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MICHAEL JAMES FREEMAN

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A CONTRIBUTION TO THE QUANTUM THEORY OF GRAVITATION

ABSTRACT

A quantum theory of gravitation is constructed, by considering the gravitational field in the linear approximation to be a special case of a rank II tensor field, which has imposed upon it the auxiliary conditions of symmetry, transversality, and tracelessness. A method proposed by Gupta of iterating the linear field equations to take into account the gravitating effect of gravitation is investigated, and it is shown that this method fails. An alternative method of iteration is proposed which yields a functional equation for the Lagrangian of the full nonlinear theory. Finally, the problem of photon-photon scattering due to the gravitational interaction is investigated. It is found that for sufficiently high frequencies this process dominates the purely electrodynamic scattering of photons by photons.

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Department of PHYSICS

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ABSTRACT

A quantum theory of gravitation is constructed, by considering the gravitational field in the linear approximation to be a rank II tensor field, which has imposed upon it the auxiliary conditions of symmetry, transversality, and tracelessness. Extensive use is made of the close analogy between the electromagnetic field as a special case of a vector field, and the gravitational field as a special case of a tensor field. This analogy includes the necessity of introducing an indefinite metric in order to make the auxiliary conditions compatible with the commutation relations.

A complete theory of gravitation must take into account the gravitating nature of gravitation and hence must be a nonlinear theory. A method proposed by Gupta of iterating the linear field equations for this purpose is investigated, and it is shown that this method fails, because the Lagrangian for the second order equations does not exist. An alternative method of iteration is proposed which avoids this problem, and which yields a functional equation for the Lagrangian of the full nonlinear theory.

Finally, the problem of photon-photon scattering due to the gravitational interaction is investigated. This is done by constructing an interaction Hamiltonian by using the principle of the compensating field and then applying the standard methods of quantum electrodynamics. It is found that for

sufficiently high frequencies this process dominates the purely electrodynamic scattering of photons by photons.

CONTENTS

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INTRODUCTION

In the past, attempts at incorporating the theory of gravitation into the body of quantum field theory have followed two different avenues of attack.

One line of approach has as starting point the classical theory of gravitation due to Einstein (1916) without approximations, and is based on the belief that canonical quantization procedures, such as those of Heisenberg and Pauli (1929) or of Peierls (1952), can be applied to such an essentially nonlinear classical theory, even though these procedures were fashioned originally for the purpose of establishing an unambiguous correspondence between essentially linear classical field theories, such as electrodynamics, and their quantum mechanical analogs. Proponents of this approach have been Arnowitt et. al. (1959, 1960, 1961), Bergmann et. al. (1956, 1958, 1960), de Witt (1961), Dirac (1964), Anderson (1964), and others. These authors have all been faced with formidable mathematical and conceptual difficulties arising from the need of casting Einstein's theory into canonical form, either in terms of a Hamiltonian or in terms of Poisson brackets, before the canonical quantization procedures can be applied. In fact, different opinions are held about what constitutes a suitable set of canonical variables in the theory of gravitation, and it is not known whether the different quantum theories resulting from this approach are physically equivalent. For example, Arnowitt and Deser (1959) prefer to cast the theory of gravitation into the so-called Palatini form (see Schroedinger, 1950) in which the components of the metric tensor and the affinities are treated as independent variables, and the connections between them are imposed later as constraints on the theory, requiring application of special mathematical techniques developed by Dirac (1949) and Bergmann (1955). De Witt (1961), on the other hand, sets up Poisson brackets only between invariants of the gravitational field, thus eliminating the need for subsidiary conditions, and carries out the transition to quantum theory at that stage, leaving open the question whether this technique leads to the same physical consequences as the Hamiltonian formalism of Arnowitt and Deser. An attempt by Feynman (1962) also belongs in this category.

The other line of approach has as starting point a linear approximation to Einstein's field equations, obtained by considering the deviations from the pseudo-euclidean metric tensor as field variables, and neglecting terms of quadratic and higher order in these variables (see Einstein, 1918). In a classic paper, Pauli and Fierz (1939) proved that the resulting field equations, in the absence of external sources, can be looked upon as dynamical equations describing the propagation and polarization properties of a massless particle of spin 2, the "graviton", in analogy to the classical Maxwell vacuum equations which can be looked upon as dynamical equations describing the propagation and polarization properties of a massless particle of spin 1, the "photon" (see, for example, Archibald, 1955). The Lorentz condition of classical electrodynamics, which restricts the gauge freedom of the electromagnetic potentials and functions as a transversality condition on the possible polarization states of the free photon, has its analog in the gravitational case in two so-called gauge conditions, due to Hilbert (1924) and Fierz (1939), which function as transversality conditions on the possible polarization states of the free graviton, leading, as in the electromagnetic case, to the elimination of all propagation modes that do not correspond to either parallel or antiparallel orientation of the spin with respect to the direction of propagation. Within the framework of this classical theory of gravitons one can perform harmonic analysis and classify multi-graviton states according to their energy, angular momentum, and parity quantum numbers, as was done, for example, by Zhirnov and Shirokov (1957), and derive selection rules governing the decay of objects into two or more gravitons (see, for example, Carswell, 1965). However, transition to a full quantum field theory of gravitation, requiring a description in terms of graviton annihilation and creation operators, runs into a characteristic difficulty, encountered already in quantum electrodynamics. When one uses as basis for the quantization procedure an action principle yielding both field equations and transversality conditions, as was done for electrodynamics by Fermi (1932) and for the linearized theory of gravitation by Pauli and Fierz (1939), one ends up, upon quantization, with an inconsistency, because the transversality conditions, if imposed as operator equations, are incompatible with the commutation relations of the theory, as was pointed out by Belinfante (1949). A way out of this difficulty was shown by Gupta (1950, 1952) and Bleuler (1950). These authors develop the quantum field theory for a generalized type of massless particle unrestricted by auxiliary conditions, and then impose the transversality conditions only for the positive frequency part and in terms of a constraint on possible state vectors. In this fashion one can satisfy the commutation relations without imposing the constraints as operator equations. The price one has to pay for this method is introduction of state vector spaces with indefinite metric, and the transversality conditions hold only as expectation values for physically permissible states. The result is a theory in which particles corresponding to polarization modes other than the purely transverse ones still do not contribute to the expectation values of dynamical observables such as the energy and momentum of the field, even though their existence as virtual particles must be conceded if the state vector space spanned by the field operators be complete.

The approach based on the linearized gravitational field equations of Einstein has the advantage of yielding at once a viable quantum field theory, permitting use of a terminology which ascribes to purely transverse gravitons the same degree of reality as that ascribed to the transverse photons of quantum electrodynamics. However this close analogy to quantum electrodynamics becomes an encumbrance when one tries to modify the initially linear quantum theory of gravitation to accommodate an essentially nonlinear feature which distinguishes gravitation from electrodynamics on account of the equivalence between energy and gravitational mass, namely the fact that the gravitational field itself must be a source of gravitation, whereas the electromagnetic field itself is not a carrier of electric currents. It is apparently mathematically impossible to construct a theory of gravitons taking into account sources, either due to other fields or due to the gravitating effect of the gravitational field itself, under simultaneous retention of the gauge conditions which are necessary for the elimination of all but the purely transverse propagation modes. This was found out by Gupta (1952, 1954) when he tried to incorporate sources into the quantum theory of gravitation. Indeed, if one linearizes Einstein's field equations with external sources, then not even in the first approximation can one satisfy the second gauge condition of Fierz. Gupta (1952) managed to repair this defect by the artifice of introducing an additional scalar field, corresponding to an additional kind of spinless graviton, and then arranging an additional constraint so that these spurious particles do not contribute to any observables of the gravitational field. If one now tries to take into account the gravitating effect of the gravitons themselves, for example by some iteration method as was proposed by Gupta (1954), then insistence on retaining the Fierz gauge conditions presents an apparently insurmountable obstacle, and Gupta was unable by his method to produce for the nonlinear theory an explicit Lagrangian which he presumed to exist under these conditions. In fact, there are reasons to believe that such a Lagrangian does not exist, as will be shown later in this work.

These considerations raise the question whether there are any stringent reasons why one could not relax the gauge conditions of Fierz when one aims at nonlinear modification of the initially linear quantum field theory of gravitation. This amounts to seriously entertaining the possibility that gravitons may exist with polarization properties other than the purely transverse modes treated exclusively in the usual versions of the linear theory. A corresponding possibility certainly does not present itself in quantum electrodynamics, because the Lorentz condition, which serves to eliminate the longitudinal and time-like polarization modes, can also be used as a convenient device for extraction of the law of conservation of charge from the field equations in terms of an identically vanishing divergence of the current four-vector. In fact, if the law of conservation of current were not true, then the field equations and the Lorentz condition would become incompatible. In a theory of gravitation which uses Einstein's theory as a guide for how to couple sources to the field, on the other hand, the first gauge condition of Fierz would require that the ordinary divergence of the energy-momentum tensor vanish, whereas the actual requirement on the energy-momentum tensor is that its covariant divergence vanish (see Landau and Lifshitz, 1962). Similarly, the second condition of Fierz would require that the trace of this tensor be zero, which is certainly not true in most cases. Thus the same reasons that lead to retention of the Lorentz condition in quantum electrodynamics and to the ensuing disqualification of longitudinal and timelike photons as real particles, argue for abolition of the gauge conditions of Fierz in any quantum theory of gravitation aimed at incorporating sources and nonlinear features of the gravitational field, and for development of a formalism that gives room to all possible polarization states of the tensor field whose quanta are to be identified as gravitons.

With these views in mind, the work reported in this thesis began with the development of a linear quantum field theory of massless tensor particles without restriction on their possible polarization states. Accordingly one has to do with 16 independent field variables, whose symmetric parts acquire physical content by identification with the deviations from the pseudo-euclidean metric field tensor. The symmetric part of the field gives rise to 10 polarization modes, corresponding in particle language to 10 types of gravitons, namely 5 modes belonging to the spin quantum number $j = 2$, 3 modes belonging to $j = 1$, and 2 independent modes each belonging to $j = 0$. The skew symmetric part of the field yields 6 more modes, corresponding to 2 independent sets of 3 gravitons each, all belonging to $j = 1$.

If one ignores the skew part, and imposes on the symmetric part the gauge conditions of Fierz, one arrives, of course, at Gupta's linear theory without sources. It is then shown how one can account for gravitating effects of these various gravitons by an iteration procedure, devised such that its results agree with those of Einstein's theory in first approximation. However, this procedure is not aimed at recovering Einstein's full nonlinear theory in the limit of infinite iteration, as was attempted by Gupta (1954). This raises the intriguing possibility that the physical consequences of the theory developed here will differ for strong fields substantially from the corresponding consequences of Einstein's theory, which has never been tested experimentally except for weak fields, and strong field solutions of Einstein's field equations thus far have been applied only speculatively to questions arising from the problem of gravitational collapse of superdense stars. The effect of the gravitons considered here on other objects, such as electrons or photons, can be treated in the usual way, for example by the method of Utiyama (1956), and one can thus predict, in principle, how unusual types of gravitons, if they exist, would manifest themselves through interaction with such other objects.

1. The Linearization of Einstein's Field Equations

All quantum theories of gravitation use as starting point the field equations due to Einstein (1916), which in absence of external sources have the form

$$R_{\mu\nu} = 0 . \tag{1.1}$$

Expressed in terms of the affinities

$$\Gamma^{\mu}_{\sigma\nu} = \tfrac{1}{2} g^{\mu\rho}\left(g_{\rho\sigma/\nu} + g_{\rho\nu/\sigma} - g_{\sigma\nu/\rho}\right) \tag{1.2}$$

the symmetric Ricci tensor

$$R_{\mu\nu} = \Gamma^{\sigma}_{\mu\nu/\sigma} - \Gamma^{\sigma}_{\mu\sigma/\nu} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\rho}_{\sigma\rho} - \Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} \tag{1.3}$$

exhibits Einstein's field equations (1.1) as a set of 10 non-linear partial differential equations of second order for the 10 independent components $g_{\mu\nu}$ of the symmetric metric tensor. (A vertical bar is understood to mean partial differentiation, thus

$$\Lambda_{\mu/\nu} \equiv \frac{\partial \Lambda_{\mu}}{\partial x^{\nu}} \quad , \text{ etc.} \tag{1.4}$$

Greek indices run from 1 to 4, repetition of an index implies summation, co-ordinates are labelled as $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$, $x^4 \equiv it$ in a particular co-ordinate system, and conventional natural units requiring $c = \hbar = 1$ are used throughout this work.) The gravitating effect of the metric field is understood to manifest itself in the motion of test particles whose worldlines are assumed to be geodesics in this metric field, so that the co-ordinates $x^\mu(s)$ of such a test particle satisfy the conditions

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma} \dot{x}^\nu \dot{x}^\sigma = 0 \tag{1.5}$$

where a dot denotes differentiation with respect to the proper time s of the particle.

If one is interested only in weak gravitational fields, characterized by not more than infinitesimal deviations from the pseudo-euclidean metric tensor

$$\delta^{\alpha\beta} = \delta_{\alpha\beta} = \begin{bmatrix} +1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{bmatrix}; \quad \delta^{\alpha\beta}\delta_{\beta\gamma} = \delta^\alpha_\gamma = \begin{cases} 0, & \alpha \neq \gamma \\ 1, & \alpha = \gamma \end{cases} \tag{1.6}$$

then one can linearize the equations (1.1), as was done first by Einstein (1918), by introducing new field variables $\gamma_{\mu\nu}$ which describe these deviations,

$$g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}, \tag{1.7}$$

and retaining only terms linear in these new variables. In this approximation the tensor $\delta_{\alpha\beta}$ is sufficient for the lowering and raising of indices, as in

$$\gamma^\sigma_\nu = \delta^{\sigma\mu}\gamma_{\mu\nu} \quad , \text{ etc.}, \tag{1.8}$$

and the field equations (1.1) reduce to

$$\gamma_{\mu\nu/\sigma\sigma} - \left(\gamma_{\sigma\nu/\sigma} - \tfrac{1}{2}\gamma_{\sigma\sigma/\nu}\right)_{/\mu} - \left(\gamma_{\sigma\mu/\sigma} - \tfrac{1}{2}\gamma_{\sigma\sigma/\mu}\right)_{/\nu} = 0 \,. \tag{1.9}$$

These equations have the interesting property of being invariant under the "gauge transformation"

$$\gamma_{\mu\nu} \to \gamma'_{\mu\nu} = \gamma_{\mu\nu} + \Lambda_{\mu/\nu} + \Lambda_{\nu/\mu} \tag{1.10}$$

with an <u>arbitrary</u> vector field Λ_μ. They can be simplified further on account of a theorem by Hilbert (1924), who observed that it is always possible to find an infinitesimal coordinate transformation, which does not destroy the infinitesimal character of the $\gamma_{\mu\nu}$, so that

$$\gamma_{\sigma\nu/\sigma} - \tfrac{1}{2}\gamma_{\sigma\sigma/\nu} = 0 . \tag{1.11}$$

These equations are called the "Hilbert gauge conditions". They amount to restricting the possible gauge functions Λ_μ introduced in (1.10) to solutions of

$$\Lambda_{\mu/\sigma\sigma} = 0 , \tag{1.12}$$

and reduce the field equations (1.9) to the form

$$\gamma_{\mu\nu/\sigma\sigma} = 0 . \tag{1.13}$$

Indeed, if one is confronted with a field $\gamma'_{\mu\nu}$ not satisfying the conditions (1.11), then a gauge transformation with Λ_μ chosen such that

$$\Lambda_{\mu/\sigma\sigma} = \gamma'_{\sigma\mu/\sigma} - \tfrac{1}{2}\gamma'_{\sigma\sigma/\mu} \tag{1.14}$$

and imposition of the restriction (1.12), will guarantee the simultaneous validity of (1.13) and (1.11). The remaining gauge freedom is sufficient, as was first shown by Fierz (1939), to permit a further specialization to fields satisfying the "first gauge condition of Fierz",

$$\gamma_{\sigma\nu/\sigma} = 0, \tag{1.15}$$

as well as the "second gauge condition of Fierz",

$$\gamma_{\sigma\sigma} = 0. \tag{1.16}$$

These conditions amount to a further restriction on the gauge function Λ_σ to solutions of

$$\Lambda_{\sigma/\sigma} = 0\,. \tag{1.17}$$

Indeed, if one is confronted within the Hilbert gauge with a field $\gamma'_{\mu\nu}$ not satisfying the condition (1.16), then a gauge transformation with Λ_μ chosen such that

$$\Lambda_{\sigma/\sigma} = \gamma'_{\sigma\sigma} \tag{1.18}$$

and imposition of the restriction (1.17) will guarantee validity of (1.16), while the validity of (1.15) follows from the Hilbert gauge condition (1.11).

The field equations (1.13), used in conjunction with the gauge conditions (1.15) and (1.16), which restrict the gauge freedom (1.10) by the conditions (1.12) and (1.17), bear a close analogy to the corresponding treatment of vacuum electrodynamics in terms of the electromagnetic potentials A_μ. Maxwell's field equations in terms of these potentials,

$$A_{\mu/\sigma\sigma} - A_{\sigma/\sigma\mu} = 0, \tag{1.19}$$

have the interesting property of being invariant under gauge transformations

$$A_\mu \rightarrow A'_\mu = A_\mu + B_{/\mu} \tag{1.20}$$

where B is an arbitrary scalar field. One can always find a gauge such that the Lorentz condition holds,

$$A_{\nu/\nu} = 0 . \tag{1.21}$$

This equation amounts to restricting the possible gauge functions B introduced in (1.20) to solutions of

$$B_{/\sigma\sigma} = 0 , \tag{1.22}$$

and reduces the field equations (1.19) to the form

$$A_{\mu/\sigma\sigma} = 0 . \tag{1.23}$$

Indeed, if one is confronted with a potential A'_σ not satisfying the condition (1.21), then a gauge transformation with B chosen such that

$$B_{/\sigma\sigma} = A'_{\sigma/\sigma} \tag{1.24}$$

and imposition of the restriction (1.22) will guarantee the simultaneous validity of (1.21) and (1.23).

2. The Quantum Mechanical Significance of Gauge Transformations

If one carries out on any quantum mechanical ψ-function a phase transformation, characterized by a parameter ϵ,

$$\psi \rightarrow \psi e^{i\epsilon\lambda(x)} \tag{2.1}$$

with an arbitrary scalar function $\lambda(x)$, then the Schroedinger equation for a free particle does not remain invariant because the derivatives of ψ transform according to

$$\frac{\partial\psi}{\partial x^\nu} \rightarrow \left(\frac{\partial}{\partial x^\nu} + i\epsilon \frac{\partial\lambda}{\partial x^\nu}\right)\psi \, . \tag{2.2}$$

However, invariance of the Schroedinger equation can be restored, if the particle is coupled to a vector field A_ν such that the derivatives of ψ occur always in the form

$$\partial_\nu\psi \equiv \left(\frac{\partial}{\partial x^\nu} - i\epsilon A_\nu\right)\psi \tag{2.3}$$

and provided any phase transformation (2.1) is accompanied by a compensating gauge transformation

$$A_\nu \rightarrow A_\nu + \frac{\partial\lambda}{\partial x^\nu} \tag{2.4}$$

of the vector field. In fact many authors look upon the requirement of phase invariance, as was done first by London (1927), as the raison d'etre for the electromagnetic field, whose potentials can be identified with the vector field A_ν introduced above, provided the parameter ϵ is taken as the electric charge of the particle described by the ψ-function. Then the gauge freedom of the electromagnetic potentials can be utilized to guarantee the invariance of the Schroedinger equation under one-parameter phase transformations of the type (2.1).

A similar situation arises if one insists, in presence of gravitational or quasi-gravitationsl inertial fields, on invariance of the <u>action principle</u> under local Lorentz transformations. If one denotes, in an underlying continuum of co-ordinates x^μ, the co-ordinates of a local inertial frame by ξ^k (k=1,2,3,4), then any infinitesimal Lorentz transformation to another local inertial frame $\bar{\xi}^k$

$$\xi^k \to \bar{\xi}^k = \xi^k + \lambda^k_\ell(x)\, \xi^\ell \tag{2.5}$$

is characterized by a six-parameter skew tensor

$$\lambda^{k\ell}(x) = -\lambda^{\ell k}(x) \tag{2.6}$$

and affects any ψ-function according to

$$\psi \to e^{\frac{1}{2}\lambda^{k\ell}(x)\Lambda_{k\ell}}\, \psi \tag{2.7}$$

where $\Lambda_{k\ell}$ is the appropriate operator representation of the transformation $\lambda^{k\ell}$ acting on the components of ψ, for which explicit expressions will be given in the next section. For the purpose of the present section it suffices to note that the derivatives $\dfrac{\partial\psi}{\partial x^\nu}$ are not invariant under the transformation

(2.7). Indeed, in case of an infinitesimal transformation

$$\psi \rightarrow \left[1 + \tfrac{1}{2}\lambda^{kl}(x)\Lambda_{kl}\right]\psi \tag{2.8}$$

the derivatives of ψ transform as

$$\frac{\partial\psi}{\partial x^\nu} \rightarrow \left[1 + \tfrac{1}{2}\lambda^{kl}(x)\Lambda_{kl}\right]\frac{\partial\psi}{\partial x^\nu} + \tfrac{1}{2}\frac{\partial\lambda^{kl}}{\partial x^\nu}\Lambda_{kl}\,\psi . \tag{2.9}$$

However, invariance of the action principle governing the dynamics of the ψ-function can be restored as was first noticed in this case by Utiyama (1956) (see also Kaempffer, 1965), if the particle is coupled to a compensating field

$$B_\nu^{kl}(x) = -B_\nu^{lk}(x) \tag{2.10}$$

such that the derivatives of ψ occur in the action principle always in the form

$$\partial_\nu\psi = \left(\frac{\partial}{\partial x^\nu} - \tfrac{1}{2}B_\nu^{kl}\Lambda_{kl}\right)\psi \tag{2.11}$$

and provided B_ν^{kl} transforms according to

$$B_\nu^{kl} \rightarrow B_\nu^{kl} + \lambda^k_{\ m}B_\nu^{ml} + \lambda^l_{\ m}B_\nu^{km} + \frac{\partial\lambda^{kl}}{\partial x^\nu} . \tag{2.12}$$

By analogy with (2.4), it is <u>this</u> transformation (2.12) for which one should reserve the term "gauge transformation" in this case. There is a connection between the derivatives $\partial_\nu\psi$ introduced in (2.11) and the covariant derivatives of ψ, which can be extracted as follows. The transformation in the neighborhood of any given continuum point x^μ to a local inertial frame with co-ordinates ξ^k can always be carried out to first order if one knows the functions

$$\frac{\partial \xi^k}{\partial x^\mu} \equiv h^k_\mu(x) \qquad \text{and} \qquad \frac{\partial x^\mu}{\partial \xi^k} \equiv f^\mu_k(x) \tag{2.13}$$

so that

$$h^k_\mu f^\nu_k = \delta^\nu_\mu \qquad \text{and} \qquad f^\mu_k h^l_\mu = \delta^l_k . \tag{2.14}$$

It should be noted that in general

$$h^k_{\mu/\nu} - h^k_{\nu/\mu} \neq 0 \tag{2.15}$$

if the $h^k_\mu(x)$ can be considered as 16 given functions, independent of the definition (2.13), representing the properties of a given gravitational or quasi-gravitational inertial field. For example, the transformations between an inertial and a rotating co-ordinate system

$$\begin{aligned}
x^1 &= \xi^1 \cos(i\omega\xi^4) - \xi^2 \sin(i\omega\xi^4) & \xi^1 &= x^1\cos(i\omega x^4) + x^2\sin(i\omega x^4) \\
x^2 &= \xi^1 \sin(i\omega\xi^4) + \xi^2 \cos(i\omega\xi^4) & \xi^2 &= -x^1\sin(i\omega x^4) + x^2\cos(i\omega x^4) \\
x^3 &= \xi^3 & \xi^3 &= x^3 \\
x^4 &= \xi^4 & \xi^4 &= x^4
\end{aligned} \tag{2.16}$$

are determined by the transformation functions

$$\begin{array}{llll}
h^1_1 = \cos(i\omega x^4) & h^2_1 = -\sin(i\omega x^4) & h^3_1 = 0 & h^4_1 = 0 \\
h^1_2 = \sin(i\omega x^4) & h^2_2 = \cos(i\omega x^4) & h^3_2 = 0 & h^4_2 = 0 \\
h^1_3 = 0 & h^2_3 = 0 & h^3_3 = 1 & h^4_3 = 0 \\
h^1_4 = i\omega[-x^1\sin(i\omega x^4) + x^2\cos(i\omega x^4)] & h^2_4 = -i\omega[x^1\cos(i\omega x^4) + x^2\sin(i\omega x^4)] & h^3_4 = 0 & h^4_4 = 1
\end{array} \tag{2.17}$$

or, alternatively, by

$$\begin{array}{llll} f_1^{\,1} = \cos(i\omega x^4) & f_1^{\,2} = \sin(i\omega x^4) & f_1^{\,3} = 0 & f_1^{\,4} = 0 \\ f_2^{\,1} = -\sin(i\omega x^4) & f_2^{\,2} = \cos(i\omega x^4) & f_2^{\,3} = 0 & f_2^{\,4} = 0 \\ f_3^{\,1} = 0 & f_3^{\,2} = 0 & f_3^{\,3} = 1 & f_3^{\,4} = 0 \\ f_4^{\,1} = -i\omega x^2 & f_4^{\,2} = i\omega x^1 & f_4^{\,3} = 0 & f_4^{\,4} = 1 \end{array} \tag{2.18}$$

The metric tensor $g_{\mu\nu}$ of the continuum, defined by

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \tag{2.19}$$

can be obtained from the functions $h_\mu^{\,k}(x)$ by noting that if one takes as inertial frame a cartesian system with

$$ds^2 = \delta_{ik}\, d\xi^i d\xi^k = \delta_{ik} \frac{\partial \xi^i}{\partial x^\mu} \frac{\partial \xi^k}{\partial x^\nu} dx^\mu dx^\nu = \delta_{ik}\, h_\mu^{\,i} h_\nu^{\,k}\, dx^\mu dx^\nu \tag{2.20}$$

one has simply

$$g_{\mu\nu} = \delta_{ik}\, h_\mu^{\,i} h_\nu^{\,k} \qquad \text{and also} \qquad g^{\mu\nu} = \delta^{ik} f_i^{\,\mu} f_k^{\,\nu} \tag{2.21}$$

allowing one to compute all other metric properties of the continuum in terms of the $h_\mu^{\,k}$. For the example given above one finds that

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & i\omega x^2 \\ 0 & 1 & 0 & -i\omega x^1 \\ 0 & 0 & 1 & 0 \\ i\omega x^2 & -i\omega x^1 & 0 & 1-\omega^2[(x^1)^2+(x^2)^2] \end{pmatrix} \tag{2.22}$$

and

$$g^{\mu\nu} = \begin{pmatrix} 1-\omega^2(x^2)^2 & \omega^2 x^1 x^2 & 0 & -i\omega x^2 \\ \omega^2 x^1 x^2 & 1-\omega^2(x^1)^2 & 0 & i\omega x^1 \\ 0 & 0 & 1 & 0 \\ -i\omega x^2 & i\omega x^1 & 0 & 1 \end{pmatrix}$$

and this tensor describes in the usual fashion (see Landau and Lifshitz, 1962) the quasi-gravitational inertial effects, such as Coriolis and centrifugal forces, in this particular rotating continuum. It will also be noticed that in the limit $\omega \to 0$ the metric tensor $g_{\mu\nu}$ as well as the h^k_μ and f^μ_k revert to diagonal form, and one anticipates that an expansion of $g_{\mu\nu}$ in the form $\delta_{\mu\nu} + \gamma_{\mu\nu}$ will correspond to similar expansions of h^k_μ and f^μ_k. The derivative (2.11) for the special case of a tensor field with components ψ^{ij} in the local inertial frame can be cast, on account of the representation (see the next section) for Λ_{kl} in this case,

$$\Lambda_{kl;mn}^{\;ij} = \delta^i_m(\delta^j_k \delta_{ln} - \delta^j_l \delta_{kn}) + \delta^j_n\left(\delta^i_k \delta_{lm} - \delta^i_l \delta_{km}\right), \tag{2.23}$$

in the form

$$\partial_\nu \psi^{ij} = \frac{\partial \psi^{ij}}{\partial x^\nu} - B_\nu^{ik}\psi^j_k - B_\nu^{jk}\psi^i_k \, . \tag{2.24}$$

