A THEORETICAL STUDY OF THE REACTION $D(p\gamma)He^3$

by

DAVID HAYWARD RENDELL

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Atomic Energy of Canada, Ltd.

Chalk River, Ontario.

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ABSTRACT

Experimental studies of the reaction $D(p\gamma)He^3$ (Fowler, et al., 1949; Wilkinson, 1952; Griffiths and Warren, 1955; Griffiths, Larson and Robertson, 1961) indicate that two different transitions contribute to the total yield. While the transition giving the larger yield has been shown, by the angular distribution and polarization of the gamma radiation, to be an electric dipole transition there is insufficient experimental evidence to determine definitely the characteristics of the other transition.

In this theoretical study the angular distributions of the gamma radiation are calculated and numerical estimates of the cross-sections are made for all transitions which might conceivably contribute to the reaction. It is concluded that the larger experimental yield comes from the electric dipole transition of a P-wave proton and ^3S deuteron to the ^2S state of He^3. The smaller part of the yield is most likely from the magnetic dipole transition of a S-wave proton and ^3S deuteron to the ^2S state of He^3 although part of the yield might be contributed by an electric dipole transition of a P-wave proton and ^3S deuteron to the ^4D state of He^3.

GRADUATE STUDIES

Field of Study: Theoretical Physics

Electromagnetic Theory	G.M. Volkoff
Theory of Measurements	A.M. Crooker
Nuclear Physics	J.B. Warren
Elementary Quantum Mechanics	F.A. Kaempffer
Theory of Relativity	P. Rastall
Group Theoretical Methods	W. Opechowski

Related Studies:

Analytic Matrix Theory	M.D. Marcus
Non-Linear Mechanics	E. Leimanis
Numerical Analysis	C. Froese

ABSTRACT

A theoretical study of the reaction $D(p\gamma)He^3$ is made in an attempt to explain the experimental data for the reaction obtained by Fowler et al. (1949), Wilkinson (1952), Griffiths and Warren (1955) and Griffiths, Larson and Robertson (1961). The angular distribution of the emitted gamma radiation, measured with respect to the incident proton beam, is predominantly proportional to $\sin^2\theta$. Measurements of the polarization of the radiation by Wilkinson (1952) indicate that the $\sin^2\theta$ component is electric dipole radiation. In addition there is a small, possibly isotropic, component. The proportion of the total yield coming from the smaller 'isotropic' component is 0.035 at a proton energy of 1 Mev, and this proportion increases with decreasing proton energy.

The $\sin^2\theta$ component has been interpreted by Griffiths and Warren as coming from an electric dipole transition from an initial state of a P-wave proton ($L = 1$, $L_z = 0$) and ^3S deuteron to the ^2S ground state of He^3. This interpretation is supported by the present calculations. They also suggest that the smaller 'isotropic' component could be either a magnetic dipole transition of S-wave protons to the ^2S state of He^3 or an electric dipole transition involving spin-orbit coupling.

In this present work the cross-sections are examined for all possible channels which might conceivably contribute to the reaction. The channels considered are

1. electric dipole transitions for
   a. P-wave protons to the ^2S state
   b. P-wave protons to the ^4D state
   c. F-wave protons to the ^4D state

2. electric quadrupole transitions for
   a. S-wave protons to the ^4D state
   b. D-wave protons to the ^2S state

3. the magnetic dipole transition for S-wave protons to the ^2S state.

Three-body wave functions are constructed, following Verde (1950) and Derrick and Blatt (1958), making use of the symmetry properties in spin space, isotropic spin space and in ordinary space. In addition to the states of total isotropic spin T = ½ considered by Derrick and Blatt the states of total isotopic spin T = 3/2 are included. The radiation matrix elements for the above channels are calculated and are expressed in terms of integrals over the three internal coordinates. These radial integrals are estimated by using very simple radial functions which are valid outside the range of the nuclear forces and which also disregard coulomb forces. The cross-sections depend on the unknown amplitudes and relative signs of the various possible symmetry states. Therefore the size, although not the angular dependance or the general energy dependance, of the cross-sections can be used only as an order-of-magnitude estimate.

By comparison of the size, angular distribution and energy dependance of the calculated cross-sections with the experimental data it is shown conclusively that the $\sin^2\theta$ component of the radiation comes from the electric dipole transition of P-wave protons to the ^2S state of He^3. The smaller 'isotropic' component of the radiation comes from either (a) an electric dipole transition of P-wave protons to the ^4D state, giving an angular distribution proportional to $1 - (1/7)\cos^2\theta$, or (b) a magnetic dipole transition of S-wave protons to the ^2S state, giving an isotropic angular distribution. The observed energy dependence of the relative yield of the small component suggests the interpretation in terms of the magnetic dipole transition. The cross-sections of the other transitions examined are too small to explain the experimental results.

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TABLE OF CONTENTS

INTRODUCTION

The reaction $D(p\gamma)He^3$ has been studied experimentally by several observers: Curran and Strothers (1939), Fowler, Lauristen and Tollestrup (1949), Wilkinson (1951), Griffiths and Warren (1955), Griffiths, Larsen and Robertson (1961). The protons are accelerated and fired at a D_2O ice or D_2 gas target. The emitted gamma rays are observed by standard scintillation counter techniques and the intensity of the emitted radiation is measured as a function of the angle with the incident proton beam (Fig. 1). In this way an attempt is made to obtain information about the bound state of He^3, the continuum states which contribute to the reaction and the detailed mechanism of the reaction.

Fowler et al. (1949) and Griffiths and Warren (1955) have measured the yield and angular distribution of the gamma radiation for proton energies from 0.2 to 2 Mev. More recent measurements have been made by Griffiths, Larson and Robertson (1961). With the direction of the proton beam as an axis, the angular distribution is mainly proportional to $\sin^2 \theta$ (Fig. 1). There is a small, possibly isotropic, component which contributes a small percentage of the total yield. Wilkinson (1952) measured the polarization of the radiation at 90° to the direction of the proton beam and found that the polarization was mainly in the plane of the reaction, i.e. the plane containing the directions of the proton beam and the observed radiation. This polarization and the $\sin^2 \theta$ distribution of the main contribution to the

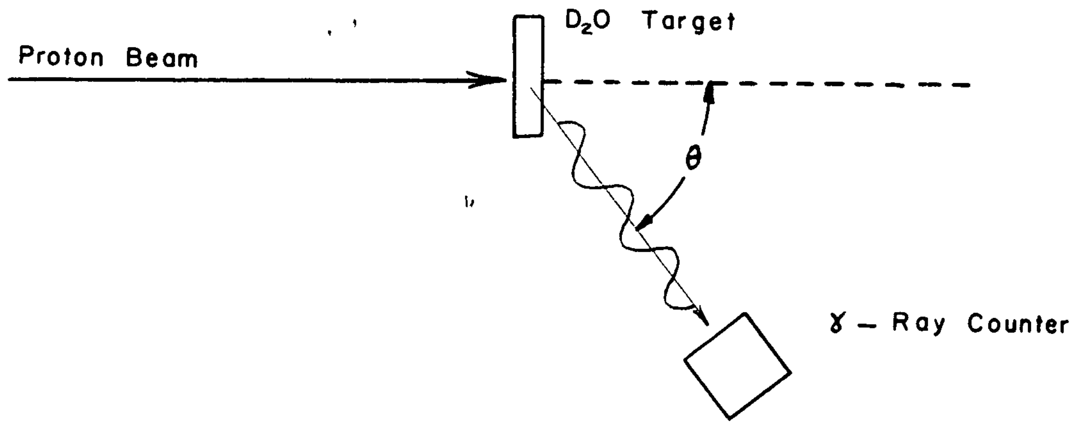

FIG. 1

Schematic experimental arrangement for studying the reaction $D(p\gamma)He^3$.

reaction indicate that the radiation is mainly electric dipole radiation. The isotropic contribution to the total yield decreases with increasing proton energy. The exact angular distribution and polarization of this smaller component have not been measured. Griffiths and Warren suggest that the simplest assumption for the reaction is an electric dipole transition from a P-wave proton ($L = 1$, $L_z = 0$) and ^3S deuteron (triplet spin state, $L = 0$) to the ^2S ground state of He^3. This would give a pure $\sin^2 \theta$ distribution. They also suggest that the smaller component could be either a magnetic dipole transition of S-wave protons to the ^2S state of He^3 or an electric dipole transition involving spin-orbit coupling which would introduce a small component proportional to $1 + \cos^2 \theta$.

The purpose of the present work was to examine in detail all channels which might conceivably contribute to the reaction and to close several gaps in the existing literature on the subject. Previous works on the nuclear reactions of three-body systems have dealt with:

1. the mirror reaction $D(n\gamma)H^3$ (Schiff, 1937; Höcker, 1942; Burhop and Massey, 1948; Verde, 1950; Austern, 1951, 1952),
2. The photodisintegration of H^3 (Verde, 1950; Delves, 1960),
3. and the photodisintegration of He^3 (Verde, 1950; Delves, 1960).

Both Verde and Delves have calculated the cross section for the electric dipole transition between the ^2P continuum and ^2S bound states of He^3 and H^3. Verde has shown that the magnetic dipole transition from a ^4S continuum state to a ^2S bound state is unimportant because it does not involve the part of the wave function which is completely symmetric in the internal coordinates. The symmetric part of the wave function is expected to be the dominant part. This supports the calculation of Burhop and Massey who found that, for the capture of neutrons in the 1 Mev range, the electric dipole transition is more important than the magnetic dipole transition. The neglect of the magnetic dipole transition by Verde and Delves may be contrasted with the estimates given without detailed explanation by Austern for the capture of thermal neutrons. Austern states that at this energy the magnetic dipole transition is the more important while the electric quadrupole transition is approximately 10^{-5} times smaller. The extent to which these estimates for nD capture may be applied to pD capture is uncertain. At low energies the coulomb barrier should cause a considerable difference between the two reactions.

In the present investigation the calculations of the above authors have been extended by calculating and numerically estimating the cross sections for the following transitions:

1. electric dipole transition for
    a. P-wave protons to the ^2S state
    b. P-wave protons to the ^4D state
    c. F-wave protons to the ^4D state

2. electric quadrupole transition for

   a. S-wave protons to the ^4D state

   b. D-wave protons to the ^2S state

3. magnetic dipole transition for S-wave protons to the ^2S state.

Chapter I contains a classification of three-body wave functions, making use of the symmetry properties of the functions in spin space, isotopic spin space and ordinary space, as done by Verde (1950) and Derrick and Blatt (1958). The classification of Derrick and Blatt is more detailed than Verde's, but does not include states of total isotopic spin $T = 3/2$. These states are included in the present work. The definition of the internal coordinates and the Euler angles differ from those of Derrick and Blatt, being less symmetrical but more convenient for the description of the continuum states for a deuteron and an unbound proton.

In the calculation of possible radiation matrix elements, contained in Chapter 2, integration and summation over the Euler angles and the spin and the isotopic spin spaces can be done in closed form leaving an integration over the three internal coordinates. Rough estimates of the radial integrals are also made in Chapter 2. To make these estimates, very simple radial functions were used: for the deuteron, an exponential function (e.g. Blatt and Weisskopf, 1952, Ch. 2), and for the third particle either a plane wave for the continuum states or an exponential function for the bound states. These functions are valid outside the range of the nuclear forces, the effects of which enter the wave functions only through the binding energies which determine the constants in the exponential terms. The effects of the coulomb repulsion are neglected.

A discussion of the results and comparison with experimental data are contained in Chapter 3. It is concluded that most important is the electric dipole transition for P-wave protons to the ^2S state, the next largest contribution coming either from an electric dipole transition for P-wave protons to the ^4D state or from a magnetic dipole transition for S-wave protons to the ^2S state. The sizes of the cross sections for the transitions depend on the amplitudes and relative signs of the various possible symmetry states. To obtain this information, it would be necessary to assume some definite spatial dependence for the nuclear forces and to examine the solutions of the twenty-four coupled Schrödinger equations for the radial functions. This was beyond the scope of the present work.

CHAPTER 1

THE THREE BODY PROBLEM

The nuclear three body problem in the non-relativistic approximation is basically the problem of finding eigensolutions of the Schrödinger equation[1]

$$(1.0.1) \quad \left\{ -\frac{1}{2M_1}\nabla_1^2 - \frac{1}{2M_2}\nabla_2^2 - \frac{1}{2M_3}\nabla_3^2 + V \right\} \Psi_E = E\,\Psi_E$$

Once Ψ_E is known, matrix elements of the radiation operators may be calculated. The equation (1.0.1) is intractable in the form in which it is written. It becomes more tractable by choosing a co-ordinate system which allows a separation of the centre-of-mass motion and a partial separation of the rigid 'body' rotations of the system. In section 1, these co-ordinates are defined and the Laplacian and angular momentum operators are derived. Section 2 contains the method of construction of antisymmetric wave functions, making use of the spin and isotopic spin formalism (Blatt & Weisskopf, 1952, p. 153). In section 3, the formal solution of the Schrödinger equation is given for unbound and bound states. These solutions are used in Chapter 2 in the calculation of the matrix elements of the radiation operators.

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(1) Natural units, $\hbar = c = 1$, are used throughout this work. This convention simplifies somewhat the appearance of the formulae.

1. Spatial Description

A. Definition of Co-ordinates

It will be assumed that protons and neutrons can be treated as equal point masses. The system of three nucleons may then be described spatially as the location of three equal point masses. It is convenient to use the three right-handed co-ordinate frames (Fig.2):

i. the laboratory frame, an inertial frame fixed in the laboratory;

ii. the centre-of-mass frame, a moving, non-rotating frame with the origin fixed at the centre of mass of the three particles;

iii. the 'body' frame, a rotating frame with the origin coinciding with the origin of the centre-of-mass frame and with the axes coinciding with the principal axes of the triangle formed by the three particles.

The notation for the cartesian components of a point is: x', y' and z' for the laboratory frame; x, y and z for the centre-of-mass frame; and X, Y and Z for the 'body' frame. The position vectors of the three particles in the laboratory frame are denoted by $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$.

The following nine co-ordinates are adopted:

i. the three co-ordinates of the position vector $\underline{s} = 1/3(\underline{r}_1 + \underline{r}_2 + \underline{r}_3)$ of the centre of mass in the laboratory frame;

ii. the three Euler angles, α, β and γ giving the rotation of the 'body' frame with respect to the centre-of-mass frame (Fig. 2);

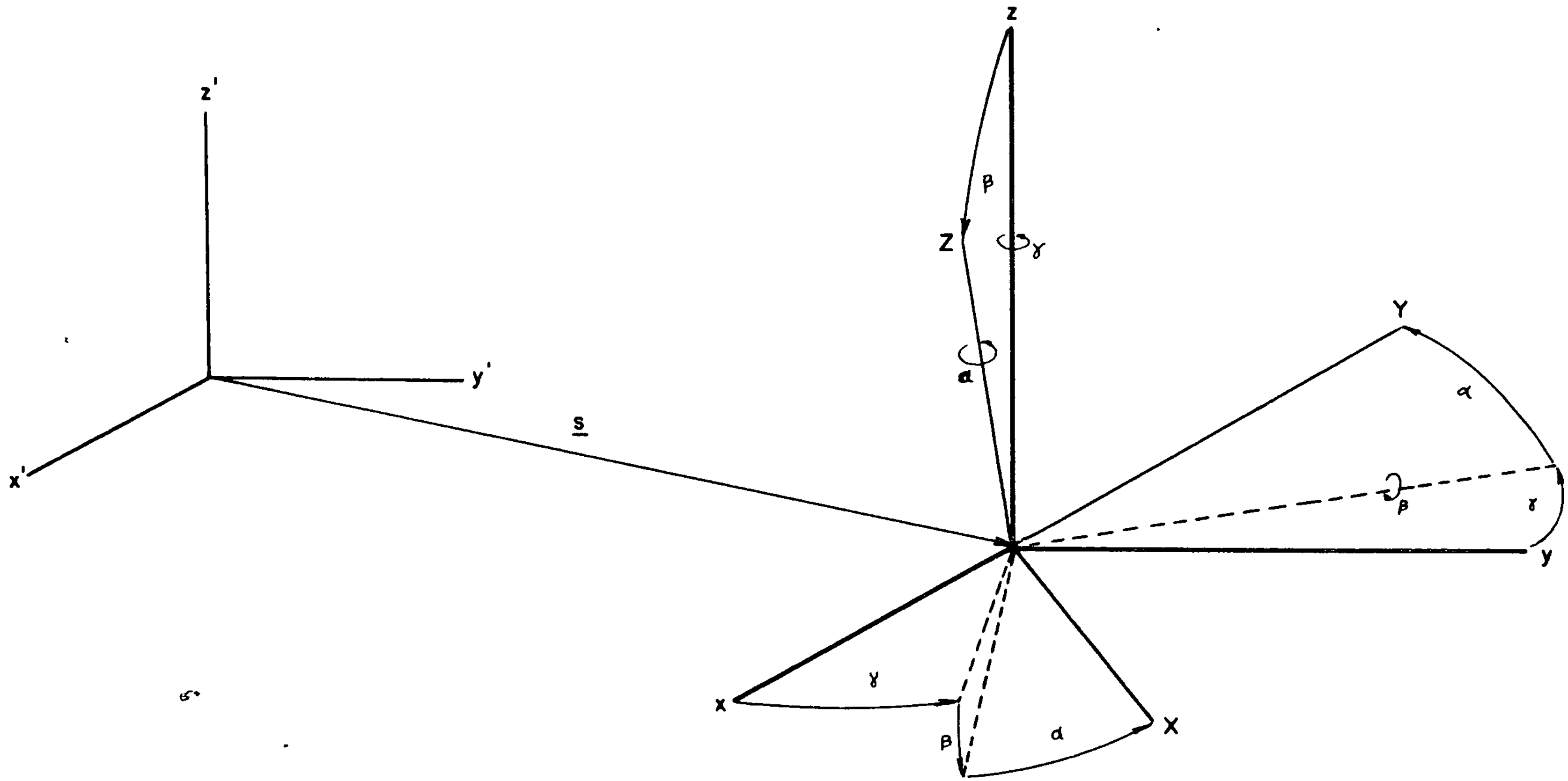

FIG 2

Showing the relationship of the three coordinate frames.

to follow page 7

iii. the three internal co-ordinates, $r = \left|\underline{r}_1 - \underline{r}_2\right|$, $q = \left|\tfrac{1}{2}\underline{r}_1 + \tfrac{1}{2}\underline{r}_2 - \underline{r}_3\right|$ and θ ($\cos\theta = \underline{r}.\underline{q}/rq$) giving the size and shape of the triangle. (Fig. 3).

The choice of Euler angles is that illustrated by Wigner (1959, p. 90) and may be defined as a positive rotation of the 'body' frame, viewed from the laboratory frame, through the angles:

γ about the Z axis,

β about the new position of the Y axis,

and α about the new position of the Z axis.

The rotations are to be performed in the indicated order. When viewed from the 'body' frame, the equivalent rotation is:

$-\alpha$ about the Z axis,

$-\beta$ about the Y axis,

and $-\gamma$ about the Z axis.

The choice of internal co-ordinates is particularly convenient for the discussion of continuum states consisting of a deuteron and an unbound proton. They are defined so that the three particles are in the XZ plane and that, for large separation of the deuteron and the proton, the particles are on the Z axis. Neglecting temporarily the effects of permuting the position of the particles (see sec. 2E), r is the distance between the neutron and the proton (Fig. 3, particles 1 and 2), q is the distance between the other proton (particle 3) and the centre of mass of the deuteron, and θ is the angle formed by the vectors $1 - C_D$ and $3 - C_D$ as indicated in Figure 3. To orient the triangle in the 'body' frame, the three principal

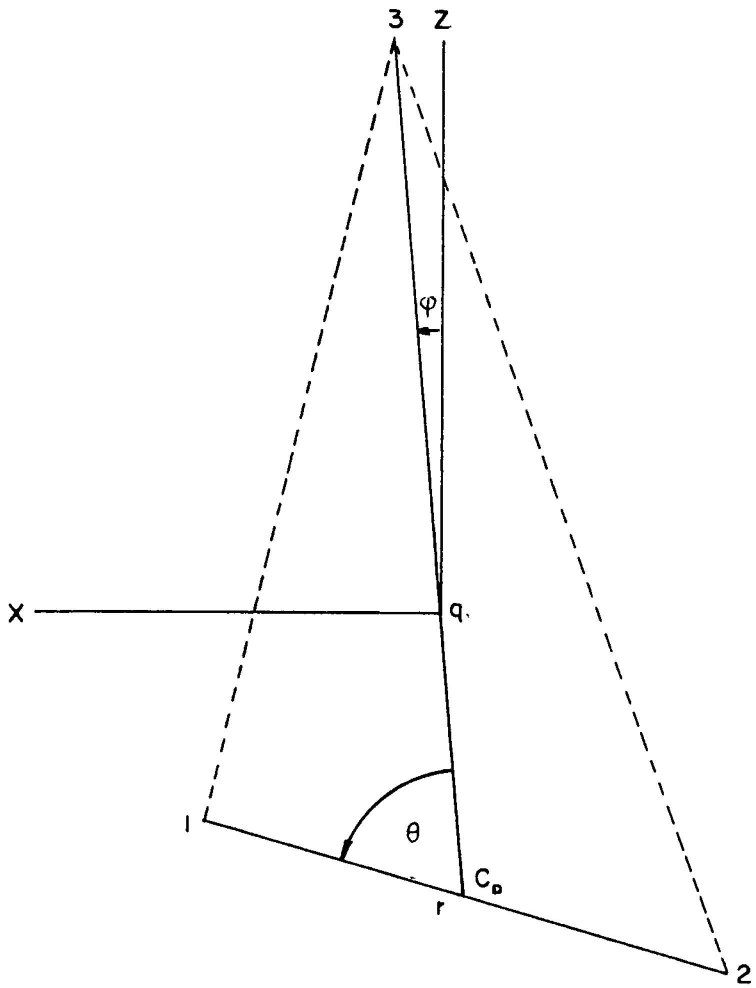

FIG. 3.

Showing the internal coordinates and the orientation of the triangle in the XZ-plane of the body frame. The Y-axis is directed out of the paper.

axes of inertia are computed. The axis associated with the largest moment of inertia is defined as the Y axis, that associated with the intermediate moment as the X axis and the remaining one as the Z axis. The directions of these axes are not yet fixed by this requirement. The positive Y axis is defined as the direction of motion of a right-handed screw rotating from particles 1, 2 and 3 in that order. The positive Z axis is defined so that the angle ϕ, the angle between $\underset{\sim}{q}$ and the Z axis (Fig. 3), is within the limits $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$. The positive X axis is fixed by the requirement that the X, Y and Z axes form a right-handed frame. Formulae for ϕ are given in section 1C.

B. Transformation Formulae

With the definitions adopted in the preceding section the following transformation rules are established by inspection.

i. Transforming from the centre-of-mass frame to the laboratory frame introduces a simple translation:

$$x' = s_x + x$$

(1.1.1) $$y' = s_y + y$$

$$z' = s_z + z$$

ii. Transforming from the 'body' frame to the centre-of-mass frame after the rotation $\{\alpha, \beta, \gamma\}$ consists of the three rotations:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ +\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & +\sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ +\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\sin\alpha\cos\beta\cos\gamma - \cos\alpha\sin\gamma & \sin\beta\cos\gamma \\ \cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\beta\sin\gamma \\ -\cos\alpha\sin\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

(1.1.2)

$$= \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

In matrix notation, $\underline{x} = R\underline{X}$. The matrix for the inverse transformation is the transpose of the above matrix, $R^{-1} = R^T$

iii. Transforming from the internal co-ordinates, r, q and θ, to the cartesian 'body' co-ordinates, X, Y and Z, of the three particles are:

(1.1.3)

$$X_1 = -\frac{q}{3}\sin\phi + \frac{r}{2}\sin(\phi+\theta) \qquad Y_1 = 0 \qquad Z_1 = -\frac{q}{3}\cos\phi + \frac{r}{2}\cos(\phi+\theta)$$

$$X_2 = -\frac{q}{3}\sin\phi - \frac{r}{2}\sin(\phi+\theta) \qquad Y_2 = 0 \qquad Z_2 = -\frac{q}{3}\cos\phi - \frac{r}{2}\cos(\phi+\theta)$$

$$X_3 = \frac{2q}{3}\sin\phi \qquad Y_3 = 0 \qquad Z_3 = \frac{2q}{3}\cos\phi$$

C. <u>Moments of Inertia</u>

To show in detail how the triangle formed by the three nucleons is located and orientated in the 'body' frame, the three principal moments of inertia are now calculated.

Since by definition the 'body' axes coincide with the principal axes of inertia, the three products of inertia (Goldstein, 1950, p. 145) are zero.

$$(1.1.4)\quad \sum_i M_i X_i Y_i = \sum_i M_i Y_i Z_i = \sum_i M_i Z_i X_i = 0$$

Since the three nucleons are defined as lying in the XZ plane, $Y_i = 0$ and the first two of the products of inertia (1.1.4) are identically zero. By expressing the third of these equations in terms of the internal co-ordinates (1.1.3) useful formulae for ϕ are obtained as follows:

$$\begin{aligned} 0 &= \sum_i M_i X_i Z_i \\ (1.1.5)\qquad &= M\left\{\tfrac{2}{3} q^2 \sin\phi\cos\phi + r^2 \sin(\phi+\theta)\cos(\phi+\theta)\right\} \\ &= M\left\{\left(\frac{q^2}{3} + \frac{r^2}{4}\cos 2\theta\right)\sin 2\phi + \left(\frac{r^2}{4}\sin 2\theta\right)\cos 2\phi\right\} \end{aligned}$$

from which

$$(1.1.6)\quad \sin 2\phi = \frac{-\frac{3}{2} r^2 \sin 2\theta}{\Lambda} \qquad\qquad \cos 2\phi = \frac{2q^2 + \frac{3}{2} r^2 \cos 2\theta}{\Lambda}$$

where

$$(1.1.7)\quad \Lambda^2 = R^4 - 48 A^2$$

$$(1.1.8)\quad R^2 = 2q^2 + \tfrac{3}{2} r^2$$

$$(1.1.9)\quad A = \tfrac{1}{2} rq \sin\theta \qquad = \text{area of triangle.}$$

For both R and Λ the positive roots are used in order that ϕ be restricted to the range $-\frac{\pi}{2} < \phi \leq \frac{\pi}{2}$, removing an ambiguity in the direction of the Z axis.

The three principal moments of inertia may now be calculated in terms of the internal co-ordinates.

$$M_X = \sum_i M_i\left(Y_i^2 + Z_i^2\right)$$

$$= M\left\{\tfrac{2}{3}q^2\cos^2\phi + \tfrac{1}{2}r^2\cos^2(\phi+\theta)\right\} \tag{1.1.10}$$

$$= M\left\{\tfrac{1}{3}q^2(1+\cos 2\phi) + \tfrac{1}{4}r^2\left(1+\cos 2\phi\cos 2\theta - \sin 2\phi\sin 2\theta\right)\right\}$$

$$= \frac{M}{6}\left(R^2+\Lambda\right)$$

$$M_Y = \sum_i M_i(Z_i^2 + X_i^2)$$

$$= M\left(\tfrac{2}{3}q^2 + \tfrac{r^2}{2}\right) = \tfrac{1}{3}MR^2 \tag{1.1.11}$$

$$M_Z = \sum_i M_i(X_i^2 + Y_i^2)$$

$$= M\left\{\tfrac{2}{3}q^2\sin^2\phi + \tfrac{1}{2}r^2\sin^2(\phi+\theta)\right\} \tag{1.1.12}$$

$$= M\left\{\tfrac{1}{3}q^2(1-\cos 2\phi) + \tfrac{1}{4}r^2\left(1-\cos 2\phi\cos 2\theta + \sin 2\phi\sin 2\theta\right)\right\}$$

$$= \frac{M}{6}\left(R^2-\Lambda\right)$$

It will be found convenient to use the quantities λ_X, λ_Y and λ_Z defined as:

$$\lambda_X = \frac{M_X}{M} = \frac{1}{6}\left(R^2+\Lambda\right)$$

$$\lambda_Y = \frac{M_Y}{M} = \frac{1}{3}R^2 \tag{1.1.13}$$

$$\lambda_Z = \frac{M_Z}{M} = \frac{1}{6}\left(R^2-\Lambda\right)$$

D. <u>Angular Momentum Operators</u>

In the centre-of-mass frame, the total orbital angular momentum is considered to be that of a rigid body rotating with the 'body' frame. This is possible, although the three nucleons do not form a rigid body, because, from the definition of the 'body' frame (sec. 1A), the total angular momentum of the three nucleons in the 'body' frame is zero. The operators for the components of the total angular momentum in the centre-of-mass frame will now be obtained in terms of the Euler angles. For later use, the operators for the components of the total angular momentum about the instantaneous positions of the body axes are also given.

The operators are obtained in the usual manner (Edmonds, 1957, p. 13) by considering two equivalent rotations:

i. the rotation $\{\alpha, \beta, \gamma\}$ through the Euler angles with the corresponding operators for infinitesimal rotations $\frac{\partial}{\partial\alpha}$, $\frac{\partial}{\partial\beta}$ and $\frac{\partial}{\partial\gamma}$,

ii. and the rotation $\{\alpha_x, \alpha_y, \alpha_z\}$ the angles being measured about the fixed centre-of-mass axes with the corresponding operators for infinitesimal rotations $\frac{\partial}{\partial\alpha_x}$, $\frac{\partial}{\partial\alpha_y}$ and $\frac{\partial}{\partial\alpha_z}$ (Fig. 4).

The latter operators are proportional to the operators for the components of the total angular momentum in the centre-of-mass frame as follows:

(1.1.14) $$L_x = -i\frac{\partial}{\partial\alpha_x} \qquad L_y = -i\frac{\partial}{\partial\alpha_y} \qquad L_z = -i\frac{\partial}{\partial\alpha_z}$$

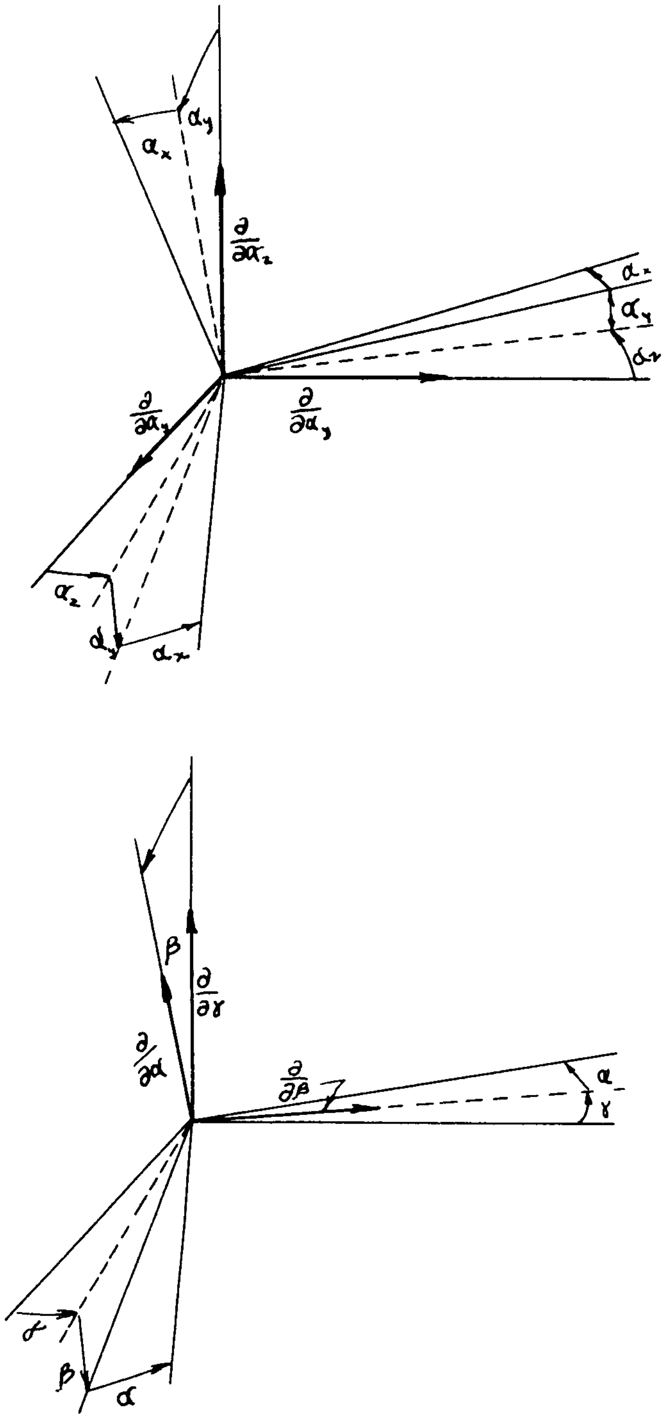

FIG 4

Illustrating the operators for infinitesimal rotations.

