THE ELECTRIC POTENTIAL IN THE NEIGHBOURHOOD OF A THIN SLIT

by

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ABSTRACT

When a slit opening is made in a plane electrode forming the boundary between two unequal electric fields, a distortion of the fields occurs. This paper studies the influence of the slit opening on the electrical potential distribution on both sides of the slit.

A theory is developed for calculating the potential at any point, and from it two methods are derived for finding curves of equal potential disturbance. Several computed curves are presented for each method. The curves suggest a simple graphical construction for approximating the potential disturbance at points not too near the slit.

Because the potential disturbance is the same at image points on either side of the slit, it is found that all the important formulas can be expressed in terms of distances, without regard to sign.

To facilitate the reproduction or extension of this work, a computer program in the widely used Fortran language is given for the simpler of the two methods.

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INTRODUCTION

The object of this thesis is to determine the theoretical potential distribution in a system of plane parallel slits maintained at fixed potentials.

Previous approaches to this problem (v.g. Boerboom, 1959) have consisted in solving for the potential on the axis of the system, and extrapolating to points off-axis using a series expansion. The problem was also tackled experimentally in a number of ways (Klemperer, 1953), the most frequent one being to measure the potential around a model immersed in an electrolyte.

As a departure from these approaches, our aim will be to develop formulae for the potential which are equally valid and equally precise at any distance from the axis, so that the study of the effect of the slit on a beam of ions need not, in the future, be limited to paraxial rays.

Where a simplification is possible on the axis of the system, it will be indicated, but will not play an essential role in the general treatment.

CHAPTER 1

THEORY OF THE SLIT PROBLEM

The General Problem

We consider a plane slit of negligible thickness compared to the slit width, and of sufficiently long length so that end effects may be disregarded.

We will call the lens axis that normal to the plane of the slit going through the geometrical center of the slit, i.e. the y-axis in Figure 1. The half-slit width will be denoted by h.

In the absence of a field component parallel to the long dimension of the slit, the potential V satisfies:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad (1)$$

We take as our boundary conditions: (I) the fixed potential on the slit, V_s, and (II) the presence of constant electric fields E_a and E_b which, at sufficiently large distances from the slit, are parallel to the lens axis and relatively undisturbed by the opening or closing of the slit, so that

$$V \longrightarrow V_s - E_a\, y \quad \text{when } y \longrightarrow +\infty$$

$$V \longrightarrow V_s - E_b\, y \quad \text{when } y \longrightarrow -\infty$$

Either field can be positive or negative, independently, and one of them may be zero.

Figure 1 summarizes the problem. We know from experimental evidence that the stronger field (in magnitude) will bulge out into the region of weaker field, much as a thin elastic membrane, stretched lightly across an opening separating two chambers at unequal pressures, would give way to the stronger pressure. (In fact the elastic membrane, since it also obeys Laplace's equation, would behave much like an equipotential in such a case.)

The Potential Disturbance

The problem is most easily solved by superposition. Consider the auxiliary problem of Figure 2(a). If U_a is the solution of that new problem, it is readily verified that, for any point of the plane,

$$V = (V_s - E_a\, y) + U_a \qquad (2\text{-}a)$$

is the solution to the general slit problem of Figure 1.

For points above the slit, the quantity $(V_s - E_a\, y)$ obviously represents the potential with the slit closed, so that U_a represents the change in potential due to the slit opening.

Such a convenient interpretation of (2-a) is not possible for the region below the slit, although the equation is still valid there. We could, however, have synthesized a different auxiliary problem, shown in

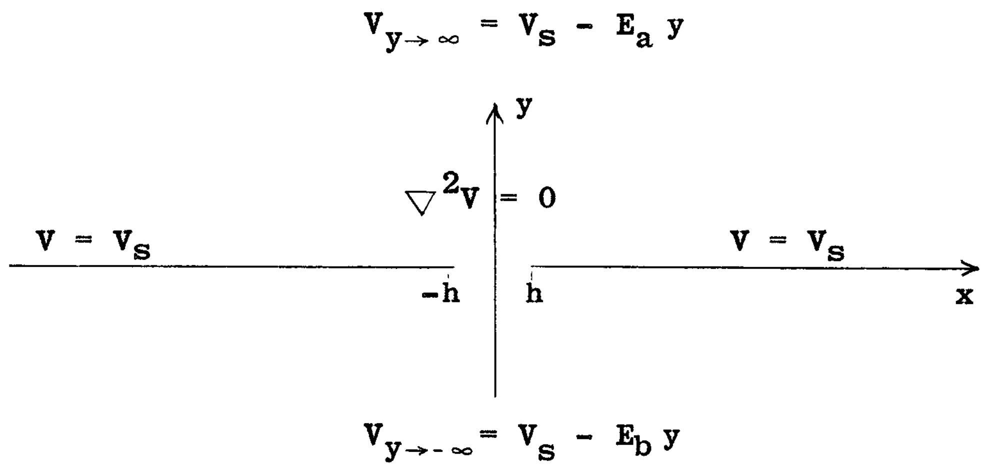
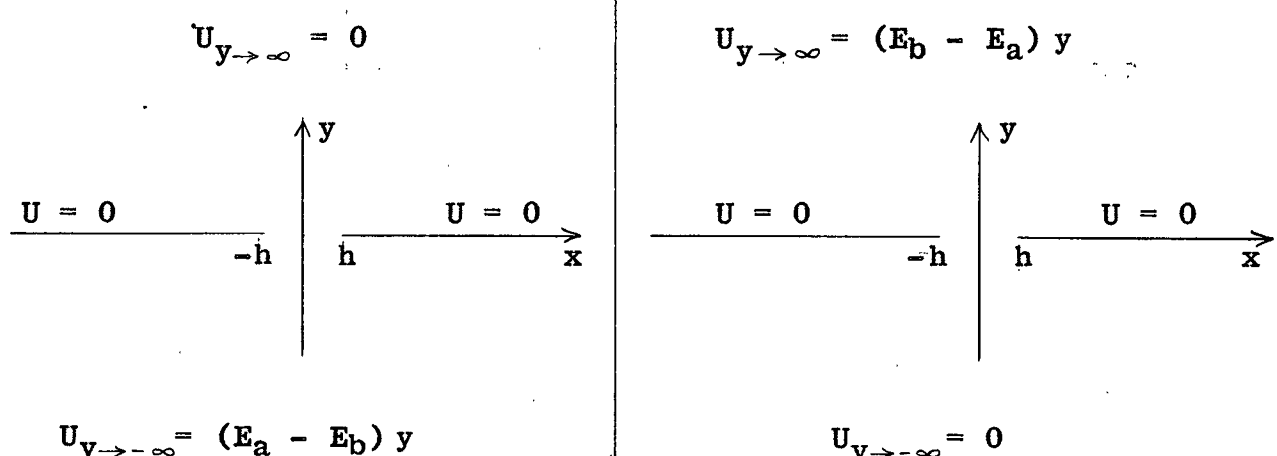


Figure 1 - General Slit Problem.
The long dimension of the slit is perpendicular to the plane of the paper.



(a) Problem with Solution U_a. (b) Problem with Solution U_b.

Figure 2 = Auxiliary Slit Problems.

Figure 2(b). If U_b is the solution of that problem, the superposition equation takes the equivalent form:

$$V = (V_s - E_b\ y) + U_b \qquad (2\text{-}b)$$

This equation is again valid for both halves of the plane. For points below the slit, however, $(V_s - E_b\ y)$ is obviously the potential with the slit closed, so that U_b represents the potential disturbance due to the slit.

The advantage of being able to separate the total potential at a point into a "slit closed" component plus a slit disturbance is so great, in practical applications, that we shall agree to use, essentially, Figure 2(a) and equation (2-a) for points above the slit, and Figure 2(b) and equation (2-b) for points below. With that convention, the superposition equation for any point of the plane takes the simple form:

$$V = (\text{Potential with slit closed}) + \triangle V \qquad (2\text{-}c)$$

where

$\triangle V = U_a(x,y)$ for points above the slit

$\triangle V = U_b(x,y)$ for points below the slit.

We can avoid the inconvenience of always carrying two functions U by a simple expedient. First, suppose we had to find the potential U at point P in the following situation (where the two equipotentials shown are assumed to lie in the undisturbed region):

.P

U = 0   U = 0

.... 800 volts

....................................1000 volts

Field: 200 volts/cm

We would not hesitate, if this were convenient, to solve instead for the potential at Q (image of P) in the following situation:

Field: -200 volts/cm

....................................1000 volts

.... 800 volts

U = 0   U = 0

.Q

(Note that the field changed sign.)

