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ABSTRACT

Under certain circumstances, a massive object travelling through a medium may experience a net average deceleration due to the gravitational interaction between it and the medium. This slowing down effect is called dynamical friction.

There are a variety of ways in which dynamical friction may be modeled, depending primarily on the nature of the medium, and on what approximations are deemed to be reasonable. This thesis is devoted to reviewing the various models, with an emphasis on the assumptions underlying each, to pinpointing the source of any discrepancies between the models, and to assessing the validity of each.

For this purpose the models are grouped into two categories. The first comprises those in which the medium consists of dust or gas. The models in the second describe the interaction between a test object and a medium consisting of other objects of its own mass. These latter are of two general types: those which employ the two-body approximation, and those which describe the interaction in terms of a stochastically fluctuating force arising from the varying distribution of field objects surrounding the test object.

The models lead to three different expressions for the dynamical friction experienced by the object: one
proportional to $T \log X$ ($X$ is the distance from the object), one proportional to $T \log vT$ ($v$ is the average velocity of the members of the system), and one finite one proportional to $T \log D_0$ ($D_0$ is the mean inter-object spacing).

It is argued that the $T \log D_0$ result is based on an analysis which does not take into account the long range nature of gravitational forces. The $T \log X$ models rely on the behaviour of the medium being stationary, but it is demonstrated that it takes an infinite amount of time to establish such behaviour, so these models are not strictly applicable to any real situation.
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INTRODUCTION

Under certain circumstances, a massive object travelling through a medium of some description may experience a deceleration due to the gravitational interaction between it and the medium. This effect is called dynamical friction. In what follows, various models of dynamical friction will be reviewed, compared, and assessed.

Gravitational dynamical friction was first examined in the 1940's and early 1950's in two different contexts. One type of model (cf. Bondi, Hoyle, Lyttleton, Dodd, McCrea, Danby, and Camm) grew primarily out of an investigation of the rate at which stars accrete material from the interstellar medium. The physics underlying the accretion process was applied to such diverse problems as terrestrial ice ages (Hoyle and Lyttleton, 1939), the formation of comets (Lyttleton, 1953), and of interest here, the deceleration of a star which is in motion with respect to the interstellar medium (cf. Bondi and Hoyle (1944), Dodd and McCrea (1952)).

This type of model of dynamical friction is reviewed in Chapter I, and is based heavily on orbit theory. The interstellar medium is taken to be infinite in extent, and composed of non-self-gravitating particles which are collisionless and have uniform density upstream of the star. These particles follow hyperbolic trajectories in the
gravitational field of the star, and in the frame of reference in which the latter is at rest, their streaming motion is assumed to be stationary. The simplest model assumes that the particles remain collisionless downstream of the star, and have no thermal motion (section (a)). In more complicated models, allowance is made for collisions along the downstream axis (section (b)), and for non-negligible thermal motion among the particles (section (c)). The expressions for the deceleration of the star yielded by these models, diverge logarithmically with distance.

Because of the difficulties involved in incorporating the thermal motion of the particles in the framework of an orbit based model, fluid mechanical models of the gravitational interaction between a massive object and a gaseous medium, have been developed (cf. Spiegel (1969), Ruderman and Spiegel (1971), Hunt (1971)). These models are modified so that they may be applied to extended objects (ie. galaxies) as well as to 'point' objects (ie. stars, blackholes).

Linearized and non-linearized fluid-mechanical analyses of dynamical friction are reviewed in sections (b) and (c) of Chapter II. In section (d), the linearized model is extended to include the self-gravity of the medium.

It is found that in the case of subsonic motion, there is no dynamical friction, while in the supersonic case, the expression for the dynamical friction diverges logarithmically as in the orbit based models.
At the same time as the orbit based models of dynamical friction were being developed, Chandrasekhar (see Chandrasekhar (1941-1944), Chandrasekhar and von Neumann, (1942)) was laying the foundations of a statistical theory of stellar encounters in which a 'test' star is viewed as acted upon by a stochastically fluctuating force which arises from the varying distribution of 'field' stars surrounding it. The fundamental ideas of this theory are described in section (a) of Chapter IV.

Although Chandrasekhar suggested how a fully stochastic model could be used to examine the dynamical friction experienced by a test star, such a calculation was not performed until many years later, by Lee (1966) and Kandrup (1980). Their approaches are described in section (b) of Chapter IV. They are based on different aspects of the statistical theory, and lead to disparate results. Lee finds that the expected change in the test object's velocity, $<\Delta V>$, is zero, while the quantity $<(\Delta V)^2>$ diverges logarithmically with time. Kandrup on the other hand, finds that both $<\Delta V>$ and $<(\Delta V)^2>$ are proportional to $T\log D_0$, where $D_0$ is the mean inter-object spacing.

Chandrasekhar (1943) did derive an expression for the deceleration of a test star, based on the approximation that it undergoes a succession of complete binary encounters with the field stars. This calculation is described in Chapter III, along with two other binary encounter models which, unlike Chandrasekhar's, do not assume complete collisions, but rather, take into account the time dependence,
particularly of distant encounters. These latter models result in expressions for the dynamical friction in which the divergence with distance is replaced by a divergence with time.

In Chapter V the various results are summarized and discussed. Suggestions as to the sources of the discrepancies between them are put forth, and the validity of each model is assessed.
CHAPTER I

DYNAMICAL FRICTION EXPERIENCED BY A MASSIVE OBJECT TRAVELLING THROUGH AN INFINITE CLOUD OF DUST

Introduction

In this chapter, models of the dynamical friction experienced by a massive object moving through an infinite cloud of dust, are presented. Their general structure is as follows. The object is considered to be at rest, while the cloud streams past it. The particles in the cloud follow hyperbolic trajectories in the gravitational field of the object, and are 'focused' along the downstream axis (see figure a-1). The change in momentum of the particles as they pass the object, is calculated using orbit theory.

The total change in the particles' component of momentum transverse to the direction of the object's motion, vanishes by symmetry. The component parallel to the direction of the object's motion does not vanish, and is accompanied by an equal and opposite change in the object's momentum in this direction, with the result that it slows down.
An alternate way of viewing this effect is in terms of the drag force exerted on the object by the region of enhanced density which forms in its wake.

A variety of different assumptions are made concerning interactions between the particles. In all cases their mutual gravitation is ignored. In section (a) the situation in which the particles are collisionless and have no thermal motion, is examined. Section (b) studies the case where thermal motion is negligible, and the particles are collisionless upstream of the object, but collide along the downstream axis. In section (c), the effect of thermal motion among the particles is discussed. Section (d) summarizes the results.
(a) Orbit Theory Based Models of Dynamical Friction:
Collisionless Particles with Negligible Thermal Motion

The simplest orbit theory based model of dynamical friction is that in which the particles do not collide and have no thermal motion: the pressure and temperature of the cloud are negligible. Such a model is presented, for example, by Dodd and McCrea (1952).

In the frame in which the object is at rest, the particles at upstream infinity are assumed to have uniform density $\rho_0$, and uniform velocity $V$, directed along the x-axis (see figure a-1). In an interval of time $\Delta t$, a wedge of particles passes through a plane $P_i$ at upstream infinity. After a time $T$ it reaches plane $P_f$ at downstream infinity, by which time its momentum has changed by $\Delta p$. The average rate of change in the momentum of the wedge is therefore $\Delta p/T$. There are $N=T/\Delta t$ such wedges between upstream infinity and downstream infinity. The total change in momentum of the cloud per unit time, is thus the change per unit time of one wedge, times the number of wedges, or

$$\frac{\Delta \rho_0 \cdot T}{\Delta t} = \frac{\Delta \rho_0}{\Delta t} = \frac{d\rho_0}{dt}$$

This is equal to the force exerted by the object on the cloud per unit time, which is the negative of the force exerted on the object by the cloud, per unit time:
\[
\frac{d \rho_c}{dt} = F_{\rho \to c} = -F_{c \to \infty}
\]

The particles describe hyperbolic orbits around the object, and recede to downstream infinity with velocity \( \dot{r}_\infty \) (see figure a-1).

At downstream infinity, they have all picked up a component of velocity transverse to the x-axis, and their component along this axis has changed. These components may be calculated using orbit theory, and the results are (cf. Dodd and McCrea):

\[
u^* \approx V - \frac{2V}{1 + \frac{\gamma^2 V^4}{G^2 M^2}}
\]

**Figure a-1.**
When the changes in momentum that accompany these changes in velocity are summed over the individual particles, the component transverse to the axis vanishes by symmetry, so only the component parallel to the axis remains. The particles passing the object thus pick up a component of momentum \( A \) (see figure a-1), so to conserve the momentum of the object-cloud system, the object must pick up an equal and opposite component, \( B \). The magnitude of this component is calculated as follows.

The amount of matter crossing an element of area, \( s ds \phi \), of a plane \( P \), upstream of the object, and perpendicular to the \( x \)-axis, per unit time, is

\[
\rho \, V \, s \, ds \, d\phi
\]

The change in momentum in this direction per unit mass, of particles starting at upstream infinity, and receding to downstream infinity, is

\[
V - V\infty
\]

so the total change in the \( x \)-component of the momentum of the cloud, per unit time, is

\[
\frac{dP_x}{dt} = \int \rho \, V \, (V - V\infty) \, s \, ds \, d\phi
\]
Substituting expression (a-1) for $v_{\perp}$, and integrating over $\phi$ from 0 to $2\pi$, and $s$ from 0 to infinity, gives

$$\frac{dp_s}{dt} = \frac{2\pi G^2 M^2 \rho_o}{V^2} \ln \left[ 1 + \frac{s^2 V^4}{G^2 M^2} \right] \bigg|_{s=0}^{s=\infty}$$

Conservation of momentum dictates that the momentum of the object must change by an amount equal and opposite (a-5). The deceleration of the object is therefore

$$\frac{dV}{dt} = -\frac{2\pi G^2 M \rho_o}{V^2} \ln \left[ 1 + \frac{s^2 V^4}{G^2 M^2} \right] \bigg|_{s=0}^{s=\infty}$$

which diverges logarithmically as the impact parameter $s$, goes to infinity.

This model is based on the assumption that the particles are collisionless, however the focusing effect of the object's gravitational field may produce a density enhancement along the downstream axis which is sufficiently large that collisions become important. In this case the assumption breaks down, and a model including collisions along the downstream axis is required. Such a model is presented in the next section.
(b) Orbit Theory Based Models of Dynamical Friction

Including Collisions along the Downstream Axis

Orbit theory based models of dynamical friction in which collisions along the downstream axis are taken to be important, while the temperature of the cloud is still assumed to be negligible, are presented by Bondi and Hoyle (1944) and Dodd and McCrea (1952). In these models, the flow upstream of the object is assumed to be collisionless. However, streams of particles are focused along the downstream axis, creating a region of enhanced density along this axis in which collisions can no longer be ignored.

The model which will be described here is simplified in the following way. To begin with, it is assumed as in the previous section, that the density at upstream infinity is uniform. This implies that particles arriving at the downstream axis collide in such a way that the component of their velocity which is perpendicular to the downstream axis is cancelled out, while the component along the axis is conserved (c.f. Dodd and McCrea). Cooperative behaviour is established as particles with velocity greater than the escape velocity from the object's gravitational field move off to infinity along the downstream axis, while those with velocity less than the escape velocity flow in towards the object, and are accreted (see figure b-1). Under these circumstances, incoming particles will collide, not only
with other incoming particles, but also with particles already in the 'accretion column', so strictly speaking, more complicated effects may be present.

Dodd and McCrea consider the case where the upstream density varies radially, and Bondi and Hoyle discuss some effects that arise in the case where an accretion column forms. Neither of these considerations influences the results significantly, so for the sake of simplicity, they will be left out of the discussion.

As in section (a), the deceleration of the object is found by calculating the change in momentum of the particles as they pass the object. The asymptotic velocity $v_{\infty}$ at downstream infinity is determined as in section (a) with one difference: the initial velocity is taken to be the velocity the particle has immediately after arriving at the downstream axis, rather than the velocity at upstream infinity. Immediately after colliding on the x-axis, a particle has velocity $V$ in the x-direction, and its component of velocity transverse to the x-axis is anhilated.

The result is

$$v_{\infty}^2 = V^2 - \frac{2GM}{x}$$

In the limit of large impact parameters, this reduces to (a-1).

Material arriving with kinetic energy less than its potential energy due to the gravitational field,

$$\frac{1}{2} m V^2 < \frac{GMm}{x}$$
or \( x < 2x_0 \), where

\[
\chi_0 = \frac{GM}{V^2}
\]

remains gravitationally bound to the object. That with kinetic energy greater than the potential energy, \( x > 2x_0 \), escapes to infinity. (See figure b-1.)

The total amount of material arriving at the downstream axis, per unit area, per unit time, is as before, (see expression (a-3))

\[
\sqrt{\rho_0} \, s \, ds \, d\phi
\]

The impact parameter \( s \), is related to the distance downstream where the particle strikes the axis by (cf. Dodd and McCrea)

\[
\chi = \frac{s^3 \, V^2}{2 \, GM}
\]

Thus, \( d\chi = \frac{V^2}{GM} s \, ds \), or with definition (b-2), \( \chi \, d\chi = s \, ds \)

\[\text{(b-5)}\]

Figure b-1.
The total rate of change of the momentum of the material is, as in section (a) (equation (a-4)),

\[
\frac{d\rho_c}{dt} = \int \rho_o \left( v - v_n \right) x_o d\phi d\chi \tag{b-6}
\]

The asymptotic velocity \( v_n \) is given by (b-1).

Assuming the dominant contribution to the drag force is due to the cumulative effect of more distant particles (ie. \( x >> x_e \)), (b-1) can be approximated as

\[
v_n = \sqrt{\left(1 - \frac{2x_o}{x}\right) v^2} \approx \sqrt{\left(1 - \frac{x_o}{x}\right)} \tag{b-7}
\]

Inserting this in (b-6) and integrating over \( x \) from \( 2x_o \) to infinity gives

\[
\frac{d\rho_c}{dt} = \frac{2\pi G^2 M^2 \rho_o}{v^2} \ln \frac{x v^2}{GM} \bigg|_{x=\infty}
\]

The change in the momentum of the object must be equal and opposite this, so

\[
\frac{dV}{dt} = - \frac{2\pi G^2 M \rho_o}{v^2} \ln \frac{x v^2}{GM} \bigg|_{x=\infty} \tag{b-8}
\]

The assumption that it is the cumulative effect of distant material that is primarily responsible for the deceleration of the object, may be justified a posteriori by comparing (b-8) with expression (a-6). Substituting (b-4) into (a-6) yields
The effect of particles near the object was taken into account in deriving this expression, and the only difference between it and (b-8) is that the argument of the logarithm is multiplied by two and increased by one.

In the case where $\chi \gg \chi_0$, 

$$\ln \left(1 + \frac{2\chi}{\chi_0}\right) \approx \ln \frac{\chi}{\chi_0}$$

so (b-9) becomes

$$\frac{dV}{dt} = -\frac{2\pi \sigma G^2 M \rho_0}{V^2} \ln \frac{\chi V^2}{G M} \bigg|_{\chi = \infty}$$

which is identical to expression (b-8). This suggests that collisions along the downstream axis play an insignificant role in the dynamical friction experienced by the object.

The main conclusion to be drawn from sections (a) and (b), are:

(1) the expression for the dynamical friction experienced by a massive object moving through a cloud of particles which is of infinite extent, and has negligible temperature, diverges logarithmically with distance;

(2) this result is insensitive to whether or not collisions occur along the downstream axis, and also to whether the object accretes any material.
It now remains to examine the case in which the temperature of the cloud is not negligible. This is discussed in the next section.
The discussion in this section will not provide an analytic expression for the dynamical friction suffered by the test object: the mathematical complexities encountered in incorporating the thermal motion of the particles, do not allow such an expression to be derived. Rather, this discussion is included because it provides a link between the models described so far, and those of the following chapters.