Since for proper rotations the operators for infinitesimal rotations can be treated as vectors, $\frac{\partial}{\partial\alpha}$, $\frac{\partial}{\partial\beta}$ and $\frac{\partial}{\partial\gamma}$ can be expressed in terms of $\frac{\partial}{\partial\alpha_x}$, $\frac{\partial}{\partial\alpha_y}$ and $\frac{\partial}{\partial\alpha_z}$ in the following manner (Fig. 4).

$$
\begin{aligned}
\frac{\partial}{\partial\alpha} &= \sin\beta\cos\gamma\frac{\partial}{\partial\alpha_x} + \sin\beta\sin\gamma\frac{\partial}{\partial\alpha_y} + \cos\beta\frac{\partial}{\partial\alpha_z}\\
\frac{\partial}{\partial\beta} &= -\sin\gamma\frac{\partial}{\partial\alpha_x} + \cos\frac{\partial}{\partial\alpha_y}\\
\frac{\partial}{\partial\gamma} &= \frac{\partial}{\partial\alpha_z}
\end{aligned}
\tag{1.1.15}
$$

The equations (1.1.15) may be inverted and then substituted in (1.1.14) to give:

$$
\begin{aligned}
L_x &= -\iota\left\{\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} - \sin\gamma\frac{\partial}{\partial\beta} - \cot\beta\cos\gamma\frac{\partial}{\partial\gamma}\right\}\\
L_y &= -\iota\left\{\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \cos\gamma\frac{\partial}{\partial\beta} - \cot\beta\sin\gamma\frac{\partial}{\partial\gamma}\right\}\\
L_z &= -\iota\left\{\frac{\partial}{\partial\gamma}\right\}
\end{aligned}
\tag{1.1.16}
$$

The equations (1.1.16) may then be transformed using (1.1.2) to give a set of operators which appear in the Laplacian.

$$
\begin{aligned}
L_4 \equiv L_X &= -\iota\left\{\cos\alpha\cot\beta\frac{\partial}{\partial\alpha} + \sin\alpha\frac{\partial}{\partial\beta} - \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right\}\\
L_5 \equiv L_Y &= -\iota\left\{-\sin\alpha\cot\beta\frac{\partial}{\partial\alpha} + \cos\alpha\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right\}\\
L_6 \equiv L_Z &= -\iota\left\{\frac{\partial}{\partial\alpha}\right\}
\end{aligned}
\tag{1.1.17}
$$

For both sets of operators (1.1.16) and (1.1.17), the operator for the square of the angular momentum is:

(1.1.18)
$$L^2 = -\left\{\frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\gamma^2}\right) + \frac{\partial^2}{\partial\beta^2} - \frac{2\cos\beta}{\sin^2\beta}\frac{\partial^2}{\partial\alpha\partial\gamma} + \cot\beta\frac{\partial}{\partial\beta}\right\}$$

The commutation rules for the two sets of operators are:

(1.1.19) $[L_x, L_y] = iL_z$ and cyclic permutations.

(1.1.20) $[L_X, L_Y] = -iL_Z$ and cyclic permutations.

The minus sign in (1.1.20) arises from the fact that L_X, L_Y, and L_Z are expressed in terms of the same Euler angles as L_x, L_y and L_z. For an observer in the 'body' frame, the rotation of the centre-of-mass frame is given by $\{\alpha', \beta', \gamma'\} = \{-\gamma, -\beta, -\alpha\}$. Thus, if the angles $\{\alpha', \beta', \gamma'\}$ are used, a set of operators $L'_X = -L_X$, $L'_Y = -L_Y$ and $L'_Z = -L_Z$ result which have the usual commutation rules.

E. The Laplacian Operator

Since the Schrödinger equation involves the Laplacian operator, this operator must be evaluated in terms of the co-ordinates used here. To do this, it is necessary to obtain the metric tensor for the co-ordinates defined in section 1A. The following notation will be used: the cartesian co-ordinates of the three nucleons in the laboratory frame (x'_1, y'_1, z'_1, x'_2, y'_2, z'_2, x'_3, y'_3, z'_3) are written in this order as η_i, i = 1 ... 9 and the co-ordinates (s_x, s_y, s_z, α, β, γ, r, q, θ) are written in this order as ρ_i, i = 1 ... 9. The transformation equations (1.1.1) to (1.1.3) are then combined into nine equations of the form

(1.1.21) $\eta_i = \eta_i(\rho_j)$

Using (1.1.21) the Jacobian of the transformation is obtained:

(1.1.22)
$$J = \det \left| \frac{\partial \eta_i}{\partial \rho_j} \right| = \sin\beta \, r^2 q^2 \sin\theta$$

The detailed expressions for (1.1.21) and (1.1.22) are given in Appendix 1.

The metric tensor consists of the elements

(1.1.23) $$g_{ij} = \frac{\partial \eta_k}{\partial \rho_i} \cdot \frac{\partial \eta_k}{\partial \rho_j}$$

and, written as a matrix, can be conveniently expressed in the following manner (Derrick, 1960).

$$|g_{ij}| = \left\| \begin{matrix} I & 0 & 0 \\ 0 & S^{-1} & 0 \\ 0 & 0 & I \end{matrix} \right\| N_{ij} \left\| \begin{matrix} I & 0 & 0 \\ 0 & (S^{-1})^T & 0 \\ 0 & 0 & I \end{matrix} \right|$$

(1.1.24)

$$I = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Where

(1.1.25) $$S = \begin{vmatrix} \cos\alpha \cot\beta & \sin\alpha & -\dfrac{\cos\alpha}{\sin\beta} \\ -\sin\alpha \cot\beta & \cos\alpha & \dfrac{\sin\alpha}{\sin\beta} \\ 1 & 0 & 0 \end{vmatrix}$$

and N is the symmetric 9 x 9 matrix:

(1.1.26)
$$|N_{ij}| = \begin{vmatrix} 3 \\ \cdot & 3 \\ \cdot & \cdot & 3 \\ \cdot & \cdot & \cdot & \lambda_X \\ \cdot & \cdot & \cdot & \cdot & \lambda_Y \\ \cdot & \cdot & \cdot & \cdot & 0 & \lambda_Z \\ \cdot & \cdot & \cdot & \cdot & -\lambda_Y \phi_r & 0 & \lambda_Y \phi_r^2 + \frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & -\lambda_Y \phi_q & 0 & \lambda_Y \phi_r \phi_q & \lambda_Y \phi_q^2 + \frac{2}{3} \\ \cdot & \cdot & \cdot & \cdot & -\lambda_Y \phi_\theta + \frac{r^2}{2} & 0 & \lambda_Y \phi_r \phi_\theta - \frac{r^2}{2}\phi_r & \lambda_Y \phi_q \phi_\theta - \frac{r^2}{2}\phi_q & \lambda_Y \phi_\theta^2 - r^2 \phi_\theta + \frac{r^2}{2} \end{vmatrix}$$

where
$$\phi_r = \frac{\partial \phi}{\partial r} = -\frac{3rq^2}{\Lambda^2} \sin 2\theta$$

(1.1.27)
$$\phi_q = \frac{\partial \phi}{\partial q} = \frac{3r^2 q}{\Lambda^2} \sin 2\theta$$

$$\phi_\theta = \frac{\partial \phi}{\partial \theta} = \frac{-3r^2}{2\Lambda^2}\left(2q^2 \cos 2\theta + \frac{3}{2} r^2\right)$$

The derivatives of ϕ obtained from (1.1.6) and λ_X, λ_Y and λ_Z are defined in (1.1.13).

The conjugate metric tensory, g^{ij}, is also needed. It is defined by $g_{ik}g^{kj} = \delta_i^j$ and the matrix representing the tensor is obtained from the reciprocals of the matrices in (1.1.24). Written in a similar manner it is:

(1.1.28)
$$|g^{ij}| = \begin{vmatrix} I & 0 & 0 \\ 0 & S^T & 0 \\ 0 & 0 & I \end{vmatrix} \; \begin{vmatrix} M^{ij} \end{vmatrix} \; \begin{vmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{vmatrix}$$

where M is the 9 x 9 symmetric matrix:

(1.1.29)

$$
\left|M^{ij}\right| = \begin{vmatrix}
\frac{1}{3} \\
\cdot & \frac{1}{3} \\
\cdot & \cdot & \frac{1}{3} \\
\cdot & \cdot & \cdot & \frac{1}{\lambda_x} \\
\cdot & \cdot & \cdot & \cdot & \frac{3R^2}{\Lambda^2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\lambda_z} \\
\cdot & \cdot & \cdot & \cdot & \frac{6rq^2\sin 2\theta}{\Lambda^2} & 0 & 2 \\
\cdot & \cdot & \cdot & \cdot & \frac{-9r^2q\sin 2\theta}{4\Lambda^2} & 0 & 0 & \frac{3}{2} \\
\cdot & \cdot & \cdot & \cdot & \frac{3(3r^2-4q^2)(1-\cos 2\theta)}{2\Lambda^2} & 0 & 0 & 0 & \frac{2}{r^2}+\frac{3}{2q^2}
\end{vmatrix}
$$

The Laplacian

(1.1.30) $$\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \rho_i}\left(\sqrt{g}\, g^{ij} \frac{\partial}{\partial \rho_j}\right)$$

can be written explicitly in terms of the ρ_i with the result (see Appendix 1)

(1.1.31) $$\nabla^2 = \frac{1}{3}\nabla_{\underline{s}}^2 + T_E + T_S$$

in which

$$\nabla_{\underline{s}}^2 = \frac{\partial^2}{\partial s_x} + \frac{\partial^2}{\partial s_y} + \frac{\partial^2}{\partial s_z}$$

(1.1.32)

$$T_E = \sum_{i=4,5,6} M^{ii}\left(\frac{L_i}{-i}\right)^2 + 2\left(\frac{L_5}{-i}\right)\sum_{i=7,8,9} M^{i6}\frac{\partial}{\partial \rho_i} + M^{0}\left(\frac{L_5}{-i}\right)$$

$$T_S = \frac{2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{3}{2q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) + \left(\frac{2}{r^2}+\frac{3}{2q^2}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)$$

$$M^{0} = \frac{36r^2q^2R^2}{\Lambda^4}\sin 2\theta\cos 2\theta$$

The usefulness of the operators L_4, L_5 and L_6 (1.1.17) is now apparent. The terms M^{ij}, M^{0} and T_S are functions of the internal co-ordinates only.

2. The Symmetries of Three-Nucleon Wave Functions

A. The Permutation Groups of Three Particles

For the classification of the three-nucleon wave functions use is made of the symmetry under permutation of co-ordinates in space, spin space and isotopic spin space. This symmetry property of nucleons is well known (Schiff, 1955, Ch. 9) and this section contains a summary of the relevant results.

The six permutations of three objects are listed in Table I. The permutations are considered as operators changing the positions of three ordered numbers. For example, the operator P_3 has the property $P_3(abc) = (132)(abc) = (cab)$, i.e. the number in the first position is replaced by that in the third, the one in the third by

TABLE I

The Representations of the Permutations

Permutation	Symbol	Representations		
		${}^1\Gamma_1$ Symmetric	${}^2\Gamma_2$ Mixed	${}^1\Gamma_3$ Antisymmetric
(1) (2) (3)	P_1	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
(123)	P_2	1	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$	1
(132)	P_3	1	$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$	1
(12) (3)	P_4	1	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	-1
(23) (1)	P_5	1	$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$	-1
(31) (2)	P_6	1	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$	-1

that in the second and the one in the second position by that in the first. The permutations form a group which is isomorphic with the group D_3 (e.g. Lomont, 1959) and has three irreducible representations ${}^1\Gamma_1$, ${}^2\Gamma_2$ and ${}^1\Gamma_3$ for which matrix representations $D^{(1)}(P_i)$, $D^{(2)}(P_i)$ and $D^{(3)}(P_i)$ are given in Table I. The three irreducible representations will be referred to as the symmetric, mixed and antisymmetric representations respectively.

When the three numbers are the arguments of a function $\phi(X_1, X_2, X_3)$, the function $P_i\,\phi(X_1, X_2, X_3)$ is interpreted (Wigner, 1959, Ch. 11) so that

$$P_i\,\phi(X_1', X_2', X_3') = \phi(X_1, X_2, X_3) \tag{1.2.1}$$

where

$$(X_1', X_2', X_3') = P_i\,(X_1, X_2, X_3) \tag{1.2.2}$$

In this way the following six functions are obtained.

$$\begin{aligned}
P_1\,\phi(X_1, X_2, X_3) &= \phi(X_1, X_2, X_3) \equiv \phi_1 \\
P_2\,\phi(X_1, X_2, X_3) &= \phi(X_3, X_1, X_2) \equiv \phi_2 \\
P_3\,\phi(X_1, X_2, X_3) &= \phi(X_2, X_3, X_1) \equiv \phi_3 \\
P_4\,\phi(X_1, X_2, X_3) &= \phi(X_2, X_1, X_3) \equiv \phi_4 \\
P_5\,\phi(X_1, X_2, X_3) &= \phi(X_1, X_3, X_2) \equiv \phi_5 \\
P_6\,\phi(X_1, X_2, X_3) &= \phi(X_3, X_2, X_1) \equiv \phi_6
\end{aligned} \tag{1.2.3}$$

When the functions ϕ_i are all independent, it is possible to form linear combinations of these so that, for the j-th representation of dimension l_j, there are l_j sets of l_j linearly independent functions which transform under permutations according to the rule:

$$P_i f_n^{(j)} = \sum_m D^{(j)}(P_i)_{mn} f_m^{(j)} \tag{1.2.4}$$

For the representations of Table I, these are:

i. Symmetric

$$f^{(1)} = \frac{1}{\sqrt{6}}\left(\phi_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6\right) \tag{1.2.5}$$

ii. Mixed

$$\begin{aligned} f_1^{(2)}(1) &= \frac{1}{\sqrt{3}}\left(\phi_1 - \tfrac{1}{2}\phi_2 - \tfrac{1}{2}\phi_3 + \phi_4 - \tfrac{1}{2}\phi_5 - \tfrac{1}{2}\phi_6\right) \\ f_2^{(2)}(1) &= \tfrac{1}{2}\left(\phi_2 - \phi_3 + \phi_5 - \phi_6\right) \end{aligned} \tag{1.2.6}$$

and

$$\begin{aligned} f_1^{(2)}(2) &= \tfrac{1}{2}\left(-\phi_2 + \phi_3 + \phi_5 - \phi_6\right) \\ f_2^{(2)}(2) &= \frac{1}{\sqrt{3}}\left(\phi_1 - \tfrac{1}{2}\phi_2 - \tfrac{1}{2}\phi_3 - \phi_4 + \tfrac{1}{2}\phi_5 + \tfrac{1}{2}\phi_6\right) \end{aligned} \tag{1.2.7}$$

iii. Antisymmetric

$$f^{(3)} = \frac{1}{\sqrt{6}}\left(\phi_1 + \phi_2 + \phi_3 - \phi_4 - \phi_5 - \phi_6\right) \tag{1.2.8}$$

Assuming the ϕ is normalized, these six functions are orthonormal.

When ϕ is invariant upon permutation of two of the coordinates, there can be only three linearly independent combinations. Assume, for example, that $\phi(X_i, X_j, X_k) = \phi(X_i, X_k, X_j)$ or i.e.

$$\phi_1 = \phi_5 \qquad \phi_2 = \phi_6 \qquad \phi_3 = \phi_4 \tag{1.2.9}$$

Then the linearly independent combinations become:

i. Symmetric

(1.2.10) $$f^{(1)} = \frac{1}{\sqrt{3}}\left(\phi_4 + \phi_5 + \phi_6\right)$$

ii. Mixed

(1.2.11) $$f_1^{(2)} = \sqrt{\frac{2}{3}}\left(\frac{1}{2}\phi_4 + \frac{1}{2}\phi_5 - \phi_6\right)$$

$$f_2^{(2)} = \frac{1}{\sqrt{2}}\left(-\phi_4 + \phi_5\right)$$

iii. Antisymmetric

(1.2.12) $$f^{(3)} = 0$$

The two sets of mixed functions become proportional, and all functions have been normalized.

When ϕ is completely invariant, only the symmetric functions remain

(1.2.13) $$f^{(1)} = \phi$$

In the following sections, combinations of properly symmetrized functions from different spaces will be needed such that the combinations transform upon permutation according to (1.2.4). With the functions in the two spaces being $f^{(1)}$, $f_i^{(2)}$, $f^{(3)}$ and $g^{(1)}$, $g_i^{(2)}$, $g^{(3)}$, the combined functions are:

i. Symmetric

(1.2.14) $$h^{(1)}(1) = f^{(1)}g^{(1)}$$

$$h^{(1)}(2) = \frac{1}{\sqrt{2}}\left(f_1^{(2)}g_1^{(2)} + f_2^{(2)}g_2^{(2)}\right)$$

$$h^{(1)}(3) = f^{(3)}g^{(3)}$$

ii. Mixed

$$h_1^{(2)}(1) = f^{(1)} g_1^{(2)}$$
$$h_2^{(2)}(1) = f^{(1)} g_2^{(2)}$$

$$h_1^{(2)}(2) = f_1^{(2)} g^{(1)}$$
$$h_2^{(2)}(2) = f_2^{(2)} g^{(1)}$$

(1.2.15)
$$h_1^{(2)}(3) = \frac{1}{\sqrt{2}}\left(-f_1^{(2)} g_1^{(2)} + f_2^{(2)} g_2^{(2)}\right)$$
$$h_2^{(2)}(3) = \frac{1}{\sqrt{2}}\left(f_1^{(2)} g_2^{(2)} + f_2^{(2)} g_1^{(2)}\right)$$

$$h_1^{(2)}(4) = -f^{(3)} g_2^{(2)}$$
$$h_2^{(2)}(4) = f^{(3)} g_1^{(2)}$$

$$h_1^{(2)}(5) = -f_2^{(2)} g^{(3)}$$
$$h_2^{(2)}(5) = f_1^{(2)} g^{(3)}$$

iii. Antisymmetric

(1.2.16)
$$h^{(3)}(1) = f^{(1)} g^{(3)}$$
$$h^{(3)}(2) = f^{(3)} g^{(1)}$$
$$h^{(3)}(3) = \frac{1}{\sqrt{2}}\left(f_1^{(2)} g_2^{(2)} - f_2^{(2)} g_1^{(2)}\right)$$

By successive application of the above, totally antisymmetric wave functions are constructed from the functions for spin, isotopic spin, internal co-ordinates and the Euler angles.

B. Spin and Isotopic Spin Functions

Linear combinations of the spin functions of the three nucleons can be formed which are eigenstates of the total spin and have definite symmetry properties. The total spin may have the values

$S = 3/2$, giving rise to a set of orthogonal quartet functions $\chi^{(1)}(S_z)$, and $S = \frac{1}{2}$, giving rise to two sets of orthogonal doublet functions $\chi_1^{(2)}(S_z)$ and $\chi_2^{(2)}(S_z)$. When two of the nucleons have parallel spins, i.e. for $S_z = +\frac{1}{2}$ or $S_z = -\frac{1}{2}$, there are three linearly independent functions for each value of S_z. From these are obtained by linear superposition, as in (1.2.10 and 11), three orthogonal functions $\left(\text{a symmetric function } \chi^{(1)}(S_z) \text{ and a pair of mixed functions } \chi_1^{(2)}(S_z) \text{ and } \chi_2^{(2)}(S_z)\right)$ which are for $S_z = +\frac{1}{2}$:

$$\chi^{(1)}(\tfrac{1}{2}) = \frac{1}{\sqrt{3}}\left(\alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3 + \alpha_1\alpha_2\beta_3\right)$$

(1.2.17a) $$\chi_1^{(2)}(\tfrac{1}{2}) = \sqrt{\tfrac{2}{3}}\left(\tfrac{1}{2}\alpha_1\beta_2\alpha_3 + \tfrac{1}{2}\beta_1\alpha_2\alpha_3 - \alpha_1\alpha_2\beta_3\right)$$

$$\chi_2^{(2)}(\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\left(-\alpha_1\beta_2\alpha_3 + \beta_1\alpha_2\alpha_3\right)$$

Similarily, when all three nucleons have parallel spins, i.e. $S_z = +3/2$ or $S_z = -3/2$, there is one linearly independent function for each value of S_z. For $S_z = +3/2$ this is

(1.2.17b) $$\chi^{(1)}(\tfrac{3}{2}) = \alpha_1\alpha_2\alpha_3$$

The single nucleon spin functions are α_i and β_i where $\sigma_z\alpha = +\alpha\,(S_3 = \frac{1}{2})$ and $\sigma_z\beta = -\beta\,(S_3 = -\frac{1}{2})$. For $S_z = -\frac{1}{2}$ and $S_z = -3/2$, the functions (1.2.17) have the same form with α_i and β_i interchanged.

The treatment of isotopic spin is exactly the same as that of the spin, except for notation. There are a set of symmetric functions $\zeta^{(1)}(T_3)$ with $T = 3/2$ and two sets of mixed functions $\zeta_1^{(2)}(T_3)$ and $\zeta_2^{(2)}(T_3)$ with $T = \frac{1}{2}$. The convention used is that for protons $\tau_3\pi = -\pi$ and for neutrons $\tau_3\nu = +\nu$ (Blatt & Weisskopf, 1952, Ch. 3). To describe the isotopic spin of two protons and one neutron, only those states with $T_3 = -\frac{1}{2}$ are needed.

$$\zeta^{(1)}(-\tfrac{1}{2}) = \frac{1}{\sqrt{3}}\left(\pi_1\nu_2\pi_3 + \nu_1\pi_2\pi_3 + \pi_1\pi_2\nu_3\right)$$

(1.2.18)
$$\zeta_1^{(2)}(-\tfrac{1}{2}) = \sqrt{\tfrac{2}{3}}\left(\tfrac{1}{2}\pi_1\nu_2\pi_3 + \tfrac{1}{2}\nu_1\pi_2\pi_3 - \pi_1\pi_2\nu_3\right)$$
$$\zeta_2^{(2)}(-\tfrac{1}{2}) = \frac{1}{\sqrt{2}}\left(-\pi_1\nu_2\pi_3 + \nu_1\pi_2\pi_3\right)$$

To obtain the isotopic spin functions for H^3 ($T_3 = +\frac{1}{2}$), interchange π and ν in the above. Unless otherwise stated, all isotopic spin functions will be those for He^3 ($T_3 = -\frac{1}{2}$).

The combined spin and isotopic spin functions can now be constructed using (1.2.14, 15 and 16). An abbreviated notation is used, the values of T and S not being included in the labelling. The symmetry properties, and the values of the spin and isotopic spin functions ξ_i are listed in Table II. The functions are:

(1.2.19)
$$\xi_1(S_3) = \chi^{(1)}(S_3)\zeta^{(1)}$$

$$\xi_{2,1}(S_3) = \chi_1^{(2)}(S_3)\zeta^{(1)}$$
$$\xi_{2,2}(S_3) = \chi_2^{(2)}(S_3)\zeta^{(1)}$$

$$\xi_3(S_3) = \frac{1}{\sqrt{2}}\left\{\chi_1^{(2)}(S_3)\zeta_1^{(2)} + \chi_2^{(2)}(S_3)\zeta_2^{(2)}\right\}$$

$$\xi_{4,1}(S_3) = \chi^{(1)}(S_3)\zeta_1^{(2)}$$
$$\xi_{4,2}(S_3) = \chi^{(1)}(S_3)\zeta_2^{(2)}$$

(1.2.19)

$$\xi_{5,1}(S_3) = \frac{1}{\sqrt{2}}\left\{-\chi_1^{(2)}(S_3)\,\zeta_1^{(2)} + \chi_2^{(2)}(S_3)\,\zeta_2^{(2)}\right\}$$

$$\xi_{5,2}(S_3) = \frac{1}{\sqrt{2}}\left\{\chi_1^{(2)}(S_3)\,\zeta_2^{(2)} + \chi_2^{(2)}(S_3)\,\zeta_1^{(2)}\right\}$$

$$\xi_6(S_3) = \frac{1}{\sqrt{2}}\left\{\chi_1^{(2)}(S_3)\,\zeta_2^{(2)} - \chi_2^{(2)}(S_3)\,\zeta_1^{(2)}\right\}$$

TABLE II

Properties of Spin-Isotopic Spin Functions

	Symmetry	Total Spin	Total Isotopic Spin
ξ_1	symmetric	3/2	3/2
$\xi_{2,i}$	mixed	1/2	3/2
ξ_3	symmetric	1/2	1/2
$\xi_{4,i}$	mixed	3/2	1/2
$\xi_{5,i}$	mixed	1/2	1/2
ξ_6	antisymmetric	1/2	1/2

C. Euler Angle Functions

In this section, functions of the Euler angles which have definite symmetry properties and parity are constructed. In order to do this the effects of the permutation and parity operators on the rotation $\{\alpha, \beta, \gamma\}$ are now determined.

When the wave functions of the deuteron and the incident proton do not overlap there can be no permutations involving the incident proton; thus the internal co-ordinates q and r are then unique (Fig. 3). The identification of q and r is made invariant upon permutation of the nucleons by the stipulation that q and r change continuously as the wave functions of the proton and deuteron overlap sufficiently to permit permutations involving the proton. The Z axis is defined so that for large separation, the incident proton is on the positive Z axis. The Z axis is invariant upon permutation of the nucleons. The Y axis, however, is dependent on the numbering of the nucleons. The permutations P_1, P_2 and P_3 do not change the Y axis but the permutations P_4, P_5 and P_6, interchanging the positions of only two of the nucleons, have the effect of reversing the positive Y axis and hence also the X axis. This may also be considered as a rotation, through an angle of 180° about the Z axis, of the triangle in the 'body' frame. If the orientation of the triangle in the centre-of-mass frame is described by the rotation $\{\alpha, \beta, \gamma\}$, then after any of the permutations P_4, P_5 or P_6, the same physical orientation of the triangle is described by the rotation $\{\alpha \pm \pi, \beta, \gamma\}$. The effect of the permutations on the Euler angles may be summarized as follows:

$$P_i\{\alpha, \beta, \gamma\} = \{\alpha, \beta, \gamma\} \qquad \text{for } i = 1, 2, 3 \tag{1.2.20}$$

$$P_i\{\alpha, \beta, \gamma\} = \{\alpha \pm \pi, \beta, \gamma\} \qquad \text{for } i = 4, 5, 6$$

When two of the principal axes become degenerate the assignment of the 'body' axes is not unique. This occurs when the three nucleons form an equilateral triangle and when the three nucleons are co-linear. A method of handling these degeneracies, which is due to Derrick (1960), is discussed in Appendix 2.

The parity operation, i.e. the inversion $x \longrightarrow -x$, $y \longrightarrow -y$ and $z \longrightarrow -z$, is equivalent to a rotation of 180° about the 'body' Y axis. The operations of finite rotations about different axes do not commute and the effect of the parity operator, Π, on the rotation $\{\alpha, \beta, \gamma\}$ must be expressed as

$$(1.2.21) \quad \Pi\{\alpha,\beta,\gamma\} = \{\pi-\alpha,\ \pi-\beta,\ \pi+\gamma\} = \{-\alpha,\ \pi+\beta,\ \gamma\}$$

These rotations are established by examining the rotation matrix $|R_{ij}|$ (1.1.2) and requiring that $R_{ij}(\alpha,\beta,\gamma) = -R_{ij}(\alpha',\beta',\gamma')$ for columns 1 and 3 when $\Pi\{\alpha,\beta,\gamma\} = \{\alpha',\beta',\gamma'\}$. Column 2 remains unchanged by this transformation but as $Y_i = 0$, this does not effect the parity operation.

Use is now made of (1.2.20 and 21) to construct normalized linear combinations of the functions $\mathcal{D}^L_{\mu'\mu}(\alpha,\beta,\gamma)$ which have definite symmetry and parity. In these functions L is the total orbital angular momentum, μ' is the 'body' Z component of L, and μ the centre-of-mass z component of L. The properties of these functions, which will be needed, are given below.

$$(1.2.22) \quad \mathcal{D}^L_{\mu'\mu}(\alpha,\beta,\gamma) = e^{i\mu'\alpha}\, d^L_{\mu'\mu}(\beta)\, e^{i\mu\gamma}$$

$$(1.2.23)\quad d^{L}_{\mu'\mu}(\beta) = \sum_{x} (-)^{x} \frac{\sqrt{(L+\mu)!\,(L-\mu)!\,(L+\mu')!\,(L-\mu')!}}{(L-\mu'-x)!\,(L+\mu'-x)!\,x!\,(x+\mu'-\mu)!} \cos\left(\frac{\beta}{2}\right)^{2L+\mu-\mu'-2x} \sin\left(\frac{\beta}{2}\right)^{2x+\mu'-\mu}$$

where the summation over x is between the zeros in the denominator.

$$d^{L}_{\mu'\mu}(\pi-\beta) = (-)^{L-\mu'} d^{L}_{\mu',-\mu}(\beta)$$

$$(1.2.24)\quad d^{L}_{\mu'\mu}(\beta) = (-)^{\mu'-\mu} d^{L}_{-\mu',-\mu}(\beta)$$

$$d^{L}_{\mu'\mu}(\pi+\beta) = (-)^{L-\mu'} d^{L}_{-\mu',\mu}(\beta)$$

The orthogenality and normalization is given by the integral

$$(1.2.25)\quad \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma\, \mathcal{D}^{L_1 *}_{\mu_1'\mu_1}(\alpha,\beta,\gamma)\, \mathcal{D}^{L_2}_{\mu_2'\mu_2}(\alpha,\beta,\gamma) = \delta_{L_1L_2}\,\delta_{\mu_1'\mu_2'}\,\delta_{\mu_1\mu_2}\,\frac{8\pi^2}{2L_1+1}$$

The properties under permutation are:

$$(1.2.26)\quad P_i \mathcal{D}^{L}_{\mu'\mu}(\alpha,\beta,\gamma) = +\mathcal{D}^{L}_{\mu'\mu}(\alpha,\beta,\gamma) \qquad \text{for } i = 1,2,3$$

$$P_i \mathcal{D}^{L}_{\mu'\mu}(\alpha,\beta,\gamma) = (-)^{\mu'} \mathcal{D}^{L}_{\mu'\mu}(\alpha,\beta,\gamma) \qquad \text{for } i = 4,5,6$$

Thus these functions are either symmetric (u' even) or antisymmetric (u' odd). The functions never transform according to the mixed representation. Under the parity operation, the properties are:

$$(1.2.27)\quad \Pi \mathcal{D}^{L}_{\mu'\mu}(\alpha,\beta,\gamma) = (-)^{L+\mu'} \mathcal{D}^{L}_{-\mu'\mu}(\alpha,\beta,\gamma)$$

The following orthogonal functions are thus defined to have even (+) or odd (-) parity and symmetry properties according to (1.2.26).

$$(1.2.28)\quad W^{L}_{\mu'\mu}(\pm) = \frac{\sqrt{2L+1}}{4\pi}\left(\mathcal{D}^{L}_{\mu'\mu}(\alpha,\beta,\gamma) \pm (-)^{L+\mu'}\mathcal{D}^{L}_{-\mu'\mu}(\alpha,\beta,\gamma)\right) \quad \text{for } \mu' \neq 0$$

$$W^{L}_{0\mu}((-)^{L}) = \frac{\sqrt{2(2L+1)}}{4\pi}\ \mathcal{D}^{L}_{0\mu}(\alpha,\beta,\gamma)$$

For some values of μ' and μ, these functions may be related to the spherical harmonics and Legendre polynomials:

$$(1.2.29)\quad W^{L}_{0\mu}((-)^{L}) = \frac{1}{\sqrt{2\pi}}\ Y_{L\mu}(\beta,\gamma)$$

$$(1.2.30)\quad W^{L}_{00}((-)^{L}) = \frac{\sqrt{2(2L+1)}}{4\pi}\ P_{L}(\cos\beta)$$

The Euler angle functions may now be combined with the spin-isotopic spin functions ξ_i in the manner given in (1.1.14, 15 and 16). These are then eigenstates of L, $L_z = \mu$, S and S_z. From these combined functions, eigenstates of J = L + S, J_z = M, L and S can be formed in the usual manner making use of Clebsch-Gordon coefficients $(L, \mu, S, S_z \mid J, M)$. For later use, all possible orthonormal eigenstates formed in this manner having J = ½ and even parity are given below. To aid in the classification and manipulation of these functions, an abbreviated notation is used, in analogy with the definition of the spin-isotopic spin states ξ_i.

