A FIELD THEORY APPROACH TO PION DEUTERON ELASTIC SCATTERING

by

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Abstract

Pion-deuteron elastic scattering is studied using a field theory of pions and nucleons. By treating the nucleons in this manner, the double-counting problem usually associated with pion multiple-scattering is avoided.

The pion-deuteron T-matrix is written as a series expansion in terms of operators between one-nucleon states. The first two terms in the series are examined. The first term yields the usual single-scattering contribution to the T-matrix. The second term in the series can be expressed as a sum of twenty terms. By making an on-shell approximation and a static approximation where physically sensible, the magnitudes of the twenty terms are compared. The dominant term is similar to the conventional double scattering term resulting from the generalized impulse approximation. There are also four other terms whose magnitude cannot be evaluated without doing numerical studies with a particular field theoretic potential.

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1 Introduction

In the next few years many experiments will be performed at the meson factories which will use the pion as a probe of nuclear structure. The pion is a particularly useful tool to investigate nuclear structure. Since it comes in three charge states, it can participate in charge exchange and double charge exchange scattering which will hopefully further the understanding of nuclear states. Also, since the pion can be absorbed or emitted by a nucleon, pion-nucleus absorption experiments can be useful probes of higher momentum components in nuclear wavefunctions.

Since the interaction of the pion with the nucleus is rather weak for pion kinetic energies below about 100 MeV, the elastic scattering of low energy pions can be utilised in a manner similar to electron scattering. However, since pions and electrons interact with nucleons via different types of forces, elastic pion scattering and electron scattering can perhaps be used as complementary methods.

The simplest way to treat pion-nucleus scattering is to use the single-scattering approximation, i.e., to assume that the pion scatters from only one nucleon in the nucleus. However, if elastic pion scattering is ever to be useful in extracting details of nuclear structure using the single-scattering approximation , it is necessary to obtain an estimate of the terms neglected by taking the single-scattering approximation.

The generalized impulse approximation [Chew and Goldberger (1952)] expresses the elastic scattering of a projectile on a nucleus as a sum of terms, the first three of which are single-scattering, double-scattering, and binding corrections. Double-scattering describes the process where the projectile scatters from one nucleon, then propogates to another nucleon and scatters again before leaving the nucleus. Binding corrections describe the effect of the nuclear potential on the projectile-nucleon scattering.

If this approach is used to describe pion-nucleus scattering, a problem arises. Since the pion is thought of as mediating the nuclear force, the projectile scattering from the nucleons is identical to the particle being exchanged between these nucleons as a part of the nuclear force. Thus it is not clear to what extent the binding corrections are already included in the double-scattering term of the generalized impulse approximation. This is usually referred to as the "double-counting problem".

This thesis avoids the double-counting problem by constructing a field theory of pions and nucleons. The nucleons are assumed to be composed of a bare nucleon core surrounded by a cloud of physical pions. By treating the nucleons in this manner, no double-counting will occur since all the pions involved in the process are accounted for explicitly.

The approach used in this thesis follows a method used by Pendleton (1963). The case chosen is elastic pion-deuteron scattering. The deuteron was chosen as the target since it is the simplest nucleus in which double-scattering and binding corrections can be non-zero. Since it contains only two nucleons, the properties of its wavefunction are perhaps the best known of any nucleus.

In chapter 2 the pion, nucleon and pion-nucleon scattering states are defined along with the basic operators used in describing the scattering. In chapter 3 approximate two-nucleon states which can be written in terms of one-nucleon operators are introduced. These states are used to write the pion-deuteron T-matrix as a series expansion in terms of operators between one-nucleon states. In chapter 4 the first two terms of the series are calculated and after making some approximations, are compared to the single- and double-scattering terms resulting from the generalized impulse approximation. Chapter 5 consists of a summary of the results.

2 Definition of Operators and States

In this section some of the operators and states which will be used in describing pion-deuteron scattering will be introduced. The zero- and one-nucleon states will be developed in a manner similar to that used by Chew and Low(1956) and Wick (1955) in describing pion-nucleon scattering. The two-nucleon states will be treated using a method first developed by Heitler and London(1927) to describe the hydrogen molecule and later applied to nucleon-nucleon problems by Cutkosky(1958) and Pendleton(1963).

The fundamental dynamical variables for the system of pions and nucleons are assumed to be boson annihilation and creation operators k and $k^\dagger$ and fermion annihilation and creation operators a and $a^\dagger$. These operators satisfy the usual commutation and anticommutation rules. That is, the boson operators satisfy

$$[k,m] = [k^\dagger,m^\dagger] = 0$$
$$[k,m^\dagger] = \delta_{mk} \qquad (2.1)$$

and the fermion operators satisfy

$$\{a,b\} = \{a^\dagger,b^\dagger\} = 0$$
$$\{a,b^\dagger\} = \delta_{ab} \qquad (2.2)$$

In addition the boson and fermion operators will commute. The delta functions above are actually products of Kroenecker delta functions of spin and isospin with Dirac delta functions of momentum. The boson field quanta will be identified with physical pions and the fermion field quanta will be identified with bare nucleons and antinucleons.

It will be assumed that a Hamiltonian can be constructed using the fundamental dynamical variables and having eigenstates corresponding to physical nucleons, pion-nucleon scattering states, deuterons, etc. These states will also be eigenstates of the baryon number operator B.

2.1 Physical States With Baryon Number Zero

The state vectors corresponding to baryon number zero will be meson states or the physical vacuum state. The meson state $|k\rangle$ represents a meson with quantum numbers (spin, isospin, momentum, etc.) labelled by k. It satisfies

$$H|k\rangle = E_k|k\rangle \tag{2.3}$$

$$\underline{P}|k\rangle = \underline{k}|k\rangle \tag{2.4}$$

$$B|k\rangle = 0 \tag{2.5}$$

$$E_k = \sqrt{|\underline{k}|^2+m_\pi^2} \tag{2.6}$$

where $\underline{P}$ is the momentum operator, B is the baryon number operator and m_π is the mass of the pion. The speed of light and $\hbar$ are taken to be one in the above equations. The physical vacuum state $|0\rangle$ is defined by

$$H|0\rangle = 0 \tag{2.7}$$

$$B|0\rangle = 0 \tag{2.8}$$

It also satisfies

$$k|0\rangle = 0 \tag{2.9}$$

$$a|0\rangle = 0 \tag{2.10}$$

As stated above it will be assumed that the boson field quanta are physical pions. That is, the physical meson state $|k\rangle$ can be thought of as resulting from the action of the meson creation operator $k^\dagger$ on the physical vacuum state $|0\rangle$

$$k^\dagger|0\rangle = |k\rangle \tag{2.11}$$

The Hamiltonian will therefore be written in the form

$$H = \Sigma_k E_k k^\dagger k + V \tag{2.12}$$

where V is an operator describing the meson-meson and meson-nucleon interaction. The summation over k in the above equation denotes a sum over all the meson quantum numbers as well as an integral over the meson momentum. (In subsection 3.2.3 a different splitting of H will be introduced. That splitting will be more useful in separating the nucleon energy part of H from the nucleon interaction part of H.)

2.2 Physical States With Baryon Number One

The state vectors corresponding to baryon number one will be either physical one-nucleon states or pion-nucleon scattering states. As both are of great importance in discussing pion-deuteron scattering, they will be treated in some detail.

2.2.1 The Physical One-Nucleon State

The physical one-nucleon state $|A\rangle$ describes a physical nucleon with quantum numbers (spin, isospin, momentum etc.) labelled by A. It satisfies

$$H|A\rangle = E_A|A\rangle \tag{2.13}$$

$$\underline{P}|A\rangle = \underline{k}_A|A\rangle \tag{2.14}$$

$$B|A\rangle = +1|A\rangle \tag{2.15}$$

$$E_A = \sqrt{|\underline{k}_A|^2+M^2} \tag{2.16}$$

where M is the observed nucleon mass.

A physical one-nucleon creation operator $A^\dagger$ is defined as follows. In the Chew-Low theory [Chew and Low(1956)] the physical one-nucleon creation operator is written as the product of a bare nucleon creation operator and an operator which produces a "cloud" of pions. The pion cloud creation operator can be written as a sum of products of pion creation operators.

In the present treatment the physical nucleon creation operator is constructed in a more general manner. The operator is written as a linear combination of the products of nucleon core creation operators and meson cloud creation operators:

$$A^{\dagger} = \Sigma\ a(AM)\ A^{\dagger}M^{\dagger} \qquad (2.17)$$

The nucleon core creation operator $A^{\dagger}$ is simply a product of bare nucleon and antinucleon creation operators with total baryon number one:

$$BA^{\dagger}|0\rangle = +1\ A^{\dagger}|0\rangle \qquad (2.18)$$

Obviously if $A^{\dagger}$ is to satisfy the above equation it must contain an odd number of bare nucleon and antinucleon creation operators. The meson cloud operator $M^{\dagger}$ consists of a product of meson creation operators.

The summation in equation (2.17) is over all nucleon core creation operators and meson cloud creation operators subject to the condition that the total charge and momentum of each term must be equal to the charge and momentum of the state $|A\rangle$.

The coefficients a(AM) in the linear combination will be the wave function of the physical nucleon in the Fock space of the cores and physical mesons.

From the form of equation (2.17) and the commutation and anticommutation relations of equations (2.1) and (2.2) it can be seen that physical nucleon operators A,B,etc. obey the following anticommutation relations:

$$\{A,B\} = \{A^{\dagger},B^{\dagger}\} = 0 \qquad (2.19)$$

However, the expression for $\{A,B^{\dagger}\}$ will be very complicated because of the necessity of anticommuting the products of the annihilation operators with the products of the creation operators in the expressions for A and $B^\dagger$. The calculations necessary to evaluate this commutator are done in an approximate manner in Appendix B.

2.2.2 The Pion-Nucleon Scattering State

The pion-nucleon scattering state $|Ak>_\pm$ represents a state composed of a physical nucleon with quantum numbers asymptotically labelled by A and a meson with quantum numbers asymptotically labelled by k, obeying either outgoing wave (+ sign) or incoming wave (- sign) boundary conditions. Since only outgoing wave states will normally be dealt with, pion-nucleon scattering states without + or - subscripts will be assumed to satisfy outgoing wave boundary conditions. The pion-nucleon scattering state $|Ak>$ satisfies

$$H|Ak>_\pm = (E_A+E_k)|Ak>_\pm \qquad (2.20)$$

$$\underline{P}|Ak>_\pm = (\underline{k}_A+\underline{k})\,|Ak>_\pm \qquad (2.21)$$

$$B|Ak>_\pm = +1\,|Ak>_\pm \qquad (2.22)$$

where E_A and E_k are as given previously.

Wick (1955) has shown that the pion-nucleon scattering state can be expressed solely in terms of the Hamiltonian, meson creation operators, and physical nucleon creation operators. Wick wrote the pion-nucleon scattering state as a state consisting of a free meson and a physical nucleon plus a scattered state

$$|Ak\rangle = k^{\dagger}|A\rangle + |Ak\rangle_s \tag{2.23}$$

An explicit expression for the scattered state $|Ak\rangle_s$ can be obtained by substituting equation (2.23) into the eigenvalue equation for H for the scattering state, equation (2.20)

$$(H-E_A-E_k)|Ak\rangle_s + Hk^{\dagger}|A\rangle - (E_A+E_k)k^{\dagger}|A\rangle = 0 \tag{2.24}$$

Equation (2.24) is simplified by writing

$$Hk^{\dagger}|A\rangle = k^{\dagger}H|A\rangle + [H,k^{\dagger}]|A\rangle \tag{2.25}$$

Using the explicit form of the Hamiltonian, equation (2.12), the commutator in the above expression becomes

$$[H,k^{\dagger}]|A\rangle = E_k\, k^{\dagger}|A\rangle + [V,k^{\dagger}]|A\rangle \tag{2.26}$$

By substituting equations (2.12), (2.25), and (2.26) into equation (2.24) the following equation is obtained

$$(H-E_A-E_k)|Ak\rangle_s + [V,k^{\dagger}]|A\rangle = 0 \tag{2.27}$$

An expression for the scattered state $|Ak\rangle_s$ is obtained by inverting the operator $(H-E_A-E_k)$ and imposing outgoing wave boundary conditions

$$|Ak\rangle_s = (E_A+E_k-H+i\varepsilon)^{-1}[V,k^{\dagger}]|A\rangle \tag{2.28}$$

Using Pendleton's notation for the vertex operator

$$[Vk] = [V,k^{\dagger}]$$
$$[kV] = [Vk]^{\dagger} \tag{2.29}$$

the outgoing wave pion-nucleon scattering state is written

$$|Ak> = k^{\dagger}|A> + (E_A+E_k-H+i\varepsilon)^{-1}[Vk]|A> \tag{2.30}$$

It should be noted that equation (2.30) is written incorrectly by Pendleton, the operator [Vk] being replaced by [kV] .

2.3 States With Baryon Number Two

2.3.1 The Physical Two-Nucleon State

The physical two-nucleon state, denoted by |AB>, represents two physical nucleons with quantum numbers asymptotically labelled by A and B. It satisfies

$$H|AB> = (E_A+E_B)|AB> \tag{2.31}$$

$$\underline{P}|AB> = (\underline{k}_A+\underline{k}_B)|AB> \tag{2.32}$$

$$\mathcal{B}|AB> = +2|AB> \tag{2.33}$$

2.3.2 The Deuteron State and the Pion-Deuteron Scattering State

The physical deuteron state is represented by the state vector $|\mathcal{D}>$. It satisfies

$$H|\mathcal{D}> = E_{\mathcal{D}}|\mathcal{D}> \tag{2.34}$$

$$\underline{P}|\mathcal{D}\rangle = \underline{k}_{\mathcal{D}}|\mathcal{D}\rangle \tag{2.35}$$

$$B|\mathcal{D}\rangle = +2|\mathcal{D}\rangle \tag{2.36}$$

$$E_{\mathcal{D}} = \sqrt{|\underline{k}_{\mathcal{D}}|^2 + M_{\mathcal{D}}^2} \tag{2.37}$$

where $M_{\mathcal{D}}$ is the observed deuteron rest mass.

The pion-deuteron scattering state $|\mathcal{D}k\rangle_{\pm}$ represents a state composed of a physical deuteron with quantum numbers asymptotically labelled by $\mathcal{D}$ and a meson with quantum numbers asymptotically labelled by k. As with the pion-nucleon scattering state, the $\pm$ subscript refers to either outgoing or incoming wave boundary conditions and a state with no subscript will be assumed to satisfy outgoing wave boundary conditions. The pion-deuteron scattering state $|\mathcal{D}k\rangle$ satisfies

$$H|\mathcal{D}k\rangle_{\pm} = (E_{\mathcal{D}}+E_k)|\mathcal{D}k\rangle_{\pm} \tag{2.38}$$

$$\underline{P}|\mathcal{D}k\rangle_{\pm} = (\underline{k}_{\mathcal{D}}+\underline{k})|\mathcal{D}k\rangle_{\pm} \tag{2.39}$$

$$B|\mathcal{D}k\rangle_{\pm} = +2|\mathcal{D}k\rangle_{\pm} \tag{2.40}$$

where $E_{\mathcal{D}}$ and E_k are as given previously.

By writing the pion-deuteron scattering state as

$$|\mathcal{D}k\rangle = k^{\dagger}|\mathcal{D}\rangle + |\mathcal{D}k\rangle_s \tag{2.41}$$

and using exactly the same procedure as was used following equation (2.23), the analogous result is obtained for the pion-deuteron scattering state

$$|\mathcal{D}k\rangle = k^{\dagger}|\mathcal{D}\rangle + (E_{\mathcal{D}}+E_k-H+i\varepsilon)^{-1}[Vk]|\mathcal{D}\rangle \tag{2.42}$$

2.4 The S-matrix and the T-matrix

The scattering operator S is defined in general as

$$S|\psi_{in}(0)\rangle = |\psi_{out}(0)\rangle \tag{2.43}$$

where $|\psi_{in}(0)\rangle$ is the time independent factor of the asymptotic limit of the state vector a long time before scattering, satisfying incoming wave boundary conditions and $|\psi_{out}(0)\rangle$ is the time independent factor of the asymptotic limit of the state vector a long time after scattering, satisfying outgoing wave boundary conditions.

For pion-nucleon scattering equation (2.43) can be written as

$$S|Bm\rangle_- = |Bm\rangle_+ \tag{2.44}$$

or, using the fact that S is unitary

$${}_+\langle Bm|S = {}_-\langle Bm| \tag{2.45}$$

The S-matrix for pion-nucleon scattering is defined as

$$\begin{aligned} S_{Bm,Ak} &= {}_+\langle Bm|S|Ak\rangle_+ \\ &= {}_-\langle Bm|Ak\rangle_+ \end{aligned} \tag{2.46}$$

by use of equation (2.45). Using equation (2.30) for $|Bm\rangle_+$ and the similar equation for $|Bm\rangle_-$

$$|Bm\rangle_- = m^\dagger|B\rangle + (E_B+E_m-H-i\varepsilon)^{-1}[Vm]|B\rangle \tag{2.47}$$

it follows that

$$|Bm\rangle_- = |Bm\rangle_+ + \{(E_B+E_m-H-i\varepsilon)^{-1}-(E_B+E_m-H+i\varepsilon)^{-1}\}[Vm]|B\rangle \tag{2.48}$$

Using the identity

$$(E_B+E_m-H-i\varepsilon)^{-1} = (E_B+E_m-H+i\varepsilon)^{-1} + 2\pi i\delta(E_B+E_m-H) \tag{2.49}$$

equation (2.48) can be written

$$|Bm\rangle_- = |Bm\rangle_+ + 2\pi i\delta(E_B+E_m-H)[Vm]|B\rangle \tag{2.50}$$

Substituting this into equation (2.46), the expression for the S-matrix becomes

$$\begin{aligned} S_{Bm,Ak} &= {}_+\langle Bm|Ak\rangle_+ - 2\pi i\delta(E_B+E_m-E_A-E_k)\langle B|[mV]|Ak\rangle_+ \\ &= \delta_{Bm,Ak} - 2\pi i\delta(E_B+E_m-E_A-E_k)\langle B|[mV]|Ak\rangle_+ \end{aligned} \tag{2.51}$$

It should be noted that the Kronecker delta function $\delta_{Bm,Ak}$ is a shorthand notation for the product of Kronecker delta functions of all the quantum numbers of the states $|Bm\rangle$ and $|Ak\rangle$ as well as a delta function of momentum.