Danby and Camm (1957) and Danby and Bray (1967) examine the case of a massive object travelling through an infinite, uniform cloud of particles which are collisionless, but which have non-negligible thermal motion. In the frame of reference in which the object is at rest, the particles at upstream infinity move with mean velocity V, and obey a Maxwellian distribution. Using orbit theory, Danby and Camm derive an expression for the perturbation in the density of the cloud, caused by the object. They assume, as in sections (a) and (b), that the flow is steady in the rest frame of the object, and they calculate the number of particles per unit time, impinging on a section of a cone as pictured in figure c-1.
In the limiting case of zero temperature, they find the expression
\[ \rho = \rho_0 \left[ \frac{2GM}{\sqrt{\alpha^2 + \sin^2 \Theta}} \right]^{\frac{3}{2}} \csc \Theta \left[ \frac{GM}{\sqrt{\alpha^2 + \sin^2 \Theta}} \right] \]

This expression has the following features.

1. Since \( \csc(0) = \infty \), the density is infinite along the downstream axis.
2. At \( \Theta = \pi \), (i.e. along the upstream axis), it becomes
\[ \rho = \rho_0 \left[ \frac{2GM}{\sqrt{\alpha^2 + 1}} \right]^{\frac{3}{2}} \left[ \frac{GM}{\sqrt{\alpha^2 + 1}} \right] \]

In the limit
\[ a \ll \frac{GM}{\sqrt{\alpha}} \]

this reduces further to
Physically, in the immediate vicinity of the object, the density falls off as $1/\sqrt{a}$. In the limit

$$c \gg \frac{C_s M}{v^2}$$

(i.e. at positions a long way upstream of the object), $\rho = \rho_0$, as required.

(3) The same limiting expressions apply at positions away from the axis.

Expression (c-1) was derived under the assumption that there are no collisions along the downstream axis: particles impinge on $a$ from both sides, and pass unperturbed through the downstream axis. This is the same situation as in section (a), and although no expression for the density of the cloud was derived there, it was suggested that the focusing effect of the object's gravitational field might produce a sufficient enhancement along the downstream axis, that collisions would become important. Expression (c-1) indicates that this is indeed true: the density is formally infinite along the downstream axis. This suggests that the models of section (b), which incorporated collisions along the downstream axis, is more accurate. However, it has already been demonstrated that the difference between the models of sections (a) and (b) is negligible. Likewise, it is straightforward to derive an expression equivalent to
(c-1), but in which particles only impinge on the cone from one side (i.e. they do not cross the axis) (c.f. Danby and Camm). Under these circumstances, the expression for the density is not altered significantly.

Expression (c-1) for the density may be used to calculate the rate of deceleration of the object. The force on the object is equal to its mass times its acceleration:

\[ F = M \frac{dV}{dt} \quad \text{so} \quad \frac{dV}{dt} = \frac{F}{M} \quad \text{c-4} \]

The force is given by

\[ F = \int \left( \frac{GM \rho \alpha^2}{c^2} \right) \alpha \sin \theta \, d\alpha \, d\theta \, d\phi \quad \text{c-5} \]

Inserting (c-1) gives

\[ \frac{dV}{dt} = (cpo) \left\{ \left[ \left( \frac{2GM}{V^2 \alpha} + \frac{\sin^2 \theta}{2} \right)^{-1/2} \csc \theta \left[ \frac{GM}{V^2 \alpha} + \frac{\sin^2 \theta}{2} \right] - 1 \right] \alpha \sin \theta \, d\alpha \, d\theta \, d\phi \right\} \]

There is no \( \phi \) dependence, so

\[ \frac{dV}{dt} = 2\pi G \rho_0 \left\{ \left[ \left( \frac{2GM}{V^2 \alpha} + \frac{\sin^2 \theta}{2} \right)^{-1/2} \csc \theta \left[ \frac{GM}{V^2 \alpha} + \frac{\sin^2 \theta}{2} \right] - 1 \right] \alpha \sin \theta \, d\alpha \, d\theta \right\} \]

To get an idea of the behaviour of the density at large distances, the integrand may be expanded in a Taylor series around \( 1/a = 0 \). The lowest order term vanishes: the first non-zero contribution is

\[ \delta \rho(a \to \infty) = \rho_0 \left[ \frac{2GM}{V^2} - \frac{1}{2} \right] \left[ \csc^2 \theta \right] \frac{1}{a} \]
It exhibits what will be seen in Chapter II, to be a characteristic $1/a$ dependence, which leads to the same logarithmic divergence already encountered.

For non-zero temperature, the expression for the density does not become infinite along the downstream axis. However, the mathematics becomes sufficiently complicated that an analytical study of the density enhancement is not feasible. A numerical analysis (c.f. Danby and Bray) yields the results pictured in figure c-2.

The shape of the isodensity contours depends mainly on the quantity

$$\alpha \equiv \sqrt{\frac{m}{kT}}$$

Here $m$ is the average mass of a particle, $k$ is Boltzman's constant, and $T$ is the temperature of the cloud at infinity.

For $\alpha=0$ (i.e. for $V=0$) the contour is spherically symmetric about the object. For $\alpha=0.5$ it is still spherically symmetric, but it is centred on a point somewhat downstream of the object. For $\alpha=4$ and greater, the contours extend only behind the object, and become more and more elongated as $\alpha$ increases.
Qualitatively, figure c-1 indicates that an object travelling at any speed will experience dynamical friction. However, in the absence of an analytic expression for either the density or velocity field of the cloud, an analytic expression for the dynamical friction cannot be derived.

Including a non-zero temperature is not expected to influence the results obtained in the last two sections, for the following reason. One of the effects of a non-zero temperature is to randomize the upstream flow of the particles, and thus decrease the frequency of collisions along the downstream axis. With no thermal motion, the particles approaching from upstream infinity with uniform velocity $V$, will be focused in such a way that they all pass exactly through the downstream axis. The density there will therefore be high, and collisions frequent. However, if the

\[ p = m = \frac{V}{\sqrt{m/kT}} \]

\[ M = M_\odot \left( \sqrt{\frac{kT}{m}} \right) = \frac{1}{\kappa} \sqrt{\frac{m}{\tau}} \]

Figure c-2.

From Danby and Bray (1967)
particles have random thermal motion upstream, some will pass a small distance away from the downstream axis, so the density will not be as high. A non-zero temperature therefore has the effect of reducing the importance of collisions along the downstream axis.

However, it has already been shown in sections (a) and (b) that collisions along the downstream axis play an insignificant role in the dynamical friction experienced by the object.

It will be seen that the claim that thermal motion among the particles does not influence the dynamical friction experienced by the object, is substantiated by the model presented in Chapter III, section (a). There a test object undergoing binary encounters with field objects of its own mass, which have peculiar velocities satisfying a Maxwellian distribution, is examined, and the result is an expression for the test object's deceleration which diverges logarithmically with distance.

The main value of the discussion in this section is the link it furnishes between orbit theory based models of dynamical friction, and fluid mechanical models, which will be presented in the next chapter. The pressure in the cloud, due to the thermal motion of the particles, may be described (as in the next chapter) by the perfect gas law:

\[ p = \frac{k}{m} \rho T \]

The quantity \( \alpha \) is then

\[ \alpha = \sqrt{\frac{p}{\rho}} \]
But \((P/\rho)\) is equal to the speed of sound, \(c\), in a gas undergoing an isothermal change, so
\[
\alpha = \frac{V}{c}
\]
In figure c-1, the spherical contours therefore correspond to subsonic motion, and the football-shaped contours, to supersonic motion.

The difficulties encountered in incorporating a non-zero temperature suggest that a more satisfactory approach might be to treat the cloud as a fluid described by the equations of fluid mechanics, rather than a collection of particles whose motions are described by orbit theory. A fluid mechanical approach to dynamical friction is explored in Chapter II.
(d) Summary

In section (a) it was found that an object travelling through a cloud of collisionless, non-self-gravitating particles which have no thermal motion, is decelerated at a rate

\[
\frac{dV}{dt} = -\frac{2\pi G^2 M \rho_0}{V^2} \ln \left[ 1 + \frac{2\pi V^2}{G M} \right] \bigg|_{x=\infty}
\]

Section (b), in which collisions along the downstream axis were included, gave the result

\[
\frac{dV}{dt} = -\frac{2\pi G^2 M \rho_0}{V^2} \ln \left[ \frac{\pi V^2}{G M} \right] \bigg|_{x=\infty}
\]

This was calculated assuming \( x \) to be large. In the limit of large \( x \), (d-1) is identical to (d-2).

The conclusions to be drawn from this are:

(i) the expression for the dynamical friction suffered by the object diverges logarithmically with distance;

(ii) collisions along the downstream axis have a negligible effect on the rate at which the object is decelerated;

(iii) the dominant contribution to the dynamical friction comes from the cumulative effect of distant
particles. Ignoring the effect of particles in the vicinity of the object does not change the results significantly.

(iv) The accretion process, which is associated with particles near the object, does not influence the rate at which the object is decelerated by dynamical friction.

(v) Although \((d-1)\) and \((d-2)\) were derived in the special case where the density at upstream infinity is uniform, it has been shown by Dodd and McCrea that allowing the density to vary radially does not alter these results.

Section (c), where a non-zero temperature was included, did not yield an expression for the dynamical friction, but for reasons already discussed, thermal motion among the particles is not expected to influence the deceleration significantly. The main value of section (c) is that it represents a link between orbit based models, and the fluid mechanical models of dynamical friction described in the next chapter.
CHAPTER II

FLUID MECHANICAL MODELS OF DYNAMICAL FRICTION

Introduction

In section (c) of Chapter I, a model of dynamical friction was presented in which the temperature of the cloud streaming past an object was included. However, because of the mathematical complexities involved in incorporating non-negligible thermal motion among the particles in an orbit-based theory, a fluid mechanical approach to dynamical friction is explored in this chapter. The basic idea of this approach is that the gravitational field of the object causes an enhancement in the density of the medium surrounding it which, under some circumstances, may exert a drag force on the object, causing it to slow down.

The equations of fluid mechanics used to describe the behaviour of the fluid in the presence of the object are presented in section (a), where the fluid is assumed to be non-viscous and non-self-gravitating.

In section (b) both subsonic and supersonic motion are examined by analysing the linearized equations of fluid
mechanics. It turns out that in the subsonic case, the density enhancement created by the object is symmetric fore and aft, so there is no net drag force. This contradicts the results of Danby and Bray (see Chapter I, section (c)), and a reason for this disparity is suggested. In the supersonic case an enhanced density wake forms behind the object, and the expression for the drag force exerted on the object by this wake diverges logarithmically with distance.

In Chapter I it was argued that it is the cumulative effect of distant material that plays the dominant role in slowing down the object, and that effects in the vicinity of the object, such as accretion, are insignificant as far as the deceleration process is concerned.

The linearized models should provide an accurate description of the behaviour of the material far from the object: there is no reason to expect non-linear effects to be important anywhere but in the immediate neighbourhood of the object, and in the vicinity of the shocks which occur in the case of supersonic motion. Neither of these regions is expected to have a significant influence on the drag force exerted by the medium on the object. However, for the sake of completeness, a non-linearized treatment is discussed briefly in section (c).

In section (d) the linearized model of section (b) is extended to include self-gravity. It is demonstrated that, although self-gravity furnishes an outer cutoff for the divergent integrals at the Jeans length, this cutoff is more mathematical than physical, and including self-gravity does
not get rid of the logarithmic divergences.

Finally, the various results are summarized and discussed in section (e).
(a) Equations of Fluid Mechanics

(i) Equations of Motion and Conservation:

The three basic equations describing the behaviour of a non-viscous, non-self-gravitating fluid in the presence of a massive object are Euler's equation (equation of motion)

\[ \frac{d\vec{v}}{dt} = -\frac{\nabla p}{\rho} + \nabla \psi \]

the equation of conservation of mass

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

and an equation of state, which will be discussed later. \( \psi \) is the gravitational field of the object of mass \( M \):

\[ \psi = \frac{GM}{\alpha} \]

and \( \frac{d}{dt} \) is the Euler derivative:

\[ \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \]

\( p, \vec{v}, \) and \( \rho \) are the pressure, velocity, and density of the fluid.

An alternate set of equations consists of the
conservation equations for mass, momentum, and energy, together with an equation of state. Conservation of momentum is found by combining conservation of mass and Euler's equation:

\[
\frac{\partial \rho \mathbf{U}}{\partial t} + \nabla \cdot \rho \mathbf{U} \mathbf{U} + \nabla p = \rho \nabla \psi \tag{a-5}
\]

To be complete, a Coriolis force

\[
F_c = -\frac{\rho u_v u_\alpha}{\alpha}
\]

and a centrifugal force

\[
F_\alpha = \frac{\rho u_\alpha^2}{\alpha}
\]

should be added to the right-hand side of the momentum conservation (or Euler's) equation.

Conservation of energy is imposed by forming an equation of energy balance comprising the internal plus kinetic energy of a fluid element, the work done on the element by pressure and by gravitational forces, and the energy leaving the fluid through conduction or radiation.

\[
\frac{\partial E}{\partial t} + \nabla \left[ (E + p) \mathbf{U} \right] = \rho \mathbf{U} \cdot \nabla \psi + \kappa \nabla^2 T \tag{a-6}
\]

Here,

\[
E = \rho u + \frac{1}{2} \rho u^2
\]
U is the internal energy of the fluid, and \( k \) is its thermal conductivity.

This latter set is particularly convenient in numerical treatments of supersonic flow, because fluxes of mass, momentum, and energy are conserved across shocks (Landau and Lifshitz, 1959) so shocks appear as steep rises over several intervals, and not as discontinuities.

Another useful expression is Bernoulli's equation, which follows from the steady state equations of conservation of mass and energy. In the absence of radiation or conduction of heat, Bernoulli's equation is

\[
\frac{E + p}{\rho} - \gamma = \text{constant}
\]

The constant on the right is determined from the boundary conditions at infinity.

(ii) Equation of State:

The equation of state, describing the relation between pressure and density in the fluid, is important in determining the nature of the density perturbation caused by the object in the spherically symmetric case, and in the region near the object in the non-linearized case.

The fluid is assumed to be a perfect monatomic gas in thermodynamic equilibrium. The variables \( p, \rho \), and \( T \) are
therefore related by the perfect gas law,

\[ p = \frac{k \rho T}{m} \]

where \( k \) is Boltzman's constant, and \( m \) is the average mass of a particle.

A second relation between these three variables may be imposed by assuming that the gas undergoes a polytropic change in the presence of the object's gravitational field. A polytropic change is a change that is carried out in such a way that

(a) the change is quasi-static (reversible): the system remains infinitesimally close to thermodynamic equilibrium at all times, and

(b) the derivative \( \frac{dQ}{dT} \) varies in a specified way throughout the change.

In a perfect monatomic gas undergoing a polytropic change, the density and pressure are related according to

\[ \frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^\Gamma \]

where \( \Gamma \) is the polytropic index, and \( p_0 \) and \( \rho_0 \) are the unperturbed pressure and density. The two particular cases of interest here are that in which the gas undergoes an adiabatic change, and that in which it undergoes an isothermal change. In the isothermal case, \( dT=0 \), so \( \frac{dQ}{dT}=\infty \), and \( \Gamma=1 \) (c.f. Cox and Giuli (1968)). Physically, the molecules in the gas radiate energy sufficiently rapidly
that the temperature of the gas remains constant throughout. In the adiabatic case $dQ=0$, so $dQ/dT=0$, and

$$\gamma = \frac{c_p}{c_v} = 5/3.$$  

$c_p$ and $c_v$ are the specific heats of the fluid at constant pressure and volume respectively (c.f. Cox and Giuli). The assumption underlying this is that cooling is unimportant: no energy is conducted or radiated by the gas as it streams past the object. Thus, the choice of polytropic index embodies assumptions concerning the radiation of energy by the molecules in the gas.

(iii) Frames of Reference:

It is assumed that the flow is steady in the frame of reference in which the object is at rest, so it is often convenient to work either in a spherical polar or a cylindrical coordinate system whose origin is fixed to the moving object (see figure a-1).

