FLUCTUATIONS IN A CONDUCTOR

by

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ABSTRACT

Applying purely thermodynamic arguments it has been shown that temperature fluctuations in a sample can be represented by introducing appropriate 'series temperature generator' or 'shunt heat current generator'. The temperature fluctuations lead to resistance fluctuations in the sample. These resistance fluctuations due to temperature fluctuations can be detected as voltage-fluctuations (temperature noise) by using a sensing direct current through the sample.

Statistical-mechanical arguments are used to obtain theoretical expressions for spectral density of heat current fluctuations in a metallic conductor in terms of the macroscopic properties of the conductor. Since the electrons are carriers of heat and electric currents in a metal, heat and electric current fluctuations are correlated. Spectral density of cross-correlation between electric and heat current is derived. Statistical considerations are extended to the calculation of the steady state spectral density of heat current fluctuations between two black bodies in radiative contact.

Temperature noise in a system in which there is only a partial correlation between temperatures at different points along the length at any time (Isothermal System) is compared with a system in which there is a complete correlation between temperatures at all points along the length at any time (Single Temperature System).

Experimental results indicate that for the metal filament used Nyquist theorem can be applied at the operating temperature. For frequencies close to the characteristic frequency of the system it is observed that there is an increase in noise temperature of the filament due to temperature noise.
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1. **CHAPTER I**

**Introduction**

1.1 **Voltage fluctuations in thermodynamic equilibrium.**

By purely thermodynamic reasoning Nyquist (1928) proved that a resistor $R$ in thermodynamic equilibrium with its surrounding at temperature $T_0$ exhibits spontaneous voltage fluctuations. Spectral density of voltage fluctuations is given by

$$S_v(f) = 4kT_0R$$  \hspace{1cm} (1.1.1)

Where $k =$ Boltzmann constant.

Einstein (1910) has derived the following entropy probability law for computing mean square fluctuations of a macroscopic variable $x$. Probability $P(x)$ that the variable $x$ lies between $x$ and $x+dx$ is

$$P(x)dx = \text{Const.} e^{\frac{\Delta S(x)/k}{}} dx$$  \hspace{1cm} (1.1.2)

where $\Delta S(x) =$ change in entropy of the system as a result of fluctuation in $x$.

Consider the following electrical system in equilibrium at a temperature $T_0$. Using (1.1.2) mean square voltage fluctuation is obtained. Langevin equation for current in the system gives voltage correlation function.

**Fig. (1.1.1)**
As far as the fluctuations in voltage $V$ are concerned these are representable by a series e.m.f generator or shunt current generator as in Fig. (1.1.1).

The generators in Fig. (1.1.1) have following properties

$$S_{v}(f) = 4kT_{0}\Re \xi(\omega)$$
$$S_{I}(f) = 4kT_{0}/R$$
$$\overline{V^{2}} = kT_{0}/C_{e}$$

(1.1.3)

1.2 Temperature fluctuations in thermodynamic equilibrium.

Consider two bodies $C_1$ and $C_2$ in thermal contact at temperature $T_0$. $C_1$, $C_2$ are the heat capacities $T_1$ and $T_2$ are instantaneous temperatures respectively.

$$\overline{T_{0}} = \langle T_{1} \rangle = \langle T_{2} \rangle$$

Fig. (1.2.1)

$$g = \text{thermal conductance between } C_1 \text{ and } C_2 = \frac{\mathcal{H}}{T}$$

where $\mathcal{H} = \text{average heat flow between } C_1, C_2 \text{ when there is a temperature difference } T$.

Using Einstein Entropy-probability law in Appendix 1(a) following results for mean square temperature fluctuations are derived.

1. $\overline{\Delta T_{1}^{2}} = kT_{0}^{2}C_{2}/(C_{1}+C_{2})$

2. $\overline{\Delta T_{2}^{2}} = kT_{0}^{2}C_{1}/(C_{1}+C_{2})$

3. $\overline{\Delta T_{1} \Delta T_{2}} = -kT_{0}^{2}/(C_{1}+C_{2})$

(1.2.1)

$\Delta T_1$ and $\Delta T_2$ are fluctuations in temperatures of the bodies characterised by the heat capacities $C_1$ and $C_2$. If $C_1/C_2 \gg 1$ i.e. the body with heat capacity $C_1$ is in contact with an infinite heat bath then
\[
\Delta T_i^2 = k T_0^2 / C_i
\]  

(1.2.2)

This is consistent with Gibbs' result that if \( E_1 \) is the energy of a system in thermodynamic equilibrium then energy fluctuations are:

\[
\text{Var} \ E_i = k T_0^2 \frac{\partial \langle E_i \rangle}{\partial T_0}
\]  

(1.2.3)

This result is valid no matter what statistics apply to the process of heat conduction between \( C_1 \) and the infinite heat bath. Equation (1.2.3) is the well known energy fluctuation theorem.

Temperature fluctuations in \( C_1 \) when in contact with an infinite heat bath are derived in Appendix 1(a). It is seen that as far as temperature fluctuations are concerned these are representable by a series "temperature generator" or a shunt heat current generator analogous to the generators in (1.1).

We have shown that

(i) \( S_\theta(f) \) = Spectral density of series "temperature generator"

\[
= 4 k T_0^2 \text{Re} \ Z_\theta(\omega)
\]  

(1.2.4)

\( Z_\theta(\omega) \) = Thermal impedance of the body \( C_1 \) to its surroundings

= \( (\theta + j \omega C_i)^{-1} \)

(ii) \( S_H(f) \) = 4 \( k T_0^2 \theta \)
1.3 **Object and scope of the thesis.**

Gill (1958) claims that temperature fluctuations and voltage fluctuations in a conductor are manifestations of the same underlying phenomena. One of the objects of this investigation is to show that such a claim is erroneous. Temperature fluctuation is a macroscopic effect and temperature noise can be detected in the following way.

\[ \Delta H \rightarrow C_{1} \rightarrow \Delta T_{1} \rightarrow \Delta R \rightarrow \frac{\Delta V}{I_0} \]  

where \( \Delta H \) = fluctuation in heat current  
\( C_{1} \) = heat capacity of the system  
\( \Delta T_{1} \) = temperature fluctuation  
\( \alpha = \frac{1}{R} \frac{dR}{dT} \) = temperature coefficient of resistance  
\( I_0 \) = biasing current  
\( V_0 \) = voltage fluctuation due to temperature fluctuation.

This investigation is concerned with theoretical and experimental aspects of temperature fluctuations. Theoretical aspects of temperature fluctuation deal with the following:

(i) **Thermodynamic Equilibrium**

General statistical considerations are used for obtaining the spectral densities of heat current fluctuations and cross-correlations between the heat and electric currents. A distributed system in which the temperature at different points are partially correlated (Isothermal System) is investigated and compared with a system in which there is a complete correlation between temperatures at different points (Single Temperature System).

(ii) **Steady State**

Temperature noise in the steady state for a single temperature system and spectral densities of heat current fluctuations between two black bodies are obtained.
CHAPTER II

Spectral density of heat current fluctuations from statistical considerations

2.1 Spectral density of electric current fluctuations.

Employing general statistical considerations Bakker and Heller (1939) have derived the spectral density of electric current fluctuations in a conductor at temperature $T_0$. Applying Lorentz theory of the electron and assuming that electric currents $I(t)$ and $I(t+\tau)$ are correlated only when $|\tau|$ is small they establish the following result.

\[ S_T(f) = 4kT_0 G(f) \]  \hspace{1cm} (2.1.1)

where $G(f) = Re \, Y_{ll}(f)$

$= \text{Real part of the conductance of the conductor at frequency } w \text{ and temperature } T_0.$

2.2 Heat and electric currents in a conductor.

In a conductor the motion of electrons leads to a simultaneous flow of electric current and heat energy. To the first approximation electric and heat current densities are dependent linearly on both the 'forces' - electric and thermal. Accordingly if $J$ is the electric current density and $q$ is the heat current density in the conductor when there is an electric field $F$ and temperature gradient $dT_0/dx$, then quite generally,

\[ J = y_{ll} U_1 + y_{l2} U_2 \]
\[ q = y_{21} U_1 + y_{22} U_2 \]  \hspace{1cm} (2.2.1)

where $y_{ab}$ $(a, b = 1, 2)$ are specific conductances

and $U_1$ is the 'electric force'

$U_2$ is the 'thermal force'.

Using Boltzmann transport equation explicit forms for $U_1$ and $U_2$ and the admittances $y_{ab}$ is evaluated. For the sake of generality we shall assume an
alternating electric field is applied.

\[ f(\mathbf{K}) = \text{Fermi Dirac Distribution function for the electrons in the presence of applied fields (steady state conditions).}\]

In the steady state, \( f(\mathbf{K}) \) varies with the same frequency as the applied field

\[ f(\mathbf{K}) = f_0(\mathbf{K}) + \frac{T_e(\mathbf{K})}{1 + j\omega T_0(\mathbf{K})} \frac{\partial f_0}{\partial E} \left[ \frac{eF + \left( \frac{T_0}{\partial x} \left( \frac{E_F}{T_0} \right) \right)}{T_0} + \frac{\partial T_0}{\partial x} \right] \]  

(2.2.2)

where \( f_0(\mathbf{K}) = \text{Fermi Dirac Distribution function for the electrons in thermodynamic equilibrium.} \)

\[ f_0(\mathbf{K}) = \frac{1}{\sqrt{\pi}} \frac{e^{-(E(\mathbf{K}) - E_F)/kT_0}}{1 + e^{-1}} \]

\( E(\mathbf{K}) = \text{energy of an electron with wave vector } \mathbf{K} \)

\( E_F = \text{Fermi energy} \)

\( T_0(\mathbf{K}) = \text{relaxation time of electrons with wave vectors in the interval } K \text{ and } K+dK \)

\( \omega = \text{angular frequency of applied field} \)

\( \nu_x = \frac{\hbar}{\pi} \frac{\partial E(\mathbf{K})}{\partial K_x} = X\text{-component of velocity of energy transport of an electron.} \)

\( \frac{\hbar}{\pi} \) is Planck's constant

Following Mott and Jones (1936) pages 305-306

\[ J = \frac{1}{4\pi^3} \int (-e) \nu_x f(\mathbf{K}) d^3K \]

\[ Q = \frac{1}{4\pi^3} \int E(\mathbf{K}) \nu_x f(\mathbf{K}) d^3K \]

\( d^3K = \text{an element of volume in } K \text{ space} \)

\(-e = \text{charge on an electron.} \)