The components ψ^{ij} are connected with the components $\psi^{\rho\sigma}$ in the underlying continuum by

$$\psi^{\rho\sigma} = f^\rho_i f^\sigma_j \psi^{ij} \qquad \text{and} \qquad \psi^{ij} = h^i_\rho h^j_\sigma \psi^{\rho\sigma} \tag{2.25}$$

allowing one to write, using $\psi^i_k = \delta_{kl}\,\psi^{il}$ and $f^\mu_k h^k_\nu = \delta^\mu_\nu$,

$$f_i^\rho f_j^\sigma \partial_\nu \psi^{ij} = \frac{\partial \psi^{\rho\sigma}}{\partial x^\nu} + \Gamma^\rho_{\tau\nu} \psi^{\tau\sigma} + \Gamma^\sigma_{\tau\nu} \psi^{\rho\tau} . \tag{2.26}$$

The symbols $\Gamma^\rho_{\tau\nu}$ are abbreviations for

$$\Gamma^\rho_{\tau\nu} \equiv f_i^\rho h^i_{\tau/\nu} - B^\rho_{\tau,\nu} \tag{2.27}$$

where

$$B^\rho_{\tau,\nu} \equiv f_i^\rho h^\ell_\tau \delta_{\ell k} B^{ik}_\nu . \tag{2.28}$$

If one now writes down equation (2.26) for the metric tensor $g^{\rho\sigma}$, for whose components δ^{ij} in the inertial frame one has by definition (2.24) and because of the antisymmetry of the fields B^{ik}_ν

$$\partial_\nu \delta^{ij} = - B^{ij}_\nu - B^{ji}_\nu = 0 , \tag{2.29}$$

one obtains for the Γ the equations

$$\frac{\partial g^{\rho\sigma}}{\partial x^\nu} + \Gamma^\rho_{\tau\nu} g^{\tau\sigma} + \Gamma^\sigma_{\tau\nu} g^{\rho\tau} = 0 . \tag{2.30}$$

They can be solved uniquely if one <u>assumes</u> that the symbols Γ are symmetric in the lower indices,

$$\Gamma^\rho_{\tau\nu} = \Gamma^\rho_{\underline{\tau\nu}} \tag{2.31}$$

with the result

$$\Gamma^\rho_{\underline{\tau\nu}} = \tfrac{1}{2} g^{\rho\sigma} \left(g_{\sigma\tau/\nu} + g_{\nu\sigma/\tau} - g_{\nu\tau/\sigma} \right) . \tag{2.32}$$

Under the assumption (2.31) the $\Gamma^\rho_{\underline{\tau\nu}}$ are thus recognized as the affinities of the metric $g^{\mu\nu}$, and this establishes equation

(2.26) as the covariant derivative of $\psi^{\rho\sigma}$,

$$f_i^{\rho} f_j^{\sigma} \partial_\nu \psi^{ij} \equiv \psi^{\rho\sigma}{}_{;\nu} \, . \tag{2.33}$$

It should be noted, however, that (2.32) is by no means the most general solution of the equation (2.30). In fact, as was already pointed out by Schroedinger (1950), if one does not assume the symmetry (2.31), the the solution of (2.30) is

$$\Gamma_{\tau\nu}^{\rho} = \Gamma_{\underline{\tau\nu}}^{\rho} - \left(C_{\tau,\nu}^{\rho} - C_{\nu,\tau}^{\rho} \right) + C_{\tau\nu,}{}^{\rho} \tag{2.34}$$

where C is a tensor, <u>arbitrary</u> except for the antisymmetry

$$C_{\mu\nu,\rho} = - C_{\nu\mu,\rho} \, . \tag{2.35}$$

By solving equations (2.27) and (2.28) one can thus express the compensating field B_ν^{kl} in terms of the metric field quantities as

$$B_\nu^{kl} = \delta^{il} f_i^{\tau} \left(h^k_{\tau/\nu} - h^k_\rho \Gamma_{\tau\nu}^{\rho} \right) \tag{2.36}$$

(For the example of the rotating system characterized by the metric tensor (2.22) one finds for the affinities (2.32)

$$\Gamma_{14}^{2} = -i\omega \, , \quad \Gamma_{24}^{1} = i\omega \, , \quad \Gamma_{44}^{1} = \omega^2 x^1 , \quad \Gamma_{44}^{2} = \omega^2 x^2 , \tag{2.37}$$

all other $\Gamma = 0$,

and thus for the components of the compensating field

$$B_\nu^{kl} = 0 \tag{2.38}$$

which is not surprising because such a quasi-gravitational inertial field can always be transformed away globally.) The obviously quite different structure of the true gauge transformation (2.12) of the compensating field B_{ν}^{kl}, as compared to the so-called gauge transformations (1.10) of the linearized field, should serve here to cast serious doubt on the fundamental nature of the transformations (1.10), and to encourage attempts at discarding in a full quantum theory of gravitation any "gauge conditions" flowing from this particular feature of the linearized theory. In fact, in the linear approximation

$$g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu} \; ; \qquad h_{\mu k} = \delta_{\mu k} + \eta_{\mu k} \tag{2.39}$$

the compensating field components are of the form

$$B_{\nu,kl} = \eta_{kl/\nu} - \tfrac{1}{2}\left(\gamma_{kl/\nu} + \gamma_{\nu k/l} - \gamma_{\nu l/k}\right) \tag{2.40}$$

and are <u>invariant</u> under the "gauge transformations" (1.10), which require η_{kl} to transform as

$$\eta_{kl} \rightarrow \eta'_{kl} = \eta_{kl} + \Lambda_{k/l} \tag{2.41}$$

contrary to the compensating fields A_{μ} which are just <u>not</u> invariant under the corresponding gauge transformation in the electromagnetic case.

3. The Requirement of Local Lorentz Invariance

Even though in presence of gravitation, locally inertial frames, i.e. co-ordinate systems tied to freely falling observers who are not rotating with respect to the distant so-called fixed stars, are in general not connected by Lorentz transformations if they are some finite distance apart, one can still insist on the requirement of local Lorentz invariance, so that coincident observers can be connected by a Lorentz transformation. This requirement is equivalent to retaining Newton's first law of motion in the form: Any two coincident inertial observers move at most with constant instantaneous velocity with respect to each other.

Reverting now for the purpose of this section to the notation introduced in Section 1, and dropping the distinction between contravariant and covariant components which is not necessary for the purpose at hand, such a local Lorentz transformation is mediated by a constant matrix $L_{\mu\nu}$ so that

$$x_\mu \to x'_\mu = L_{\mu\nu} x_\nu \; . \tag{3.1}$$

To obtain the representations of this transformation it is sufficient to consider an infinitesimal transformation

$$L_{\mu\nu} = \delta_{\mu\nu} + \lambda_{\mu\nu} \tag{3.2}$$

where the matrix elements $\lambda_{\mu\nu}$ satisfy $|\lambda_{\mu\nu}| << 1$, and the skew-symmetry relation

$$\lambda_{\mu\nu} = -\lambda_{\nu\mu} \tag{3.3}$$

on account of the invariance of

$$x'_\mu x'_\mu = x_\mu x_\mu = x^2 + y^2 + z^2 - t^2. \tag{3.4}$$

The inverse transformation is then simply

$$L^{-1}_{\mu\nu} = \delta_{\mu\nu} - \lambda_{\mu\nu} , \tag{3.5}$$

and since the (k=1,2,3) are real, and x_4 is pure imaginary, the $\lambda_{jk}, \lambda_{44}$ are real, and the λ_{4k} are pure imaginary. Now consider a multicomponent field ψ_A which under the transformation (3.1) transforms into ψ'_A. One defines the representation $\Lambda_{\mu\nu}$ of $\lambda_{\mu\nu}$ by writing

$$\psi_A \rightarrow \psi'_A = \psi_A + \tfrac{1}{2} \lambda_{\mu\nu} \Lambda_{\mu\nu;A,B} \psi_B . \tag{3.6}$$

(By convention, the index A is understood to run from 1 to 4 in case of a vector field, and in case of a 2nd rank tensor field $\psi_A = \psi_{\alpha\beta}$ the index A may be thought of to run from 1 to 16 according to the correspondence

A	: 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	(3.7)
$\alpha\beta$	: 11	12	13	14	21	22	23	24	31	32	33	34	41	42	43	44,	

with similar conventions for spinor fields and tensor fields of higher rank.) By placing a probability interpretation on ψ_A, one requires conservation of probability,

$$\psi^{*\prime}_A \psi'_A = \psi^*_A \psi_A \tag{3.8}$$

which upon substitution of (3.6), neglecting terms quadratic in the $\lambda_{\mu\nu}$, becomes

$$\lambda^*_{\mu\nu} \Lambda^*_{\mu\nu;A,B} + \lambda_{\mu\nu} \Lambda_{\mu\nu;B,A} = 0. \qquad (3.9)$$

On account of the reality properties of the $\lambda_{\mu\nu}$ this means that $i\Lambda_{\mu\nu}$, written as a matrix in the indices A,B, is hermitean for $(\mu\nu) = (jk), (44)$,

$$i\Lambda_{\mu\nu;A,B} = -i\Lambda^*_{\mu\nu;B,A} \quad \text{for } (\mu\nu) = (jk),\ (44) \qquad (3.10a)$$

where as for $(\mu\nu) = (4k), (j4)$ the matrix $\Lambda_{\mu\nu}$ is hermitean,

$$\Lambda_{\mu\nu;A,B} = \Lambda^*_{\mu\nu;B,A} \quad \text{for } (\mu\nu) = (4k), (j4). \qquad (3.10b)$$

The generators $\Lambda_{\mu\nu}$ quite generally satisfy the C.R.'s (see for example, Lomont (1959))

$$[\Lambda_{\mu\nu}, \Lambda_{\rho\sigma}] = \tfrac{1}{2} C_{\mu\nu,\rho\sigma;\varkappa\tau} \Lambda_{\varkappa\tau} \qquad (3.11)$$

with

$$C_{\mu\nu,\rho\sigma;\varkappa\tau} = \delta_{\mu\rho}\delta_{\varkappa\nu}\delta_{\tau\sigma} + \delta_{\nu\sigma}\delta_{\varkappa\mu}\delta_{\tau\rho} - \delta_{\nu\rho}\delta_{\varkappa\mu}\delta_{\tau\sigma} - \delta_{\mu\sigma}\delta_{\varkappa\nu}\delta_{\tau\rho}. \qquad (3.12)$$

In a three-vector notation they may be written

$$\underline{\Sigma} \times \underline{\Sigma} = i\underline{\Sigma}; \quad \underline{\Sigma} \times \underline{K} = i\underline{K}; \quad \underline{K} \times \underline{K} = -i\underline{\Sigma} \qquad (3.13)$$

where

$$i\Lambda_{12} \equiv \Sigma_3;\ i\Lambda_{23} \equiv \Sigma_1;\ i\Lambda_{31} \equiv \Sigma_2;\ \Lambda_{14} \equiv K_1;\ \Lambda_{24} \equiv K_2;\ \Lambda_{34} \equiv K_3. \qquad (3.14)$$

The vector $\underline{\Sigma}$ can be interpreted as angular momentum operator, generating the pure rotations in space, and the vector $\underline{K}$ as center of mass co-ordinate operator, generating the pure Lorentz transformations.

For later reference, the following three examples of these representations are useful.

(i) The Vector Field

By definition, any vector field A_μ transforms according to

$$A_\mu \to A_\mu' = A_\mu + \lambda_{\mu\nu} A_\nu . \tag{3.15}$$

Identification with the definition (3.6) gives the conditions

$$\tfrac{1}{2} \lambda_{\rho\sigma} \Lambda_{\rho\sigma;\mu,\nu} = \lambda_{\mu\nu} \tag{3.16}$$

which are solved by

$$\Lambda_{\rho\sigma;\mu,\nu} = \delta_{\rho\mu} \delta_{\nu\sigma} - \delta_{\rho\nu} \delta_{\sigma\mu} . \tag{3.17}$$

(ii) The Tensor Field

By definition, any tensor field $\gamma_{\mu\nu}$ transforms according to

$$\gamma_{\mu\nu} \to \gamma_{\mu\nu}' = L_{\mu\rho} L_{\nu\sigma} \gamma_{\sigma\rho} = \gamma_{\mu\nu} + \left(\delta_{\mu\rho} \lambda_{\nu\sigma} + \delta_{\nu\sigma} \lambda_{\mu\rho} \right) \gamma_{\rho\sigma} . \tag{3.18}$$

Identification with the definitiom (3.6) gives the conditions

$$\tfrac{1}{2} \lambda_{\kappa\tau} \Lambda_{\kappa\tau;\mu\nu,\rho\sigma} = \delta_{\mu\rho} \lambda_{\nu\sigma} + \delta_{\nu\sigma} \lambda_{\mu\rho} \tag{3.19}$$

which are solved by

$$\Lambda_{\kappa\tau;\mu\nu,\rho\sigma} = \delta_{\mu\rho} \left(\delta_{\kappa\nu} \delta_{\tau\sigma} - \delta_{\tau\nu} \delta_{\kappa\sigma} \right) + \delta_{\nu\sigma} \left(\delta_{\kappa\mu} \delta_{\tau\rho} - \delta_{\tau\mu} \delta_{\kappa\rho} \right) . \tag{3.20}$$

(iii) The Spinor Field

A four component spinor ψ transforms under an infinitesimal Lorents transformation (see Pauli (1958)) according to

$$\psi \rightarrow \psi' = S\psi \qquad \text{with } S = I + \tfrac{1}{8}[\gamma_\varkappa, \gamma_\tau]\lambda_{\varkappa\tau} \tag{3.21}$$

where the γ_τ are the well-known Dirac matrices operating on the spinor indices of ψ . Identification with the definition (3.6) gives at once the representation

$$\Lambda_{\varkappa\tau} = \tfrac{1}{4}[\gamma_\varkappa, \gamma_\tau] . \tag{3.22}$$

4. Classification of Polarization States of a Vector Field in Terms of Spin Quantum Numbers

In the linearized quantum theory of gravitation one aims at the closest possible analogy with electrodynamics. Accordingly, it is desirable to classify the possible polarization states of gravitons in close analogy to the classification of photons into transverse, longitudinal, and time-like photons (see, for example, Akhiezer and Berestetskii, 1964). It is the purpose of this section to briefly summarize the well-known description of the polarization states of a vector field, for given propagation vector $\underline{\varkappa}$, in terms of the eigenvectors of the operator representing an infinitesimal rotation around the direction of $\underline{\varkappa}$, and to introduce a notation that lends itself for an easy adaptation to the case of the tensor field which is the object of the next section.

Consider a rotation about the x_3- axis which is generated by the operator (see equations (3.14) and (3.17), with μ labelling rows and ν labelling columns)

$$\Sigma_3 \equiv i\Lambda_{12} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.1}$$

The eigenvalues of Σ_3 are obtained in the usual way by setting the determinant

$$\det(\Sigma_3-\lambda I) = \begin{vmatrix} -\lambda & i & 0 & 0 \\ -i & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = \lambda^2(\lambda^2-1)=0 \qquad (4.2)$$

yielding the solutions λ =1,0,-1;0, corresponding to a reduction of Σ_3 into a direct sum of a (3x3) operator representing a spin j=1 and a (1x1) operator representing a spin j=0. Classifying the respective eigenvectors according to the quantum numbers (j,m), one has a set of four such eigenvectors (1,1), (1,0), (1,-1); (0,0) of Σ_3 which upon normalization may be written as the columns of the unitary matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}. \qquad (4.3)$$

In order to find the eigenvectors of the operator $\Sigma\cdot\underline{n}$, which generates the rotation about the $\underline{\kappa}$ -axis, it is useful to first write all three generators $\Sigma_1 \equiv i\Lambda_{23}$, $\Sigma_2 \equiv i\Lambda_{31}$, $\Sigma_3 \equiv i\Lambda_{12}$ in the representation in which Σ_3 is diagonal, which is brought about by the unitary transformation matrix S according to

$$\sigma_1 \equiv S^{-1}\Sigma_1 S, \quad \sigma_2 \equiv S^{-1}\Sigma_2 S, \quad \sigma_3 \equiv S^{-1}\Sigma_3 S = \mathrm{diag}(1,0,-1,0). \qquad (4.4)$$

In this representation one has

$$\underline{\sigma}\cdot\underline{\varkappa} = \frac{1}{2}\begin{pmatrix} 2\varkappa_3 & -\sqrt{2}\varkappa_+ & 0 & 0 \\ -\sqrt{2}\varkappa_- & 0 & \sqrt{2}\varkappa_+ & 0 \\ 0 & \sqrt{2}\varkappa_- & -2\varkappa_3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with } \varkappa_\pm \equiv \varkappa_1 \pm i\varkappa_2 \tag{4.5}$$

and each σ_k is, of course, again a direct sum of the respective operators representing a spin j=1 and a spin j=0. The operator (4.5) has again the eigenvalues 1,0,-1;0, and its normalized eigenvectors $\eta(j,m)$, classified according to spin quantum numbers j and m, are (with $\omega \equiv [\varkappa_1^2+\varkappa_2^2+\varkappa_3^2]^{1/2}$)

$$\eta(1,\pm 1) = \frac{1}{2\omega\sqrt{\omega^2-\varkappa_3^2}}\begin{bmatrix} \mp\varkappa_+(\omega\pm\varkappa_3) \\ \sqrt{2}(\omega^2-\varkappa_3^2) \\ \pm\varkappa_-(\omega\mp\varkappa_3) \\ 0 \end{bmatrix};\ \eta(1,0) = \frac{1}{\sqrt{2}\,\omega}\begin{bmatrix} \varkappa_+ \\ \sqrt{2}\varkappa_3 \\ \varkappa_- \\ 0 \end{bmatrix};\ \eta(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.6}$$

The eigenvectors of $\underline{\Sigma}\cdot\underline{\varkappa}$ are then obtained by inverting the transformation (4.4), thus

$$\epsilon_\varkappa(1) \equiv S\eta(1,1) = \left[2\omega^2(\omega^2-\varkappa_3^2)\right]^{-1/2}\begin{pmatrix} -\varkappa_1\varkappa_3 - i\omega\varkappa_2 \\ -\varkappa_2\varkappa_3 + i\omega\varkappa_1 \\ \omega^2-\varkappa_3^2 \\ 0 \end{pmatrix}$$

$$\epsilon_\varkappa(2) \equiv S\eta(1,-1) = \left[2\omega^2(\omega^2-\varkappa_3^2)\right]^{-1/2}\begin{pmatrix} -\varkappa_1\varkappa_3 + i\omega\varkappa_2 \\ -\varkappa_2\varkappa_3 - i\omega\varkappa_1 \\ \omega^2-\varkappa_3^2 \\ 0 \end{pmatrix} \tag{4.7}$$

$$\epsilon_\mu(3) \equiv S\eta(1,0) = \frac{1}{\omega}\begin{pmatrix} \varkappa_1 \\ \varkappa_2 \\ \varkappa_3 \\ 0 \end{pmatrix}$$

$$\epsilon_\mu(4) \equiv S\eta(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The "polarization state vectors" $\epsilon_\mu(s)$ (s=1,2,3,4), characterized by the "polarization quantum number" s, have the properties

$$\epsilon_\mu^*(1) = \epsilon_\mu(2)\,; \qquad \epsilon_\mu^*(3) = \epsilon_\mu(3)\,; \qquad \epsilon_\mu^*(4) = \epsilon_\mu(4), \tag{4.8}$$

$$\varkappa_\mu \epsilon_\mu(s) = \begin{cases} 0 & \text{for } s=1,2 \\ \omega & \text{for } s=3 \\ i\omega & \text{for } s=4 \end{cases} \qquad \text{where } \varkappa_\mu \equiv (\underline{\varkappa}, i\omega) \tag{4.9}$$

$$\epsilon_\mu^*(s)\,\epsilon_\mu(s') = \delta_{ss'} \tag{4.10}$$

$$\sum_s \epsilon_\mu^*(s)\,\epsilon_\nu(s) = \delta_{\mu\nu} \tag{4.11}$$

$$\epsilon_\mu(-\underline{\varkappa},s) = \begin{cases} \epsilon_\mu(\underline{\varkappa},2) = \epsilon_\mu^*(\underline{\varkappa},1) & \text{for } s=1 \\ \epsilon_\mu(\underline{\varkappa},1) = \epsilon_\mu^*(\underline{\varkappa},2) & \text{for } s=2 \\ -\epsilon_\mu(\underline{\varkappa},3) = -\epsilon_\mu^*(\underline{\varkappa},3) & \text{for } s=3 \\ \epsilon_\mu(\underline{\varkappa},4) = \epsilon_\mu^*(\underline{\varkappa},4) & \text{for } s=4. \end{cases} \tag{4.12}$$

It is the property (4.9) which allows one to look upon the vectors $\epsilon_\mu(1)$ and $\epsilon_\mu(2)$ as representing "transverse" polarization states, the vector $\epsilon_\mu(3)$ as a "longitudinal" polarization state, and the vector $\epsilon_\mu(4)$ a "time-like" polarization state. For some purposes it will be found useful to express the $\epsilon_\mu(s)$ in terms of polar angles θ, ϕ defined by the direction of $\underline{\varkappa}$,

$$\underline{\varkappa} = (\omega \sin\theta \cos\phi,\ \omega \sin\theta \sin\phi,\ \omega\cos\theta) \tag{4.13}$$

with the result

$$\epsilon_\mu(1) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\theta\cos\phi - i\sin\phi \\ -\cos\theta\sin\phi + i\cos\phi \\ -\sin\theta \\ 0 \end{pmatrix} \qquad \epsilon_\mu(3) = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \\ 0 \end{pmatrix}$$

$$\epsilon_\mu(2) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\theta\cos\phi + i\sin\phi \\ -\cos\theta\sin\phi - i\cos\phi \\ -\sin\theta \\ 0 \end{pmatrix} \qquad \epsilon_\mu(4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.14}$$

An additional property of the transverse polarization vectors is

$$\begin{aligned} \underline{\varkappa} \times \underline{\epsilon}(\underline{\varkappa},1) &= i\omega\,\underline{\epsilon}(\underline{\varkappa},1) \\ \underline{\varkappa} \times \underline{\epsilon}(\underline{\varkappa},2) &= -i\omega\,\underline{\epsilon}(\underline{\varkappa},2) \end{aligned} \tag{4.15}$$

5. Classification of Polarization States of a Tensor Field in Terms of Spin Quantum Numbers

The tensor field may be treated in a fashion analogous to the vector field, provided that the 16 components of the tensor field are treated as components of a 16 dimensional column vector, according to the labelling scheme given in (3.7). The (16x16) operator generating a rotation about the x_3 -axis (see (3.20) with $\mu\nu$ labelling rows and $\rho\sigma$ labelling columns) is given by

$$\Sigma_3 \equiv i\Lambda_{12} = \begin{pmatrix} X & iI & 0 & 0 \\ -iI & X & 0 & 0 \\ 0 & 0 & X & 0 \\ 0 & 0 & 0 & X \end{pmatrix} \qquad (5.1)$$

where I ≡ diag (1,1,1,1) and X is the matrix (4.1). Note that

$$i\Lambda_{12;\mu\nu,\rho\sigma} = \delta_{\mu\rho} \otimes i\Lambda_{12;\nu,\sigma} + i\Lambda_{12;\mu,\rho} \otimes \delta_{\nu\sigma} \qquad (5.2)$$

where $\otimes$ symbolizes a direct product. The eigenvalues of Σ_3 are solutions of

$$\det(\Sigma_3 - \lambda I) = \lambda^6(\lambda^2-1)^4(\lambda^2-4) = 0, \qquad (5.3)$$

namely λ =2,1,0,-1,-2;1,0,-1;0;0;1,0,-1;1,0,-1 corresponding to a reduction of Σ_3 into a direct sum of a (5x5) operator representing a spin j=2, three (3x3) operators representing spins j=1, and two (1x1) operators representing spins j=0. Classifying the respective eigenvectors according to quantum numbers (j,m) one has a set of sixteen such eigenvectors (2,2), (2,1),(2,0),(2,-1),(2,-2); (1,1),(1,0),(1,-1); (0,0); (0,0); (1,1),(1,0),(1,-1); (1,1),(1,0),(1,-1) which, upon normalization, may be written as the columns of the unitary matrix

$$S=\frac{1}{2}\begin{bmatrix}
1 & 0 & -\sqrt{2/3} & 0 & 1 & 0 & 0 & 0 & 1/\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
-i & 0 & 0 & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 \\
-1 & 0 & -\sqrt{2/3} & 0 & -1 & 0 & 0 & 0 & 1/\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & i & 0 & 0 & 0 & 0 & 0 & -i & 0 & i \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & -i & 0 & i & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & -i & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{2/3} & 0 & 0 & 0 & 0 & 0 & 1/\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -i & 0 & i & 0 & 0 & 0 & 0 & 0 & i & 0 & -i \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \qquad (5.4)$$

Remembering that these columns, with the labelling convention (3.7) represent 2nd rank tensors, one can see that the first nine of these tensors are symmetric and traceless, the tenth is diagonal and has a trace, and the remaining six are skew (and hence traceless). Defining the σ_k analogously to the vector case, S provides a breakdown of $\sigma\cdot\varkappa$ into blocks, corresponding to a reduction into a direct sum of the type (j=2) ⊕ (j=1) ⊕ (j=0) ⊕ (j=0) ⊕ (j=1) ⊕ (j=1) similar to the reduction of Σ_3 namely

$$2\sigma\cdot\varkappa = \begin{bmatrix} 4\varkappa_3 & -2\varkappa_+ & 0 & 0 & 0 \\ -2\varkappa_- & 2\varkappa_3 & -\sqrt{6}\varkappa_+ & 0 & 0 \\ 0 & -\sqrt{6}\varkappa_- & 0 & \sqrt{6}\varkappa_+ & 0 \\ 0 & 0 & \sqrt{6}\varkappa_- & -2\varkappa_3 & 2\varkappa_3 \\ 0 & 0 & 0 & 2\varkappa_- & -4\varkappa_3 \end{bmatrix} \oplus \begin{bmatrix} 2\varkappa_3 & -\sqrt{2}\varkappa_+ & 0 \\ -\sqrt{2}\varkappa_- & 0 & \sqrt{2}\varkappa_+ \\ 0 & \sqrt{2}\varkappa_- & -2\varkappa_3 \end{bmatrix}$$

$$\oplus [0] \oplus [0] \oplus \begin{bmatrix} 2\varkappa_3 & -\sqrt{2}i\varkappa_+ & 0 \\ \sqrt{2}i\varkappa_- & 0 & \sqrt{2}i\varkappa_+ \\ 0 & -\sqrt{2}i\varkappa_- & -2\varkappa_3 \end{bmatrix} \oplus \begin{bmatrix} 2\varkappa_3 & -\sqrt{2}\varkappa_+ & 0 \\ -\sqrt{2}\varkappa_- & 0 & \sqrt{2}\varkappa_+ \\ 0 & \sqrt{2}\varkappa_- & -2\varkappa_3 \end{bmatrix} \qquad (5.5)$$

The operator (5.5) again has eigenvalues 2,1,0,-1,-2;1,0,-1; 0;0;1,0,-1;1,0,-1 and its normalized eigenvectors classified according to (j,m) are given in Appendix A and labelled

$$\underbrace{\begin{array}{ll} \eta(j,m): & j=2\ \ m=2,1,0,-1,-2 \\ & j=1\ \ m=1,0,-1 \\ & j=0\ \ m=0 \end{array}}_{\text{symmetric, traceless}} \qquad \underbrace{\xi(0,0)}_{\text{trace}} \quad \underbrace{\begin{array}{l} \zeta(1,m):\ m=1,0,-1 \\ \gamma(1,m):\ m=1,0,-1 \end{array}}_{\text{skew}} \qquad (5.6)$$

The eigenvectors of $\Sigma\cdot\varkappa$ are also given in Appendix A, and are labelled

$$\begin{aligned} S\,\eta(j,m) &\equiv \epsilon_{\mu\nu}(9-j[j+1]-m) \\ S\,\xi(0,0) &\equiv \epsilon_{\mu\nu}(10) \\ S\,\zeta(1,m) &\equiv \epsilon_{\mu\nu}(12-m) \\ S\,\gamma(1,m) &\equiv \epsilon_{\mu\nu}(15-m) \end{aligned} \qquad (5.7)$$