![Figure a-1](image-url)
The \((a, \theta)\) coordinates are related to the \((x, r)\) coordinates according to

\[
\begin{align*}
  a^2 &= r^2 + r^2 \\
  \sin^2 \theta &= \frac{r^2}{a^2} = \frac{r^2}{x^2 + r^2}
\end{align*}
\]
(b) Linearized Fluid Mechanics

The natural starting place for a fluid mechanical study of dynamical friction, is with a linearized treatment. In the frame of reference in which the medium is at rest and the object is moving with speed \( V \), Euler's equation (a-1), the equation of conservation of mass (a-2), and the polytropic equation of state (a-7), may be written

\[
\frac{d\mathbf{v}}{dt} + \mathbf{v} \cdot \nabla \rho = \nabla \psi \tag{b-1}
\]

\[
\frac{\partial \rho}{\partial t} + \rho (\mathbf{v} \cdot \nabla) = 0 \tag{b-2}
\]

\[
\frac{\rho}{\rho_0} = \left( \frac{\rho}{\rho_0} \right)^\Gamma \tag{b-3}
\]

These are combined by taking \( \nabla \cdot (b-1) \), \( \frac{\partial}{\partial t} (b-2) \), and using \( (b-3) \) to eliminate pressure. The quantities \( \rho, \rho_0 \), and \( \mathbf{v} \) are perturbed:

\[
\rho = \rho_0 + \delta \rho \tag{b-4}
\]

\[
\rho = \rho_0 + \delta \rho \tag{b-4}
\]

\[
\mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v} \tag{b-4}
\]
\( p_0, \hat{v}_0, \) and \( \rho_0 \) are constants. It is assumed that \( \delta p \ll p_0 \) and \( \delta \rho \ll \rho_0 \), and that there is a frame of reference in which the unperturbed medium is at rest: \( \hat{v}_0 = 0 \). When the polytropic relation is perturbed this way, it becomes

\[
\delta p = c^2 \delta \rho \tag{b-5}
\]

which is independent of the polytropic index. Here the expression for the speed of sound in a fluid,

\[
c^2 = \frac{\Gamma p^*}{\rho^0} \tag{b-6}
\]

has been used. The linearized model will therefore yield the same result, independent of whether the medium is assumed to undergo an adiabatic or an isothermal change in the presence of the object. Using expression (a-10), the linearized equation becomes

\[
\frac{1}{c^2} \frac{\partial^2 \delta p}{\partial t^2} - \nabla^2 \delta p = - \frac{\rho_0}{c^2} \nabla^2 \psi \tag{b-7}
\]

Figure b-1 defines the notation.
In this coordinate system, Poisson's equation is

$$\nabla^2 \psi = -2GM \frac{\delta(r)}{r} \delta(x)$$  \hspace{1cm} (b-8)  

where $x = z - \nu t$. Inserting this and the cylindrical Laplacian into (b-7), and defining

$$\Omega^2 \equiv 1 - \frac{\nu^2}{c^2}$$  \hspace{1cm} (b-9)  

yields

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \Omega^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{2GM \rho}{c^2} \frac{\delta(r)}{r} \delta(x)$$ \hspace{1cm} (b-10)  

Symmetry considerations dictate that there is no angular dependence. Equation (b-10) can be solved using Fourier-Hankel transforms (cf. Dokuchaev (1963)) (see appendix A), and the results are as follows.
(i) Subsonic Case:

In the case where the object is moving subsonically, $\mathcal{R} > 0$, so

$$\delta \rho = \frac{2GM\rho_0}{c^2} \frac{1}{(x^2 + y^2 + r^2)^{3/2}}$$  \hspace{1cm} \text{b-11}

The isodensity surfaces (pictured in figure b-2) are ellipsoids centred on the object, with major axis perpendicular to the object's direction of motion. The eccentricity of the ellipsoids is $\epsilon = V/c$, so as $V$ approaches the speed of sound in the medium, the isodensity contours become increasingly eccentric. For $V=0$ the perturbation becomes spherically symmetric about the object.

It is interesting to compare this with the results of Danby and Camm (1957), and Danby and Bray (1967) in section (c) of Chapter I. Expression (c-1), derived by Danby and Camm for the density enhancement in a medium with no thermal motion, was not symmetric about the object. Including pressure results in an expression which, in the subsonic case, is symmetric about the object. Furthermore, Danby and Bray's numerical results indicate that when thermal motion is included, the subsonic contours (i.e. $V \sqrt{m/kT} < 1$) are not centred on the object, whereas in the fluid mechanical treatment, they are.

These differences arise because in the present case, the trajectories of the fluid elements, and hence the density, are determined by $\nabla \rho$ as well as $\nabla \psi$, while in
the orbit theory based model, even when the thermal motion was included, the trajectories of the particles were determined only by $\nabla \psi$.

(iii) Supersonic Case:

In the case where the object is moving supersonically ($\beta^2 < 0$), the density perturbation is

$$\delta \rho = \begin{cases} \frac{2GM\rho}{c^2} \frac{1}{(x^2 + \beta^2 r^2)^{1/2}} & x^2 < \beta^2 r^2 \\ 0 & x^2 > \beta^2 r^2 \end{cases} \quad \text{b-13}$$
The isodensity surfaces are hyperboloids with the object at the centre (see figure b-3). The eccentricity is again equal to $V/c$.

Danby and Bray's results are suggestive of this in that their contours extend only downstream of the object, and become increasingly narrow with large $V$.

Figure b-4 shows the change in the isodensity contours of the density perturbation as $V$ goes from zero to infinity.
One feature which was not present in the subsonic case is the singularity at \( x^2 = -x_0^2 \). It indicates the presence of a shock, which is a line with slope \( \frac{x}{r} = \left( \frac{\gamma}{2} - 1 \right)^{-\frac{1}{2}} \), passing through the origin, as pictured in figure b-3. The linearized treatment contains insufficient information to determine whether it is a tail or a bow shock.

(iii) Drag Force:

In both the subsonic and supersonic case, the radial dependence of the density perturbation is roughly \( 1/a \) (see figure a-1), and is independent of the polytropic exponent. In the subsonic case, the perturbation is symmetric fore and aft, and there is no net drag force exerted on the object. In the supersonic case, an enhanced density wake forms behind the object.

It decelerates the object at a rate given by

\[
\frac{dV}{dt} = \frac{F_2}{M} = \frac{G \, \frac{Sp \, x^2}{(x^2 + r^2)^{3/2}}} \\
\]

Inserting (b-13) for the density, and integrating over all volume, yields (see appendix B for calculation):

\[
\frac{dV}{dt} = -\frac{\pi G^2 M \rho_0}{\sqrt{1}} \left( 1 - \frac{2 \, \Omega^2}{\sqrt{1 + \Omega^2}} \right) \ln r \bigg|_{0}^{\infty} 
\]

The linearized model thus indicates that in the case of subsonic motion, a test object experiences no dynamical friction, while in the case of supersonic motion, the expression for its deceleration diverges logarithmically
with distance, as in the orbit based models.

There is no reason to expect non-linear effects to be important far from the object: the linearized model provides an accurate description of the outer regions of the density enhancement. A non-linearized model will still be examined, however since the results of Chapter I indicate that it is the cumulative effects of distant particles that play an important role in decelerating the object, it is not expected that any non-linear effects will significantly influence the expression for the dynamical friction already obtained.
The results of section (b), where a linearized treatment of dynamical friction was presented, showed that the density perturbation created by an object moving subsonically is symmetric, while in the supersonic case, an enhanced density wake forms downstream of the object. The radial dependence of the perturbation is independent of the polytropic exponent. Although the linearized treatment provides an accurate description of the perturbation far from the object, it is interesting to see how the results obtained from a non-linearized analysis differ from those of the linearized case.

In view of the difficulties involved in a full, non-linearized treatment of the dynamical interaction between a moving object and the medium through which it moves, it is instructive to investigate the simplified case in which the object is at rest with respect to the medium. Although there is obviously no dynamical friction in this case, the results might be suggestive of what to expect in terms of the radial dependence of the density perturbation caused by a moving object.

To begin with, the implications of the assumption that the medium undergoes a polytropic change in the presence of the object's gravitational field, are examined.

A polytropic change is one that is carried out
infinitesimally close to thermodynamic equilibrium, and one of the conditions of thermodynamic equilibrium is hydrostatic equilibrium: outward directed pressure gradients must balance inward directed gravitational forces. In the limiting case where the object is at rest with respect to the medium, and everything is spherically symmetric, this condition determines the radial dependence of the density enhancement as follows (cf. Cox and Giuli (1968)). Consider a fluid element a distance \(a\) from the object, as pictured in figure c-1.

The pressure gradient gives rise to a force on the surface \(A\) of the element which is

\[ dF_r =dp \, dA \]  \hspace{1cm} c-1
The gravitational force on the element due to the object is

\[ \frac{dF_g}{dA} = \frac{GM}{a^2} \frac{dm}{dA} \quad dm = \rho \, da \, dA \]

\[ \frac{dF_g}{dA} = \frac{GM\rho}{a^2} \, da \, dA \quad \text{c-2} \]

The condition for hydrostatic equilibrium is that these two forces be equal and opposite:

\[ \frac{dF_p}{dF_g} = \frac{\alpha^2}{GM\rho} \frac{dp}{da} = -1 \quad \text{c-3} \]

Eliminating pressure using the polytropic relation (a-9) gives

\[ \alpha^2 \rho^{\frac{\Gamma-2}{\Gamma}} = -\frac{GM\rho_o^{\Gamma}}{\Gamma \rho_o} = -\frac{GM\rho_o^{\Gamma-1}}{c^2} \quad \text{c-4} \]

Here the expression for the speed of sound in a fluid,

\[ c^2 = \frac{\Gamma \rho_o}{\rho_o} \quad \text{c-5} \]

has been used. In the adiabatic case, \( \Gamma=5/3 \), and solving for \( \rho \) yields

\[ \rho = \rho_o \left[ \frac{2}{3} \frac{GM}{c^2 \alpha} + \text{constant} \right]^{3/2} \quad \text{c-6} \]

In the isothermal case, \( \Gamma=1 \) and solving for \( \rho \) gives
It is interesting to contrast these results with the results obtained assuming negligible pressure and temperature. In this case, the particles in the medium fall freely in the gravitational field of the object, so their motion is one of radial streaming towards the object. The radial acceleration is \( \frac{dv_a}{dt} = \frac{v_a^2}{a} \), and is due to the gravitational force:

\[
\frac{dv_a}{dt} = \frac{F_a}{M} = \frac{GM}{a^2}
\]

so the radial velocity of the particles is

\[
v_a = \left( \frac{GM}{a} \right)^{\frac{1}{2}}
\]

Conservation of mass requires that

\[
\frac{d}{da} (a^2 \rho v_a) = 0
\]

so

\[
\rho v_a \sim \frac{1}{a^2}
\]

and

\[
\rho \sim \frac{1}{a^{3/2}}
\]
The radial dependence is the same as in the case where pressure gradients and gravitational forces are exactly balanced, so that there is no inflow of material towards the object. This indicates that, as the results of Chapter I suggested, the accretion process has no significant effect on the shape of the density enhancement created by the object.

Obviously there is no drag force on the object if it is at rest relative to the medium, so in order to study dynamical friction, the case where the object is moving must be examined. Hunt (1971), Spiegel (1970), and Ruderman and Spiegel (1971) present non-linearized analyses of the behaviour of a fluid in the presence of a gravitating object moving with velocity $V$ with respect to the medium.

Hunt (1971) does a numerical study of the above situation, for both subsonic and supersonic motion, in the case where the medium undergoes an adiabatic change. He uses conservation equations (a-2), (a-5), and (a-6) (with no thermal conductivity term), and equation of state (a-8) with \[ c_r - c_v = \frac{\kappa}{\gamma} \]

\[
\rho = (\gamma - 1) \left[ E - \frac{1}{2} \rho \left( \nu^2 + \nu_0^2 \right) \right]
\]

(E is the sum of internal and kinetic energies.) He considers the object to be a point source, and integrates the full time-dependent fluid equations from a given initial
state until a steady state is achieved (in the frame of reference in which the object is at rest.)

As in the linearized treatment, he finds that in the subsonic case, the isodensity contours are symmetric about the object, while in the supersonic case, a density enhancement forms downstream.

His results do not yield an exact analytic expression for this density enhancement. An approximate expression describing the inner region may be derived, and it agrees with the spherically symmetric case already discussed: the perturbation falls off as

\[ \rho \sim \frac{1}{r^{\nu_2}} \]

If the density continued to fall off at this rate in the outer regions of the perturbation, the model would yield an expression for the drag force with no divergence. However, there is no reason to expect that it does this: it has already been mentioned that non-linear effects are not expected to persist at large distances from the object (or from the shock).

In connection with the latter, Hunt's results indicate that the shock is a bow shock: the linearized treatment did not yield enough information to distinguish whether it preceded or followed the object.
Spiegel (1970) and Ruderman and Spiegel (1971) also present a non-linearized examination of the behaviour of a gas streaming past a gravitating object. They assume that the gas upstream is cold, and they adopt an orbit theory description. In particular they use the expression given by Danby and Camm (1957) (see Chapter I, equation (c-1)) to describe the density of the incident flow.

Downstream of the object they use the equations of fluid mechanics to describe the flow. In particular they use the steady state conservation equations (a-2) and (a-5). In the latter they include Coriolis and centrifugal forces, but ignore the gravitational field of the object, as they assume that the fluid elements arrive on the accretion axis sufficiently far downstream that the gravitational forces due to the object are negligible. (Their model is concerned specifically with an extended object such as a galaxy, rather than with an object which can be treated as a point source.) They also use a polytropic equation of state, derived from (a-9).

The downstream and upstream flows are matched across a conical shock, as suggested by the linearized treatment. For physical reasons, pertaining mainly to the nature of the object, they assume the shock is a tail shock, rather than a bow shock as Hunt's analysis indicates.

The expression they find for the density perturbation is

$$\rho \sim \frac{1}{2} \rho_0 \frac{R}{a} F(\theta)$$

(c-14)
R is the radius of the object. $F(\theta)$ depends on the polytropic index, but the radial dependence does not. The density falls off as $1/a$ as in the linearized case, no matter whether the fluid is assumed to be undergoing an adiabatic or an isothermal change. It is easy to see that, as in the linearized case, the $1/a$ behaviour of the density perturbation will lead to an expression for the dynamical friction which diverges logarithmically with distance.

The results presented in this section indicate that, although a non-linearized treatment provides a more accurate description of the density enhancement in the neighbourhood of the object, and of the shock that forms in the case of supersonic motion, it does not yield any new information as far as the dynamical friction experienced by the object is concerned.

In the next section, the linearized model of section (b) is extended to include the self-gravity of the medium.
(d) A Fluid Mechanical Model of Dynamical Friction
Incorporating the Self-Gravity of the Medium

The problem of integrals which diverge logarithmically with distance, encountered here in the context of a massive object interacting gravitationally with its surroundings, also arises in connection with the interaction between a charged particle and the electromagnetic plasma through which it moves.

In the latter case, the presence of oppositely charged particles produces a screening effect (Debye screening) which limits the size of the region which a test particle can influence, or be influenced by. There is therefore a natural cutoff for the diverging integrals at the Debye length.

The similarity between systems of charged particles interacting through Coulomb forces, and systems of massive objects interacting through gravitational forces, has prompted some people to try to find a gravitational analog of Debye screening.

In electromagnetic plasmas, it is the presence of oppositely charged particles that is responsible for Debye screening. In gravitational 'plasmas' however, all the particles have the same 'charge'. Although some attempts have been made to treat objects as 'positive' masses, and empty spaces as 'negative' masses, such negative masses
could not effectively shield objects from the gravitational fields of other objects. No matter how rarefied (ie. how 'negative') a volume of space between two massive objects is, it cannot prevent the objects from feeling each other's presence. It is therefore questionable whether a gravitational equivalent of Debye screening, based on the idea of positive and negative masses, is possible.

The claim is sometimes made (cf. Ruderman and Spiegel) that including the self-gravity of the medium produces a shielding effect similar to Debye screening. In what follows, it is shown that, although there is no evidence for such a screening effect, including the self-gravitation of the medium does imply a cutoff for the diverging integrals. However, as will be pointed out, this is a mathematical cutoff, and does not have the same physical significance as the Debye length.

Consider a self-gravitating medium with density $\rho$, velocity $\vec{v}$, and pressure $p$. The equations of fluid mechanics describing this medium are

\[ \frac{d\vec{v}}{dt} = -\frac{\nabla p}{\rho} + \nabla \phi_{self} \quad \text{(d-1)} \]

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \quad \text{(d-2)} \]

\[ \frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^\gamma \quad \text{(d-3)} \]
These are the same as those in section (a), except that the gravitational potential term on the right hand side of Euler's equation is due here to the self-gravity of the medium, rather than to the presence of a massive object.

