THE ANALYTIC PROPERTIES OF THE SCATTERING AMPLITUDE FOR INTERACTION VIA NONLOCAL POTENTIALS

by

RONALD STUART DAVIS

B.Sc., University of Alberta, 1963

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in the department

of

PHYSICS

We accept this thesis as conforming to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

April 1965

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Physics

The University of British Columbia,
Vancouver 8, Canada

Date 17 April, 1965.

ABSTRACT

The derivation of a partial-wave amplitude for scattering by a separable, nonlocal potential given by McMillan in Nuovo Cimento 29, 4153 (1963) is reviewed. Using his results, an exact expression for the amplitude is derived for a potential of the form $-g\ V(r)V(r')$, where $V(r) = r^{a}\ e^{-\mu r}$, and its analytic properties are studied. The asymptotic behaviour of the amplitude as $|\ell| \to \infty$ (where ℓ is the usual angular-momentum parameter) is derived, and is shown to permit a Sommerfeld-Watson transformation to be performed on the series expression for the total scattering amplitude in terms of the partial-wave amplitudes. By means of this transformation, a double-dispersion relation is derived for the total amplitude in both the complex-energy and complex-cos θ planes. Explicit forms are derived for the weight functions, and the convergence of the integrals involved is studied. In addition to the usual branch cuts along the positive, real energy and cos θ axes, an extra cut along the negative , real energy axis is found which is not present for the local case. Its origin is traced to the fact that the Wronskian of two solutions of the nonlocal radical Schroedinger equation is not necessarily a constant, as it is in the purely local case; and to the conditions necessary to ensure convergence of the extra integral in the nonlocal Schroedinger equation.

ACKNOWLEDGEMENTS

I am indebted to Dr. J. M. McMillan for suggesting the problem and for generous assistance with it. This work was supported by the National Research Council of Canada.

TABLE OF CONTENTS

Chapter 1. Introduction

In this thesis, a detailed investigation is made of the analytic properties of the partial-wave and total amplitude for scattering by a particular class of separable, nonlocal potentials. The advantage of such potentials, as shown by McMillan (1963), is that it is possible to obtain an explicit, closed form for the partial-wave scattering amplitude. For the particular type of potential used here, this amplitude has a relatively simple form, so that it is possible to derive an explicit form for the total scattering amplitude, and to express it in a double dispersion relation.

Separable, nonlocal potentials are not entirely devoid of physical significance. Lomon and McMillan (1963) have shown that the Boundary Condition Model for the nuclear-nuclear interaction can be reformulated in terms of separable potential, and Tabakin (1964), assuming the nuclear-nuclear potential to be separable, has obtained a detailed fit to the available nuclear-nuclear scattering data up to 310 mev.

Cushing (1963) and Mitra (1963), using potentials similar to those used here, have also proven double-dispersion relations. This work goes beyond theirs, however, by presenting explicit, closed forms for the spectral functions involved, and by investigating the convergence of the spectral integrals. It also extends the work of others by applying the method of Bottino (1962) to an investigation of the origin of certain analytic properties of the scattering amplitude for general nonlocal potentials.

Chapter 2 - A closed Form for the Partial-Wave Scattering Amplitude

In this chapter, the integral equations given by McMillan (1963) for the scattering wave function and the partial-wave amplitude for scattering via a separable, nonlocal potential are derived.

The Schroedinger equation with a nonlocal potential is

$$\nabla^2 \psi(\vec{r}) + k^2 \psi(\vec{r}) - \int_{\substack{\text{all real}\\ \vec{r}'}} d\vec{r}' \; V(\vec{r},\vec{r}') \, \psi(\vec{r}') = 0 \tag{2.1}$$

and the corresponding integral equation for the scattering-wave function is

$$\psi(\vec{r}) = \exp(i\,\vec{k}\cdot\vec{r}) + \int_{\substack{\text{all real}\\ \vec{r}' \text{ and } \vec{r}''}} d\vec{r}' \int d\vec{r}'' \; G(\vec{r},\vec{r}') \, V(\vec{r}',\vec{r}'') \, \psi(\vec{r}'') \tag{2.2}$$

where the Green's function G is

$$G(\vec{r},\vec{r}') = \int_{\substack{\text{all real}\\ \vec{k}'}} d\vec{k}' \; \frac{\exp(i\,\vec{k}'\cdot(\vec{r}-\vec{r}'))}{k'^2-k^2+i\epsilon} \; . \tag{2.3}$$

The first term in equation (2.2) represents an incoming plane wave and the second term gives the effect due to the potential. The Green's function given by equation (2.3) contains $+ i\epsilon$ to yield outgoing waves at infinity. Equation (2.2) is a generalization of the integral equation for scattering given in Morse (1953), page 1077, and may be derived similarly. A much more elegant derivation is given by McMillan (1961), chapter 3.1 and appendix I.

Equation (2.2) may be expressed as a radial equation by expanding each of the quantities concerned in terms of spherical harmonics; the appropriate expansions are[1]:

2.2

$$\psi(\vec{r}) = 4 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^{\ell} \psi_{\ell}(r) Y_{\ell m}^{*}(\hat{k}) Y_{\ell m}(\hat{r}),$$

$$V(\vec{r},\vec{r}\,') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} V_{\ell}(r,r') Y_{\ell m}(\hat{r}) Y_{\ell m}^{*}(\hat{r}'), \qquad (2.4)$$

$$\exp(i\,\vec{k}\cdot\vec{r}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} i^{\ell} j_{\ell}(kr) Y_{\ell m}^{*}(\hat{k}) Y_{\ell m}(\hat{r}),$$ -Blatt (1952) Appendix A

which define the new functions $\psi_{\ell}(r)$ and $V_{\ell}(r,r')$ · ($j_{\ell}(kr)$ is the usual spherical Bessel function). When the integrals of equations (2.2) and (2.3) are expressed in spherical polar form by means of the expressions (2.4), the angular portions of the integrations may all be separated and evaluated by means of the orthonormality relation for spherical harmonics (Dicke(1961), page 189):

$$\int_{\text{sphere}} d\hat{r}\; Y_{\ell m}^{*}(\hat{r}) Y_{\ell' m'}(\hat{r}) = \delta_{\ell\ell'}\,\delta_{mm'}\,. \qquad (2.5)$$

Equation (2.3) thus leads to a family of equations of the form

$$\psi_{\ell}(r) = j_{\ell}(kr) + \int_0^{\infty} dr'\, r'^2\, G_{\ell}(r,r') \int_0^{\infty} dr''\, r''^2\, V_{\ell}(r',r'')\, \psi_{\ell}(r'') \qquad (2.6)$$

where

$$G_{\ell}(r,r') = \int_0^{\infty} dk'\, k'^2\, \frac{j_{\ell}(k'r) j_{\ell}(k'r')}{k'^2 - k^2 + i\epsilon} \qquad (2.7)$$

and $\ell = 0, 1, 2, \ldots \infty$ (the "physical" values of ℓ.).

The method of McMillan (1961), appendix I, may be used to show

---

[1] $\hat{a}$ denotes the spherical polar coordinate angles of $\vec{a}$ in some arbitrary coordinate system; a denotes the magnitude of $\vec{a}$.

that

$$G_\ell(r,r') = -\,ik\; j_\ell(kr_<)\; h_\ell^{(1)}\,(kr_>) \tag{2.8}$$

where $\quad r_{\gtrless} = \begin{matrix}\max\\ \min\end{matrix}\,(r,r')$.

An expression for the partial-wave scattering amplitude $a_\ell(k)$ may be derived from the asymptotic form of $\psi_\ell(r)$. Substituting expression (2.8) for the Green's function in equation (2.6) gives

$$\psi_\ell(r) = j_\ell\,(kr) - ik\; h_\ell^{(1)}\,(kr) \int_0^r dr'\; r'^2 \int_0^\infty dr''\; r''^2\; j_\ell(kr')$$

$$\times V_\ell(r',r'')\; \psi_\ell(r'')$$

$$- ik\; j_\ell(kr) \int_r^\infty dr'\; r'^2 \int_0^\infty dr''\; r''^2\; h_\ell^{(1)}\,(kr')$$

$$\times V_\ell(r',r'')\; \psi_\ell(r'') \;\ldots \tag{2.9}$$

As $r \to \infty$, the integral in the last term of (2.9) vanishes and, using

$$h_\ell^{(1)}\,(kr) \to \frac{i^{-\ell-1}}{kr}\; \exp(i\,kr) \qquad \text{as} \qquad r \to \infty, \quad \text{(Morse, (1953))}$$

the second term becomes

$$i^{-\ell}\,\frac{\exp(i\,kr)}{r} \int_0^\infty dr'\, r'^2 \int_0^\infty dr''\; r''^2\; j_\ell(kr')\; V_\ell(r',r'')\; \psi_\ell(r'') \tag{2.10}$$

$$\text{as} \quad r \to \infty,$$

which is part of the outgoing, spherical, scattered wave with angular momentum equal to ℓ. The coefficient of $i^{-\ell}\,\exp(i\,kr)/r$ is

defined to be the partial-wave scattering amplitude, i.e.,

$$a_\ell(k) = \int_0^\infty dr' \; r'^2 \int_0^\infty dr'' \; r''^2 \; j_\ell(kr') \; V_\ell(r',r'') \; \psi_\ell(r'') \; . \qquad (2.11)$$

The equations up to this point may be converted to the corresponding expressions for the more familiar local potential by writing

$$V(\vec{r},\vec{r}\,') = V(r) \; \delta(\vec{r}-\vec{r}\,')$$

or, using equation (2.4),

$$V_\ell(r,r') = \frac{V(r)}{r^2} \; \delta(r-r')$$

since

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(r-r')}{r^2} \; \delta(\hat{r}-\hat{r}) = \frac{\delta(r-r')}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=\ell}^{\ell} Y_{\ell m}^{*}(\hat{r}) \; Y_{\ell m}(\hat{r}\,') .$$

The condition is now imposed that the potential be separable; that is, that it may be written in the form

$$V_\ell(r,r') = - g \; V_\ell(r) \; V_\ell(r') \qquad (2.12)$$

where g is a constant. This form is chosen primarily for mathematical convenience, since under this condition, equation (2.6) can be solved <u>exactly</u>, as is now shown.

Substituting expression (2.12) for $V_\ell(r,r')$ in equation (2.6) yields

$$\psi_\ell(r) = j_\ell(kr) - g\int_0^\infty dr'\, r'^2\, G_\ell(r,r')\, V_\ell(r')\int_0^\infty dr''\, r''^2\, V_\ell(r'') \times\psi_\ell(r'') \ldots \tag{2.13}$$

The double integral thus becomes a product of two single integrals and the kernel of the equation becomes degenerate. Let the second integral now be denoted

$$A = \int_0^\infty dr''\, r''^2\, V_\ell(r'')\, \psi_\ell(r''). \tag{2.14}$$

Substituting the right-hand side of equation (2.13) for $\psi_\ell(r'')$ in (2.14) yields

$$A = \int_0^\infty dr\, r^2\, V_\ell(r)\, j_\ell(kr) - gA\int_0^\infty dr\, r^2\, V_\ell(r)\int_0^\infty dr'\, r'^2\, G_\ell(r,r')\, V_\ell(r'),$$

which may be solved algebraically to yield

$$A = \frac{\int_0^\infty dr\, r^2\, j_\ell(kr)\, V_\ell(r)}{1 + g\int_0^\infty dr\, r^2 \int_0^\infty dr'\, r^2\, G_\ell(r,r')V_\ell(r)V_\ell(r')}\,. \tag{2.15}$$

Substituting this expression for the second integral of equation (2.13) yields

$$\psi_\ell(r) = j_\ell(kr) - \frac{\int_0^\infty dr'\, r'^2 G_\ell(r,r')V_\ell(r')\int_0^\infty dr''\, r''^2 V_\ell(r'')\, j_\ell(kr'')}{1 + g\int_0^\infty dr'\, r'^2 V_\ell(r')\int_0^\infty dr''\, r''^2 G_\ell(r',r'')V_\ell(r'')} \tag{2.16}$$

Now new functions are defined:

$$V_\ell(k) = \int_0^\infty dr\ r^2\ V_\ell(r)\ j_\ell(kr), \tag{2.17}$$

and

$$D_\ell(k) = 1 + g\int_0^\infty dr'r'^2\ V_\ell(r')\int_0^\infty dr''r''^2\ G_\ell(r',r'')\ V_\ell(r'') \tag{2.18}$$

$$= 1 + \frac{2g}{\pi}\int_0^\infty \frac{dq\ q^2\ V_\ell^{\ 2}(q)}{k^2 - q^2} \tag{2.19}$$

Equation (2.16) may then be written

$$\psi_\ell(r) = j_\ell(kr) - \frac{g\ V_\ell(k)}{D_\ell(k)}\int_0^\infty dr'r'^2\ G_\ell(r,r')\ V_\ell(r'). \tag{2.20}$$

Then

$$a_\ell(k) = \int_0^\infty dr'r'^2\int_0^\infty dr''r''^2\ j_\ell(kr')\ V_\ell(r',r'')\ \psi_\ell(r'') \tag{2.11}$$

$$= -g\int_0^\infty dr'r'^2\ j_\ell(kr')\ V_\ell(r')\int_0^\infty dr''r''^2\ V_\ell(r'')\ \psi_\ell(r'') \tag{2.21}$$

$$= g\ \frac{V_\ell^{\ 2}(k)}{D_\ell(k)} \tag{2.22}$$

where (2.12), (2.14), (2.17), and (2.19) have been used.