Similarly, whenever we need to solve the problem of Figure 2(a) for points above the x-axis, we could solve instead the problem of Figure 2(b) for the <u>image</u> points below the axis. Consequently, knowing U_b for the lower half of Figure 2(b) is all that would be needed for a knowledge of the potential distribution over the whole plane. Alternatively, we could use U_a and solve only for the upper half of the plane.

In short, the problem is simplified because the potential disturbance obeys the relations:

$$\triangle V = U_a(x,y) = U_b(x,-y) \quad \text{for points above the slit} \qquad (3\text{-a})$$

$$\triangle V = U_a(x,-y) = U_b(x,y) \quad \text{for points below the slit} \qquad (3\text{-b})$$

At this stage, there is little to choose between solving U_a for positive y's, or U_b for negative y's, except that a solution involving only positive coordinates appears desirable. However, we shall see later that there is a definite advantage in solving for U_b and that, furthermore, the final expression for the potential disturbance can be expressed in terms of the absolute values of the coordinates. We shall therefore solve for U_b.

The solution was originally worked out for the whole plane. We now know that this was not strictly necessary. Nevertheless, the solution will be presented in its entirety because in one important case, $E_b = 0$, the shape of U_b is the shape of the actual potential V.

Finally, since we have no further use for U_a, we shall drop the subscript, with the understanding that

$$U \equiv U_b$$

Solution of the Auxiliary Problem

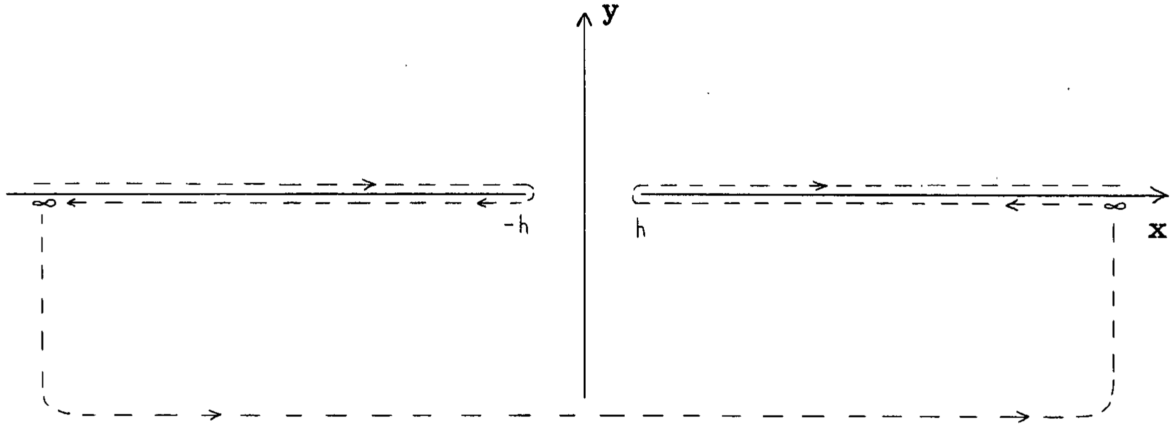
The problem having been reduced to that of Figure 2(b), it is fairly natural to seek a transformation which will facilitate its solution. After several transformations were tried, one was suggested by Dr. J.C. Savage which is probably the simplest of all.

Let Z = x + iy be the complex variable which has the plane of the figure as its domain. Using the Schwarz-Christoffel transformation (Morse and Feshback, 1953, p. 445), we can map the contour (degenerate polygon) shown dotted in Figure 3(a) onto the horizontal axis of a new plane W = u + iv, in such a way that the "interior" of the polygon (i.e. the whole Z plane) maps onto the upper half of the W plane, Figure 3(b).

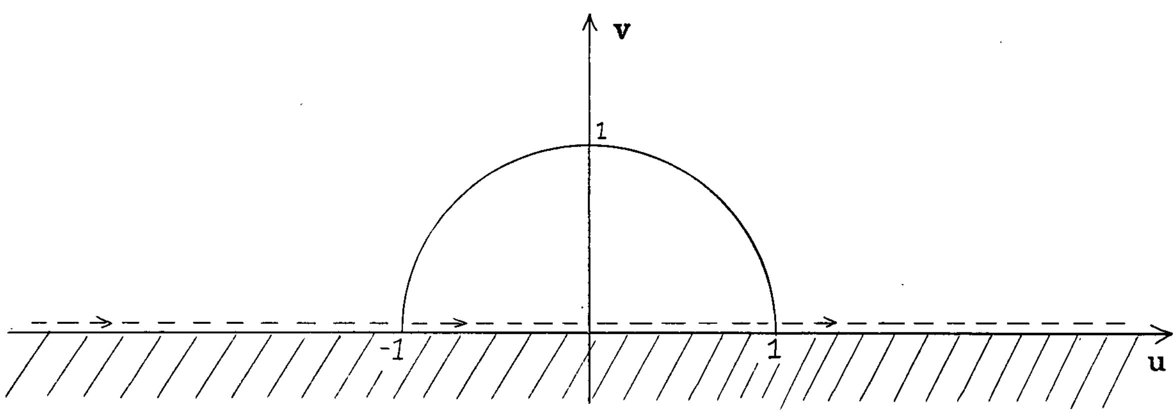
As in all such transformations, the correspondence of any three points in the two planes may be arbitrarily prescribed. This is equivalent to specifying, for the transformed figure, the position of one point; the scale or distance between two chosen points, and the orientation, all three of which are obviously arbitrary.

Let us impose that the point at infinity in the upper half of the Z plane transform to the point at infinity in W, and that the slit edges transform to (-1,0) and (1,0). From symmetry considerations, we may derive immediately that the fourth vertex of our degenerate polygon, the point at infinity in the lower half of the Z plane, must fall at the origin in the W plane.

Under those conditions, the Schwarz-Christoffel transformation takes the particularly simple form:



(a) Z Plane



(b) W Plane

Figure 3 - Transformation $Z = \frac{h}{2}\ (W + \frac{1}{W})$.

$$Z = Z_o + k \int (W + 1)^1 (W - 0)^{-2} (W - 1)^1 \, dW$$

$$= Z_o + k \int \frac{W^2 - 1}{W^2} \, dW$$

$$= Z_o + k \left(W + \frac{1}{W} \right)$$

To find Z_o and k, we substitute the values of Z and W corresponding to the slit edges. The resulting equations give $Z_o = 0$, $k = h/2$. The transformation equation which effects the desired mapping is then:

$$Z = \frac{h}{2}\left(W + \frac{1}{W}\right)$$

or

$$x + iy = \frac{h}{2}\left(u + iv + \frac{1}{u + iv}\right)$$

or, equating real and imaginary parts:

$$x = \frac{h}{2}\left(u + \frac{u}{u^2 + v^2}\right) \quad ; \quad y = \frac{h}{2}\left(v - \frac{v}{u^2 + v^2}\right) \qquad \begin{matrix}(4\text{-a})\\(4\text{-b})\end{matrix}$$

We may note at this time that although u and v are associated with a system of axes, they have the dimension of x/h and y/h, that is, they are dimensionless.

Aside from the defining characteristics, the following are the salient features of the transformation:

(i) Every point in the lower half of the (x,y) plane corresponds to a point inside the unit semi-circle on the upper (u,v) plane.

(ii) The portion of the x-axis from -h to h has been stretched, so to speak, to become the curved boundary of that semi-circle. In particular, the origin in (x,y) has moved to (0,1) in (u,v).

(iii) The contour in Figure 3(a) which has become the u-axis in Figure 3(b) is precisely the one which joins all the points at zero potential in the problem we are considering.

It is this last fact which makes the transformation particularly suited for the problem: one equipotential, U = 0, has been straightened out in the new plane. It will shortly be seen that this is, in fact, the case for all equipotentials.

The transformation we have used being conformal, the following statements hold:

1) If U is harmonic in x and y, it is also harmonic in u and v, i.e. the differential equation (1) becomes, in the transformed plane,

$$\nabla^2 U(u,v) = \frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2} = 0 \qquad (5)$$

2) Boundary conditions of the type we are considering go over unchanged into the new plane, that is, they are

merely expressed in terms of the new variables u and v.

3) The field lines are orthogonal to the equipotentials in both the (x,y) and the (u,v) planes.

