathcal{M}$,

$$
\begin{equation*}
\mathcal{D}_{\mu}(x) \xrightarrow{\Lambda} \mathcal{D}_{\mu}^{\prime}\left(x^{\prime}\right)=\Lambda_{\mu}^{\nu}\left(x^{\prime}\right)\left[\mathcal{D}_{\nu}\left(x^{\prime}\right)+\operatorname{tr} \Lambda^{-1}\left(x^{\prime}\right) \partial_{\nu}^{\prime} \Lambda\left(x^{\prime}\right)\right] \tag{7.12}
\end{equation*}
$$

where

$$
\begin{equation*}
\Lambda^{-1}(x)=\left[\frac{\partial x^{\prime \mu}(x)}{\partial x^{\nu}}\right] \in G L(2 n, \mathbb{R}) \tag{7.13}
\end{equation*}
$$

is the induced change of basis transformation on the tangent bundle.
Since $H$ is a Morse function, we can apply the Morse lemma [92] to the correction terms (7.7), i.e. there exists a sufficiently small neighbourhood $U_{p}$ about each critical point $p$ in which the Hamiltonian looks like a "harmonic oscillator",

$$
\begin{equation*}
H(x)=H(p)-\left(x^{1}\right)^{2}-\left(x^{2}\right)^{2}-\ldots-\left(x^{\lambda(p)}\right)^{2}+\left(x^{\lambda(p)+1}\right)^{2}+\ldots+\left(x^{2 n}\right)^{2} \quad, \quad x \in U_{p} \tag{7.14}
\end{equation*}
$$

so that the critical point $p$ is at $x=0$ in this open set in $\mathcal{M}$. We shall call these harmonic coordinates, and this result simply means that the symmetric matrix $\mathcal{H}(x)$ can be diagonalized constantly in an entire neighbourhood of the critical point. Given that the quantity (7.7) must be independent of coordinates (although not manifestly), we can evaluate it in a harmonic coordinate system. Then the Gaussian deviation function $g(x ; p)$ vanishes identically in the neighbourhood $U_{p}$ and only the $j=0$ term contributes to (7.7). Then the series (7.3) is simply

$$
\begin{equation*}
Z(T)=\left.\left(\frac{2 \pi i}{T}\right)^{n} \sum_{p \in \mathcal{M}_{V}}(-i)^{\lambda(p)} \mathrm{e}^{i T H(p)} A_{0}(p)\left(\mathrm{e}^{-\frac{\mathcal{H}(p)^{\mu \nu}}{2 T} \mathcal{D}_{\mu}(x) \mathcal{D}_{\nu}(x)}\right) \cdot 1\right|_{x=p} \tag{7.15}
\end{equation*}
$$

It follows that if the 2-loop term vanishes in the entire neighbourhood of the critical point $p$ (and not just at $x=p$ ), i.e.

$$
\begin{equation*}
\mathcal{H}(p)^{\mu \nu} \mathcal{D}_{\mu}(x) \mathcal{D}_{\nu}(x) \equiv 0 \quad \text { for } \quad x \in U_{p} \tag{7.16}
\end{equation*}
$$

then, as all higher-loop terms in these coordinates can be written as derivative operators acting on the 2-loop contribution $A_{1}(p)$, all corrections to the semi-classical approximation vanish. Thus, for the sake of localization arguments, we shall examine only the 2-loop term $A_{1}(p)$, and when we require that corrections to the WKB approximation vanish, we shall require them to vanish in an entire neighbourhood of each critical point. This feature is actually anticipated, because, as we shall see, the condition (7.16) imposes some conditions that a given dynamical system must obey, and if the higher-order terms in the loop-expansion weren't related to this one in some way, then the vanishing of corrections could in principle impose an infinite set of conditions on the dynamical system. This would then greatly limit the possibilities for localization.

We now covariantize the expression (7.7) for $\ell=1$. We expand out the 3 terms there in higher-order derivatives of $H$ and the connection $\gamma$, noting that only third- and higher-order derivatives of $g(x ; p)$ when evaluated at $x=p$ are non-vanishing. After some
algebra, we arrive at

$$
\begin{align*}
A_{1}(p)= & \frac{A_{0}(p)}{2} \mathcal{H}(p)^{\mu \nu}\left\{\mathcal{D}_{\mu}(x) \gamma_{\nu}(x)-\frac{\mathcal{H}(p)^{\lambda \rho}}{4}\left(\partial_{\mu} \partial_{\lambda} \partial_{\nu} \partial_{\rho} H(x)+4 \gamma_{\mu}(x) \partial_{\lambda} \partial_{\nu} \partial_{\rho} H(x)\right.\right. \\
& \left.\left.+\frac{\mathcal{H}(p)^{\alpha \beta}}{12}\left[3 \partial_{\mu} \partial_{\lambda} \partial_{\nu} H(x) \partial_{\rho} \partial_{\alpha} \partial_{\beta} H(x)+2 \partial_{\mu} \partial_{\lambda} \partial_{\alpha} H(x) \partial_{\nu} \partial_{\rho} \partial_{\beta} H(x)\right]\right)\right\}\left.\right|_{x=p} \tag{7.17}
\end{align*}
$$

It is easily checked, after some algebra, that this expression is indeed invariant under local diffeomorphisms of $\mathcal{M}$. To manifestly covariantize it, we introduce an arbitrary torsionfree connection $\Gamma_{\mu \nu}^{\lambda}$ on the tangent bundle $T \mathcal{M}$. For now, we need not assume that $\Gamma$ is the Levi-Civita connection associated with a Riemannian metric on $\mathcal{M}$. Indeed, as the original dynamical problem is defined only in terms of a symplectic geometry, not a Riemannian geometry, the expression (7.17) should be manifestly covariant in its own right without reference to any geometry that is external to the problem. All that is required is some connection that specifies parallel transport along the fibers of the tangent bundle and allows us to extend derivatives of quantities to an entire neighbourhood, rather than just at a point, in a covariant way.

The Hessian of $H$ can be written in terms of this connection and the associated covariant derivative as

$$
\begin{equation*}
\mathcal{H}(x)_{\mu \nu}=\nabla_{\mu} \nabla_{\nu} H(x)+\Gamma_{\mu \nu}^{\lambda}(x) \partial_{\lambda} H(x) \tag{7.18}
\end{equation*}
$$

and, using $d \equiv \nabla-\Gamma$, we can write the third and fourth order derivatives appearing in (7.17) in terms of $\nabla$ and $\Gamma$ by taking derivatives of (7.18). Substituting these complicated expressions into (7.17), after a long and quite tedious calculation we arrive at a manifestly covariant expression for the 2-loop correction,

$$
\begin{align*}
A_{1}(p)= & \frac{A_{0}(p)}{8} \mathcal{H}(p)^{\mu \nu}\left\{\frac { \mathcal { H } ( p ) ^ { \lambda \rho } \mathcal { H } ( p ) ^ { \alpha \beta } } { 3 } \left[3 \nabla_{\mu} \nabla_{\nu} \nabla_{\lambda} H(x) \nabla_{\alpha} \nabla_{\beta} \nabla_{\rho} H(x)\right.\right. \\
& \left.+2 \nabla_{\mu} \nabla_{\lambda} \nabla_{\alpha} H(x) \nabla_{\nu} \nabla_{\rho} \nabla_{\beta} H(x)\right]-\mathcal{H}(p)^{\lambda \rho} \nabla_{\rho} \nabla_{\mu} \nabla_{\nu} \nabla_{\lambda} H(x)  \tag{7.19}\\
& \left.+4\left(\nabla_{\mu}+\Delta_{\mu}-\mathcal{H}(p)^{\lambda \rho} \nabla_{\rho} \nabla_{\mu} \nabla_{\lambda} H(x)\right) \Delta_{\nu}(x)+R_{\mu \nu}(\Gamma)\right\}\left.\right|_{x=p}
\end{align*}
$$

where

$$
\begin{equation*}
R_{\mu \nu}(\Gamma) \equiv R_{\mu \lambda \nu}^{\lambda}=\partial_{\nu} \Gamma_{\mu \lambda}^{\lambda}-\partial_{\lambda} \Gamma_{\mu \nu}^{\lambda}+\Gamma_{\mu \lambda}^{\alpha} \Gamma_{\alpha \nu}^{\lambda}-\Gamma_{\mu \nu}^{\alpha} \Gamma_{\alpha \lambda}^{\lambda} \tag{7.20}
\end{equation*}
$$

is the (symmetric) Ricci curvature tensor of $\Gamma$ and we have introduced the 1-form $\Delta=$ $\Delta_{\mu}(x) d x^{\mu}$ with the local components

$$
\begin{equation*}
\Delta_{\mu}(x)=\gamma_{\mu}(x)-\Gamma_{\mu \lambda}^{\lambda}(x)=\nabla_{\mu} \log h_{L}(x) \tag{7.21}
\end{equation*}
$$

It is intriguing that the covariantization of the 2-loop expression simply involves replacing ordinary derivatives $d$ with covariant ones $\nabla$, non-covariant connection terms $\gamma$ with the 1-form $\Delta$, and then the remainder terms from this process are simply determined by the curvature of the Christoffel connection $\Gamma$ which realizes the covariantization. Note that if $\Gamma$ is in addition chosen as the Levi-Civita connection compatible with $g$, i.e. $\nabla g=0$, then $\Gamma_{\mu \lambda}^{\lambda}=\partial_{\mu} \log \sqrt{\operatorname{det} g}$ and the 1-form components (7.21) become $\Delta_{\mu}=\partial_{\mu} \log \sqrt{\operatorname{det}\left(g^{-1} \cdot \omega\right)}$.

The expression (7.19) in general is extremely complicated. However, besides being manifestly independent of the choice of coordinates, (7.19) is independent of the chosen connection $\Gamma$, because by construction it simply reduces to the original connectionindependent term (7.17). We can exploit this degree of freedom by choosing a connection that simplifies the correction (7.19) to a form that is amenable to explicit analysis. To motivate a specific choice of connection, consider the following situation. Suppose that $\Gamma$ is the Levi-Civita connection associated to some globally-defined metric tensor $g$ on $\mathcal{M}$, and consider the rank $(1,1)$ tensor field

$$
\begin{equation*}
J_{\mu}^{\nu}=\sqrt{\frac{\operatorname{det} \omega}{\operatorname{det} g}} g_{\mu \lambda} \omega^{\lambda \nu} \tag{7.22}
\end{equation*}
$$

In 2-dimensions, it is easily seen that (7.22) defines a linear isomorphism $J: T \mathcal{M} \rightarrow T \mathcal{M}$ satisfying $J^{2}=\mathbf{- 1}$. In general, if such a linear transformation $J$ exists then it is called $J$ an almost complex structure of the manifold $\mathcal{M}[35,51]$. This means that there is a local basis of tangent vectors in which the only non-vanishing components of $J$ are

$$
\begin{equation*}
J_{\nu}^{\mu}=i \delta_{\nu}^{\mu} \quad, \quad J_{\bar{\nu}}^{\bar{\mu}}=-i \delta_{\bar{\nu}}^{\bar{\mu}} \tag{7.23}
\end{equation*}
$$

so that there is "almost" a separation of the tangent bundle into holomorphic and antiholomorphic components. However, an almost complex structure does not necessarily lead to a complex structure - there are certain sufficiency requirements to be met before $J$ can be used to define local complex coordinates in which the overlap transition functions can be taken to be holomorphic [17]. One such case is when $J$ is covariantly constant, $\nabla J=0$ - actually this condition only ensures that a sub-collection of subsets of the differentiable structure determine a local complex structure (but recall that any Riemann surface is a complex manifold). Again in 2-dimensions this means that then $\nabla \omega=0$ and the pair $(g, \omega)$ define a Kähler structure on $\mathcal{M}$ (again note that any 2-dimensional symplectic manifold is automatically a Kähler manifold for some metric defined by $\omega$ ).

Given these facts, suppose now that $g$ and $\omega$ define a Kähler structure on the $2 n$ dimensional manifold $\mathcal{M}$ with respect to an almost complex structure $J$, i.e. $\operatorname{det} \omega=$ $\operatorname{det} g, g$ is Hermitian with respect to $J$,

$$
\begin{equation*}
g_{\mu \nu}=J_{\mu}^{\lambda} g_{\lambda \rho} J_{\nu}^{\rho} \tag{7.24}
\end{equation*}
$$

and $\omega$ is determined from $g$ by (7.22). In the local coordinates (7.23), this means the usual Kähler conditions that we encountered before, i.e. $g_{\mu \nu}=g_{\bar{\mu} \bar{\nu}}=0, g_{\mu \bar{\nu}}=g_{\bar{\nu} \mu}^{*}$ and $\omega_{\mu \bar{\nu}}=-i g_{\mu \bar{\nu}}$. In this case, the flows of $g$ under the action of the Hamiltonian vector field V,

$$
\begin{equation*}
\left(\mathcal{L}_{V} g\right)_{\mu \nu}=g_{\mu \lambda} \nabla_{\nu} \omega^{\lambda \rho} \partial_{\rho} H+g_{\nu \lambda} \nabla_{\mu} \omega^{\lambda \rho} \partial_{\rho} H+\omega^{\lambda \rho}\left(g_{\mu \lambda} \nabla_{\nu} \nabla_{\rho} H+g_{\nu \lambda} \nabla_{\mu} \nabla_{\rho} H\right) \tag{7.25}
\end{equation*}
$$

can be written using the almost complex structure as the anti-commutator

$$
\begin{equation*}
\mathcal{L}_{V} g=[\nabla \nabla H, J]_{+} \tag{7.26}
\end{equation*}
$$