Now suppose something perturbs the medium slightly. \( p \), \( \rho \), and \( \bar{v} \) are changed as in the linearized case in section (b):

\[
\begin{align*}
p & = p_0 + \delta p \\
\rho & = \rho_0 + \delta \rho \\
\bar{v} & = \bar{v}_0 + \delta \bar{v}
\end{align*}
\]

(d-1) through (d-3) are combined and linearized as in section (b), and Poisson's equation,

\[
\nabla^2 \psi_{\text{self}} = \mu \pi G \delta \rho
\]

is used.

The result is

\[
\left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{4\pi G \rho_0}{c^2} \right] \delta \rho = 0
\]

This equation (the Jeans wave equation), is essentially a Klein-Gordon equation with a negative mass term. The general solution is

\[
i (k \bar{R} + \omega t) \delta \rho = k e
\]
Inserting this in (d-6) yields the dispersion relation

\[ \omega^2 - c_s^2 k^2 + \frac{4\pi G \rho_0}{c^2} = 0 \]  

d-8

which determines the frequency \( \omega \) and the wave number \( k \) of the perturbation.

There are several possibilities for the temporal and spatial behaviour of the perturbation. Saslaw (1968) for instance, considers a perturbation which grows exponentially with time, and claims that there is a collective shielding effect that causes the perturbation to die off quickly at distances greater than the Jeans length. He derives an expression describing the spatial dependence of the density perturbation which is a damped sinusoid, and it is not clear that this substantiates his claim.

The physical explanation he offers for the shielding is that choosing a specific time dependence forces the perturbation to be spatially coherent, and this coherenence can only be enforced in a region that is small enough so that information can propagate through it in a time less than the characteristic time scale on which the perturbation grows.

For further discussion of Jeans theory of gravitational instabilities, see for example Layzer (1964).

For the present purposes, the type of behaviour that is of interest, is that in which the perturbation is marginally
stable (time independent). Any perturbation which is not time-independent leads to behaviour which is not stationary in the rest frame of the object, and thus invalidates the assumption of stationarity, upon which the present discussion is based.

For a time-independent perturbation,

\[ \omega = 0 \]

In this case (d-8) becomes

\[ \frac{L}{k} = \frac{\mu \pi G \rho}{c^2} = \frac{L}{k_j} \sim \frac{1}{\lambda_j^2} \]

\(k_j\) is the Jeans number, and \(\lambda_j\) is the Jeans length.

The latter corresponds to the characteristic size of the largest possible volume of self-gravitating fluid which would be stable under some perturbing influence. The Jeans length thus furnishes an upper cutoff for the diverging integrals. It does not, however, have the same physical significance as the Debye length. Whereas the Debye length delimits the region which a test particle is capable of effecting or being affected by, the Jeans length delimits the largest region in which an assumption underlying the model, namely that the flow is stationary in the object's rest frame, remains valid. It does not represent a region outside of which the object's influence cannot be felt. In this sense, it is a mathematical cutoff rather than a physical one.
Instead of considering separately the case of a massive object with a non-self-gravitating medium, and that of a self-gravitating medium with no massive object, it is interesting to include both in the same model. Rather than simply providing a mathematical upper cutoff for a diverging integral, self-gravity might have the effect of stopping the integral from diverging in the first place.

Including both effects in the same linearized model involves solving the equation (see appendix C)

\[ \frac{1}{c^2} \frac{\partial^2 \delta \rho}{\partial t^2} - \nabla^2 \delta \rho - \frac{k^2}{c^2} \delta \rho = \frac{2\pi GM\rho}{c^2} \frac{\delta(r)}{\delta(x)} \]  

Again the assumption has been made that the perturbation is time independent.

Equation (d-10) is solved for the case of an object travelling subsonically, in appendix C. The result is

\[ \delta \rho = \frac{GM\rho}{c^2} \frac{1}{(x^2 + \rho^2 r^2)^{3/2}} \cos \left[ \frac{r^2}{\rho^2} \left( x^2 + \rho^2 r^2 \right)^{1/2} \right] \]  

This differs from the non-self-gravitating case by a factor

\[ \frac{1}{2} \cos \left[ \frac{r^2}{\rho^2} \left( x^2 + \rho^2 r^2 \right)^{1/2} \right] \]  

The supersonic enhancement is not as easily found. However, in the case in which self gravity is neglected, the angular dependence of the subsonic and supersonic perturbations is the same. The only difference between them is that in the supersonic case, there is a shock at
\( \chi = \frac{2}{3} \), upstream of which, the density is unenhanced.

Since there is no reason to expect that including self-gravity will alter this aspect of the non-self-gravitating results, expression (d-11) may be used to examine the drag force exerted on the object by the medium. For this purpose, it is convenient to switch to spherical polar coordinates (see section (a)(iii)). Expression (d-11) then becomes

\[
\delta \rho = \frac{GM \rho_0}{c^2} \frac{1}{a \sqrt{1 - \frac{v^2}{c^2} \sin^2 \theta}} \cos \left[ \frac{k_3}{\sqrt{1 - \frac{v^2}{c^2}} \sin \theta} \right] \tag{d-12}
\]

which is of the form

\[
\delta \rho \sim \frac{1}{a} \cos k \alpha
\]

The deceleration is given as before by

\[
\frac{dV}{dt} = \left( \frac{F}{M} \right) \quad \text{d Vol} \tag{d-13}
\]

Supressing the constants and the angular dependence, this gives

\[
\frac{dV}{dt} \sim \int_0^\infty \frac{1}{a^2} \left( \frac{1}{a} \cos k \alpha \right) a^2 \, da = \int_0^\infty \frac{1}{a} \cos k \alpha \, da \tag{d-14}
\]
This integral diverges at \( a=0 \), but if an appropriate inner cutoff (c.f. Chapter I) is introduced, then (d-14) yields a finite expression for the deceleration of the object. However, as has been noted already, including self-gravity means that only systems smaller than the Jeans length may be considered: any larger system is unstable against the object's perturbing influence, and may not be described using a model which assumes stationary behaviour in the object's rest frame.

The conclusion is therefore that including the self-gravity of an infinite medium does not lead to a finite expression for the dynamical friction suffered by a test object. At best it furnishes an outer cutoff at the Jeans length, but this cutoff is mathematical rather than physical.
In this chapter, fluid mechanical models of dynamical friction were studied. It was assumed that the medium could be modeled as a non-viscous, compressible fluid. Its unperturbed density was assumed to be uniform, and in the frame of reference in which the object travels with velocity $V$, the medium was assumed to be at rest.

In section (b) the linearized equations of fluid mechanics, describing a non-self-gravitating fluid, were analysed. The subsonic density enhancement turns out to be symmetric, while in the supersonic case, an enhanced wake forms along the downstream axis. In both cases the density falls off roughly as $1/a$, independent of whether the medium is assumed to undergo an adiabatic or an isothermal change, in the presence of the object's gravitational field. The supersonic perturbation exerts a drag force on the object which decelerates it at the rate

$$\frac{dV}{dt} = -\frac{4\pi G^2 M a}{V^2} \left(1 - \frac{2 a^2}{\sqrt{1 + 2 a^2}}\right) \log r \bigg|_0^\infty e^{-1}$$

In section (c) a non-linearized treatment was reviewed briefly. To begin with, the spherically symmetric case was examined. It was found that, when pressure gradients are considered to be negligible compared to gravitational
forces, the density falls off as $\rho \sim \rho_0^{1/2}$. When pressure gradients and gravitational forces exactly balance, the density enhancement falls off as $\rho \sim (\frac{1}{3} + \frac{1}{2})^{1/2}$ in the adiabatic case, and $\rho \sim e^{1/\gamma}$ in the isothermal case. The fact that in the adiabatic case, the radial dependence of the density enhancement is the same whether or not there is an inflow of material towards the object suggests that, at least in the adiabatic case, the accretion process does not influence the dynamical friction that the object experiences. The conclusion of the orbit based models of Chapter I, namely that it is the cumulative effect of distant material that plays the dominant role in decelerating the object, is thus reinforced by the adiabatic results.

A review of two non-linearized analyses of the fluid equations in the case of a moving object, indicated that, in the region near the object, the radial dependence of the enhancement depends on the polytropic exponent, but that in the region far from the object, it does not. Non-linearized effects should not be important far from the object, and as indicated in Chapter I, the dominant influence in decelerating the object is due to the cumulative effect of material far from the object. The non-linearized analysis therefore yields no new information as far as dynamical friction is concerned.

In section (d), the linearized model of section (b) was extended to include self-gravity. It was demonstrated that including self-gravity does not provide shielding effect analogous to Debye screening, and does not stop the
integrals from diverging. It does however, provide a cutoff at the Jeans length, which delimits the maximum size of a volume of self-gravitating gas that would be stable against perturbations. This cutoff is really more mathematical than physical, in that any larger region would be unstable under the perturbing influence of the object, and the assumption of stationary behaviour underlying the model, could not be maintained.

The principle conclusions to be drawn from this are that an object moving subsonically through a non-viscous medium experiences no dynamical friction, while the expression for the deceleration of an object moving supersonically diverges logarithmically with distance, independent of whether or not the pressure or self-gravity of the medium is included, and independent of whether the medium is assumed to undergo an adiabatic or an isothermal change in the presence of the object.
CHAPTER III

MODELS OF DYNAMICAL FRICTION BASED ON BINARY ENCOUNTERS

Introduction

In Chapters I and II, models of dynamical friction were presented, in which a test object was assumed to be travelling through a medium composed of dust or gas. This chapter and the next examine models of dynamical friction where the medium through which the test object travels is composed of 'field' objects of roughly the same mass as the test object. It is convenient to separate these models into two types: those which employ the two-body approximation, and those which do not. The present chapter is devoted to the former type.

The basic assumption underlying these models is that a test object travelling through a system of field objects, may be viewed as undergoing binary encounters with field objects. In section (a), a model is presented in which it is assumed that the collisions are all complete, and all the momentum transfer occurs at the point of closest approach. The result of this model is the same as those of the orbit
based models of Chapter I. This is to be expected since the expressions for the dynamical friction of the test object, found in Chapter I were independent of the mass of the field particles: all field particles follow the same trajectories, regardless of their mass. Especially in the limit of distant material, the test object should have no way of distinguishing whether the mass it feels is smoothed out, as dust or gas, or clumped into other field objects.

It is argued however, at the end of section (a), that the orbit models, the fluid models, and this simple binary encounter model all portray a stationary situation which, if collisions with infinite impact parameters are included, takes an infinite time to establish.

To avoid this, the time dependence, particularly of distant encounters, must be included. Models incorporating this time dependence are presented in section (b). They result in expressions for the deceleration of the test object, in which the divergence with distance is replaced by a divergence with time.

The results are summarized in section (c).
(a) A Two-Body Calculation of Dynamical Friction in which Complete Collisions are Assumed

In this section, a method due to Chandrasekhar, for calculating the dynamical friction experienced by a test object undergoing successive binary encounters with field objects, is presented. It involves a calculation of the net change in the velocity of a test object (mass $M$, velocity $V$) passing through a spherically symmetric distribution of non-interacting field objects (mass $m$, velocity $v$), whose peculiar velocities obey a Maxwellian distribution.

This is equivalent to the situation described in Chapter I, section (c), in which the test object was assumed to travel through a system of particles whose thermal velocities obey a Maxwellian distribution. The result is, as in Chapter I, an expression for the deceleration of the test object which diverges logarithmically with impact parameter.

The change in velocity that the test object suffers during a single binary encounter with a field object is determined using orbit theory, and the effect of the entire system is calculated by summing over the individual encounters in a suitable way. The details of the calculation are left out of the present discussion: they may be found in Chandrasekhar (1943a), and in his book *Principles of Stellar Dynamics*.

During a single binary encounter, a test object
experiences velocity changes $v_\parallel$ and $v_\perp$, parallel to and perpendicular to its direction of motion respectively. These increments in velocity may be calculated using standard orbit theory (see Stellar Dynamics). When summed over a large number of such encounters, $v_\perp$ vanishes as expected from symmetry considerations, but $v_\parallel$ does not.

The net change in the velocity of the test object, $\Delta v$, due to a succession of binary encounters, is calculated for a time interval $\Delta t$ which is long compared with the duration of a single collision (i.e., long enough for the most distant encounters to be completed), but short compared with the time required to significantly alter the test object's velocity. This calculation involves integrations over three angles parametrizing the encounter, over the impact parameter, and over the initial velocities of the field objects.

To facilitate the integrations over the angles, several simplifications are made. To begin with, it is assumed that the distribution of the velocities of the field objects is spherically symmetric. (Chandrasekhar (1943a) discusses briefly the task of generalizing this to a random distribution.)

In the integral over impact parameters, it is assumed that

$$S \gg \frac{GM}{v^2} \quad \frac{GM}{(v-v')^2} \quad \frac{GM}{(v+v')^2}$$

Physically, as in Chapter I, this amounts to considering
only encounters with impact parameter much greater than the distance at which a field object with mass $M$, travelling with velocity $v \ (v-V), \ (v+V)$, just remains gravitationally bound to the test object.

This excludes close encounters, but it is argued that such encounters are sufficiently infrequent, that the approximation is justified. A more detailed argument to this effect is included in section (b).

Only the 'dominant' term in the integration is retained (see Stellar Dynamics, pp.62-64). This leads to the significant result that, to this accuracy, the only contributions to the deceleration of the test object, are from field objects whose initial velocity is less than that of the test object. In other words, in order to experience a deceleration, the test object must be travelling faster than the average velocity $\bar{v}$ of objects in the system:

$$v > \bar{v}$$

According to kinetic theory, the average speed of the members of a system is given by

$$\bar{v} = \sqrt{\frac{KT}{m}}$$

Therefore, in order to be decelerated, the test object's velocity must satisfy the inequality

$$v > \sqrt[2]{\frac{KT}{m}}$$
But since (c.f. Chapter I, section (c))
\[ \sqrt{\frac{kT}{m}} \sim \sqrt{\frac{P}{\rho}} \sim c \]

this inequality is simply
\[ \sqrt{\frac{\rho}{\gamma}} > c \]

Thus, to a good approximation, the object only experiences a deceleration if it is travelling supersonically. This is the same result as that yielded by the fluid mechanical model of Chapter II.

Finally, it is assumed that the field objects' velocities obey a Maxwellian distribution, and that the unperturbed number of field objects per unit volume, \( N \), is constant.

The resulting expression for the net change in velocity of the test object in the interval of time \( \Delta t \), is
\[ \Delta \nu = -\frac{4\pi NM(m+M)G^2}{\nu^2} \ln \left[ \frac{\sqrt{\nu^2}}{G(m+M)} \right] \bigg|_{S=\infty}^{a-14} \]

\( \sqrt{\nu^2} \) is the mean square velocity of the field particles, and

\[ \kappa \equiv \phi(\alpha) - \alpha \phi'(\alpha) \]

Here, \( \alpha \) is the same quantity encountered in Chapter I,
section (c):

\[ \alpha = \sqrt{\frac{m}{kT}} \]

and \( \Phi(\alpha) \) is the error function:

\[ \Phi(\alpha) = \frac{2}{\sqrt{\pi}} \int_{0}^{\alpha} e^{-x^2} \, dx \]

This may be compared with the equivalent expression derived in Chapter I, by writing \( mN = \rho_0 \), and assuming \( m \ll M \).

Then (a-7) becomes

\[ \frac{dV}{dt} = -\frac{2\pi G^2 M \rho_0 K}{V^2} \left\{ \ln \left[ \frac{S \sqrt{\nu^2}}{G(m+M)} \right] \right\}_{s=\infty} \]

which is the same as Chapter I, expression (a-6), for \( s >> GM/V^2 \), except that the mean velocity of the field objects is \( \sqrt{\nu^2} \), whereas in (a-6) it was \( V \), and the factor \( K \) has been introduced.

This factor \( K \) depends on the same quantity that cropped up in the orbit based model, when the thermal motion of the particles was included (see Chapter I(c)). An analytic expression for the deceleration of the test object was not obtained in that case, but it was argued that including the thermal motion of the field particles was not expected to alter the results significantly.