The general expression for the S-matrix is

$$S_{FI} = \delta_{FI} - 2\pi i\delta(E_F-E_I)T_{FI} \tag{2.52}$$

where I refers to the initial state, F refers to the final state and T_{FI} is the T-matrix. Thus, comparing equations (2.51) and (2.52), the T-matrix for pion-nucleon scattering can be written

$$T_{Bm,Ak} = \langle B|[mV]|Ak\rangle_+ \tag{2.53}$$

Similarily, using equation (2.42) for $|\mathcal{D}k\rangle_+$ and the corresponding equation for $|\mathcal{D}k\rangle_-$, the T-matrix for pion-deuteron scattering can be written

$$T_{\mathcal{D}'m,\mathcal{D}k} = \langle \mathcal{D}'|[mV]|\mathcal{D}k\rangle_+ \tag{2.54}$$

The objective now is to calculate the right hand side of eq. (2.54). To do this, the deuteron state and the pion-deuteron scattering state will be expanded in terms of a particular set of two-nucleon states introduced in the next chapter.

3 Meson Exchange Series

In order to proceed from this point, a complete set of states with baryon number two is introduced. Rather than choosing physical two-nucleon states which cannot readily be expressed in terms of one-nucleon creation operators, a set of states is constructed explicitly in terms of one-nucleon creation operators and meson creation operators following a method used by Cutkosky(1958) and based on the work of Heitler and London(1927). These states will be called Cutkosky states.

Since Cutkosky states are written in terms of physical nucleon and meson creation operators, matrix elements between these states will be able to be reduced to matrix elements between one-nucleon states. Thus, by using these states to expand the pion-deuteron T-matrix, it will be possible to express the T-matrix entirely in terms of one-nucleon matrix elements.

3.1 Cutkosky States with Baryon Number Two

The simplest Cutkosky state is that which consists of two nucleons with no mesons present. This state, called an unexcited Cutkosky state and denoted by $|AB\}$, is defined by

$$|AB\} = A^{\dagger}B^{\dagger}|0\rangle \tag{3.1}$$

It satisfies

$$\underline{P}|AB\} = (\underline{k}_A+\underline{k}_B)|AB\} \tag{3.2}$$

$$B|AB\} = +2|AB\} \tag{3.3}$$

However, unlike the physical two-nucleon state $|AB\rangle$, the Cutkosky state $|AB\}$ is not an eigenstate of the Hamiltonian. This is shown explicitly in equation (3.44). Using the definition of the state $|AB\}$ and the fact that physical nucleon operators anticommute, it is obvious that the state $|AB\}$ is antisymmetric

$$|AB\} = -|BA\} \tag{3.4}$$

Singly excited Cutkosky states, i.e. those in which an extra meson is present, are defined by

$$|ABk\} = k^\dagger A^\dagger B^\dagger|0\rangle + (Ak)^\dagger_s B^\dagger|0\rangle + A^\dagger(Bk)^\dagger_s|0\rangle \tag{3.5}$$

These states satisfy

$$\underline{P}|ABk\} = (\underline{k}_A+\underline{k}_B+\underline{k})|ABk\} \tag{3.6}$$

$$B|ABk\} = +2|ABk\} \tag{3.7}$$

These states are not eigenstates of the Hamiltonian as is shown in equation (3.46).

The general excited Cutkosky state with baryon number two, $|ABM\}$, where M represents a product of meson operators, can be defined in a similar manner but it will not be needed in the single- and double-scattering calculations. General Cutkosky states (either excited or unexcited) will be denoted by $|U\}$, $|V\}$, $|W\}$ etc.

One of the fundamental assumptions of this method is that the set of all Cutkosky states form a complete set. In other words, it is assumed that any physical state with baryon number two can be expressed as a linear combination of the states |U}. In particular a physical two-nucleon state should be able to be written as

$$|AB\rangle = |AB\} + \Sigma\ \chi_M |ABM\} \qquad (3.8)$$

where the coefficients χ_M are to be determined. For the calculations to be done it is hoped that equation (3.8) will converge quickly. Since the deuteron is rather weakly bound, a good approximation to the deuteron state should be able to be obtained just by taking the noninteracting term and the terms in the summation in equation (3.8) for which M contains only one meson operator (i.e. singly excited states).

As the Cutkosky states defined above are not orthogonal, orthonormal Cutkosky states, denoted by |U), |V), |W) etc., are defined (following Pendleton(1963)) by the equation

$$|W) = \Sigma\ F_{uw} |U\} \qquad (3.9)$$

The summation in equation (3.9) runs over all two-nucleon Cutkosky states. The orthonormal states are introduced for computational convenience. Using the orthonormal states the unit operator can be written in the form

$$1 = \Sigma\ |U)(U| \qquad (3.10)$$

which will be used in the calculation of matrix elements involved in single- and double-scattering expressions. Equation (3.10) is not true if the complete set of states is not orthonormal.

Using equation (3.10) and the orthonormality of the states defined in equation (3.9), an expression for the matrix F is derived as follows.

$$\begin{aligned}\delta_{wv} &= (V|W) \\ &= \Sigma\Sigma\, F_{xv}^{*}\, F_{uw}\, \{X|U\}\end{aligned} \qquad (3.11)$$

Defining the matrix G by the equation

$$\delta_{xu} + G_{xu} = \{X|U\} \qquad (3.12)$$

equation (3.11) can be written

$$\delta_{wv} = \Sigma\, F_{uv}^{*}\, F_{uw} + \Sigma\Sigma\, F_{xv}^{*}\, G_{xu}\, F_{uw} \qquad (3.13)$$

Writing the above equation in matrix form yields the equation

$$1 = F^{\dagger}F + F^{\dagger}GF \qquad (3.14)$$

where the adjoint of the matrix F is defined by $F^{\dagger}_{uv} = F^{*}_{vu}$. Equation (3.14) can be solved for $F^{\dagger}F$. The solution is

$$F^{\dagger}F = (1+G)^{-1} \qquad (3.15)$$

The above equation can be solved for F. By choosing F to be Hermitian a solution can be written

$$F = (1+G)^{-\frac{1}{2}} = 1 - \frac{1}{2}G + \frac{3}{8}G^2 + \ldots \tag{3.16}$$

The expansion using the binomial theorem will be used to calculate approximations to . It will be shown that for single- and double-scattering calculations, no more than the first two terms in the expansion will be needed. For example, by taking the first two terms only, the state $|AB)$ may be written

$$|AB) = \frac{3}{2}|AB\} - \frac{1}{2}\sum_U |U\}\{U|AB\} \tag{3.17}$$

It will be useful to define an operator $K^\dagger$ as follows

$$K^\dagger|U) = |Uk) \tag{3.18}$$

This operator is not easily expressed in terms of physical nucleon and meson creation operators. Taking the first two terms only in eq. (3.16) it follows that

$$\begin{aligned} K^\dagger|AB) &= |ABk) \\ &= \frac{3}{2}|ABk\} - \frac{1}{2}\sum_U |U\}\{U|ABk\} \end{aligned} \tag{3.19}$$

Because of the second term in the above equation, the relationship between $K^\dagger$ and physical nucleon and meson creation operators is very complicated.

3.2 Meson Operator Identities and Cutkosky Matrix Elements

In order to evaluate the pion-deuteron T-matrix, the deuteron state and the pion-deuteron scattering state will be expanded in terms of the orthonormal Cutkosky states. Thus the T-matrix will be written in terms of matrix elements between orthonormal Cutkosky states. Using equations (3.9), (3.12), and (3.16) these matrix elements can be written in terms of matrix elements between nonorthonormal Cutkosky states. The purpose of this section is to evaluate Cutkosky matrix elements and present meson operator identities which will prove to be useful in evaluating and simplifying these matrix elements.

3.2.1 Operator Identities

The meson operator identities will only be stated here, the proofs being left to Appendix A. The first identity gives an expression for the commutator of a product of meson creation operators $M^\dagger$ and a product of meson annihilation operators Q. It will be written in two forms

$$[M^\dagger,Q] = -\Sigma_R' \, [[r,M^\dagger]][[Q,r^\dagger]]/n(R)! \tag{3.20}$$

or

$$QM^\dagger = \Sigma_R[[r,M^\dagger]][[Q,r^\dagger]]/n(R)! \tag{3.21}$$

where

$$[[r,M^\dagger]] = [r_1,[r_2,[\ldots[r_n,M^\dagger]\ldots]]] \tag{3.22}$$

$$[[M,r^\dagger]] = [[r,M^\dagger]]^\dagger \tag{3.23}$$

n(R)=n=number of operators in R

R represents a product of meson annihilation operators. The prime on the summation in equation (3.20) indicates that the unit operator is to be excluded from the sum over all R. When R=1 the nested commutator of equation (3.22) is defined to be just $M^{\dagger}$.

The next three identities will be useful in removing meson operators from one nucleon matrix elements. These identities are

$$r|A\rangle = -(H+E_r-E_A-i\varepsilon)^{-1}[rV]|A\rangle \tag{3.24}$$

$$r(H-E)^{-1} = (H+E_r-E)^{-1}r - (H+E_r-E)^{-1}[rV](H-E)^{-1} \tag{3.25}$$

$$r|Ak\rangle = \delta_{kr}|A\rangle - (H+E_r-E_A-E_k-i\varepsilon)^{-1}[rV]|A\rangle \tag{3.26}$$

where r is a meson annihilation operator and where E_r, E_A, and E_k have been defined previously. The proofs of these identities are given in Appendix A.

3.2.2 Cutkosky Overlap Matrix Elements and Diagrams

The matrix elements between Cutkosky states are evaluated in Appendix B. Aside from the general result for the matrix element {AB|CD}, only those matrix elements which will be needed to calculate single- and double-scattering will be presented below. The simplest matrix element is that between two unexcited Cutkosky states, |AB} and |CD}. It can be written

$$\{AB|CD\} = \sum_{R,Q} \langle A|Q^{\dagger}R|C\rangle\langle B|R^{\dagger}Q|D\rangle/n(R)!n(Q)!$$
$$-\sum_{R,Q} \langle B|Q^{\dagger}R|C\rangle\langle A|R^{\dagger}Q|D\rangle/n(R)!n(Q)! \tag{3.27}$$

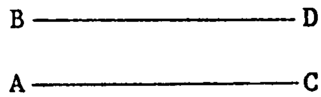
where $R^{\dagger}$ and $Q^{\dagger}$ represent products of meson annihilation operators. The unit operator is included in the above summations. Equation (3.27) is not an exact result. The terms neglected above are those in which the core operators in a particular term of the product AB are equal to the core operators in a particular term of the product CD but the core operator from A is not equal to the core operator from C or the core operator from D. This type of term can be thought of as describing the exchange of one or more pairs of bare nucleons and antinucleons between nucleons in the initial state to form the nucleons in the final state. A more complete explanation of the above is given in Appendix B.

The above expansion of a Cutkosky matrix element will be called a meson exchange series. The series is ordered by the number of mesons exchanged, i.e., by the total number of meson operators in the product $Q^{\dagger}R$. The n meson exchange contribution to a matrix element will be denoted by a bracketed superscript n.

Thus the zero meson exchange contribution to the matrix element $\{AB|CD\}$ is

$$\begin{aligned} \{AB|CD\}^{(0)} &= \langle A|C\rangle\langle B|D\rangle - \langle B|C\rangle\langle A|D\rangle \\ &\equiv \delta_{AB,CD} \end{aligned} \tag{3.28}$$

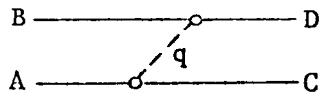
This can be represented diagramatically as



The one-meson exchange contribution is

$$\{AB|CD\}^{(1)} = \sum_q \langle A|[Vq]|C\rangle\langle B|[qV]|D\rangle(E_C+E_q-E_A)^{-1}(E_B+E_q-E_D)^{-1} \quad (3.29)$$
$$+ \text{ antisym.}$$

This is represented diagramatically as



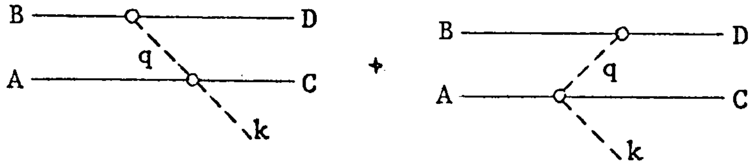
The terms in equation (3.29) represented by 'antisym.' are those necessary to make the right hand side of the equation antisymmetric with respect to exchange of A and B or C and D. The meson annihilation and creation operators have been removed from equation (3.29) by use of equation (3.25). The solid lines represent physical nucleons and the dotted lines represent mesons. The diagrams are drawn with the initial state on the right and the final state on the left. For example, the first diagram for $\{AB|CD\}^{(1)}$ corresponds to a nucleon in state D emitting a meson in state q and becoming a nucleon in state B. The meson is then absorbed by a nucleon in state C becoming a nucleon in state A. There is no rule for obtaining the energy denominators from inspection of the diagram since the same diagram can represent different matrix elements with different energy denominators.

If one or both of the Cutkosky states are excited the results are

$$\{AB|CDk\}^{(0)} = 0 \quad (3.30)$$

$$\{AB|CDk\}^{(1)} = \sum_q \langle A|[Vq]|Ck\rangle\langle B|[qV]|D\rangle(E_C+E_k+E_q-E_A)^{-1}(E_B+E_q-E_D)^{-1}$$
$$+ \sum_q \langle A|[qV]|Ck\rangle\langle B|[Vq]|D\rangle(E_C+E_k-E_q-E_A+i\epsilon)^{-1}(E_B-E_q-E_D)^{-1}$$
$$+ \text{antisym.}$$

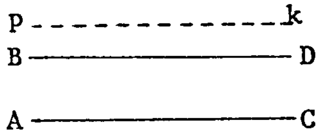
=



(3.31)

$$\{ABp|CDk\}^{(0)} = \delta_{kp}\delta_{AB,CD}$$

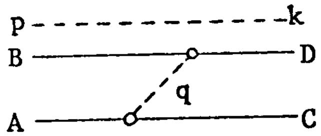
=



(3.32)

$$\{ABp|CDk\}^{(1)} = \delta_{kp}\ \{AB|CD\}^{(1)} + \text{other terms.}$$

=



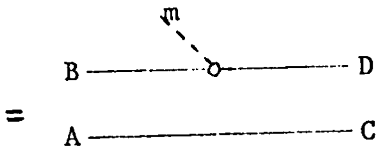
(3.33)

The other terms in equation (3.33) are those in which three distinct mesons, p, k, and q, are present

3.2.3 Cutkosky Matrix Elements of the Vertex Operator and the Hamiltonian

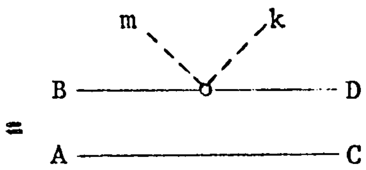
The matrix elements of the vertex operator between Cutkosky states are given by

$$\{AB|[mV]|CD\}^{(0)} = \delta_{AC}\langle B|[mV]|D\rangle + \text{antisym.}$$



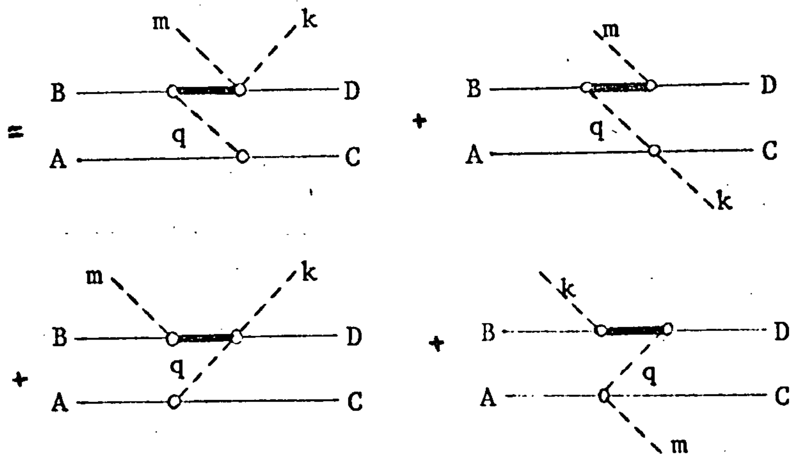
(3.34)

$$\{AB|[mV]|CDk\}^{(0)} = \delta_{AC}\langle B|[mV]|Dk\rangle + \text{antisym.}$$



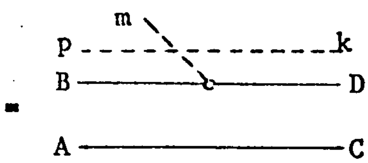
(3.35)

$$\begin{aligned}\{AB|[mV]|CDk\}^{(1)} &= \sum_q \langle A|[qV]|C\rangle\langle B|[Vq](H+E_q-E_B)^{-1}[mV]|Dk\rangle(E_A+E_q-E_C)^{-1}\\ &+ \sum_q \langle A|[qV]|Ck\rangle\langle B|[Vq](H+E_q-E_B)-1[mV]|D\rangle(E_A+E_q-E_C-E_k-i\varepsilon)^{-1}\\ &+ \sum_q \langle A|[Vq]|C\rangle\langle B|[mV](H+E_q-E_D-E_k-i\varepsilon)^{-1}[qV]|Dk\rangle(E_C+E_q-E_A)^{-1}\\ &+ \sum_q \langle A|[Vq]|Ck\rangle\langle B|[mV](H+E_q-E_D)^{-1}[qV]|D\rangle(E_C+E_k+E_q-E_A)^{-1}\\ &+ \text{antisym.}\end{aligned}$$



(3.36)

$$\{ABp|[mV]|CDk\}^{(0)} = \delta_{kp}\ \{AB|[mV]|CD\}^{(0)} + \text{antisym.}$$



(3.37)

The heavy solid line indicates that the intermediate state is to be summed over a complete set of states with baryon number one.

That is, to evaluate the matrix elements in equation (3.36), a complete set of states with baryon number one must be inserted into the second matrix element for each term. Note that in the diagrams drawn, the ordering of vertices has no significance if the vertices are not directly connected by either a dotted line or a solid line. For example, in the first diagram for equation (3.36), the vertices representing the emission of the meson q and the absorption of the meson k are not directly connected and thus they need not be drawn as occuring at the same instant.