Thus if $V$ is a global Killing vector of a Kähler metric on $\mathcal{M}$, then the covariant Hessian of $H$ is also Hermitian with respect to $J$, as in (7.24). Since the Kähler metric of a Kähler manifold is essentially the unique Hermitian rank $(2,0)$ tensor, it follows that the
covariant Hessian is related to the Kähler metric by a transformation of the form

$$
\begin{equation*}
\nabla_{\mu} \nabla_{\nu} H=K_{\mu}^{\lambda} g_{\lambda \rho} K_{\nu}^{\rho} \tag{7.27}
\end{equation*}
$$

where $K$ is some non-singular $(1,1)$ tensor which commutes with $J$. In 2-dimensions, the Hermiticity conditions imply that both the Hessian and $g$ have only 1 degree of freedom and (7.27) gets replaced by the much simpler condition

$$
\begin{equation*}
\nabla \nabla H=\mathcal{G} g \tag{7.28}
\end{equation*}
$$

where $\mathcal{G}(x)$ is some globally-defined $C^{\infty}$-function on $\mathcal{M}$.
Of course from the fundamental equivariant localization principle we know that this implies the vanishing of the 2-loop correction term, i.e. the Duistermaat-Heckman theorem. We shall soon see this in a more transparent way which isn't based on the cohomological principle. Indeed, from the analysis of the last 2 Chapters we have seen that most of the localizable examples fall into these Kähler-type situations. But we what we are really interested in is the symmetry this implies between the Hessian and metric tensors. In the above sense, the Hessian essentially defines a metric on $\mathcal{M}$. This is also apparent in the correction term (7.19), where the inverse Hessians contract with the other tensorial terms to form scalars, i.e. the Hessians in that expression act just like metrics. This suggests that the non-degenerate Hessian of $H$ could be used to define a metric which is compatible with the connection $\Gamma$ used in (7.19). Of course, this in general cannot be done globally on the manifold $\mathcal{M}$, because the signature of $\mathcal{H}(x)$ varies over $\mathcal{M}$ in general, but for a $C^{\infty}$ Hamiltonian $H$ it can at least be done locally in a sufficiently small neighbourhood surrounding each critical point. For now, we concentrate on the 2-dimensional case. Then motivated by the situation above, in general we define a metric tensor $g$ that is $a b$ initio proportional to the covariant Hessian as in (7.28), for which the connection $\Gamma$ used in the covariant derivatives $\nabla$ is the Levi-Civita connection for $g$. This means that, given a Hamiltonian $H$ on $\mathcal{M}$, we try to solve the coupled non-linear
partial differential equations

$$
\begin{gather*}
\mathcal{G}(x) g_{\mu \nu}(x)=\partial_{\mu} \partial_{\nu} H(x)-\Gamma_{\mu \nu}^{\lambda}(x) \partial_{\lambda} H(x) \\
\Gamma_{\mu \nu}^{\lambda}=\frac{1}{2} g^{\lambda \rho}\left(\partial_{\mu} g_{\rho \nu}+\partial_{\nu} g_{\rho \mu}-\partial_{\rho} g_{\mu \nu}\right) \tag{7.29}
\end{gather*}
$$

consistently for $g$ (or $\Gamma$ ).
This may seem somewhat peculiar, and indeed impossible in the general case. But let us give some indication as to why it should be possible to solve (7.29) for $g$ or $\Gamma$ 'almost' all of the time. The covariant constancy condition on $g$ in (7.29) implies that

$$
\begin{equation*}
\partial_{\mu} \mathcal{G}=R_{\mu}^{\lambda} \partial_{\lambda} H=R \partial_{\mu} H / 2 \tag{7.30}
\end{equation*}
$$

where $R=g^{\mu \nu} R_{\mu \nu}$ is the scalar curvature of $g$. (7.30) follows from the defining identity for the Riemann curvature tensor,

$$
\begin{equation*}
\nabla_{\mu} \nabla_{\nu} \nabla_{\lambda} H=\nabla_{\nu} \nabla_{\mu} \nabla_{\lambda} H+R_{\lambda \mu \nu}^{\rho} \nabla_{\rho} H \tag{7.31}
\end{equation*}
$$

Given $H$, (7.30) determines $\mathcal{G}$ locally in terms of $g$. This means that the above ansatz can be written as an equation for the associated connection coefficients $\Gamma_{\mu \nu}^{\lambda}$,

$$
\begin{equation*}
\nabla_{\lambda} \nabla_{\mu} \nabla_{\nu} H=R_{\mu \nu} \nabla_{\lambda} H \tag{7.32}
\end{equation*}
$$

If these equations are going to determine a well-defined 2-dimensional metric tensor locally, then that metric will admit a local isothermal form (5.43). Recalling from Chapter 5 that this implies that the only non-vanishing components of the Ricci curvature tensor are

$$
\begin{equation*}
R_{z \bar{z}}=\partial_{z} \Gamma_{\bar{z} \bar{z}}^{\bar{z}}=\partial_{\bar{z}} \Gamma_{z z}^{z}=R_{\bar{z} \bar{z}} \tag{7.33}
\end{equation*}
$$

we find after a little bit of algebra that (7.32) is solved by the connection ${ }^{3}$

$$
\begin{equation*}
\Gamma_{z z}^{z}=\partial_{z} \log \partial_{z} H+f(z) / \partial_{z} H \tag{7.34}
\end{equation*}
$$

[^36]where $f(z)$ is an arbitrary locally-defined holomorphic function on $\mathcal{M}$. Substituting (7.34) back into (7.32), we find that in general only $f(z) \equiv 0$ is fully consistent with the set of equations (7.29). However, in general we find that the required relations in (7.33) are not satisfied. But they are satisfied for any Hamiltonian $H=H(z \bar{z})$ which is a function only of the radius of the isothermal coordinate system $(z, \bar{z})$. As all Morse functions admit a form of this sort (c.f. eq. (7.14)), this metric can be locally constructed for a 'large' class of Hamiltonian functions. We shall see various examples of this later on in this Chapter.

In the case that $H=H(z \bar{z})$, we can solve (7.29) using the identity $\Gamma_{z z}^{z}=\partial_{z} \log g_{z \bar{z}}$ to get the metric in isothermal coordinates as

$$
\begin{equation*}
g_{z \bar{z}}(z, \bar{z})=H^{\prime}(z \bar{z}) \tag{7.35}
\end{equation*}
$$

The main advantage of using the inductively-defined metric in (7.29) is that all third order derivatives of $H$ in (7.19) now vanish when evaluated at $p \in \mathcal{M}_{V}$. The fourth order derivatives contribute curvature terms according to (7.32), which are then cancelled by the curvature tensor already present in (7.19). The final result is an expression involving only the Liouville and Levi-Civita connections, which after some algebra we find can be written in the simple form

$$
\begin{equation*}
A_{1}(p)=\frac{A_{0}(p)}{\mathcal{G}} \omega^{z \bar{z}} \partial_{z} \partial_{\bar{z}}\left(g^{z \bar{z}} \omega_{z \bar{z}}\right) \tag{7.36}
\end{equation*}
$$

Therefore, requiring that this correction term vanish leads to the condition that

$$
\begin{equation*}
\partial_{z} \partial_{\bar{z}}\left(g^{z \bar{z}} \omega_{z \bar{z}}\right)=0 \tag{7.37}
\end{equation*}
$$

in some neighbourhood of the given critical point $p$, so that

$$
\begin{equation*}
\omega_{z \bar{z}}(z, \bar{z})=g_{z \bar{z}}(z, \bar{z})[f(z)+\bar{f}(\bar{z})] \tag{7.38}
\end{equation*}
$$

However, the holomorphic function $f(z)$ can really only contribute to $\omega$ here in the constant term of its Taylor expansion. This is because there is another set of special
coordinates, namely Darboux coordinates, that exist for which the function $\omega_{z \bar{z}}(z, \bar{z})$ in (7.38) must become a constant locally (in possibly some smaller neighbourhood than we are already in). We see from (7.38) that the Jacobian for the Darboux coordinate transformation would involve the inverse of the combination involving $f(z)$ there. Even if this Jacobian would turn out to be non-singular, it does not appear that $H$ will have a nice smooth form in those Darboux coordinates. For instance, if $g_{z \bar{z}}=1$, so that the Hamiltonian $H=z \bar{z}$ is expressed in local harmonic coordinates, then the transformation $z \rightarrow w(z, \bar{z})$ to Darboux coordinates has Jacobian

$$
\begin{equation*}
\frac{\partial w}{\partial z} \frac{\partial \bar{w}}{\partial \bar{z}}-\frac{\partial w}{\partial \bar{z}} \frac{\partial \bar{w}}{\partial z}=f(z)+\bar{f}(\bar{z}) \tag{7.39}
\end{equation*}
$$

If we take, for example, $f(z)=-2 z$ in (7.39), then the local diffeomorphism $w$ is $w(z, \bar{z})=$ $z-\bar{z}-z \bar{z}$ and the Hamiltonian becomes $H(w, \bar{w})=-\frac{1}{2}(w+\bar{w})$. Thus this set of coordinates leads to a degenerate Hamiltonian function, a rather undesirable feature. Thus we take $f(z)$ to be a real-valued constant, so that $\omega$ and $g$ locally define a Kähler structure.

The Lie derivative (7.25) of the metric (7.29) is now easily seen to be zero in a neighbourhood of the critical point. Conversely, if the Lie derivative of the metric in (7.29) vanishes on $\mathcal{M}$, then it induces a Kähler structure (i.e. $\nabla \omega=0$ ). This is not that surprising, given the way things have turned out. If we solve the Hamiltonian equations in the local isothermal coordinate system above, we find that the Hamiltonian vector field is just locally the rotation generator $V^{z}=i z$. Recalling from Section 3.3 the proof of the Duistermaat-Heckman theorem using solely symplectic geometry arguments, we see that this is the same sort of mechanism that occured there. The main feature there of the localization was the possibility of simultaneously choosing harmonic and Darboux coordinates. This same feature occurs similarly above, when we map onto local Darboux coordinates. The new insight gained here is the geometric manner in which this occurs - the vanishing of the loop expansion beyond leading order gives the dynamical system a local Kähler structure (see (7.16)). Whether or not this extends to a global geometry
depends on many things. First of all, a non-trivial Kähler structure on $\mathcal{M}$ exists (i.e. $[\omega] \neq 0$ ) only if all the even degree cohomology groups of $\mathcal{M}$ are non-trivial. Thus the topology of $\mathcal{M}$ restricts the possibilities for extending the above structure to a globally defined Kähler geometry on the whole of $\mathcal{M}$, and in this way the loop expansion probes the topology of $\mathcal{M}$. Secondly, the coefficient function $\mathcal{G}(x)$ in (7.28) must be so that the metric defined by that equation has a constant signature on the whole of $\mathcal{M}$. If $H$ has odd Morse indices $\lambda(p)$, then it is impossible to choose the function $\mathcal{G}$ in (7.28) such that, say, $g$ has a uniform Euclidean signature on the whole of $\mathcal{M}$. But if the correction terms above vanish, then Kirwan's theorem implies that $H$ has only even Morse indices and it may be possible to extend this geometry globally. In this way, the examination of the vanishing of the loop expansion beyond leading order gives insights into some novel geometrical structures on the phase space representing symmetries of the localization. We shall encounter similar sorts of geometric structures in the next Section. Moreover, if such a metric is globally-defined on $\mathcal{M}$, then $\mathcal{L}_{V} g=0$, and these classes of localizable systems fall into the same framework as those we studied before.

For $n>1$ degree of freedom, the equation (7.28) should be replaced by the more complicated relation (7.27) appropriate to higher dimensions. However, this relation does not kill off the Hamiltonian terms in (7.19) as nicely as it did in 2-dimensions, and it is difficult to make general statements in higher-dimensions. Nonetheless, the covariant form (7.19) of the loop-correction terms still indicates some novel properties of the ways in which localization can occur. In particular, note that the symplectic connection (7.10) is reminescent of the connection that appears when one constructs the Fubini-Study metric using the geometry of a holomorphic line bundle $L \rightarrow \mathcal{M}$ [35]. If we choose such a line bundle over $\mathcal{M}$ and view the Liouville density (7.11) as a metric in the fibers of this bundle, then from it one can construct a Kähler structure on $\mathcal{M}$ from the curvature 2 -forms of the associated connections (7.10), i.e.

$$
\begin{equation*}
\Omega=-i(\partial+\bar{\partial}) \bar{\gamma}=-i \partial \bar{\partial} \log h_{L} \tag{7.40}
\end{equation*}
$$

where we have restricted to the holomorphic and anti-holomorphic components of the connection (7.10). If $\mathcal{M}$ itself is already a Kähler manifold, then whether or not (7.40) agrees with the original Kähler 2-form will depend on where it sits in the DeRham cohomology of $\mathcal{M}$. If we further adjust the Christoffel connection $\Gamma$ so that it is related to the Liouville connection by the boundary condition $\Gamma_{\mu \lambda}^{\lambda}=\gamma_{\mu}$, i.e. $\Delta=0$, then the correction term (7.19) will involve only derivatives on the Hamiltonian, but now the vanishing of the correction term can be related to the geometry of a line bundle $L \rightarrow \mathcal{M}$. It would certainly be interesting to examine more geometric implications implied by the covariantized loop expansion by generalizing the arguments above. The key step is to produce 'Morse theory' type arguments, i.e. extract global information about a manifold from local properties of a $C^{\infty}$-function (or other differentiable structures), but for Hamiltonians with multi-critical points this is usually immediate in order for the above construction to work in each patch $U_{p}$. This is as far as we shall go with a general geometric interpretation of localization from the covariant loop expansion. The above discussion shows what deep structures one may uncover from such an analysis.

### 7.3 Conformal Symmetry and Kähler Structures

In this Section we shall show, directly from the loop-expansion, that it is possible to extend the fundamental symmetry requirement of the localization theorems we encountered earlier on $[105,106]$. In the next Section we shall put this into the context of a new generalized sort of localization principle. Let $g$ be a globally-defined metric tensor on $\mathcal{M}$, and consider its flows under the Hamiltonian vector field $V$. Instead of the usual assumption that $V$ be an infinitesimal isometry generator for $g$, we weaken this requirement and assume that instead $V$ is globally an infinitesimal generator of conformal transformations with respect to $g$, i.e.

$$
\begin{equation*}
\mathcal{L}_{V} g=\Upsilon g \tag{7.41}
\end{equation*}
$$

where $\Upsilon(x)$ is some $C^{\infty}$-function on $\mathcal{M}$. Intuitively, this means that the diffeomorphisms generated by $V$ preserves angles in the space, but not distances. The function $\Upsilon$ can be explicitly determined by contracting both sides of (7.41) with $g^{-1}$ to get

$$
\begin{equation*}
\Upsilon=\nabla_{\mu} V^{\mu} / n=\nabla_{\mu} \omega^{\mu \nu} \partial_{\nu} H / n \tag{7.42}
\end{equation*}
$$

which we note vanishes on the critical point set $\mathcal{M}_{V}$ of the Hamiltonian $H$. This implies, in particular, that either $\Upsilon \equiv 0$ almost everywhere on $\mathcal{M}$ (in which case $V$ is an isometry of $g$ ) or $\Upsilon(x)$ is a non-constant function on $\mathcal{M}$ corresponding to non-homothetic transformations (i.e. constant rescalings of $g$ are not possible under the flow of a Hamiltonian vector field). Killing vector fields in this context arise as those which are covariantly divergence-free, $\nabla_{\mu} V^{\mu}=0$.

