The University of British Columbia

FACULTY OF GRADUATE STUDIES

PROGRAMME OF THE

FINAL ORAL EXAMINATION

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

of

ERNEST GERHARD ENNS

B.Sc., The University of British Columbia, 1961

MONDAY, APRIL 26, 1965, AT 3:30 P.M.

IN ROOM 10, HEBB (Physics)

COMMITTEE IN CHARGE

Chairman: I. McT. Cowan

R. E. Burgess	G. P. Erickson
J. W. Bichard	S. W. Nash
F. L. Curzon	G. B. Porter

External Examiner: K. M. van Vliet

Department of Electrical Engineering

University of Minnesota

NON-EQUILIBRIUM PARTICLE NUMBER FLUCTUATIONS

ABSTRACT

The fluctuations of the constituent populations in a multi-level system are considered as applied to a three and four level semi-conductor model. It is shown that under certain conditions the fluctuations of the electron numbers in the conduction band can be super-poisson. The general auto correlation function for a three level system is derived. For certain limits this correlation function is the sum of two damped sinusoidal terms. It is speculated that the phenomena of oscillating chemical reactions can be explained by this correlation function.

The photon distribution as a function of position within an active medium is derived. A loaded cavity width is defined and shown to have a lower bound consistent with the usual cavity width $c(1-R)/2nL$. The loaded cavity width is found generally to be a function of the cavity amplification and of the mirror reflectivities.

The distribution of photoelectrons emitted from a detector of area A and resolving time T due to an incident light beam is derived. By using a binomial rather than a deterministic quantum efficiency, an additional term is obtained in the

auto-correlation function. The resulting spectral density of the photocurrent fluctuations is shown to be the sum of a Poisson particle noise and wave interference term. Several examples are discussed including an intensity modulated light beam.

GRADUATE STUDIES

Field of Study: Physics

Elementary Quantum Mechanics	F. A. Kaempffer
Waves	R. W. Stewart
Electromagnetic Theory	G. M. Volkoff
Quantum Theory of Solids	R. Barrie
Noise in Physical Systems	R. E. Burgess
Statistical Mechanics	R. Barrie

Related Studies:

Transients in Linear Systems	E. V. Bohn

NON-EQUILIBRIUM PARTICLE NUMBER FLUCTUATIONS

BY

ERNEST GERHARD ENNS

B.Sc.(Hons.) U.B.C. 1961

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

In the Department of PHYSICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April 1965

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of PHYSICS

The University of British Columbia,
Vancouver 8, Canada

Date April 28, 1965

ABSTRACT

The fluctuations of the constituent populations in a multi-level system are considered as applied to a three and four level semiconductor model. It is shown that under certain conditions the fluctuations of the electron numbers in the conduction band can be super-poisson, defined as having a variance greater than the mean number. The general autocorrelation function for a three level system is obtained. For certain limits, this correlation function is the sum of two damped sinusoidal terms. This would indicate there is an oscillatory interaction between the population numbers.

The photon distribution as a function of position within a one-dimensional active medium is derived. When two partially reflecting mirrors are situated at both ends of the medium, the stationary photon distribution obtained is a function of the cavity amplification and the mirror reflectivities.

The distribution of photo-electrons emitted from a detector of area A and resolving time T when illuminated by an incident light beam is derived. By using a binomial rather tha a deterministic quantum efficiency an additional term is obtained in the autocorrelation function. The resulting spectral density of the photo-current fluctuations is shown to be the sum of a Poisson particle noise and wave interference term. Several examples of different spectral line shapes are discussed. Also considered is an intensity modulated light beam.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I should like to thank Professor R.E. Burgess, my supervisor, for his invaluable guidance and and assistance during the course of my studies.

I am indebted to the National Research Council of Canada for financial support in the form of three studentships.

CHAPTER 1

INTRODUCTION

1-1 Number fluctuations in Multi-level systems:

The fluctuations of a c urrent passed through a semiconductor can be attributed to a combination of thermal effects and to population fluctuations in the electronic energy levels. The thermal noise due to the velocity distribution of the electrons can be calculated from Nyquist's theorem. The transport noise is due to the fluctuations of the numbers of electrons in the various electronic energy bands. These fluctuations are completely determined by the transition probabilities between the various energy levels.

We will be concerned with determining the variances and covariances of the fluctuations in the level populations. In particular we will be interested in the non-equilibrium steady state when the system is driven by some external source.

The general theory for the fluctuations of numbers in multi-level systems has been derived by van Vliet et al (1956, 1965). We have used the Langevin approach to derive the second moments and the spectral density of the number fluctuations. We believe this gives the required results more directly without having to expand the generalized Fokker-Planck equation as done by van Vliet.

Previous to an article by van Vliet (1964), it was believed that the variances of the numbers of electrons and holes could never be super-poisson, defined as being greater than the mean number. Van Vliet showed that for a particular three level model, super-poisson fluctuations were possible. We have extended his treatment of this model by analyzing the fluctuations of the trapped electrons and the fluctuations of the holes in the valence band. The correlation between the conduction electrons and the trapped electrons is examined and a simple criterion for the onset of super-poisson fluctuations in the conduction band is obtained.

The analysis was extended by adding an additional impurity level into the model. Because of the complexity of the expressions for the second moments, the fluctuations were only evaluated in two special limits.

The general auto-correlation function for a three level system has been derived. It is shown that under certain conditions it contains damped sinusoidal terms. This implies there is an oscillating interaction between the levels under consideration. For the semiconductor systems considered, this condition can never arise. It is speculated that the phenomena of oscillating chemical reactions can be explained by this correlation function. It is shown that in a certain limit for the chemical kinetics assumed the condition for oscillation can indeed arise.

1-2 Incoherent emission from an active medium

In chapter 3, a one dimensional homogeneous medium is assumed capable of absorption, induced emission and spontaneous emission of photons. A differential difference equation is derived for the joint probability of having n and m photons travelling in opposing directions respectively within the medium. By introducing partially reflecting mirrors at both ends of the medium we form a cavity enclosing an active medium. An expression for the fluctuations of the photon numbers as a function of position within the cavity is obtained. In this analysis we are just dealing with photon numbers and do not account for the phase and coherence of the reflected waves. Our results therefore do not include the coherent emission line.

A general expression for the loaded cavity width of the emission line is evaluated using the usual definition of the Quality factor "Q" of the cavity. It is shown that this reduces to the usual cavity widths assumed in the literature for the case of almost perfectly reflecting mirrors. Upper and lower bounds for the cavity width are also determined for various mirror reflectivities.

1-3 Photon Counting

The distribution of the photoelectrons emitted from a detector with response time T is evaluated. Mandel (1958) derived the distribution for the number of photoelectrons emitted in a time interval T. In a further paper with Wolf(1961) he derives the autocorrelation function for the photoelectron fluctuations in a time interval T. In both of these derivations a deterministic quantum efficiency is assumed. By using a stochastic binomial quantum efficiency we obtain an additional term in the autocorrelation function and show it corresponds to the pure shot noise spectrum of the photoelectric fluctuations. The spectral density of the photoelectric current is therefore the sum of a shot noise and a wave interference term. This wave interference spectrum which is the convolution of the spectral line shape with itself was first derived by Forrester(1961)

Examples of the photon counting spectrum of various spectral line shapes are considered. We also investigate the special case of an intensity modulated light beam and the resulting spectrum.

The photoelectron distribution calculated is shown to be the boson distribution for C cells in phase space, C is shown to be the product of the number of coherence areas on the detector surface and the number of correlation times in the measuring interval T.

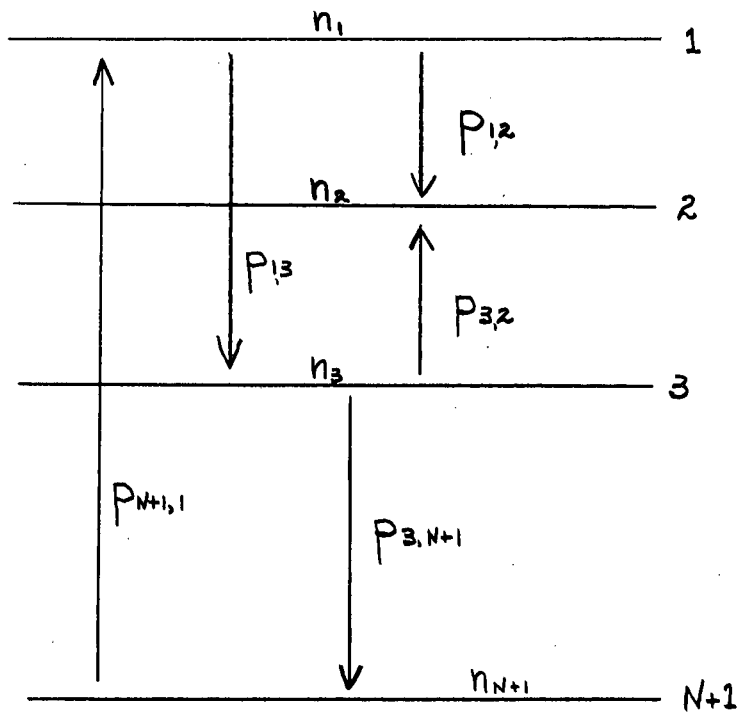

FIGURE 2-1

N + 1 LEVEL SYSTEM INCLUDING SEVERAL TRANSITION PROBABILITIES

CHAPTER 2

NUMBER FLUCTUATIONS IN MULTILEVEL SYSTEMS

2-1 Derivation of Spectral Densities and Covariances

The fluctuations of carrier numbers in a semiconductor are important in determining the fluctuations of the current under the influence of an electric field. In this chapter we will be concerned with the fluctuations of electrons or holes due to transitions between electronic energy bands. We will not deal with velocity fluctuations or scattering within a band as this gives rise to the thermal noise which is calculable by Nyquist's theorem.

The fluctuations of numbers in a multilevel system due to inter-band transitions has been generally solved by van Vliet and Blok(1956) for an N+1 level system. We will summarize his method and results and give an alternative derivation.

Consider an N+1 level system (refer to figure 2-1) where n_i is the occupation number of the i th level and $p_{ij}(\Delta t)$ is the probability of a transition from level i to level j in a time interval $t, t+\Delta t$.

van Vliet and Blok wrote down the generalized time dependent Fokker-Planck equation and by assuming that all numbers n_i under consideration were large, linearized the transition probabilities. This is valid provided the fluctuations about the mean values are small. Then bya matrix transformation they were able to show that the

joint probability distribution for the number of particles in the various levels was a multi-variate gaussian distribution. Taking the limit as time $t \to \infty$ they eliminated the exponential or transient terms to obtain the stationary distribution function.

The covariances resulting from this calculation are then found from the matrix equation:

$$\mathbf{AC} + \mathbf{C}\tilde{\mathbf{A}} = -\mathbf{B}$$

where $\mathbf{A} = \{a_{ij}\}$ $\mathbf{B} = \{b_{ij}\}$ and $\mathbf{C} = \{c_{ij}\}$

The transpose is denoted by $\tilde{a}_{ij} = a_{ji}$

The various matrix elements are then:

$$a_{ij} = \sum_{k=1}^{N+1} \left(\frac{\partial p_{ki}}{\partial n_j} - \frac{\partial p_{ik}}{\partial n_j} \right)_0 \qquad i, j = 1, 2, \cdots N \text{ and } p_{ii} = 0$$

$$b_{ij} = -(p_{ij} + p_{ji})_0 \qquad i \neq j$$

$$b_{ii} = 2 \sum_{k=1}^{N+1} (p_{ik})_0$$

$$c_{ij} = \overline{\Delta n_i \, \Delta n_j}$$

The subscript "o" means the expression in the preceding bracket is evaluated at $n_j = \overline{n_j}$, the mean occupation number where the bar implies an ensemble average.

Using the above results, van Vliet and Fassett (1965) derived the spectral density matrix of the carrier fluctuations:

$$\mathbf{S}(f) = 2\mathrm{Re}(\mathbf{A} + j\omega\mathbf{I})^{-1} \mathbf{B} \, (\tilde{\mathbf{A}} - j\omega\mathbf{I})^{-1}$$

where A and B are the matrices previously defined.

We will derive the above spectral density matrix directly by the Langevin approach. Then by a simple integral theorem we will find the covariances.

If we have N + 1 levels as in figure 2-1, then by neutrality $\sum_i n_i$ = constant and we will only have N independent variables.

The kinetic equations for the stochastic variables are

$$\frac{dn_i}{dt} = \sum_{j=1}^{N+1} (p_{ji} - p_{ij}) \qquad i = 1, 2, \cdots N$$

where $p_{ij}(\Delta t)$ is again the transition probability from level i to level j in the time interval t,t+Δt. For a stationary process p_{ij} will not be an explicit function of time. Thus p_{ij} will be a function only of the N independent n_i.

Linearizing the above equation by expanding it about the mean values of the n_i (denoted by the subscript "o") and adding the Langevin random fluctuating source terms, we have:

$$\frac{dn_i}{dt} = \sum_{k=1}^{N} \sum_{j=1}^{N+1} \left(\frac{\partial p_{ji}}{\partial n_k} - \frac{\partial p_{ij}}{\partial n_k} \right)_o (n_k - \bar{n}_k) + \sum_{j=1}^{N+1} (p_{ji} - p_{ij})_t$$

If we denote the Fourier transform by $\mathcal{F}$ and

$$\mathcal{F}(n_k - \bar{n}_k) = S_k$$

$$\mathcal{F}\left[\sum_{j=1}^{N+1} (p_{ji} - p_{ij})_t \right] = f_{i\omega}$$

and $$\sum_{j=1}^{N+1} \left(\frac{\partial p_{ji}}{\partial n_k} - \frac{\partial p_{ij}}{\partial n_k} \right)_o = a_{ik}$$

then $$-j\omega S_i = \sum_{k=1}^{N} a_{ik} S_k + f_{i\omega} \qquad i = 1, 2, \cdots N$$

In matrix notation, where $A = \{a_{ik}\}$ and $\alpha_{ki} = a_{ik}$

Define the vectors $$\underline{S} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_N \end{pmatrix}, \qquad \underline{f}_\omega = \begin{pmatrix} f_{1\omega} \\ f_{2\omega} \\ \vdots \\ f_{N\omega} \end{pmatrix}$$

then

$$(A + j\omega I)\underline{S} = -\underline{f}_\omega$$

$$\tilde{\underline{S}}(\tilde{A} + j\omega I) = -\tilde{\underline{f}}_\omega$$

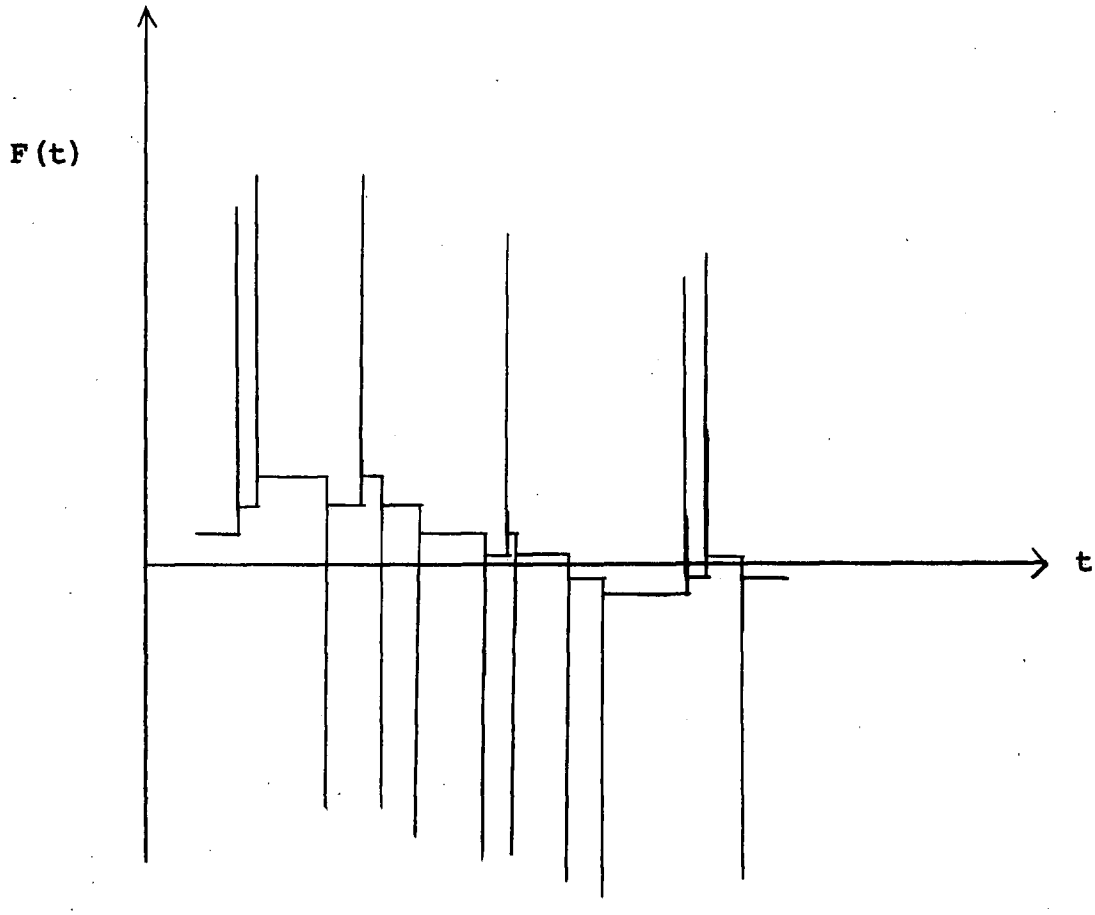

FIGURE 2-2

LANGEVIN SOURCE TERM

The spectral density matrix is defined by:

$$S(f) = \{S_{ij}(f)\} = 2\,\mathrm{Re}\,(\underline{\xi}\,\underline{\tilde{\xi}}^{*})$$

Thus

$$S(f) = 2\,\mathrm{Re}\,(A+j\omega I)^{-1} B\,(\tilde{A}-j\omega I)^{-1}$$

where we have let $B = \{b_{ij}\} = \underline{f}_{\omega}\,\underline{\tilde{f}}^{*}_{\omega}$

thus $$b_{ij} = \sum_{r=1}^{N+1}\sum_{s=1}^{N+1} \mathcal{F}(p_{ri}-p_{ir})_t\ \mathcal{F}^{*}(p_{sj}-p_{js})_t$$

the product of the Fourier transforms of the stochastic source terms.