Defining now

$$N_\ell(k) = g\ V_\ell^{\ 2}(k), \tag{2.23}$$

equations (2.19) and (2.22) may be rewritten

$$D_\ell(k) = 1 - \frac{2}{\pi}\int_0^\infty \frac{dq\ q^2\ N_\ell(q)}{q^2 - k^2} \tag{2.24}$$

and

$$a_\ell(k) = \frac{N_\ell(k)}{D_\ell(k)}\ . \tag{2.25}$$

In appendix I it is shown that the integrals in equations (2.17) and (2.19) converge for physical values of ℓ if

$$V_\ell(r) = \phi\,(r - {}^{5/2}) \qquad \text{as } r \to 0 \qquad \text{(I.3)}$$

$$\text{and} \qquad \text{as } r \to \infty. \qquad \text{(I.4)}$$

Chapter 3: The Analytic Properties of $V_\ell(k)$ and of $V_\ell^2(k)$ for a Particular Potential.

In this chapter, several closed forms for $V_\ell(k)$ are derived in terms of hypergeometric and Legendre functions for the special case

$$V_\ell(r) = r^a \exp(-\mu r), \tag{3.1}$$

where a and μ are real and $\mu > 0$. By means of these formulae, the analytic properties of $V_\ell(k)$ and $V_\ell^2(k)$ are derived. It will be seen that the general properties of $a_\ell(k)$ derived in McMillan (1963) appear explicitly in the present case, this being one of the main motivations of the present work. A particular case of (3.1) has been studied by Cushing (1963), and comparisons with his work will be pointed out in due course.

According to Bateman (1954), Volume II, page 29:

$$\int_0^\infty dx\, x^{\mu-3/2} \exp(-\alpha x)\, J_\nu(xy)(xy)^{1/2}$$

$$= \frac{y^{\nu+1/2}\,\Gamma(\mu+\nu)}{2^\nu \alpha^{\mu+\nu}\,\Gamma(\nu+1)} \quad F\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \nu+1; -\frac{y^2}{\alpha^2}\right) \tag{3.2}$$

$$\equiv \frac{\sqrt{y}\,\Gamma(\mu+\nu)}{(\alpha^2+y^2)^{\mu/2}} \quad P_{\mu-1}^{-\nu}\left(\frac{\alpha}{\sqrt{\alpha^2+y^2}}\right) \tag{3.3}$$

where F is a "hypergeometric function", defined by

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k\,(b)_k}{(c)_k\,k!} \quad z^k \tag{3.4}$$

where $(a)_k = \dfrac{\Gamma(a+k)}{\Gamma(a)} = \prod_{n=0}^{k-1} (a+n)$ (3.5)

(See, for instance, EI 2.1.1 (1) ff.[1]), and where $P^{-\nu}_{\mu-1}$ is an associated Legendre function (EI chapter 3).

The most useful properties of the hypergeometric function in the present work are that

$$F(a,b;\ c;\ 0) = 1, \tag{3.6}$$

and that the series defining it converges and is an analytic function of z if $|z| < 1$. Also, there are several analytic continuations available for it, some of which are used in the following.

Using (3.1), (3.2), and (3.3),

$$V_\ell(k) = \frac{k^\ell \sqrt{\pi}\ \Gamma(a+\ell+3)}{2^{\ell+1}\mu^{a+\ell+3}\Gamma(\ell+3/2)} \quad F\left(\frac{a+\ell+3}{2}, \frac{a+\ell+4}{2}; \ell+3/2; -\frac{k^2}{\mu^2}\right) \tag{3.7}$$

$$= \sqrt{\frac{\pi}{2k}}\ \frac{\Gamma(a+\ell+3)}{(\mu^2+k^2)^{\frac{a+5/2}{2}}}\ P^{-\ell-1/2}_{a+3/2}\left(\frac{\mu}{\sqrt{\mu^2+k^2}}\right) \tag{3.8}$$

[1] The abbreviations EI and EII are used for volumes 1 and 2 respectively of "Higher Transcendental Functions" by the makers of references (Bateman (1954)). The subsequent numerals are equation numbers in those works.

Equation (3.7) is valid only in the domain $|k^2| < \mu^2$ because this is the region in which the series defining the hypergeometric function always converges. However, there are several analytic continuations available for it which we now use.

Using the analytic continuation given by EI 2.10 (4) on the hypergeometric function in equation (3.7) yields

$$V_\ell(k) = \frac{\sqrt{\pi}\,\Gamma(a+\ell+3)}{2^{\ell+1}} \left(\frac{\Gamma(-a-2)\, i^{a+\ell+3}}{\Gamma(\frac{\ell-a}{2})\Gamma(\frac{\ell-a-1}{2}) k^{a+3}} F\left(\frac{a+\ell+3}{2}, \frac{a+\ell+4}{2}; a+3; \frac{k^2+\mu^2}{k^2}\right) \right.$$

$$\left. + \frac{\Gamma(a+2)\, i^{\ell-a} k^a}{\Gamma(\frac{a+\ell+3}{2})\Gamma(\frac{a+\ell+4}{2}) \mu^{a+3}} \left(1+\frac{k^2}{\mu^2}\right)^{-a-2} F\left(\frac{\ell-a}{2}, \frac{-a-\ell-1}{2}; -a-1; \frac{k^2+\mu^2}{k^2}\right) \right) \qquad (3.9)$$

This continuation is valid in the region in which $\left|\frac{k^2+\mu^2}{k^2}\right| < 1$; that is, $\mathrm{Re}(k^2) < -\frac{\mu^2}{2}$. However, according to the restrictions specified with EI 2.10 (4), this region is cut in general along the negative real k^2 axis from $-\mu^2$ to $-\infty$. (It will be seen later that the singularity at $k^2 = -\mu^2$ may sometimes be a pole.)

Similarly, substitution of EI 2.10 (6) into equation (3.7) reveals

$$V_\ell(k) = \frac{k^\ell \sqrt{\pi}\,\Gamma(a+\ell+3)}{2^{\ell+1}\mu^{a+\ell+3}\Gamma(\ell+3/2)} \left(1+\frac{k^2}{\mu^2}\right)^{-\frac{a+\ell+3}{2}} F\left(\frac{a+\ell+3}{2}, \frac{\ell-a-1}{2}; \ell+\frac{3}{2}; \frac{k^2}{k^2+\mu^2}\right). \qquad (3.10)$$

This continuation is valid in the region $\mathrm{Re}(k^2) > -\mu^2/2$. The cut due to the factor $(1 + k^2/\mu^2)^{-\frac{a+\ell+3}{2}}$ is outside this region.

3.4

Another continuation, based on EI 2.10 (2), reveals that

$$V_\ell(k) = \frac{\pi\,\Gamma(a+\ell+3)}{2^{\ell+1}} \left(\frac{k^{-a-3}}{\Gamma(\frac{a+\ell+4}{2})\,\Gamma(\frac{\ell-a}{2})} F(\frac{a+\ell+3}{2}, \frac{a-\ell+2}{2}; \frac{1}{2}; -\frac{\mu^2}{k^2}) \right.$$

$$\left. - \frac{2\,\mu k^{-a-4}}{\Gamma(\frac{a+\ell+3}{2})\,\Gamma(\frac{\ell-a-1}{2})} F(\frac{a+\ell+4}{2}, \frac{a-\ell+3}{2}; \frac{3}{2}; -\frac{\mu^2}{k^2}) \right) \tag{3.11}$$

This is valid and analytic for all $|k^2| > \mu^2$ except along the negative real k^2 axis.

Thus, analytic continuations valid both inside and outside the circle $|k^2| = \mu^2$, and both to the left and to the right of the straight line $\mathrm{Re}(k^2) = -\mu^2/2$ have been found. With their assistance, the properties of $V_\ell(k)$ as a function of complex k shall now be studied.

As $k \to 0$ in equation (3.3), the hypergeometric function approaches 1; and hence,

$$V_\ell(k) \propto k^\ell \quad \text{as} \quad k \to 0. \tag{3.12}$$

Thus, $V_\ell(k)$ has a branch point at $k = 0$ for non-integral ℓ, and a pole there for ℓ a negative integer; $V_\ell^2(k)$ has a branch point at $k^2 = 0$ when 2ℓ is not an integer, and a pole there when 2ℓ is a negative integer. The branch line corresponding to this singularity is generally considered to go from the origin along the positive real axis to infinity.

Equation (3.5) shows a branch point at $k^2 = -\mu^2$ of the form

$$(k^2 + \mu^2)^{-a-2}$$

The branch line for this point is considered to run from $k^2 = -\mu^2$ to $k^2 = -\infty$ along the negative real k^2 axis. The branch point seems to disappear when a is an integer but it is impossible to infer this with certainty from equation (3.5) because the right-hand side assumes the form $\infty - \infty$ for this case. Further information concerning the behaviour of the function at this point may be obtained by means of EI 3.9.2 (19),

$$P_\nu^\mu(z) \to \frac{2^\nu}{\sqrt{\pi}} \frac{\Gamma(\nu + 1/2)}{\Gamma(\frac{1}{2} + \nu - \mu)} z^\nu \qquad \text{as} \quad |z| \to \infty,$$

which is valid for $\mathrm{Re}(\nu) > -1/2$. Substituting this into equation (3.8) yields

$$V_\ell(k) \to \frac{2^{a+1}}{\sqrt{k}} \mu^{-a-5/2} \Gamma(a+2)(1 + \frac{k^2}{\mu^2})^{-a-2}, \tag{3.13}$$

valid for $a > -2$. Thus, as expected, $V_\ell(k)$ has no branch point at $k^2 = -\mu^2$ for integral $a > -2$, but rather a pole of order $a + 2$. Further, EI 3.2 (24) and EI 3.2 (9) conspire to say

$$P_\nu^\mu(z) = \frac{2^\mu}{\Gamma(1-\mu)} \frac{z^{\mu+\nu}}{(z^2-1)^{\mu/2}} F(\frac{-\mu-\nu}{2}, \frac{1-\mu-\nu}{2}; 1-\mu; 1-\frac{1}{z^2}). \tag{3.14}$$

Substituting into equation (3.7) yields

$$V_\ell(k) = k^\ell \sqrt{\frac{\pi}{2}} \frac{\Gamma(a+\ell+3)}{\mu^{a+\ell+3}} (-\frac{\mu^2}{k^2})^{\frac{1}{2}(\ell+1/2)} (1 + \frac{k^2}{\mu^2})^{-\frac{1}{2}(a+5/2)}$$

$$\times P_{-a-5/2}^{-\ell-1/2} ((1+\frac{k^2}{\mu^2})^{-1/2}). \tag{3.15}$$

Again using EI 3.9.2 (19), which is now valid for $a < -2$,

$$V_\ell(k) \to \frac{1}{2^{a+1}} \frac{\Gamma(a+\ell+3)}{\sqrt{k}\,\mu^{a+5/2}} \frac{\Gamma(-a-2)}{\Gamma(-a+\ell-1)} \left(1 + \frac{k^2}{\mu^2}\right)^{-a-2}$$

$$\text{as} \quad k^2 \to -\mu^2, \qquad (3.16)$$

which shows that $V_\ell(k)$ has no branch point at $k^2 = -\mu^2$, but rather a zero of order $-a-2$ for integral $a < -2$. The structure of $V_\ell(k)$ when $a = -2$ remains uninvestigated.