S states (L = 0)

$$\mu_1 = W^{0}_{00}(+)\ \xi_3\ (S_3)$$

$$(1.2.31)\quad \mu_{2,1} = W^{0}_{00}(+)\ \xi_{2,1}\ (S_3)$$

$$\mu_{2,2} = W^{0}_{00}(+)\ \xi_{2,2}\ (S_3)$$

$$\mu_{3,1} = W^{0}_{00}(+)\ \xi_{5,1}\ (S_3)$$

$$\mu_{3,2} = W^{0}_{00}(+)\ \xi_{5,2}\ (S_3)$$

$$\mathcal{W}_4 = W^0_{00}(+)\xi_6(S_3)$$

P states (L = 1)

$$\mathcal{W}_5 = \sum_{\mu,S_3}\left(1,\mu,\tfrac{1}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_6(S_3)$$

$$\mathcal{W}_{6,1} = \sum_{\mu,S_3}+\left(1,\mu,\tfrac{1}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_{2,2}(S_3)$$
$$\mathcal{W}_{6,2} = \sum_{\mu,S_3}-\left(1,\mu,\tfrac{1}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_{2,1}(S_3)$$

$$\mathcal{W}_{7,1} = \sum_{\mu,S_3}\left(1,\mu,\tfrac{1}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_{5,2}(S_3)$$
(I.2.32)
$$\mathcal{W}_{7,2} = \sum_{\mu,S_3}-\left(1,\mu,\tfrac{1}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_{5,1}(S_3)$$

$$\mathcal{W}_8 = \sum_{\mu,S_3}\left(1,\mu,\tfrac{1}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_3(S_3)$$

$$\mathcal{W}_{9,1} = \sum_{\mu,S_3}\left(1,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_{4,2}(S_3)$$
$$\mathcal{W}_{9,2} = \sum_{\mu,S_3}-\left(1,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_{4,1}(S_3)$$

$$\mathcal{W}_{10} = \sum_{\mu,S_3}\left(1,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^1_{1\mu}(+)\xi_1(S_3)$$

D states (L = 2)

$$\mathcal{W}_{11} = \sum_{\mu,S_3}\left(2,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^2_{0\mu}(+)\xi_1(S_3)$$

$$\mathcal{W}_{12} = \sum_{\mu,S_3}\left(2,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^2_{2\mu}(+)\xi_1(S_3)$$

(I.2.33)
$$\mathcal{W}_{13,1} = \sum_{\mu,S_3}\left(2,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^2_{0\mu}(+)\xi_{4,1}(S_3)$$
$$\mathcal{W}_{13,2} = \sum_{\mu,S_3}\left(2,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^2_{0\mu}(+)\xi_{4,2}(S_3)$$

$$\mathcal{W}_{14,1} = \sum_{\mu,S_3}\left(2,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^2_{1\mu}(+)\xi_{4,2}(S_3)$$
$$\mathcal{W}_{14,2} = \sum_{\mu,S_3}-\left(2,\mu,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)W^2_{1\mu}(+)\xi_{4,1}(S_3)$$

$$(1.2.33)\quad \mathscr{W}_{15,1} = \sum_{\mu,S_3} (2\mu,\tfrac{3}{2},S_3|\tfrac{1}{2},M)\, W^2_{2\mu}(+)\, \xi_{4,1}(S_3)$$

$$\mathscr{W}_{15,1} = \sum_{\mu,S_3} (2\mu,\tfrac{3}{2},S_3|\tfrac{1}{2},M)\, W^2_{2\mu}(+)\, \xi_{4,2}(S_3)$$

$$\mathscr{W}_{16} = \sum_{\mu,S_3} (2\mu,\tfrac{3}{2},S_3|\tfrac{1}{2},M)\, W^2_{1\mu}(+)\, \xi_{1}(S_3)$$

The states $\mathscr{W}_1$ to $\mathscr{W}_8$ are doublet states (S = ½) and $\mathscr{W}_9$ to $\mathscr{W}_{16}$ are quartet states (S = 3/2). The states $\mathscr{W}_1$, $\mathscr{W}_5$, and $\mathscr{W}_{11}$ are symmetric on permutation of angle, spin and isotopic spin co-ordinates, whereas states $\mathscr{W}_2$, $\mathscr{W}_3$, $\mathscr{W}_6$, $\mathscr{W}_7$, $\mathscr{W}_9$, $\mathscr{W}_{13}$, $\mathscr{W}_{14}$, and $\mathscr{W}_{15}$ transform according to mixed representation, and states $\mathscr{W}_4$, $\mathscr{W}_8$, $\mathscr{W}_{10}$, and $\mathscr{W}_{16}$ are antisymmetric. Similar sets can be constructed for any value of J, however they are of no interest for the present problem as the ground state of He^3 has J = ½ and even parity. Also it is more convenient to use eigenstates of L, μ, S and S_z in describing the continuum states.

D. Internal Functions

Functions of the internal co-ordinates may be constructed from an arbitrary function in the way indicated in (1.2.5 to 8) or (1.2.10 and 11) to form symmetric, mixed or antisymmetric functions. These must then be combined with spin angle functions, in the manner of (1.2.16) to form functions which are completely antisymmetric on permutation of all the co-ordinates of any two nucleons. The construction of these functions is given for use in the following sections, assuming that the initial function, ϕ , is symmetric on interchange of the co-ordinates of two nucleons. Denoting this symmetry by the notation, $\phi\ (1, 23) = \phi\ (1, 32)$, the symmetrical functions are (1.2.10 and 11):

$$f^{(1)} = \frac{1}{\sqrt{3}}\left(\phi_4(2,13) + \phi_5(1,23) + \phi_6(3,12)\right)$$

(1.2.34)
$$f_1^{(2)} = \sqrt{\frac{2}{3}}\left(\frac{1}{2}\phi_4(2,13) + \frac{1}{2}\phi_5(1,23) - \phi_6(3,12)\right)$$
$$f_2^{(2)} = \frac{1}{\sqrt{2}}\left(-\phi_4(2,13) + \phi_5(1,23)\right)$$

Assuming ϕ to be a product of a deuteron function and a function describing the third nucleon, the functions (1.2.34) have the asymtotic forms,

$$f^{(1)} \longrightarrow \frac{1}{\sqrt{3}}\ \phi_6(3,12) = \frac{1}{\sqrt{3}}\ u(r)\, v(q)$$

(1.2.35)
$$f_1^{(2)} \longrightarrow -\sqrt{\frac{2}{3}}\ \phi_6(3,12) = -\sqrt{\frac{2}{3}}\ u(r)\, v(q)$$
$$f_2^{(2)} \longrightarrow 0$$

when separation of particle 3 from particles 1 and 2 is large compared with the size of the deuteron. Here u(r) is the internal deuteron function and v(q) is the incident proton function. This asymptotic form is used in examining the continuum states in the following section and also in the approximate calculations in Chapter 2.

3. Wave Functions for Continuum and Bound States

A. Separation of the Schrödinger Equation

The Schrödinger equation is now re-written using the results of the preceding section. Using (1.1.30), the equation is:

(1.3.1) $\left\{-\frac{1}{2M}\left(\frac{1}{3}\nabla_{s}^{2}+T_{E}+T_{S}\right)+V\right\}\Psi_{E}=E\Psi_{E}$

The function Ψ_E is a function of the spin, the isotopic spin and the nine spatial co-ordinates. In the absence of external forces, the potential operator, V, does not depend on the centre-of-mass co-ordinate, $\underline{s}$. The centre-of-mass motion is then easily separated by writing $\Psi_E = \Psi''(\underline{s})\Psi'$ where Ψ' is independent of $\underline{s}$. The resulting equations are:

(1.3.2) $\frac{-1}{6M}\nabla_{s}^{2}\Psi(\underline{s})=E''\Psi(\underline{s})$

(1.3.3) $\left\{\frac{-1}{2M}\left(T_{E}+T_{S}\right)+V\right\}\Psi'=E'\Psi$

where E'' + E' = E.

Using the results of section 2, Ψ' may be formally written in the following manner,

(1.3.4) $\Psi'=\sum_{L,\mu',j}a(L,\mu',j)\,F(L,\mu',j)\,W_{\mu'\mu}^{L}(\pm)\,\xi_{j}$

where a(L, μ', j) are numerical co-efficients, F(L, μ', j) are internal functions (1.2.14), $W^L_{\mu'\mu}(\pm)$ are Euler angle functions (1.2.26) and ξ_j are spin-isotopic spin functions (1.2.19). When (1.3.4) is substituted in (1.3.3), both sides of the resulting equation multiplied on the left by $W^{L*}_{\mu'\mu}(\pm)\xi_j^{\dagger}$, and the integration over the Euler angles and summation over the spin and isotopic spin co-ordinates performed, a series of coupled differential equations for the radial function F(L, μ', j) result. The coupling arises from the off-diagonal matrix elements of the kinetic energy operator, $\frac{-1}{2M}\left(T_E + T_S\right)$, and of the potential energy operator, V. In Appendix 3 are tables of matrix elements for the kinetic energy operator and for spin and isotopic spin operators associated with central forces. In Appendix 4, some of the coupled differential equations for the radial functions are given. The precise form of these equations is not needed for the present work. Only the symmetry properties and the asymptotic form of the radial functions are used here.

B. Continuum States

For the continuum states, it is required that the wave functions, for large separation of the incident proton and the deuteron, become a simple product wave function of a deuteron in a ^3S state and an incident wave for the proton.

The spin function must be either $\chi^{(1)}(S_3)$ or $\chi_1^{(2)}(S_3)$ as only these two have the form of a triplet spin function, for nucleons 1 and 2, combined with a single particle spin function, for nucleon 3. The form of $\chi_2^{(2)}(S_3)$ is that of a singlet spin function and a single particle spin function. The isotopic spin function can only be $\zeta_2^{(2)}$ as the deuteron is in an isotopic spin singlet state. Thus the spin-isotopic spin function must be either

(1.3.5a) $\chi_1^{(2)}(S_3)\,\zeta_2^{(2)} = \frac{1}{\sqrt{2}}\left(\xi_{5,2} + \xi_6\right)$

or

(1.3.5b) $\chi^{(1)}(S_3)\,\zeta_2^{(2)} = \xi_{4,2}$

The relative amplitudes of the doublet and quartet states are $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{2}{3}}$ respectively so that, for large separation, the spin-isotopic spin function must be:

$$(1.3.6) \quad \left\{\sqrt{\tfrac{1}{3}}\cdot\sqrt{\tfrac{1}{2}}\left(\xi_{5,2}+\xi_{6}\right)+\sqrt{\tfrac{2}{3}}\,\xi_{4,2}\right\}$$

The asymptotic spatial functions are required to be of the form:

$$(1.3.7) \quad u(r)\sum_{L=0}^{\infty} i^{L}\sqrt{4\pi(2L+1)}\;v(q)\,Y_{L0}(\beta)$$

where u(r) is the S-state deuteron function with the normalization

$$(1.3.8) \quad \int_0^{\pi}\sin\theta\,d\theta\int_0^{\infty}r^2dr\left|u(r)\right|^2=1$$

and the terms in the summation represent a wave function normalized to unit density at infinity. The constants are such that if the functions $v_L(q)$ are replaced by the spherical Bessel functions $j_L(kq)$, the summation becomes the well-known expansion of a plane wave:

$$(1.3.9) \quad e^{ikq\cos\beta}=\sum_{L=0}^{\infty} i^{L}\sqrt{4\pi(2L+1)}\;j_L(kq)\,Y_{L0}(\beta)$$

The terms in this expansion are referred to as S, P, D, F ... waves for L = 0, 1, 2, 3 ... respectively.

For properly symmetrized racial functions which have the above asymptotic limits the notation is $G_i(L)$ where i refers to the spin-isotopic spin function ξ_i and L is the orbital angular momentum. The continuum wave functions must be of the following form:

For the doublet state

$$(1.3.10) \quad \sum_{L=0}^{\infty} i^L \sqrt{6\pi(2L+1)}\; W_{00}^{L}\left((-)^L\right)\left\{\frac{1}{\sqrt{2}} G_{5,2}(L)\xi_{5,1} - \frac{1}{\sqrt{2}} G_{5,1}(L)\xi_{5,2} + G_6(L)\xi_6\right\}$$

and for the quartet state

$$(1.3.11) \quad \sum_{L=0}^{\infty} i^L \sqrt{6\pi(2L+1)}\; W_{00}^{L}\left((-)^L\right)\left\{G_{4,2}(L)\xi_{4,1} - G_{4,1}(L)\xi_{4,2}\right\}$$

the asymptotic expressions (1.2.35) were used.

In Appendix 4, details are given for the derivation of the differential equations for the radial functions and it is shown that the above asymptotic solutions are possible for q greater than the size of the deuteron.

C. Bound States

The bound state , assumed to be the ground state of He^3 must have J = ½ and even parity. This is accomplished by using the functions $\mathcal{W}_i$ defined in (1.2.31, 32 and 33). For each of these functions there is a properly symmetrized radial function, F_i, so that, when combined with $\mathcal{W}_i$, a totally antisymmetric function is formed. The bound state is then a linear combination of the $F_i\mathcal{W}_i$ as follows:

$$(1.3.12) \quad \begin{aligned} &F_1\mathcal{W}_1 + \frac{1}{\sqrt{2}}\left(F_{2,2}\mathcal{W}_{2,1} - F_{2,1}\mathcal{W}_{2,2}\right) + \frac{1}{\sqrt{2}}\left(F_{3,2}\mathcal{W}_{3,1} - F_{3,1}\mathcal{W}_{3,2}\right) \\ &+ F_4\mathcal{W}_4 + F_5\mathcal{W}_5 + \frac{1}{\sqrt{2}}\left(F_{6,2}\mathcal{W}_{6,1} - F_{6,1}\mathcal{W}_{6,2}\right) + \frac{1}{\sqrt{2}}\left(F_{7,2}\mathcal{W}_{7,1} - F_{7,1}\mathcal{W}_{7,2}\right) \\ &+ F_8\mathcal{W}_8 + \frac{1}{\sqrt{2}}\left(F_{9,2}\mathcal{W}_{9,1} - F_{9,1}\mathcal{W}_{9,2}\right) + F_{10}\mathcal{W}_{10} + F_{11}\mathcal{W}_{11} + F_{12}\mathcal{W}_{12} \\ &+ \frac{1}{\sqrt{2}}\left(F_{13,2}\mathcal{W}_{13,1} - F_{13,1}\mathcal{W}_{13,2}\right) + \frac{1}{\sqrt{2}}\left(F_{14,2}\mathcal{W}_{14,1} - F_{14,1}\mathcal{W}_{14,2}\right) \\ &+ \frac{1}{\sqrt{2}}\left(F_{15,2}\mathcal{W}_{15,1} - F_{15,1}\mathcal{W}_{15,2}\right) + F_{16}\mathcal{W}_{16} \end{aligned}$$

The functions F_4, F_8, F_{10} and F_{16} are symmetric, the pairs $F_{2,i}$, $F_{3,i}$, $F_{6,i}$, $F_{7,i}$, $F_{9,i}$, $F_{13,i}$, $F_{14,i}$ and $F_{15,i}$ are mixed and F_1, F_5, F_{11}, and F_{12} are antisymmetric. The combinations $F_i \mu_i$ are orthogonal as the functions μ_i are orthonormal, and each enters with an amplitude a_i in the following sense:

for F_i symmetric or antisymmetric,

$$(1.3.13)\quad \int (F_i \mu_i)^* (F_i \mu_i)\, d\tau = \int |F_i|^2 d\tau = a_i^2$$

and for F_i mixed,

$$(1.3.14)\quad \frac{1}{2}\int \left(F_{i,2}\mu_{i,1} - F_{i,1}\mu_{i,2}\right)^* \left(F_{i,2}\mu_{i,1} - F_{i,1}\mu_{i,2}\right) d\tau = \frac{1}{2}\int \left(|F_{i,1}|^2 + |F_{i,2}|^2\right) d\tau = a_i^2$$

with

$$(1.3.15)\quad \int |F_{i,1}|^2 d\tau = \int |F_{i,2}|^2 d\tau = a_i^2$$

where

$$(1.3.16)\quad \sum_{i=1}^{16} a_i^2 = 1$$

D. <u>Estimates of Importance for Bound States</u>

It is possible to make qualitative estimates for the relative importance of the sixteen states in the bound state of He^3. These fall into four groups.

i. Using a similar classification of state for H^3 and He^3, but including only the states of isotopic spin, $T = ½$, Derrick, Mustard and Blatt (1961) have made a variational calculation for the binding energies of H^3 and He^3.

Their results were negative in that the potential used, that of Brueckner and Gammel (1955) could not fit the binding energy and the coulomb energy difference. The results did show, however, that the total probability of the P states was of the order of 7×10^{-4}. It is believed that this probability would not change by several orders of magnitude if a more accurate nuclear force were used. For this reason it was felt that the P states could be safely neglected in the present calculations. Derrick, Mustard and Blatt also obtained a 7 per cent probability for the D states, the remainder being S states.

ii. The states may be divided into two groups, one of $T = ½$ and the other of $T = 3/2$. The ground states of the isotopic spin quartet Li^3, He^3, H^3 and n^3 are expected to have a similar energy level, neglecting the coulomb force. Li^3 and n^3 do not form bound states and one expects the lowest energy, $T = 3/2$, states in H^3 and He^3 to be in the continuum and well above the observed ground state energies. The most important states in the ground state of H^3 and He^3 would then be $T = ½$ states.

iii. Of the possible transitions examined in Chapter III, only one has an angular distribution proportional to $\sin^2\theta$ to agree with the experimental data. This is an electric dipole transition from a P-wave proton to the S state of

He^3. This, combined with the estimates of Derrick, Mustard and Blatt (1961), indicates that the S states are the most important.

iv. For the S states, the expectation value of the magnetic moment operator:

$$\sum_{i=1}^{3}\left\{\mu_P\,\tfrac{1}{2}(1-\tau_{i3})\,\sigma_{i3}+\mu_N\,\tfrac{1}{2}(1+\tau_{i3})\,\sigma_{i3}\right\} \tag{1.3.17}$$

may be calculated using tables of matrix elements for these operators listed in Appendix 4. The result is:

$$\langle\mu\rangle=\left(\tfrac{4}{3}\mu_P-\tfrac{1}{3}\mu_N\right)a_1^2+\left(\tfrac{2}{3}\mu_P+\tfrac{1}{3}\mu_N\right)(a_2^2+a_3^2)+(\mu_N)(a_4^2) \tag{1.3.18}$$

$$=(4.36)a_1^2+(1.22)(a_2^2+a_3^2)+(-1.91)(a_4^2)\quad\text{nuclear magnetons}$$

The experimental value is $u(He^3) = -2.13$ n.m.

It is seen that the best fit is obtained for $a_1 = a_2 = a_3 = 0$, $a_4 = 1$. This implies that the states $F_4 \mathcal{W}_4$, with a symmetric radial function is the most important state. Some admixture of D states is needed to provide complete agreement with experiment.

Thus it is concluded that the state with the largest probability is the radially symmetrical state $F_4 \mathcal{W}_4$. There will be admixtures of the other S states and D states but with much smaller probability. The P states may be neglected.

CHAPTER 2

EVALUATION OF THE MATRIX ELEMENTS

1. Introduction

In this chapter the multipole matrix elements given in Appendix 5 are evaluated using the wave functions of Chapter 1 for the continuum and bound nucleon states. In the integrals for the multipole moments, the factors involving $r_L^L Y_{L,m}(\theta_L, \phi_L)$ are expressed in terms of the internal co-ordinates and the Euler angles (see App. 6). The integration over the Euler angles and the summation over the spin and isotopic spin co-ordinates may be performed exactly, leaving an integration over the internal co-ordinates. These radial integrals cannot be evaluated until the correct internal functions are known or until approximate forms for the internal functions are chosen. In section 3, approximate internal functions are chosen which enable rough estimates of the radial integrals and of the cross-sections to be made. In the multipole moments, the radial integrals are independent of the value of m (the Wigner-Eckhart theorem, see e.g. Edmonds 1957, p. 73). Thus the angular distribution of the emitted radiation for each multipole transition may be evaluated without knowing the radial integrals. This is done by making use of the angular distributions listed in (A5.1.21).

The well-known selection rules involving parity, orbital angular momentum and spin limit the possible transitions. The matrix elements must be invariant under the parity operation. Examination of the integrals for the multipole moments shows that for electric multipole radiation (Blatt and Weisskopf, 1952, p. 587):

(2.1.1) $\pi_a \pi_b \pi_l = +1$

and for magnetic multipole radiation

(2.1.2) $\pi_a \pi_b \pi_l = -1$

where π_a , π_b and π_l are the parities of the final and initial nucleon states and of the operator Y_{lm}, respectively. The selection rules for orbital angular momentum are shown by the following integral over the Euler angles, derived in Appendix 6,

(2.1.3)
$$\left\langle W^{L_a}_{\mu'_a \mu_a}(\pi_a) \middle| W^{l*}_{m'm}(\pi_l) \middle| W^{L_b}_{o,o}(\pi_b) \right\rangle = \pi_l (-)^{l-m'} \frac{\sqrt{2(2L_a+1)(2l+1)(2L_b+1)}}{4\pi} \begin{pmatrix} l & L_b & L_a \\ m' & o & -\mu'_a \end{pmatrix} \begin{pmatrix} l & L_b & L_a \\ -m & o & -\mu_a \end{pmatrix}$$

where
$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

is the Wigner 3-j symbol (Edmonds, 1957, p. 46) and is related to the Clebsch-Gordan co-efficients by

(2.1.4)
$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-)^{l_1-l_2-m_3} \frac{1}{\sqrt{2l_3+1}} \left(l_1 m_1 l_2 m_2 \middle| l_3 -m_3\right)$$

Convenient tables of the 3-j symbols are given by Rotenberg et al, (1959). This integral is zero unless $\pi_a \pi_l \pi_b = 1$. The Euler angle functions W come from the final nucleon state, the operator $r_i^l Y^*_{lm}(\theta_i \phi_i)$ and the initial nucleon state respectively. The interaction H' (A6.1.32) does not contain any spin operators and so does not mix states of different S or S_z. The proton operator contained in it mixes states of different T as shown in (A3.2.13 and 14) and (A3.2.18 and 19). The interaction H'' (A5.1.33) containing the magnetic moment operator mixes states of different S and of different T as shown in (A3.2.23 - 27).

2. Exact Evaluation in Terms of Radial Integrals

A. Electric Dipole Transition, P State to S State

In evaluating the electric dipole moment $Q_{1,m}$ (A5.2.13) the initial state is the P-wave part of (1.3.10). The quartet state (1.3.11) will not contribute to the reaction. Including the factor $\sqrt{\frac{2}{3}}$ for the relative amplitude of the doublet state, the initial state is

$$(2.2.1)\qquad \psi_b = i\sqrt{4\pi}\; W^1_{00}(-)\left\{\frac{1}{\sqrt{2}}\left(G_{5,2}(1)\,\xi_{5,1} - G_{5,1}(1)\,\xi_{5,2}\right) + G_6(1)\,\xi_6\right\}_b$$

The final state is the L = 0 part of (1.3.12),

$$(2.2.2)\qquad \psi_a = W^0_{00}(+)\left\{F_1\,\xi_3 + \frac{1}{\sqrt{2}}\left(F_{2,2}\,\xi_{2,1} - F_{2,1}\,\xi_{2,2}\right) + \frac{1}{\sqrt{2}}\left(F_{3,2}\,\xi_{5,1} - F_{3,1}\,\xi_{5,2}\right) + F_4\,\xi_6\right\}_a$$

The factor $r_i\, Y^*_{l,m}(\theta_i, \phi_i)$ is expressed in terms of the internal coordinates and the Euler angle functions W. The result is (App. 6)

$$(2.2.3)\qquad r_i\, Y_{l,m}(\theta_i,\phi_i) = \sqrt{2\pi}\; W^1_{0m}(-)\, Z_i - \sqrt{2\pi}\; W^1_{1m}(-)\, X_i$$

where X_i and Z_i are given in (1.1.3) in terms of the internal coordinates. The integration over the Euler angles is now performed using (2.1.3). The only non-zero terms occur in $Q_{1,0}$ and come from the first term in (2.2.3). Performing these operations and introducing the notation $Q_{1,m}(P\text{-}S,\ S_z)$ for the (1,m) electric dipole moment between the P and S states for an initial spin value of S_z gives

$$(2.2.4)\quad Q_{1m}(P\text{-}S,S_z) = i e\sqrt{3}.2\pi\sum_i\int W_{00}^{0}(+)\left\{\;\right\}_a^{*}\cdot\tfrac{1}{2}(1-\tau_{i3})\left(W_{0m}^{1*}(-)Z_i - W_{1m}^{1*}(-)X_i\right)W_{00}^{1}(+)\left\{\;\right\}_b d\tau$$

from which

$$(2.2.5)\quad Q_{1,0}(P\text{-}S,\pm\tfrac{1}{2}) = i e\sqrt{3}.2\pi\left\langle W_{\infty}^{0}(+)\middle|W_{00}^{1*}(-)\middle|W_{\infty}^{1}(-)\right\rangle I_1$$

$$= i e\sqrt{\tfrac{3}{2}}\, I_1$$

where

$$(2.2.6)\quad I_1 = \sum_i\int\left\{\;\right\}_a^{*}\tfrac{1}{2}(1-\tau_{i3})Z_i\left\{\;\right\}_b d\tau$$

In I_1, the summation over i, the spin and isotopic spin co-ordinates may be performed using the matrix elements listed in (A3.2.18 and 19) to give

$$(2.2.7)\quad \begin{aligned} I_1 = {} & \tfrac{1}{\sqrt{2}}\left\langle F_1\middle|\tfrac{1}{2}Z_3\middle|G_{5,2}(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_1\middle|\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_{5,1}(1)\right\rangle + \tfrac{1}{2}\left\langle F_{2,2}\middle|-\tfrac{1}{2}Z_3\middle|G_{5,2}(1)\right\rangle \\ & -\tfrac{1}{2}\left\langle F_{2,2}\middle|\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_{5,1}(1)\right\rangle + \tfrac{1}{\sqrt{2}}\left\langle F_{2,2}\middle|\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_6(1)\right\rangle \\ & -\tfrac{1}{2}\left\langle F_{2,1}\middle|\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_{5,2}(1)\right\rangle + \tfrac{1}{2}\left\langle F_{2,1}\middle|\tfrac{1}{2}Z_3\middle|G_{5,1}(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_{2,1}\middle|-\tfrac{1}{2}Z_3\middle|G_6(1)\right\rangle \\ & +\tfrac{1}{\sqrt{2}}\left\langle F_{3,2}\middle|-\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_6(1)\right\rangle - \tfrac{1}{2}\left\langle F_{3,1}\middle|\tfrac{1}{2}Z_3\middle|G_6(1)\right\rangle \\ & +\tfrac{1}{\sqrt{2}}\left\langle F_4\middle|-\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_{5,2}(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_4\middle|\tfrac{1}{2}Z_3\middle|G_{5,1}(1)\right\rangle \end{aligned}$$

Thus the reduction of the matrix element to a series of radial integrals is complete. Nothing further can be done to the expression I_1, until either exact or approximate forms are known for the radial functions.

The partial cross-section arising from $Q_{1,0}(P\text{-}S,\ S_z)$ is given by (A5.1.35) and (A5.2.19), (2.2.8) $$\sigma_{1,0}\left(P\text{-}S, \pm\tfrac{1}{2}\right) = \frac{16\pi}{9}\cdot\frac{k^3}{V}\left|Q_{1,0}\left(P\text{-}S,\pm\tfrac{1}{2}\right)\right|^2 = \frac{8\pi}{3}\cdot\frac{e^2k^3}{V}\left(I_1\right)^2$$

For an unpolarized proton beam and target, this should be averaged over initial spin values giving

(2.2.9) $$\sigma_{1,0}(P\text{-}S) = \frac{1}{2}\sum_{S_3=\pm\frac{1}{2}}\sigma_{1,0}(P\text{-}S,S_3) = \frac{8\pi}{3}\cdot\frac{e^2k^3}{V}\left(I_1\right)^2$$

The total cross-section is obtained by summing over m.

(2.2.10) $$\sigma(P\text{-}S) = \sum_{m=-1}^{1}\sigma_{1,m}(P\text{-}S) = \sigma_{1,0}(P\text{-}S)$$

The angular distribution of the radiation from this transition is given by $\left|\underline{X}_{1,0}\right|^2$ (A5.1.21) and is

(2.2.11) $$\left|\underline{X}_{1,0}\right|^2 = \frac{3}{8\pi}\left(1-\cos^2\theta\right) = \frac{3}{8\pi}\sin^2\theta$$

This is the only one of the transitions studied which has an angular distribution proportional to $\sin^2\theta$.