The relaxation time constants, obtained from the macroscopic mass action equation are much longer than any time constants contained in the stochastic source term

$$F(t) = \sum_j (p_{ji}-p_{ij})_t$$

where

$$\frac{dn_i}{dt} - \sum_j (p_{ji}-p_{ij}) = F(t)$$

The spectrum of the source term can therefore be considered as white for the purposes of our calculation. This also implies there will be no measurable correlation between different elements of the source term. A graph of what F(t) might look like is plotted in figure 2-2. A positive or negative delta function pulse indicates the increase or decrease of n_i by one electron. The exaggerated flat plateau between events indicates whether $p_{ji}(n_k) - p_{ij}(n_k)$ is positive or negative. Thus a positive plateau indicates that a downward transition is slightly favoured and vice-versa. This maintains our steady state. Since the fluctuations of the n_k are small, the graph of F(t) is basically just a series of positive and negative delta functions. The a.c. spectrum of F(t) is invariant to whether the

pulses are all positive, all negative or both. The mean rate of occurence of an event is the constant $(p_{ij} + p_{ji})_0$, therefore the spectrum of $(p_{ij} - p_{ji})_t$ is $2(p_{ij} + p_{ji})_0$ and:

$$\mathcal{F}(p_{ri})_t \, \mathcal{F}^*(p_{sj})_t = (p_{ri})_0 \, \delta_{rs} \, \delta_{ij} \qquad \delta_{ij} = 1 \,,\ i = j \qquad \delta_{ij} = 0 \,,\ i \neq j$$

Simplifying B, one obtains:

$$b_{ii} = \sum_{r=1}^{N+1} (p_{ri} + p_{ir})_0$$

$$= 2 \sum_{r=1}^{N+1} (p_{ir})_0 \qquad i = 1, 2, \cdots N$$

since the net increase $\sum_{r=1}^{N+1} (p_{ri} - p_{ir})$ of any level is zero for a stationary system.

$$b_{ij} = b_{ji} = -(p_{ij} + p_{ji})_0 \qquad i \neq j$$

The choice of notation is now evident, as the matrices A and B defined in this derivation are the same as those quoted in the previous results. The spectral density matrix is thus obtained directly without a knowledge of the covariances.

The second moments can now readily be obtained if we use the identity:

$$\mathrm{Re} \int_0^\infty (M \pm j\omega I)^{-1} d\omega = I\pi/2 \quad \text{if } \sum_i m_{ii} > 0$$

$$= -I\pi/2 \quad \text{if } \sum_i m_{ii} < 0$$

where $M = \{m_{ij}\}$ is any real square matrix independent of ω . If we define:

$$G(f) = S(f) + j\hat{S}(f) = 2\,\mathrm{Re}\,(A + j\omega I)^{-1} B (\bar{A} - j\omega I)^{-1}$$

then

$$\mathrm{Re} \int_0^\infty (A + j\omega I) G(f)\, df = 2\,\mathrm{Re} \int_0^\infty B(\bar{A} - j\omega I)^{-1} df$$

$$\mathrm{Re} \int_0^\infty G(f)(\bar{A} - j\omega I) df = 2\,\mathrm{Re} \int_0^\infty (A + j\omega I)^{-1} B \, df$$

But $\int_0^\infty S(f)\, df = C$ as previously defined.

Therefore:

$$AC - \int_0^\infty \omega \hat{S}(f)\, df = -B/2$$

$$C\tilde{A} + \int_0^\infty \omega \hat{S}(f)\, df = -B/2$$

The negative sign on the right hand side was used because:

$$\sum_i a_{ii} = \sum_i \sum_j \left(\frac{\partial p_{ji}}{\partial n_i} - \frac{\partial p_{ij}}{\partial n_i} \right)_0 < 0$$

since p_{ji} is a decreasing function of n_i and p_{ij} is an increasing function of n_i.

Adding the two equations we obtain the desired result:

$$AC + C\tilde{A} = -B$$

2 - 2 Two level system

For a two level system we obtain Burgess' (1956) well known g-r theorem. In this case:

$$p_{12} = r(n) \qquad \text{and} \qquad p_{21} = g(n)$$

The spectrum is given by:

$$S_n(f) = \frac{4\, r(n_0)}{\omega^2 + \left[g'(n_0) - r'(n_0) \right]^2}$$

The auto-correlation function defined by:

$$\psi_n(\tau) = \overline{n_t n_{t+\tau}} - \bar{n}^2 = \int_0^\infty S_n(f) \cos \omega\tau \, df$$

is $$\psi_n(\tau) = \frac{r(n_0) \exp\left[-\left(r'(n_0) - g'(n_0) \right) \right] \tau}{\left(r'(n_0) - g'(n_0) \right)}$$

and the variance:

$$\text{var } n = \psi_n(0) = \frac{r(n_0)}{r'(n_0) - g'(n_0)}$$

2-3 The spectral density and auto-correlation function for a three level system.

The complexity of the spectra and covariances increases notable with the addition of every new level. We will therefore limit our discussion to two and three level systems and a few particular cases of a four level system.

Consider the general three level system where $p_{ij} = p_{ij}(n_1, n_2)$ and $n_1 + n_2 + n_3 = \text{constant}$. For this system our spectral density matrix will be:

$$S(f) = \frac{2J}{\omega^4 + \omega^2(a_{11}^2 + a_{22}^2 + 2a_{12}a_{21}) + (a_{11}a_{22} - a_{12}a_{21})^2}$$

where

$$J_{11} = b_{11}\omega^2 + b_{11}a_{22}^2 - 2b_{12}a_{12}a_{22} + b_{22}a_{12}^2$$

$$J_{12} = J_{21} = b_{12}\omega^2 - b_{11}a_{21}a_{22} + b_{12}(a_{11}a_{22} + a_{12}a_{21}) - b_{22}a_{11}a_{12}$$

$$J_{22} = b_{22}\omega^2 + b_{11}a_{21}^2 - 2b_{12}a_{11}a_{21} + b_{22}a_{11}^2$$

and

$$b_{11} = 2\left[p_{12} + p_{13}\right]_0$$

$$b_{12} = b_{21} = -\left[p_{12} + p_{21}\right]_0$$

$$b_{22} = 2\left[p_{21} + p_{23}\right]_0$$

and

$$a_{11} = \left(\frac{\partial p_{21}}{\partial n_1} - \frac{\partial p_{12}}{\partial n_1}\right)_0 + \left(\frac{\partial p_{31}}{\partial n_1} - \frac{\partial p_{13}}{\partial n_1}\right)_0$$

$$a_{12} = \left(\frac{\partial p_{21}}{\partial n_2} - \frac{\partial p_{12}}{\partial n_2}\right)_0 + \left(\frac{\partial p_{31}}{\partial n_2} - \frac{\partial p_{13}}{\partial n_2}\right)_0$$

$$a_{21} = \left(\frac{\partial p_{12}}{\partial n_1} - \frac{\partial p_{21}}{\partial n_1}\right)_0 + \left(\frac{\partial p_{32}}{\partial n_1} - \frac{\partial p_{23}}{\partial n_1}\right)_0$$

$$a_{22} = \left(\frac{\partial p_{12}}{\partial n_2} - \frac{\partial p_{21}}{\partial n_2}\right)_0 + \left(\frac{\partial p_{32}}{\partial n_2} - \frac{\partial p_{23}}{\partial n_2}\right)_0$$

The correlation function matrix $\psi(\tau) = \{\psi_{ij}(\tau)\}$

where $$\psi_{ij}(\tau) = \overline{n_{it}\, n_{jt+\tau}} - \overline{n_i}\,\overline{n_j}$$
$$= \int_0^\infty S_{ij}(f)\cos\omega\tau\, df$$

Then find:

$$\psi(\tau) = \frac{e^{-|p|\tau}}{4|p|q(p^2+q^2)}\left[q(\beta+\alpha(p^2+q^2))\cos q\tau + |p|(\beta-\alpha(p^2+q^2))\sin q\tau\right]$$

where

$$\Gamma = \alpha\omega^2 + \beta$$

and $$p^2+q^2 = a_{11}a_{22} - a_{12}a_{21}$$
$$2p = a_{11} + a_{22}$$

We have written $\psi(\tau)$ in the above fashion to illustrate the fact that q may be real.

$$q^2 = -\left[(a_{11}-a_{22})^2 + 4a_{12}a_{21}\right]/4$$

Thus for q to be real, we must have:

$$(a_{11} - a_{22})^2 < -4a_{12}a_{21}$$

Generally this inequality is not valid and q is a pure imaginery number making $\psi(\tau)$ the sum of two exponential terms. If q is real however, this would indicate we have an oscillating interaction between the three occupation numbers.

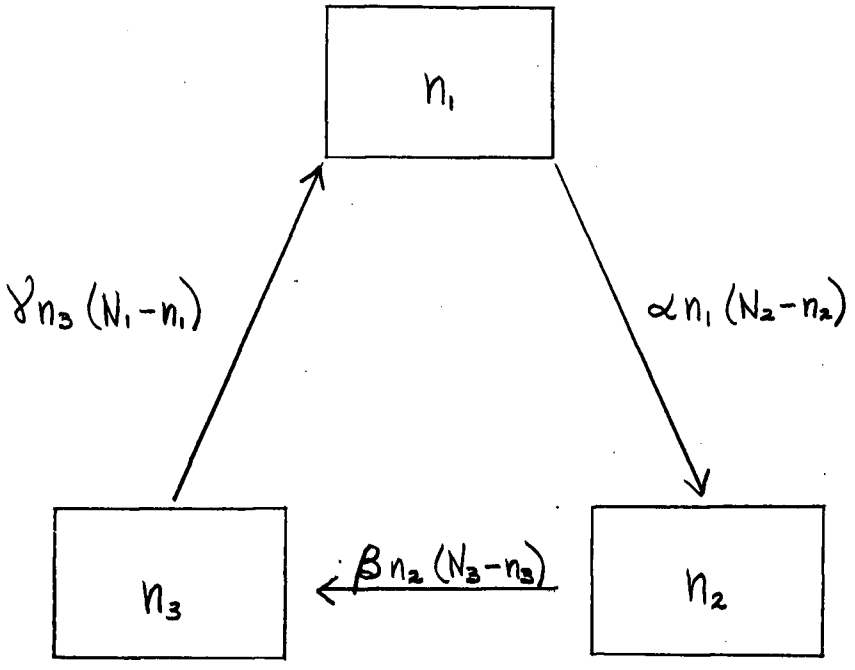

FIGURE 2-3

STEADY STATE CHEMICAL

REACTION RATES

2-4 Oscillating chemical reactions

Bak(1963), in his review of recent contributions to the theory of chemical kinetics discusses the inadequacy of the present theories to deal with oscillating chemical reactions. The kinetic equations are the same as those under discussion. In this case the driving force would likely be an external heat source as the reactions would naturally be endothermic.

An example of a system where q could be real would be the following. (Refer to figure 2-3) Let n_1, n_2 and n_3 be the number of molecules of the three constituents of interest. Again $n_1 + n_2 + n_3 =$ constant and we have only two independent variables.

The reaction changing n_1 to n_2 and n_2 to n_3 etc. will be assumed to be favoured over any reverse reaction. This would be the case when a by-product formed in the reaction n_1 to n_2 was used in the interaction n_2 to n_3. We will ignore by-products and catalysts which are undoubtedly necessary to make the reaction work, but do not enter explicitly into the kinetics. We will assume the net rate changing n_1 to n_2 as $\propto n_1(N_2 - n_2)$. When $n_2 = N_2$ we will have detailed balance. Similarly for the other two transitions. We will examine the case when we have cyclic transitions $(N_i - n_i) > 0$ and real q resulting in an oscillating reaction.

In the steady state:

$$K = \alpha n_1 (N_2 - n_2) = \beta n_2 (N_3 - n_3) = \gamma n_3 (N_1 - n_1)$$

where all n_1 , n_2 and n_3 used hereafter are mean values. Then the matrix elements are:

$$a_{11} = -K\left(\frac{1}{n_1} + \frac{1}{N-n_1} + \frac{1}{n_3}\right) \qquad a_{12} = K\left(\frac{1}{N_2 - n_2} - \frac{1}{n_3}\right)$$

$$a_{21} = K\left(\frac{1}{n_1} - \frac{1}{N_3 - n_3}\right) \qquad a_{22} = -K\left(\frac{1}{n_2} + \frac{1}{N_2 - n_2} + \frac{1}{N_3 - n_3}\right)$$

and

$$-4q^2 = K^2\left[\left(\frac{1}{n_1} + \frac{1}{N_1 - n_1} + \frac{1}{n_3} - \frac{1}{n_2} - \frac{1}{N_2 - n_2} - \frac{1}{N_3 - n_3}\right)^2 + 4\left(\frac{1}{N_2 - n_2} - \frac{1}{n_3}\right)\left(\frac{1}{n_1} - \frac{1}{N_3 - n_3}\right)\right]$$

For the second term in this expression to be dominant and negative, we must have $n_i \ll (N - n_i)$, which implies we are far from equilibrium and detailed balance does not hold. Then for q to be real we can have

$$n_2 \propto n_3 \qquad \text{and} \qquad n_3 \ll n_1$$

then $$q^2 = K^2 / n_1 n_3$$

The oscillation frequency can be determined from the power spectrum:

$$S(f) = \frac{2(\alpha\omega^2 + \beta)}{\omega^4 + 2\omega^2(p^2 - q^2) + (p^2 + q^2)^2}$$

Maximizing S(f) with respect to ω , we find that:

$$S(f)_{max.} = \alpha / (\omega_0^2 + (p^2 - q^2))$$

where the characteristic frequency ω_0 is:

$$\omega_0^2 = -\beta/\alpha + \left[(\beta/\alpha)^2 - 2(\beta/\alpha)(p^2 - q^2) + (p^2 + q^2)^2\right]^{1/2}$$

For the case under discussion, the frequency of oscillation for all n_i is the same since $B/2 \gg (p^2-q^2)$ and $(B/2)(p^2-q^2) \gg (p^2+q^2)^2$.

Thus $$\omega_o^2 \simeq -(p^2-q^2)$$

In this particular case we find:

$$\omega_o = K/(n_1 n_3)^{1/2}$$

ω_o is therefore a function of the molecular concentrations and of the transition rates.

We have thus shown that q can be real, resulting in an oscillating reaction about some characteristic frequency ω_o. We will not discuss other limits for which q may be real as the purpose of this discussion was merely to show that indeed q can be real.

2-5 The second moments of a three level system and an example

The covariances for a three level system are:

$$\text{cov}(n_i n_j) = \psi_{ij}(0) = \frac{\beta_{ij} + \alpha_{ij}(p^2+q^2)}{4|p|(p^2+q^2)}$$

More explicitly, they become:

$$\text{var } n_1 = \frac{b_{11}(a_{11}a_{22} - a_{12}a_{21} + a_{22}^2) - 2b_{12}a_{12}a_{22} + b_{22}a_{12}^2}{2|a_{11}+a_{22}|(a_{11}a_{22} - a_{12}a_{21})}$$

$$\text{cov}(n_1 n_2) = \frac{-b_{11}a_{21}a_{22} + 2b_{12}a_{11}a_{22} - b_{22}a_{11}a_{12}}{2|a_{11}+a_{22}|(a_{11}a_{22} - a_{12}a_{21})}$$

$$\text{var } n_2 = \frac{b_{11}a_{21}^2 - 2b_{12}a_{11}a_{21} + b_{22}(a_{11}a_{22} - a_{12}a_{21} + a_{11}^2)}{2|a_{11}+a_{22}|(a_{11}a_{22} - a_{12}a_{21})}$$

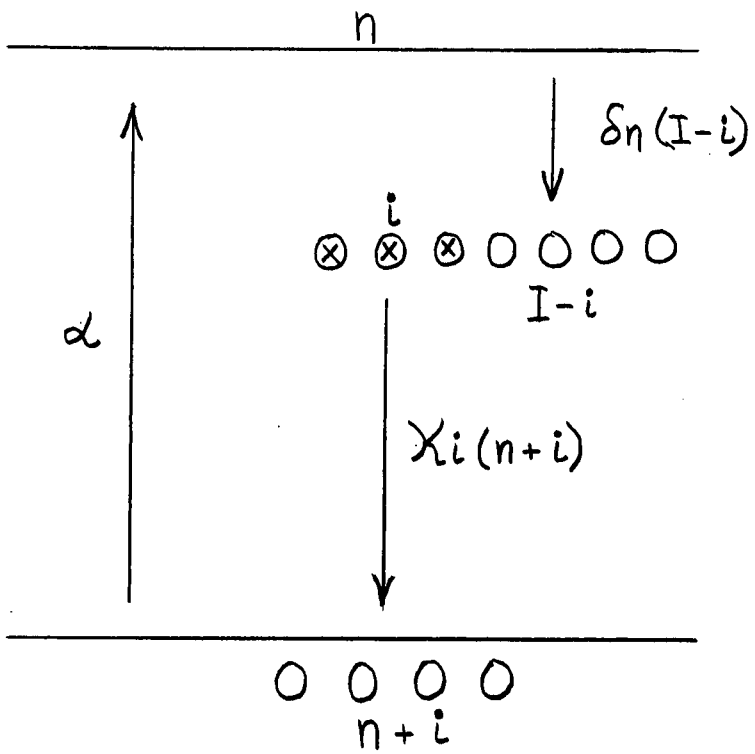

FIGURE 2-4

CYCLIC TRANSITIONS IN A PUMPED PHOTOCONDUCTOR

We will now consider a three level system where the variance of the fluctuations in one level can be much greater than the mean value. We will call these super-poisson fluctuations.

Consider the model of a three level photoconductor in the steady state by far from equilibrium. The transition probabilities p_{13}, p_{21} and p_{32} will be considered negligible compared to all other transitions. The transition probabilities for this semiconductor model are depicted in figure 2-4. The p_{31} transition is optically induced and is a function of the source intensity and absorption efficiency of the sample only. n, i and p are the carrier numbers in the conduction band, the trapping level and the valence band respectively. I is the number of traps in a unit volume. δ and χ are proportional to the capture cross-sections for the n and p carriers, and will be constant in the steady state.

This particular model has been discussed by van Vliet(1964) and Cole(1965). Van Vliet introduces an approximation into his analysis which invalidates several limits he discusses. (Van Vliet, 1964: figure 4 is incorrect. The lower bound for the fluctuations never approaches zero) We will correct this error and also discuss the correlation and fluctuations in the trapping level. Cole, in his critique of van Vliet's work, derived the correct expression for the fluctuations of n from a bivariate difference equation. This difference equation is however just a particular case of the general Fokker-

Planck equation. Cole has unknowingly performed the same linearization van Vliet generalized for N levels. Thus Cole's claim that his method is superior to van Vliet's is invalid.