The only remaining singularity in the complex-k^2 plane is at infinity and equation (3.11) shows the behaviour of $V_\ell(k)$ at this point. Since the hypergeometric functions in (3.11) both approach 1 as k^2 approaches infinity; in the general case,

$$V_\ell(k) \propto k^{-a-3} \qquad \text{as} \quad k \to \infty. \qquad (3.17)$$

If either of the quantities $(a + \ell + 4)/2$ or $(\ell - a)/2$ is a non-positive integer here, the first term in (3.11) disappears, and

$$V_\ell(k) \propto k^{-a-4} \qquad \text{as} \quad |k| \to \infty.$$

Thus, the point at infinity will be either a branch point or a pole, depending on the value of a.

In summary, $V_\ell(k)$ has a singularity of the form k^ℓ at $k = 0$, one of the form $(k^2 + \mu^2)^{-a-2}$ at $k^2 = -\mu^2$, and one of the form k^{-a-3} or, in special cases, k^{-a-4}, at $k^2 = \infty$. This agrees with the properties of $a_\ell(k)$ at $k^2 = 0$ and at $k^2 = -\mu^2$ derived in McMillan (1963).

Considering now $V_\ell(k)$ as a function of complex ℓ, equation (3.7) shows two possible sources of singularities in the ℓ plane

for $|k^2| < \mu^2$: the hypergeometric function; which has simple poles for non-positive-integral values of the third parameter, i.e. $\ell = -3/2, -5/2, \ldots$; (see EI, page 57) and the gamma function in the numerator, which has simple poles for non-positive integers, i.e. $\ell = -a-3, -a-4 \ldots$ (see EI, page 2.) The former may be seen to be illusory in $V_\ell(k)$ by means of EI 2.1.3 (16):

$$\frac{F(a,b;c;z)}{\Gamma(c)} = (a)_{1-c}\,(b)_{1-c}\,z^{1-c}\,\frac{F(a+1-c,\ b+1-c;\ 2-c;\ z)}{\Gamma(2-c)}; \qquad (3.18)$$

showing that $F(a,b;c;z)\,/\,\Gamma(c)$ is finite for all c, and thus that

$$\frac{F\left(\frac{a+\ell+3}{2},\ \frac{a+\ell+4}{2};\ \ell+\frac{3}{2};\ -\frac{k^2}{\mu^2}\right)}{\Gamma(\ell+3/2)}$$

is finite for all values of ℓ. Thus the only source of singularities in the complex-ℓ plane in equation (3.7) is the gamma function in the numerator of equation (3.7). It gives a simple pole for

$$a + \ell + 3 = -n$$

$$\text{or} \qquad \ell = -(n + a + 3)$$

where n is a non-negative integer. These same poles are exhibited by all the analytic continuations except when k^2 is at a singularity in the complex-k^2 plane. Each of these first-order poles of $V_\ell(k)$ provides a second-order pole in $V_\ell^{\,2}(k)$. These poles are mentioned in McMillan (1963).

Most of the gamma functions involved in the equations for $V_\ell(k)$ have essential singularities at $\ell = \infty$. The behaviour of $V_\ell(k)$ as $\ell \to \infty$ is thus rather complicated, and shall be treated separately in the next chapter.

Chapter 4. The Asymptotic Behaviour of $a_\ell(k)$ as $|\ell| \to \infty$.

In order to derive a double-dispersion relation for the total scattering amplitude, the Sommerfeld-Watson transformation on the series expression for the total scattering amplitude will be employed, as described by Omnès (1963). In this chapter, restrictions on the potential shall be found such that the behaviour of $a_\ell(k)$ as $|\ell| \to \infty$ permits the Sommerfeld-Watson transformation to be employed.

The series expression for the total scattering amplitude in terms of the partial-wave amplitudes is

$$f(k^2, \cos(\theta)) = \sum_{\ell=0}^{\infty} (2\ell + 1)\, a_\ell(k)\, P_\ell(+\cos(\theta)). \tag{4.1}$$

Using the Sommerfeld-Watson transformation, this may be written

$$\begin{aligned} f(k^2, \cos(\theta)) = \frac{i}{2} \int_{\ell = -1/2-i\infty}^{-1/2+i\infty} d\ell \, \frac{(2\ell+1)\, a_\ell(k)\, P_\ell(-\cos(\theta))}{\sin(\pi\ell)} \\ + \sum_j \frac{(2\alpha_j(k^2) + 1)\, \beta_j(k^2)\, P_{\alpha_j(k^2)}(-\cos(\theta))}{\sin(\pi\, \alpha_j(k^2))} \end{aligned} \tag{4.2}$$

where $\alpha_j(k^2)$ is the site of the jth Regge pole of $a_\ell(k)$ in the half-plane $\mathrm{Re}(\ell) > -1/2$, and $\beta_j(k^2) = \lim_{\ell \to a_j(k^2)} (\ell - \alpha_j(k^2)) \times a_\ell(k)$, the residue of the jth pole. This transformation is valid if $a_\ell(k)$ satisfies the following two conditions, given by Squires (1962):

(1) $a_\ell(k) = \emptyset(\ell^n)$ as $|\ell| \to \infty$, where n is a constant and

$$-\pi/2 < \arg(\ell) < \pi/2; \tag{4.3}$$

(2) $a_\ell(k) = \emptyset(\ell^{-3/2})$ as $\ell \to \pm i\infty$

Conditions shall now be placed on $V(r)$ as given by (3.1) in order that conditions (4.3) may be satisfied. It will be found that the behaviour of $V(r)$ as $r \to 0$ must be severely restricted.

In order to investigate the behaviour of $a_\ell(k)$ as $|\ell| \to \infty$, EI 2.3.2 (16) is used:

$$\left(\frac{z-1}{2}\right)^{-a-\lambda} F(a+\lambda,\ a-c+1+\lambda;\ a-b+1+2\lambda;\ \frac{2}{1-z})$$

$$= \frac{2^{a+b}}{\sqrt{\lambda}} \left[\frac{\Gamma(a-b+1+2\lambda)}{\Gamma(a-c+1+\lambda)\Gamma(c-b+\lambda)}\right] \Gamma(1/2)\ (z-\sqrt{z^2-1})^{a+\lambda}$$

$$\times\ ((1-z+\sqrt{z^2-1})\ \exp(\mp i\pi))^{1/2-c}\ (1+z-\sqrt{z^2-1})^{c-a-b-1/2}$$

$$\times\ (1+\emptyset(\lambda^{-1})) \qquad \text{as} \quad |\lambda| \to \infty, \tag{4.4}$$

where the upper or lower sign in $\exp(\mp i\pi)$ is chosen according as $\mathrm{Im}(z) \gtrless 0$.

The expression in brackets can be tidied up somewhat by using EI 1.18 (4):

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\ (1+\emptyset(z^{-1})) \qquad \text{as} \qquad |z| \to \infty$$

and Stirling's formula (EI 1.18 (2)),

$$\Gamma(z) = e^{-z}\, z^{z-1/2} \sqrt{2\pi}\ (1+\phi(z^{-1})) \qquad \text{as} \qquad |z| \to \infty,$$

to obtain $2^{2\lambda-1/2}\sqrt{\pi\lambda}$.

Setting $\lambda = \ell/2$, a(in equation (4.4)) $= \frac{a+3}{2}$

$$b = \frac{a+2}{2}, \qquad c = a + 3,$$

and $\frac{2}{1-z} = \frac{k^2}{k^2+\mu^2}$ in equation (4.4);

substituting into equation (3.7); and using EI 1.18 (4) again;

$$V_\ell(k) = \frac{\pi\, 2^{a+3/2}}{k^{a+3}}\ \ell^{a+3/2}\ (\sqrt{z^2-1} - z)^{\frac{a+\ell+3}{2}}$$

$$\times\ ((1 - z + \sqrt{z^2-1})\ \exp(\mp i\pi))^{-a-5/2}(1+\phi(\ell^{-1}))$$

as $|\ell| \to \infty$. (4.5)

Now the quantity

$$w(z) = \sqrt{z^2-1} - z \tag{4.6}$$

which appears in (4.5) has the happy property that it has magnitude no greater than 1 for all values of z. This may be seen by considering (4.6) to be a conformal-mapping function, the inverse mapping of which is

$$2z = -(w + 1/w),$$

which maps the unit circle and its interior in the w plane onto the entire z plane (Churchill (1960)). Thus, the magnitude of

$w(z)$, the quantity raised to the power ℓ in (4.5), never exceeds 1. Therefore, $V_\ell(k)$ decreases exponentially as $\mathrm{Re}(\ell)$ approaches $+\infty$ for all values of k.

Since $V_\ell(k)$ decreases uniformly for all values of k, and, in particular, for the values occurring along the path of integration used in $D_\ell(k)$, it follows that the integral therein disappears exponentially. This implies that the denominator approaches 1, and that $a_\ell(k)$ behaves as $V_\ell^2(k)$ for large $\mathrm{Re}(\ell)$. The resulting exponential disappearance of $a_\ell(k)$ amply satisfies Squires' first condition.

It will be noticed that one gets into trouble if $w(z)$ happens to wander into its other branch; since

$$(z + \sqrt{z^2 - 1})(z - \sqrt{z^2 - 1}) = 1,$$

it follows that the other branch of $w(z)$ is always on or outside the unit circle. This branch must be excluded by avoiding the branch line from $z = -1$ to $z = +1$. This corresponds to a branch line for the hypergeometric function in equation (4.4) from

$$\frac{2}{1-z} = 1 \qquad \text{to} \qquad \frac{2}{1-z} = \infty,$$

the usual branch line of the hypergeometric function. It corresponds to a branch line in $a_\ell(k)$ from $k^2 = -\mu^2$ to $k^2 = -\infty$, which branch line has already been recognized; and thus the necessity of avoiding the spurious branch of $w(z)$ creates no new problems.

In determining the behaviour of $V_\ell(k)$ as $\mathrm{Im}(\ell) \to \pm\infty$ with

$Re(\ell)$ held constant, it is unnecessary to consider quantities raised to the power ℓ in equation (4.5) because the magnitude of such a quantity is a function only of the real part of the power. The significant behaviour in this case is thus given by the factor $\ell^{a+3/2}$. As before, $a_\ell(k) \propto V_\ell^{\,2}(k)$ for large $Im(\ell)$ provided $a < -3/2$. Thus, Squires' second requirement implies

$$2a + 3 \leq -3/2$$

or

$$a \leq -9/4.$$

The behaviour when $a \geq -3/2$ violates Squires' 2nd requirement since, in this case, $D_\ell(k)$ grows no faster than $N_\ell(k)$ as $Im(\ell) \to \pm\infty$.

In order that the integral in $D_\ell(k)$ converge, it is necessary to impose the requirement $a > -5/2$. Thus, a has been severely restricted; in order for equation (4.2) to be valid in the potential used here, it must satisfy

$$-5/2 < a \leq -9/4. \tag{4.7}$$

Cushing (1963) found it necessary to use an ℓ-dependent potential in order to be able validly to perform the Sommerfeld-Watson transformation. The present work is not inconsistent with his, however, since he used a potential which is $\emptyset(r^{-3/2})$ for small r, and which is thus not sufficiently singular at the origin. Thus, the

number of parameters required in the potential is reduced from denumerable infinity to three; g, μ, and a.

The validity of (4.5) thus having been secured, it is now used to investigate the analytic properties of $f(k^2, \cos\theta)$ as a function of the complex variables k^2 and $\cos\theta$.

Chapter 5: The Analytic Properties of $f(k^2, \cos\theta)$ in the Complex - $\cos\theta$ Plane.

In this chapter, the analytic properties of the scattering amplitude as a function of the cosine of the scattering angle are studied, and the function is expressed as a dispersion integral. The results given here are consistent with the earlier work by Cushing (1963) and Mitra (1963) but differ in two respects from the earlier works;

(1) The convergence, or otherwise, of the dispersion integrals is determined in the present effort,

(2) An explicit form is given for the weight function in the integrals in the present work.