In the above analysis, the peculiar motions of field objects were included, and the results are, except for \( K \), the same as the results found by ignoring the peculiar
motions. Thus, the claim of Chapter I, section (c), that thermal motion among the field particles does not influence significantly the dynamical friction suffered by the test object, is substantiated.

Lee (1968) also calculates the dynamical friction suffered by a test object using a method based on summing over independent binary encounters, and like Chandrasekhar, he derives an expression which diverges logarithmically with distance.

The orbit models of Chapter I, the fluid models of Chapter II, and the binary encounter model just discussed, are all based on the assumption that the flow of the medium in the rest frame of the test object is stationary.

The implications of this assumption are most easily seen in the context of the orbit models. (Refer to the discussion at the beginning of Chapter I, section (a).) If one imagines the flow of particles past the object to be 'turned on' at some particular time, then in order for the stationary situation described in the model to be established, the first wedge of particles must have completed their trajectories from upstream to downstream infinity (i.e. between the asymptotes of their trajectories). Particles with small impact parameters will reach their downstream asymptote relatively quickly. Particles with large impact parameters will take longer. It follows that, if impact parameters out to infinity are considered, then it will take an infinite amount of time for these particles to
reach their downstream asymptotes, and thus, an infinite amount of time must pass before a stationary situation is established.

In the context of the binary encounter models, the interval of time required for infinitely distant encounters to be completed, is infinite.

Hunt (1971) notes the same phenomenon in connection with his non-linearized fluid mechanical analysis. In integrating the time dependent fluid equations from a given initial state, until a steady state is reached, he finds that for small $r$, the solutions become steady quite quickly, while for large $r$, they take a proportionally long time.

The next section describes modified binary encounter models which circumvent this problem by taking into account the time dependence of distant collisions.
(b) Time Dependent Collisions

As pointed out in the previous section, to establish a stationary regime in which all encounters may be considered to be complete, requires an infinite amount of time. In this sense, the stationary models can never strictly apply to any real situation.

To get around this problem, the time dependence, particularly of distant encounters, must be incorporated in the model. Ostriker and Davidsen (1968), and Henon (1958) present such models.

At this stage, a new feature, representing an evolution towards a fully statistical model, is introduced: rather than determining the change in the velocity of a test object, it is the expectation value of this quantity that is calculated. The basic structure of this calculation is as follows (c.f. Henon (1958)).

The initial state of the system of objects is described by the function

\[ p = a(\tau) \, d\tau \]

which gives the probability that at time \( t=0 \), there is an object whose coordinates in phase space lie between \( \tau \) and \( \tau + d\tau \).

The change in the velocity of the test object after a
time $T_r$ due to its interaction with a single field object, is given by

$$\Delta \vec{V} = \vec{S}(\tau)$$

where $\vec{S}(\tau)$ is some function of the field object's position and velocity.

In reality, the test object is acted upon simultaneously by a large number of field objects. However, to avoid the complexities of a full N-body problem, two approximations are made: the interaction of the field objects with one another is ignored, and it is assumed that the total change, $\Delta \vec{V}$ in the test object's velocity, may be calculated by adding together the changes due to the individual encounters:

$$\Delta \vec{V} = \sum_{\text{field objects}} \vec{S}(\tau)$$

The former approximation, concerning the self-interaction of the medium, was examined in Chapter II, section (d) in the context of a fluid mechanical model, and will be discussed further in Chapter V. The latter approximation is justified in the case of distant encounters, where the individual changes are small, but not in the case of close encounters. (The terms 'close' and 'distant' are defined further on.) However, given that the probability of an object undergoing more than one close encounter is remote, the approximation may still be used.
To calculate the expectation value of $\Delta \vec{V}$, a function $g(\tau)$ is defined. It represents the perturbation in the test object's velocity due to the element of phase space $d\tau$. Thus, $g(\tau)$ is equal to $\xi(\tau)$ if $d\tau$ contains a field object (probability $p$), and zero if it does not (probability $1-p$). Equation (b-3) may be rewritten:

$$\Delta \vec{V} = \int g(\tau)$$

(b-4)

where the integration extends over all phase space.

The expectation value of $\Delta \vec{V}$ is therefore

$$\langle \Delta \vec{V} \rangle = \int \langle g(\tau) \rangle$$

(b-5)

Since

$$\langle g(\tau) \rangle = p \xi(\tau)$$

(b-6)

(b-5) becomes (with (b-1)),

$$\langle \Delta \vec{V} \rangle = \int g(\tau) a(\tau) d\tau$$

(b-7)

Thus,

$$\langle \Delta V_k \rangle = \int g_x(\tau) a(\tau) d\tau$$

(b-8)

Similar expressions may be obtained for $\Delta V_\phi$ and $\Delta V_r$.

The second moments, $\langle \Delta V \cdot \Delta V \rangle$, may also be calculated.
For example, from (b-4),

\[ (\Delta V_x)^2 = \left[ \int g_x(\tau) \right]^2 \]  

(b-9)

So

\[ \langle (\Delta V_x)^4 \rangle = \int \int \langle g_x(\tau_1) g_x(\tau_2) \rangle + \int \langle g_x^2(\tau) \rangle \]  

(b-10)

It was assumed that the field objects are non-interacting, so their velocities and positions must be uncorrelated. \( g_x(\tau) \) and \( g_x(\tau_2) \) are therefore independent, and

\[ \langle (\Delta V_x)^4 \rangle = \langle g_x(\tau_1) \rangle \langle g_x(\tau_2) \rangle + \int \langle g_x^2(\tau) \rangle \]

\[ = \langle \Delta V \rangle^2 + \int \langle g_x^2(\tau) \rangle \]  

(b-11)

It turns out (c.f. Henon) that, for times small compared with the relaxation time of the system, \( \langle \Delta V \rangle \) is negligible compared with \( \langle (\Delta V)^2 \rangle \). From the definition of \( g(\tau) \),

\[ \langle g_x^2(\tau) \rangle = \rho \int x^2(\tau) \]

(b-12)

Expression (b-11) therefore becomes

\[ \langle (\Delta V_x)^4 \rangle = \int g_x^2(\tau) \alpha(\tau) d\tau \]  

(b-13)
Similar expressions hold for the other second moments.

It should be noted that $<(\Delta V_x)^2>$ represents the dispersion of $\Delta V$ around its expectation value, and is not the same as $<\Delta(V_x^2)>$, the expectation value of the change in the object's squared velocity (i.e., its kinetic energy). The dispersion $<(\Delta V)^2>$ is related to the diffusion process, or the randomization of the test object's peculiar velocity.

The quantity of interest here is $<\Delta V>$, the change in the test object's velocity in its direction of motion. To find $<\Delta V_x>$, an expression for $\mathcal{S}_x(\tau)$, the change in the velocity due to a single encounter, must be calculated, and then the integration over all phase space must be performed.

Two different methods are employed by Henon to determine $\mathcal{S}_x(\tau)$, depending on whether the encounters in question are close or distant. For close encounters, the impact parameter, $s$, is small compared with the distance travelled by the field object in time $T$:

$$s \ll \omega T \quad b.14$$

$$(w=|\vec{v}_f-\vec{v}_b|)$$. In this case, the velocities at the beginning and end of the interaction are approximately equal to their asymptotic velocities at infinity, and orbit theory is used (c.f. Chapter I).

For distant encounters, a perturbation technique is employed, which is accurate as long as the deflections produced by individual encounters are very small. Since
\[ S = \frac{G(M+m)}{\omega^2} \]

is the impact distance at which a maximum deflection occurs, the perturbation technique may be used for impact parameters such that

\[ S \gg \frac{G(M+m)}{\omega^2} \]

In general however,

\[ \omega T \gg \frac{G(M+m)}{\omega^2} \]

so the regions in which the two methods are valid overlap. A distance \( l \) is therefore chosen such that

\[ \frac{G(M+m)}{\omega^2} \ll l \ll \omega T \]

and this distance is used as the change over point between the two methods. It turns out that \( l \) does not appear in the final results, so its value is unimportant.

The perturbation technique is as follows. \( X \) and \( x \) are the coordinates of the test object (mass \( M \)) and a field object (mass \( m \)) respectively, at some time \( t \). They are expanded in a series with respect to the gravitational constant \( G \):
\[ x = x_0 + x_1G + x_2G^2 + \ldots \]  
\[ \dot{x} = \dot{x}_0 + \dot{x}_1G + \dot{x}_2G^2 + \ldots \]  

Equivalent expressions hold for the y-direction.

If \( G \) were zero (i.e., if \( M \) and \( m \) did not interact), then \( x, \dot{x}, y, \) and \( \dot{y} \) would be

\[ x = \dot{x} = 0 \]
\[ \dot{x} = \dot{x}_0 + \omega t \]
\[ y = 0 \]
\[ \dot{y} = \dot{y}_0. \]

Here \( M \) is assumed to remain at the origin of the coordinate system, \( \omega \) is the relative velocity of \( M \) and \( m \), which is in the \( x \) direction, and \( x \) is \( m \)'s initial position.

The equation describing \( M \)'s motion is

\[ \ddot{x} = \frac{Gm(y - \dot{y})}{\left[ (x - \dot{x})^2 + (y - \dot{y})^2 \right]^{3/2}} \]

Equivalent expressions hold for \( \dot{x}, \ddot{x}, \) and \( \ddot{y} \).

Expressions (b-19) are inserted into (b-21): the higher order terms decrease very rapidly when expression (b-16) is satisfied.

The function \( q_x(\tau) \) is the change after time \( T \), in the test object's velocity due to a single encounter. For a
distant encounter, it is therefore given by (from (b-19) and (b-20)),

$$\dot{\mathcal{J}} = \dot{X}(\tau) = G_\nu X_\nu(\tau) + G^2 \dot{X}_\nu(\tau) + \cdots$$

where $\dot{X}_\nu$, $\dot{X}_\nu$, ..., are found by integrating the expressions for $\dot{X}_\nu$, $\dot{X}_\nu$, ..., which are the coefficients of successive powers of $G$, found from (b-21).

Having found an expression for $\dot{\mathcal{J}}(\tau)$, Henon proceeds to perform the integrations over all phase space. One set of integrals extends over impact parameters $0<s<1$, and the other over $1<s<\infty$.

The former set contains logarithmic terms in which $l$ comes into the argument. The latter set also contains logarithmic terms, and $l$ and $wT$ are present in the argument. When the expressions for $s<1$ and $s>1$ are added, $l$ cancels out, and the results are, after retaining only the dominant terms,

$$\langle \Delta V_\nu \rangle = \frac{-16\pi^2 G^2 m(M+m)T}{V^3} \ln \left[ \frac{2 \bar{V}^3 T}{e^2 G(M+m)} \right] \int_0^V a(\nu) \nu^3 d\nu$$

$$\langle (\Delta V_\nu)^2 \rangle = \frac{32\pi^2 G^2 m^2 T}{3 \bar{V}^3} \ln \left[ \frac{2 \bar{V}^3 T}{e^2 G(M+m)} \right] \left\{ \int_0^V a(\nu) \nu^4 d\nu + \int_0^\infty a(\nu) \nu d\nu \right\}$$

Here, $\bar{V}$ is the mean value of the field object's velocities.

The divergence with impact parameter has been replaced by a divergence with time, and the rate of deceleration of
the test object is proportional to $T \log T$, rather than just to $T$.

Ostriker and Davidsen arrive at similar results, but rather than using a perturbation technique, they take into account explicitly the time dependence of distant encounters. They assume that the momentum transfer in a single distant encounter, in an interval of time $T$, is given by

$$\Delta p(T) = \int_0^T F(t) \, dt$$

$b-15$

$F$ is the force exerted on the field object by the test object, but here it is taken to be a function of time:

$$F(t) = \frac{G m M (r^2 + \dot{r}^2)}{(r^2 + 2\dot{r} \cdot \dot{\eta} + \dot{\eta}^2)^{\nu/2}}$$

$b-26$

Close encounters are treated as in section (a), where all the momentum transfer is assumed to occur at the point of closest approach.

The major approximation underlying this method, is that the field objects follow linear trajectories. This allows considerable mathematical simplification, and although it leads to an inaccurate treatment of close encounters, it is argued that this doesn't matter since, as demonstrated in previous chapters using a stationary approach, it is the cumulative effect of distant material that is important. The
linear approximation is more accurate in the case of distant encounters, and Ostriker and Davidsen claim that it can be shown that the difference between the results using linear orbits, and those using hyperbolic orbits, is negligible.

It will be seen that this claim is substantiated in the case of \( \langle (\Delta V)^2 \rangle \), by the agreement between Ostriker and Davidsen's results, and those of Henon, which were found by taking into account the deviations of field objects from linear trajectories. It is not substantiated in the case of \( \langle \Delta V \rangle \).

The quantity \( \langle (\Delta p)^2 \rangle \) is calculated by summing over time dependent encounters described by (b-25) and (b-26), as follows. Expression (b-26) is inserted into (b-25), the integration is performed, and the result is squared. The field objects are assumed to be randomly distributed, with constant number density \( N \). Their velocities are assumed to satisfy the spherically symmetric distribution

\[
\hat{f}(\hat{r}, \hat{v}) = \frac{N}{\pi \bar{v}^2} \delta(\hat{r} - \hat{v})
\]

where \( \bar{v} \) is the average velocity of the field objects. The integration

\[
\langle (\Delta p)^2 \rangle = \int |\Delta p|^2 f(\tau) d\tau
\]

is then performed, using asymptotic expansions in the parameter
\( \beta = \frac{2GM}{v^3T} \)

\((\Delta p)^2\) is given by (b-25) and (b-26). \(\beta\) represents the coordinates of a field object in phase space (the test object is initially at the origin), and \(f(z)\) is the probability distribution in phase space, described above.

The result, keeping only the dominant term, is

\[
\langle (\Delta p)^2 \rangle = \frac{8\pi N M^4 G^3 T}{v^2} \left[ \ln \frac{v^3 T}{2GM} + O(1) \right]
\]

The contribution for close encounters is demonstrated by Ostriker and Davidsen to be negligible compared to this.

Expression (b-30) contains no factor \(K\), since the velocities of the field objects were not assumed to be Maxwellian. If the mean velocity of the objects, \(\bar{v}\), is taken to be \(V\), (as it was in section (a)), then (b-30) agrees with (a-12) with one notable difference: as in Henon's results, the divergence with distance has been replaced by a divergence with time.

It should be remarked that there is a misleading aspect to Ostriker and Davidsen's work. It was pointed out previously that \(\langle (\Delta V)^2 \rangle\) is the dispersion of \(\Delta V\) around its expectation value, \(\langle \Delta V \rangle\), while \(\langle \Delta (V^1) \rangle\), an entirely different quantity, represents the expected value of the net change in the object's kinetic energy. Ostriker and Davidsen
however, take $\langle (\Delta p)^{\Delta} \rangle$ to be the change in the energy of the test object due to a single encounter, but then erroneously associate $\langle (\Delta p)^{\Delta} \rangle$ with the total change in its energy due to all encounters.

Ostriker and Davidsen's method could be used to calculate an expression for $\langle \Delta V \rangle$ simply by integrating the quantity $\Delta p$ over all phase space, rather than $(\Delta p)^{\Delta}$. It is easily seen however, that such a calculation would lead to the result

$$\langle \Delta V \rangle = 0$$

The density of the system was assumed to be initially uniform, and the approximation that the field objects follow linear trajectories means that it must remain uniform. The net force acting of the test object therefore cancels out by symmetry.
(c) Summary

Calculating the dynamical friction experienced by a test object, using the binary encounter method and assuming stationary behaviour (ie. complete encounters), leads as expected, to the same results as Chapters I and II: the rate at which a test object is decelerated diverges logarithmically with distance. These results are independent of the mass of the field objects (particles).