Using equations (4), the boundary conditions transform as follows:

$$\left.\begin{aligned} U_{slit} &= 0 \\ U_{y\to-\infty} &= 0 \end{aligned}\right\} \Longrightarrow U(u,0) = 0 \qquad (6\text{-a})$$

$$U_{y\to+\infty} = (E_b - E_a)\, y \Longrightarrow U_{v\to+\infty} = \frac{h(E_b - E_a)}{2}\, v \qquad (6\text{-b})$$

A solution satisfying (5) and boundary conditions (6) is, by inspection,

$$U = \frac{h(E_b - E_a)}{2}\, v \qquad (7)$$

The situation is actually entirely similar to that between two condenser plates perpendicular to the v-axis. We know that in such cases the potential is uniquely defined, so that (7) is the unique solution. We also note that equipotentials, in the transformed plane, are simply lines of constant v.

Having solved the problem in the (u,v) plane, we now go back to the (x,y) plane. For points on axis, this is easily done, for x = 0 implies u = 0, and introducing the latter into (4-b) yields:

$$v^2 - 2\,\frac{y}{h}\,v - 1 = 0$$

from which, if we discard negative values of v,

$$v = \frac{y}{h} + \sqrt{\left(\frac{y}{h}\right)^2 + 1} \qquad \underline{\text{On axis}} \qquad (8)$$

The formula is simple and convenient for points on axis, but our general solution will not depend on it.

Eliminating $\frac{1}{u^2 + v^2}$ from equations (4), we get the interesting relation:

$$\frac{x}{u} + \frac{y}{v} = h \qquad (9)$$

Eliminating u from (9) and (4-b), and using the simplified notation:

$$X \equiv x/h \quad , \qquad Y \equiv y/h \qquad (10)$$

we finally get:

$$v^4 - (4Y)v^3 + (5Y^2 + X^2 - 1)v^2 - 2Y(Y^2 + X^2 - 1)v - Y^2 = 0 \qquad (11)$$

The solution to this equation, that is the value of v corresponding to a given (X,Y), when substituted in (7), would thus give the potential U at that point. The presence of four roots, and the reasons for rejecting all but the positive real root, which is unique, are discussed in Appendix A.

Since finding the roots of a quartic is rather involved, we discuss in the next section a more convenient solution scheme, which consists essentially in refining to any accuracy desired the value of v obtainable from a graph of lines of constant v in the (X,Y) plane.

It is interesting to note that on the unit circle

$$X^2 + Y^2 = 1,$$

equation (11) takes the simple form:

$$v^4 - 4Yv^3 + 4Y^2v^2 - Y^2 = 0$$

Alternate Solution for the Potential at a Point

Assume that, from the results to be presented later, a fairly good approximation to v at a point (X,Y), say v_o, is known. We look into the possibility of refining this value to any accuracy desired.

One scheme would be to apply the Newton-Raphson iteration (Hildebrand, 1956, p. 447) to equation (11). This would converge at all points where the expression in the left-hand side of (11) does not have a zero or near-zero derivative with respect to v. (With an arbitrary v_o this method is, in fact, one of the standard methods of solving (11).)

A somewhat simpler iteration can be derived by going back to the transformation equations themselves, rather than to (11).

From equation (4-b) we may write

$$u = v\sqrt{\frac{1}{v(v-2Y)} - 1} \qquad (12\text{-a})$$

and from equation (9), which combines (4-a) and (4-b):

$$v = \frac{u\ Y}{u-X} \tag{12-b}$$

where, as before, $X \equiv x/h$ and $Y \equiv y/h$.

The first approximation v_o can be inserted into (12-a) to obtain u_o, and the latter used in (12-b) to obtain the improved value of v. The two steps, (12-a) and (12-b), are repeated, using always the latest values of the variables, until v changes by a negligible amount from one iterate to the next.

Will the iteration sometimes diverge, or oscillate? This question is taken up in Appendix B, and it is there concluded that the iteration can be expected to converge at all points of interest except very near the slit edges.

Extension to Slit Systems

The only restrictive condition we have imposed in the treatment of the single slit is boundary condition (II), concerning the fields E_a and E_b. Hence, when studying a system of slits, the method can be applied to each slit individually, provided the following undisturbed field assumption holds:

> Between any two slits, there is a region, enclosing the lens axis, where the electric field, and hence the equipotentials, are relatively undisturbed by the opening or closing of either slit.

For an entry or an exit slit, this condition will usually be satisfied on one side of the slit. Between two slits, it will be seen in Chapter 2 that the validity of the assumption is readily checked using the graphs which will be presented.

CHAPTER 2

COMPUTER SOLUTIONS

Introduction

Our general problem was about a slit having a given potential, a given electric field on each side, and a given slit width. In going from V to U_b in equation (2-b), we generalized the problem to one common to all slits having the same slit width and the same field difference. In going from $U_b \equiv U$ to v in equation (7), we generalized to all slits having the same slit width, and by introducing the variables X and Y in (10), we generalized to all slits. This can be seen from (11), which is free of any particular slit parameter. Thus, this last equation contains the essence of our slit problem, and we need only solve it once for all slits.

The solution was carried out using the Alwac III-E computer at this university. Rather than find v at specified points, it was judged preferable to find the loci of several curves of constant v in the (X,Y) plane.

This can be done by solving (11) either as a cubic in Y, for arbitrary X's:

$$Y^3+\left(\frac{1}{2v}-\frac{5v}{2}\right)Y^2+(2v^2-1+X^2)Y+\frac{v}{2}(1-v^2-X^2) = 0 \qquad (11')$$

or as a quadratic in X, for arbitrary Y's:

$$X^2 = \frac{2vY^3+(1-5v^2)Y^2+2v(2v^2-1)Y+v^2(1-v^2)}{v(v-2Y)} \qquad (11'')$$

Lines of Constant v by the Cubic Method

The first computer program actually solved the cubic. This made it easier to select the arbitrary values of the independent variable which would give points at approximately the desired spacing for plotting, particularly in the upper portion of Figure 2(b). However, for finding additional lines of constant v when a few are known, solving for X is to be preferred, being much simpler to program.

For solving the cubic, Bairstow's method (Hildebrand, 1956, p. 472) was used in a floating-point program incorporating a modified version of Computing Centre Library routine Z-3F, written by A.C.R. Newberry. Only the real roots were output. The program has been deposited at the Computing Centre under the title: "Equipotentials Around a Slit by the Cubic Method".

Much was learned about the difficulties of solving polynomial equations by iterative methods! Some of these difficulties, together with their cures, are discussed in Appendix C.

As an X-Y plotter was available, a plotting program was written which would accept the floating-point results tape of the first program, extract only the desired variables, convert the quantities to fixed-point, round to three digits and prepare a plotting tape for the X-Y plotter. This useful program was later generalized and made available as Computing Centre Library routine U-52.

The results in graphical form appear in Figure 4. Sample numerical results are given in Table 1, and the same results after processing by the plotting program are shown in Table 2. The fact that in Table 1 the output subroutine used was putting out a constant number of digits should not be taken as meaning that the accuracy was a constant number of significant figures.

There was a double real root, of value $Y = v$, which often appeared at $X = 0$. Using the standard theory of cubic equations, one can show that (looking only at the coefficients of the equation) this double root must indeed appear when $X = 0$, in addition to the normal, single real root. However, equation (9), when solved for u, carries with it the condition $Y \neq v$. The condition could have been built into the program, in which case it might have prevented the output of the double root, were it not for the fact that the accumulated round-off error usually made the two roots slightly unequal and slightly different from v. In fact, it sometimes made them slightly complex, in which case they would not be output. When they were output, the amount by which they differed from each other gave some clue as to the accuracy of the process, at least in that region.