B. <u>Electric Dipole Transition, P State to D State</u>

The method of evaluating the electric dipole moment is exactly the same as in the preceding section, except that only the quartet continuum state contributes. Including the factor $\sqrt{\frac{2}{3}}$ for the relative amplitude of the quartet spin state, the initial state is, from (1.3.11),

(2.2.12) $$\Psi_b = i2\sqrt{3\pi}\,W'_{00}(-)\left\{G_{4,2}(1)\,\xi_{4,1} - G_{4,1}(1)\,\xi_{4,2}\right\}_b$$

and the final state is the L = 2 part of (1.3.12)

$$\Psi_a = \left\{ F_{11}\mathcal{W}_{11} + F_{12}\mathcal{W}_{12} + \frac{1}{\sqrt{2}}\left(F_{13,2}\mathcal{W}_{13,1} - F_{13,1}\mathcal{W}_{13,2}\right) + \frac{1}{\sqrt{2}}\left(F_{14,2}\mathcal{W}_{14,1} - F_{14,1}\mathcal{W}_{14,2}\right) + \frac{1}{\sqrt{2}}\left(F_{15,2}\mathcal{W}_{15,1} - F_{15,1}\mathcal{W}_{15,2}\right) + F_{16}\mathcal{W}_{16} \right\} \tag{2.2.13}$$

The electric dipole moment becomes

$$\begin{aligned} Q_{1m}(P\text{-}D, S_3) &= ie\,2\pi\sqrt{6}\sum_i \int \Psi_a^{*} \tfrac{1}{2}(1-\tau_{i3})\left(W^{1*}_{0m}(-)Z_i - W^{1*}_{1m}(-)X_i\right) W^{1}_{00}(-)\left\{\;\right\}_b d\tau \\ &= ie\,2\pi\sqrt{6}\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)\left\{\left\langle W^{2}_{0,-m}(+)\middle|W^{1*}_{0,m}(-)\middle|W^{1}_{00}(-)\right\rangle I_2 - \left\langle W^{2}_{1,-m}(+)\middle|W^{1*}_{1,m}(-)\middle|W^{1}_{00}(-)\right\rangle I_3\right\} \\ &= ie\,3\sqrt{15}\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)\begin{pmatrix}1 & 1 & 2\\ 0 & -m & m\end{pmatrix}\left\{\begin{pmatrix}1 & 1 & 2\\ 0 & 0 & 0\end{pmatrix} I_2 + \begin{pmatrix}1 & 1 & 2\\ 0 & 1 & -1\end{pmatrix} I_3\right\} \end{aligned} \tag{2.2.14}$$

where the Clebsch-Gordon co-efficient comes from the $\mathcal{W}$ functions (1.2.33) and

$$\begin{aligned} I_2 &= \sum_i \int \left\{F_{11}\xi_1 + \frac{1}{\sqrt{2}}\left(F_{13,2}\xi_{4,1} - F_{13,1}\xi_{4,2}\right)\right\}^{*} \left(\tfrac{1}{2}(1-\tau_{i3})Z_i\right)\left\{G_{4,2}(1)\xi_{4,1} - G_{4,1}(1)\xi_{4,2}\right\} d\tau \\ &= \left\langle F_{11}\middle|\tfrac{1}{\sqrt{2}}Z_3\middle|G_{4,2}(1)\right\rangle - \left\langle F_{11}\middle|\tfrac{1}{\sqrt{6}}(-Z_1+Z_2)\middle|G_{4,1}(1)\right\rangle + \tfrac{1}{\sqrt{2}}\left\langle F_{13,2}\middle|-\tfrac{1}{2}Z_3\middle|G_{4,2}(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_{13,2}\middle|\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_{4,1}(1)\right\rangle \\ &\quad - \tfrac{1}{\sqrt{2}}\left\langle F_{13,1}\middle|\tfrac{1}{2\sqrt{3}}(-Z_1+Z_2)\middle|G_{4,2}(1)\right\rangle + \tfrac{1}{\sqrt{2}}\left\langle F_{13,1}\middle|\tfrac{1}{2}Z_3\middle|G_{4,1}(1)\right\rangle \end{aligned} \tag{2.2.15}$$

and

$$\begin{aligned} I_3 &= \sum_i \int \left\{\frac{1}{\sqrt{2}}\left(F_{14,2}\xi_{4,2} + F_{14,1}\xi_{4,1}\right) + F_{16}\xi_1\right\}^{*} \left(\tfrac{1}{2}(1-\tau_{i3})X_i\right)\left\{G_{4,2}(1)\xi_{4,1} - G_{4,1}(1)\xi_{4,2}\right\} d\tau \\ &= \tfrac{1}{\sqrt{2}}\left\langle F_{14,2}\middle|\tfrac{1}{2\sqrt{3}}(-X_1+X_2)\middle|G_{4,2}(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_{14,2}\middle|\tfrac{1}{2}X_3\middle|G_{4,1}(1)\right\rangle + \tfrac{1}{\sqrt{2}}\left\langle F_{14,1}\middle|-\tfrac{1}{2}X_3\middle|G_{4,2}(1)\right\rangle \\ &\quad - \tfrac{1}{\sqrt{2}}\left\langle F_{14,1}\middle|\tfrac{1}{2\sqrt{3}}(-X_1+X_2)\middle|G_{4,1}(1)\right\rangle + \left\langle F_{16}\middle|\tfrac{1}{\sqrt{2}}X_3\middle|G_{4,2}(1)\right\rangle - \left\langle F_{16}\middle|\tfrac{1}{\sqrt{6}}(-X_1+X_2)\middle|G_{4,1}(1)\right\rangle \end{aligned} \tag{2.2.16}$$

The partial cross-section becomes

(2.2.17) $$\sigma_{l,m}\left(P\text{-}D, S_3\right) = 240\pi \cdot \frac{e^2k^3}{V} \cdot \left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)^2 \begin{pmatrix} 1 & 1 & 2 \\ 0 & -m & m \end{pmatrix}^2 \left\{ \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} I_2 + \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix} I_3 \right\}^2$$

In the averaging over S_z, each cross-section $\sigma_{l,m}(P\text{-}D)$ contains, apart from the common factors, the factor

(2.2.18) $$\frac{1}{4}\sum_{S_3}\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)^2 \begin{pmatrix} 1 & 1 & 2 \\ 0 & -m & m \end{pmatrix}^2 = \frac{1}{2}\sum_{S_3} \begin{pmatrix} 2 & \frac{3}{2} & \frac{1}{2} \\ m & S_3 & -M \end{pmatrix}^2 \begin{pmatrix} 1 & 1 & 2 \\ 0 & -m & m \end{pmatrix}^2$$

$$= \frac{3}{300} \quad \text{for} \quad m = \pm 1$$

$$= \frac{4}{300} \quad \text{for} \quad m = 0$$

In the summation over m to give the total cross-section this factor becomes 1/30. The total cross-section becomes

(2.2.19) $$\sigma\left(P\text{-}D\right) = \frac{4\pi}{5} \cdot \frac{e^2k^3}{V} \cdot \left\{\frac{2}{\sqrt{3}} I_2 - I_3\right\}^2$$

Adding the angular distribution $\left|X_{Lm}\right|^2$ (A5.1.21) with the relative weights (3:4:3) for m = (-1:0:1) gives an angular distribution proportional to

(2.2.20) $$\left(1 - \frac{1}{7}\cos^2\theta\right)$$

C. Electric Dipole Transition, F State to L State

The calculations for this transition are exactly the same as those in section B. Including the factor $\sqrt{\frac{2}{3}}$ for the relative amplitude of the quartet state, the initial state is, from (1.3.11),

(2.2.21) $$\Psi_b = -i2\sqrt{7\pi}\; W_{00}^{3}(-)\left\{G_{4,2}(3)\,\zeta_{4,1} - G_{4,1}(3)\,\zeta_{4,2}\right\}$$

and the final state is Ψ_a in (2.2.13). The electric dipole moment is (cf. (2.2.14, 15 and 16))

(2.2.22) $$Q_{lm}(F\text{-}D, S_3) = -ie\,7\sqrt{15}\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)\begin{pmatrix}3 & 1 & 2\\ 0 & -m & m\end{pmatrix}$$

where

(2.2.23) $I_4 = I_2$ (2.2.15) with $|G_l(1)\rangle$ replaced by $|G_l(3)\rangle$

and

(2.2.24) $I_5 = I_3$ (2.2.16) with $|G_l(1)\rangle$ replaced by $|G_l(3)\rangle$

The partial cross-section is

(2.2.25) $$\sigma_{lm}(F\text{-}D, S_3) = \frac{3920\pi}{3}\cdot\frac{e^2k^3}{v}\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)\begin{pmatrix}3 & 1 & 2\\ 0 & -m & m\end{pmatrix}^2\left\{\begin{pmatrix}3 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}I_4+\begin{pmatrix}3 & 1 & 2\\ 0 & -1 & 1\end{pmatrix}I_5\right\}^2$$

In the averaging over S_z, each cross-section $\sigma_{lm}(F\text{-}D)$ contains, apart from the common factors, the factor

(2.2.26) $$\frac{1}{4}\sum_{S_3}\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)^2\begin{pmatrix}3 & 1 & 2\\ 0 & -m & m\end{pmatrix}^2 = \frac{1}{2}\sum_{S_3}\begin{pmatrix}2 & \frac{3}{2} & \frac{1}{2}\\ -m & S_3 & -M\end{pmatrix}^2\begin{pmatrix}3 & 1 & 2\\ 0 & -m & m\end{pmatrix}^2$$
$$= \frac{1}{350} \quad \text{for} \quad m = \pm 1$$
$$= \frac{3}{350} \quad \text{for} \quad m = 0$$

In the summation over m to give the total cross-section this factor becomes 1/70. The total cross-section is

(2.2.27) $$\sigma(F\text{-}D) = \frac{8}{15}\pi\cdot\frac{e^2k^3}{v}\left\{\sqrt{3}\,I_4 + I_5\right\}^2$$

Adding the angular distribution $|X_{lm}|^2$ (A5.1.21) with the relative weights (1:3:1) for m = (-1:0:1) gives an angular distribution proportional to

(2.2.28) $\left(1-\frac{1}{2}\cos^2\theta\right)$

D. Electric Quadrupole Transition, D State to S State

Only the doublet part of the initial state will contribute so the initial state is, including the factor $\frac{1}{\sqrt{3}}$ for the relative amplitude of the doublet state,

(2.2.29) $$\Psi_b = -\sqrt{10\pi}\; W^2_{oo}(+)\left\{\frac{1}{\sqrt{2}}\left(G_{5,2}(2)\,\zeta_{5,1} - G_{5,1}(2)\,\zeta_{5,2}\right) + G_6(2)\,\zeta_6\right\}_b$$

and the final state is Ψ_a in (2.2.2).

The factor $r_i^2\, Y_{2,m}(\theta_i,\phi_i)$ appearing in Q_{2m} is expressed in terms of the internal co-ordinates and the Euler angles (App. 6).

(2.2.30) $$r_i^2 Y_{2,m}(\theta_i,\phi_i) = \sqrt{\frac{\pi}{2}}\, W^2_{om}(+)\left(2Z_i^2 - X_i^2\right) - \sqrt{6\pi}\, W^2_{1,m}(+)\left(X_i Z_i\right) + \sqrt{\frac{3\pi}{2}}\, W^2_{2,m}(+)\left(X_i\right)^2$$

The evaluation of Q_{2m} (D-S, S_z) proceeds in the same way as the evaluation in section 2A. The only non-zero term is

(2.2.31) $$\begin{aligned} Q_{2,0}\left(D\text{-}S, \pm\tfrac{1}{2}\right) &= -e\sqrt{5}\,\pi\left\langle W^0_{oo}(+)\middle| W^{2*}_{oo}(+)\middle| W^2_{oo}(+)\right\rangle I_6 \\ &= -\frac{e}{2}\sqrt{\frac{5}{2}}\, I_6 \end{aligned}$$

where

(2.2.32) $$\begin{aligned} I_6 &= \sum_i \int \left\{\ \right\}^*_a \tfrac{1}{2}\left(1-\tau_{i3}\right)\left(2Z_i^2 - X_i^2\right)\left\{\ \right\}_b d\tau \\ &= \frac{1}{\sqrt{2}}\left\langle F_1\middle|-f_1^{(2)}\middle|G_{5,2}(2)\right\rangle - \frac{1}{\sqrt{2}}\left\langle F_1\middle|f_2^{(2)}\middle|G_{5,1}(2)\right\rangle + \frac{1}{2}\left\langle F_{2,2}\middle|f_1^{(2)}\middle|G_{5,2}(2)\right\rangle - \frac{1}{2}\left\langle F_{2,2}\middle|f_2^{(2)}\middle|G_{5,1}(2)\right\rangle \\ &+ \frac{1}{\sqrt{2}}\left\langle F_{2,2}\middle|f_2^{(2)}\middle|G_6(2)\right\rangle - \frac{1}{2}\left\langle F_{2,1}\middle|f_2^{(2)}\middle|G_{5,2}(2)\right\rangle + \frac{1}{2}\left\langle F_{2,1}\middle|-f_1^{(2)}\middle|G_{5,1}(2)\right\rangle - \frac{1}{\sqrt{2}}\left\langle F_{2,1}\middle|f_1^{(2)}\middle|G_6(2)\right\rangle \\ &+ \frac{1}{2}\left\langle F_{3,2}\middle|f^{(1)}\middle|G_{5,2}(2)\right\rangle - \frac{1}{\sqrt{2}}\left\langle F_{3,2}\middle|-f_2^{(2)}\middle|G_6(2)\right\rangle + \frac{1}{2}\left\langle F_{3,1}\middle|f^{(1)}\middle|G_{5,1}(2)\right\rangle - \frac{1}{\sqrt{2}}\left\langle F_{3,1}\middle|-f_1^{(2)}\middle|G_6(2)\right\rangle \\ &+ \frac{1}{\sqrt{2}}\left\langle F_4\middle|-f_2^{(2)}\middle|G_{5,2}(2)\right\rangle - \frac{1}{\sqrt{2}}\left\langle F_4\middle|-f_1^{(2)}\middle|G_{5,1}(2)\right\rangle + \left\langle F_4\middle|f^{(1)}\middle|G_6(2)\right\rangle \end{aligned}$$

and $f^{(1)}$, $f_1^{(2)}$, $f_2^{(3)}$ are defined as in (A3.2.20) with

(2.2.33) $f(\tau_i) = (2Z_i^2 - X_i^2)$

The partial cross-section is

(2.2.34) $\sigma_{2,0}\left(D\text{-}S, \pm\frac{1}{2}\right) = \frac{\pi}{30} \cdot \frac{e^2 k^5}{V} I_6^2$

Averaging over spins gives

(2.2.35) $\sigma_{2,0}(D\text{-}S) = \frac{1}{2}\sum_{S_3} \sigma_{2,0}(D\text{-}S, S_3) = \sigma_{2,0}\left(D\text{-}S, \pm\frac{1}{2}\right)$

and the total cross section is

$$(2.2.36)\quad \begin{aligned} \sigma(D\text{-}S) &= \sum_m \sigma_{2,m}(D\text{-}S) \\ &= \frac{\pi}{30} \cdot \frac{e^2 k^5}{V} I_6^2 \end{aligned}$$

The angular distribution is proportional to

(2.2.37) $\left(\cos^2\theta - \cos^4\theta\right)$

E. Electric Quadrupole Transition, S State to D State

Only the quartet part of the S state will contribute in transitions to the (quartet) D state. Including the spin factor $\sqrt{\frac{2}{3}}$, the initial state is

(2.2.38) $\Psi_b = 2\sqrt{\pi}\, W_{00}^{0}(+) \left\{G_{4,2}(0)\, \zeta_{4,1} - G_{4,1}(0)\, \zeta_{4,2}\right\}_b$

and the final state is Ψ_d in (2.2.13). The integrals for $Q_{2,m}(S\text{-}D, S_z)$ are

$$
\begin{aligned}
(2.2.39)\quad Q_{2,m}(S\text{-}D,S_3) &= e2\pi\sum_i\int \psi_d^{*}\left(\frac{1}{\sqrt{2}}W^{2\,*}_{0,m}(+)\,(2Z_i^2-X_i^2)-\sqrt{6}\,W^{2\,*}_{1,m}(+)(X_iZ_i)+\sqrt{\frac{3}{2}}\,W^{2\,*}_{2,m}(+)(X_i^2)\right)W^{0}_{\infty}(+)\left\{\right\}_b d\tau \\
&= e2\pi\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)\left\{\frac{1}{\sqrt{2}}\left\langle W^{2}_{1,-m}(+)\middle|W^{0}_{\infty}(+)\middle|W^{0}_{\infty}(+)\right\rangle I_7-\sqrt{6}\left\langle W^{2}_{1,-m}(+)\middle|W^{2\,*}_{1,m}(+)\middle|W^{0}_{\infty}(+)\right\rangle I_8+\sqrt{\frac{3}{2}}\left\langle W^{2}_{2,-m}\middle|W^{2\,*}_{2,m}\middle|W^{0}_{\infty}\right\rangle I_9\right\} \\
&= e\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)(-)^m\left\{\tfrac{1}{2}I_7-\sqrt{3}\,I_8+\sqrt{\tfrac{3}{2}}\,I_9\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
I_7 &= \sum_i\int\left\{F_{11}\xi_1+\frac{1}{\sqrt{2}}\left(F_{13,2}\xi_{4,1}-F_{13,1}\xi_{4,2}\right)\right\}\cdot\tfrac{1}{2}(1-\tau_{i3})f_i(1)\left\{\right\}_b d\tau \\
(2.2.40)\quad &= \left\langle F_{11}\middle|-\sqrt{2}f_1^{(2)}\middle|G_{4,2}(0)\right\rangle-\left\langle F_{11}\middle|\sqrt{2}f_2^{(2)}\middle|G_{4,1}(0)\right\rangle+\frac{1}{\sqrt{2}}\left\langle F_{13,2}\middle|f^{(1)}+f_1^{(2)}\middle|G_{4,2}(0)\right\rangle-\frac{1}{\sqrt{2}}\left\langle F_{13,2}\middle|f_2^{(2)}\middle|G_{4,1}(0)\right\rangle \\
&\quad -\frac{1}{\sqrt{2}}\left\langle F_{13,1}\middle|f_2^{(2)}\middle|G_{4,2}(0)\right\rangle+\frac{1}{\sqrt{2}}\left\langle F_{13,1}\middle|f^{(1)}-f_1^{(2)}\middle|G_{4,1}(0)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
I_8 &= \sum_i\int\left\{\frac{1}{\sqrt{2}}\left(F_{14,2}\xi_{4,2}+F_{14,1}\xi_{4,1}\right)+F_{16}\xi_1\right\}\cdot\tfrac{1}{2}(1-\tau_{i3})f_i(2)\left\{\right\}_b d\tau \\
(2.2.41)\quad &= \frac{1}{\sqrt{2}}\left\langle F_{14,2}\middle|f_2^{(2)}\middle|G_{4,2}(0)\right\rangle-\frac{1}{\sqrt{2}}\left\langle F_{14,2}\middle|f^{(1)}-f_1^{(2)}\middle|G_{4,1}(0)\right\rangle+\frac{1}{\sqrt{2}}\left\langle F_{14,1}\middle|f^{(1)}+f_1^{(2)}\middle|G_{4,2}(0)\right\rangle \\
&\quad -\frac{1}{\sqrt{2}}\left\langle F_{14,1}\middle|f_2^{(2)}\middle|G_{4,1}(0)\right\rangle+\left\langle F_{16}\middle|-\sqrt{2}f_1^{(2)}\middle|G_{4,2}(0)\right\rangle-\left\langle F_{16}\middle|\sqrt{2}f_2^{(2)}\middle|G_{4,1}(0)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
I_9 &= \sum_i\int\left\{F_{12}\xi_1+\frac{1}{\sqrt{2}}\left(F_{15,2}\xi_{4,1}-F_{15,1}\xi_{4,2}\right)\right\}\cdot\tfrac{1}{2}(1-\tau_{i3})f_i(3)\left\{\right\}_b d\tau \\
(2.2.42)\quad &= \left\langle F_{12}\middle|-\sqrt{2}f_1^{(2)}\middle|G_{4,2}(0)\right\rangle-\left\langle F_{12}\middle|\sqrt{2}f_2^{(2)}\middle|G_{4,1}(0)\right\rangle+\frac{1}{\sqrt{2}}\left\langle F_{15,2}\middle|f^{(1)}+f_1^{(2)}\middle|G_{4,2}(0)\right\rangle \\
&\quad -\frac{1}{\sqrt{2}}\left\langle F_{15,2}\middle|f_2^{(2)}\middle|G_{4,2}(0)\right\rangle-\frac{1}{\sqrt{2}}\left\langle F_{15,1}\middle|f_2^{(2)}\middle|G_{4,2}(0)\right\rangle+\frac{1}{\sqrt{2}}\left\langle F_{15,1}\middle|f^{(1)}-f_1^{(2)}\middle|G_{4,1}(0)\right\rangle
\end{aligned}
$$

with $f^{(1)}$, $f_1^{(2)}$ and $f_2^{(2)}$ as in (A3.2.20) and

$$
\begin{aligned}
& f(r_i) = (2Z_i^2-X_i^2) && \text{in } I_7 \\
(2.2.43)\quad & f(r_i) = X_iZ_i && \text{in } I_8 \\
& f(r_i) = X_i^2 && \text{in } I_9
\end{aligned}
$$

The partial cross-section is

$$(2.2.44)\quad \sigma_{2,m}(S\text{-}D,S_3) = \frac{4\pi}{75}\cdot\frac{e^2k^5}{v}\cdot\left(2,-m,\tfrac{3}{2},S_3\middle|\tfrac{1}{2},M\right)^2\left\{\tfrac{1}{2}I_7-\sqrt{3}\,I_8+\tfrac{\sqrt{3}}{2}I_9\right\}^2$$

Averaging over S_z,

$$(2.2.45)\quad \frac{1}{4}\sum_{S_3}\left(2,-m,\tfrac{3}{2};S_3\middle|\tfrac{1}{2},M\right)^2 = \frac{1}{2}\sum_{S_3}\begin{pmatrix}2 & \frac{3}{2} & \frac{1}{2}\\ -m & S_3 & -M\end{pmatrix}^2$$

$$= \frac{1}{10} \quad \text{for all } m$$

In the summation over m to give the total cross-section, this factor becomes ½. The total cross-section is

$$(2.2.46)\quad \sigma\,(S\text{-}D) = \frac{2\pi}{75}\cdot\frac{e^2k^5}{v}\left\{\tfrac{1}{2}I_7-\sqrt{3}\,I_8+\tfrac{\sqrt{3}}{2}I_9\right\}^2$$

Adding the angular distributions $\left|\underline{X}_{2m}\right|^2$ with the relative weights (1:1:1:1:1) for m = (-2:-1:0:1:2) gives an isotopic distribution.

E. <u>Magnetic Dipole Transition, S State to S State</u>

The magnetic dipole moment (A5.2.25)

$$(2.2.47)\quad M'_{1,m} = -\frac{e}{2M}\sum_i\int r_i\,Y^*_{1m}(\theta_i\phi_i)\,\nabla\left\{\psi^*_a\,\mu_k\,\underline{\sigma}_k\,\psi_b\right\}d\tau$$

can be simplified by a partial integration to give

$$(2.2.48)\quad M'_{1,m} = \frac{e}{2M}\sum_i\int\left\{\nabla\left(r_i\,Y^*_{1m}(\theta_i\phi_i)\right)\right\}\cdot\left\{\psi_a\,\mu_k\,\underline{\sigma}_k\,\psi_b\right\}d\tau$$

The gradient term in (2.2.48) can be simplified,

$$(2.2.49)\quad \nabla r\,Y_{1,m} = \sqrt{\frac{3}{4\pi}}\,\underline{X}_m$$

where $\underline{\chi}_m$ are the vectors

$$\underline{\chi}_1 = -\frac{1}{\sqrt{2}}\left(\underline{e}_x + i\underline{e}_y\right)$$

(2.2.50) $$\underline{\chi}_0 = \underline{e}_z$$

$$\underline{\chi}_{-1} = \frac{1}{\sqrt{2}}\left(\underline{e}_x - i\underline{e}_y\right)$$

Expressed in terms of (2.2.50) the spin is

(2.2.51) $$\underline{\sigma} = -\sqrt{2}\,\underline{\chi}_{+1}\,\underline{\sigma}_- + \sqrt{2}\,\underline{\chi}_{-1}\,\underline{\sigma}_+ + \underline{\chi}_0\,\underline{\sigma}_0$$

The integral (2.2.48) then becomes

(2.2.52) $$M'_{lm} = \sqrt{\frac{3}{4\pi}}\cdot\frac{e}{2M}\sum_i \int \left(\underline{\chi}^*_m\cdot\underline{\sigma}_i\,\mu_i\right)\Psi_a\,\Psi_b\,d\tau$$

The spatial integral contains only the radial parts of Ψ_a and Ψ_b, with no factors (e.g. X_1, Z_1, ...) depending on the radial co-ordinates. The function Ψ_a (2.2.2) is a doublet S state of energy $-B_{He^3}$ and Ψ_b contains both doublet and quartet S states of positive energy describing the same system. Because of the orthogonality of wave functions differing only in energy, the doublet part of Ψ_b will not contribute to the integral. The initial state is therefore taken to be the quartet S state as in (2.2.2). These states are

(2.2.53) $$\Psi_a = W^0_{00}(+)\left\{F_1\xi_3 + \frac{1}{\sqrt{2}}\left(F_{2,2}\xi_{2,1} - F_{2,1}\xi_{2,2}\right) + \frac{1}{\sqrt{2}}\left(F_{3,2}\xi_{5,1} - F_{3,1}\xi_{5,2}\right) + F_4\xi_6\right\}_a$$

$$\Psi_b = 2\sqrt{\pi}\,W^0_{00}(+)\left\{G_{4,2}(0)\,\xi_{4,1} - G_{4,1}(0)\,\xi_{4,2}\right\}_b$$

The magnetic dipole moments become

$$(2.2.54)\quad \begin{aligned} M'_{1,0}(S\text{-}S,S_3) &= \sqrt{\tfrac{3}{4}}\cdot\frac{e}{M}\cdot\sum_i\int\{\}^*_a\mu_i\sigma_{i0}\{\}_b d\tau \\ M'_{1,\pm1}(S\text{-}S,S_3) &= \mp\sqrt{\tfrac{3}{2}}\cdot\frac{e}{M}\cdot\sum_i\int\{\}^*_a\mu_i\sigma_{i\mp}\{\}_b d\tau \end{aligned}$$

The summations over i, spin and isotopic spin are obtained using the matrix elements (A3.2.23-27). The results are

$$(2.2.55)\quad \begin{aligned} M'_{1,0}\left(S\text{-}S,\pm\tfrac{1}{2}\right) &= \pm\frac{1}{\sqrt{6}}\cdot\frac{e}{M}\left(\mu_P-\mu_N\right)I_{10} \\ M'_{1,\pm1}\left(S\text{-}S,\pm\tfrac{3}{2}\right) &= \pm\frac{1}{2}\cdot\frac{e}{M}\left(\mu_P-\mu_N\right)I_{10} \\ M'_{1,\pm1}\left(S\text{-}S,\pm\tfrac{1}{2}\right) &= \mp\frac{1}{2\sqrt{3}}\cdot\frac{e}{M}\left(\mu_P-\mu_N\right)I_{10} \end{aligned}$$

where

$$(2.2.56)\quad I_{10} = \left\langle F_{2,1}\middle|G_{2,1}(0)\right\rangle + \left\langle F_{2,2}\middle|G_{2,2}(0)\right\rangle + \left\langle F_{3,1}\middle|G_{2,1}(0)\right\rangle + \left\langle F_{3,2}\middle|G_{2,2}(0)\right\rangle$$

The partial cross-sections, averaged over initial spin states become

$$(2.2.57)\quad \begin{aligned} \sigma_{lm}(S\text{-}S) &= \frac{16\pi}{9}\cdot\frac{k^3}{v}\cdot\frac{1}{4}\sum_{S_3}{}'\left|M'_{lm}(S\text{-}S,S_3)\right|^2 \\ &= \frac{4\pi}{27}\cdot\frac{e^2}{M^2v}\,k^3\left(\mu_P-\mu_N\right)^2 I_{10}^2 \end{aligned}$$

The total cross-section is

$$(2.2.58)\quad \sigma(S\text{-}S) = \frac{4\pi}{9}\cdot\frac{e^2}{M^2v}\,k^3\left(\mu_P-\mu_N\right)^2 I_{10}^2$$

All m values enter with equal weight so the emitted radiation is completely isotropic.

3. Approximation for the Radial Integrals

In order to obtain an estimate of the relative magnitudes of the six cross-sections of the preceding section, the radial integrals $I_1 \ldots I_{10}$ must be evaluated. The exact evaluation of these integrals would require the radial functions $F_1 \ldots F_{16}$. To obtain the radial functions, it would be necessary to assume forms for the nuclear forces and solve the coupled differential equations. This was beyond the scope of this present work. It was therefore decided to use very simple approximations for the radial functions which would provide a rough estimate for the magnitude and energy dependence of the cross-sections, to show which reactions would be most important in the low energy range. More elaborate calculations could then be made for the cross-sections of those reactions which appear to be most important.

The main approximation is to use the asymptotic form (1.2.35) of the radial functions at all distances, and to use functions valid outside the range of the nuclear forces and also neglecting the coulomb forces. In the notation of (1.2.35) the function ϕ_6 is

(2.3.1) $$\phi_6 = u(r)\, v(q)$$

For the continuum states

(2.3.2) $$v(q) = j_l\,(kq)$$

and for the bound states

(2.3.3) $$v(q) = v_S(q) = \sqrt{2\beta_2}\,\frac{e^{-\beta_2 q}}{q}$$

The deuteron function $u(r)$ is, for the continuum S states,

(2.3.4) $$u(r) = u'_S(r) = \sqrt{\beta_1}\,\frac{e^{-\beta_1 r}}{r}$$

For the bound S states,

(2.3.5) $$u(r) = u'_S(r) = \sqrt{\beta'_1}\,\frac{e^{-\beta'_1 r}}{r}$$

and for the bound D states,

(2.3.6) $$u(r) = u'_D(r) = \sqrt{\beta'_1}\,C\,\frac{e^{-\beta'_1 r}}{r}\left(1 + \frac{3}{\beta'_1 r} + \frac{3}{(\beta'_1 r)^2}\right)$$

The function $u_D(r)$ is a deuteron D state function (Blatt and Weisskopf, 1952, p. 103) for which the integration cannot be extended into $r = 0$ but has the normalization

(2.3.7) $$\int_0^{\pi}\int_R^{\infty} u'^2_D(r)\, r^2 \sin\theta\, dr\, d\theta = 1$$

Integrals over r in which $u_D(r)$ appears are evaluated over the range $R \leq r \leq \infty$. The final results are not sensitive to the value of R. The normalizations of $u_S(r)$ and $v_S(q)$ are

(2.3.8) $$\int_0^{\pi}\int_0^{\infty} u^2_S(r)\, r^2 \sin\theta\, dr\, d\theta = 1 \qquad \text{and} \qquad \int_0^{\infty} v^2_S(q)\, q^2\, dq = 1$$

The constants β_1, β'_1, β_2 and k are obtained from the binding energies of deuteron and He^3 and the kinetic energy of the continuum state (App. 4). To determine the range parameter β'_1, an equal division of the binding energy of He^3 between the three nucleons is assumed. This assumption gives a deuteron function which is more tightly bound in the bound state of He^3 than in the continuum state.

$$(2.3.9)\qquad \beta_1^{2} = MB_D \qquad \beta_2^{2} = \tfrac{4}{9} MB_{He}$$
$$\beta_1'^{2} = \tfrac{2}{3} MB_{He} \qquad k^{2} = \tfrac{4}{3} ME_c$$

In using the approximation (1.2.35) for the bound state functions, there arises a question regarding the correct normalization. For example

$$(2.3.10)\qquad F_4 \approx \frac{a_4}{\sqrt{3}}\, u_S(r)\, v_S(q)$$

has the normalization

$$(2.3.11)\qquad \int F_4^2 \, d\tau = \frac{a_4^2}{3}$$

This factor of $\frac{1}{\sqrt{3}}$ in (2.3.10) appears in all the radial terms. It was decided that the approximate functions, whatever their form, must have unit normalization. Accordingly in all the approximate approximations of the bound state radial functions the factor $1/\sqrt{3}$ is omitted. This, however, does not affect the relative sizes of the estimated cross-sections. Thus the functions become

$$F_1,\ \frac{F_{2,2}}{\sqrt{2}},\ \frac{F_{3,2}}{\sqrt{2}},\ F_{11},\ F_{12},\ \frac{F_{13,2}}{\sqrt{2}},\ \frac{F_{14,2}}{\sqrt{2}},\ \frac{F_{15,2}}{\sqrt{2}} = 0$$

$$(2.3.13)\qquad -\frac{F_{2,1}}{\sqrt{2}},\ -\frac{F_{3,1}}{\sqrt{2}},\ F_4 = a_i\, u_S'(r)\, v_S(q)$$

$$-\frac{F_{13,1}}{\sqrt{2}},\ -\frac{F_{14,1}}{\sqrt{2}},\ -\frac{F_{15,1}}{\sqrt{2}},\ F_{16} = a_i\, u_D'(r)\, v_S(q)$$

$$\frac{G_{4,2}(l)}{\sqrt{2}},\ \frac{G_{5,2}(l)}{\sqrt{2}} = 0$$

$$-\frac{G_{4,1}(l)}{\sqrt{2}},\ -\frac{G_{5,1}(l)}{\sqrt{2}},\ G_6(l) = \frac{1}{\sqrt{3}}\, u_S(r)\, j_l(kq)$$

In the integrations, there is one further approximation made. The factors Z_i, X_i, $(2Z_i^2 - X_i^2)$ ... appearing in the integrals contain terms depending on the angle ϕ (1.1.6). The dependence of ϕ on r, q and θ is rather complex. However in the range of validity of the approximate radial functions, i.e. $q > r >$ (deuteron size), ϕ is slowly varying and approximately zero. For example for $q = 1.5\ r$, $|\phi| < 9.6^\circ$ for $0 \leq \theta < \pi$. It was therefore decided to place $\phi = 0$ in these approximate calculations. The effect of this approximation on the calculation was felt to be considerably less than the effect of using the asymtotic radial functions (1.2.35).

The factors Z_i, X_i, $(2Z_i^2 - X_i^2)$ ... appearing in I_1 ... I_9 are expressed in terms of the internal co-ordinates using (1.1.3). The results and approximate forms used are given below.