In writing matrix elements of the Hamiltonian, it will prove to be useful to write the Hamiltonian as

$$H = H_o + H' \tag{3.38}$$

where H_o is defined by

$$H_o|AB\} = (E_A+E_B)|AB\} \tag{3.39}$$

$$H_o|ABk\} = (E_A+E_B+E_k)|ABk\} \tag{3.40}$$

with similar relations for multiply excited Cutkosky states. The above equations can be written generally as

$$H_o|U\} = E_u|U\} \tag{3.41}$$

for any Cutkosky state $|U\}$.

Using equation (3.41), it follows that

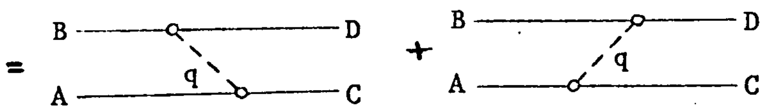
$$\{U|H_o|W\} = \frac{1}{2}(E_u+E_w)\{U|W\} \tag{3.42}$$

As is shown below, the remainder H' has the very important property of having no zero-meson exchange contributions to its matrix elements. This separation of H will be useful when evaluating approximations to the pion-deuteron T-matrix.

The matrix elements of the Hamiltonian between Cutkosky states are given below. Note that the diagrams are drawn for the matrix elements of H' only.

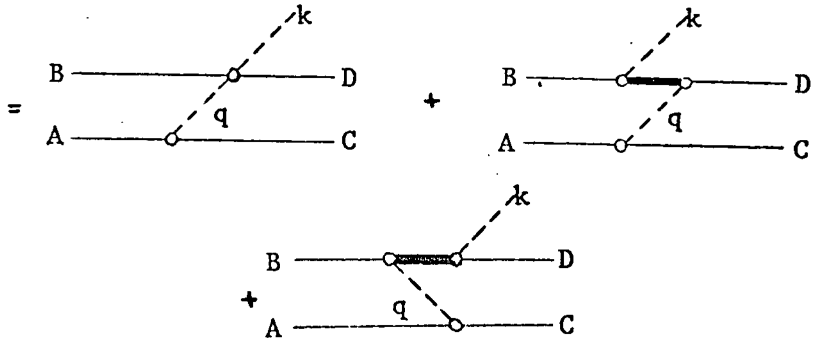
$$\{AB|H|CD\}^{(0)} = \frac{1}{2}\,(E_A+E_B+E_C+E_D)\,\delta_{AB,CD} \tag{3.43}$$

$$\begin{aligned}\{AB|H|CD\}^{(1)} &= \frac{1}{2}\,(E_A+E_B+E_C+E_D)\{AB|CD\}^{(1)} \\ &+ \frac{1}{2}\sum_q \langle A|[qV]|C\rangle\langle B|[Vq]|D\rangle(E_A+E_q-E_C)^{-1} \\ &+ \frac{1}{2}\sum_q \langle A|[Vq]|C\rangle\langle B|[qV]|D\rangle(E_C+E_q-E_A)^{-1} \\ &+ \text{antisym.}\end{aligned} \tag{3.44}$$



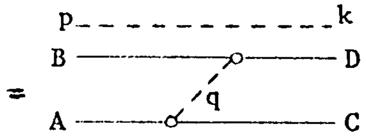
$$\{AB|H|CDk\}^{(0)} = 0 \tag{3.45}$$

$$
\begin{aligned}
\{AB|H|CDk\}^{(1)} &= \frac{1}{2}\,(E_A+E_B+E_C+E_D+E_k)\{AB|CDk\}^{(1)} \\
&+ \sum_q \langle A|[Vq]|C\rangle\langle B|[qV]|Dk\rangle\; a_1(E) \\
&+ \sum_q \langle A|[Vq]|C\rangle\langle B|[Vk](a_2(E)(H+E_q-E_D-i\varepsilon)^{-1}-a_3(E)(H+E_k\ E_B+i\varepsilon)^{-1})[qV]|D\rangle \\
&+ \sum_q \langle A|[qV]|C\rangle\langle B|[Vq](H-E_D-E_k-i\varepsilon)^{-1}(H+E_q-E_B+i\varepsilon)^{-1}[Vk]|D\rangle\; a_4(E) \\
&+ \text{antisym.}
\end{aligned}
$$



(3.46)

$$\{ABp|H|CDk\}^{(1)} = \delta_{pk}\,\{AB|(H+E_k)|CD\}^{(1)} + \text{other terms}$$



(3.47)

The factors $a_i(E)$ $i=1,\ldots,4$ in equation (3.46) represent functions of the energies of the particles invclved. The other terms in equation (3.47) are those containing three mesons p, k, and the exchanged meson q.

Matrix elements between orthonormal Cutkosky states can be calculated in terms of the above matrix elements using equations (3.9), (3.11), and (3.16). One important result is the matrix element of the Hamiltonian between two orthonormal Cutkosky states. From equations (3.38) and (3.42) it is evident that the matrix elements of H between nonorthonormal states may be written

$$\{U|H|W\} = \frac{1}{2}(E_u+E_w)(\delta_{uw} + G_{uw}) + \{U|H'|W\} \tag{3.48}$$

where G_{uw} is defined by equation (3.11). Using equations (3.9) and (3.16) it follows that

$$\begin{aligned}(V|H|X) &= \frac{1}{2}\sum_{u,w} F^{*}_{uv}\,(1+G)_{uw}\,(E_u+E_w)\,F_{wx} + (V|H'|X)\\ &= \frac{1}{2}\sum_{u,w} (1+G)^{-\frac{1}{2}}_{vu}\,(1+G)_{uw}\,(E_u+E_w)\,(1+G)^{-\frac{1}{2}}_{wx} + (V|H'|X)\\ &= E_v\,\delta_{vx} + (V|H'|X)\end{aligned} \tag{3.49}$$

Thus H_o is diagonal in the orthonormal Cutkosky states and H' has no zero meson exchange contribution to its matrix element.

3.3 Meson Exchange Series for Pion-Deuteron Scattering

The pion-deuteron T-matrix given by equation (2.54) can be evaluated by inserting complete sets of orthonormal Cutkosky states and evaluating the resulting matrix elements using the expressions given in sections 3.1 and 3.2. In order to proceed from this point it is necessary to express $(U|\mathcal{D}>$ and $(U|\mathcal{D}k>$, where $|U)$ is an orthonormal Cutkosky state, in terms of known deuteron wavefunctions. This is done in the next two subsections.

3.3.1 The Deuteron State Vector

The deuteron state vector $|\mathcal{D}>$ can be expanded in terms of the orthonormal Cutkosky states using equation (3.9)

$$|\mathcal{D}> = \Sigma|U)(U|\mathcal{D}> \tag{3.50}$$

where $(U|\mathcal{D}>$ is the deuteron wavefunction in the orthonormal Cutkosky representation. The complete deuteron wavefunction $(U|\mathcal{D}>$ incorporates the meson degrees of freedom implicitly. However, known deuteron wavefunctions do not include all these degrees of freedom. Thus an expression for the exact deuteron wavefunction is needed in terms of an approximate deuteron wavefunction which hopefully is known. This expression should take into account explicitly the meson degrees of freedom in the complete deuteron wavefunction.

At this point, projection operators are defined for the unexcited and excited orthonormal Cutkosky states. These projection operators are denoted by P and P' respectively. A state vector $|\mathcal{D}_0>$ is defined which satisfies the following equations

$$PHP|\mathcal{D}_0> = E_{\mathcal{D}}|\mathcal{D}_0> \qquad (3.51)$$

$$P'|\mathcal{D}_0> = 0 \qquad (3.52)$$

By writing the deuteron state vector as

$$|\mathcal{D}> = |\mathcal{D}_0> + |\mathcal{D}_1> \qquad (3.53)$$

an expression can be obtained for the remainder $|\mathcal{D}_1>$. Using the fact that $P+P'=1$, the eigenvalue equation for H for the deuteron state may be written

$$(P+P')H(P+P')|\mathcal{D}> = E_{\mathcal{D}}|\mathcal{D}> \qquad (3.54)$$

Using equations (3.51)-(3.53) the above equation can be written

$$H|\mathcal{D}_1\rangle + P'HP|\mathcal{D}_0\rangle = E_{\mathcal{D}}|\mathcal{D}_1\rangle \tag{3.55}$$

By inverting the operator $(H-E_{\mathcal{D}})$ an expression is obtained for $|\mathcal{D}_1\rangle$. Thus from equation (3.53) the deuteron state vector may be written

$$|\mathcal{D}\rangle = \{1-(H-E_{\mathcal{D}})^{-1}P'HP\}|\mathcal{D}_0\rangle \tag{3.56}$$

Using equation (3.38) the operator $(H-E_{\mathcal{D}})^{-1}$ may be expanded to give the following expression for $|\mathcal{D}\rangle$

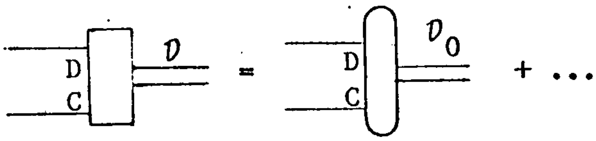
$$|\mathcal{D}\rangle = \{1-\sum_{n=0}^{\infty}[(H_o-E_{\mathcal{D}})^{-1}H']^n(H_o-E_{\mathcal{D}})^{-1}P'HP\}|\mathcal{D}_0\rangle \tag{3.57}$$

The above equation expresses the exact deuteron state vector in terms of an approximate deuteron state vector satisfying equations (3.51) and (3.52). Since matrix elements of H' do not have any zero-meson exchange terms, the above expression can be evaluated to any order of meson exchange. Note that since H_o is diagonal in the orthonormal Cutkosky states

$$P'HP = P'H'P \tag{3.58}$$

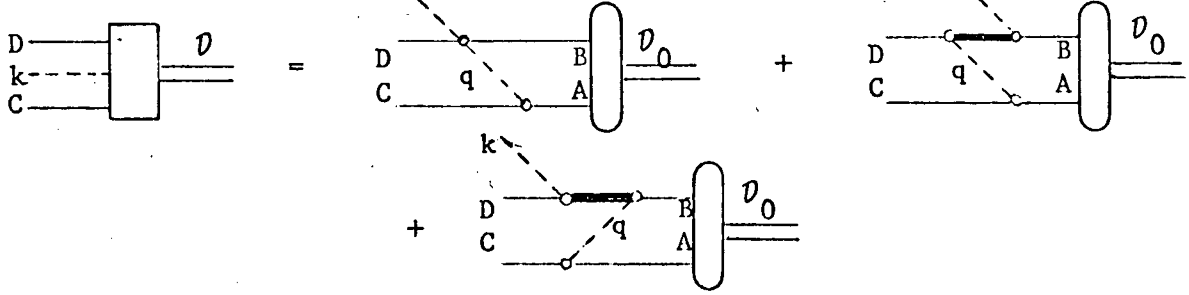
One might wish to identify $(U|\mathcal{D}_o\rangle$ with the conventional deuteron wavefunction in momentum space. Although this is a rather arbitrary assumption, it does seem reasonable. The conventional deuteron wavefunction describes a system composed of only two nucleons, the mesons not taken into account explicitly. The unexcited orthonormal Cutkosky states correspond to orthonormal states which, in the zero-meson exchange approximation, consist of two noninteracting nucleons with no mesons explicitly present. The properties of $|\mathcal{D}_0>$ stated in equations (3.52) and (3.53) thus correspond to the desired properties of the conventional deuteron wavefunction. That is, $|\mathcal{D}_0>$ is not coupled to states consisting of more than two nucleons (in the zero-meson exchange approximation) and $|\mathcal{D}_0>$ is an eigenstate with the correct eigenvalue of the Hamiltonian in the reduced space of the unexcited orthonormal Cutkosky states.

Using equation (3.57) diagrams can be drawn relating the deuteron state $|\mathcal{D}>$ to the approximate deuteron state $|\mathcal{D}_0>$. For example $(CD|\mathcal{D}>$ can be drawn as



The first term above represents the zero-meson exchange contribution and the terms neglected represent multiple-meson exchange contributions. The one-meson exchange contribution vanishes.

The one-meson exchange approximation to $(CDk|\mathcal{D}>$ can be drawn as



3.3.2 The Pion-Deuteron Scattering State Vector

The pion-deuteron scattering state vector can be expanded in terms of the orthonormal Cutkosky states using equation (3.9)

$$|\mathcal{D}k\rangle = \Sigma|U)(U|\mathcal{D}k\rangle \tag{3.59}$$

Since the wavefunction $(U|\mathcal{D}k\rangle$ is not known, equation (3.59) is written in the form of a series, the first term of which is obtained from the impulse approximation. The impulse approximation assumes that the scattering takes place so rapidly that it can be approximated by a pion scattering from a free nucleon. Thus the amplitude for finding a particular pion-two-nucleon state in the decomposition of the pion-deuteron scattering state is approximated by the amplitude for finding that two-nucleon state in the decomposition of the deuteron state.

Thus equation (3.59) can be written

$$|\mathcal{D}k\rangle = \Sigma|Uk)(U|\mathcal{D}\rangle + |S\rangle \tag{3.60}$$

where the first term represents the impulse approximation to the state $|\mathcal{D}k\rangle$ and $|S\rangle$ represents the remainder. Using the operator $K^{\dagger}$ defined in eq. (3.18), equation (3.60) can be written

$$|\mathcal{D}k\rangle = \Sigma K^{\dagger}|U)(U|\mathcal{D}\rangle + |S\rangle \tag{3.61}$$

or in shortened notation

$$|\mathcal{D}k\rangle = K^{\dagger}|\mathcal{D}\rangle + |S\rangle \tag{3.62}$$

where the presence of the unit operator has been assumed.

An expression for $|S\rangle$ can be obtained from the eigenvalue equation for H for the pion-deuteron scattering state, equation (2.38), by substituting for $|\mathcal{D}k\rangle$ from equation (3.62) giving the equation

$$(H-E_s)|S\rangle = (-HK^{\dagger}+E_sK^{\dagger})|\mathcal{D}\rangle \tag{3.63}$$

where

$$E_s = E_{\mathcal{D}} + E_k \tag{3.64}$$

By inverting $(H-E_s)$ with outgoing wave boundary conditions, the expression for $|S\rangle$ becomes

$$|S\rangle = (E_s-H+i\varepsilon)^{-1}(HK^{\dagger}-E_sK^{\dagger})|\mathcal{D}\rangle \tag{3.65}$$

Using equation (3.38) and writing

$$(E_s-H+i\varepsilon)^{-1} = \sum_{n=0}^{\infty}[(E_s-H_o+i\varepsilon)^{-1}H']^n(E_s-H_o+i\varepsilon)^{-1} \tag{3.66}$$

the pion-deuteron scattering state may be written

$$|\mathcal{D}k\rangle = K^{\dagger}|\mathcal{D}\rangle + \sum_{n=0}^{\infty}[(E_s-H_o+i\varepsilon)^{-1}H']^n(E_s-H_o+i\varepsilon)^{-1}(HK^{\dagger}-E_sK^{\dagger})|\mathcal{D}\rangle \tag{3.67}$$

This may be put in a more convenient form by noting that

$$
\begin{aligned}
(HK^{\dagger}-E_SK^{\dagger})|\mathcal{D}> &= H'K^{\dagger}|\mathcal{D}> + H_oK^{\dagger}|\mathcal{D}> - E_{\mathcal{D}}K^{\dagger}|\mathcal{D}> - E_kK^{\dagger}|\mathcal{D}> \\
&= H'K^{\dagger}|\mathcal{D}> + \sum_U (E_U+E_k)K^{\dagger}|U)(U|\mathcal{D}> - K^{\dagger}H|\mathcal{D}> - E_kK^{\dagger}|\mathcal{D}> \\
&= H'K^{\dagger}|\mathcal{D}> + \sum_U K^{\dagger}H_o|U)(U|\mathcal{D}> - K^{\dagger}H|\mathcal{D}> \\
&= (H'K^{\dagger} - K^{\dagger}H')|\mathcal{D}>
\end{aligned}
\tag{3.68}
$$

Thus the pion-deuteron scattering state may be written

$$
|\mathcal{D}k> = K^{\dagger}|\mathcal{D}> + \sum_{n=0}^{\infty} [(E_S-H_o+i\varepsilon)^{-1}H']^n(E_S-H_o+i\varepsilon)^{-1}(H'K^{\dagger}-K^{\dagger}H')|\mathcal{D}> \tag{3.69}
$$

The state $|\mathcal{D}>$ is known from section 3.3.1 and thus the pion-deuteron scattering state can be evaluated in terms of the approximate deuteron state $|\mathcal{D}_0>$ whose wavefunction is assumed to be known.

The above equation appears to resemble the expression for the pion-deuteron scattering state derived in section 2.3.2. However they are quite different. Equation (2.42) separates the pion-deuteron state into an asymptotically free state and a scattering state. That type of expansion is not particularly useful for obtaining approximations to the pion-deuteron state. On the other hand, equation (3.62) contains more than just the asymptotically free state. It also contains the interaction of the meson k with the nucleons in the Cutkosky state $|U)$. For example, the state $K^{\dagger}|AB>$, in the zero-meson exchange approximation, is equal to the state $|ABk\}$ which according to equation (3.5) contains the interaction of the meson k with both physical nucleons A and B. Thus the expansion of equation (3.69) should be useful in approximating the pion-deuteron state.

3.3.3 The Pion-Deuteron T-matrix

Having obtained expressions for the deuteron state and the pion-deuteron scattering state in the previous two subsections, the pion-deuteron T-matrix can be written solely in terms of matrix elements between Cutkosky states and approximate deuteron wavefunctions. Using equations (2.54), (3.57) and (3.69) the T-matrix becomes

$$\begin{aligned} T_{\mathcal{D}'m,\mathcal{D}k} = \langle \mathcal{D}_o' | & \left\{ 1 - PH'P' \sum_{n=0}^{\infty} [(H_o - E_{\mathcal{D}})^{-1} H']^n (H_o - E_{\mathcal{D}})^{-1} \; [mV] \right\} x \\ & x \left\{ K^{\dagger} + \sum_{n=0}^{\infty} [(E_s - H_o + i\varepsilon)^{-1} H']^n (E_s - H_o + i\varepsilon)^{-1} (H' K^{\dagger} - K^{\dagger} H') \right\} x \\ & x \left\{ 1 - \sum_{n=0}^{\infty} [(H_o - E_{\mathcal{D}})^{-1} H']^n (H_o - E_{\mathcal{D}})^{-1} P' H' P \right\} | \mathcal{D}_o \rangle \end{aligned} \tag{3.70}$$

The above expression is exact. No approximations have been made in writing the above equation. In order to evaluate the T-matrix many assumptions will be made. The infinite series will all be truncated to take into account no more than one-meson exchange terms for the T-matrix. In the next chapter the zero- and one-meson exchange terms of the T-matrix will be calculated and discussed.