The sixteen polarization tensors $\epsilon_{\mu\nu}(s)$ (s=1,...16), characterized by the polarization quantum number s, can be decomposed into linear combinations of products of the four polarization vectors $\epsilon_{\mu}(s)$ treated in the preceding section, as follows:

$$\left.\begin{aligned} \epsilon_{\mu\nu}(1) &= \epsilon_{\mu}(1)\,\epsilon_{\nu}(1) \\ \epsilon_{\mu\nu}(5) &= \epsilon_{\mu}(2)\,\epsilon_{\nu}(2) \end{aligned}\right\}\ \text{transverse-transverse} \tag{5.8}$$

$$\left.\begin{aligned} \epsilon_{\mu\nu}(2) &= \frac{1}{\sqrt{2}}\left[\epsilon_{\mu}(1)\,\epsilon_{\nu}(3) + \epsilon_{\mu}(3)\,\epsilon_{\nu}(1)\right] \\ \epsilon_{\mu\nu}(4) &= \frac{1}{\sqrt{2}}\left[\epsilon_{\mu}(2)\,\epsilon_{\nu}(3) + \epsilon_{\mu}(3)\,\epsilon_{\nu}(2)\right] \end{aligned}\right\}\ \text{transverse-longitudinal}$$

$$\epsilon_{\mu\nu}(3) = \frac{1}{\sqrt{6}}\left[2\epsilon_{\mu}(3)\,\epsilon_{\nu}(3) - \epsilon_{\mu}(1)\,\epsilon_{\nu}(2) - \epsilon_{\mu}(2)\,\epsilon_{\nu}(1)\right]\ \text{mixed}$$

$$\left.\begin{aligned} \epsilon_{\mu\nu}(6) &= \frac{1}{\sqrt{2}}\left[\epsilon_{\mu}(1)\,\epsilon_{\nu}(4) + \epsilon_{\mu}(4)\,\epsilon_{\nu}(1)\right] \\ \epsilon_{\mu\nu}(8) &= \frac{1}{\sqrt{2}}\left[\epsilon_{\mu}(2)\,\epsilon_{\nu}(4) + \epsilon_{\mu}(4)\,\epsilon_{\nu}(2)\right] \end{aligned}\right\}\ \text{transverse-time-like}$$

$$\epsilon_{\mu\nu}(9) = \frac{1}{2\sqrt{3}}\left[\begin{aligned} &\epsilon_{\mu}(1)\,\epsilon_{\nu}(2) + \epsilon_{\mu}(2)\,\epsilon_{\nu}(1) \\ &\qquad + \epsilon_{\mu}(3)\,\epsilon_{\nu}(3) - 3\,\epsilon_{\mu}(4)\,\epsilon_{\mu}(4) \end{aligned}\right]\ \text{mixed}$$

$$\epsilon_{\mu\nu}(10) = \frac{1}{2}\left[\begin{aligned} &\epsilon_{\mu}(1)\,\epsilon_{\nu}(2) + \epsilon_{\mu}(2)\,\epsilon_{\nu}(1) \\ &\qquad + \epsilon_{\mu}(3)\,\epsilon_{\nu}(3) + \epsilon_{\mu}(4)\,\epsilon_{\nu}(4) \end{aligned}\right]\ \text{mixed}$$

$$\left.\begin{aligned} \epsilon_{\mu\nu}(11) &= \frac{i}{\sqrt{2}}\left[\epsilon_{\mu}(1)\,\epsilon_{\nu}(3) - \epsilon_{\mu}(3)\,\epsilon_{\nu}(1)\right] \\ \epsilon_{\mu\nu}(13) &= -\frac{i}{\sqrt{2}}\left[\epsilon_{\mu}(2)\,\epsilon_{\nu}(3) - \epsilon_{\mu}(3)\,\epsilon_{\nu}(2)\right] \end{aligned}\right\}\ \text{transverse-longitudinal}$$

$$\epsilon_{\mu\nu}(12) = \frac{i}{\sqrt{2}}\left[\epsilon_\mu(1)\epsilon_\nu(2) - \epsilon_\mu(2)\epsilon_\nu(1)\right] \quad \text{transverse-transverse}$$

$$\left.\begin{aligned} \epsilon_{\mu\nu}(14) &= \frac{1}{\sqrt{2}}\left[\epsilon_\mu(1)\epsilon_\nu(4) - \epsilon_\mu(4)\epsilon_\nu(1)\right] \\ \epsilon_{\mu\nu}(16) &= \frac{1}{\sqrt{2}}\left[\epsilon_\mu(2)\epsilon_\nu(4) - \epsilon_\mu(4)\epsilon_\nu(2)\right] \end{aligned}\right\} \text{transverse-time-like}$$

$$\left.\begin{aligned} \epsilon_{\mu\nu}(15) &= \frac{1}{\sqrt{2}}\left[\epsilon_\mu(3)\epsilon_\nu(4) - \epsilon_\mu(4)\epsilon_\nu(3)\right] \\ \epsilon_{\mu\nu}(7) &= \frac{1}{\sqrt{2}}\left[\epsilon_\mu(3)\epsilon_\nu(4) + \epsilon_\mu(4)\epsilon_\nu(3)\right] \end{aligned}\right\} \text{longitudinal-time-like}$$

The polarization tensors have been named according to the combination of polarization vectors from which they have been formed. They have the additional properties

$$\epsilon^*_{\mu\nu}(1) = \epsilon_{\mu\nu}(5), \qquad \epsilon^*_{\mu\nu}(2) = \epsilon_{\mu\nu}(4), \quad \epsilon^*_{\mu\nu}(6) = \epsilon_{\mu\nu}(8), \qquad (5.9)$$

$\epsilon^*_{\mu\nu}(11) = \epsilon_{\mu\nu}(13), \qquad \epsilon^*_{\mu\nu}(14) = \epsilon_{\mu\nu}(16),$ all others are real

$\varkappa_\mu \epsilon_{\mu\nu}(s) = 0$	s=1,5	$\varkappa_\mu \epsilon^*_{\mu\nu}(s) = 0$	s=1,5
$\frac{\omega}{\sqrt{2}}\epsilon_\nu(1,2)$	s=2,4	$\frac{\omega}{\sqrt{2}}\epsilon^*_\nu(1,2)$	s=2,4
$\frac{2\omega}{\sqrt{6}}\epsilon_\nu(3)$	s=3	$\frac{2\omega}{\sqrt{6}}\epsilon_\nu(3)$	s=3
$\frac{i\omega}{\sqrt{2}}\epsilon_\nu(1,2)$	s=6,8	$\frac{i\omega}{\sqrt{2}}\epsilon^*_\nu(1,2)$	s=6,8
$\frac{i}{\sqrt{2}}\varkappa^*_\nu$	s=7	$\frac{i}{\sqrt{2}}\varkappa^*_\nu$	s=7
$\frac{1}{2\sqrt{3}}(\underline{\varkappa}_\nu - 3i\omega)$	s=9	$\frac{1}{2\sqrt{3}}(\underline{\varkappa}_\nu - 3i\omega)$	s=9
$\frac{1}{2}\varkappa_\nu$	s=10	$\frac{1}{2}\varkappa_\nu$	s=10

$\mp\frac{i\omega}{\sqrt{2}}\,\epsilon_\nu(1,2)$	s=11,13	$\pm\frac{i\omega}{\sqrt{2}}\,\epsilon_\nu^*(1,2)$	s=11,13
0	s=12	0	s=12
$-\frac{i\omega}{\sqrt{2}}\,\epsilon_\nu(1,2)$	s=14,16	$-\frac{i\omega}{\sqrt{2}}\,\epsilon_\nu^*(1,2)$	s=14,16
$-\frac{i}{\sqrt{2}}\,\chi_\nu$	s=15	$-\frac{i}{\sqrt{2}}\,\chi_\nu$	s=15

$$\epsilon^*_{\mu\nu}(s)\,\epsilon_{\mu\nu}(s') = \delta_{ss'} \qquad (5.11)$$

$$\sum_{s=1}^{16} \epsilon^*_{\mu\nu}(s)\,\epsilon_{\rho\sigma}(s) = \delta_{\mu\rho}\,\delta_{\nu\sigma} \qquad (5.12)$$

Other properties such as the expression of $\epsilon_{\mu\nu}(s)$ in terms of polar angles and the values of $\epsilon_{\mu\nu}(-\underline{\kappa},s)$ may be found from (5.8) and the appropriate result from Section 4.

6. The Dynamical Attributes of Classical Gravitons Flowing From an Action Principle

The source-free gravitational field in linear approximation, as was explained in section 1, is described by a tensor field of the second rank, $\gamma_{\mu\nu}(x)$ which satisfies the wave equation

$$\Box\gamma_{\mu\nu} \equiv \gamma_{\mu\nu/\sigma\sigma} = \frac{\partial^2\gamma_{\mu\nu}}{\partial x_k \partial x_k} - \frac{\partial^2\gamma_{\mu\nu}}{\partial t^2} = 0, \tag{6.1}$$

the symmetry condition

$$\gamma_{\mu\nu} - \gamma_{\nu\mu} = 0 \tag{6.2}$$

and the two gauge conditions of Fierz

$$\gamma_{\mu\nu/\nu} = 0 \tag{6.3a}$$

$$\gamma_{\sigma\sigma} = 0. \tag{6.3b}$$

The dynamical properties of such a field will be examined in this section and in section 7. In this section, the three conditions (6.2), (6.3) will be ignored, and $\gamma_{\mu\nu}$ will represent a general tensor field satisfying only the wave equation (6.1) and the reality conditions γ_{ik}, γ_{44} real, γ_{4k}, γ_{k4} imaginary which follow from the reality conditions on the $\lambda_{\mu\nu}$ stated after (3.5). This corresponds to a study of "generalized electrodynamics" which is described by a vector field satisfying only the wave equation (1.23). Generalized electrodynamics becomes the same as Maxwell's electrodynamics only when, to (1.23), is added the Lorentz condition (1.21). Thus, the gravitational analog of the Lorentz condition is the set of conditions (6.2) and (6.3).

The field equations (6.1) may be "derived" from an action principle, using the Lagrangian density

$$L = -\tfrac{1}{2}\gamma_{\mu\nu/\sigma}\gamma_{\mu\nu/\sigma} \quad \text{(const.)} \tag{6.4}$$

where a constant must be included for dimensional consistency, but will be ignored for now. If one treats the $\gamma_{\mu\nu}$ as 16 independent field variables, then the action principle

$$\delta\left(\int dx\, L\right) = 0 \tag{6.5}$$

yields, in the usual manner, the Euler-Lagrange equations

$$\frac{\delta L}{\delta \gamma_{\mu\nu}} \equiv \frac{\partial L}{\partial \gamma_{\mu\nu}} - \frac{\partial}{\partial x_\sigma}\left[\frac{\partial L}{\partial \gamma_{\mu\nu/\sigma}}\right] = \Box\gamma_{\mu\nu} = 0 \tag{6.6}$$

where $dx = d^4x = dx_1 dx_2 dx_3 dt$ (see Roman, 1964, Ch. IV, § 1). From Noether's theorem, the invariance of the action

$$\mathcal{L} \equiv \int dx L \tag{6.7}$$

under various transformations implies the existence of conserved quantities as follows. Suppose $L = L(\psi_A, \psi_{A/\sigma})$ where ψ_A is an arbitrary field. If $\mathcal{L}$ is invariant under space or time translations, then the momentum

$$P_k \equiv i\int dx \frac{\partial L}{\partial \psi_{A/4}} \psi_{A/k} \tag{6.8}$$

or the energy

$$H \equiv \int d\underline{x} \left(\frac{\partial L}{\partial \psi_{A/4}} \psi_{A/4} - L \right) \tag{6.9}$$

respectively is conserved, where $d\underline{x} = dx_1 dx_2 dx_3$. If $\mathcal{L}$ is invariant under rotations, then the total angular momentum

$$J_k \equiv L_k + S_k \tag{6.10}$$

is conserved, where

$$L_k \equiv -\epsilon_{ijk} \int d\underline{x}\, x_i \mathcal{P}_j \tag{6.11}$$

is the orbital angular momentum, which depends on the choice of the co-ordinate system (with $\mathcal{P}_j$ the momentum density, and ϵ_{ijk} the usual completely antisymmetric tensor density) and S_k is the spin angular momentum

$$S_k \equiv -i \epsilon_{ijk} \int d\underline{x} \frac{\partial L}{\partial \psi_{A/4}} \Lambda_{ij;A,B} \psi_B \tag{6.12}$$

with $\Lambda_{ij;A,B}$ defined by (3.6). The momentum, energy, and spin are, in terms of the tensor field $\gamma_{\mu\nu}$,

$$P_k = -i \int d\underline{x}\, \gamma_{\mu\nu/4} \gamma_{\mu\nu/k} \tag{6.13}$$

$$H = \tfrac{1}{2} \int d\underline{x} \left(\gamma_{\mu\nu/i} \gamma_{\mu\nu/i} - \gamma_{\mu\nu/4} \gamma_{\mu\nu/4} \right) \tag{6.14}$$

$$S_k = i \epsilon_{ijk} \int d\underline{x} \left(\gamma_{\mu j} \gamma_{\mu i/4} - \gamma_{\mu i} \gamma_{\mu j/4} + \gamma_{j\mu} \gamma_{i\mu/4} - \gamma_{i\mu} \gamma_{j\mu/4} \right). \tag{6.15}$$

At this point the development of Section 5 enables one to make a Fourier decomposition of $\Upsilon_{\mu\nu}(x)$, in complete analogy to the corresponding expansion of electrodynamics, by using the polarization tensors $\epsilon_{\mu\nu}(s)$ (see definition (5.7)) as follows

$$\Upsilon_{\mu\nu}(x) = \frac{1}{\sqrt{V}} \sum_{\underline{\varkappa}} \sum_{s=1}^{16} \frac{1}{\sqrt{2\omega}} \left[\epsilon_{\mu\nu}(s) b(\underline{\varkappa}, s) e^{i\varkappa x} + \epsilon^{*}_{\mu\nu}(s) b^{\dagger}(\underline{\varkappa}, s) e^{-i\varkappa x} \right] \tag{6.16}$$

where V is a normalization volume, which can be extended to infinity by means of the correspondence

$$\frac{1}{V} \sum_{\underline{\varkappa}} \rightarrow \frac{1}{(2\pi)^3} \int d\underline{\varkappa}, \tag{6.17}$$

the terms in the exponentials are four vector products

$$\varkappa x = \underline{\varkappa} \cdot \underline{x} - \omega t \tag{6.18}$$

so that (6.16) automatically satisfies the wave equation (6.1), the $b(\underline{\varkappa}, s)$ and $b^{\dagger}(\underline{\varkappa}, s)$ are Fourier amplitudes defined by (6.16), and the factor $(2\omega)^{-1/2}$ may be thought of as part of this definition for reasons which will become obvious later. A constant to match the dimensionality has been omitted for the present. The reality conditions on the $\Upsilon_{\mu\nu}$ require that the Fourier amplitudes satisfy the relations

$$\left[b^{\dagger}(\underline{\varkappa}, s) \right]^{*} \begin{cases} b(\underline{\varkappa}, s) & s=1\text{-}5, 9, 10, 11\text{-}13 \\ -b(\underline{\varkappa}, s) & s=6\text{-}8; 14\text{-}16. \end{cases} \tag{6.18}$$

If one substitutes the expansion (6.16) into (6.14), the energy becomes

$$H = \frac{-1}{2V}\int d\underline{x} \sum_{\underline{\varkappa}\,\underline{\varkappa}'} \sum_{s\,s'} \frac{(\varkappa_i\varkappa_i' + \omega\omega')}{2\sqrt{\omega\omega'}} \Big[\epsilon_{\mu\nu}(s)\,\epsilon_{\mu\nu}(s')\,b(\underline{\varkappa},s)\,b(\underline{\varkappa}',s')\,e^{i(\varkappa+\varkappa')x}$$

$$+ \epsilon^*_{\mu\nu}(s)\,\epsilon^*_{\mu\nu}(s')\,b^\dagger(\underline{\varkappa},s)\,b^\dagger(\underline{\varkappa}',s')\,e^{-i(\varkappa+\varkappa')x} \tag{6.19}$$

$$- \epsilon^*_{\mu\nu}(s)\,\epsilon_{\mu\nu}(s')\,b^\dagger(\underline{\varkappa},s)\,b(\underline{\varkappa}',s')\,e^{-i(\varkappa-\varkappa')x}$$

$$- \epsilon^*_{\mu\nu}(s')\,\epsilon_{\mu\nu}(s)\,b^\dagger(\underline{\varkappa}',s')\,b(\underline{\varkappa},s)\,e^{i(\varkappa-\varkappa')x} \Big].$$

Because of the relation

$$\int d\underline{x}\, e^{\pm i(\varkappa+\varkappa')x} = e^{\mp i(\omega+\omega')t} \int d\underline{x}\, e^{\pm i(\underline{\varkappa}+\underline{\varkappa}')\cdot\underline{x}}$$

$$= e^{\mp i(\omega+\omega')t}(2\pi)^3\,\delta(\underline{\varkappa}+\underline{\varkappa}') \tag{6.20}$$

and the fact that

$$\varkappa_i\varkappa_i' + \omega\omega' = -\varkappa_i\varkappa_i + \omega^2 = 0 \text{ when } \varkappa_i' = -\varkappa_i\,, \tag{6.21}$$

the first two terms vanish; so, using (6.17),

$$H = \sum_{\underline{\varkappa}} \sum_{s,s'} \frac{\omega}{2}\Big[\epsilon^*_{\mu\nu}(s)\,\epsilon_{\mu\nu}(s')\,b^\dagger(\underline{\varkappa},s)\,b(\underline{\varkappa},s') + \epsilon^*_{\mu\nu}(s)\,\epsilon_{\mu\nu}(s')\,b^\dagger(\underline{\varkappa},s')\,b(\underline{\varkappa},s)\Big] \tag{6.22}$$

and, making use of the orthonormality relation (5.11), one obtains

$$H = \sum_{\underline{\varkappa},s} \omega\, b^\dagger(\underline{\varkappa},s)\,b(\underline{\varkappa},s) \equiv \sum_{\underline{\varkappa},s} \omega\, n(\underline{\varkappa},s) \tag{6.23}$$

where $n(\underline{\varkappa},s)$ may be thought of as the number of "classical gravitons" of momentum $\underline{\varkappa}$ and polarization s. A similar calculation for the momentum yields

$$P_k = \sum_{\underline{\varkappa},s} \varkappa_k\, b^\dagger(\underline{\varkappa},s)\,b(\underline{\varkappa},s) = \sum_{\underline{\varkappa},s} \varkappa_k\, n(\underline{\varkappa},s)\,. \tag{6.24}$$

The spin is more complicated, a similar calculation yields

$$S_k = 2i\epsilon_{ijk} \sum_{\underline{\kappa}} \sum^{*}_{s,s'} \Big[\epsilon^{*}_{\mu i}(s)\,\epsilon_{\mu j}(s')\, b^{+}(\underline{\kappa},s)\, b(\underline{\kappa},s') - \epsilon^{*}_{\mu j}(s')\,\epsilon_{\mu i}(s)\, b^{+}(\underline{\kappa},s')\, b(\underline{\kappa},s) \Big] \tag{6.25}$$

where the symbol $\sum^{*}_{s,s'}$ means a summation only over s and s' either <u>both</u> less than 11 (symmetric) or <u>both</u> greater than 10 (skew). An explicit substitution using relations (5.8) and the polarization vectors (4.7) yields, suppressing the $\underline{\kappa}$

$$\begin{aligned} S_k = \sum_{\underline{\kappa}} \Big\{ & \frac{\kappa_k}{\omega}\Big(2[n(1)-n(5)] + [n(2)-n(4)] + [n(6)-n(8)] + [n(11)-n(13)] \\ & + [n(14)-n(16)]\Big) + R_k\Big[b^{+}(1)b(2) - b^{+}(4)b(5) + \sqrt{\tfrac{3}{2}}\, b^{+}(2)b(3) \\ & - \sqrt{\tfrac{3}{2}}\, b^{+}(3)b(4) - \tfrac{1}{\sqrt{2}} b^{+}(6)b(7) - \tfrac{1}{\sqrt{2}} b^{+}(7)b(8) - \tfrac{1}{\sqrt{2}} b^{+}(11)b(12) \\ & + \tfrac{1}{\sqrt{2}} b^{+}(12)b(13) + \tfrac{1}{\sqrt{2}} b^{+}(14)b(15) - \tfrac{1}{\sqrt{2}} b^{+}(15)b(16)\Big] + R_k^{*}\Big[b^{+}(2)b(1) \\ & - b^{+}(5)b(4) + \sqrt{\tfrac{3}{2}}\, b^{+}(3)b(2) - \sqrt{\tfrac{3}{2}}\, b^{+}(4)b(3) + \tfrac{1}{\sqrt{2}} b^{+}(7)b(6) \\ & - \tfrac{1}{\sqrt{2}} b^{+}(8)b(7) - \tfrac{1}{\sqrt{2}} b^{+}(12)b(11) + \tfrac{1}{\sqrt{2}} b^{+}(13)b(12) + \tfrac{1}{\sqrt{2}} b^{+}(15)b(14) - \tfrac{1}{\sqrt{2}} b^{+}(16)b(15)\Big]\Big\} \end{aligned} \tag{6.26}$$

where

$$R_1 = \frac{\kappa_1\kappa_3 - i\omega\kappa_2}{\omega(\omega^2-\kappa_3^2)^{1/2}}, \qquad R_2 = \frac{\kappa_2\kappa_3 + i\omega\kappa_1}{\omega(\omega^2-\kappa_3^2)^{1/2}}, \qquad R_3 = \frac{-(\omega^2-\kappa_3^2)^{1/2}}{\omega}$$

7. The Restrictions Imposed by the Auxiliary Conditions

The linearized, source-free gravitational field is described by a tensor field $\gamma_{\mu\nu}$ which satisfies, in addition to the wave equation (6.1), the condition of symmetry (6.2) and the Fierz gauge conditions (6.3). Applying these conditions to the expansion (6.16), one obtains from (6.2)

$$\frac{1}{\sqrt{V}} \sum_{\kappa,s} \frac{1}{\sqrt{2\omega}} \Big\{ \left[\epsilon_{\mu\nu}(s) - \epsilon_{\nu\mu}(s)\right] b(\kappa,s)\, e^{i\kappa x} + \left[\epsilon^{*}_{\mu\nu}(s) - \epsilon^{*}_{\nu\mu}(s)\right] b^{\dagger}(\kappa,s)\, e^{-i\kappa x} \Big\} = 0 \tag{7.1}$$

and since, for s=11, ...16,

$$\epsilon_{\mu\nu}(s) - \epsilon_{\nu\mu}(s) = 2\epsilon_{\mu\nu}(s) \neq 0, \tag{7.2}$$

it follows that

$$b(s) = b^{\dagger}(s) = 0 \text{ for } s=11,\ldots 16. \tag{7.3}$$

Similarly, from the condition (6.3a) one obtains

$$\sum_{s} \kappa_{\nu} \epsilon_{\mu\nu}(s)\, b(s) = 0, \tag{7.4}$$

which is a set of four conditions (μ =1,2,3,4). Using the relations (5.10), one finds that they become

$$\begin{aligned}
&b(2) + i b(6) + i\left[b(11) + b(14)\right] = 0 \\
&b(4) + i b(8) - i\left[b(13) - b(16)\right] = 0 \\
&b(7) - i\sqrt{\tfrac{3}{2}}\, b(9) + \tfrac{i}{\sqrt{2}}\, b(10) - b(15) = 0 \\
&b(7) - \tfrac{i}{\sqrt{6}}\, b(9) - \tfrac{2i}{\sqrt{3}}\, b(3) - \tfrac{i}{\sqrt{2}}\, b(10) + b(15) = 0.
\end{aligned} \tag{7.5}$$

Similarly, condition (6.3b) yields

$$b(10) = b^{\dagger}(10) = 0. \tag{7.6}$$

Combining these conditions, one arrives at

$$\begin{aligned} &b(2) + i b(6) = 0 \\ &b(4) + i b(8) = 0 \\ &b(7) - i\sqrt{\tfrac{3}{2}}\, b(9) = 0 \\ &b(3) - \tfrac{1}{\sqrt{2}}\, b(9) = 0 \\ &b(s) = 0, \quad s = 10, \dots 16 \end{aligned} \tag{7.7}$$

and from them, one obtains for the classical "graviton numbers" $n(s) = b^{\dagger}(s)b(s)$ the constraints

$$\begin{aligned} &n(2) + n(6) = 0 \\ &n(4) + n(8) = 0 \\ &n(3) + n(7) + n(9) = 0 \\ &n(s) = 0, \quad s=10,\dots 16. \end{aligned} \tag{7.8}$$

These conditions are the complete analog of the effect the Lorentz condition has in electrodynamics, which, in terms of Fourier amplitudes, amounts to the constraints

$$b(3) + ib(4) = 0, \quad n(3) + n(4) = 0 \tag{7.9}$$

where polarizations 3 and 4 refer respectively to longitudinal and time-like photons. The energy and momentum are modified by (7.7) to become

$$H = \sum_{\mathcal{K}} \omega \left[n(1) + n(5) \right] \tag{7.10a}$$

$$P_k = \sum_{\underline{x}} \varkappa_k [n(1) + n(5)] \tag{7.10b}$$

which is an important result, namely, that only the two polarizations of symmetric transverse-transverse gravitons contribute to the energy and momentum. These may be defined respectively as right hand and left hand circular polarizations. The spin is also modified by (7.7) to yield

$$S_k = \sum_{\underline{x}} \left\{ \frac{\varkappa_k}{\omega} 2[n(1) - n(5)] + R_k [b^\dagger(1) b(2) - b^\dagger(4) b(5)] \right. \\ \left. + R_k^* [b^\dagger(2) b(1) - b^\dagger(5) b(4)] \right\} \tag{7.11}$$

In the next section, it will be shown that the coefficients of R_k and R_k^* in (7.11) are gauge dependent and hence are of no physical significance. Thus, the spin (in the direction of propagation) may be identified with the difference of the number of right hand circularly polarized gravitons $n(1)$ and the number of left hand circularly polarized gravitons $n(5)$ times the spin of a single graviton which is 2, in complete analogy with the electromagnetic field whose spin is given by the number of RH minus number of LH circularly polarized photons, times the spin of a single photon (which is 1).

Consider a pure Lorentz transformation in the x_1 direction with $\lambda_{14} = -\lambda_{41} = v$ (see equation (3.2)), where $v^2 \ll 1$ so that

$$\begin{aligned} x_1' &= x_1 + v x_4 \\ x_2' &= x_2 \\ x_3' &= x_3 \\ x_4' &= x_4 - v x_1 \, . \end{aligned} \tag{7.12}$$

Before the behaviour of the gravitational field under such a transformation is considered, the behaviour of the electromagnetic field will be examined. The electromagnetic field is a vector field and hence must transform according to

$$\begin{aligned} A_1' &= A_1 + v A_4 \\ A_2' &= A_2 \\ A_3' &= A_3 \\ A_4' &= A_4 - v A_1 \end{aligned} \tag{7.13}$$

by the rule (3.15). Besides being a vector field, $A_\mu(x)$ must also satisfy the Lorentz condition

$$A_{\sigma/\sigma} = 0 \tag{7.14}$$

in the new system, since (7.14) is a scalar. In order to see how the Fourier amplitudes of such a field are affected by this transformation, one must make the decomposition

$$A_\mu(x) = \frac{1}{\sqrt{V}} \sum_{\underline{\kappa}} \frac{1}{\sqrt{2\omega}} \sum_{s=1}^{4} \left[\epsilon_\mu(s) b(\underline{\kappa},s) e^{i\kappa x} + \epsilon_\mu^*(s) b^\dagger(\underline{\kappa},s) e^{-i\kappa x} \right] \tag{7.15}$$

where the polarization vectors $\epsilon_\mu(s)$ were introduced in Section 4. It is helpful to take the simple case of a plane wave, with propagation vector $\underline{k}$ = (0,0,w) and with RH circular polarization (s=1), so that

$$A_\mu(x) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2w}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} b(\underline{k},1) e^{ikx} + \text{c.c.} \tag{7.16}$$

In components, the transformed field becomes

$$\begin{aligned} A_1'(x') &= \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2w}} \frac{1}{\sqrt{2}} b(\underline{k},1) e^{ikx'} + \text{c.c.} \\ A_2'(x') &= \;'' \quad \frac{-i}{\sqrt{2}} \quad '' \qquad\qquad '' \\ A_3'(x') &= 0 \\ A_4'(x') &= \;'' \quad \frac{-v}{\sqrt{2}} \quad '' \qquad\qquad '' \;. \end{aligned} \tag{7.17}$$

Now the vector product kx' may be written as k'x if one defines k' to be a new propagation vector

$$k' = (ivw,\ 0,\ w,\ iw). \tag{7.18}$$

This enables one to determine the Fourier amplitudes of the transformed field, $b'(\underline{\kappa},s)$, defined by

$$A_\mu'(x') = \frac{1}{\sqrt{V}} \sum_{\underline{\kappa}} \frac{1}{\sqrt{2\omega}} \sum_{s=1}^{4} \left[\epsilon_\mu(s) b'(\underline{\kappa},s) e^{i\kappa x} + \epsilon_\mu^*(s) b'^\dagger(\underline{\kappa},s) e^{-i\kappa x} \right]. \tag{7.19}$$

First of all, it is clear that $b'(\underline{\kappa},s)$ vanishes unless $\underline{\kappa}=\underline{k}'$. Next, expressions for $\epsilon_\mu(\underline{k}',s)$ are found, using definition (4.7) to be

$$\begin{aligned} \epsilon_\mu(\underline{k}',1) &= \frac{1}{\sqrt{2}}(1,-i,-iv,0) & \qquad \epsilon_\mu(\underline{k}',3) &= (iv,0,1,0) \\ \epsilon_\mu(\underline{k}',2) &= \frac{1}{\sqrt{2}}(1,i,-iv,0) & \qquad \epsilon_\mu(\underline{k}',4) &= (0,0,0,1) \end{aligned} \tag{7.20}$$

and, solving the equations (7.17) for the b'($\underline{k}$',s), one obtains

$$\begin{aligned} b'(1) &= b(1) \\ b'(2) &= 0 \\ b'(3) &= \frac{iv}{\sqrt{2}}\, b(1) \\ b'(4) &= -\frac{v}{\sqrt{2}}\, b(1) \end{aligned} \tag{7.21}$$

where the important facts are that the amplitudes for longitudinal and timelike photons still obey

$$b'(3) + ib'(4) = 0, \qquad n'(3) + n'(4) = 0 \tag{7.22}$$

which is required by the Lorentz condition, and that the amplitudes for transverse photons are invariant.