Substituting for \( f(\mathbf{K}) \) from (2.2.2) in (2.2.3) leads to

\[ J = k_e e^2 \left[ F + \frac{T_0}{\partial x} \left( \frac{E_F}{T_0} \right) \right] + \frac{k_e}{T_0} \frac{\partial T_0}{\partial x} \]

\[ Q = -k_e e^2 \left[ F + \frac{T_0}{\partial x} \left( \frac{E_F}{T_0} \right) \right] - \frac{k_e}{T_0} \frac{\partial T_0}{\partial x} \]  

(2.2.4)
where \( K_n = \frac{1}{4\pi^2} \int \frac{\sigma_T(\vec{K})}{1+j\omega \tau(\vec{K})} \, \delta^2 \vec{E}^n(\vec{K}) \, \frac{\partial f_0}{\partial E} \, d^3K \)

Comparing (2.2.4) with (2.2.1) leads to the following equalities
\[
\begin{align*}
\gamma_{11} &= k_e e^2 \\
\gamma_{12} &= -k_1 e = \gamma_{21} \\
\gamma_{22} &= k_2 \\
U_1 &= F + \frac{T_0}{e} \frac{\partial}{\partial x} \left( \frac{E_F}{\eta} \right) \\
U_2 &= -\frac{1}{T_0} \frac{\partial T_0}{\partial x}
\end{align*}
\] (2.2.5)

when \( U_1 = 0 \)
\[
q = -\frac{k_2}{T_0} \frac{\partial T_0}{\partial x} = -\gamma_1 \frac{\partial T_0}{\partial x}
\] (2.2.6)

where \( \gamma_1 = \frac{k_2}{T_0} = \frac{\gamma_{22}}{T_0} = \) thermal conductivity of the specimen conductor when \( U_1 = 0 \).

2.3 Spectral density of heat current fluctuations in a conductor.

In this section Bakker and Heller type approach is extended to the calculation of spectral density of heat current fluctuations in a conductor at temperature \( T_0 \).

Consider a conductor of length \( L \) and area of cross section \( A \). \( N_{Kx}(t) \) is the number of electrons in the conductor at a certain time \( t \) with their wave vectors lying between \( \vec{K} \) and \( \vec{K}+d\vec{K} \).

\( P(\tau;\vec{K}) \) is the probability that an electron which has not suffered a collision at \( t \) continues to move freely in the time interval \( t \) and \( t+\tau \).

It is assumed that \( P \) is independent of \( t \), i.e. the collision process is a Poisson Process.

\[
P(\tau;\vec{K}) = e^{\exp(-\frac{\tau}{\eta T_0}(\vec{K}))}
\] (2.3.1)
\[ \text{d}H(t) = \text{fluctuation of heat current along x direction due to } N_R(t). \]
\[ \frac{\text{d}H(t)}{L} = \frac{E^{(2)}}{L^2} V_{x} N_R(t) \]  
\[ (2.3.2) \]

\[ \frac{\text{d}H(t+\tau)}{L} = \frac{E^{(2)}}{L^2} V_{x} N_R(t+\tau) \]  
\[ (2.3.3) \]

\[ \frac{\text{d}H(t)}{H(t+\tau)} = \frac{E^{(2)}}{L^2} V_{x}^2 \mathcal{P}(\tau, t) \mathcal{V}_{N_{R}} \]  
\[ (2.3.4) \]

If \( \eta_k \) is the occupancy number of the electrons then
\[ N_{R} = \text{AL} \eta_{R} \frac{d^3 k}{4 \pi^3} \]  
\[ (2.3.5) \]

and
\[ \mathcal{V}_{N_{R}} = \text{AL} \mathcal{V}_{N_{R}} \eta_{R} \frac{d^3 k}{4 \pi^3} \]  
\[ (2.3.6) \]

\[ \frac{\text{d}H(t)}{H(t+\tau)} = \frac{\text{AL}^{-1}}{4 \pi^3} \frac{E^{(2)}}{L^2} V_{x}^2 \mathcal{P}(\tau, t) \mathcal{V}_{N_{R}} \eta_{R} \frac{d^3 k}{4 \pi^3} \]  
\[ (2.3.7) \]

Auto correlation function for the heat currents at \( t \) and \( t+\tau \) due to all electrons:
\[ \frac{H(t)}{H(t+\tau)} = \frac{\text{AL}^{-1}}{4 \pi^3} \int \frac{E^{(2)}}{L^2} V_{x}^2 \mathcal{P}(\tau, t) \mathcal{V}_{N_{R}} \eta_{R} \frac{d^3 k}{4 \pi^3} \]  
\[ (2.3.8) \]

Using Wiener-Khinchin theorem, spectral density of heat current fluctuations is
\[ S_H(f) = 4 \int_0^\infty \frac{H(t)}{H(t+\tau)} \cos \omega \tau \ d \tau \]  
\[ (2.3.9) \]

Substituting in (2.3.9) from (2.3.8) and integrating w.r.t leads to
\[ S_H(f) = 4 \frac{\text{AL}^{-1} R T_o}{4 \pi^3} \int \frac{E^{(2)}}{1 + \omega^2 \sigma_o^2} \mathcal{P}(\tau, t) \mathcal{V}_{N_{R}} \eta_{R} \frac{d^3 k}{4 \pi^3} \]  
\[ (2.3.10) \]
where \( V_{AR} \eta_R = -kT_0 \frac{df}{dF} \)  \( \text{ (Tolman R.C. (1946))} \)

\[ S_{H} (f) = 4kT_0 Re Y_{12} \]  \( \text{(2.3.11)} \)

where \( Y_{12} = AL^{-1} \eta_{12} \)

Motion of electrons in the conductor leads to a simultaneous flow of electric and heat currents. The spectral density of cross correlation between the electric and heat current is calculated as follows:

\[
\begin{align*}
\frac{d H(t)}{dt} &= \frac{E(t)}{L} \eta_{12} N_R(t) \\
\frac{d I(t)}{dt} &= -\frac{e}{L} \eta_{12} N_R(t) \\
\frac{d H(t)I(t)}{dt} &= -\frac{e}{L} \frac{E(t)}{N_R(t)} \eta_{12}^2 N_R(t)N_R(t) \tag{2.3.12}
\end{align*}
\]

Here on following the procedures leading to equations (2.3.4) to (2.3.9) leads to

\[ S_{H}(f) = 4kT_0 Re Y_{12}(f) \]  \( \text{(2.3.13)} \)

where \( Y_{12}(f) = AL^{-1} \eta_{12}(f) = -K_1 e AL^{-1} \)

The negative sign is expected as the heat current and electric current carried by the electron are in opposite directions.

Equations (2.2.1), (2.1.1), (2.3.11) and (2.3.14) suggest that for a conductor in thermodynamic equilibrium at a temperature \( T_0 \), following 4 terminal network fully describes its response to external forces and its properties noise-wise.
(i) $\Psi = 0$ i.e. open circuit $(2,2)$. Corresponding to this situation

$$\left( \mathcal{J} \right)_{\Psi=0} = y_{\parallel} F$$

and the spectral density of current fluctuations is given by (Bakker and Heller)

$$S_{\mathcal{I}}(\mathcal{f}) = 4K T_0 \text{Re} Y_{\parallel}(\mathcal{f})$$

where

$$Y_{\parallel}(\mathcal{f}) = AL^{-1} y_{\parallel}(\mathcal{f})$$

(ii) Open Circuit $(1,1)$ i.e. $J=0$ then from (2.2.1)

$$\left( \Phi \right)_{J=0} = \left( \begin{array}{c} y_{22} - \frac{y_{12}^2}{y_{\parallel}^2} \end{array} \right) \Phi_{2} = y_{201} \Phi_{1}$$

$$= -\left( k_2 - \frac{k_1^2}{k_0} \right) \frac{1}{T_0} \frac{dT_0}{dx} = -K \frac{dT_0}{dx}$$

Substitution for $y_{a,b}$ is made from (2.2.5).

$$K = \text{Conventional Thermal Conductivity of the Conductor}$$

$$= \frac{1}{T_0} \left( k_2 - \frac{k_1^2}{k_0} \right)$$

If $H_0$ is the heat current flow when $I=0$.

Spectral density of $H_0$ fluctuations is

$$S_{H_0}(\mathcal{f}) = 4K T_0 \text{Re} y_{201}$$

where $y_{201} = \text{Admittance between the terminals (2,2) when (1,1) are 'open circuited'}$

$$y_{201} = \left( k_2 - \frac{k_1^2}{k_0} \right) = K T_0$$

hence;

$$S_{H_0}(\mathcal{f}) = 4K T_0^2 \text{Re} y_L$$

$$y_L = AL^{-1}K = \text{Thermal Conductance of the Conductor at } T_0.$$

Equation (2.3.14) giving the spectral density of cross correlation between the currents $H$ and $I$ is an example of the generalised Nyquist theorem (Takahasi, 1941) in which the elements of the admittance matrix do not all have the same dimension.
Spectral density of radiant heat current fluctuations between two black bodies in radiative contact

Two black bodies with surface areas $A_1$ and $A_2$ are maintained at temperatures $T_1$ and $T_2$ respectively, $T_1 > T_2$. There is a net flow of radiant energy from $A_1$ to $A_2$ which is supplied by a power source in $A_1$.

First consider the radiant energy emitted by $A_1$. Let $\eta_\nu(t)$ be the occupancy of a cell in boson phase-space. The number of photons in the frequency range $\nu$ and $\nu + d\nu$ emitted from $A_1$ to $A_2$ in time $t$ is (Planck, M. 1913):

$$N_\nu(t) = \frac{2A}{C^2} t \eta_\nu(t) \nu^2 d\nu$$  \hspace{1cm} (2.4.1)

where $A = \int \int \frac{dA_1 dA_2 \cos \beta_1 \cos \beta_2}{r^2}$

$t \gg$ transit time of photons from $A_1$ to $A_2$

$C =$ speed of light

$$\text{Var } N_\nu = \frac{2A}{C^2} t (\text{Var } \eta_\nu) \nu^2 d\nu$$ \hspace{1cm} (2.4.2)

If $E_\nu$ is the energy flow from the surface of $A_1$ in time $t$ due to photons in the frequency range $\nu$ and $\nu + d\nu$

$$E_\nu = h \nu N_\nu$$

and

$$\text{Var } E_\nu = (h \nu)^2 \text{Var } N_\nu$$ \hspace{1cm} (2.4.3)

Using (2.4.2), (2.4.3) becomes

$$\text{Var } E_\nu = \frac{2Ah^2 t}{C} (\text{Var } \eta_\nu) \nu^4 d\nu$$ \hspace{1cm} (2.4.4)
If \( S_{H^V}(f) \) = spectral density of heat current fluctuations for the radiant energy due to photons of frequency \( V \) and \( V + dV \) then
\[
S_{H^V}(f) = \frac{2 \text{ Var } E_V}{t} \quad \text{(Appendix 2a) \ (2.4.5)}
\]
Substituting for Var \( E_V \) from (2.4.4)
\[
S_{H^V}(f) = \frac{2 A h^2}{c^2} (\text{Var } \eta_V) \nu^4 d\nu \quad \text{(2.4.6)}
\]
since the radiation of different frequencies are statistically independent, the spectral density of radiant heat current fluctuations from \( A_1 \) to \( A_2 \) due to photons of all frequencies is obtained by integrating (2.4.6).
\[
\int_s \int_0^{\infty} \frac{2 A h^2}{c^2} (\text{Var } \eta_V) \nu^4 d\nu
\]
Any photon emitted by \( A_1 \) and reabsorbed by it will not give rise to any fluctuations.