4 The Meson Exchange Series For The T-matrix

4.1 Zero-Meson Exchange Contribution to the T-matrix

The expression for the pion-deuteron T-matrix is evaluated by inserting complete sets of orthonormal Cutkosky states into equation (3.70) then writing the resulting matrix elements as meson exchange series.

There is only one zero-meson exchange term in equation (3.70). It can be written

$$T^{(o)}_{\mathcal{D}'m,\mathcal{D}k} = \sum_{U,W} \langle \mathcal{D}'_o|U)(U|[mV]K^{\dagger}|W)^{(o)}(W|\mathcal{D}_o\rangle \tag{4.1}$$

Using equation (3.52), the above equation can be written

$$T^{(o)}_{\mathcal{D}'m,\mathcal{D}k} = \frac{1}{4}\sum_{A,B,C,D} \langle \mathcal{D}'_o|AB)(AB|[mV]|CDk)^{(o)}(CD|\mathcal{D}_o\rangle \tag{4.2}$$

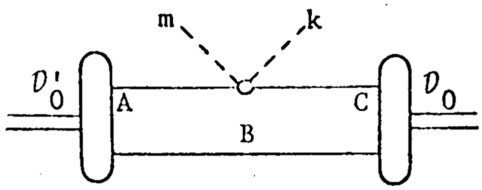
The normalization factor of 1/4 takes into account the fact that $|AB\} = -|BA\}$ and $|CD\} = -|DC\}$. Using the equations of section 3.1 and 3.2.2,

$$\begin{aligned}(AB|[mV]|CDk)^{(o)} &= \sum_{U,W} F^{*(o)}_{U,AB}\ \{U|[mV]|W\}^{(o)}\ F^{(o)}_{W,CDk} \\ &= \{AB|[mV]|CDk\}^{(o)}\end{aligned} \tag{4.3}$$

Using equations (3.35) and (4.3), equation (4.2) can be written

$$T^{(o)}_{\mathcal{D}'m,\mathcal{D}k} = \sum_{A,B,C} \langle \mathcal{D}'_o|AB)\ \langle A|[mV]|Ck\rangle\ (CB|\mathcal{D}_o\rangle \tag{4.4}$$

The above equation is drawn below.



Equation (4.4) can be rewritten in terms of the conventional deuteron momentum space wavefunction and pion nucleon T-matrices. This is done in Appendix D. The result in the laboratory frame is expressed below

$$T_L^{(0)}(\underline{k}_{\mathcal{D}'},\underline{m};\ 0,\underline{k};\ M',M)\ \delta\ (\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k})$$

$$= \sum_{\ell,\ell'=-1}^{+1} \int d^3\kappa\ \phi_{\ell'}^{M'*}\ (\underline{\kappa}+\tfrac{1}{2}(\underline{k}-\underline{m}))\,\phi_{\ell}^{M}(\underline{\kappa})\{T_{\ell'\ell}^{\pi p}+T_{\ell'\ell}^{\pi n}\}\delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k}) \tag{4.5}$$

The wavefunction ϕ_{ℓ}^{M} $(\underline{\kappa})$ is just the component of the conventional deuteron wavefunction with total spin projection M and total nucleon spin projection ℓ . The T-matrices $T_{\ell'\ell}^{\pi p}$ and $T_{\ell'\ell}^{\pi n}$ represent averages over nucleon spin orientations of pion-proton and pion-neutron T-matrices, respectively. The explicit expressions for ϕ_{ℓ}^{M} $(\underline{\kappa})$, $T_{\ell'\ell}^{\pi p}$ and $T_{\ell'\ell}^{\pi n}$ are given in Appendix D.

Thus the zero-meson exchange contribution to the T-matrix corresponds to the usual single scattering contribution to pion-deuteron elastic scattering. A fairly extensive numerical study of this contribution has been performed by McMillan and Landau (1974).

4.2 One-Meson Exchange Contribution to the T-matrix

The one-meson exchange contribution to the pion-deuteron T-matrix given in equation (3.70) can be written

$$
\begin{aligned}
T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = {} & \sum_{U,W} \langle \mathcal{D}'_o|U)(U|[mV]K^\dagger|W)^{(1)}(W|\mathcal{D}_o\rangle \\
& - \sum_{U,W} \sum_{\substack{X \\ \text{excited}}} \langle \mathcal{D}'_o|U)(U|H'|X)^{(1)}(X|[mV]K^\dagger|W)^{(o)}(W|\mathcal{D}_o\rangle(E_X-E_{\mathcal{D}'})^{-1} \\
& + \sum_{U,W} \sum_X \langle \mathcal{D}'_o|U)(U|[mV]|X)^{(o)}(X|(H'K^\dagger-K^\dagger H')|W)^{(1)}(W|\mathcal{D}_o\rangle(E_S-E_X+i\varepsilon)^{-1} \\
& - \sum_{U,W} \sum_{\substack{X \\ \text{excited}}} \langle \mathcal{D}'_o|U)(U|[mV]K^\dagger|X)^{(o)}(X|H'|W)^{(1)}(W|\mathcal{D}_o\rangle(E_X-E_{\mathcal{D}})^{-1}
\end{aligned}
\tag{4.6}
$$

The fourth term above can be rewritten by noting

$$
K^\dagger(E_X - E_{\mathcal{D}})^{-1}|X) = -(E_S - E_X + i\varepsilon)^{-1}K^\dagger|X) \tag{4.7}
$$

The third and fourth terms in equation (4.6) can now be combined to give the following expression for the T-matrix

$$
\begin{aligned}
T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = {} & \sum_{U,W} \langle \mathcal{D}'_o|U)(U|[mV]K^\dagger|W)^{(1)}(W|\mathcal{D}_o\rangle \\
& - \sum_{U,W} \sum_{\substack{X \\ \text{excited}}} \langle \mathcal{D}'_o|U)(U|H'|X)^{(1)}(X|[mV]K^\dagger|W)^{(o)}(W|\mathcal{D}_o\rangle(E_X-E_{\mathcal{D}'})^{-1} \\
& + \sum_{U,W} \sum_X \langle \mathcal{D}'_o|U)(U|[mV]|X)^{(o)}(X|H'K^\dagger|W)^{(1)}(W|\mathcal{D}_o\rangle(E_S-E_X+i\varepsilon)^{-1} \\
& - \sum_{U,W} \sum_{\substack{X \\ \text{unexcited}}} \langle \mathcal{D}'_o|U)(U|[mV]K^\dagger|X)^{(o)}(X|H'|W)^{(1)}(W|\mathcal{D}_o\rangle(E_S-E_X+i\varepsilon)^{-1}
\end{aligned}
\tag{4.8}
$$

Using equation (3.52) the sums over the states $|U\}$ and $|W\}$ may be rewritten yielding

$$T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = \frac{1}{4}\sum_{ABCD}\langle\mathcal{D}'_o|AB)(CD|\mathcal{D}_o\rangle \times \Big\{ (AB|[mV]|CDk)^{(1)}$$

$$- \sum_{\substack{X \\ \text{excited}}} (AB|H'|X)^{(1)}(X|[mV]|CDk)^{(o)}(E_X-E_{\mathcal{D}'})^{-1}$$

$$+ \sum_X (AB|[mV]|X)^{(o)}(X|H'|CDk)^{(1)}(E_S-E_X+i\varepsilon)^{-1}$$

$$- \sum_{\substack{X \\ \text{unexcited}}} (AB|[mV]|Xk)^{(o)}(X|H'|CD)^{(1)}(E_S-E_X+i\varepsilon)^{-1}\Big\} \tag{4.9}$$

As explained in the previous section, the factor of 1/4 is a normalization factor resulting from the antisymmetry of the Cutkosky states.

To proceed, the matrix elements between the orthonormal Cutkosky states are expressed in terms of matrix elements between the non-orthonormal states. The zero-meson exchange matrix elements in equation (4.9) are easily expressed in terms of the non-orthonormal states as was done in equation (4.3). The same technique can be used for the one-meson exchange matrix elements of H', since matrix elements of H' do not have any zero-meson exchange contributions. However, since matrix elements of the vertex operator do have zero-meson exchange contributions, it is more complicated to express the one-meson exchange matrix element of the vertex operator between orthonormal states in terms of matrix elements between non-orthonormal states.

Using equation (3.9) the matrix element of the vertex operator may be written

$$(AB|[mV]|CDk) = \sum_{U,W} F^*_{U,AB}\,\{U|[mV]|W\}\,F_{W,CDk} \tag{4.10}$$

The one-meson exchange contribution to the above matrix element is obtained by writing each of the three matrix elements on the right hand side as a sum of zero- and one-meson exchange contributions then taking those products which yield a net one-meson exchange. That is,

$$\begin{aligned}(AB|[mV]|CDk)^{(1)} &= \sum_{U,W} F^{*(o)}_{U,AB}\,\{U|[mV]|W\}^{(1)}\,F^{(o)}_{W,CDk} \\ &+ \sum_{U,W} F^{*(1)}_{U,AB}\,\{U|[mV]|W\}^{(o)}\,F^{(o)}_{W,CDk} \\ &+ \sum_{U,W} F^{*(o)}_{U,AB}\,\{U|[mV]|W\}^{(o)}\,F^{(1)}_{W,CDk}\end{aligned} \tag{4.11}$$

Using equations (3.12) and (3.16), it follows that

$$\begin{aligned}F^{(o)}_{U,W} &= \delta_{UW} \\ F^{(1)}_{U,W} &= -\frac{1}{2}\,\{U|W\}^{(1)}\end{aligned} \tag{4.12}$$

Thus equation (4.11) becomes

$$\begin{aligned}(AB|[mV]|CDk)^{(1)} &= \{AB|[mV]|CDk\}^{(1)} \\ &-\frac{1}{2}\sum_{U}\{AB|U\}^{(1)}\,\{U|[mV]|CDk\}^{(o)} \\ &-\frac{1}{2}\sum_{W}\{AB|[mV]|W\}^{(o)}\,\{W|CDk\}^{(1)}\end{aligned} \tag{4.13}$$

From the above discussion, it follows that

$$T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = \frac{1}{4}\sum_{ABCD}\langle\mathcal{D}'_o|AB)(CD|\mathcal{D}_o\rangle\Big\{\{AB|[mV]|CDk\}^{(1)}$$

$$-\frac{1}{2}\sum_U\{AB|U\}^{(1)}\{U|[mV]|CDk\}^{(o)}$$

$$-\frac{1}{2}\sum_U\{AB|[mV]|U\}^{(o)}\{U|CDk\}^{(1)}$$

$$-\sum_{\substack{U\\ \text{excited}}}\{AB|H'|U\}^{(1)}\{U|[mV]|CDk\}^{(o)}(E_U-E_{\mathcal{D}'})^{-1}$$

$$+\sum_U\{AB|[mV]|U\}^{(o)}\{U|H'|CDk\}^{(1)}(E_{\mathcal{D}}+E_k-E_U+i\varepsilon)^{-1}$$

$$-\sum_{\substack{U\\ \text{unexcited}}}\{AB|[mV]|Uk\}^{(o)}\{U|H'|CD\}^{(1)}(E_{\mathcal{D}}+E_k-E_U+i\varepsilon)^{-1}\Big\} \quad (4.14)$$

In order to evaluate the T-matrix as expressed above, it will be necessary to truncate the summations over the intermediate states $|U\}$. This will be done by disregarding all terms in which more than three mesons are present. That is, the only terms retained in the above summations will be those which contain the initial and final mesons, labelled by k and m respectively, and one exchanged meson, labelled by q.

When truncating the above summations, one must be careful to include all terms which contain three mesons. For example, in the second and third terms, the summation should include those states $|U\}$ which are singly excited as well as unexcited Cutkosky states. If $|U\}=|FGp\}$ in the second term, then by using equation (3.37) it follows that

$$\{AB|FGp\}^{(1)}\{FGp|[mV]|CDk\}^{(o)} = \{AB|FGk\}^{(1)}\{FG|[mV]|CDk\}^{(o)} \quad (4.15)$$

$$+ \text{ terms with 4 mesons present}$$

Similarily, if $|U\}=|FGp\}$ in the third term, then by using equation (3.33) it follows that

$$\{AB|[mV]|FGp\}^{(o)}\{FGp|CDk\}^{(1)} = \{AB|[mV]|FGk\}^{(o)}\{FG|CD\}^{(1)} \tag{4.16}$$

$$+ \text{ terms with 4 mesons present}$$

When $|U\}=|FGp\}$ in the fourth term using equation (3.37) will give

$$\{AB|H'|FGp\}^{(1)}\{FGp|[mV]|CDk\}^{(o)} = \{AB|H'|FGk\}^{(1)}\{FG|[mV]|CD\}^{(o)} \tag{4.17}$$

$$+ \text{ terms with 4 mesons present}$$

The fifth term will also have some terms coming from $|U\}=|FGp\}$ since by using equation (3.47)

$$\{AB|[mV]|FGp\}^{(o)}\{FGp|H'|CDk\}^{(1)} = \{AB|[mV]|FGk\}^{(o)}\ \{FG|H'|CD\}^{(1)} \tag{4.18}$$

$$+ \text{ terms with 4 mesons present}$$

Note that the part of the fifth term written above exactly cancels the sixth term of equation (4.14).

Thus, keeping only those terms containing three mesons, the T-matrix in equation (4.14) can be written

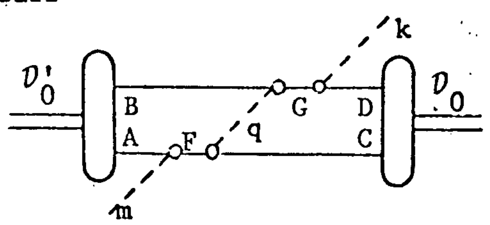
$$\begin{aligned} T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = \frac{1}{4}\sum_{ABCD}\langle\mathcal{D}'_o|AB)(CD|\mathcal{D}_o\rangle \Big\{ & \{AB|[mV]|CDk\}^{(1)} \\ & - \frac{1}{4}\sum_{F,G}\{AB|FG\}^{(1)}\{FG|[mV]|CDk\}^{(o)} \\ & - \frac{1}{4}\sum_{F,G}\{AB|FGk\}^{(1)}\{FG|[mV]|CD\}^{(o)} \\ & - \frac{1}{4}\sum_{F,G}\{AB|[mV]|FG\}^{(o)}\{FG|CDk\}^{(1)} \\ & - \frac{1}{4}\sum_{F,G}\{AB|[mV]|FGk\}^{(o)}\{FG|CD\}^{(1)} \\ & - \frac{1}{2}\sum_{F,G}\{AB|H'|FGk\}^{(1)}\{FG|[mV]|CD\}^{(o)}(E_F+E_G+E_k-E_{\mathcal{D}'})^{-1} \\ & + \frac{1}{2}\sum_{F,G}\{AB|[mV]|FG\}^{(o)}\{FG|H'|CDk\}^{(1)}(E_{\mathcal{D}}+E_k-E_F-E_G+i\varepsilon)^{-1}\Big\} \end{aligned} \tag{4.19}$$

The T-matrix can now be expressed in terms of one-nucleon matrix elements using the equations of subsections 3.2.2 and 3.2.3. The final result can be written in the form

$$T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = T^{+}_{diff} + T^{-}_{diff} + T^{+}_{same} + T^{-}_{same} \tag{4.20}$$

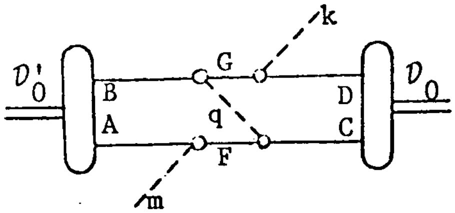
The terms on the right hand side of the above equation are expressed in terms of one-nucleon matrix elements in Appendix C.

The terms denoted by T^{+}_{diff} represent processes where the initial meson is absorbed on one nucleon of the deuteron, the final meson is emitted by the other nucleon of the deuteron, and the exchanged meson is emitted by the first nucleon and absorbed by the second nucleon. The subscript "diff" indicates that the initial and final mesons are absorbed and emitted by different nucleons. There are four terms represented by T^{+}_{diff} as the exchanged meson can be emitted before or after the initial meson is absorbed and then can be absorbed on the second nucleon either before or after the final meson is emitted. One of the four terms of T^{+}_{diff} is drawn below.



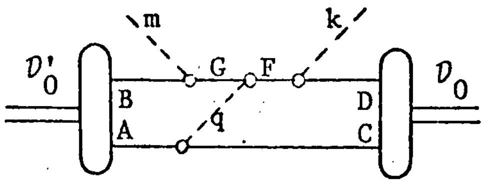
The single solid lines denote physical nucleons and the dashed lines denote mesons. The double solid lines denote the deuteron. The diagram is drawn with the initial state on the left and the final state on the right. The diagram drawn above corresponds to the second term in equation (C.2).

The terms denoted by T^-_{diff} are similar to the terms denoted by T^+_{diff} except that the terms of T^-_{diff} have the exchanged meson emitted by the second nucleon and absorbed by the first nucleon. As above, there are four terms represented by T^-_{diff} and one of these terms is drawn below.



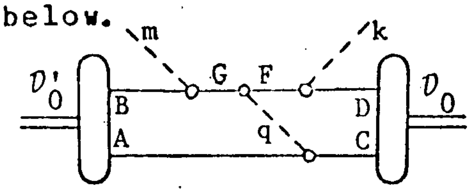
The diagram corresponds to the second term of equation (C.3)

The terms denoted by T^+_{same} represent processes where the final meson is emitted by the same nucleon which absorbs the initial meson and the exchanged meson is emitted by the first nucleon and absorbed by the second nucleon. There are six terms represented by T^+_{same} corresponding to the six different orderings of the absorption of the initial meson and the emission of the exchanged and final mesons. One of the six terms of T^+_{same} is drawn below.



The diagram corresponds to the fourth term of equation (C.4).