In I_1, I_2, I_3, I_4 and I_5

(2.3.13)
$$Z_3 = \tfrac{2}{3} q \cos\phi \approx \tfrac{2}{3} q$$
$$(-Z_1 + Z_2) = -r\cos(\phi+\theta) \approx -r\cos\theta$$
$$X_3 = \tfrac{2}{3} q \sin\phi \approx 0$$
$$(-X_1 + X_2) = -r\sin(\phi+\theta) \approx -r\sin\theta$$

In I_6 and I_7

(2.3.14)
$$f^{(1)} = \tfrac{4}{9} q^2 \left(3\cos^2\phi - 1\right) + \tfrac{1}{3} r^2 \left(3\cos^2(\phi+\theta) - 1\right) \approx \frac{8q^2}{9} + \frac{r^2}{3}\left(3\cos^2\theta - 1\right)$$

$$f_1^{(2)} = -\tfrac{2}{9} q^2 \left(3\cos^2\phi - 1\right) + \frac{r^2}{6}\left(3\cos^2(\phi+\theta) - 1\right) \approx -\tfrac{4}{9} q^2 + \frac{r^2}{6}\left(3\cos^2\theta - 1\right)$$
$$f_2^{(2)} = \frac{rq}{3\sqrt{3}}\left(2\cos\phi\cos(\phi+\theta) - \sin\phi\sin(\phi+\theta)\right) \approx \frac{2rq}{3\sqrt{3}}\cos\theta$$

In I_8

(2.3.15)

$$f^{(1)} = \frac{4}{9}q^2 \sin\phi\cos\phi + \frac{r^2}{3}\sin(\phi+\theta)\cos(\phi+\theta) \approx \frac{r^2}{3}\sin\theta\cos\theta$$

$$f_1^{(2)} = -\frac{2}{9}q^2\sin\phi\cos\phi + \frac{r^2}{6}\sin(\phi+\theta)\cos(\phi+\theta) \approx \frac{r^2}{6}\sin\theta\cos\theta$$

$$f_2^{(2)} = \frac{1}{6\sqrt{3}}rq\left(\sin\phi\cos(\phi+\theta) + \sin(\phi+\theta)\cos\phi\right) \approx \frac{1}{6\sqrt{3}}rq\sin\theta$$

In I_9

(2.3.16)

$$f^{(1)} = \frac{4}{9}q^2\sin^2\phi + \frac{r^2}{3}\sin^2(\phi+\theta) \approx \frac{r^2}{3}\sin^2\theta$$

$$f_1^{(2)} = -\frac{2}{9}q^2\sin^2\phi + \frac{r^2}{6}\sin^2(\phi+\theta) \approx \frac{r^2}{6}\sin^2\theta$$

$$f_2^{(2)} = \frac{1}{3\sqrt{3}}rq\sin\phi\sin(\phi+\theta) \approx 0$$

The evaluation of the approximate integrals is now straightforward. The integrals $\int e^{-\beta_2 q} j_l(kq)q^n dq$ are taken from Morse and Feshbach (1953, p. 1575). Taking only the non-zero terms, the results are:

(2.3.17)
$$I_1 \approx \tfrac{1}{2}\left\langle F_{2,1}\middle|\tfrac{1}{2}Z_3\middle|G_{5,1}(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_{2,1}\middle|-\tfrac{1}{2}Z_3\middle|G_6(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_{3,1}\middle|\tfrac{1}{2}Z_3\middle|G_6(1)\right\rangle - \tfrac{1}{\sqrt{2}}\left\langle F_4\middle|\tfrac{1}{2}Z_3\middle|G_{5,1}(1)\right\rangle$$
$$= \frac{1}{3\sqrt{3}}(a_3+a_4)\int u_S'(r)\,v_S(q)(q)\,u_S(r)\,j_1(kq)\,r^2q^2\sin\theta\,dr\,dq\,d\theta$$
$$= \frac{4}{3}\sqrt{\frac{2}{3}}(a_3+a_4)\frac{\sqrt{\beta_1\beta_1'\beta_2}}{(\beta_1+\beta_1')}\cdot\frac{k}{(k^2+\beta_2^2)^2}$$

(2.3.18)
$$I_2 \approx \frac{1}{\sqrt{2}}\left\langle F_{13,1}\middle|\tfrac{1}{2}Z_3\middle|G_{4,1}(1)\right\rangle$$
$$= \frac{1}{3}\sqrt{\frac{2}{3}}\,a_{13}\int u_D(r)\,v_S(q)(q)\,u_S(r)\,j_1(kq)\,r^2q^2\sin\theta\,dr\,dq\,d\theta$$
$$= \frac{8}{3\sqrt{3}}\,a_{13}\,R(1)\sqrt{\beta_2}\cdot\frac{k}{(k^2+\beta_2^2)^2}$$

(2.3.19)
$$I_3 \approx -\frac{1}{\sqrt{2}}\left\langle F_{14,1}\middle|\frac{1}{2\sqrt{3}}(-X_1+X_2)\middle|G_{4,1}(1)\right\rangle - \left\langle F_{16}\middle|\frac{1}{\sqrt{6}}(-X_1+X_2)\middle|G_{4,1}(1)\right\rangle$$
$$= \frac{1}{3\sqrt{2}}\left(a_{14}-\sqrt{2}\,a_{16}\right)\int u_D(r)\,v_S(q)(r\sin\theta)\,u_S(r)\,j_1(kq)\,r^2q^2\sin\theta\,dr\,dq\,d\theta$$
$$= \frac{\pi}{9}\left(a_{14}-\sqrt{2}\,a_{16}\right)R(2)\frac{\sqrt{\beta_2}}{\beta_1'}\cdot\frac{k}{(k^2+\beta_2^2)^{3/2}}\,F\left(\tfrac{3}{2},\tfrac{1}{2},\tfrac{5}{2},3\right)$$

$$(2.3.20)\quad I_4 \approx \frac{1}{\sqrt{2}}\left\langle F_{13,1}\left|\tfrac{1}{2}Z_3\right|G_{4,1}(3)\right\rangle$$
$$= \frac{1}{3}\sqrt{\frac{2}{3}}\, a_{13}\int u_D(r)\, v_S(q)\chi(q)\, u_S(r)\, j_3(kq)\, r^2q^2 \sin\theta\, dr\, dq\, d\theta$$
$$= \frac{16}{21\sqrt{3}}\, a_{13}\, R(1)\sqrt{\beta_2}\ \frac{k^3}{(k^2+\beta_2^{\,2})^3}\ F(3,1,\tfrac{9}{2},3)$$

$$(2.3.21)\quad I_5 \approx -\frac{1}{\sqrt{2}}\left\langle F_{14,1}\left|\frac{1}{2\sqrt{3}}(-X_1+X_2)\right|G_{4,1}(3)\right\rangle - \left\langle F_{16}\left|\frac{1}{\sqrt{6}}(-X_1+X_2)\right|G_{4,1}(3)\right\rangle$$
$$= \frac{1}{3\sqrt{2}}\left(a_{14}-\sqrt{2}\,a_{16}\right)\int u_D(r)\, v_S(q)(r\sin\theta)\, u_S(r)\, j_3(kq)\, r^2q^2\sin\theta\, dr\, dq\, d\theta$$
$$= \frac{4\pi}{135}\left(a_{14}-\sqrt{2}\,a_{16}\right)\frac{R(2)\sqrt{\beta_2}}{\beta_1'}\ \frac{k^3}{(k^2+\beta_2^{\,2})^{\frac{5}{2}}}\ F(\tfrac{5}{2},\tfrac{3}{2},\tfrac{9}{2},3)$$

$$(2.3.22)\quad I_6 \approx \tfrac{1}{2}\left\langle F_{2,1}\left|-f_1^{(2)}\right|G_{5,1}(2)\right\rangle - \frac{1}{\sqrt{2}}\left\langle F_{2,1}\left|f_1^{(2)}\right|G_6(2)\right\rangle + \tfrac{1}{2}\left\langle F_{3,1}\left|f^{(1)}\right|G_{5,1}(2)\right\rangle - \frac{1}{\sqrt{2}}\left\langle F_{3,1}\left|-f_1^{(2)}\right|G_6(2)\right\rangle$$
$$-\frac{1}{\sqrt{2}}\left\langle F_4\left|-f_1^{(2)}\right|G_{5,1}(2)\right\rangle + \left\langle F_4\left|f^{(1)}\right|G_6(2)\right\rangle$$
$$= \frac{4}{3\sqrt{3}}\left(a_3+a_4\right)\int u_S'(r)\, v_S(q)\left(q^2+\tfrac{r^2}{4}(3\cos^2\theta-1)\right) u_S(r)\, j_2(kq)\, r^2q^2\sin\theta\, dr\, dq\, d\theta$$
$$= \frac{64}{3}\sqrt{\frac{2}{3}}\left(a_3+a_4\right)\frac{\beta_1\beta_1'\beta_2}{(\beta_1+\beta_1')}\cdot\frac{k^2}{(k^2+\beta_2^{\,2})^3}$$

$$(2.3.23)\quad I_7 \approx \frac{1}{\sqrt{2}}\left\langle F_{13,1}\left|f^{(1)}-f_1^{(2)}\right|G_{4,1}(0)\right\rangle$$
$$= \frac{4}{3}\sqrt{\frac{2}{3}}\, a_{13}\int u_D(r)\, v_S(q)\left(q^2+\tfrac{r^2}{4}(3\cos^2\theta-1)\right) u_S(r)\, j_0(kq)\, r^2q^2\sin\theta\, dr\, dq\, d\theta$$
$$= \frac{32}{\sqrt{3}}\, a_{13}\, R(1)\sqrt{\beta_2}\cdot\frac{1}{(k^2+\beta_2^{\,2})^2}\ F(2,-1,\tfrac{3}{2},3)$$

$$(2.3.24)\quad I_8 \approx -\frac{1}{\sqrt{2}}\left\langle F_{14,1}\left|f_2^{(2)}\right|G_{4,1}(0)\right\rangle - \left\langle F_{16}\left|\sqrt{2}f_2^{(2)}\right|G_{4,1}(0)\right\rangle$$
$$= \frac{-(a_{14}-\sqrt{2}\,a_{16})}{9\sqrt{2}}\int u_D(r)\, v_S(q)(rq\sin\theta)\, u_S(r)\, j_0(kq)\, r^2q^2\sin\theta\, dr\, dq\, d\theta$$
$$= -\frac{\pi}{9}\left(a_{14}-\sqrt{2}\,a_{16}\right)\frac{R(2)\beta_2^{3/2}}{\beta_1'}\ \frac{1}{(k^2+\beta_2^{\,2})^2}$$

$$(2.3.25)\quad I_9 \approx \frac{1}{\sqrt{2}}\left\langle F_{15,1}\left|f^{(1)}-f_1^{(2)}\right|G_{4,1}(0)\right\rangle$$
$$= \frac{1}{3\sqrt{6}}\, a_{15}\int u_D(r)\, v_S(q)(r^2\sin^2\theta)\, u_S(r)\, j_0(kq)\, r^2q^2\sin\theta\, dr\, dq\, d\theta$$
$$= \frac{4}{9\sqrt{3}}\, a_{15}\, R(3)\frac{\sqrt{\beta_2}}{\beta_1'^{\,2}}\cdot\frac{1}{(k^2+\beta_2^{\,2})}$$

$$(2.3.26)\quad I_{10} \approx \langle F_{2,1} | G_{2,1}(0) \rangle + \langle F_{3,1} | G_{2,1}(0) \rangle$$

$$= \frac{2}{\sqrt{3}} (a_2 + a_3) \int u_S'(r)\, v_S(q)\, u_S(r)\, J_0(kq)\, r^2 q^2 \sin\theta \, dr\, dq\, d\theta$$

$$= 4\sqrt{\frac{2}{3}} \cdot (a_2 + a_3) \quad \frac{\sqrt{\beta_1 \beta_1' \beta_2}}{(\beta_1 + \beta_1')} \cdot \frac{1}{(k^2 + \beta_2^2)}$$

where

$$(2.3.27)\quad \begin{aligned} R(1) &= \int_R^\infty u_D'(r)\, u_S(r)\, r^2 dr \\ R(2) &= \beta_1' \int_R^\infty u_D'(r)\, u_S(r)\, r^3 dr \\ R(3) &= \beta_1'^2 \int_R^\infty u_D'(r)\, u_S(r)\, r^4 dr \end{aligned}$$

are factors depending on the lower limit of the integrations over r. Some numerical values of these are given in Table III.

TABLE III

Numerical Values of the Integrals R(i)

$x = \beta_1' R$	R(1)	R(2)	R(3)
0.4	0.30	0.22	0.20
0.6	0.28	0.28	0.33
0.8	0.26	0.32	0.44
1.0	0.23	0.34	0.54
1.2	0.21	0.32	0.63

F(a,b,c,z) is the hypergeometric function

$$(2.3.28)\quad F(a,b,c,z) = 1 + \frac{ab}{c} z + \frac{a(a+1)\, b(b+1)\, z^2}{2!\, c(c+1)} + \ldots$$

$$z = \frac{k^2}{(k^2 + \beta_2^2)}$$

These are now substituted in the expressions for the total cross-sections. All energy dependent functions have been written in terms of the nucleon momentum k .

$$(2.3.29)\quad \sigma_{E_1}(P\text{-}S) = \frac{8\pi}{9}\cdot\frac{e^2}{M^2}\cdot\frac{\beta_1\beta_1'\beta_2}{(\beta_1+\beta_1')^2}\cdot\frac{k(k^2+\beta_2'^2)^3}{(k^2+\beta_2^2)^4}\cdot\left\{a_3+a_4\right\}^2$$

$$(2.3.30)\quad \sigma_{E_1}(P\text{-}D) = \frac{32\pi}{45}\cdot\frac{e^2\beta_2 k(k^2+\beta_2'^2)^3}{M^2\,(k^2+\beta_2^2)^4}\left\{a_{13}R_{(1)}-\left(a_{14}-\sqrt{2}a_{16}\right)\frac{\pi}{16}\,\frac{R_{(2)}}{\beta_1'}\left(k^2+\beta_2^2\right)^{\frac{1}{2}}F\left(\tfrac{3}{2},\tfrac{1}{2},\tfrac{5}{2},3\right)\right\}^2$$

$$(2.3.31)\quad \sigma_{E_1}(F\text{-}D) = \frac{64\pi}{735}\cdot\frac{e^2}{M^2}\cdot\frac{\beta_2 k(k^2+\beta_2'^2)^3}{(k^2+\beta_2^2)^4}\left\{a_{13}R_{(1)}F\left(3,1,\tfrac{9}{2},3\right)+\left(a_{14}-\sqrt{2}a_{16}\right)\frac{\pi}{20}\,\frac{R_{(2)}}{\beta_1'}\left(k^2+\beta_2^2\right)^{\frac{1}{2}}F\left(\tfrac{5}{2},\tfrac{3}{2},\tfrac{9}{2},3\right)\right\}^2$$

$$(2.3.32)\quad \sigma_{E_1}(D\text{-}S) = \frac{64\pi}{15}\cdot\frac{e^2}{M^4}\cdot\frac{\beta_1\beta_1'\beta_2}{(\beta_1+\beta_1')^2}\,\frac{k^3(k^2+\beta_2'^2)^3}{(k^2+\beta_2^2)^6}\left\{a_3+a_4\right\}^2$$

$$(2.3.33)\quad \sigma_{E_2}(S\text{-}D) = \frac{9\pi}{25}\cdot\frac{e^2}{M^4}\cdot\frac{\beta_2\,(k^2+\beta_2'^2)^5}{k\,(k^2+\beta_2^2)^4}\cdot\left\{a_{13}R_{(1)}\left(1-\frac{4}{3}\frac{k^2}{(k^2+\beta_2^2)}\right)+\frac{\pi}{48}\left(a_{14}-\sqrt{2}a_{16}\right)\frac{R_{(2)}\beta_2}{\beta_1'}+a_{15}\frac{\sqrt{3}}{72}\,\frac{R_{(3)}}{\beta_1'^2}\left(k^2+\beta_2^2\right)\right\}^2$$

$$(2.3.34)\quad \sigma_{M_1}(S\text{-}S) = \frac{4\pi}{3}\cdot\frac{e^2}{M^4}\cdot\frac{\beta_1\beta_1'\beta_2}{(\beta_1+\beta_1')^2}\cdot\left(\mu_P-\mu_N\right)^2\frac{(k^2+\beta_2'^2)^3}{k(k^2+\beta_2^2)^2}\cdot\left\{a_2+a_3\right\}^2$$

To obtain numerical values of these cross-sections, they are now calculated assuming a laboratory proton energy of 1 Mev. The values used are (App. 4)

(2.3.35)

$$E_C = 2/3\ E_L = 0.667\ \text{Mev} = 0.338 \times 10^{11}\ \text{cm}^{-1}$$

$$\beta_1 = 2.32 \times 10^{12}\ \text{cm}^{-1}$$

$$\beta_1' = 3.53 \times 10^{12}\ \text{cm}^{-1}$$

$$\beta_2 = 2.79 \times 10^{12}\ \text{cm}^{-1}$$

$$\beta_2' = 4.20 \times 10^{12}\ \text{cm}^{-1}$$

$$k = 1.46 \times 10^{12}\ \text{cm}^{-1}$$

$$e^2 = 1/137$$

$$M = 4.76 \times 10^{13}\ \text{cm}^{-1}$$

$$\mu_P = 2.79\ \text{n.m.}$$

$$\mu_N = -1.91\ \text{n.m.}$$

The results are tabulated below.

TABLE IV

Numerical Values of the Cross-sections

Multipolarity of Radiation	Cross-Section at $E_L = 1$ Mev. in 10^{-30} cm^2	Angular Distribution
Electric dipole	$\sigma_{E_1}(P-S) = 6.4 \left\{a_3 + a_4\right\}^2$	$\sin^2\theta$
Electric dipole	$\sigma_{E_1}(P-D) = 22 \left\{(23)a_{13} - (07)(a_{14} - \sqrt{2}\, a_{16})\right\}^2$	$1 - \frac{1}{7}\cos^2\theta$
Electric dipole	$\sigma_{E_1}(F-D) = 0.12 \left\{(.28)a_{13} + (.06)(a_{14} - \sqrt{2}\, a_{16})\right\}^2$	$1 - \frac{1}{2}\cos^2\theta$
Electric quadrupole	$\sigma_{E_2}(D-S) = 0.012 \left\{a_3 + a_4\right\}^2$	$\cos^2\theta - \cos^4\theta$
Electric quadrupole	$\sigma_{E_2}(S-D) = 0.10 \left\{(17)a_{13} + (.02)(a_{14} - \sqrt{2}\, a_{16}) + (01)a_{15}\right\}^2$	isotropic
Magnetic dipole	$\sigma_{M_1}(S-S) = 5.0 \left\{a_2 + a_3\right\}^2$	isotropic

For transitions going to the D state of He^3, the value $\beta_1' R = 1$ was arbitrarily assumed. Examination of Table III shows that the cross-sections concerned are relatively insensitive to the choice of β_1' R.

The results of (2.3.29 - 34) and Table IV are discussed and compared with the experimental results in Chapter 3.

CHAPTER 3

DISCUSSION AND COMPARISON WITH EXPERIMENT

In the cross-sections for the reactions studied in Chapter 2, the angular distributions obtained are independent of any approximations made for the radial functions. The magnitudes of the cross-sections given in (2.2.29-34) are greatly dependent on the form of the radial functions and of the amplitudes, a_i, of the states and it is seen that considerable cancellation may occur depending on the relative signs of the amplitudes. From the arguments presented in Chapter 1, section 3D, it is expected that the S state of He^3 having a symmetrical radial function, $\mu_4 F_4$, would be the most important state, i.e. $a_4^2 \sim 1$ and for all other states $a_i^2 \ll 1$. Thus in the cross-section $\sigma_{E_1}(P\text{-}S)$ and $\sigma_{E_2}(D\text{-}S)$ it might be expected that the amplitude factor $\{a_3 + a_4\}^2 \sim 1$ and all other amplitude factors $\{\ \}^2 \ll 1$ (Table IV). These arguments, while only indicating expected orders of magnitude for these factors, are useful in comparing the numerical results of Table IV with the experimental data.

The most recent data with which the results of Chapter 2 may be compared are those of Griffiths, Larson and Robertson (1961). This data is tabulated in Table V. In obtaining the cross-sections for the $\sin^2\theta$ component (σ_{E_1}) and the 'isotropic' component ($\sigma_?$) from the measured total cross-section ($\sigma = \sigma_{E_1} + \sigma_?$), it was assumed that the smaller component was completely isotropic.

TABLE V

Summary of Experimental Data

Energy (E_L)	Angular Distribution	σ_{E_1} in 10^{-30} cm.^{2}	$\sigma_?$	$\sigma_? / \sigma_{E_1}$
0.3 Mev	$\sin^2\theta + (.079 \pm .010)$	0.803	0.095	0.12
0.6 Mev	$\sin^2\theta + (.032 \pm .004)$			
1.0 Mev	$\sin^2\theta + (.024 \pm .003)$	3.13	0.11	0.035

Comparing the angular distributions it is seen that the main component, proportional to $\sin^2\theta$ can only arise from an electric dipole transition from a P-wave proton to the S state of He^3. The calculated size of $\sigma_{E_1}(P\text{-}S)$ at E_L = 1 Mev is, assuming $\{a_3 + a_4\}^2 = 1$, remarkably close to the experimental value considering the roughness of the approximations made. For further comparisons the calculated value of $\sigma_{E_1}(P\text{-}S)$ at E_L = 0.3 Mev is $\sigma_{E_1}(P\text{-}S) = 5.7 \times 10^{-30}\{a_3 + a_4\}^2 \text{cm.}^2$ It must be remembered that the effect of the coulomb repulsion has been ignored in these approximations. The effect of this repulsion increases as the energy decreases and would, if included, lower the calculated value of the cross-sections. For example, the simple coulomb penetraction factor (Blatt and Weisskopf, 1952, p. 87)

$$C^2 = \frac{2\pi\eta}{e^{2\pi\eta} - 1} \qquad (3.1.1)$$

$$\eta = \frac{e^2}{v}$$

has the values C^2 = 0.26 at E_L = 0.3 Mev and C^2 = 0.59 at E_L = 1.0 Mev.

In attempting to identify the origin of the smaller component of the reaction, the contributions from the electric dipole transition from an F-wave proton to the D state of He^3, and from both of the electric quadrupole transitions may be neglected at this energy. The cross-sections

$$(3.1.2) \quad \sigma_{E_1}(F\text{-}D) = 0.12 \times 10^{-30}\left\{(28)a_{13} + (.06)(a_{14} - \sqrt{2}\,a_{16})\right\}^2 cm^2$$
$$= 0.0093 \times 10^{-30}\left\{a_{13} + (.22)(a_{14} - \sqrt{2}\,a_{16})\right\}^2 cm^2$$

and

$$(3.1.3) \quad \sigma_{E_2}(S\text{-}D) = 0.10 \times 10^{-30}\left\{(.17)a_{13} + (.02)(a_{14} - \sqrt{2}\,a_{16}) + (.01)a_{15}\right\}^2 cm^2 \cdot$$
$$= 0.003 \times 10^{-30}\left\{a_{13} + (.10)(a_{14} - \sqrt{2}\,a_{16}) + (.06)a_{15}\right\}^2 cm^2$$

are both far too small to explain the observed value of $\sigma_? = 0.11 \times 10^{-30}\ cm^2$. The cross-section

$$(3.1.4) \quad \sigma_{E_2}(D\text{-}S) = 0.012 \times 10^{-30}\left\{a_3 + a_4\right\}^2$$

is small, but more important, the angular distribution, proportional to $\cos^2\theta - \cos^4\theta = \sin^{+2}\theta\cos^2\theta$ is zero at $\theta = 0$ and would not appear as an 'isotropic' contribution. It is possible that this reaction could contribute to the total cross-section but the contribution would be small and difficult to detect as $\dfrac{\sigma_{E_2}(D\text{-}S)}{\sigma_{E_1}(P\text{-}S)} = 0.016$ at $E_L = 1$ Mev. The accuracy quoted by Griffiths, et al. is, at this energy, σ total $= (3.24 \pm 0.35) \times 10^{-30}\ cm^2$. or $\pm$ 11%.

Either of the remaining cross-sections

$$(3.1.5) \quad \sigma_{E_1}(P\text{-}D) = 2.2 \times 10^{-30}\left\{(.23)a_{13} - (.07)(a_{14} - \sqrt{2}\,a_{16})\right\}^2 \text{cm}^2$$
$$= 1.2 \times 10^{-30}\left\{a_{13} - (.28)(a_{14} - \sqrt{2}\,a_{16})\right\}^2 \text{cm}^2$$

or

$$(3.1.6) \quad \sigma_{M_1}(S\text{-}S) = 5.0 \times 10^{-30}\left\{a_2 + a_3\right\}^2 \text{cm}^2$$

could be the correct size to explain the smaller component, for a suitable choice of the amplitudes a_i. It is thus necessary to look at the energy dependence of these two cross-sections, or rather the energy dependence of the ratios $\frac{\sigma_{E_1}(P\text{-}D)}{\sigma_{E_1}(P\text{-}S)}$ and $\frac{\sigma_{M_1}(S\text{-}S)}{\sigma_{E_1}(P\text{-}S)}$. These ratios are

$$(3.1.7) \quad \frac{\sigma_{E_1}(P\text{-}D)}{\sigma_{E_1}(P\text{-}S)} = \frac{16}{5}\,\frac{\left\{a_{13}R_{(1)} - (a_{14} - \sqrt{2}\,a_{16})\frac{\pi}{16}\,\frac{R(2)}{\beta_1}(k^2+\beta_2^2)^{\frac{1}{2}}F\left(\frac{3}{2},\frac{1}{2},\frac{5}{2},3\right)\right\}^2}{\left\{a_3 + a_4\right\}^2}$$

and

$$(3.1.8) \quad \frac{\sigma_{M_1}(S\text{-}S)}{\sigma_{E_1}(P\text{-}S)} = \frac{3}{2}\,\frac{(\mu_P - \mu_N)^2(k^2+\beta_2^2)^2\left\{a_2 + a_3\right\}^2}{M^2k^2\left\{a_3 + a_4\right\}^2}$$

The ratio (3.1.7) is independent of energy, as would be expected, both cross-sections arising from electric dipole transitions for P-wave protons. The ratio (3.1.8) decreases with increasing energy, the values for E_L = 0.3 Mev and 1 Mev being

$$(3.1.9) \quad \frac{\sigma_{M_1}(S\text{-}S)}{\sigma_{E_1}(P\text{-}S)} = 1.8\,\frac{\left\{a_2+a_3\right\}^2}{\left\{a_3+a_4\right\}^2} \qquad \text{for } E_L = 0.3 \text{ Mev.}$$

$$= .075\,\frac{\left\{a_2+a_3\right\}^2}{\left\{a_3+a_4\right\}^2} \qquad \text{for } E_L = 1.0 \text{ Mev}$$

Between these two energies this ratio decreases by a factor of 2.4. Experimentally, the ratio $\sigma_?/\sigma_{E_1}$, decreases by a factor of 3.4. If the smaller component of the experimental cross-section is identified with $\sigma_{M_1}(S\text{-}S)$, it would be necessary to assume that $\{a_2 + a_3\}^2 \sim (.05)\{a_3 + a_4\}^2$ thus supporting the qualitative arguments concerning the amplitudes a_i.

It thus is more probable that the small 'isotropic' component of the reaction comes mainly from the magnetic dipole transition of S-wave protons to the S state of He^3. The observed cross-section is small because the reaction does not involve the radially symmetric state, $\omega_4 F_4$, even before any approximations are made. However, the electric dipole transition of P-wave protons to the D state of He^3 could also contribute to observed cross-section. Only these two transitions appear likely to contribute to the smaller component of the cross-section.

The numerical values of the cross-sections as given in Table IV depend on the particular choice of β_1' and β_2 in (2.3.5). The original numerical estimates were made using

$$\begin{aligned} \beta_1' &= \beta_1 = 2.32 \times 10^{12}\,\mathrm{cm}^{-1} \\ \beta_2 &= 4.20 \times 10^{12}\,\mathrm{cm}^{-1} \end{aligned} \tag{3.1.10}$$

The cross-sections obtained differed from those in Table IV, the differences coming mainly from the variation of β_2. The differences were not sufficient to alter the conclusions of this chapter.

To extend these calculations in order to obtain more valuable estimates for the three cross-sections of interest $\sigma_{E_1}(P-S)$, $\sigma_{E_1}(P-D)$ and $\sigma_{M_1}(S-S)$ it would be possible to use more accurate radial functions which include the effects of the coulomb repulsion and also, for the continuum states, the scattering phase shifts. This would not remove the problem of the amplitudes of the bound states and until numerical estimates for the a_i can be made the absolute magnitude of any calculated cross-section will be uncertain. The programme for future work should thus include some method for deriving numerical estimates for the amplitudes of the bound state.