The analytic properties of the scattering amplitude in the complex-$\cos\theta$ plane may be obtained from equation (4.2) by means of the argument of Bottino (1962), pages 988 to 989.

$$f(k^2, \cos(\theta)) = \frac{i}{2} \int_{\ell=-1/2-i\infty}^{-1/2+i\infty} d\ell \, \frac{(2\ell+1)\, a_\ell(k)\, P_\ell(-\cos(\theta))}{\sin(\pi\ell)} + \text{pole terms.} \tag{4.2}$$

There it is shown that the integral converges for all values of $\cos\theta$ and thence that $f(k^2, \cos\theta)$ is analytic in the domain of $P_\ell(-\cos\theta)$. In general, $P_\nu^\mu(z)$ has two branch lines in the z plane, one along the real axis from -1 to 1 and another from $-\infty$ to -1. There are no other singularities. The branch line from -1 to 1 disappears when μ is an even integer. Thus, $f(k^2, \cos\theta)$ is analytic for

$\cos\theta$ not on the positive real axis from 1 to ∞. Since Squires' conditions (inequations (4.3)) are sufficient for convergence of the integral in (4.2), the same result may be obtained similarly in the present case.

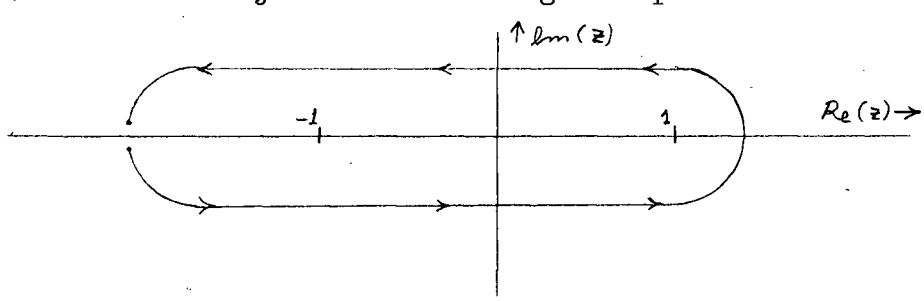
A dispersion relation for $f(k^2, \cos\theta)$ as a function of $\cos\theta$ will now be derived, using the method of Mitra (1963). Using Cauchy's theorem,

$$P_\ell(-z) = \frac{1}{2\pi i}\int_{t=-\infty}^{-1} dt\,\frac{P_\ell(t+i\epsilon)-P_\ell(t-i\epsilon)}{t+z} + \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\varphi=0}^{2\pi} d(Te^{i\varphi})\,\frac{P_\ell(Te^{i\varphi})}{Te^{i\varphi}+z} \tag{5.1}$$

Considering now only the first integral, the discontinuity across the branch line, i.e. $P_\ell(t+i\epsilon) - P_\ell(t-i\epsilon)$, may be obtained by means of EI 3.7(6), which yields

$$P_\ell(z) = \frac{1}{\pi}\int_0^\pi du\,(z+\sqrt{z^2-1}\,\cos u)^\ell, \tag{5.2}$$

valid for all values of the parameters. In this expression, let z be varied continuously from a value just below the negative real axis to a value just above along the path shown below:



Since this path avoids the branch line of $\sqrt{z^2 - 1}$, the phase of $z + \sqrt{z^2 - 1}\ \cos u$ passes through 2π radians, and that of the integrand passes through $2\pi\ell$ radians.

$\therefore$ in $-\infty < t \leq -1$,

$$P_\ell(t + i\epsilon) - P_\ell(t - i\epsilon)$$

$$= \frac{1}{\pi}\int_0^\pi du\ (t + \sqrt{t^2 - 1}\cos u)^\ell\ (1 - \exp(2\pi i\ell))$$

$$= \frac{\exp(i\pi\ell) - \exp(-i\pi\ell)}{\pi}\int_\pi^0 du\ (-t + \sqrt{t^2-1}\cos u)^\ell$$

$$= -2i\ \sin(\pi\ell)\ P_\ell(-t). \tag{5.3}$$

Substituting into (5.1),

$$P_\ell(-z) = -\frac{\sin(\pi\ell)}{\pi}\int_{t=1}^\infty dt\ \frac{P_\ell(t)}{t - z} + \text{Integral around infinite circle.} \tag{5.4}$$

Mitra (1963) has not studied the validity of this representation: for it to be valid, the integrals of (5.4) must converge.

The integral along the branch line shall be considered first. The lower limit of the integral gives no trouble because $P_\ell(t)$ is nonsingular at $t = 1$. As for the infinite tail; since the coefficient of $P_\ell(t)$ is $\emptyset(t^{-1})$ as $t \to \infty$, the integral will converge at the upper limit provided

$$P_\ell(t) = \emptyset(1) \quad \text{as} \quad t \to \infty. \tag{5.5}$$

It is shown in appendix II that

$$P_\ell(t) = \max\,(\phi(t^\ell),\ \phi(t^{-\ell-1})) \qquad \text{as} \qquad |t| \to \infty. \qquad \text{(II.1)}$$

Thus, the integral converges for $-1 < \mathrm{Re}\,(\ell) < 0$.

The neglect of the contribution of the infinite circle in equation (5.1) may now be justified. The contribution is

$$\phi\,(P_\ell(T)) \qquad \text{as} \qquad |T| \to \infty.$$

Hence, the contribution may be neglected if condition (5.5) is satisfied for any arg(t). A review of the derivation of equation (II.1) shows that the latter is valid regardless of arg(t), and thus that the contribution of the infinite circle may be neglected for $-1 < \mathrm{Re}\,(\ell) < 0$.

Thus, the dispersion relation (5.4) is valid for all values of ℓ occurring in equation (4.2); it can also be used in the Regge poleterms provided $-1 < \alpha_j(k^2) < 0$. These terms will however be left in "undispersed" form, hence this restriction need not be made in the following.

Hence, this representation of $P_\ell(-\cos(\theta))$ may be substituted into equation (4.2) to yield

$$f(k^2, \cos\,(\theta)) = -\frac{i}{2\pi} \int_{-1/2-i\infty}^{-1/2+i\infty} d\ell\ (2\ell+1)\ a_\ell(k) \int_{t=1}^{\infty} dt\ \frac{P_\ell(t)}{t-\cos(\theta)}$$

$$+ \text{ Regge pole terms} \qquad (5.6)$$

$$= \int_{t=1}^{\infty} \frac{dt\ \sigma(k,t)}{t-\cos(\theta)} \qquad + \text{ Regge pole terms,}$$

where

$$\sigma(k,t) = -\frac{i}{2\pi} \int_{-1/2-i\infty}^{-1/2+i\infty} d\ell\ (2\ell+1)\ a_\ell(k)\ P_\ell(t), \tag{5.7}$$

which is the desired spectral representation in $\cos(\theta)$.

The convergence of the integral representing $\sigma(k,t)$ will now be investigated. EI 3.6.1 (3) says

$$P_\ell(t) = F(\ell + 1, -\ell\ ;\ 1;\ \tfrac{1-t}{2}).$$

and the behaviour of this function for large ℓ may be investigated by means of EI 3.2.2 (17):

$$F(a + \lambda,\ b - \lambda;\ c;\ \tfrac{1-z}{2})$$

$$= \frac{\Gamma(1-b+\lambda)\Gamma(c)}{\Gamma(1/2)\Gamma(c-b+\lambda)}\ \frac{2^{a+b-1}}{\sqrt{\lambda}}\ (1-z+\sqrt{z^2-1})^{-c+1/2}(1+z+\sqrt{z^2-1})^{c-a-b-1/2}$$

$$\times\ ((z + \sqrt{z^2 - 1})^{\lambda-b}$$

$$+ \exp(\pm\ i\pi(c-1/2))(z-\sqrt{z^2-1})^{-a-\lambda})(1+\emptyset(\lambda^{-1}))$$

as $|\lambda| \to \infty$. Setting $a = 1$, $b = 0$, $c = 1$, and $z = t$;

$$F(\ell+1, -\ell;\ 1;\ \tfrac{1-t}{2}) = P_\ell(t) = \frac{2}{\sqrt{\pi}}\ \frac{(t^2-1)^{-1/4}}{\sqrt{\ell}}\ ((t+\sqrt{t^2-1})^{\ell}$$

$$\pm i\ (t - \sqrt{t^2 - 1})^{-\ell-1}) \times (1 + \emptyset\ (\ell^{-1})).$$

Since $\ell = \emptyset\ (\mathrm{Im}(\ell))$ as $\mathrm{Im}(\ell) \to \pm\infty$ with $\mathrm{Re}(\ell)$ held constant,

$$(2\ell + 1) = \emptyset\ (\ell),$$

$$a_\ell(k) = \emptyset\ (\ell^{2a+3}) \qquad \text{from equation (4.5),}$$

and $P_\ell(t) = \emptyset\ (\ell^{-1/2})$ for $Re(\ell)$ held constant; the integrand is $\emptyset\ (Im(\ell)^{2a+7/2})$ as $Im(\ell) \to \pm\infty$. A sufficient condition for convergence is thus

$$2a + 7/2 < -1,$$

or $$a < -9/4.$$

(This condition becomes necessary for $k^2=0$ and for $t=\infty$. Otherwise, $a < -7/4$ is necessary and sufficient.) This further strengthens the restrictions imposed on a earlier by excluding the case $a = -9/4$.

This dispersion representation reflects the property of the scattering amplitude in the local case, that the integral term decreases as $\cos(\theta) \to \infty$, and that the behaviour of the amplitude is dominated by the rightmost Regge pole term $\alpha_R(k)$ (see, for instance, the paper by Mandelstam in Theoretical Physics (1963), page 413). In both cases, the amplitude behaves like $(\cos\theta)^{\alpha_R(k)}$ as $\cos(\theta) \to \infty$.

Chapter 6. The Analytic Properties of $f(k^2, \cos\theta)$ in the k^2 plane.

In this chapter, a double-dispersion relation is derived for $f(k^2, \cos(\theta))$. It takes the form

$$f(k^2, \cos\theta) = \int_{t=1}^{\infty} \frac{dt}{t-\cos\theta} \int_{q^2=-\mu^2}^{-\infty} \frac{d(q^2)}{k^2-q^2} \rho_-(q^2,t)$$

$$+ \int_{t=1}^{\infty} \frac{dt}{t-\cos\theta} \int_{q^2=0}^{\infty} \frac{d(q^2)}{k^2-q^2} \rho_+(q^2,t) \qquad (6.1)$$

$$+ \text{ Regge pole terms.}$$

As in chapter 5, the form derived is identical to that of Mitra and Cushing. Here however the convergence of the integrals involved is examined and explicit, closed forms for the weight functions are derived.

In equation (5.6), the only function of k^2 other than in the pole terms is $a_\ell(k)$, which appears in the definition of $\sigma(k,t)$, the weight function. $a_\ell(k)$ has two sources of branch lines in the complex $-k^2$ plane: the function $V_\ell^2(k)$, which is cut from $k^2 = -\mu^2$ to $k^2 = -\infty$ and also along the entire positive real axis; and the integral in $D_\ell(k)$, which has a branch line along the positive real axis.

For the discontinuity across the negative real k^2 axis, equation (3.9), which is valid for $\mathrm{Re}(k^2) < -\mu^2/2$, may be used to give

$$a_\ell(\sqrt{k^2 + i\epsilon}) - a_\ell(\sqrt{k^2 - i\epsilon}) = g\,\frac{V_\ell^{\,2}(\sqrt{k^2 + i\epsilon}) - V_\ell^{\,2}(\sqrt{k^2 - i\epsilon})}{D_\ell(k)}$$

$$= g\left[\frac{k^a\sqrt{\pi}\,\Gamma(a+\ell+3)\,i^{\ell-a}}{2^{\ell+1}\,\mu^{2a+3}}\;\frac{\Gamma(a+2)}{\Gamma(\frac{a+\ell+3}{2})\Gamma(\frac{a+\ell+4}{2})}\;\left(1+\frac{k^2}{\mu^2}\right)^{-a-2}\right.$$

$$\left.\times\; F\left(\frac{\ell-a}{2},\ \frac{-a-\ell-1}{2}\ ;\ -a-1\ ;\ \frac{k^2+\mu^2}{k^2}\right)\right]^2\;\frac{1-\exp(-4\pi i(a+2))}{D_\ell(k)}\,. \tag{6.2}$$

The contribution of this singularity to $a_\ell(k)$ is given by

$$\frac{1}{2\pi i}\int_{k'^2=-\mu^2}^{-\infty} d(k'^2)\;\frac{a_\ell(\sqrt{k'^2-i\epsilon}) - a_\ell(\sqrt{k'^2-i\epsilon})}{k^2-k'^2}\,. \tag{6.3}$$

Since $|a_\ell(\sqrt{k'^2+i\epsilon}) - a_\ell(\sqrt{k'^2-i\epsilon})| \le |a_\ell(\sqrt{k'^2+i\epsilon})| + |a_\ell(\sqrt{k'^2-i\epsilon})|$, the integral in (6.3) is $\emptyset(a_\ell(k)k^{-2})$ as $k \to \infty$. Equation (3.17) reveals that

$$V_\ell^{\,2}(k) = \emptyset(k^{-2a-6}) \quad \text{as} \quad k \to \infty; \tag{3.17}$$

and the behaviour of $a_\ell(k)$ is similarly bounded because $D_\ell(k)$ approaches 1 as k^2 approaches $-\infty$. Thus, the integral is $\emptyset(k^{-2a-8})$ for large k, and the infinite tail of integrand (6.3) converges for

$$-2a - 8 < -1,$$

$$\text{or} \qquad a > -7/2.$$

This imposes no new restriction on a.