The treatment of the gravitational field proceeds in the same manner. The condition of symmetry and the gauge conditions must remain true in the transformed co-ordinate system, because they are expressed by tensor, vector, and scalar equations. Using the symmetry condition one can obtain the transformation law for the field components from equation (3.18)

$$\begin{aligned}
\gamma_{11}' &= \gamma_{11} + 2v\gamma_{14} & \gamma_{23}' &= \gamma_{23} \\
\gamma_{12}' &= \gamma_{12} + v\gamma_{24} & \gamma_{24}' &= \gamma_{24} - v\gamma_{12} \\
\gamma_{13}' &= \gamma_{13} + v\gamma_{34} & \gamma_{33}' &= \gamma_{33} \\
\gamma_{14}' &= \gamma_{14} - v(\gamma_{11} - \gamma_{44}) & \gamma_{34}' &= \gamma_{34} - v\gamma_{31} \\
\gamma_{22}' &= \gamma_{22} & \gamma_{44}' &= \gamma_{44} - 2v\gamma_{14}\,.
\end{aligned} \tag{7.23}$$

Then, if one takes the example of a plane gravitational wave with propagation vector $\underline{k} = (0, 0, w)$ and with RH circular polarization (s=1), the field may be written

$$\gamma_{\mu\nu}(x) = \frac{1}{\sqrt{2wV}}\;\frac{1}{2}\begin{pmatrix} 1 & -i & 0 & 0 \\ -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} b(\underline{k},1)\, e^{ikx} + \quad \text{c.c.} \tag{7.24}$$

As in the electromagnetic case, the decomposition

$$\gamma_{\mu\nu}'(x') = \frac{1}{\sqrt{V}}\sum_{\underline{\kappa}}\frac{1}{\sqrt{2\omega}}\sum_{s=1}^{16}\left[\epsilon_{\mu\nu}(s)\, b'(\underline{\kappa},s)\, e^{i\kappa x} + \epsilon^{*}_{\mu\nu}(s)\, b'^{\dagger}(\underline{\kappa},s)\, e^{-i\kappa x}\right] \tag{7.25}$$

yields the result that the $b'(\underline{\kappa},s)$ vanish except when $\underline{\kappa} = \underline{k}'$, and one may obtain expressions for the $\epsilon_{\mu\nu}(\underline{k}',s)$ from those in (7.20) for the $\epsilon_{\mu}(\underline{k}',s)$ and the relations (5.8). Then the component equations of (7.24) may be solved to yield

$$b'(1) = b(1),\; b'(2) = ivb(1),\; b'(6) = -vb(1) \tag{7.26}$$

with all other b'(s) = 0. So, it is still true that

$$b'(2) + ib'(6) = 0, \quad n'(2) + n'(6) = 0, \tag{7.27}$$

and the amplitudes of the purely transverse gravitons have been left invariant by the transformation. It follows that the energy, momentum, and gauge-independent spin are also left invariant.

8. Utilization of the Remaining Gauge Freedom

It was shown in Section 1 that the linearized gravitational field equations (6.1) and the auxiliary conditions (6.2), (6.3) are invariant under gauge transformations of the type

$$\gamma_{\mu\nu} \rightarrow \gamma'_{\mu\nu} = \gamma_{\mu\nu} + \Lambda_{\mu|\nu} + \Lambda_{\nu|\mu} \tag{8.1}$$

where Λ_μ is a vector field satisfying

$$\Box \Lambda_\mu = 0 \tag{8.2a}$$

$$\Lambda_{\sigma|\sigma} = 0 \tag{8.2b}$$

and the conditions that Λ_k are real, Λ_4 imaginary because of the reality conditions imposed on $\gamma_{\mu\nu}$. Note that the "gauge field" Λ_μ satisfies all the requirements of the electromagnetic field A_μ. Thus Λ_μ may be expanded in a Fourier series

$$\Lambda_\mu = \frac{-i}{\sqrt{V}} \sum_{\underline{\varkappa}} \frac{1}{\sqrt{2\omega}} \frac{1}{\omega} \left[c_\mu(\underline{\varkappa}) e^{i\varkappa x} - c^\dagger_\mu(\underline{\varkappa}) e^{-i\varkappa x} \right] \tag{8.3}$$

where the Fourier amplitudes c_μ and $c^\dagger_\mu$ satisfy the reality conditions

$$(c^\dagger_k)^* = c_k \,, \qquad (c^\dagger_4)^* = -c_4 \tag{8.4}$$

and the condition

$$\varkappa_\mu c_\mu(\underline{\varkappa}) = 0 \tag{8.5}$$

due to the restriction (8.2b). Using this expansion and that of the field, (6.16), one arrives at the expression for the transformed field:

$$\gamma'_{\mu\nu} = \gamma_{\mu\nu} + \Lambda_{\mu/\nu} + \Lambda_{\nu/\mu} =$$

$$\frac{1}{\sqrt{V}}\sum_{\underline{\kappa}} \frac{1}{\sqrt{2\omega}}\left\{ \left[\sum_{s'} \epsilon_{\mu\nu}(s') b(\underline{\kappa}, s') + \frac{\kappa_\nu}{\omega} C_\mu(\underline{\kappa}) + \frac{\kappa_\mu}{\omega} C_\nu(\underline{\kappa})\right] e^{i\kappa x} + \left[\sum_{s'} \epsilon^*_{\mu\nu}(s') b^\dagger(\underline{\kappa}, s') + \frac{\kappa_\nu}{\omega} C^\dagger_\mu(\underline{\kappa}) + \frac{\kappa_\mu}{\omega} C^\dagger_\nu(\underline{\kappa})\right] e^{-i\kappa x} \right\}. \tag{8.6}$$

If one multiplies the contents of the first square bracket by $\epsilon^*_{\mu\nu}(s)$ and those of the second by $\epsilon_{\mu\nu}(s)$, and uses the orthonormality property (5.11), one finds that

$$\begin{aligned} b'(s) &= b(s) + \frac{\kappa_\nu}{\omega}\epsilon^*_{\mu\nu}(s) C_\mu + \frac{\kappa_\mu}{\omega}\epsilon^*_{\mu\nu}(s) C_\nu \\ b^{\dagger\prime}(s) &= b^\dagger(s) + \frac{\kappa_\nu}{\omega}\epsilon_{\mu\nu}(s) C^\dagger_\mu + \frac{\kappa_\mu}{\omega}\epsilon_{\mu\nu}(s) C^\dagger_\nu \end{aligned} \tag{8.7}$$

where $b'(s)$, $b^{\dagger\prime}(s)$ are the Fourier amplitudes of the transformed field. For s=1,5 the transversality condition on $\epsilon_{\mu\nu}(s)$ causes the gauge terms to vanish, leaving

$$b'(1) = b(1), \qquad b'(5) = b(5). \tag{8.8}$$

Hence, the energy and momentum must be gauge-invariant

$$\begin{aligned} H' &= \sum_{\underline{\kappa}} \omega[n'(1) + n'(5)] = \sum_{\underline{\kappa}} \omega[n(1) + n(5)] = H \\ P'_k &= \sum_{\underline{\kappa}} \kappa_k [n'(1) + n'(5)] = \sum_{\underline{\kappa}} \kappa_k [n(1) + n(5)] = P_k. \end{aligned} \tag{8.9}$$

for s=2,4, the transversality conditions (5.10) yield

$$b'(2,4) = b(2,4) + \sqrt{2}\,\epsilon^*_\mu(1,2) C_\mu \tag{8.10}$$

where $\epsilon_\mu(1,2)$ are the transverse polarization vectors introduced in Section 4. These may be used to make the decomposition

$$\zeta_\mu(\underline{\kappa}) = \sum_{r=1}^{4} \epsilon_\mu(r)\, C(\underline{\kappa}, r) \tag{8.11}$$

whereby equations (8.10) become

$$b'(2) = b(2) + \sqrt{2}\, C(1), \qquad b'(4) = b(4) + \sqrt{2}\, C(2). \tag{8.12}$$

Using this gauge freedom, one can make b'(2) and b'(4) vanish, by choosing

$$C(1) = -\frac{i}{\sqrt{2}}\, b(2), \qquad C(2) = -\frac{i}{\sqrt{2}}\, b(4) \tag{8.13}$$

so that the spin, from (7.11), becomes

$$S_k = 2 \sum_{\underline{\kappa}} \frac{\kappa_k}{\omega} \left[n(1) - n(5) \right] \tag{8.14}$$

in the new gauge, as was promised in Section 7. Although the spin is not gauge-invariant, the total angular momentum, orbital plus spin, is gauge invariant. This may be shown as follows. The orbital angular momentum (see (6.11) and (6.13)) is given by

$$L_k = i \epsilon_{ijk} \int d\underline{x}\; x_i\, \gamma_{\mu\nu/4}\, \gamma_{\mu\nu/j} \tag{8.15}$$

Application of the gauge transformation (8.1) yields

$$\begin{aligned} L_k' = L_k + i \epsilon_{ijk} \int d\underline{x}\; x_i \Big[& \gamma_{\mu\nu/4} \Lambda_{\mu/\nu j} + \gamma_{\mu\nu/4} \Lambda_{\nu/\mu j} \\ & + \Lambda_{\mu/\nu 4} \gamma_{\mu\nu/j} + \Lambda_{\mu/\nu 4} \Lambda_{\mu/\nu j} + \Lambda_{\mu/\nu 4} \Lambda_{\nu/\mu j} \\ & + \Lambda_{\nu/\mu 4} \gamma_{\mu\nu/j} + \Lambda_{\nu/\mu 4} \Lambda_{\mu/\nu j} + \Lambda_{\nu/\mu 4} \Lambda_{\nu/\mu j} \Big], \end{aligned} \tag{8.16}$$

and using the symmetry of $\gamma_{\mu\nu}$, this becomes

$$\mathcal{L}_k' = \mathcal{L}_k + 2i\,\epsilon_{ijk}\int d\underline{x}\; x_i\Big[\gamma_{\mu\nu/4}\Lambda_{\mu/\nu j} + \gamma_{\mu\nu/j}\Lambda_{\mu/\nu 4} + \Lambda_{\mu/\nu 4}\Lambda_{\mu/\nu j} + \Lambda_{\mu/\nu 4}\Lambda_{\nu/\mu j}\Big]. \tag{8.17}$$

Now, using the conditions (8.2) on Λ_μ as well as the Fierz condition (6.3a) on $\gamma_{\mu\nu}$, one may write this in the form

$$\mathcal{L}_k' = \mathcal{L}_k + 2i\,\epsilon_{ijk}\int d\underline{x}\; x_i\left[\frac{\partial M}{\partial x_j} + x_i\frac{\partial M_{aj}}{\partial x_a}\right] \tag{8.18}$$

where

$$M \equiv \gamma_{\mu\nu/4}\Lambda_{\mu/\nu} + \Lambda_{\mu/4}\Lambda_{\mu/44} + \Lambda_{a/b}\Lambda_{b/a4} + \Lambda_{4/4}\Lambda_{4/44} \tag{8.19}$$

and
$$M_{aj} \equiv \gamma_{\mu a/j}\Lambda_{\mu/4} - \gamma_{\mu 4/j}\Lambda_{\mu/a} + \Lambda_{\mu/aj}\Lambda_{\mu/4} + \Lambda_{a/4j}\Lambda_{4/4} - \Lambda_{b/a}\Lambda_{4/bj} + \Lambda_\mu(\gamma_{\mu 4/aj} - \gamma_{\mu a/4j}) + \Lambda_b(\Lambda_{4/baj} - \Lambda_{a/b4j}).$$

An integration by parts (where it is assumed that all fields vanish at the boundaries) yields

$$\mathcal{L}_k' = \mathcal{L}_k - 2i\,\epsilon_{ijk}\int d\underline{x}\,\left[M\delta_{ij} + M_{aj}\delta_{ia}\right] \tag{8.20}$$

The first term vanishes because it is symmetric in i and j, leaving

$$\mathcal{L}_k' = \mathcal{L}_k - 2i\,\epsilon_{ijk}\int d\underline{x}\, M_{ij} = \mathcal{L}_k - 2i\,\epsilon_{ijk}\int d\underline{x}\left[\gamma_{\mu i/j}\Lambda_{\mu/4} + \gamma_{\mu i/4}\Lambda_{\mu/j}\right] \tag{8.21}$$

where some terms have been dropped because of the antisymmetry and more integrations by parts. The spin angular momentum (see (6.15)), utilizing the symmetry of $\gamma_{\mu\nu}$, may be written

$$S_k = 2i\epsilon_{ijk}\int d\underline{x}\, \gamma_{\mu j}\gamma_{\mu i/4}\,, \tag{8.22}$$

which, after the gauge transformation (8.1) becomes

$$\begin{aligned} S_k' = S_k + 2i\epsilon_{ijk}\int d\underline{x}\, \big[&\gamma_{\mu i/4}\Lambda_{\mu/j} + \gamma_{\mu i/4}\Lambda_{j/\mu} + \gamma_{\mu j}\Lambda_{\mu/i4} \\ &+\Lambda_{\mu/i4}\Lambda_{\mu/j} + \Lambda_{\mu/i4}\Lambda_{j/\mu} + \Lambda_{i/\mu 4}\gamma_{\mu j} \\ &+\Lambda_{i/\mu 4}\Lambda_{\mu/j} + \Lambda_{i/\mu 4}\Lambda_{j/\mu}\big]\,. \end{aligned} \tag{8.23}$$

The fourth term, $\Lambda_{\mu/i4}\Lambda_{\mu/j}$, vanishes after an integration by parts because of the antisymmetry of i and j. The eighth term, $\Lambda_{i/\mu 4}\Lambda_{j/\mu}$, may be written

$$\begin{aligned} &\Lambda_{i/a4}\Lambda_{j/a} + \Lambda_{i/44}\Lambda_{j/4} = \\ &\Lambda_{i/a4}\Lambda_{j/a} - \Lambda_{i/aa}\Lambda_{j/4} = \\ &\Lambda_{i/a4}\Lambda_{j/a} + \Lambda_{i/a}\Lambda_{j/a4} \end{aligned} \tag{8.24}$$

which is symmetric in i and j and vanishes. The second and sixth terms may be written

$$\begin{aligned} &\gamma_{\mu i/4}\Lambda_{j/\mu} + \gamma_{\mu j}\Lambda_{i/\mu 4} = \\ &\gamma_{ai/4}\Lambda_{j/a} + \gamma_{4i/4}\Lambda_{j/4} + \gamma_{aj}\Lambda_{i/a4} + \gamma_{4j}\Lambda_{i/44} = \\ &\gamma_{ai/4}\Lambda_{j/a} - (\gamma_{ai/a}\Lambda_{j/4} + \gamma_{aj/a}\Lambda_{i/4}) - \gamma_{4j}\Lambda_{i/aa} = \\ &\gamma_{ai/4}\Lambda_{j/a} - \gamma_{4j}\Lambda_{i/aa} = \\ &-\gamma_{ai/4a}\Lambda_j - \gamma_{4j/aa}\Lambda_i = \\ &\Lambda_i(\gamma_{aj/a4} - \gamma_{4j/aa}) = \\ &-\Lambda_i(\gamma_{4j/44} - \gamma_{4j/aa}) = -\Lambda_i\gamma_{4j/\mu\mu} = 0\,. \end{aligned} \tag{8.25}$$

A similar calculation reduces the fifth and seventh terms to zero. This leaves the first and third terms, and an integration by parts and permutation of i and j changes the third term so that

$$S_k' = S_k + 2i\epsilon_{ijk}\int d\underline{x}\left[\gamma_{\mu i/4}\Lambda_{\mu/j} + \gamma_{\mu i/j}\Lambda_{\mu/4}\right]. \tag{8.26}$$

Finally, (8.26) may be combined with (8.21) to yield

$$J_k' = L_k' + S_k' = L_k + S_k = J_k . \tag{8.27}$$

At this point it is interesting to note that the requirement of gauge invariance alone is sufficient to restrict the Lagrangian so that the field equations must necessarily be wave equations, provided that the auxiliary conditions are taken into account. This was shown for the case of a symmetric tensor field by Wyss (1965). Consider first the electromagnetic (vector) field. The most general Lagrangian must consist of a bilinear combination of the field variables and their first derivatives. The possible invariants are

$$\begin{aligned}
I_1 &= A_\mu A_\mu \\
I_2 &= A_{\mu/\nu} A_{\mu/\nu} \\
I_3 &= A_{\mu/\mu} A_{\nu/\nu} \\
I_4 &= A_{\mu/\nu} A_{\nu/\mu} = I_3 + \frac{\partial}{\partial x_\nu}\left[A_\mu A_{\nu/\mu} - A_\nu A_{\mu/\mu}\right].
\end{aligned} \tag{8.28}$$

Since I_4 differs from I_3 only by a four-divergence, it may be omitted from separate consideration so that the most general Lagrangian

$$\mathcal{L} = c_1 I_1 + c_2 I_2 + c_3 I_3 . \tag{8.29}$$

This yields, upon variation with respect to A_μ , the Euler-Lagrange equations

$$2c_1 A_\mu - 2c_2 A_{\mu/\nu\nu} - 2c_3 A_{\nu/\nu\mu} = 0 . \tag{8.30}$$

Requiring invariance of (8.30) under the gauge transformation

$$A_\mu \rightarrow A_\mu + B_{/\mu} , \tag{8.31}$$

one obtains the conditions

$$\begin{aligned} &2c_1 B_{/\mu} - 2(c_2+c_3) B_{/\mu\nu\nu} = 0 \qquad \text{or} \\ &c_1 = 0, \quad c_3 = -c_2 . \end{aligned} \tag{8.32}$$

This reduces the Lagrangian to

$$\mathcal{L} = c_2 (A_{\mu/\nu} A_{\mu/\nu} - A_{\mu/\mu} A_{\nu/\nu}) \tag{8.33}$$

and the field equations to

$$A_{\mu/\nu\nu} - A_{\nu/\nu\mu} = 0 . \tag{8.34}$$

When the Lorentz condition is imposed, these reduce to wave equations

$$\Box A_\mu = 0. \tag{8.35}$$

For the general tensor field, one finds fourteen invariants:

$$
\begin{array}{ll}
I_1 = \gamma_{\mu\nu}\gamma_{\mu\nu} & I_6 = \gamma_{\mu\nu/\sigma}\gamma_{\mu\sigma/\nu} \\
I_2 = \gamma_{\mu\nu}\gamma_{\nu\mu} & I_7 = \gamma_{\mu\nu/\sigma}\gamma_{\sigma\mu/\nu} \\
I_3 = \gamma_{\mu\mu}\gamma_{\nu\nu} & I_8 = \gamma_{\mu\nu/\sigma}\gamma_{\sigma\nu/\mu} \\
I_4 = \gamma_{\mu\nu/\sigma}\gamma_{\mu\nu/\sigma} & I_9 = \gamma_{\mu\mu/\sigma}\gamma_{\nu\nu/\sigma} \\
I_5 = \gamma_{\mu\nu/\sigma}\gamma_{\nu\mu/\sigma} & I_{10} = \gamma_{\mu\mu/\sigma}\gamma_{\nu\sigma/\nu}
\end{array}
$$

$$
\begin{aligned}
I_{11} &= \gamma_{\mu\nu/\nu}\gamma_{\mu\sigma/\sigma} = I_6 + \frac{\partial}{\partial x_\sigma}\left(\gamma_{\mu\sigma}\gamma_{\mu\nu/\nu} - \gamma_{\mu\nu}\gamma_{\mu\sigma/\nu}\right) \\
I_{12} &= \gamma_{\mu\nu/\nu}\gamma_{\sigma\mu/\sigma} = I_7 + \frac{\partial}{\partial x_\sigma}\left(\gamma_{\mu\sigma}\gamma_{\nu\mu/\nu} - \gamma_{\mu\nu}\gamma_{\sigma\mu/\nu}\right) \\
I_{13} &= \gamma_{\mu\nu/\mu}\gamma_{\sigma\nu/\sigma} = I_8 + \frac{\partial}{\partial x_\sigma}\left(\gamma_{\sigma\nu}\gamma_{\mu\nu/\mu} - \gamma_{\mu\nu}\gamma_{\sigma\nu/\mu}\right) \\
I_{14} &= \gamma_{\nu\sigma/\sigma}\gamma_{\mu\mu/\nu} = I_{10} + \frac{\partial}{\partial x_\sigma}\left(\gamma_{\nu\sigma}\gamma_{\mu\mu/\nu} - \gamma_{\sigma\nu}\gamma_{\mu\mu/\nu}\right)
\end{aligned}
\tag{8.36}
$$

Since $I_{11}, I_{12}, I_{13}, I_{14}$ differ from I_6, I_7, I_8, I_{10} by a four-divergence, respectively, the most general Lagrangian may be expressed as a linear combination of the first ten

$$\mathcal{L} = \sum_{i=1}^{10} c_i I_i \;. \tag{8.37}$$

Variation of (8.37) with respect to $\gamma_{\mu\nu}$ yields the Euler-Lagrange equations

$$
\begin{aligned}
&2c_1\gamma_{\mu\nu} + 2c_2\gamma_{\nu\mu} + 2c_3\delta_{\mu\nu}\gamma_{\sigma\sigma} - 2c_4\gamma_{\mu\nu/\sigma\sigma} - 2c_5\gamma_{\nu\mu/\sigma\sigma} \\
&-2c_6\gamma_{\mu\sigma/\nu\sigma} - c_7\gamma_{\sigma\mu/\nu\sigma} - c_7\gamma_{\nu\sigma/\mu\sigma} - 2c_8\gamma_{\sigma\nu/\mu\sigma} - 2c_9\delta_{\mu\nu}\gamma_{\rho\rho/\sigma\sigma} \\
&\quad - c_{10}\gamma_{\sigma\sigma/\mu\nu} - c_{10}\delta_{\mu\nu}\gamma_{\rho\sigma/\rho\sigma} = 0 .
\end{aligned}
\tag{8.38}
$$

Requiring invariance of (8.38) under the gauge transformation (8.1) one obtains the conditions

$$c_1 + c_2 = 0, \quad c_3 = 0, \quad 2(c_4 + c_5 + c_6) + c_7 = 0,$$
$$2(c_4 + c_5 + c_8) + c_7 = 0, \quad c_6 + c_7 + c_8 + c_{10} = 0, \tag{8.39}$$
$$2c_9 + c_{10} = 0$$

If $c_1, c_4, c_5,$ and c_6 are taken to be independent, $\mathcal{L}$ may be written

$$\begin{aligned} \mathcal{L} = {} & c_1 \gamma_{\mu\nu}(\gamma_{\mu\nu} - \gamma_{\nu\mu}) + c_4\big[\gamma_{\mu\nu|\sigma}(\gamma_{\mu\nu|\sigma} - 2\gamma_{\sigma\mu|\nu}) + \\ & \gamma_{\mu\mu|\sigma}(2\gamma_{\nu\sigma|\nu} - \gamma_{\nu\nu|\sigma})\big] + c_5\big[\gamma_{\mu\nu|\sigma}(\gamma_{\nu\mu|\sigma} - 2\gamma_{\sigma\mu|\nu}) + \\ & \gamma_{\mu\mu|\sigma}(2\gamma_{\nu\sigma|\nu} - \gamma_{\nu\nu|\sigma})\big] + c_6 \gamma_{\mu\nu|\sigma}(\gamma_{\mu\sigma|\nu} - 2\gamma_{\sigma\mu|\nu} + \gamma_{\sigma\nu|\mu}) \end{aligned} \tag{8.40}$$

and the field equations become

$$\begin{aligned} & c_1(\gamma_{\mu\nu} - \gamma_{\nu\mu}) - c_4\big[\gamma_{\mu\nu|\sigma\sigma} + \gamma_{\sigma\mu|\sigma\nu} + \gamma_{\nu\sigma|\mu\sigma} - \gamma_{\sigma\sigma|\mu\nu} \\ & - \delta_{\mu\nu}(\gamma_{\rho\sigma|\rho\sigma} + \gamma_{\rho\rho|\sigma\sigma})\big] - c_5\big[\gamma_{\nu\mu|\sigma\sigma} + \gamma_{\sigma\mu|\sigma\nu} + \gamma_{\nu\sigma|\mu\sigma} \\ & - \gamma_{\sigma\sigma|\mu\nu} - \delta_{\mu\nu}(\gamma_{\rho\sigma|\rho\sigma} + \gamma_{\rho\rho|\sigma\sigma}) - c_6(\gamma_{\sigma\mu|\nu\sigma} + \gamma_{\mu\sigma|\nu\sigma} \\ & + \gamma_{\nu\sigma|\mu\sigma} + \gamma_{\sigma\nu|\mu\sigma}) = 0. \end{aligned} \tag{8.41}$$

Imposition of the symmetry condition reduces them to

$$\begin{aligned} & -(c_4 + c_5)\big[\gamma_{\mu\nu|\sigma\sigma} + \gamma_{\sigma\mu|\sigma\nu} + \gamma_{\nu\sigma|\mu\sigma} - \gamma_{\sigma\sigma|\mu\nu} \\ & -\delta_{\mu\nu}(\gamma_{\rho\sigma|\rho\sigma} + \gamma_{\rho\rho|\sigma\sigma})\big] - 2c_6(\gamma_{\sigma\mu|\sigma\nu} + \gamma_{\sigma\nu|\sigma\mu}) = 0, \end{aligned} \tag{8.42}$$

and with the Fierz gauge conditions, they become wave equations

$$\Box \gamma_{\mu\nu} = 0. \tag{8.43}$$

9. Transition to Quantum Field Theory

The classical field theory of gravitation, developed up to this point, may be turned into a quantum field theory in the same way that classical electrodynamics is turned into quantum electrodynamics. The fourier amplitudes $b(\underline{\kappa},s)$ and $b^{\dagger}(\underline{\kappa},s)$ become respectively the annihilation and creation operators for a graviton of momentum $\underline{\kappa}$ and polarization s. These operators are required to obey the usual commutation relations for bosons:

$$\begin{aligned}
\left[b(\underline{\kappa},s), b(\underline{\kappa}',s')\right] &= \left[b^{\dagger}(\underline{\kappa},s), b^{\dagger}(\underline{\kappa}',s')\right] = 0 \\
\left[b(\underline{\kappa},s), b^{\dagger}(\underline{\kappa}',s')\right] &= \delta_{\underline{\kappa}\underline{\kappa}'}\delta_{ss'} .
\end{aligned} \tag{9.1}$$

They operate on a vector space of graviton number states, where $|n_{\underline{\kappa},s}\rangle$ denotes the state containing $n_{\underline{\kappa},s}$ gravitons of momentum $\underline{\kappa}$ and polarization s. The general state is a linear combination of direct products of states $|n_{\underline{\kappa},s}\rangle$ over all possible values of $\underline{\kappa}$ and s. The normalization rule for these states is

$$\langle n_{\underline{\kappa},s}|n_{\underline{\kappa}',s'}\rangle = \delta_{\underline{\kappa}\underline{\kappa}'}\delta_{ss'} . \tag{9.2}$$

The usual representation of $b(\underline{\kappa},s)$ and $b^{\dagger}(\underline{\kappa},s)$ will be used, namely where

$$\begin{aligned}
b^{\dagger}|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\
b|n\rangle &= \sqrt{n}\,|n-1\rangle
\end{aligned} \tag{9.3}$$

($\underline{\kappa}$ and s are suppressed except where needed)

from which it follows that $|n\rangle$ is an eigenstate of the number operator $\tilde{n}$

$$\tilde{n}|n\rangle \equiv b^{\dagger}b|n\rangle = n|n\rangle . \tag{9.4}$$

The hermitean conjugate operators b and $b^{\dagger}$ may be explicitly represented by the matrices

$$b = \begin{vmatrix} 0 & \sqrt{1} & & & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \sqrt{4} \\ & & & & \ddots \end{vmatrix} \qquad b^{\dagger} = \begin{vmatrix} 0 & & & & \\ \sqrt{1} & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ & & & \sqrt{4} & 0 \\ & & & & \ddots \end{vmatrix} \tag{9.5}$$

where $|n\rangle$ is a column vector containing a 1 in the $(n+1)^{th}$ position and zeros elsewhere. The expectation value of any b or $b^{\dagger}$ must vanish

$$\langle b\rangle_n \equiv \langle n|b|n\rangle = \langle n|\sqrt{n}|n-1\rangle = \sqrt{n}\langle n|n-1\rangle = 0 \tag{9.6}$$

because of the normalization condition (9.2).

