OSCILLATIONS OF THE EARTH'S OUTER ATMOSPHERE
AND MICROPULSATIONS

by

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M.A. The University of Toronto, 1959

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FACULTY OF GRADUATE STUDIES

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FINAL ORAL EXAMINATION
FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

of

KARL OSKAR WESTPHAL
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OSCILLATIONS OF THE EARTH'S OUTER ATMOSPHERE AND MICROPULSATIONS

ABSTRACT

Micropulsations of the Earth's magnetic field are closely related to the properties of the upper atmosphere and probably interstellar space as well. A model which would explain the observed data in a satisfactory manner would also give information on the properties of the outer atmosphere which at the present time are difficult to obtain otherwise. However, in spite of the vast amount of observational data now available, a satisfactory theoretical analysis of the problem has not yet been given. To a large extent this is due to mathematical and computational difficulties.

Geomagnetic micropulsations may be described as magnetohydrodynamic oscillations of which two different modes, namely poloidal and toroidal oscillations, may exist, and which in general are coupled. In this work only the toroidal mode, which may be understood as oscillations of a line of force, is considered. If the phenomenon is studied under the simplifying assumption $\frac{\partial}{\partial \phi} = 0$ where $\phi$ is the longitude it is possible to obtain the eigenvalues of the oscillating lines of force in simple terms by treating the problem in cylindrical rather than in spherical polar coordinates. For a constant and variable charge density distribution, the eigenfrequencies are obtained as functions of the latitude.

For a variable charge density distribution, the agreement between theory and observation is good in middle and low latitudes. However, as the latitude increases towards the poles the model gives periods which tend to infinity.

As a result of recent studies it appears that the geomagnetic field does not extend as far into outer space as has been assumed, but that to a first approximation it is confined to a cavity. For this reason the equation of toroidal oscillations is applied to a compressed dipole field. Assuming both a constant and a variable charge density distribution the eigenperiods of the deformed magnetic lines of force are obtained, using an electronic computer.

The results for a variable charge density distribution agree with observational data in the polar regions but not in middle and low latitudes. This may imply that the charge density distribution is in error and it is hoped that further rocket and satellite data will settle this point.

PUBLICATIONS


GRADUATE STUDIES

Field of Study: Geomagnetic Micropulsations

Nuclear Physics .............................................. J. B. Warren
Advanced Geophysics ........................................ J. A. Jacobs
Isotope Geology .................................................. R. D. Russell

Related Studies:

Applied Electromagnetic Theory .................. G. B. Walker
Computational Methods .................................. C. Froese
Programming of Digital Computers ............. J. R. H. Dempster
Using Maxwell's equations of electrodynamics and the linearized fundamental equation of hydrodynamics neglecting all but the ponderomotive force, the two differential equations characterizing toroidal and poloidal modes of oscillations are obtained. Neglecting the coupling between these modes the toroidal mode which appears to be connected with the phenomenon of geomagnetic micropulsations is studied in detail.

Substituting for the constant magnetic field the undeformed dipole field of the Earth the eigenperiods of the oscillating lines of force are computed assuming a constant charge density distribution. Using numerical methods the eigenperiods are also obtained in the case of a variable charge density.

Since the Earth's dipole field is presumably deformed by the solar wind a compressed dipole field is introduced into the equation of toroidal oscillations. The eigenperiods of the oscillating lines of force are obtained in this case, assuming a constant charge density distribution. For the case of a variable charge density a numerical method is described which could yield the eigenperiods.
ACKNOWLEDGEMENTS

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CHAPTER I

INTRODUCTION

During the last ten years pulsations of the geomagnetic field have received increasing attention. A great amount of recorded data from stations all over the world has accumulated since such pulsations were first observed by Balfour Stewart in 1861. Amplitudes range from a fraction of a gamma (1 gamma ($\gamma$) = $10^{-5}$ gauss) to as much as a few tens of a gamma, and frequencies from 0.01 to 10 cps.

The main reason for studying the phenomenon is that the generation of such pulsations is closely related to the properties of the upper atmosphere and probably interstellar space. A model which would explain the observed data in a satisfactory manner would also give information on the properties of the outer atmosphere which at the present time are difficult to obtain otherwise. However, in spite of the vast amount of observed data now available, a satisfactory theoretical analysis of the problem has not yet been given. To a large extent this is due to mathematical and computational difficulties.
In principle geomagnetic pulsations may be described as magneto-hydrodynamic oscillations of which two different modes, namely poloidal and toroidal oscillations can exist and which in general are coupled. In this thesis we deal only with the toroidal mode which may be understood as oscillations of a line of force. Considering the phenomenon to take place in a meridian plane it is possible to obtain the eigenvalues of the oscillating lines of force by treating the problem in cylindrical rather than in spherical polar coordinates. For a given charge density distribution the eigenfrequencies are obtained as functions of the latitude.

In the first attempt to treat this hydromagnetic problem the Earth's main field is taken to be that of a geocentric dipole. This has the unfortunate consequence that the eigenfrequencies tend towards infinity as the point of intersection of the lines of force with the Earth's surface approach very high latitudes - which is not in agreement with the observations. By confining the dipole field to a sphere of finite radius which may be the case on account of the presence of the solar wind which carries the field along, it is possible to remove the discrepancies.

From the experimental data obtained at different latitudes but equal longitudes it is possible to determine those eigenperiods which show a maximum frequency occurrence and by comparing this frequency-latitude dependence with
those calculated it appears that the model of an oscillating magnetic line of force is feasible.

Before discussing the theoretical approach to the problem of geomagnetic micropulsations which is based on the concept of magneto-hydrodynamic waves it is instructive to outline this phenomenon, which was discovered by Alfvén in 1942, for a rather simple case.

If an electrically conducting medium moves in the presence of a magnetic field, an electric field is induced in the medium, the current being at right angles to both the direction of motion of the medium and the magnetic field. However, an electric current in a magnetic field is acted upon by a mechanical force perpendicular to both the current and the magnetic field, i.e. either in the direction of motion or against it. It follows from simple energy considerations that the mechanical force must be directed so as to impede the original movement. If the original motion was at right angles to the magnetic field the disturbance will propagate in the form of a wave along the magnetic line of force.

In order that such an interaction between electromagnetic and hydrodynamic phenomena exist it was shown by Lundquist (1952) that the inequality

\[ B \sigma \sqrt{\frac{\mu_0}{\rho}} \gg 1 \]
must be satisfied, where L is the linear dimension of a liquid conductor, of density \( \rho \) and permeability \( \mu_0 \) in the presence of a magnetic field B. (mks units are used throughout this thesis.) It is easily seen that under normal laboratory conditions this criterion is not satisfied and magneto-hydrodynamic effects are not observed.

The situation, however, is completely different in problems of cosmic physics. Because of the enormous dimensions involved in such cases Lundquist's criterion is easily satisfied and interactions between electromagnetic and hydrodynamic phenomena may be considerable. It is for this reason that nearly all attempts which have been made in recent years to explain the phenomenon of geomagnetic micropulsations apply the principles of magneto-hydrodynamics to the outer atmosphere.

The suggestion that geomagnetic micropulsations might be explained in terms of magneto-hydrodynamic oscillations was first advanced by Dungey (1954b). Under the assumption that the outer atmosphere is a very good conductor he derived from Maxwell's equations and from the fundamental equation of fluid dynamics two coupled partial differential equations governing the two components \( E_\varphi \) and \( V_\varphi \), viz.:

\[
\left\{- \frac{\rho}{\mu_0 H_0} \frac{d}{dt} \frac{d}{dr} + \frac{d^2}{dr^2} + \frac{\sin \varphi}{r^2} \frac{d}{d\varphi} \left( \frac{1}{\sin \varphi} \frac{d}{d\varphi} \right) \right\} (r \sin \varphi E_\varphi) = \mu_0 \sin \varphi \left\{ H_r \frac{d}{d\varphi} - H_\varphi \frac{d}{dr} \right\} \left( \frac{1}{r \sin \varphi} \frac{dV_\varphi}{d\varphi} \right)
\]
and

\[
\left\{ \frac{\rho}{\mu_0} \frac{\partial^2}{\partial t^2} - \frac{1}{(r \sin \phi)^2} \left[ \left( \mathbf{H}_o \cdot \nabla \right) (r \sin \phi)^2 \left( \mathbf{H}_o \cdot \nabla \right) + H_o^2 \frac{\partial^2}{\partial \phi^2} \right] \right\} \left( \frac{V_\phi}{r \sin \phi} \right)
\]

(2)

From (1) and (2) which are termed the equations of poloidal and toroidal oscillations the field vectors \( \mathbf{E} \{ E_r, E_\phi, E_\psi \} \) and \( \mathbf{V} \{ V_r, V_\phi, V_\psi \} \) are determined in a unique way. However, these two equations are far too complicated to be of much use in studying geomagnetic micropulsations. For this reason one assumes axial symmetry, i.e. one supposes that the phenomenon takes place only in the plane of a meridian – an assumption which seems to be confirmed by the local time dependence. Under this assumption the coupling term vanishes \( \left( \frac{\partial}{\partial \phi} \right) = 0 \) splitting the coupled system into two separate equations, viz.:

\[
\left\{ \frac{-\rho}{\mu_0 H_o^2} \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial r^2} + \frac{\sin \phi}{r^2} \frac{\partial}{\partial \phi} \left( \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \right) \right\} (r \sin \phi E_\phi) = 0 \quad (3)
\]

and

\[
\left\{ \frac{\rho}{\mu_0} \frac{\partial^2}{\partial t^2} - \frac{1}{(r \sin \phi)^2} \left[ \left( \mathbf{H}_o \cdot \nabla \right) (r \sin \phi)^2 \left( \mathbf{H}_o \cdot \nabla \right) \right] \right\} \left( \frac{V_\phi}{r \sin \phi} \right) = 0 \quad (4)
\]

The first attempt to compute from equation (4) the eigenperiods of the toroidal oscillations was made by Dungey himself. Taking the magnetic field of the Earth \( \mathbf{H}_o \) to be
that of a dipole he obtained under the assumption of a constant charge density \( \rho \) \( = 10^{-18} \text{ kg/m}^3 \) for the fundamental period the approximation

\[
T_1 = \frac{0.6}{\cos \theta \lambda_0} \text{ [sec]} \quad (\lambda > 30^\circ) \quad (5)
\]

where \( \lambda_0 \) is the latitude at which a particular magnetic line of force intersects the surface of the earth. Since this is a boundary value problem one in general assumes that the hydro-magnetic wave is reflected at the point of intersection.

Evaluating equation (5) for particular latitudes gives the results shown in table 1.

<table>
<thead>
<tr>
<th>( \lambda_0 )</th>
<th>( T_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>10 sec</td>
</tr>
<tr>
<td>55°</td>
<td>54 sec</td>
</tr>
<tr>
<td>65°</td>
<td>11 min</td>
</tr>
<tr>
<td>70°</td>
<td>55 min</td>
</tr>
</tbody>
</table>

A different method of approach to the solution of the boundary value problem which is in particular permissible for obtaining the periods of the higher modes of oscillations has been taken by Kato and Watanabe (1956). Again assuming a constant charge density distribution, they obtained for the period of the \( n \)th order oscillation
\[ T_n = \frac{2.89 \times 10^7 \times \sin \lambda_0}{n \times \cos \theta \lambda_0} \sqrt{\rho_0} \int_0^1 (1 - ax^2)^3 \, dx \]  

(6)

where \( \rho_0 \) the charge density has been taken equal to its value at the most distant point and

\[ a = \sin^2 \lambda_0 \]

The results for the fundamental mode as obtained from equation (6) are summarized in table 2.

<table>
<thead>
<tr>
<th>( \lambda_0 )</th>
<th>( \rho_0 = 0.8 \times 10^{-18} \text{ kg/m}^3 )</th>
<th>( 1.6 \times 10^{-18} \text{ kg/m}^3 )</th>
<th>( 2.4 \times 10^{-18} \text{ kg/m}^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>6 sec</td>
<td>9 sec</td>
<td>11 sec</td>
</tr>
<tr>
<td>55°</td>
<td>33 sec</td>
<td>47 sec</td>
<td>57 sec</td>
</tr>
<tr>
<td>65°</td>
<td>381 sec</td>
<td>538 sec</td>
<td>659 sec</td>
</tr>
<tr>
<td>70°</td>
<td>2071 sec</td>
<td>2929 sec</td>
<td>3585 sec</td>
</tr>
</tbody>
</table>

In equation (4) the only spacial derivatives which occur are those in the operator \( (\vec{H}_0 \cdot \nabla) \) operating on the function \( v_y \); this however can be interpreted as the derivative of \( v_y \) in the direction of \( \vec{H}_0 \). Therefore equation (4) can be interpreted as a wave equation where the disturbance propagates along a line of force. Choosing for \( \vec{H}_0 \) a dipole field, equation (4) represents the wave equation governing the time-space relationship of a disturbance which propagates along
the magnetic lines of force of the dipole. It is then in this
sense that one speaks of oscillations of the lines of force
which, it must be emphasized, applies in its truest sense
only to the case of an infinitely conducting medium.

This concept of oscillating magnetic lines of force has
been utilized by Obayashi and Jacobs (1958) for computing
the charge density distribution of the outer atmosphere making
use of the observed periods of micropulsations. To carry out
the analysis it was however necessary to make an assumption
of the functional relationship between the height and the
charge density.

In order to facilitate the solution of equation (4)
Siebert (private communication) chose a charge density dis­
tribution of a mathematical form which reduces equation (4)
to the simple differential equation of a harmonic oscillator.
This method of approach however leads to results which are in
disagreement with the observed data indicating that this
approach is not appropriate.

Reviewing the theoretical work which has been done on
the problem so far one arrives at the conclusion that the
magneto-hydrodynamical treatment of the problem leads to
equations which in their general form are too complex to be
treated mathematically. Even under the reasonable assumption
that the phenomenon takes place in a meridional plane only,
the resulting equations are quite intractable - even, if in
addition, a constant charge density distribution and a simple dipole field of the Earth are assumed.

Since coupling effects, which manifest themselves through the presence of terms involving \( \partial / \partial \rho \) in the case of spherical polar coordinates, are initially neglected it seems feasible to investigate the problem anew by working in cylindrical rather than spherical polar coordinates and dropping terms containing \( \partial / \partial z \). The equations then take on a simpler form in which a mathematical treatment is possible even for the case of a non-uniform charge density distribution and a deformed dipole field.
1. The Basic Equations

In order to relate the phenomena of electrodynamics and hydrodynamics, Maxwell's equations and the basic equations of hydrodynamics are used. It follows from a consideration of the orders of magnitude of the quantities involved that in problems of cosmic physics, displacement currents are negligible in comparison with the conduction current. Therefore Maxwell's first equation is simply

$$\nabla \times \vec{H} = \vec{j}$$  \hspace{1cm} (7)

where $\vec{H}$ is the magnetic field strength and $\vec{j}$ the current density. In addition $\vec{H}$ satisfies the equation

$$\nabla \cdot \vec{H} = 0$$  \hspace{1cm} (8)

Maxwell's second equation which connects the electric field intensity $\vec{E}$ with the change of the magnetic flux $\vec{B}$ is

$$\nabla \times \vec{E} = -\dot{\vec{B}}$$  \hspace{1cm} (9)
where

\[ \mathbf{B} = \mu_0 \mathbf{H} \]  \hspace{1cm} (10)

Ohm's law for a medium moving with the velocity \( \mathbf{V} \) is

\[ \mathbf{j} = \sigma ( \mathbf{E} + \mathbf{V} \times \mathbf{B} ) \]  \hspace{1cm} (11)

Considering the outer atmosphere as a perfect conductor \((\sigma \to \infty)\) this becomes

\[ \mathbf{E} = - \mathbf{V} \times \mathbf{B} \]  \hspace{1cm} (12)

The basic equation of hydrodynamics can be written in the form

\[ \rho \frac{d\mathbf{V}}{dt} = \mathbf{G} + \mathbf{j} \times \mathbf{B} \]  \hspace{1cm} (13)

where \( \rho \) is the density, \( \mathbf{V} \) the velocity, and \( \mathbf{G} \) the sum of all external, non-magnetic forces. The term \( \mathbf{j} \times \mathbf{B} \) is the mechanical force exerted by the magnetic field \( \mathbf{B} \) on a volume element carrying the current density \( \mathbf{j} \). \( \frac{d}{dt} \) represents the mobile operator

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \]  \hspace{1cm} (14)
It has been shown by Dungey (1954a,b) that for hydro-magnetic waves in the outer atmosphere the effect of viscosity which is usually the most important cause of attenuation, can be neglected when considering long period oscillations in the presence of the earth's magnetic field. Also Plumpton and Ferraro (1953) have shown that the periods of oscillation of a gravitating liquid star in the presence of a central magnetic pole and are not much reduced if the gravitational field is neglected. Since the magnetic pressure is much greater than the gas pressure and the pressure gradient due to the disturbance is small compared with the magnetic force, $\vec{G} = 0$ and equation (13) reduces to

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B}$$

These are the basic equations which will be applied to the problems of geomagnetic pulsations.