The singularity of the integrand at $k^2 = -\mu^2$ is

$\emptyset((k^2+\mu^2)^{-2a-4})$. Consequently, this portion of the integral converges for

$$-2a-4 > -1,$$

$$\text{or} \qquad a < -3/2.$$

Again, a is not further restricted.

Equation (3.16), which is valid for $\mathrm{Re}(k^2) > -\frac{\mu^2}{2}$, reveals that the phase of $V_\ell(k)$ changes by $2\pi\ell$ radians as k passes from one side of the positive real axis to the other. Thus, the phase of $V_\ell^2(\sqrt{k^2})$ changes by the same amount as k^2 crosses its positive real axis.

The integral in $D_\ell(k)$ can be written in the form

$$\frac{1}{2\pi i} \int_0^\infty \frac{d(q^2)}{(k^2-q^2)} \, 2g\, i\, q\, V_\ell^2(q), \tag{6.4}$$

showing that there is a branch line along its path of integration, the positive real k^2 axis, with discontinuity

$$D_\ell(\sqrt{k^2+i\epsilon}) - D_\ell(\sqrt{k^2-i\epsilon}) = 2\, g\, i\, k\, V_\ell^2(k) = 2\, i\, k\, N_\ell(k). \tag{6.5}$$

Therefore, along the positive real k^2 axis,

$$\begin{aligned} & a_\ell(\sqrt{k^2+i\epsilon}) - a_\ell(\sqrt{k^2-i\epsilon}) \\ &= \frac{N_\ell(k)}{D_\ell(k)} - \frac{N_\ell(k)\, \exp(2\pi i\, \ell)}{D_\ell(k) + 2\, i\, k\, N_\ell(k)} \\ &= a_\ell(k)\, \left(1 - \frac{D_\ell(k)\, \exp(2\pi i\, \ell)}{D_\ell(k) + 2\, i\, k\, N_\ell(k)}\right) \\ &= a_\ell(k)\, \left(1 - \frac{\exp(2\pi i\, \ell)}{1 + 2\, i\, k\, a_\ell(k)}\right). \end{aligned} \tag{6.6}$$

The contribution of this branch line to $a_\ell(k)$ is thus given by

$$\frac{1}{2\pi i}\int_0^\infty \frac{d(q^2)}{k^2-q^2}\; a_\ell(q)\;\left(1-\frac{\exp(2\pi i\ell)}{1+2\,i\,k\,a_\ell(q)}\right), \tag{6.7}$$

the convergence of which must now be established.

In order to establish the behaviour of $a_\ell(k)$ as $k^2 \to +\infty$, it is necessary to determine that of $D_\ell(k)$. The behaviour of the latter is not entirely obvious since, in the limit under consideration, k^2 goes to infinity along the contour of integration used in defining $D_\ell(k)$. However, from equation (3.17) and the work of Lanz (1964), it follows that

$$D_\ell(k) = 1 + \emptyset\,(k^{-1}) \qquad \text{as} \qquad k \to +\infty. \tag{6.8}$$

Hence, the behaviour of $a_\ell(k)$ is identical to that of $V_\ell^{\,2}(k)$ for large k^2; that is, it approaches $\emptyset((k^2)^{-a-3})$. Since this is a decreasing function, the integrand in (6.7) also has the same behaviour as $V_\ell^{\,2}(k)$ for large k. Hence, the integrand is $\emptyset((k^2)^{-a-4})$; the integral converges for

$$-a-4 < -1 \quad \text{or}$$
$$a > -3;$$

and again no new restrictions on a are necessary.

As was shown in the derivation of the $\cos(\theta)$ - spectral representation, the condition for convergence of the integral in

(6.7) is sufficient for disappearance of the contribution of the infinite circle to the Cauchy integral for $a_\ell(k)$.

The result is that the cut structure of $a_\ell(k)$ may be validly represented by

$$a_\ell(k) = \frac{1}{2\pi i}\left[\int_{k'^2=-\mu^2}^{-\infty} \frac{d(k'^2)}{k^2-k'^2}\,\gamma_-(k'^2) + \int_{k'^2=0}^{\infty} \frac{d(k'^2)}{k^2-k'^2}\,\gamma_+(k'^2)\right]$$

where γ_- is given by expression (6.2) and γ_+ by (6.6).

Substituting into equation (5.6) and interchanging the order of integration over ℓ and k^2 yeilds the result

$$f(k^2, \cos(\theta)) = \int_{t=1}^{\infty} \frac{dt}{t-\cos(\theta)} \int_{q^2=-\mu^2}^{-\infty} \frac{d(q^2)}{k^2-q^2}\,\varrho_-(q^2,t)$$

$$+ \int_{t=1}^{\infty} \frac{dt}{t-\cos(\theta)} \int_{q^2=0}^{\infty} \frac{d(q^2)}{k^2-q^2}\,\varrho_+(q^2,t) + \text{Regge pole terms} \qquad (6.9)$$

where

$$\varrho_-(k^2,t) = -\frac{1}{4\pi^2}\int_{-1/2-i\infty}^{-1/2+i\infty} d\ell\,(2\ell+1)\,P_\ell(t)\left(a_\ell(\sqrt{k'^2+i\epsilon})-a_\ell(\sqrt{k'^2-i\epsilon})\right)$$

$$= -\frac{g}{4\pi^2}\int_{-1/2-i\infty}^{-1/2+i\infty} d\ell\,(2\ell+1)\,P_\ell(t)\,\frac{1-\exp(-4\pi i(a+2))}{D_\ell(k)}$$

$$\times\left[\frac{k^a\sqrt{\pi}\,\Gamma(a+\ell+3)}{2^{\ell+1}\,\mu^{2a+3}}\,\frac{i^{\ell-a}\,\Gamma(a+2)}{\Gamma(\frac{a+\ell+3}{2})\Gamma(\frac{a+\ell+4}{2})}\left(1+\frac{k^2}{\mu^2}\right)^{-a-2}\right.$$

$$\left.\times\,F\left(\frac{\ell-a}{2},\ \frac{-a-\ell-1}{2};\ -a-1;\ \frac{k^2+\mu^2}{k^2}\right)\right]^2 \qquad (6.10)$$

and

$$\varrho_+(k^2,t) = -\frac{1}{4\pi^2}\int_{-1/2-i\infty}^{-1/2+i\infty} d\ell\ (2\ell+1)\ P_\ell(t)(a_\ell(k+i\epsilon)-a_\ell(k-i\epsilon))$$

$$= -\frac{1}{4\pi^2}\int_{-1/2-i\infty}^{-1/2+i\infty} d\ell\ (2\ell+1)\ P_\ell(t)\ a_\ell(k)(1-\frac{\exp(2\pi i\ell)}{1+2ik\ a_\ell(k)})\ . \tag{6.11}$$

By the same method as was used in discussing formula (6.3) it may be shown that the integrals in equations (6.10) and (6.11) are both

$$\emptyset\ (\ (2\ell+1)\ P_\ell(t)\ a_\ell(k)\ )\qquad \text{as}\qquad \ell \to \pm i\ \infty.$$

The integral of this latter quantity has been shown to converge for $a < -9/4$ in the discussion following equation (5.7). The validity of equation (6.9) is thus proven.

Thus, by omitting pole terms from the dispersion integral, the need for any subtractions is eliminated.

The question arises of whether or not the left-hand cuts of the pole terms may cancel that of the dispersion integral in other than special cases. This possibility may be obviated by means of the considerations mentioned at the end of chapter 5. The contribution to the amplitude decreases, and that of most pole terms increases, with increasing $Re(\cos(\theta))$. Thus, if the contribution to the cut from the integral happens to cancel that from the poles for one

particular value of $\cos\theta$, the latter will yet dominate the former for a sufficiently large value of $\cos\theta$.

Chapter 7. The Origin of the "Extra" Branch Cut in the k Plane.

In this chapter, the investigation by Bottino (1962) of the analytic properties of the scattering amplitude as a function of complex k for fixed angle shall be followed in a more general fashion, suitable for both local and nonlocal potential scattering. It turns out that in the nonlocal case the scattering amplitude has a cut not only along the negative imaginary k axis, <u>but also along the positive imaginary axis</u>; that is, that it has a cut on both sheets of the complex $-k^2$ plane. The reason for this difference is shown in detail below.

The radial Schroedinger equation says

$$\psi''(z) + \left(\frac{\ell(\ell+1)}{z^2} + k^2\right) \psi(z) + \int_0^\infty dz'\, z'^2\, V_\ell(z,z')\, \psi(z') = 0.$$

For large z, the centrifugal term disappears, and if the potential $V(z,z')$ disappears faster than $\psi(z)$ for large z, then the equation approaches the form

$$\psi''(z) + k^2\, \psi(z) = 0.$$

Thus, for large z, the general solution approaches the form

$$A \exp(i\, k\, z) + B \exp(-\, i\, k\, z).$$

For a potential satisfying

$$V(z,z') = \mathscr{o}(\exp(-\,\mu\, z))$$

for large z, the solution will thus always approach the above form for $|\mathrm{Im}\,(k)| < \mu$.

Four solutions of the radial Schroedinger equation may be defined (for convenience, the symbol $\lambda = \ell + 1/2$ will be used in the following):

$$\varphi\,(\lambda,\ k,\ z),\ \varphi\,(-\lambda,\ k,\ z),\ f\,(\lambda,\ k,\ z),\ f\,(\lambda,\ ke^{-i\pi},\ z);$$

or, for the sake of brevity,

$$\varphi_+,\ \varphi_-,\ f_+,\ f_-\ ;$$

where

$$\left.\begin{aligned} \varphi(\lambda,\ k,\ z) &\sim z^{\lambda+1/2} \quad \text{as} \quad z \to 0, \\ f\,(\lambda,\ k,\ z) &\sim \exp(-i\,k\,z) \quad \text{as} \quad z \to \infty. \end{aligned}\right\} \qquad (7.1)$$

The f's are known as the "Jost solutions".

In accordance with Bottino (1962), the Jost fuctions $f(\pm\lambda,\pm k)$ are defined by the relation:

$$\varphi(\pm\lambda,k,z) \sim \frac{f(\pm\lambda,k)\exp(-ikz)}{2\,i\,k} + \frac{f(\pm\lambda,-k)\exp(ikz)}{2\,i\,k} \quad \text{as } z \to \infty. \qquad (7.2)$$

This is equivalent to equation (2.9) of Bottino (1962).

It shall prove useful to obtain an expression for the quantity

$$f(\lambda,k)\ f(-\lambda,-k) - f(-\lambda,k)\ f(\lambda,-k)$$

in terms of the four solutions defined in (7.1). In Bottino (1962) it is shown that, for the local case,

$$f(\lambda,k)\ f(-\lambda,-k) - f(-\lambda,k)\ f(\lambda,-k) = 4\,i\,\lambda\,k. \qquad (7.3)$$

However, it will be seen that, in the nonlocal case, this important identity does not hold.

Now let

$$\left.\begin{aligned} \varphi_+ &= A\, f_+ + B\, f_- \,, \\ \varphi_- &= C\, f_+ + D\, f_- \,. \end{aligned}\right\} \tag{7.4}$$

Since $f_\pm \sim \exp(\mp i\, k\, z)$ as $z \to \infty$, the quantities A,B,C, and D become Jost functions as $z \to \infty$, provided the solutions φ and f exist. In the local case, the coefficients A,B,C, and D are constants, since a linear relationship exists among any three pairwise linearly independent solutions of the local equation; but with a nonlocal potential this may not be the case.