If \( H_{\nu_1 \nu_2} \) = heat current composed of radiation of frequency between \( \nu \) and \( \nu + d\nu \) radiated from \( A_1 \) to \( A_2 \)
\[
H_{\nu_1 \nu_2} = \frac{E_V}{t} = \frac{2 A h}{c^2} \eta_\nu(t_1) \nu^3 d\nu \quad \text{(2.4.8)}
\]
Total heat current from \( A_1 \) to \( A_2 \) is the integral overall frequencies of the radiation in equation (2.4.8)
\[
H_{12} = \frac{2 A h}{c^2} \int_0^{\infty} \eta_\nu \nu^3 d\nu \quad \text{(2.4.9)}
\]
Average heat flow from \( A_1 \) to \( A_2 \) is
\[
\overline{H_{12}} = 2 \cdot \frac{A h}{c^2} \int_0^{\infty} \eta_\nu(t_1) \nu^3 d\nu
\]
and
\[
\frac{d \overline{H_{12}}}{d \tau_1} = 2 \cdot \frac{A h}{c^2} \int_0^{\infty} \frac{d \overline{H_\nu}}{d \tau_1} \nu^3 d\nu \quad \text{(2.4.10)}
\]
It is assumed that the distribution function for photons emitted by $A_1$ is one appropriate to its steady state temperature (Tolman R.C., 1946, page 630).

$$\bar{n}_\nu = \left[ \exp \left( \frac{\hbar \nu}{k T_1} \right) - 1 \right]^{-1}$$

$$\frac{d\bar{n}_\nu}{d T_1} = \nabla T \nu \left( \frac{\hbar \nu}{k T_1} \right)$$ (2.4.11)

Substituting in (2.4.10) from (2.4.11)

$$\frac{d\bar{H}_{12}}{d T_1} = \frac{2 A h^2}{k T_1^2} \int_0^\infty \nu^4 (\nabla T \nu) d\nu$$ (2.4.12)

Comparing (2.4.7) with (2.4.12) one can easily write

$$\mathcal{S}_{H_{12}} = 2 k T_1^2 \frac{d\bar{H}_{12}}{d T_1}$$ (2.4.12)

Since $A$ is symmetric in suffixes 1 and 2 introducing an appropriate geometrical factor which determines the fraction of photons emitted by $A_2$ and absorbed by $A_1$ and following arguments similar to the ones outlines above, one can show that $\mathcal{S}_{H_{21}}$, the spectral density of heat current fluctuations from $A_2$ to $A_1$ is given by

$$\mathcal{S}_{H_{21}} = 2 k T_2^2 \frac{d\bar{H}_{21}}{d T_2}$$ (2.4.13)

Spectral density of total heat current fluctuations, is

$$\mathcal{S}_H = \mathcal{S}_{H_{12}} + \mathcal{S}_{H_{21}}$$

Radiations from $A_1$ and $A_2$ being independent (no reflections).

$$\mathcal{S}_H(\nu) = 2 k \left( \frac{T_1^2}{d T_1} \frac{d\bar{H}_{12}}{d T_1} + T_2^2 \frac{d\bar{H}_{21}}{d T_2} \right)$$ (2.4.14)

If the two bodies $A_1$ and $A_2$ are in thermodynamic equilibrium with each other then

$$\langle T_1 \rangle = \langle T_2 \rangle = T_0$$

and

$$\mathcal{S}_H(\nu) = 4 k T_0^2 \frac{d}{d T_0}$$ (2.4.15)

where $g$ = conventional thermal conductance between the two bodies $A_1$ and $A_2$

$$g \equiv \frac{\bar{H}_{12} - \bar{H}_{21}}{T_1 - T_2} = \frac{d\bar{H}_{12}}{d T_1} = \frac{d\bar{H}_{21}}{d T_2}$$ appendix (2b).
Steady state calculation of heat current fluctuations becomes extremely complicated when the bodies $A_1$ and $A_2$ are not black. Reflection from the bodies will destroy the statistical independence of the radiant heat currents.
CHAPTER III

Temperature fluctuations in a conductor in Thermodynamic Equilibrium, Isothermal Case

3.1 Equivalence between heat flow equations and transmission-line equations.

Consider a one dimensional non-inductive transmission line.

Let
\[ R_e = \text{Resistance per unit length} \]
\[ G_e = \text{Conductance per unit length} \]
\[ C_e = \text{Capacity per unit length} \]

then
\[ Y_e(\omega) = \text{shunt admittance per unit length} \]
\[ = \left( j\omega C_e + G_e \right) \]

and
\[ Z_e(\omega) = \text{series Impedance per unit length} \]
\[ = R_e \]

If \( I(x) \) and \( V(x) \) are current and voltage respectively at some point \( x \) then
\[ - \frac{dI}{dx} = Y_e(\omega) V \]
\[ - \frac{dV}{dx} = Z_e(\omega) I \]

and
\[ \frac{d^2V}{dx^2} = Z_e(\omega) Y_e(\omega) V = R_e \left( G_e + j\omega C_e \right) V \]
\[ \frac{d^2I}{dx^2} = Z_e(\omega) Y_e(\omega) I = R_e \left( G_e + j\omega C_e \right) I \]

Consider a thermal conductor of length \( L \) and area of cross-section \( A \).

The specimen is in thermodynamic equilibrium at temperature \( T_0 \). Microscopic temperature fluctuations will be present along the length of the specimen, except at the two ends. Two ends of the specimen are held constant at \( T_0 \).

\[ \begin{array}{c|c}
T_0 & \downarrow \lambda \\
\hline
K & \rightarrow \\
\hline
x=0 & \rightarrow \\
\hline
x=L & T_0 \\
\end{array} \]

Fig. 3.1.1
Heat balance equation, neglecting lateral temperature gradient in the specimen compared to the longitudinal temperature gradient, is given by:

\[ \lambda k \frac{\partial^2 \theta}{\partial x^2} - A \lambda \theta = CA \frac{\partial \theta}{\partial t} \]  

(3.1.2)

where

- \( \lambda \) = lateral conductivity
- \( K \) = longitudinal conductivity
- \( C \) = heat capacity per unit volume

Response of (3.1.2) to a sinusoidal signal at frequency \( \omega \) is

\[ \frac{\partial^2 \theta}{\partial x^2} = (q_\theta + j \omega C_\theta) R_\theta \theta \]  

(3.1.3)

where

- \( q_\theta \) = lateral heat conductance per unit length = \( A \lambda \)
- \( R_\theta \) = longitudinal thermal resistance per unit length = \( (K A)^{-1} \)
- \( C_\theta \) = heat capacity per unit length = \( CA \)

If \( h \) is heat current,

\[ \frac{\partial^2 h}{\partial x^2} = (q_\theta + j \omega C_\theta) R_\theta h \]  

(3.1.4)

Comparing (3.1.3) and (3.1.4) with (3.1.1) it is seen that equations of voltage and current distributions in a RCG transmission line and those for temperature and heat current distributions in a specimen with lateral and longitudinal heat current flows are similar, i.e. (3.1.3) and (3.1.4) can be obtained by replacing

- (i) \( V \) by \( \theta \)
- (ii) \( i \) by \( h \)
- (iii) \( R_e \) by \( R_\theta \)
- (iv) \( G_e \) by \( q_\theta \)
- (v) \( C_e \) by \( C_\theta \)

in (3.1.1).
Thus a conductor with heat flow described by (3.1.2) has an electrical transmission line as its analogue.

3.2 Generalisation of Nyquist theorem.

Consider a 4 terminal network in thermodynamic equilibrium at temperature $T_o$.

![Fig.(3.2.1)](image)

Impedance matrix is defined as follows:

$$
V_1 = I_1 Z_{11} + I_2 Z_{12}
$$

$$
V_2 = I_1 Z_{21} + I_2 Z_{22}
$$

(3.2.1)

Reciprocal of the Impedance Matrix defines the Admittance Matrix:

$$
I_1 = V_1 Y_{11} + V_2 Y_{12}
$$

$$
I_2 = V_1 Y_{21} + V_2 Y_{22}
$$

(3.2.2)

Twiss, R.Q (1955) has shown that the spectral density of cross correlations between the voltages $V_1$ and $V_2$ and the currents $I_1$ and $I_2$ is:

$$
S_{V_1 V_2}^{(f)} = \langle V_1 V_2^* \rangle = 2 k T_o (Z_{12} + Z_{21}^*)
$$

$$
S_{I_1 I_2}^{(f)} = \langle I_1 I_2^* \rangle = 2 k T_o (Y_{12} + Y_{21}^*)
$$

(3.2.3)

The results (3.2.3) were first derived by Takahasi (1941).

3.3 Spectral density of Resistance fluctuations, Isothermal System.

For calculating the spectral density of resistance fluctuations in the specimen equivalent "temperature generators" in series with thermal resistance and "heat current generators" in shunt with the thermal conductance are introduced. These generators are assumed to be uniformly distributed along
the length of the specimen, the generators are not correlated and their spectral densities of fluctuations in temperature and heat current are given by (1.2.4).

The generalisation of Nyquist theorem provides an elegant tool for computing temperature correlations in the specimen.