2. The equations of small magneto-hydromagnetic oscillations

To obtain the equations of small oscillations it is necessary to eliminate from the system all but one dependent vector variable. Because of the nonlinearity of the equations this will be impossible in general. However it proves possible in our case to obtain two coupled partial differential equations governing one component of each of the vectors $\vec{E}$ and $\vec{V}$. 
To this end one has to substitute equations (7) and (10) into equation (15) which yields

\[ \rho \frac{d\vec{V}}{dt} = -\mu_0 \vec{H} \times (\nabla \times \vec{H}) \]  

(16)

where \( \vec{H} \) is the sum of a constant field \( \vec{H}_0 \) and a small disturbance \( \vec{h} \) introduced through the motion of the medium. Since

\[ |\vec{H}_0| \gg |\vec{h}| \quad \text{and} \quad \nabla \times \vec{H}_0 = 0 \]

equation (16) becomes, under the assumption that \( |\vec{V}| \gg |(\nabla \cdot \vec{V})\vec{V}| \),

\[ \rho \frac{\partial \vec{V}}{\partial t} = -\mu_0 \vec{H}_0 \times (\nabla \times \vec{h}) \]  

(17)

Taking the vector product of this equation with \( \vec{B} = \mu_0 (\vec{H}_0 + \vec{h}) \) and taking the time derivative it follows that

\[ \rho \frac{\partial}{\partial t} (\frac{\partial \vec{V}}{\partial t} \times \vec{B}) = -\mu_0 \vec{H}_0 \times (\nabla \times \vec{h}) \times \vec{H}_0 \]  

(18)

or by virtue of equation (9)

\[ \rho \frac{\partial}{\partial t} (\frac{\partial \vec{V}}{\partial t} \times \vec{B}) = \mu_0 \vec{H}_0 \times (\nabla \times \nabla \times \vec{E}) \times \vec{H}_0 \]  

(19)
Carrying out the differentiation on the left-hand side of equation (19) one obtains

\[ \rho \frac{\partial}{\partial t} \left( \frac{\partial \vec{v}}{\partial t} \times \vec{B} \right) = \rho \left( \frac{\partial^2 \vec{v}}{\partial t^2} \times \vec{B} + \frac{\partial \vec{v}}{\partial t} \times \frac{\partial \vec{B}}{\partial t} \right) \]

However since \( \vec{E} = -\vec{V} \times \vec{B} \),

\[ \frac{\partial \vec{E}}{\partial t} = -\frac{\partial \vec{V}}{\partial t} \times \vec{B} - \nabla \times \frac{\partial \vec{B}}{\partial t} \quad ; \quad \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{\partial^2 \vec{V}}{\partial t^2} \times \vec{B} - 2 \frac{\partial \vec{V}}{\partial t} \times \frac{\partial \vec{B}}{\partial t} - \nabla \times \frac{\partial^2 \vec{B}}{\partial t^2} \]

Introducing the last expression into the expanded left-hand side of equation (19)

\[ \rho \frac{\partial}{\partial t} \left( \frac{\partial \vec{v}}{\partial t} \times \vec{B} \right) = -\rho \left( \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\partial \vec{V}}{\partial t} \times \frac{\partial \vec{B}}{\partial t} + \nabla \times \frac{\partial^2 \vec{B}}{\partial t^2} \right) \]

\[ = -\rho \left\{ \frac{\partial^2 \vec{E}}{\partial t^2} + \frac{\partial}{\partial t} \left( \nabla \times \frac{\partial \vec{B}}{\partial t} \right) \right\} = -\rho \frac{\partial^2 \vec{E}}{\partial t^2} + \rho \frac{\partial}{\partial t} \left( \nabla \times \nabla \times \vec{E} \right) \]

is obtained. Since \(|\vec{E}|\) is proportional \(|\vec{V}|\) the second term in the last expression is of the order \(V^2\) and can be neglected.

Hence

\[ \rho \frac{\partial}{\partial t} \left( \frac{\partial \vec{v}}{\partial t} \times \vec{B} \right) = -\rho \frac{\partial^2 \vec{E}}{\partial t^2} \]
Therefore equation (19) becomes

\[- \rho \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_o \mathbf{H}_o \times (\nabla \times \nabla \times \vec{E}) \times \mathbf{H}_o \quad (20)\]

i.e.

\[- \rho \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_o \left( \mathbf{H}_o \cdot \mathbf{H}_o \right) \nabla \times \nabla \times \vec{E} - \mu_o \left( \mathbf{H}_o \cdot \nabla \times \nabla \times \vec{E} \right) \mathbf{H}_o \quad (21)\]

which is a vector wave equation for \( \vec{E} \). Transforming equation (21) into cylindrical coordinates \((r, \theta, z)\), since \( \mathbf{H}_o = \{H_r, H_\theta, 0\} \), the second term on the right-hand side will vanish in the case of the z-component. Thus the z-component of equation (21) becomes after expanding \( \nabla \times \nabla \times \vec{E} \) in the cylindrical coordinates

\[- \rho \frac{\partial^2 E_z}{\partial t^2} = \mu_o \frac{H_z^2}{r} \left[ \left\{ \frac{\partial E_r}{\partial \phi} - \frac{\partial E_\theta}{\partial r} \right\} + r \frac{\partial^2 E_r}{\partial \phi \partial r} - r \frac{\partial^2 E_\theta}{\partial r^2} - \frac{1}{r} \frac{\partial E_r}{\partial \phi} + \frac{\partial^2 E_\phi}{\partial \phi^2} + \frac{\partial^2 E_\theta}{\partial \phi \partial \phi} \right] \quad (22)\]

Since from equation (12)

\[E_r = \mu_o V_\phi H_\theta \quad \text{and} \quad E_\phi = - \mu_o V_z H_r\]

equation (22) can be further written

\[\left\{ - \frac{\rho}{\mu_o H_o^2} \frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2} \frac{\partial^2}{\partial z^2} \right) \right\} E_z = \frac{\mu_o}{r} \left[ \frac{\partial(V_\phi H_\theta)}{\partial \phi} + \frac{\partial^2(V_\phi H_\theta)}{\partial \phi^2} + \frac{\partial^2(V_\phi H_\theta)}{\partial \phi \partial \phi} \right] \quad (23)\]
Again, since

\[ \frac{\partial (V_2 H_\varphi)}{\partial z} = H_\varphi \frac{\partial V_2}{\partial z} \]

\[ r \frac{\partial^2 (V_2 H_\varphi)}{\partial z \partial r} = r \left[ \frac{\partial V_2}{\partial z} \frac{\partial H_\varphi}{\partial r} + H_\varphi \frac{\partial^2 V_2}{\partial r \partial z} \right] \]

and

\[ \frac{\partial^2 (V_2 H_r)}{\partial z \partial \varphi} = \frac{\partial V_2}{\partial z} \frac{\partial H_r}{\partial \varphi} + H_r \frac{\partial^2 V_2}{\partial \varphi \partial z} \]

the right-hand side of equation (23) becomes

\[ \frac{\mu_0}{r} \left\{ \frac{\partial V_2}{\partial z} \left[ H_\varphi + r \frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial \varphi} \right] + r H_\varphi \frac{\partial^2 V_2}{\partial r \partial z} - H_r \frac{\partial^2 V_2}{\partial \varphi \partial z} \right\} \]  \hspace{1cm} (24)

Since the constant field \( H_0 \) obeys \( \nabla \times \vec{H}_0 = 0 \) which in cylindrical coordinates is equivalent to

\[ H_\varphi + r \frac{\partial H_\varphi}{\partial r} - \frac{\partial H_r}{\partial \varphi} = 0 \]  \hspace{1cm} (25)

equation (23) by virtue of expression (24) and equation (25) finally becomes

\[ \left\{ \frac{-P}{\mu_0 H_0^2 \frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^2} \frac{\partial^2}{\partial z^2}} \right\} E_2 = \frac{\mu_0}{r} \left\{ r H_\varphi \frac{\partial}{\partial r} - H_r \frac{\partial}{\partial \varphi} \right\} \left( \frac{\partial V_2}{\partial z} \right) \]  \hspace{1cm} (26)
This is the differential equation of poloidal oscillations and one notices that on the right-hand side the z-component of the velocity field appears. Therefore the term on that side represents the coupling between the two modes $E_z$ and $V_z$ which is necessary to describe the magneto-hydrodynamic problem in a unique way.

To obtain a time-space relationship for the z-component of the velocity field it is necessary to eliminate the vector of the magnetic disturbance $\vec{h}$ from the system represented by equations (7) to (15). To this end one has to consider equation (16) which, neglecting terms of higher order than the first, becomes

$$\frac{\rho}{\mu_0} \frac{\partial \vec{V}}{\partial t} = - \vec{H} \times (\nabla \times \vec{H})$$

(27)

Using the identity

$$\vec{H} \times (\nabla \times \vec{H}) = \frac{1}{2} \nabla H^2 - (\vec{H} \cdot \nabla) \vec{H}$$

(28)

and writing $\vec{H} = \vec{H}_0 + \vec{h}$ one obtains for the right-hand side of equation (27)

$$\vec{H} \times (\nabla \times \vec{H}) = \frac{1}{2} \nabla \{ H_0^2 + 2 (\vec{H}_0 \cdot \vec{h}) + h^2 \} - \{ (\vec{H}_0 + \vec{h}) \cdot \nabla \} (\vec{H}_0 + \vec{h})$$

$$= \frac{1}{2} \nabla H_0^2 + \nabla (\vec{H}_0 \cdot \vec{h}) + \frac{1}{2} \nabla h^2 - (\vec{H}_0 \cdot \nabla) \vec{H}_0 - (\vec{h} \cdot \nabla) \vec{H}_0 - (\vec{H}_0 \cdot \nabla) \vec{h} - (\vec{h} \cdot \nabla) \vec{h}$$
Neglecting higher terms in \( \hat{h} \) and applying the identity (28) equation (27) becomes

\[
\frac{\rho}{\mu_0} \frac{\partial \hat{v}}{\partial t} = (\hat{H}_o \cdot \nabla) \hat{h} + (\hat{h} \cdot \nabla) \hat{H}_o - \nabla(h_o \cdot h) \tag{29}
\]

Since the cylindrical coordinates for any two vectors \( \hat{A} \) and \( \hat{B} \)

\[
\{ (\hat{A} \cdot \nabla) \hat{B}_z \} = A \cdot \nabla B_z \quad \text{and} \quad \{ \hat{H}_o \}_z = 0
\]

the z-component of (29) finally becomes

\[
\frac{\rho}{\mu_0} \frac{\partial V_z}{\partial t} = \hat{H}_o \cdot \nabla h_z - \frac{\partial}{\partial z} (\hat{H}_o \cdot \hat{h}) \tag{30}
\]

To eliminate \( h_z \) in the last equation one has to substitute equation (12) and (10) into equation (9) and one obtains

\[
\frac{\partial \hat{h}}{\partial t} = \nabla \times (\nabla \times \hat{H}) \tag{31}
\]

Expanding the right-hand side yields

\[
\frac{\partial \hat{h}}{\partial t} = (\hat{H} \cdot \nabla) \hat{V} - (\nabla \cdot \nabla) \hat{H} - \hat{H} \nabla \cdot \hat{V} + \nabla \nabla \cdot \hat{H}
\]

which since \( \nabla \cdot \hat{H} = 0 \) becomes

\[
\frac{\partial \hat{h}}{\partial t} = (\hat{H} \cdot \nabla) \hat{V} - (\nabla \cdot \nabla) \hat{H} - \hat{H} \nabla \cdot \nabla \hat{V} \tag{32}
\]
Since \( H_0 \) is constant in time, \( H_z = 0 \), and \( |H_0| >> |h| \) the z-component of equation (32) is

\[
\frac{\partial h_x}{\partial t} = \vec{H}_0 \nabla V_z
\]  

(33)

To eliminate the last term on the right-hand side of (30) the \( r \)- and \( \phi \)- components of Maxwell's equation

\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}
\]

are used which after introducing (12) becomes

\[
\mu_0 \frac{\partial h_x}{\partial t} = -\mu_0 H_r \frac{\partial V_z}{\partial r} - \frac{1}{r} \frac{\partial E_z}{\partial \phi}
\]  

(34)

and

\[
\mu_0 \frac{\partial h_z}{\partial t} = -\mu_0 \frac{\partial V_z}{\partial z} + \frac{\partial E_z}{\partial r}
\]  

(35)

Multiplying equation (34) by \( H_r \) and equation (35) by \( H_\phi \) and adding yields, with \( H_r^2 + H_\phi^2 = H_0^2 \),

\[
\mu_0 \frac{\partial (H_r \vec{h})}{\partial t} = -\mu_0 H_0^2 \frac{\partial V_z}{\partial z} - \left( \frac{H_r}{r} \frac{\partial}{\partial \phi} - H_\phi \frac{\partial}{\partial r} \right) E_z
\]  

(36)

In order to obtain an equation which contains only \( E_z \) and \( V_z \), one differentiates equation (30) with respect to time and
substitutes equation (36), giving

\[
\frac{\rho}{\mu_0} \frac{\partial^2 V_z}{\partial t^2} = \nabla \cdot \mathbf{H}_0 + \nabla \times \mathbf{H}_0 \frac{\partial^2 V_z}{\partial z^2} + \frac{1}{\mu_0} \left( \frac{H_r}{r} \frac{\partial}{\partial \varphi} - H_y \frac{\partial}{\partial r} \right) \frac{\partial E_z}{\partial z} \tag{37}
\]

Substituting equation (33) finally yields

\[
\frac{\rho}{\mu_0} \frac{\partial^2 V_z}{\partial t^2} = (\nabla \cdot \mathbf{H}_0) (\nabla \cdot \mathbf{V}_z) + \nabla \times \mathbf{H}_0 \frac{\partial^2 V_z}{\partial z^2} + \frac{1}{\mu_0} \left( \frac{H_r}{r} \frac{\partial}{\partial \varphi} - H_y \frac{\partial}{\partial r} \right) \frac{\partial E_z}{\partial z} 
\]

which can be rewritten in the form

\[
\left\{ \frac{\rho}{\mu_0} \frac{\partial^2}{\partial t^2} - \left[ (\nabla \cdot \mathbf{H}_0) (\nabla \cdot \mathbf{V}_z) + H_0^2 \frac{\partial^2}{\partial z^2} \right] \right\} V_z = \frac{1}{\mu_0} \left( \frac{H_r}{r} \frac{\partial}{\partial \varphi} - H_y \frac{\partial}{\partial r} \right) \frac{\partial E_z}{\partial z} \tag{38}
\]

Equation (38) is for toroidal oscillations.

Equations (26) and (38), together with the boundary conditions, describe the behaviour of an infinitely conducting medium of cylindrical symmetry due to a disturbance.

Apart from the fact that these two equations are too complicated to be solved in their general form we are concerned here only with phenomena in the meridional plane.