We shall prove that with the conformal symmetry requirement (7.41), the covariant 2-loop correction vanishes and so the partition function undergoes a DuistermaatHeckman type localization, although the symmetry causing this is much different than that encountered in the earlier Chapters. For simplicity, we shall prove this for a 2dimensional symplectic manifold - in the next Chapter we shall see that it generalizes to higher dimensions. To do this, let $\Gamma$ above be the Levi-Civita connection associated with the metric $g$ which obeys (7.41) everywhere on $\mathcal{M}$. We write out the 3 components of the conformal Killing equations (7.41). Notice that, in contrast to the ordinary Killing equations, one of these will be an identity since one of the Killing equations tells us that $V$ is covariantly divergence-free with respect to $g$ (see (5.45)). After some manipulation with the other 2 equations, it is not difficult to see that the components of the Riemann moment map have the tensorial structure

$$
\begin{equation*}
\left(\mu_{V}\right)_{\mu}^{\nu}=\frac{\omega_{12}}{2 g_{12}}\left(\nabla_{1} V^{1}-\nabla_{2} V^{2}\right) g_{\mu \lambda} \omega^{\lambda \nu}+\frac{1}{2}\left(\nabla_{\lambda} V^{\lambda}\right) \delta_{\mu}^{\nu} \tag{7.43}
\end{equation*}
$$

so that

$$
\begin{equation*}
\operatorname{det} \mu_{V}=\left(\left(\nabla_{1} V^{1}-\nabla_{2} V^{2}\right) / 2 g_{12}\right)^{2} \operatorname{det} g+\left(\nabla_{\lambda} V^{\lambda}\right)^{2} / 4 \tag{7.44}
\end{equation*}
$$

and substituting these into the covariant Hamiltonian equations in the form

$$
\begin{equation*}
\nabla_{\mu} \nabla_{\nu} H=V^{\lambda} \nabla_{\nu} \omega_{\nu \lambda}+\omega_{\nu \lambda} \nabla_{\mu} V^{\lambda} \tag{7.45}
\end{equation*}
$$

we arrive at an expression for the covariant derivatives of $H$,

$$
\begin{equation*}
\nabla_{\mu} \nabla_{\nu} H=V^{\lambda} \nabla_{\mu} \omega_{\nu \lambda}-\left(\operatorname{tr} \mu_{V} / 2\right) \omega_{\mu \nu}-\Sigma \sqrt{\operatorname{det} \omega} g_{\mu \nu} \tag{7.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\Sigma=\sqrt{\frac{\operatorname{det} \mu_{V}-\left(\operatorname{tr} \mu_{V} / 2\right)^{2}}{\operatorname{det} g}} \tag{7.47}
\end{equation*}
$$

is a scalar density of weight 1.
From (7.46) the covariant derivatives of $H$ appearing in (7.19) are now easily found on the critical point set $\mathcal{M}_{V}$ to be

$$
\begin{gather*}
\nabla_{\lambda} \nabla_{\mu} \nabla_{\nu} H(p)=-\Sigma \nabla_{\mu} \sqrt{\operatorname{det} \omega} g_{\lambda \nu} \nabla_{\mu}\left(\operatorname{tr} \mu_{V} / 2\right)-\nabla_{\lambda}(\Sigma \sqrt{\operatorname{det} \omega}) g_{\mu \nu} \\
\nabla_{\lambda} \nabla_{\rho} \nabla_{\mu} \nabla_{\nu} H(p)=-\Sigma \nabla_{\rho} \nabla_{\mu} \sqrt{\operatorname{det} \omega} g_{\nu \lambda}-\nabla_{\lambda}\left(\Sigma \nabla_{\mu} \sqrt{\operatorname{det} \omega}\right) g_{\rho \nu}  \tag{7.48}\\
-\nabla_{\lambda}\left(\operatorname{tr} \mu_{V} / 2\right) \nabla_{\mu} \omega_{\rho \nu}-\nabla_{\lambda} \nabla_{\rho}(\Sigma \sqrt{\operatorname{det} \omega}) g_{\mu \nu}
\end{gather*}
$$

Furthermore, since $\nabla H(p)=0$ for $p \in \mathcal{M}_{V}$ we have

$$
\begin{equation*}
\mathcal{H}(p)^{\mu \nu} \nabla_{\lambda} \nabla_{\mu} \nabla_{\nu} H(p)=\mathcal{H}(p)^{\mu \nu} \nabla_{\mu} \nabla_{\lambda} \nabla_{\nu} H(p)=\mathcal{H}(p)^{\mu \nu} \nabla_{\mu} \nabla_{\nu} \nabla_{\rho} H(p) \tag{7.49}
\end{equation*}
$$

which using (7.48) gives

$$
\begin{equation*}
\nabla_{\mu} \sqrt{\operatorname{det} \omega}(p)=0 \quad \text { and } \quad \nabla_{\mu} \Sigma(p)=0 \tag{7.50}
\end{equation*}
$$

Note that the first condition in (7.50) is just $\Delta(p)=0$. In addition, we can use the commutation properties of covariant derivatives,

$$
\begin{equation*}
\mathcal{H}(p)^{\mu \nu} \mathcal{H}(p)^{\lambda \rho}(p)\left(\nabla_{\mu} \nabla_{\nu} \nabla_{\lambda} \nabla_{\rho} H-\nabla_{\mu} \nabla_{\lambda} \nabla_{\nu} \nabla_{\rho} H\right)(p)=\mathcal{H}(p)^{\mu \nu} R_{\mu \nu}(p) \tag{7.51}
\end{equation*}
$$

to eliminate the curvature terms in (7.19). Combining this with (7.48) and (7.50), we find at $A_{1}(p)=0$ at each critical point $p \in \mathcal{M}_{V}$. Requiring that (7.41) hold globally on $\mathcal{M}$
then ensures that the 2-loop correction vanishes in neighbourhoods of each of the critical points, and thus that the stationary-phase approximation to the partition function (3.51) is exact, as claimed.

Note that the localization here has occured rather non-trivially, i.e. the vanishing of the 2-loop correction is not an immediate consequence of (7.41), as would be the case if, say, $A_{1}(p)$ could be written as some sort of combination of the terms $\mathcal{L}_{V} g-\Upsilon g$. To explore the precise mechanism for localization here, consider the above derived identities in the case when $V$ is an isometry of $g$, i.e. $\operatorname{tr} \mu_{V}=0$. Then the Killing equation $\mathcal{L}_{V} g=0$ is equivalent to the Hermiticity condition (7.24) with

$$
\begin{equation*}
J_{\mu}^{\nu}=\left(\operatorname{det} \mu_{V}\right)^{-1 / 2}\left(\mu_{V}\right)_{\mu}^{\nu} \tag{7.52}
\end{equation*}
$$

which we find then coincides with the almost complex structure (7.22). In general, (7.52) is not covariantly conserved, but (7.50) shows that $\nabla J=0$ on the critical point set of the Hamiltonian. This gives a nice geometric interpretation to the mechanism behind the standard localization property in the above context. In that case (i.e. when $\mathcal{L}_{V} g=0$ globally on $\mathcal{M}$ ), the Riemann moment map $\mu_{V}=\nabla V$ defines a local Kähler geometry about each critical point $p \in \mathcal{M}_{V}$. If the topology of $\mathcal{M}$ allows this to be globally extended away from $\mathcal{M}_{V}$, then the Riemannian geometry so introduced induces a global Kähler structure with respect to the canonical symplectic structure of $\mathcal{M}^{4}$.

Indeed, in the Kähler case when $\nabla \omega=0$, (7.45) shows that the almost complex structure (7.22),(7.52) coincides with the covariantly constant Kähler one there and moreover that the covariant Hessian of $H$ coincides with the Kähler metric up to a proportionality term $K$ or $\mathcal{G}$ as in the last Section. Now, though, these proportionality functions are explicitly determined from (7.45) in terms of the Riemann moment map, e.g. we find

[^37]$\mathcal{G}=\sqrt{\operatorname{det} \mu_{V}}$. In particular, on a homogenous Kähler manifold, i.e. one with a constant scalar curvature $R$, we can in addition integrate up (7.30) which then determines the moment map as a linear functional of the Hamiltonian $H$,
\[

$$
\begin{equation*}
\mathcal{G}(x)=C_{0}+R H(x) / 2=\sqrt{\operatorname{det} \mu_{V}(x)} \tag{7.53}
\end{equation*}
$$

\]

This was observed for the height function on the sphere at the end of Section 5.5. Again, these Kähler structures that we find from the vanishing of the 2-loop terms is completely consistent with most of the localizable examples we have found. But the above shows the way that the vanishing of the higher-order terms in the loop expansion really lead to (at least local) Kähler structures. It is in this way that the localization of the partition function probes the topology and geometry of $\mathcal{M}$ and thus can lead to interesting topological quantum field theories.

The above comments are true for globally-defined systems with $\mathcal{L}_{V} g=0$, which as we saw in earlier Chapters almost uniquely fixed $\omega$ to be the Kähler 2-form associated with $g$. The induced Kähler structures above however do not come about completely when the general conformal Killing equation (7.41) holds, i.e. $\operatorname{tr} \mu_{V} \neq 0$, so that $V$ is a conformal Killing vector. Now, we do not get a Kähler structure on $\mathcal{M}$, because $\nabla \omega=0$ would automatically imply the vanishing of the function $\Upsilon(x)$ in (7.41). Thus this new sort of novel Hamiltonian dynamics is not associated with any others that we have encountered thus far, such as the homogenous phase spaces associated with coadjoint orbits of Lie groups. This conformal symmetry of the dynamics really represents some new classes of dynamical systems whose partition functions are given exactly by the semi-classical approximation.

This sort of dynamics is rather intriguing. In order to construct examples of systems with a non-trivial conformal symmetry, one has to look at spaces which have a Riemannian metric $g$ for which the Hamiltonian vector field $V$ is a generator of both the conformal group $\operatorname{Conf}(\mathcal{M}, g)$ and the symplectomorphism group $\operatorname{Sp}(\mathcal{M}, \omega)$. From the past few Chapters we have a relatively good idea of what the latter group looks like.

The conformal group for certain Riemannian manifolds is also well-understood [47]. For instance, the conformal group of a flat Euclidean space of dimension at least 3 is locally isomorphic to $S O(d+1,1)$, where $d$ is its dimension. The (global) conformal group of the Riemann sphere $\mathbb{C} \cup\{\infty\}$ was encountered already in Chapter 6 (albeit in a different context), namely the group $S L(2, \mathbb{C}) / \mathbb{Z}_{2} \simeq S O(3,1)$ of projective conformal transformations. In these cases, the conformal group consists of the usual isometries of the space, along with dilatations or scale transformations (e.g. translations of $r$ in $z=r \mathrm{e}^{i \theta}$ ) and the $d$ dimensional subgroup of so-called special conformal transformations. More interesting is the case of the flat complex plane $\mathbb{C}$. Here the conformal algebra is infinite-dimensional and its Lie algebra is just the classical Virasoro algebra [47]. Indeed, the conformal Killing equations in this case are just the first set of equations in (5.45) (the other one represents the divergence-free condition $\operatorname{tr} \mu_{V}=0$ ). This means that the conformal Killing vectors in this situation are the holomorphic functions $V^{z}=f(z), V^{\bar{z}}=\bar{f}(\bar{z})$. The Hamiltonian flows of these vector fields are therefore the arbitrary analytic coordinate transformations

$$
\begin{equation*}
\dot{z}=f(z) \quad, \quad \dot{\bar{z}}=\bar{f}(\bar{z}) \tag{7.54}
\end{equation*}
$$

The conformal flows (7.54) determine a very intriguing sort of dynamics for the Hamiltonian system.

Unfortunately, it appears difficult to construct examples of such conformal dynamical systems. For instance, in the simple example above, the conformal transformation $z \rightarrow$ $f(z)$ must also be a canonical transformation of some symplectic structure on $\mathcal{M}=\mathbb{C}$. A little experimentation shows that one encounters unavoidable singularities in the solutions of $\mathcal{L}_{V} \omega=0$ for $\omega$ (e.g. a $\frac{1}{z \bar{z}}$ behaviour near the origin), which then lead to highly singular Hamiltonian functions on $\mathbb{C}$ [105]. Some more sophisticated spaces need to be considered, and this construction could entail either starting from the conformal group of a given Riemannian space and finding a non-singular symplectic structure, or conversely starting from a given symplectomorphism group of a symplectic manifold and trying to construct a given conformal geometry on that space. Both approaches appear to be rather difficult.

In any case, this new feature may shed light on the localization properties of many new integrable field theories, and as well one might be able to develop other more general formalisms and interpretations for the path integral other than the usual homogenous spaces that we studied before. This indeed opens up a whole new aspect of localization theory which encompasses a much larger class of integrable systems.

### 7.4 The Extended Localization Principle

The last few Sections have dealt primarily with attempting to deduce geometric features of a localizable system from the covariant loop expansion. We found that in fact there was a very intimate connection with the possible Kähler structures that the phase space can possess, which leads to the hope that one can construct topological field theories which probe deeper, more complicated geometrical and topological characteristics of a manifold. We shall now take a rather different approach [106] to examining corrections to the Duistermaat-Heckman formula which will accomplish a number of things. First of all, in this Section we shall establish the conformal symmetry property in a very general setting [105], much in the same way that the canonical localization theorems followed from more general principles of equivariant cohomology. Then in the next Section we shall present an alternative to the conventional loop-expansion which focuses on more geometrical and topological features of the phase space of the given dynamical system. This will, among other things, put Kirwan's theorem, whose proof is based on a rather formal complex analytic argument [72], into a much clearer topological perspective. It will also give a much simpler way to compute corrections to the lowest order terms of the stationary-phase series for $H$ which does not involve having to carry out the evaluation of the cumbersome derivative expressions in (7.3) and (7.4).

Throughout this Section we will work within the rather general framework of Sections 2.4 and 2.5, and, in particular, we consider the integral $\mathcal{Z}(s)$ in (2.108). Again we assume that $\alpha$ is any equivariantly-closed differential form under the flows of an (arbitrary) vector
field $V$, but now we do not require that $\beta$ be an equivariant differential form, i.e. $\mathcal{L}_{V} \beta \neq 0$ in general. We also assume, for full generality, that the manifold $\mathcal{M}$ can have a boundary $\partial \mathcal{M}$. Then the second line of (2.109) and the identity

$$
\begin{equation*}
\mathcal{Z}(0)=\lim _{s \rightarrow \infty} \mathcal{Z}(s)-\int_{0}^{\infty} d s \frac{d}{d s} \mathcal{Z}(s) \tag{7.55}
\end{equation*}
$$

implies that the integral $\int_{\mathcal{M}} \alpha$ can be determined through the identity

$$
\begin{equation*}
\int_{\mathcal{M}} \alpha=\lim _{s \rightarrow \infty} \int_{\mathcal{M}} \alpha \mathrm{e}^{-s D_{V} \beta}+\int_{0}^{\infty} d s\left\{\oint_{\partial \mathcal{M}} \alpha \beta \mathrm{e}^{-s D_{V} \beta}+s \int_{\mathcal{M}} \alpha \beta\left(\mathcal{L}_{V} \beta\right) \mathrm{e}^{-s D_{V} \beta}\right\} \tag{7.56}
\end{equation*}
$$

There are a couple of things to be learned from the general expression (7.56), both of which can be thought of as extensions of the equivariant localization principle of Section 2.4. The first is the effect of the boundary contribution in (7.56). If we choose $\beta \in \Lambda_{V} \mathcal{M}$, then the second integral on the right-hand side of (7.56) vanishes. If we furthermore assume that the group action represented by the flows of the vector field $V$ preserves the boundary of $\mathcal{M}$ (i.e. $g \cdot \partial \mathcal{M}=\partial \mathcal{M}$ ) and that the action is free on $\partial \mathcal{M}$, then the $s$-integral of the boundary term in (7.56) can be carried out explicitly and we find

$$
\begin{equation*}
\int_{\mathcal{M}} \alpha=\lim _{s \rightarrow \infty} \int_{\mathcal{M}} \alpha \mathrm{e}^{-s D_{V} \beta}-\oint_{\partial \mathcal{M}} \frac{\beta \wedge \alpha}{D_{V} \beta} \tag{7.57}
\end{equation*}
$$

In particular, taking $\beta=i_{V} g$ to be the metric-dual 1-form of $V$ with respect to an invariant Riemannian geometry of $\mathcal{M}$, the large-s limit in (7.57) localizes to the usual Berline-Vergne localization formula (2.122) and (7.57) is therefore an extension of that localization formula to manifolds with boundary. In this context $\beta$ is the connection 1form for the induced group action on the boundary $\partial \mathcal{M}$, because as we have seen $d \beta$ is the moment map for this action. This boundary term can be determined using the Jeffrey-Kirwan-Kalkman residue that was introduced when we discussed the Witten localization formula back in Section 3.8, i.e. the coefficient of $\frac{1}{\phi}$ in the quantity $(\beta \wedge \alpha) / D_{V} \beta$, where $\phi$ is the element of the symmetric algebra $S\left(\mathbf{g}^{*}\right)$ representing the given circle action [68].