Since v is linearly related to the potential $U \equiv U_b$, Figure 4 gives us a clear picture of the potential distribution around any slit which is field-free on one side, the lower side in this case. (The increase in v is not

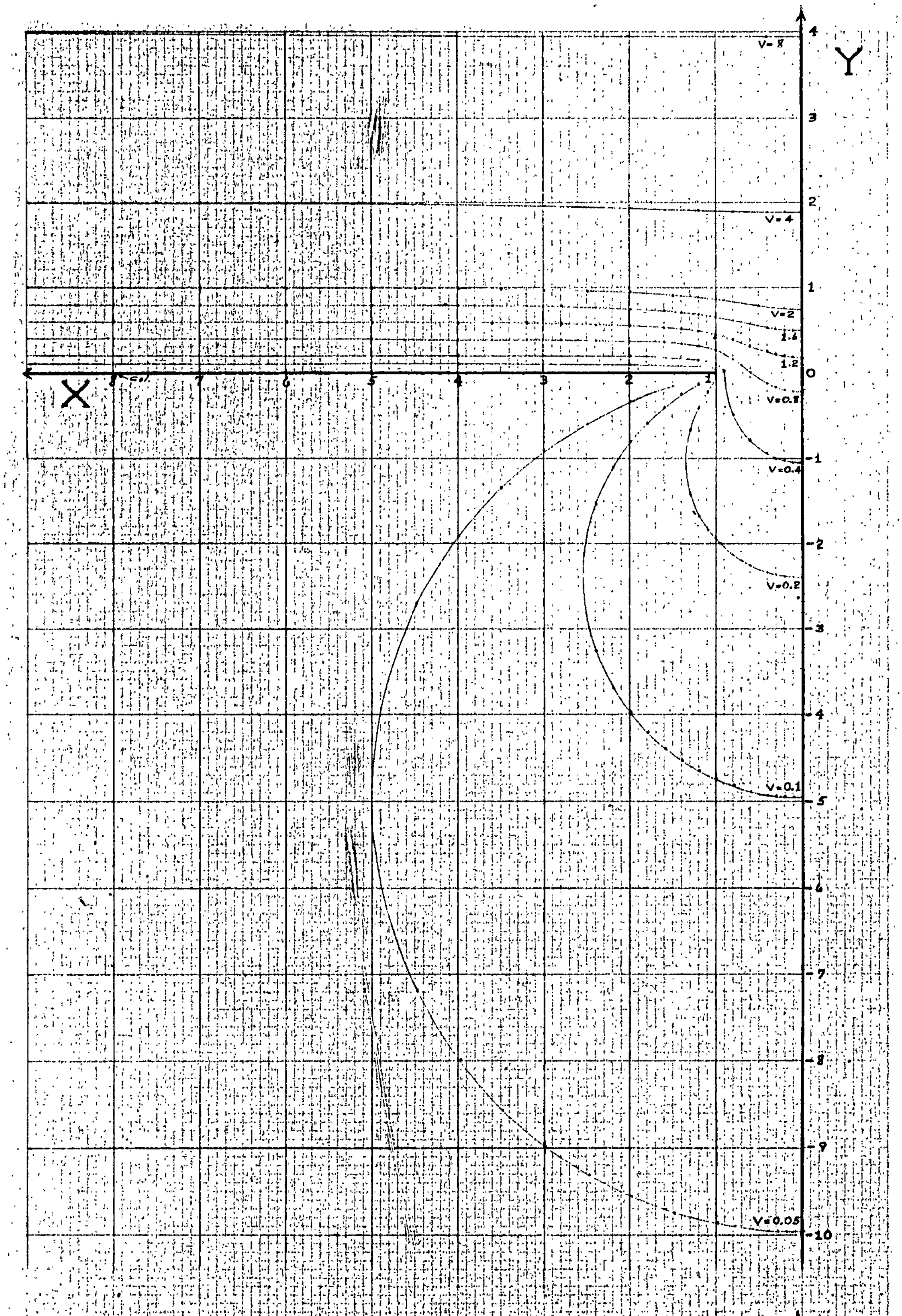


Figure 4 - Lines of Constant v Obtained by the Cubic Method.

Table 1 - Sample Computer Results for the Cubic Method.

v= 4.00,-00,

X	a Y	b Y	c Y
0.000000, 00,	-9.875001,-00, 4.002847,-00,	3.100000, 01, 3.997153,-00,	-3.000001, 01, 1.875001,-00,
2.000000,-01,	-9.875001,-00, 1.876099,-00,	3.104001, 01,	-3.008001, 01,
4.000000,-01,	-9.875001,-00, 1.879296,-00,	3.116000, 01,	-3.032001, 01,
8.000000,-01,	-9.875001,-00, 1.890722,-00,	3.164000, 01,	-3.128001, 01,
1.200000,-00,	-9.875001,-00, 1.905901,-00,	3.244000, 01,	-3.288001, 01,
1.600000,-00,	-9.875001,-00, 1.921512,-00,	3.356000, 01,	-3.512001, 01,
2.000000,-00,	-9.875001,-00, 1.935518,-00,	3.500001, 01,	-3.800001, 01,

v= 2.00,-00,

X	a Y	b Y	c Y
0.000000, 00,	-4.750001,-00, 2.000691,-00,	7.000001,-00, 1.999310,-00,	-3.000001,-00, 7.500005,-01,
5.000000,-01,	-4.750001,-00, 7.862715,-01,	7.250001,-00,	-3.250001,-00,
1.000000,-00,	-4.750001,-00, 8.585593,-01,	8.000001,-00,	-4.000001,-00,
1.500000,-00,	-4.750001,-00, 9.140259,-01,	9.250001,-00,	-5.250001,-00,
2.000000,-00,	-4.750001,-00, 9.456300,-01,	1.100000, 01,	-7.000002,-00,

v= 1.60,-00,

X	a Y	b Y	c Y
0.000000, 00,	-3.687500,-00, 4.875002,-01,	4.120000,-00,	-1.248000,-00,
1.000000,-01,	-3.687500,-00, 4.900160,-01,	4.130000,-00,	-1.256000,-00,
2.000000,-01,	-3.687500,-00, 4.974554,-01,	4.160000,-00,	-1.280000,-00,
3.000000,-01,	-3.687500,-00, 5.094860,-01,	4.210000,-00,	-1.320000,-00,
4.000000,-01,	-3.687500,-00, 5.255378,-01,	4.279999,-00,	-1.376000,-00,
6.000000,-01,	-3.687500,-00, 5.662458,-01,	4.480000,-00,	-1.536000,-00,
8.000000,-01,	-3.687500,-00, 6.111181,-01,	4.760000,-00,	-1.760000,-00,
1.000000,-00,	-3.687500,-00, 6.521030,-01,	5.120000,-00,	-2.048000,-00,
1.200000,-00,	-3.687500,-00, 6.851294,-01,	5.560000,-00,	-2.400000,-00,
1.400000,-00,	-3.687500,-00, 7.100671,-01,	6.080000,-00,	-2.816000,-00,
1.600000,-00,	-3.687500,-00, 7.284992,-01,	6.680000,-00,	-3.295999,-00,
2.000000,-00,	-3.687500,-00, 7.524209,-01,	8.119999,-00,	-4.448000,-00,

v= 1.20,-00,

X	a Y	b Y	c Y
0.000000, 00,	-2.583333,-00, 1.833334,-01,	1.880000,-00,	-2.640000,-01,
1.000000,-01,	-2.583333,-00, 1.873574,-01,	1.890000,-00,	-2.700000,-01,
2.000000,-01,	-2.583333,-00, 1.993387,-01,	1.920000,-00,	-2.880000,-01,
3.000000,-01,	-2.583333,-00, 2.189651,-01,	1.970000,-00,	-3.180000,-01,
4.000000,-01,	-2.583333,-00, 2.455850,-01,	2.040000,-00,	-3.600000,-01,
6.000000,-01,	-2.583333,-00, 3.144243,-01,	2.240000,-00,	-4.800000,-01,
8.000000,-01,	-2.583333,-00, 3.887601,-01,	2.520000,-00,	-6.480000,-01,
1.000000,-00,	-2.583333,-00, 4.500001,-01,	2.880000,-00,	-8.640000,-01,
1.200000,-00,	-2.583333,-00, 4.925296,-01,	3.320000,-00,	-1.128000,-00,
1.400000,-00,	-2.583333,-00, 5.205729,-01,	3.840000,-00,	-1.440000,-00,
1.600000,-00,	-2.583333,-00, 5.393014,-01,	4.440000,-00,	-1.800000,-00,
2.000000,-00,	-2.583333,-00, 5.614562,-01,	5.880000,-00,	-2.664000,-00,

v= 8.00,-01,

X	a Y	b Y	c Y
0.000000, 00,	-1.375000,-00, -2.249999,-01,	2.799999,-01,	1.440000,-01,
1.000000,-01,	-1.375000,-00, -2.190389,-01,	2.899999,-01,	1.400001,-01,
2.000000,-01,	-1.375000,-00, -2.010081,-01,	3.199999,-01,	1.280000,-01,
3.000000,-01,	-1.375000,-00, -1.704859,-01,	3.699999,-01,	1.080000,-01,
4.000000,-01,	-1.375000,-00, -1.268733,-01,	4.399999,-01,	8.000005,-02,
6.000000,-01,	-1.375000,-00, -1.192093,-01,	6.399999,-01,	4.470349,-08,
8.000000,-01,	-1.375000,-00, 1.527293,-01,	9.199998,-01,	-1.119999,-01,
1.000000,-00,	-1.375000,-00, 2.581455,-01,	1.280000,-00,	-2.559999,-01,
1.200000,-00,	-1.375000,-00, 3.109926,-01,	1.720000,-00,	-4.320000,-01,
1.400000,-00,	-1.375000,-00, 3.388174,-01,	2.240000,-00,	-6.400000,-01,
1.600000,-00,	-1.375000,-00, 3.551542,-01,	2.840000,-00,	-8.799998,-01,
2.000000,-00,	-1.375000,-00, 3.727187,-01,	4.280000,-00,	-1.456000,-00,