APPENDIX 1

THE CO-ORDINATE TRANSFORMATION

The detailed expression for the transformations $\eta_i(\rho_j)$ defined in (1.1.21) is

$$
(A1.1.1)\quad
\begin{aligned}
\eta_1 &= x_1' = s_x + R_{11}X_1 + R_{13}Z_1 \\
\eta_2 &= y_1' = s_y + R_{21}X_1 + R_{23}Z_1 \\
\eta_3 &= z_1' = s_z + R_{31}X_1 + R_{33}Z_1 \\
\eta_4 &= x_2' = s_x + R_{11}X_2 + R_{13}Z_2 \\
\eta_5 &= y_2' = s_y + R_{21}X_2 + R_{23}Z_2 \\
\eta_6 &= z_2' = s_z + R_{31}X_2 + R_{33}X_2 \\
\eta_7 &= x_3' = s_x + R_{11}X_3 + R_{13}Z_3 \\
\eta_8 &= y_3' = s_y + R_{21}X_3 + R_{23}Z_3 \\
\eta_9 &= z_3' = s_z + R_{31}X_3 + R_{33}Z_3
\end{aligned}
$$

The definitions of $\underline{s}$, R_{ij} and X_i and Z_i are given in (1.1.1, 2 and 3) respectively. From this, the elements of the Jacobian determinant are obtained by inspection. The resulting 9 x 9 determinant is then evaluated by standard methods to give

$$
(A1.1.2)\quad J = \det\left|\frac{\partial \eta_i}{\partial \rho_i}\right| = \sin\beta\, r^2 q^2 \sin\theta
$$

The columns of the matrix $\left|\frac{\partial \eta_i}{\partial \rho_i}\right|$ are then multiplied together, term by term, to give the symmetric metric tensor,

$$
(A1.1.3)\quad |g_{ij}| = \left|\frac{\partial \eta_k}{\partial \rho_i}\;\frac{\partial \eta_k}{\partial \rho_j}\right|
$$

(A1.1.3)

$$
= \left|\begin{array}{ccccccccc}
3 & & & & & & & & \\
\cdot & 3 & & & & & & & \\
\cdot & \cdot & 3 & & & & & & \\
\cdot & \cdot & \cdot & \lambda_Z & & & & & \\
\cdot & \cdot & \cdot & \cdot & \lambda_X \sin^2\alpha + \lambda_Y \cos^2\alpha & & & & \\
\cdot & \cdot & \cdot & \lambda_Z \cos\beta & (-\lambda_X + \lambda_Y)\sin\alpha\cos\alpha\sin\beta & g_{66} & & & \\
\cdot & \cdot & \cdot & \cdot & N_{75}\cos\alpha & N_{75}\sin\alpha\sin\beta & N_{77} & & \\
\cdot & \cdot & \cdot & \cdot & N_{85}\cos\alpha & N_{85}\sin\alpha\sin\beta & N_{87} & N_{88} & \\
\cdot & \cdot & \cdot & \cdot & N_{95}\cos\alpha & N_{95}\sin\alpha\sin\beta & N_{97} & N_{98} & N_{99}
\end{array}\right|
$$

$$
= \left|\begin{array}{ccc} I & & \\ & S^{-1} & \\ & & I \end{array}\right| \left| N_{ij} \right| \left|\begin{array}{ccc} I & & \\ & S^{-1^T} & \\ & & I \end{array}\right|
$$

where $g_{66} = \lambda_X \cos^2\alpha \sin^2\beta + \lambda_Y \sin^2\alpha \sin^2\beta + \lambda_Z \cos^2\beta$

(A1.1.4)

$$N_{75} = \sum_{i=1}^{3}\left(X_i \frac{\partial Z_i}{\partial r} - Z_i \frac{\partial X_i}{\partial r}\right)$$

$$N_{85} = \sum\left(X_i \frac{\partial Z_i}{\partial q} - Z_i \frac{\partial X_i}{\partial q}\right)$$

$$N_{95} = \sum\left(X_i \frac{\partial Z_i}{\partial \theta} - Z_i \frac{\partial X_i}{\partial \theta}\right)$$

$$N_{77} = \sum\left(\frac{\partial X_i}{\partial r}\right)^2 + \left(\frac{\partial Z_i}{\partial r}\right)^2$$

$$N_{87} = \sum\left(\frac{\partial X_i}{\partial r}\right)\left(\frac{\partial X_i}{\partial q}\right) + \left(\frac{\partial Z_i}{\partial r}\right)\left(\frac{\partial Z_i}{\partial q}\right)$$

$$N_{97} = \sum\left(\frac{\partial X_i}{\partial r}\right)\left(\frac{\partial X_i}{\partial \theta}\right) + \left(\frac{\partial Z_i}{\partial r}\right)\left(\frac{\partial Z_i}{\partial \theta}\right)$$

$$N_{88} = \sum\left(\frac{\partial X_i}{\partial q}\right)^2 + \left(\frac{\partial Z_i}{\partial q}\right)^2$$

$$N_{98} = \sum\left(\frac{\partial X_i}{\partial q}\right)\left(\frac{\partial X_i}{\partial \theta}\right) + \left(\frac{\partial Z_i}{\partial q}\right)\left(\frac{\partial Z_i}{\partial \theta}\right)$$

$$N_{99} = \sum\left(\frac{\partial X_i}{\partial \theta}\right)^2 + \left(\frac{\partial Z_i}{\partial \theta}\right)^2$$

This form of metric tensor is quite general in that the internal co-ordinates (r, q, θ) may be replaced by any other set of three co-ordinates in which the 'body' co-ordinates X_i and Z_i may be expressed, subject to the condition that $X_i = 0$. The conjugate metric tensor $|g^{ij}|$ is now obtained from the reciprocals of the three matrices in which $|g_{ij}|$ is expressed. The result is quoted in (1.1.28). On multiplication of the three matrices in (1.1.28) the Laplacian

(A1.1.5) $$\nabla^2 = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \rho_i}\left(\sqrt{g}\; g^{ij} \frac{\partial}{\partial \rho_j}\right) = g^{ij} \frac{\partial^2}{\partial \rho_i \partial \rho_j} + \frac{g^{ij}}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial \rho_i} \frac{\partial}{\partial \rho_j} + \frac{\partial g^{ij}}{\partial \rho_i} \frac{\partial}{\partial \rho_j}$$

may be evaluated. The evaluation is aided by noting that the rows of the matrix S (1.1.25) are the co-efficients of the derivatives in the operators L_4, L_5 and L_6 (1.1.17). This is the reason for introducing these operators. The terms are examined in groups of three as follows.

i. $i = 1, 2, 3$ $\qquad j = 1, 2, 3$

These are the only terms involving the centre-of-mass co-ordinate $\underline{s}$. The result is

(A1.1.6) $$\frac{1}{3} \frac{\partial^2}{\partial s_x^2} + \frac{1}{3} \frac{\partial^2}{\partial s_y^2} + \frac{1}{3} \frac{\partial^2}{\partial s_z^2} = \frac{1}{3} \nabla_s^2$$

ii. $i = 4, 5, 6$ $\qquad j = 4, 5, 6$

These terms, including derivatives of the Euler angles only, are similar to the conjugate metric tensor for an asymmetric top. The result is

(A1.1.7) $$M^{44}\left(\frac{L_4}{-i}\right)^2 + M^{55}\left(\frac{L_5}{-i}\right)^2 + M^{66}\left(\frac{L_6}{-i}\right)^2$$

iii. $i = 7, 8, 9$ $\qquad$ $j = 7, 8, 9$

The metric tensor is diagonal in these terms and gives directly

$$(A1.1.8) \quad \frac{2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{3}{2q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) + \left(\frac{2}{r^2}+\frac{3}{2q^2}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) = T_S$$

iv. $i = 4, 5, 6$ $\qquad$ $j = 7, 8, 9$

In the expansion of ∇^2 in (A1.1.5) the first term,

$g^{ij}\frac{\partial^2}{\partial\rho_i\partial\rho_j}$, gives directly

$$(A1.1.9) \quad \left(\frac{L_5}{-i}\right)\left\{M^{57}\frac{\partial}{\partial r} + M^{58}\frac{\partial}{\partial q} + M^{59}\frac{\partial}{\partial\theta}\right\}$$

Neglecting those terms for which the derivative of $\sqrt{g}\, g^{ij}$ is zero, the last two terms involve

$$(A1.1.10) \quad \left\{M^{57}\frac{\partial}{\partial r} + M^{58}\frac{\partial}{\partial q} + M^{59}\frac{\partial}{\partial\theta}\right\}\left\{\frac{1}{\sin\beta}\frac{\partial}{\partial\alpha}\left(-\sin\beta\cot\beta\sin\alpha\right) + \frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\cos\beta\sin\alpha\right)\right\} = 0$$

v. $i = 7, 8, 9$ $\qquad$ $j = 4, 5, 6$

As in iv. the first term gives

$$(A1.1.11) \quad \left(\frac{L_5}{-i}\right)\left\{M^{57}\frac{\partial}{\partial r} + M^{58}\frac{\partial}{\partial q} + M^{59}\frac{\partial}{\partial\theta}\right\}$$

while the last two terms become

$$(A1.1.12) \quad \left(\frac{L_5}{-i}\right)\frac{1}{r^2q^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2q^2\sin\theta\, M^{57}\right) + \frac{\partial}{\partial q}\left(r^2q^2\sin\theta\, M^{58}\right) + \frac{\partial}{\partial\theta}\left(r^2q^2\sin\theta\, M^{59}\right)\right\} = \left(\frac{L_5}{-i}\right)M^0$$

thus defining the term M^0.

Collecting together these various terms gives the Laplacian as expressed in (1.1.31).

APPENDIX 2

DEGENERACY OF THE PRINCIPAL AXES

The two situations for which the principal axes are degenerate are the co-linear position and the equilateral position. This has been discussed by Derrick (1960). By requiring the radial functions to be continuous for these two positions, divergences in the kinetic energy integrals are avoided. The co-linear position occurs when $\theta = 0^{\circ}$, 180° or alternatively when $A = \frac{1}{2}rq\sin\theta = 0$. The equilateral position occurs when both $\theta = 90^{\circ}$ and $q = \frac{1}{2}\sqrt{3}r$ or alternatively when both $F = \frac{1}{\sqrt{3}}\,(2q^{2} - 3/2r^{2}) = 0$ and $G = 2rq\cos\theta = 0$. It is convenient to use A, F and G as the internal co-ordinates instead of r, q and θ and then require that the radial functions have formal relations as $A \longrightarrow 0$ and as $F \longrightarrow 0$, $G \longrightarrow 0$. The conjugate metric tensor $\bar{g}^{\,ij}$ may be written as in (1.1.29) with the matrix $\bar{M}^{\,ij}$ given by

(A2.1.1)
$$\left|\bar{M}^{ij}\right| = \left|\begin{array}{ccccccccc}
\frac{1}{3} & & & & & & & & \\
\cdot & \frac{1}{3} & & & & & & & \\
\cdot & \cdot & \frac{1}{3} & & & & & & \\
\cdot & \cdot & \cdot & \frac{6}{R^2}+\Lambda & & & & & \\
\cdot & \cdot & \cdot & \cdot & \frac{3R^2}{\Lambda} & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{6}{R^2}-\Lambda & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{R^2}{4} & & \\
\cdot & \cdot & \cdot & \cdot & \frac{24\sqrt{3}\,AG}{\Lambda^2} & \cdot & \cdot & 4R^2 & \\
\cdot & \cdot & \cdot & \cdot & \frac{-24\sqrt{3}\,AF}{\Lambda^2} & \cdot & \cdot & \cdot & 4R^2
\end{array}\right|$$

where

$$\Lambda^2 = 3\left(F^2+G^2\right)$$

(A2.1.2) $$R^4 = 48\,A^2 + 3\left(F^2+G^2\right)$$

$$\sqrt{g} = \sqrt{3}\,\sin\beta\,\frac{A}{R^2}$$

The Laplacian is

(A2.1.3) $$\nabla^2 = \frac{1}{3}\nabla_s^2 + \overline{T}_E + \overline{T}_S$$

where

$$\frac{1}{3}\nabla_s^2 = \frac{1}{3}\left(\frac{\partial^2}{\partial s_x^2} + \frac{\partial^2}{\partial s_y^2} + \frac{\partial^2}{\partial s_z^2}\right)$$

(A2.1.4) $$\overline{T}_E = \sum \overline{M}^{ii}\left(\frac{L_i}{-i}\right)^2 + 2\left(\frac{L_S}{-i}\right)\left\{\overline{M}^{58}\frac{\partial}{\partial F} + \overline{M}^{59}\frac{\partial}{\partial G}\right\}$$

$$\overline{T}_S = \frac{R^2}{4A}\frac{\partial}{\partial A}\left(\frac{A\partial}{\partial A}\right) + 4R^2\frac{\partial^2}{\partial F^2} + 4R^2\frac{\partial^2}{\partial G^2}$$

The quantities F and G form a pair of functions of mixed symmetry, A is symmetric on permutation of the co-ordinates. Using the rules of (1.2.14-16) other symmetric, mixed and antisymmetric functions may be built up from A, F and G. Thus using the pair (F, G) for both pairs of functions in (1.2.15(3)), another mixed pair is

(A2.1.5) $$\left(-F^2+G^2,\ 2FG\right)$$

Combining (A2.1.5) with (F, G) according to (1.2.16(3)) an antisymmetric function is

(A2.1.6) $$3F^2G - G^3$$

In the same manner, two symmetric functions are

$$(A2.1.7)\quad \left(F^2+G^2\right) \qquad \text{and} \qquad \left(F^3-3FG^2\right)$$

By continuing this process, other antisymmetric, mixed and symmetric functions may be constructed but it can be shown that all may be expressed in terms of the functions given in (A2.1.5, 6 and 7).

Thus when the co-ordinates A, F and G are used the most generalized symmetrized functions must be of the form:

Symmetric

$$(A2.1.7a)\quad f^{(1)} = g_1\left(A, (F^2+G^2), (F^3-3FG^2)\right)$$

Mixed

$$(A2.1.7b)\quad \begin{aligned} f_1^{(2)} &= F\, g_2\left(A, (F^2+G^2), (F^3-3FG^2)\right) + (-F^2+G^2)\, g_3\left(A, (F^2+G^2), (F^3-3FG^2)\right) \\ f_2^{(2)} &= G\, g_2\left(A, (F^2+G^2), (F^3-3FG^2)\right) + (2FG)\, g_3\left(A, (F^2+G^2), (F^3-3FG^2)\right) \end{aligned}$$

Antisymmetric

$$(A2.1.7c)\quad f^{(3)} = \left(3F^2G-G^3\right)\, g_4\left(A, (F^2+G^2), (F^3-3FG^2)\right)$$

where the functions g_i are symmetric.

APPENDIX 3

MATRIX ELEMENTS

1. Matrix Elements of Kinetic Energy Operator

The evaluation of the kinetic energy matrix elements requires the derivatives of the functions $\mathcal{D}^L_{\mu'\mu}(\alpha,\beta,\gamma)$ with respect to the operators L_4, L_5 and L_6. These are

$$\left(\frac{L_4}{-i}\right)\mathcal{D}^L_{\mu'\mu}(\alpha,\beta,\gamma)=\frac{i}{2}\left\{\sqrt{(L+\mu')(L-\mu'+1)}\;\mathcal{D}^L_{\mu'-1,\mu}(\alpha,\beta,\gamma)+\sqrt{(L-\mu')(L+\mu'+1)}\;\mathcal{D}^L_{\mu'+1,\mu}(\alpha,\beta,\gamma)\right\}$$

(A3.1.1) $$\left(\frac{L_5}{-i}\right)\mathcal{D}^L_{\mu'\mu}(\alpha,\beta,\gamma)=\frac{1}{2}\left\{\sqrt{(L+\mu')(L-\mu'+1)}\;\mathcal{D}^L_{\mu'-1,\mu}(\alpha,\beta,\gamma)-\sqrt{(L-\mu')(L+\mu'+1)}\;\mathcal{D}^L_{\mu'+1,\mu}(\alpha,\beta,\gamma)\right\}$$

$$\left(\frac{L_6}{-i}\right)\mathcal{D}^L_{\mu'\mu}(\alpha,\beta,\gamma)=i\mu'\,\mathcal{D}^L_{\mu'\mu}(\alpha,\beta,\gamma)$$

The derivatives of the functions $W^L_{\mu'\mu}(\pm)$ can now be evaluated and are listed in Table AI.

TABLE AI

Derivatives of the Functions $W^L_{\mu'\mu}(\pm)$ for L = 0, 1, 2

	$\left(\frac{L_4}{-i}\right)$	$\left(\frac{L_5}{-i}\right)$	$\left(\frac{L_6}{-i}\right)$	$\left(\frac{L_4}{-i}\right)^2$	$\left(\frac{L_5}{-i}\right)^2$	$\left(\frac{L_6}{-i}\right)^2$
$W^0_{00}(+)$	0	0	0	0	0	0
$W^1_{0\mu}(-)$	$iW^1_{1\mu}(+)$	$-W^1_{1\mu}(-)$	0	$-W^1_{0\mu}(-)$	$-W^1_{0\mu}(-)$	0
$W^1_{1\mu}(+)$	$iW^1_{0\mu}(-)$	0	$W^1_{1\mu}(-)$	$-W^1_{1\mu}(+)$	0	$-W^1_{1\mu}(+)$
$W^1_{1\mu}(-)$	0	$W^1_{0\mu}(-)$	$W^1_{1\mu}(+)$	0	$-W^1_{1\mu}(-)$	$-W^1_{1\mu}(-)$
$W^2_{0\mu}(+)$	$i\sqrt{3}\,W^2_{1\mu}(-)$	$-\sqrt{3}\,W^2_{1\mu}(+)$	0	$-3W^2_{0\mu}(+)-\sqrt{3}\,W^2_{2\mu}(+)$	$-3W^2_{0\mu}(+)+\sqrt{3}\,W^2_{2\mu}(+)$	0
$W^2_{1\mu}(+)$	$iW^2_{2\mu}(-)$	$\sqrt{3}\,W^2_{0\mu}(+)-W^2_{2\mu}(+)$	$iW^2_{1\mu}(-)$	$-W^2_{1\mu}(+)$	$-4W^2_{1\mu}(+)$	$-W^2_{1\mu}(+)$
$W^2_{1\mu}(-)$	$i\sqrt{3}\,W^2_{0\mu}(+)+iW^2_{2\mu}(+)$	$-W^2_{2\mu}(-)$	$iW^2_{1\mu}(+)$	$-4W^2_{1\mu}(-)$	$-W^2_{1\mu}(-)$	$-W^2_{1\mu}(-)$
$W^2_{2\mu}(+)$	$iW^2_{1\mu}(-)$	$W^2_{1\mu}(+)$	$2iW^2_{2\mu}(-)$	$\sqrt{3}\,W^2_{0\mu}(+)-W^2_{2\mu}(+)$	$\sqrt{3}\,W^2_{0\mu}(+)-W^2_{2\mu}(+)$	$-4W^2_{2\mu}(+)$
$W^2_{2\mu}(-)$	$iW^2_{1\mu}(+)$	$W^2_{1\mu}(-)$	$2iW^2_{2\mu}(+)$	$-W^2_{2\mu}(-)$	$-W^2_{2\mu}(-)$	$-4W^2_{2\mu}(-)$

The matrix elements between the W states of the kinetic energy operator, $\left\langle W^{L_1}_{\mu_1'\mu}(\pm) \middle| T_E + T_S \middle| W^{L_2}_{\mu_2'\mu}(\pm) \right\rangle$ are given below, taking the states in the order given in Table AI.

(A3.1.2)

$$\begin{vmatrix} 0 & & & & & & & & \\ & A_1 & 0 & B_1 & & & & & \\ & 0 & A_2 & 0 & & & & & \\ & -B_1 & 0 & A_3 & & & & & \\ & & & & A_4 & \sqrt{3}B_1 & 0 & +B_2 & 0 \\ & & & & -\sqrt{3}B_1 & A_5 & 0 & B_1 & 0 \\ & & & & 0 & 0 & A_6 & 0 & +B_1 \\ & & & & +B_2 & -B_1 & 0 & A_7 & 0 \\ & & & & 0 & 0 & -B_1 & 0 & A_7 \end{vmatrix}$$

where

(A3.1.3)

$$\begin{aligned}
A_1 &= T_S - M^{44} - M^{45} \\
A_2 &= T_S - M^{44} - M^{66} \\
A_3 &= T_S - M^{55} - M^{66} \\
A_4 &= T_S - 3M^{44} - 3M^{55} \\
A_5 &= T_S - M^{44} - 4M^{55} - M^{66} \\
A_6 &= T_S - 4M^{44} - M^{55} - M^{66} \\
A_7 &= T_S - M^{44} - M^{55} - 4M^{66} \\
B_1 &= 2M^{75}\frac{\partial}{\partial r} + 2M^{85}\frac{\partial}{\partial q} + 2M^{95}\frac{\partial}{\partial \theta} + M^{0} \\
B_2 &= \sqrt{3}\,M^{44} + \sqrt{3}\,M^{55}
\end{aligned}$$

These matrix elements do not contain the factor $\left(\frac{-1}{2M}\right)$.

2. Matrix Elements of Spin and Isotopic Spin Operators

A. Introduction

In the calculation of the radiation matrix elements and in the separation of the Schrödinger equation, the matrix elements between the spin-isotopic spin states of the following operators are needed.

Spin operators σ_{i3} , $\sigma_{i+} = \frac{1}{2}(\sigma_x + i\sigma_y)$ and $\sigma_{i-} = \frac{1}{2}(\sigma_x - i\sigma_y)$

Proton operators $\frac{1}{2}(1 - T_{i3})$

Neutron operator $\frac{1}{2}(1 + T_{i3})$

Magnetic moment operator $\frac{1}{2}(1 - T_{i3})\sigma_i \mu_P$ and $\frac{1}{2}(1 + T_{i3})\sigma_i \mu_N$

Singlet operator $\frac{1}{4}(1 - \underline{\sigma}_i \cdot \underline{\sigma}_j)$

Triplet operator $\frac{1}{4}(3 + \underline{\sigma}_i \cdot \underline{\sigma}_j)$

These matrix elements may be evaluated easily from first principles and the results are tabulated in matrix form below. The elements of the matrices are $\langle \xi_i(S_3) | (\text{Operator}) | \xi_j(S_3) \rangle$, ξ_i labelling the rows and ξ_j labelling the columns. For the three pairs of mixed functions ξ_2 , ξ_4 and ξ_6 , the matrix elements are taken in the order $\xi_{i,1}$, $\xi_{i,2}$. For each type of operator there are three sets of matrix elements, e.g. there are three proton operators $\frac{1}{2}(1 - T_{13})$, $\frac{1}{2}(1 - T_{23})$ and $\frac{1}{2}(1 - T_{33})$ with their corresponding matrix elements $\langle \xi_i(S_3) | \frac{1}{2}(1 - T_{13}) | \xi_j(S_3) \rangle$, $\langle \xi_i(S_3) | \frac{1}{2}(1 - T_{23}) | \xi_j(S_3) \rangle$ and $\langle \xi_i(S_3) | \frac{1}{2}(1 - T_{33}) | \xi_j(S_3) \rangle$. These three matrices are connected by unitary transformations, $U(P_i)$, in which the representations of the permutation, $D^{\nu}(P_i)$ form the diagonal blocks and all the off-diagonal blocks are zero.

(A3.2.1) $U(P_i) = \mathrm{diag}\left(D^{(1)}(P_i), D^{(2)}(P_i), D^{(1)}(P_i), D^{(2)}(P_i), D^{(2)}(P_i), D^{(3)}(P_i)\right)$

Thus, for example

(A3.2.2) $\left|\frac{1}{2}(1-T_{13})\right| = U(P_4)^+ \left|\frac{1}{2}(1-T_{13})\right| U(P_4)$

where $P_4 = (12)$ is the permutation interchanging particles 1 and 2, and $\left|\frac{1}{2}(1-T_{i3})\right|$ is the matrix of the matrix elements $\left\langle \xi_i(S_3)\left|\frac{1}{2}(1-T_{i3})\right|\xi_j(S_3)\right\rangle$. Thus one of the matrices may be calculated from first principles and the other two sets obtained by use of the transformation (A3.2.1). For the operators $\frac{1}{2}(1-T_{i3})$, $\frac{1}{2}(1+T_{i3})$, σ_{i0} and $\sigma_{i\pm}$ only the matrices for i = 3 are given and for the operators $\underline{\sigma}_i \cdot \underline{\sigma}_j$ only the matrix for i = 1 and j = 2 are given.

B. <u>The Spin Operators</u>

The matrix elements of the operators σ_{10} and $\sigma_{1\pm}$ are required for the magnetic moment operators.

(A3.2.3) $\underline{\mu} = \left(\mu_P \frac{1}{2}(1-T_3) + \mu_N \frac{1}{2}(1+T_3)\right)\underline{\sigma}$

The properties of these operators on the single nucleon spin functions, α and β, are

(A3.2.4) $\sigma_+ \alpha = 0 \qquad \sigma_0 \alpha = \alpha \qquad \sigma_- \alpha = \beta$

$\sigma_+ \beta = \alpha \qquad \sigma_0 \beta = -\beta \qquad \sigma_- \beta = 0$

The matrices for $\left\langle \xi_i(\frac{3}{2})\middle|\sigma_{30}\middle|\xi_j(\frac{3}{2})\right\rangle$, $\left\langle \xi_i(\frac{1}{2})\middle|\sigma_{30}\middle|\xi_j(\frac{1}{2})\right\rangle$, $\left\langle \xi_i(\frac{3}{2})\middle|\sigma_{3+}\middle|\xi_j(\frac{1}{2})\right\rangle$ and $\left\langle \xi_i(\frac{1}{2})\middle|\sigma_{3+}\middle|\xi_j(-\frac{1}{2})\right\rangle$ are:

(A3.2.5)
$$\left\langle \xi_i(\tfrac{3}{2})\middle|\sigma_{30}\middle|\xi_j(\tfrac{3}{2})\right\rangle = \begin{array}{c c} \begin{array}{ccc} (1) & (4) & \end{array} & \\ \left|\begin{array}{ccc} 1 & 0 & 0 \\ & & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right| & \begin{array}{c} (1) \\ \\ \\ (4) \end{array} \end{array}$$

(A3.2.6)

$$\left\langle \xi_L\left(\tfrac{1}{2}\right)\middle|\sigma_{30}\middle|\xi_J\left(\tfrac{1}{2}\right)\right\rangle = \begin{array}{|ccccccccc|c}
(1) & (2) & & (3) & (4) & & (5) & & (6) & \\
\frac{1}{3} & & & & & & & & & (1) \\
\frac{2\sqrt{2}}{3} & \frac{1}{3} & & & & & & & & (2) \\
0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & \frac{1}{3} & & & & & & (3) \\
0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & & & & & (4) \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & & & & \\
0 & 0 & 0 & \frac{2}{3} & -\frac{2}{3} & 0 & \frac{1}{3} & & & (5) \\
0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} & & \\
0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & -\frac{2}{3} & \frac{1}{3} & (6)
\end{array}$$

(A3.2.7)

$$\left\langle \xi_L\left(\tfrac{3}{2}\right)\middle|\sigma_{3+}\middle|\xi_J\left(\tfrac{1}{2}\right)\right\rangle = \begin{array}{|ccccccccc|c}
(1) & (2) & & (3) & (4) & & (5) & & (6) & \\
\frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & & & & & & & & (1) \\
 & 0 & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & & \frac{1}{\sqrt{3}} & & & (4) \\
 & 0 & 0 & 0 & & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} &
\end{array}$$

(A3.2.8)

$$\left\langle \xi_L\left(\tfrac{1}{2}\right)\middle|\sigma_{3+}\middle|\xi_J\left(-\tfrac{1}{2}\right)\right\rangle = \begin{array}{|ccccccccc|c}
(1) & (2) & & (3) & (4) & & (5) & & (6) & \\
\frac{2}{3} & & & & & & & & & (1) \\
\frac{\sqrt{2}}{3} & \frac{1}{3} & & & & & & & & (2) \\
0 & 0 & -1 & & & & & & & \\
0 & 0 & 0 & -\frac{1}{3} & & & & & & (3) \\
0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & & & & & (4) \\
0 & 0 & 0 & 0 & & \frac{2}{3} & & & & \\
0 & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} & 0 & -\frac{1}{3} & & & (5) \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & -\frac{1}{3} & & \\
0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & -\frac{1}{3} & (6)
\end{array}$$

The matrices with blank upper halves are symmetric. The matrices for other values of S_z and for σ_{3-} are obtained from the above by using the identities:

$$\text{(A3.2.9)}\quad \begin{aligned}
\left\langle \xi_i(-\tfrac{3}{2}) \middle| \sigma_{30} \middle| \xi_j(-\tfrac{3}{2}) \right\rangle &= -\left\langle \xi_i(\tfrac{3}{2}) \middle| \sigma_{30} \middle| \xi_j(\tfrac{3}{2}) \right\rangle \\
\left\langle \xi_i(-\tfrac{1}{2}) \middle| \sigma_{30} \middle| \xi_j(-\tfrac{1}{2}) \right\rangle &= -\left\langle \xi_i(\tfrac{1}{2}) \middle| \sigma_{30} \middle| \xi_j(\tfrac{1}{2}) \right\rangle \\
\left\langle \xi_i(-\tfrac{1}{2}) \middle| \sigma_{3+} \middle| \xi_j(-\tfrac{3}{2}) \right\rangle &= +\left\langle \xi_j(\tfrac{3}{2}) \middle| \sigma_{3+} \middle| \xi_i(\tfrac{1}{2}) \right\rangle \\
\left\langle \xi_i(\tfrac{1}{2}) \middle| \sigma_{3-} \middle| \xi_j(\tfrac{3}{2}) \right\rangle &= +\left\langle \xi_j(\tfrac{3}{2}) \middle| \sigma_{3+} \middle| \xi_i(\tfrac{1}{2}) \right\rangle \\
\left\langle \xi_i(-\tfrac{1}{2}) \middle| \sigma_{3-} \middle| \xi_j(\tfrac{1}{2}) \right\rangle &= +\left\langle \xi_j(\tfrac{1}{2}) \middle| \sigma_{3+} \middle| \xi_i(-\tfrac{1}{2}) \right\rangle \\
\left\langle \xi_i(-\tfrac{3}{2}) \middle| \sigma_{3-} \middle| \xi_j(-\tfrac{1}{2}) \right\rangle &= +\left\langle \xi_i(\tfrac{3}{2}) \middle| \sigma_{3+} \middle| \xi_j(\tfrac{1}{2}) \right\rangle
\end{aligned}$$

C. Proton and Neutron Operators

The proton operator $\frac{1}{2}(1-\tau_{i3})$ and the neutron operator $\frac{1}{2}(1+\tau_{i3})$ occur in the magnetic moment operator (A3.2.3), in the coulomb potential operator:

$$\text{(A3.2.10)}\quad \tfrac{1}{2}(1-\tau_{i3})\,\tfrac{1}{2}(1-\tau_{j3})\,\frac{e^2}{r_{ij}}$$

and also in the radiation operator in the integrals for the electric multipole radiation:

$$\text{(A3.2.11)}\quad e\,\tfrac{1}{2}(1-\tau_{i3})\,f(r_i)$$

where $f(r_i)$ is a function of the spatial co-ordinates of the i-th nucleon. The properties of these operators in the single particle isotopic spin functions, ν and π, are

$$\text{(A3.2.12)}\quad \begin{aligned}
\tfrac{1}{2}(1-\tau_3)\nu &= 0 \qquad & \tfrac{1}{2}(1+\tau_3)\nu &= \nu \\
\tfrac{1}{2}(1-\tau_3)\pi &= \pi \qquad & \tfrac{1}{2}(1+\tau_3)\pi &= 0
\end{aligned}$$

The matrices for $\left\langle \xi_L\left(\tfrac{3}{2}\right) \middle| \tfrac{1}{2}(1-T_{33}) \middle| \xi_J\left(\tfrac{3}{2}\right)\right\rangle$, $\left\langle \xi_L\left(\tfrac{1}{2}\right) \middle| \tfrac{1}{2}(1-T_{33}) \middle| \xi_J\left(\tfrac{1}{2}\right)\right\rangle$, $\left\langle \xi_L\left(\tfrac{3}{2}\right) \middle| \tfrac{1}{2}(1+T_{33}) \middle| \xi_J\left(\tfrac{3}{2}\right)\right\rangle$ and $\left\langle \xi_L\left(\tfrac{1}{2}\right) \middle| \tfrac{1}{2}(1+T_{33}) \middle| \xi_J\left(\tfrac{1}{2}\right)\right\rangle$ are:

(A3.2.13)
$$\left\langle \xi_L\left(\tfrac{3}{2}\right) \middle| \tfrac{1}{2}(1-T_{33}) \middle| \xi_J\left(\tfrac{3}{2}\right)\right\rangle = \begin{array}{c|ccc|c}
 & (1) & (4) & & \\
\hline
 & \frac{2}{3} & & & (1) \\
 & \frac{1}{3}\sqrt{2} & \frac{1}{3} & & (4) \\
 & 0 & 0 & 1 & \\
\end{array}$$

(A3.2.14)
$$\left\langle \xi_L\left(\tfrac{1}{2}\right) \middle| \tfrac{1}{2}(1-T_{33}) \middle| \xi_J\left(\tfrac{1}{2}\right)\right\rangle = \begin{array}{c|ccccccccc|c}
 & (1) & (2) & & (3) & (4) & & (5) & & (6) & \\
\hline
 & \frac{2}{3} & & & & & & & & & (1) \\
 & 0 & \frac{2}{3} & & & & & & & & (2) \\
 & 0 & 0 & \frac{2}{3} & & & & & & & \\
 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & & & & & & (3) \\
 & \frac{1}{3}\sqrt{2} & 0 & 0 & 0 & \frac{1}{3} & & & & & (4) \\
 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\
 & 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{2}{3} & & & (5) \\
 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} & & \\
 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & (6) \\
\end{array}$$

(A3.2.15)
$$\left\langle \xi_L\left(\tfrac{3}{2}\right) \middle| \tfrac{1}{2}(1+T_{33}) \middle| \xi_J\left(\tfrac{3}{2}\right)\right\rangle = \begin{array}{c|ccc|c}
 & (1) & (4) & & \\
\hline
 & \frac{1}{3} & & & (1) \\
 & -\frac{1}{3}\sqrt{2} & \frac{2}{3} & & (4) \\
 & 0 & 0 & 0 & \\
\end{array}$$

(A3.2.16)

$$\left\langle \xi_{\iota}\left(\tfrac{1}{2}\right)\middle|\tfrac{1}{2}(1+\tau_{33})\middle|\xi_{j}\left(\tfrac{1}{2}\right)\right\rangle = \begin{array}{c} \begin{array}{ccccccccc} (1) & (2) & & (3) & (4) & & (5) & & (6) \end{array} \\ \left|\begin{array}{ccccccccc} \frac{1}{3} & & & & & & & & \\ 0 & \frac{1}{3} & & & & & & & \\ 0 & 0 & \frac{1}{3} & & & & & & \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} & & & & & \\ -\frac{\sqrt{2}}{3} & 0 & 0 & 0 & \frac{2}{3} & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & & \\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} \end{array}\right| \end{array} \begin{array}{c} \\ (1) \\ (2) \\ \\ (3) \\ (4) \\ \\ (5) \\ \\ (6) \end{array}$$

For $S_z = -½$ and $S_z = -3/2$ the matrices are the same as for $S_z = ½$ and $S_z = 3/2$ respectively. Because the functions ξ_{ι} describe two neutrons and one proton, there is the identity:

$$\text{(A3.2.17)} \quad \left\langle \xi_{\iota}(S_3)\middle|\tfrac{1}{2}(1-\tau_{l3})\tfrac{1}{2}(1-\tau_{m3})\middle|\xi_j(S_3)\right\rangle = \left\langle \xi_{\iota}(S_3)\middle|\tfrac{1}{2}(1+\tau_{n3})\middle|\xi_j(S_3)\right\rangle$$

where (l m n) is any permutation of (1 2 3).