Next, assume for the moment that everything is pretty much arbitrary, except that the zero locus $\mathcal{M}_{V}$ consists of only isolated fixed points. We can then work out the
large-s limit integral in (7.56) in the same way as in Section 2.5 to arrive at the same expression (2.121), except that now the 2-form $\Omega_{V}=d \beta$ there is given quite arbitrarily as

$$
\begin{equation*}
\Omega_{V}=2 g \cdot \mu_{V}-\mathcal{L}_{V} g \tag{7.58}
\end{equation*}
$$

and the formula (7.56) becomes

$$
\begin{align*}
\int_{\mathcal{M}} \alpha & =(-2 \pi)^{n / 2} \sum_{p \in \mathcal{M}_{V}} \frac{\alpha^{(0)}(p)}{\operatorname{det} d V(p) \mid} \operatorname{Pfaff}\left(d V(p)-g(p)^{-1} \mathcal{L}_{V} g(p) / 2\right) \\
& +\int_{0}^{\infty} d s \oint_{\partial \mathcal{M}} \alpha \beta \mathrm{e}^{-s D_{V} \beta}+\int_{0}^{\infty} d s s \int_{\mathcal{M}} \alpha \beta\left(\mathcal{L}_{V} \beta\right) \mathrm{e}^{-s D_{V} \beta} \tag{7.59}
\end{align*}
$$

If we now assume that $\partial \mathcal{M}=\emptyset$ then the first integral on the right-hand side of (7.59) vanishes. If we further assume that instead of the usual Killing equation, the Riemannian metric $g$ is only conformally Lie-derived by the vector field $V$ as in (7.41), then the second integral on the right-hand side here also vanishes because then $\mathcal{L}_{V} \beta=\Upsilon \beta$ and so the integrand there involves the exterior product of the 1 -form $\beta$ with itself. Thus the integral $\int_{\mathcal{M}} \alpha$ again localizes onto the zero locus $\mathcal{M}_{V}$ of the given group action on $\mathcal{M}$. In particular, if $\Upsilon(p)=\operatorname{tr} d V(p) / n=0$ for all $p \in \mathcal{M}_{V}$ (as is the case for a Hamiltonian vector field $V$ ), then the localization formula (7.59) is identical to the Berline-Vergne localization formula (2.122).

Thus the localization properties of equivariant cohomology are even stronger than we saw before. Indeed, the result carried out in (2.109) still applies to any differential form that lives in the subset

$$
\begin{equation*}
\Lambda_{\text {conf }} \mathcal{M} \equiv\left\{\beta \in \Lambda \mathcal{M}: \mathcal{L}_{V} \beta=\Upsilon \beta \text { for some } \Upsilon \in C^{\infty}(\mathcal{M})\right\} \tag{7.60}
\end{equation*}
$$

of the exterior algebra $\Lambda \mathcal{M}$. Then the algebra of equivariant differential forms $\Lambda_{V} \mathcal{M}$ represents a very small subalgebra of the set $\Lambda_{\text {conf }} \mathcal{M}$ of differential forms which are invariant under the group action represented by $V$ up to rescaling by a $C^{\infty}$-function on $\mathcal{M}$. This construction can clearly be generalized to the case of a non-abelian group action on $\mathcal{M}$ as well, where $\mathcal{L}_{V}$ above would get replaced by $\phi^{a} \otimes \mathcal{L}_{V^{a}}$. This can be
thought of as an extended localization priniciple for equivariant cohomology. The crucial difference, however, is that it is not a cohomological symmetry that can be interpreted as the independence of an integral on the choice of equivariant cohomology class, as the differential forms $\beta \in \Lambda_{\text {conf }} \mathcal{M}$ here are not equivariant differential forms and so do not lie in the domain of some exterior derivative operator. Note this establishes quite generally the conformal symmetry arguments of the last Section. It would be interesting to see if there is some deeper sort of cohomological structure connected with this generalized conformal symmetry. Indeed, it is quite intriguing that there is a large mixture of topological (i.e. an equivariant cohomology class $[\alpha] \in H_{V}(\mathcal{M})$ ) and geometrical (i.e. conformally-invariant localization forms $\beta$ ) symmetries that are ultimately responsible for localization, and not merely just the previous equivariant cohomological symmetries. In this sense, the localization properties of equivariant cohomology are very strong ${ }^{5}$.

The fact that this conformal symmetry does not lead directly to an immediate equivariant cohomological structure is itself interesting and makes this requirement rather different than other geometrical alternatives to the Lie derivative condition $\mathcal{L}_{V} g=0$ that have been considered. For instance, in [69], Kärki and Niemi considered the alternative condition

$$
\begin{equation*}
V^{\lambda} \nabla_{\lambda} V^{\mu}=0 \tag{7.61}
\end{equation*}
$$

to the Killing equation, which means that the Hamiltonian flows are geodetic to $g$. After some algebra, it is straightforward to show that (7.61) is equivalent to [69]

$$
\begin{equation*}
D_{V}\left(K_{V} / 2+\Omega_{V}\right)=0 \tag{7.62}
\end{equation*}
$$

so that the dynamical systems $\left(\frac{1}{2} K_{V}, \Omega_{V}\right)$ and $(H, \omega)$ determine a bi-Hamiltonian structure. Moreover, in this case it is also possible to solve the equivariant Poincaré lemma

[^38][69], just as we did in Section 3.6. Thus given that $\frac{1}{2} K_{V}+\Omega_{V}$ is an equivariantly-closed differential form, the condition (7.61) has the potential of leading to possibly new localization formulas.

However, there are 2 things to note about the geometric condition (7.61). The first is its connection with a non-trivial conformal Killing equation $\mathcal{L}_{V} g=\Upsilon g$, which follows from the identity

$$
\begin{equation*}
V^{\nu} g^{\alpha \mu}\left(\mathcal{L}_{V} g\right)_{\mu \nu}=V^{\lambda} \nabla_{\lambda} V^{\alpha}+g^{\alpha \mu} \nabla_{\mu} K_{V} / 2 \tag{7.63}
\end{equation*}
$$

Contracting both sides of (7.63) with $g_{\alpha \rho} V^{\rho}$ leads to

$$
\begin{equation*}
\Upsilon K_{V}=2 g_{\mu \nu} V^{\nu} V^{\lambda} \nabla_{\lambda} V^{\mu} \tag{7.64}
\end{equation*}
$$

when (7.41) holds. This implies that if (7.61) is satisfied, then $\Upsilon \equiv 0$ away from the zeroes of $V$. Thus $\Upsilon \equiv 0$ everywhere on $\mathcal{M}$ and so the geometric condition (7.61) can only be compatible with the Killing equation, and not the inhomogeneous conformal Killing equation.

Secondly, the exact 2-form $\Omega_{V} \equiv d i_{V} g$ is degenerate on $\mathcal{M}$, because an application of the Leibniz rule and Stokes' theorem gives

$$
\begin{equation*}
n!\int_{\mathcal{M}} d^{2 n} x \sqrt{\operatorname{det} \Omega_{V}(x)}=\int_{\mathcal{M}} \Omega_{V}^{n}=\int_{\mathcal{M}} d\left(i_{V} g \wedge \Omega_{V}^{n-1}\right)=0 \tag{7.65}
\end{equation*}
$$

when $\partial \mathcal{M}=\emptyset$. Thus $\operatorname{det} \Omega_{V}(x)=0$ on some submanifold of $\mathcal{M}$, and thus the Hamiltonian system determined by $\left(\frac{1}{2} K_{V}, \Omega_{V}\right)$ is degenerate. As mentioned in Section 3.6, this isn't so crucial so long as the support of $\operatorname{det} \Omega_{V}(x)$ is a submanifold of $\mathcal{M}$ of dimension at least 2. It would certainly be interesting to investigate these geometric structures in more detail and see what localization schemes they lead to.

### 7.5 Poincaré Duality and Corrections to the Duistermaat-Heckman Formula

We now go back and look at an arbitrary dynamical system, and assume for now that the symplectic manifold $\mathcal{M}$ can have a non-empty boundary $\partial \mathcal{M}$. Given that, as always,
$\alpha=\mathrm{e}^{i T(H+\omega)} /(i T)^{n}$ is equivariantly-closed with respect to the Hamiltonian vector field $V$, we can apply the extended localization formula (7.59) to give

$$
\begin{align*}
Z(T)= & \left(\frac{2 \pi i}{T}\right)^{n} \sum_{p \in \mathcal{M}_{V}}(-i)^{\lambda(p)} \sqrt{\frac{\operatorname{det} \omega(p)}{\operatorname{det} \mathcal{H}(p)}} \mathrm{e}^{i T H(p)} \sqrt{\operatorname{det}\left(1-\mathcal{H}^{-1} \omega g^{-1} \mathcal{L}_{V} g / 2\right)(p)} \\
& +\frac{1}{(i T)^{n}} \int_{0}^{\infty} d s \oint_{\partial \mathcal{M}} \frac{\mathrm{e}^{i T H-s K_{V}}}{(n-1)!} g(V, \cdot) \wedge\left(i T \omega-s \Omega_{V}\right)^{n-1} \\
& +\frac{1}{(i T)^{n}} \int_{0}^{\infty} d s s \int_{\mathcal{M}} \frac{\mathrm{e}^{i T H-s K_{V}}}{(n-1)!} g(V, \cdot) \wedge\left(\mathcal{L}_{V} g\right)(V, \cdot) \wedge\left(i T \omega-s \Omega_{V}\right)^{n-1} \tag{7.66}
\end{align*}
$$

which holds for an arbitrary Riemannian metric $g$ on $\mathcal{M}$. Again, we see explicitly how the conformal Lie derivative condition (7.41) collapses this expression down to the Duistermaat-Heckman formula (3.62) when $\partial \mathcal{M}=\emptyset$, as we saw by more explicit means in Section 7.3 above. Note that we cannot naively carry out the $s$-integrations quite yet, because the function $K_{V}=g(V, V)$ has zeroes on $\mathcal{M}$.

The expression (7.66), although quite complicated, shows explicitly how the Lie derivative conditions make the semi-classical approximation to the partition function exact. This is in contrast to the loop-expansion we studied earlier, where the corrections to the Duistermaat-Heckman formula were not just some combinations of Lie derivatives. (7.66) therefore represents a sort of resummation of the loop-expansion that explicitly takes into account the geometric symmetries that make the 1 -loop approximation exact. We shall see soon that it is quite consistent with the results predicted from the loop-expansion, and moreover that it gives many new insights.

In particular, the formula (7.66) suggests a geometric approach to the evaluation of corrections to the Duistermaat-Heckman formula in the cases where it is known to fail. Recall that there is always locally a metric tensor on $\mathcal{M}-\mathcal{M}_{V}$ for which $V$ is a Killing vector (see the discussion at the beginning of Section 3.6). For the systems where the semi-classical approximation is not exact, there are global obstructions (usually from the topology of $\mathcal{M}$ ) in extending these locally invariant metric tensors to globally-defined
geometries on the phase space which are invariant under the full group action generated by the Hamiltonian vector field $V$ on $\mathcal{M}$, i.e. there are no globally defined single-valued Riemannian geometries on $\mathcal{M}$ for which $V$ is globally a Killing vector. This means that although the Killing equation $\mathcal{L}_{V} g=0$ can be solved for $g$ locally on patches covering the manifold, there is no way to glue the patches together to give a single-valued invariant geometry on the whole of $\mathcal{M}$ (c.f. Section 5.7). We shall now use the expression (7.66) in this sense to evaluate the corrections to the sum over critical points there, and we shall see that not only does this method encompass much more of the loop-expansion than the term-by-term analysis of the last Section, but it also characterizes the nonexactness of the Duistermaat-Formula in a much more transparent and geometric way than Kirwan's theorem. In this way we will obtain an explicit geometric picture of the failure of the Duistermaat-Heckman theorem and in addition a systematic, geometric method for approximating the integral (3.51). Furthermore, the ensuing analysis will show explicitly the reasons that for certain dynamical systems there are no globally defined Riemannian metric on the given symplectic manifold for which any given vector field with isolated zeroes is a Killing vector, and as well this will give another geometric description of the integrability properties of the given dynamical system.

The idea is to define a set of patches covering $\mathcal{M}$ in each of which we can solve the Killing equations for $g$, but for which the gluing of these patches together to give a globally defined metric tensor is highly singular. The non-triviality that occurs when these subsets are patched back together will then represent the corrections to the DuistermaatHeckman formula, and from our earlier arguments we know that this will be connected with the integrability of the Hamiltonian system. We introduce a set of preferred coordinates $x^{\prime \prime}$ for the vector field $V$ following Section 5.2. In general, this diffeomorphism can only be defined locally on patches over $\mathcal{M}$ and the failure of this coordinate transformation in producing globally-defined $C^{\infty}$-coordinates on $\mathcal{M}$ gives an analytic picture of why the Hamiltonian vector field fails to generate global isometries. Notice in particular
that these coordinates are only defined on $\mathcal{M}-\mathcal{M}_{V}$. In this way we shall see geometrically how Kirwan's theorem restricts dynamical systems whose phase spaces have non-trivial odd-degree homology and explicitly what type of flow the Hamiltonian vector field generates.