v= 4.00,-01,

X	a Y	b Y	c Y
0.000000, 00,	2.500000,-01, -1.050000,-00,	-6.800001,-01,	1.680000,-01,
1.000000,-01,	2.500000,-01, -1.044034,-00,	-6.700001,-01,	1.660000,-01,
2.000000,-01,	2.500000,-01, -1.025882,-00,	-6.400001,-01,	1.600000,-01,
3.000000,-01,	2.500000,-01, -9.947247,-01,	-5.900001,-01,	1.500000,-01,
4.000000,-01,	2.500000,-01, -9.489776,-01,	-5.200001,-01,	1.360000,-01,
6.000000,-01,	2.500000,-01, -8.000001,-01,	-3.200001,-01,	9.600000,-02,
8.000000,-01,	2.500000,-01, -4.944831,-01,	-4.000014,-02,	4.000002,-02,
1.000000,-00,	2.500000,-01, 9.114424,-02,	3.200001,-01,	-3.200002,-02,
2.000000,-00,	2.500000,-01, 1.858283,-01,	3.320001,-00,	-6.320001,-01,

Note: a, b and c are the coefficients of the cubic to be solved, normalized so that the coefficient of Y^3 is one.

Following each number is the applicable power of ten.

v= 2.00,-01,

X	a Y	b Y	c Y
0.000000, 00,	2.000000,-00, -2.400000,-00,	-9.200000,-01,	9.600000,-02,
1.000000,-01,	2.000000,-00, -2.396297,-00,	-9.100001,-01,	9.500000,-02,
2.000000,-01,	2.000000,-00, -2.385126,-00,	-8.800000,-01,	9.200000,-02,
3.000000,-01,	2.000000,-00, -2.366297,-00,	-8.300001,-01,	8.700000,-02,
4.000000,-01,	2.000000,-00, -2.339476,-00,	-7.600001,-01,	8.000001,-02,
6.000000,-01,	2.000000,-00, -2.259585,-00,	-5.600000,-01,	6.000000,-02,
8.000000,-01,	2.000000,-00, -2.137967,-00,	-2.800001,-01,	3.200001,-02,
1.000000,-00,	2.000000,-00, 2.886690,-02,	8.000014,-02, -7.076613,-02,	-4.000008,-03, -1.958101,-00,
1.100000,-00,	2.000000,-00, -2.258507,-01,	2.899999,-01, -1.834489,-00,	-2.499999,-02, 6.033969,-02,
1.200000,-00,	2.000000,-00, -4.000000,-01,	5.200000,-01, -1.671780,-00,	-4.800000,-02, 7.177973,-02,
1.300000,-00,	2.000000,-00, -6.555531,-01,	7.700000,-01, -1.422717,-00,	-7.299999,-02, 7.827020,-02,
1.400000,-00,	2.000000,-00, 8.251858,-02,	1.040000,-00,	-1.000000,-01,
1.500000,-00,	2.000000,-00, 8.552385,-02,	1.330000,-00,	-1.290000,-01,
2.000000,-00,	2.000000,-00, 9.284401,-02,	3.080001,-00,	-3.040000,-01,

Table 2 - Results for v = 0.2 after Processing by the Plotting Program.

999,-02,	999,-02,		
000	-240		
010	-240		
020	-239		
030	-237		
040	-234		
060	-226		
080	-214		
100	003	-007	-196
110	-023	-183	006
120	-040	-167	007
130	-066	-142	008
140	008		
150	009		
200	009		

Note: The numerical heading, also output by the program, indicates the maximum value attainable in the columns below. This information serves only to situate the orders of magnitude, and is not fed to the X-Y plotter.

linear from one curve to another. It is logarithmic from $v = 0.05$ to 1.6, with an extra curve for $v = 1.2$, then it is logarithmic again from $v = 2$ to $v = 8$.)

With the slit closed, v is not defined in the lower half of Figure 4, except through a limiting process. By analogy with the potential, let us define $v = 0$ below the closed slit. This allows us to speak of a disturbance in v as well as of a disturbance in the potential.

It was suspected, and then verified numerically, that for image points on either side of the slit (as well as on either side of the axis), the disturbance in v is the same, including sign. It was actually after this fact had been discovered that the full symmetry of the problem, as discussed under "The Potential Disturbance", became apparent.

That the disturbance in v must be positive over the whole plane is obvious from the following consideration: with the opening of the slit, all curves of constant v have slumped downwards; hence every point has, going through itself, a curve of larger v. Using (7), this means that the potential disturbance is of the same sign on both sides of the slit, the disturbance being an increase if the <u>slit constant</u>

$$S = \frac{h}{2} (E_b - E_a) \qquad (13)$$

is positive, and a decrease otherwise.

It now becomes apparent that there is an advantage in having lines of constant v with positive labels. Had we solved the problem of Figure 2(a) for U_a, we would have been led to a transformation where lines of negative v enter the upper part of the x-y plane. However, this would have been compensated by the field difference being reversed in the expression for the slit constant, so that the result would have been the same.

In a configuration where the slit axis is not in the direction shown in our figures, there must still be defined on it a positive direction in terms of which the fields derive their signs. Let the plane of the slit divide the space into a negative region (i.e. a region where points on the axis have negative coordinates with respect to the centre of the slit) and a positive region. Then the equivalent definition of the slit constant is

$$S = \frac{h}{2} \left(E_{\text{negative region}} - E_{\text{positive region}}\right) \qquad (13')$$

Lines of Constant v by the Quadratic Method

From now on, lines of constant v will often be called equipotentials for convenience.

The original objection to using (11'') for finding equipotentials was that, in the upper part of the plane of Figure 2(b), the selection of the arbitrary Y's for calculating a given equipotential is rather critical, it being only known that they must be in the range:

$$\frac{v^2 - 1}{2v} \leqslant Y < \frac{v}{2} \qquad (14)$$

The lower bound is the value of Y where the equipotential crosses the axis, and the upper bound is the value of Y where x, and hence u, tends to infinity. (See equation (4-b).)

Now that some equipotentials are known, this selection is much easier. Furthermore, we have seen that we do not really need a plot of the equipotentials in the upper portion of the Figure, if all we are interested in is the potential disturbance.

A second equipotentials program was written which finds equipotentials or portions of equipotentials by repeatedly solving (11''). It was designed with the potential disturbance in mind, but it may also be used to solve the complete problem of Figure 2(b) by a suitable choice of the input parameters.

The program was written in a simple Fortran language (with a few additions peculiar to Alwac Fortran) in order that it may be processed by a variety of machines. The program is very straightforward, the only complications being introduced by safeguards and conveniences. Details will be found in Appendix D.