For this reason the simplifying assumption \( \frac{\partial}{\partial z} = 0 \) is made, i.e. the two modes of oscillation are decoupled, and equations (26) and (38) become

\[
\left\{ -\frac{\rho}{\mu_0 H_0^2} \frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \right\} E_z = 0 \tag{39}
\]
It has been stated earlier that a knowledge of $E_z$ and $V_z$ is sufficient for the determination of the other field components. Assuming that the field quantities depend on $t$ through the factor $\exp(i\omega t)$, equation (9) yields

$$-i\omega\mu_0 h_r = \left\{ \nabla \times \vec{E} \right\}_r = \frac{i}{r} \frac{\partial E_z}{\partial \phi}$$

(41)

and

$$-i\omega\mu_0 h_\phi = \left\{ \nabla \times \vec{E} \right\}_\phi = -\frac{\partial E_z}{\partial r}$$

(42)

while for computing the components of $\vec{V}$, equation (17) gives

$$i\omega \rho V_r = -\mu_0 H_\phi \left\{ \nabla \times \vec{h} \right\}_z$$

(43)

and

$$i\omega \rho V_\phi = \mu_0 H_r \left\{ \nabla \times \vec{h} \right\}_z$$

(44)

However because of equations (9) and (21),

$$\left\{ \nabla \times \vec{h} \right\}_z = \frac{i}{\mu_0 \omega} \left\{ \nabla \times \nabla \times \vec{E} \right\}_z = \frac{i\omega \rho}{(\mu_0 H_0)^2} E_z$$
and the $r$- and $\dot{\phi}$- components of $\vec{V}$ reduce to

$$V_r = -\frac{H_\phi}{\mu_0 H_0} E_z$$

(45)

and

$$V_\phi = \frac{H_r}{\mu_0 H_0} E_z$$

(46)

If on the other hand the $z$-component of the velocity field is known then from equation (12)

$$E_r = \mu_0 H_0 V_z$$

(47)

and

$$E_\phi = -\mu_0 H_r V_z$$

(48)

while from equation (33)

$$i \omega h_z = H_r \frac{\partial V_z}{\partial r} + \frac{H_\phi}{r} \frac{\partial V_z}{\partial \phi}$$

(49)

It follows from equations (41) to (46) that for the poloidal mode of oscillation the following set of quantities are involved

$$(0, 0, E_z); (h_r, h_\phi, 0); (V_r, V_\phi, 0)$$
while for the toroidal mode of oscillations, equations (47) to (49) show a relationship between

\[(E_r, E_j, 0); (0, 0, h_z); (0, 0, V_z)\]

Hence from a knowledge of \(E_z\) and \(V_z\) the two vector fields \(\vec{E}\) and \(\vec{V}\) may be determined.

3. The equation of toroidal oscillations when the constant field is a dipole field

Considering the motion along a line of force given by

\[r = r_0 \sin^2 \theta\]  \( (50) \)

where \(\theta\) is the co-latitude and \(r_0\) the value at the furthest point (\(\theta = 90^\circ\)) the expression \((\vec{H}_o \cdot \nabla)\) is given by

\[(\vec{H}_o \cdot \nabla) = H_r \frac{\partial}{\partial r} + \frac{H_\theta}{r} \frac{\partial}{\partial \theta} = \frac{H_\theta}{r_o \sin^2 \theta} \frac{d}{d \theta} \]  \( (51) \)

Since the \(\theta\)-component of a dipole field is given by

\[H_\theta = \frac{M}{\mu_o r^3} \sin \theta = \frac{M}{\mu_o r_o^3 \sin^3 \theta} \]

where \(M\) is the magnetic dipole moment, equation (51) becomes

\[(\vec{H}_o \cdot \nabla) = \frac{M}{\mu_o r_0^3 \sin^2 \theta} \frac{d}{d \lambda}\]  \( (52) \)
and therefore

\[
(\mathbf{H}_0 \cdot \nabla)(\mathbf{H}_0 \cdot \nabla) = \frac{\mathbf{M}^2}{\mu_0 r_0^3 \sin^2 \gamma} \frac{d}{d \gamma} \left[ \frac{1}{\sin^2 \gamma} \frac{d}{d \gamma} \right]
\]  

(53)

Taking the time dependance in equation (40) to be of the form \(\exp(i \omega t)\) the equation of toroidal oscillations in the presence of a dipole field becomes

\[
\frac{1}{\sin^2 \gamma} \frac{d}{d \gamma} \left[ \frac{1}{\sin^2 \gamma} \frac{dV_z}{d \gamma} \right] + \frac{\mu_0 r_0^3 \rho \omega^2}{\mathbf{M}^2} V_z = 0
\]

(54)

It is frequently convenient to express lengths in units of the Earth's radius, and one thus writes \(r_0 = \gamma_0 a\). Remembering that the magnetic dipole moment of the Earth is given by

\[
\mathbf{M} = \mathbf{H}_0 a^3
\]

where \(a\) is the radius of the Earth and \(\mathbf{H}_0\) the maximum value of the magnetic field intensity at the magnetic equator, equation (54) finally becomes

\[
\frac{1}{\sin^2 \gamma} \frac{d}{d \gamma} \left[ \frac{1}{\sin^2 \gamma} \frac{dV_z}{d \gamma} \right] + \frac{\mu_0 a^2 \gamma_0^2 \rho \omega^2}{H_0^2} V_z = 0
\]

(55)

This is an ordinary homogeneous differential equation of the second order. Associating the periods of geomagnetic micropulsations with the eigenvalues of equation (55) one has to find \(\omega^2\) such that \(V_z\) satisfies the boundary conditions
at the ends of the interval of integration.

It will prove very convenient in comparing results to incorporate the charge density distribution \( \rho \) in \( \omega^2 \) by putting \( \rho \omega^2 = \bar{\omega}^2 \). In this case equation (55) becomes

\[
\frac{1}{\sin^2 \vartheta} \frac{d}{d \vartheta} \left[ \frac{1}{\sin^2 \vartheta} \frac{d V_z}{d \vartheta} \right] + \frac{\mu_0 a^2 \nu_s^2 \omega_\star^2}{H_0^2} V_z = 0. \tag{56}
\]

However this is only legitimate if the charge density distribution along the line of force is constant and not a function of the co-latitude \( \vartheta \).

4. The equation of toroidal oscillations in the case of a compressed dipole field

Parker (1958) has considered in some detail the interaction of the "solar wind" with the Earth's geomagnetic field. The solar wind is the name given to the outward streaming of gas in all directions from the sun with velocities in the range 500-1500 km/sec. The solar wind will compress or sweep away the outer geomagnetic field down to a level where the energy density of the field is equal to the kinetic energy density of the solar wind. This results in a deformation of the dipole field which under these circumstances may as a first approximation be considered as being confined to a spherical cavity of radius \( R_0 \).

Postulating that the r-component of the magnetic field
vanishes at \( r = R_0 \), it is shown in appendix I that

\[
H_r = -\frac{M}{\mu_0} \left( \frac{2}{r^3} - \frac{2}{R_0^3} \right) \cos \varphi \quad (57)
\]

and

\[
H_\varphi = \frac{M}{\mu_0} \left( \frac{1}{r^3} + \frac{2}{R_0^3} \right) \sin \varphi \quad (58)
\]

Using the differential equation of the lines of force, viz.,

\[
\frac{dr}{H_r} = \frac{r d\varphi}{H_\varphi} \quad (59)
\]

it is further shown in appendix I that the lines of force of a compressed dipole field are the solutions of the cubic equation

\[
r^3 + \left( \frac{a^3}{R_0^3} \right) \frac{\sin^2 \varphi}{\sin^2 \varphi_0} \quad r = R_0^3 \quad (60)
\]

where \( a \) is the radius of the Earth, \( R_0 \) the radius of the confining cavity and \( \varphi_0 \) the co-latitude of the intersection with the Earth of the line of force which touches the cavity in the equatorial plane. Introducing the radius of the Earth as the unit of length equation (60) becomes

\[
\psi^3 + (\alpha^3 - 1) \frac{\sin^2 \varphi_0}{\sin^2 \varphi} \psi = \alpha^3 \quad (61)
\]
where

\[ \psi_c = \frac{r}{\alpha} \quad \text{and} \quad \alpha = \frac{R_o}{\alpha} \]

By letting \( \alpha \) tend to large values equation (61) becomes

\[ \frac{\psi_c^3}{\alpha^3} + \frac{\sin^2 \psi_c}{\sin^2 \phi} \psi_c = 1 \]

which finally in the limit \( \alpha \to \infty \) becomes

\[ \psi_c = \frac{\sin^2 \phi}{\sin^2 \phi} = \psi_o \sin^2 \phi \]

which is the equation of the lines of force for the ordinary dipole field expressed in units of earth radii.

Using a digital computer \( \psi_c \) has been evaluated from equation (61) as a function of the co-latitude \( \phi \) for several values of \( \alpha \). The results are plotted in fig. 2-4.

To consider the effect of a compressed dipole field with field components characterized by equations (57) and (58) one has to go back to the operator \( \nabla \cdot \mathbf{H}_o \) which along a line of force is given by

\[ \mathbf{H}_o \cdot \nabla = \frac{H_o}{r} \frac{d}{d\phi} \]

Introducing for \( H_o \) the expression (58) one obtains
Field pattern of a magnetic dipole
Fig 2.
field pattern of a compressed magnetic dipole
(size of cavity 12 Earth radii)
field pattern of a compressed magnetic dipole

(size of cavity 8 Earth radii)
Fig. 4

Field pattern of a compressed magnetic dipole

(size of cavity 4 Earth radii)
\[ (\vec{H}_0 \cdot \nabla) = \frac{M}{\mu_0} \left( \frac{1}{r^4} + \frac{2}{r R_0^3} \right) \sin \vartheta \frac{d}{d \vartheta} \]

and hence

\[ (\vec{H}_0 \cdot \nabla)(\vec{H}_0 \cdot \nabla) = \frac{M^2}{\mu_0^2} \left( \frac{1}{r^4} + \frac{2}{r R_0^3} \right) \sin \vartheta \frac{d}{d \vartheta} \left[ \left( \frac{1}{r^4} + \frac{2}{r R_0^3} \right) \sin \vartheta \frac{d}{d \vartheta} \right] \quad (62) \]

Introducing this result into equation (40) yields

\[ \left( \frac{1}{r^4} + \frac{2}{r R_0^3} \right) \sin \vartheta \frac{d}{d \vartheta} \left[ \left( \frac{1}{r^4} + \frac{2}{r R_0^3} \right) \sin \vartheta \frac{d V_z}{d \vartheta} \right] + \frac{\mu_0 \rho \omega^2}{M^2} V_z = 0 \quad (63) \]

Taking the radius of the Earth as the unit of length equation (63) can be written

\[ \left( \frac{1}{\gamma_c^4} + \frac{2}{\gamma_c \alpha^3} \right) \sin \vartheta \frac{d}{d \vartheta} \left[ \left( \frac{1}{\gamma_c^4} + \frac{2}{\gamma_c \alpha^3} \right) \sin \vartheta \frac{d V_z}{d \vartheta} \right] + \frac{\mu_0 \alpha^2 \rho \omega^2}{H_0^2} V_z = 0 \quad (64) \]

letting \( \alpha \) tend to infinity and remembering then that \( \gamma_c \to \gamma \) equation (64) reduces to (56).
CHAPTER III

SOLUTIONS

1. Solution for the normal dipole field with constant charge density

As pointed out previously the differential equation of toroidal oscillations in the case of a constant charge density distribution may be written

\[ \frac{1}{\sin \gamma} \frac{d}{d\gamma} \left[ \frac{1}{\sin \gamma} \frac{d}{d\gamma} V_\gamma \right] + \frac{\mu_0 \alpha^2 \psi_0^2 \omega^2}{\mu_0^2} V_\gamma = 0 \]  \hspace{1cm} (56)

Changing the independent variable by putting

\[ \sin \gamma \, d\gamma = d\chi \]  \hspace{1cm} (65)

equation (56) becomes

\[ \frac{\alpha^2 V_\gamma}{d\chi^2} + \frac{\mu_0 \alpha^2 \psi_0^2 \omega^2}{\mu_0^2} V_\gamma = 0 \]  \hspace{1cm} (66)

for which a solution is easily found.

Since one is particularly interested in finding the
eigenvalues of this equation it is necessary to specify the boundary conditions which most likely apply to the physical situation.

It has been shown by Dungey (1954) that hydromagnetic waves having long periods are almost perfectly reflected when they hit the ionosphere. Considering the fact that the ionosphere is very close to the surface of the Earth compared with the distances over which the lines of force extend one can assume without introducing much inaccuracy that the point of reflection is situated at the surface of the Earth. Therefore

\[ E_r = E_\varphi = 0 \quad \text{at} \quad \varphi = \varphi_0 \]

From equation (12) it then follows that

\[ \nabla_z = 0 \quad \text{at} \quad \varphi = \varphi_0 \]

To find the corresponding value of \( x \) equation (65) must be integrated, i.e.

\[
\int_{X_0}^{X_1} dx = \int_{\varphi_0}^{\varphi_1} \sin \varphi d\varphi
\]

where \( \varphi_1 = 180 - \varphi_0 \). Since the integrand on the right-hand side is symmetrical about the point \( \varphi = \pi/2 \) this
becomes

\[
\int_{x_0}^{\pi/2} d\gamma = \Delta x = 2 \int_{\gamma_0}^{\pi/2} \sin \gamma d\gamma
\]

which after integration gives

\[
\Delta x = \frac{2}{35} \cos \gamma_0 \left( 5 \sin^6 \gamma_0 + 6 \sin^4 \gamma_0 + 8 \sin^2 \gamma_0 + 16 \right)
\]

Equation (67) associates the interval from \( \gamma_0 \) to \( \gamma_1 \) with an interval extending from \( x_0 \) to \( x_1 \). Letting the point \( \gamma = \pi/2 \) correspond to \( x = 0 \) it follows that the interval \( \gamma_0 \) to \( \gamma_1 = 180^\circ - \gamma_0 \) corresponds to the interval from \(-x_0\) to \(+x_0\). Hence

\[
\chi_0 = -\frac{1}{35} \cos \gamma_0 \left( 5 \sin^6 \gamma_0 + 6 \sin^4 \gamma_0 + 8 \sin^2 \gamma_0 + 16 \right)
\]

Using the Alwac III E this expression has been computed for \( 0 \leq \gamma_0 \leq 90 \) in steps of 5° and the result is given in table 3. Fig. 5 shows the plot of \( x_0 \) against \( \gamma_0 \).

The solution of equation (66) is then

\[
V_2 = A \cos \left( \frac{\alpha \psi_0^4 \omega \sqrt{\mu_0}}{H_0} X \right)
\]

and

\[
V_2 = B \sin \left( \frac{\alpha \psi_0^4 \omega \sqrt{\mu_0}}{H_0} X \right)
\]
Table 3

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Because of the boundary condition it follows from equations (69a) and (69b) that

$$\frac{a\sqrt{\mu_o}}{H_o} \nu_0^4 \omega |X_o| = n \frac{\pi}{2}$$  \hspace{1cm} (70)

With $a = 6.371 \times 10^6$ (m), $\nu_o = 1.256 \times 10^{-6}$ (voltsec/amp. meter), $H_o = 0.312 \times 10^{-4}$ (voltsec/m²)

$$\frac{a\sqrt{\mu_o}}{H_o} = 2.288 \times 10^8 \left[ \frac{\nu_0}{\omega n \gamma_2} \right]$$

Equation (70) can thus be written
Fig. 5

\[ |X| = \frac{1}{15} \cos \frac{\theta_0}{2} (\sin \theta_0^2 + 6 \sin \theta_0^2 + 8 \sin^2 \theta_0 + 16) \]
\[ \tilde{\omega} = \frac{n \pi}{\sqrt{5.376 \times 10^6 x x_x^y |X_0|}} \]  

(71)

where \(|X_0|\) is given by equation (68) and

\[ \psi_0 = \frac{1}{\sin^2 \psi_0} \]

In equation (71) the odd values of \(n\) correspond to a cosine oscillation while even values of \(n\) are associated with a sine oscillation.