The Wronskian of two functions, say $g(z)$ and $h(z)$, is defined by

$$W(g(z),\, h(z)) = g(z)\, \frac{dh(z)}{dz} - h(z)\, \frac{dg(z)}{dz} \,.$$

Now let $g(z)$ and $h(z)$ be two solutions of the equation;

$$g''(z) + \int_0^\infty dz'\, R(z,z')\, g(z') = 0$$

$$h''(z) + \int_0^\infty dz'\, R(z,z')\, h(z') = 0.$$

Multiplying the first equation by $h(z)$ and the second by $g(z)$ and sub tracting yields

$$\frac{d}{dz}\, W(g,h) = \int_0^\infty dz'\, R(z,z')\, [g(z)\, h(z') - h(z)\, g(z')].$$

In general, therefore, the Wronskian is a function of z. If, however,

$$R(z,z') = R(z)\ \delta(z-z'),$$

as in the local case, then the Wronskian is a constant. Thus appears an important difference between the properties of the solutions of the local and nonlocal Schroedinger equations.

From equations (7.4),

$$\begin{aligned}
W(f_+,\varphi_+) &= W(f_+,\ Af_+ + Bf_-) = BW(f_+,f_-),\\
W(f_+,\varphi_-) &= DW(f_+,f_-),\\
W(f_-,\varphi_+) &= -\ AW(f_+,f_-),\\
W(f_-,\varphi_-) &= -\ CW(f_+,f_-).
\end{aligned}$$

$$\left.\begin{aligned}
\therefore\ \varphi_+ &= -\ \frac{W(f_-,\varphi_+)}{W(f_+,f_-)}\ f_+ + \frac{W(f_+,\varphi_+)}{W(f_+,f_-)}\ f_-\ ,\\
\varphi_- &= -\ \frac{W(f_-,\varphi_-)}{W(f_+,f_-)}\ f_+ + \frac{W(f_+,\varphi_-)}{W(f_+,f_-)}\ f_-\ .
\end{aligned}\right\}\qquad (7.5)$$

Taking the limit $z \to \infty$ in equations (7.5) and using equations (7.1) and (7.2) reveals that

$$\left.\begin{aligned}
\frac{f(\lambda,\pm k)}{2\ i\ k} &= \lim_{z\to\infty} \pm\ \frac{W(f_\pm,\varphi_+)}{W(f_+,f_-)}\\
\text{and}\qquad & \\
\frac{f(-\lambda,\pm k)}{2\ i\ k} &= \lim_{z\to\infty} \pm\ \frac{W(f_\pm,\varphi_-)}{W(f_+,f_-)}
\end{aligned}\right\}\qquad (7.6)$$

The denominators of these quantities may be computed with the aid of equation (7.1):

$$\lim_{z\to\infty} W(f_+,f_-) = W(\exp(-ikz),\ \exp(ikz)) = 2ik. \tag{7.7}$$

Thus,

$$\left.\begin{aligned} f(\lambda,\pm k) &= \pm \lim_{z\to\infty} W(f_\pm,\varphi_+) \\ &\text{and} \\ f(-\lambda,\pm k) &= \pm \lim_{z\to\infty} W(f_\pm,\varphi_-). \end{aligned}\right\} \tag{7.8}$$

This shows that the Jost functions exist provided f and φ exist. One sees also that in the local case, these expressions are identical with the definition (2.1) of Bottino (1962).

Substituting equations (7.5) into the expression $W(\varphi_+,\varphi_-)$ yields

$$\frac{W(\varphi_+,\varphi_-)}{W(f_+,f_-)} = \frac{W(f_+,\varphi_+)}{W(f_+,f_-)}\ \frac{W(f_-,\varphi_-)}{W(f_+,f_-)} - \frac{W(f_-,\varphi_+)}{W(f_+,f_-)}\ \frac{W(f_+,\varphi_-)}{W(f_+,f_-)}$$

and thus, using equations (7.6),

$$\frac{f(\lambda,k)\ f(-\lambda,-k) - f(-\lambda,k)\ f(\lambda,-k)}{(2\ i\ k)^2} = \lim_{z\to\infty} \frac{W(\varphi_+,\varphi_-)}{W(f_+,f_-)}$$

$$= \frac{1}{2ik} \lim_{z\to\infty} W(\varphi_+,\varphi_-), \tag{7.9}$$

using also (7.7).

In the local case, $W(\varphi_+,\varphi_-)$ is constant, and then

$$\lim_{z\to\infty} W(\varphi_+,\varphi_-) = \lim_{z\to 0} W(\varphi_+,\varphi_-) = W(z^{\lambda+1/2},\ z^{-\lambda+1/2}) = -\ 2\lambda \tag{7.10}$$

so that

$$f(\lambda,k)\ f(-\lambda,-k) - f(-\lambda,k)\ f(\lambda,-k) = 4\ i\ \lambda\ k, \tag{7.11}$$

which is equation (2.5) of Bottino (1962). Thus, in the local case, the quantity

$$f(\lambda,k)\ f(-\lambda,-k) - f(-\lambda,k)\ f(\lambda,-k)$$

is analytic over the entire λ and k planes, even when the functions $\varphi_{\pm}$ and $f_{\pm}$ are not.

Using the usual relation for the S matrix, which may be derived in the nonlocal case in exactly the same manner as in the local case,

$$S(\lambda,k) = \frac{f(\lambda,k)}{f(\lambda,-k)} \quad \exp\ (i\pi\ (\lambda - 1/2)),$$

assuming the Sommerfeld-Watson transformation may be performed. (The validity of this step in the local Yukawa case is shown by Bottino (1962), in a particular nonlocal case in earlier chapters) yields

$$f(E,\cos(\theta)) = \frac{1}{2k} \int_{-i\infty}^{i\infty} d\lambda\ \frac{\lambda\ P_{\lambda-1/2}(\cos(\theta))}{\cos(\pi\lambda)} \quad \exp(-i\pi(\lambda+1/2))(S(\lambda,k)-1)$$

$$+ \text{ pole terms}$$

$$= \frac{1}{2k} \int_{-i\infty}^{i\infty} d\lambda\ \frac{\lambda\ P_{\lambda-1/2}(\cos(\theta))}{\cos(\pi\lambda)} \left[- \frac{f(\lambda,k)}{f(\lambda,-k)} - \sin(\ \lambda)\right.$$

$$\left. + i\ \cos\ (\pi\lambda)\right] + \text{ pole terms.} \tag{7.12}$$

Now only that component of the integrand which is an even function of

λ contributes to $f(E, \cos(\theta))$.

According to EI 3.3.1 (1), $P_{\lambda-1/2}(\cos(\theta))$ is an even function of λ; further, $\cos(\pi\lambda)$ is famous for its evenness, and λ is the very paragon of oddity. Therefore,

$$\frac{\lambda P_{\lambda-1/2}(\cos(\theta))}{\cos(\pi\lambda)}$$

is odd, and only the odd part of

$$\left(-\frac{f(\lambda,k)}{f(\lambda,-k)} - \sin(\pi\lambda) + i\cos(\pi\lambda)\right)$$

contributes. In taking the odd part, $i\cos(\pi\lambda)$ drops out, $\sin(\pi\lambda)$ is unaltered, and $-f(\lambda,k)/f(\lambda,-k)$ becomes, using equation (7.9),

$$-\frac{f(\lambda,k)\,f(-\lambda,-k) - f(-\lambda,k)\,f(\lambda,-k)}{2\,f(-\lambda,-k)\,f(\lambda,-k)} = \frac{\lim_{z\to\infty} W(\varphi_+,\varphi_-)}{4ik\,f(-\lambda,-k)f(\lambda,-k)}. \quad (7.13)$$

Thus,

$$f(E, \cos(\theta)) = -\frac{1}{2k}\int_{-i\infty}^{i\infty} d\lambda\, \frac{\lambda\, P_{\lambda-1/2}(\cos(\theta))}{\cos(\pi\lambda)} \left(\sin(\pi\lambda) + \frac{\lim_{z\to\infty} W(\varphi_+,\varphi_-)}{4ik\,f(-\lambda,-k)f(\lambda,-k)}\right) + \text{pole terms.} \quad (7.14)$$

(The pole terms are inexplicably missing from equation (6.16) of Bottino (1962)).

In the local case, equation (7.10) may be used to show that $\lim_{z\to\infty} W(\varphi_+,\varphi_-) = -2\lambda$, and thus that the only source of singularities

in the k plane from the integral in (7.7) are the Jost functions in the denominator. In the nonlocal case, however, the φ functions in the numerator may contribute additional singularities to the scattering amplitude.

From equations (7.8), it may be seen that the domain of analyticity of $f(\pm\lambda, -k)$ is the intersection of the domains of f_- and $\varphi_\pm$ in the λ and k planes.

Integral equations shall now be set up for the solutions φ and f in the same manner as in Appendix I of the Bottino effort, and shall be used to show analyticity of the solutions in a restricted portion of the λ and k planes.

Each of the integral equations shall be written in the form

$$g(\lambda, k, z) = g_0(\lambda, k, z) + \int_0^\infty dz' \, L(\lambda, k, z, z') \, g(\lambda, k, z'). \quad (7.15)$$

This is identical in form to equation (II.1) in Bottino (1962). Thus, one may apply to the present case also their subsequent argument to the effect that, for the function $g(\lambda, k, z)$ to be analytic in a given region, it is sufficient for the free eigenfunctions of the original equation, i.e. g_0, to be analytic in the region, and for an upper bound to exist for the integral

$$\int_0^\infty dz' \left| L(\lambda, k, z, z') \frac{M(\lambda, k, z')}{M(\lambda, k, z)} \right| \quad (7.16)$$

where $|g_0(\lambda, k, z)| \leq M(\lambda, k, z)$ for all λ, k, z in the region.

The integral equation for $\psi_\ell(r)$ derived by McMillan(1963) for the nonlocal case is

$$\psi_\ell(z) = j_\ell(kz) - ik \int_0^\infty dz''\, z''^2 \int_0^\infty dz'\, z'^2\, j_\ell(kz''_<) h_\ell^{(1)}(kz''_>) \, V_\ell(z'',z')\, \psi_\ell(z')$$

where $z''_< = \min(z, z'')$,

$z''_> = \max(z, z'')$.

Thus, for small z,

$$\psi_\ell(z) = j_\ell(kz)\Big(1 - ik \int_0^\infty dz''\, z''^2 \int_0^\infty dz'\, z'^2\, h_\ell^{(1)}(kz'')\, V_\ell(z'',z')\, \psi_\ell(z')\Big) \tag{7.17}$$

and since $j_\ell(z) \sim \frac{\pi^{1/2}}{2\Gamma(\ell+3/2)} \left(\frac{z}{2}\right)^\ell$ as $z \to 0$,

$$\psi_\ell(z) \sim z^\ell \frac{k^\ell}{2^{\ell+1}} \frac{\pi^{1/2}}{\Gamma(\ell+3/2)} \Big(1 - ik \int_0^\infty dz''\, z''^2 \int_0^\infty dz'\, z'^2\, h_\ell^{(1)}(kz'')\, V_\ell(z'',z')\, \psi_\ell(z'')\Big) \text{ as } z \to 0.$$

The coefficient of z^ℓ is not a function of z. Thus, solution $\psi_\ell(z)$ of equation (2.20) may be identified with $\varphi(\lambda, k, z)$ up to a normalization constant, which is immaterial for present purposes.

For the purpose of deriving the analytic properties of φ in the nonlocal case, formula (7.16) becomes

$$|k| \int_0^\infty dz' \int_0^\infty dz'' \left| z''^2\, z'^2\, j_\ell(kz''_<)\, h_\ell^{(1)}(kz''_>)\, V_\ell(z'',z') \right| \times \frac{M(\ell, k, z')}{M(\ell, k, z)} \tag{7.18}$$

where $M(\ell, k, z) \geq |j_\ell(kz)|$.

$j_\ell(kz)$ has singularities only for $kz = 0$ and $kz = \infty$ (EII page 4), and the potential will be assumed to be equally well behaved in both space variables; hence, the convergence of the integral need be investigated only for the end points of the integrations. Any local component of the potential, which will manifest itself as a contribution to the nonlocal potential of the form $V(z'')\delta(z''-z')/z''^2$, may be treated separately, and will yield the result of Bottino (1962), i.e. that the φ solution is analytic for all k and for $Re(\lambda) \geq 0$.