The terms denoted by T^-_{same} are similar to the terms denoted by T^+_{same} except that the terms of T^-_{same} have the exchanged meson emitted by the second nucleon and absorbed by the first nucleon. As above, there are six terms represented by T^-_{same} and one of these terms is drawn below.



The diagram corresponds to the fourth term of equation (C.5).

It is worthwhile at this point to outline the approximations which have been made in expressing the pion-deuteron T-matrix in the form given by equation (4.20) and Appendix C.

When matrix elements between Cutkosky states were evaluated in subsections 3.2.2 and 3.2.3, all core exchange terms were neglected. That is, all terms which could not be expressed in terms of one-nucleon matrix elements were assumed to be negligible. Also, the expressions for the matrix elements of the Hamiltonian and the vertex operator neglected the meson-meson interaction, V', and any terms in V which were not linear in the meson creation or annihilation operators.

A major assumption made in writing equation (4.20) was the neglecting of all terms in which more than three mesons were present. This was done firstly by taking the one-meson exchange approximation to all terms in the T-matrix as given by equation (3.67) and secondly by truncating all the summations over intermediate states such that only three mesons were present in the final expression for the T-matrix.

There is another major assumption which must be made if equation (4.20) is to be identified with the pion-deuteron T-matrix. One must assume that when the deuteron state vector is written as in equation (3.53), then the conventional deuteron wavefunction in momentum space corresponds to

It is very difficult, without specifying V, to draw any conclusions about the one-meson exchange approximation to the T-matrix when it is written as in equation (4.20). If one has some potential V then, in principle, equation (4.20) could be evaluated numerically. However, this will not be done here. Instead, in the next section, some further approximations will be made which will enable one to draw some conclusions without doing any numerical calculations.

4.3 The Static On-Shell Approximation

In a calculation of the type done in the previous sections, it is useful to get some idea of the relative sizes of the various terms in equation (4.20). In order to do this, some approximations will be made.

The first approximation will be to assume that the kinetic energy of the nucleons in the deuteron is small compared to the total energy of the incoming pion. Also, assume that the kinetic energy of the deuteron is small compared to the pion total energy. These assumptions will be called the static approximation.

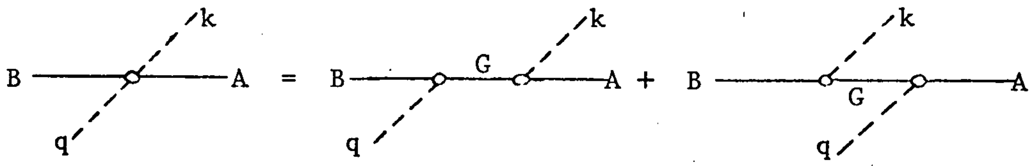
It is also necessary to obtain an approximation to the pion-nucleon T-matrix. Using equations (2.42) and (3.24), the pion-nucleon T-matrix as given by equation (2.53) can be written

$$\begin{aligned} T_{Bq,Ak} &= -\langle B|[qV](H-E_A-E_k-i\varepsilon)^{-1}[Vk]|A\rangle \\ &\quad -\langle B|[Vk](H+E_k-E_B+i\varepsilon)^{-1}[qV]|A\rangle \end{aligned} \tag{4.21}$$

Each matrix element above can be simplified by inserting a complete set of one-nucleon states and pion-nucleon scattering states and truncating the summation by neglecting all scattering states. With this assumption the T-matrix can be written

$$T_{Bq,Ak} = -\sum_G \langle B|[qV]|G\rangle\langle G|[Vk]|A\rangle\ (E_G-E_A-E_k-i\varepsilon)^{-1}$$
$$-\sum_G \langle B|[Vk]|G\rangle\langle G|[qV]|A\rangle\ (E_G+E_k-E_B+i\varepsilon)^{-1} \qquad (4.22)$$

This can be represented diagramatically as follows



One further assumption is necessary to simplify the terms represented by T^{+}_{diff} and T^{+}_{same}. Most of the ten terms represented by T^{+}_{diff} and T^{+}_{same} are proportional to a factor of $(E_B+E_q-E_D-E_k-i\varepsilon)^{-1}$ (or a similar term with G interchanged with either B or D). The approximation made here is to assume that the scattering involving the four particles in the above energy denominator takes place on-shell, i.e. assume that energy is conserved in the scattering. This is equivalent to neglecting the principal value integral when the energy denominator above is split in the following manner

$$(E_B+E_q-E_D-E_k-i\varepsilon)^{-1} = \pi i\delta(E_B+E_q-E_D-E_k) + P(E_B+E_q-E_D-E_k)^{-1} \qquad (4.23)$$

Making use of these approximations the terms represented by T^{+}_{diff} and T^{+}_{same} become

$$T^{+}_{diff} + T^{+}_{same} = \frac{1}{2} \sum_{ABCDq} \langle \mathcal{D}_o' | AB)(CD| \mathcal{D}_o \rangle \quad \times$$

$$\times \Big\{ -\pi i \langle A|[mV]|Cq\rangle\langle B|[qV]|Dk\rangle \;\; \delta(E_B+E_q-E_D-E_k) \;\; (2 - \frac{E_{\mathcal{D}}-2M}{E_k})$$

$$- 4\pi i \sum_G \langle A|[Vq]|G\rangle\langle G|[mV]|C\rangle\langle B|[qV]|Dk\rangle \;\; \delta(E_B+E_q-E_D-E_k)(\frac{E_{\mathcal{D}}-2M}{E_k}) \quad \times$$

$$\times \; (E_G+E_q-E_A+i\varepsilon)^{-1}$$

$$+ \frac{\pi i}{2} \sum_G \langle A|[Vq]|C\rangle\langle B|[mV]|G\rangle\langle G|[qV]|Dk\rangle \;\; \delta(E_G+E_q-E_D-E_k)(\frac{E_{\mathcal{D}}-2M}{E_k}) \quad \times$$

$$\times \; (E_C+E_q-E_A+i\varepsilon)^{-1}$$

$$+ \frac{\pi i}{2} \sum_G \langle A|[Vq]|C\rangle\langle B|[qV]|Gk\rangle\langle G|[mV]|D\rangle \;\; \delta(E_B+E_q-E_G-E_k)(\frac{E_{\mathcal{D}}-2M}{E_k}) \quad \times$$

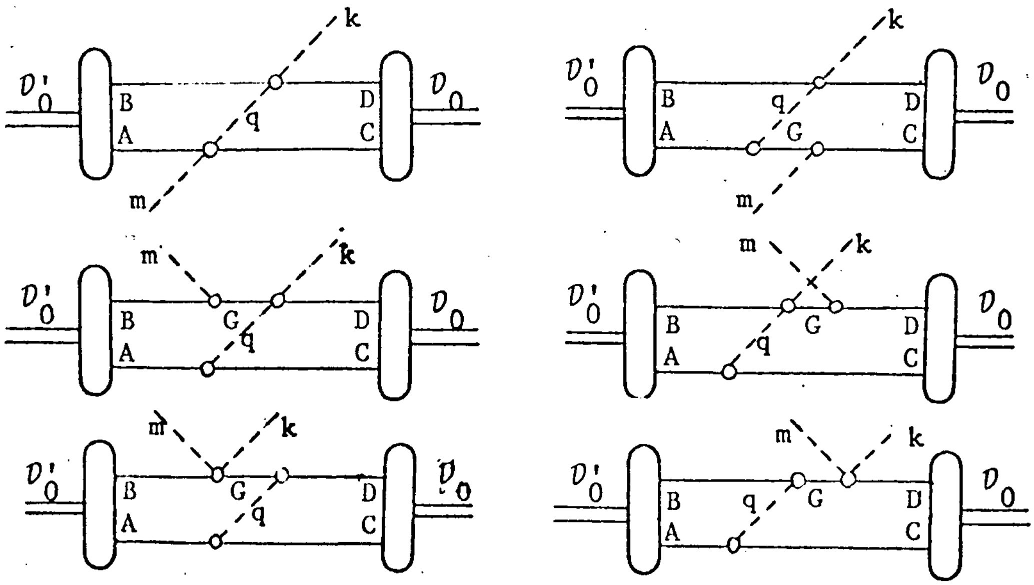
$$\times \; (E_C+E_q-E_A+i\varepsilon)^{-1}$$

$$- \sum_G \langle A|[Vq]|C\rangle\langle B|[qV]|G\rangle\langle G|[mV]|Dk\rangle \;\; (E_C+E_q-E_A+i\varepsilon)^{-1}(E_B+E_q-E_G-i\varepsilon)^{-1}$$

$$- \sum_G \langle A|[Vq]|C\rangle\langle B|[mV]|Gk\rangle\langle G|[qV]|D\rangle \;\; (E_C+E_q-E_A+i\varepsilon)^{-1}(E_G+E_q-E_D-i\varepsilon)^{-1} \Big\}$$

(4.24)

These terms can be represented diagramatically as below



The first two terms result from T^+_{diff} and the other four result from T^+_{same}. In the static approximation (E_D-2M) is just the binding energy of the deuteron (about 2.2 MeV). Thus, making use of equation (4.22) and assuming all matrix elements of the form $\langle A|[Vq]|G\rangle$, $\langle G|[mV]|C\rangle$ etc. are of the same order of magnitude, it follows that the second, third and fourth terms are about two orders of magnitude smaller than the first term (assuming an incident pion kinetic energy of 50 MeV or greater). It is not obvious that the fifth and sixth terms are small compared to the first term. This could only be investigated if one did numerical calculations with some potential V.

When simplifying terms represented by T^-_{diff} and T^-_{same} , it is not valid to make both the static approximation and the on-shell approximation. The energy denominator associated with most of these terms is proportional to $(E_B-E_q-E_D-E_k-i\varepsilon)^{-1}$ (or a similar term with G interchanged with either B or D). Since the energies in the above energy denominator are total energies, not just kinetic energies, it follows that the energy of the nucleon after absorbing the mesons k and q will be about 300 MeV greater than the energy of the nucleon before the interaction. Thus the static approximation will not be made when dealing with these terms.

Making only the on-shell approximation, the terms represented by T^-_{diff} and T^-_{same} can be written

$$T^{-}_{diff} + T^{-}_{same} = \frac{1}{2} \sum_{ABCDq} \langle \mathcal{D}'_o | AB)(CD | \mathcal{D}_o \rangle \quad \times$$

$$\times \Big\{ \frac{\pi i}{2} \langle Aq | [mV] | C \rangle \langle B | [Vq] | Dk \rangle \; \delta(E_B - E_q - E_D - E_k)$$

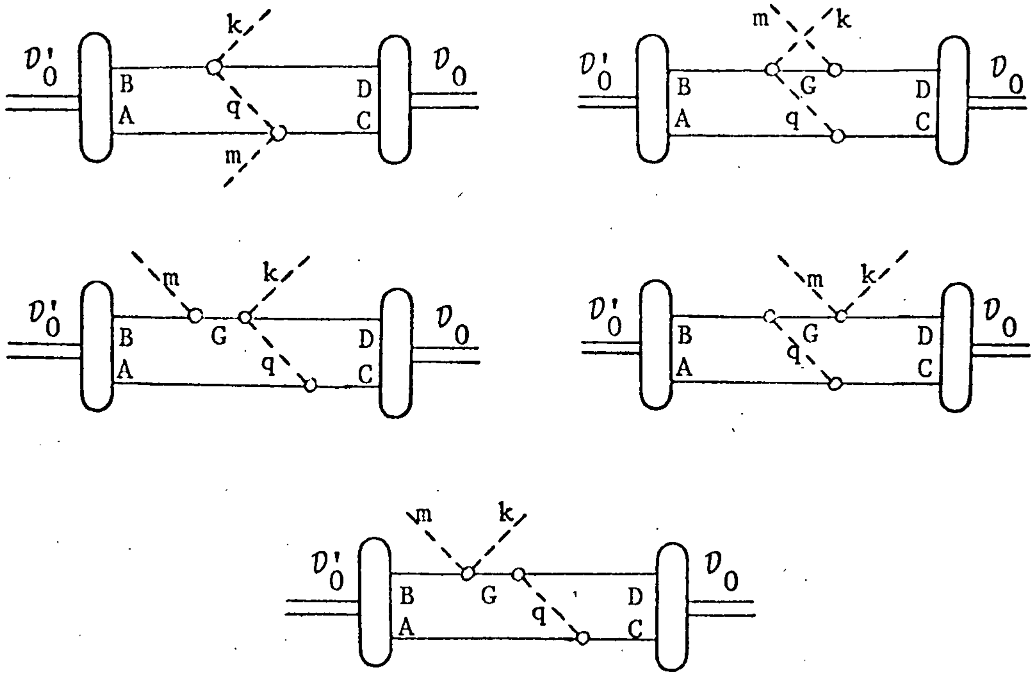
$$- \frac{\pi i}{2} \sum_G \langle A | [qV] | C \rangle \langle G | [mV] | D \rangle \langle B | [Vq] | Gk \rangle \; \delta(E_B - E_q - E_G - E_k)(E_C - E_q - E_A + i\varepsilon)^{-1}$$

$$- \frac{\pi i}{2} \sum_G \langle A | [qV] | C \rangle \langle B | [mV] | G \rangle \langle G | [Vq] | Dk \rangle \; \delta(E_G - E_q - E_D - E_k)(E_C - E_q - E_A + i\varepsilon)^{-1}$$

$$+ \sum_G \langle A | [qV] | C \rangle \langle B | [Vq] | G \rangle \langle G | [mV] | Dk \rangle \; (E_C - E_q - E_A + i\varepsilon)^{-1}(E_B - E_q - E_G - i\varepsilon)^{-1}$$

$$- \sum_G \langle A | [qV] | C \rangle \langle B | [mV] | Gk \rangle \langle G | [Vq] | D \rangle \; (E_C - E_q - E_A + i\varepsilon)^{-1}(E_G - E_q - E_D - i\varepsilon)^{-1} \Big\} \tag{4.25}$$

These terms can be represented diagramatically as follows

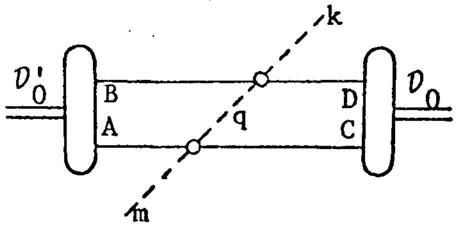


The first three terms of equation (4.25) will be small compared to the corresponding terms in equation (4.24) since the nucleon kinetic energy of over 300 MeV will yield a relative momentum of about 400 MeV/c for the nucleons in the deuteron. Since the deuteron wavefunction is very small for such large relative momentum, the first three terms of equation (4.25) can be neglected in comparison to the first term of equation (4.24). As in equation (4.24) the size of the last two terms of equation (4.25) can only be estimated numerically given some potential V.

Thus making an on-shell approximation and a static approximation where physically sensible, the dominant term of $T^{(1)}_{\mathcal{D}'m,\mathcal{D}k}$ can be written

$$T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = -\pi i \sum_{ABCDq} \langle \mathcal{D}'_o|AB)(CD|\mathcal{D}_o\rangle\langle A|[mV]|Cq\rangle\langle B|[qV]|Dk\rangle\, \delta(E_B+E_q-E_D-E_k) \tag{4.26}$$

In writing this equation it has been assumed that the binding energy of the deuteron is negligible compared to the total energy of the incoming pion. The above equation can be represented diagramatically as below



In addition to the term given in equation (4.26) there will be the last two terms of both equations (4.24) and (4.25) whose contributions can only be evaluated numerically.

The above equation can be written in terms of the conventional deuteron momentum space wavefunction and pion nucleon T-matrices. This is done in Appendix D. The result in the laboratory frame is expressed below

$$T_L^{(1)}\,(\underline{k}_{\mathcal{D}'},\underline{m};\,0,\underline{k};\,M',M)\quad\delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k})$$

$$= -\pi i \sum_{\ell,\ell'=-1}^{+1} \int d^3\kappa\, d^3q\;\; \phi^{M'^{*}}_{\ell'}\left(\underline{\kappa}+\underline{q}-\tfrac{1}{2}(\underline{k}-\underline{m})\right)\; \phi^{M}_{\ell}(\underline{\kappa})\;\; T_{\ell'\ell}\;\; \delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k}) \tag{4.27}$$

The T-matrix $T_{\ell'\ell}$ represents an average over nucleon spins and pion and nucleon isospins of a product of two pion nucleon T-matrices. The explicit expression for $T_{\ell'\ell}$ is given in Appendix D. Equation (4.27) agrees with the double-scattering result obtained by Pendleton (1963).

5 Conclusions

Using a field theory approach, the pion-deuteron elastic scattering T-matrix was expanded in a meson exchange series and the first two terms of this series were examined. The reason for using this approach was to avoid the double-counting problems usually associated with multiple scattering corrections in pion-deuteron scattering.

In section 4.1, the first term in the expansion of the T-matrix (called the zero-meson exchange term) was evaluated. This term was shown to be the usual single-scattering approximation to the T-matrix (equation (4.5)).

In section 4.2, the next order of terms in the expansion (called the one-meson exchange term) was evaluated. Keeping only those terms containing one exchanged meson and the initial and final mesons, the T-matrix was expressed as a sum of twenty terms, each written as products of matrix elements of the vertex operator between one-nucleon states (equation (4.21)).

In order to get an indication as to which of these twenty terms were important, an on-shell approximation was made as well as a static approximation where physically sensible. With these approximations, the one-meson exchange contribution to the T-matrix was written in a form which facilitated comparison of the magnitudes of the various terms in the expression for the T-matrix (equations (4.25) and (4.26)).

Assuming an incident pion kinetic energy of 50 MeV, it was shown that one of the terms was about two orders of magnitude greater than most of the remaining terms. This term was shown to be similar to the conventional double-scattering term resulting from the generalized impulse approximation. There were four additional terms whose magnitude could not be easily compared to the double-scattering term. These four terms represented single-scattering processes with a meson exchanged between the nucleons either before or after the scattering. The size of these terms could only be evaluated numerically using a particular field theoretic potential V.

Pendleton(1963) has calculated the double-scattering contribution (equation (4.28)) for a pion kinetic energy of 142 MeV. Carlson(1970) has calculated the double-scattering contributions at pion kinetic energies ranging from 61 to 300 MeV. It would perhaps be worthwhile to calculate other terms contained in either section 4.2 or 4.3. Using a Chew-Low Hamiltonian [Chew and Low(1956)], numerical results could be obtained which could be compared with the double-scattering contribution usually calculated.