Since \(\bar{\omega} = \omega \sqrt{\rho} \times \left(\frac{2\pi}{T}\right) \sqrt{\rho}\), one obtains for the fundamental eigenperiod (n = 1)

\[ T_1 = 9.152 \times 10^8 \times \psi_0^4 \times |X_0| \times \sqrt{\rho}^{-1} \text{ [sec]} \]  

(72)

The eigenperiods of the oscillating lines of force have been calculated from equation (72) and plotted in fig. 6 using a constant charge density distribution of \(6.5 \times 10^{-19}\) (kg/m\(^3\)). The neutral particle density has not been included in \(\rho\) because the mean free path is so long that only charged particles will contribute to the hydromagnetic wave motion.
Table 4

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<td>50°</td>
<td>3.074</td>
</tr>
<tr>
<td>15°</td>
<td>1.896x10^4</td>
<td>55°</td>
<td>1.727</td>
</tr>
<tr>
<td>20°</td>
<td>2038</td>
<td>60°</td>
<td>1.036</td>
</tr>
<tr>
<td>25°</td>
<td>375.0</td>
<td>65°</td>
<td>0.651</td>
</tr>
<tr>
<td>30°</td>
<td>97.61</td>
<td>70°</td>
<td>0.418</td>
</tr>
<tr>
<td>35°</td>
<td>31.95</td>
<td>75°</td>
<td>0.267</td>
</tr>
<tr>
<td>40°</td>
<td>12.97</td>
<td>80°</td>
<td>0.157</td>
</tr>
<tr>
<td>45°</td>
<td>5.972</td>
<td></td>
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</tr>
</tbody>
</table>

2. Solution for the normal dipole field with variable charge density

In the preceding section the eigenperiods of the toroidal oscillations of a conducting medium in the presence of a dipole were obtained assuming a constant charge density distribution. Because of this simplification it was possible to obtain the eigenperiods without making any approximations. However, in general, the assumption of a constant charge density will not be the case. Under these circumstances one has to go back to equation (54) where $\rho$ is a function of the distance from the Earth's surface. Since equation (54) governs the oscillations along the lines of force $\rho$ will be a function of both $r_o$ and $\phi$. For the calculations the
Fig. 6.

Fundamental period, constant charge density normal dipole field.

Dungey's values
charge density distribution which was proposed by Dessler (1958) in his calculations of the propagation velocity of sudden commencements was used. This distribution is shown in fig. 7.

To obtain the eigenvalues $\omega$ of equation (54) one has to resort to numerical methods. The differential equation is replaced by finite differences and the problem is then to find the eigenvalues of a matrix.

To this end one writes equation (54) in the form

$$\frac{d^2 V_z}{dx^2} + G(x, \omega^2) V_z = 0 \quad (73)$$

where the independent variable has been changed using the transformation (68) and where

$$G(x, \omega^2) = \frac{\mu_0 \tau \rho(x)}{M^2} \omega^2 \quad (74)$$

As in the previous section one takes vanishing values of $V_z$ at the boundaries $x_0$ and $x_1$. In order to apply algebraic methods to the problem one divides the interval $(x_0, x_1)$ into $n$ parts each of length

$$h = \frac{x_1 - x_0}{n} = \frac{2 |x_0|}{n}$$
Charge Density Distribution as function of height above the Earth's surface

Fig. 7
The last form is possible because of the symmetry about the point \( x = 0 \). The second derivative in equation (73) at points inside \((x_0, x_1)\) is replaced by a difference expression resulting from a truncated power series, viz.,

\[
\frac{d^2 V_z (\alpha)}{d \alpha^2} = \frac{V_z (\alpha - h) - 2 V_z (\alpha) + V_z (\alpha + h)}{h^2} \tag{75}
\]

The differential equation can then be approximated by the following linear and homogeneous recursion relation,

\[
V_z (\alpha - h) - 2 V_z (\alpha) + V_z (\alpha + h) + h G (\alpha, \omega^2) V_z (\alpha) = 0 \tag{76}
\]

in which terms of the order \( h^4 \) are ignored. This recursion formula has to hold at \( n-1 \) interior points resulting in \( n-1 \) simultaneous linear equations. Starting at the left-hand end of the interval, the following system of equations results

\[
V_z (x_0) - 2 V_z (x_0 + h) + V_z (x_0 + 2h) + h^2 G (x_0 + h, \omega^2) V_z (x_0 + h) = 0
\]

\[
V_z (x_0 + h) - 2 V_z (x_0 + 2h) + V_z (x_0 + 3h) + h^2 G (x_0 + 2h, \omega^2) V_z (x_0 + 2h) = 0
\]

With the boundary conditions at \( x_0 \) and \( x_1 \) it follows that for a non-trivial solution of this system of linear equations
must have

\[ \begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & A_2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & A_3 & 1 & 0 & \cdots \\ 0 & 0 & \cdots & A_{n-1} \end{vmatrix} = 0 \]  

(77)

with

\[ A_j = \hbar^2 G (X_0 + jh, \omega^2) - 2. \]

which, because of the relation (74), becomes

\[ A_j = \hbar^2 \frac{\mu_0 r_s^8 \rho(x_0 + jh)}{M^2} \omega^2 - 2 \]

(78)

This may be written in abbreviated form

\[ A_j = a_j \omega^2 - 2 \]

(79)

where because of the dependence on \( \rho(\xi) \), the \( a_j \) are functions of \( \xi \) along a line of force.

Inserting the last expression into equation (77) the problem can be considered to be an eigenvalue problem of the following kind
\[ a_1 \omega^{2-2} \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots \\ 1 & a_2 \omega^{2-2} & 1 & 0 & \cdots \\ 0 & 1 & a_3 \omega^{2-2} & 1 & 0 & \cdots \\ \end{pmatrix} = 0 \] (80)

which is identical to the matrix problem

\[ A \omega^2 - B = 0 \]

where \( A \) and \( B \) are the matrices

\[
A = \begin{pmatrix}
  a_1 & 0 & 0 & 0 & \cdots \\
  0 & a_2 & 0 & 0 & \cdots \\
  0 & 0 & a_3 & 0 & \cdots \\
  \vdots & & & & & \\
  0 & & & & a_n
\end{pmatrix} \quad (81)
\]

and

\[
B = \begin{pmatrix}
  2 & -1 & 0 & 0 & 0 & \cdots \\
  -1 & 2 & -1 & 0 & 0 & \cdots \\
  0 & -1 & 2 & -1 & 0 & \cdots \\
  \vdots & & & & & \\
  0 & & & & -1 & 2
\end{pmatrix} \quad (82)
\]
Thus in matrix notation the problem may be written

\[ B \mathbf{X} = \omega^2 A \mathbf{X} \quad (83) \]

Since this equation is not in the usual form suitable for diagonalizing a transformation has to be performed. As shown in appendix III equation (83) may be written

\[ \mathbf{M} \mathbf{Y} = \omega^2 \mathbf{Y} \quad (84) \]

Because of the symmetry of \( \mathbf{M} \) the eigenvalues are real as would be expected from the nature of the problem.

The next step is to compute the values of \( a_j \) in equation (79) which according to equation (78) will depend on \( r_0 \), i.e. the line of force, and on the charge density distribution which here appears as a function of \( x_j = x_0 + jh \). However this distribution as already mentioned is given as a function of altitude above the surface of the Earth. Since furthermore the altitude of a particular point above the Earth's surface along a certain line of force is given by

\[ H = r_0 \sin^2 \varphi - \alpha \quad (85) \]

it follows that for a given \( x_j \) the corresponding value of \( \sin^2 \varphi \) has to be calculated. To accomplish this one returns
to equation (68) which for an arbitrary $x$ may be written

$$x_0 = -\frac{1}{35} (1 - S)^\frac{1}{2} \left\{ 5S^3 + 6S^2 + 8S + 16 \right\}$$  \hspace{1cm} (86)

where $S = \sin^2 \varphi \hspace{0.5cm} (0 \leq S \leq 1)$. Because of the symmetry about the point $x = 0$, i.e. $\varphi = \pi/2$, equation (86) need be solved for one half of the interval only. Choosing the left-hand half of the interval where $x < 0$, one obtains from equation (86)

$$|x_0| - \hat{j}h = -\frac{1}{35} (1 - S)^\frac{1}{2} \left\{ 5S^3 + 6S^2 + 8S + 16 \right\}$$  \hspace{1cm} (87)

where $1 \leq \hat{j} \leq \frac{1}{2} (n-2)$, $n$ being an even number of intervals. Since for each different line of force $|x_0|$ as well as $h$ are different it is obvious that equation (87) has to be solved a great number of times. For this reason a programme was written for the Alwac III E using the Newton-Raphson iteration method for solving this equation with respect to $s$ for arbitrary values of the left-hand side, i.e. for values in the range $0 < |x_0| - \hat{j}h < |x_0|$. A short description of the method as applied to this particular problem is given in appendix IV. Table (III-a) gives the solutions of equation (87) for the particular value of $|x_0| - \hat{j}h$ needed in our case. Since $s = \sin^2 \varphi$, the altitude $H$ above the surface of the earth is readily calculated (Table III-b).
With these values the charge density may be determined.

Once the relation between $\rho(|x_0| - jh)$ and $\rho(r)$ has been found for each value of the argument, values of $a_j$ may be readily obtained from the equation

$$a_j = \frac{\hbar^2 \mu_0 r^6 \rho(|x_0| - jh)}{M^2} \quad (88)$$

where $\mu_0 = 1.256 \times 10^{-6}$ (volt-sec/amp-meter), $M^2 = 65.028 \times 10^{30}$ (volt$^2$·sec$^2$·meter$^2$) and $r_0 = a/sin^2 \omega_0$

In calculating the values of $a_j$ one notes the symmetry of $H$ (equation 85) about the center point $\omega = \pi/2$ ($|x_0| - jh = 0$) For this reason $\rho(|x_0| - jh)$ will also be symmetric about the same point, which implies that the number of points for which $\rho(|x_0| - jh)$ and therefore the expression (88) have to be calculated, is approximately halved. Since in the present case it was found sufficient to subdivide the entire interval into eight parts, one has to compute values of $a_j$ at three points and the center point of the interval only. The resulting values of $A_j$ are compiled in table 5 - from them one may calculate the elements of the matrix $M$ as given in appendix II. These matrices which in the present case are of order seven have been diagonalized with the help of the Alwac III E using the Jacobi method. The matrices as well as their eigenvalues are given in appendix IV. The periods of the fundamental and the next higher mode are given in table 6.
Table 5

<table>
<thead>
<tr>
<th>$\theta_o$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>14.039x10^5</td>
<td>14.039x10^5</td>
<td>14.039x10^5</td>
<td>14.039x10^5</td>
</tr>
<tr>
<td>20°</td>
<td>1.6230x10^4</td>
<td>1.6230x10^4</td>
<td>1.6230x10^4</td>
<td>1.6230x10^4</td>
</tr>
<tr>
<td>25°</td>
<td>6.0841x10^2</td>
<td>5.6616x10^2</td>
<td>5.4926x10^2</td>
<td>5.4926x10^2</td>
</tr>
<tr>
<td>30°</td>
<td>6.5847x10^1</td>
<td>5.4395x10^1</td>
<td>5.1532x10^1</td>
<td>5.0960x10^1</td>
</tr>
<tr>
<td>35°</td>
<td>6.6500x10</td>
<td>6.0166x10</td>
<td>5.7000x10</td>
<td>5.5733x10</td>
</tr>
<tr>
<td>40°</td>
<td>1.3394x10</td>
<td>1.1878x10</td>
<td>1.1373x10</td>
<td>1.1221x10</td>
</tr>
<tr>
<td>45°</td>
<td>3.7511x10^-1</td>
<td>3.1081x10^-1</td>
<td>2.9473x10^-1</td>
<td>2.8937x10^-1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\theta_o$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50°</td>
<td>1.5053x10^-1</td>
<td>1.1076x10^-1</td>
<td>1.0082x10^-1</td>
<td>0.97985x10^-1</td>
</tr>
<tr>
<td>55°</td>
<td>12.005x10^-2</td>
<td>5.5187x10^-2</td>
<td>4.6587x10^-2</td>
<td>4.4795x10^-2</td>
</tr>
<tr>
<td>60°</td>
<td>9.3571x10^-1</td>
<td>0.70985x10^-1</td>
<td>0.34202x10^-1</td>
<td>0.30330x10^-1</td>
</tr>
<tr>
<td>65°</td>
<td>5.0936x10</td>
<td>5.7303x10</td>
<td>2.1648x10</td>
<td>1.6554x10</td>
</tr>
<tr>
<td>70°</td>
<td>1.4209x10^-1</td>
<td>0.3157x10^-1</td>
<td>0.14472x10^-1</td>
<td>0.11314x10^-1</td>
</tr>
<tr>
<td>75°</td>
<td>1.2196x10^-1</td>
<td>0.83446x10^-1</td>
<td>0.53491x10^-1</td>
<td>0.44933x10^-1</td>
</tr>
<tr>
<td>80°</td>
<td>2.6061x10^-2</td>
<td>3.8719x10^-2</td>
<td>4.2144x10^-2</td>
<td>4.2442x10^-2</td>
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</table>
Table 6

<table>
<thead>
<tr>
<th>$\phi$</th>
<th>$T_1$</th>
<th>$T_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>$1.907\times10^4$ sec.</td>
<td>$0.9727\times10^4$ sec.</td>
</tr>
<tr>
<td>20°</td>
<td>$2.052\times10^3$</td>
<td>$1.046\times10^3$</td>
</tr>
<tr>
<td>25°</td>
<td>$3.804\times10^2$</td>
<td>$1.965\times10^2$</td>
</tr>
<tr>
<td>30°</td>
<td>$1.174\times10^2$</td>
<td>$6.184\times10^1$</td>
</tr>
<tr>
<td>35°</td>
<td>$3.884\times10^1$</td>
<td>$2.028\times10^1$</td>
</tr>
<tr>
<td>40°</td>
<td>$1.735\times10^1$</td>
<td>9.049</td>
</tr>
<tr>
<td>45°</td>
<td>8.871</td>
<td>4.673</td>
</tr>
<tr>
<td>50°</td>
<td>5.254</td>
<td>2.837</td>
</tr>
<tr>
<td>55°</td>
<td>3.759</td>
<td>2.235</td>
</tr>
<tr>
<td>60°</td>
<td>6.513</td>
<td>5.370</td>
</tr>
<tr>
<td>65°</td>
<td>$1.556\times10^1$</td>
<td>$1.265\times10^1$</td>
</tr>
<tr>
<td>70°</td>
<td>$2.908\times10^1$</td>
<td>$2.188\times10^1$</td>
</tr>
<tr>
<td>75°</td>
<td>$4.104\times10^1$</td>
<td>$2.440\times10^1$</td>
</tr>
<tr>
<td>80°</td>
<td>$3.230\times10^1$</td>
<td>$1.574\times10^1$</td>
</tr>
</tbody>
</table>

At this point the accuracy of the approximations should be considered - in other words was the number of intervals great enough to yield the lowest eigenvalue even in the most unfavorable case with sufficient accuracy? With a fixed
Fig. 8.

Fundamental period, variable charge density normal dipole field.
number of intervals the most unfavorable situation occurs
when the entire interval has its greatest value since then $|h|$ has its greatest value too which means that the truncation error is a maximum. To answer this question one goes back to the case of constant $\rho$ which permitted a solution of equation (54) in closed form. In this case the determinant (77) by virtue of equation (78) can be written in the form

$$
\begin{vmatrix}
\gamma^2 - 2 & 1 & 0 & 0 & \cdots \\
1 & \gamma^2 - 2 & 1 & 0 & \cdots \\
0 & 1 & \gamma^2 - 2 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{vmatrix} = 0 \quad (89)
$$

where

$$
\gamma^2 \equiv h^2 \frac{\mu \rho}{r_b} \frac{\omega^2}{M^2} \quad (90)
$$

It is shown in appendix $\Xi$ that $\gamma^2$ which has to satisfy the determinantal equation (89) is given by

$$
\gamma^2_k = 2 + 2 \cos \frac{\pi k}{M+1} \quad (91)
$$

where $M$ is the order of the determinant and $k = 1, 2, \cdots, M$. Since in our case $M = 7$ the following seven values of $\gamma^2_k$
are obtained, viz.,

\[
\begin{align*}
\gamma_1^z &= 3.84776 & \gamma_0^z &= 2.00000 \\
\gamma_2^z &= 3.41422 & \gamma_5^z &= 1.23464 \\
\gamma_3^z &= 2.76536 & \gamma_6^z &= 0.58578 \\
\gamma_7^z &= 0.15224
\end{align*}
\]

The most unfavorable situation occurs when \( \gamma_0 = 15^\circ \). In this case

\[
\gamma^z = h^2 \frac{M_0 r^9 \rho}{M^2} \omega^2 = 14.039 \times 10^5 \omega^2
\]

(92)

from table 5 where \( \rho \) was taken to be \( 6.5 \times 10^{-19} \) (kg/m\(^3\)).