The operators (A3.2.11), when summed over all three nucleons have the matrix elements:

(A3.2.18)

$$\left\langle \xi_{\iota}\left(\tfrac{3}{2}\right)\middle|\sum_k \tfrac{1}{2}(1-\tau_{k3})f(r_k)\middle|\xi_j\left(\tfrac{3}{2}\right)\right\rangle = \begin{array}{c} \begin{array}{ccc} (1) & & (4) \end{array} \\ \left|\begin{array}{ccc} f^{(1)} & & \\ -\sqrt{2}f_1^{(2)} & f^{(1)}+f_1^{(2)} & \\ \sqrt{2}f_2^{(2)} & f_2^{(2)} & f^{(1)}-f_1^{(2)} \end{array}\right| \end{array} \begin{array}{c} \\ (1) \\ (4) \\ \\ \end{array}$$

(A3.2.19)

$$\left\langle \xi_i\left(\tfrac{1}{2}\right)\middle|\sum_k \tfrac{1}{2}(1-\tau_{k3})f(r_k)\middle|\xi_j\left(\tfrac{3}{2}\right)\right\rangle = \begin{array}{ccccccccc|c} (1) & (2) & & (3) & (4) & & (5) & & (6) & \\ \hline f^{(1)} & & & & & & & & & (1) \\ 0 & f^{(1)} & & & & & & & & \\ 0 & 0 & f^{(1)} & & & & & & & (2) \\ 0 & -f_1^{(2)} & f_2^{(2)} & f^{(1)} & & & & & & (3) \\ \sqrt{2}f_1^{(2)} & 0 & 0 & 0 & f^{(1)}+f_1^{(2)} & & & & & \\ \sqrt{2}f_2^{(2)} & 0 & 0 & 0 & f_2^{(2)} & f^{(1)}-f_1^{(2)} & & & & (4) \\ 0 & f_1^{(2)} & f_2^{(2)} & -f_1^{(2)} & 0 & 0 & f^{(1)} & & & \\ 0 & f_2^{(2)} & -f_1^{(2)} & f_2^{(2)} & 0 & 0 & 0 & f^{(1)} & & (5) \\ 0 & f_2^{(2)} & f_1^{(2)} & 0 & 0 & 0 & -f_2^{(2)} & -f_1^{(2)} & f^{(1)} & (6) \end{array}$$

where $f^{(1)} = \frac{2}{3}\left(f(r_1) + f(r_2) + f(r_3)\right)$

(A3.2.20) $f_1^{(2)} = \frac{1}{6}\left(f(r_1) + f(r_2) - 2f(r_3)\right)$

$f_2^{(2)} = \frac{1}{2\sqrt{3}}\left(-f(r_1) + f(r_2)\right)$

$f^{(1)}$ is symmetric and $(f_1^{(2)}, f_2^{(2)})$ form a mixed pair.

D. Magnetic Moment Operators

The matrix elements of the magnetic moment operator (A3.2.3) can now be evaluated by multiplying the matrices for the spin operators by the matrices for the proton and neutron operators:

(A3.2.21) $$\left\langle \xi_i(S_3')\middle|\mu_k\middle|\xi_j(S_3)\right\rangle = \sum_l \left\langle \xi_i(S_3')\middle|\underline{\sigma}_k\middle|\xi_l(S_3)\right\rangle \left\langle \xi_l(S_3)\middle|\tfrac{1}{2}(1-\tau_{k3})\mu_P + \tfrac{1}{2}(1+\tau_{k3})\mu_N\middle|\xi_j(S_3)\right\rangle$$

using the notation

(A3.2.22)

$$\mu_{k0} = \sigma_{k0}\left(\tfrac{1}{2}(1-\tau_{k3})\mu_P + \tfrac{1}{2}(1+\tau_{k3})\mu_N\right)$$

$$\mu_{k\pm} = \sigma_{k\pm}\left(\tfrac{1}{2}(1-\tau_{k3})\mu_P + \tfrac{1}{2}(1+\tau_{k3})\mu_N\right)$$

Summed over all three nucleons, the matrix elements for $\left\langle \xi_i\left(\tfrac{3}{2}\right)\middle|\sum_k \mu_{k0}\middle|\xi_j\left(\tfrac{3}{2}\right)\right\rangle$, $\left\langle \xi_i\left(\tfrac{1}{2}\right)\middle|\sum_k \mu_{k0}\middle|\xi_j\left(\tfrac{1}{2}\right)\right\rangle$, $\left\langle \xi_i\left(\tfrac{3}{2}\right)\middle|\sum_k \mu_{k+}\middle|\xi_j\left(\tfrac{1}{2}\right)\right\rangle$ and $\left\langle \xi_i\left(\tfrac{1}{2}\right)\middle|\sum_k \mu_{k+}\middle|\xi_j\left(-\tfrac{1}{2}\right)\right\rangle$ are:

(A3.2.23)

$$\left\langle \xi_i\left(\tfrac{3}{2}\right)\middle|\sum_k \mu_{k0}\middle|\xi_j\left(\tfrac{3}{2}\right)\right\rangle = \begin{array}{cc} \begin{array}{ccc} (1) & (4) & \end{array} & \\ \left|\begin{array}{ccc} 3\mu(1) & & \\ 0 & 3\mu(1) & \\ 0 & 0 & 3\mu(1) \end{array}\right| & \begin{array}{c} (1) \\ (4) \\ \end{array} \end{array}$$

$$(A3.2.24)\quad \left\langle \xi_i\left(\tfrac{1}{2}\right)\middle|\sum_k \mu_{k0}\middle|\xi_j\left(\tfrac{1}{2}\right)\right\rangle = \begin{array}{cc} \begin{array}{ccccccccc} (1) & (2) & & (3) & (4) & & (5) & & (6) \end{array} & \\ \left|\begin{array}{ccccccccc} \mu(1) & & & & & & & & \\ 0 & \mu(1) & & & & & & & \\ 0 & 0 & \mu(1) & & & & & & \\ \frac{2\sqrt{2}}{3}\mu(2) & 0 & 0 & \mu(3) & & & & & \\ 0 & \frac{2}{3}\mu(2) & 0 & 0 & \mu(1) & & & & \\ 0 & 0 & \frac{2}{3}\mu(2) & 0 & 0 & \mu(1) & & & \\ 0 & \frac{2}{3}\mu(2) & 0 & 0 & \frac{2}{3}\mu(2) & 0 & \mu(1) & & \\ 0 & 0 & \frac{2}{3}\mu(2) & 0 & 0 & \frac{2}{3}\mu(2) & 0 & \mu(1) & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu_N \end{array}\right| & \begin{array}{c} (1) \\ (2) \\ \\ (3) \\ (4) \\ \\ (5) \\ \\ (6) \end{array} \end{array}$$

$$(A3.2.25)\quad \left\langle \xi_i\left(\tfrac{3}{2}\right)\middle|\sum_k \mu_{k+}\middle|\xi_j\left(\tfrac{1}{2}\right)\right\rangle = \begin{array}{cc} \begin{array}{ccccccccc} (1) & (2) & & (3) & (4) & & (5) & & (6) \end{array} & \\ \left|\begin{array}{ccccccccc} \sqrt{3}\mu(1) & 0 & 0 & -\sqrt{\frac{2}{3}}\mu(2) & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu(2)}{\sqrt{3}} & 0 & 0 & \sqrt{3}\mu(1) & 0 & -\frac{\mu(2)}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & -\frac{\mu(2)}{\sqrt{3}} & 0 & 0 & \sqrt{3}\mu(1) & 0 & -\frac{\mu(2)}{\sqrt{3}} & 0 \end{array}\right| & \begin{array}{c} (1) \\ (4) \\ \\ \end{array} \end{array}$$

$$(A3.2.26)\quad \left\langle \xi_i(\tfrac{1}{2}) \middle| \sum_k \mu_{k+} \middle| \xi_j(-\tfrac{1}{2}) \right\rangle = \begin{array}{ccccccccc|c}
(1) & (2) & & (3) & (4) & & (5) & & (6) & \\
\hline
2\mu(1) & & & & & & & & & (1) \\
0 & \mu(1) & & & & & & & & (2) \\
0 & 0 & \mu(1) & & & & & & & \\
-\frac{\sqrt{2}}{3}\mu(2) & 0 & 0 & -\mu(3) & & & & & & (3) \\
0 & \frac{1}{3}\mu(2) & 0 & 0 & 2\mu(1) & & & & & (4) \\
0 & 0 & \frac{1}{3}\mu(2) & 0 & 0 & 2\mu(1) & & & & \\
0 & \frac{2}{3}\mu(2) & 0 & 0 & \frac{1}{3}\mu(2) & 0 & -\mu(1) & & & (5) \\
0 & 0 & \frac{2}{3}\mu(2) & 0 & 0 & \frac{1}{3}\mu(2) & 0 & -\mu(1) & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_N & (6)
\end{array}$$

where

$$\mu(1) = \frac{2}{3}\mu_P + \frac{1}{3}\mu_N$$

$$(A3.2.27)\quad \mu(2) = \mu_P - \mu_N$$

$$\mu(3) = \frac{4}{3}\mu_P - \frac{1}{3}\mu_N$$

Matrix elements not listed may be obtained from the identification:

$$\left\langle \xi_i(-\tfrac{3}{2}) \middle| \sum_k \mu_{k0} \middle| \xi_j(-\tfrac{3}{2}) \right\rangle = -\left\langle \xi_i(\tfrac{3}{2}) \middle| \sum_k \mu_{k0} \middle| \xi_j(\tfrac{3}{2}) \right\rangle$$

(A3.2.28)

$$\left\langle \xi_i(-\tfrac{1}{2}) \middle| \sum_k \mu_{k0} \middle| \xi_j(-\tfrac{1}{2}) \right\rangle = -\left\langle \xi_i(\tfrac{1}{2}) \middle| \sum_k \mu_{k0} \middle| \xi_j(\tfrac{1}{2}) \right\rangle$$

$$\left\langle \xi_i(-\tfrac{1}{2}) \middle| \sum_k \mu_{k+} \middle| \xi_j(-\tfrac{3}{2}) \right\rangle = \left\langle \xi_j(\tfrac{3}{2}) \middle| \sum_k \mu_{k+} \middle| \xi_i(\tfrac{1}{2}) \right\rangle$$

$$\left\langle \xi_i(S_z-1) \middle| \sum_k \mu_{k-} \middle| \xi_j(S_z) \right\rangle = \left\langle \xi_j(S_z) \middle| \sum_k \mu_{k+} \middle| \xi_i(S_z-1) \right\rangle$$

E. The Singlet and Triplet Operators

In the phenomenological description of the interaction of two nucleons, part of the potential is taken to depend on the relative spin state of the two nucleons. There are two possible spin states, S = 0 (singlet) and S = 1 (triplet) and the spin operators which select these states are $\frac{1}{4}(1 - \sigma_i \cdot \sigma_j)$ and $\frac{1}{4}(3 + \sigma_i \cdot \sigma_j)$ respectively. That is, if $\chi_{ij}(0)$ and $\chi_{ij}(1)$ are the singlet and triplet spin functions for the two nucleons,

$$\text{(A3.2.29)}\quad \tfrac{1}{4}(1 - \sigma_i\ \sigma_j)\chi_{ij}(0) = \chi_{ij}(0) \qquad \tfrac{1}{4}(1 - \sigma_i\cdot\sigma_j)\chi_{ij}(1) = 0$$

$$\tfrac{1}{4}(3 + \sigma_i\ \sigma_j)\chi_{ij}(0) = 0 \qquad \tfrac{1}{4}(3 + \sigma_i\cdot\sigma_j)\chi_{ij}(1) = \chi_{ij}(1)$$

The matrix elements of the operator $\frac{1}{4}(1 - \sigma_i\cdot\sigma_j)$ and $\frac{1}{4}(3 + \sigma_i\ \sigma_j)$ are,

$$\text{(A3.2.30)}\quad \left\langle \xi_i(\pm\tfrac{3}{2}) \middle| \tfrac{1}{4}(1 - \sigma_1\ \sigma_2) \middle| \xi_j(\pm\tfrac{3}{2}) \right\rangle = 0$$

$$\text{(A3.2.31)}\quad \left\langle \xi_i(\pm\tfrac{1}{2}) \middle| \tfrac{1}{4}(1 - \sigma_1\cdot\sigma_2) \middle| \xi_j(\pm\tfrac{1}{2}) \right\rangle = \begin{array}{|ccccccccc|l}
(1) & (2) & & (3) & (4) & & (5) & & (6) & \\
0 & & & & & & & & & (1)\\
0 & 0 & & & & & & & & (2)\\
0 & 0 & 1 & & & & & & & \\
0 & 0 & 0 & \frac{1}{2} & & & & & & (3)\\
0 & 0 & 0 & 0 & 0 & & & & & (4)\\
0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & & & (5)\\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & & \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & (6)
\end{array}$$

$$\text{(A3.2.32)}\quad \left\langle \xi_i(\pm\tfrac{3}{2}) \middle| \tfrac{1}{4}(3 + \sigma_1\cdot\sigma_2) \middle| \xi_j(\pm\tfrac{3}{2}) \right\rangle = 1$$

$$(A3.2.33)\quad \left\langle \xi_i\left(\pm\tfrac{1}{2}\right) \middle| \tfrac{1}{4}\left(3+\sigma_1.\sigma_2\right) \middle| \xi_j\left(\pm\tfrac{1}{2}\right) \right\rangle = \begin{array}{|ccccccccc|c}
\multicolumn{1}{c}{(1)} & \multicolumn{2}{c}{(2)} & \multicolumn{1}{c}{(3)} & \multicolumn{2}{c}{(4)} & \multicolumn{2}{c}{(5)} & \multicolumn{1}{c}{(6)} & \\
1 & & & & & & & & & (1) \\
0 & 1 & & & & & & & & \\
0 & 0 & 0 & & & & & & & (2) \\
0 & 0 & 0 & \frac{1}{2} & & & & & & (3) \\
0 & 0 & 0 & 0 & 1 & & & & & \\
0 & 0 & 0 & 0 & 0 & 1 & & & & (4) \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & & & (5) \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & (6)
\end{array}$$

APPENDIX 4

RADIAL DIFFERENTIAL EQUATIONS

The derivation of the differential equations for the radial functions can be done once the matrix elements of the operators in the Hamiltonian have been evaluated. Examples of these matrix elements are given in Appendix 3. In this appendix, the equations for the S states of the three nucleon systems are obtained. The Hamiltonian used is

$$(A4.1.1) \quad H = \frac{-1}{2M}\left(T_E + T_S\right) + \sum_{i<j}\left(\tfrac{1}{4}(1-\sigma_i\cdot\sigma_j)V_S(r_{ij}) + \tfrac{1}{4}(3+\sigma_i\cdot\sigma_j)V_T(r_{ij})\right) + \sum_{i<j}\tfrac{1}{4}(1-\tau_{i3})(1-\tau_{j3})V_C(r_{ij})$$

where $V_S(r)$ is the potential between two nucleons in a singlet spin state ($S = 0$), $V_T(r)$ is the potential between two nucleons in a triplet spin state ($S = 1$) and $V_C(r) = e^2/r$ is the coulomb potential between the two protons. The equations can be written more concisely by introducing

$$V^{(1)} = \left(V(r_{13}) + V(r_{23}) + V(r_{12})\right)$$

$$(A4.1.2) \quad V_1^{(2)} = \tfrac{1}{2}\left(V(r_{13}) + V(r_{23}) - 2V(r_{12})\right)$$

$$V_2^{(2)} = \frac{\sqrt{3}}{2}\left(-V(r_{13}) + V(r_{23})\right)$$

where (see Fig. 3)

$$r_{13} = \left(q^2 - rq\cos\theta + \frac{r^2}{4}\right)^{\frac{1}{2}}$$

$$(A4.1.3) \quad r_{23} = \left(q^2 + rq\cos\theta + \frac{r^2}{4}\right)^{\frac{1}{2}}$$

$$r_{12} = r$$

and there is no antisymmetric combination $V^{(3)}$ assuming that the potential $V(r)$ depends only on the magnitude of r.

In writing down the equations use can be made of the symmetry properties of the equations. Thus, for example, in the equation for the antisymmetric function, F_1, any coupling to the mixed functions $F_{3,1}$, $F_{3,2}$ must enter through the antisymmetric combination $(V_1^{(2)} F_{3,2} - V_2^{(2)} F_{3,1})$. Then it is only necessary to obtain the numerical constant which can generally be done by examining a single matrix element. If V is the triplet potential, then the coupling of the states $F_1 W^{\circ}_{oo}(+) \xi_3$ and $\frac{1}{\sqrt{2}}\left(F_{3,2}\xi_{5,1} - F_{3,1}\xi_{5,2}\right) W^{\circ}_{oo}(+)$ by this potential is obtained from the matrix element (A3.2.33).

(A4.1.4) $$\left\langle W^{\circ}_{\infty}(+)\xi_3 \left| \tfrac{1}{4}(3+\sigma_1\ \sigma_2) \right| W^{\circ}_{\infty}(+)\xi_{5,1} \right\rangle = \left\langle \xi_3 \left| \tfrac{1}{4}(3+\sigma_1\ \sigma_2)\right| \xi_{5,1}\right\rangle = -\tfrac{1}{2}$$

From this it can be seen that the coupling term must be

(A4.1.5) $$\frac{1}{2\sqrt{2}}\left(V_{1T}^{(2)} F_{3,2} - V_{2T}^{(2)} F_{3,1}\right)$$

In this way the following equations are obtained.

(A4.1.6)
$$\begin{aligned}
&\left\{T_S + V^S - E\right\}F_1 + \tfrac{1}{3\sqrt{2}}\left\{V_{1C}^{(2)}F_{2,2} - V_{2C}^{(2)}F_{2,1}\right\} + \tfrac{1}{\sqrt{2}}\left\{V'F_{3,2} - V''F_{3,1}\right\} = 0\\
&\left\{T_S + V^S - E\right\}F_{2,1} - \tfrac{\sqrt{2}}{3}\left\{V_{2C}^{(2)}F_1 + V_{1C}^{(2)}F_4\right\} + \tfrac{1}{2}\left\{\left(V_{1T}^{(2)} - V_{1S}^{(2)}\right)F_{2,1} - \left(V_{2T}^{(2)} - V_{2S}^{(2)}\right)F_{2,2}\right\} = 0\\
&\left\{T_S + V^S - E\right\}F_{2,2} + \tfrac{\sqrt{2}}{3}\left\{V_{1C}^{(2)}F_1 - V_{2C}^{(2)}F_4\right\} - \tfrac{1}{2}\left\{\left(V_{1T}^{(2)} - V_{1S}^{(2)}\right)F_{2,2} + \left(V_{2T}^{(2)} - V_{2S}^{(2)}\right)F_{2,1}\right\} = 0\\
&\left\{T_S + V^S - E\right\}F_{3,1} - \sqrt{2}\left\{V''F_1\right\} + \sqrt{2}\left\{V'F_4\right\} = 0\\
&\left\{T_S + V^S - E\right\}F_{3,2} + \sqrt{2}\left\{V'F_1\right\} + \sqrt{2}\left\{V''F_4\right\} = 0\\
&\left\{T_S + V^S - E\right\}F_4 - \tfrac{1}{3\sqrt{2}}\left\{V_{1C}^{(2)}F_{2,1} + V_{2C}^{(2)}F_{2,2}\right\} + \tfrac{1}{\sqrt{2}}\left\{V'F_{3,1} + V''F_{3,2}\right\} = 0
\end{aligned}$$

where

(A4.1.7)
$$\begin{aligned}
V^S &= \tfrac{1}{2}V_T^{(1)} + \tfrac{1}{2}V_S^{(1)} + \tfrac{1}{2}V_C^{(1)}\\
V' &= \tfrac{1}{2}V_{1T}^{(2)} - \tfrac{1}{2}V_{1S}^{(2)} + \tfrac{1}{3}V_{1C}^{(2)}\\
V'' &= \tfrac{1}{2}V_{2T}^{(2)} - \tfrac{1}{2}V_{2S}^{(2)} + \tfrac{1}{3}V_{2C}^{(2)}
\end{aligned}$$

Outside the range of the forces, all these equations go into the form

$$\left\{T_s - E\right\} F = 0$$

(A4.1.8)

$$\left\{\frac{2}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{3}{2q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) + \left(\frac{2}{r^2}+\frac{3}{2q^2}\right)\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + 2ME\right\} F = 0$$

For q ≫ r, $\left(\frac{2}{r^2}+\frac{3}{2q^2}\right)$ may be replaced by $2/r^2$ and the equation is then separable by writing $F = u(r)\,v(q)\,w(\theta)$. The separated equations are

(A4.1.9) $$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{m}{r^2} + ME'\right\} u(r) = 0$$

(A4.1.10) $$\left\{\frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) + \frac{4}{3}ME''\right\} v(q) = 0$$

(A4.1.11) $$\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + m\right\} w(\theta) = 0$$

where $E' + E'' = E$.

For the continuum states, the requirement that u(r) be the asymtotic deuteron S state function leads to $m = 0$, $w(\theta)$ = constant and

(A4.1.12) $$E' = -B_D$$

where B_D = 2.226 Mev is the binding energy of the deuteron. The solution of (A4.1.9) is

(A4.1.13) $$u(r) = c_1 \frac{e^{-\beta_1 r}}{r}$$

with

(A4.1.14)

$$\beta_1^2 = MB_D$$

$$\beta_1 = 232 \times 10^{12}\ \mathrm{cm}^{-1}$$

The total energy is $E = -B_D + E_C$ where E_C is the kinetic energy in the centre-of-mass frame. Thus $E'' = E_C$ and a solution of (A4.1.10) is

(A4.1.15) $v(q) = j_0(kq) = \frac{\sin kq}{kq}$

with

(A4.1.16)
$$k^2 = \frac{4}{3} M E_C$$
$$k = 1.47 \times 10^{12}\,\text{cm}^{-1} \quad \text{for } E_C = \frac{2}{3}\,\text{Mev}$$

Examination of the continuum states for higher angular momentum $l = 1, 2, \ldots$ with the same approximations leads to the equation for $v(q)$,

(A4.1.17) $$\left\{\frac{1}{q^2}\frac{\partial}{\partial q}\left(q^2\frac{\partial}{\partial q}\right) - \frac{l(l+1)}{q^2} + k^2\right\} v(q) = 0$$

for which the solutions are the spherical Bessel functions $j_l(kg)$.

For the ground state of He^3, the total energy is

(A4.1.18) $$E = -B_{He^3}$$

where $B_{He^3} = 7.73$ Mev is the binding energy of He^3. The division of this energy between E' and E'' in (A4.1.9 and 10) is somewhat arbitrary. It was decided to set

(A4.1.19)
$$E' = -2/3\ B_{He^3}$$
$$E'' = -1/3\ B_{He^3}$$

representing an equal division of the binding energy between the three particles.

With this choice, the solution of (A4.1.9) is

(A4.1.20) $$u'(r) = C_1'\,\frac{e^{-\beta_1 r}}{r}$$

with

$$\beta_1'^2 = \frac{2}{3} MB_{He^3} \qquad \text{(A4.1.21)}$$
$$\beta_1' = 3.53 \times 10^{12} \text{ cm}^{-1}$$

and the solution of (A4.1.10) is

$$\text{(A4.1.22)} \quad V(q) = C_2 \frac{e^{-\beta_2 q}}{q}$$

with

$$\beta_2^2 = \frac{4}{9} MB_{He^3} \qquad \text{(A4.1.23)}$$
$$\beta_2 = 2.79 \times 10^{12} \text{ cm}^{-1}$$

APPENDIX 5

INTERACTION WITH THE ELECTROMAGNETIC FIELD

1. Explanation of the Interaction

The manner in which the electromagnetic field is handled must be described before discussing the interaction of the electromagnetic field with the nucleons. In the following the convention $\hbar = c = 1$ will be used. Thus energy, E, momentum vectors, $\underline{k}$ and k, the nuclear mass, M and time have the dimensions of reciprocal lengths. The conversion factors are

$$(A5.1.1)\quad E\,(cm^{-1}) = \frac{E\,(Mev)}{197 \times 10^{-11}\ Mev\,cm}$$

$$M\,(cm^{-1}) = \frac{939\ Mev}{197 \times 10^{-11}\ Mev\,cm} = 476 \times 10^{13}\,cm^{-1}$$

Only the radiation field will be considered, consisting of transverse waves, for which the coulomb (or solenoidal) guage may be used, i.e.

$$(A5.1.2)\quad \nabla\ \underline{A} = 0$$

and

$$(A5.1.3)\quad \begin{aligned} \underline{E} &= -\frac{\partial}{\partial t}\,\underline{A} \\ \underline{H} &= \nabla \times \underline{A} \end{aligned}$$

where $\underline{A}$ is the vector potential and $\underline{E}$, $\underline{H}$ are the electric and magnetic fields. Any static interactions arising from the scalar potential ϕ are assumed to be included in the Hamiltonian for the nuclear system. In both the classical and quantum mechanical treatment of the field, it is thought of as being enclosed in a large but finite cavity having perfectly reflecting surfaces. The boundary conditions then require that the tangential component of $\underline{E}$ is zero, and this permits an enumeration of the allowed modes of the cavity. The electromagnetic field can then be expressed as a linear superposition of the allowed modes. There are many ways in which the allowed modes may be described, e.g. plane waves, spherical waves, cylindrical waves, etc. The method used for any particular problem should be that which allows greatest simplification and ease of calculation. The nucleon states with which the field interacts are eigenstates of angular momentum and have zero linear momentum in the centre-of-mass frame. The electromagnetic field will therefore be expanded in terms of spherical waves contained in a spherical cavity centred at the origin of the centre-of-mass frame. This can be done for either the classical (Blatt and Weisskopf, 1952, App. B) or quantum mechanical (Heitler, 1954, App. 1) description of the field. The vector potential may then be written as

(A5.1.4) $\underline{A} = \sum_{k,l,m} \left(a_{lm}(k+)\, \underline{A}_{lm}(k+) + a_{lm}(k-)\, \underline{A}_{lm}(k-) \right) e^{-ikt}$ + complex conjugate

where

(A5.1.5) $\underline{A}_{lm}(k+) = \sqrt{\frac{4\pi}{R}}\, j_l(kr)\, \underline{X}_{lm}$

$\underline{A}_{lm}(k-) = i\sqrt{\frac{4\pi}{kR}}\, \nabla \times \left(j_l(kr)\, \underline{X}_{lm} \right)$

(A5.1.6) $\underline{X}_{lm} = \frac{\underline{L}\, Y_{lm}}{\sqrt{l(l+1)}} = \frac{-i(\underline{r} \times \nabla)}{\sqrt{l(l+1)}}\, Y_{lm}$

(A5.1.7) $j_l(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{l+\frac{1}{2}}(kr) \longrightarrow \frac{1}{kr}\sin\left(kr - \frac{l\pi}{2}\right)$ for $kr \gg l$

$\longrightarrow \frac{(kr)^l}{(2l+1)!!}$ for $kr \ll l$

and $a_{lm}(k^{\pm})$, $a^{*}_{lm}(k^{\pm})$ are amplitudes which, in the quantum theory, become annihilation and creation operators for a photon of energy k, angular momentum l and $l_z = m$. The Y_{lm} are normalized spherical harmonics expressed in polar co-ordinates (θ, ϕ)

(A5.1.8) $$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(l-m)!}{(l+m)!}} \cdot \frac{1}{2^l l!} \sin^m\theta \frac{d^{l+m}}{d\cos\theta^{l+m}} (-\sin^2\theta)^l e^{im\phi}$$

$$Y^{*}_{lm}(\theta,\phi) = (-)^m Y_{l,-m}(\theta,\phi)$$

The vectors $\underline{X}_{lm}$ are normalized vector spherical harmonics,

(A5.1.9) $$\int \underline{X}^{*}_{l'm'} \cdot \underline{X}_{lm} \, d\Omega = \delta_{l'l} \delta_{m'm}$$

Each of the terms $\underline{A}_{lm}(k^{\pm})$ are solutions of the source-free wave equation,

(A5.1.10) $$(\nabla^2 + k^2) \underline{A}_{lm}(k^{\pm}) = 0$$

and the factors $\sqrt{\frac{4\pi}{R}}$ and $\sqrt{\frac{4\pi}{kR}}$ have been included in order that the terms $A_{lm}(k^{\pm})$ have the normalization,

(A5.1.11) $$\int \left| A_{lm}(k\pm) \right|^2 d\tau = \frac{2\pi}{k}$$

In order to perform this integration, the asymtotic form of $j_l(kr)$ for $kr \gg 1$ was used for all values of r. The accuracy of this approximation increases as $R \longrightarrow \infty$ and a greater proportion of the field is contained in the region for which $kr \gg 1$.

The boundary conditions at the surface of the sphere require that $E_{tang} = 0$, this is the same as requiring that $A_{tang} = 0$. Using the approximation for $kr \gg 1$, this leads to

(A5.1.12a) $\sin\left(kR - \frac{l\pi}{2}\right) = 0$

for $A_{lm}(k,+)$, or

(A5.1.12b) $\cos\left(kR - \frac{l\pi}{2}\right) = 0$

for $A_{lm}(k,-)$. Thus

(A5.1.13) $k_n R - \frac{l\pi}{2} = n\pi$ and $k_n R - \frac{l\pi}{2} = \left(n+\frac{1}{2}\right)\pi$ respectively

where n is an integer. It is of interest to know the number of allowed modes, Δn, in the range k_n to $k_n + \Delta k$. In the limit as $R \longrightarrow \infty$, both k and n may be considered to be continuous variables. Thus the density of states, ρ_k, of given l and m may be defined to be

(A5.1.14) $\rho_{k,l,m} dk \equiv \frac{dn}{dk} \cdot dk = \frac{R}{\pi} \cdot dk$

Alternatively, the density, ρ_E, i.e. the number of modes of given l, m in the interval dE, may be defined

(A5.1.15) $\rho_{E,l,m} dE = \frac{dn}{dE} \cdot dE = \frac{R}{\pi} \cdot dE$

The terms $\underline{A}_{lm}(k,-)$ for l = 1, 2, 3 ... are referred to as electric dipole, quadrupole, octopole, ... radiation as these are the same as the asymtotic forms, at large distances from the source, of the radiation from oscillating electric dipoles, quadrupoles, octopoles, ... For example, in the radiation from an electric dipole (Panofsky and Phillips, 1955, p. 222) the magnetic field is proportional to

(A5.1.16) $\underline{H} = C \frac{e^{ikr}}{kr} \sin\theta \, \underline{e}_\phi = C\left(\frac{\cos kr}{kr} + \frac{i \sin kr}{kr}\right) \sin\theta \, \underline{e}_\phi$

where $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_\phi$ are the unit vectors in spherical polar co-ordinates. Examining the magnetic field arising from $A_{1,0}(k,-)$ gives, assuming $kr \gg 1$,

$$\begin{aligned}(A5.1.17)\quad \underline{H} &= C\,\frac{\sin(kr-\frac{\pi}{2})}{kr}\,\underline{X}_{1,0} \\ &= C'\,\frac{\sin(kr-\frac{\pi}{2})}{kr}\left(-Y_{1,-1}(\theta,\phi)\,\underline{\chi}_{+1} + Y_{1,1}(\theta,\phi)\,\underline{\chi}_{-1}\right) \\ &= C''\,\frac{\sin(kr-\frac{\pi}{2})}{kr}\,\sin\theta\,\underline{e}_\phi\end{aligned}$$

where $\underline{\chi}_{+1}$, $\underline{\chi}_0$, $\underline{\chi}_{-1}$, are the unit vectors,

$$\begin{aligned}(A5.1.18)\quad \underline{\chi}_{+1} &= \frac{-1}{\sqrt{2}}(\underline{e}_x + i\underline{e}_y) \\ \underline{\chi}_0 &= \underline{e}_z \\ \underline{\chi}_{-1} &= \frac{1}{\sqrt{2}}(\underline{e}_x - i\underline{e}_y)\end{aligned}$$

and are related to $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_\phi$ by

$$\begin{aligned}\underline{\chi}_{+1} &= -\frac{e^{i\phi}}{\sqrt{2}}(\sin\theta\,\underline{e}_r + \cos\theta\,\underline{e}_\theta + i\underline{e}_\phi) \\ (A5.1.19)\qquad \underline{\chi}_0 &= (\cos\theta\,\underline{e}_r - \sin\theta\,\underline{e}_\theta) \\ \underline{\chi}_{-1} &= \frac{e^{-i\phi}}{\sqrt{2}}(\sin\theta\,\underline{e}_r + \cos\theta\,\underline{e}_\theta - i\underline{e}_\phi)\end{aligned}$$

Comparing (A5.1.17) with the part of (A5.1.16) regular at the origin, it is seen that, apart from a constant phase factor, the two fields are the same.