![Diagram](3.3.1)

Analogous to (3.2.3) spectral density of cross correlations in temperature

$$S_{\theta_1 \theta_2}(f) = \langle \theta_1(x_1) \theta_2^*(x_2) \rangle = 4kT_0^2Re Z_{\theta_1 \theta_2}$$ (3.3.1)

Appendix 3(a), equation (3.a.13)

from appendix 3a for the ends of the conductor clamped at $T_0$.

$$Z_{\theta_1 \theta_2} = Z_{\theta \phi} \frac{\sinh \gamma \theta y_{\theta} x_1 \sinh \gamma \phi y_{\phi} (L-x_2)}{\sinh \gamma \phi y_{\phi} L} x_1 \leq x_2$$ (3.3.2)

where

- $Z_{\theta \phi}$ = Characteristic thermal impedance of the specimen
  - $Re \left( \frac{1}{y_{\theta \phi}} \right)$
  - $\gamma_{\theta \phi} = Re \left( \frac{1}{y_{\theta \phi}} \right)$

= thermal propagation constant of the specimen

If

- $R$ = D.C. electrical resistance of the specimen
  - $R = \frac{1}{A} \int_0^L s(x_1) dx_1$

- $\Delta R = \frac{1}{A} \int_0^L \Delta s(x_1) dx_1$

- $\alpha = \frac{1}{s_0} \frac{d s_0}{dT_0}$ = temperature coefficient of Resistance at $T_0$

- $s_0$ = electrical resistivity of the specimen at $T_0$

- $\Delta R = \frac{1}{A} \int_0^L \alpha s_0 \theta(x_1) dx_1$ (3.3.4)
\[ S_R(f) = \langle \Delta R^2 > = \text{spectral density of resistance fluctuations} \]
\[ = \left( \frac{\alpha^2 g_0^2}{A^2} \right) 4k T_0^2 \int_0^l \text{Re} \ Z_\theta(x) dx_1 dx_1 \]  
\[ (3.3.5) \]
Substituting for \( \langle \theta(x) \theta^*(x') \rangle \) from equation (3.3.1).

Finally on integration in (3.3.5)
\[ S_R(f) = 4k T_0^2 \left( \frac{\alpha^2 g_0^2}{A^2} \right) \text{Re} \left[ L R_\theta v_\theta^{-2} \left\{ 1 - (\bar{v}_L)^{-1} \tan h \left( \theta_L/2 \right) \right\} \right] \]

Putting \( R_\theta = S_0 l / A \) and \( \mu = \rho L / 2 \)
\[ S_R(f) = k T_0^2 \alpha^2 R_\theta^2 v_\theta^2 \text{Re} \left[ \mu^{-2} \left( 1 - \mu^2 \tan h \mu \right) \right] \]  
\[ (3.3.6) \]

If \( I_o \) is a sensing current through the specimen then the spectral density of voltage fluctuations due to temperature fluctuations: \( \theta^{(1)} \)

is given by
\[ S_V(f) = I_o^2 S_R(f) \]
\[ = k T_0^2 \alpha^2 v_\theta^2 L R_\theta \text{Re} \left( \frac{f(\mu)}{\mu^2} \right) \]
\[ (3.3.7) \]

where \( v_\theta = \text{D.C. voltage drop across the specimen} \)
\[ f(\mu) = \mu^{-2} \left( 1 - \mu^2 \tan h \mu \right) \]

Real part of \( f(\mu) = \bar{R}_\mu^{-1} \left[ \cos \bar{\delta} - \bar{v}_3^1 \bar{v}_3 \cos \left( \bar{\delta}_3 - (3 \bar{\delta}/2) \right) \right] \]  
\[ (3.3.8) \]

where
\[ \bar{\mu}^2 = \frac{R_\theta e^{i \delta}}{\gamma_1} \]
\[ \gamma_1 = \frac{R_0}{2} \]
\[ R_\theta = \left( (\theta_0 R_\theta L^2)/4 \right) \left( 1 + \omega^2 \eta^2 \right)^{3/2} \]
\[ \eta = \frac{C_0}{\theta_0} \]
\[ \tan \delta = \omega \eta \quad \delta \leq \pi/2 \]
\[ \gamma_3^2 = \frac{(\cosh \alpha_{1L} - \cos \alpha_{1L})}{(\cosh \alpha_{1L} + \cos \alpha_{1L})} \]
\[ \tan \delta_3 = \frac{\sin (\alpha_{2L})}{\sinh \alpha_{1L}} \]
\[ \alpha_1 = \left( \frac{\theta_0 R_\theta}{2} \right)^{1/2} \left[ \frac{1 + \omega^2 \eta^2}{(1 + \omega^2 \eta^2)^{3/2}} \right] \]
\[ \alpha_2 = \left( \frac{\theta_0 R_\theta}{2} \right)^{1/2} \left[ \frac{1 + \omega^2 \eta^2}{(1 + \omega^2 \eta^2)^{3/2}} \right] \]
As shown in Appendix 3(b) \( \Re f(\mu) \) has following properties:

(i) if \( |\mu| \ll \omega \ll 1 \), \( \Re f(\mu) = \frac{1}{2} \)

(ii) if \( |\mu| \gg 1 \)

\[
\Re f(\mu) = \frac{4}{L^2 g_0 R_0} \frac{1}{1 + \omega^2 \sigma^2}
\]

for \( g_0 R_0 L^2 = 4 \), \( \Re f(\mu) \) has the following form:
4.1 Single temperature system.

In Chapter III an expression for spectral density of voltage fluctuations due to temperature fluctuations was obtained. In the model considered there Macroscopic and Microscopic temperatures were not the same. This resulted in temperature correlations, spectral density of which is given by \( (3.3.1) \).

In the 'single temperature' system it is assumed that both the temperatures, Microscopic and Macroscopic, of the specimen are the same.

4.2 Temperature noise.

Let \( V \) and \( I \) be the instantaneous values of current and voltage applied to the system characterised by a single temperature. If the rate of loss of heat is determined only by the excess temperature, then heat balance equation is

\[
C_i \frac{d \theta}{dt} + Q \theta = P = IV
\]

where \( \theta = \) instantaneous excess of temperature \( = T - T_o \)

\( C_i = \) heat capacity of the system

\( Q = \frac{dP}{d\theta} = \) 'local' thermal conductance (i.e. at the operating point.)

A.C. Impedance of the system under operating temperature \( T_o \) is given by;

Burgess, R.E. (1955)

\[
Z(\omega) = R_o \left[ \frac{\frac{Q + \alpha P_o + j\omega C_i}{Q - \alpha P_o + j\omega C_i}} \right]
\]

where \( R_o = \frac{V_o}{I_o} = \) D.C. resistance at the operating point.

\( P_o = V_o I_o = \) Steady state power dissipated in the system.

\( \alpha = \left. \frac{dR_o}{dt} \right|_{T_o} = \) Temperature coefficient of resistance at \( T_o \).

\[
\frac{dV}{dI} = Z(0) = R_o \left[ \frac{\frac{Q + \alpha P_o}{Q - \alpha P_o}} \right]
\]

(4.2.3)
The non-linearity of I-V characteristic of the specimen is purely a thermal effect. If \( \alpha = 0 \) then \( \frac{dV}{dI} = R_0 \). For frequencies \( \omega \gg \frac{q-xP_0}{C_1} \), due to its thermal inertia the specimen is not able to follow rapid variations in temperatures.

A.C. Resistance of the specimen at angular frequency \( \omega \) is

\[
R(\omega) = R_0 \left[ 1 + \frac{2\alpha P_0}{(\theta-\alpha P_0)(1+\omega^2\sigma_0^2)} \right]
\tag{4.2.4}
\]

and the A.C. reactance is given by

\[
X(\omega) = -R_0 \frac{2\alpha P_0}{\theta-\alpha P_0} \frac{\omega \sigma_0}{1+\omega^2\sigma_0^2}
\tag{4.2.4}
\]

where

\[
\sigma_0 = C_1/\left(\theta-\alpha P_0\right)
\tag{4.2.4}
\]

Eliminating \((1 + \omega^2\sigma_0^2)^{-1}\) from (4.2.4) (a) and (b), (Burgess, R.E.)

\[
R(\omega) = R_0 - \frac{X(\omega)}{\omega \sigma_0}
\tag{4.2.5}
\]

Thus for a system characterized by a single time constant a plot of \( R(\omega) \) and \( \frac{X(\omega)}{\omega} \) is linear. Intercept on the \( \frac{X(\omega)}{\omega} = 0 \) axis gives the D.C. resistance and slope of the linear plot gives the reciprocal of time constant \( \sigma_0 \).

A.C. Responsivity of the single temperature system:

Consider the following set up:

![Fig.(4.2.1)](image-url)
R is the instantaneous resistance of the single temperature system.

V, I are the voltage and current respectively.

There is a fluctuation in heat current from the system to the ambient and vice-versa.

\[ R_L = \text{Load resistance}. \]

\[ e_R \quad \text{and} \quad e_L \] are the equivalent Johnson noise e.m.f. generators in series with the specimen resistance R and the load resistance \( R_L \).

To a first approximation the A.C. components of the equation (4.2.1) satisfy the following equation of heat balance.

\[
(j \omega c_1 + q) \Theta = I_o V_1 + I_1 V_o + H_1
\]  

\[
\alpha \Theta = \frac{V_1 I_o - I_1 V_o}{P_o}
\]

further, from Kirchoff's law

\[
I_1 R_L + V_1 + e_R + e_L = 0
\]

Substituting for \( \Theta \) in (4.2.6) from (4.2.7) and evaluating \( V/I_1 \) from (4.2.8) leads to

\[
I_1 = - (e_R + e_L) + \frac{\alpha V_o H_1}{\overline{Z} + R_L}
\]

equation (4.2.9) states that the current fluctuation \( I_1 \) is the result of three noise e.m.f.'s acting in the circuit. The e.m.f. generator due to temperature fluctuations of the system has the value \( e_H \) given by

\[
e_H = - \frac{\alpha V_o H_1}{\overline{Z} + R_L}
\]

Current fluctuation due to heat current fluctuations only is \( I_{1\theta} \) where

\[
I_{1\theta} = - \frac{\alpha V_o}{\overline{Z} + R_L}
\]
and the spectral density of voltage fluctuations due to heat current fluctuations

\[ S_V(f) = \left( \frac{R_L}{R_0+l} \right)^2 \frac{\alpha^2 V_0^2 S_{H_1}(f)}{(\omega^2 Q_0^2 + 1)^2} \]

\[ S_V(f) = \left( \frac{R_L}{R_0+l} \right)^2 \frac{\alpha^2 V_0^2 S_{H_1}(f)}{\left[ (\omega^2 + \alpha^2 R_0 \frac{R_0-l}{R_0+l})^2 + \omega^2 \xi_1^2 \right]} \]  (4.2.12)

The term \( \eta = \frac{R_0 - R_L}{R_0+l} \) in (4.2.11) takes into account the electro-thermal feedback. Electro-thermal interaction is due to the fact that the power dissipated in the specimen is a function of time because the resistance is changing. The time constant \( \xi_1 = C_1/(\alpha + \alpha P_0 \eta) \) is a function of operating condition.