Recall that the coordinate functions $x^{\prime \prime}$ map the constant coordinate lines $\left(x_{0}^{2}, \ldots, x_{0}^{2 n}\right) \in$ $\mathbb{R}^{2 n-1}$ onto the integral curves of the isometry defined by the classical Hamilton equations of motion $\dot{x}^{\mu}(t)=V^{\mu}(x(t))$, i.e. in the coordinates $x^{\prime \prime}(x)$, the flows generated by the Hamiltonian vector field look like

$$
\begin{equation*}
x^{\prime \prime 1}(t)=x_{0}^{1}+t \quad ; \quad x^{\prime \prime \mu}(t)=x_{0}^{\mu} \quad, \quad \mu>1 \tag{7.67}
\end{equation*}
$$

In general, the coordinate transformation function will have singularities associated with the fact that there is no Riemannian metric tensor on $\mathcal{M}$ for which the Lie derivative condition $\mathcal{L}_{V} g=0$ holds. Otherwise, if these transformation functions were globally defined on $\mathcal{M}-\mathcal{M}_{V}$, then we could take the metric on $\mathcal{M}$ to be any one whose components in the $x^{\prime \prime}$-coordinates are independent of $x^{\prime \prime 1}$, and thereby solving the Killing equations directly and hence from (7.66) the WKB approximation would be exact. For a non-integrable system, there must therefore be some sort of obstructions to defining the $x^{\prime \prime}$-coordinate system globally over $\mathcal{M}$. In light of the above comments, these singularities will partition the manifold up into patches $P$, each of which is a $2 n$-dimensional contractable submanifold of $\mathcal{M}$ with boundaries $\partial P$ some other $(2 n-1)$-dimensional submanifolds of $\mathcal{M}$ induced by the constant coordinate line transformation from $\mathbb{R}^{2 n-1}$ above. By dropping some of these coordinate surfaces if necessary, we can assume that these patches induced from the singularities of the above coordinate transformation form a disjoint cover of the manifold $\mathcal{M}, \mathcal{M}=\bigsqcup_{P} P^{6}$. Then we can write the partition function as

$$
\begin{equation*}
Z(T)=\left.\sum_{P} \int_{P} \alpha\right|_{P} \tag{7.68}
\end{equation*}
$$

[^39]where as usual $\alpha$ is the equivariantly-closed differential form (3.56).
By the choice of the patches $P$, in their interior there is a well-defined (bounded) translation action generated by $V^{\prime \prime \mu}$. Since the patches $P$ are diffeomorphic to rectangles in $\mathbb{R}^{2 n}$, we can place a Euclidean metric on them,
\[

$$
\begin{equation*}
g_{P}=\mathrm{e}^{\varphi_{P}\left(x^{\prime \prime}\right)} d x_{\mu}^{\prime \prime} \otimes d x^{\mu} \tag{7.69}
\end{equation*}
$$

\]

where the conformal factor $\varphi_{P}\left(x^{\prime \prime}\right)$ is a globally-defined real-valued $C^{\infty}$-function on $P$. If we choose it so that it is independent of the coordinate $x^{\prime \prime 1}$, then the metric (7.69) satisfies the Killing equation on $P$. Thus on each patch $P$, by the given choice of coordinates, we can solve the Lie derivative constraint, even though this cannot be extended to the whole of $\mathcal{M}$. Then each integral over $P$ in (7.68) can be written using the formula (7.66), restricted to the patch $P$, to get

$$
\begin{align*}
\left.\int_{P} \alpha\right|_{P}= & \left(\frac{2 \pi i}{T}\right)^{n} \sum_{p \in \mathcal{M}_{V} \cap P}(-i)^{\lambda(p)} \sqrt{\frac{\operatorname{det} \omega(p)}{\operatorname{det} \mathcal{H}(P)}} \mathrm{e}^{i T H(p)}  \tag{7.70}\\
& +\left.\frac{1}{(i T)^{n}} \int_{0}^{\infty} d s \oint_{\partial P} \frac{\mathrm{e}^{i T H-s K_{V}}}{(n-1)!} g(V, \cdot) \wedge\left(i T \omega-s \Omega_{V}\right)^{n-1}\right|_{\partial P}
\end{align*}
$$

The first term here, when (7.70) is substituted back into (7.68), represents the lowestorder term in the semi-classical expansion of the partition function over $\mathcal{M}$, i.e. the Duistermaat-Heckman term $Z_{0}(T)$ in (3.62), while the boundary terms give the general corrections to this formula and represent the non-triviality that occurs rendering inexact the stationary-phase approximation. The result is

$$
\begin{equation*}
Z(T)=Z_{0}(T)+\delta Z(T) \tag{7.71}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta Z(T)=\left.\frac{1}{(i T)^{n}} \int_{0}^{\infty} d s \sum_{P} \oint_{\partial P} \frac{\mathrm{e}^{i T H-s K_{V}}}{(n-1)!} g(V, \cdot) \wedge\left(i T \omega-s \Omega_{V}\right)^{n-1}\right|_{\partial P} \tag{7.72}
\end{equation*}
$$

The contributions from the patch terms in (7.72) therefore represent an alternative geometric approach to the loop-expansion of the earlier Sections of this Chapter.

To evaluate the correction term $\delta Z(T)$, we recall from Section 5.2 (eqs. (5.33),(5.37)) that the coordinate functions $\chi^{\mu}(x)$ for $\mu=2, \ldots, 2 n$ are local conserved charges of the Hamiltonian system, i.e.

$$
\begin{equation*}
\left\{\chi^{\mu}, H\right\}_{\omega}=V^{\nu} \partial_{\nu} \chi^{\mu}=0 \tag{7.73}
\end{equation*}
$$

Thus we can take one of them, say $\chi^{2}$, to be a functional of the Hamiltonian, which we choose to be $x^{\prime \prime 2}(x)=\chi^{2}(x)=\sqrt{H(x)}$, where by adding an irrelevant constant to $H$ we may assume that it is a positive function on the (compact) manifold $\mathcal{M}$. Then, using the metric tensor transformation law, we find that the metric (7.69) when written back into the original (unprimed) coordinates has the form

$$
\begin{equation*}
g_{P}=\mathrm{e}^{\varphi_{P}(x)}\left(\frac{1}{\left(V^{\lambda} \partial_{\lambda} \chi^{1}\right)^{2}} \partial_{\mu} \chi^{1} \partial_{\nu} \chi^{1}+\frac{1}{4 H} \partial_{\mu} H \partial_{\nu} H+\sum_{\alpha>2} \partial_{\mu} \chi^{\alpha} \partial_{\nu} \chi^{\alpha}\right) d x^{\mu} \otimes d x^{\nu} \tag{7.74}
\end{equation*}
$$

so that the metric-dependent quantities appearing in (7.72) can be written as

$$
\begin{gather*}
g_{P}(V, \cdot)=\frac{\mathrm{e}^{\varphi_{P}(x)}}{V^{\lambda}(x) \partial_{\lambda} \chi^{1}(x)} \partial_{\mu} \chi^{1}(x) d x^{\mu} \quad,\left.\quad K_{V}(x)\right|_{P}=g_{P}(V, V)=\mathrm{e}^{\varphi_{P}(x)}  \tag{7.75}\\
\left.\Omega_{V}\right|_{P}=\frac{\mathrm{e}^{\varphi_{P}(x)}}{2\left(V^{\lambda} \partial_{\lambda} \chi^{1}\right)^{2}}\left\{\partial_{\lambda} \chi^{1}\left(\partial_{\mu} V^{\lambda} \partial_{\nu} \chi^{1}-\partial_{\nu} V^{\lambda} \partial_{\mu} \chi^{1}\right)\right. \\
\left.+V^{\lambda} \partial_{\lambda} \chi^{1}\left(\partial_{\mu} \varphi_{P} \partial_{\nu} \chi^{1}-\partial_{\nu} \varphi_{P} \partial_{\mu} \chi^{1}\right)\right\} d x^{\mu} \wedge d x^{\nu} \tag{7.76}
\end{gather*}
$$

When these expressions are substituted back into the correction term (7.72), we find that the integrands of $\delta Z(T)$ depend only on the coordinate function $\chi^{1}(x)$. This is not surprising, since the only effect of the other coordinate functions, which define local action variables of the dynamical system, is to make the effect of the partitioning of $\mathcal{M}$ into patches above non-trivial, reflecting the fact that the system is locally integrable, but not globally (otherwise, the partition function localizes).

In general, the correction term (7.72) is extremely complicated, but we recall that there is quite some freedom left in the choice of $\chi^{1}$. All that is required of this function is that it have no critical points in the given coordinate neighbourhood. We can therefore choose it appropriately so as to simplify the correction $\delta Z(T)$ somewhat. Given this
choice, in general singularities will appear from the fact that it cannot be defined globally on $\mathcal{M}$, and we can use these identifications to identify the specific regions $P$ above. The form of the function $\chi^{1}$ is at the very heart of this approach to evaluating corrections to the Duistermaat-Heckman formula. We shall see how this works in some explicit examples in the next Section. Notice that a similar phenomenon to what occured in Section 5.7 has happened here - the function $K_{V}$ in (7.75) is non-zero, as the zeroes of the vector field $V$ have been absorbed into the metric term $g_{P}(V, \cdot)$ thereby making it singular. We can therefore now carry out the explicit $s$-integral in (7.72), as the singularities on $\mathcal{M}_{V}$ are already present in the integrand there. Although this may seem to make everything hopelessly singular, we shall see that they can be regulated with special choices of the function $\chi^{1}$ thereby giving workable forms. We shall see in fact that when such divergences do occur, they are actually predicted by Kirwan's theorem which we recall dictates also when the full stationary-phase series diverges for a given function $H$.

There does not seem to be any immediate way of simplifying the patch corrections $\delta Z(T)$ above due to the complicated nature of the integrand forms. However, as usual in 2-dimensions things can be simplified rather nicely and the analysis reveals some very interesting properties of this formalism which could be generalized to higher-dimensional symplectic manifolds. To start, we notice that in 2-dimensions, if $\mathcal{M}$ is a compact manifold, then the union above over all of the patch boundaries $\partial P \subset \mathcal{M}$ will in general form a sum over 1-cycles $a_{\ell} \in H_{1}(\mathcal{M} ; \mathbb{Z})$. Next, we substitute (7.75) and (7.76) into (7.72) with $n=1$, and after working out the easy $s$-integration we find that the 2 -dimensional correction terms can be written as

$$
\begin{equation*}
\delta Z(T)=\frac{1}{i T} \sum_{\ell} \oint_{a_{\ell}} \frac{\mathrm{e}^{i T H(x)}}{V^{\lambda}(x) \partial_{\lambda} \chi^{1}(x)} \partial_{\mu} \chi^{1}(x) d x^{\mu} \tag{7.77}
\end{equation*}
$$

As for the function $\chi^{1}$, we need to choose one which is independent of the other coordinate transformation function $\chi^{2}$ to ensure that these 2 functions truly do define a (local) diffeomorphism of $\mathcal{M}$. The simplest choice, as far as the evaluation of (7.77) is concerned,
is to choose $\chi^{1}$ as the solution of the first-order linear partial differential equation

$$
\begin{equation*}
V^{1}(x) \partial_{1} \chi^{1}(x)=V^{2}(x) \partial_{2} \chi^{1}(x) \tag{7.78}
\end{equation*}
$$

With this choice of $\chi^{1}$, the functions $\chi^{1}$ and $\chi^{2}$ are independent of each other wherever $\partial_{\mu} \chi^{\nu}(x) \neq 0, \mu, \nu=1,2$, which follows from working out the Jacobian for the coordinate transformation defined by $\chi^{\mu}$ and using their defining partial differential equations above.

With this and the Hamiltonian equations $d H=-i_{V} \omega$, the correction terms (7.77) become

$$
\begin{equation*}
\delta Z(T)=-\left.\frac{1}{2 i T} \oint_{a_{\ell}} F\right|_{a_{\ell}} \tag{7.79}
\end{equation*}
$$

where we have introduced the 1-form

$$
\begin{equation*}
F=\omega_{12}(x) \mathrm{e}^{i T H(x)}\left(\frac{1}{\partial_{2} H(x)} d x^{1}-\frac{1}{\partial_{1} H(x)} d x^{2}\right) \tag{7.80}
\end{equation*}
$$

The expression (7.79) leads to a nice geometric interpretation of the corrections above to the Duistermaat-Heckman formula. To each of the homology cycles $a_{\ell} \in H_{1}(\mathcal{M} ; \mathbb{Z})$, there corresponds a cohomology class $\eta_{\ell} \in H^{1}(\mathcal{M} ; \mathbb{R})$, called their Poincaré dual [27], which has the property that it localizes integrals of 1 -forms $\alpha \in \Lambda^{1} \mathcal{M}$ to $a_{\ell}$, i.e.

$$
\begin{equation*}
\left.\oint_{a_{\ell}} \alpha\right|_{a_{\ell}}=\int_{\mathcal{M}} \alpha \wedge \eta_{\ell} \tag{7.81}
\end{equation*}
$$

Defining

$$
\begin{equation*}
\eta=\sum_{\ell} \eta_{\ell} \in H^{1}(\mathcal{M} ; \mathbb{R}) \tag{7.82}
\end{equation*}
$$

we see that the correction term (7.79) can be written as

$$
\begin{equation*}
\delta Z(T)=-\frac{1}{i T} \int_{\mathcal{M}} F \wedge \eta \tag{7.83}
\end{equation*}
$$

Noting also that the original partition function itself can be written as

$$
\begin{equation*}
Z(T)=\frac{1}{2} \int_{\mathcal{M}} F \wedge d H \tag{7.84}
\end{equation*}
$$

it then follows from $Z(T)=Z_{0}(T)+\delta Z(T)$ that

$$
\begin{equation*}
\int_{\mathcal{M}} F \wedge(i T d H+2 \eta)=-4 \pi \sum_{p \in \mathcal{M}_{V}}(-i)^{\lambda(p)} \sqrt{\frac{\operatorname{det} \omega(p)}{\operatorname{det} \mathcal{H}(p)}} e^{i T H(p)} \tag{7.85}
\end{equation*}
$$

Thus in this sense, the partition function represents intersection numbers of $\mathcal{M}$ associated to the homology cycles $a_{\ell}$.

This last equation is particularly interesting. It shows that the corrections to the Duistermaat-Heckman formula generate the Poincaré duals to the homology cycles which signify that the Hamiltonian vector field does not generate a globally well-defined group action on $\mathcal{M}$. When the correction 1-form $\eta / i T$ is added to the 1 -form $d H=-\omega(V, \cdot)$ which defines the flow of the Hamiltonian vector field on $\mathcal{M}$, the resulting 1-form is enough to render the Duistermaat-Heckman formula exact for the new "effective" partition function. This means that although the initial Hamiltonian flow $d H$ doesn't 'close enough' to satisfy the conditions required for the Duistermaat-Heckman theorem, adding the cohomological Poincaré dual to the singular homology cycles of the flow is enough to close the flows so that the partition now is given exactly by the lowest-order term $Z_{0}(T)$ of its semi-classical expansion. One now can solve for the vector field $W$ satisfying the "renormalized" Hamiltonian equations

$$
\begin{equation*}
d H+2 \eta / i T=-\omega(W, \cdot) \tag{7.86}
\end{equation*}
$$

We can consider $W$ as a "renormalization" of the Hamiltonian vector field $V$ which renders the stationary-phase series convergent and the Duistermaat-Heckman formula exact. Note that since the symplectic form $\omega$ hence defines a cohomology class in $H^{2}(\mathcal{M} ; \mathbb{R})$, this just corresponds to choosing a different representative in $H^{1}(\mathcal{M} ; \mathbb{R})$ for $\omega(V, \cdot)$ (recall $\left.\eta \in H^{1}(\mathcal{M} ; \mathbb{R})\right)$. Thus in our approach here, the corrections to the Duistermaat-Heckman formula computes (possibly) non-trivial cohomology classes of the manifold $\mathcal{M}$ and expresses geometrically what is missing from the original dynamical system that prevents its saddle-point approximation from being exact. The explicit constructions of the Poincaré
duals above are well-known [27] - one takes the embedding $\alpha_{\ell}: S^{1} \rightarrow \mathcal{M}$ of $S^{1}$ in $\mathcal{M}$ which corresponds to the loop $a_{\ell}$, and constructs its DeRham current which is the Dirac delta-function 1-form $\delta^{(1,1)}\left(x, \alpha_{\ell}(y)\right) \in \Lambda^{1} \mathcal{M}(x) \otimes \Lambda^{1} \mathcal{M}(y)$ with the property (7.81).