Figure 5 shows the result of using the program to plot essentially the complete curves, and thus avoid hand plotting, by specifying small enough intervals for the

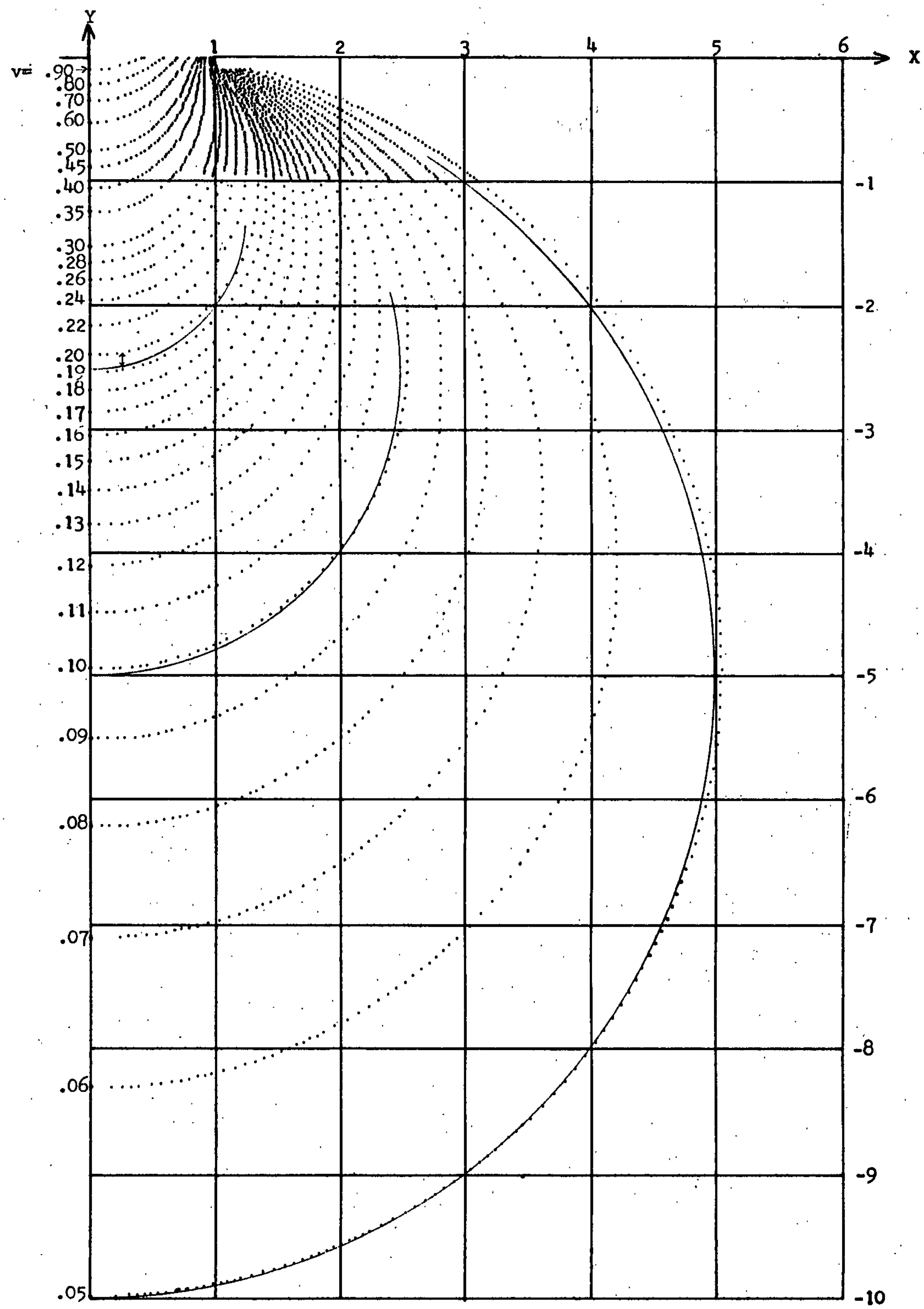


Figure 5 - Lines of Constant v Obtained by the Quadratic Method.

Curves for v < .20 were arbitrarily stopped at Y = -0.01. The solid lines were drawn with a compass.

independent variable Y. The solid curves shown are not part of the results, and should be disregarded for the moment.

The Potential Disturbance Curves as Circles

It was first noticed by Dr. R.D. Russell that, away from the slit, the curves of equal potential disturbance can be approximated fairly accurately with a compass, and that there seems to be a fairly simple law for the position of the centre along the axis.

This can also be seen from the equation of the equipotentials. Dividing (11') by Y and rearranging:

$$(1-\frac{v}{2Y})X^2+Y^2+(\frac{1}{2v}-\frac{5v}{2})Y+\left[2v^2-1+\frac{v}{2Y}(1-v^2)\right] = 0 \qquad (11''')$$

In regions where $|Y|>1$ and $v<1$, $\frac{v}{Y}$ and v^2 are small compared to 1, and v is small compared to $\frac{1}{v}$. Neglecting such terms:

$$X^2 + Y^2 + \frac{1}{2v} Y - 1 = 0 \qquad (15)$$

Comparing this with the equation of a circle of radius R with centre at (0, -C):

$$X^2 + (Y + C)^2 = R^2$$

$$X^2 + Y^2 + 2CY + C^2 - R^2 = 0 \qquad (16)$$

we have:

$$C = \frac{1}{4v} \qquad (17)$$

and

$$R = \sqrt{1 + \frac{1}{16v^2}} \cong \frac{1}{4v} \qquad (18)$$

Using $C = R = \frac{1}{4v}$ gives the solid curves of Figure 5. The agreement, one or more slit width away from the slit, is remarkably good.

Simplifying the Potential Disturbance Calculation

Consider the information available from Figure 5. For any point $(X_1, -Y_1)$ below the slit, we would calculate the potential disturbance as:

$$\triangle V = U_b(X_1, -Y_1) = U(X_1, -Y_1) = S \cdot v(X_1, -Y_1) \qquad (19\text{-a})$$

where S is the slit constant defined above.

For the image point (X_1, Y_1) above the slit, we would make use of (3-a) and calculate

$$\triangle V = U_a(X_1, Y_1) = U_b(X_1, -Y_1) = S \cdot v(X_1, -Y_1) \qquad (19\text{-b})$$

Thus, whether the point is above or below the slit, the graph would actually be consulted without regard to the sign of the coordinates and the result multiplied by S to obtain the potential disturbance.

This being so, one does not change the end result by labeling the axes in terms of absolute values, and this was done in Figure 6. (The change of quadrant is easily accomplished at the plotter.)

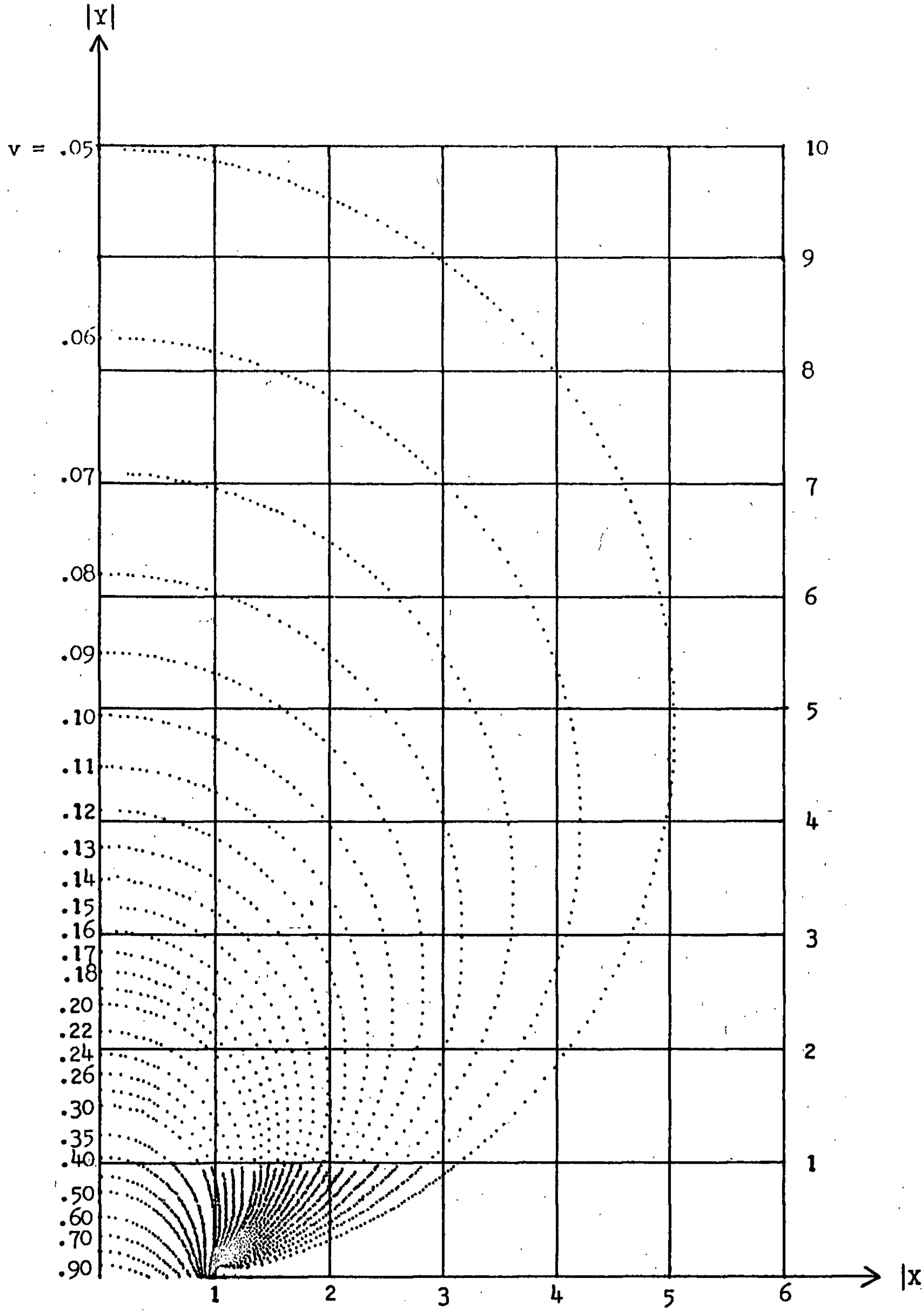


Figure 6 - Graph for Calculating the Potential Disturbance.

$$\triangle V = S \cdot v$$

CONCLUSIONS

A convenient general expression for the potential disturbance due to a slit at a point (X,Y) is

$$\triangle V = S \cdot v\ (|X|, |Y|) \qquad (20)$$

where

S = the slit constant defined by (13'), namely,
$\frac{h}{2}(E_{negative\ region} - E_{positive\ region})$
in which h is the half-slit width;

v = the quantity which remains constant along the curves of Figure 6.