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APPENDIX A Meson Operator Identities

The identity for the commutator of a product of meson creation operators $M^\dagger$ and a product of meson destruction operators Q is

$$[M^\dagger,Q] = -\Sigma_R' \, [[r,M^\dagger]][[Q,r^\dagger]]/n(R)! \qquad (A.1)$$

It can be proved by induction on the number of operators in the product Q.

Let $M^\dagger$ be an arbitrary product of meson creation operators and let Q=q. Then

$$\begin{aligned}[q,M^\dagger] &= \Sigma_r \, [r,M^\dagger][q,r^\dagger] \\ &= \Sigma_R' \, [[r,M^\dagger]][[q,r^\dagger]]/n(R)! \qquad (A.2)\end{aligned}$$

Thus equation (A.1) is valid for n(Q)=1.

Assume the identity is true for arbitrary Q. It must now be shown that the identity is true for Qk. The proof of this will be done only for the case $k \notin Q$. The proof for $k \in Q$ will be outlined only as it is conceptually similar to the case $k \notin Q$ although it is more complicated algebraically. For $k \notin Q$ the proof proceeds as follows.

Using the relation

$$[M^\dagger,Qk] = [M^\dagger,Q]k + [M^\dagger,k]Q + [[k,M^\dagger],Q] \qquad (A.3)$$

and assuming equation (A.1) is true for Q, the commutator

$[M^\dagger, Qk]$ may be written

$$-[M^\dagger,Qk] = \sum_R{}' \, [[r,M^\dagger]][[Q,r^\dagger]]k/n(R)! + \sum_R{}' \, [[r,M^\dagger]][[k,r^\dagger]]Q/n(R)! + \sum_R{}' \, [[r,[k,M^\dagger]]][[Q,r^\dagger]]/n(R)! \quad (A.4)$$

Denoting by R_o those products R which do not contain k and defining $S_1 = R_o k$, equation (A.4) can be written

$$-[M^\dagger,Qk] = \sum_{R_o}{}' \, [[r,M^\dagger]][[Qk,r^\dagger]]/n(R_o)! + [k,M^\dagger][Qk,k] + \sum_{S_1}{}'' \, [[s,M\ ]][[Qk,s^\dagger]]/n(S_1)! \quad (A.5)$$

The double prime on the summation indicates that the unit operator and single meson operators are excluded from the summations. The factor of $n(S_1)^{-1}$ comes from the fact that there are $n(S_1)$ distinguishable ways of placing k in a given product R_o.

Relabelling the sums in equation (A.5) yields the desired result. Thus equation (A.1) has been shown, by induction, to be true for a set Q containing an arbitrary number of operators no two of which are identical and an arbitrary set M.

To prove equation (A.1) in general it is assumed that it is true for $[M^\dagger,Qk^j]$ where $k \in Q$. Then it is shown that it is true for $[M^\dagger,Qk^{j+1}]$. Using the following identity

$$[M^\dagger,Qk^{j+1}] = [M^\dagger,Qk^j]k + [M^\dagger,k]Qk^j + [[k,M^\dagger],Qk^j] \quad (A.6)$$

an equation similar to equation (A.4) can be written. Then the sums over R are separated into sums over R_n where R_n contains the operator k exactly n times. Denote by S_o the product of the operators not equal to k in R_n. By writing the sums over R_n as sums over S_o with the n operators k explicitly written in the nested commutators, it is a matter of straightforward manipulation to prove that equation (A.1) is true for $[M^\dagger,Qk^{j+1}]$ if it is true for $[M^\dagger,Qk^j]$. Thus equation (A.1) is true for any M and Q.

The identities involving meson annihilation operators, equations (3.24)-(3.26), can be proved as follows. Since the one nucleon state is an eigenstate of H it follows that

$$r(H-E_A)|A> = 0 \tag{A.7}$$

From equation (2.12)

$$[r,H] = E_r r + [rV] \tag{A.8}$$

Thus equation (A.7) can be written

$$(H-E_A)r|A> + E_r r|A> + [rV]|A> = 0 \tag{A.9}$$

Inverting the operator $(H-E_A+E_r)$ the desired identity is obtained

$$r|A> = -(H+E_r-E_A-i\varepsilon)^{-1}\ [rV]|A> \tag{A.10}$$

When two or more meson operators are to be removed, the commutator of a meson annihilation operator with $(H-E)^{-1}$ will be needed. Using equation (A.8), $r(H-E)$ may be written

$$r(H-E) = (H-E+E_r)r + [rV] \tag{A.11}$$

Multiplying on the right by $(H-E)^{-1}$ and on the left by $(H-E+E_r)^{-1}$, the desired result is obtained

$$r(H-E)^{-1} = (H+E_r-E)^{-1} - (H+E_r-E)^{-1}\,[rV]\,(H-E)^{-1} \tag{A.12}$$

The third operator identity is used in removing meson annihilation operators from matrix elements involving pion-nucleon scattering states. From equation (2.12) it follows that

$$[rk^\dagger,H] = (E_r-E_k)rk^\dagger - r[Vk] + [rV]k^\dagger \tag{A.13}$$

Using this identity the equation

$$rk^\dagger(H-E_C)|C\rangle = 0 \tag{A.14}$$

may be written

$$(H+E_r-E_C-E_k)rk^\dagger|C\rangle - r[Vk]|C\rangle + [rV]k^\dagger|C\rangle = 0 \tag{A.15}$$

Now, since

$$(H+E_r-E_C-E_k)\delta_{kr}|C\rangle = (H-E_C)|C\rangle = 0 \tag{A.16}$$

the following result can be obtained from equation (A.14)

$$rk^{\dagger}|C\rangle - (H+E_r-E_C-E_k-i\varepsilon)^{-1}r[Vk]|C\rangle$$
$$= \delta_{kr}|C\rangle - (H+E_r-E_C-E_k-i\varepsilon)^{-1}[rV]k^{\dagger}|C\rangle \quad (A.17)$$

Outgoing wave boundary conditions are imposed when inverting $(H+E_r-E_C-E_k)$. Using the definition of the pion nucleon scattering state and equation (A.12) and (A.17) the desired identity may be written

$$r|Ck\rangle = \delta_{kr}|C\rangle - (H+E_r-E_C-E_k-i\varepsilon)^{-1}[rV]|Ck\rangle \quad (A.18)$$

APPENDIX B Cutkosky Matrix Elements

B.1 Unexcited Overlap Matrix Elements

The simplest Cutkosky matrix element is that which gives the overlap of two unexcited Cutkosky states $|AB\}$ and $|CD\}$. Using the definition of Cutkosky states this matrix element may be written

$$\{AB|CD\} = \langle o|BAC^{\dagger}D^{\dagger}|o\rangle \tag{B.1}$$

Using equation (2.16) the operators A, B, C, and D may be written

$$\begin{aligned} A &= \sum_{\mathcal{A},K} a^{*}(\mathcal{A}K)\mathcal{A}K \\ B &= \sum_{\mathcal{B},L} b^{*}(\mathcal{B}L)\mathcal{B}L \\ C^{\dagger} &= \sum_{\mathcal{C},M} c(\mathcal{C}M)\mathcal{C}^{\dagger}M^{\dagger} \\ D^{\dagger} &= \sum_{\mathcal{D},N} d(\mathcal{D}N)\mathcal{D}^{\dagger}N^{\dagger} \end{aligned} \tag{B.2}$$

Using these expressions for the physical nucleon operators, the matrix element can be written

$$\{AB|CD\} = \sum_{\mathcal{A},K}\sum_{\mathcal{B},L}\sum_{\mathcal{C},M}\sum_{\mathcal{D},N} a^{*}(\mathcal{A}K)b^{*}(\mathcal{B}L)c(\mathcal{C}M)d(\mathcal{D}N)\langle o|\mathcal{B}L\mathcal{A}K\mathcal{C}^{\dagger}M^{\dagger}\mathcal{D}^{\dagger}N^{\dagger}|o\rangle \tag{B.3}$$

In order to express $\{AB|CD\}$ as products of one nucleon matrix elements the terms in the summation in equation (B.3) are separated into four types. The first type of term consists of those terms in the summation in which $\mathcal{DC}\neq\mathcal{BA}$ (In all statements in this appendix regarding the equality or inequality of products of bare nucleon or antinucleon operators, the ordering of the operators will be ignored. To be completely accurate the foregoing inequality should be written $DC\neq\pm BA$ but since the signs have no effect on the classification of terms, they will be omitted.) These terms do not contribute to the summation since at least one of the operators in $C^{\dagger}D^{\dagger}$ is not in $B^{\dagger}A^{\dagger}$ (or vice versa) and thus will anticommute with all the operators in BA (or $C^{\dagger}D^{\dagger}$) making the term zero by annihilating on the vacuum state.

The second type of term consists of those terms in the summation in which $C=A$ and $D=B$. The quadruple summation over all these terms will be abbreviated $\sum_{(AC,BD)}$. The third type of term consists of those terms in the summation in which $C=B$ and $D=A$. The quadruple summation over all these terms will be abbreviated $\sum_{(BC,AD)}$. If a term in equation (B.3) is to be non-zero then it can be shown that no two operators in A and B or in C and D can be identical. If, for example, the operators $a\varepsilon A$ and $b\varepsilon B$ are identical then the matrix element on the right hand side of equation (E.3) can be written

$$\langle o|\ldots ba\ldots|o\rangle = \langle o|\ldots ab\ldots|o\rangle \tag{E.4}$$

But since a and b anticommute

$$\langle o|\ldots ba\ldots|o\rangle = -\langle o|\ldots ab\ldots|o\rangle \tag{E.5}$$

thus proving that the operators in A and B or in C and D must be distinct if the term is to be non-zero. Thus for non-zero terms of the second type C and B will have no operators in common nor will A and D. Similarily, for non-zero terms of the third type A and C will have no operators in common nor will B and D.

The fourth type of term consists of the remainder of the terms in equation (B.3), i.e. those terms for which $CD{=}AB$ but which are neither of the second or third type. These terms can be thought of as describing the exchange of bare nucleon-antinucleon pairs between the physical nucleon cores $C^\dagger$ and $D^\dagger$ to form the physical nucleon cores $A^\dagger$ and $B^\dagger$. These terms should have little effect for two reasons. Firstly, although the masses of the bare nucleons are not known, it will be assumed that the bare nucleon-antinucleon pair will have a sufficiently large mass that these terms can be thought of as describing short range forces which will not be significant for medium energy scattering. Secondly, these terms will only arise when at least two of the nucleon cores consist of three or more bare nucleons and antinucleons. It will be assumed that the wavefunction of the physical nucleon will be small in that part of the Fock space representing three or more bare nucleons and antinucleons. Thus in all calculations of Cutkosky matrix elements these 'core exchange' terms will be neglected.

With the above arguments, equation (B.3) can be written (ignoring core exchange terms) as

$$\begin{aligned}\{AB|CD\} = &\sum_{(AC,BD)} \langle o|BLAKC^\dagger M^\dagger D^\dagger N^\dagger|o\rangle \\ &+ \sum_{(BC,AD)} \langle o|BLAKC^\dagger M^\dagger D^\dagger N^\dagger|o\rangle\end{aligned} \qquad (B.6)$$

where each term in the summations is to be multiplied by $a^*(AK)$

etc. Since $[L,K]=0$ and $\{A,B\}=0$ this can be written as

$$\{AB|CD\} = -\sum_{(AC,BD)} \langle o|AKBLC^{\dagger}M^{\dagger}D^{\dagger}N^{\dagger}|o\rangle + \sum_{(BC,AD)} \langle o|BLAKC^{\dagger}M^{\dagger}D^{\dagger}N^{\dagger}|o\rangle$$

$$= \sum_{(AC,BD)} \langle o|AKC^{\dagger}LM^{\dagger}BD^{\dagger}N^{\dagger}|o\rangle - \sum_{(BC,AD)} \langle o|BLC^{\dagger}KM^{\dagger}AD^{\dagger}N^{\dagger}|o\rangle \quad \text{(B.7)}$$

the second equality resulting from the fact that for terms of the second type $\{B,C^{\dagger}\}=0$ and for terms of the third type $\{A,C^{\dagger}\}=0$. Using equation (3.21) this can be written as

$$\{AB|CD\} = \sum_{R}\sum_{(AC,BD)} \langle o|AKC^{\dagger}[[r,M^{\dagger}]][[L,r^{+}]]BD^{\dagger}N^{\dagger}|o\rangle/n(R)!$$

$$- \sum_{R}\sum_{(BC,AD)} \langle o|BLC^{\dagger}[[r,M^{\dagger}]][[K,r^{+}]]AD^{\dagger}N^{\dagger}|o\rangle/n(R)! \quad \text{(B.8)}$$

In order to express the above matrix element in terms of one nucleon matrix elements, the unit operator is inserted between the nested commutators. The unit operator can be written

$$1 = \sum_{E,Q} E^{\dagger}Q^{\dagger}|o\rangle\langle o|EQ/n(Q)!n(E)! \quad \text{(B.9)}$$

where Q is a product of meson annihilation operators and E is a product of bare nucleon and antinucleon annihilation operators satisfying the condition

$$BE^{\dagger}|o\rangle = +1E^{\dagger}|o\rangle \quad \text{(B.10)}$$

where B is the baryon number operator. Terms in which E is the unit operator are also to be included in equation (E.9).

Thus the matrix element {AB|CD} may be written

$$\{AB|CD\} = \sum_{E,Q,R} \sum_{(AC,BD)} \langle o|AKC^{\dagger}[[r,M^{\dagger}]]E^{\dagger}Q^{\dagger}|o\rangle\langle o|EQ[[L,r^{+}]]BD^{\dagger}N^{\dagger}|o\rangle/n(R)!$$

$$\times\ n(Q)!n(E)!$$

$$- \sum_{E,Q,R} \sum_{(BC,AD)} \langle o|BLC^{\dagger}[[r,M^{\dagger}]]E^{\dagger}Q^{\dagger}|o\rangle\langle o|EQ[[K,r^{+}]]AD^{\dagger}N^{\dagger}|o\rangle/n(R)!$$

$$\times\ n(Q)!n(E)! \qquad \text{(E.11)}$$

Now, all terms in which $E\neq 1$ will vanish. For example, in the first term if $E^{\dagger}\neq 1$ then it either has an operator in common with A (in which case the operator is in $C^{\dagger}$ which causes the matrix element to vanish because of the anticommutation of identical operators) or it does not have an operator in common with A (in which case this operator will anticommute with A and annihilate on the vacuum state). A similar argument holds for the second term of equation (E.11).

Also, $Q^{\dagger}$ commutes with $[[r,M^{\dagger}]]$ since $[[r,M^{\dagger}]]$ contains only meson creation operators. Similarily Q commutes with both $[[L,r^{\dagger}]]$ and $[[K,r^{\dagger}]]$. Since $r|0\rangle=0$ for any meson annihilation operator,

$$[[r,M^{\dagger}]]|o\rangle = RM^{\dagger}|o\rangle$$

$$\langle o|[[L,r^{+}]] = \langle o|LR^{\dagger} \qquad \text{(E.12)}$$

$$\langle o|[[K,r^{+}]] = \langle o|KR^{\dagger}$$

Thus equation (E.11) can be written

$$\{AB|CD\} = \sum_{Q,R} \sum_{(AC,BD)} \langle o|\mathcal{A}KQ^{\dagger}RC^{\dagger}M^{\dagger}|o\rangle\langle o|\mathcal{B}LR^{\dagger}Q\mathcal{D}^{\dagger}N^{\dagger}|o\rangle/n(R)!n(Q)!$$
$$- \sum_{Q,R} \sum_{(BC,AD)} \langle o|\mathcal{B}LQ^{\dagger}RC^{\dagger}M^{\dagger}|o\rangle\langle o|\mathcal{A}KR^{\dagger}Q\mathcal{D}^{\dagger}N^{\dagger}|o\rangle/n(R)!n(Q)! \qquad (E.13)$$

The restrictions on the summations in the above equation can now be removed without affecting the result. That is, terms in which $\mathcal{CD}$ $\mathcal{AB}$ will be zero as will all terms in which $\mathcal{C}\neq\mathcal{A}$ and $\mathcal{D}\neq\mathcal{B}$ or $\mathcal{B}\neq\mathcal{C}$ and $\mathcal{A}\neq\mathcal{D}$. Also, terms of the third type can be added to the first summation (since they will be zero) and terms of the second type can be added to the second summation (since they will be zero). Terms of the second and third type in which $\mathcal{A}=\mathcal{B}=\mathcal{C}=\mathcal{D}$ will cancel between the first and second summation. Performing these summations, the overlap matrix element can be written

$$\{AB|CD\} = \sum_{R,Q} \langle A|Q^{\dagger}R|C\rangle\langle B|R^{\dagger}Q|D\rangle/n(R)!n(Q)!$$
$$- \sum_{R,Q} \langle B|Q^{\dagger}R|C\rangle\langle A|R^{\dagger}Q|D\rangle/n(R)!n(Q)! \qquad (E.14)$$

Thus the matrix element between two unexcited Cutkosky states has been written as a sum of products of matrix elements of physical one nucleon states. The only approximation in this result is that core exchange terms have been ignored.

B.2 Unexcited Interaction Matrix Elements

Interaction matrix elements are matrix elements of the Hamiltonian, H, or the vertex operator, [mV], evaluated between Cutkosky states. The matrix element of the Hamiltonian will be evaluated first as it uses all the ideas necessary to evaluate matrix elements of the vertex operator.

Using the definition of Cutkosky states the matrix element of the Hamiltonian between unexcited states can be written

$$\{AB|H|CD\} = \langle o|BAHC^{\dagger}D^{\dagger}|o\rangle \tag{E.15}$$

The general method used is to commute H to the right past either $C^{\dagger}$ or $D^{\dagger}$ and then separate the resulting Cutkosky matrix elements into one nucleon matrix elements following the methods of the preceding section. Since the final result should be symmetric with respect to initial and final states the process should be repeated commuting H with either A or B. The final result will be the average of the two results obtained above.