The kinetic equations are:

$$\frac{dn}{dt} = \alpha - \delta n(I-i)$$

$$\frac{di}{dt} = \delta n(I-i) - \chi_1(n+i)$$

The linearized matrix elements are:

$$a_{11} = -\delta(I-i_0) \qquad a_{12} = \delta n_0$$

$$a_{21} = \delta(I-i_0) - \chi i_0 \qquad a_{22} = -\delta n_0 - \chi(n_0+2i_0)$$

$$b_{11} = b_{22} = 2\alpha \qquad b_{12} = b_{21} = -\alpha$$

where $$\alpha = \delta n_0(I-i_0) = \chi i_0(n_0+i_0)$$

Then find that the fluctuations of the conduction band carriers are given by:

$$\text{var } n/n_0 = 1 + \frac{i_0 n_0\left[-(I-i_0)(n_0^2+2i_0n_0) + i_0^2(n_0+i_0)\right]}{(I-i_0)^2(n_0^3+5i_0n_0^2+7i_0^2n_0+2i_0^3) + (I-i_0)i_0n_0(2n_0^2+6i_0n_0+3i_0^2) + (i_0n_0)^2(n_0+i_0)}$$

Using the ratios $(I-i_0)/i_0 = k$ and $n_0/i_0 = q$ we have in van Vliets notation:

$$\text{var } n/n_0 = 1 + \frac{-kq^2(q+2) + q(q+1)}{k^2(q^3+5q^2+7q+2) + k(2q^3+6q^2+3q) + q^2(q+1)}$$

We have written the variance in the above form to facilitate recognizing conditions for super-poisson and

sub-poisson fluctuations. Similarly:

$$\mathrm{cov}(n,i) = \frac{n_0 k[(1+q) - qk(q+2)]}{k^2(q^3+5q^2+7q+2) + k(2q^3+6q^2+3q) + q^2(q+1)}$$

$$\mathrm{var}\, i/i_0 = 1 - \frac{k^2(q^2+4q+1) + k(q^3+5q^2+3q) + q^2(q+1)}{k^2(q^3+5q^2+7q+2) + k(2q^3+6q^2+3q) + q^2(q+1)}$$

$$\mathrm{var}\, p/p_0 = 1 - \frac{k(q+1)[k(2q+1) + q(q+1)]}{k^2(q^3+5q^2+7q+2) + k(2q^3+6q^2+3q) + q^2(q+1)}$$

$$\mathrm{cov}(n,p) = \frac{n_0(q+1)[k^2(q^2+3q+2) + k(q^2+3q+1) + q(q+1)]}{k^2(q^3+5q^2+7q+2) + k(2q^3+6q^2+3q) + q^2(q+1)}$$

Written in this form, it is immediately obvious that the fluctuations of i and p are sub-poissonian. Also the correlation between n and p is always positive.

2-6 Super-poisson fluctuations

Super poisson fluctuations occur in the conduction band when:

$$k^{-1} > (q+1) - (q+1)^{-1}$$

This is also just the condition for the covariance cov(n,i) to be positive.

Thus when $\mathrm{var}(n) > n_0$ then $\mathrm{cov}(n,i) > 0$

and when $\mathrm{var}(n) < n_0$ then $\mathrm{cov}(n,i) < 0$

When n is super-poissonian, an increase in n tends to increase i and when n is sub-poissonian, an increase

in n will cause i to decrease.

From the steady state rate equations:

$$\alpha = \delta n_0 (I - i_0) = \chi i_0 (n_0 + i_0)$$

we obtain

$$\delta k q = \chi (q+1)$$

or

$$\left(\frac{kq}{q+1}\right) = \frac{\chi}{\delta} = y$$

the ratio of the capture cross-sections of electrons and holes by the trapping level respectively. For super-poisson fluctuations therefore:

$$y = \left(\frac{kq}{q+1}\right) < \frac{1}{(q+2)}$$

y will thus always be less than one-half when $\text{var}n/n_0 \geq 1$.

When the traps are almost full $(I - i_0) \ll i_0$ and when the density of carriers in the conduction band is much less than that in the trap level $n_0 \ll i_0$, then the electrons in the conduction band will exhibit giant fluctuations, which we define as $\text{var}\, n \gg n_0$

In this case $\text{var}\, n/n_0 \propto \frac{1}{q}$ since $q \ll 1$

This can be indefinitely great subject to the limitations of our linearized theory. Thus we can write:

$$\text{var}\, n = i_0$$

When the conduction carriers exhibit giant fluctuations, namely when $q \ll 1$ and $k \ll 1$, then:

$$\text{var}\, i/i_0 = \frac{k^2}{(2k+q)(k+q)}$$

$$\text{var}\, p/p_0 = \frac{k+q}{(2k+q)}$$

Depending on the relative magnitude of q and k we obtain the following two limits:

i) when $k \ll q$:

$$\text{var } i/i_o = (k/q)^2$$

$$\text{var } p/p_o = 1 - (k/q)$$

Thus the trap carrier fluctuations tend to zero as the traps become filled and the hole fluctuations in the valence band tend to become poisson.

ii) when $k \gg q$

$$\text{var } i/i_o = \frac{1}{2} - \frac{3}{4}(q/k)$$

$$\text{var } p/p_o = \frac{1}{2} + \frac{1}{4}(q/k)$$

In this limit the variance of both i and p approaches one-half the mean value as the number of electrons in the conduction band decreases.

2-7 Bounds for the second moments

Another case of interest is when $q \gg 1$ and $k \gg 1$ Then:

$$\text{var } n/n_o = 1 - 1/k$$

$$\text{var } i/i_o = 1 - 1/q - 1/k$$

$$\text{var } p/p_o = 1 - 2/q - 1/k$$

For this limit the "bottleneck" effect of the traps is minimized and all fluctuations become poisson.

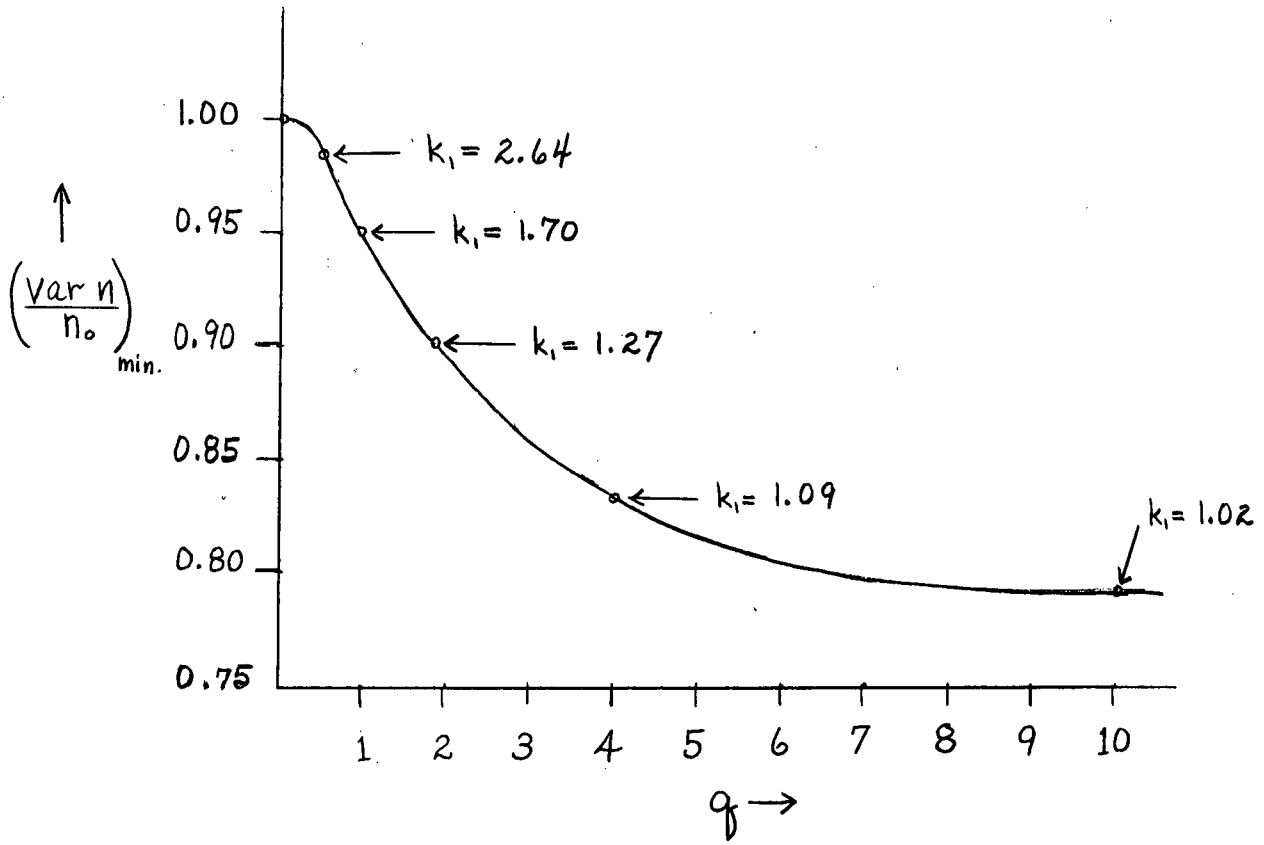

FIGURE 2-5

GRAPH SHOWING THE MINIMUM VALUE OF VAR n/n_0 WITH RESPECT TO k AS A FUNCTION OF q

van Vliet asserted that the conduction carrier fluctuations tended to zero in the high input region, namely when $q \gg 1$. On examination of this limit, we found the lower bound $var(n)/n_o > 0.75$ This may be seen as follows:

Let $$var\, n/n_o = 1 + f(k,q)/g(k,q)$$

Minimizing with respect to the independent variable k requires that: $$q\frac{\partial f}{\partial k} = f\frac{\partial g}{\partial k}$$

Thus $$\left(var\, n/n_o\right)_{min} = 1 + \frac{\left(\frac{\partial f}{\partial k}\right)}{\left(\frac{\partial g}{\partial k}\right)}$$

$$= 1 - \frac{q^2(q+2)}{2k_1(q^3+5q^2+7q+2) + (2q^3+6q^2+3q)}$$

where

$$k_1^2 q(q+2)^2(q^2+3q+1) - 2k_1(q+1)(q+2)(q^2+3q+1) - q(q+1)^2(q^2+3q+3) = 0$$

There is only one positive root for this quadratic and it is bounded by $1 < k_1 < \infty$. For any value of q this equation then gives us the value of k that minimizes the fluctuations of n. (Refer to figure 2-5) The minimum value of 0.75 is obtained when the input light intensity is great, $q \to \infty$, and the traps are half filled, k=1.

It is easily verified that the fluctuations of i and p have the following bounds:

$$0 < \frac{var\, i}{i_o} < 1$$

and

$$\frac{1}{2} < \frac{var\, p}{p_o} < 1$$

	$k \ll 1$	$k = 1$	$k \gg 1$
$q \ll 1$	1	$0.71q^{\frac{1}{2}}$	$0.71q^{\frac{1}{2}}(1-2qk)/k$
$q = 1$	$0.71k^{\frac{1}{2}}$	-0.06	-0.26
$q \gg 1$	$(k/q)(1-kq)$	$-0.41q^{-\frac{1}{2}}$	$-q^{-\frac{1}{2}}$

FIGURE 2-6a

TABLE OF Γ_{hi} FOR VARIOUS LIMITS

	$k \ll 1$	$k = 1$	$k \gg 1$
$q \ll 1$	1	$2.12q^{\frac{1}{2}}$	$1.41q^{\frac{1}{2}}$
$q = 1$	1	0.83	0.72
$q \gg 1$	1	1	1

FIGURE 2-6b

TABLE OF Γ_{hp} FOR VARIOUS LIMITS

2-8 Magnitude of cross-correlations

It is of interest to examine the correlation coefficients r_{ni} and r_{np} where we define:

$$r_{xy} = \frac{cov(x,y)}{[(var\,x)(var\,y)]^{1/2}}$$

By the Schwarz inequality $|r_{xy}| \leq 1$

r_{xy} is a measure of the degree of correlation between the two stochastic variables x and y. Refer to figure 2-6 for a tabulation of r_{ni} and r_{np} for various limits of q and k.

From the table of r_{ni} we note that there is complete positive correlation between n and i only when $var\,n/n_o \gg 1$. For all other cases the correlation between n and i is relatively small.

From the table of r_{np} we observe that there is complete positive correlation between the conduction electrons and the holes in the valence band when either the traps are virtually empty or when the density of conduction carriers is much greater than the density of trap carriers.

2-9 Four level system

We will now consider some special cases of a four-level photoconductor. (Refer to figure 2-7) Again we will discuss a driven system, one in which a steady state has been reached but where we are so far from thermal equil-

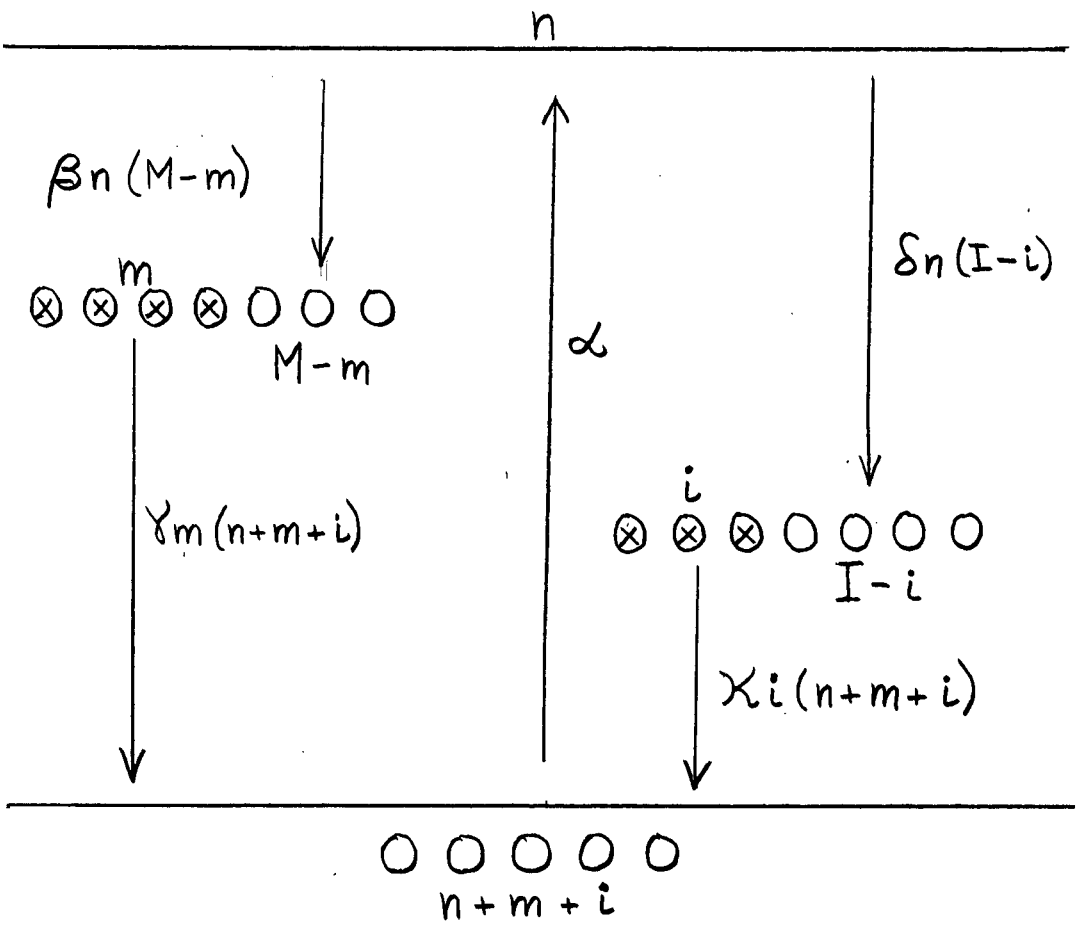

FIGURE 2-7

CYCLIC TRANSITIONS IN A FOUR LEVEL PHOTOCONDUCTOR

ibrium that detailed balance does not hold. We will assume the traps are sufficiently diffuse to eliminate the possibility of p_{23} and p_{32} transitions. This will also eliminate the possibility of "hopping" from one trap to another within an impurity level. All transition rates not labelled in the diagram are assumed to be much smaller than those labelled. The p_{+1} transition is pumped and the conduction electrons now have two possible paths to return to the valence band. For the above kinetics to be strictly valid we should also state that the drift velocities for the electrons and holes are assumed to satisfy the following inequalities respectively.

$$V_e \gg d_2 \beta (M - m_0)$$

$$V_h \gg d_3 \chi i_0$$

d_2 and d_3 are the mean distances between traps within levels 2 and 3 respectively. This is to ensure that the electron or hole effectively "sees" all the available traps before making a transition.

As a consequence of stationarity:

$$\alpha = \beta n_0 (M - m_0) + \delta n_0 (I - i_0)$$

$$\beta n_0 (M - m_0) = \gamma m_0 (n_0 + m_0 + i_0)$$

$$\delta n_0 (I - i_0) = \chi i_0 (n_0 + m_0 + i_0)$$

where the subscript "o" indicates the most probable value which is very near the mean value for large numbers.

The linearized matrix elements in this case are:

$$a_{11} = -\beta(M-m_0) - \delta(I-i_0) \qquad a_{12} = \beta n_0 \qquad a_{13} = \delta n_0$$

$$a_{21} = \beta(M-m_0) - \gamma m_0 \qquad a_{22} = -\beta n_0 - \gamma(n_0+l_0+2m_0) \qquad a_{23} = -\gamma m_0$$

$$a_{31} = \delta(I-i_0) - \chi i_0 \qquad a_{32} = -\chi i_0 \qquad a_{33} = -\delta n_0 - \chi(n_0+m_0+2i_0)$$

$$b_{11} = 2\alpha \qquad b_{12} = -\beta n_0(M-m_0) \qquad b_{13} = -\delta n_0(I-i_0)$$

$$b_{21} = -\beta n_0(M-m_0) \qquad b_{22} = 2\beta n_0(M-m_0) \qquad b_{23} = 0$$

$$b_{31} = -\delta n_0(I-i_0) \qquad b_{32} = 0 \qquad b_{33} = 2\delta n_0(I-i_0)$$

Solving for the matrix elements of C using:

$$AC + C\tilde{A} = -B$$

we obtain six equations in six unknowns since C is symmetric. If we let:

$$m_0/(M-m_0) = r\,, \qquad l_0/(I-i_0) = s\,, \qquad m_0/n_0 = p, \qquad l_0/n_0 = q$$

and

$$k = \delta n_0(I-i_0)\Big/\beta n_0(M-m_0)\,,$$

the ratio of the net flux through the two possible return paths.

The six equations thus obtained can be found in Appendix I. These equations are too complicated to solve exactly, thus we will only consider two special cases.

i) $k \gg 1$, $s, q \gg 1$ and $s, q > r, p$

In this case the transition through level 3 is dominant. The traps in level 3 are almost filled and

the density of trap 3 carriers is greater than the density of conduction electrons. If the transition through level 2 were missing, this would correspond to the giant fluctuation case of the three level system.

On solving the six equations we find:

$$C_{11} = \text{var } n = n_0 q k/s \qquad C_{12} = \text{cov}(n,m) = n_0 q/s$$

$$C_{13} = \text{cov}(n,i) = 2n_0 q/s \qquad C_{22} = \text{var } m = \frac{m_0(q+s)}{s(r+1)}$$

$$C_{33} = \text{var } i = i_0(2q+s)/s^2$$

If q and s are of the same magnitude, then the fluctuations of n are super-poisson, and the fluctuations of i are sub-poisson.

It is interesting to note that when the traps in level 2 are mostly empty corresponding to $r<1$, then m is also super-poissonian. This would suggest that like in the three level case, the filling up of the traps in the major transition route causes the super-poisson fluctuations in the conduction band. The fluctuations in level 2, the minor transition route have little effect on the conduction electrons but themselves follow the fluctuations in the conduction band.