Under constant current conditions, i.e. \( R_L \gg |Z(\omega)| \)

\[ S_V(f) = \frac{\alpha^2 V_0^2 S_{H_1}(f)}{(\omega^2 Q_0^2 + 1)^2} \]  (4.2.13)

4.3 Discussion of temperature fluctuations in

(1) Isothermal System (Chapter III) and Single Temperature System

For the sake of comparison we will regard the Isothermal and the single temperature systems to be in thermodynamic equilibrium at a temperature \( T_0 \).

Isothermal System:

Two ends of the specimen conductor, i.e. \( X = 0 \) and \( X = L \) are 'clamped' at a temperature \( T_0 \). Temperature fluctuation (spectral density) at a given point on the conductor is given by setting \( X = X_1 = X \) in (3.3.2). At the given point and for fixed parameters of the system spectral density of temperature fluctuations is a maximum when \( \omega \xi^2 << 1 \). With this approximation
where $M^2 = \lambda L^2 / k$

$\eta = L \eta_e = AL \lambda$

The expression (4.3.1) vanishes at the two ends $x=0$ and $x=L$. This is expected because the two ends are clamped at $T_0$. (4.3.1) has a maximum at $x=L/2$.

$$S_{\theta}(\xi) = \frac{4kT_0^2}{\eta} (M/2) \tanh (M/2) \quad \omega \sigma << 1$$

**Single temperature system:**

The single temperature system is characterised by a complete correlation between the temperatures at different points at any instant of time. This situation represents a "thermally short" conduction. Spectral density of temperature fluctuations at all points is the same. In thermodynamic equilibrium $\alpha P_0 / \eta = 0$. With this value (4.2.10) becomes

$$S_{\theta}(\xi) = 4kT_0^2/\eta \quad \omega \sigma << 1$$

A plot of spectral density of temperature fluctuations in the two systems as a function of distance looks like Fig.(4.3.1).
Supposing a sensing current $I_0$ is passed through the two systems, spectral density of the voltage fluctuations due to temperature fluctuations in each case is given by the following:

**Isothermal model**

$$S_v(f) = kT_o^2 \alpha^2 V_o^2 L R_\theta \Re f(\omega)$$  \hspace{1cm} (3.3.7)

**Single temperature model**

$$S_v(f) = \frac{4kT_o^2 \alpha^2 V_o^2}{\eta (1+\omega^2 \tau^2)}$$  \hspace{1cm} (4.2.13)

where $$\eta = \kappa / g = C_0 / g_0$$ and $\alpha P_0 / g << 1$

for $\omega \tau << 1$

$$\frac{S_v(f)}{S_0(f)} = 1 - 2M^{-1} \tanh (M/2)$$  \hspace{1cm} (4.3.4)

For a specimen of fixed length $L$ increasing the longitudinal conductivity $K$ and or decreasing the lateral conductivity $\lambda$ reduces the voltage fluctuations due to temperature fluctuations as compared to that in a single temperature system.

When $\omega \tau \gg 1$ $$\frac{S_v(f)}{S_0(f)} \approx 1$$

Temperature fluctuations in the Isothermal system are equivalent to fluctuations in temperature of a 'string' of single temperature systems. Temperature fluctuations at points farther from the thermally clamped ends of the system are not sensitive to the boundary condition. Fig.(4.3.2) shows the spectral density of temperature fluctuations in the two systems.

![Fig.(4.3.2)](image-url)}
5.1 Selection of Sample

In a metal current noise is not detectable because the number of electrons is fixed. Only two sources of noise are present. They are (i) Johnson Noise and (ii) Temperature Noise. The experimental investigation outlined below is therefore concerned with a metallic conductor. The choice of sample is dictated by the following requirements.

(i) Material of the sample be known.
(ii) Uniformly high temperature be attainable. This will increase the heat current fluctuations.
(iii) Sufficiently high d.c. resistance at the operating point. This helps in improving the noise figure of the noise detecting circuit.
(iv) \( \frac{I_0}{V_0} \frac{dV}{dI} \) should be significantly larger than unity so that a check can be made for the appropriate expression for Johnson noise.
(v) Sufficiently small thermal time constant at the operating point. It is the reciprocal of the time constant that determines the frequency below which temperature noise increases.

Pihlites Axial leaded style (Type 13-7) manufactured by Kay Electric Co. were found acceptable. The lead material of the pinlite is platinum and the filament material is tungsten.

The lamps were examined with a microscope for uniformity of temperature at a suitable value of current. It was observed, in case of some of the lamps, that at a certain bias current the resistance dropped. This on visual check led to the observation that decrease in resistance of the lamp was due to shorting out of adjacent coils of the filament. This was the case with a non-uniformly coiled lamp, Fig.(5.1.1).
5.2 Determination of Parameters

In the measurements outlined below the sample was immersed in an oil-bath contained in a metallic container. This was done to assure that the ends of the filament were thermally clamped to the bath temperature and the metallic container provided an electric shield.

(i) Measurement of $R_0$, $P_0$, $T_0$ and $\alpha$.

D.C. resistance $R_0$ of the sample was measured using a standard wheatstone's bridge circuit for different currents through it. Power dissipated in the lamp was noted. Knowing the resistivity of tungsten as a function of temperature Smithels. (1955) the temperature at the operating points was determined. Temperature coefficient of resistance at the operating temperature was then derived from these tables for tungsten at the operating temperature.

(ii) Measurement of A.C. Impedance

Following bridge circuit was used for measuring the a.c. resistance and reactance of the lamp at the operating point. All the bridge components were carefully shielded against extraneous pickup.

![Bridge Circuit Diagram](image)

Fig.(5.2.1)

$E_1$ is a model 202C Hewlett-Packard low frequency oscillator

$C_1$, $C_2$ are fixed capacitors
$C_3, R_3$ are standard capacitance box and General Radio resistance box respectively. They constitute the variable arm of the bridge.

$L_q$ is a bridge transformer.  

$R$ is the pinlite lamp under investigation.

$D$ is the detecting circuit consisting of Keithley Low Noise pre-amplifier connected to a General Radio Wave Analyser (Type No. 736-A).

$B$ is a variable output d.c. supply.

$A$ is a calibrated ammeter.

Balance condition corresponds to

$$R_s(\omega) = R_e Z(\omega) = \frac{R_3}{(1+\omega^2\sigma_3^2)}$$

$$X_s(\omega) = -(R_3 \omega \sigma_3)/(1+\omega^2\sigma_3^2) = \Im Z(\omega)$$

$$\sigma_3 = C_3 R_3$$

$$\omega = 2\pi f$$

An advantage of using the circuit of Fig. (5.2.1) is that no d.c. passes through the other arms of the bridge excepting $R$. Use of Wave-Analyser as the null detector has the advantage of eliminating any possible harmonics at null point. Band width of the Wave Analyser was $\Delta f = 44/5$.

Performance of the bridge was checked by replacing $R$ by a fixed carbon resistor at room temperature and its a.c. impedance measured over the frequency range $f = 200/5$ to $f = 500/5$. The a.c. bridge readings agreed within $5\%$ with the value of the resistor determined by a d.c. Wheatstone bridge.

5.3 Noise Measurement

(i) Assembly

A pre-amplifier using E810F Philips tube was assembled. Choice of the tubes was dictated by its high value of trans-conductance, which implies low noise tubes. Careful shielding of the input circuit from the
rest of the circuit was necessary to prevent oscillations. Following is a block diagram of the noise detecting circuit used.

![Block Diagram of Noise Detecting Circuit](image)

**Fig. (5.3.1)**

1-1 is the input to noise measuring circuit.

W-S is a wafer switch. This connects the input of the pre-amplifier to the lamp or $R_{std}$.

$R_{std}$ is a General Radio Decade Resistance box kept at room temperature.

W-A is a narrow band $(\Delta f = 4\%$) General Radio Wave Analyser (Type 736-A).

Wirewound resistors are used as source resistance to ensure that Johnson noise is the only source of noise in the power supply.

Source resistance is very much greater than $|Z(\omega)|$. This is done to obtain a constant current source.

(ii) Measurement of Equivalent Noise Resistance ($R_{eq}$).

Equivalent noise resistance of the noise measuring circuit at a frequency is obtained as follows. The input (1,1) is shorted. The mean r.m.s. output $V_{sh}$ is noted in W-A at the chosen frequency. Then a variable Resistance Standard, $R_{std}$, at room temperature is introduced at the input (1,1) and its value adjusted till the average indication of the r.m.s. output $V_{std}$ in W-A is such that $V_{std}/V_{sh} = \sqrt{2}$. The value of $R_{std}$ corresponding to this situation is $R_{eq}(f)$, of the noise measuring circuit.
\( R_{eq}(f) \) of the noise detecting circuit was measured at \( f=40c/s, f=100c/s, f=1000c/s \) and \( f=5000c/s \). The procedure used was to plot \( V_{std}^2 \) as a function of \( R_{std} \) at each of these frequencies. It was observed that the plot was linear and therefore W-A readings were r.m.s. readings for the values of \( R_{std} \) used.

Values obtained for \( R_{eq}(f) \) are as follows:

\[
\begin{align*}
R_{eq}(f) &= 3500 \text{ ohms} \quad f=40c/s. \\
R_{eq}(f) &= 850 \text{ ohms} \quad f=40c/s. \\
R_{eq}(f) &= 275 \text{ ohms} \quad f=1Kc/s \text{ and } 5Kc/s.
\end{align*}
\]

These figures show that \( 1/f \) noise becomes apparent in the system at frequencies below about \( 300c/s \).

(iii) Measurement of noise in the sample.

(a) For checking the accuracy of noise measurements, the lamp in Fig.(5.3.1) was replaced by a fixed resistor at room temperature. W-S was switched to 3 and the mean output meter reading \( V_{ns} \) in W-A was noted. Then W-S was switched to 2 and the value of \( R_{std} \) adjusted till the mean output meter reading \( V_{std} \) was equal to \( V_{ns} \). It was noted that at \( f=40c/s \) the random error in resistance measurement was about \( 20\% \) while at \( f=5Kc/s \) the error in measurement was less than \( 10\% \). This is because the equivalent noise resistance of the noise detecting circuit increases with decreasing frequency.