Using the fact that $C^{\dagger}$ and $D^{\dagger}$ anticommute

$$HC^{\dagger}D^{\dagger}|o\rangle = (C^{\dagger}HD^{\dagger}-D^{\dagger}HC^{\dagger})|o\rangle - [H,D^{\dagger}]C^{\dagger}|o\rangle + [H,C^{\dagger}]D^{\dagger}|o\rangle - HC^{\dagger}D^{\dagger}|o\rangle \tag{B.16}$$

Since $C^{\dagger}|0\rangle$ and $D^{\dagger}|0\rangle$ are eigenstates of the Hamiltonian

$$(C^{\dagger}HD^{\dagger}-D^{\dagger}HC^{\dagger})|o\rangle = (E_C+E_D)C^{\dagger}D^{\dagger}|o\rangle \tag{E.17}$$

Defining

$$H_{\pi} = \sum_{k} E_k\, k^{\dagger}k \tag{E.18}$$

the last three terms of equation (B.16) can be written

$$\begin{aligned} &- [H,D^{\dagger}]C^{\dagger}|o\rangle + [H,C^{\dagger}]D^{\dagger}|o\rangle - HC^{\dagger}D^{\dagger}|o\rangle \\ &= H_{\pi}C^{\dagger}D^{\dagger}|o\rangle + D^{\dagger}H_{\pi}C^{\dagger}|o\rangle - C^{\dagger}H_{\pi}D^{\dagger}|o\rangle + VC^{\dagger}D^{\dagger}|o\rangle + D^{\dagger}VC^{\dagger}|o\rangle - C^{\dagger}VD^{\dagger}|o\rangle \end{aligned} \tag{B.19}$$

The terms involving H_{π} in equation (E.19) will cancel as can be verified by writing $C^{\dagger}$ and $D^{\dagger}$ explicitly and evaluating those terms.

Thus the matrix element of the Hamiltonian can be written

$$\{AB|H|CD\} = (E_C+E_D)\ \{AB|CD\} + \langle o|BAVC^\dagger D^\dagger|o\rangle + \langle o|BAD^\dagger VC^\dagger|o\rangle - \langle o|BAC^\dagger VD^\dagger|o\rangle \tag{B.20}$$

Using the notation of equations (B.2) a typical term of the three matrix elements involving V is

$$\langle o|\mathcal{B}L\mathcal{A}KVC^\dagger M^\dagger \mathcal{D}^\dagger N^\dagger|o\rangle + \langle o|\mathcal{B}L\mathcal{A}K\mathcal{D}^\dagger N^\dagger VC^\dagger M^\dagger|o\rangle - \langle o|\mathcal{B}L\mathcal{A}KC^\dagger M^\dagger V\mathcal{D}^\dagger N^\dagger|o\rangle \tag{B.21}$$

where the coefficients $a^*(\mathcal{A}K)$ etc. have been omitted.

To proceed, V is written as a sum of five terms. V can be written in terms of the fundamental dynamical variables as

$$V = \sum_{\mathcal{V},\mathcal{W},S,T} v(\mathcal{V},\mathcal{W},S,T)\,\mathcal{V}^\dagger \mathcal{W} S^\dagger T \tag{B.22}$$

where $\mathcal{V}^\dagger$ and $\mathcal{W}$ are either unit operators or are products of odd numbers of bare nucleon and antinucleon operators and $S^\dagger$ and T are products of meson operators. The coefficients $v(\mathcal{V},\mathcal{W},S,T)$ will depend upon the type of interaction chosen. Now, the sum in equation (B.22) is broken up into five terms. The first term, V', consists of all those terms in which $\mathcal{V}=\mathcal{W}=1$. These will be terms describing the meson-meson interaction. The other four terms are denoted by V''_C, V''_D, V''_R and V''_o. V''_C consists of all those terms in which $\mathcal{W}=C$ and V''_D consists of all those terms in which $\mathcal{W}=\mathcal{D}$. V''_R consists of those terms in which $\mathcal{W}$ is partly contained in C and partly contained in $\mathcal{D}$ while V''_o consists of those terms in which $\mathcal{W}$ contains some bare nucleon or antinucleon operators not in $\mathcal{C}$ or $\mathcal{D}$.

Thus, writing

$$V = V' + V''_C + V''_D + V''_R + V''_O \tag{B.23}$$

some simplifications can be made immediately in the matrix elements involving V. The first matrix element involving V in equation (B.21) becomes

$$\begin{aligned}<o|\mathcal{B}LAKVC^\dagger M^\dagger \mathcal{D}^\dagger N^\dagger|o> &= <o|\mathcal{B}LAKV'C^\dagger M^\dagger \mathcal{D}^\dagger N^\dagger|o> \\ &+ <o|\mathcal{B}LAK(V''_C+V''_D)C^\dagger M^\dagger \mathcal{D}^\dagger N^\dagger|o> + <o|\mathcal{B}LAKV''_R C^\dagger M^\dagger \mathcal{D}^\dagger N^\dagger|o>\end{aligned} \tag{B.24}$$

The matrix element involving V''_o will vanish since at least one of the annihilation operators in $\mathcal{W}$ will anticommute with all the operators in $C^\dagger$ and $\mathcal{D}^\dagger$ and annihilate on the vacuum state. Since

$$V''_R C^\dagger|o> = V''_R \mathcal{D}^\dagger|o> = o \tag{B.25}$$

the matrix element involving V''_R will not be able to be expressed as products of one-nucleon matrix elements. Since this matrix element only appears when either $C^\dagger$ or $\mathcal{D}^\dagger$ (or both) contain three or more bare nucleon and antinucleon operators, this matrix element will be neglected on the basis that the nucleon cores are primarily composed of single bare nucleons.

The second matrix element in equation (B.21) becomes

$$<o|\mathcal{B}LAK\mathcal{D}^\dagger N^\dagger VC^\dagger M^\dagger|o> = <o|\mathcal{B}LAK\mathcal{D}^\dagger N^\dagger V'C^\dagger M^\dagger|o> + <o|\mathcal{B}LAK\mathcal{D}^\dagger N^\dagger V''_C C^\dagger M^\dagger|o> \tag{B.26}$$

The matrix element involving V''_o will vanish for the same reason as in equation (E.24) and the matrix element involving V''_R will vanish as a result of equation (E.25). Since $\mathcal{C}^\dagger$ and $\mathcal{D}^\dagger$ cannot have any operators in common (otherwise the matrix element would vanish as a result of the anticommutation of identical bare nucleon operators) V''_D cannot contain any terms in which $\mathcal{W}=\mathcal{C}$. Thus the matrix element involving V''_D will vanish.

By similar arguments

$$<o|\mathcal{B}LAK\mathcal{C}^\dagger M^\dagger V\mathcal{D}^\dagger N^\dagger|o> = <o|\mathcal{B}LAK\mathcal{C}^\dagger M^\dagger V'\mathcal{D}^\dagger N^\dagger|o> + <o|\mathcal{B}LAK\mathcal{C}^\dagger M^\dagger V''_D\mathcal{D}^\dagger N^\dagger|o> \quad (E.27)$$

Thus the matrix elements involving V can be written

$$\begin{aligned}&<o|\mathcal{B}LAK(V''_C+V')\mathcal{C}^\dagger M^\dagger\mathcal{D}^\dagger N^\dagger|o> + <o|\mathcal{B}LAK\mathcal{D}^\dagger N^\dagger(V''_C+V')\mathcal{C}^\dagger M^\dagger|o> \\ &+ <o|\mathcal{B}LAK(V''_D+V')\mathcal{C}^\dagger M^\dagger\mathcal{D}^\dagger N^\dagger|o> - <o|\mathcal{B}LAK\mathcal{C}^\dagger M^\dagger(V''_D+V')\mathcal{D}^\dagger N^\dagger|o> \\ &- <o|\mathcal{B}LAKV'\mathcal{C}^\dagger M^\dagger\mathcal{D}^\dagger N^\dagger|o> \end{aligned} \quad (E.28)$$

The terms in the above equation containing V''_C can be written

$$\begin{aligned}&- <o|\mathcal{B}LAK(V''_C+V')\mathcal{D}^\dagger N^\dagger\mathcal{C}^\dagger M^\dagger|o> + <o|\mathcal{B}LAK\mathcal{D}^\dagger N^\dagger(V''_C+V')\mathcal{C}^\dagger M^\dagger|o> \\ &= - <o|\mathcal{B}LAK\mathcal{D}^\dagger[V''_C+V',N^\dagger]\mathcal{C}^\dagger M^\dagger|o>\end{aligned} \quad (E.29)$$

In writing the above equation use has been made of the fact that since each term in V''_C contains an even number of bare nucleon and antinucleon operators and since $\mathcal{W}$ cannot contain any operators which are in $\mathcal{D}$, then $[\mathcal{D},V''_C]=0$.

Now since each term of V''_C+V' contains meson annihilation operators, equation (3.20) is used to write

$$[N^\dagger,V''_C+V'] = -\sum_R{}'[[r,N^\dagger]][[V''_C+V',r^+]]/n(R)! \quad (E.30)$$

Thus the terms involving V_C'' become

$$-\sum_R{}' \langle o|BLAK[[r,D^\dagger N^\dagger]][[V_C''+V',r^+]]C^\dagger M^\dagger|o\rangle/n(R)! \tag{E.31}$$

Using methods identical to those used in evaluation of the overlap matrix element, the above expression can be written

$$-\sum_R{}' \sum_{Q,P} \langle o|AKP^\dagger QRD^\dagger N^\dagger|o\rangle\langle o|BLQ^\dagger P[(V_C''+V')R]C^\dagger M^\dagger|o\rangle$$
$$+\sum_R{}' \sum_{Q,P} \langle o|BLP^\dagger QRD^\dagger N^\dagger|o\rangle\langle o|AKQ^\dagger P[(V_C''+V')R]C^\dagger M^\dagger|o\rangle \tag{E.32}$$

each term being divided by $n(Q)!n(P)!$. The Generalized vertex operators $[VR]$ and $[RV]$ are defined by

$$[VR] = [[V,r^+]]/n(R)!$$
$$[RV] = [VR]^\dagger \tag{E.33}$$

Performing the sums over A,B, etc. and using the fact that

$$[(V_C''+V')R]C^\dagger M^\dagger|o\rangle = [VR]C^\dagger M^\dagger|o\rangle \tag{E.34}$$

the above expression can be written

$$-\sum_R{}' \sum_{Q,P} \langle A|P^\dagger QR|D\rangle\langle B|Q^\dagger P[VR]|C\rangle/n(Q)!n(P)!$$
$$+\sum_R{}' \sum_{Q,P} \langle B|P^\dagger QR|D\rangle\langle A|Q^\dagger P[VR]|C\rangle/n(Q)!n(P)! \tag{E.35}$$

Similarily the terms involving V_D'' yield the expression

$$\sum_{R}{}' \sum_{Q,P} \langle A|P^{\dagger}QR|C\rangle\langle B|Q^{\dagger}P[VR]|D\rangle/n(Q)!n(P)!$$
$$- \sum_{R}{}' \sum_{Q,P} \langle B|P^{\dagger}QR|C\rangle\langle A|Q^{\dagger}P[VR]|D\rangle/n(Q)!n(P)! \qquad (E.36)$$

By defining

$$H_{AC,BVD} = \sum_{R}{}' \sum_{Q,P} \langle A|P^{\dagger}QR|C\rangle\langle B|Q^{\dagger}P[VR]|D\rangle/n(Q)!n(P)! \qquad (E.37)$$

the Hamiltonian matrix element can be written

$$\{AB|H|CD\} = (E_C+E_D)\{AB|CD\} + H_{AC,BVD} - H_{BC,AVD} + H_{BD,AVC} - H_{AD,BVC}$$
$$- \{AB|V'|CD\} + \text{term involving } V_R'' \qquad (E.38)$$

In order to make the derivation symmetrical with respect to initial and final states the whole procedure is repeated starting with the analogue of equation (B.16)

$$\langle o|BAH = \langle o|(BHA-AHB) - \langle o|A[B,H] + \langle o|B[A,H] - \langle o|BAH \qquad (E.39)$$

Using the same method the result is obtained

$$\{AB|H|CD\} = (E_A+E_B)\{AB|CD\} + H_{AVC,BD} - H_{BVC,AD} + H_{BVD,AC} - H_{AVD,BC}$$
$$- \{AB|V'|CD\} + \text{term involving } V_R'' \qquad (E.40)$$

where

$$H_{AVC,BD} = \sum_{R}{}' \sum_{Q,P} \langle A|[RV]P^{\dagger}Q|C\rangle\langle B|R^{\dagger}Q^{\dagger}P|D\rangle/n(P)!n(Q)! \qquad (E.41)$$

The final expression for the Hamiltonian matrix element is

$$\{AB|H|CD\} = \frac{1}{2}(E_A+E_B+E_C+E_D)\{AB|CD\} + \frac{1}{2}(H_{AC,BVD} + H_{BVD,AC} - H_{BC,AVD} - H_{AVD,BC} + H_{BD,AVC} + H_{AVC,BD} - H_{AD,BVC} - H_{BVC,AD}) - \{AB|V'|CD\} + \text{term involving } V_R'' \quad (E.42)$$

The expression for the Cutkosky matrix element of the vertex operator, [mV], follows the above derivation starting at equation (B.20) with V replaced by [mV]. The second and third matrix elements in equation (B.20) will not be present in this derivation which will result in the unit operator being included in all the summations of the resulting expression (the unit operator is included because equation (3.21) is used rather than equation (3.20)). Since the energy terms come from manipulations preceding equation (B.20) the result for the vertex operator will not include these terms.

Thus the final expression for the matrix element of the vertex operator [mV] is

$$\{AB|[mV]|CD\} = \frac{1}{2}(V_{AC,BVD} + V_{BVD,AC} - V_{BC,AVD} - V_{AVD,BC} + V_{BD,AVC} + V_{AVC,BD} - V_{AD,BVC} - V_{BVC,AD}) - \{AB|[mV']|CD\} + \text{term involving } [mV_R''] \quad (B.43)$$

where
$$V_{AC,BVD} = \sum_{R,Q,P} \langle A|P^{\dagger}QR|C\rangle\langle B|Q^{\dagger}P[mVR]|D\rangle/n(P)!n(Q)! \quad (E.44)$$

and
$$V_{AVC,BD} = \sum_{R,Q,P} \langle A|[mRV]P^{\dagger}Q|C\rangle\langle B|R^{\dagger}Q^{\dagger}P|D\rangle/n(P)!n(Q)! \quad (E.45)$$

B.3 Excited Overlap Matrix Elements

Matrix elements between excited Cutkosky states involve some additional complications. However the basic method of expanding these matrix elements into matrix elements of one nucleon states is the same as for the unexcited Cutkosky states.

The overlap matrix element calculated here will be between two singly excited Cutkosky states. Since this calculation exhibits all the complicating features of overlap matrix elements between excited states, the overlap matrix element between any Cutkosky states can be calculated using methods based on the following calculations.

Using the definition of singly excited Cutkosky states the matrix element can be written

$$\begin{aligned}\{ABm|CDk\} = &<o|BAmk^{\dagger}C^{\dagger}D^{\dagger}|o> + <o|BAm(Ck)_s^{\dagger}D^{\dagger}|o> + <o|BAmC^{\dagger}(Dk)_s^{\dagger}|o> \\ &+ <o|B(Am)_s k^{\dagger}C^{\dagger}D^{\dagger}|o> + <o|B(Am)_s(Ck)_s^{\dagger}D^{\dagger}|o> + <o|B(Am)_s C^{\dagger}(Dk)_s^{\dagger}|o> \\ &+ <o|(Bm)_s Ak^{\dagger}C^{\dagger}D^{\dagger}|o> + <o|(Bm)_s A(Ck)_s^{\dagger}D^{\dagger}|o> + <o|(Bm)_s AC^{\dagger}(Dk)_s^{\dagger}|o>\end{aligned} \tag{B.46}$$

The first matrix element on the right hand side of the above equation can be written

$$\begin{aligned}<o|BAmk^{\dagger}C^{\dagger}D^{\dagger}|o> = &\ \delta_{km}<o|BAC^{\dagger}D^{\dagger}|o> + <o|B[A,k^{\dagger}]C^{\dagger}[m,D^{\dagger}]|o> \\ &+ <o|B[A,k^{\dagger}][m,C^{\dagger}]D^{\dagger}|o> + <o|[B,k^{\dagger}]AC^{\dagger}[m,D^{\dagger}]|o> + <o|[B,k^{\dagger}]A[m,C^{\dagger}]D^{\dagger}|o>\end{aligned} \tag{B.47}$$

and matrix elements involving only one meson operator can be written, for example,

$$<o|BAm(Ck)_s^{\dagger}D^{\dagger}|o> = <o|BA(Ck)_s^{\dagger}[m,D^{\dagger}]|o> + <o|BA[m,(Ck)_s^{\dagger}]D^{\dagger}|o> \tag{B.48}$$

Thus every matrix element in equation (B.46) can be written as $<0|F_1F_2F_3^{\dagger}F_4^{\dagger}|0>$ where the operators $F_i^{\dagger}$ $(i=1,\dots,4)$ can all be expressed in the form

$$F_i^{\dagger} = \sum_{F,M} f_i(FM)F^{\dagger}M^{\dagger} \tag{B.49}$$

and $F^{\dagger}$ and $M^{\dagger}$ satisfy equations (2.18)-(2.18).

Thus all matrix elements resulting from equation (B.46) by using equations (B.47) and (B.48) and similar equations can be expanded in terms of one nucleon matrix elements using the tecniques of section B.1. The resulting expression can be written

$$\{ABm|CDk\} = \delta_{km}\{AB|CD\} + \sum_{R,Q} \langle A|Q^\dagger R|C\rangle\{-\delta_{km}\langle B|R^\dagger Q|D\rangle + \langle Bm|R^\dagger Q|Dk\rangle$$

$$+ \langle Bm|R^\dagger[k^\dagger,Q]|D\rangle + \langle B|[R^\dagger,m]Q|Dk\rangle + \langle B|[R^\dagger,m][k^\dagger,Q]|D\rangle\}$$

$$+ \sum_{R,Q} \langle Am|Q^\dagger R|C\rangle + \langle A|[Q^\dagger,m]R^\dagger|C\rangle\}\{\langle B|R^\dagger Q|Dk\rangle + \langle B|R^\dagger[k^\dagger,Q]|D\rangle\}$$

$$+ \text{antisym.} \qquad (B.50)$$

The sign of the summations in the above equation is determined by noting that the expressions will be antisymmetric with respect to exchange of A and B or C and D. When the matrix elements are simplified in the above equation by use of equations (3.24)-(3.26), the δ-function in the summations will vanish and all the terms in equation (B.50) will be able to be written as matrix elements involving one nucleon states and/or pion-nucleon scattering states and vertex operators.