The correlation between the n and m, and n and i electrons expressed in terms of the correlation coefficients is:

$$r_{nm} = \left[\frac{q(r+1)}{pk(s+q)}\right]^{1/2} \qquad r_{ni} = \left[\frac{4s}{k(2q+s)}\right]^{1/2}$$

Again if $r<1$ and $p \ll 1$, the correlation between levels 1 and 2 could be considerable as expected if the fluctuations in level 2 are to follow those in level 1. r_{n1} , however, is small in this case while it was large in the 3 level model.

ii) $k \ll 1$, $s, q \gg 1$, and $s, q > r, p$

In this case the transition through level 2 is dominant due to the relatively small capture cross-sections between levels 1 and 3 and levels 3 and 4. For all sub-cases considered, namely:

A) $r, p \ll 1$

B) $r, p > 1$

a) $qr \gg kps$

b) $qr \ll kps$

it was always found that $\text{var } n/n_0 \simeq 1$

Thus in this case the fluctuations in the conduction band are virtually independent of the fluctuations in the minor transition trapping level.

CHAPTER 3

INCOHERENT EMISSION FROM AN ACTIVE MEDIUM

3-1 Derivation of photon distribution

Consider a substance, homogeneous in its bulk properties, in interaction with a radiation field. Depending on whether our system is a gas or a semiconductor, let N_2 denote the number of gas atoms or electrons per unit volume respectively in the upper energy level E_2. Similarly let N_1 denote the constituent density in the lower energy level E_1. We will only consider transitions between levels E_1 and E_2, assuming these are the only two levels in the system capable of supporting an inverted population. Physically both energy levels E_1 and E_2 will have a finite width ΔE_1 and ΔE_2 which will be a contributing factor to the width of the emission spectrum. We will examine the characteristics of both the internal and emitted radiation fields.

Consider a one dimensional system of length L, cross-sectional area A and thickness $d \ll L$ so that we may ignore angular effects. We can then write a Fokker-Planck equation for $P_{n,m,x}$, the probability of having n and m photons of energy $\mathcal{E} \geq (E_2 - E_1)_{min}$ travelling in a positive and negative x-direction respectively in a volume $A(dx)$ at a position x where $0 \leq x \leq L$. The transition probabilities, namely absorption, spontaneous emission and induced emission will be not only functions

of (Δx) but also of the electron or gas density in the appropriate levels. We will only consider a stationary system in the sense that at any position x, the average photon density is not a function of time. Thus we will not consider transient "switch on" or warm up" effects. We will also assume that the steady state values of the densities N_1 and N_2 are sufficiently large to be virtually independent of the photon density at any position x. In this analysis we are just dealing with photon numbers and do not account for the phase and coherence of the reflected waves. Our results therefore do not include the coherent emission line.

We are now able to derive a differential difference equation for $P_{n,m,x}$. $P_{n,m,x}$, however, is a function of both $P_{n,m,x-\Delta x}$ and $P_{n,m,x+\Delta x}$. To facilitate writing $P_{n,m,x}$ we will transform to a time scale using $(\Delta x) = c(\Delta t)$ The $P_{n,m,t}$ so defined is therefore completely determined be $P_{n,m,t-\Delta t}$. Thus we are dealing with a first order Markov process.

For the sake of mathematical symmetry and the possibility of other applications, we will include in the following list of transition probabilities, several not realizable in the two systems aforementioned.

Subscripts of $+$ or $-$ will refer to the x direction of the particle under consideration. Multiple subscripts will refer to the direction of the incoming particle and

the emitted particle or particles respectively.

Consider the following transition probabilities for an incident particle on an increment of width (Δx).

Absorption: $\alpha_+(\Delta x)$, $\alpha_-(\Delta x)$

This will include bulk absorption and surface losses due to angular scattering.

Spontaneous Emission: $\varepsilon_+(\Delta x)$, $\varepsilon_-(\Delta x)$

Induced Emission: $\beta_{+++}(\Delta x)$, $\beta_{++-}(\Delta x)$, $\beta_{+--}(\Delta x)$
$\beta_{---}(\Delta x)$, $\beta_{-+-}(\Delta x)$, $\beta_{-++}(\Delta x)$

Transmission: $1-\gamma_+(\Delta x)$, $1-\gamma_-(\Delta x)$

Reverse Scattering: $\sigma_{+-}(\Delta x)$, $\sigma_{-+}(\Delta x)$

Keeping only first order terms in (Δt), we can write:

$$
\begin{aligned}
P_{n,m,t+\Delta t} = {} & (n+1)\alpha_+ c(\Delta t) P_{n+1,m,t} + (n+1)\beta_{+--} c(\Delta t) P_{n+1,m-2,t} \\
& + (n+1)\sigma_{+-} c(\Delta t) P_{n+1,m-1,t} + n\beta_{+--} c(\Delta t) P_{n,m-1,t} \\
& + (n-1)\beta_{+++} c(\Delta t) P_{n-1,m,t} + (m+1)\alpha_- c(\Delta t) P_{n,m+1,t} \\
& + (m+1)\beta_{-++} c(\Delta t) P_{n-2,m+1,t} + (m+1)\sigma_{-+} c(\Delta t) P_{n-1,m+1,t} \\
& + m\beta_{-+-} c(\Delta t) P_{n-1,m,t} + (m-1)\beta_{---} c(\Delta t) P_{n,m-1,t} \\
& + \varepsilon_+ c(\Delta t) P_{n-1,m,t} + \varepsilon_- c(\Delta t) P_{n,m-1,t} \\
& + (1-n\gamma_+ c(\Delta t))(1-m\gamma_- c(\Delta t))(1-\varepsilon_+ c(\Delta t))(1-\varepsilon_- c(\Delta t)) P_{nmt}
\end{aligned}
$$

In the limit as $(\Delta t)\to 0$, we have in the space coordinate frame:

$$\frac{dP_{n,m,x}}{dx} = (n+1)\beta_{+--}P_{n+1,m-2,x} + (n+1)\sigma_{+-}P_{n+1,m-1,x}$$
$$+ (n+1)\alpha_{+}P_{n+1,m,x} + \left[(m-1)\beta_{---} + \varepsilon_{-} + n\beta_{++-}\right]P_{n,m-1,x}$$
$$- \left[n\gamma_{+} + \varepsilon_{+} + m\gamma_{-} + \varepsilon_{-}\right]P_{n,m,x}$$
$$+ (m+1)\beta_{-++}P_{n-2,m+1,x} + (m+1)\sigma_{-+}P_{n-1,m+1,x}$$
$$+ (m+1)\alpha_{-}P_{n,m+1,x} + \left[(n-1)\beta_{+++} + \varepsilon_{+} + m\beta_{-+-}\right]P_{n-1,m,x}$$

If we introduce the bivariate generating function:

$$G(r,s;x) = \sum_{n}\sum_{m} s^{n} r^{m} P_{n,m,x}$$

Then the above differential difference equation can be written as a partial differential equation in $G(r,s;x)$. Multiplying the above equation by $s^{n}r^{m}$ and summing over all m and n, we find that:

$$\frac{\partial G}{\partial x} = G\left[\varepsilon_{-}(r-1) + \varepsilon_{+}(s-1)\right]$$
$$+ \frac{\partial G}{\partial r}\left[\beta_{---}r^{2} + \beta_{+-}rs + \beta_{++}s^{2} + \sigma_{+}s - \gamma_{-}r + \alpha_{-}\right]$$
$$+ \frac{\partial G}{\partial s}\left[\beta_{+++}s^{2} + \beta_{++-}rs + \beta_{+--}r^{2} + \sigma_{-}r - \gamma_{+}s + \alpha_{+}\right]$$

If we substitute: $y = s - 1$ and $z = r - 1$

and let $\Gamma(y,z;x) = \ln G(y,z;x)$

Then:

$$\frac{\partial \Gamma}{\partial x} = (\varepsilon_{+}y + \varepsilon_{-}z)$$
$$+ \frac{\partial \Gamma}{\partial z}\left[z^{2}\beta_{---} + yz\beta_{+-} + y^{2}\beta_{++} - c_{1}z + c_{2}y\right]$$
$$+ \frac{\partial \Gamma}{\partial y}\left[y^{2}\beta_{+++} + yz\beta_{++-} + z^{2}\beta_{+--} - a_{1}y + a_{2}z\right]$$

where $a_1 = \beta_{+--} + \alpha_+ + \sigma_{+-} - \beta_{+++}$ $\quad a_2 = 2\beta_{+--} + \beta_{++-} + \sigma_{+-}$

$c_1 = \beta_{-++} + \alpha_- + \sigma_{-+} - \beta_{---}$ $\quad c_2 = 2\beta_{-++} + \beta_{-+-} + \sigma_{-+}$

and $\alpha_+ + \beta_{+++} + \beta_{++-} + \beta_{+--} + \sigma_{+-} = \gamma_+$

$\alpha_- + \beta_{---} + \beta_{-++} + \beta_{-+-} + \sigma_{-+} = \gamma_-$

3-2 Calculation of the first and second moments:

Generally the explicit evaluation of this partial differential equation is not possible. The various moments of n and m however may be found using:

$$\left(\frac{\partial \Pi}{\partial y}\right)_{\substack{y=0\\z=0}} = \overline{n_x} \qquad \left(\frac{\partial \Pi}{\partial z}\right)_{\substack{y=0\\z=0}} = \overline{m_x}$$

$$\left(\frac{\partial^2 \Pi}{\partial y^2}\right)_{\substack{y=0\\z=0}} = \text{Var } n_x - \overline{n_x} \qquad \left(\frac{\partial^2 \Pi}{\partial z^2}\right)_{\substack{y=0\\z=0}} = \text{Var } m_x - \overline{m_x}$$

$$\left(\frac{\partial^2 \Pi}{\partial y \partial z}\right)_{\substack{y=0\\z=0}} = \text{COV}(n_x, m_x)$$

The average value of any function f(n,m;x) is defined as:

$$\overline{f(n,m;x)} = \sum_n \sum_m f_{nmx} P_{nmx}$$

All moments of n and m can therefore be found from the partial differential equation by differentiating the appropriate number of times with the variables in question.

For simplicity, consider the general partial differential equation of the above form.

$$\frac{\partial \Pi(y,z;x)}{\partial x} = A_{y,z} + B_{y,z} \frac{\partial \Pi(y,z;x)}{\partial z} + C_{y,z} \frac{\partial \Pi(y,z;x)}{\partial y}$$

where $A_{00} = B_{00} = C_{00} = 0$

Differentiating with respect to y or z, the letting y = z = 0, we obtain the following two differential equations for $\overline{m_x}$ and $\overline{n_x}$ respectively, the average values of n and m at position x.

$$\left.\begin{aligned} \frac{d\overline{m_x}}{dx} &= \frac{\partial A}{\partial z} + \frac{\partial B}{\partial z}\overline{m_x} + \frac{\partial C}{\partial z}\overline{n_x} \\ \frac{d\overline{n_x}}{dx} &= \frac{\partial A}{\partial y} + \frac{\partial B}{\partial y}\overline{m_x} + \frac{\partial C}{\partial y}\overline{n_x} \end{aligned}\right]_{y=z=0}$$

Similarly we obtain the differential equations for the second moments in terms of:

$$\omega = \text{var } m_x - \overline{m_x}$$

$$V = \text{var } n_x - \overline{n_x}$$

and $$k = \text{cov}(n_x, m_x)$$

Therefore:

$$\frac{d\omega}{dx} = 2\frac{\partial C}{\partial z}k + 2\frac{\partial B}{\partial z}\omega + \frac{\partial^2 C}{\partial z^2}\overline{n_x} + \frac{\partial^2 B}{\partial z^2}\overline{m_x} + \frac{\partial^2 A}{\partial z^2}\Bigg]_{y=z=0}$$

$$\frac{dV}{dx} = 2\frac{\partial C}{\partial y}V + 2\frac{\partial B}{\partial y}k + \frac{\partial^2 C}{\partial y^2}\overline{n_x} + \frac{\partial^2 B}{\partial y^2}\overline{m_x} + \frac{\partial^2 A}{\partial y^2}\Bigg]_{y=z=0}$$

$$\frac{dk}{dx} = \frac{\partial C}{\partial z}V + \left(\frac{\partial C}{\partial y} + \frac{\partial B}{\partial z}\right)k + \frac{\partial B}{\partial y}\omega + \frac{\partial^2 C}{\partial y \partial z}\overline{n_x} + \frac{\partial^2 B}{\partial y \partial z}\overline{m_x} + \frac{\partial^2 A}{\partial y \partial z}\Bigg]_{y=z=0}$$

For a particular system, these coupled equations could readily be solved using Laplace transforms.

3-3 Equilibrium Fluctuations:

For the equilibrium case each increment of volume A(dx) will emit as many particles as it absorbs. By detailed balance the above statement holds for either

direction. The above equations will thus be independent of x in equilibrium. Then in Jacobian notation:

$$\bar{n} = \frac{\partial(A,B)}{\partial(z,y)} \Big/ \frac{\partial(B,C)}{\partial(z,y)}\Bigg]_{y=z=0} \qquad \bar{m} = \frac{\partial(C,A)}{\partial(z,y)} \Big/ \frac{\partial(B,C)}{\partial(z,y)}\Bigg]_{\substack{y=0\\z=0}}$$

Letting $M = \begin{pmatrix} v & k \\ k & w \end{pmatrix}$ $\qquad \alpha = \begin{pmatrix} \frac{\partial C}{\partial y} & \frac{\partial C}{\partial z} \\ \frac{\partial B}{\partial y} & \frac{\partial B}{\partial z} \end{pmatrix}_{\substack{y=0\\z=0}}$

and $\beta = \begin{pmatrix} \frac{\partial^2 I}{\partial y^2} & \frac{\partial^2 I}{\partial y \partial z} \\ \frac{\partial^2 I}{\partial y \partial z} & \frac{\partial^2 I}{\partial z^2} \end{pmatrix}_{\substack{y=0\\z=0}}$

where $I_{yz} = A_{yz} + B_{yz}\bar{m} + C_{yz}\bar{n}$

and $\det(\alpha) = \frac{\partial(C,B)}{\partial(y,z)} = \frac{\partial(B,C)}{\partial(z,y)}$

Then the matrix equation for the covariances is:

$$M\alpha + \alpha M = -\beta \qquad \alpha_{ij} = \alpha_{ji}$$

In equilibrium, the transition probabilities will be the same for particles propagating in the positive or negative x direction. In that case:

$$M\alpha = \alpha M$$

thus $$2M\alpha = -\beta$$

The moments of interest are then:

$$\text{var } n = \text{var } m = \bar{n} + \frac{\bar{n}\left[a_1\beta_{+++} + 2a_2\beta_{++-} + a_1\beta_{+--}\right]}{2(a_1^2 - a_2^2)}$$

$$\text{cov}(n,m) = \frac{\bar{n}\left[a_2\beta_{+++} + 2a_1\beta_{++-} + a_2\beta_{+--}\right]}{2(a_1^2 - a_2^2)}$$

where $\bar{n} = \bar{m} = \varepsilon_+/(a_1 - a_2)$

The fluctuations of the total number of particles in a volume $V = A(dx)$ is:

$$\text{var}(n + m) = \text{var } n + \text{var } m + 2\text{cov}(n,m)$$

For our system in equilibrium this reduces to:

$$\text{var}(n+m) = 2n_0 + \frac{n_0}{(a_1 - a_2)}\left[\beta_{+++} + 2\beta_{++-} + \beta_{+--}\right]$$

where $(a_1 - a_2) = \alpha_+ - (\beta_{+++} + \beta_{++-} + \beta_{+--})$

is positive in equilibrium.

If we write $\text{var } n = n(1 + k_1)$

and $\text{var}(n + m) = 2n(1 + k_2)$

then $k_2 = k_1 + \frac{\text{cov}(n,m)}{n_0}$

k_2 will equal k_1 only when there is no correlation between the n and m particles as is the case for photons.

3-4 Non-equilibrium photon distribution function

For photons in a gas or semiconductor, the transition probabilities β_{+--} and β_{-++} must be zero as the inducing photon does not undergo any transition. Also for a highly collimated beam, the probability of reverse scattering into a small solid angle is negligible. The direction of the induced emission will also be the same as the direction of the incident radiation. (Fowler,

1929; Sobel'man and Tyutin, 1963) Thus β_{++-} and β_{-+-} must also be zero.

In this case the cov(n,m) = 0 and the partial differential equation is separable. In the original G(r,s;x) notation we have:

$$\frac{\partial G}{\partial x} = G\left[\varepsilon_-(r-1) + \varepsilon_+(s-1)\right] + \frac{\partial G}{\partial r}\left[\beta_-(r^2-r) + \alpha_-(1-r)\right]$$

$$+ \frac{\partial G}{\partial s}\left[\beta_+(s^2-s) + \alpha_+(1-s)\right]$$

where we have dropped the multiple subscripts.

Let $G(r,s;x) = G_1(s;x)G_2(r;x)$

then $$\frac{\partial G_1(s;x)}{\partial x} = G_1\varepsilon_+(s-1) + \frac{\partial G_1}{\partial s}\left[\beta_+(s^2-s) + \alpha_+(1-s)\right]$$

$$\frac{\partial G_2(r;x)}{\partial x} = G_2\varepsilon_-(r-1) + \frac{\partial G_2}{\partial r}\left[\beta_-(r^2-r) + \alpha_-(1-r)\right]$$

For a generating function $\gamma_{x,s}$ if:

$$\frac{\partial \gamma_{xs}}{\partial x} = f(s)\gamma_{x,s} + g(s)\frac{\partial \gamma_{x,s}}{\partial s}$$

then $$\gamma_{xs} = H(\xi_{xs})\, e^{\int_0^x f(s_{x's})\,dx'}$$

where $\xi_{x,s}$ may be found from $$\int_s^{\xi_{x,s}} \frac{du}{g(u)} = x$$

and H(s) is the input photon distribution, namely $\gamma_{s,0}$

For the derivation of these relationships refer to Appendix II.