The fields, being linear combinations of the b's, are now operators as well. The commutator of the fields at two space-time points x and x' may now be calculated. Let

$$\begin{aligned} \gamma_{\mu\nu}(x) &= \frac{1}{\sqrt{V}}\sum_{\underline{\kappa},s}\frac{1}{\sqrt{2\omega}}\left[\epsilon_{\mu\nu}(s)b(\underline{\kappa},s)e^{i\kappa x}+\epsilon^{*}_{\mu\nu}(s)b^{\dagger}(\underline{\kappa},s)e^{-i\kappa x}\right] \\ \gamma_{\rho\sigma}(x') &= \frac{1}{\sqrt{V}}\sum_{\underline{\kappa}',s'}\frac{1}{\sqrt{2\omega'}}\left[\epsilon_{\rho\sigma}(s')b(\underline{\kappa}',s')e^{i\kappa' x'}+\epsilon^{*}_{\rho\sigma}(s')b^{\dagger}(\underline{\kappa}',s')e^{-i\kappa' x'}\right]. \end{aligned} \tag{9.7}$$

Then, the linear property of the commutator yields

$$
\begin{aligned}
&[\gamma_{\mu\nu}(x), \gamma_{\rho\sigma}(x')] = \\
&\frac{1}{V}\sum_{\underline{\varkappa},s}\sum_{\underline{\varkappa}',s'} \frac{1}{2\sqrt{\omega\omega'}}\Big\{ \epsilon_{\mu\nu}(s)\,\epsilon_{\rho\sigma}(s')\, e^{i(\varkappa x+\varkappa' x')}\,[b(\underline{\varkappa},s), b(\underline{\varkappa}',s')] + \\
&\qquad \epsilon^{*}_{\mu\nu}(s)\,\epsilon^{*}_{\rho\sigma}(s')\, e^{-i(\varkappa x+\varkappa' x')}\,[b^{\dagger}(\underline{\varkappa},s), b^{\dagger}(\underline{\varkappa}',s')] + \\
&\qquad \epsilon_{\mu\nu}(s)\,\epsilon^{*}_{\rho\sigma}(s')\, e^{i(\varkappa x-\varkappa' x')}\,[b(\underline{\varkappa},s), b^{\dagger}(\underline{\varkappa}',s')] + \\
&\qquad \epsilon^{*}_{\mu\nu}(s)\,\epsilon_{\rho\sigma}(s')\, e^{-i(\varkappa x-\varkappa' x')}\,[b^{\dagger}(\underline{\varkappa},s), b(\underline{\varkappa}',s')]\Big\} = \\
&\frac{1}{V}\sum_{\underline{\varkappa},s}\frac{1}{2\omega}\Big\{\epsilon_{\mu\nu}(s)\,\epsilon^{*}_{\rho\sigma}(s)\, e^{i\varkappa(x-x')} - \epsilon^{*}_{\mu\nu}(s)\,\epsilon_{\rho\sigma}(s')\, e^{-i\varkappa(x-x')}\Big\} = \\
&\qquad \delta_{\mu\rho}\delta_{\nu\sigma}\frac{1}{V}\sum_{\underline{\varkappa}}\frac{1}{2\omega}\left[e^{i\varkappa(x-x')} - e^{-i\varkappa(x-x')}\right] = \\
&\delta_{\mu\rho}\delta_{\nu\sigma}\frac{i}{(2\pi)^3}\int d\underline{\varkappa}\;\frac{\sin[\varkappa(x-x')]}{\omega} = \\
&\qquad -i\delta_{\mu\rho}\delta_{\nu\sigma}D(x-x') \text{ where } D(x) \equiv \frac{-1}{(2\pi)^3}\int\frac{d\underline{\varkappa}}{\omega}\sin\varkappa x\,.
\end{aligned}
\qquad (9.8)
$$

The properties of D(x), a function singular on the light cone and zero everywhere else, are well known from quantum electrodynamics.

The three auxiliary conditions on $\gamma_{\mu\nu}$ must now be reinterpreted in the context of the quantum field theory. The obvious interpretation, namely that of operator conditions, will now be shown to be incompatible with the field commutation relations (9.8).

<u>The symmetry condition</u> $\gamma_{\mu\nu} - \gamma_{\nu\mu} = 0$

Consider the commutator

$$
\begin{aligned}
&[\gamma_{\mu\nu}(x) - \gamma_{\nu\mu}(x),\, \gamma_{\rho\sigma}(x')] = \\
&[\gamma_{\mu\nu}(x), \gamma_{\rho\sigma}(x')] - [\gamma_{\nu\mu}(x), \gamma_{\rho\sigma}(x')] = \\
&-iD(x-x')\{\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\nu\rho}\delta_{\mu\sigma}\}
\end{aligned}
\qquad (9.9)
$$

according to (9.8), which does <u>not</u> vanish everywhere. Thus

$\gamma_{\mu\nu}(x) - \gamma_{\nu\mu}(x)$ cannot be zero.

<u>The first gauge condition</u> $\gamma_{\mu\nu/\nu} = 0$

Consider the commutator

$$\left[\gamma_{\mu\nu/\nu}(x), \gamma_{\rho\sigma}(x')\right] = \frac{\partial}{\partial x_\nu}\left[\gamma_{\mu\nu}(x), \gamma_{\rho\sigma}(x')\right] = -i\delta_{\mu\rho}\frac{\partial}{\partial x_\sigma}D(x-x') \tag{9.10}$$

which does not vanish everywhere. Thus, $\gamma_{\mu\nu/\nu}$ cannot be zero.

<u>The second gauge condition</u> $\gamma_{\nu\nu} = 0$

Consider the commutator,

$$\left[\gamma_{\nu\nu}(x), \gamma_{\rho\sigma}(x')\right] = -i\delta_{\rho\sigma}D(x-x') \tag{9.11}$$

which does not vanish everywhere. Thus, $\gamma_{\nu\nu}$ cannot be zero. The conclusion is that none of the three auxiliary conditions may be interpreted as operator equations; another interpretation will be given in the next section.

The dynamical quantities H, P_k, and S_k become operators in the quantum theory. The calculations in Section 6, which gave these quantities in terms of the Fourier amplitudes, are not valid in the quantum theory since they have been made under the assumption that $b(\underline{\varkappa},s)$ and $b^\dagger(\underline{\varkappa},s)$ commute. Thus, the correct energy operator should be

$$\begin{aligned} H &= \tfrac{1}{2}\sum_{\underline{\varkappa},s}\omega\left[b^\dagger(\underline{\varkappa},s)b(\underline{\varkappa},s) + b(\underline{\varkappa},s)b^\dagger(\underline{\varkappa},s)\right] \\ &= \sum_{\underline{\varkappa},s}\omega\,\tilde{n}(\underline{\varkappa},s) + \sum_{\underline{\varkappa},s}\omega \end{aligned} \tag{9.12}$$

The second term may be thought of as the zero point energy and neglected, as in quantum electrodynamics so the energy operator is then

$$H = \sum_{\underline{\kappa},s} \omega \tilde{n}(\underline{\kappa},s) \tag{9.13}$$

whose expectation value, in a state containing $n_{\underline{\kappa},s}$ gravitons of momentum $\underline{\kappa}$ and polarization s, is

$$\langle H \rangle = \omega n_{\underline{\kappa},s} \tag{9.14}$$

which corresponds to an energy of ω per graviton. Similar considerations apply to the momentum and spin, and the operator expressions for them are just like the classical expressions (6.24) and (6.26), with the amplitudes regarded as operators. Further consideration of these quantities will be given in Section 12.

In some formulations of a quantum theory of gravitation the commutator of the energy density operator with itself does not vanish for space-like separations (Schwinger, 1963). It will now be shown that this is not a problem for the present theory. Consider the commutator $[\mathcal{H}(x), \mathcal{H}(x')]$, where

$$\mathcal{H}(x) = \tfrac{1}{2}\left(\gamma_{\mu\nu/i}\gamma_{\mu\nu/i} - \gamma_{\mu\nu/4}\gamma_{\mu\nu/4}\right). \tag{9.15}$$

This commutator may be written as a sum of four terms,

$[\mathcal{H}(x), \mathcal{H}(x')] = A_1 + A_2 + A_3 + A_4$, where

$$\begin{aligned} A_1 &= \tfrac{1}{4}\left[\gamma_{\mu\nu/i}(x)\gamma_{\mu\nu/i}(x),\ \gamma_{\rho\sigma/j}(x')\gamma_{\rho\sigma/j}(x')\right] \\ A_2 &= -\tfrac{1}{4}\left[\gamma_{\mu\nu/i}(x)\gamma_{\mu\nu/i}(x),\ \gamma_{\rho\sigma/4}(x')\gamma_{\rho\sigma/4}(x')\right] \\ A_3 &= \tfrac{1}{4}\left[\gamma_{\mu\nu/4}(x)\gamma_{\mu\nu/4}(x),\ \gamma_{\rho\sigma/4}(x')\gamma_{\rho\sigma/4}(x')\right]. \end{aligned} \tag{9.16}$$

$$A_4 = -\tfrac{1}{4}\left[\mathcal{Y}_{\mu\nu/4}(x)\mathcal{Y}_{\mu\nu/4}(x), \mathcal{Y}_{\rho\sigma/j}(x')\mathcal{Y}_{\rho\sigma/j}(x')\right].$$

By use of the properties of the commutator, one finds that

$$4A_1 = \mathcal{Y}_{\mu\nu/i}(x)\left[\mathcal{Y}_{\mu\nu/i}(x), \mathcal{Y}_{\rho\sigma/j}(x')\mathcal{Y}_{\rho\sigma/j}(x')\right]$$

$$+ \left[\mathcal{Y}_{\mu\nu/i}(x), \mathcal{Y}_{\rho\sigma/j}(x')\mathcal{Y}_{\rho\sigma/j}(x')\right]\mathcal{Y}_{\mu\nu/i}(x)$$

and $\left[\mathcal{Y}_{\mu\nu/i}(x), \mathcal{Y}_{\rho\sigma/j}(x')\mathcal{Y}_{\rho\sigma/j}(x')\right] =$ (9.17)

$$\left[\mathcal{Y}_{\mu\nu/i}(x), \mathcal{Y}_{\rho\sigma/j}(x')\right]\mathcal{Y}_{\rho\sigma/j}(x') + \mathcal{Y}_{\rho\sigma/j}(x')\left[\mathcal{Y}_{\mu\nu/i}(x), \mathcal{Y}_{\rho\sigma/j}(x')\right].$$

Thus, A_1 is proportional to

$$\left[\mathcal{Y}_{\mu\nu/i}(x), \mathcal{Y}_{\rho\sigma/j}(x')\right] = \frac{\partial^2}{\partial x_i \partial x_j'}\left[\mathcal{Y}_{\mu\nu}(x), \mathcal{Y}_{\rho\sigma}(x')\right] = \tag{9.18}$$

$$\frac{\partial^2}{\partial x_i \partial x_j'}\left\{-i\,\delta_{\mu\rho}\delta_{\nu\sigma}D(x-x')\right\}.$$

Similarly, the other coefficients are also each proportional to some

$$\frac{\partial^2}{\partial x_\mu \partial x_\nu'} D(x-x') \; . \tag{9.19}$$

But, since D(x-x') may be written as $\frac{1}{2\pi}\delta\left[(x-x')^2\right]\varepsilon(t-t')$ and the derivative of $\delta\left[(x-x')^2\right]$ with respect to any x_μ or x_ν' is proportional to $\delta'\left[(x-x')^2\right]$, expression (9.19) is proportional to either $\delta\left[(x-x')^2\right]$ or its derivative, both of which vanish everywhere except on the light cone. Thus, each of the $A's$ and hence the commutator $\left[\mathcal{H}(x), \mathcal{H}(x')\right]$ vanish except on the light cone.

10. Accommodation of the Auxiliary Conditions by Introduction of an Indefinite Metric

There are two problems in the quantum field theory developed in the last section. The first, as has been shown, is that the auxiliary conditions on the field may not be interpreted as operator conditions. The second is that since b and $b^\dagger$ are hermitean conjugates, matrix elements for any component of the field make up a hermitean matrix. However, for those components whose classical counterparts are pure imaginary, the matrix elements should make up a skew-hermitean matrix. The solution to both these difficulties may be guessed from the analogy with electrodynamics. The analogous problem there is solved by use of an indefinite metric (see Gupta, 1950) and this method works equally well in the case of gravitation.

The metric operator η is defined to satisfy

$$\begin{aligned} [\eta, b(s)] &= 0, \quad s=1\text{-}5, 9\text{-}13 \\ [\eta, b(s)]_+ &= \eta b(s) + b(s)\eta = 0, \quad s=6\text{-}8, 14\text{-}16 \end{aligned} \tag{10.1}$$

and also the requirement

$$\eta^\dagger = \eta, \quad \eta^2 = \eta^\dagger \eta = I, \tag{10.2}$$

i.e. that η be hermitean and unitary. Then (10.1) remains true if b is replaced by $b^\dagger$. It follows from this definition and from the definition of $\epsilon_{\mu\nu}(s)$ that

$$[\eta, \gamma_{\mu\nu}(x)] = 0 \qquad \text{when } (\mu\nu)=(ab),(44)$$
$$[\eta, \gamma_{\mu\nu}(x)]_+ = 0 \qquad \text{when } (\mu\nu)=(a4),(4a). \tag{10.3}$$

From now on, products in the state vector space will be redefined as

$$\langle a|b\rangle \to \langle a|\eta|b\rangle \tag{10.4}$$

and the expectation values will become

$$\langle A\rangle_a = \langle a|\eta A|a\rangle . \tag{10.5}$$

Consider now a matrix element of $\gamma_{\mu\nu}$. For $(\mu\nu)$ = (ab),(44)

$$\langle b|\eta\gamma_{\mu\nu}|a\rangle^* = \langle a|\gamma_{\mu\nu}^\dagger\eta^\dagger|b\rangle = \langle a|\gamma_{\mu\nu}\eta|b\rangle = \langle a|\eta\gamma_{\mu\nu}|b\rangle \tag{10.6}$$

so the matrix elements are hermitean, corresponding to a real $\gamma_{\mu\nu}$ in the classical theory. But for $(\mu\nu)$ = (a4),(4a)

$$\langle b|\eta\gamma_{\mu\nu}|a\rangle^* = \langle a|\gamma_{\mu\nu}\eta|b\rangle = -\langle a|\eta\gamma_{\mu\nu}|b\rangle \tag{10.7}$$

so these matrix elements are skew-hermitean, corresponding to a pure imaginary $\gamma_{\mu\nu}$ in the classical theory. An explicit representation of η is easy to construct. For a given $\underline{k}$, η is a direct product of identity matrices for s=1-5,9-13, and of matrices of the form diag (1,-1,1,-1,...) for s=6-8,14-16, the time-like gravitons. The norm of a state containing n of some particular polarization of time-like gravitons is then

$$\langle n|\eta|n'\rangle = \delta_{nn'}(-1)^n . \tag{10.8}$$

States of negative norm cannot be understood in any way consistent with the usual probability interpretation of quantum mechanics; therefore, since such states have been introduced because of the metric operator, they will have to be eliminated. It is the auxiliary conditions which will do the job.

Since the auxiliary conditions are incompatible with the commutation relations if they are interpreted as operator conditions, a weaker interpretation will be made. The auxiliary conditions need hold only for the expectation values of the operators, according to

$$\begin{aligned} \langle a|\eta(\gamma_{\mu\nu}-\gamma_{\nu\mu})|a\rangle &= 0 \\ \langle a|\eta\,\gamma_{\mu\nu|\nu}|a\rangle &= 0 \\ \langle a|\eta\,\gamma_{\nu\nu}|a\rangle &= 0\,. \end{aligned} \tag{10.9}$$

These are restrictions not on the operators themselves, but on the admissible states, $|a\rangle$. It will be shown that all states of negative norm are eliminated by conditions (10.9). It is sufficient to satisfy (10.9) that the conditions on the states be replaced by

$$\begin{aligned} (\gamma_{\mu\nu}-\gamma_{\nu\mu})^{(+)}|a\rangle &= 0 \\ \gamma^{(+)}_{\mu\nu|\nu}|a\rangle &= 0 \\ \gamma^{(+)}_{\nu\nu}|a\rangle &= 0 \end{aligned} \tag{10.10}$$

where the (+) refers to the positive frequency part (annihilation operators only) of each Fourier decomposition. These conditions amount to the following restrictions on the states involving the annihilation operators (see Section 7)

$$
\begin{aligned}
&\left[b(2) + i b(6)\right]|a\rangle = 0 \\
&\left[b(4) + i b(8)\right]|a\rangle = 0 \\
&\left[b(7) - i\sqrt{\tfrac{3}{2}}\, b(9)\right]|a\rangle = 0 \\
&\left[b(3) - \tfrac{1}{\sqrt{2}}\, b(9)\right]|a\rangle = 0 \\
&b(s)|a\rangle = 0, \quad s = 10, \cdots 16.
\end{aligned}
\tag{10.11}
$$

Similar restrictions hold for the "bra" states, for example

$$\langle a|\left[b^\dagger(2) - i b^\dagger(6)\right] = \langle a|\left[b^\dagger(2) - i b^\dagger(6)\right]\eta = \langle a|\eta\left[b^\dagger(2) + i b^\dagger(6)\right] = 0. \tag{10.12}$$

11. Determination of Admissible States in the Fierz Gauge

It was pointed out in the last section that an admissible state $|a\rangle$ is one satisfying the conditions (10.11). It is sufficient to deal with gravitons of a fixed momentum $\underline{k}$ because the general state will simply be a direct product

$$|a\rangle = \prod_{\underline{k}} |a_{\underline{k}}\rangle . \tag{11.1}$$

The state $|a_{\underline{k}}\rangle$ may be split into a direct product

$$\begin{aligned} |a_{\underline{k}}\rangle = |a_{1,5}\rangle \otimes |a_{2,6}\rangle \otimes |a_{4,8}\rangle \otimes |a_{3,7,9}\rangle \otimes |a_{10}\rangle \\ \otimes |a_{11}\rangle \otimes \cdots \otimes |a_{16}\rangle . \end{aligned} \tag{11.2}$$

There are no restrictions on the transverse-transverse states (s=1,5) so that $|a_{1,5}\rangle$ may be written as the linear combination

$$|a_{1,5}\rangle = \sum_{n(1),n(5)} d_{n(1),n(5)} |n(1),n(5)\rangle . \tag{11.3}$$

For the trace and skew polarizations (s=10,11,...16) the condition

$$b(s)|a_s\rangle = 0 \tag{11.4}$$

implies that $|a_s\rangle$ can only be the vacuum state $|0_s\rangle$ for each of these polarizations.

The states $|a_{2,6}\rangle$ containing transverse-longitudinal and transverse-time-like gravitons may be found in the same way as states containing longitudinal and time-like photons are found in electrodynamics. The state $|a_{2,6}\rangle$ may be written as a linear combination

$$|a_{2,6}\rangle = \sum_{n(2),n(6)} C_{n(2),n(6)} |n(2),n(6)\rangle \tag{11.5}$$

where the states $|n(2),n(6)\rangle$ have the normalization

$$\langle n(2),n(6)|\eta|n'(2),n'(6)\rangle = (-1)^{n(6)} \delta_{n(2)n'(2)} \delta_{n(6)n'(6)} \, . \tag{11.6}$$

Then the condition

$$[b(2)+ib(6)]|a_{2,6}\rangle = 0 \tag{11.7}$$

gives a set of relations between the coefficients $C_{n(2),n(6)}$

$$\sqrt{n(2)+1}\, C_{n(2)+1,n(6)} + i\sqrt{n(6)+1}\, C_{n(2),n(6)+1} = 0 . \tag{11.8}$$

This means that there is exactly <u>one</u> admissible state for each total number of gravitons with the s=2,6 polarizations, and these states may be explicitly listed

$$\begin{aligned} |a^{(0)}_{2,6}\rangle &= |0,0\rangle \\ |a^{(1)}_{2,6}\rangle &= |1,0\rangle + i\,|0,1\rangle \end{aligned} \tag{11.9}$$

and in general,

$$\begin{aligned} |a^{(n)}_{2,6}\rangle &= |n,0\rangle + i\sqrt{\binom{n}{1}}\,|n-1,1\rangle + \cdots + i^r\sqrt{\binom{n}{r}}\,|n-r,r\rangle \\ &\quad + \cdots + i^n|0,n\rangle \\ &= \frac{1}{\sqrt{n!}}[b^\dagger(2)+ib^\dagger(6)]^n|0,0\rangle . \end{aligned}$$

The inner product of two such admissible states $\langle a^{(n)}_{2,6}|\eta|a^{(n')}_{2,6}\rangle$ vanishes for $n \neq n'$, and when $n = n'$,

$$\langle a_{2,6}^{(n)}|\eta|a_{2,6}^{(n)}\rangle = \frac{1}{n!}\langle 0,0|\eta\{[b(2)+ib(6)][b^\dagger(2)+ib^\dagger(6)]\}^n|0,0\rangle$$
$$= \frac{1}{n!}\langle 0,0|\eta\{b(2)b^\dagger(2)-b(6)b^\dagger(6)\}^n|0,0\rangle = 0 \tag{11.10}$$

except when n=0, where it equals one. Thus the condition (11.7) eliminates all states with negative norm. Then $|a_{2,6}\rangle$ may finally be written

$$|a_{2,6}\rangle = |0,0\rangle + \sum_{n>0} C_{2,6}^{(n)}|a_{2,6}^{(n)}\rangle \tag{11.11}$$

and of these states, only the vacuum has a nonvanishing norm.

Exactly the same procedure may be followed for the states $|a_{4,8}\rangle$, with the result that

$$|a_{4,8}\rangle = |0,0\rangle + \sum_{n>0} C_{4,8}^{(n)}|a_{4,8}^{(n)}\rangle \tag{11.12}$$

where $|a_{4,8}^{(n)}\rangle$ has zero norm for $n > 0$.

Finally, the states containing symmetric spin zero gravitons $|a_{3,7,9}\rangle$ may be written as linear combinations

$$|a_{3,7,9}\rangle = \sum_{n(3)\,n(7)\,n(9)} C_{n(3),n(7),n(9)}\,|n(3),n(7),n(9)\rangle \tag{11.13}$$

where the states $|n(3),n(7),n(9)\rangle$ have the normalization

$$\langle n(3),n(7),n(9)|\eta|n'(3),n'(7),n'(9)\rangle =$$
$$(-1)^{n(7)}\delta_{n(3)n'(3)}\,\delta_{n(7)\,n'(7)}\,\delta_{n(9)\,n'(9)}\,. \tag{11.14}$$

The conditions

$$\left[b(7) - i\sqrt{\tfrac{3}{2}}\, b(9)\right] |a_{3,7,9}\rangle = 0$$
$$\left[b(3) - \tfrac{1}{\sqrt{2}}\, b(9)\right] |a_{3,7,9}\rangle = 0 \tag{11.15}$$

give two sets of relations among the coefficients $C_{n(3),n(7),n(9)}$

$$\sqrt{n(7)+1}\, C_{n(3),n(7)+1,n(9)} - i\sqrt{\tfrac{3}{2}}\sqrt{n(9)+1}\, C_{n(3),n(7),n(9)+1} = 0 \tag{11.16}$$
$$\sqrt{n(3)+1}\, C_{n(3)+1,n(7),n(9)} - \tfrac{1}{\sqrt{2}}\sqrt{n(9)+1}\, C_{n(3),n(7),n(9)+1} = 0 .$$

From these, one may infer that there can be exactly one admissible state for each total number of gravitons with the s=3,7,9 polarizations:

$$|a^{(0)}_{3,7,9}\rangle = |0,0,0\rangle$$
$$|a^{(1)}_{3,7,9}\rangle = |1,0,0\rangle + \sqrt{3}\, i\,|0,1,0\rangle + \sqrt{2}\,|0,0,1\rangle \tag{11.17}$$

and, in general,

$$|a^{(n)}_{3,7,9}\rangle = \frac{1}{\sqrt{n!}}\left[b^\dagger(3) + \sqrt{3}\, i\, b^\dagger(7) + \sqrt{2}\, b^\dagger(9)\right]^n |0,0,0\rangle .$$

The inner product of two such states $\langle a^{(n)}_{3,7,9}|\eta|a^{(n')}_{3,7,9}\rangle$ vanishes for $n \neq n'$, and when $n = n'$,

$$\begin{aligned}\langle a^{(n)}_{3,7,9}|\eta|a^{(n)}_{3,7,9}\rangle &= \frac{1}{n!}\langle 0,0,0|\,\eta \left\{\left[b(3) + \sqrt{3}\, i\, b(7) + \sqrt{2}\, b(9)\right]\right.\\ &\qquad \left.\left[b^\dagger(3) + \sqrt{3}\, i\, b^\dagger(7) + \sqrt{2}\, b^\dagger(9)\right]\right\}^n |0,0,0\rangle\\ &= \frac{1}{n!}\langle 0,0,0|\,\eta\left\{b(3)b^\dagger(3) - 3b(7)b^\dagger(7) + 2b(9)b^\dagger(9)\right\}^n |0,0,0\rangle\end{aligned} \tag{11.18}$$

= 0 except when n=0, when it equals one.

Again, all states of negative norm have been eliminated, and $|a_{3,7,9}\rangle$ may be written

$$|a_{3,7,9}\rangle = |0,0,0\rangle + \sum_{n>0} C^{(n)}_{3,7,9}\, |a^{(n)}_{3,7,9}\rangle \tag{11.19}$$

where $|a_{3,7,9}^{(n)}\rangle$ has zero norm for $n > 0$.

Finally, (11.2) may be written as

$$|a_{\underline{x}}\rangle = \sum_{n(1)n(5)} d_{n(1),n(5)} |n(1),n(5)\rangle \otimes \sum_{n\geq 0} C_{2,6}^{(n)} |a_{2,6}^{(n)}\rangle \otimes \sum_{n\geq 0} C_{4,8}^{(n)} |a_{4,8}^{(n)}\rangle \otimes \sum_{n\geq 0} C_{3,7,9}^{(n)} |a_{3,7,9}^{(n)}\rangle \otimes |0_{10}\rangle \otimes \cdots |0_{16}\rangle \tag{11.20}$$

and it is clear that the only contributions to a nonvanishing norm come from the purely transverse states, s=1,5.