There is one crucial point that needs to be addressed before we turn to some explicit examples. In general we shall see that there are essentially 2 types of homology cycles that appear in the above when examining the singularities of the diffeomorphisms $\chi^{\mu}$ that prevent them from being global coordinate transformations of $\mathcal{M}$. The first type we shall call 'pure singular cycles'. These arise solely as a manifestation of the choice of equation satisfied by $\chi^{1}$. The second type shall be refered to as 'critical cycles'. These are the cycles on which at least one of the components of the Hamiltonian vector field vanish, $V^{\mu}(x)=0$ for $\mu=1$ or 2 . On these latter cycles the above integrals in $\delta Z(T)$ become highly singular and require regularization. Notice in particular that if, say, $V^{1}(x)=0$ but $V^{2}(x) \neq 0$ on some cycle $a_{\ell}$, then the equations (7.73) and (7.78) which determine the functions $\chi^{\mu}$ implies that $\partial_{2} \chi^{\mu}(x)=0$ while leaving the derivatives $\partial_{1} \chi^{\mu}(x)$ undetermined. Recall that it was precisely at these points where the Jacobian of the coordinate transformation defined by $\chi^{\mu}$ vanished.

In this case one must regulate the 1 -form $F$ defined above by letting $\partial_{1} \chi^{1}$ and $\partial_{2} \chi^{1}$ both approach zero on this cycle $a_{\ell}$ in a correlated manner so as to cancel the resulting divergence in the integrand of (7.77). Note that this regularization procedure now requires that $x^{1}$ and $x^{2}$ transform identically, particularly under rescalings, so that the tensorial properties of the differential form $F$ are unaffected by this definition. In this case, the 1-form $F$ which appears above gets replaced by the 1 -form

$$
\begin{equation*}
\left.F\right|_{a_{\ell}}=-\frac{1}{V^{2}(x)}\left(d x^{1}+d x^{2}\right) \mathrm{e}^{i T H(x)}=\frac{\omega_{12}(x)}{\partial_{1} H(x)}\left(d x^{1}+d x^{2}\right) \mathrm{e}^{i T H(x)} \tag{7.87}
\end{equation*}
$$

which follows from the general expression (7.77). This procedure for defining $F$ can be thought of as a quantum field theoretic ultraviolet regularization for the higher-loop corrections to the partition function. In general, we shall always obtain such singularities corresponding to the critical points of the Hamiltonian because, as mentioned before,
the diffeomorphism equations above become singular at the points where $V^{\mu}(x)=0$. Note that (7.87) will also diverge when the cycle $a_{\ell}$ crosses a critical point, i.e. on $a_{\ell} \cap \mathcal{M}_{V}$. Such singularities, as we shall see, will be just a geometric manifestation of Kirwan's theorem and the fact that in general the stationary-phase expansion does not converge for the given Hamitltonian system. We shall also see that in general the pure singular cycles do not contribute to the corrections, as anticipated, as they are only a manifestation of the particular coordinate system used, of which the covariant corrections should be independent. It is only the critical cycles that contribute to the corrections and mimick in some sense the sum over critical points series for the partition function.

### 7.6 Examples

In this Section we illustrate some of the formalism of this Chapter with 2 classes of explicit examples. The first class we shall consider is the height function of a Riemann surface, a set of examples which we have become well-acquainted with. These were first studied in Section 3.5, and in the case of the Riemann sphere we have little to add at this point since the height function (3.70) localizes. The only points we wish to make are that its partition function (3.71) represents the equivariant cohomology classes in ${ }^{7}$

$$
\begin{equation*}
H_{U(1)}^{2}\left(S^{2}\right)=\mathbb{Z} \oplus \mathbb{Z} \tag{7.88}
\end{equation*}
$$

Intuitively, (7.88) follows from the fact that the single Lie algebra generator $\phi \in \mathbb{R}$ and the invariant volume form of $S^{2}$ are linearly independent. Furthermore, the covariant Hessian with respect to the standard Kähler geometry of $S^{2}$ (see Section 5.5) is related to the Kähler metric $g_{S^{2}}$ by

$$
\begin{equation*}
\nabla \nabla h_{\Sigma^{0}}=2 \frac{1-z \bar{z}}{(1+z \bar{z})^{3}} d z \otimes d \bar{z}=2\left(1-h_{\Sigma^{0}}\right) g_{S^{2}} \tag{7.89}
\end{equation*}
$$

[^40]which is in agreement with the analysis of Sections 7.2 and 7.3. This shows the precise mechanism (i.e. the Hessian of $h_{\Sigma^{0}}$ generates covariantly the Kähler structure of $S^{2}$ ) that makes the loop corrections vanish.

An interesting check of the above formalisms is provided by a modified version of the height function $h_{\Sigma^{0}}$ which is the quadratic functional

$$
\begin{equation*}
h_{\Sigma^{0}}^{(2)}=h_{\Sigma^{0}}-h_{\Sigma^{0}}^{2}=(1-\cos \theta)-(1-\cos \theta)^{2}=-\frac{2 z \bar{z}}{1+z \bar{z}}-\left(\frac{2 z \bar{z}}{1+z \bar{z}}\right)^{2} \tag{7.90}
\end{equation*}
$$

which has the same critical behaviour as $h_{\Sigma^{0}}$. Now we find that the metric equations (7.29) are solved by (7.35), as anticipated, so that

$$
\begin{equation*}
\nabla \nabla h_{\Sigma^{0}}^{(2)}=\frac{2 z \bar{z}}{(1+z \bar{z})^{3}}(z \bar{z}-1) d z \otimes d \bar{z}=g_{z \bar{z}} d z \otimes d \bar{z} \tag{7.91}
\end{equation*}
$$

As (7.91) does not coincide with the standard Kähler geometry of $S^{2}$, the 1-loop approximation to the partition function in this case is not exact, as expected. However, the partition function still localizes, in the sense that it can be computed via the Gaussian integral transform

$$
\begin{equation*}
Z(T)=\int_{\mathcal{M}} d \mu_{L} \mathrm{e}^{i T\left(H-H^{2}\right)}=\int_{-\infty}^{\infty} \frac{d \phi}{\sqrt{2 \pi i}} \mathrm{e}^{-i \phi^{2} / 2} \int_{\mathcal{M}} d \mu_{L} \mathrm{e}^{i(T-2 i \sqrt{T} \phi) H} \tag{7.92}
\end{equation*}
$$

of the usual equivariant characteristic classes. Thus since (7.90) is a functional of an isometry generator (i.e. a conserved charge), it is still localizable, as anticipated from the discussions in Sections 4.7 and 4.8. This is also consistent with the formalism of Section 7.5 above. In this case, the preferred coordinates for the Hamiltonian vector field are $\theta$ and $x=\phi /(1-\cos \theta)$. Although these coordinates are singular at the poles of $S^{2}$ (i.e. the critical points of (7.90)), the correction terms $\delta Z(T)$ do not localize onto any cycles and just represent the terms in the characteristic class expansion for $Z(T)$ here. This just reflects the fact that $S^{2}$ is simply connected, and also that the geometric terms $\delta Z(T)$ detect the integrability features of a dynamical system (as (7.90) is of course an integrable Hamiltonian).

Next, we consider the height function on the torus, with the Kähler geometry in Section 6.2 adjusted so that $\varphi=0$ in (6.9) and $v=1$ in (6.35). The covariant Hessian of the Hamiltonian (3.75) in this case is

$$
\begin{align*}
\mathcal{H}\left(\phi_{1}, \phi_{2}\right)= & \operatorname{Im} \tau \cos \phi_{1} \cos \phi_{2} d \phi_{1} \otimes d \phi_{1}-2 \operatorname{Im} \tau \sin \phi_{1} \sin \phi_{2} d \phi_{1} \otimes d \phi_{2}  \tag{7.93}\\
& +\left(r_{1}+\operatorname{Im} \tau \cos \phi_{1}\right) \cos \phi_{2} d \phi_{2} \otimes d \phi_{2}
\end{align*}
$$

Clearly in the complex coordinatization used to define the Kähler structure this Hessian is not of the standard Hermitian form and the analysis used to show the exactness of the stationary phase approximation in the case of the height function on $S^{2}$ using the loop-expansion will not work here. Indeed, we do not expect that any metric on $T^{2}$ to be defined from the covariant Hessian as we did in Section 7.2, and we already know that the Duistermaat-Heckman formula is not exact for this example. This is because of the saddle-points at $\left(\phi_{1}, \phi_{2}\right)=(0, \pi)$ and $(\pi, \pi)$. The Hessian at these points will always determine an indefinite metric which is not admissible as a globally-defined geometry on the torus.

This is also apparent from examination of the connection (7.10) and its associated Fubini-Study geometry defined by (7.40). In this case $\gamma \equiv 0$ and the curvature (7.40) is trivial. The 2-form $\Omega$ does not determine the same cohomology class as the Kähler 2-form of $\Sigma^{1}$ does, so that there is not enough "mixing" of the Hessian and Liouville terms in the loop expansion to cancel out higher-order corrections. For the sphere, a little bit of algebra shows that the Fubini-Study metric coincides with the standard Kähler metric and thus the appropriate mixing is there to make the dynamics integrable. It is the lack of formation of a non-trivial Kähler structure on the torus here that makes almost all dynamical systems on it non-integrable.

Although the failure of the Duistermaat-Heckman theorem in this case can be understood in terms of the non-trivial first homology of $T^{2}$ via Kirwan's theorem, we can examine analytically the obstructions in extending the Hamiltonian vector field (3.77) to a global isometry of the standard Kähler metric (6.9) of $T^{2}$ which defines the unique

Riemannian geometry for equivariant localization on the torus. We shall find that the local translation action defined by the vector field (3.77) cannot be extended globally in a smooth way to the whole of $T^{2}$. The set of coordinates $(x, y)$ on the torus in which the components of the Hamiltonian vector field are $V^{x}=1$ and $V^{y}=0$ as prescribed before are first defined by taking $\chi^{2}\left(\phi_{1}, \phi_{2}\right)$ to be the square root of the height function (3.75) and $\chi^{1}\left(\phi_{1}, \phi_{2}\right)$ to be the $C^{\infty}$-function with non-vanishing first order derivatives which is the solution of the partial differential equation (7.78). In the case at hand (7.78) can be written as

$$
\begin{equation*}
-\left(r_{1}+\operatorname{Im} \tau \cos \phi_{1}\right) \frac{\partial \chi^{1}}{\partial \phi_{1}}=\operatorname{Im} \tau \sin \phi_{1} \cot \phi_{2} \frac{\partial \chi^{1}}{\partial \phi_{2}} \tag{7.94}
\end{equation*}
$$

which is solved by

$$
\begin{equation*}
\chi^{1}\left(\phi_{1}, \phi_{2}\right)=\log \left(r_{1}+\operatorname{Im} \tau \cos \phi_{1}\right)-\log \left(\cos \phi_{2}\right) \tag{7.95}
\end{equation*}
$$

and integrating (7.95) as in (5.37) yields the desired set of coordinates $(x, y)$. This gives

$$
\begin{gather*}
x\left(\phi_{1}, \phi_{2}\right)=\frac{1}{2 \operatorname{Im} \tau}\left[\frac{2 \operatorname{Im} \tau}{\sqrt{r_{2}|\operatorname{Re} \tau|} \sin \phi_{2}} \arctan \left(\sqrt{\frac{|\operatorname{Re} \tau|}{r_{2}}} \tan \frac{\phi_{1}}{2}\right)-\frac{\log \left(\tan \frac{\phi_{2}}{2}\right)}{\cos \phi_{1}}\right] \\
y\left(\phi_{1}, \phi_{2}\right)=\sqrt{r_{2}-\left(r_{1}+\operatorname{Im} \tau \cos \phi_{1}\right) \cos \phi_{2}} \tag{7.96}
\end{gather*}
$$

which hold provided that $\operatorname{Re} \tau \neq 0$.
In the coordinates defined by the diffeomorphism (7.96) the Hamiltonian vector field generates the local action of the group $\mathbb{R}^{1}$ of translations in $x$. Clearly, however, this diffeomorphism cannot be extended globally to the whole of $T^{2}$ because it has singularities along the coordinate circles

$$
\begin{array}{cc}
a_{1}=\left\{(\pi / 2, \phi) \in T^{2}\right\} & , \quad a_{2}=\left\{(3 \pi / 2, \phi) \in T^{2}\right\} \\
b_{1}=\left\{(\phi, 0) \in T^{2}\right\} & , \quad b_{2}=\left\{(\phi, \pi) \in T^{2}\right\} \tag{7.98}
\end{array}
$$

This means that $V_{\Sigma^{1}}$ cannot globally generate isometries of any Riemannian geometry on $T^{2}$. Although translations in the coordinate $x$ represent some unusual local symmetry
of the torus, it shows that the existence of non-trivial homology cycles on $T^{2}$ lead to singularities in the circle action of the Hamiltonian vector field on $T^{2}$. These singularities do not appear on the Riemann sphere because any closed loop on $S^{2}$ is contractable, so that the singular circles above collapse to points which can be identified with the critical points of the Hamiltonian function. In fact, as we saw in Chapter 6, the only equivariant Hamiltonians on the torus are precisely those which generate translations along the homology cycles of $T^{2}$, and so we see that the Hamiltonian (3.75) generates a circle action that is singular along those cycles which are exactly those needed for a globally equivariantly-localizable system on the torus. This is equivalent to the fact the flow generated by $V_{\Sigma^{1}}$ bifurcates at the saddle points of $h_{\Sigma^{1}}$, and the above shows analytically why there is no single-valued, globally-defined Riemannian geometry on the torus for which the height function $h_{\Sigma^{1}}$ generates isometries.