Solving for G (s;x):

then $$g(s) = \beta_+ s^2 - (\alpha_+ + \beta_+)s + \alpha_+$$

$$f(s) = \varepsilon_+(s-1)$$

Thus $$\int_{s}^{S_{x,s}} \frac{du}{\beta_+ u^2 - (\alpha_+ + \beta_+) u + \alpha_+} = x$$

$$S_{x,s} = 1 + \frac{(\alpha_+ - \beta_+)(s-1)/\beta_+}{(s-1)(1 - \exp(\alpha_+ - \beta_+)x) + \frac{\alpha_+ - \beta_+}{\beta_+} \exp(\alpha_+ - \beta_+)x}$$

Also: $$\int_0^x f(S_{x',s})\, dx' = \frac{\varepsilon_+(\alpha_+ - \beta_+)(s-1)}{\beta_+} \int_0^x \frac{dx'}{(s-1)(1 - \exp(\alpha_+ - \beta_+)x) + \frac{\alpha_+ - \beta_+}{\beta_+} \exp(\alpha_+ - \beta_+)}$$

$$= \ln\left[1 + \frac{(1-s)\beta_+}{\alpha_+ - \beta_+}\left(1 - \exp -(\alpha_+ - \beta_+)x\right)\right]^{-\varepsilon_+/\beta_+}$$

If we have an input photon distribution with effective temperature T_{10} for frequency ν, then the generating function for the boson distribution in N cells is

$$H(s) = \left[1 + b_{10}(1-s)\right]^{-N}$$

where $$b_{10} = \left[\exp\left(h\nu/kT_{10}\right) - 1\right]^{-1}$$

Thus

$$H(S_{x,s}) = \left[1 + \frac{b_{10}(1-s)\exp -(\alpha_+ - \beta_+)x}{1 + \frac{\beta_+}{\alpha_+ - \beta_+}(1-s)(1 - \exp -(\alpha_+ - \beta_+)x)}\right]^{-N}$$

The required solution is therefore:

$$G_1(s;x) = \left[1 + \frac{b_{10}(1-s)\exp -(\alpha_+ - \beta_+)x}{1 + \frac{\beta_+}{\alpha_+ - \beta_+}(1-s)(1 - \exp -(\alpha_+ - \beta_+)x)}\right]^{-N} \times$$

$$\left[1 + \frac{(1-s)\beta^+}{\alpha^+ - \beta^+}\left(1 - \exp -(\alpha_+ - \beta_+)x\right)\right]^{-\varepsilon_+/\beta_+}$$

N is the number of cells in boson phase space of the input photons. Since it is proportional to the volume it will not change throughout our system. Therefore $N = \varepsilon_+/\beta_+$ and

$$G_1(s;x) = \left[1 + (1-s)\left\{ b_{10} \exp -(\alpha_+-\beta_+)x + \frac{\beta_+}{\alpha_+-\beta_+}\left(1-\exp -(\alpha_+-\beta_+)x\right)\right\}\right]^{-N}$$

Where now $b_1(x)$ is the effective boson factor at position x of the n photons.

Then

$$b_1(x) = b_{10}\exp -(\alpha_+-\beta_+)x + \frac{\beta_+}{\alpha_+-\beta_+}\left(1-\exp -(\alpha_+-\beta_+)x\right)$$

Similarly for the m photons:

$$b_2(x) = b_{2L}\exp(\alpha_--\beta_-)(x-L) + \frac{\beta_-}{\alpha_--\beta_-}\left(1-\exp(\alpha_--\beta_-)(x-L)\right)$$

For a highly attenuating medium, namely when $\alpha_\pm \gg \beta_\pm$ the effective boson factors quickly lose all information about the input and finally reduce to:

$$b_1(L) = b_{1m} = \beta_+/(\alpha_+-\beta_+) \qquad b_2(0) = b_{2m} = \beta_-/(\alpha_--\beta_-)$$

where we define b_{1m} and b_{2m} as the effective boson factors of the medium. Also the gain of the medium to position x may be written as:

$$g_1(x) = \exp -(\alpha_+-\beta_+)x$$

$$g_2(x) = \exp(\alpha_--\beta_-)(x-L)$$

Then we have with Burgess (1961):

$$b_1(x) = b_{10}\,g_1(x) + b_{1m}\left(1-g_1(x)\right)$$

$$b_2(x) = b_{2L}\,g_2(x) + b_{2m}\left(1-g_2(x)\right)$$

where $g_i(x) < 1$ and $b_i(m) > 0$ if $\alpha > \beta$

and $g_i(x) > 1$ and $b_i(m) < 0$ if $\alpha < \beta$

Thus for an amplifying medium the induced emission must exceed the absorption. This corresponds to the medium being represented by a negative temperature.

In a gas or semiconductor laser for example, $g_1(x)$ and $g_2(L-x)$ will have to be equal. This is because the medium is homogeneous and the doppler broadened width of the emission spectrum will be the same as viewed from either end of the device. This is of course assuming the doppler broadening is greater than the Lorentz broadening.

If the energy difference between the two levels under consideration is $h\nu_0$, then the incoming photon must be of frequency $\nu_0\sqrt{\frac{1+\beta}{1-\beta}}$ or $\nu_0\sqrt{\frac{1-\beta}{1+\beta}}$ depending on whether the photon is travelling in the same or opposite direction respectively of the interacting electrons or atoms. $\beta = vn'/c$ where v is the velocity of the electron or atom and n' is the refractive index of the medium. If the velocity distribution of the particles is Maxwellian, then $\overline{v^2} = kT/m$ where m is the mass of the electron or gas atom. If $\beta \ll 1$, then the doppler width will have equal contributions from particles travelling in the same or opposite direction to the interacting photons.

Thus $$(\Delta\nu)_{Doppler} = \frac{2\nu_0 n'}{c}\sqrt{\frac{kT}{m}}$$

If we have a superimposed drift velocity V_D, for example due to an electric field applied along a semiconductor specimen, then the resultant velocity is $u = v \pm V_D$ as seen by an n or m photon. Then $\overline{u^2} = \overline{v^2} + V_D^2 = \frac{kT}{m} + V_D^2$ Again we have equal contributions to $(\Delta\nu)_D$ from electrons travelling in the same or opposite direction to the inducing photons. In this case:

$$(\Delta\nu)_{Doppler} = \frac{2\nu_0}{c}\sqrt{\frac{kT}{m} + V_D^2}$$

and the line is broadened.

The generating function derived for the photons in the medium is strictly correct only for a single frequency. It is however a good approximation for a narrow spectral line where the transition probabilities are independent of the frequency. To show this, let $P(n)$ be the probability of having n photons in a narrow frequency interval $(\Delta\nu)$. If we consider the discrete case where we have k different frequencies in $(\Delta\nu)$, then:

$$P(n) = \sum_{n_1}\sum_{n_2}\cdots\sum_{n_{k-1}} P_1(n_1)P_2(n_2)\cdots P_k(n_k)$$

subject to the constraint that $\sum_{i=1}^{k} n_i = n$

Then the generating function for $P(n)$ is:

$$G(s) = \sum_n s^n P(n) = \prod_{i=1}^{k} G_i(s)$$

For bosons: $$G_i(s) = (1 + b_i - b_i s)^{-N_i}$$

For $G(s)$ to have the form $G(s) = (1 + b - bs)^{-N}$,

all b_i must be the same $b_1 = b_2 = \cdots b_K = b$ and $N = \sum N_i$

Therefore for the above analysis to be strictly valid, we must be able to approximate the spectral line shape by a rectangular shape.

3-5 Cavity Line Width

Let us now introduce partially reflecting boundaries into our system at $x = 0$ and $x = L$, thus forming a cavity. Let p_1 and p_2 be the probabilities that a photon is reflected at $x = 0$ and $x = L$ respectively.

When photons are reflected or transmitted at a boundary they undergo a binomial process. Any statistical distribution of particles remains invariant in form or stable when subjected to a binomial process. (Refer to Appendix III) Thus bosons subjected to a binomial process are still bosons. If p is the probability of reflection and the incident photons have a boson factor b, then the reflected and transmitted photons will have boson factors pb and $(1-p)b$ respectively.

Let $\theta_1 = \alpha_+ - \beta_+$ and $\theta_2 = \alpha_- - \beta_-$

then
$$b_1(x) = b_{10} \exp(-\theta_1 x) + \frac{\beta_+}{\theta_1}\left(1 - \exp(-\theta_1 x)\right)$$

$$b_2(x) = b_{2L} \exp \theta_2 (x-L) + \frac{\beta_-}{\theta_2}\left(1 - \exp \theta_2 (x-L)\right)$$

When the system has reached a steady state, then:

$$b_{10} = p_1 b_{20}$$

$$b_{2L} = p_2 b_{1L}$$

Solving for b_{20} and b_{1L}, we find:

$$b_{20} = p_2\left(\beta_+/\theta_1\right)\frac{e^{-\theta_2 L}\left(1-e^{-\theta_1 L}\right)+\left(\beta_-/\theta_2\right)\left(1-e^{-\theta_2 L}\right)}{1-p_1p_2\exp-(\theta_1+\theta_2)L}$$

$$b_{1L} = p_1\left(\beta_-/\theta_2\right)\frac{e^{-\theta_1 L}\left(1-e^{-\theta_2 L}\right)+\left(\beta_+/\theta_1\right)\left(1-e^{-\theta_1 L}\right)}{1-p_1p_2\exp-(\theta_1+\theta_2)L}$$

For simplicity, we will only consider the symmetric case, when $\theta=\theta_1=\theta_2$ and $\beta=\beta_+=\beta_-$

Then

$$b_{20} = \frac{\beta}{\theta}\frac{\left(1-\exp(-\theta L)\right)\left(1+p_2\exp(-\theta L)\right)}{1-p_1p_2\exp(-2\theta L)}$$

$$b_{1L} = \frac{\beta}{\theta}\frac{\left(1-\exp(-\theta L)\right)\left(1+p_1\exp(-\theta L)\right)}{1-p_1p_2\exp(-2\theta L)}$$

For an attenuating medium, $\theta>0$ while for an amplifying medium $\theta<0$. b_{20} and b_{1L}, however, must always be positive.

Thus $$1-p_1p_2\exp(-2\theta L)>0$$

or $$\theta L > \frac{1}{2}\ln(p_1p_2)$$

In the steady state this will be the lower bound for θL or the upper bound for the amplification. If θL was less than the above minimum, for example in the initial warm up period in a laser, then the photon density would build up until the losses became sufficiently large such that $\theta L > \frac{1}{2}\ln(p_1p_2)$.

The boson factors for the output photons from the cavity will be:

At $x = 0$ $\qquad b_0 = (1-p_1)\, b_{20}$

$x = L$ $\qquad b_L = (1-p_2)\, b_{1L}$

The quality factor or "Q" of the loaded cavity may be found from the usual definition.

$$Q = \frac{\omega_0}{(\Delta\omega)_{cav.}} = 2\pi \left(\frac{\text{energy stored in the cavity}}{\text{energy lost/cycle}} \right)$$

For a narrow spectral line each photon will have energy approximately equal to $h\omega_0$ where ω_0 is the central frequency. Then:

$$\text{The energy stored in the cavity} = h\omega_0 N \left[\int_0^L b_1(x)\,dx + \int_0^L b_2(x)\,dx \right]$$

$$\text{The energy lost/cycle} = \frac{hcN}{n'} \left[b_0 + b_L \right]$$

where n' is the refractive index of the medium.

The cavity width $(\Delta\omega)_c$ is then:

$$(\Delta\omega)_c = \frac{c}{2\pi n'} \frac{b_0 + b_L}{\left[\int_0^L b_1(x)\,dx + \int_0^L b_2(x)\,dx \right]}$$

where

$$\int_0^L b_1(x)\,dx = \frac{\beta}{\theta^2}\left(e^{-\theta L} - 1\right) - \frac{b_{10}}{\theta}\left(e^{-\theta L} - 1\right) + \beta L/\theta$$

$$\int_0^L b_2(x)\,dx = \frac{\beta}{\theta^2}\left(e^{-\theta L} - 1\right) - \frac{b_{2L}}{\theta}\left(e^{-\theta L} - 1\right) + \beta L/\theta$$

Thus find:

$$(\Delta\omega)_c = \left(\frac{c}{2\pi n'}\right) \frac{(1 - e^{-\theta L})\left\{2 + (p_1+p_2)(e^{-\theta L} - 1) - 2p_1p_2 e^{-\theta L}\right\}}{2L\left(1 - p_1p_2 e^{-2\theta L}\right) + \frac{(e^{-\theta L} - 1)}{\theta}\left[2 + (p_1+p_2)(e^{-\theta L} - 1) - 2p_1p_2 e^{-\theta L}\right]}$$

When $p_1 = p_2 = p$, we have equal outputs from both ends of the cavity and $(\Delta\nu)_c$ simplifies to:

$$(\Delta\nu)_c = \frac{c(1-p)}{2\pi n' L} \cdot \frac{1}{1 - (1-p)\left(\frac{1}{L\theta} - \frac{1}{(e^{\theta L}-1)}\right)}$$

where $$\ln p < \theta L < \infty$$

The function $$S(\theta L) = \frac{1}{\theta L} - \frac{1}{(e^{\theta L}-1)}$$

is a monotonically decreasing function of θL , where

$$S(-\infty) = 1, \qquad S(0) = \frac{1}{2} \qquad S(\infty) = 0$$

The upper and lower bounds for $(\Delta\nu)_c$ are therefore:

$$\frac{c(1-p)}{2\pi n' L} < (\Delta\nu)_c < \frac{c}{2\pi n' L} \ln\left(\frac{1}{p}\right)$$

$(\Delta\nu_c)$ is therefore generally dependent on the active medium, its lower bound being the usual cavity width

$$\frac{c(1-p)}{2\pi n' L}$$

CHAPTER 4

PHOTON COUNTING

4-1. Introduction

In this chapter we will be concerned with calculating the fluctuations of the photoelectron numbers emitted from a detector due to an incident light source. We will also calculate the spectral density of the photoelectric current which would be the quantity of interest in any experimental measurements. We will illustrate our results by assuming various spectral line shapes for the incident light beam.

We will use the theory of analytic signals to describe the electric field on the detector surface. (Deutsch, 1962) Thus consider a quasi-monochromatic light wave incident on a photodetector of area A. Let E_t denote the electric vector on the detector surface at time t. Then

$$\vec{E}_t = \vec{u}_t + j\vec{v}_t$$

where u_t and v_t are Hilbert transform pairs, namely

$$v_t = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u_{t'} dt'}{t'-t}$$

P denoting the Cauchy principal value at $t' = t$. Thus if $u(\omega)$ is the Fourier transform of u_t, then the Fourier transform of v_t is:

$$V(\omega) = -j\,u(\omega) \qquad \omega > 0$$
$$= 0 \qquad \omega = 0$$
$$= j\,u(\omega) \qquad \omega < 0$$

The Fourier transform of E_t is therefore:

$$E_\omega = 2\,u(\omega) \qquad \omega > 0$$
$$= u(\omega) \qquad \omega = 0$$
$$= 0 \qquad \omega < 0$$

Thus using analytic signals, E_ω has positive frequency components only.

We will assume the electric vector components are gaussian distributed. For black body radiation this can be found by considering an ensemble of harmonic oscillators. This has been generally justified for "random waves" by various authors by appealing to the Central Limit theorem.

We will only consider plane polarized light in this discussion. It is shown in Appendix IV that for unpolarized light we have equal and additive contributions to the spectral density from the two resolved components of the electric field vector. A partially plane polarized source will also give additive contributions to the spectral density as we can consider the source as a superposition of an unpolarized and a polarized beam.

The area of coherence, defined as the area of the detector over which the radiation field is closely correlated is given by λ^2/Ω where λ is the wave-

length of the incident radiation and Ω is the solid angle subtended by the source at the detector. (Forrester, 1961) If C_1 is the number of coherence areas on our detector surface of area A, then $C_1 = A\Omega/\lambda^2$ We will be concerned with the photon and photoelectron statistics in the following analysis. Thus if there are n photons incident on the detector in a time T, then $n = \sum_{i=1}^{C_1} n_i$ where n_i are the number of photons incident in the i^{th} coherence area. Since the n_i in different coherence areas are uncorrelated, we have the following averages:

$$\langle n \rangle = N \langle n_i \rangle$$

$$\text{var } n = N \text{ var } n_i$$

We need therefore analyze only one coherence area, all areas being additive in the cumulants of which the mean and variance are the most important for present work.

Thus consider a beam of plane polarized light incident on one coherence area of the detector. Define the intensity of the light on the detector at time t as $I_t = E_t E_t^* = u^2 + v^2$. We will assume there is no reflection at the detector.

4-2 The first and second moments of the photon and photoelectron fluctuations

If $p_n(t,T)$ is the probability that n photons are incident on one coherence area of the photodetector in a

time interval t - T, t; where T is the detector resolving time, then:

$$p_1(t, dt) = \alpha I_t \, dt$$

where $\alpha = \lambda^2/\Omega R h\nu$

and $\lambda^2/\Omega = A/C_1$ ~ coherence area

$R = (\mu_0/\varepsilon_0)^{1/2}$ impedance of space

h Planck's constant

ν frequency of the input light

On the assumption of a Markov process:

$$\frac{\partial P_n(t, dt)}{dt} = \left[P_{n-1}(t, dt) - P_n(t, dt)\right] \alpha I_t$$

thus:

$$p_n(t, T) = \frac{1}{n!}\left[\alpha \int_{t-T}^{t} I_t' dt'\right]^n \exp\left[-\alpha \int_{t-T}^{t} I_t' dt'\right]$$

We will assume the detector response has a uniform memory or sampling time T. The mean value of n in a single system of an ensemble in the time interval T is:

$$\sum_n n \, p_n(t, T) = \overline{n_{t,T}} = \alpha \int_{t-T}^{t} I_t' dt'$$

$\overline{n_{tT}}$ will in general be a fluctuating number, the fluctuations being determined by the relative magnitude of T and τ_c, the correlation time for the input spectral line defined by: (Mandel, 1959)

$$\tau_c = \int_{-\infty}^{\infty} \psi_I(\tau) \, d\tau \Big/ \psi_I(0)$$

where $$\psi_I(\tau) = \langle I_t I_{t+\tau} \rangle - \langle I \rangle^2$$

is the autocorrelation function of the intensity

We will denote by a $\langle\ \rangle$ an ensemble average, thus the averaged quantity:

$$\langle \overline{n_{t\tau}} \rangle = \langle n_\tau \rangle = \left\langle \alpha \int_{t-\tau}^{t} I_t' dt' \right\rangle$$

$$= \alpha \langle I \rangle \tau$$

Similarly the second moment for a single photon counting system is:

$$\sum_n n^2 p_n(t,\tau) = \overline{n_{t,\tau}^2} = \alpha \int_{t-\tau}^{t} I_t' dt' + \alpha^2 \int_{t-\tau}^{t}\int_{t-\tau}^{t} I_u I_v \, du\, dv$$

Taking the ensemble average, we find:

$$\langle \overline{n_{t\tau}^2} \rangle = \langle n_\tau \rangle + \alpha^2 \int_{t-\tau}^{t}\int_{t-\tau}^{t} \langle I_u I_v \rangle \, du\, dv$$

$$\text{var } n_\tau = \langle n_\tau \rangle + \alpha^2 \int_0^\tau \int_0^\tau \psi_I(u-v)\, du\, dv$$

$$= \langle n_\tau \rangle + 2\alpha^2 \int_0^\tau (\tau - \tau)\, \psi_I(\tau)\, d\tau$$

We thus have super-poisson fluctuations, the second term in the variance due to the wave nature of the incident light. It is shown in Appendix V that this super-poisson character of the variance can be simulated by modulating a beam of poisson particles.

The above expression is therefore the variance of the number of photons incident on the detector area λ^2/Ω

in a time interval T, given the autocorrelation function of the incident light. This expression is similar to that derived by Mandel(1958) for the fluctuations of photoelectrons. His expression is not strictly valid as he assumes a deterministic rather than a stochastic quantum efficiency.