Hence for the lowest and next higher eigenvalue one obtains

\[
\omega_1^2 = 10.84 \times 10^{-8} \text{ (sec}^{-2}\text{)} \text{ and } \omega_2^2 = 41.72 \times 10^{-8} \text{ (sec}^{-2}\text{)}.
\]

On the other hand in the case of constant \( \rho \), the lowest and next higher eigenvalue can be obtained from equation (71) which, with \( \rho = 6.5 \times 10^{-19} \) (kg/m\(^3\)), yields

\[
\omega_1^2 = 10.98 \times 10^{-8} \text{ (sec}^{-2}\text{)} \text{ and } \omega_2^2 = 43.92 \text{ (sec}^{-2}\text{)}.
\]

Thus the difference between the approximative solution and the exact solution is within reasonable limits. This difference
becomes smaller as \( \phi \) increases since the number of intervals is kept constant which implies that \( h^2 \) becomes smaller. It therefore can be concluded that subdividing the entire interval into eight parts yields sufficient accuracy.

Using the values of \( \gamma_k \) given by equation (91) and diagonalizing the matrices in the case of constant \( \rho \) one can check the accuracy of the computer calculations by comparing the results obtained from combining equations (91) and (92) with those obtained from matrix diagonalization. This was possible only for the cases \( \phi = 15^\circ \) and \( \phi = 20^\circ \) since then \( a_j \) is essentially constant. It was found that the agreement was very good if the off-diagonal elements did not exceed 0.00005.

3. Solution for the compressed dipole field with constant charge density

In the preceding two sections the eigenperiods of toroidal oscillations were obtained under the assumption that the magnetic field was a dipole field. As pointed out earlier this is not strictly true. For this reason in this section the eigenperiods of toroidal oscillations are studied under the assumption that the dipole field has been deformed, the extent of the deformation being governed by the radius of the cavity in which the field is confined.

The differential equation which governs the motion under
these circumstances was derived in chapter II-4 and is

\[
\left( \frac{1}{V_c^4} + \frac{2}{V_c^2} \right) \sin \theta \left( \frac{dV_z}{d\theta} \right) + \sin \theta \left( \frac{dV_z}{d\theta} \right) + \frac{\mu_0 \alpha^2 \rho \omega^2}{H_o^2} V_z = 0 \tag{64}
\]

where \( V_c \) is obtained from the equation

\[
V_c^3 + (\alpha^3 - 1) \frac{\sin^2 \phi}{\sin^2 \phi} V_c = \alpha^3 \tag{61}
\]

Before computing the eigenvalues of equation (64) we will obtain an approximate solution. As may be seen from the plots of the magnetic lines of force in figures 2-4 \( V_c \) very soon approaches a constant value for lines of force intersecting the Earth at very low co-latitudes (small \( \phi_0 \)). The limiting value taken in this case is approximately the radius of the cavity \( \alpha \). Therefore with \( V_c = \alpha \) equation (64) becomes

\[
\frac{3 \sin \theta}{\alpha^4} \left( \frac{d}{d\theta} \right) \left[ \frac{3 \sin \theta}{\alpha^4} \left( \frac{dV_z}{d\theta} \right) \right] + \frac{\mu_0 \alpha^2 \rho \omega^2}{H_o^2} V_z = 0 \tag{93}
\]

which may be written in the form

\[
\frac{1}{\csc \theta} \left( \frac{d}{d\theta} \right) \left[ \frac{1}{\csc \theta} \left( \frac{dV_z}{d\theta} \right) \right] + \frac{\mu_0 \alpha^2 \rho \omega^2}{g H_o^2} V_z = 0 \tag{94}
\]

In order to reduce equation (94) to a simpler form one may change, as has been done before, the independent variable by
putting

$$\csc \varphi \, d \varphi = d \chi$$  \hspace{1cm} (95)

Then equation (94) becomes

$$\frac{d^2 \varphi_z}{d \chi^2} + \frac{\mu \alpha^2 \phi \omega^2}{g \bar{H}^2} \varphi_z = 0$$ \hspace{1cm} (96)

As before one assumes perfect reflection at the end points as the boundary conditions, viz.:

$$\varphi_z = 0 \quad \text{at} \quad \varphi = \varphi_o$$

The relation between \( \varphi_o \) and \( x_o \) is found by integrating equation (95), i.e.

$$\int_{x_0}^{x_1} d \chi = \int_{\varphi_0}^{\varphi_1} \csc \varphi \, d \varphi$$

where, as before, \( \varphi_1 = 180^\circ - \varphi_0 \). Because of symmetry this becomes

$$\int_{x_0}^{x_1} d \chi = \Delta x = 2 \int_{\varphi_0}^{\pi_2} \csc \varphi \, d \varphi$$

or after integrating
\[ \Delta X = -2 \log \tan \frac{\theta_0}{2} \] 

(97)

Since \( \theta_0 < \frac{\pi}{2} \), \( \Delta X \) will be a positive quantity as it was in equation (67). Therefore equation (97) associates the interval from \( \theta_0 \) to \( \theta_i \) with an interval extending from \( x_0 \) to \( x_1 \). By similar reasoning to that given in section 1 it follows that

\[ x_0 = \log \tan \frac{\theta_0}{2} \] 

(98)

The solution of equation (96) is

\[ V_z = A \cos \left( \frac{\alpha \chi \omega \sqrt{\mu_0}}{3 H_0} x \right) \] 

(99a)

and

\[ V_z = B \sin \left( \frac{\alpha \chi \omega \sqrt{\mu_0}}{3 H_0} x \right) \] 

(99b)

where \( \tilde{\omega} = \omega \sqrt{\rho} \). Using the boundary conditions it follows from these two equations that

\[ \frac{\alpha \chi \tilde{\omega} \sqrt{\mu_0}}{3 H_0} |x_0| = n \frac{\pi}{2} \] 

(100)
Since
\[ \frac{\alpha \sqrt{\mu_o}}{3 H_o} = 0.7626 \times 10^8 \left[ \frac{m \frac{\pi}{2}}{Am \frac{\pi}{2}} \right] \]
equation (100) can be written
\[ \tilde{\omega} = \frac{n \pi}{1.5252 \times 10^8 \times \alpha^4 \times |x_0|} \] (101)

where \( a \) is the radius of the limiting cavity expressed in units of Earth radii and \( x_0 \) is given by equation (98). However in this type of problem one is more interested in the eigenperiod than in the angular frequency. From equation (101) the period of the \( n \)-th oscillation is given by
\[ T_n = \frac{3.050 \times 10^8 \times \alpha^4 \times |x_0| \times \sqrt{\rho}}{n} \] [sec] (102)

However it should be kept in mind that this formula applies only to those magnetic lines of force which intersect the Earth at very low co-latitudes (\( \nu_o \leq 20^\circ \)) and in addition the radius of the cavity should not exceed six Earth radii in order to guarantee that the assumption \( \nu_c = \alpha = \text{constant} \) is approximately satisfied. Taking \( \nu_o = 10^\circ \), \( \rho = 6.5 \times 10^{-19} \text{ kg/m}^3 \) equation (102) yields
\[ T \hat{=} \frac{109}{\text{sec for } \alpha = 4 \text{ and } T \hat{=} \frac{174}{8} \text{ sec for } \alpha = 8.} \]
In the majority of cases however it will not be permissible to assume that \( \psi_c \) is constant. For these cases equation (64) has to be solved rigorously using lines of force given by equation (61). In the first section as well as in this section we changed the independent variable by putting

\[
f(\psi) \, d\psi = dx
\]

In the first section, \( f(\psi) = \sin^7 \psi \) while in this section \( f(\psi) = \csc \psi \). It thus appears logical to put

\[
\left( \frac{1}{\psi^4} + \frac{2}{\psi^3 \alpha^2} \right) \sin \psi = \frac{\alpha^3 + 2 \psi^3}{\psi^4 \alpha^3} \sin \psi \equiv \frac{1}{f(\psi)}
\]

i.e.

\[
f(\psi) = \frac{\alpha^3 \psi^4}{(\alpha^3 + 2 \psi^3) \sin \psi}
\]

Equation (64) then takes the simple form

\[
\frac{d^2 V_z}{dx^2} + \frac{\mu_o \alpha^2 \rho}{H_o} \frac{\omega^2}{H_o} V_z = 0
\] (103)

The solution of this equation is

\[
V_z = A \cos \left( \frac{\alpha \omega \sqrt{\mu_o}}{H_o} x \right)
\] (104a)
and

\[ V_z = B \sin \left( \frac{\alpha \tilde{\omega} \sqrt{\mu_o}}{H_o} X \right) \]  \hspace{1cm} (104b)

where, as before, \( \tilde{\omega} = \omega \sqrt{\rho} \). With the same boundary conditions as used before, one obtains from these equations

\[ \tilde{\omega}_n = \omega_n \sqrt{\rho} = \frac{n \frac{\pi}{4.576 \times 10^8 |X_o|}}{X_o} = \frac{6.864 \times 10^9 n}{[\text{sec}^{-1}]} \]  \hspace{1cm} (105)

where

\[ \frac{\alpha \sqrt{\mu_o}}{H_o} = 2.288 \times 10^8 \left[ \frac{n \gamma}{A m \rho^{1/2}} \right] \]

From equation (105) the eigenperiods are readily obtained, viz.

\[ T_n = \frac{9.152 \times 10^8 |X_o| \sqrt{\rho}}{n} \]  \hspace{1cm} [\text{sec}]  \hspace{1cm} (106)

In particular the fundamental eigenperiod \( (n = 1) \) is given by

\[ T_1 = 9.152 \times 10^8 |X_o| \sqrt{\rho} \]  \hspace{1cm} [\text{sec}]  \hspace{1cm} (107)

Comparing the equation for the eigenperiods of the oscillating magnetic lines of force in the case of an
ordinary dipole field (equ. 72) with that of a compressed dipole field (equ. 107), one sees that they are of the same form and would be identical if the factor \( v_o^4 \) could be incorporated into \( |\chi_o| \). For this reason we might expect that both expressions would yield the same result if the presence of the limiting cavity does not influence the shape of the magnetic lines of force too much. This would be the case for those lines of force intersecting the Earth at high co-latitudes.

In order to evaluate equation (106) it is necessary to find the relation between \( v_o \) and \( |\chi_o| \). In the present case the change of independent variable was accomplished by the transformation

\[
dX = \frac{\alpha^3 v_c^4}{(\alpha^3 + 2 v_c^3) \sin \varphi} d\varphi
\]

(108)

where \( v_c \) has to be determined from the cubic equation

\[
v_c^3 + (\alpha^3 - 1) \frac{\sin^2 v_o}{\sin^2 \varphi} v_c = \alpha^3
\]

(61)

Since interval \( \Delta X = X_1 - X_0 \) is associated with the interval \( \Delta \varphi = \varphi_1 - \varphi_0 \), integration of equation (108) yields

\[
\Delta X = \int_{\varphi_0}^{\varphi_1} \frac{\alpha^3 v_c^4}{(\alpha^3 + 2 v_c^3) \sin \varphi} d\varphi = 2 \int_{v_0}^{v_1} \frac{\alpha^3 v_c^4}{(\alpha^3 + 2 v_c^3) \sin \varphi} d\varphi
\]

(109)
The last expression is due to symmetry considerations. Thus

\[ X_0 = \int_{\psi_0}^{\pi/2} \frac{\alpha^3 \psi^4_c}{(\alpha^3 + 2 \psi^3_c) \sin \delta} \, d\psi \quad (110) \]

As opposed to the integral which gave \( x_0 \) in section 1 of this chapter, it is not possible here to evaluate the integral (110) in terms of known functions. In this case one has to resort to numerical methods in order to obtain \( X_0 = X_0(\alpha, \psi_0) \).

The computation of the integral was performed using Simpson's rule with an interval length of one degree. Since under these circumstances the integrand had to be evaluated at many interior points of the given interval from \( \psi_0 \) to \( 90^\circ \), a computer programme for the ALWAC III E was written to let the machine take care of this laborious step. Appendix VI contains numerical values of the integrand. The evaluation of \( \psi_c \) from equation (61) was carried out using the Newton-Raphson method taking as a first approximation for \( \psi_c \) the radius of the limiting cavity. Table 7 gives the numerical values of the integral. These are plotted in Fig. 9 as a function of \( \psi_0 \) for \( \alpha = 12 \), \( \alpha = 8 \) and \( \alpha = 4 \).

As pointed out already in the short discussion following the derivation of formula (106) for the eigenperiods, the quantities \( \psi_0^4 \times |X_0| \) in equation (72) and \( |X_0| \) in equation (106) correspond to one another. In order to study
this relationship more closely values of the expression $|x_0| \times y_c^4 (\gamma=50)$ were computed and the results tabulated in table 8 and plotted in Fig. 10.

<table>
<thead>
<tr>
<th>Table 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>10</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>30</td>
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<td>40</td>
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<td>50</td>
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<tr>
<td>60</td>
</tr>
<tr>
<td>70</td>
</tr>
<tr>
<td>80°</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
</tr>
<tr>
<td>10°</td>
</tr>
<tr>
<td>20°</td>
</tr>
<tr>
<td>30°</td>
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<tr>
<td>40°</td>
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<td>50°</td>
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<tr>
<td>60°</td>
</tr>
<tr>
<td>70°</td>
</tr>
<tr>
<td>80°</td>
</tr>
</tbody>
</table>
\[
X_0(\gamma_0, \alpha) = \int_{\gamma_0}^{\gamma} \frac{\alpha^3 \, \kappa_0^4}{(\alpha^3 + 2 \nu^3) \sin \gamma} \, d\gamma
\]

Fig. 9.
We can now evaluate the expression (106) for the eigen-periods of toroidal oscillations of a magnetic line of force belonging to a compressed dipole field. Taking the value $\rho = 6.5 \times 10^{-19} \text{[kg/m}^3\text{]}$ for the charge density of the conducting medium the following fundamental periods are obtained.

<table>
<thead>
<tr>
<th>$\omega_0$</th>
<th>$T_i(\alpha = 12)$ (sec)</th>
<th>$T_i(\alpha = 2)$ (sec)</th>
<th>$T_i(\alpha = 4)$ (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4228</td>
<td>1072</td>
<td>92.00</td>
</tr>
<tr>
<td>20</td>
<td>725.0</td>
<td>309.4</td>
<td>43.30</td>
</tr>
<tr>
<td>30</td>
<td>74.30</td>
<td>56.48</td>
<td>17.69</td>
</tr>
<tr>
<td>40</td>
<td>11.07</td>
<td>10.27</td>
<td>6.136</td>
</tr>
<tr>
<td>50</td>
<td>2.686</td>
<td>2.619</td>
<td>2.113</td>
</tr>
<tr>
<td>60</td>
<td>0.9112</td>
<td>0.9016</td>
<td>0.8182</td>
</tr>
<tr>
<td>70</td>
<td>0.3690</td>
<td>0.3670</td>
<td>0.3490</td>
</tr>
<tr>
<td>80</td>
<td>0.1403</td>
<td>0.1399</td>
<td>0.1355</td>
</tr>
</tbody>
</table>

The results of table 9 are plotted in Fig. 11 and show clearly the influence of the finite cavity.

4. **Solution for the compressed dipole field with variable charge density**

In the preceding section the fundamental periods of toroidal oscillations of a conducting medium in the presence of a compressed dipole field were obtained assuming a constant
Fig. II.

Fundamental period, constant charge density
compressed dipole field
charge density distribution. However, in general, this assumption will not be valid.