The latter figure can be obtained by repeatedly solving either (11') or (11'') for positive X's and negative Y's, and reversing the sign of Y when plotting. For points away from the slit, the figure can be approximated by circles of radii 1/4v with centres at (0,1/4v).

No matter how v is obtained, its accuracy can be increased, if desired, by using the iteration based on equations (12-a) and (12-b), provided Y is made negative. Alternately, we can re-write the iteration in terms of absolute values:

$$u_o = v_o \sqrt{\frac{1}{v_o(v_o+2\,|Y|)} - 1} \qquad (21\text{-a})$$

$$v = \frac{u_o\ |Y|}{|X| - u_o} \qquad (21\text{-b})$$

For points close to the lens (Y) axis, a convenient starting value v_o can also be obtained from (8), without reference to any graph. In terms of absolute values, that equation becomes:

$$v_o = \sqrt{Y^2 + 1} - |Y| \qquad (22)$$

APPENDIX A

DISCUSSION OF THE ROOTS OF EQUATION (11)

The transformation equations we have used are already cubic in u and v. Equation (4-b), for example, may be re-written:

$$\frac{h}{2} v^3 - y v^2 + \frac{h}{2} (u^2 - 1) v - (u^2 y) = 0$$

For a given u and a given y, this would yield only one real root v, since the mapping is one-to-one. (The root may be a triple real root for certain choices of u and y.) One of the steps of our solution, namely equation (9), gave u as a single-valued function of v, but in substituting into the above equation to obtain (11), we squared it, thus introducing an <u>extraneous</u> root.

One is thus certain of finding two real roots and two complex roots (or possibly a triple real root and one other real root) when solving (11) for v. However, an application of Descarte's rule of signs (Dickson, 1939, p. 77), for all possible sign combinations of the quantities Y, $(5Y^2 + X^2 - 1)$ and $(Y^2 + X^2 - 1)$, shows that there will always be <u>one negative</u> real root, which is obviously the extraneous root and is to be rejected. This leaves only one other real root as the solution.

APPENDIX B

COMMENTS ON THE PROPOSED ITERATION

Given an approximate value v_o of v at (X,Y), equations (12-a) and (12-b) were suggested as the basis of an iteration for increasing the accuracy of v.

From an examination of the equations, the following points can be made:

1) Equation (12-b) will have difficulty defining v near the slit edges, since both the numerator and the denominator tend to zero there. Such a situation could cause loss of significance and oscillation. However, it is to be expected that any method would have a similar difficulty, due to the extreme bunching of the equipotentials at the slit edges, and to the fact that we can only calculate with finite precision.

2) Equation (12-a) could cause divergence as $v \rightarrow 2Y$. From Figure 4, this only happens in the upper portion of the plane when either $Y \rightarrow +\infty$ or $X \rightarrow \pm\infty$. Both of these regions are uninteresting.

3) If v is taken inside the root sign, it will be seen that a similar situation to 1) above exists when both v and Y tend to zero. This only occurs on the surface of the slit, again an uninteresting region.

4) The root in (12-a) will be imaginary if

$$\frac{1}{v(v-2Y)} < 1$$

or, since v and (v-2Y) are always positive, if

$$v(v - 2Y) > 1$$

or

$$v^2 - 2Yv - 1 > 0$$

In theory this is impossible, for equation (4-b) can be re-written:

$$v^2 - 2vY - 1 = u^2 \left(\frac{2Y}{v} - 1\right)$$

Since one always has $v > 2Y$, the right-hand side is necessarily negative, except on the Y-axis where, as we have seen in deriving (8), the left-hand side is exactly zero.

In practice, however, an initial approximation to v which is too far wrong could conceivably make the root imaginary. To prevent this, it is merely necessary to ensure that

$$\frac{v_o^2 - 1}{2v_o} \leqslant Y \tag{23}$$

5) In an electronic computer, the test for imaginary roots takes the form: Is the quantity under the radical negative? It is conceivable that very close to the Y-axis (where the quantity under the radical is zero) round-off error may swing that quantity negative. This possibility should be kept in mind when doing the test, and imaginary roots, if any, replaced with zero without attempting the square root.

APPENDIX C

DIFFICULTIES ENCOUNTERED WITH THE "CUBIC" METHOD

Although Bairstow's method, which was used to solve (11') repeatedly for Y, is acknowledged as having good convergence properties, there was often, at first, some difficulty in converging, which necessitated the operator's intervention during the execution of the program. With experience, this was largely eliminated, as will be seen.

Two types of difficulties arose. In one, the successive iterates would seem to converge normally for a while. Then suddenly the next iterate would jump far away from the previous estimates, and the process would have to start all over again. This phenomenon, when it occurs in connection with the Newton-Raphson method, indicates a zero or near zero derivative of the function near the roots. With Bairstow's method, the meaning is not so graphically clear, the quantity which is zero or near zero in such a case being the Jacobian of two functions defining the next pair of roots. Nevertheless, because the phenomenon does occur, it is useful to impose a limit on the corrections which the program applies to the previous iterate in forming the new iterate. This prevents the latter from jumping too far, but it does not guarantee that the process will not repeat exactly and indefinitely. In the latter case, the remedy, short of modifying the method, is either to increase the tolerance (difference which may be considered negligible between two successive iterates), or give the operator some means of monitoring the computation and of "forcing out" the root at its point of closest approach, whenever necessary. The latter course was resorted to, but roots which were forced out were output in red by the program, as a precaution.

The other type of difficulty which sometimes arose was a small amplitude oscillation about the true solution. This is due to the fact that numbers have to be rounded off during the computations, and may be likened to the inability of a servo system to zero itself because of the presence of electrical noise. However, not all equations of a given degree are equally affected; the influence of round-off depends on the coefficients of the equation. Thus, increasing the tolerance (decreasing the sensitivity) is a poor remedy here, since many equations which would be capable of meeting the smaller tolerance would not be asked to do so.

A remedy which was found very effective is based on the observation that, in most cases of oscillation, the quantity T which is compared with the tolerance alternates in sign, from one iterate to the next, but otherwise remains constant. The assumption was made that, when this happens, the process has reached its maximum accuracy for the particular equation being solved and the round-off of the particular machine being used. The program was modified so that the normal test (Is $T <$ tolerance?) was immediately followed, when the answer was negative, by a second test: Is the present value of T equal in absolute value to the T of the previous iterate? When the answer to this question was positive, the roots were output, along with the value of T (to guard against accidental coincidence, and to indicate by how much the tolerance was not met).

The results have fully justified the assumption made above, in that no point which was output as a result of this second test appeared out of line on the graph. This simple modification to the program cured nearly all cases of oscillation.

APPENDIX D

FORTRAN-TYPE EQUIPOTENTIALS PROGRAM

The program shown in Figure 7 will be accepted by the Alwac III-E Fortran (Altran I) written by Werner Dettwiler of U.B.C.

Given a value of v , the program will generate a sequence of Y's and, by repeatedly solving equation (11''), it will produce curves of constant v in the X-Y plane like the ones shown in Figure 5.

It should be noted that, in the potential disturbance region, Y is negative. To reproduce the graph of Figure 6, the curves should be calculated with negative Y's, and the sign of Y reversed when plotting.