Similarly the terms $\underline{A}_{lm}(k,+)$ are referred to as magnetic multipole radiation as the fields arising from these terms have the form of the fields arising from magnetic multipoles at large distances from the origin. Both electric and magnetic multipole radiation contributes to the energy density terms proportional to

$\left|\underline{A}_{lm}(k,\pm)\right|^2$ in which the angular dependence is contained in

$$(A5.1.20)\quad \left|\underline{X}_{lm}\right|^2 = \frac{(l+m)(l-m+1)}{2l(l+1)}\left|Y_{l,m-1}\right|^2 + \frac{m^2}{l(l+1)}\left|Y_{lm}\right|^2 + \frac{(l-m)(l+m+1)}{2l(l+1)}\left|Y_{l,m+1}\right|^2$$

$\left|\underline{X}_{lm}\right|^2$ will be called the angular distribution of the corresponding multipole radiation. For l = 1 and l = 2 these angular distributions are,

$$(A5.1.21)\quad \begin{aligned} \left|\underline{X}_{1,0}\right|^2 &= \frac{3}{8\pi}\left(1-\cos^2\theta\right) \\ \left|\underline{X}_{1,\pm1}\right|^2 &= \frac{3}{16\pi}\left(1+\cos^2\theta\right) \\ \left|\underline{X}_{2,0}\right|^2 &= \frac{15}{8\pi}\left(\cos^2\theta-\cos^4\theta\right) \\ \left|\underline{X}_{2,\pm1}\right|^2 &= \frac{5}{16\pi}\left(1-3\cos^2\theta+4\cos^4\theta\right) \\ \left|\underline{X}_{2,\pm2}\right|^2 &= \frac{5}{16\pi}\left(1-\cos^4\theta\right) \end{aligned}$$

The distinction between electric and magnetic multipoles of the same order cannot be made by measuring the angular distribution. The polarization must be measured to make this distinction. An observer, looking along the radius vector towards the origin, would measure the relative magnitude of electric vector along the $\underline{e}_\theta$ and $\underline{e}_\phi$ direction. The expected polarization of a given multipole radiation may be obtained by expressing the fields in terms of the components along $\underline{e}_r$, $\underline{e}_\theta$ and $\underline{e}_\phi$ instead of $\underline{\chi}_+$, $\underline{\chi}_0$ and $\underline{\chi}_-$. In the expansion of the fields in spherical waves, the direction of the polarization vector is not constant throughout space as it is for the plane wave expansion. At any given point, however, the polarization vectors of electric and magnetic multipole radiations of the same order are at right angles.

In the quantum mechanical description of the electromagnetic field in terms of photons, it is well-known that, as photons are bosons, the number of photons in a given state may be zero or any given positive integer. In the spherical wave representation, the observables which may be simultaneously diagonalized are k, l, $m = l_z$ and the parity ($\pm$). These are the quantities which will be used to label the states. Had the plane wave representation been used the observables which could be simultaneously diagonalized would have been k_x, k_y, k_z and the polarization vector. The electromagnetic field is described by the number of photons $n_{lm}(k,\pm)$ in any given state, the corresponding state vector being $|\ldots, n_{lm}(k,\pm), \ldots\rangle$. The basic state is the vacuum state $|0\rangle$ in which all occupation numbers are zero. From $|0\rangle$, any state may be constructed by repeated application of the creation operator $a^*_{lm}(k\pm)$, for a photon in the state $(k,l,m,\pm)$. The corresponding annihilation operator is $a_{lm}(k\pm)$. These operators have the usual commutation rules and the usual properties when applied to a state vector.

(A5.1.22)
$$\left[a_{l'm'}(k'\pm),\, a^*_{lm}(k\pm)\right] = \delta_{l'l}\,\delta_{m'm}\,\delta_{k'k}$$
$$\left[a_{l'm'}(k'+),\, a^*_{lm}(k-)\right] = \left[a_{l'm'}(k'-),\, a^*_{lm}(k+)\right] = 0$$

(A5.1.23)
$$a_{lm}(k\pm)\,|\ldots, n_{lm}(k\pm), \ldots\rangle = \sqrt{n_{lm}(k\pm)}\,|\ldots, n_{lm}(k\pm)-1, \ldots\rangle$$
$$a^*_{lm}(k\pm)\,|\ldots, n_{lm}(k\pm), \ldots\rangle = \sqrt{n_{lm}(k\pm)+1}\,|\ldots, n_{lm}(k\pm)+1, \ldots\rangle$$

The operators for occupation number, angular momentum, m, and energy can be expressed in terms of the creation and annihilation operators. For example, the number operator is

(A5.1.24) $N_{lm}(k\pm) = a^*_{lm}(k\pm)\, a_{lm}(k\pm)$

with eigenvalues

(A5.1.25) $\langle N_{lm}(k\pm)\rangle = n_{lm}(k\pm) = 0, 1, 2, ..$

and the energy operator is

(A5.1.26)
$$\mathcal{H}_{op} = \frac{1}{8\pi}\int\left(\underline{E}^2_{op} + \underline{H}^2_{op}\right) d\tau$$

$$= \sum_{k,l,m,\pm}\left(a_{lm}(k\pm)\, a^*_{lm}(k\pm) + a^*_{lm}(k\pm)\, a_{lm}(k\pm)\right)\frac{k^2}{4\pi}\int\left|\underline{A}_{lm}(k\pm)\right|^2 d\tau$$

$$= \sum_{k,l,m,\pm}\frac{k}{2}\left(a_{lm}(k\pm)\, a^*_{lm}(k\pm) + a^*_{lm}(k\pm)\, a_{lm}(k\pm)\right)$$

with total energy

(A5.1.27) $\langle \mathcal{H}_{op}\rangle = \sum_{k,l,m,\pm} k\left(n_{lm}(k\pm) + \frac{1}{2}\right)$

To introduce the interaction of the electromagnetic field with the nucleon system, the method due to Dirac (1931) will be used. The wave function $\Psi = \Psi' e^{i(\beta_1+\beta_2+\quad)}$ of a system of charged particles is separated into a function Ψ' of definite amplitude and phase, and a phase factor $e^{i(\beta_1+\beta_2+\ldots)}$ for which β_i is a function of the co-ordinates of the i-th particle. The β_i need not have definite value. The derivatives $\left(\frac{\partial\beta_i}{\partial x_i}, \frac{\partial\beta_i}{\partial y_i}, \frac{\partial\beta_i}{\partial z_i}, \frac{\partial\beta_i}{\partial t}\right)$ have a definite value although the condition $\frac{\partial^2\beta}{\partial x\partial y} = \frac{\partial^2\beta}{\partial y\partial x}$ need not be satisfied. This is equivalent to the requirement that only the difference in phase between two neighbouring points be definite. Integrating the phase change for a single particle around a closed four dimensional curve given by the use of Stoke's theorem, (A5.1.28) $$\int (\text{grad}\ \beta_i) \cdot d\underline{s} = \int \text{curl}\ (\text{grad}\ \beta_i) \cdot d\underline{S}$$

where the line element $\underline{ds}$ = (dx,dy,dz,-dt) is a 4-vector, the surface element $\underline{dS}$ = (dxdy, dxdz,-dxdt,dydz,dydt,dzdt) is a 6-vector and $(\text{grad.}\ \beta_i) = \left(\frac{\partial \beta_i}{\partial x}, \frac{\partial \beta_i}{\partial y}, \frac{\partial \beta_i}{d z}, \frac{-\partial \beta_i}{\partial t}\right)$. If the gradient of any scalar is added to (grad β_i) the integral will be invariant. This is a similar transformation to the gauge transformation for the electromagnetic field with the potential (A_x,A_y,A_z, ϕ). If it is required that the interaction of the electromagnetic field be gauge invariant, this will be accomplished if the derivatives of β_i at each point are identified with the electromagnetic potential at that point.

(A5.1.29) $$\left(\frac{\partial \beta_i}{\partial x}, \frac{\partial \beta_i}{\partial y}, \frac{\partial \beta_i}{\partial z}, \frac{-\partial \beta_i}{\partial t}\right) = (e A_x, e A_y, e A_z, e\phi)$$

If Ψ satisfies a wave equation involving the momentum operator $\underline{p} = -i\nabla$ and the energy operator $E = i\frac{\partial}{\partial t}$, then Ψ' will satisfy the same equation with $\underline{p}$ replaced by ($\underline{p} + \nabla\beta$) and $\underline{E}$ replaced by ($E - \frac{\partial \beta}{\partial t}$). Thus in an electromagnetic field, the potential enters through

(A5.1.30) $$(\underline{p} + e\underline{A}) \qquad \text{and} \qquad (E - e\phi)$$

Thus, e.g., the Schrödinger equation becomes

(A5.1.31) $$\left\{\sum_i \left[\frac{1}{2M_i}(\underline{p}_i + e_i\underline{A}_i)^2 + V_i - (E_i - e_i\phi_i)\right]\right\}\Psi = 0$$

$$\left\{\sum_i \left[\frac{1}{2M_i}\underline{p}_i^2 + \frac{e_i}{2M}(\underline{p}_i\cdot\underline{A}_i + \underline{A}_i\cdot\underline{p}_i) + \frac{e_i^2}{2M_i}\underline{A}_i^2 + e_i\phi_i + V_i - E_i\right]\right\}\Psi = 0$$

where e_i is the charge on the i-th particle and the potentials are evaluated at the position of the i-th particle. In first order perturbation theory, only terms to the first order in e are used. This is generally valid because of the smallness of the coupling parameter $e^2 = 1/137$. If the matrix elements, taken between stationary states of the system, of the perturbation

$$(A5.1.32)\quad H' = \sum_i \frac{e_i}{2M_i}\left(\underline{p}_i \cdot \underline{A}_i + \underline{A}_i \cdot \underline{p}_i\right)$$

be zero for any reason, it would then be necessary to examine the matrix elements in e^2. This, for example, is the case when examining the Stark effect. In the present problem, the matrix elements of H' are not zero and it is not necessary, in the first approximation, to examine higher order terms. The transitions between stationary states caused by H' are accompanied by the emission or absorption of a single photon. If the particles have magnetic moments $\underline{\mu}_i = \mu_i \underline{\sigma}_i$, there will be an additional perturbation

$$(A5.1.33)\quad H'' = -\sum_i \underline{\mu}_i \cdot \underline{H}_i$$

where the magnetic field $\underline{H}_i$ is evaluated at the i-th nucleon.

Standard time-independent perturbation theory is used so the transition probability is given by

$$(A5.1.34)\quad w = 2\pi \rho_E \left|H\right|^2$$

The final state of the nucleon system has only a single energy so that the density of energy levels ρ_E is given by the density of energy levels for the electromagnetic field (A5.1.15). The total cross-section is obtained by dividing the transition probability by the flux of incident particles. The incident wave is normalized to unit density at infinity, thus the flux is equal to the relative velocity of the incident proton and the deuteron. This is, non-relativistically,

(A5.1.35) $v = \frac{3k}{2M}$

where 2/3 M is the reduced mass of the system and **k** is the momentum given by

(A5.1.36) $\frac{3k^2}{4M} = E_C$

and E_C is the kinetic energy in the centre-of-mass frame. This is related to the laboratory energy E_L of the proton by

(A5.1.37) $E_C = \frac{2}{3} E_L$

The energy of the emitted photon is determined by E_C and the difference in binding energies of He^3 and deuteron (B_{He^3} and B_D),

(A5.1.38) $k = E_C + (B_{He^3} - B_D)$

Defining the quantity β_2' by

(A5.1.39) $\beta_2'^2 = \frac{4}{3} M (B_{He^3} - B_D)$

the photon energy may be expressed in terms of **k** and β_2',

(A5.1.40) $k = \frac{3}{4M} (k^2 + \beta_2'^2)$

2. Derivation of the Matrix Elements

A. Electric Multipole Matrix Elements

The matrix elements arising from the perturbation H' (A5.1.32) and giving rise to electric multipole radiation can now be obtained. The use of the coulomb gauge allows a simplification of H',

$$\begin{aligned} H' &= -\frac{e}{2M}\left(\underline{A}_{op}\,\underline{p} + \underline{p}\,\underline{A}_{op}\right) \\ &= -\frac{e}{2M}\left(2\underline{A}_{op}\,\underline{p} - i\left(\nabla\,\underline{A}_{op}\right)\right) \\ &= -\frac{e}{M}\,\underline{A}_{op}\,\underline{p} \end{aligned} \qquad (A5.2.1)$$

Using ψ_a and ψ_b as the final and initial particle wave functions, and $|1_{lm}(k-)\rangle$ and $|0_{lm}(k-)\rangle$ as the final and initial photon state functions, the matrix element becomes

$$\begin{aligned} H_{ab} &= -\frac{e}{M}\int\left\langle 1_{lm}(k-)\right|\psi_a\left(a^*_{lm}(k-)\,\underline{A}^*_{lm}(k-)\right)\cdot\underline{p}\left|0_{lm}(k-)\right\rangle\psi_b\,d\tau \\ &= -\frac{e}{M}\cdot\sqrt{\frac{1}{2k}}\cdot -i\sqrt{\frac{8\pi}{R}}\cdot -i\cdot\int\psi_a^*\left(\nabla\times j_l(kr)\,\underline{X}_{lm}\right)\cdot\nabla\psi_b\,d\tau \end{aligned} \qquad (A5.2.2)$$

An integration by parts enables this to be written

$$\begin{aligned} H_{ab} &= \frac{e}{2M}\sqrt{\frac{4\pi}{kR}}\int\left(\nabla\times j_l(kr)\underline{X}_{lm}\right)\cdot\left(\psi_a^*\nabla\psi_b - (\nabla\psi_a)^*\psi_b\right)d\tau \\ &= ie\sqrt{\frac{4\pi}{kR}}\int\left(\nabla\times j_l(kr)\,\underline{X}_{lm}\right)\cdot\underline{j}_{ab}\,d\tau \end{aligned} \qquad (A5.2.3)$$

where

$$\underline{j}_{ab} = \frac{-i}{2M}\left(\psi_a^*\nabla\psi_b - (\nabla\psi_a)^*\psi_b\right) \qquad (A5.2.4)$$

is the probability current density and is related to the probability density

$$\rho_{ab} = \psi_a^*\psi_b \qquad (A5.2.5)$$

by

(A5.2.6) $\nabla \underline{j}_{ab} = \iota k \rho_{ab}$

Integrating by parts again, (A5.2.3) becomes

(A5.2.7) $H_{ab} = \iota e \sqrt{\frac{4\pi}{kR}} \int (j_l \underline{X}_{lm}) \cdot (\nabla \times \underline{j}_{ab}) \, d\tau$

The evaluation of the integral is (A5.2.7) now follows the method of Blatt and Weisskopf (1952, p. 806). It is assumed that the wavelength of the radiation is long compared with the size of the source so that the approximation

(A5.2.8) $j_l(kr) = \frac{(kr)^l}{(2l+1)!!}$

may be used. Using the definition of $\underline{X}_{lm}$ (A5.1.20) and the hermitian property of $\underline{L}$, (A5.2.7) becomes

(A5.2.9) $H_{ab} = \iota e \sqrt{\frac{4\pi}{kR}} \frac{k^l}{(2l+1)!!} \frac{1}{\sqrt{l(l+1)}} \int r^l Y_{lm}^* \underline{L} \cdot (\nabla \times \underline{j}_{ab}) \, d\tau$

Evaluation of the integral gives

(A5.2.10) $-k(l+1) \int r^l Y_{lm}^* \psi_a \psi_b \, d\tau$

The matrix element is then,

(A5.2.11) $H_{ab} = \frac{-\iota e}{(2l+1)!!} \sqrt{\frac{4\pi k}{R}} \cdot \sqrt{\frac{l+1}{l}} \, k^l \int r^l Y_{lm}^* \psi_a^* \psi_b \, d\tau$

$$= \frac{-\iota}{(2l+1)!!} \sqrt{\frac{4\pi k}{R}} \cdot \sqrt{\frac{l+1}{l}} \, k^l Q_{lm}$$

where

(A5.2.12) $Q_{lm} = e \int r^l Y_{lm}^* \psi_a^* \psi_b \, d\tau$

These are referred to as the electric 2^l-pole moments.

When the system contains several nucleons, the integrals Q_{lm} (A5.2.12) are replaced by

$$Q_{lm} = e \sum_i \int \psi_a^* \tfrac{1}{2}(1 - \tau_{i3})\, r_i^l\, Y_{lm}(\theta_i \phi_i)\, \psi_b\, d\tau \tag{A5.2.13}$$

where use is made of the isotopic spin formalism and $\frac{1}{2}(1 - \tau_{i3})$ selects the charged particles. The integral includes integration over the spatial co-ordinates and summation over the spin and isotopic spin co-ordinates. The transition probability (A5.1.34) then becomes

$$\begin{aligned} w &= 2\pi\, \rho_E \left| H_{ab} \right|^2 \\ &= \frac{8\pi(l+1)}{l\left((2l+1)!!\right)^2}\, k^{2l+1} \left| Q_{lm} \right|^2 \end{aligned} \tag{A5.2.14}$$

Here the density of energy levels is the density for the radiation field (A5.1.15) as the final state of the nucleons is the ground state of He^3 which has only a single energy level.

The interaction H'' (A5.1.33) due to the magnetic moment can also give rise to electric multipole radiation. The matrix element is

$$\begin{aligned} H_{ab} &= -\int \left\langle 1_{lm}(k,-) \right| \psi_a^* \left\{ a_{lm}^*(k,-)\, \nabla \times \underline{A}_{lm}^*(k,-) \right\} \cdot \underline{\mu} \left| 0_{lm}(k,-) \right\rangle \psi_b\, d\tau \\ &= \frac{-e}{2M}\, \mu \int \left\langle 1_{lm}(k,-) \right| \psi_a^* \left\{ a_{lm}^*(k,-)\, \nabla \times \underline{A}_{lm}^*(k,-) \right\} \cdot \sigma \left| 0_{lm}(k,-) \right\rangle \psi_b\, d\tau \end{aligned} \tag{A5.2.15}$$

where μ is expressed in terms of the nuclear magneton. This integral may be evaluated in a manner similar to that used for the evaluation of (A5.2.2). The result is:

$$H_{ab} = \frac{-ie}{(2l+1)!!} \sqrt{\frac{4\pi k}{R}} \cdot \sqrt{\frac{l+1}{l}}\; k^l\, Q'_{lm} \tag{A5.2.16}$$

where, adapting the notation of Blatt and Weisskopf,

$$(A5.2.17)\quad Q'_{lm} = \frac{-ik}{(l+1)}\,\frac{e}{2M}\,\mu\int r^{l}\,Y^{*}_{lm}\,\nabla\cdot\left(\psi^{*}_{a}\,(\underline{r}\times\underline{\sigma})\,\psi_{b}\right)d\tau$$

For a system of protons and neutrons this becomes, using the expression (1.3.17) for the magnetic moment of the operator,

$$(A5.2.18)\quad Q_{lm} = \frac{-ik}{(l+1)}\cdot\frac{e}{2M}\sum_{i}\int r_{i}^{l}\,Y^{*}_{lm}(\theta_{i}\phi_{i})\,\nabla\left\{\psi^{*}_{a}\left(\tfrac{1}{2}(1-\tau_{i3})\mu_{P}+\tfrac{1}{2}(1+\tau_{i3})\mu_{N}\right)\underline{r}_{i}\times\underline{\sigma}_{i}\,\psi_{b}\right\}d\tau$$

Including both of these electric multipole moments, the transition probability (A5.2.14) becomes

$$(A5.2.19)\quad w = \frac{8\pi\,(l+1)}{l\,((2l+1)!!)^{2}}\;k^{2l+1}\left|Q_{lm}+Q'_{lm}\right|^{2}$$

B. <u>Magnetic Multipole Matrix Elements</u>

Both of the interactions H' and H'' (A5.1.32 and 33) may give rise to magnetic multipole radiation. The calculation of the matrix elements proceeds similarly to that for the electric multipole matrix elements. The result for the interaction H' is

$$(A5.2.20)\quad \begin{aligned} H_{ab} &= \frac{-e}{M}\int\left\langle 1_{lm}(k+)\left|\psi^{*}_{a}\left(a^{*}_{lm}(k+)\,\underline{A}_{lm}(k+)\right)\rho\right|0_{lm}(k+)\right\rangle\psi_{b}\,d\tau \\ &= \frac{-i}{(2l+1)!!}\sqrt{\frac{4\pi k}{R}}\sqrt{\frac{(l+1)}{l}}\;k^{l}\,M_{lm}\end{aligned}$$

where

$$(A5.2.21)\quad M_{lm} = \frac{-1}{(l+1)}\,\frac{e}{M}\int r^{l}\,Y^{*}_{lm}\,\nabla\left(\psi^{*}_{a}\,\underline{L}\,\psi_{b}\right)d\tau$$

The result for the interaction H'' is

$$(A5.2.22)\quad H_{ab} = -\int \left\langle 1_{lm}(k+) \middle| \psi_a^* \left(a_{lm}(k+) \nabla \times \underline{A}_{lm}(k+)\right) \cdot \underline{\mu} \middle| O_{lm}(k+) \right\rangle \psi_b \, d\tau$$

$$= \frac{-i}{(2l+1)!!} \sqrt{\frac{4\pi k}{R}} \sqrt{\frac{l+1}{l}}\, k^l M'_{lm}$$

where

$$(A5.2.23)\quad M'_{lm} = \frac{-e}{2M} \mu \int r^l Y_{lm} \nabla \cdot \left(\psi_a^* \underline{\sigma} \psi_b\right) d\tau$$

The integrals are referred to as the magnetic multipole moments. For systems of several nucleons they become

$$(A5.2.24)\quad M_{lm} = \frac{-1}{(l+1)} \cdot \frac{e}{M} \cdot \sum_i \int r_i^l Y_{lm}^* (\theta_i \phi_i) \nabla \cdot \left(\psi_a^* \tfrac{1}{2}(1-\tau_{i3}) \underline{L}_i \psi_b\right) d\tau$$

and

$$(A5.2.25)\quad M'_{lm} = \frac{-e}{2M} \sum_i \int r_i^l Y_{lm}^* (\theta_i \phi_i) \nabla \cdot \left\{\psi_a^* \left(\tfrac{1}{2}(1-\tau_{i3})\mu_P + \tfrac{1}{2}(1+\tau_{i3})\mu_N\right) \underline{\sigma} \psi_b\right\} d\tau$$

The transition probability has the same form as (A5.2.19)

$$(A5.2.26)\quad w = \frac{8\pi(l+1)}{l((2l+1)!!)^2} \cdot k^{2l+1} \left| M_{lm} + M'_{lm} \right|^2$$

APPENDIX 6

EULER ANGLE FUNCTIONS

In the evaluation of the integrals for the multipole moments, it is necessary to express $r^l Y_{lm}(\theta, \phi)$ in terms of the Euler angles and the internal co-ordinates. This is done by using the representation of $r^l Y_{lm}(\theta, \phi)$ in cartesian co-ordinates and then using the transformation (1.1.2) to express these in terms of Euler angles and the 'body' co-ordinates X, Y and Z (1.1.3). Thus for $l = 1$, $m = 1$

$$
\begin{aligned}
r Y_{1,1} &= -\sqrt{\frac{3}{8\pi}}\,(x + iy) \\
&= -\sqrt{\frac{3}{8\pi}}\left\{(R_{11} + iR_{21})X + (R_{13} + iR_{23})Z\right\} \\
&= -\sqrt{\frac{3}{8\pi}}\left\{\left(e^{i\alpha}\frac{(1+\cos\beta)}{2}e^{i\gamma} - e^{-i\alpha}\frac{(1-\cos\beta)}{2}e^{i\gamma}\right)X + (\sin\beta\, e^{i\gamma})Z\right\} \qquad \text{(A6.1.1)} \\
&= -\sqrt{\frac{3}{8\pi}}\left\{\left(\mathcal{D}^1_{11}(\alpha\beta\gamma) - \mathcal{D}^1_{-11}(\alpha\beta\gamma)\right)X + \left(-\sqrt{2}\,\mathcal{D}^1_{01}(\alpha\beta\gamma)\right)Z\right\} \\
&= -\sqrt{2\pi}\,W^1_{11}(-)X + \sqrt{2\pi}\,W^1_{01}(-)Z
\end{aligned}
$$

Similar calculations for m = 0 and m = -1 lead to the general result

$$
r Y_{1m} = \sqrt{2\pi}\,W^1_{0m}(-)Z - \sqrt{2\pi}\,W^1_{1m}(-)X \qquad \text{(A6.1.2)}
$$

For $r^2 Y_{2,m}$ the calculation may be done in an exactly similar manner to give the result

$$
r^2 Y_{2m} = \sqrt{\frac{\pi}{2}}\,W^2_{0m}(+)\left(2Z^2 - X^2\right) - \sqrt{6\pi}\,W^2_{1m}(+)\left(XZ\right) + \sqrt{\frac{3\pi}{2}}\,W^2_{2m}(+)\left(X^2\right) \qquad \text{(A6.1.3)}
$$

Alternatively, this result may be obtained from the calculations for(A6.1.2) by using the formula given below for the product of two functions. For example

(A6.1.4)
$$r^2 Y_{2,2} = \sqrt{\frac{15}{32\pi}}\ (x+iy)^2$$

$$= \sqrt{\frac{15}{32\pi}}\left\{\left(\mathcal{D}^1_{11}(\alpha\beta\gamma) - \mathcal{D}^1_{-1,1}(\alpha\beta\gamma)\right)X + \left(-\sqrt{2}\,\mathcal{D}^1_{01}(\alpha\beta\gamma)\right)Z\right\}^2$$

$$= \sqrt{\frac{15}{32\pi}}\left\{\left(\mathcal{D}^2_{2,2}(\alpha\beta\gamma) + \mathcal{D}^2_{-2,2}(\alpha\beta\gamma) - \frac{2}{\sqrt{6}}\mathcal{D}^2_{02}(\alpha\beta\gamma)\right)X^2 - 2\left(\mathcal{D}^2_{1,2}(\alpha\beta\gamma) - \mathcal{D}^2_{-1,2}(\alpha\beta\gamma)\right)XZ + \frac{4}{\sqrt{6}}\mathcal{D}^2_{0,2}(\alpha\beta\gamma)Z^2\right\}$$

$$= \sqrt{\frac{\pi}{2}}\, W^2_{02}(+)\left(2Z^2 - X^2\right) - \sqrt{6\pi}\, W^2_{12}(+)(XZ) + \sqrt{\frac{3\pi}{2}}\, W^2_{2,2}(+)\,(X^2)$$

In the evaluation of the integrals for the multipole moments, products of the Euler angle functions appear such as $W^{l_1}_{m'_1,m_1}(\pi_1)\,W^{l_2}_{0,m_2}(\pi_2)$ and it is helpful to express these as a sum of single W functions. Use is made of the formula for the product of two functions $\mathcal{D}^{l}_{m'm}$ (Edmonds, 1957, p. 61)

(A6.1.5)
$$\mathcal{D}^{l_1}_{m'_1 m_1}\mathcal{D}^{l_2}_{m'_2 m_2} = \sum_{l,m',m}(2l+1)\begin{pmatrix} l_1 & l_2 & l \\ m'_1 & m'_2 & m \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix}\mathcal{D}^{j\,*}_{m'm}$$

and the complex conjugate

(A6.1.6)
$$\mathcal{D}^{l\,*}_{m'm}(\alpha\beta\gamma) = (-)^{m'-m}\,\mathcal{D}^{l}_{-m'\,-m}(\alpha\beta\gamma)$$

For the W functions, the complex conjugate is

(A6.1.7)
$$W^{l\,*}_{m'm}(\pi) = \sqrt{\frac{2l+1}{4\pi}}\left(\mathcal{D}^{l\,*}_{m'm} + \pi(-)^{l+m'}\mathcal{D}^{l\,*}_{-m'm}\right)$$

$$= \sqrt{\frac{2l+1}{4\pi}}\left((-)^{m'-m}\mathcal{D}^{l}_{-m'-m} + \pi(-)^{l-m}\mathcal{D}^{l}_{m'-m}\right)$$

$$= \pi(-)^{l-m}\sqrt{\frac{2l+1}{4\pi}}\left(\mathcal{D}^{l}_{m'-m} + \pi(-)^{l+m'}\mathcal{D}^{l}_{m'-m}\right)$$

$$= \pi(-)^{l-m}\,W^{l}_{m'-m}(\pi) \qquad \text{for } l \text{ integral}$$

where π = parity of the function (1.2.26). For m' = 0, $\pi = (-)^{l}$.

The product of two W functions is

$$W^{l_1}_{m_1'm_1}(\pi_1)W^{l_2}_{0,m_2}(\pi_2) = \frac{\sqrt{2(2l_1+1)(2l_2+1)}}{16\pi^2}\left(\mathcal{D}^{l_1}_{m_1'm_1} + \pi_1(-)^{l+m'}\mathcal{D}^{l_1}_{-m_1'm_1}\right)\mathcal{D}^{l_2}_{om_2}$$

(A6.1.8)
$$= \frac{\sqrt{2(2l_1+1)(2l_2+1)}}{16\pi^2}\sum(2l+1)\begin{pmatrix} l_1 & l_2 & l \\ m_1' & 0 & m \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix}\left\{\mathcal{D}^{l*}_{m'm} + \pi(-)^{l_2}(-)^{l+m'}\mathcal{D}^{l*}_{-m'm}\right\}$$

$$= \frac{\sqrt{2(2l_1+1)(2l_2+1)}}{4\pi}\sum\sqrt{2l+1}\begin{pmatrix} l_1 & l_2 & l \\ m_1' & 0 & m \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix}W^{l*}_{m'm}(\pi_1\pi_2)$$

This result is valid for $m_1' = 0$ but is not valid when neither $m_1' = 0$ nor $m_2' = 0$. In the latter case different terms appear. Using (A6.1.7 and 8) and the orthonormality of these functions gives

(A6.1.9)
$$\left\langle W^{l_1}_{m_1'm_1}(\pi_1)\middle|W^{l_2 *}_{m_2'm_2}(\pi_2)\middle|W^{l_3}_{om_3}(\pi_3)\right\rangle = \pi_l(-)^{l_2-m_2'}\frac{\sqrt{2(2l_1+1)(2l_2+1)(2l_3+1)}}{4\pi}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & m_3 \end{pmatrix}$$

BIBLIOGRAPHY

Austern, N., 1951 Phys. Rev. 83, 672
1952 Phys. Rev. 85, 147

Blatt, J. M. & Weisskopf, V. W., 1952 Theoretical Nuclear Physics, (John Wiley & Sons, New York)

Brueckner, K. A. & Gammel, J. L., 1958 Phys. Rev. 109, 1023

Burhop, E. & Massey, H. S. W., 1948 Proc. Roy. Soc. A192, 156

Curran, S. C. & Strothers, J., 1939 Proc. Roy. Soc. A172, 72

Delves, L. M., 1960 Phys. Rev. 118, 1318

Derrick, G. H. & Blatt, J. M., 1958 Nucl. Phys. 8, 310

Derrick, G. H., 1960 Nucl. Phys. 16, 405
1960 Nucl. Phys. 18, 303

Derrick, G. H., Mustard, D. & Blatt, J. M., 1961 Phys. Rev. Lett. 6, 69

Dirac, P. A. M., 1931 Proc. Roy. Soc. A133, 60

Edmonds, A. R., 1957 Angular Momentum in Quantum Mechanics, (Princeton University Press, Princeton, N.J.)

Fowler, W. A., Lauristen, C. C. & Tollestrup, A. V., 1949 Phys. Rev. 76, 1767

Goldstein, H., 1950 Classical Mechanics, (Addison-Wesley Publishing Co., Cambridge, Mass.)

Griffiths, G. M. & Warren, J. B., 1955 Proc. Roy. Soc. A67, 781

Griffiths, G. M., Larsen, E. & Robertson, L., 1961 to be published

Heitler, W., 1954 Quantum Theory of Radiation, (3rd edition), (Oxford University Press)

Höcker, K. H., 1942 Phys. Z. 43, 239

Lomont, J. S., 1959 Applications of Finite Groups, (Academic Press, New York)

Morse, P. M. & Feshbach, H., 1953 Methods of Theoretical Physics (McGraw-Hill Book Co., New York)

Panofsky, W. & Phillips, M., 1955 Classical Electricity and Magnetism, (Addison-Wesley Publishing Co., Cambridge, Mass.)

Rotenberg, M., Bivens, R., Metropolis, N. & Wooten, J. K., 1959 The 3-j and 6-j Symbols, (The M.I.T. Press, Cambridge, Mass.)

Schiff, L. I., 1937 Phys. Rev. 52, 242
1955 Quantum Mechanics, (McGraw-Hill Book Co., New York)

Verde, M., 1950 Helv. Phys. Acta 23, 453
1951 Helv. Phys. Acta 24, 298

Wigner, E. P., 1959 Group Theory, (Academic Press, New York)