However, the fact that it takes an infinite amount of time for a stationary situation to be established, is not entirely satisfactory, so models taking into account the time dependence, particularly of distant encounters, were examined. These led to expressions for the dynamical friction experienced by the test object in which the divergence with distance is replaced by a divergence with time, so the test object is decelerated at a rate proportional to $T \log T$, rather than just to $T$. It was demonstrated that the expression for $<(\Delta V)^2>$, the dispersion of $\Delta V$ about its mean, is insensitive to whether or not the trajectories of the field objects are assumed to be linear. Assuming linear trajectories leads to the result

$$<(\Delta V)> = 0$$

as expected from symmetry considerations.
There is another school which questions the use of the assumption of binary encounters altogether. Instead, it is claimed that the test object should be viewed as acted upon by a stochastically fluctuating force arising from the varying complexion of field objects in the test object's immediate neighbourhood. Such a 'stochastic' model is described in the next chapter.
CHAPTER IV

STOCHASTIC MODELS OF DYNAMICAL FRICITION

Introduction

In Chapter III, the dynamical friction suffered by a test object travelling through a system of field objects was calculated, assuming that the test object could be viewed as undergoing binary encounters with the field objects.

An alternate approach, developed primarily by Chandrasekhar, is to view the test object as acted upon by a stochastically fluctuating force $F$, arising from the varying complexion of field objects surrounding it. Although it is impossible to predict exactly what $F$ will be at any particular instant, it lends itself well to a statistical description. It is therefore the aim of this approach to discover the statistical properties of $F$, and to use them to analyse the dynamical interaction between the test object and the rest of the system.

The test object in this approach is a randomly chosen member of a system of similar objects, whereas in the models of Chapters I and II, it was an 'interloper'. The question
therefore arises whether the situation described by this approach is equivalent to that of the previous chapters.

To answer this question, it is instructive to consider the binary encounter models. The equivalence between the binary encounter models and the orbit based and fluid mechanical models has already been demonstrated (c.f. Chapter III): the test object is incapable of distinguishing between a medium consisting of dust or gas, and one consisting of massive field objects. On the other hand, the binary encounter models and the stochastic models are equivalent: in both cases the medium consists of objects of roughly the same mass as the test object, and the arguments concerning the deceleration of the latter apply to any member of the system. The test object has no special qualities which set it apart from the field objects.

Since the binary encounter models describe the same situation as that described by the orbit based and fluid mechanical models, and also by the stochastic model, it follows that the stochastic model is indeed equivalent to the models of Chapters I and II, in which the test object is an interloper rather than a member of the system.

There is an extensive literature describing the statistical approach (cf. Chandrasekhar (1941-44), Chandrasekhar and von Neumann (1942), Kandrup (1980), Lee (1968)): none of the details will be included in this discussion. Since it draws heavily on ideas from the theory of Brownian motion, a distillation of the fundamentals of this theory is presented in section (a).
Section (b) demonstrates how they may be applied to a gravitational system, and in particular, used to derive an expression for the dynamical friction experienced by a test object.

The results are summarized in section (c).
(a) Fundamental Notions of the Theory of Brownian Motion

Many of the basic ideas of the statistical approach are borrowed from the theory of Brownian motion, so before proceeding to discuss the properties of the stochastically fluctuating force in the context of a gravitational system, the fundamentals of Brownian motion will be reviewed briefly.

The theory of Brownian motion was originally developed to describe the motion of a 'Brownian' particle (large on a molecular scale) moving through a fluid. The Brownian particle collides with the molecules in the fluid, and thus suffers a succession of random changes in velocity. The sequence of collisions that the Brownian particle undergoes constitutes a Markoff process: the future 'state' of the particles is dependent only on its present state, and not on its past history.

If \( F(T) \) is the force acting on it at time \( t \), then the force it experiences at time \( t + \Delta t \), \( F(t + \Delta t) \), is to a high degree, independent of \( F(t) \). In particular, the correlation between \( F(t) \) and \( F(t + \Delta t) \) may be shown to be proportional to \( e^{-t/\tau} \), where \( \tau \) is the mean life of the force (c.f. Doob, 1942).

A standard starting place for a study of Brownian motion is with the assumption that the motion of the Brownian particle may be described as a diffusion process:
there is a function \( W(v,t;v_0,t_0) \), which describes the probability that a particle will have velocity \( v \) at time \( t \), given that it had velocity \( v_0 \) at time \( t_0 \), and which satisfies the diffusion equation:

\[
\frac{\partial W}{\partial t} = q \nabla^2 \frac{\partial W}{\partial v}
\]

where \( q \) is the diffusion coefficient:

\[
q \equiv \frac{1}{6} \left\langle |F|^2 T(F) \right\rangle
\]

It is easily verified that the Gaussian distribution,

\[
W = \frac{1}{\sqrt{2\pi q t}} e^{-\frac{|\vec{v} - \vec{v}_0|^2}{4qt}}
\]

is a solution of equation (a-1). The expectation value \( \langle (\Delta v)^2 \rangle \), may be calculated:

\[
\left\langle (\Delta v)^2 \right\rangle = \int (\Delta v)^2 W(v,t;v_0,t_0) \, dv
\]

Inserting (a-3) into (a-4), performing the integration, and using definition (a-2), the result is

\[
\left\langle (\Delta v)^2 \right\rangle = \left\langle F^2 T(F) \right\rangle t
\]

However this has an unpleasant feature: the expectation value of the squared change in the particle's velocity becomes arbitrarily large for long intervals of time.
To get around this problem, the standard procedure is to introduce dynamical friction. In particular, it is assumed that the rate of change in velocity of the particle is the sum of two parts:

\[
\frac{d\vec{v}}{dt} = \vec{F}(+) - \beta \vec{v}
\]

This is the Langevin equation. The first term is due to the statistically fluctuating force, and the second represents a deceleration due to dynamical friction. (\( \beta \) is the coefficient of dynamical friction.)

It should be noted that there are two effects operating. One is responsible for the evolution of the quantity \(<\Delta v>\), and leads to a systematic deceleration of the test object. This is dynamical friction. The other is responsible for the evolution of the quantity \(<(\Delta v)^2>\). This is not dynamical friction: it corresponds to the dispersion of \(\Delta v\) around its mean value, and is related to the diffusion process in velocity space. It represents the randomization of the test object's peculiar velocity, and not a net change in the object's kinetic energy. This latter is \(<\Delta(v^2)>\), which is a different quantity.

In Brownian motion the physical effect responsible for dynamical friction is viscosity. The fluctuating force is responsible for the change in \(<(\Delta v)^2>\), and is governed by a probability distribution \(\Psi(v,v')\), which gives the probability of the particle undergoing a transition from one velocity to another, and which is assumed to be Gaussian.
Note that this is different from $W(v,t)$, which governs the probability of the occurrence of a velocity $v$ at a given time. The latter may be calculated using the former, from the integral equation

$$W(v_j + \Delta v) = \int_{-\infty}^{+\infty} W(v - \Delta v, t) \Psi(v - \Delta v) \, d(\Delta v)$$

This integral equation is used to derive a modified form of diffusion equation (a-1), namely the Fokker-Planck equation. Finally, the requirement that the Maxwellian distribution be an exact solution of the Fokker-Planck equation in the limit $t \to \infty$, imposes the relation between $q$ and $p$:

$$\frac{q}{p} = \frac{1}{3} \left| \mathbf{v} \right|^2$$

Although only a few of the rudimentary notions and equations of Brownian theory have been sketched, they should be sufficient to provide a basis for the discussion of gravitational Brownian motion, in the next section.
Returning to the gravitational case, it is clear that it differs from Brownian motion on a molecular level in some fundamental ways. To begin with, as mentioned in the previous section, dynamical friction in molecular Brownian motion is due to viscosity: the Brownian particle is slowed down by colliding with molecules which lie in its path. Accordingly, it feels a force $F$ only when it comes into direct contact with a molecule.

The force felt by a massive test object, on the other hand, is due to a long range interaction. Dynamical friction does not arise from the test object colliding with field objects: the systems generally of interest are sufficiently diffuse that the probability of this happening is vanishingly small. Rather, the object is decelerated by the drag force exerted on it by a region of enhanced density which forms in its wake (c.f. Chapters I through III). This is a long range interaction.

One of the manifestations of its long range nature is the correlation between two forces acting at times $t$ and $t_0$: $F_t$ and $F_{t_0}$. Whereas in the molecular case, this correlation decreases as $\sim t^{-1}$, in the gravitational case, the correlation only decreases as $1/t$ (see Chandrasekhar, 1944).
In a first attempt at a statistical analysis of a gravitational system, Chandrasekhar (1943), and Chandrasekhar and von Neumann (1942) calculated expressions for \( W(F) \), the probability of an instantaneous force \( F \) acting on the test object, and \( T(F) \), the mean life of such a force. The latter involves the quantity \( W(\dot{F}) \): the probability that the force experienced by the object has an instantaneous rate of change \( \dot{F} = \frac{dF}{dt} \).

Kandrup (1980) calculates an expression for the dynamical friction suffered by a test object using these two quantities, in the following way. From definition (a-2),

\[
\varrho = \langle F^2 T(F) \rangle \quad b-1
\]

This expectation value is given by

\[
\langle F^2 T(F) \rangle = \int W(F) F^2 T(F) dF \quad b-2
\]

The expressions for \( W(F) \) and \( T(F) \) derived by Chandrasekhar and Chandrasekhar and von Neumann, are inserted into (b-2), and the integration is performed. The resulting expression for the diffusion coefficient is

\[
\varrho = 1.48 \frac{G^3 m^3 n}{\sqrt{\langle u^2 \rangle}} \left[ \ln \left( \frac{2.23 D_\infty \langle u^2 \rangle}{2 Gm} \right) \right] \quad b-3
\]

\( D_\infty \) is the mean inter-object separation. It was not introduced as a cutoff for a diverging integral.
Relation (a-8) yields an expression for the coefficient of dynamical friction:

$$\beta = \frac{3a}{10\eta^2} \quad b-4$$

The equations

$$\langle (\Delta r)^2 \rangle = 6q\Delta t \quad b-5$$

$$\langle \Delta r \rangle = \beta v\Delta t \quad b-6$$

follow from the integrated form of the Langevin equation (c.f. Kandrup):

$$\Delta r = \Delta r(t) - \beta v\Delta t \quad b-7$$

Inserting (b-3) and (b-4) into (b-5) and (b-6) respectively, yields

$$\langle \Delta r \rangle = -u_n u_n \frac{G^2 m^2 n \Delta t}{\langle v^2 \rangle} \ln \left[ \frac{2.23 \Delta x \langle v^2 \rangle}{2Gm} \right] \quad b-8$$

$$\langle (\Delta r)^2 \rangle = 2.96 \frac{G^2 m^2 n \Delta t}{\langle v^2 \rangle} \ln \left[ \frac{2.23 \Delta x \langle v^2 \rangle}{2Gm} \right] \quad b-9$$

These expressions are finite: no divergences were
encountered in arriving at them.

These results contradicts all those obtained so far, which indicate that it is the cumulative effect of distant material which is the dominant factor in decelerating the object. Expressions (b-8) and (b-9) imply that only field objects within about the mean inter-object distance (ie. only the nearest neighbours) influence the test object.

Kandrup's calculation made use of the quantities $W(F)$ and $T(F)$. The latter was calculated on the premise that a gravitational Markoff process possesses the usual characteristic that correlations between subsequent forces die of as $e^{-t/T}$. However, as Chandrasekhar pointed out in 1944,

"While the specification of these moments of $F$ are sufficient for the purposes of determining the instantaneous rates of change of $F$ that are to be expected, they are very far from providing all the information that is necessary for a complete statistical description of the fluctuating force acting on a star. For the entire stochastic variation of $F$ with time can be described fully only in terms of the average force $F_+$ acting at any later time $t$, given that a force of some prescribed intensity acted at time $t=0$. In other words, we need a complete 'integration' of the stochastic equations of $F$.

In this paper, Chandrasekhar calculates the required autocorrelation function $W(F_+,F_0)$, the probability that a test object will feel a force $F_+$ at time $t$, given that it felt a force $F_0$ at time $t=0$. It has the characteristic
already mentioned, that instead of decreasing as $e^{-t/D}$, it falls off as $1/t$ at large $t$.

Lee (1968) points out that

"This fantastic 'memory' of a force is incompatible with the assumption of brief encounters and with the use of the D cutoff."

The effect of a given encounter dies off so slowly that the next encounter may begin before it has ended. Thus, as argued in section (b) of Chapter III, distant encounters cannot be considered to be complete. Lee uses the autocorrelation function to find an expression for the dynamical friction suffered by a test object, in the following way.

The force acting on a test object at some initial time is

$$ F_0 = \sum \frac{\gamma M_i \dot{r}_i}{|r_i|^3} $$

Assuming that distant field objects travel along linear trajectories, the force acting at some later time $t$ is

$$ F_+ = \gamma M \sum \frac{\dot{r}_i + \dot{v}_i^+}{|\dot{r}_i + \dot{v}_i^+|^3} $$

The autocorrelation function and its first and second moments are calculated using Chandrasekhar's method. The first moment, $<F_+>$, turns out to be zero. This is expected: as a consideration of the orbit based or fluid mechanical models indicates, if the motion of the particles (or fluid
elements) were unperturbed, no density enhancement would form in the wake of the object, and there could be no dynamical friction.

The second moment, \( <F_x, F_x> \), does not vanish, and can be used to calculate the quantity \( <(\Delta v)^2> \): the dispersion of \( \Delta v \) about its mean.

The change in the object’s velocity in an interval of time \( T \), is given by

\[
\Delta v = \int_0^T F_x \, dt
\]

Squaring this,

\[
|\Delta v|^2 = \int_0^T \int_0^T dt \, dt' \, F_x \cdot F_x'
\]

\[
= 2 \int_0^T \int_t^T dt' \, F_x \cdot F_x'
\]

The expectation value is therefore

\[
<(\Delta v)^2> = 2 \int_0^T \int_t^T dt' \, \langle F_x \cdot F_x' \rangle
\]

\( <F_x F_x'> \) depends on the interval of time \( s \) between \( t' \) and \( t \). This allows (b-14) to be written (c.f. Lee)

\[
<(\Delta v)^2> = 2 \int_0^T ds \, (T-s) \langle F_0 \cdot F_s \rangle
\]
Inserting the expression for $<F_0 F_s>$ derived according to Chandrasekhar's method (c.f. Lee), and integrating over $s$, yields

$$
\langle (\Delta v)^2 \rangle \sim 4\pi m^2 G^2 T \ln \left[ \frac{T^2}{T_o^2} \right] \int \frac{f(|\nu|)}{|\nu|} \, d\nu
$$

$f(|\nu|)$ describes the initial distribution of velocities in the system, and is left unspecified. $T$ is an 'appropriate' inner cutoff. This expression is of the form $T \log T$, in agreement with those derived in the time dependent binary encounter models.
In this chapter, a stochastic model of the evolution of a system of gravitating objects, developed primarily by Chandrasekhar, was presented. The model borrows many of its ideas from the theory of Brownian motion, but also differs from the latter in some fundamental ways.

In this approach, a test object is viewed as acted upon by a stochastically fluctuating force $F$, arising from the varying complexion of field objects surrounding it. Early attempts at analysing the behaviour of the test object under the influence of the fluctuating force, involved the quantities $W(F)$ and $T(F)$. The former is the probability of the object experiencing an instantaneous force $F$. The latter is the duration of the force, and depends on the moments of the bivariate distribution $W(F,F)$. $\dot{F}$ is the instantaneous rate of change of $F$.

An analysis based on these quantities yields expressions for the dynamical friction, $<\Delta V>$, and the quantity $<(\Delta V)^2>$, which are proportional to $T\log D_0$, where $D_0$ is the mean inter-object separation (c.f. Kandrup, 1980). This contradicts the results obtained up to this point, in that it implies that it is the material in the immediate vicinity of the object that plays the dominant role in decelerating it, rather than the cumulative effect of distant material.
However, Chandrasekhar (1944) points out that a complete analysis must be based not on $W(F)$ and $T(F)$, but on $W(F_+, F_0)$, the probability of the test object experiencing a force $F_+$ at time $t$, given that it experienced a force $F_0$ at time $t$. Chandrasekhar derives an expression for this function, and Lee (1968) uses it to find the dynamical friction experienced by the object. He assumes that field objects follow linear trajectories, which leads to the result that there is no dynamical friction. However, the quantity $<(Δv)^2>$ does not vanish, and the expression for it that he arrives at is of the form $T\log T$, in agreement with the time dependent binary encounter models of chapter III.