Statements or portions of statements containing lower case letters control the output format in Altran. If they are omitted, the resulting program will work on IBM 1620 Fortran (Sept. 1961), and possibly on other systems as well. Computers like the IBM 704, 709 or 7090 would require a Format Statement for each READ or PUNCH statement, and a format statement number to be inserted between READ and the first comma; similarly for PUNCH.

If Altran is used, the spacing shown between the statement numbers and the statements must be achieved through the use of the tabulator.

For controlling the point density on the curves, the equipotentials are arbitrarily divided into three regions: two regions adjacent to the Y and the X axes, respectively, and a connecting link. The three regions are defined by:

(i) $X < Xcriterion$

(ii) $X \geq Xcriterion$ and $Y < Ycriterion$

(iii) $X \geq Xcriterion$ and $Y \geq Ycriterion$

where Xcriterion and Ycriterion are input parameters. Typical values would be 0.5 and -1 respectively.

In the first region, the regular increase in Y from one point to the next is

$$\triangle Y = StepA$$

In the second region it is StepB, and in the third region, StepC. Typical values for the three step sizes would be 0.02, 0.1 and 0.05 respectively.

Once these conventions have been established through the first five input data, they hold for the duration of the program. However, if at any time the increase <u>in X</u> between the last output and the coordinate just calculated exceeds StepB, the output is suppressed, $\triangle Y$ is cut in half, and the quadratic is solved again with the reduced value of Y. This halving of $\triangle Y$ could conceivably occur several times in a row between two outputs, but after each output the program reverts to the standard conventions above.

StepB is thus intended as the maximum increment in either coordinate from one point to the next.

The last quantity in the first set of data, Xmax, is a convenience only, and is only useful when it is desired to plot over a relatively narrow range of X, less than the maximum spread of the equipotentials needed. After each calculation, the output is suppressed if X>Xmax, but the solution otherwise continues normally. If this feature is not needed, any large number can be input for Xmax.

The program will then ask for sets of three data, each set representing the specification for one equipotential. The first quantity is the value of v. This is followed by the two levels of Y between which the equipotential is to be traced.

For an equipotential to start on the Y-axis, it would at first seem necessary to specify for Yinitial exactly the value $\frac{v^2-1}{2v}$. However, a value truncated to the nearest hundredth, say, would be preferable, since Y could then advance in a sequence of round numbers convenient for plotting. To permit this, the program will calculate the value of Y on axis, then compare it with Yinitial. If the difference is smaller than or equal to 0.1 (indicating that the user effectively wants to start on axis), the program will itself output the coordinates of the point on axis, then proceed to Yinitial.

Finally, if the progression in Y is such that it would overshoot Ymax, the program will calculate the point at Ymax before proceeding to the next equipotential.

Two safeguards have been incorporated:

(a) If Ymax $> \frac{v}{2} - 0.01$ (algebraically), the program replaces Ymax with the latter value.

(b) If Yinitial $< \frac{v^2-1}{2v}$, then Yinitial is made equal to the latter value. (This could happen if in the preparation of the data the quantity $\frac{v^2-1}{2v}$ was rounded rather than truncated, to obtain Yinitial for a curve starting on axis.) Note that in such a case the progression in Y would not be, in general, a sequence of convenient round numbers.

```
        READ, STEPA, XCRIT, STEPB, YCRIT, STEPC, XMAX
10      READ, V, YINIT, YMAX
        .df50
        .df6a
        PUNCH, V, 2cr
        .f746
        .df52
        .df70
        .f746
        .df58
        .f746
        .df6e
        .df50
        PUNCH, 10feeds
        VSQ = V*V
        A = 1.0 - 5.0*VSQ
        B = 2.0*V*(2.0*VSQ - 1.0)
        C = VSQ*(1.0 - VSQ)
        YTOP = V / 2.0  -  0.01
        TESTM = YMAX - YTOP
        IF (TESTM) 25,25,20
20      YMAX = YTOP
25      YAXIS = (VSQ - 1.0) / (2.0*V)
        XAXIS = 0.0
        TESTA = YINIT - YAXIS
        IF (TESTA) 30,50,40
30      YINIT = YAXIS
        GO TO 50
40      TEST = TESTA - 0.1
        IF (TEST) 50,50,60
50      PUNCH, 1cr, YAXIS, 1tab, XAXIS, 2feeds
60      Y = YINIT
65      XLAST = 0.0
        S = -1.0
        GO TO 110
70      S = 0.0
        TESTD = XLAST - XCRIT
        IF (TESTD) 80,90,90
80      DELY = STEPA
        GO TO 100
90      TESTD = Y - YCRIT
        IF (TESTD) 93,95,95
93      DELY = STEPB
        GO TO 100
95      DELY = STEPC
100     Y = Y + DELY
110     X = SQR((((2.0*V*Y + A)*Y + B)*Y + C) / (V*(V - 2.0*Y)))
        IF (S) 160,120,160
```

(Continued Next Page)

Figure 7 - Altran Equipotentials Program for the Quadratic Method.

```
120     XINCR = X - XLAST
        IF (XINCR) 130,140,140
130     XINCR = -XINCR
140     TESTI = XINCR - STEPB
        IF (TESTI) 150,150,145
145     DELY = DELY / 2.0
        Y = Y - DELY
        GO TO 110
150     TESTY = Y - YMAX
        IF (TESTY) 160,155,190
155     S = 1.0
160     TESTX = X - XMAX
        IF (TESTX)  170,170,175
170     PUNCH, 1cr, Y, 1tab, X, 2feeds
175     IF (S) 180,180,210
180     XLAST = X
        GO TO 70
190     Y = YMAX
200     S = 1.0
        GO TO 110
210     .df5c
        PUNCH, 3cr, 50feeds
        GO TO 10
        END
```

Table 3 - Input Data Used for Calculating the Points of Figure 5.

0.02	0.5	0.1	-1.0	0.02	6.0

.90	-0.10	0.0
.80	-0.22	0.0
.70	-0.36	0.0
.60	-0.53	0.0
.50	-0.75	0.0
.45	-0.88	0.0
.40	-1.05	0.0
.35	-1.25	0.0
.30	-1.51	0.0
.28	-1.64	0.0
.26	-1.79	0.0
.24	-1.96	0.0
.22	-2.16	0.0
.20	-2.40	0.0
.19	-2.53	-0.1
.18	-2.68	-0.1
.17	-2.85	-0.1
.16	-3.04	-0.1
.15	-3.25	-0.1
.14	-3.50	-0.1
.13	-3.78	-0.1
.12	-4.10	-0.1
.11	-4.49	-0.1
.10	-4.95	-0.1
.09	-5.51	-0.1
.08	-6.21	-0.1
.07	-7.10	-0.1
.06	-8.30	-0.1
.05	-9.97	-0.1

Note: The meaning of each quantity is explained in Appendix D. The sequence is:
StepA, Xcriterion, StepB, Ycriterion, StepC, Xmax
followed by sets of
v, Yinitial, Ymax.

Table 4 - Sample Computer Results for the Quadratic Method.

v 8.000000e-01

Y	X
-2.2500e-01	0.0000e 00
-2.2000e-01	9.1600e-02
-2.1000e-01	1.5838e-01
-1.9000e-01	2.4113e-01
-1.7000e-01	3.0131e-01
-1.5000e-01	3.5081e-01
-1.3000e-01	3.9374e-01
-1.1000e-01	4.3212e-01
-9.0000e-02	4.6715e-01
-7.0000e-02	4.9962e-01
-5.0000e-02	5.3007e-01
-3.0000e-02	5.5894e-01
-9.9999e-03	5.8656e-01
0.0000e 00	6.0000e-01

v 7.000000e-01

Y	X
-3.6429e-01	0.0000e 00
-3.6000e-01	8.2355e-02
-3.5000e-01	1.5000e-01
-3.3000e-01	2.3128e-01
-3.1000e-01	2.8966e-01
-2.9000e-01	3.3729e-01
-2.7000e-01	3.7827e-01
-2.5000e-01	4.1461e-01
-2.3000e-01	4.4749e-01
-2.1000e-01	4.7765e-01
-1.9000e-01	5.0562e-01
-1.7000e-01	5.3179e-01
-1.5000e-01	5.5646e-01
-1.3000e-01	5.7987e-01
-1.1000e-01	6.0224e-01
-9.0000e-02	6.2374e-01
-7.0000e-02	6.4454e-01
-5.0000e-02	6.6480e-01
-3.0000e-02	6.8468e-01
-9.9998e-03	7.0434e-01
0.0000e 00	7.1414e-01

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