In order to treat the case of a variable charge density in the presence of a compressed dipole field one has to go back to equation (103) from section 3, viz.

\[ \frac{d^2 V_x}{d y^2} + \frac{\mu_o a^2 \rho \omega^2}{H_o} V_x = 0 \]  \hspace{1cm} (103)

where the independent variable x has been replaced by y. It was shown in the last section how the interval \((\phi_o, \pi/2\)) is transformed by means of the integral (110) into an interval \((y_o, 0)\). Furthermore the evaluation of this integral has shown that the variable y takes a far wider range of values than the corresponding variable x of the normal dipole field. For this reason the matrix method cannot be applied to equation (103) immediately unless one uses a very high order matrix (up to 1,000) which is prohibitive. Since in general the length of the interval is irrelevant for the calculation of the eigenvalue it seems appropriate to perform a second transformation which compresses the interval \((y_o, 0)\) into \((x_o, 0)\) where \(x_o\) may be chosen arbitrarily. However the success of such a transformation depends largely upon the smooth behaviour of the charge density. For the new interval \((x_o, 0)\) the matrix method already described can be applied without difficulty.
In order to illustrate the method it will be assumed that a knowledge of the charge density at 7 intermediate points is sufficient to determine the fundamental eigenperiod of the oscillating line of force with reasonable accuracy. Since it has been shown that with a 7 order matrix and \(|h| < 0.11\) the lowest eigenvalue is approximated with sufficient accuracy, \(h\) will be taken to be 0.1 and thus \(x_0 = 0.4\). Since in most cases \(|y_0| \gg x_0\) it seems feasible to put

\[ y = \beta x \]  \hspace{1cm} (111)

where \(\beta\) is a constant factor depending upon \(\alpha\) and \(y_0\). \(\beta\) is determined from the relation

\[ \beta = \frac{y_0}{0.4} \]

where \(y_0\) is found from table 7 \((y_0 \approx x_0)\). Confining the calculation to the case where \(\alpha = 4\) only, the following values of \(\beta\) are obtained:
Table 10

<table>
<thead>
<tr>
<th>θ°</th>
<th>( \beta(\alpha=4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>3.1170x10^2</td>
</tr>
<tr>
<td>20°</td>
<td>1.4672x10^2</td>
</tr>
<tr>
<td>30°</td>
<td>5.9948x10^1</td>
</tr>
<tr>
<td>40°</td>
<td>20.792</td>
</tr>
<tr>
<td>50°</td>
<td>7.1595</td>
</tr>
<tr>
<td>60°</td>
<td>2.7733</td>
</tr>
<tr>
<td>70°</td>
<td>1.1826</td>
</tr>
</tbody>
</table>

Introducing \( y = \beta x \) into equation (103) yields

\[
\frac{d^2 V_x}{dx^2} + \beta^2 \frac{\mu_0 \alpha^2 \rho \omega^2}{H_o^2} V_x = 0 \tag{112}
\]

where the independent variable \( x \) now extends from \(-x_o\) to 0 (because of symmetry about the origin).

In order to apply the same numerical methods to equation (112) as those used in section 2 one has to evaluate the expression

\[
\frac{h^2 \beta^2}{\mu_0 \alpha^2 \rho(x)} \tag{113}
\]

at the intermediate points \( x_o = |x_o| - jh \) subject to the same symmetry properties as mentioned in section 2. In the
present case the intermediate points are \( x_j = 0.3; 0.2; 0.1; 0.0 \). Since \( \rho \) is given as a function of altitude above the surface of the Earth (fig. 7) one has to associate each \( x_j \) with \( y_j \) by means of the relation (111) and then, by using the graph in fig. 9, find the corresponding co-latitude. Having found the co-latitude \( \nu \), the corresponding \( \nu_c \) of the compressed dipole field can be obtained from table VI-a of the appendix. By evaluating

\[
H = 6.370 (\nu_c - 1) \text{[km]} \quad (114)
\]

the height of a particular point corresponding to \( x_j \) can be found, and hence the expression (113) can be calculated. The resulting values of \( A_j \) are compiled in table 11 - from them one may calculate the elements of the matrix \( M \) as given in appendix II. As before the matrices are diagonalized with the help of the AlvacIII E using the Jacobi method. The matrices as well as their eigenvalues are given in appendix VII. The periods of the fundamental modes are given in table 12 and are plotted in fig. 11.
Table 11

<table>
<thead>
<tr>
<th>$\phi_0$</th>
<th>$A_1$</th>
<th>$A_2$</th>
<th>$A_3$</th>
<th>$A_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>$8.39570\times10^2\omega^2$</td>
<td>$5.44448\times10^2\omega^2$</td>
<td>$4.35050\times10^2\omega^2$</td>
<td>$3.56181\times10^2\omega^2$</td>
</tr>
<tr>
<td>20°</td>
<td>$1.94940\times10^2\omega^2$</td>
<td>$1.94768\times10^2\omega^2$</td>
<td>$1.35288\times10^2\omega^2$</td>
<td>$0.924468\times10^2\omega^2$</td>
</tr>
<tr>
<td>30°</td>
<td>$1.59980\times10^2\omega^2$</td>
<td>$0.715206\times10^2\omega^2$</td>
<td>$0.331253\times10^2\omega^2$</td>
<td>$0.188212\times10^2\omega^2$</td>
</tr>
<tr>
<td>40°</td>
<td>$6.22804\times10^2\omega^2$</td>
<td>$13.5701\times10^2\omega^2$</td>
<td>$0.334225\times10^2\omega^2$</td>
<td>$0.0566430\times10^2\omega^2$</td>
</tr>
<tr>
<td>50°</td>
<td>$1.04729\times10^2\omega^2$</td>
<td>$13.2925\times10^2\omega^2$</td>
<td>$0.384064\times10^2\omega^2$</td>
<td>$0.060420\times10^2\omega^2$</td>
</tr>
<tr>
<td>60°</td>
<td>$7.50754\times10^2\omega^2$</td>
<td>$19.4097\times10^2\omega^2$</td>
<td>$3.66222\times10^2\omega^2$</td>
<td>$2.92977\times10^2\omega^2$</td>
</tr>
<tr>
<td>70°</td>
<td>$3.66222\times10^2\omega^2$</td>
<td>$3.66222\times10^2\omega^2$</td>
<td>$3.66222\times10^2\omega^2$</td>
<td>$3.66222\times10^2\omega^2$</td>
</tr>
</tbody>
</table>

Table 12

<table>
<thead>
<tr>
<th>$\phi_0$</th>
<th>Ti (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10°</td>
<td>111</td>
</tr>
<tr>
<td>20°</td>
<td>124</td>
</tr>
<tr>
<td>30°</td>
<td>252</td>
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<td>40°</td>
<td>245</td>
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<tr>
<td>50°</td>
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<tr>
<td>60°</td>
<td>69</td>
</tr>
<tr>
<td>70°</td>
<td>46</td>
</tr>
</tbody>
</table>
DISCUSSION OF THE RESULTS

The aim of this thesis was to calculate the eigenperiods of geomagnetic micropulsations and in particular to investigate their dependence on latitude. The main reason for carrying out the study was the great discrepancy between the observed and calculated values especially when the points of observation are in the polar regions. This discrepancy casts much doubt on the model put forward by Dungey (1954b). For this reason the principles of magnetohydrodynamics were applied to a somewhat different model which, because of the greater simplicity of the resulting equations, promised an easier solution.

From Maxwell's equations and the basic equation of hydrodynamics, two partial differential equations were obtained in cylindrical coordinates. The use of this system of coordinates seems feasible since the phenomenon of geomagnetic micropulsations appears to be confined to meridional planes.* Mathematically this can be expressed by disregarding the coupling terms contained in both equations. Since this is equivalent to considering the problem in a plane it should in principle make no difference whether cylindrical

* The question of the dependence of micropulsations on G.M.T. or L.M.T. is still rather open.
or spherical coordinates are used. On account of cylindrical symmetry solutions are obtained more easily.

The eigenperiods of the toroidal oscillations were obtained as a function of co-latitude first assuming a medium of constant charge density in a normal dipole field. As can be seen from fig. 6 the eigenperiods tend towards infinity as zero co-latitude is approached, a result which has not been observed. The resulting high values are entirely due to the fact that the lines of force in the case of a normal dipole extend very far into outer space. The eigenperiods calculated by Dungey using spherical coordinates agree very well with those obtained in this thesis which justifies the use of cylindrical coordinates in describing the problem. Figure 6 also shows that the eigenperiods at high co-latitudes tend to zero. Since toroidal oscillations can be understood as oscillating lines of force it follows that the period must tend to zero as the length of the line of force decreases as it does in this model at higher co-latitudes. This is the principal difficulty of this model whether spherical or cylindrical coordinates are used.

Since the charge density distribution of the medium above the surface of the Earth is probably not constant the above model was modified using a variable charge density as proposed by Dessler (1958). A comparison of figures 6 and 8 shows the effect of a variable charge density on the
fundamental period. The values on the left of fig. 6 and fig. 8 agree exactly since the major part of the magnetic lines of force run through regions in which the charge density can be considered to be constant and equal to that used in the previous calculation. At higher co-latitudes, the magnetic lines of force spend longer in regions of lower altitude in which the charge density is higher than that further out. This results in a somewhat slower drop of the calculated eigenperiods with increasing co-latitude. Above co-latitude 55° the periods increase, the increase in charge density completely compensating for the shortening of the lines of force.

So far all the results were obtained on the assumption that the Earth's magnetic field is that of a geocentric dipole. Recent investigations (Parker, 1958) however have shown that this is not so. One of the more important features of the field is that it does not extend as far into outer space as was originally thought. To a first approximation on the daylight side of the Earth it appears to be confined to a cavity of variable radius (4-12 Earth radii). This may be attributed to the "solar wind" which compresses the field on the daylight side of the Earth. Since the form of the field has an important effect on the eigenperiods of geomagnetic micropulsations a compressed dipole field was substituted into the toroidal equation and the eigenperiods computed. The results of this calculation have been plotted in fig. 12.
One of the important features is that the large values of the eigenperiods have vanished, the eigenperiods assuming finite values as the co-latitude approaches zero. To obtain the limit one cannot use equation (98). For this purpose however $X_0$ in fig. 9 as computed from integral (110) is nearly equal to $|X_0| \times \nu_{oc}^4$ in fig. 10 where $|X_0|$ is to be taken from equation (68) on putting $\nu_{oc} = 0$. Using the limiting value for $|X_0|$ at $\nu_{oc} = 0$ and the fact that $\nu_{oc}$ tends to $\alpha$, the radius of the cavity, the eigenperiod of the fundamental mode takes the limiting value

$$T_l = 4 \times 10^8 \times 10^8 \times \alpha^4 \times \sqrt{\rho} \text{ [sec]}$$

which, with $\alpha = 4$ and $\rho = 6.5 \times 10^8$ gives the value 86 seconds. The small disagreement between the above value and $T_l$ for $\alpha = 4$ in table 9 is due to the use of the approximation rather than the integral. As one would expect the radius of the cavity influences the size of the eigenperiods considerably, in particular those which are associated with lines of force intersecting the Earth's surface at low co-latitudes. As higher co-latitudes are approached the eigenperiods corresponding to different values of $\alpha$ almost coincide with those obtained in the case of a normal dipole field. Physically this can be understood from figures 2, 3 and 4. The lines of force which intersect the Earth's surface at
high co-latitudes \((\lambda > 45^\circ - 50^\circ)\) hardly suffer any deformation even for \(\alpha = 4\) and hence have the same shape as those of a normal dipole with the result that the eigenperiods are equal to those of a normal dipole. As a consequence the eigenperiods tend to zero as the equator is approached, a fact which is in disagreement with the observed data.

Finally it seemed promising to calculate the eigenperiods for a compressed field assuming a variable charge density distribution. In that way one could expect to incorporate both the advantages of model 2 and model 3 (sections 2 and 3 of chapter III) into one model which would remove the trend to infinity at low co-latitudes and at the same time raise the small values of the eigenvalues at high co-latitudes. The calculation was carried out for the case \(\alpha = 4\) and the result indicated in fig. 11. The result as it stands is not too satisfactory. One would have expected from simple considerations that at low co-latitudes the curve would approximately follow that obtained for the compressed dipole field with constant charge density while at high co-latitudes the behaviour of the curve would be similar to that of a normal dipole with variable charge density distribution. Instead the curve rises slightly with increasing co-latitude (up to about 30\(^\circ\)) and there decreases continuously. Before drawing any conclusions from this result one must consider the following point. As already mentioned the interval over
which the differential equation had to be integrated was quite large a difficulty which was overcome by introducing a second variable. Subdividing the interval into 8 parts and associating the equally spaced points with points along the lines of force, those points near the ends of the lines of force are favored. Since these points are situated in that part of the medium having the higher charge density an incorrect picture may easily be obtained. In order to avoid this the interval should be subdivided into more parts in order to obtain a better representation with respect to the charge density distribution along a line of force. Since for an interval of length 0.1 a seven order matrix gives the lowest eigenvalue with sufficient accuracy in the case of constant charge density one can be sure that any change in the eigenperiod with increase in the number of intervals is entirely due to the increased information on the charge density along the lines of force. The number of points which would be sufficient could only be found by reflecting the calculation using 16 intervals instead of 8 as in the present case. Before actually performing the calculation it would be advisable to supplement table 7 by calculating the values of \( x \) for \( \psi = 15^\circ, 25^\circ, \ldots, 75^\circ \). That would allow higher accuracy in the inverse interpolation for determining the value of \( \psi \) which belongs to a certain value of \( x \).

In view of the difficulties involved in carrying out the
improvement described above it seems feasible to consider a different method of approach. Assuming that the disturbance travels along a line of force the time required to travel from one end point to the other is given by

\[ T_1 = 2 \int_0^{\frac{\pi}{2}} \frac{ds}{V} \]

where \( V \) is the Alfven velocity given by

\[ V = \frac{H_0 \sqrt{\mu_0}}{V} \left[ \frac{\omega \rho}{\omega_c} \right] \]

This is the formulation which has been used by Jacobs and Obayashi (1958) to find the charge density from a knowledge of the period of micropulsations. The integration has to be carried out along the line of force using the compressed dipole field. The difficulty lies in the evaluation of the integral and arises mainly from the fact that the equation of a line of force is a cubic equation which makes the line element \( ds \) more complicated. Using modern computing devices however the problem should be tractable.

So far the discussion has been concerned only with the results of the calculations. Since it was our aim to determine a model which would explain the phenomenon of micropulsations it is necessary to consider the experimental data which have been obtained. The usual procedure to obtain
Fig. 12
Comparison of experimental and theoretical values.

References
Kato & Saita (1959)
Campbell (1959)
Maple (1959)
Berthold (1960)
Scholle & Veldkamp (1955)
Duffus & Shand (1958)
the dependence of the eigenperiods on the co-latitude is to take those eigenperiods which exhibit a maximum frequency of occurrence at a certain latitude. Only data which have been observed simultaneously at stations lying on the same meridian should be used. In spite of the greatly increased efforts which have been put into research on micropulsations, data satisfying these requirements are very difficult to obtain. For this reason it is necessary to resort to data obtained under less rigorous conditions. In fig. 12 experimental values of periods are plotted against geomagnetic co-latitude. The tendency of the eigenperiods to decrease with increasing co-latitude is clearly apparent. For comparison some of the calculated eigenperiods are plotted in the same diagram. The agreement between the observed and calculated values is not too satisfactory either for the compressed dipole field with constant charge density, or for the compressed field with variable charge density. In the first case the magnitude as well as the slope do not agree; the disagreement in the second case is only with respect to magnitude. Apart from the fact that the model may not be appropriate the discrepancy might be due to an incorrect charge density distribution or to the wrong choice of $\alpha$. Since the value of $\alpha$ is not critical in the range of co-latitudes under discussion (see fig. 10) the major cause for the discrepancy has to be sought in the charge density
distribution. From single measurements in low co-latitudes which yield eigenperiods of the order of 100 seconds, it appears almost certain from our model that during disturbed conditions \( \propto \) must be around 4 in agreement with other calculations. The theoretical explanation of geomagnetic micropulsations in terms of toroidal oscillations of a compressed dipole field is promising and mathematically tractable if cylindrical coordinates are used. Further studies on the subject using the methods described in this thesis are strongly recommended.
The field intensity of a magnetic dipole can be derived from the potential

\[ V = -\frac{M}{\mu_0 r^2} \cos \varphi \]

where \( \varphi \) is the co-latitude and \( M \) the magnetic moment of the dipole. In order to allow for the influence of the solar wind, one modifies equation (I-1) according to Obayashi (1960) superimposing a potential associated with a constant field. Hence one writes

\[ V = -\frac{M}{\mu_0} \left( \frac{1}{r^2} + A \right) \cos \varphi \]

where the constant \( A \) is chosen so that the \( r \)-component of the magnetic field vanishes at \( r = R_0 \) (compressed field).