(b) Lamp current was adjusted to 6.3 m.a. The choice of biasing current is dictated by the need to keep the evaporation rate small enough while obtaining a high enough uniform temperature. As outlined in (a) above W-A was set to 40c/s. W-S was switched to 3 and the mean output meter reading \( V_n \) was noted. Then W-S was switched to 2 and the value of \( R_{std} \) adjusted till the mean output meter reading \( V_{std} \) was equal to \( V_n \).

Keeping the current constant the procedure was repeated for various frequencies up to \( 5Kc/s \).
I-V Characteristic of the filament

Figure (5.4.1)
$R_0(\omega)$ vs $|X_0(\omega)|/\omega$ at the operating point.
5.4 Results

(i) D.C. and A.C. Data

\[ R_a = \text{Resistance of the lamp at the ambient temperature} \]
\[ 296^\circ K = 30 \text{ ohms.} \]

\[ I_o = \text{Current through the lamp at the operating point} = 6.3 \text{ mA.} \]

\[ R_o = \text{D.C. resistance at the operating} = 206 \text{ ohm.} \]

\[ P_o = \text{D.C. Power dissipated in the steady state} = 8.2 \times 10^{-3} \text{ watt.} \]

\[ R_o/R_a = 6.8 \]

Using Smithels (1955) page 638, from the resistivity-temperature relationship for tungsten the ratio \( R_o/R_a = 6.8 \) corresponds to the operating temperature

\[ T_o = 1440^\circ K \]

From the I-V characteristic Fig. (5.4.1) \( \frac{dV}{dI} \) at the operating point is equal to 400 ohm.

A.C. Impedance measurements at low frequency \( f = 20c/s \) gives

\[ \text{Re}(Z) = 395 \pm 15 \text{ ohm.} \]

Thus \( \frac{dV}{dI} \) obtained from I-V characteristic (at the operating point and the values of \( \text{Re} Z(f) \) for \( f=20c/s \) agree within the experimental error in measurement.

A plot of \( R_s(\omega) \) and \( \left| X_s(\omega) \right|/\omega \) is linear, Fig. (5.4:2). The slope of the graph gives the value of characteristic time \( \tau_o = 5 \times 10^{-3} \text{ sec.} \) This implies that the lamp at the operating point can be treated as a system having a single characteristic time.
Power dissipated as a function of Temperature

Figure (5.1.3)
(ii) Discussion of D.C. and A.C. Data

For a single temperature system (4.2.3)

\[
\frac{dV}{dI} = R_0 \left( \frac{1+\left(\alpha \frac{P_0}{Q}\right)}{1-\left(\alpha \frac{P_0}{Q}\right)} \right)
\]

Substituting the measured values of \( \frac{dV}{dI} \) and \( R_0 \) gives

\[ \alpha \frac{P_0}{Q} = 0.32 \]

As a check the value of \( \alpha \frac{P_0}{Q} \) can be independently calculated by determining the individual parameters involved.

\( \alpha \) is found from Smithells (loc. cit) at the operating point.

\[ \alpha = 8.3 \times 10^{-4} \text{ K}^{-1} \]

Knowing \( P_0 \) as a function of \( T \) Fig. (5.4.3) \( \frac{\partial Q}{\partial T_0} \) at \( T_0 \) is calculated

\[ \frac{\partial Q}{\partial T_0} = 1.86 \times 10^{-5} \text{ W K}^{-1} \text{ K}^{-1} \]

Using the individual values of \( \alpha \), \( P_0 \) and \( Q \) gives

\[ \alpha \frac{P_0}{Q} = 0.37 \]

The two values of \( \alpha \frac{P_0}{Q} \) obtained from independent means show fairly close agreement providing a confirmation of the lamp at the operating point as a Single Temperature system.

Thus by two different criteria (plot of \( R_5(\omega) \) vs \( \omega \) and the \( \alpha \frac{P_0}{Q} \) comparison) we have corroborated the essential feature of the model used in the theory. It is also noted from Fig. (5.4.3) that the power dissipated tends to be proportional to \( T^4 \) for temperatures in the neighbourhood of \( T_0 \).
Noise Temperature of filament as a function of frequency
Experimental Curve (Dotted line)
Theoretical Curve (solid line)
5.5 Noise Data

We define a noise temperature $T_n(f)$ of the lamp at the operating point as a function of frequency as follows:

$$T_n(f) = \frac{Q_{\text{total}}(f)}{4kR_0 \Delta f} \quad (5.5.1)$$

where $Q_{\text{total}}(f)$ is mean square voltage fluctuation in the lamp at frequency $f$.

$k = \text{Boltzmann constant}$

$R_0 = \text{D.C. resistance of the lamp at } T_0 \text{ K}.$

Experimental values of $T_n(f)$ as a function of frequency are plotted in Fig.(5.5.1).

Discussion of noise data.

We have shown that the lamp at the operating point acts as a Single Temperature System. Total noise in the system is composed of two components (i) Johnson noise, (ii) Temperature noise. We assume that as far as temperature fluctuations due to radial heat losses are concerned the two components are independent. In the theoretical expression for noise temperature $T_n(f)$ we add the mean square voltage fluctuations due to each

$$Q_{\text{total}}(f) = Q_J(f) + Q_{\text{temp}}(f) \quad (5.5.2)$$

We adopt the view that for a metal

$$Q_J(f) = \text{Johnson noise of the conductor} = 4kT_0 R_0 \Delta f \quad (5.5.3)$$

Since the filament of the lamp is in radiative contact with the ambient at $296^0\text{K}$ and total emissivity of tungsten at the operating temperature is 0.2, Smithels (1955) page 666, we assume that equation (2.4.14) is valid. At the operating temperature $(T_0/T_a)^5 \gg 1$. Hence the dominating term in $S_H(f)$ in (2.4.14) is

$$S_H(f) = \frac{2kT_0^2}{f} \quad (5.5.4)$$
Temperature noise from (4.2.13) and substituting for \( S_H(f) \) from (5.5.4) is
\[
Q_{\text{temp}}(f) = \frac{2kT_o^2}{\xi} \frac{\alpha^2 v_o^2}{(1 - \alpha \phi/g)^2} \frac{df}{1 + \omega^2 \xi_o^2}
\]
(5.5.5)
we define \( T_n(f) \) as before:
\[
T_n(f) = \frac{Q_{\text{med}}(f)}{4kT_o \xi}
\]
\[
T_n(f) = T_o \left[ 1 + \frac{1}{2} \frac{(\alpha \phi/g)}{(1 - \alpha \phi/g)^2} \frac{\alpha T_o}{1 + \omega^2 \xi_o^2} \right]
\]
(5.5.6)
(mean value of \( \alpha \phi/g = 0.34 \))

Substituting for the parameters in (5.5.6) at the operating point yields
\[
T_n(f) = T_o \left[ 1 + \frac{0.45}{1 + \omega^2 \xi_o^2} \right]
\]
(5.5.7)

In Fig. (5.5.1) theoretical value of \( T_n(f) \) is plotted as a function of frequency.

In passing, we note that Brophy (1963) has reported measurements of noise in a similar sample of pinlite operating under similar conditions. He has used expressions for temperature fluctuations and heat current fluctuations that are not appropriate to the experimental conditions to which they are presumed to apply. For example, at the high temperature (1600°K) at which his noise measurements are made, contrary to his assumption, Newton's law of cooling does not apply. Electro-thermal interaction, our factor \( (1 - \alpha \phi/g) \), at the operating point in his experiments does not appear in his expression for temperature noise. His expression for Nyquist noise is erroneous for he has assumed that Nyquist noise is given by
\[
4kT_o \Re Z(\omega)
\]
in the metal. It is worth while to recall at this point that it is purely the thermal inertia of the system at the operating point that gives rise to \( Z(\omega) \) different from \( R_o \). His experimental result
\( T_n(0)/T_o(0) = 15 \) is probably due to a component of \( 1/f \) noise. It is noted that the product of mean square noise voltage and frequency in the range \( f=0 \) to \( f=30 \text{c/s} \) is appreciably constant.

Now reverting to the discussion of our own data, it is observed that for frequencies much less than the characteristic frequency \( f_c = \sqrt{]}2\pi f_o} \), the upper limit to the theoretical \( T_n \) is

\[
\left( T_n \right)_{\text{MAX}} = 2100^\circ K
\]

and \( T_n = T_o \) for frequencies in the range \( 800 \text{c/s} \) to \( 5 \text{Kc/s} \).

Comparing the experimental value of \( T_n(f) \) with the theoretical value in Fig. (5.5.1) we observe that the experimental curve agrees with the theoretical curve for frequencies in the range \( 800 \text{c/s} \) to \( 5 \text{Kc/s} \).

In this frequency range the experimental noise temperature \( T_n \) of the filament is observed to be \( 1550^\circ K \). This agrees with the conventional temperature (Thermodynamic) of the filament at the operating point within the experimental error. This implies that the temperature noise is absent at higher frequencies and the conventional Nyquist theorem \( (4kT_oR_o) \) is applicable to the filament at \( 1450^\circ K \).

There is a definite increase in the experimental noise temperature for frequencies less than \( 800 \text{c/s} \). The rise is of the type expected from the appearance of temperature fluctuations at low frequencies. However, the frequency at which this appears is higher by a factor of about 5 than the characteristic frequency \( (\sqrt{]}2\pi f_o} \) derived from impedance data. This discrepancy remains unexplained.
6.1 Conclusions

Experimental results indicate that for frequencies much greater than the characteristic frequency the noise temperature of the pinlite filament is in close agreement with the conventional (thermodynamic) temperature. This would imply that the electrons in the conductor are essentially in equilibrium with the lattice and that temperature fluctuations are negligible at frequencies greater than the characteristic frequency.

Results for noise temperature for frequencies of the order of characteristic frequency seem to indicate a definite increase in noise temperature. The results though not conclusive make a strong case for the existence of temperature fluctuations. However, there remains an unexplained discrepancy between the frequency at which these fluctuations begin to appear and the theoretical characteristic frequency.

6.2 Recommendations for future projects

Pinlites being coiled filaments with essential non-uniformities are not necessarily the most elegant samples for measurement of temperature fluctuations at frequencies for which small enough noise figure of the noise detecting circuit is attainable. The only parameter an experimenter can hope to adjust is the temperature of the filament. This again is limited by the need to keep the rate of evaporation small.

Following experimental projects are expected to lead to conclusive experiments regarding temperature fluctuations.