The local circle action defined by the diffeomorphism (7.96) however partitions the torus into 4 open sets $P_{i}$ which are the disjoint sets that remain when one removes the 2 canonical homology cycles discussed above. Each of these sets $P_{i}$ is diffeomorphic to an open rectangle in $\mathbb{R}^{2}$ on which the Hamiltonian vector field $V_{\Sigma^{1}}$ generates a global $\mathbb{R}^{1}$-action. Thus the above formalism implies that the corrections to the DuistermaatHeckman formula for the partition function in this case is given by (7.77) evaluated on the pure singular cycles $a_{1}$ and $a_{2}$ above, and on the critical cycles $b_{1}$ and $b_{2}$ (see the last Section). Summing the 2 contributions from the 1 -form $F$ in (7.80) along the pure homology cycles shows immediately that $\left.\oint_{a_{1}} F\right|_{a_{1}}+\left.\oint_{a_{2}} F\right|_{a_{2}}=0$, as anticipated. As for the integrals along the critical cycles, taking proper care of orientations induced by the contractable patches, we find that the contributions from $b_{1}$ and $b_{2}$ are the same and that the corrections can be written as

$$
\begin{equation*}
\delta Z_{T^{2}}(T)=-\frac{1}{i T \operatorname{Im} \tau}\left(\mathrm{e}^{i T\left(r_{2}-r_{1}\right)} \int_{0}^{\pi} d \phi \frac{\mathrm{e}^{-i T \operatorname{Im} \tau \cos \phi}}{\sin \phi}-\mathrm{e}^{i T\left(r_{2}+r_{1}\right)} \int_{0}^{\pi} d \phi \frac{\mathrm{e}^{i T \operatorname{Im} \tau \cos \phi}}{\sin \phi}\right) \tag{7.99}
\end{equation*}
$$

After a change of variables we find that the integrals in (7.99) can be expressed in terms
of the exponential integral function [50]

$$
\begin{equation*}
\operatorname{Ei}(x)=-f_{-x}^{\infty} d t \frac{\mathrm{e}^{-t}}{t} \tag{7.100}
\end{equation*}
$$

which diverges for $x \leq 0$. Here the integral denotes a Cauchy principal value integration. After some algebra we find

$$
\begin{align*}
\delta Z_{T^{2}}(T)= & -\frac{1}{i T \operatorname{Im} \tau}\left[\mathrm { e } ^ { i T ( r _ { 2 } - r _ { 1 } ) } \left\{\frac{\mathrm{e}^{i T \operatorname{Im} \tau}}{2}\left(\operatorname{Ei}(-2 i T \operatorname{Im} \tau)-\operatorname{Ei}\left(-2 i T \operatorname{Im} \tau \cos ^{2} \frac{y}{2}\right)\right)\right.\right. \\
& \left.-\frac{\mathrm{e}^{-i T \operatorname{Im} \tau}}{2}\left(\operatorname{Ei}\left(2 i T \operatorname{Im} \tau \sin ^{2} \frac{\epsilon}{2}\right)-\operatorname{Ei}(2 i T \operatorname{Im} \tau)\right)\right\} \\
& -\mathrm{e}^{i T\left(r_{2}+r_{1}\right)}\left\{\frac{\mathrm{e}^{-i T \operatorname{Im} \tau}}{2}\left(\operatorname{Ei}(2 i T \operatorname{Im} \tau)-\operatorname{Ei}\left(2 i T \operatorname{Im} \tau \cos ^{2} \frac{y}{2}\right)\right)\right. \\
& \left.\left.-\frac{\mathrm{e}^{i T \operatorname{Im} \tau}}{2}\left(\mathrm{Ei}\left(-2 i T \operatorname{Im} \tau \sin ^{2} \frac{\epsilon}{2}\right)-\operatorname{Ei}(-2 i T \operatorname{Im} \tau)\right)\right\}\right] \tag{7.101}
\end{align*}
$$

where $y=\pi-\epsilon$ and $\epsilon \rightarrow 0$ is used to regulate the divergence of the integrals in (7.99) at $\phi=0$ and $\phi=\pi$.

The correction term (7.101) tells us quite a bit. First of all, note that it is a sum of 4 terms which can be identified with the contributions from the critical points of the Hamiltonian $h_{\Sigma^{1}}$. However, these terms are resummed, since the above correction terms take into account the full loop corrections to the Duistermaat-Heckman formula. Next, the terms involving $\epsilon$ are divergent, and the overall divergence of $\delta Z_{T^{2}}(T)$ is anticipated from Kirwan's theorem, which says that the full saddle-point series for this Hamiltonian diverges. The exponential integral function can be expanded as the series [50]

$$
\begin{equation*}
\mathrm{Ei}(x)=\gamma+\log x+\sum_{n=1}^{\infty} \frac{x^{n}}{n n!} \tag{7.102}
\end{equation*}
$$

where $\gamma$ is the Euler number. Thus the divergent pieces in (7.101) can be explicitly expanded in powers of $\frac{1}{T}$, giving a much simpler way to read off the coefficients of the loop-expansion (note the enormous complexity of the series coefficients in (7.3) for this Hamiltonian - a direct signal of the messiness of its stationary-phase series). Finally,
the finite terms (those independent of the regulator $\epsilon$ ), can be evaluated for $T=-i$ and $\tau=1+i$, and we find $\delta Z_{T^{2}}=123.086$. From Section 3.5 we saw that the exact value of the partition function for this dynamical system was 2117.12 , while the DuistermaatHeckman formula gave $Z_{0}=1849.327$. Thus $Z_{0}+\delta Z_{T^{2}}=1972.41$, which is a better approximation to the partition function than the Duistermat-Heckman formula. Of course, given the large divergence of the stationary phase series, we do not expect that the finite contributions in (7.101) will give the exact result for the partition function, but we certainly do get much closer. As the function $\chi^{1}$ which generates the set of preferred coordinates is by no means unique, perhaps a refined definition of it could lead to a better approximation $Z_{0}+\delta Z$. Then, however, we lose a lot of the geometrical interpretation of the corrections that we gave in the last Section.

The second set of examples we consider here are the potential problems (5.165) defined on the plane $\mathbb{R}^{2}$, where $U(q)$ is a $C^{\infty}$ potential which is a non-degenerate function. In this case the equation (7.78) becomes

$$
\begin{equation*}
p \frac{\partial \chi^{1}}{\partial q}=-U^{\prime}(q) \frac{\partial \chi^{1}}{\partial p} \tag{7.103}
\end{equation*}
$$

which is solved by

$$
\begin{equation*}
\chi^{1}(q, p)=p^{2} / 2-U(q) \tag{7.104}
\end{equation*}
$$

Then proceeding as above the local coordinates $(\bar{x}, \bar{y})$ in which the Hamiltonian vector field generates translations is

$$
\begin{equation*}
\bar{x}(q, p)=\frac{1}{p U^{\prime}(q)}\left(q U^{\prime}(q)-p^{2}\right) \quad, \quad \bar{y}(q, p)=\sqrt{\frac{p^{2}}{2}+U(q)} \tag{7.105}
\end{equation*}
$$

Thus here there are only critical 'cycles' given by the infinite lines

$$
\begin{equation*}
P=\left\{(0, q) \in \mathbb{R}^{2}\right\} \quad, \quad \mathcal{U}_{i}=\left\{\left(p, q_{i}\right) \in \mathbb{R}^{2}\right\} \tag{7.106}
\end{equation*}
$$

where $q_{i}$ are the extrema of the potential $U(q)$.

Since for the Darboux Hamiltonian (5.165), $V^{p}$ and $V^{q}$ vanish on the 'cycles' $P$ and $\mathcal{U}_{i}$ respectively, we must use the renormalized version of (7.80), namely (7.87). Combining (7.87) with (7.79) we find, for a symmetric potential, that the corrections are

$$
\begin{equation*}
\delta Z_{\mathbf{R}^{2}}(T)=-\frac{1}{i T}\left\{\int_{0}^{\infty} d q \frac{\mathrm{e}^{i T U(q)}}{U^{\prime}(q)}-\left(\sum_{q_{i}} \mathrm{e}^{i T U\left(q_{i}\right)}\right) \int_{0}^{\infty} d p \frac{\mathrm{e}^{i T p^{2} / 2}}{p}\right\} \tag{7.107}
\end{equation*}
$$

and we note the manner in which the divergences are cancelled here. From this we immediately see that for the harmonic oscillator potential $U(q)=a q^{2}$, the corrections (7.107) vanish (note that the integration measures in (7.107) contain implicit factors of $\omega_{12}$ that maintain covariance). Similarly, it is easily verified, by a simple change of variables, that for a potential of the form $U(q)=a q+b q^{2}$ these correction terms vanish, again as expected. Finally, for a quartic potential $U(q)=\frac{q^{2}}{2}+\frac{q^{4}}{4}$, a numerical integration of (7.107) for $T=i$ gives $\delta Z_{\mathbf{R}^{2}}=-0.538$ and the Duistermaat-Heckman formula yields $Z_{0}=2 \pi$. A numerical integration of the original partition function gives $Z=4.851$, which differs from the value $Z_{0}+\delta Z_{\mathbf{R}^{2}}=5.745$. The corrections do not give the exact value here, but again at least they are a better approximation than the Duistermaat-Heckman formula. Of course, here the expression (7.107) is rather formal, since our derivations of the last Section mostly assumed a compact phase space. Again, a refinement of the preferred coordinates could lead to a better approximation. The method of the last Section has therefore "stripped" off any potentially divergent contributions to the loopexpansion but at the same time approximated the partition function in a much better way. Nevertheless, these last few examples illustrate the applicability and the complete consistency of the geometric approach of the last Section to the saddle-point expansion. Indeed, we see that it reproduces the precise features of the loop-expansion but avoids many of the cumbersome calculations in evaluating (7.3).

We would next like to check if, following the analysis of Section 5.7, if there are any conformally-invariant geometries for this dynamical system when the potential $U(q) \geq 0$ is bounded from below. In the harmonic-polar coordinates (5.166), the conformal Killing equations (7.41) can be determined by setting the right-hand sides of the Killing equations
(5.168) equal to instead $\left(\nabla_{\theta} V^{\theta}\right) g_{\mu \nu}=\left(\partial_{\theta} V^{\theta}+\Gamma_{\theta \theta}^{\theta} V^{\theta}\right) g_{\mu \nu}$. After some algebra, we find that they generate the 2 equations

$$
\begin{align*}
\partial_{\theta} \log \left(\frac{\left(V^{\theta}\right)^{2} g_{\theta \theta}}{g_{r r}}\right) & =2 \frac{g_{r \theta}}{g_{r r}} \partial_{r} \log V^{\theta}  \tag{7.108}\\
\partial_{\theta} \log \left(V^{\theta} \frac{g_{\theta \theta}}{g_{r \theta}}\right) & =\frac{g_{\theta \theta}}{g_{r \theta}} \partial_{r} \log V^{\theta} \tag{7.109}
\end{align*}
$$

(7.109) can be formally solved as

$$
\begin{equation*}
g_{r \theta}=-V^{\theta} g_{\theta \theta} \int_{\theta_{0}}^{\theta} d \theta^{\prime} \partial_{r} V^{\theta^{\prime}}+f(r) \tag{7.110}
\end{equation*}
$$

from which we see that again single-valuedness $g_{r \theta}(r, \theta+2 \pi)=g_{r \theta}(r, \theta)$ holds only when (5.171) is true, i.e. when $U(q)$ is the harmonic oscillator potential with $V^{\theta}=1$. Even for the harmonic oscillator, the equations (7.108) and (7.109) only seem to admit radiallysymmetric solutions $g_{\mu \nu}=g_{\mu \nu}(r)$ so that $V^{\theta}=1$ is a global isometry of $g$. Thus, even though we lose the third equation in (5.168) which established the results of Section 5.7 using the Killing equations, we still arrive at the conclusion that there are no singlevalued metric tensors obeying the conformal Lie derivative requirement for essentially all potentials which are bounded from below (and the harmonic oscillator only seems to generate isometries). Thus the conformal symmetry requirement in the case at hand does not lead to any new localizable systems.

Finally, we examine what can be learned in these cases from the vanishing of the 2-loop correction (7.16) in harmonic coordinates. In these coordinates, the connection 1-form (7.10) has components

$$
\begin{equation*}
\gamma_{p}=0 \quad, \quad \gamma_{y}=\frac{d q}{d y} \tag{7.111}
\end{equation*}
$$

and the condition (7.16) reads

$$
\begin{equation*}
\frac{d}{d y} \gamma_{y}=-\gamma_{y}^{2} \tag{7.112}
\end{equation*}
$$

There are 2 solutions to (7.112). Either $\gamma_{y}=0$, in which case $U(q)$ is the harmonic oscillator potential, or $\gamma_{y}=(y+a)^{-1}$, where $a$ is an integration constant. This latter
solution, however, yields $q(y)=C_{1} y^{2}+a y+C_{0}$, which gives a potential $U(q)$ which is not globally defined as a $C^{\infty}$-function on $\mathbb{R}^{2}$. Thus the only potential which is bounded from below that leads to a localizable partition function is that of the simple-harmonic oscillator. This example illustrates how the deep geometric analyses of this Chapter serve of use in examining the localizability properties of dynamical systems. As for these potential problems, it could prove of use in examining the localization features of other more complicated integrable systems [49].

### 7.7 Generalizations to Path Integrals

The generalization of the loop expansion to functional integrals is not yet known in the literature, although some formal suggestive techniques for carrying out the full semiclassical expansion can be found in [76] and [116]. It would be of utmost interest to carry out an analysis along the lines of this Chapter for path integrals for several reasons. There the appropriate loop space expansion should again be covariantized, but this time the functional result need not be fully independent of the loop space coordinates. This is because the quantum corrections could cause anomalies for many of the symmetries of the classical theory (i.e. of the classical partition function). It would be interesting to know if the extended localization principle could be generalized to the loop space. There the quite large algebra $\Lambda_{\text {conf }} \mathcal{M}$, and hence the far more numerous possibilities for localization, would have to represent some new sort of symmetry of the path integral. As this symmetry in the finite-dimensional case is not represented by a nilpotent operator, such as an exterior derivative, one would need some sort of generalized supersymmetry arguments to establish the localization with these sorts of symmetries.

Much of the general analysis we have carried out throughout this Thesis has been restricted to 2 -dimensional phase spaces. In order to establish to what extent the localization formulas for path integrals are trustworthy and serve as reliable calculational
tools, other (higher-dimensional) examples need to be worked out. The isometric analysis that we have carried out does not straightforwardly generalize to higher-dimensional symplectic manifolds, because there a Riemannian metric tensor can have more than 1 degree of freedom (in addition to its possible modular degrees of freedom), and the structure of the isometry groups becomes rather involved as well (as they can then have up to $n(2 n+1)$ generators in general and $\mathcal{M}$ can contain several smaller maximallysymmetric subspaces). Another path that is of interest to explore is when, instead of circular actions, one considers the Poisson action of some non-abelian Lie group acting on the phase space. Then the non-abelian generalizations of the equivariant localization formulas, discussed a bit in Sections 3.8 and 4.8, might lead to richer structures in the quantum representations discussed earlier and one might then obtain intriguing path integral representations of the groups involved.

In any case, we have shown that the equivariant localization formalism is an excellent, conceptual geometric arena for studies of supersymmetric and topological field theories, and more generally of (quantum) integrability. Given that the Hamiltonians in an integrable hierarchy are functionals of action variables alone [87], the equivariant localization formalism might yield a geometric characterization of quantum integrability, and perhaps some deeper connection between quantum-integrable bosonic theories and supersymmetric quantum field theories. This is particularly interesting from the point of view of examining corrections to the localization formulas, which in this Chapter we have seen reflect global properties of the theory. This would be of particular interest to analyse more closely, as it could then lead to a unified description of localization in the symplectic loop space, the supersymmetric loop space and in topological quantum field theory.

Quite generally though, one also has to keep in mind that the loop space localization formulas are rather formal. We have overlooked several formal functional aspects, such as difficulties associated with the definition of the path integral measure. There may be
anomalies associated with the argument in Section 4.3 that the path integral is independent of the limiting parameter $\lambda \in \mathbb{R}$, for instance the supersymmetry may be broken in the quantum theory (e.g. by a scale anomaly in the rescaling of the phase space metric $g \rightarrow \lambda \cdot g$ ). The same sort of anomalies could also break the large conformal symmetry we have found for the classical theory above, although it doesn't look like they would have anything to do with supersymmetry breaking. However, even if the localization formulas are not correct as they stand, it would then be interesting to uncover the reasons for that. This could then provide one with a systematic geometric method for analysing corrections to the WKB approximation.