12. Further Exploration of the Remaining Gauge Freedom

In order to determine the meaning of the gauge freedom (see Section 8) in the quantum theory of gravitation, one must first consider the expectation values of the operators $b(s)$ in the admissible states described in the preceding section. Consider the operators $b(2)$ and $b(6)$. Since they operate on the states $|a_{2,6}^{(n)}\rangle$ according to

$$b(2)\,|a_{2,6}^{(n)}\rangle = \sqrt{n}\,|a_{2,6}^{(n-1)}\rangle$$
$$b(6)\,|a_{2,6}^{(n)}\rangle = i\sqrt{n}\,|a_{2,6}^{(n-1)}\rangle\,, \tag{12.1}$$

the expectation value

$$\begin{aligned}\langle b(2)\rangle &= \langle a_{2,6}|\eta\, b(2)|a_{2,6}\rangle \\ &= \langle a_{2,6}^{(0)}|\eta\, b(2)\sum_{n>0} C_{2,6}^{(n)}|a_{2,6}^{(n)}\rangle \\ &\qquad + \sum_{n'>0}\sum_{n>0} C_{2,6}^{(n')*} C_{2,6}^{(n)}\langle a_{2,6}^{(n')}|\eta\, b(2)|a_{2,6}^{(n)}\rangle = C_{2,6}^{(1)}\,.\end{aligned} \tag{12.2}$$

Similarly, the other expectation values are found to be

$$\begin{aligned}&\langle b(6)\rangle = iC_{2,6}^{(1)} &&\langle b(3)\rangle = C_{3,7,9}^{(1)} \\ &\langle b(4)\rangle = C_{4,8}^{(1)} &&\langle b(7)\rangle = \sqrt{3}\,i\,C_{3,7,9}^{(1)} \\ &\langle b(8)\rangle = iC_{4,8}^{(1)} &&\langle b(9)\rangle = \sqrt{2}\,C_{3,7,9}^{(1)} \\ &\langle b(s)\rangle = 0\,, \quad s = 10, \cdots 16\,.\end{aligned} \tag{12.3}$$

Then the expectation value of the field may be calculated:

$$\langle \gamma_{\mu\nu}(x)\rangle = \langle a|\eta\,\gamma_{\mu\nu}(x)|a\rangle =$$

$$= \frac{1}{\sqrt{V}} \sum_{\varkappa} \frac{1}{\sqrt{2\omega}} \sum_{s=1}^{16} \left[\epsilon_{\mu\nu}(s) \langle b(s) \rangle e^{i\varkappa x} + \epsilon^{*}_{\mu\nu}(s) \langle b^{\dagger}(s) \rangle e^{-i\varkappa x} \right]$$

$$= \frac{1}{\sqrt{V}} \sum_{\varkappa} \frac{1}{\sqrt{2\omega}} \Big\{ \Big(\epsilon_{\mu\nu}(1) \langle b(1) \rangle + \epsilon_{\mu\nu}(5) \langle b(5) \rangle + C^{(1)}_{2,6} \left[\epsilon_{\mu\nu}(2) + i\, \epsilon_{\mu\nu}(6) \right]$$
$$+ C^{(1)}_{4,8} \left[\epsilon_{\mu\nu}(4) + i\, \epsilon_{\mu\nu}(8) \right] + C^{(1)}_{3,7,9} \left[\epsilon_{\mu\nu}(3) + \sqrt{3}\, i\, \epsilon_{\mu\nu}(7) + \sqrt{2}\, \epsilon_{\mu\nu}(9) \right] \Big) e^{i\varkappa x} \tag{12.4}$$
$$+ \langle h.c. \rangle \Big\} .$$

Letting

$$\langle \Upsilon_{\mu\nu} \rangle_{Tr.} = \frac{1}{\sqrt{V}} \sum_{\varkappa} \frac{1}{\sqrt{2\omega}} \Big\{ \left[\epsilon_{\mu\nu}(1) \langle b(1) \rangle + \epsilon_{\mu\nu}(5) \langle b(5) \rangle \right] e^{i\varkappa x} + \langle h.c. \rangle \Big\} \tag{12.5}$$

be the purely transverse part of the field, and simplifying the linear combinations of polarization tensors, one obtains

$$\langle \Upsilon_{\mu\nu} \rangle = \langle \Upsilon_{\mu\nu} \rangle_{Tr.} + \frac{1}{\sqrt{V}} \sum_{\varkappa} \frac{1}{\sqrt{2\omega}} \Big\{ \Big(\frac{\varkappa_\nu}{\omega} \Big[\frac{\omega}{\sqrt{2}} C^{(1)}_{2,6}\, \epsilon_\mu(1) + \frac{\omega}{\sqrt{2}} C^{(1)}_{4,8}\, \epsilon_\mu(2)$$
$$+ \frac{\omega}{\sqrt{2}} \frac{\sqrt{3}}{2} C^{(1)}_{3,7,9}\, \varkappa_\mu \Big] + \frac{\varkappa_\mu}{\omega} \Big[\frac{\omega}{\sqrt{2}} C^{(1)}_{2,6}\, \epsilon_\nu(1) + \frac{\omega}{\sqrt{2}} C^{(1)}_{4,8}\, \epsilon_\nu(2) \tag{12.6}$$
$$+ \frac{\omega}{\sqrt{2}} \frac{\sqrt{3}}{2} C^{(1)}_{3,7,9}\, \varkappa_\nu \Big] \Big) e^{i\varkappa x} + \langle h.c. \rangle \Big\} .$$

A comparison of (12.6) with the classical expression (8.6) shows that

$$\zeta_\mu(\varkappa) = \frac{\omega}{\sqrt{2}} \Big[C^{(1)}_{2,6}\, \epsilon_\mu(1) + C^{(1)}_{4,8}\, \epsilon_\mu(2) + \frac{\sqrt{3}}{2} C^{(1)}_{3,7,9}\, \varkappa_\mu \Big] \tag{12.7}$$

which satisfies the requirement $\varkappa_\mu \zeta_\mu = 0$. Thus, the quantum mechanical representation of the gauge freedom is found in the possible admixtures of virtual graviton states. The higher coefficients $C^{(n)}_{2,6}$ $(n>1)$ etc. would be involved in the expectation values of products of the fields. In addition to the above there exists the possibility of <u>operator</u> gauge functions (the so-called Landau gauge transformation), which have not been investigated in this work.

Consider now the energy and momentum operators.

$$\underset{\sim}{P}_{\mu} = \sum_{\underset{\sim}{\kappa}} \sum_{s=1}^{16} \underset{\sim}{\kappa}_{\mu}\, n(\underset{\sim}{\kappa},s) \tag{12.8}$$

where $P_4 \equiv iH$. In order to obtain the expectation value $\langle \underset{\sim}{P}_{\mu} \rangle = \langle a|\eta \underset{\sim}{P}_{\mu}|a\rangle$, one must first compute the expectation values of the various operators $n(s)$,

$$\langle n(s)\rangle = \langle a|\eta\, b^{\dagger}(s) b(s)|a\rangle \,. \tag{12.9}$$

For s=10,...16 it follows from equation (11.4) that $\langle n(s)\rangle$ =0. Consider the case s=2:

$$\langle a_{2,6}^{(n')}|\eta\, b^{\dagger}(2) b(2)|a_{2,6}^{(n)}\rangle = \sqrt{nn'}\,\langle a_{2,6}^{(n'-1)}|\eta|a_{2,6}^{(n-1)}\rangle \tag{12.10}$$

which vanishes unless n=n'=1. Thus,

$$\langle n(2)\rangle = |C_{2,6}^{(1)}|^2 . \tag{12.11}$$

Similar calculations yield

$$\begin{aligned} \langle n(6)\rangle &= -|C_{2,6}^{(1)}|^2 & \langle n(3)\rangle &= |C_{3,7,9}^{(1)}|^2 \\ \langle n(4)\rangle &= |C_{4,8}^{(1)}|^2 & \langle n(7)\rangle &= -3|C_{3,7,9}^{(1)}|^2 \\ \langle n(8)\rangle &= -|C_{4,8}^{(1)}|^2 & \langle n(9)\rangle &= 2|C_{3,7,9}^{(1)}|^2 \end{aligned} \tag{12.12}$$

from which one immediately obtains the relations

$$\begin{aligned} \langle n(2)\rangle + \langle n(6)\rangle &= 0 \\ \langle n(4)\rangle + \langle n(8)\rangle &= 0 \\ \langle n(3)\rangle + \langle n(7)\rangle + \langle n(9)\rangle &= 0 \,. \end{aligned} \tag{12.13}$$

Then the expectation value of the energy and momentum

$$\langle P_\mu \rangle = \sum_{\varkappa} \sum_{s=1}^{16} \varkappa_\mu \langle n(\varkappa, s) \rangle = \sum_{\varkappa} \varkappa_\mu \left[\langle n(1) \rangle + \langle n(5) \rangle \right] . \tag{12.14}$$

Thus, only the purely transverse gravitons contribute to the expectation values of the energy and momentum.

Similarly, for the expectation value of the spin, one obtains (see (7.11))

$$\begin{aligned} \langle S_k \rangle = \sum_{\varkappa} \Big\{ \frac{\varkappa_k}{\omega} 2 \left[\langle n(1) \rangle - \langle n(5) \rangle \right] + R_k \left[\langle b^\dagger(1) b(2) \rangle - \right. \\ \left. \langle b^\dagger(4) b(5) \rangle \right] + R_k^* \left[\langle b^\dagger(2) b(1) \rangle - \langle b^\dagger(5) b(4) \rangle \right] \Big\} \\ = \sum_{\varkappa} \Big\{ \frac{\varkappa_k}{\omega} 2 \left[\langle n(1) \rangle - \langle n(5) \rangle \right] + R_k \left[C^{(1)}_{2,6} \langle b^\dagger(1) \rangle - C^{(1)*}_{4,8} \langle b(5) \rangle \right] \\ + R_k^* \left[C^{(1)*}_{2,6} \langle b(1) \rangle - C^{(1)}_{4,8} \langle b^\dagger(5) \rangle \right] \Big\} . \end{aligned} \tag{12.15}$$

The gauge-independent part of this expression is proportional to the difference between the numbers of R.H. and L.H. circularly polarized transverse-transverse gravitons. As in the classical theory, the other terms are assumed to be of no physical significance since they depend upon the choice of the gauge functions.

13. The Gravitational Analog of Calkin's Transformation

In electromagnetic theory, a transformation among the electric and magnetic fields of the form

$$
\begin{aligned}
\underline{E}' &= \underline{E}\cos\theta + \underline{H}\sin\theta \\
\underline{H}' &= -\underline{E}\sin\theta + \underline{H}\cos\theta
\end{aligned}
\tag{13.1}
$$

leaves Maxwell's equations invariant. This transformation is called a duality rotation, see Wheeler (1962). The resulting conserved quantity for an infinitesimal transformation, parametrized by $\delta\theta$, is minus twice the spin component in the direction of propagation,

$$-2\delta\theta\,[n(R)-n(L)] \tag{13.2}$$

for a transverse field. This was pointed out by Calkin (1965).

Conversely, one may consider the spin operator S_3 for a quantized gravitational field, and ask what sort of transformation is generated by this operator. Consider the spin operator for some fixed propagation direction $\underline{k}$ which defines the direction of the x_3 co-ordinate,

$$S_3 = 2\,[b^\dagger(1)b(1) - b^\dagger(5)b(5)] \tag{13.3}$$

and the infinitesimal unitary transformation

$$U = I + i\delta\theta S_3 = I + 2i\delta\theta\,[b^\dagger(1)b(1) - b^\dagger(5)b(5)] \tag{13.4}$$

generated by S_3 which transforms the field according to

$$\gamma'_{\mu\nu} = U\gamma_{\mu\nu}U^{-1} = \gamma_{\mu\nu} + i\delta\theta\,[S_3, \gamma_{\mu\nu}]. \tag{13.5}$$

Using the expansion of the field operator (9.7), one may work out this commutator

$$\begin{aligned}
[S_3, \gamma_{\mu\nu}] = \frac{2}{\sqrt{V}} \sum_{\underline{\varkappa}} \frac{1}{\sqrt{2\omega}} \sum_{s} \Big\{ \epsilon_{\mu\nu}(s) \Big([b^\dagger(\underline{k},1)b(\underline{k},1), b(\underline{\varkappa},s)] \\
- [b^\dagger(\underline{k},5)b(\underline{k},5), b(\underline{\varkappa},s)]\Big) e^{i\varkappa x} \\
+ \epsilon^*_{\mu\nu}(s)\Big([b^\dagger(\underline{k},1)b(\underline{k},1), b(\underline{\varkappa},s)] \\
- [b^\dagger(\underline{k},5)b(\underline{k},5), b^\dagger(\underline{\varkappa},s)]\Big) e^{-i\varkappa x} \Big\}
\end{aligned} \tag{13.6}$$

Now,

$$\begin{aligned}
[b^\dagger(\underline{k},1)b(\underline{k},1), b(\underline{\varkappa},s)] = b^\dagger(\underline{k},1)[b(\underline{k},1), b(\underline{\varkappa},s)] \\
+ [b^\dagger(\underline{k},1), b(\underline{\varkappa},s)]b(\underline{k},1) = -\delta_{\underline{\varkappa}\underline{k}}\delta_{s1}\, b(\underline{k},1)
\end{aligned} \tag{13.7}$$

and $[b^\dagger(\underline{k},1)b(\underline{k},1), b^\dagger(\underline{\varkappa},s)] = \delta_{\underline{\varkappa}\underline{k}}\delta_{s1}\, b^\dagger(\underline{k},1)$

so (13.5) becomes

$$\begin{aligned}
b'(\underline{k},1) &= b(\underline{k},1)(1-2i\delta\theta) \\
b'(\underline{k},5) &= b(\underline{k},5)(1+2i\delta\theta)
\end{aligned} \tag{13.8}$$

with all other $b'(\underline{\varkappa},s) = b(\underline{\varkappa},s)$.

In order to see how this transformation is analogous to Calkin's transformation (13.1), one must define analogs to the electric and magnetic field "vectors", namely the "tensors"

$$\begin{aligned}
E_{ab} &\equiv -i\left(\gamma_{3b/4} - \gamma_{34/b}\right) \\
H_{ab} &\equiv \epsilon_{bcd}\,\gamma_{ad/c}.
\end{aligned} \tag{13.9}$$

The expectation values of these operators, in dyadic form, satisfy close analogs of Maxwell's vacuum equations:

$$\underline{\nabla}\cdot\underline{\underline{E}} = 0 \qquad \underline{\nabla}\cdot\underline{\underline{H}} = 0$$
$$\langle \underline{\nabla}\times\underline{\underline{E}} + \dot{\underline{\underline{H}}}\rangle = 0 \qquad \langle \underline{\nabla}\times\underline{\underline{H}} - \dot{\underline{\underline{E}}}\rangle = 0 \tag{13.10}$$

where $(\underline{\nabla}\cdot\underline{\underline{E}})_a \equiv E_{ab/b}$, $(\underline{\nabla}\times\underline{\underline{E}})_{ab} \equiv \epsilon_{bcd}E_{ad/c}$, etc.

Using expansion (9.7) again, one finds that these operators have the form

$$E_{ab} = \frac{1}{\sqrt{V}}\sum_{\kappa,s}\frac{1}{\sqrt{2\omega}}\left\{[i\omega\epsilon_{ab}(s) - \kappa_b\epsilon_{a4}(s)]b(s)e^{i\kappa x} + \text{h.c.}\right\}$$
$$H_{ab} = \frac{1}{\sqrt{V}}\sum_{\kappa,s}\frac{1}{\sqrt{2\omega}}\left\{i\kappa_c\epsilon_{bcd}\epsilon_{ad}(s)b(s)e^{i\kappa x} + \text{h.c.}\right\} \tag{13.11}$$

and restricting them to the case of a plane wave propagating in the $\underline{k}$ direction (which means that it is expectation values that are being considered), one obtains

$$E_{ab} = \frac{1}{\sqrt{2wV}}\left\{iw\epsilon_{ab}(1)b(1) + iw\epsilon_{ab}(5)b(5)\right\}e^{ikx} + \text{h.c.}$$
$$H_{ab} = \frac{1}{\sqrt{2wV}}\left\{i\kappa_c\epsilon_{bcd}\epsilon_{ad}(1)b(1) + i\kappa_c\epsilon_{bcd}\epsilon_{ad}(5)b(5)\right\}e^{ikx} + \text{h.c.} \tag{13.12}$$

and, using relations (5.8) and (4.15), one can simplify H_{ab} to

$$H_{ab} = \frac{1}{\sqrt{2wV}}\left\{-w\epsilon_{ab}(1)b(1) + w\epsilon_{ab}(5)b(5)\right\}e^{ikx} + \text{h.c.} \tag{13.13}$$

Now, it is easily seen that the transformation (13.8) is equivalent to the transformation

$$E'_{ab} = E_{ab} - 2\delta\theta H_{ab}$$
$$H'_{ab} = H_{ab} + 2\delta\theta E_{ab}. \tag{13.14}$$

In other words, the infinitesimal gravitational analog of Calkin's transformation

$$E' = E + H\delta\theta \qquad H' = H - E\delta\theta \tag{13.15}$$

is generated by the operator $-\frac{1}{2}S_3$.

14. Proof That a Lagrangian for the Iterated Field Equations Proposed by Gupta Does Not Exist

A linear field theory cannot properly describe gravitation because it fails to take into account the gravitating effect of gravitation itself, i.e. the fact that the gravitational field contains energy, which, because of its equivalence to mass, must itself be a source of gravitation. This is a non-linear effect.

Gupta (1954) has outlined an iteration procedure which, beginning with the type of linear theory described in this paper, supposedly enables one to take into account the gravitating effects of gravitation to any desired order, and which becomes identical to Einstein's theory in the limit of infinitely many iterations. This procedure is supposed to work as follows.

First, it is helpful when external sources of gravitation are present to make a change of variables from the $\gamma_{\mu\nu}$ which have been used up till now. If one replaces the $\gamma_{\mu\nu}$ by the new variables $U_{\mu\nu}$, defined by

$$\begin{aligned} \gamma_{\mu\nu} &= 2k\left(U_{\mu\nu} - \tfrac{1}{2} U_{\sigma\sigma}\delta_{\mu\nu}\right) \\ U_{\mu\nu} &= \tfrac{1}{2k}\left(\gamma_{\mu\nu} - \tfrac{1}{2}\gamma_{\sigma\sigma}\delta_{\mu\nu}\right) \end{aligned} \tag{14.1}$$

one finds that the linear homogeneous field equations and auxiliary conditions remain true for the $U_{\mu\nu}$

$$\begin{aligned} \Box U_{\mu\nu} &= 0 \\ U_{\mu\nu} - U_{\nu\mu} &= 0 \\ U_{\mu\nu/\nu} &= 0 \\ U_{\sigma\sigma} &= 0 \end{aligned} \tag{14.2}$$

so that everything stated so far about the $\gamma_{\mu\nu}$ is also true for the $U_{\mu\nu}$. The advantage of using the $U_{\mu\nu}$ is that with an external source, represented by an energy-momentum tensor $T_{\mu\nu}$, the linear approximation of Einstein's theory gives as inhomogeneous field equations

$$\Box U_{\mu\nu} = k T_{\mu\nu} . \tag{14.3}$$

The coupling constant k is related to Newton's gravitational constant G by

$$k = \sqrt{8\pi G} . \tag{14.4}$$

Then, in the absence of external sources, the true (nonlinear) equation describing a gravitational field is

$$\Box U_{\mu\nu} = k t_{\mu\nu} \tag{14.5}$$

where $t_{\mu\nu}$ is the energy-momentum tensor of the gravitational field. As a first approximation to obtaining a Lagrangian for the equation (14.5) one takes

$$\mathcal{L}^{(1)} = -\tfrac{1}{2} U_{\mu\nu/\sigma} U_{\mu\nu/\sigma} \tag{14.6}$$

given previously in (6.4). From this is calculated the energy-momentum tensor

$$\begin{aligned} t^{(1)}_{\mu\nu} &= \mathcal{L}^{(1)} \delta_{\mu\nu} - \frac{\partial \mathcal{L}^{(1)}}{\partial U_{\alpha\beta/\mu}} U_{\alpha\beta/\nu} \\ &= U_{\alpha\beta/\mu} U_{\alpha\beta/\nu} - \tfrac{1}{2} \delta_{\mu\nu} U_{\alpha\beta/\sigma} U_{\alpha\beta/\sigma} . \end{aligned} \tag{14.7}$$

Gupta obtains the second approximation by shifting (14.7) to the left hand side of the field equations

$$\Box U_{\mu\nu} - k\left(U_{\alpha\beta/\mu}U_{\alpha\beta/\nu} - \tfrac{1}{2}\delta_{\mu\nu}U_{\alpha\beta/\sigma}U_{\alpha\beta/\sigma}\right) = 0 . \tag{14.8}$$

Then, (14.8) must be the Euler-Lagrange equations of a Lagrangian $\mathcal{L}'$ which Gupta writes as

$$\mathcal{L}' = -\tfrac{1}{2} U_{\mu\nu/\sigma}U_{\mu\nu/\sigma} + f_3 \tag{14.9}$$

where f_3 consists of one or more terms such that each term is a product of three factors, each factor being either U or its derivative, and each term containing a total of two derivatives.

This method of iteration fails, because, as it will now be shown, <u>no such f_3 exists</u>. There are fifty-seven possible terms which could contribute to f_3, which are listed in Appendix B, using the notation

$$(\mu\nu) \equiv U_{\mu\nu}\ , \qquad (\mu\nu/\alpha) \equiv U_{\mu\nu/\alpha}. \tag{14.10}$$

This list exhausts all possibilities, since a term containing a second derivative would differ from a linear combination of the L_i $(i=1,2,\ldots 57)$ by a four-divergence, for example

$$(\mu\nu/\alpha\beta)(\mu\alpha)(\nu\beta) = -L_7 - L_{33} + \frac{\partial}{\partial x_\beta}\left[(\mu\nu/\alpha)(\mu\alpha)(\nu\beta)\right]. \tag{14.11}$$

Then, f_3 may be written as

$$f_3 = \sum_{k=1}^{57} C_k L_k \tag{14.12}$$

and the coefficients C_k are determined by the fact that the variation of f_3 with respect to $U_{\mu\nu}$ must yield, from (14.8)

$$\frac{\delta f_3}{\delta U_{\mu\nu}} = -k\left[(\alpha\beta/\mu)(\alpha\beta/\nu) - \tfrac{1}{2}\delta_{\mu\nu}(\alpha\beta/\sigma)(\alpha\beta/\sigma)\right]. \qquad (14.13)$$

The conditions among these coefficients (see Appendix B) include

$$\begin{aligned} &C_1 + C_2 - C_{29} - C_{30} - C_{35} - C_{36} = -k \qquad && (\alpha\beta/\mu)(\alpha\beta/\nu) \\ &-2(C_1 + C_2) = 0 && (\alpha\beta)(\mu\nu/\alpha\beta) \\ &-(C_{29} + C_{30} + C_{35} + C_{36}) = 0 && (\alpha\beta)(\alpha\beta/\mu\nu) \end{aligned} \qquad (14.14)$$

where the coefficients of the terms on the right have been selected. These equations are clearly incompatible. Since (14.12) was the most general Lagrangian, it can only be concluded that f_3 and hence L' does not exist.

This proof leaves open the possibility that equations (14.8) could be obtained from a Lagrangian which was other than cubic in the field variables. However, even if such a Lagrangian could be found, to be used in this procedure it would have to be expanded as a power series in k, and only terms containing k to the zeroth or first power would be retained for the iteration. But then, the Lagrangian would have to be of the form (14.9) which has been shown not to exist.

15. Attempt at Incorporating the Gravitating Effects of Gravitation by an Iteration Procedure

Although the method of iteration proposed by Gupta, outlined in the last section, fails, it is possible to develop an iteration procedure which will work along very similar lines. The first order equations are given by (14.2) and will be written as

$$\begin{aligned} &\Box U^{(1)}_{\mu\nu} = 0, \quad \text{with} \\ &A^{(1)}_{\mu\nu} \equiv U^{(1)}_{\mu\nu} - U^{(1)}_{\nu\mu} = 0 \\ &B^{(1)}_{\mu} \equiv U^{(1)}_{\mu\nu/\nu} = 0 \\ &C^{(1)} \equiv U^{(1)}_{\nu\nu} = 0. \end{aligned} \tag{15.1}$$

They are obtained from the Lagrangian $\mathcal{L}^{(1)}$ given by (14.6) and from which may be constructed the energy-momentum tensor $t^{(1)}_{\mu\nu}$ given by (14.7).

The second order equations are written as

$$\Box U^{(2)}_{\mu\nu} - k t^{(1)}_{\mu\nu} = 0 \tag{15.2}$$

but this equation will now be understood to have a different meaning. Instead of understanding that the variables $U^{(2)}_{\mu\nu}$ appear in the expression for $t^{(1)}_{\mu\nu}$ as well as in $\Box U^{(2)}_{\mu\nu}$ as Gupta did, one can interpret the $t^{(1)}_{\mu\nu}$ as a function of the co-ordinates x_μ, in terms of the known functions $U^{(1)}_{\mu\nu}$. Consider what happens to the auxiliary conditions for a solution of (15.2):

$$\begin{aligned} &A^{(2)}_{\mu\nu} \equiv U^{(2)}_{\mu\nu} - U^{(2)}_{\nu\mu} \\ &\Box A^{(2)}_{\mu\nu} = k\left(t^{(1)}_{\mu\nu} - t^{(1)}_{\nu\mu}\right) = 0 \end{aligned} \tag{15.3}$$

Although $\Box A^{(2)}_{\mu\nu}$ vanishes, there is nothing to insure that $A^{(2)}_{\mu\nu}$ itself vanishes.

Similarly,

$$\Box B^{(2)}_{\mu} \equiv \Box U^{(2)}_{\mu\nu/\nu} = k t^{(1)}_{\mu\nu/\nu} = 0 \tag{15.4}$$

but this does not insure that $B^{(2)}_{\mu}$ itself be zero. Finally

$$\Box C^{(2)} \equiv \Box U^{(2)}_{\nu\nu} = k t^{(1)}_{\nu\nu} = -k\, U^{(1)}_{\alpha\beta/\nu} U^{(1)}_{\alpha\beta/\nu} \neq 0 \tag{15.5}$$

so $C^{(2)}$ cannot vanish in general.

There is no problem in obtaining a "Lagrangian" for equations (15.2); it must of course be an explicit (as well as implicit) function of the co-ordinates

$$\mathcal{L}^{(2)} = -\tfrac{1}{2} U^{(2)}_{\alpha\beta/\sigma} U^{(2)}_{\alpha\beta/\sigma} - k\, U^{(2)}_{\alpha\beta} t^{(1)}_{\alpha\beta} . \tag{15.6}$$

Notice that $\mathcal{L}^{(2)}$ is not a true Lagrangian, since its dependence on the co-ordinates through $t^{(1)}_{\alpha\beta}$ contains a dependence on a <u>particular</u> solution of (15.1). However, $\mathcal{L}^{(2)}$ and the other $\mathcal{L}^{(n)}$ will be called "Lagrangians" because they are used to generate the energy-momentum tensor for each order, just as an ordinary Lagrangian may be so used.