(1) A thin straight wire of a metal, with diameter of a few microns and length a few cms. To increase the characteristic frequency $\omega$ will have to be increased. This can be done by choosing a material of high emissivity.
It is advisable to work under conditions of radiative cooling only. This provides the possibility of operating point at which \( \frac{V}{V_0} \frac{dV}{dI} \) is much greater than unity.

(ii) Using a sensitive infra-red detector, spectral density of heat current fluctuations from a black body can be measured. This experiment will provide a check for the expression for non-thermodynamic equilibrium fluctuations in heat current between two black bodies derived in (2.4).

Following theoretical problems suggest themselves.

(i) Extension of general statistical considerations for obtaining an expression for heat current fluctuations in a conductor with a temperature gradient and also when both electrons and phonons are the carriers of heat energy.

(ii) Extension of the arguments outlined in (2.4) to spectral density of heat current fluctuations between two non-black bodies in radiative contact.
APPENDIX 1(a)

Temperature fluctuations in Thermodynamic Equilibrium

Two bodies with heat capacities $C_1$ and $C_2$ and instantaneous temperatures $T_1$ and $T_2$ respectively are in thermal contact. $q$ is the thermal conductance. $T_0$ is the thermodynamic equilibrium temperature of the system.

$$q = \frac{\Delta T}{\Delta \theta}$$

$H$ = nett heat flow between the two bodies

$\Theta$ = difference in temperature between the bodies leading a flow of heat $H$.

![Diagram](image)

Total change in entropy of the system due to a spontaneous flow of heat from $T_2$ to $T_1$ is calculated as follows

$$T_1 = T_0 + \Delta T_1$$
$$T_2 = T_0 + \Delta T_2$$
$$\Delta T = \Delta T_1 \left( \frac{\Delta T_2}{\Delta T_1} - 1 \right)$$

(1.a.1)

Spontaneous flow of heat from $A_2$ to $A_1$ results in no nett change in energy

$$\Delta S = \int_{T_0}^{T_2} \frac{C_1 dT_1}{T_1} + \int_{T_0}^{T_2} \frac{C_2 dT_2}{T_2} = \text{total change in entropy} = -\frac{1}{2T_0^2} \frac{C_1 \Delta T_1^2}{C_2} \left(C_1 + C_2 \right)$$

(1.a.2)

Probability of a fluctuation in temperature $\Delta T_1$ in $C_1$ is

$$W(\Delta T_1) \propto e^{-\frac{\Delta S}{k}} \propto e^{-\frac{\Delta T_1^2 C_1(C_1+C_2)}{2RT_0^2C_2}}$$

(1.a.3)

Yields

$$\Delta T_1^2 = \frac{(RT_0^2C_2)}{C_1(C_1+C_2)}$$

Covariance

$$\frac{\Delta T_1 \Delta T_2}{\Delta T_1^2} = \frac{-kT_0^2}{C_1+C_2}$$

(1.a.4)
Negative sign of the covariance implies that there is anti correlation between the temperatures of the two bodies. This is a consequence of the fact that the spontaneous flow of heat is not accompanied by any nett change in energy of the system.

Spectral density of temperature fluctuations

Consider a body of heat capacity $C_1$ in thermal contact with an infinite heat bath at temperature $T_0$. $\Theta(t)$ is an instantaneous temperature difference between the body $C_1$ and the ambient.

Heat balance equation is

$$C_1 \frac{d\Theta}{dt} + q \Theta = H(t) \quad (1.a.5)$$

$H(t) =$ fluctuating heat current with a sharp correlation function

and $\overline{H(t)} = 0$

Solution of (1.a.5) is $\Theta(t) = \Theta(0) e^{\frac{-\beta_t}{C_1}} \left( e^{\frac{-\beta_t}{C_1}} - e^{\frac{-\beta_t}{C_1}} \int H(t) \, dt \right)$

$$\beta_t = \frac{q}{C_1}, \quad \frac{\Theta(t)}{\Theta(t+\delta)} = e^{\frac{-\beta_t}{\Theta(t)}} \quad (1.a.6)$$

Spectral density of temperature fluctuations is (Wiener-Khinchin theorem)

$$S_\Theta(\omega) = 4 \int_0^\infty e^{-\frac{\beta_t}{\Theta(t)}} \cos \omega \delta \, d\delta$$

$$= \frac{4K T_0^2}{\delta \left( 1 + \omega^2 \delta^2 \right)} \quad (1.a.7)$$

where $\Theta(t) = K T_0 / C_1$, from $\Theta(t) = \Delta T$, and $C_2 = \infty$ in (1.a.3)

$\delta = C_1 / T$

As far as fluctuations in temperature in $C_1$ are concerned analogous to electrical case (1.1), we introduce series temperature generators and shunt heat current generators to give (1.a.7)

\[ \text{Fig. (1.a.2)} \]
Spectral densities of the series temperature generator and shunt heat current generator are given by

\[ S_\theta(f) = 4kT_0^2 \Re \, Z_\theta(\omega) \]

\[ Z_\theta(\omega) = (q + j\omega c_i)^{-1} \]

\[ S_H(f) = 4kT_0^2 q \]

Thus in thermodynamic equilibrium we have an analogue of Nyquist theorem for heat current and temperature fluctuations.
APPENDIX 2(a)

Relationship between variance and spectral density

Consider the following system. It has perfect memory up to a certain time interval $T$ and no memory later. $T$ is the sampling time.

$I(t)$ is the frequency response of the system to a unit delta function.

\[ A(f) = \int_0^\infty I(t) e^{-j\omega t} dt = \int_T^{-j\omega T} e^{-j\omega t} dt = \frac{e^{-j\omega T} - 1}{-j\omega} \quad (2.a.1) \]

Let \[ y(t) = \int_{t-T}^t X(u) du \]

Then \[ S_y(f) = \text{spectral density of fluctuations in } y \]

\[ = |A(f)|^2 S_x(f) \quad (2.a.2) \]

where \[ S_x(f) = \text{spectral density of fluctuations in } X. \]

\[(\text{Var}_T y) = \text{Variance of } y \text{ as measured in the sampling time } T. \]

\[ = \int_0^\infty S_y(f) df = \int_0^\infty S_x(f) |A(f)|^2 df \quad \text{from (2.a.2)}. \]

For sampling time $T \gg \text{longest characteristic time } \tau_{\text{max}} \text{ of the system (i.e. in the case treated in (2.4) } T \gg \text{max transit time of photons between } A_1 \text{ and } A_2) \]

\[ S_y(f) \text{ is uniform} \]

\[ (\text{Var}_T y) = \int_0^\infty S_x(f) |A(f)|^2 df \quad (2.a.3) \]

This simplifies to

\[ S_x(f) = \frac{2(\text{Var}_T y)}{T} \quad (2.a.4) \]
(2.a.3) is quite a general statement. If \( S_x(f) \) has the following form

\[
\begin{align*}
\mathcal{S}_x(f) \\
\frac{1}{f_{\min}} \quad f \quad \frac{1}{f_{max}}
\end{align*}
\]

Fig. (2.a.1)

then for (2.a.3) to be valid sampling time \( T \) should satisfy the following inequality:

\[
\frac{1}{f_{\min}} \geq T \geq \frac{1}{f_{max}}
\]
APPENDIX 2(b)

Thermal Conductance in Equilibrium

Consider the two bodies $A_1$ and $A_2$ in thermodynamic equilibrium. $\overline{H_0}$ is the heat flow in each direction. Temperature of $A_1$ is raised so that

$$\overline{H_{12}} - \overline{H_0} = \frac{\partial \overline{H_{12}}}{\partial T_1} (T_1 - T_0)$$

(2.b.1)

Similarly

$$\overline{H_{21}} - \overline{H_0} = \frac{\partial \overline{H_{21}}}{\partial T_2} (T_2 - T_0)$$

if $T_1 - T_0 = T_2 - T_0$  $\overline{H_{12}} = \overline{H_{21}}$  (thermodynamic equilibrium)

Yields,  $\frac{\partial \overline{H_{12}}}{\partial T_1} = \frac{\partial \overline{H_{21}}}{\partial T_2}$

(2.b.2)

from (2.b.1) and (2.b.2)

$$\overline{H_{12}} - \overline{H_{21}} = \frac{\partial \overline{H_{12}}}{\partial T_1} (T_1 - T_2)$$

hence

$$\frac{\overline{H_{12}} - \overline{H_{21}}}{T_1 - T_2} = \frac{\partial \overline{H_{12}}}{\partial T_1} \equiv Q = \frac{\partial \overline{H_{21}}}{\partial T_2}$$

(2.b.3)

$Q$ is the conventional thermal conductance.
APPENDIX 3(a)

Distributed fluctuation generator approach for a transmission line in thermodynamic equilibrium.

Method:

E.m.f. generators in series with electrical resistance and current generators in parallel with electrical conductance are introduced uniformly along the line. Spectral density of these generators are given by 1.1.5. Voltage and current components due to the two types of generators at given points are obtained. Replacing voltage by temperature current by heat current equivalent expressions for temperature correlations is written down. Appropriate value for spectral density of the series temperature generators and shunt heat current generators, proved in 1.2.3, is used.

Outline of procedure.

\[
\begin{align*}
\mathcal{T}_b & \quad \mathcal{R}_e \\
\mathcal{C}_e & \quad \mathcal{G}_e \\
\mathcal{I}_1 & \quad \mathcal{I}_2 \\
X=0 & \quad X=x_1 \quad X=x_2 \quad X=L
\end{align*}
\]

Fig.(3.a.1)

Assumptions: (i) The generators \( \mathcal{E}(x) \), \( \mathcal{I}(x) \), \( \mathcal{E}(x) \) and \( \mathcal{I}(x) \) are completely uncorrelated.