Using the results of Appendix III, we can readily determine the fluctuations of the emitted photoelectrons.

The generating function for $p_n(t,T)$ for a single system is:

$$G_p(s) = \sum_n s^n \, p_n(t,T)$$

$$= \exp\left[\overline{n}_T (s-1)\right]$$

If the detector has a quantum efficiency η and assuming that for every incident photon we have either zero or one photoelectron emitted, then we can write:

$$G_Q(s) = G_p(1-\eta+\eta s)$$

where $G_Q(s)$ is the generating function for $Q_m(t,T)$ the probability that m photoelectrons are emitted in a time interval t - T, t. Thus:

$$G_Q(s) = G_p(1-\eta+\eta s)$$

From the well known properties of the generating function, we obtain for the ensemble averages:

$$\langle m_T \rangle = \eta \langle n_T \rangle$$

$$\text{Var } m_T = \eta \langle n_T \rangle + 2(\alpha\eta)^2 \int_0^T (T-\tau)\, \psi_I(\tau)\, d\tau$$

4-3 Derivation of the photon distribution

The distribution of photons in an observing interval T is:

$$P_T(n) = \langle p_n(t,T) \rangle$$

$$= \left\langle \frac{1}{n!}\left(\alpha \int_{t-T}^{t} I_{t'}dt'\right)^n \exp\left(-\alpha \int_{t-T}^{t} I_{t'}dt'\right) \right\rangle$$

Let $G_T(s)$ be the generating function for $P_T(n)$

Then $$G_T(s) = \sum s^n P_T(n)$$

$$G_T(s) = \left\langle e^{\alpha(s-1)\int_{t-T}^{t} I_{t'}dt'} \right\rangle$$

If we let $$y = \int_{t-T}^{t} I_{t'}dt'$$

then $$G_T(s) = \int_0^\infty \exp[\alpha(s-1)y]\, P(y)\, dy$$

where P(y) is the probability distribution for the stochastic variable y. We will derive P(y) for the two limits; $T \ll \tau_c$ and $T \gg \tau_c$:

i) $T \ll \tau_c$

In a time $T \ll \tau_c$, I_t is a slowly varying function and we can write $y = IT$.

If we first only consider one coherence area, then:

$$I = EE^* = u^2 + v^2$$

where

$$P(\theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\theta^2}{2\sigma^2}} \qquad \theta = u, v$$

The average intensity on one coherence area is $I_o = 2\sigma^2$

If we let $z_1 = u^2$ and $z_2 = v^2$, then

$$P(z_{1,2}) = \frac{\exp[-z_{1,2}/2\sigma^2]}{\sigma\sqrt{2\pi z_{1,2}}}$$

and

$$P(I) = \frac{\exp[-I/I_0]}{\pi I_0} \int_0^I \frac{dz}{[z(I-z)]^{1/2}}$$

Therefore $P(I) = \frac{1}{I_0} \exp[-I/I_0]$

the intensity distribution on one coherence area. For C_1 coherence areas:

$$I = \sum_{i=1}^{C_1} I_i$$

where $$P(I_i) = \frac{\exp[-I_i/I_0]}{I_0}$$

Therefore since all areas are independent:

$$P(I)dI = P(I_1)dI_1\, P(I_2)dI_2 \ldots P(I_{C_1})dI_{C_1}$$

subject to the constraint $\sum_i I_i = I$

If we take the Laplace transform of $P(I)$:

$$S(u) = \int_0^\infty P(I) \exp(-uI)\, dI$$

then $$S_i(u) = \int_0^\infty P(I_i) \exp(-uI_i) dI_i = (1+uI_0)^{-1}$$

and $$S(u) = [S_i(u)]^{C_1} = (1+uI_0)^{-C_1}$$

is therefore the Laplace transform of the gamma distribution. Therefore when $T \ll \tau_c$:

$$G_T(s) = \int_0^\infty \exp[\alpha(s-1)IT]\, P(I)\, dI$$

$$= S[\alpha(1-s)T]$$

Thus: $$G_T(s) = \left(1 + \alpha I_0 T (1-s)\right)^{-c_1}$$

$$= \left(1 + \langle n_T \rangle (1-s)\right)^{-c_1}$$

where $\langle n_T \rangle$ is the mean number of photons incident on one coherence area in a time T.

ii) $T \gg \tau_c$

In this case there will only be correlation between photons arriving within a time $k\tau_c$ of each other, where k is a constant to be determined. Therefore:

$$y = k\tau_c \sum_{j=1}^{c_2} I_j$$

where $$c_2 = T/k\tau_c$$

If $$y = k\tau_c z$$

then

$$\Psi(u) = \int_0^\infty P(z) e^{-uz} dz$$

$$= \left[\Xi(u)\right]^{c_2}$$

$$= \left(1 + uI_0\right)^{-c_1 c_2}$$

In this case: $$G_T(s) = \int_0^\infty \exp\left[\alpha(s-1)k\tau_c z\right] P(z) dz$$

Thus: $$G_T(s) = \left[1 + \alpha I_0 k\tau_c (1-s)\right]^{-c_1 c_2}$$

We therefore obtain the boson distribution of having n photons distributed among $C = c_1 c_2$ cells, namely

$$P_T(n) = \frac{\Gamma(C+n)}{\Gamma(C)\, n!} \left(\frac{1}{1+b}\right)^C \left(\frac{b}{1+b}\right)^n$$

where $b = \alpha I_0 k \tau_c$, the mean number of photons in one cell.

The fluctuations of the n_T photons can be written as

$$\text{Var } n_T = \langle n_T \rangle (1 + b)$$

where

$$b = \langle n_T \rangle / C$$

Consequently

$$\text{Var } m_T = \langle m_T \rangle (1 + \eta b)$$

The super-poisson fluctuations of the photons thus reappear slightly reduced in the fluctuations of the photoelectrons. From the second moments previously derived, we deduce:

$$C = C_1 \frac{\langle I \rangle^2 T^2}{2} \left[\int_0^T (T - \tau) \psi_I(\tau) \, d\tau \right]$$

If our observing time $T \ll \tau_c$, then $\psi_I(\tau) \simeq \langle I \rangle^2$ throughout the domain of integration. Therefore:

$$C = C_1 \frac{\langle I \rangle^2 T^2}{2} \left[\langle I \rangle^2 \int_0^T (T - \tau) \, d\tau \right]^{-1}$$

and $C = C_1$ or $C_1 = 1$

Thus as expected for very short measuring times all photons arriving in a single coherence area occupy the same cell in phase space.

We will evaluate C explicitly for various $\psi_I(\tau)$

i) If the input spectral line shape is gaussian, then

$$\psi_I(\tau) = \langle I \rangle^2 \exp\left[-\pi \tau^2 / \tau_c^2 \right]$$

The number of cells C is given by:

$$C = C_1 \left[\frac{\tau_c}{T} \operatorname{erf}\left(\sqrt{\pi}\, T/\tau_c\right) - \frac{\tau_c^2}{\pi T^2}\left(1 - \exp\left(-\pi T^2/\tau_c^2\right)\right)\right]^{-1}$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2} dx$$

As expected, when $T \ll \tau_c$, then $C = C_1$. As our observing length increases beyond our coherence length, then only photons within a distance $L_c = c\,\tau_c$ of each other are correlated. Finally when $T \gg \tau_c$, we obtain $C = C_1 T/\tau_c$. Therefore $C_2 = T/\tau_c$ or $k = 1$. C is therefore the product of the number of coherence times τ_c in our measuring interval T and the number of coherence areas on our detector surface.

ii) Similarly if our line shape is Lorentzian, then

$$\psi_I(\tau) = \langle I \rangle^2 \exp\left[-2|\tau|/\tau_c\right]$$

and

$$C = C_1 \left[\left(\frac{\tau_c}{T}\right) - \frac{\tau_c^2}{2T^2}\left(1 - \exp\left(-2T/\tau_c\right)\right)\right]^{-1}$$

Again we find that when $T \ll \tau_c$, then $C = C_1$ and when $T \gg \tau_c$, then $C = C_1 T/\tau_c$. In this case also C is the product of the number of coherence times in our measuring interval T and the number of coherence areas on our detector surface.

4-4 Spectral density of the photoelectric current

Physical measurements on an incident light beam can be made indirectly by examining the spectral density of the fluctuations in the resultant photoelectric current in the detector circuit.

If $n_{t,T}$ is the number of photons incident on one coherence area of the detector in the time interval $t - T$, t; then let $m_{t,T}$ be the corresponding number of photoelec trons emitted within the same time interval. The photocurrent averaged over the interval of duration T terminating at t is therefore:

$$J_t = \frac{e}{T} m_{t,T}$$

where e is the electronic charge.

If we let $$\langle n_{t,T}\, n_{t+\tau,T}\rangle - \langle n_T\rangle^2 = \zeta(\tau)$$

and $$\langle m_{t,T}\, m_{t+\tau,T}\rangle - \langle m_T\rangle^2 = q(\tau)$$

Then $$\langle J_t\, J_{t+\tau}\rangle - \langle J_t\rangle^2 = \psi_J(\tau) = \left(\frac{e}{T}\right)^2 q(\tau)$$

We must now relate the photoelectron autocorrelation function to the photon autocorrelation function. This requires the consideration of the two cases, namely when $|\tau| \geq T$ and when $|\tau| \leq T$

i) When $|\tau| \geq T$, we are averaging the product of two separate samples of photons or photoelectrons. Let $P(n_1, n_2)$ be the probability of having n_1 photons in the sample t, $t + T$ and n_2 photons in the sample $t+\tau$, $t+\tau+T$. Let $Q(m_1, m_2)$ be the corresponding probability for the photoelectrons. Then since we are dealing with two non-

overlapping samples, we will have:

$$Q(m_1, m_2) = \sum_{n_1} \sum_{n_2} P(n_1, n_2)\, B_{n_1}(m_1)\, B_{n_2}(m_2)$$

where $$B_n(m) = \binom{n}{m} \eta^m (1-\eta)^{n-m}$$

is the binomial distribution.

Then $$\sum_m s^m B_n(m) = (1-\eta+\eta s)^n$$

where η is the quantum efficiency of the detector.

The bivariate generating functions for $P(n_1, n_2)$ and $Q(m_1, m_2)$ are defined as follows:

$$G_P(r,s) = \sum_{n_1} \sum_{n_2} P(n_1, n_2)\, r^{n_1} s^{n_2}$$

$$G_Q(r,s) = \sum_{m_1} \sum_{m_2} Q(m_1, m_2)\, r^{m_1} s^{m_2}$$

Then find that:

$$G_Q(r,s) = \sum_{n_1} \sum_{n_2} P(n_1, n_2)(1-\eta+\eta r)^{n_1} (1-\eta+\eta s)^{n_2}$$

$$= G_P(1-\eta+\eta r,\ 1-\eta+\eta s)$$

Then from the properties of the generating function;

$$\langle m_1 m_2 \rangle = \left. \frac{\partial^2 G_Q(r,s)}{\partial r\, \partial s} \right]_{\substack{r=1 \\ s=1}} = \eta^2 \langle n_1 n_2 \rangle$$

Therefore if $|\tau| \geq T$

$$q(\tau) = \eta^2 f(\tau)$$

In this case:

$$\langle n_{t,T}, n_{t+\tau,T} \rangle = \left\langle \sum_{n_1} \sum_{n_2} n_1 n_2\, p_{n_1}(t,T)\, p_{n_2}(t+\tau, T) \right\rangle$$

where $p_n(t,T)$ is the Poisson distribution originally defined. Therefore:

$$f(\tau) = h(T,T,\tau) = \alpha^2 \int_0^T du \int_0^T \psi_I(v-u+\tau)\, dv$$

where the two T 's in $h(T,T,\tau)$ represent the two upper limits of integration respectively and τ is self evident.

ii) When $|\tau| \leq T$, we are averaging the product of two overlapping samples. Let $n_1 = \xi_1 + \xi_2$ and $n_2 = \xi_2 + \xi_3$ where n_1 and n_2 are as previously defined. ξ_2 is the number of photons contained in the overlapping volume of the two samples. Similarly for the photoelectrons, let $m_1 = \eta_1 + \eta_2$ and $m_2 = \eta_2 + \eta_3$

Then again:

$$\langle n_1 \rangle = \langle n_2 \rangle = \langle n_T \rangle$$
$$\langle m_1 \rangle = \langle m_2 \rangle = \langle m_T \rangle$$

and

$$f(\tau) = \operatorname{cov}(\xi_1,\xi_2) + \operatorname{cov}(\xi_1,\xi_2) + \operatorname{cov}(\xi_2,\xi_3) + \operatorname{var}\xi_2$$
$$g(\tau) = \operatorname{cov}(\eta_1,\eta_2) + \operatorname{cov}(\eta_1,\eta_3) + \operatorname{cov}(\eta_2,\eta_3) + \operatorname{var}\eta_2$$

From the generating function on the previous page:

$$\langle m_T(m_T-1)\rangle = \left.\frac{\partial^2 G_Q(r,s)}{\partial s^2}\right]_{\substack{r=1\\s=1}} = \left.\frac{\partial^2 G_Q(r,s)}{\partial r^2}\right]_{\substack{r=1\\s=1}}$$

$$= \eta^2 \langle n_T(n_T-1)\rangle$$

Therefore $\operatorname{var} m_T = \eta^2 \operatorname{var} n_T + \langle n_T \rangle \eta(1-\eta)$

This is the expression for the fluctuations of the photoelectrons as a function of the photon fluctuations.

Evaluating the covariances in $f(\tau)$, one finds:

$$\operatorname{cov}(\xi_1,\xi_2) = \operatorname{cov}(\xi_2,\xi_3) = h(\tau, T-\tau, \tau)$$
$$\operatorname{cov}(\xi_1,\xi_3) = h(T,\tau,T)$$

$$\text{var } S_2 = h(T-\tau, T-\tau, 0) + \langle S_2 \rangle$$

It is readily verified that:

$$h(T,T,\tau) = 2h(\tau, T-\tau, \tau) + h(\tau,\tau,\tau) + h(T-\tau, T-\tau, 0)$$

Therefore:

$$f(\tau) = h(T,T,\tau) + \langle S_2 \rangle$$

Since $$\text{cov}(\xi_i \xi_j) = \eta^2 \text{cov}(S_i S_j) \qquad i \neq j$$

and $$\text{var } \xi_2 = \eta^2 \text{var } S_2 + \langle S_2 \rangle \eta(1-\eta)$$

we find: $$g(\tau) = \eta^2 \left[h(T,T,\tau) + \langle S_2 \rangle \right] + \langle S_2 \rangle \eta(1-\eta)$$

But $$\langle S_2 \rangle = \left(\frac{T - |\tau|}{T} \right) \langle n_T \rangle$$

Therefore the general photocurrent autocorrelation function is:

$$\psi_I(\tau) = \left(\frac{e}{T}\right)^2 \left[\eta^2 h(T,T,\tau) + \eta \langle n_T \rangle \left(\frac{T - |\tau|}{T} \right) \right] \quad \text{if } |\tau| \leq T$$

$$= \left(\frac{e}{T}\right)^2 \eta^2 h(T,T,\tau) \qquad \text{if } |\tau| \geq T$$

Mandel and Wolf(1961) obtained the term involving $h(T,T,\tau)$ as valid for all τ due to using a deterministic rather than stochastic quantum efficiency.

The spectral density of the photoelectron fluctuations due to an incident light beam is therefore:

$$S_I(f) = 2 \int_{-\infty}^{\infty} \psi_I(\tau)\, e^{-j\omega\tau}\, d\tau$$

The first term of $\psi_I(\tau)$ valid for all τ can be simplified to:

$$\frac{\langle I\rangle^2}{T^2}\int_{-T}^{T}(T-|x|)\,\gamma(x+\tau)\,\gamma^*(x+\tau)\,dx$$

where $\gamma(\tau)$ is the normalized correlation function for the electric field defined in Appendix IV.

Thus:

$$S_I(f) = \frac{2e\langle I\rangle}{T^2}\int_{-T}^{T}(T-|\tau|)\,e^{-j\omega\tau}\,d\tau + \frac{2\langle I\rangle^2}{T^2}\int_{-T}^{T}(T-|x|)\,dx\int_{-\infty}^{\infty}\gamma(x+\tau)\,\gamma^*(x+\tau)\,e^{-j\omega\tau}\,d\tau$$

By Parseval's theorem:

$$\int_{-\infty}^{\infty}\gamma(x+\tau)\,\gamma^*(x+\tau)\,e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}G_1(u,x)\,G_2(f-u,x)\,du$$

where

$$G_1(f,x) = \int_{-\infty}^{\infty}\gamma(x+\tau)\,e^{-j\omega\tau}\,d\tau = e^{j\omega x}\,F(f)$$

and F(f) is the normalized spectral line shape derived in Appendix IV. Similarly;

$$G_2(-f,x) = e^{-j\omega x}\,F(f)$$

The spectral density becomes:

$$S_I(f) = 2e\langle I\rangle\left[\frac{2}{(\omega T)^2}(1-\cos\omega T)\right] + \frac{2\langle I\rangle^2}{T^2}\int_{-T}^{T}(T-|x|)\,e^{j\omega x}\,dx\int_{0}^{\infty}F(v)\,F(v+f)\,dv$$

Or

$$S_I(f) = 2e\langle I\rangle\left[\frac{2}{(\omega T)^2}(1-\cos\omega T)\right] + 2\langle I\rangle^2\left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}\right)^2\int_0^\infty F(v)\,F(v+f)\,dv$$

Frequencies of interest will be much less than the inverse of the detector response time. Thus we can write for $(\omega T) \ll 1$,

$$S_I(f) = 2e\langle I\rangle + 2\langle I\rangle^2\int_0^\infty F(v)F(v+f)\,dv$$

The shot noise term may be associated with the particle fluctuation nature of the light while the convolution term is associated with the wave interference nature of the incident light. The second or photoelectric mixing term of $S_I(f)$ was derived by Forrester (1961) using somewhat implausible arguments.

As stated the above derivation is valid only for light incident on one coherence area. Due to the independence of the fluctuations in different areas we can write:

$$S_I(f) = C_1\,S_{I_i}(f)$$

and

$$\langle I\rangle = C_1\langle I_i\rangle$$

where $\langle I_i\rangle$ is the average photocurrent contribution from the i^{th} coherence area and $\langle I\rangle$ is the total mean photocurrent. Thus the spectral density for the total photocurrent in the detector circuit is:

$$S_I(f) = 2e\langle I\rangle + \frac{2\langle I\rangle^2}{C_1}\int_0^\infty F(v)\,F(v+f)\,dv$$

To determine the relative magnitude of the two terms, consider the case of black body radiation in a frequency interval (ΔW), for example a spectral line from a gas discharge. The wave interference term divided by shot noise is therefore:

$$S = \frac{\langle I\rangle}{e\,C_1(\Delta W)}$$

$$= \frac{\eta\langle n_T\rangle}{C_1 T(\Delta W)} = \frac{\eta c A\,\rho(W)}{C_1 h W}$$

where $\rho(W) = \Omega\, h W^3\, b(W)/c^3$ and $A/C_1 = \lambda^2/\Omega$

Therefore $S = \eta\, b(W)$ where $b(W) = \left(e^{\frac{hv}{KT}} - 1\right)^{-1}$

For a source temperature of $10{,}000^\circ K$ and wavelength $\lambda = 6000$ Ang. and a quantum efficiency of unity we find $S = 0.1$. This is also valid for unpolarized light because of the additive nature of the spectral density. For thermal sources in the optical range therefore the wave interference term will be undetectable.