First, to find conditions for convergence of the inner integral; using (7.17) and stipulating $k \neq 0$, the integrand becomes

$$\emptyset(z''^2 \; z''^\ell \; V_\ell(z'',z') \;) \quad \text{as} \quad z'' \to 0.$$

For $V_\ell(z'',z') = \varnothing(z''^{-5/2})$, this portion of the integral converges for $Re(\ell) \geq -1/2$, or $Re(\lambda) \geq 0$.

For the infinite tail of the inner integral, from Morse (1953), page 622,

$$h_\ell(z) \sim \frac{1}{z} \exp(i \; (z - \frac{\pi}{2} \; (\ell+1))) \quad \text{as} \quad z \to \infty. \tag{7.19}$$

Therefore, the integrand is

$$\emptyset(z''^2 \; z''^{-1} \; \exp(i \; k \; z'') \; V(z'',z')) \quad \text{as} \quad z'' \to \infty.$$

For $V(z'',z') = \emptyset \; (\exp(-\mu \; z''))$ as $z'' \to \infty$, the integrand is

$$\emptyset \; (z'' \; \exp(z''(i \; k - \mu))).$$

Therefore, the integral converges for $\mathrm{Re}(i\,k - \mu) < 0$, or $\mathrm{Im}(k) > -\mu$. Thus, the inner integral is found to converge for $\mathrm{Re}(\lambda) \geq 0$, $\mathrm{Im}\,(k) > -\mu$.

For treatment of the outer integral, let

$$M(\ell,\ k,\ z') = |j_\ell(kz')|. \tag{7.20}$$

Thus, provided the inner integral converges, the integrand of the outer is

$$\emptyset\ (z'^2 |j_\ell(kz')|\ V_\ell(z'',z'))\quad \text{as}\quad z' \to 0.$$

Precisely as in the discussion of the inner integral; for $V(x'',z') = \mathscr{o}(z'^{-5/2})$, the integral converges for $\mathrm{Re}(\lambda) \geq 0$.

The infinite tail of the outer integral provides a strengthening of the requirements.

$$M(\ell,k,z) = |j_\ell(kz)| \sim \left|\frac{1}{z}\cos(z - \frac{\pi}{2}(\ell+1))\right| \quad \text{as}\quad z \to \infty.$$

$$= \left|\frac{1}{2z}\left(\exp(i(kz - \frac{\pi}{2}(\ell+1))) + \exp(-i(kz - \frac{\pi}{2}(\ell+1)))\right)\right|$$

and the integrand is

$$\emptyset(z'^2 \max\ (|\exp(i\ kz')|\ ,\ |\exp(-\ i\ kz')|)\ V(z'',z')).$$

For $V(z'',z') = \emptyset\ (\exp(-\ \mu z'))$ as $z' \to \infty$, this quantity is

$$\emptyset(z'^2 \max(\ |\exp(z'(-\ \mu + ik))|\ ,\ |\exp(z'(-\ \mu - ik))|\ )).$$

Therefore, it converges for $-\ \mu < \mathrm{Im}(k) < \mu$.

Thus, the function φ is analytic for $\mathrm{Re}(\lambda) \geq 0$, $k \neq 0$, and $-\mu < \mathrm{Im}(k) < \mu$; provided the potential is no more singular than $z^{-5/2}$ as either space variable approaches 0, and provided it decays like $\exp(-\mu z)$ as either variable approaches ∞. This contrasts with the result of Bottino (1962) for the local case, in which φ was found to be analytic over the entire k plane (with the same restriction on λ). This result applies to a local potential which is $o(z^{-2})$ as $z \to 0$.

The reason for the loss of analyticity is that, in the local case, an integral equation for the solution $\varphi(z)$ may be found which involves only an integration from 0 to z. (See, for example, equation (1.4) of Bottino (1962)). In the nonlocal case, however, there is an extra integral, the potential integral itself, which runs all the way from 0 to ∞. Hence, extra restrictions must be imposed to ensure convergence of the infinite tail.

An integral equation for the Jost solution is

$$f(\lambda,k,z) = \exp(-i\,kz) - \frac{1}{2ik}\int_z^\infty d\xi\,\left(\exp(ik\xi)\exp(-ikz) - \exp(-ik\xi)\,\exp(ikz)\right)$$

$$\times \int_0^\infty d\xi'\left(\frac{\lambda^2-1/4}{\xi^2}\,\delta(\xi-\xi') + \xi'^2 V(\xi,\xi')\,f(\lambda,\,k,\,\xi')\right)$$

$$= \exp(-ikz) - \frac{1}{2ik}\int_0^\infty d\xi'\,f(\lambda,k,\xi')\int_z^\infty d\xi\left(\exp(ik\xi)\exp(-ikz) - \exp(-ik\xi)\,\exp(ikz)\right)$$

$$\times\left(\frac{\lambda^2-1/2}{\xi^2}\,\delta(\xi-\xi') + \xi'^2\,V(\xi,\xi')\right).$$

This may be derived in exactly the same manner as that used in appendix I of Bottino (1962) to derive his equation (1.6). It is identical in form to equation (7.15) with

$$L = -\frac{1}{2ik}\int_z^\infty d\xi\,(\exp(ik\xi)\,\exp(-ikz) - \exp(-ik\xi)\,\exp(ikz))$$
$$\left(\frac{\lambda^2-1/4}{\xi^2}\,\delta(\xi-\xi') + \xi'^2\,V(\xi,\xi')\right).$$

For $f(\lambda, k, z)$ to be analytic, it is sufficient for there to exist an upper bound for the integral

$$\int_0^\infty d\xi' \int_z^\infty d\xi\,\left|(\exp(ik\xi)\,\exp(-ikz) - \exp(-ik\xi)\,\exp(ikz))\xi'^2\right|$$
$$\times\frac{M(\lambda, k, \xi')}{M(\lambda, k, z)} \times \left|\frac{\lambda^2-1/4}{\xi^2}\,\delta(\xi-\xi') + \xi'^2\,V(\xi,\xi')\right| \qquad (7.21)$$

where $M(\lambda, k, \xi') = |\exp(-i\,k\,\xi')|$.

The centrifugal contribution must be considered separately but, since it is identical to the corresponding contribution in the local case, Bottino's discussion in appendix II shows that its contribution is bounded for all λ and $\mathrm{Im}(K) < 0$. Any local component of the potential, which will be manifested as a similar delta-function term, may also be included in this result provided it is no more singular than z^{-2} at the origin.

The remaining integral of equation (7.21) is not singular except at $\xi = \infty$, $\xi' = \infty$, and $\xi' = 0$, assuming the potential is well

behaved at all other points in the integration.

Considering first the inner integral, the integrand is

$$\emptyset\ (\max\ (|\exp(ik\xi)|,\ |\exp(-ik\xi)|)\ V(\xi,\xi'))\quad \text{as}\quad \xi \to \infty.$$

If

$$V(\xi,\xi') = \emptyset\ (\exp(-\ \mu\ \xi))\quad \text{as}\quad \xi \to \infty,$$

the quantity is

$$\emptyset(\max\ (|\exp((ik - \mu)\ \xi\ )|,\ |\exp((-ik - \mu)\ \xi\ )|)).$$

This quantity decays exponentially, and hence the integral converges, for

$$-\ \mu < \mathrm{Im}\ (K) < \mu.$$

If the inner integral converges, then the outer integrand is

$$\emptyset(\xi'^{2}\ V(\xi,\xi'))\quad \text{as}\quad \xi' \to 0.$$

If $$V(\xi,\xi') = o(\xi'^{-5/2})\quad \text{as}\quad \xi' \to 0,$$

then the bottom end of the outer integral always converges.

Under the same supposition, the outer integrand is

$$\emptyset(\ \exp(-ik\xi')\ V(\xi,\xi'))\quad \text{as}\quad \xi' \to 0,$$

which decays nicely for $\mathrm{Im}(k) < \mu$.

Thus, considering only the nonlocal contribution, $f(\lambda,\ k,\ z)$

is analytic for all λ, and for $-\mu < \mathrm{Im}(k) < \mu$. However, in order that the centrifugal contribution and any other local contribution no more singular than it may converge, one must strengthen the restriction on k to $-\mu < \mathrm{Im}(k) < 0$.

The regions of analyticity of the solutions may be extended further. Let $z = \rho \exp(i\sigma)$, ρ and σ real, and assume that the potential may be analytically continued into the complex z plane. Then the radial Schroedinger equation may be written

$$\ddot{\psi}(z) - \frac{\lambda^2 - 1/2}{\rho^2} \psi(z) + k^2 \exp(2i\sigma)\, \psi(z) - \exp(2i\sigma) \int_0^\infty dz' z'^2$$

$$V(\rho \exp(i\sigma), z')\, \psi(z') = 0. \qquad (7.22)$$

This is just the original equation with a complex potential and complex k. If the potential $V(z,\xi)$ is $\emptyset(\exp(-\mu z))$ for large z; in order that the general solution may approach the form

$$\psi' = \alpha(\sigma) \exp(-ik\rho) + \beta(\sigma) \exp(ik\rho),$$

it is necessary to impose the restriction $|\sigma| < \pi/2$ in order that the last term may disappear for large z.

As before, four solutions may be defined. The new φ solution has the behaviour

$$\varphi'(\lambda, k', \rho) \xrightarrow[\rho \to 0]{} \rho^{\lambda + 1/2}.$$

The new Jost solution is defined by

$$f'(\lambda,k'\rho) \xrightarrow[\rho\to\infty]{} \exp(-i\ k'\ \rho)$$

where $k' = k \exp(i\sigma)$.

The original Jost solution $f(\lambda, k, z)$, defined for z real, may be analytically continued into the complex-z plane provided the potential itself may be so continued. Its behaviour will then become

$$f(\lambda, k, z) \xrightarrow[|z|\to\infty]{} \alpha(\sigma) \exp(-ikz) + \beta(\sigma) \exp(ikz).$$

Since $f(\lambda, k, z) \exp(-ikz)$ is analytic in the complex-z plane,

$$\alpha(\sigma) \exp(-2\ ikz) + \beta(\sigma)$$

must be analytic for large ρ. For $\mathrm{Im}(k) < 0$, the first term disappears for large $|z|$ and $\beta(\sigma)$ itself must be analytic for large ρ. Since it is not a function of ρ, it must therefore be a constant. Since $\beta(\sigma) = 0$ for $\rho = 0$,

$$f(\lambda, k, z) \xrightarrow[|z|\to\infty]{} \alpha(\sigma) \exp(-ikz).$$

Since $\exp(ikz)$ is itself analytic, $\alpha(\sigma)$ must also be analytic, and therefore constant. $\therefore$ $f(\lambda, k, z) \xrightarrow[z\to\infty]{} \exp(-ikz)$ for all z;

$$\therefore\ f(\lambda, k, z) = f'(\lambda, k', \rho). \qquad (7.23)$$

The situation for the φ solutions is much simpler. The boundary conditions for φ and φ' are identical up to a phase factor, and hence so are the functions themselves:

$$\varphi'(\lambda, k, \rho) = \exp(-i\ \sigma(\lambda+1/2))\ \varphi(\lambda, k, z). \qquad (7.24)$$

Substituting from equations (7.21) and (7.22) into equation (7.8), and using the bilinearity of the Wronskian,

$$f'(\lambda,k') = \exp(-\sigma(\lambda+1/2))\ f(\lambda,k), \tag{7.25}$$

(where $f'(\lambda,k')$ is the Jost function for equation 7.12). Thus, the old Jost function is analytic if the new one is.

The only change introduced in the previous analysis of analyticity of the various solutions by making z complex is that, in order to obtain convergence of the infinite tails of the integrals involved, the quantity which must decay exponentially is not $\exp\ (z(-\ \mu \pm ik))$ as $z\to\infty$, but $\exp(\rho(\cos(\sigma) + i\ \sin(\sigma))(-\ \mu \pm i(k))$ as $\rho\to\infty$. The real part of the coefficient of ρ is

$$\mathrm{Re}((\cos(\sigma) + i\ \sin\ (\sigma))\ (-\ \mu \pm ik))$$

$$= -\ \mu\ \cos(\sigma) \mp (\cos(\sigma)\ \mathrm{Im}(k) + \sin\ (\sigma)\ \mathrm{Re}(k)).$$

Thus, the solutions φ and f are analytic only if there exists a σ such that $|\sigma| < \pi/2$, and

$$-\ \mu < \mathrm{Im}(k) + \mathrm{Re}(k)\ \tan(\sigma) < \mu. \tag{7.26}$$

$\tan(\sigma)$ may take on any finite, real value. Thus, (7.26) can always be satisfied when $\mathrm{Re}(k) \neq 0$. If $\mathrm{Re}(k) = 0$, it reduces to

$$-\ \mu < \mathrm{Im}(k) < \mu.$$

Thus, there remain in general branch lines for each solution along

both imaginary axes from $\pm i\mu$ to $\pm i\infty$.