From $\mathcal{L}^{(2)}$ can be constructed the energy-momentum tensor

$$\begin{aligned} t^{(2)}_{\mu\nu} &= -\delta_{\mu\nu}\left(\tfrac{1}{2} U^{(2)}_{\alpha\beta/\sigma} U^{(2)}_{\alpha\beta/\sigma} + k\, U^{(2)}_{\alpha\beta} t^{(1)}_{\alpha\beta}\right) + U^{(2)}_{\alpha\beta/\mu} U^{(2)}_{\alpha\beta/\nu} \\ &= t^{(1)\to(2)}_{\mu\nu} - k\,\delta_{\mu\nu} U^{(2)}_{\alpha\beta} t^{(1)}_{\alpha\beta} \end{aligned} \tag{15.7}$$

where

$$t^{(1)\to(n)}_{\mu\nu} \equiv U^{(n)}_{\alpha\beta/\mu} U^{(n)}_{\alpha\beta/\nu} - \tfrac{1}{2}\delta_{\mu\nu} U^{(n)}_{\alpha\beta/\sigma} U^{(n)}_{\alpha\beta/\sigma} . \tag{15.8}$$

Then, the iteration may be repeated to yield the third order equations

$$\Box U^{(3)}_{\mu\nu} - k t^{(2)}_{\mu\nu} = 0, \quad \text{or}$$
$$\Box U^{(3)}_{\mu\nu} - k t^{(1)\to(2)}_{\mu\nu} + k^2 \delta_{\mu\nu} U^{(2)}_{\alpha\beta} t^{(1)}_{\alpha\beta} \tag{15.9}$$

where the $U^{(3)}_{\mu\nu}$ appears only in the first term, the other terms being explicit functions of the co-ordinates. Again the auxiliary conditions do not hold. All that can be said is that

$$\Box A^{(3)}_{\mu\nu} = 0$$
$$\Box B^{(3)}_{\mu} = -k^2 U^{(2)}_{\alpha\beta} t^{(1)}_{\alpha\beta/\mu} \neq 0 \tag{15.10}$$
$$\Box C^{(3)} = -k U^{(2)}_{\alpha\beta/\sigma} U^{(2)}_{\alpha\beta/\sigma} - 4k^2 U^{(2)}_{\alpha\beta} t^{(1)}_{\alpha\beta} \neq 0 .$$

The "Lagrangian" and energy-momentum tensor for this order are respectively

$$\mathcal{L}^{(3)} = -\tfrac{1}{2} U^{(3)}_{\alpha\beta/\sigma} U^{(3)}_{\alpha\beta/\sigma} - k U^{(3)}_{\alpha\beta} t^{(2)}_{\alpha\beta}$$
$$t^{(3)}_{\mu\nu} = t^{(1)\to(3)}_{\mu\nu} - k U^{(3)}_{\alpha\beta} t^{(2)}_{\alpha\beta} \delta_{\mu\nu} . \tag{15.11}$$

The pattern for successive iterations is clear, and the n^{th} order equations

$$\Box U^{(n)}_{\mu\nu} - k t^{(n-1)}_{\mu\nu} = 0 \tag{15.12}$$

may be derived from the n^{th} order "Lagrangian"

$$\mathcal{L}^{(n)} = -\tfrac{1}{2} U^{(n)}_{\alpha\beta/\sigma} U^{(n)}_{\alpha\beta/\sigma} - k U^{(n)}_{\alpha\beta} t^{(n-1)}_{\alpha\beta} , \tag{15.13}$$

whose energy-momentum tensor is

$$t^{(n)}_{\mu\nu} = t^{(1)\to(n)}_{\mu\nu} - k \delta_{\mu\nu} U^{(n)}_{\alpha\beta} t^{(n-1)}_{\alpha\beta} . \tag{15.14}$$

Substituting the definition of $t_{\mu\nu}^{(n-1)}$ into (15.13), one obtains the functional equation

$$\mathcal{L}^{(n)} = -\tfrac{1}{2} U^{(n)}_{\alpha\beta/\sigma} U^{(n)}_{\alpha\beta/\sigma} - k U^{(n)}_{\alpha\beta}\left[\mathcal{L}^{(n-1)}\delta_{\alpha\beta} - \frac{\partial \mathcal{L}^{(n-1)}}{\partial U^{(n-1)}_{\rho\sigma/\beta}} U^{(n-1)}_{\rho\sigma/\alpha}\right] \qquad (15.15)$$

where it is to be understood that the expression in the square brackets is an explicit function of the co-ordinates. Although there is no information about the convergence of this procedure, one may assume that the field equations obtained by varying $\mathcal{L}^{(n)}$ do not differ significantly from those obtained by varying $\mathcal{L}^{(n-1)}$ when $n \gg 1$. Then, if this assumption is valid, one may drop the distinction between $\mathcal{L}^{(n)}$ and $\mathcal{L}^{(n-1)}$ in equation (15.15) and just write

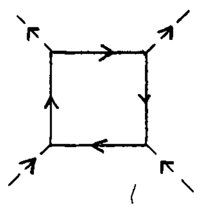
$$\mathcal{L} + \tfrac{1}{2} U_{\alpha\beta/\sigma} U_{\alpha\beta/\sigma} + k U_{\alpha\alpha}[L] - k U_{\alpha\beta}\left[\frac{\partial \mathcal{L}}{\partial U_{\rho\sigma/\beta}} U_{\rho\sigma/\alpha}\right] = 0 \qquad (15.16)$$

where $\mathcal{L}$ is the Lagrangian and $[L]$ is the expression for $\mathcal{L}$ evaluated as an explicit function of the co-ordinates. Then one obtains a functional equation for L, the Lagrangian for a fully nonlinear theory. It is not known how to solve this equation. Thus the solutions of equation (15.16), if they exist, would yield field equations for a source-free, gravitating, gravitational field which in linear approximation have solutions compatible with all known experimental results. Because the auxiliary conditions would not be true in such a nonlinear theory, it must be concluded that polarization modes other than the purely transverse ones would be possible, <u>including the skew modes</u>. This should be a very interesting field for investigation. It would also be very useful to study whether solutions of these equations are also solutions of Einstein's field equations.

16. The Scattering of Photons by Photons due to the Gravitational Interaction

The considerations of Section 2 show how an interaction Hamiltonian for the electromagnetic-gravitational interaction may be constructed, using the principle of the compensating field. Once this construction has been made, such quantities as the cross-section for photon-photon scattering due to the exchange of a graviton can be calculated by the usual methods of quantum electrodynamics. The content of this section follows closely a development by Kaempffer (1967).

The motivation for investigating such a small quantity as the photon-photon cross-section can be seen from the following dimensional considerations: In quantum electrodynamics the lowest order contribution to photon-photon scattering comes from the fourth order term whose Feynman graph is (Karplus and Neuman, 1951)



(16.1)

where the dotted lines represent photons and the solid lines represent electrons or positrons, and the cross-section is on the order of

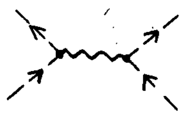
$$\left(\frac{e^2}{\hbar c}\right)^4\left(\frac{\hbar}{m_e c}\right)^2\left(\frac{\hbar\omega}{m_e c^2}\right)^6 \sim 10^{-152}\,\omega^6 \;\;(cm)^2 \qquad (16.2)$$

for low frequencies, where $\hbar\omega < m_e c^2$.

For extremely high frequencies, where $\hbar\omega \gg m_e c^2$, the cross-section cannot depend on m_e and must depend, for the n^{th} order term, upon the inverse square of the frequency as

$$\sigma_n \sim \left(\frac{e^2}{\hbar c}\right)^n \frac{c^2}{\omega^2} \, . \tag{16.3}$$

On the other hand, the contribution to this cross-section from the graviton exchange interaction, which can be represented by the Feynman diagram



$\sim\!\sim \equiv$ graviton (16.4)

in the second order, must depend on frequency as

$$\sigma \sim \left(\frac{G\hbar\omega^2}{c^5}\right)^2 \frac{c^2}{\omega^2} \sim 10^{-152}\,\omega^2 \quad (cm)^2 \tag{16.5}$$

since there is no mass factor which can be included. Why the gravitational constant appears to the second power will be shown in the calculation. Then, clearly, when the frequency reaches a certain critical value, the gravitational term will become dominant. This frequency is about 10^{41} sec^{-1}, which corresponds to an energy equivalent to the annihilation of 10^{-7} grams of matter. Thus, for extremely high energies, there is reason to believe that the gravitational interaction plays an important role in the scattering of photons.

Now, the interaction Hamiltonian is constructed in the following manner. One starts with the Hamiltonian for the free electromagnetic field,

$$H_0 = \tfrac{1}{2}\int d\underline{x}\, A_{a/b} A_{a/b} \tag{16.6}$$

in which the gauge is chosen so that $A_4 = 0$, and transverse photons only are being considered. To take into account the gravitational interaction, one simply replaces the ordinary derivative $A_{a/b}$ by the special derivative $\partial_b A_a$, introduced in (2.11). The form of this derivative, in the linear approximation, is given in (2.40) as

$$\begin{aligned}\partial_b A_a &\equiv A_{a/b} + B_{b,ac} A_c \\ &= A_{a/b} + \left[\eta_{ac/b} - \tfrac{1}{2}\left(\gamma_{ac/b} + \gamma_{ba/c} - \gamma_{bc/a}\right)\right] A_c .\end{aligned} \tag{16.7}$$

Now, in this approximation, η_{ac} may be replaced by $\tfrac{1}{2}\gamma_{ac}$ since only symmetric, in fact only purely transverse gravitons will be considered. Then, one may write

$$\begin{aligned}\partial_b A_a &= A_{a/b} + \left(\gamma_{ba/c} - \gamma_{bc/a}\right) A_c \\ &= A_{a/b} + \tfrac{1}{2} E_{b[a/c]} A_c\end{aligned} \tag{16.8}$$

where

$$E_{b[a/c]} \equiv 2\left(\gamma_{ba/c} - \gamma_{bc/a}\right). \tag{16.9}$$

Now one may expand A_a as a sum of photon creation and annihilation operators

$$\begin{aligned}A_a = \frac{1}{\sqrt{V}} \sum_{k} \sum_{\sigma=1}^{2} \frac{1}{\sqrt{2\Omega}} \Big\{ &\epsilon_a(\underline{k},\sigma)\, a(\underline{k},\sigma)\, e^{ikx} \\ &+ \epsilon_a^*(\underline{k},\sigma)\, a^\dagger(\underline{k},\sigma)\, e^{-ikx} \Big\}\end{aligned} \tag{16.10}$$

where the $\epsilon_a(\underline{k},\sigma)$ are the polarization vectors introduced in Section 4, and $a(\underline{k},\sigma)$ and $a^\dagger(\underline{k},\sigma)$ respectively annihilate and create a photon of momentum $\underline{K}$, polarization σ. Also, one has

$$\gamma_{ab} = \frac{k}{\sqrt{V}} \sum_{\kappa} \sum_{s=1}^{2} \frac{1}{\sqrt{2\omega}} \left[\epsilon_{ab}(\kappa,s) b(\kappa,s) e^{i\kappa x} + \text{h.c.} \right] \tag{16.11}$$

where the constant k has finally been introduced for dimensional reasons, as was promised in Section 6, and the polarization s=5 is now being written s=2. Using the various properties of the $\epsilon_{ab}(\kappa,s)$ and the $\epsilon_a(\kappa,s)$, one then finds that (16.9) may be written as

$$E_{b[a/c]} = -\frac{k}{\sqrt{V}} \sum_{\kappa,s=1}^{2} (-1)^s \sqrt{\frac{\omega}{2}}\, \epsilon_{acm} \left[\epsilon_b(\kappa,s) \epsilon_m(\kappa,s) b(\kappa,s) e^{i\kappa x} + \text{h.c.} \right]. \tag{16.12}$$

So, the interaction part of the Hamiltonian may be written as

$$\mathcal{H}' = \tfrac{1}{4} \left\{ A_{a/b} E_{b[a/c]} A_c + E_{b[a/c]} A_c A_{a/b} \right\}. \tag{16.13}$$

The A_c and the $A_{a/b}$ may be commuted since the operators which do not commute will be eliminated by the relation

$$\epsilon_{abc} \epsilon_a(k,\sigma) \epsilon_b^*(k,\sigma) = 0 \tag{16.14}$$

and $\mathcal{H}'$ is seen to be already in normal order, and may be written as

$$\mathcal{H}'(x) = \tfrac{1}{2} N\left[A_{a/b}(x) E_{b[a/c]}(x) A_c(x) \right] \tag{16.15}$$

where the operator N places all creation operators to the left of all annihilation operators.

Now, the full machinery of quantun electrodynamics may be used to evaluate, for example, the cross-section for photon-photon scattering. The scattering operator

$$S = T\left[\exp -i \int \mathcal{H}'(x)\, dx \right] \tag{16.16}$$

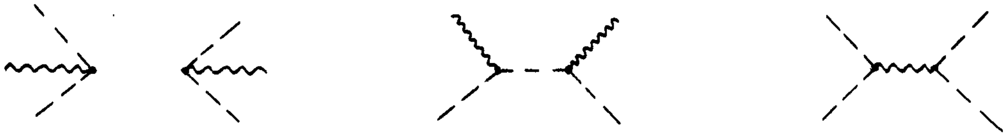
is expanded as

$$S = I - i\int T[\mathcal{H}'(x)]dx - \tfrac{1}{2}\iint T[\mathcal{H}'(x_1)\mathcal{H}'(x_2)]dx_1dx_2 + \cdots \tag{16.17}$$

where T is the time-ordering operator. The lowest order term which can contribute to a scattering process is the second order term, which may be written

$$S_2 = -\tfrac{1}{8}\iint T\Big\{N\big[A_{a_1|b_1}(x_1)E_{b_1[a_1/c_1]}(x_1)A_{c_1}(x_1)\big]\cdot N\big[A_{a_2/b_2}(x_2)E_{b_2[a_2/c_2]}(x_2)A_{c_2}(x_2)\big]\Big\}dx_1dx_2. \tag{16.18}$$

Wick's theorem allows one to write (16.18) as the sum of six terms whose graphs are



$S_2^{(I)}$ $S_2^{(II)}$ $S_2^{(III)}$

$S_2^{(IV)}$ $S_2^{(V)}$ $S_2^{(VI)}$

The photon-photon scattering term is $S_2^{(III)}$, which can be written

$$\overbrace{E_{b_1[a_1/c_1]}(x_1)E_{b_2[a_2/c_2]}}(x_2) = \langle 0|T[E_{b_1[a_1/c_1]}(x_1)E_{b_2[a_2/c_2]}(x_2)]|0\rangle. \tag{16.20}$$

The following identities allow one to simplify this propagator:

$$\sum_{s=1}^{2}\epsilon_{b_1}(\kappa,s)\epsilon^*_{b_2}(\kappa,s) = \delta_{b_1b_2} - \frac{\kappa_{b_1}\kappa_{b_2}}{\omega^2}$$

$$\epsilon_{a_1b_1c_1}\epsilon_{a_2b_2c_2}\left(\delta_{c_1c_2} - \frac{\kappa_{c_1}\kappa_{c_2}}{\omega^2}\right) = \frac{1}{\omega^2}\Big\{\delta_{a_1a_2}\kappa_{b_1}\kappa_{b_2} + \delta_{b_1b_2}\kappa_{a_1}\kappa_{a_2} \tag{16.21}$$

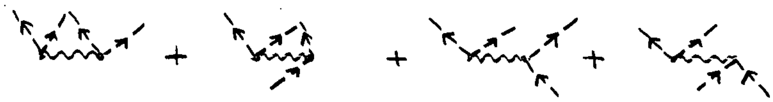
$$-\delta_{a_1 b_2}\varkappa_{b_1}\varkappa_{a_2} - \delta_{b_1 a_2}\varkappa_{a_1}\varkappa_{b_2}\Big\}$$

so that (16.20) becomes

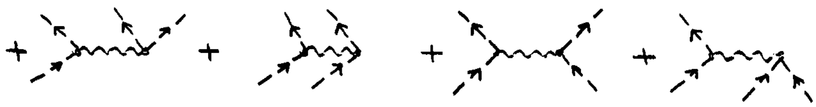
$$\frac{k^2}{i(2\pi)^4}\int_F \frac{1}{\omega^2}\left(\delta_{b_1 b_2} - \frac{\varkappa_{b_1}\varkappa_{b_2}}{\omega^2}\right)\Big(\delta_{a_1 a_2}\varkappa_{c_1}\varkappa_{c_2} + \delta_{c_1 c_2}\varkappa_{a_1}\varkappa_{a_2} - \delta_{a_1 c_2}\varkappa_{c_1}\varkappa_{a_2} - \delta_{c_1 a_2}\varkappa_{a_1}\varkappa_{c_2}\Big)e^{i\varkappa(x_1-x_2)}d\varkappa \qquad (16.22)$$

where F is the Feynman contour on the $\varkappa_4$ -plane; which includes one of the poles $\varkappa_4 = \omega$, $\varkappa_4 = -\omega$, depending on the time-order of x_1 and x_2. Then $S_2^{(III)}$ may be split into sixteen fundamental processes, whose graphs are:

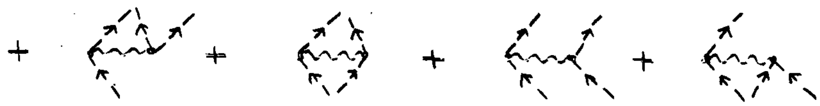
$S_2^{(III)}$ =



(1) (2) (3) (4)

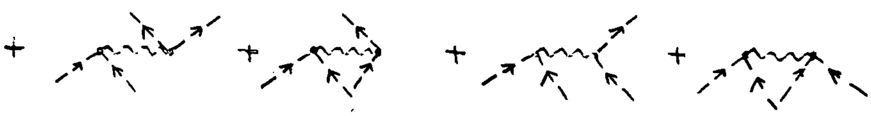


(5) (6) (7) (8)



(16.23)

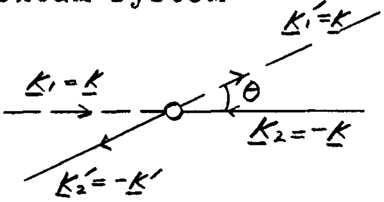
(9) (10) (11) (12)



(13) (14) (15) (16)

where the convention used is that photon lines from A_c are drawn from SW to NE, those from $A_{3/6}$ are drawn from SE to NW, and creation (annihilation) operators are represented by a line leaving (entering) a vertex. The processes which can contribute to photon-photon scattering are those numbered (4), (6), (7), (10), (11), and (13).

The matrix elements may best be evaluated in the centre-of-momentum system



$$\Omega_1 = \Omega_2 = \Omega_1' = \Omega_2' = \Omega$$

$$\underline{k}\cdot\underline{k}' = \Omega^2 \cos\theta \tag{16.24}$$

where they may be written as

$$\langle \underline{k}',\sigma_1'; -\underline{k}',\sigma_2' | S_2^{(III)} | \underline{k},\sigma_1; -\underline{k},\sigma_2 \rangle . \tag{16.25}$$

It can be seen that the 4th and 13th diagrams do not contribute, and the remaining four are identical. Thus, the complete matrix element is

$$\frac{(2\pi)^4 k^2}{i\Omega^2 V^2}\left(\frac{\cos\theta+1}{\cos\theta-1}\right) M\, \delta\left(K_1'+K_2'-K_1-K_2\right) \tag{16.26}$$

where

$$\begin{aligned} M = {} & (\epsilon_1^{*\prime}\cdot\epsilon_2^{*\prime})(\underline{k}\cdot\epsilon_1)(\underline{k}'\cdot\epsilon_2) + (\epsilon_1\cdot\epsilon_2)(\underline{k}\cdot\epsilon_1^{*\prime})(\underline{k}\cdot\epsilon_2^{*\prime}) \\ & + (\epsilon_1^{*\prime}\cdot\epsilon_2)(\underline{k}'\cdot\epsilon_1)(\underline{k}\cdot\epsilon_2^{*\prime}) + (\epsilon_1\cdot\epsilon_2^{*\prime})(\underline{k}\cdot\epsilon_1^{*\prime})(\underline{k}'\cdot\epsilon_2) \end{aligned} \tag{16.27}$$

and

$$\epsilon_1 \equiv \epsilon(\underline{k},\sigma_1) \qquad \epsilon_1' \equiv \epsilon(\underline{k}',\sigma_1')$$

$$\epsilon_2 \equiv \epsilon(-\underline{k},\sigma_2) \qquad \epsilon_2' \equiv \epsilon(-\underline{k}',\sigma_2')$$

The usual phase-space considerations (see Jauch and Rohrlich, 1955) enable one to compute the differential cross-section

$$\sigma ds = \frac{G^2}{16\Omega^2}\, ds \left(\frac{\cos\theta + 1}{\cos\theta - 1}\right)^2 \overline{|M|}^2 \tag{16.28}$$

where M is found by averaging over initial spins and summing over final spins,

$$\overline{|M|}^2 \equiv \frac{1}{4}\sum_{\sigma_1}\sum_{\sigma_2}\sum_{\sigma_1'}\sum_{\sigma_2'} |M|^2 = \Omega^4 \sin^4\theta\,[2(1-\cos\theta) + \cos^2\theta]\,. \tag{16.29}$$

Then the differential cross-section is

$$\sigma ds = \frac{G^2\Omega^2}{16} ds\,(\cos\theta + 1)^4 [2(1-\cos\theta) + \cos^2\theta] \tag{16.30}$$

and the total cross-section is given by

$$\sigma_{tot} \equiv \int_0^{2\pi}\int_0^{\pi} \sigma ds = \frac{20}{21}\pi G^2 \Omega^2 \tag{16.31}$$

or, in cgs units,

$$\sigma_{tot} = \frac{20}{21}\pi \left(\frac{G\hbar\Omega^2}{c^5}\right)^2 \frac{c^2}{\Omega^2} \quad (cm)^2\,. \tag{16.32}$$

Other quantities may be calculated in a similar fashion, and should prove to be a worthwhile subject of investigation.

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Appendix A. Normalized Eigenvectors of Spin Projection Operators for the Tensor Field

The eigenvectors of the operator $\underline{\sigma}\cdot\underline{\kappa}$ introduced in (5.5), labelled according to spin quantum numbers (j,m), are:

$$\zeta(2,\pm 2) = \frac{1}{4\omega^2(\omega^2-\kappa_3^2)} \begin{bmatrix} \kappa_+^2(\omega\pm\kappa_3)^2 \\ \mp 2\kappa_+(\omega\pm\kappa_3)(\omega^2-\kappa_3^2) \\ \sqrt{6}(\omega^2-\kappa_3^2)^2 \\ \pm 2\kappa_-(\omega\mp\kappa_3)(\omega^2-\kappa_3^2) \\ \kappa_-^2(\omega\mp\kappa_3)^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\eta(2,\pm 1) = \frac{1}{2\omega^2(\omega^2-\varkappa_3^2)^{1/2}} \begin{bmatrix} \pm\varkappa_\mp^2(\omega\pm\varkappa_3) \\ \pm\varkappa_+(\omega\pm\varkappa_3)(2\varkappa_3\mp\omega) \\ -\sqrt{6}\,\varkappa_3(\omega^2-\varkappa_3^2) \\ \mp\varkappa_-(\omega\mp\varkappa_3)(2\varkappa_3\pm\omega) \\ \mp\varkappa_-^2(\omega\mp\varkappa_3) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\eta(2,0) = \frac{1}{2\omega^2} \begin{bmatrix} \sqrt{3}\,\chi_+^2 \\ 2\sqrt{3}\,\chi_+\chi_3 \\ \sqrt{2}\,(3\chi_3^2 - \omega^2) \\ 2\sqrt{3}\,\chi_-\chi_3 \\ \sqrt{3}\,\chi_-^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \eta(1,\pm 1) = \frac{1}{2\omega(\omega^2 - \chi_3^2)^{1/2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mp\chi_+(\omega \pm \chi_3) \\ \sqrt{2}\,(\omega^2 - \chi_3^2) \\ \pm\chi_-(\omega \mp \chi_3) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\eta(1,0) = \frac{1}{\sqrt{2}\,\omega}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \chi_+ \\ \sqrt{2}\,\chi_3 \\ \chi_- \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \eta(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \xi(0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\zeta(1,\pm 1) = \frac{1}{2\omega(\omega^2-\varkappa_3^2)^{1/2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mp i\varkappa_+(\omega\pm\varkappa_3) \\ \sqrt{2}\,(\omega^2-\varkappa_3^2) \\ \mp i\varkappa_-(\omega\mp\varkappa_3) \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \zeta(1,0) = \frac{1}{\sqrt{2}\,\omega} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ i\varkappa_+ \\ \sqrt{2}\,\varkappa_3 \\ -i\varkappa_- \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Upsilon(1,\pm 1) = \frac{1}{2\omega(\omega^2-\varkappa_3^2)^{1/2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \mp\varkappa_+(\omega\pm\varkappa_3) \\ \sqrt{2}\,(\omega^2-\varkappa_3^2) \\ \pm\varkappa_-(\omega\mp\varkappa_3) \end{bmatrix} \qquad \Upsilon(1,0) = \frac{1}{\sqrt{2}\,\omega} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \varkappa_+ \\ \sqrt{2}\,\varkappa_3 \\ \varkappa_- \end{bmatrix}$$

The eigenvectors of the operator $\underline{\Sigma}\cdot\underline{\varkappa}$ are obtained from those of $\underline{\sigma}\cdot\underline{\varkappa}$ by the operation S (see definition (5.7)) with the result

$$\epsilon_{\mu\nu}(1,5) = \frac{1}{2\omega^2(\omega^2-\varkappa_3^2)} \begin{bmatrix} \varkappa_1^2\varkappa_3^2 - \omega^2\varkappa_2^2 \pm 2i\omega\varkappa_1\varkappa_2\varkappa_3 \\ \varkappa_1\varkappa_2(\omega^2+\varkappa_3^2) \mp i\omega\varkappa_3(\varkappa_1^2-\varkappa_2^2) \\ (\omega^2-\varkappa_3^2)(-\varkappa_1\varkappa_3 \mp i\omega\varkappa_2) \\ 0 \\ \varkappa_1\varkappa_2(\omega^2+\varkappa_3^2) \mp i\omega\varkappa_3(\varkappa_1^2-\varkappa_2^2) \\ \varkappa_2^2\varkappa_3^2 - \omega^2\varkappa_1^2 \mp 2i\omega\varkappa_1\varkappa_2\varkappa_3 \\ (\omega^2-\varkappa_3^2)(-\varkappa_2\varkappa_3 \pm i\omega\varkappa_1) \\ 0 \\ (\omega^2-\varkappa_3^2)(-\varkappa_1\varkappa_3 \mp i\omega\varkappa_2) \\ (\omega^2-\varkappa_3^2)(-\varkappa_2\varkappa_3 \pm i\omega\varkappa_1) \\ (\omega^2-\varkappa_3^2)^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\epsilon_{\mu\nu}(3,4) = \frac{1}{2\omega^2(\omega^2-\varkappa_3^2)^{1/2}} \begin{bmatrix} -2\varkappa_3\varkappa_1^2 \mp 2i\omega\varkappa_1\varkappa_2 \\ -2\varkappa_1\varkappa_2\varkappa_3 \pm i\omega(\varkappa_1^2-\varkappa_2^2) \\ \varkappa_1(\omega^2-2\varkappa_3^2) \mp i\omega\varkappa_2\varkappa_3 \\ 0 \\ -2\varkappa_1\varkappa_2\varkappa_3 \pm i\omega(\varkappa_1^2-\varkappa_2^2) \\ -2\varkappa_3\varkappa_2^2 \pm 2i\omega\varkappa_1\varkappa_2 \\ 2(\omega^2-2\varkappa_3^2) \pm i\omega\varkappa_1\varkappa_3 \\ 0 \\ \varkappa_1(\omega^2-2\varkappa_3^2) \mp i\omega\varkappa_2\varkappa_3 \\ \varkappa_2(\omega^2-2\varkappa_3^2) \pm i\omega\varkappa_1\varkappa_3 \\ 2\varkappa_3(\omega^2-\varkappa_3^2) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\epsilon_{\mu\nu}(3) = \frac{1}{\sqrt{6}\,\omega^2}\begin{bmatrix} 3\varkappa_1^2-\omega^2 \\ 3\varkappa_1\varkappa_2 \\ 3\varkappa_1\varkappa_3 \\ 0 \\ 3\varkappa_1\varkappa_2 \\ 3\varkappa_2^2-\omega^2 \\ 3\varkappa_2\varkappa_3 \\ 0 \\ 3\varkappa_1\varkappa_3 \\ 3\varkappa_2\varkappa_3 \\ 3\varkappa_3^2-\omega^2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \epsilon_{\mu\nu}(6,8) = \frac{1}{2\omega(\omega^2-\varkappa_3^2)^{1/2}}\begin{bmatrix} 0 \\ 0 \\ 0 \\ -\varkappa_1\varkappa_3 \mp i\omega\varkappa_2 \\ 0 \\ 0 \\ 0 \\ -\varkappa_2\varkappa_3 \pm i\omega\varkappa_1 \\ 0 \\ 0 \\ 0 \\ \omega^2-\varkappa_3^2 \\ -\varkappa_1\varkappa_3 \mp i\omega\varkappa_2 \\ -\varkappa_2\varkappa_3 \pm i\omega\varkappa_1 \\ \omega^2-\varkappa_3^2 \\ 0 \end{bmatrix}$$

$$\epsilon_{zero}(7) = \frac{1}{\sqrt{2}\,\omega}\begin{bmatrix} 0 \\ 0 \\ 0 \\ \varkappa_1 \\ 0 \\ 0 \\ 0 \\ \varkappa_2 \\ 0 \\ 0 \\ 0 \\ \varkappa_3 \\ \varkappa_1 \\ \varkappa_2 \\ \varkappa_3 \\ 0 \end{bmatrix} \quad \epsilon_{zero}(9) = \frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3 \end{bmatrix} \quad \epsilon_{zero}(10) = \tfrac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\epsilon_{\mu\nu}(11,13) = \frac{1}{2\omega(\omega^2-\kappa_3^2)^{1/2}} \begin{bmatrix} 0 \\ \omega^2-\kappa_3^2 \\ \kappa_2\kappa_3 \mp i\omega\kappa_1 \\ 0 \\ -(\omega^2-\kappa_3^2) \\ 0 \\ -\kappa_1\kappa_3 \mp i\omega\kappa_2 \\ 0 \\ -\kappa_2\kappa_3 \pm i\omega\kappa_1 \\ \kappa_1\kappa_3 \pm i\omega\kappa_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \epsilon_{\mu\nu}(12) = \frac{1}{\sqrt{2}\,\omega} \begin{bmatrix} 0 \\ \kappa_3 \\ -\kappa_2 \\ 0 \\ -\kappa_3 \\ 0 \\ \kappa_1 \\ 0 \\ \kappa_2 \\ -\kappa_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\epsilon_{\mu\nu}(14,16) = \frac{1}{2\omega(\omega^2-\varkappa_3^2)^{1/2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\varkappa_1\varkappa_3 \mp i\omega\varkappa_2 \\ 0 \\ 0 \\ 0 \\ -\varkappa_2\varkappa_3 \pm i\omega\varkappa_1 \\ 0 \\ 0 \\ 0 \\ \omega^2-\varkappa_3^2 \\ \varkappa_1\varkappa_3 \pm i\omega\varkappa_2 \\ \varkappa_2\varkappa_3 \mp i\omega\varkappa_1 \\ -(\omega^2-\varkappa_3^2) \\ 0 \end{bmatrix} \qquad \epsilon_{\mu\nu}(15) = \frac{1}{\sqrt{2}\,\omega} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varkappa_1 \\ 0 \\ 0 \\ 0 \\ \varkappa_2 \\ 0 \\ 0 \\ 0 \\ \varkappa_3 \\ -\varkappa_1 \\ -\varkappa_2 \\ -\varkappa_3 \\ 0 \end{bmatrix}$$

Appendix B. The Fifty-Seven Possible Terms in the Lagrangian f_3 and the Result of their Variation.