The ideas in this Chapter are a small step forward in this direction. In particular, it would be interesting to generalize the construction of Section 7.5, as this is the one that is intimately connected to the integrability features of the dynamical system. The Poincaré duality interpretation there is one possible way that the construction could generalize to path integrals. For path integrals, we would expect the feature of an invariant metric tensor that cannot be extended globally to manifest itself as a local supersymmetry of the theory which is dynamically broken globally on the loop space. This has been discussed recently by Niemi and Palo [99] in the context of the supersymmetric non-linear sigmamodel. Another place where the metric could enter into a brakdown of the localization formulas is when the localization 1-form $\psi \sim i_{W} g$ does not lead to a homotopically-trivial element under the supersymmetry transformation described by $Q_{S}$. Then additional input into the localization formalism should be required on a topologically non-trivial phase space to ensure that $Q_{S} \psi$ indeed does reside in the trivial homotopy class. These inputs could follow from an appropriate loop space extension of the correction terms $\delta Z(T)$ discussed above, which will then always reflect global properties of the quantum theory.

Other directions could also entail examining the connections between equivariant localization and other ideas we have encountered in this Thesis. One is the Parisi-Sourlas
supersymmetry that we encountered in the evaluation of the Niemi-Tirkkonen localization formula for the height function on the sphere, although this feature seems to be more intimately connected to the Kähler geometry of $S^{2}$, as we showed above. The Kähler symmetries we constantly found would be a good probe of the path integral correction formulas, and it would interesting if they could also be generalized to some sort of supersymmetric structure. Another line one could take is examining further the connection between localization and the Lagrangian anti-field formalism first discussed in [93]. Indeed, the manner in which the equivariant exterior derivative acts is more so like the situation in Lagrangian BRST quantization [17]. This is were the possible connections with topological field theories lies. Finally, the connection between localization and the constructive Mathai-Quillen formalism is yet to be clarified, as the latter relies on quite different cohomological symmetries than the ordinary BRST supersymmetries responsible for equivariant localization [98]. This might give a more direct connection between localization and some of the more modern theories of quantum integrability [30], such as $R$-matrix formulations and the Yang-Baxter equation. This has been discussed somewhat in [49]. These connections are all important and should be found in order to have full understandings of the structures of topological and integrable quantum field theories from the point of view of loop space equivariant localization.

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[^0]:    ${ }^{1}$ The exact solvability features of path integrals in this context is similar to the solvability features of the Schrödinger equation in quantum mechanics when there is a large symmetry group of the problem. For instance, the $O(4)$ symmetry of the 3 -dimensional Coulomb problem is what makes the hydrogen atom an exactly solvable quantum system [83]:

[^1]:    ${ }^{1}$ The topology on the total spaces $E=\bigsqcup_{x \in \mathcal{M}} \pi^{-1}(x)$ of fiber bundles is usually taken as the induced

[^2]:    topology from the erection of points from $\mathcal{M}$. However, Atiyah has shown that continuous functors, when applied to sets of vector bundles, yield vector bundles, which immediately gives (2.4) and its topology in a much neater way.
    ${ }^{2}$ Actually, one properly defines vector and tensor fields as 'sections' of the associated bundles, i.e. smooth maps $s: \mathcal{M} \rightarrow E$ which take a point $x \in \mathcal{M}$ into the fiber $\pi^{-1}(x)$ over $x$. Although we shall be a bit abusive in our discussion by considering these as genuine functions on $\mathcal{M}$, for simplicity and ease of notation, it should be kept in mind that it is only locally where these objects admit such a functional interpretation.

[^3]:    ${ }^{3}$ Actually, in the mathematics literature the term 'nilpotent' usually means that some power of that quantity vanishes. The proper terminology for the property (2.19) would be to say that $d$ is of order 2. However, following the standard supersymmetry and topological field theory terminology, we shall throughout refer to this property as nilpotency.
    ${ }^{4}$ These constructions can in fact all be generalized to an arbitrary topological space using the so-called DeRham-Sullivan complex.

[^4]:    ${ }^{5}$ The integral over $\mathcal{M}$ of a $p$-form with $p<\operatorname{dim} \mathcal{M}$ is always understood here to be zero.
    ${ }^{6}$ Again, the notion of integration and Stokes' theorem generalize to an arbitrary topological space via the DeRham-Sullivan complex.

[^5]:    ${ }^{7}$ In fact, with some other technical restrictions, this defines a principal $G$-bundle $\mathcal{M} \rightarrow \mathcal{M} / G$.

[^6]:    ${ }^{8}$ Occasionally, for ease of notation, we shall denote exterior products of differential forms as though they were just ordinary multiplication of functions. For instance, we define $\alpha^{\wedge n} \equiv \alpha^{n}$.

[^7]:    ${ }^{9}$ We shall assume here that $n$ is even. This restriction is by no means necessary but it will allow us to shorten some of the arguments in this section.

[^8]:    ${ }^{1}$ Usually one argues that the phase will concentrate around the points where $H$ is minimized since this should be the dominant contribution for $T \rightarrow \infty$. However, the localization is properly determined by the points where $d H(p)=0$ since the contribution from other extrema turn out to be of the same order of magnitude as those from the minima [61].

[^9]:    ${ }^{2}$ In general, if $\mathcal{M}$ is path-connected, as we always assume here, then $H_{0}(\mathcal{M} ; \mathbb{Z})=\mathbb{Z}$ and if $\mathcal{M}$ is closed, then $H_{2 n}(\mathcal{M} ; \mathbb{Z})=\mathbb{Z}$. The intermediate homology groups depend on whether or not $\mathcal{M}$ has 'holes' in it or not.

[^10]:    ${ }^{3}$ In algebraic geometry one would therefore say that $\mathbb{C}^{+}$is the Teichmüller space of the torus. The Teichmüller space of a simply-connected Riemann surface is a point, so that there is a unique complex structure (i.e. a unique way of defining complex coordinates) in genus 0 . This is a consequence of the celebrated Riemann uniformization theorem. We refer to [92] and [121] for an elementary introduction to Teichmüller spaces in algebraic geometry, while a more extensive treatment can be found in [62].

[^11]:    ${ }^{4}$ All numerical integrations in this thesis were performed using the mathematical software package MATHEMATICA.

[^12]:    ${ }^{5}$ Here we assume that $\Omega_{V}$ is non-degenerate on $\mathcal{M}$ except possibly on submanifolds of $\mathcal{M}$ of codimension at least 2, since when it is degenerate some of the equations in (3.92) should be considered as constraints. On these submanifolds, the Hamiltonian $K_{V}$ must then vanish in order to keep the equations of motion non-singular [5].

[^13]:    ${ }^{1}$ More precisely, the operators ( $\hat{p}, \hat{q}$ ) generate the universal enveloping algebra of an extended affine Lie algebra which is usually identified as the Heisenberg algebra.

[^14]:    ${ }^{2}$ Strictly speaking, these function spaces should be properly defined as distribution spaces in light of the discussion which follows.

[^15]:    ${ }^{3}$ Note that the transition from the multiple integral representation in (4.18) to the representation (4.19),(4.20) in terms of phase space paths requires that these trajectories can at least be approximated by piecewise-linear functions.

[^16]:    ${ }^{4}$ Actually, in supersymmetric quantum field theories the BRST transformations of operators and fields are represented by a graded BRST commutator [ $\left.Q_{S}, \cdot\right]$. This commutator in the case at hand can be represented by the Poisson structure of the phase space as follows. We introduce periodic trajectories $\lambda_{\mu}(t)$ on $L \mathcal{M}$ conjugate to $x^{\mu}(t)$ and anticommuting periodic paths $\bar{\eta}_{\mu}(t)$ conjugate to $\eta^{\mu}(t)$, i.e.

    $$
    \left\{\lambda_{\mu}(t), x^{\nu}\left(t^{\prime}\right)\right\}_{\omega}=\left\{\bar{\eta}_{\mu}(t), \eta^{\nu}\left(t^{\prime}\right)\right\}_{\omega}=\delta_{\mu}^{\nu} \delta\left(t-t^{\prime}\right)
    $$

    which are to be identified as the Poisson algebra realization of the operators $\lambda_{\mu}(t) \sim \frac{\delta}{\delta x^{\mu}(t)}$ and $\bar{\eta}_{\mu}(t) \sim$ $\frac{\delta}{\delta \eta^{\mu}(t)}$ acting in the usual way. This gives a Poisson bracket realization of the actions of the operators $d_{L}$ and $i_{S}$, and then the action of $Q_{S}$ is represented by the BRST commutator $\left\{Q_{S}, \cdot\right\}_{\omega}$. In the following, one can keep in mind this representation which maintains a complete formal analogy with supersymmetric theories.

[^17]:    ${ }^{5}$ We also require that the combination (4.59) be such that it determines a homotopically trivial element as above, so that it introduces no extra topological effects into the path integral (4.58) when evaluated on contractable loops. For the most part, we shall be rather cavalier about this requirement and discuss it only towards the end of this thesis.

[^18]:    ${ }^{6}$ Some features of the space of $T$-periodic classical trajectories for both energy conserving and nonconserving Hamiltonian systems have been discussed recently by Niemi and Palo in [96, 99].

[^19]:    ${ }^{7}$ This analogy, as well as the localization of the quantum partition function in general, requires that boundary conditions for the path integral be selected which respect the pertinent supersymmetry. We shall say more about this requirement later on.

[^20]:    ${ }^{8}$ The Callias index theorem is the analog of the Atiyah-Singer index theorem for a Dirac operator on an odd-dimensional non-compact manifold. Basically, one computes this index from the path integral for a higher-dimensional Atiyah-Singer index by introducing a simple first class constraint that eliminates the extra dimensions. The ensuing BRST-quantized canonical action then admits a superloop space interpretation as above to which the localization techniques become directly applicable.

[^21]:    ${ }^{9}$ This term is analogous to the instanton term $F \wedge F$ in 4-dimensional Yang-Mills theory which can be represented as a locally exact 1 -form and is therefore non-trivial only for space-times which have non-contractable loops [118].

[^22]:    ${ }^{1}$ The coset obtained by quotienting a Lie group by a maximal torus is often called a flag manifold.

[^23]:    ${ }^{2}$ Here the complexification of the group $G$ is defined by exponentiating the complexification $\mathbf{g} \otimes \mathbb{C}$ of the finite-dimensional vector space $g$.

[^24]:    ${ }^{3}$ The derivation of Stone has been generalized by Perret [109] to the Weyl-Kac character formula for Kac-Moody algebras (i.e. loop groups).

[^25]:    ${ }^{4}$ Here and in the following we shall always ignore the discrete isometries of $(\mathcal{M}, g)$, such as reflections, since these cannot be represented as continuous flows of vector fields on $\mathcal{M}$ and so are not of particular use to us.

[^26]:    ${ }^{5}$ The fact that this holds globally follows from an application of the classical Riemann-Roch theorem, or the more modern Atiyah-Singer index theorem [35, 51, 92].

[^27]:    ${ }^{6}$ In the Kähler polarization, the correct physical zero-point energy $E_{0}=\frac{1}{2}$ of the harmonic oscillator is obtained only with a specific choice of regularization scheme for the fluctuation determinant [84]. This is similar to the Weyl shift effect discussed in section 5.1 above in the path integral evaluation of character formulas.

[^28]:    ${ }^{7}$ Taking $V \rightarrow 0$ in this sort of localization formula represents the Atiyah-Singer index theorem for the elliptic Dolbeault exterior derivative. In this context, it is usually referred to as the Riemann-RochHirzebruch index theorem [35, 92].

[^29]:    ${ }^{8}$ Of course, we could alternatively obtain the Weyl character formula using instead the $G$-index localization formula (5.23) without having to perform this Weyl shift.

[^30]:    ${ }^{1}$ Here a homotopy class of curves $\left[C_{x}\right]$ can be identified with an element of $\pi_{1}(\mathcal{M})$ by choosing another basepoint $x_{0}^{\prime}$ and a grid of standard paths from $x_{0}^{\prime}$ to any other point in $\mathcal{M}$. Then the associated homotopy class is represented by the loop $\left[x_{0}^{\prime}, x_{0}\right] \cup C_{x} \cup\left[x, x_{0}^{\prime}\right]$.

[^31]:    ${ }^{2}$ For an exposition of the various equivalent ways, such as above, of describing compact Riemann surfaces in different geometric forms, see [91].

[^32]:    ${ }^{3}$ This definition could also be applied to the full quantum propagator $\mathcal{K}\left(x^{\prime}, x ; T\right)$ between 2 phase space points. Then the sum in (6.27) is over all homotopy classes of curves [ $C_{x x^{\prime}}$ ] from $x$ to $x^{\prime}$ which are identified with elements of $\pi_{1}(\mathcal{M})$ using a standard mesh of paths.

[^33]:    ${ }^{4}$ That the 2 associated tori are conformally isomorphic can be seen intuitively by representing each as a parallelogram in the complex plane and tracing out this transformation.

[^34]:    ${ }^{1}$ The extension to degenerate Hamiltonians is fairly straightforward. In what follows all statements made concerning the structure of the discrete critical point set $\mathcal{M}_{V}$ of $H$ will then apply to the full critical submanifold.

[^35]:    ${ }^{2}$ See [61] for the generalization of this formula to the case where $H$ is a degenerate function.

[^36]:    ${ }^{3}$ Note that this connection, and the metric that follows from it below, is singular at the critical points of $H$. This is merely another manifestation of the coordinate singularities that we encountered in Section 5.7 from a naive choice of coordinate neighbourhood on $\mathcal{M}$. As we saw there, this does not change the overall conclusions.

[^37]:    ${ }^{4}$ The existence of an almost complex structure $J$ for which the symplectic 2-form $\omega$ is Hermitian and for which the associated Kähler metric $g=J \cdot \omega$ is positive-definite is not really an issue for a symplectic manifold [135]. Such a $J$ always exists (and is unique up to homotopy) because the Siegal upper-half plane is contractible. Thus the existence of a Kähler structure for which $\Delta=\nabla \log h_{L}=0$ is not a problem. However, for the Killing equation for $g=J \cdot \omega$ to hold, $J$ itself must be invariant under the flows of the Hamiltonian vector field $V$, i.e. $\mathcal{L}_{V} J=0$.

[^38]:    ${ }^{5}$ The fact that a conformal symmetry leads to a localization onto $\mathcal{M}_{V}$ as before is not that surprising in light of the proof of the equivariant localization principle of section 2.4. It is essentially a consequence of the fact that the differential form $\beta$ above is a connection 1-form that specifies a splitting of the tangent bundle involving a component over $\mathcal{M}_{V}$. This is implicit in the proof by Atiyah and Bott in [8] using the Weil algebra.

[^39]:    ${ }^{6}$ Here we assume that $\mathcal{M}$ is compact, but we shall see that this formalism can also be extended to $\mathbb{R}^{2 n}$.

[^40]:    ${ }^{7}$ Equivariant cohomology groups are usually computed using so-called classifying bundles of Lie groups (the topological definition of equivariant cohomology) - see [92], for example.