Therefore, since

\[ H_r = \frac{\partial V}{\partial r} = -\frac{M}{\mu_0} \left( -\frac{2}{r^3} + A \right) \cos \varphi, \]

one takes

\[ A = \frac{2}{R_0^3} \]

(I-3)
With this value of $A$, the $r$- and $\psi$-components of the compressed dipole field are

$$H_r = \frac{\partial V}{\partial r} = -\frac{M}{\mu_0} \left( -\frac{2}{r^3} + \frac{2}{R_0^3} \right) \cos \psi$$  \hspace{1cm} (I-4)$$

and

$$H_\psi = \frac{1}{r} \frac{\partial V}{\partial \psi} = \frac{M}{\mu_0} \left( \frac{1}{r^3} + \frac{2}{R_0^3} \right) \sin \psi$$  \hspace{1cm} (I-5)$$

The lines of force are obtained by integrating the differential equation

$$\frac{dr}{H_r} = \frac{rd\psi}{H_\psi}$$

which upon introducing the equations (I-4) and (I-5) becomes

$$\frac{r^3 dr}{2(1-r^3/R_0^3) \cos \psi} = \frac{r^4 d\psi}{(1+2r^3/R_0^3) \sin \psi}$$

i.e.

$$\frac{\left(1+2r^3/R_0^3\right)dr}{r(1-r^3/R_0^3)} = 2 \cot \psi d\psi$$  \hspace{1cm} (I-6)$$

Integrating this equation yields

$$\ln \frac{r}{1-r^3/R_0^3} = 2 \ln \sin \psi + C$$
To determine the constant $C$ in equation (I-7) we postulate that the line of force intersects the surface of the Earth $(r = a)$ at $\varphi = \varphi_0$.

Thus

$$C = \frac{a}{(1 - \frac{a^{3/2}}{R_0^{3/2}}) \sin^2 \varphi_0}$$

and equation (I-7) becomes

$$r^3 + \frac{(1 - \frac{a^{3/2}}{R_0^{3/2}})}{\frac{a}{R_0^{3/2}}} \frac{\sin^2 \varphi_0}{\sin^2 \varphi} r = R_0^3$$
APPENDIX II

Letting $X = A^{-1/2}Y$, where $A^{-1/2}$ is defined later, equation (83) becomes

$$BA^{-1/2}Y = \omega^2A^{-1/2}Y$$

Multiplying this from the left-hand side one obtains

$$A^{-1/2}BA^{-1/2}Y = \omega^2A^{-1/2}AA^{-1/2}Y = \omega^2Y$$

or

$$MY = \omega^2Y$$

where one has abbreviated

$$M = A^{-1/2}BA^{-1/2}$$

With

$$A = \begin{pmatrix} \sqrt{a_1} & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \sqrt{a_2} & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{a_n} \end{pmatrix}$$

$$A^{-1/2} = \begin{pmatrix} \sqrt{a_1} & 0 & 0 & 0 & \cdots & \cdots \\ 0 & \sqrt{a_2} & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{a_n} \end{pmatrix}$$
the real symmetric matrix $M$ becomes

$$
\begin{pmatrix}
\frac{2}{\alpha_1} & -\frac{1}{\sqrt{\alpha_1 \alpha_2}} & 0 & 0 & \cdots & \cdots & \cdots \\
-\frac{1}{\sqrt{\alpha_1 \alpha_2}} & \frac{2}{\alpha_2} & -\frac{1}{\sqrt{\alpha_2 \alpha_3}} & 0 & \cdots & \cdots & \cdots \\
0 & -\frac{1}{\sqrt{\alpha_2 \alpha_3}} & \frac{2}{\alpha_3} & -\frac{1}{\sqrt{\alpha_3 \alpha_4}} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -\frac{1}{\sqrt{\alpha_{n-1} \alpha_n}} & \frac{2}{\alpha_n} & \cdots & \cdots \\
\end{pmatrix}
$$
APPENDIX III

In order to apply the Newton-Raphson method, equation (87) has to be written in the form

\[
f(s) = \frac{1}{3\sigma} (\sigma - s)^3 (5\sigma^3 + 6\sigma^2 + 8\sigma + 16) - (y_{X_0} - y_jh) = 0 \quad (III-1)
\]

Taking the derivative of \( f(s) \) with respect to \( s \) yields

\[
f'(s) = -\frac{1}{2} \frac{s^3}{(\sigma - s)^{3/2}} \quad (III-2)
\]

If therefore an approximative solution \( s_i \) of equation (III-1) is known an improved solution \( s_{i+1} \) is obtained from

\[
s_{i+1} = s_i - \frac{f(s_i)}{f'(s_i)} \quad (III-3)
\]

As may be seen from equation (III-2) \( f'(s) \) will never vanish in the interval \( 0 < s < 1 \) and hence the method may be used quite safely to obtain a solution of any desired accuracy.

Leaving out all details concerned with scaling, etc. the "flow chart" looks as follows
\begin{align*}
\text{Inp.} & \quad 1 \times 1 - j^h \\
\text{Inp.} & \quad S_i \\
\text{Comp.} & \quad 5 S_i \\
\text{Comp.} & \quad 5 S_i + 6 \\
\text{Comp.} & \quad (5 S_i + 6) S_i \\
\text{Comp.} & \quad (5 S_i + 6) S_i + 8 \\
\text{Comp.} & \quad [(5 S_i + 6) S_i + 8] S_i \\
\text{Comp.} & \quad [(5 S_i + 6) S_i + 8] S_i + 16 \\
\text{Comp.} & \quad (1 - S)^{1/2} \\
\text{Comp.} & \quad \gamma_{35}(1 - S)^{1/2} \frac{[(5 S_i + 6) S_i + 8] S_i + 16}{S_i^3 / (1 - S)^{1/2}} \\
\text{Comp.} & \quad S_i = S_i - \frac{\gamma_{35}(1 - S) S_i^{1/2} \frac{[(5 S_i + 6) S_i + 8] S_i + 16}{S_i^3 / (1 - S)^{1/2}}}{S_i^3 / (1 - S)^{1/2}} \\
\text{Accuracy Test} & \\
\text{Outp.} & \quad 1 \times 0 - j^h \\
\text{Outp.} & \quad S_i^{+1} \\
\text{Comp.} & \quad S_i^3 \\
\text{Comp.} & \quad S_i^3 / (1 - S_i)^{1/2} \\
\text{Comp.} & \quad \gamma_{35}(1 - S)^{1/2} \frac{[(5 S_i + 6) S_i + 8] S_i + 16}{S_i^3 / (1 - S)^{1/2}}.
\end{align*}
The necessary starting values $s_i$ were obtained from the graph in fig. 5. For speeding up the input both $|x_0| - jh$ and $s_i$ were punched out on tape.

The following table contains the solutions of equation (87) which were required for the computations of the matrix elements. The pairs of values are arranged in groups each corresponding to a certain co-latitude of intersection.
Table III-a

<table>
<thead>
<tr>
<th>φ₀</th>
<th>x₀ - j₀</th>
<th>s</th>
<th>φ₀</th>
<th>x₀ - j₀</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>0.342855 s=0.83872 ; 0.228570 =0.94129 ; 0.114285 =0.98658 ;</td>
<td>50°</td>
<td>0.327425 s=0.85836 ; 0.218283 =0.94706 ; 0.109142 =0.98779 ;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>20°</td>
<td>0.342839 s=0.83884 ; 0.220559 =0.94130 ; 0.114280 =0.98659 ;</td>
<td>55°</td>
<td>0.314405 s=0.87306 ; 0.209603 =0.95163 ; 0.104802 =0.98877 ;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25°</td>
<td>0.342754 s=0.83886 ; 0.228503 =0.94133 ; 0.114251 =0.98659 ;</td>
<td>60°</td>
<td>0.294476 s=0.89284 ; 0.196317 =0.95811 ; 0.098158 =0.99017 ;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30°</td>
<td>0.342447 s=0.83927 ; 0.228298 =0.94145 ; 0.114149 =0.98662 ;</td>
<td>65°</td>
<td>0.266161 s=0.91639 ; 0.177441 =0.96633 ; 0.088720 =0.99200 ;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>35°</td>
<td>0.341575 s=0.84045 ; 0.227717 =0.94178 ; 0.113858 =0.98669 ;</td>
<td>70°</td>
<td>0.228556 s=0.94130 ; 0.152371 =0.97563 ; 0.076185 =0.99413 ;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40°</td>
<td>0.339506 s=0.84321 ; 0.226337 =0.94258 ; 0.113169 =0.98685 ;</td>
<td>75°</td>
<td>0.181626 s=0.96460 ; 0.121084 =0.98489 ; 0.060542 =0.99631 ;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>45°</td>
<td>0.335244 s=0.84873 ; 0.223496 =0.94419 ; 0.111748 =0.98719 ;</td>
<td>80°</td>
<td>0.126380 s=0.98349 ; 0.084254 =0.99280 ; 0.042127 =0.99822 ;</td>
<td></td>
<td></td>
</tr>
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</table>
Table III-b

<table>
<thead>
<tr>
<th>$\phi_0$</th>
<th>H</th>
<th>s</th>
<th>$\phi_0$</th>
<th>H</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>82259km</td>
<td>0.83872</td>
<td>50°</td>
<td>3240km</td>
<td>0.85836</td>
</tr>
<tr>
<td></td>
<td>92319</td>
<td>0.94129</td>
<td></td>
<td>4233</td>
<td>0.94706</td>
</tr>
<tr>
<td></td>
<td>96761</td>
<td>0.98658</td>
<td></td>
<td>4689</td>
<td>0.98779</td>
</tr>
<tr>
<td></td>
<td>98077</td>
<td>1.0000</td>
<td></td>
<td>4826</td>
<td>1.0000</td>
</tr>
<tr>
<td>20°</td>
<td>40736km</td>
<td>0.83874</td>
<td></td>
<td>2178km</td>
<td>0.87306</td>
</tr>
<tr>
<td></td>
<td>46496</td>
<td>0.94130</td>
<td></td>
<td>2948</td>
<td>0.95163</td>
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<tr>
<td></td>
<td>49040</td>
<td>0.98659</td>
<td></td>
<td>3311</td>
<td>0.98877</td>
</tr>
<tr>
<td></td>
<td>49793</td>
<td>1.0000</td>
<td></td>
<td>3421</td>
<td>1.0000</td>
</tr>
<tr>
<td>25°</td>
<td>24486km</td>
<td>0.83886</td>
<td></td>
<td>1451km</td>
<td>0.89284</td>
</tr>
<tr>
<td></td>
<td>28255</td>
<td>0.94133</td>
<td></td>
<td>2304</td>
<td>0.95811</td>
</tr>
<tr>
<td></td>
<td>29921</td>
<td>0.98659</td>
<td></td>
<td>2390</td>
<td>0.99017</td>
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<tr>
<td></td>
<td>30414</td>
<td>1.0000</td>
<td></td>
<td>2390</td>
<td>1.0000</td>
</tr>
<tr>
<td>30°</td>
<td>15686km</td>
<td>0.83927</td>
<td></td>
<td>959km</td>
<td>0.91639</td>
</tr>
<tr>
<td></td>
<td>18371</td>
<td>0.94145</td>
<td></td>
<td>1359</td>
<td>0.96633</td>
</tr>
<tr>
<td></td>
<td>19558</td>
<td>0.98662</td>
<td></td>
<td>1564</td>
<td>0.99200</td>
</tr>
<tr>
<td></td>
<td>19910</td>
<td>1.0000</td>
<td></td>
<td>1628</td>
<td>1.0000</td>
</tr>
<tr>
<td>35°</td>
<td>10414km</td>
<td>0.84045</td>
<td></td>
<td>633km</td>
<td>0.94130</td>
</tr>
<tr>
<td></td>
<td>12437</td>
<td>0.94178</td>
<td></td>
<td>889</td>
<td>0.97563</td>
</tr>
<tr>
<td></td>
<td>13334</td>
<td>0.98669</td>
<td></td>
<td>1026</td>
<td>0.99413</td>
</tr>
<tr>
<td></td>
<td>13600</td>
<td>1.0000</td>
<td></td>
<td>1070</td>
<td>1.0000</td>
</tr>
<tr>
<td>40°</td>
<td>7039km</td>
<td>0.84321</td>
<td></td>
<td>422km</td>
<td>0.96460</td>
</tr>
<tr>
<td></td>
<td>8618</td>
<td>0.94258</td>
<td></td>
<td>565</td>
<td>0.98489</td>
</tr>
<tr>
<td></td>
<td>9322</td>
<td>0.98685</td>
<td></td>
<td>646</td>
<td>0.99631</td>
</tr>
<tr>
<td></td>
<td>9531</td>
<td>1.0000</td>
<td></td>
<td>672</td>
<td>1.0000</td>
</tr>
<tr>
<td>45°</td>
<td>4782km</td>
<td>0.84873</td>
<td></td>
<td>292km</td>
<td>0.98349</td>
</tr>
<tr>
<td></td>
<td>6037</td>
<td>0.94419</td>
<td></td>
<td>355</td>
<td>0.99280</td>
</tr>
<tr>
<td></td>
<td>6602</td>
<td>0.98719</td>
<td></td>
<td>392</td>
<td>0.99822</td>
</tr>
<tr>
<td></td>
<td>6770</td>
<td>1.0000</td>
<td></td>
<td>404</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
As can be seen from table 5 a common factor in the $a_j$ which is of course different for the different lines of force appears quite frequently. For this reason it seems feasible to incorporate this factor into the value of $\omega^2$. One then has the following matrices and their eigenvalues:

### $\phi = 15^\circ$

$$
\begin{pmatrix}
1.42460 & -0.71230 & 0 & 0 & 0 & 0 & 0 \\
-0.71230 & 1.42460 & -0.71230 & 0 & 0 & 0 & 0 \\
0 & -0.71230 & 1.42460 & -0.71230 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.71230 & 1.42460 & -0.71230 & 0 \\
0 & 0 & 0 & 0 & -0.71230 & 1.42460 & -0.71230 \\
0 & 0 & 0 & 0 & 0 & -0.71230 & 1.42460
\end{pmatrix}
$$

**Scaling:** $10^6 \omega^2 = \lambda^2$

**Eigenvalues:** $\lambda_1^2 = 0.10844$; $\lambda_2^2 = 0.41725$; $\lambda_3^2 = 0.87944$

### $\phi = 20^\circ$

$$
\begin{pmatrix}
1.232286 & -0.616143 & 0 & 0 & 0 & 0 & 0 \\
-0.616143 & 1232286 & -0.616143 & 0 & 0 & 0 & 0 \\
0 & -0.616143 & 1232286 & -0.616143 & 0 & 0 & 0 \\
0 & 0 & -0.616143 & 1232286 & -0.616143 & 0 & 0 \\
0 & 0 & 0 & -0.616143 & 1232286 & -0.616143 & 0 \\
0 & 0 & 0 & 0 & -0.616143 & 1232286 & -0.616143 \\
0 & 0 & 0 & 0 & 0 & -0.616143 & 1232286
\end{pmatrix}
$$

**Scaling:** $10^4 \omega^2 = \lambda^2$

**Eigenvalues:** $\lambda_1^2 = 0.093802$; $\lambda_2^2 = 0.360924$; $\lambda_3^2 = 0.760715$
\[ \phi = 25^\circ \]

\[
\begin{align*}
0.657451 & -0.340771 & 0 & 0 & 0 & 0 & 0 \\
-0.340771 & 0.706514 & -0.358650 & 0 & 0 & 0 & 0 \\
0 & -0.358650 & 0.728252 & -0.364128 & 0 & 0 & 0 \\
0 & 0 & -0.364128 & 0.728252 & -0.364128 & 0 & 0 \\
0 & 0 & 0 & -0.364128 & 0.728252 & -0.358650 & 0 \\
0 & 0 & 0 & 0 & -0.358650 & 0.706514 & -0.340771 \\
0 & 0 & 0 & 0 & 0 & -0.340771 & 0.657451
\end{align*}
\]