The first eighteen contain i) $(\mu\nu)$

ii) some permutation of α, β, μ

iii) some permutation of α, β, ν

$L_1 = (\mu\nu)(\alpha\beta/\mu)(\alpha\beta/\nu)$

$L_2 = (\mu\nu)(\alpha\beta/\mu)(\beta\alpha/\nu)$

$L_3 = (\mu\nu)(\alpha\beta/\mu)(\alpha\nu/\beta)$

$L_4 = (\mu\nu)(\alpha\beta/\mu)(\beta\nu/\alpha)$

$L_5 = (\mu\nu)(\alpha\beta/\mu)(\nu\alpha/\beta)$

$L_6 = (\mu\nu)(\alpha\beta/\mu)(\nu\beta/\alpha)$

$L_7 = (\mu\nu)(\alpha\mu/\beta)(\alpha\beta/\nu)$

$L_8 = (\mu\nu)(\beta\mu/\alpha)(\alpha\beta/\nu)$

$L_9 = (\mu\nu)(\mu\alpha/\beta)(\alpha\beta/\nu)$

$L_{10} = (\mu\nu)(\mu\beta/\alpha)(\alpha\beta/\nu)$

$L_{11} = (\mu\nu)(\alpha\mu/\beta)(\alpha\nu/\beta)$

$L_{12} = (\mu\nu)(\alpha\mu/\beta)(\beta\nu/\alpha)$

$L_{13} = (\mu\nu)(\alpha\mu/\beta)(\nu\alpha/\beta)$

$L_{14} = (\mu\nu)(\alpha\mu/\beta)(\nu\beta/\alpha)$

$L_{15} = (\mu\nu)(\mu\alpha/\beta)(\alpha\nu/\beta)$

$L_{16} = (\mu\nu)(\mu\alpha/\beta)(\beta\nu/\alpha)$

$L_{17} = (\mu\nu)(\mu\alpha/\beta)(\nu\alpha/\beta)$

$L_{18} = (\mu\nu)(\mu\alpha/\beta)(\nu\beta/\alpha)$

The next nine contain i) $(\mu\nu)$

ii) some permutation of α, α, μ

iii) some permutation of β, β, ν

$L_{19} = (\mu\nu)(\alpha\alpha/\mu)(\beta\beta/\nu)$

$L_{20} = (\mu\nu)(\alpha\alpha/\mu)(\beta\nu/\beta)$

$L_{21} = (\mu\nu)(\alpha\alpha/\mu)(\nu\beta/\beta)$

$L_{22} = (\mu\nu)(\alpha\mu/\alpha)(\beta\beta/\nu)$

$L_{23} = (\mu\nu)(\alpha\mu/\alpha)(\beta\nu/\beta)$

$L_{24} = (\mu\nu)(\alpha\mu/\alpha)(\nu\beta/\beta)$

$L_{25} = (\mu\nu)(\mu\alpha/\alpha)(\beta\beta/\nu)$

$L_{26} = (\mu\nu)(\mu\alpha/\alpha)(\beta\nu/\beta)$

$L_{27} = (\mu\nu)(\mu\alpha/\alpha)(\nu\beta/\beta)$

The next eighteen contain i) $(\mu\nu)$

ii) some permutation of μ, ν, α

iii) some permutation of α, β, β

$L_{28} = (\mu\nu)(\mu\nu/\alpha)(\beta\beta/\alpha)$

$L_{29} = (\mu\nu)(\mu\nu/\alpha)(\beta\alpha/\beta)$

$L_{30} = (\mu\nu)(\mu\nu/\alpha)(\alpha\beta/\beta)$

$L_{31} = (\mu\nu)(\mu\alpha/\nu)(\beta\beta/\alpha)$

$L_{32} = (\mu\nu)(\mu\alpha/\nu)(\beta\alpha/\beta)$

$L_{33} = (\mu\nu)(\mu\alpha/\nu)(\alpha\beta/\beta)$

$L_{34} = (\mu\nu)(\nu\mu/\alpha)(\beta\beta/\alpha)$

$L_{35} = (\mu\nu)(\nu\mu/\alpha)(\beta\alpha/\beta)$

$L_{36} = (\mu\nu)(\nu\mu/\alpha)(\alpha\beta/\beta)$

$L_{37} = (\mu\nu)(\nu\alpha/\mu)(\beta\beta/\alpha)$

$L_{38} = (\mu\nu)(\nu\alpha/\mu)(\beta\alpha/\beta)$

$L_{39} = (\mu\nu)(\nu\alpha/\mu)(\alpha\beta/\beta)$

$L_{40} = (\mu\nu)(\alpha\mu/\nu)(\beta\beta/\alpha)$

$L_{41} = (\mu\nu)(\alpha\mu/\nu)(\beta\alpha/\beta)$

$L_{42} = (\mu\nu)(\alpha\mu/\nu)(\alpha\beta/\beta)$

$L_{43} = (\mu\nu)(\alpha\nu/\mu)(\beta\beta/\alpha)$

$L_{44} = (\mu\nu)(\alpha\nu/\mu)(\beta\alpha/\beta)$

$L_{45} = (\mu\nu)(\alpha\nu/\mu)(\alpha\beta/\beta)$

The last twelve contain $(\alpha\alpha)$

$L_{46} = (\alpha\alpha)(\mu\nu/\beta)(\mu\nu/\beta)$

$L_{47} = (\alpha\alpha)(\mu\nu/\beta)(\nu\mu/\beta)$

$L_{48} = (\alpha\alpha)(\mu\nu/\beta)(\mu\beta/\nu)$

$L_{49} = (\alpha\alpha)(\mu\nu/\beta)(\nu\beta/\mu)$

$L_{50} = (\alpha\alpha)(\mu\nu/\beta)(\beta\mu/\nu)$

$L_{51} = (\alpha\alpha)(\mu\nu/\beta)(\beta\nu/\mu)$

$L_{52} = (\alpha\alpha)(\beta\beta/\mu)(\nu\nu/\mu)$

$L_{53} = (\alpha\alpha)(\beta\beta/\mu)(\nu\mu/\nu)$

$L_{54} = (\alpha\alpha)(\beta\beta/\mu)(\mu\nu/\nu)$

$L_{55} = (\alpha\alpha)(\beta\mu/\beta)(\nu\mu/\nu)$

$L_{56} = (\alpha\alpha)(\beta\mu/\beta)(\mu\nu/\nu)$

$L_{57} = (\alpha\alpha)(\mu\beta/\beta)(\mu\nu/\nu)$

The variation of f_3 (see equation (14.13)) yields

$$c_1 \left\{(\alpha\beta/\mu)(\alpha\beta/\nu) - 2(\alpha\beta)(\mu\nu/\alpha\beta) - (\mu\nu/\beta)(\alpha\beta/\alpha) - (\mu\nu/\alpha)(\alpha\beta/\beta)\right\} +$$

$$c_2 \left\{(\alpha\beta/\mu)(\alpha\beta/\nu) - 2(\alpha\beta)(\nu\mu/\alpha\beta) - (\nu\mu/\beta)(\alpha\beta/\alpha) - (\nu\mu/\alpha)(\alpha\beta/\beta)\right\} +$$

$$c_3 \left\{(\alpha\beta/\mu)(\alpha\nu/\beta) - (\alpha\beta)(\mu\beta/\nu\alpha) - (\alpha\beta/\alpha)(\mu\beta/\nu) - (\alpha\nu)(\mu\beta/\alpha\beta) - (\alpha\nu/\beta)(\mu\beta/\alpha)\right\} +$$

$$c_4 \left\{(\alpha\beta/\mu)(\beta\nu/\alpha) - (\alpha\beta)(\nu\beta/\mu\alpha) - (\alpha\beta/\alpha)(\nu\beta/\mu) - (\beta\nu)(\alpha\mu/\beta\alpha) - (\beta\nu/\alpha)(\alpha\mu/\beta)\right\} +$$

$$c_5 \left\{(\alpha\beta/\mu)(\nu\alpha/\beta) - (\alpha\beta)(\beta\mu/\nu\alpha) - (\alpha\beta/\alpha)(\beta\mu/\nu) - (\alpha\mu)(\nu\beta/\alpha\beta) - (\alpha\mu/\beta)(\nu\beta/\alpha)\right\} +$$

$$c_6\left\{(\alpha\beta/\mu)(\nu\beta/\alpha) - (\alpha\beta)(\beta\nu/\mu\alpha) - (\alpha\beta/\alpha)(\beta\nu/\mu) - (\beta\mu)(\alpha\nu/\alpha\beta) - (\beta\mu/\alpha)(\alpha\nu/\beta)\right\} +$$

$$c_7\left\{(\alpha\mu/\beta)(\alpha\beta/\nu) - (\alpha\beta)(\mu\alpha/\nu\beta) - (\alpha\beta/\beta)(\mu\alpha/\nu) - (\nu\alpha)(\mu\beta/\alpha\beta) - (\nu\alpha/\beta)(\mu\beta/\alpha)\right\} +$$

$$c_8\left\{(\beta\mu/\alpha)(\alpha\beta/\nu) - (\alpha\beta)(\nu\alpha/\mu\beta) - (\alpha\beta/\beta)(\nu\alpha/\mu) - (\nu\beta)(\alpha\mu/\beta\alpha) - (\nu\beta/\alpha)(\alpha\mu/\beta)\right\} +$$

$$c_9\left\{(\mu\alpha/\beta)(\alpha\beta/\nu) - (\alpha\beta)(\alpha\mu/\nu\beta) - (\alpha\beta/\beta)(\alpha\mu/\nu) - (\mu\alpha)(\nu\beta/\alpha\beta) - (\mu\alpha/\beta)(\nu\beta/\alpha)\right\} +$$

$$c_{10}\left\{(\mu\beta/\alpha)(\alpha\beta/\nu) - (\alpha\beta)(\alpha\nu/\mu\beta) - (\alpha\beta/\beta)(\alpha\nu/\mu) - (\mu\beta)(\alpha\nu/\beta\alpha) - (\mu\beta/\alpha)(\alpha\nu/\beta)\right\} +$$

$$c_{11}\left\{(\alpha\mu/\beta)(\alpha\nu/\beta) - (\alpha\nu)(\mu\alpha/\beta\beta) - (\alpha\nu/\beta)(\mu\alpha/\beta) - (\nu\alpha)(\mu\alpha/\beta\beta) - (\nu\alpha/\beta)(\mu\alpha/\beta)\right\} +$$

$$c_{12}\left\{(\alpha\mu/\beta)(\beta\nu/\alpha) - (\beta\nu)(\alpha\beta/\mu\alpha) - (\beta\nu/\alpha)(\alpha\beta/\mu) - (\nu\alpha)(\beta\alpha/\mu\beta) - (\nu\alpha/\beta)(\beta\alpha/\mu)\right\} +$$

$$c_{13}\left\{-(\nu\alpha)(\alpha\mu/\beta\beta) - (\alpha\mu)(\nu\alpha/\beta\beta) - (\alpha\mu/\beta)(\nu\alpha/\beta)\right\} +$$

$$c_{14}\left\{(\alpha\mu/\beta)(\nu\beta/\alpha) - (\nu\alpha)(\alpha\beta/\mu\beta) - (\nu\alpha/\beta)(\alpha\beta/\mu) - (\beta\mu)(\alpha\beta/\nu\alpha) - (\beta\mu/\alpha)(\alpha\beta/\nu)\right\} +$$

$$c_{15}\left\{(\mu\alpha/\beta)(\alpha\nu/\beta) - (\mu\alpha)(\nu\alpha/\beta\beta) - (\mu\alpha/\beta)(\nu\alpha/\beta) - (\alpha\nu)(\alpha\mu/\beta\beta) - (\alpha\nu/\beta)(\alpha\mu/\beta)\right\} +$$

$$c_{16}\left\{(\mu\alpha/\beta)(\beta\nu/\alpha) - (\mu\alpha)(\beta\alpha/\nu\beta) - (\mu\alpha/\beta)(\beta\alpha/\nu) - (\beta\nu)(\beta\alpha/\mu\alpha) - (\beta\nu/\alpha)(\beta\alpha/\mu)\right\} +$$

$$c_{17}\left\{(\mu\alpha/\beta)(\nu\alpha/\beta) - (\mu\alpha)(\alpha\nu/\beta\beta) - (\mu\alpha/\beta)(\alpha\nu/\beta) - (\alpha\mu)(\alpha\nu/\beta\beta) - (\alpha\mu/\beta)(\alpha\nu/\beta)\right\} +$$

$$c_{18}\left\{(\mu\alpha/\beta)(\nu\beta/\alpha) - (\mu\alpha)(\alpha\beta/\nu\beta) - (\mu\alpha/\beta)(\alpha\beta/\nu) - (\beta\mu)(\beta\alpha/\nu\alpha) - (\beta\mu/\alpha)(\beta\alpha/\nu)\right\} +$$

$$c_{19}\left\{(\alpha\alpha/\mu)(\beta\beta/\nu) - 2(\alpha\beta)(\sigma\sigma/\alpha\beta)\delta_{\mu\nu} - (\alpha\beta/\alpha)(\sigma\sigma/\beta)\delta_{\mu\nu} - (\alpha\beta/\beta)(\sigma\sigma/\alpha)\delta_{\mu\nu}\right\} +$$

$$c_{20}\left\{(\alpha\alpha/\mu)(\beta\nu/\beta) - (\alpha\nu)(\beta\beta/\alpha\mu) - (\alpha\nu/\mu)(\beta\beta/\alpha) - (\alpha\beta)(\sigma\beta/\sigma\alpha)\delta_{\mu\nu} - (\alpha\beta/\alpha)(\sigma\beta/\sigma)\delta_{\mu\nu}\right\} +$$

$$c_{21}\left\{(\alpha\alpha/\mu)(\nu\beta/\beta) - (\beta\mu)(\alpha\alpha/\beta\nu) - (\beta\mu/\nu)(\alpha\alpha/\beta) - (\alpha\beta)(\beta\sigma/\sigma\alpha)\delta_{\mu\nu} - (\alpha\beta/\alpha)(\beta\sigma/\sigma)\delta_{\mu\nu}\right\} +$$

$$c_{22}\left\{(\alpha\mu/\alpha)(\beta\beta/\nu) - (\nu\beta)(\alpha\alpha/\beta\mu) - (\nu\beta/\mu)(\alpha\alpha/\beta) - (\alpha\beta)(\sigma\alpha/\sigma\beta)\delta_{\mu\nu} - (\alpha\beta/\beta)(\sigma\alpha/\sigma)\delta_{\mu\nu}\right\} +$$

$$c_{23}\left\{(\alpha\mu/\alpha)(\beta\nu/\beta) - (\nu\beta)(\alpha\beta/\alpha\mu) - (\nu\beta/\mu)(\alpha\beta/\alpha) - (\beta\nu)(\alpha\beta/\alpha\mu) - (\beta\nu/\mu)(\alpha\beta/\alpha)\right\} +$$

$$c_{24}\left\{(\alpha\mu/\alpha)(\nu\beta/\beta) - (\nu\alpha)(\alpha\beta/\beta\mu) - (\nu\alpha/\mu)(\alpha\beta/\beta) - (\beta\mu)(\alpha\beta/\alpha\nu) - (\beta\mu/\nu)(\alpha\beta/\alpha)\right\} +$$

$$c_{25}\left\{(\mu\alpha/\alpha)(\beta\beta/\nu) - (\mu\alpha)(\beta\beta/\alpha\nu) - (\mu\alpha/\nu)(\beta\beta/\alpha) - (\beta\alpha)(\beta\sigma/\sigma\alpha)\delta_{\mu\nu} - (\beta\alpha/\alpha)(\beta\sigma/\sigma)\delta_{\mu\nu}\right\} +$$

$$c_{26}\left\{(\mu\alpha/\alpha)(\beta\nu/\beta) - (\mu\alpha)(\beta\alpha/\beta\nu) - (\mu\alpha/\nu)(\beta\alpha/\beta) - (\beta\nu)(\beta\alpha/\alpha\mu) - (\beta\nu/\mu)(\beta\alpha/\alpha)\right\} +$$

$$c_{27}\left\{(\mu\alpha/\alpha)(\nu\beta/\beta) - (\mu\alpha)(\alpha\beta/\beta\nu) - (\mu\alpha/\nu)(\alpha\beta/\beta) - (\beta\mu)(\beta\alpha/\alpha\nu) - (\beta\mu/\nu)(\beta\alpha/\alpha)\right\} +$$

$$c_{28}\left\{-(\mu\nu)(\beta\beta/\alpha\alpha) - (\beta\sigma)(\beta\sigma/\alpha\alpha)\delta_{\mu\nu} - (\beta\sigma/\alpha)(\beta\sigma/\alpha)\delta_{\mu\nu}\right\} +$$

$$c_{29}\left\{-(\mu\nu)(\beta\alpha/\beta\alpha) - (\alpha\beta)(\alpha\beta/\nu\mu) - (\alpha\beta/\mu)(\alpha\beta/\nu)\right\} +$$

$$c_{30}\left\{(\mu\nu/\alpha)(\alpha\beta/\beta) - (\mu\nu)(\alpha\beta/\beta\alpha) - (\mu\nu/\alpha)(\alpha\beta/\beta) - (\alpha\beta)(\alpha\beta/\mu\nu) - (\alpha\beta/\mu)(\alpha\beta/\nu)\right\} +$$

$$c_{31}\left\{(\mu\alpha/\nu)(\beta\beta/\alpha) - (\mu\alpha)(\beta\beta/\nu\alpha) - (\mu\alpha/\alpha)(\beta\beta/\nu) - (\beta\sigma)(\beta\alpha/\sigma\alpha)\delta_{\mu\nu} - (\beta\sigma/\alpha)(\beta\alpha/\sigma)\delta_{\mu\nu}\right\} +$$

$$c_{32}\left\{(\mu\alpha/\nu)(\beta\alpha/\beta) - (\mu\alpha)(\beta\nu/\beta\alpha) - (\mu\alpha/\alpha)(\beta\nu/\beta) - (\alpha\beta)(\alpha\nu/\beta\mu) - (\alpha\beta/\mu)(\alpha\nu/\beta)\right\} +$$

$$c_{33}\left\{(\mu\alpha/\nu)(\alpha\beta/\beta) - (\mu\alpha)(\nu\beta/\beta\alpha) - (\mu\alpha/\alpha)(\nu\beta/\beta) - (\alpha\beta)(\alpha\mu/\beta\nu) - (\alpha\beta/\nu)(\alpha\mu/\beta)\right\} +$$

$$c_{34}\Big\{(\nu\mu/\alpha)(\beta\beta/\alpha) - (\nu\mu)(\beta\beta/\alpha\alpha) - (\nu\mu/\alpha)(\beta\beta/\alpha) - (\beta\sigma)(\sigma\beta/\alpha\alpha)\delta_{\mu\nu} - (\beta\sigma/\alpha)(\sigma\beta/\alpha)\delta_{\mu\nu}\Big\} +$$

$$c_{35}\Big\{- (\nu\mu)(\beta\alpha/\beta\alpha) - (\alpha\beta)(\beta\alpha/\nu\mu) - (\alpha\beta/\mu)(\beta\alpha/\nu)\Big\} +$$

$$c_{36}\Big\{- (\nu\mu)(\alpha\beta/\beta\alpha) - (\alpha\beta)(\beta\alpha/\mu\nu) - (\alpha\beta/\nu)(\beta\alpha/\mu)\Big\} +$$

$$c_{37}\Big\{(\nu\alpha/\mu)(\beta\beta/\alpha) - (\alpha\mu)(\beta\beta/\nu\alpha) - (\alpha\mu/\alpha)(\beta\beta/\nu) - (\beta\sigma)(\sigma\alpha/\beta\alpha)\delta_{\mu\nu} - (\beta\sigma/\alpha)(\sigma\alpha/\beta)\delta_{\mu\nu}\Big\} +$$

$$c_{38}\Big\{(\nu\alpha/\mu)(\beta\alpha/\beta) - (\alpha\mu)(\beta\nu/\beta\alpha) - (\alpha\mu/\alpha)(\beta\nu/\beta) - (\alpha\beta)(\beta\nu/\alpha\mu) - (\alpha\beta/\mu)(\beta\nu/\alpha)\Big\} +$$

$$c_{39}\Big\{(\nu\alpha/\mu)(\alpha\beta/\beta) - (\alpha\mu)(\nu\beta/\beta\alpha) - (\alpha\mu/\alpha)(\nu\beta/\beta) - (\alpha\beta)(\beta\mu/\alpha\nu) - (\alpha\beta/\nu)(\beta\mu/\alpha)\Big\} +$$

$$c_{40}\Big\{(\alpha\mu/\nu)(\beta\beta/\alpha) - (\nu\alpha)(\beta\beta/\mu\alpha) - (\nu\alpha/\alpha)(\beta\beta/\mu) - (\beta\sigma)(\alpha\beta/\sigma\alpha)\delta_{\mu\nu} - (\beta\sigma/\alpha)(\alpha\beta/\sigma)\delta_{\mu\nu}\Big\} +$$

$$c_{41}\Big\{(\alpha\mu/\nu)(\beta\alpha/\beta) - (\nu\alpha)(\beta\mu/\beta\alpha) - (\nu\alpha/\alpha)(\beta\mu/\beta) - (\alpha\beta)(\nu\alpha/\beta\mu) - (\alpha\beta/\mu)(\nu\alpha/\beta)\Big\} +$$

$$c_{42}\Big\{(\alpha\mu/\nu)(\alpha\beta/\beta) - (\nu\alpha)(\mu\beta/\beta\alpha) - (\nu\alpha/\alpha)(\mu\beta/\beta) - (\alpha\beta)(\mu\alpha/\beta\nu) - (\alpha\beta/\nu)(\mu\alpha/\beta)\Big\} +$$

$$c_{43}\Big\{(\alpha\nu/\mu)(\beta\beta/\alpha) - (\alpha\nu)(\beta\beta/\mu\alpha) - (\alpha\nu/\alpha)(\beta\beta/\mu) - (\beta\sigma)(\alpha\sigma/\beta\alpha)\delta_{\mu\nu} - (\beta\sigma/\alpha)(\alpha\sigma/\beta)\delta_{\mu\nu}\Big\} +$$

$$c_{44}\Big\{(\alpha\nu/\mu)(\beta\alpha/\beta) - (\alpha\nu)(\beta\mu/\beta\alpha) - (\alpha\nu/\alpha)(\beta\mu/\beta) - (\alpha\beta)(\nu\beta/\alpha\mu) - (\alpha\beta/\mu)(\nu\beta/\alpha)\Big\} +$$

$$c_{45}\Big\{(\alpha\nu/\mu)(\alpha\beta/\beta) - (\alpha\nu)(\mu\beta/\beta\alpha) - (\alpha\nu/\alpha)(\mu\beta/\beta) - (\alpha\beta)(\mu\beta/\alpha\nu) - (\alpha\beta/\nu)(\mu\beta/\alpha)\Big\} +$$

$$c_{46}\Big\{(\alpha\sigma/\beta)(\alpha\sigma/\beta)\delta_{\mu\nu} - 2(\alpha\alpha)(\mu\nu/\beta\beta) - 2(\alpha\alpha/\beta)(\mu\nu/\beta)\Big\} +$$

$$c_{47}\Big\{(\alpha\sigma/\beta)(\sigma\alpha/\beta)\delta_{\mu\nu} - 2(\alpha\alpha)(\nu\mu/\beta\beta) - 2(\alpha\alpha/\beta)(\nu\mu/\beta)\Big\} +$$

$$c_{48}\Big\{(\alpha\sigma/\beta)(\alpha\beta/\sigma)\delta_{\mu\nu} - 2(\alpha\alpha)(\mu\beta/\nu\beta) - 2(\alpha\alpha/\beta)(\mu\beta/\nu)\Big\} +$$

$$c_{49}\Big\{ (\alpha\sigma/\beta)(\sigma\beta/\alpha)\delta_{\mu\nu} - (\alpha\alpha)(\nu\beta/\mu\beta) - (\alpha\alpha/\beta)(\nu\beta/\mu) - (\alpha\alpha)(\beta\mu/\nu\beta) - (\alpha\alpha/\beta)(\beta\mu/\nu) \Big\} +$$

$$c_{50}\Big\{ (\alpha\sigma/\beta)(\beta\alpha/\sigma)\delta_{\mu\nu} - (\alpha\alpha)(\beta\mu/\nu\beta) - (\alpha\alpha/\beta)(\beta\mu/\nu) - (\alpha\alpha)(\nu\beta/\mu\beta) - (\alpha\alpha/\beta)(\nu\beta/\mu) \Big\} +$$

$$c_{51}\Big\{ (\alpha\sigma/\beta)(\beta\sigma/\alpha)\delta_{\mu\nu} - 2(\alpha\alpha)(\beta\nu/\mu\beta) - 2(\alpha\alpha/\beta)(\beta\nu/\mu) \Big\} +$$

$$c_{52}\Big\{ - 2(\alpha\alpha)(\beta\beta/\sigma\sigma)\delta_{\mu\nu} - (\alpha\alpha/\sigma)(\beta\beta/\sigma)\delta_{\mu\nu} \Big\} +$$

$$c_{53}\Big\{ - (\alpha\alpha)(\beta\beta/\nu\mu) - (\alpha\alpha/\mu)(\beta\beta/\nu) - (\alpha\alpha)(\sigma\beta/\sigma\beta)\delta_{\mu\nu} \Big\} +$$

$$c_{54}\Big\{ - (\alpha\alpha)(\beta\beta/\mu\nu) - (\alpha\alpha/\nu)(\beta\beta/\mu) - (\alpha\alpha)(\beta\sigma/\sigma\beta)\delta_{\mu\nu} \Big\} +$$

$$c_{55}\Big\{ (\beta\alpha/\beta)(\sigma\alpha/\sigma)\delta_{\mu\nu} - 2(\alpha\alpha)(\beta\nu/\beta\mu) - 2(\alpha\alpha/\mu)(\beta\nu/\beta) \Big\} +$$

$$c_{56}\Big\{ (\beta\alpha/\beta)(\alpha\sigma/\sigma)\delta_{\mu\nu} - (\alpha\alpha)(\beta\mu/\beta\nu) - (\alpha\alpha/\nu)(\beta\mu/\beta) - (\alpha\alpha)(\nu\beta/\beta\mu) - (\alpha\alpha/\mu)(\nu\beta/\beta) \Big\} +$$

$$c_{57}\Big\{ (\alpha\beta/\beta)(\alpha\sigma/\sigma)\delta_{\mu\nu} - 2(\alpha\alpha)(\mu\beta/\beta\nu) - 2(\alpha\alpha/\nu)(\mu\beta/\beta) \Big\} =$$

$$= k\Big\{ \tfrac{1}{2}(\alpha\beta/\sigma)(\alpha\beta/\sigma)\delta_{\mu\nu} - (\alpha\beta/\mu)(\alpha\beta/\nu) \Big\}$$