Since the advent of lasers, we have access to light sources with very high effective temperatures. This has made possible spectral density measurements where the wave interference contribution is dominant compared to the shot noise.

We will define a bandwidth of our input spectral line as:

$$(\Delta f) = \left[\int_0^\infty F^2(f)\, df\right]^{-1}$$

where F(f) is the normalized input spectrum. It is readily verified, see Appendix IV, that $(\Delta f)\tau_c = 1$.

If our input line shape is:

i) Gaussian:

then
$$F(f) = \frac{\exp}{\sigma\sqrt{2\pi}}\left[-(f-f_0)^2/2\sigma^2\right]$$

where $(\Delta f) = 2\sqrt{\pi}\,\sigma$ and f_0 is the central frequency of the line. We assume that $(\Delta f) \ll f_0$

If B is the bandwidth at half intensity, then

$$(\Delta f) = \left(\frac{\pi}{2\ln 2}\right)^{1/2} B \simeq 1.5\, B$$

Thus:
$$S_I(f) = 2e\langle I\rangle + \frac{2\langle I\rangle^2}{(\Delta f)} \exp\left[-\pi f^2/(\Delta f)^2\right]$$

ii) Rectangular:

Then
$$F(f) = (\Delta f)^{-1} \quad \text{for} \quad |f-f_0| \le (\Delta f)/2$$
$$= 0 \quad \text{for} \quad |f-f_0| > (\Delta f)/2$$

and
$$S_I(f) = 2e\langle I\rangle + \frac{2\langle I\rangle^2}{(\Delta f)^2}\left[(\Delta f) - f\right] \quad \text{for } 0 \le f \le (\Delta f)$$
$$= 2e\langle I\rangle \quad \text{for} \quad f \ge (\Delta f)$$

For $f \gg (\Delta f)$ there will be only the pure shot noise component present.

iii) Lorentzian

$$F(f) = \frac{2B}{\pi\left[4(f-f_0)^2 + B^2\right]}$$

where B is the bandwidth at half intensity.

In this case

$$(\Delta f) = \pi B$$

and

$$S_I(f) = 2e\langle I\rangle + \frac{2\langle I\rangle^2 B}{\pi}\left(\frac{1}{f^2+B^2}\right)$$

In a gas laser, the mirror spacing is usually sufficiently large to allow the excitation of several longitudinal modes within the doppler broadened line width. If for example we had N longitudinal modes each with a gaussian lineshape, then

$$F(f) = \sum_{i=1}^{N} \frac{\beta_i \exp\left[-(f-f_i)^2/2\sigma_i^2\right]}{\sigma_i\sqrt{2\pi}}$$

where β_i determines the relative intensity of each line. Then

$$\Sigma \beta_i = 1$$

since F(f) is the normalized line shape.

f_i = central frequency of the i^{th} mode

$(\Delta f_i) = 2\sqrt{\pi}\,\sigma_i$ is the linewidth of the i^{th} mode

From the definition of (Δf) we thus obtain:

$$(\Delta f)^{-1} = \sum_i \sum_j \frac{\beta_i \beta_j \exp\left[-(f_i-f_j)^2/2(\sigma_i^2+\sigma_j^2)\right]}{\left[2\pi(\sigma_i^2+\sigma_j^2)\right]^{1/2}}$$

For two lines of equal intensity and width:

$$(\Delta f) = 4\sigma\sqrt{\pi}\Big/\left(1+\exp\left[-\frac{(f_2-f_1)^2}{4\sigma^2}\right]\right)$$

When the lines are will separated, (Δf) is just the sum of the two individual line widths. The spectral density is then:

$$S_I(f) = 2e\langle I\rangle + \frac{2\langle I\rangle^2}{[2\pi]^{1/2}} \sum_i \sum_j \frac{\beta_i \beta_j \exp\left[-\frac{(f+f_i-f_j)^2}{2(\sigma_i^2+\sigma_j^2)}\right]}{[\sigma_i^2+\sigma_j^2]^{1/2}}$$

When $i = j$, the wave interference term is just the sum of N individual self-mixing spectra. The $i \neq j$ terms correspond to the photoelectric mixing between different modes and are centered at the corresponding difference frequencies.

4-5 Photon counting for a modulated light source

If we modulate sinusoidally the input light intensity, then the intensity at the detector is:

$$I_t' = a I_t \left[1 + m \cos \omega_1 t\right]$$

where I has all the properties previously defined. $a(1+m)$ and m are less than unity and $\omega_1 = 2\pi f_1$ is the modulating frequency.

For a single spectral line this modulation has the effect of producing two sidebands each with intensity $(m/2)^2 I_o$ where I_o is the intensity of the central line. For the resolution of these three lines, the modulation frequency $f_1 > (\Delta f)$ the linewidth of the incident beam.

The average photocurrent is therefore:

$$\langle I\rangle = e\eta\alpha a \langle I\rangle$$

The wave interference autocorrelation function of the photocurrent after uniform averaging over T is:

$$\psi_I(\tau) = \frac{\langle I\rangle^2 m^2}{(\omega_1 T)^2}(1-\cos\omega_1 T)\cos\omega_1\tau + \frac{\langle I\rangle^2}{T^2}\int_{-T}^{T}(T-|x|)\left(1+\frac{m^2}{2}\cos\omega_1(x+\tau)\right)\gamma(x+\tau)\gamma^*(x+\tau)\,dx$$

The spectral density is:

$$S_I(f) = 2e\langle I\rangle + \frac{2\langle I\rangle^2 m^2}{(\omega_1 T)^2}(1-\cos\omega_1 T)\,\delta(f-f_1) + 2\langle I\rangle^2\left(1-\frac{m^2}{2}\right)\left(\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}\right)^2\int_0^\infty F(v)F(v+f)\,dv + \frac{2\langle I\rangle^2 m^2}{T^2}\int_{-T}^{T}(T-|x|)\,dx\int_{-\infty}^{\infty}\cos^2\frac{\omega_1}{2}(x+\tau)\,\gamma(x+\tau)\gamma^*(x+\tau)e^{-j\omega\tau}\,d\tau$$

Again using Parseval's theorem:

$$\int_{-\infty}^{\infty}\cos^2\frac{\omega_1}{2}(x+\tau)\,\gamma(x+\tau)\gamma^*(x+\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}H_1(u;x)\,H_2(f-u;x)\,du$$

where $$H_1(f;x) = \int_{-\infty}^{\infty}\cos\frac{\omega_1}{2}(x+\tau)\,\gamma(x+\tau)e^{-j\omega\tau}\,d\tau$$

and $$H_2(f;x) = \int_{-\infty}^{\infty}\cos\frac{\omega_1}{2}(x+\tau)\,\gamma^*(x+\tau)e^{-j\omega\tau}\,d\tau$$

$H_1(f;x)$ as a function of the input light spectrum is therefore:

$$H_1(f;x) = \frac{e}{2}^{-j\omega x}\left[F(f-f_1/2) + F(f+f_1/2)\right]$$

$$H_2(-f;x) = \frac{e}{2}^{-j\omega x}\left[F(f-f_1/2) + F(f+f_1/2)\right]$$

Using the properties of F(f), we find:

$$\int_{-\infty}^{\infty} H_1(u;x)\, H_2(f-u;x)\, du = \left[\exp[\omega x]/4\right]\left[2\int_0^{\infty} F(v)F(v+f)\,dv + \int_0^{\infty} F(v)F(v+|f-f_1|)\,dv + \int_0^{\infty} F(v)F(v+f+f_1)\,dv\right]$$

Thus:

$$S_I(f) = 2e\langle I\rangle\left[\frac{2}{(\omega T)^2}(1-\cos\omega T)\right] + \frac{2\langle I\rangle^2 m^2}{(\omega_1 T)^2}(1-\cos\omega_1 T)\,\delta(f-f_1) + 2\langle I\rangle^2\left(\frac{\sin(\omega T/2)}{\omega T/2}\right)^2\int_0^{\infty} F(v)F(v+f)\,dv + \frac{\langle I\rangle^2 m^2}{2}\left(\frac{\sin(\omega T/2)}{\omega T/2}\right)^2\left[\int_0^{\infty} F(v)F(v+|f-f_1|)\,dv + \int_0^{\infty} F(v)F(v+f+f_1)\,dv\right]$$

Again for frequencies of interest, $\omega T \ll 1$ and:

if $\omega_1 T \ll 1$;

$$S_I(f) = 2e\langle I\rangle + \langle I\rangle^2 m^2\delta(f-f_1) + 2\langle I\rangle^2\int_0^{\infty} F(v)F(v+f)\,dv + \frac{\langle I\rangle^2 m^2}{2}\left[\int_0^{\infty} F(v)F(v+|f-f_1|)\,dv + \int_0^{\infty} F(v)F(v+f+f_1)\,dv\right]$$

If the input lineshape is gaussian, then again

$$F(f) = \frac{\exp\left[-(f-f_0)^2/2\sigma^2\right]}{\sigma\sqrt{2\pi}} \quad \text{where} \quad (\Delta f) = 2\sqrt{\pi}\,\sigma$$

The spectral density of the photocurrent is:

$$S_I(f) = 2e\langle I\rangle + \langle I\rangle^2 m^2\delta(f-f_1) + \frac{2\langle I\rangle^2}{(\Delta f)}\left[e^{-\pi f^2/(\Delta f)^2} + \left(\frac{m}{2}\right)^2 e^{-\pi(f-f_1)^2/(\Delta f)^2} + \left(\frac{m}{2}\right)^2 e^{-\pi(f+f_1)^2/(\Delta f)^2}\right]$$

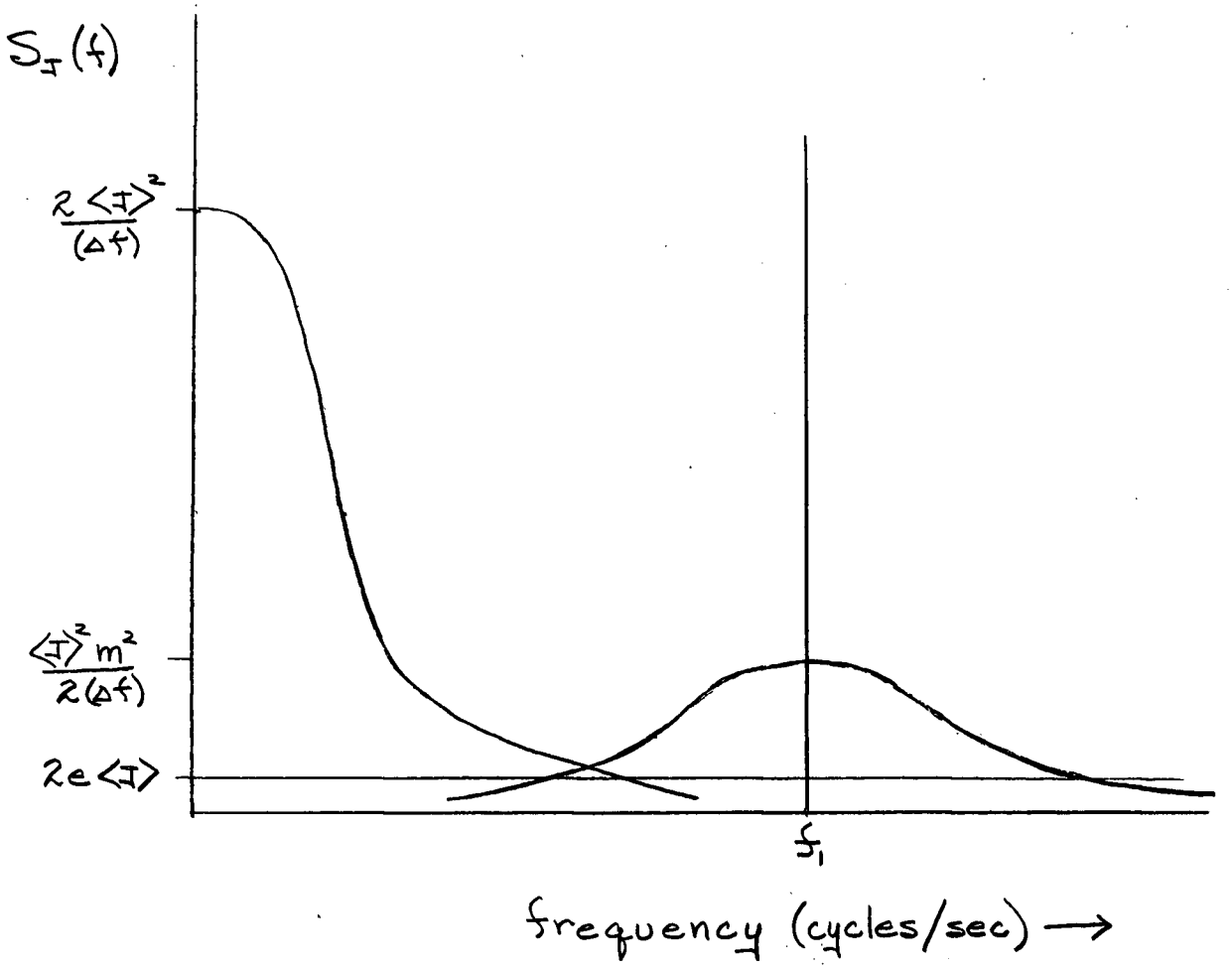

FIGURE 4-1

SUPERIMPOSED SPECTRA FOR A MODULATED LIGHT SOURCE WITH A DOMINANT WAVE INTERFERENCE TERM

The third term in the spectrum is the sum of the self-mixing terms of the optical line and its two sidebands. The last two terms correspond to the mixing of the input spectral line with its two optical sidebands. Since the three spectral lines are derived from the same input spectrum, all components of frequency difference f_1 are completely coherent. This results in the delta function at the modulation frequency f_1 in the detector circuit. (see figure 4-1)

When the wave interference term is negligible as compared to the shot noise, for example from a black body source, then the photocurrent spectrum is:

$$S_I(f) = 2e\langle I\rangle + m^2\langle I\rangle \delta(f-f_1)$$

Thus a modulated black body source can be used for the transmission of information.

CHAPTER 5

CONCLUSIONS

An alternative derivation of the number fluctuations in a multilevel system has been obtained using the Langevin approach. van Vliets analysis of a three level semiconductor model has been extended and a lower bound of 0.75 was obtained for the ratio of the variance to the mean number of the conduction electrons. A special case of a four level system was discussed. It was shown that in certain limits super-poisson fluctuations of the conduction electron numbers could occur.

A general expression for the autocorrelation function for a three level system was derived. It was demonstrated that in certain limits this was the sum of two sinusoidally damped terms. A general criterion for the appearance of oscillatory correlation is a profitable topic for future investigation.

Another important area for further work is the examination of non-stationary systems. This would require a modification of the present theory in terms of the time-dependent Fokker-Planck formulation.

The fluctuations of the photon numbers as a function of position within a cavity enclosing an active homogeneous medium were derived. The width of the emission line was obtained and for small mirror reflectivities

shown to be a function of the cavity amplification. For reflectivities almost equal to unity, the width is virtually independent of the amplification and reduces to the unloaded cavity width, $c(1-\bar{P})/2\pi n'L$.

Further analysis would require the consideration of wave interference effects and some specific coupling between the photons and gas atoms.

The spectral density of the fluctuations in a photoelectric current was derived by considering a plane polarized light beam incident on a detector of area A and resolving time T. By using a stochastic quantum efficiency, a term additional to that derived by Mandel (1958) is obtained in the autocorrelation function. The spectral density derived from this correlation function is then the sum of a particle noise and wave interference term.

The distribution of photons incident on a detector of area A in a time interval T has the boson distribution for C cells in phase space. C is found to be the product of the number of coherence areas on the detector surface and the ratio of the observing interval T to the reciprocal of the line width of the incident radiation.

Further work would consist of deriving the spectral density of the photocurrent fluctuations for any arbitrary polarization of the incident light.

APPENDIX I

The six equations for evaluating the covariances for the four level system are:

$$-qp(k+1)C_{11} + rqC_{12} + kpsC_{13} = m_0 q(k-1)$$

$$pq(p+q)C_{11} - q\left[(p+pk+r+1)(p+q+1) + p\right]C_{12} - pqC_{13} + rqC_{22} + kpsC_{23} = m_0 q(p+q+1)$$

$$pq(p+q)C_{11} - qpC_{12} - p\left[(q+qk+s+1)(p+q+1) + q\right]C_{13} + psC_{33} + (rq/k)C_{23} = i_0 p(p+q+1)$$

$$kp\left[q(p+q)C_{12} - qC_{22} - \{(s+1)(q+p+1)+q\}C_{23}\right] + q\left[p(p+q)C_{13} - pC_{33} - \{(r+1)(p+q+1)+p\}C_{23}\right] = 0$$

$$p(p+q)C_{12} - \{(r+1)(p+q+1)+p\}C_{22} - pC_{23} = -m_0(p+q+1)$$

$$q(p+q)C_{13} - \{(s+1)(p+q+1)+q\}C_{33} - qC_{23} = -i_0(p+q+1)$$

APPENDIX II

We will extent the analysis of Bartlett (1951) for a Markov Chain to include spontaneous emission. First of all, however, we will briefly review Bartlett's results.

At t=0, we have a one particle input into our system. Each particle can generate a distribution of particles, whose generating function is $G(s)$. If events occur at discrete times (Δt) apart, then $S_n(s)$ is the generating function of the system after the n^{th} event, namely at a time $n(\Delta t)$.

Thus $$S_0(s) = s \qquad S_1(s) = G(s)$$

$$S_2(s) = G(G(s)) = S_1(G(s)) = G(S_1(s))$$

Similarly:

$$S_{n+1}(s) = S_n(G(s)) = G(S_n(s))$$

As $\Delta t \to 0$ we can write

$$G(s) - s = q(s)\,\Delta t$$

and $$S_{t+\Delta t}(s) = S_t(G(s)) = G(S_t(s))$$

Expanding in powers of (Δt).

$$S_{ts} + \frac{\partial S}{\partial t}(\Delta t) = S_{ts} + \frac{\partial S}{\partial s} q(s)\,(\Delta t)$$

and

$$S_{ts} + \frac{\partial S}{\partial t}(\Delta t) = S_{ts} + q(S_t(s))\,\Delta t$$

We therefore obtain Bartlett's results:

$$\frac{\partial S_{ts}}{\partial t} = q(s)\frac{\partial S}{\partial s}_{ts}$$

$$\frac{\partial S_{ts}}{\partial t} = q(S_{ts})$$

To find S_{ts} , we then simply evaluate

$$\int_{s}^{S_{ts}} \frac{du}{g(u)} = t$$

The former analysis has not included the possibility of spontaneous emission at any time t. Let F(s) be the generating function of the distribution of particles generated spontaneously in a time (Δt) . Let us now denote the total generating function of the system after the n^{th} event as γ_{ns}.