The next and final chapter contains a summary of the various results, a discussion of the discrepancies between them, and suggestions for further investigations.
CHAPTER V

SUMMARY AND CONCLUSIONS

Introduction

Under certain circumstances, a massive object travelling through a medium may experience a deceleration due to the dynamical interaction between it and the medium. In Chapters I through IV, a variety of different models of this effect were presented.

Section (a) of this chapter summarizes the results yielded by the various models. Section (b) contains a discussion of the discrepancies between these results, and assesses the validity of each of the models. Finally, in section (c), points requiring further investigation are suggested.
(a) Summary of Results

Chapters I and II described models of the dynamical friction experienced by a massive object travelling through a medium consisting of field particles much less massive than itself. In particular, the models of Chapter I were based primarily on orbit theory. In the simplest of these models, the field particles were assumed to be collisionless, and have no thermal motion, and their mutual gravitation was ignored. Some modifications were made to include collisions downstream of the object, where the particles, following hyperbolic trajectories in the object's gravitational field, were focused along the downstream axis. Further modifications were made to include thermal motion.

The mathematical complexities involved in incorporating thermal motion suggest that a fluid mechanical approach might be more appropriate, so several analyses of the interaction between a gaseous medium and a massive object were discussed in Chapter II. Both the linearized and non-linearized equations of fluid mechanics were examined, and the linearized case was extended to include the self-gravity of the medium.

In all of the models, the medium's behaviour was assumed to be stationary in the object's rest frame.

The simple orbit based models yielded an expression for the dynamical friction suffered by a test object which
depends on the object's mass and velocity, and on the
density of the medium. Including thermal motion introduced a
dependence on the quantity $V/\sqrt{kT/m}$, however the
mathematical complexities were such that an analytic
expression for the object's deceleration could not be
derived.

The fluid mechanical approach yielded two results: for
subsonic motion, the object experiences no dynamical
friction, while in the case of supersonic motion, the
expression for its deceleration is the same as that of the
orbit based models, including a dependence on the ratio $V/c$,
equivalent to the dependence on $V/\sqrt{kT/m}$.

In all cases the expressions diverge logarithmically
with distance.

Chapters III and IV dealt with the situation in which
the medium through which a test object travels consists of
objects of roughly the same mass as itself. In Chapter III,
the interaction between the test object and field objects
was described in terms of binary encounters.

The simplest of these models assumed that the
encounters are complete, and that the field objects' peculiarity velocities obey a Maxwellian distribution. The
result of this model is equivalent to the results of the
orbit based models, including the dependence on the quantity $\sqrt{m/kT}$. Furthermore, the model indicates that an object
will only be decelerated if its initial velocity is greater
than the average velocity of the objects in the system. It
was demonstrated (see Chapter III) that this is equivalent
to the fluid mechanical result that only an object travelling supersonically will experience dynamical friction.

In all the models mentioned so far, a stationary flow in the object's rest frame is assumed. However, it was argued in Chapter III that such a stationary flow takes an infinite time to establish. Binary encounter models taking into account the time dependence of distant encounters were therefore examined. These led to expressions for the dynamical friction in which the divergence with distance is replaced by a divergence with time.

Chapter IV presented a stochastic model of dynamical friction, which is based largely on ideas borrowed from the theory of Brownian motion. The test object is viewed as acted upon by a stochastically fluctuating force arising from the varying distribution of field objects surrounding it. An early attempt at analysing the properties of the fluctuating force yielded expressions for $W(F)$, the probability of the test object experiencing an instantaneous force $F$, and $T(F)$, the duration of that force. This latter quantity depends on the bivariate distribution $W(F, F')$: the probability that the test object will feel an instantaneous force $F$, whose instantaneous rate of change is $F'$.

In contradiction to all the other results, the expression for the dynamical friction of the test object derived using $W(F)$ and $T(F)$ is finite: the deceleration is proportional to $T \log D_0$, where $D_0$ is the mean inter-object separation.
It was pointed out by Chandrasekhar however, that $W(F)$ and $W(F,F)$ are not sufficient to describe the evolution of the velocity distribution. What is needed is the autocorrelation function, $W(F_t,F_{t'})$: the probability that the test object will feel a force $F_t$ at time $t$, given that it felt a force $F_{t'}$ at time $t'$. Using this function takes into account the long range nature of gravitational forces. A derivation of the dynamical friction based on this quantity results in an expression for the quantity $<(\Delta V)^2>$ which is proportional to $T\log T$. The dynamical friction, $<\Delta V>$, is zero, due to the assumption that the field objects follow linear trajectories.

Thus, there are two predominant results: those which are of the form $T\log X$, and those which are of the form $T\log T$. Both of these indicate that it is the cumulative effect of distant matter that plays the dominant role in altering the test object's motion. In addition, there is one contradictory result, which is a finite expression indicating that only material near the object, in particular only its nearest neighbours, have any effect in slowing it down.

Reasons for the differences between these results are discussed in the next section.
(b) Discussion of the Differences between the Results

Three types of expression for the deceleration of an object due to dynamical friction have been obtained: those of the form $T \log X$, those of the form $T \log T$, and one of the form $T \log D_0$.

The difference between the $T \log D_0$ and $T \log T$ expression has already been discussed (see Chapter IV): it is due to basing the derivation on the quantities $W(F)$ and $T(F)$, rather than on the autocorrelation function $W(F_+, F_0)$.

$T(F)$ is derived on the assumption that gravitational Brownian motion is characterized by the same correlation between forces acting at subsequent times, as molecular Brownian motion: namely, the correlation dies off as $e^{-\tau/T}$. As shown by Chandrasekhar, $W(F_+, F_0)$ for a gravitational system falls off only as $1/t$, and as pointed out by Lee, this is not comensurate with a short distance cutoff such as $D_0$.

Furthermore, no agreement would be possible between a $T \log D_0$ result and the results of a fluid mechanical or orbit based model. In these, there is only one object: $D_0$, the average inter-object separation, has no meaning in the context of a model describing a single object travelling through a cloud of dust or gas.

The discrepancy between the $T \log X$ and $T \log T$ results can be traced to one particular assumption: the $T \log X$ models all
assume that the flow of the medium past the test object is stationary in the test object's rest frame. In the TlogT binary encounter models, no such assumption is made: the time dependence of distant encounters is explicitly included. In the TlogT stochastic model, time dependence is incorporated by analysing the autocorrelation function $W(F^+,F_0)$ rather than the quantities $W(F)$ and $T(F)$. The former describes the correlation between forces acting over widely separated times, while the latter two depend only on the instantaneous force and its instantaneous rate of change.

It was demonstrated in Chapter III that stationary behaviour takes an infinite amount of time to establish, so the TlogX models cannot strictly be applied to any real situation.

It should be noted that all these models assume a medium which has an initially uniform density. In the orbit and fluid mechanical models, the density becomes non-uniform as an enhanced region forms downstream of the object. The change in the momentum of the test object may be found by calculating the drag force exerted on it by this enhancement.

In Ostriker and Davidsen's time dependent encounter model, and in the stochastic models, the assumption is made that distant field objects follow linear trajectories. With no perturbation of their orbits, there can be no enhancement downstream of the test object, no drag force on it, and therefore no dynamical friction.
However $<(\Delta v)^2>$, the dispersion of $\Delta v$ about its mean, does not vanish on assuming linear trajectories, and it has been demonstrated by the authors of the time dependent binary encounter models (c.f. Henon, Ostriker and Davidsen; also Lee) that calculating the quantity $<(\Delta V)^2>$ using the approximation that the distant field objects travel along unperturbed trajectories, does not alter the resulting expression significantly.

If correlations between field objects (i.e. self-gravity) are included, then the TlogX and TlogT results can be shown to be equivalent in the following way. It has already been demonstrated (c.f. Chapter II) that including the self-gravity of the medium in the case of a stationary fluid mechanical model, introduces an outer cutoff at the Jeans length. This is a mathematical cutoff in the sense that it delimits the largest possible region in which the assumption of stationary behaviour remains valid. However, it has physical significance in the sense that, according to Jeans instability theory, no stable self-gravitating systems with dimensions larger than the Jeans length can exist.

Henon and Ostriker and Davidsen, (also see Prigogine and Severne (1966)), give arguments to the effect that, in the expressions which diverge logarithmically with time, the Jeans time, $T_J$, which is the Jeans length divided by the characteristic velocity of particles in the system, provides an upper cutoff. The arguments are as follows.

Because of the time dependence in the argument of the
logarithm, the perturbation in the velocity of a test object is not simply directly proportional to time, but increases faster. If the field objects are initially independent, correlations between them develop, and collective behaviour ensues. However, collective behaviour on a scale greater than the Jeans length results in instabilities.

Starting with a large, homogeneous population of objects, the time scale for forming a gravitationally bound system (whose dimensions are determined by the Jeans length) is the Jeans time. After the system has formed, the objects will orbit it on a time scale $T_J$. After several crossing times, the correlations existing at the time of formation, $T_J$, will have been destroyed. This suggests that cumulative effects leading to the $T \log T$ behaviour, happen only for a time $T < T_J$, and that for $T > T_J$, the velocity of the test object changes at a rate directly proportional to $T$, and equal to the expression given by the formulas if $T$ in the argument of the logarithm is replaced with $T_J$.

The Jeans time therefore comes in as an outer cutoff, yielding an expression equivalent to that in which the divergence was with distance, and the Jeans length was introduced as an outer cutoff.

In the next section, possible directions for further investigation of dynamical friction are suggested.
(c) Suggestions for Further Investigations

In the previous section, a brief argument was given, indicating why, in the case of the stochastic model, the TlogT expression for dynamical friction was to be considered more accurate than the TlogD expression. A more thorough investigation of the stochastic model is needed to substantiate this argument.

Likewise, a closer examination of the role of self-gravity and correlations, is needed to tighten the connection between the TlogT and TlogX results.

A further study of the role of assumptions concerning the Markoffian nature of gravitational encounters, could prove interesting. Chandrasekhar's early work leans quite heavily on a Markoffian analysis of the type used in Brownian motion. Agekyan, whose work was not discussed here, has developed a stochastic model similar to Chandrasekhar's. This model is based on the assumption of a Markoffian encounter process, and it, like Chandrasekhar's early work, leads to results which indicate the existence of a cancellation of the effects of distant material, so that only nearby objects contribute to the evolution of the quantities $<\Delta V>$ and $<(\Delta V)^2>$. 

The work of Prigogine and Severne (1966) was not discussed here either. However, it is based on the premise that in gravitational systems, the collision process is
non-Markoffian. This is essentially the same assumption as in the time dependent binary encounter models, where collisions are no longer considered to be complete and independent. Chandrasekhar's later work seems to follow these lines as well, concentrating on the autocorrelation function \( W(F_+, F_0) \), which indicates that the correlation between two forces acting at different times decreases only as \( 1/t \), rather than as \( e^{-\tau} \), as in a usual Markoff process.

A possibility for getting rid of the divergence altogether, which has not been considered here, is to include the expansion of the Universe. This is motivated in part by the similarity between the present situation and Olber's paradox. The former deals with \( 1/r \) forces, while the latter deals with radiation whose intensity falls off as \( 1/r \). Olber's paradox is dispelled by including the expansion of the Universe.
APPENDIX A

Solution of the Linearized Equation Describing a Perturbation Created by a Massive Object Travelling through a Non-Self-Gravitating Gas

Starting with equation (b-10) of Chapter II,

\[ \frac{\partial^2 \delta \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \delta \rho}{\partial r} + \sigma^2 \frac{\partial^2 \delta \rho}{\partial x^2} = -\frac{2G\mathcal{M}_0}{c^2} \frac{\delta(r) \delta(x)}{r} \]  \hspace{1cm} (A-1)

let

\[ B = \frac{2G\mathcal{M}_0}{c^2} \]  \hspace{1cm} (A-2)

and

\[ \sigma^2 = 1 - \frac{v^2}{c^2} \]  \hspace{1cm} (A-3)

Then \((A-1)\) is

\[ \frac{\partial^2 \delta \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \delta \rho}{\partial r} + \sigma^2 \frac{\partial^2 \delta \rho}{\partial x^2} = -B \frac{\delta(r) \delta(x)}{r} \]  \hspace{1cm} (A-4)

Let

\[ g(r, x) = -B \frac{\delta(r) \delta(x)}{r} \]  \hspace{1cm} (A-5)
Expanding \( \rho \) in Fourier-Hankel integrals gives

\[
\bar{\rho}(r, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} r \rho(r, x) e^{-ixs} J_0(kr) \, dr \, dk
\]

\[
\rho(r, x) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{k} \bar{\rho}(r, s) e^{ixs} J_0(kr) \, dk \, ds
\]

Inserting (A-5) and (A-6) into (A-4):

\[
\begin{align*}
\int \left[ \bar{k} \bar{\rho} e^{-ixs} \frac{\partial^2 J_0}{\partial r^2} \, dk \, ds \right] + \int \left[ \bar{k} \bar{\rho} e^{-ixs} \frac{1}{r} \frac{\partial J_0}{\partial r} \, dk \, ds \right] + \int \left[ \bar{k} \bar{\rho} e^{-ixs} \frac{1}{r^2} \, dk \, ds \right] \\
= \int \left[ k \bar{\rho} e^{-ixs} J_0 \, dk \, ds \right]
\end{align*}
\]

Taking \( d/dk \) and \( d/dx \), and cancelling the \( k \)'s and \( e^{-ixs} \)'s yields

\[
\bar{\rho} \frac{\partial^2 J_0}{\partial r^2} + \bar{\rho} \frac{1}{r} \frac{\partial J_0}{\partial r} - \bar{\rho} \frac{e^{-ixs}}{r^2} \bar{\rho} J_0 = \bar{\rho} J_0
\]

\[
\bar{\rho} \left[ -k \left( \frac{\partial}{\partial r} J_0 - \frac{1}{r} J_0 \right) + \left( - \frac{k}{r} J_0 \right) - \frac{e^{-ixs}}{r^2} J_0 \right] = \bar{\rho} J_0
\]

\[
\bar{\rho} = \frac{\bar{\rho} J_0}{-k^2 - \frac{e^{-ixs}}{r^2}}
\]

A-8
However,

\[ \bar{g}(k, \beta) = \frac{1}{2\pi} \int \left( \frac{ixs}{\sqrt{k^2 + \beta^2}} \right) g(r, x) e^{-ikr} dr dx \]

\[ = \frac{1}{2\pi} \int -\frac{B}{r} \delta(r) \delta(x) e^{ixs} dr dx \]

\[ = -\frac{B}{2\pi} \]

so

\[ \bar{\rho} = \frac{B}{2\pi} \frac{1}{k^2 + \beta^2} \]

(A-9)

Using (A-6) to invert (A-9),

\[ \rho(r, x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{B}{2\pi} \left( \frac{e^{-ixs} J_0(kr)}{k^2 + \beta^2} \right) dk dx \]  

(A-10)
There are two cases: (a) subsonic ($\omega > 0$), and (b) supersonic ($\omega < 0$)

(a) Subsonic case:

Consider the integral over $k$:

$$\int_{0}^{\infty} \frac{J_0(kr)}{kr + x^2 s^2} dk = K_0(rs)$$

[Watson p.165]

$$K_0(rs) = \frac{i\pi}{2} \left( J_0(i\pi sr) + i Y_0(i\pi sr) \right)$$

(A-10) becomes

$$\rho(r, x) = \frac{B}{2} \left[ i \int_{0}^{\infty} (\cos xs) J_0(i\pi rs) ds - \int_{0}^{\infty} (\cos xs) Y_0(i\pi rs) ds \right]$$

Since integrals over infinite limits, of odd functions are zero, this becomes

$$\rho(r, x) = \frac{B}{2} \left[ i \int_{0}^{\infty} (\cos xs) J_0(i\pi rs) ds - \int_{0}^{\infty} (\cos xs) Y_0(i\pi rs) ds \right]$$

$$\begin{cases} 
- \frac{i}{\left( x^2 + \pi^2 r^2 \right)^{1/2}} & 0 < x < i\pi r \\
0 & x > i\pi r 
\end{cases}$$

[Reihl p.206 #102]
For all cases,

\[
\delta \rho(r, x) = \frac{2GM\rho_0}{c^2} \frac{1}{(x^2 + \omega^2 r^2)^{1/2}}
\]

The isodensity surfaces are represented by

\[
x^2 + \left(1 - \frac{\omega^2}{c^2}\right)r^2 = \text{constant}
\]

These are ellipses with minor axis parallel to the x-axis.