**Scaling:** \(2 \times 10^2 \omega^2 = \lambda^2\)

**Eigenvalues:** \(\lambda_1^2 = 0.054579\); \(\lambda_2^2 = 0.204527\); \(\lambda_3^2 = 0.426525\)

\[ \phi = 30^\circ \]

\[
\begin{align*}
0.607469 & -0.334182 & 0 & 0 & 0 & 0 & 0 \\
-0.334182 & 0.735362 & -0.377756 & 0 & 0 & 0 & 0 \\
0 & -0.377756 & 0.776217 & -0.390280 & 0 & 0 & 0 \\
0 & 0 & -0.390280 & 0.784929 & -0.390280 & 0 & 0 \\
0 & 0 & 0 & -0.390280 & 0.776217 & -0.377756 & 0 \\
0 & 0 & 0 & 0 & -0.377756 & 0.735362 & -0.334182 \\
0 & 0 & 0 & 0 & 0 & -0.334182 & 0.607469
\end{align*}
\]

**Scaling:** \(2 \times 10^1 \omega^2 = \lambda^2\)

**Eigenvalues:** \(\lambda_1^2 = 0.057250\); \(\lambda_2^2 = 0.206443\); \(\lambda_3^2 = 0.422851\)

\[ \phi = 35^\circ \]

\[
\begin{align*}
0.601504 & -0.316186 & 0 & 0 & 0 & 0 & 0 \\
-0.316186 & 0.664827 & -0.341521 & 0 & 0 & 0 & 0 \\
0 & -0.341521 & 0.701754 & -0.354844 & 0 & 0 & 0 \\
0 & 0 & -0.354844 & 0.717708 & -0.354844 & 0 & 0 \\
0 & 0 & 0 & -0.354844 & 0.701754 & -0.341521 & 0 \\
0 & 0 & 0 & 0 & -0.341521 & 0.664827 & -0.316186 \\
0 & 0 & 0 & 0 & 0 & -0.316186 & 0.601504
\end{align*}
\]

**Scaling:** \(2 \omega^2 = \lambda^2\)

**Eigenvalues:** \(\lambda_1^2 = 0.052352\); \(\lambda_2^2 = 0.191944\); \(\lambda_3^2 = 0.401217\)
\[ \mathbf{w} = \mathbf{A} \mathbf{w} \]

### Eigenvalues: $A^2.1^2 = 0.131089$ ; $A.2^2 = 0.482142$ ; $A.3^2 = 1.00078$

### Scaling: $w^2 = \lambda^2$

### Eigenvalues: $A^2.0.050164$ ; $A.2^2 = 0.180788$ ; $A.3^2 = 0.371086$

### Scaling: $10^{-1}w^2 = \lambda^2$

### Eigenvalues: $A^2.1^2 = 0.143013$ ; $A.2^2 = 0.490445$ ; $A.3^2 = 0.990559$

### Scaling: $10^{-1}w^2 = \lambda^2$
$$\theta_0 = 55^\circ$$

\[
\begin{bmatrix}
0.333194 & -0.245714 & 0 & 0 & 0 & 0 & 0 \\
-0.245714 & 0.724808 & -0.394438 & 0 & 0 & 0 & 0 \\
0 & -0.394438 & 0.858609 & -0.437807 & 0 & 0 & 0 \\
0 & 0 & -0.437807 & 0.892957 & -0.437807 & 0 & 0 \\
0 & 0 & 0 & -0.437807 & 0.858609 & -0.394438 & 0 \\
0 & 0 & 0 & 0 & -0.394438 & 0.724808 & -0.245714 \\
0 & 0 & 0 & 0 & 0 & -0.245714 & 0.333194
\end{bmatrix}
\]

Scaling: \(2 \times 10^{-2} \omega^2 = \lambda^2\)

Eigenvalues: \(\lambda_1^2 = 0.055870\); \(\lambda_2^2 = 0.158036\); \(\lambda_3^2 = 0.303705\)

$$\theta_0 = 60^\circ$$

\[
\begin{bmatrix}
0.427483 & -0.776024 & 0 & 0 & 0 & 0 & 0 \\
-0.776024 & 5.63499 & -4.059018 & 0 & 0 & 0 & 0 \\
0 & -4.059018 & 11.69522 & -6.209676 & 0 & 0 & 0 \\
0 & 0 & -6.209676 & 13.18825 & -6.209676 & 0 & 0 \\
0 & 0 & 0 & -6.209676 & 11.69522 & -4.059018 & 0 \\
0 & 0 & 0 & 0 & -4.059018 & 5.63499 & -0.776024 \\
0 & 0 & 0 & 0 & 0 & -0.776024 & 0.427483
\end{bmatrix}
\]

Scaling: \(2 \times 10^{-1} \omega^2 = \lambda^2\)

Eigenvalues: \(\lambda_1^2 = 0.18612\); \(\lambda_2^2 = 0.27378\); \(\lambda_3^2 = 0.153292\)

$$\theta_0 = 65^\circ$$

\[
\begin{bmatrix}
0.0392650 & -0.0585326 & 0 & 0 & 0 & 0 & 0 \\
-0.0585326 & 0.349022 & -0.283924 & 0 & 0 & 0 & 0 \\
0 & -0.283924 & 0.923873 & -0.528251 & 0 & 0 & 0 \\
0 & 0 & -0.528251 & 1.208170 & -0.528251 & 0 & 0 \\
0 & 0 & 0 & -0.528251 & 0.923873 & -0.283924 & 0 \\
0 & 0 & 0 & 0 & -0.283924 & 0.349022 & -0.0585326 \\
0 & 0 & 0 & 0 & 0 & -0.0585326 & 0.0392650
\end{bmatrix}
\]

Scaling: \(10^{-1} \omega^2 = \lambda^2\)

Eigenvalues: \(\lambda_1^2 = 0.016305\); \(\lambda_2^2 = 0.024666\); \(\lambda_3^2 = 0.118095\)
Scaling: \( 10^1 \omega^2 = \lambda^2 \)

Eigenvalues: \( \lambda_1^2 = 0.4669 \); \( \lambda_2^2 = 0.8250 \); \( \lambda_3^2 = 2.3658 \)

Scaling: \( 10^1 \omega^2 = \lambda^2 \)

Eigenvalues: \( \lambda_1^2 = 0.234426 \); \( \lambda_2^2 = 0.662810 \); \( \lambda_3^2 = 1.45850 \)

Scaling: \( \omega^2 = \lambda^2 \)

Eigenvalues: \( \lambda_1^2 = 0.037834 \); \( \lambda_2^2 = 0.159243 \); \( \lambda_3^2 = 0.352488 \)
APPENDIX V

Putting \( y - 2 = x \) the determinant in (89) becomes

\[
D_n = \begin{vmatrix}
x & 1 & 0 & 0 & \ldots \\
1 & x & 1 & 0 & \ldots \\
0 & 1 & x & 1 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{vmatrix} = 0
\]

This determinant is of the \( n \)th order and can be expanded to obtain a recursion formula, viz.

\[ D_n = xD_{n-1} - D_{n-2} \quad (V-1) \]

where \( D_{-n} = 0, \ D_0 = 1, \ D_1 = x \). In order to find a solution of equation (V-1), one writes

\[ D_n = u^n \sum_{k} a_k u^k \quad (V-2) \]

Substitution then yields, after cancelling the terms on both sides,
\[ u^n = xu^{n-1} - u \]  \hspace{1cm} (V-3)

Therefore for \( n = 2 \)

\[ u^2 = xu - 1 \]

or

\[ x = D_1 = u + 1/u = u^{-1}(1+u^2) \]  \hspace{1cm} (V-4)

From this it follows from equation (V-1), that

\[ D_2 = xD_1 - 1 = u^{-2}(1+u^2+u^4) \]

Therefore by induction

\[ D_n = u^{-n} (1+u^2+u^4+\cdots+u^{2n}) \]  \hspace{1cm} (V-5)

\[ = u^{-n} \left\{ \frac{u^{2n+2} - 1}{u^2 - 1} \right\} \]  \hspace{1cm} (V-6)

As required this expression will vanish if the numerator is
zero without the denominator being zero at the same time.

Hence the solution is

\[ u = e^{i \frac{\pi k}{M+1}} \quad k = 1, 2, \ldots, M \quad (V-7) \]

Substituting (V-7) into equation (V-4) one obtains

\[ \chi = u + \frac{1}{u} = 2 \cos \frac{\pi k}{M+1} \]

or since

\[ \chi = 2 \]

\[ \chi^2 - 2 \]

As an illustrative example let us take the simple case given by

\[ \begin{vmatrix} \chi^2 - 2 & 1 \\ 1 & \chi^2 - 2 \end{vmatrix} = 0 \]

Expanding yields the equation

\[ (\chi^2 - 2)^2 - 1 = 0 \]
which has the solutions $\gamma_1^2 = 3$ and $\gamma_2^2 = 1$. On the other hand from equation (V-8) with $M = 2$ for the order of the determinant one obtains

\[ \gamma_1^2 = 2 + 2 \cos \frac{\pi}{3} = 3 \]
\[ \gamma_2^2 = 2 + 2 \cos 2\frac{\pi}{3} = 1 \]
To evaluate the integral (110) using Simpson's rule it is necessary to compute values of the integrand at equally spaced points depending on the length of the interval $\Delta \mathcal{S}$. Since the integrand contains powers of $\psi_c$ being the solutions of the cubic equation (61) it was essential to incorporate a subroutine into the program which by means of the Newton-Raphson method solved equation (61) whenever necessary. The program was written in such a way that after inputting $\mathcal{S}_0$, $\Delta \mathcal{S}$ and $\psi_c$ the values of $\psi_c$ and those of the integrand (I) were computed and their numerical values printed out. The advancing from $\mathcal{S}$ to $\mathcal{S} + \Delta \mathcal{S}$ was done automatically. The results of the computation for different lines of force are given in table VI-a.
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APPENDIX VII

\( \delta = 10^\circ \)

\[
\begin{bmatrix}
0.238217 & -0.147909 & 0 & 0 & 0 & 0 & 0 \\
-0.147909 & 0.367344 & -0.205472 & 0 & 0 & 0 & 0 \\
0 & -0.205472 & 0.459717 & -0.254037 & 0 & 0 & 0 \\
0 & 0 & -0.254037 & 0.561512 & -0.254037 & 0 & 0 \\
0 & 0 & 0 & -0.254037 & 0.459717 & -0.205472 & 0 \\
0 & 0 & 0 & 0 & -0.205472 & 0.367344 & -0.147909 \\
0 & 0 & 0 & 0 & 0 & -0.147909 & 0.238217 \\
\end{bmatrix}
\]

Scaling: \( 10^1 \omega^2 = \lambda^2 \)

Eigenvalues:
\( \lambda_1^2 = 0.031931 \); \( \lambda_2^2 = 0.096597 \); \( \lambda_3^2 = 0.201148 \)

\( \delta = 20^\circ \)

\[
\begin{bmatrix}
0.057226 & -0.121296 & 0 & 0 & 0 & 0 & 0 \\
-0.121296 & 1.028404 & -0.616504 & 0 & 0 & 0 & 0 \\
0 & -0.616504 & 1.478328 & -0.894182 & 0 & 0 & 0 \\
0 & 0 & -0.894182 & 2.163406 & -0.894182 & 0 & 0 \\
0 & 0 & 0 & -0.894182 & 1.478328 & -0.616504 & 0 \\
0 & 0 & 0 & 0 & -0.616504 & 1.028404 & -0.121296 \\
0 & 0 & 0 & 0 & 0 & -0.121296 & 0.057226 \\
\end{bmatrix}
\]

Scaling: \( 10^1 \omega^2 = \lambda^2 \)

Eigenvalues:
\( \lambda_1^2 = 0.025495 \); \( \lambda_2^2 = 0.037008 \); \( \lambda_3^2 = 0.225847 \)
\[ \delta_0 = 30^\circ \]

\[
\begin{array}{cccccccc}
0.01250 & -0.09349 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.09349 & 2.79640 & -2.05449 & 0 & 0 & 0 & 0 & 0 \\
0 & -2.05449 & 6.03768 & -4.00495 & 0 & 0 & 0 & 0 \\
0 & 0 & -4.00495 & 10.62632 & -4.00495 & 0 & 0 & 0 \\
0 & 0 & 0 & -4.00495 & 6.03768 & -2.05449 & 0 & 0 \\
0 & 0 & 0 & 0 & -2.05449 & 2.79640 & -0.09349 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.09349 & 0.01250 & 0 \\
\end{array}
\]

**Scaling:** \( 10^1 \omega^2 = \lambda^2 \)

**Eigenvalues:** \( \lambda_1^2 = 0.00620 \); \( \lambda_2^2 = 0.00830 \); \( \lambda_3^2 = 0.74523 \)

\[ \delta_0 = 40^\circ \]

\[
\begin{array}{cccccccc}
0.01318 & -0.08326 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.08326 & 2.10324 & -4.22665 & 0 & 0 & 0 & 0 & 0 \\
0 & -4.22665 & 33.97541 & -23.75178 & 0 & 0 & 0 & 0 \\
0 & 0 & -23.75178 & 66.41804 & -23.75178 & 0 & 0 & 0 \\
0 & 0 & 0 & -23.75178 & 33.97541 & -4.22665 & 0 & 0 \\
0 & 0 & 0 & 0 & -4.22665 & 2.10324 & -0.08326 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.08326 & 0.01318 & 0 \\
\end{array}
\]

**Scaling:** \( 10^1 \omega^2 = \lambda^2 \)

**Eigenvalues:** \( \lambda_1^2 = 0.00655 \); \( \lambda_2^2 = 0.00877 \); \( \lambda_3^2 = 0.97673 \)

\[ \delta_0 = 50^\circ \]

\[
\begin{array}{cccccccc}
0.00321 & -0.01016 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.01016 & 0.12845 & -0.69173 & 0 & 0 & 0 & 0 & 0 \\
0 & -0.69173 & 14.90035 & -11.46859 & 0 & 0 & 0 & 0 \\
0 & 0 & -11.46859 & 35.30886 & -11.46859 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & -0.69173 & 0.12845 & -0.01016 & 0 \\
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\end{array}
\]

**Scaling:** \( \omega^2 = \lambda^2 \)

**Eigenvalues:** \( \lambda_1^2 = 0.00157 \); \( \lambda_2^2 = 0.00210 \); \( \lambda_3^2 = 0.06522 \)
\( \lambda_0 = 60^\circ \)

\[
\begin{array}{ccccccc}
0.01909 & -0.02680 & 0 & 0 & 0 & 0 & 0 \\
-0.02680 & 0.15046 & -0.35889 & 0 & 0 & 0 & 0 \\
0 & -0.35889 & 3.42428 & -5.32325 & 0 & 0 & 0 \\
0 & 0 & -5.32325 & 33.10140 & -5.32325 & 0 & 0 \\
0 & 0 & 0 & 5.32325 & 3.42428 & -0.35889 & 0 \\
0 & 0 & 0 & 0 & -0.35889 & 0.15046 & -0.02680 \\
0 & 0 & 0 & 0 & 0 & -0.02680 & 0.01909 \\
\end{array}
\]

Scaling: \( \omega^2 = \lambda^2 \)

Eigenvalues: \( \lambda_1^2 = 0.00829; \lambda_2^2 = 0.01197; \lambda_3^2 = 0.08253 \)

\( \lambda_0 = 70^\circ \)

\[
\begin{array}{ccccccc}
0.266399 & -0.082840 & 0 & 0 & 0 & 0 & 0 \\
-0.082840 & 0.103041 & -0.118609 & 0 & 0 & 0 & 0 \\
0 & -0.118609 & 0.546117 & -0.305288 & 0 & 0 & 0 \\
0 & 0 & -0.305288 & 0.682647 & -0.305288 & 0 & 0 \\
0 & 0 & 0 & -0.305288 & 0.546117 & -0.118609 & 0 \\
0 & 0 & 0 & 0 & -0.118609 & 0.103041 & -0.082840 \\
0 & 0 & 0 & 0 & 0 & -0.082840 & 0.266399 \\
\end{array}
\]

Scaling: \( \omega^2 = \lambda^2 \)

Eigenvalues: \( \lambda_1^2 = 0.018424; \lambda_2^2 = 0.044137; \lambda_3^2 = 0.211214 \)
REFERENCES


Parker, E.N. (1958) Interaction of the solar wind with the geomagn. field, Physics of Fluids 1, 171-187.