(ii) Response of the system is linear.

\[
\begin{align*}
\mathcal{I}(x) &= \text{amplitude of shunt (electric) current generator} \\
\mathcal{I}_1 &= \text{total (electric) current at a point } X_1 \\
\mathcal{I}_2 &= \text{total (electric) current at a point } X_2 \\
V_1 \text{ and } V_2 \text{ are voltages at the points } X_1 \text{ and } X_2 \text{ respectively} \\
\mathcal{E}(x) &= \text{amplitude of series e.m.f generator} \\
\mathcal{Y}_e(\omega) &= \text{admittance per unit length} \\
\mathcal{Z}_e(\omega) &= \text{impedance per unit length} \\
\gamma &= \text{propagation constant} = \mathcal{Y}_e \mathcal{Z}_e \\
\mathcal{Z}_e &= \text{characteristic impedance} = Z_e / \gamma
\end{align*}
\]
Series e.m.f generators and shunt current generators contribute to current and voltage at any given point on the line. These contributions are calculated from the following formulae.

\[-\frac{dV}{dx} = Z_e(e) i + e(x)\]
\[-\frac{dI}{dx} = V_e(e) V + I(x)\]  \hspace{1cm} (3.1)

Boundary condition is \(V(0) = V(L) = 0\)

**Part I: Current and voltage at two points due to an e.m.f generator.**

'Switch off' the shunt current generators.

\[E_e(x_1, x') = \text{current at a point } x = x_1 \text{ due to a shunt current source at } x'\]

Superscript 'e' refers to component due to series e.m.f generator and 's' refers to component due to shunt current generators.

\[E_e(x_1, x') = e(x') \frac{\cosh \gamma(L-x') \cosh \gamma x_1}{\sinh \gamma L} X_1 \leq x'\]
\[E_e(x_2, x') = e(x') \frac{\cosh \gamma(L-x_2) \cosh \gamma x'}{\sinh \gamma L} X_2 > x'\]
\[V_1^e(x, x') = -e(x') \frac{\sinh \gamma x_1 \cosh \gamma (L-x')}{\sinh \gamma L} X_1 \leq x'\]
\[V_2^e(x, x') = e(x') \frac{\sinh \gamma (L-x_2) \cosh \gamma x'}{\sinh \gamma L} X_2 > x'\]  \hspace{1cm} (3.2)
Currents and voltages at $X_1$ and $X_2$ due to a shunt current generator.

\[
\begin{array}{c}
\text{Part II} \\
\text{Fig. (3.a.3)} \\
\end{array}
\]

\[\therefore \begin{align*}
\dot{I}^S_{S}(x_{1},x_{1}') &= \frac{I(x') \cosh \gamma x_{1} \sinh \gamma (x_{1}' - L)}{\sinh \gamma L} \\
\dot{I}^S_{S}(x_{2},x_{1}') &= \frac{I(x') \cosh \gamma (L-x_{2}) \sinh \gamma x_{1}'}{\sinh \gamma L} \\
V^S_{1}(x_{1},x_{1}') &= \frac{\sinh \gamma x_{1} \sinh \gamma (L-x_{1}')}{\sinh \gamma L} I(x') \\
V^S_{2}(x_{2},x_{1}') &= \frac{\sinh \gamma (L-x_{2}) \sinh \gamma x_{1}'}{\sinh \gamma L} I(x')
\end{align*}\]

Total voltage at $X_1$ and $X_2$ is obtained by adding the components due to shunt current generators and series e.m.f generators and then integrating over $X_1'$. Thus taking into account all the generators.

\[V_{1}(x_{1}) = \int_{x_{1}}^{x_{1}'} V^{S}_{1}(x_{1},x_{1}') dx_{1} + \int_{x_{1}}^{L} V^{S}_{1}(x_{1},x_{1}') dx_{1} + \int_{0}^{x_{1}} V^{e}_{1}(x_{1},x_{1}') dx_{1} + \int_{x_{1}}^{L} V^{e}_{1}(x_{1},x_{1}') dx_{1}
\]  

\[\text{(3.a.4)}\]
Similarly an expression for $\bar{V}_2(x_1) \ (\text{complex conjugate of } V_2(x_1))$ is obtained.

Integral expressions for $V_1(x_1)$ and $\bar{V}_2(x_1)$ are multiplied and bearing in mind the assumption (i) is a delta function.

\[
\begin{align*}
S(f) = \langle I(x') I^*(x'') \rangle &= 4 k T_o G \delta(x''-x') \\
\langle V_1 \bar{V}_2 \rangle &\text{ is obtained.}
\end{align*}
\]

\[
S(f) = \langle e(x') e^*(x'') \rangle = 4 k T_o R e \delta(x''-x')
\]

\[
S(f) = \langle e(x') I^*(x'') \rangle = 0
\]

\[
\bar{Z}_{12} \text{ for a RCG transmission line shorted at } x=0 \text{ and } x=L
\]

\[
\begin{align*}
V_1 &= I_1 \bar{Z}_{11} + I_2 \bar{Z}_{12} \\
V_2 &= I_1 \bar{Z}_{21} + I_2 \bar{Z}_{22} \\
\bar{Z}_{12} &= (\bar{Z}_{11} - \bar{Z}_{1SC2}) \bar{Z}_{22}
\end{align*}
\]

where

\[
\begin{align*}
\bar{Z}_{1SC2} &= \frac{Z_c \sinh \gamma x_1 \sinh \gamma (x_2-x_1)}{\sinh \gamma L} \quad x_2 > x_1 \\
&= \left( \frac{V_1}{I_2} \right) V_2 \to 0 \\
\bar{Z}_{12} &= \frac{Z_c \sinh \gamma x_1 \sinh \gamma (L-x_2)}{\sinh \gamma L} \quad x_2 > x_1
\end{align*}
\]
A careful algebra then leads to the following

$$\langle V_1 V_2^* \rangle = \sum_{V_1, V_2}^{(e)} V_1 \cdot 4 R T_0 R e(Z_{12})$$  \hspace{1cm} (3.a.8)

Spectral density of current correlations.

Total current at $x_1$ and $x_2$, as in the preceding case is obtained by adding the components due to series e.m.f. generators and shunt current generators and then integrating over $x$.

From (3.a.2) and (3.a.3)

$$\mathcal{I}_1(x_1) = \sum_{x_1 \leq x_1} \mathcal{I}^S(x_1, x') dx' + \sum_{x_1 \geq x_1} \mathcal{I}^S(x_1, x') dx' + \sum_{0}^{L} \mathcal{I}^E(x_1, x') dx'$$

Similarly an integral expression for $\mathcal{I}_2(x_2)$ is obtained.

As before, bearing in mind the relations (3.a.5) an expression for $\langle \mathcal{I}_1 \mathcal{I}_2^* \rangle$ is obtained.

Admittance matrix is defined by:

$$\mathcal{Y}_{11} = \mathcal{Y}_{11} V_1 + \mathcal{Y}_{12} V_2$$

$$\mathcal{Y}_{22} = \mathcal{Y}_{21} V_1 + \mathcal{Y}_{22} V_2$$

$$\frac{\mathcal{I}_1}{\mathcal{Y}_{11}} = V_2 \quad \text{when} \quad V_1 = 0$$

$$\mathcal{Y}_{22} = \frac{V_2}{\mathcal{I}_2} = \mathcal{Z}_c \left[ \text{tanh} \, \gamma (L - x_2) + \text{tanh} \, \gamma x_2 \right]$$

$$\frac{\mathcal{I}_1}{\mathcal{I}_2} = \frac{\cosh \, \gamma x_1}{\cosh \, \gamma x_2}$$

\hspace{1cm} (3.a.10)
\[ y_{12} = \left( \frac{I_1}{V_1} \right) = \frac{\cosh \gamma x_1 \cosh \gamma (L-x_2)}{\zeta \sinh \delta L} x_2 > x_1 \] (3.1.11)

Careful algebra leads to the following

\[ \langle I_1, I_2^* \rangle = \sum_{x_1} = 4 k T_0 R_0 y_{12} \] (3.1.12)

**Temperature Correlations**

For calculating the spectral density of temperature correlations following replacements are made in the preceding analysis.

- \( \mathcal{E}(x) \) by \( \mathcal{H}(x) \)
- \( T(x) \) by \( H(x) \)
- \( V_1 \) by \( \Theta_1 \)
- \( I_1 \) by \( \Theta_1 \)
- \( Z_6(\omega) \) by \( Z_6(\omega) \)
- \( y_{\Theta}(\omega) \) by \( y_{\Theta}(\omega) \)

The heat and current generator and temperature generator at any given point are not correlated; also the heat current generators and temperature generators at different points are not correlated with themselves nor with each other;

\[ \langle H(x') H(x'') \rangle = 4 k T_0^2 R_0 \delta(x''-x') \]
\[ \langle \Theta(x') \Theta(x'') \rangle = 4 k T_0^2 R_0 \delta(x''-x') \]
\[ \langle \Theta(x') H(x) \rangle = 0 \] (3.1.13)

For spectral density of temperature correlations

\[ \langle \Theta_1(x_1) \Theta_1^*(x_2) \rangle = 4 k T_0^2 R_0 Z_{\Theta,12} \]

where

\[ Z_{\Theta,12} = Z_{\Theta} \frac{\sinh \gamma_0 x_1 \sinh \gamma_0 (L-x_2)}{\sinh \delta_0 L} \quad x_1 \leq x_2 \]
APPENDIX 3(b)

Properties of \( f(\mu) \)

\[
|\mu| << 1 \quad \omega \delta << 1
\]

\[
f(\mu) = \frac{1}{\gamma_0} \left[ e^{-i \delta} - \frac{2}{15} \gamma_0^{2} \gamma s + \frac{17}{315} \gamma_0^{3} \gamma s \right]
\]

\[
\delta = \omega \delta
\]

expanding \( e^\delta \) and \( e^{2i \delta} \) up to terms in \( \omega^3\delta^3 \)

\[
\text{Re} \, f(\mu) = \frac{1}{3} - \frac{\gamma_0}{15} \left[ (2 - \frac{17}{21} \gamma_s) + \omega^2 \delta^2 \left( \frac{34}{21} \gamma_0 - 1 \right) \right]
\]

\[
\text{Im} \, f(\mu) = \frac{\gamma_0 \omega \delta}{15} \left[ (\frac{34}{21} \gamma_0 - 2) + \omega^2 \delta^2 \left( 1 - \frac{68}{21} \gamma_0 \right) \right]
\]

\[
\lim_{\omega \delta \to 0} \frac{|\mu|}{\omega \delta} \Rightarrow 0
\]

\[
\text{Re} \, f(\mu) = \frac{1}{3}
\]

(3.2.1)

\( \text{Re} \, f(\mu) \), for \( \omega \delta << 1 \) has an upper bound = \( 1/3 \).

\[
|\mu| >> 1
\]

\[
f(\mu) = \mu^{-2} \left[ 1 - \mu^{-1} \tan \mu \right]
\]

\[
= \frac{4}{L^2 \rho_0 \delta_0} \left( \frac{q_0 - j \omega \rho_0 \theta}{1 + \omega^2 \delta^2} \right)
\]

\[
\text{Re} \, f(\mu) = \frac{4}{L^2 \rho_0 \delta_0} \frac{1}{1 + \omega^2 \delta^2}
\]

(3.2.2)
BIBLIOGRAPHY

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