(b) Supersonic case:

The integral
has a singularity at \( \xi = \pm \omega \) in the complex plane. The integral is taken in the sense of its principal value, and the non-singular part is added to the Fourier-Hankel transform.

\[
\int_{0}^{\infty} \frac{\hat{f}(k) J_0(kr)}{k^2 + \omega^2 \xi^2} \, dk
\]

The integral becomes

\[
i \pi \int_{0}^{\infty} \hat{f}(k) (\delta(k^2 - \omega^2 \xi^2) J_0(kr) \, dk
\]

\[
\mathcal{H} = \frac{i \pi}{2} \left( J_0(\omega \xi) - i Y_0(\omega \xi) \right)
\]

\[
\mathcal{H} = \left[ -\frac{1}{2} J_0(-\omega r) + \frac{1}{2} J_0(\omega r) \right] = 0
\]

(A-10) becomes

\[
\rho(\xi, \chi) = \frac{i \hat{g}(\xi) \chi}{4} \int_{-\infty}^{\infty} e^{-i \chi s} \left( J_0(\omega \xi) - Y_0(\omega \xi) \right) \, ds
\]

Dropping odd terms as before,

\[
\rho(\xi, \chi) = \frac{B}{2} \left[ i \int_{0}^{\infty} (\cos \chi s) J_0(\omega \xi) \, ds - \int_{0}^{\infty} (\cos \chi s) Y_0(\omega \xi) \, ds \right]
\]
\[ I = \begin{cases} 
\frac{-i}{[\chi^2 + \Omega^2 r^2]^{1/2}} & \text{for } x < \Omega r \\
0 & \text{for } x > \Omega r 
\end{cases} \]

\[ II = \begin{cases} 
0 & \text{for } 0 < x < \Omega r \\
-\frac{1}{i (\chi^2 + \Omega^2 r^2)^{1/2}} & \text{for } x > \Omega r 
\end{cases} \]

Ignoring the imaginary solution leaves

\[ \delta \rho (r, \chi) = \begin{cases} 
\frac{2GM\rho_0}{c^2} \frac{1}{(\chi^2 + \Omega^2 r^2)^{1/2}} & \chi^2 < \Omega^2 r^2 \\
0 & \chi^2 > \Omega^2 r^2 
\end{cases} \]
Drag Force exerted on a Test Object by a Non-Self-Gravitating Fluid (Linearized Model)

In the supersonic case, the density enhancement created by an object in a non-self-gravitating fluid is (see Appendix A):

\[
\delta \rho = \begin{cases} 
\frac{2GM \rho_0}{c^2} & x < 2r \\
0 & x > 2r
\end{cases}
\]

The drag force exerted on the object by this enhancement is given by

\[
F = Ma = M \frac{dV}{dt}
\]

so the object's deceleration is

\[
\frac{dV}{dt} = \begin{cases} 
\frac{F}{M} \, dV_{vol} & x < 2r \\
\frac{G \delta \rho}{[x^2 + r^2]^{1/2}} \, dV_{vol} & x < 2r
\end{cases}
\]
Inserting \((B-1)\),

\[
\frac{dV}{dt} = \frac{\mu \pi G^2 M}{c^2} \int_0^\infty \frac{z - Vt}{[(z-Vt)^2 + \omega^2 \tau^2]} \left[ \frac{z - Vt}{[(z-Vt)^2 + \omega^2 \tau^2]} \right] \, r \, dr \, dz
\]

Performing the \(z\)-integration (remembering that \(x = z - Vt\)),

\[
I = \left\{ \frac{\pi^2 \omega}{[x^2 + \omega^2 \tau^2]^{3/2}} \right\} \, dx = \frac{1}{\tau (1 - \omega^2)} \left[ \frac{(x^2 + \omega^2 \tau^2)^{1/2}}{(x^2 + \tau^2)^{3/2}} \right]_0^\infty
\]

Inserting the limits gives

\[
I = \frac{1}{\tau (1 - \omega^2)} \left[ \frac{\sqrt{2} \omega}{[\omega^2 + 1]^{3/2}} - \lim_{\epsilon \to 0} \frac{1 + \frac{2 \sqrt{\omega^2 + \omega^2 \tau^2} \epsilon^{1/2}}{1 + \frac{2 \sqrt{\omega^2 + \omega^2 \tau^2} \epsilon^{1/2}} + \frac{(\omega^2 \tau^2) \epsilon^{1/2}}{1 + \frac{(\omega^2 \tau^2) \epsilon^{1/2}}}} \right]
\]

Expanding the second term in a Taylor series around \(\epsilon = 0\) yields

\[
I = \left\{ \frac{\pi^2 \omega}{[x^2 + \omega^2 \tau^2]^{3/2}} \right\} \, dx = \frac{1}{\tau (1 - \omega^2)} \left[ \frac{(x^2 + \omega^2 \tau^2)^{1/2}}{(x^2 + \tau^2)^{3/2}} \right]_0^\infty
\]

Performing the \(r\)-integration leaves

\[
\frac{dV}{dt} = -\frac{\mu \pi G^2 M \rho}{\sqrt{\lambda}} \left( 1 - \sqrt{\frac{2 \omega^2 \tau^2}{1 + \omega^2 \tau^2}} \right) \log r \bigg|_0^\infty
\]

Thus, the rate at which the object is decelerated diverges logarithmically with distance.
APPENDIX C

Solution of the Equation Describing the Density Enhancement Created by an Object Travelling Through a Self-Gravitating Medium

The equation describing the behaviour of a self-gravitating medium in the presence of a massive object is (for a marginally stable perturbation, \( w = 0 \)):

\[
\frac{1}{c^2} \frac{\partial^2 \delta \rho}{\partial t^2} - \nabla^2 \delta \rho - h_{ij} \delta \rho = \frac{2GM\rho_0}{c^2} \frac{\delta(r)}{r} \delta(x) \tag{C-1}
\]

where

\[
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \phi^2}
\]

\[\chi \equiv z - \gamma t\]

(C-1) is

\[
\frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \rho}{\partial r} + \left( 1 - \frac{\gamma^2}{c^2} \right) \frac{\partial^2 \rho}{\partial x^2} + h_{ij} \rho = -\frac{2GM\rho_0}{c^2} \frac{\delta(r)}{r} \delta(x)
\]

Let

\[\Omega^2 \equiv 1 - \frac{\gamma^2}{c^2}\]
\[ q(r, x) = \frac{\delta(r)}{r} \delta(x) \]

\[ B = \frac{2GM_0}{c^2} \]

So

\[ \frac{\partial^2 \rho}{\partial r^2} + \frac{1}{r} \frac{\partial \rho}{\partial r} + \omega^2 \frac{\partial^2 \rho}{\partial x^2} + k_y^2 \rho = - B \frac{\delta(r)}{r} \delta(x) \quad C-2 \]

Using Fourier-Hankel transforms:

\[ \rho(r, x) = \int_0^\infty \int_{-\infty}^{\infty} k \rho(k, \delta) e^{-i\lambda \delta} J_0(kr) \, dk \, ds \quad C-3 \]

\[ \frac{\partial \rho}{\partial r} = \int_0^\infty \int_{-\infty}^{\infty} k \rho(k, \delta) e^{-i\lambda \delta} \frac{\delta J_0}{\partial r} \, dk \, ds \]

\[ \frac{\partial^2 \rho}{\partial r^2} = \int_0^\infty \int_{-\infty}^{\infty} k \rho(k, \delta) e^{-i\lambda \delta} \delta^2 J_0 \, dk \, ds \]

\[ \frac{\partial^3 \rho}{\partial x^3} = \int_0^\infty \int_{-\infty}^{\infty} -k \rho(k, \delta) e^{-i\lambda \delta} \delta^2 J_0 \, dk \, ds \]

Inserting these in (C-2), taking \( d/dk \) and \( d/dx \), and cancelling \( k \, e^{-i\lambda \delta} \) yields

\[ \rho \left[ \frac{\partial^2 J_0}{\partial r^2} + \frac{1}{r} \frac{\partial J_0}{\partial r} - \delta^2 \rho J_0 + k_y^2 J_0 \right] = -B \rho J_0 \quad C-4 \]
With

\[ \frac{\partial}{\partial r} J_0(kr) = -k J_1(kr) \]

\[ \frac{\partial^2}{\partial r^2} J_0(kr) = -k \left[ k J_0 - \frac{1}{r} J_1 \right] \]

(C-4) becomes

\[ \bar{\rho} = \frac{\rho \tilde{\tilde{\mathcal{g}}}}{[r^2 k^2 + k^2 - k_0^2]} \]

Now,

\[ \tilde{\tilde{\mathcal{g}}} = \frac{1}{2\pi} \int_{0}^{\infty} r f(r, x) e^{(ix^5)} J_0(kr) \, dr \, dx \]

\[ = \frac{1}{2\pi} \int_{0}^{\infty} r \frac{\delta(r) \delta(x)}{r} e^{(ix^5)} J_0(kr) \, dr \, dx \]

\[ = \frac{1}{2\pi} \]
So
\[ \vec{P} = \frac{B}{2\pi} \frac{1}{s^2 \omega^2 + k^2 - k_3^2} \]

Inverting the transform:
\[ \rho(r,\chi) = \int_0^\infty \int_{-\omega}^{\omega} \frac{B}{2\pi} \frac{k}{s^2 \omega^2 + k^2 - k_3^2} \exp(-i \chi s) J_0(kr) \, dk \, ds \]

There are two cases: the subsonic case \((\omega > 0)\), and the supersonic case \((\omega < 0)\).

(a) Subsonic Case:
(i) \( \chi \)-integration:
\[ \rho(r,\chi) = \frac{B}{2\pi} \int_0^\infty k J_0(kr) \left\{ \int_{-\omega}^{\omega} \frac{\exp(-i \chi s)}{s^2 \omega^2 + (k^2 - k_3^2)} \, ds \right\} \, dk \]

\[ I \equiv \int_{-\omega}^{\omega} \frac{\exp(-i \chi s)}{s^2 \omega^2 + (k^2 - k_3^2)} \, ds \]

Let \( \omega^2 = k^2 - k_3^2 \)

\[ = \int_{-\omega}^{\omega} \frac{\cos \chi s - \omega \sin \chi s}{s^2 \omega^2 + \omega^2} \, ds \]

[Schwam 15-40]
\[
I = \frac{2}{\omega^2} \left[ \frac{\pi}{2 \left( \frac{a^4}{\omega^2} \right)^{1/4}} e^{-x \left( \frac{a^4}{\omega^2} \right)^{1/4}} \right]
\]

So

\[
\rho(r,x) = \frac{2}{2\sqrt{\pi}} \left. \frac{1}{\sqrt{r}} \right|_{0}^{\infty} \int \frac{\rho_{\infty}(kr)}{\left( k^2 - k_x^2 \right)^{1/2}} \frac{1}{\left( k^2 - k_x^2 \right)^{1/2}} e^{-\frac{x}{4\omega^2} \left( k^2 - k_x^2 \right)^{1/2}} dk
\]

(ii) \text{k-integration:}

\text{Rewrite (C-8):}

\[
\rho(r,x) = \frac{2}{2\sqrt{\pi}} \left. \frac{1}{\sqrt{r}} \right|_{0}^{\infty} \int \frac{\rho_{\infty}(kr)}{\left( k^2 - k_x^2 \right)^{1/2}} \frac{1}{\left( k^2 - k_x^2 \right)^{1/2}} e^{-\frac{x}{4\omega^2} \left( k^2 - k_x^2 \right)^{1/2}} dk
\]

\text{There are two cases: (I) } k^2 < k_x^2 \text{, and (II) } k^2 > k_x^2 \text{.}

(I) \text{ } k^2 < k_x^2 :
(C-9) becomes

$$
\rho = \frac{B}{2\sqrt{\pi^2 \alpha^3}} \int_0^{k_3} \left( \frac{kr}{r} \right)^{1/4} J_0(kr) \frac{kr}{(k_3^2 - k_3^4)^{1/4}} e^{-\frac{\lambda}{k_3} i (k_3^2 - \lambda^2)^{1/2}} dk
$$

$$
= \frac{B}{2\sqrt{\pi^2 \alpha^3 \sqrt{r}}} \int_0^{k_3} \left( \frac{kr}{r} \right)^{1/4} J_0(kr) \frac{kr}{(k_3^2 - \lambda^2)^{1/4}} \left[ \cos \frac{\lambda}{\sqrt{\alpha^3}} (k_3^2 - \lambda^2)^{1/2} - i \sin \frac{\lambda}{\sqrt{\alpha^3}} (k_3^2 - \lambda^2)^{1/2} \right] dk
$$

Keeping only the real part,

$$
\rho = -\frac{B}{2\sqrt{\pi^2 \alpha^3 \sqrt{r}}} \int_0^{k_3} \left( \frac{kr}{r} \right)^{1/4} J_0(kr) \frac{kr}{(k_3^2 - \lambda^2)^{1/4}} \sin \frac{\lambda}{\sqrt{\alpha^3}} (k_3^2 - \lambda^2)^{1/2} dk \quad \text{C-10}
$$

\[\text{[E.M.O.T. (B) p.35 #23]}\]

$$
\rho = -\frac{B \sqrt{\pi \alpha^3}}{2\sqrt{r}} \frac{1}{(x_2^2 + r^2)^{1/4}} \left[ \frac{x_2}{(x_2^2 + r^2)^{1/4}} \right] \frac{b}{2} \left[ \frac{x_2}{(x_2^2 + r^2)^{1/4}} \right] \quad \text{C-11}
$$

Now,
\[ Y_{\frac{1}{2}} (u) = \frac{J_{\nu_k} (u) \cos \frac{\theta}{2} - J_{-\nu_k} (u)}{\sin \frac{\theta}{2}} \]

\[ = -J_{-\frac{1}{2}} (u) \]

So

\[ \rho = \frac{8 \sqrt{\pi} k_3}{2 \sqrt{\pi} \sqrt{\beta}} \frac{1}{(x^2 + \beta^2 \gamma^2)^{\frac{1}{4}}} \left( \frac{2}{\pi k_3 (x^2 + \beta^2 \gamma^2)^{\frac{1}{2}}} \right)^{\frac{1}{2}} \cos \left[ k_3 \left( \frac{x^2 + \beta^2 \gamma^2}{\beta^2} \right)^{\frac{1}{4}} \right] \]

\[ \delta \rho (r, x) = \frac{G M \rho_0}{c^2} \frac{1}{(x^2 + \beta^2 \gamma^2)^{\frac{1}{2}}} \cos \left[ \frac{k_3}{\sqrt{\beta^2}} \left( \frac{x^2 + \beta^2 \gamma^2}{\beta^2} \right)^{\frac{1}{4}} \right] \]

\[ (x > \lambda_3) \]

(II) \( k^2 > k_3^2 \):

\[ \rho = \frac{B}{2 \sqrt{\pi} \sqrt{\beta}} \int_{k_3}^{\infty} \left( (\kappa \gamma)^{\frac{1}{2}} J_0(\kappa r) \right) \frac{\kappa^{1/2}}{(k^2 - k_3^2)^{1/2}} \frac{x}{\sqrt{\beta}} \left( k^2 - k_3^2 \right)^{\frac{1}{4}} d\kappa \]
\[ \delta \rho = \frac{c \rho_0}{c^2} \frac{1}{(x^2 + \frac{\rho_2 r^2}{r^2})^{1/2}} \cos \frac{k_3}{\sqrt{\rho_2}} (x^2 + \frac{\rho_2 r^2}{r^2})^{1/2} \]

\[(x < \lambda_3)\]

The density perturbation is the same at distances greater than the Jeans length as at distances less than the Jeans length, and is

\[ \delta \rho (r, x) = \frac{c \rho_0}{c^2} \frac{1}{(x^2 + \frac{\rho_2 r^2}{r^2})^{1/2}} \cos \frac{k_3}{\sqrt{\rho_2}} (x^2 + \frac{\rho_2 r^2}{r^2})^{1/2} \]

This differs from the non-self-gravitating case by the factor

\[ \frac{1}{2} \cos \frac{k_3}{\sqrt{\rho_2}} (x^2 + \frac{\rho_2 r^2}{r^2})^{1/2} \]
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