Precisely as in the case discussed Bottino (1962), the local and centrifugal contributions give rise to an additional branch line for the Jost solution along the positive imaginary k axis which may start even closer to the origin.

Thus it is seen that two circumstances conspire to engender the extra branch cut of the scattering amplitude in the complex-k plane:

I The inconstancy of a Wronskian of two solutions of the nonlocal equation.

If the Wronskian of two solutions were constant, equation (7.10) could be used to obtain an expression for $W(\varphi_+,\varphi_-)$ which is analytic even when the φ solutions themselves are not. Thus, the numerator of the parenthetic expression in equation (7.14) could not have any singularities. (The denominator even in the local case has a cut along the negative imaginary k axis.)

II The extra integral which is necessary to accommodate the nonlocal potential.

In order that the potential integral in formulae (7.18) and (7.21) may converge, it is necessary to introduce the restriction $-\mu < \mathrm{Im}(k) < \mu$ so that neither $\exp(ikz)$ nor $\exp(-ikz)$ may explode for large z faster than $\exp(-\mu z)$ damps the integrand. Even when the analytic continuation based on equation (7.12) is employed, this restriction in general engenders branch cuts along both imaginary k axes in both the φ and the f solutions. Thus, both the

numerator and the denominator of the parenthetic effort in equation (7.14) have branch lines along <u>both</u> imaginary axes.

Chapter 8. Summary.

In this thesis, a detailed investigation is made of the analytic properties of the partial-wave and total amplitude for scattering by a particular class of separable, nonlocal potentials. The advantage of studying separable potentials lies in the fact that the partial-wave amplitude can be written in closed form as has been shown by McMillan (1963), and the particular class of these potentials studied here is physically reasonable for strong interactions in the sense that the potentials involve an exponential decay. They also yield a relatively simple form for the partial-wave amplitude.

The analytic properties of the partial-wave amplitude given by McMillan (1963) for the more general case have been shown explicitly for the case considered here, but the present work goes beyond his by proving a double dispersion relation for the total scattering amplitudes using a technique involving complex angular momentum first exploited by Regge (see Bottino (1962)). This extension has also been performed by Mitra (1963) and Cushing (1963) for a more restricted class of separable potentials, but neither has investigated the convergence of the spectral integrals obtained.

The analytic properties of the scattering amplitude found for the case considered here are essentially the same as those found for the local potential case by, for example, Bottino (1962), except that the total scattering amplitude in the present case contains an extra branch point the origin of which, as is shown in detail in the last

chapter, lies in the fact that a Wronskian for the nonlocal radial Schroedinger equation is not in general a constant, together with the fact that the extra integral which accommodates the nonlocal potential converges only for a restricted class of solutions. Thus, it is a general property of nonlocal potentials that the scattering amplitude has a branch line along the negative real axis of both sheets of the complex-energy plane.

Appendix I. Convergence of the Integrals defining $D_\ell(k)$ and $V_\ell(k)$.

In this appendix, it is shown that the integrals

$$V_\ell(k) = \int_0^\infty dr\ r^2\ j_\ell(kr)\ V_\ell(r) \tag{I.1}$$

and

$$D_\ell(k) = \int_0^\infty dq\ q^2\ \frac{V_\ell^2(q)}{k^2-q^2} \tag{I.2}$$

converge if

$$V_\ell(r) = o(r^{-5/2}) \qquad \text{as} \qquad r \to 0 \tag{I.3}$$

$$\text{and as} \qquad r \to \infty. \tag{I.4}$$

The relevant properties of the spherical Bessel functions according to Morse (1953) page 1573 are

$$j_\ell(z) \sim \frac{\sqrt{\pi}}{2\Gamma(\ell+3/2)} \left(\frac{z}{2}\right)^\ell \qquad \text{as} \qquad z \to 0 \tag{I.5}$$

$$\sim \frac{1}{z} \cos\left(z - \frac{\pi}{2}(\ell+1)\right) \qquad \text{as} \qquad z \to \infty. \tag{I.6}$$

The convergence of the integral in (I.1) is first considered. The worst case for convergence of the lower end is when $\ell = 0$; In this case (for small r), $j_\ell(kr)$ approaches a finite constant. Thus, the requirement for convergence at the lower limit is

$$r^2\ V_\ell(r) = o(r^{-1}) \qquad \text{as} \qquad r \to 0,$$

or

$$V_\ell(r) = o(r^{-3}) \quad \text{as} \quad r \to 0. \tag{I.7}$$

Since $j_\ell(kr) = O(r^{-1})$ as $r \to \infty$, the requirement for convergence at the upper limit of (I.1) is

$$r\, V_\ell(r) = o(r^{-1}) \quad \text{as} \quad r \to \infty,$$

or

$$V_\ell(r) = o(r^{-2}) \quad \text{as} \quad r \to \infty. \tag{I.8}$$

Convergence of the integral in (I.2) will now be studied. It will be found that somewhat stronger restrictions on the potential will be necessary to ensure convergence.

From (I.5) and (I.6),

$$\left| j_\ell(z) \right| \leq \frac{C}{z^b}, \; 0 \leq z < \infty$$

for $\ell \geq 0$, where C is an arbitrary, finite constant and $0 \leq b \leq 1$. Thus,

$$\left| V_\ell(k) \right| \leq \left| \int_0^\infty dr\, r^2 \frac{C}{(kr)^b} V_\ell(r) \right| = \left| \frac{C}{k^b} \int_0^\infty dr\, r^{2-b} V_\ell(r) \right|, \tag{I.9}$$

and

$$\left| D_\ell(k) \right| \leq 1 + C \int_0^\infty \frac{dq\, q^2}{q^2 - k^2} \left(\frac{1}{q}\right)^{2b}.$$

Thus, the integrand in (I.2) is bounded by q^{-2b}, provided the integral in (I.9) converges; that is, provided

$$V_\ell(r) = o(r^{b-3}) \quad \text{as} \quad r \to 0 \tag{I.10}$$

$$\text{and} \quad \text{as} \quad r \to \infty.$$

For convergence at the lower limit of the integral in (I.2), where the integrand may be singular, the most unfavourable value of k^2 is zero, and this value is assumed in this paragraph. The integrand is then $\emptyset(q^{-2b})$ as $q \to 0$, and therefore converges for $b < 1/2$. Thus, the integral in (I.9) must converge for some b such that $0 \leq b < 1/2$. With these bounds on b, (I.10) leads to

$$V_\ell(r) = \begin{cases} o(r^{-3}) & \text{as} \quad r \to 0 \\ o(r^{-5/2}) & \text{as} \quad r \to \infty. \end{cases} \tag{I.11}$$

Convergence of the infinite tail of (I.2) is now studied. Using again (I.9), as $q \to \infty$ the integrand becomes bounded by q^{-2b}, and the integral converges for $b > 1/2$. Thus, in order that the tail of (I.2) converge, it is sufficient that (I.9) converge for some b such that $1/2 < b \leq 1$. (I.10) thus leads to

$$\begin{aligned} V_\ell(r) &= o(r^{-5/2}) \qquad \text{as} \quad r \to 0 \\ &= o(r^{-2}) \qquad \text{as} \quad r \to \infty. \end{aligned} \tag{I.12}$$

In order to satisfy all the restrictions imposed in (I.7), (I.8), (I.10), and (I.11), $V_\ell(r)$ must satisfy

$$\begin{aligned} V_\ell(r) = o(r^{-5/2}) \qquad &\text{as} \quad r \to 0 \\ \text{and} \qquad &\text{as} \quad r \to \infty, \end{aligned} \tag{I.13}$$

as given by McMillan (1963).

In the local case, by a similar argument, the requirement turns out to be

$$\begin{aligned} V_\ell(r) = o(r^{-2}) \qquad &\text{as} \quad r \to 0 \\ \text{and as} \quad &r \to \infty. \end{aligned}$$

Appendix II. The Asymptotic Behaviour of $P_\ell(t)$ as $|t| \to \infty$.

EI 3.9.2 (19) and EI 3.9.2 (20) yield

$$P_\ell(t) = \max\left(\emptyset(t^\ell),\ \emptyset(t^{-\ell-1})\right) \quad \text{as} \quad |t| \to \infty. \tag{II.1}$$

when $Re(\ell) \neq -1/2$. In this appendix, it isshown that this latter condition may be relaxed.

EI 3.3.1 (8) says

$$P_\ell(t) = \frac{\tan(\pi\ell)}{\pi}\left(Q_\ell(t) - Q_{-\ell-1}(t)\right); \tag{II.2}$$

and EI 3.9.2 (21) yields

$$Q_\ell(t) = \emptyset(t^{-\ell-1}) \quad \text{as} \quad |t| \to \infty \tag{II.3}$$

for all ℓ. Combining the two confirms equation (9.8) for $Re(\ell) = -1/2$ and $Im(\ell) \neq 0$.

For the case $\ell = -1/2$, EI 3.14 (5) may be used:

$$P_{-1/2}(\cos\theta) = \frac{2}{\pi} K\left(\sin(\theta/2)\right) \tag{II.4}$$

where K is the complete elliptic integral of the first kind. Substituting into EII 13.8 (1) yields

$$P_{-1/2}(1+2z) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1+z\sin^2\varphi}} \tag{II.5}$$

specializing to the case z real and positive,

$$P_{-1/2}(1+2z) = \frac{2}{\pi}\int_{\varphi=0}^{\operatorname{asin}(z^{-1/2})} \frac{d\varphi}{\sqrt{1+z\sin^2\varphi}}$$

$$+ \frac{2}{\pi\sqrt{z}}\int_{\varphi=\operatorname{asin}(z^{-1/2})}^{\pi/2} d\varphi\left(-\sin(\varphi) + \frac{1}{2\sin(\varphi)\sqrt{z}}\right.$$

$$\left. - \phi(z^{-3/2})\right). \qquad \text{(II.6)}$$

Although that last integrand is singular at $\varphi = \operatorname{asin}(z^{-1/2})$ because the binomial expansion used doesn't converge there, the singularity is removable because the integrand approaches $2^{-1/2}$ at this point. Thus, both integrals are bounded and, using

$$\operatorname{asin}(x) \sim x \quad \text{as} \quad x \to 0,$$

equation (II.1) is proven for this case also.

Bibliography

Bateman Manuscript Project (Erdelyi et al), Higher Transcendental Functions, McGraw-Hill, 1953.

Bateman Manuscript Project, Tables of Integral Transforms, McGraw-Hill, 1954.

Blatt, J.M., and V.J. Weisskopf, Theoretical Nuclear Physics, Wiley, 1952.

Bottino, A., A.M. Longoni, T. Regge, Nuovo Cimento, 23, 954, 1962.

Churchill, R.V., Complex Variables and Applications, McGraw-Hill, 1960.

Cushing, J.T., Nuovo Cimento, 27, 2364, 1963.

Dicke, R.H., and I.R. Wittke, Introduction to Quantum Mechanics, Addison-Wesley, 1961.

Lanz, L. and G.M. Prosperi, Nuovo Cimento, 33, 3481, 1964.

Lomon, E. and M. McMillan, Annals of Physics, 23, 439, 1963.

McMillan, M., Ph.D. Thesis, McGill University, 1961.

McMillan, M., Nuovo Cimento, 29, 4153, 1963.

Mitra, A.N., Physical Review, 130, 2117. 1963.

Morse, P.M., and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill, 1953.

Omnès, R., and M. Froissart, Mandelstam theory and Regge Poles, Benjamin, 1963.

Salam, A., Director, Theoretical Physics, International Atomic Energy Agency, Vienna, 1963.

Squires, E.J., Nuovo Cimento, 25, 242, 1962.

Tabakin, F., Annals of Physics, 30, 51, 1964.