Then again for a one particle input: $\gamma_0(s) = s$

$$\gamma_1(s) = \gamma_0(G(s))\, F(s)$$

$$\gamma_{n+1}(s) = \gamma_n(G(s))\, F(s)$$

Expanding in (Δt) , we obtain in the limit as the differential equation.

$$\frac{\partial \gamma_{ts}}{\partial t} = f(s)\,\gamma_{ts} + g(s)\,\frac{\partial \gamma_{ts}}{\partial s}$$

where we have used:

$$G(s) - s = g(s)(\Delta t)$$

$$F(s) - 1 = f(s)(\Delta t)$$

To solve explicitly for γ_{ts} we must proceed as follows:

$$\gamma_1(s) = S_1(s)\, F(s)$$

$$\gamma_2(s) = S_2(s)\, F(S_1(s))\, F(s)$$

thus $$\gamma_{n+1}(s) = S_{n+1}(s) \prod_{r=0}^{n} F(S_r(s))$$

or taking the logarithm:

$$\log \gamma_{n+1}(s) = \log S_{n+1}(s) + \sum_{r=0}^{n} \log F(S_r(s))$$

Again using $$F(s) - 1 = f(s)\,\Delta t$$

and also: $\log(a+b) = \log a + b/a$ if $a \gg b$

we find upon expansion in (Δt):

$$\log \gamma_{ts} + \frac{1}{\gamma_{ts}} \frac{\partial \gamma}{\partial t}(\Delta t) = \log \varsigma_{ts} + \frac{1}{\varsigma_{ts}} \frac{\partial \varsigma}{\partial t}(\Delta t) + \sum_n f(\varsigma_{n\Delta t})\,\Delta t$$

In the limit as $\Delta t \to 0$ we find

$$\gamma_{ts} = \varsigma_{ts} \exp \int_0^t f(\varsigma_{s\tau})\, d\tau$$

Obviously for no spontaneous emission, f(s) = 0, and $\gamma_{ts} = \varsigma_{ts}$ as previously derived.

If instead of a one particle input, we have an input distribution generating function H(s), then:

$$\gamma_0(s) = H(s)$$
$$\gamma_1(s) = H(\varsigma_1(s))\, F(s)$$
$$\gamma_2(s) = H(\varsigma_2(s))\, F(\varsigma_1(s))\, F(s)$$

thus $$\gamma_{n+1}(s) = H(\varsigma_{n+1}(s)) \prod_{r=0}^{n} F(\varsigma_r(s))$$

Taking logs and expanding as before, we find that:

$$\gamma_t(s) = H(\varsigma_{ts}) \exp \int_0^t f(\varsigma_{s\tau})\, d\tau$$

where as before $\gamma_t(s)$ satisfies the differential equation

$$\frac{\partial \gamma_{ts}}{\partial t} = f(s)\,\gamma_{ts} + g(s) \frac{\partial \gamma_{ts}}{\partial s}$$

APPENDIX III

Proof of the stability of any statistics when subjected to a binomial process.

If the distribution of the input particles is $P(m)$ and the distribution of the output particles is $Q(n)$ after having been subjected to a binomial process, then

$$Q(n) = \sum_{m=n}^{\infty} P(m)\, B_m(n)$$

where $B_m(n)$ is the binomial distribution:

$$B_m(n) = \binom{m}{n} p^n (1-p)^{m-n}$$

where p is the probability for an m particle to become an n particle. If:

$$G_p(s) = \sum_m s^m P(m)$$

and

$$G_Q(s) = \sum_n s^n Q(n)$$

then

$$G_Q(s) = \sum_{n=0}^{\infty} s^n \sum_{m=0}^{\infty} P(m)\, B_m(n)$$

$$= \sum_{m=0}^{\infty} P(m)(1-p+ps)^m$$

or

$$G_Q(s) = G_p(1-p+sp)$$

Thus the generating functions remain invariant in form. The moments are therefore:

$$\left\langle \frac{n!}{(n-r)!} \right\rangle = \left[\frac{\partial^r G_Q(s)}{\partial s^r} \right]_{s=1} = p^r \left\langle \frac{m!}{(m-r)!} \right\rangle$$

If $G_p(s)$ is a boson distribution generating function, then

$$G_p(s) = (1+b-bs)^{-N}$$

thus

$$G_Q(s) = (1+bp-bps)^{-N}$$

and the effective boson factor b has become pb.

APPENDIX IV

Derivation of $\psi_I(\tau)$ and $S_I(f)$:

The instantaneous intensity $I_t = \vec{E}_t \cdot \vec{E}_t^*$

where $\vec{E}_t = \vec{u}_t + j\vec{v}_t$

and u_t and v_t are Hilbert transform pairs:

Also $\vec{u}_t = \vec{i}\, u_{1t} + \vec{j}\, u_{2t}$

$\vec{v}_t = \vec{i}\, v_{1t} + \vec{j}\, v_{2t}$

where $\vec{i}$ and $\vec{j}$ are unit vectors in the x and y directions respectively. We will assume the light is incident in the z direction. Also u_k and v_k are gaussian random variables with zero mean.

Therefore if E_1 and E_2 are the x and y components of the electric vector on the detector, then:

$$E_{1t} = u_{1t} + j v_{1t}$$

$$E_{2t} = u_{2t} + j v_{2t}$$

Define an electric field correlation coefficient:

$$\gamma_{ik}(\tau) = \langle E_{it} E^*_{k\,t+\tau} \rangle \Big/ \left(\langle I_i \rangle \langle I_k \rangle\right)^{1/2}$$

$$= \langle (u_{it} + j v_{it})(u_{k\,t+\tau} - j v_{k\,t+\tau}) \rangle \Big/ \left(\langle I_i \rangle \langle I_k \rangle\right)^{1/2}$$

where $\langle I_i \rangle$ is the intensity due to light polarized in the i direction. From the conjugate properties of u and v, we have: (Mandel and Wolf, 1961)

$$\langle u_{i,t}\, u_{k,t+\tau} \rangle = \langle v_{i,t}\, v_{k,t+\tau} \rangle$$

$$\langle u_{i,t}\, v_{k,t+\tau} \rangle = -\langle u_{k,t+\tau}\, v_{i,t} \rangle$$

Therefore:

$$\gamma_{ik}(\tau) = 2\left[\langle u_{it}u_{kt+\tau}\rangle - j\langle u_{it}v_{kt+\tau}\rangle\right] \Big/ \left(\langle I_i\rangle\langle I_k\rangle\right)^{1/2}$$

and $$\gamma_{12}(\tau) = \gamma_{21}^{*}(\tau)$$

Also: $$\langle I_t\rangle = 2\left[\langle u_1^2\rangle + \langle u_2^2\rangle\right]$$

$$\langle I_t I_{t+\tau}\rangle = \left\langle \sum_{i=1}^{2}\sum_{k=1}^{2}\left(u_{it}^2 + v_{it}^2\right)\left(u_{kt+\tau}^2 + v_{kt+\tau}^2\right)\right\rangle$$

If the z_i $(i=1,2,3,4)$ form a multivariate gaussian distribution, then we have the following identity.

$$\langle z_1z_2z_3z_4\rangle = \langle z_1z_2\rangle\langle z_3z_4\rangle + \langle z_1z_3\rangle\langle z_2z_4\rangle + \langle z_1z_4\rangle\langle z_2z_3\rangle$$

Thus $$\langle u_{it}^2 u_{kt+\tau}^2\rangle = \langle u_i^2\rangle\langle u_k^2\rangle + 2\langle u_{it}u_{kt+\tau}\rangle^2$$

The other products may be similarly reduced. Again using the conjugate properties of u_t and v_t, we find:

$$\langle I_t I_{t+\tau}\rangle = 4\left[\langle u_1^2\rangle + \langle u_2^2\rangle\right]^2 + 4\sum_{i=1}^{2}\sum_{k=1}^{2}\left[\langle u_{it}u_{kt+\tau}\rangle^2 + \langle u_{it}v_{kt+\tau}\rangle^2\right]$$

The autocorrelation function:

$$\psi_I(\tau) = \langle I_t I_{t+\tau}\rangle - \langle I\rangle^2$$

is then $$\psi_I(\tau) = 4\sum_i\sum_k\left[\langle u_{it}u_{kt+\tau}\rangle^2 + \langle u_{it}v_{kt+\tau}\rangle^2\right]$$

or $$\psi_I(\tau) = \sum_i\sum_k \left|\gamma_{ik}(\tau)\right|^2\langle I_i\rangle\langle I_k\rangle$$

The spectral density defined by:

$$S_I(f) = 2\int_{-\infty}^{\infty}\psi_I(\tau)\,e^{-j\omega\tau}\,d\tau$$

is $$S_I(f) = 2 \sum_i \sum_k \langle I_i \rangle \langle I_k \rangle \int_{-\infty}^{\infty} \gamma_{ik}(\tau)\, \gamma_{ik}^*(\tau)\, e^{-j\omega\tau}\, d\tau$$

Then using Parseval's theorem:

$$\int_{-\infty}^{\infty} \gamma_{ik}(\tau)\, \gamma_{ik}^*(\tau)\, e^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} F_{ik}(x)\, F_{ik}'(f-x)\, dx$$

where $$F_{ik}(f) = \int_{-\infty}^{\infty} \gamma_{ik}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$F_{ik}'(f) = \int_{-\infty}^{\infty} \gamma_{ik}^*(\tau)\, e^{-j\omega\tau}\, d\tau$$

Let $$\theta_{ik}(\tau) = \langle u_{it}\, u_{k\,t+\tau} \rangle$$

then using the properties of analytic signals (Deutsch, 1962)

$$\left[\langle I_i \rangle \langle I_k \rangle\right]^{1/2} F_{ik}(f) = 2 \int_{-\infty}^{\infty} \theta_{ik}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad f > 0$$

$$= \int_{-\infty}^{\infty} \theta_{ik}(\tau)\, e^{-j\omega\tau}\, d\tau \qquad f = 0$$

$$= 0 \qquad f < 0$$

Similarly $$F_{ik}'(-f) = F_{ik}(f)$$

Therefore: $$\int_{-\infty}^{\infty} \gamma_{ik}(\tau)\, \gamma_{ik}^*(\tau)\, e^{-j\omega\tau} = \int_{f}^{\infty} F_{ik}(x)\, F_{ik}(x-f)\, dx$$

$$= \int_{0}^{\infty} F_{ik}(x)\, F_{ik}(x+f)\, dx$$

Thus: $$S_I(f) = 2 \sum_i \sum_k \langle I_i \rangle \langle I_k \rangle \int_0^{\infty} F_{ik}(x)\, F_{ik}(x+f)\, dx$$

For unpolarized light $\gamma_{12}(\tau) = 0$, and:

$$S_I(f) = 2 \sum_k \langle I_k \rangle^2 \int_0^{\infty} F_{kk}(x)\, F_{kk}(x+f)\, dx$$

The intensity on the detector is $I_t = I_{1t} + I_{2t}$

where $$\langle I_k \rangle = \langle E_k^2 \rangle = \langle I_k \rangle \gamma_{kk}(0) = \int_0^\infty F_{kk}(f)\, df$$

Thus $F_{11}(f)$ and $F_{22}(f)$ are the input spectral line shapes for the two polarization components. For unpolarized light $F_{11}(f) = F_{22}(f)$ and we have two equal contributions to the spectral density.

For a plane polarized beam in the x-direction we have (dropping the suffices):

$$\psi_I(\tau) = \langle I \rangle^2 \gamma(\tau)\, \gamma^*(\tau)$$

where $$\gamma(\tau) = \langle E_t E^*_{t+\tau} \rangle \Big/ \langle E_t E^*_t \rangle$$

$$\gamma(\pm\infty) = 0 \qquad \gamma(0) = 1$$

Therefore $\psi_I(0) = \langle I \rangle^2$ as expected since the intensity is exponentially distributed when considering a single coherence area.

The wave interference spectrum is:

$$S_I(f) = 2\langle I \rangle^2 \int_0^\infty F(x) F(x+f)\, dx$$

where F(f) is the normalized input spectral line shape.

Proof that $\tau_c(\Delta f) = 1$

We defined $$\tau_c = \int_{-\infty}^{\infty} \psi_I(\tau)\, d\tau \Big/ \psi_I(0)$$

and $$\Delta f = \left[\int_0^\infty F^2(f)\, df \right]^{-1}$$

Using the relationship between the autocorrelation function and the spectral density, namely:

$$\psi_I(\tau) = \int_0^\infty S_I(f) \cos \omega\tau \, df$$

then

$$\tau_c = \frac{\int_{-\infty}^{\infty} d\tau \int_0^\infty S_I(f) \cos \omega\tau \, df}{\psi_I(0)}$$

$$= \frac{\int_0^\infty S_I(f)\, \delta(f)\, df}{\psi_I(0)}$$

$$= \frac{S_I(0)}{2\,\psi_I(0)}$$

$$= \int_0^\infty F^2(f)\, df$$

and

$$(\Delta f)\, \tau_c = 1$$

APPENDIX V

Modulated poisson input:

Given a source which emits particles at random, the probability of an emission in a time interval $t, t+\Delta t$ being $\omega\Delta t$. Then the statistics of the number of particles emitted in a time interval T is poisson. Then insert a chopper into the path of the particles such that it alternately transmits the particles for a time T_1 and absorbs them for a time T_2 . A detector after the chopper counts the particles. By examining the first and second moments of the number of particles incident on the detector in a time T we can show that the resulting fluctuations are always super-poisson.

We will examine two cases:

1. T_1 and T_2 are constant
2. T_1 and T_2 are stochastic

1. The probability that n particles are emitted in a time T is:

$$P_T(n) = (\omega T)^n \exp(-\omega T)/n!$$

The generating function for $P_T(n)$ is:

$$G_T(s) = \sum_n s^n P(n) = \exp(\omega T(s-1))$$

Then if $T_0 = T_1 + T_2$, then $G_T(s) = \left[G_{T_1}(s) G_{T_2}(s)\right]^N G_{t_1}(s) G_{t_2}(s)$

where $N = \left[T/T_0\right]'$ and t_1 and t_2 are the remaining partial intervals that were included in the time T.

$\left[x\right]'$ denotes the greatest integer less than or equal to x.

If $Q_T(m)$ is the probability of detecting m particles in a time T, then the generating function for $Q_T(m)$ is:

$$H_T(s) = [G_{T_1}(s)]^N G_{t_1}(s)$$

or

$$H_T(s) = \exp\left(w(s-1)[NT_1 + t_1]\right)$$

The moments of m for this particular measuring interval are:

$$\overline{m} = w[NT_1 + t_1]$$

$$\overline{m(m-1)} = w^2[NT_1 + t_1]^2$$

t will, however, depend on where we start our measuring interval. If we take an ensemble average over all possible measuring intervals, then:

$$\langle m \rangle = w[NT_1 + \langle t_1 \rangle]$$

and

$$\text{var}\, m = \langle m \rangle + w^2 \,\text{var}\, t_1$$

If we let $S = T - NT_0$ and $T_1 < T_2$:

then find: $\langle t_1 \rangle = ST_1/T_0$

$$\text{var}\, t_1 = \frac{S^2}{3T_0^2}\left[3T_1T_2 - ST_0\right] \qquad 0 \le S \le T_1$$

$$= \frac{T_1^2}{3T_0^2}\left[3S(T_0 - S) - T_1T_0\right] \qquad T_1 \le S \le T_2$$

and

$$\text{var}\, t_1\Big]_S = \text{var}\, t_1\Big]_{T_0 - S}$$

Since $\text{var}\, t_1 = \text{var}\, t_2$ the case for $T_2 < T_1$ is easily evaluated.

2. T_1 and T_2 are stochastic:

If $\lambda_1 \Delta t$ is the probability that the pulse T will terminate between t and $t+\Delta t$ if it is on at t; then the probability of having a pulse of duration T is:

$$P(T_1) = \lambda_1 \exp(-\lambda_1 T_1)$$

Similarly $$P(T_2) = \lambda_2 \exp(-\lambda_2 T_2)$$

Then using the previous approach:

$$G_T(s) = \prod_i \left[G_{T_{1i}} G_{T_{2i}} \right]$$

where $$\sum_i T_{1i} + T_{2i} = T$$

and $$H_T(s) = \prod_i G_{T_{1i}} = \exp\left[w(s-1) \sum T_{1i} \right]$$

Therefore: $$\overline{m} = w \sum_i T_{1i}$$

$$\overline{m(m-1)} = w^2 \sum_i \sum_j T_{1i} T_{1j}$$

Again taking an ensemble average and noting that T_{1i} and T_{1j} are independent if $i \neq j$:

$$\langle m \rangle = w N \langle T_1 \rangle$$

$$\langle m(m-1) \rangle = w^2 \left[N \langle T_1^2 \rangle + N(N-1) \langle T_1 \rangle^2 \right]$$

where $$N = T / (\langle T_1 \rangle + \langle T_2 \rangle)$$

We obtain:

$$\text{Var } m = \langle m \rangle + w^2 N \text{ var } T_1$$

or

$$\text{var } m = \langle m \rangle (1 + \langle m \rangle / N)$$

In both of these esamples, the modulation increases the ratio of the variance to the mean. This modulation has the effect of sending packets or bunches of particles to the detector. Thus the super-poisson nature of bosons is attributed to bunching.

At present it is not known whether there is any possible operation that performed on the poisson particles would produce sub-poisson fluctuations in the output particles.

BIBLIOGRAPHY

Bak, T. 1963. Contributions to the theory of Chemical Kinetics, p. 31, W.A.Benjamin Inc. N.Y.

Bartlett, M.S. 1951. Proc. Camb. Phil. Soc. 47, 821.

Burgess, R. E. 1956. Proc. Phys. Soc. B, 69, 1020.

Burgess, R. E. 1961. J. Phys. Chem. Sol. 22, 371.

Cole, E.A.B. 1965. Proc. Phys. Soc. 85, 135.

Deutsch, R. 1962. Nonlinear Transformations of Random Processes, chapt. 1, Prentice-Hall, N.J.

Forrester, A.T. 1961. J. Opt. Soc. Am. 51, 253.

Fowler, R.H. 1929. Statistical Mechanics, p. 483, Cambridge University Press

Mandel, L. 1958, Proc. Phys. Soc. 72, 1037.

Mandel, L. 1959, Proc. Phys. Soc. 74, 233.

Mandel, L., Wolf, E. 1961. Phys. Rev. 124, 1696.

Sobel'man, I.I., Tyutin, I.V. 1963, Soviet Physics Uspeckhi, 6, 267.

van Vliet, K. M., Blok, J. 1956. Physica 23, 231.

van Vliet, K. M. 1964. Phys. Rev. 133, 1182

van Vliet, K. M., Fassett, J. R. 1965. Fluctuation Phenomena in Solids p. 268, Burgess, R. E. Ed., Academic Press, N.Y.