B.4 Excited Interaction Matrix Elements

The interaction matrix element calculated here will be between a singly excited Cutkosky state and an unexcited Cutkosky state. The more complicated matrix elements will not be given here since they are not used in the calculations of the pion-deuteron T-matrix. However the method of evaluating them is similar to the techniques used below and in section B.3.

Using the definition of Cutkosky states the matrix element of the Hamiltonian between an excited and an unexcited state can be written

$$\{AB|H|CDk\} = -<o|BAHC^{\dagger}D^{\dagger}k^{\dagger}|o> + <o|BAH(Ck)^{\dagger}D^{\dagger}|o> + <o|BAHC^{\dagger}(Dk)^{\dagger}|o> \quad (B.51)$$

The second and third terms above can be evaluated using the methods of section B.2. The results are

$$\begin{aligned} <o|BAH(Ck)^{\dagger}D^{\dagger}|o> &= \frac{1}{2}(E_A+E_B+E_C+E_D+E_k)<o|BA(Ck)^{\dagger}D^{\dagger}|o> \\ &+ \frac{1}{2}\sum_R{}' \sum_{Q,M} <A|M^{\dagger}QR|Ck><B|Q^{\dagger}M[VR]|D>/n(Q)!n(M)! \\ &+ \frac{1}{2}\sum_R{}' \sum_{Q,M} <A|R^{\dagger}Q^{\dagger}M|Ck><B|[RV]M^{\dagger}Q|D>/n(Q)!n(M)! \quad (B.52) \\ &+ \text{antisym.} \end{aligned}$$

$$\begin{aligned} <o|BAHC^{\dagger}(Dk)^{\dagger}|o> &= \frac{1}{2}(E_A+E_B+E_C+E_D+E_k)<o|BAC^{\dagger}(Dk)^{\dagger}|o> \\ &+ \frac{1}{2}\sum_R{}' \sum_{Q,M} <A|M^{\dagger}QR|C><B|Q^{\dagger}M[VR]|Dk>/n(Q)!n(M)! \\ &+ \frac{1}{2}\sum_R{}' \sum_{Q,M} <A|R^{\dagger}Q^{\dagger}M|C><B|[RV]M^{\dagger}Q|Dk>/n(Q)!n(M)! \quad (B.53) \\ &+ \text{antisym.} \end{aligned}$$

The first term in equation (B.51) can be written

$$\begin{aligned} <o|BAHC^{\dagger}D^{\dagger}k^{\dagger}|o> &= <o|B[A,k^{\dagger}]HC^{\dagger}D^{\dagger}|o> + <o|[B,k^{\dagger}]AHC^{\dagger}D^{\dagger}|o> \\ &+ E_k<o|BAk^{\dagger}C^{\dagger}D^{\dagger}|o> + <o|BA[Vk]C^{\dagger}D^{\dagger}|o> \quad (B.54) \end{aligned}$$

All the above terms can be evaluated using the methods of the previous sections. The only differences are that the energy term in the first two matrix elements above becomes $\frac{1}{2}(E_A+E_B-E_k+E_C+E_D)$

and there are extra matrix elements involving $[Vk]$ since

$$\langle o|[A,k^\dagger]H = (E_A - E_k)\ \langle o|[A,k^\dagger] - \langle o|A[Vk] \qquad \text{(B.55)}$$

The final result for the Hamiltonian matrix element is

$$\begin{aligned}
\{AB|H|CDk\} = {} & \tfrac{1}{2}(E_A+E_B+E_C+E_D+E_k)\ \{AB|CDk\} \\
& + \frac{1}{2}\sum_R{}' \sum_{Q,M} \langle A|M^\dagger QR|Ck\rangle\langle B|Q^\dagger M[VR]|D\rangle + \langle A|R^\dagger Q^\dagger M|Ck\rangle\langle B|[RV]M^\dagger Q|D\rangle \\
& \qquad + \langle A|M^\dagger QR|C\rangle\langle B|Q^\dagger M[VR]|Dk\rangle + \langle A|R^\dagger Q^\dagger M|C\rangle\langle B|[RV]M^\dagger Q|Dk\rangle \\
& \qquad - \langle A|k^\dagger M^\dagger QR|C\rangle\langle B|Q^\dagger M[VR]|D\rangle - \langle A|k^\dagger R^\dagger Q^\dagger M|C\rangle\langle B|[RV]M^\dagger Q|D\rangle \\
& \qquad - \langle A|M^\dagger QR|C\rangle\langle B|k^\dagger Q^\dagger M[VR]|D\rangle - \langle A|R^\dagger Q^\dagger M|C\rangle\langle B|k^\dagger [RV]M^\dagger Q|D\rangle \\
& \qquad - \tfrac{1}{2}\langle A|M^\dagger QR|C\rangle\langle B|Q^\dagger M[VkR]|D\rangle - \tfrac{1}{2}\langle A|R^\dagger Q^\dagger M|C\rangle\langle B|[RVk]M^\dagger Q|D\rangle \\
& + \frac{1}{4}\sum_R \sum_{Q,M} -\langle A|M^\dagger QR|C\rangle\langle B|Q^\dagger M[VkR]|D\rangle - \langle A|R^\dagger Q^\dagger M|C\rangle\langle B|[RVk]M^\dagger Q|D\rangle \qquad \text{(B.56)} \\
& + \text{antisym.}
\end{aligned}$$

Although it appears as though the last two terms will give zero meson exchange terms, these terms will cancel with some of the one meson exchange terms.

The derivation for the matrix elements of the vertex operator $[mV]$ between excited and unexcited or excited Cutkosky states will not be given here as it can be evaluated using the techniques of section B.2.

APPENDIX C The T-matrix in the One-Meson Exchange Approximation

In section 4.2, the T-matrix in the one-meson exchange approximation was written as below

$$T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = T^{+}_{diff} + T^{-}_{diff} + T^{+}_{same} + T^{-}_{same} \tag{C.1}$$

The terms on the right hand side of the above equation are written in terms of one-nucleon matrix elements in the following equations.

$$\begin{aligned} T^{+}_{diff} = \frac{1}{2} \sum_{\substack{ABCD\\FGq}} & <\mathcal{D}'_o|AB)(CD|\mathcal{D}_o> \quad \times \\ & \times\ (<A|[mV]|F><F|[Vq]|C><B|[Vk]|G><G|[qV]|D>\ a_1(E) \\ & + <A|[mV]|F><F|[Vq]|C><B|[qV]|G><G|[Vk]|D>\ a_2(E) \\ & + <A|[Vq]|F><F|[mV]|C><B|[Vk]|G><G|[qV]|D>\ a_3(E) \\ & + <A|[Vq]|F><F|[mV]|C><B|[qV]|G><G|[Vk]|D>\ a_4(E)\ ) \end{aligned} \tag{C.2}$$

$$\begin{aligned} T^{-}_{diff} = \frac{1}{2} \sum_{\substack{ABCD\\FGq}} & <\mathcal{D}'_o|AB)(CD|\mathcal{D}_o> \quad \times \\ & \times\ (<A|[mV]|F><F|[qV]|C><B|[Vk]|G><G|[Vq]|D>\ a_5(E) \\ & + <A|[mV]|F><F|[qV]|C><B|[Vq]|G><G|[Vk]|D>\ a_6(E) \\ & + <A|[qV]|F><F|[mV]|C><B|[Vk]|G><G|[Vq]|D>\ a_7(E) \\ & + <A|[qV]|F><F|[mV]|C><B|[Vq]|G><G|[Vk]|D>\ a_8(E)\ ) \end{aligned} \tag{C.3}$$

$$T^{+}_{\text{same}} = \frac{1}{2} \sum_{\substack{ABCD\\FGq}} \langle \mathcal{D}'_o | AB)(CD | \mathcal{D}_o\rangle \quad x$$

$$x \; (\langle A|[Vq]|C\rangle\langle B|[Vk]|F\rangle\langle F|[qV]|G\rangle\langle G|[mV]|D\rangle \; a_9(E)$$

$$+\langle A|[Vq]|C\rangle\langle B|[qV]|F\rangle\langle F|[Vk]|G\rangle\langle G|[mV]|D\rangle \; a_{10}(E)$$

$$+\langle A|[Vq]|C\rangle\langle B|[mV]|F\rangle\langle F|[Vk]|G\rangle\langle G|[qV]|D\rangle \; a_{11}(E)$$

$$+\langle A|[Vq]|C\rangle\langle B|[mV]|F\rangle\langle F|[qV]|G\rangle\langle G|[Vk]|D\rangle \; a_{12}(E)$$

$$+\langle A|[Vq]|C\rangle\langle B|[qV]|F\rangle\langle F|[mV]|G\rangle\langle G|[Vk]|D\rangle \; a_{13}(E)$$

$$+\langle A|[Vq]|C\rangle\langle B|[Vk]|F\rangle\langle F|[mV]|G\rangle\langle G|[qV]|D\rangle \; a_{14}(E) \;) \qquad (C.4)$$

$$T^{-}_{\text{same}} = \frac{1}{2} \sum_{\substack{ABCD\\FGq}} \langle \mathcal{D}'_o | AB)(CD | \mathcal{D}_o\rangle \quad x$$

$$x \; (\langle A|[qV]|C\rangle\langle B|[Vk]|F\rangle\langle F|[Vq]|G\rangle\langle F|[mV]|D\rangle \; a_{15}(E)$$

$$+\langle A|[qV]|C\rangle\langle B|[Vq]|F\rangle\langle F|[Vk]|G\rangle\langle G|[mV]|D\rangle \; a_{16}(E)$$

$$+\langle A|[qV]|C\rangle\langle B|[mV]|F\rangle\langle F|[Vk]|G\rangle\langle F|[Vq]|D\rangle \; a_{17}(E)$$

$$+\langle A|[qV]|C\rangle\langle B|[mV]|F\rangle\langle F|[Vq]|G\rangle\langle G|[Vk]|D\rangle \; a_{18}(E)$$

$$+\langle A|[qV]|C\rangle\langle B|[Vq]|F\rangle\langle F|[mV]|G\rangle\langle G|[Vk]|D\rangle \; a_{19}(E)$$

$$+\langle A|[qV]|C\rangle\langle B|[Vk]|F\rangle\langle F|[mV]|G\rangle\langle G|[Vq]|D\rangle \; a_{20}(E) \;) \qquad (C.5)$$

where

$$a_1(E) = \frac{1}{v_1(v_2+v_3)} \left(\frac{1}{v_2} - \frac{v_1+v_2+v_3}{v_3(v_1+v_2+v_3+v_5)} \right) \qquad (C.6)$$

$$a_2(E) = - \frac{2v_1+v_5}{v_1 v_3 (v_2+v_3)(v_1+v_2+v_3+v_5)} \qquad (C.7)$$

$$a_3(E) = - \frac{1}{v_1(v_2+v_3)} \left(\frac{1}{v_2} - \frac{v_1+v_2+v_3}{v_3(v_1+v_2+v_3+v_4)} \right) \qquad (C.8)$$

$$a_4(E) = + \frac{2v_1+v_4}{v_1v_3(v_2+v_3)(v_1+v_2+v_3+v_4)} \tag{C.9}$$

$$a_5(E) = - \frac{1}{v_1v_2(v_2+v_3)} \tag{C.10}$$

$$a_6(E) = \frac{1}{v_1(v_2+v_3)(v_1+v_2+v_3+v_5)} \left(\frac{2v_1+v_5}{v_3} + \frac{v_1-v_2-v_3}{v_2} \right) \tag{C.11}$$

$$a_7(E) = \frac{1}{v_1v_2(v_2+v_3)} \tag{C.12}$$

$$a_8(E) = - \frac{1}{v_1(v_2+v_3)(v_1+v_2+v_3+v_4)} \left(\frac{2v_1+v_4}{v_3} + \frac{v_1-v_2-v_3}{v_2} \right) \tag{C.13}$$

$$a_9(E) = \frac{1}{v_1(v_2+v_3)} \left(\frac{1}{v_2} - \frac{v_1+v_2+v_3}{v_3(v_1+v_2+v_3+v_4)} \right) \tag{C.14}$$

$$a_{10}(E) = \frac{1}{v_1v_3} \left(\frac{1}{v_2} - \frac{2v_1+v_4}{(V_2+v_3)(v_1+v_2+v_3+v_4)} \right) \tag{C.15}$$

$$a_{11}(E) = - \frac{1}{v_1(v_2+v_3)} \left(\frac{2}{v_2} + \frac{2v_2+2v_3+v_5}{v_3(v_1+v_2+v_3+v_5)} \right) \tag{C.16}$$

$$a_{12}(E) = \frac{v_5}{v_1v_3(v_2+v_3)(v_1+v_2+v_3+v_5)} \tag{C.17}$$

$$a_{13}(E) = - \frac{1}{v_1v_2v_3} \tag{C.18}$$

$$a_{14}(E) = \frac{1}{v_1 v_2 v_3} \tag{C.19}$$

$$a_{15}(E) = \frac{1}{v_1 v_2 (v_2+v_3)} \tag{C.20}$$

$$a_{16}(E) = -\frac{1}{v_1 (v_2+v_3)(v_1+v_2+v_3+v_4)} \left(\frac{3v_1+v_2+v_3+2v_4}{v_3} + \frac{2v_1+v_4}{v_2} \right) \tag{C.21}$$

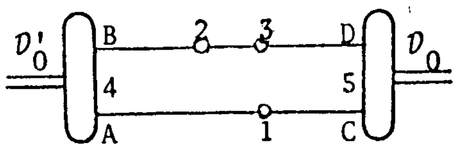
$$a_{17}(E) = -\frac{1}{v_1 v_3 (v_2+v_3)} \tag{C.22}$$

$$a_{18}(E) = -\frac{1}{v_1 (v_2+v_3)(v_1+v_2+v_3+v_5)} \left(\frac{2v_2+2v_3+v_5}{v_3} - \frac{v_1-v_2-v_3}{v_2} \right) \tag{C.23}$$

$$a_{19}(E) = \frac{1}{v_1 v_2 v_3} \tag{C.24}$$

$$a_{20}(E) = \frac{1}{v_1 v_2 v_3} \tag{C.25}$$

The energy functions $v_1, \ldots, v_5$ are obtained as outlined below. If the vertex involving the meson m (the final emitted meson) is removed from each diagram, then all diagrams will appear similar to the diagram below.



The energy function is given by

$$v_i = E_{Fi} - E_{Ii} \tag{C.26}$$

where the indices i are as specified in the diagram above and where

E_{Ii} = sum of energy of particles going into vertex i

and E_{Fi} = sum of energy of particles coming from vertex i.

Note that since time is assumed to be increasing towards the left in the above diagram, E_{Fi} corresponds to the energy of the particles to the left of the vertex i and E_{Ii} corresponds to the energy of the particles to the right of the vertex i.

Since the functions v_1, v_2, and v_3 can vanish, a term $-i\varepsilon$ should be added to each of these energy denominators. This will insure that the meson at each of these vertices will obey the appropriate boundary conditions.

APPENDIX D

D.1 The Deuteron Wavefunction

The deuteron wavefunction $(AB|\mathcal{D}_o>$ is related to the conventional deuteron momentum space wavefunction of McMillan and Landau(1974) in the following way

$$(AB|\mathcal{D}_o> = \sqrt{2}\,\chi_o(\tau_A,\tau_B)\ \delta\ (\underline{k}_{\mathcal{D}}-\underline{K})\ \phi\ (\underline{\kappa},\sigma_A,\sigma_B,M) \tag{D.1}$$

where
$$\underline{K} = \underline{k}_A + \underline{k}_B \tag{D.2}$$

$$\underline{\kappa} = \frac{1}{2}\ (\underline{k}_A - \underline{k}_B) \tag{D.3}$$

and M is the projection of the deuteron spin. The function is the singlet isospin function as given by Blatt and Weisskopf (1952). The quantities τ and σ represent the isospin and spin, respectively, of the nucleons.

It will prove to be convenient to separate the spin functions from the above deuteron wavefunction. Thus the wavefunction will be written

$$\phi(\underline{\kappa},\sigma_A,\sigma_B,M) = \Sigma_\ell\ \phi_\ell^M\ (\underline{\kappa})\ \chi_{1\ell}(\sigma_A,\sigma_B) \tag{D.4}$$

where $\chi_{1\ell}$ are the triplet spin functions as given by Blatt and Weisskopf(1952) and

$$\phi_\ell^M\ (\underline{\kappa}) = U\ (\kappa)\ Y_{oo}\ (\hat{\kappa})\ \delta_{M\ell} + W\ (\kappa)\ Y_{2,M-\ell}\ (\hat{\kappa})\ C_{2\ \ 1\ 1}^{M-\ell\,|\,\ell\ M} \tag{D.5}$$

The functions $Y_{mn}(\hat{k})$ are spherical harmonics of the unit vector $\hat{k}$ and the coefficient in the last term is the conventional Clebsch-Gordan coefficient. The functions U and W are the S and D state momentum space wavefunctions used by McMillan and Landau (1974).

D.2 Single-Scattering

In section 4.1 the pion-deuteron single-scattering T-matrix is written

$$T^{(o)}_{\mathcal{D}'m,\mathcal{D}k} = \sum_{A,B,C} \langle \mathcal{D}_o' | AB) \; \langle A | [mV] | Ck \rangle \; (CB | \mathcal{D}_o \rangle \tag{D.6}$$

Delta functions of momentum can be extracted from the pion-deuteron and the pion-nucleon T-matrices. By explicitly writing the above summation as a product of sums over spin and isospin and an integral over momentum, the above equation can be written

$$T^{(0)}(\mathcal{D}'m,\mathcal{D}k) \; \delta\,(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k}_{\mathcal{D}}-\underline{k}) = \sum_{\tau_A,\tau_B,\tau_C=-\frac{1}{2}}^{+\frac{1}{2}} \; \sum_{\sigma_A,\sigma_B,\sigma_C=-\frac{1}{2}}^{+\frac{1}{2}} \int d^3k_A d^3k_B d^3k_C \; \langle \mathcal{D}_o' | AB) T(Am,Ck)$$

$$\times \; (CB | \mathcal{D}_o \rangle \; \delta\,(\underline{k}_A+\underline{m}-\underline{k}_C-\underline{k}) \tag{D.7}$$

Using equations (D.1) and (D.4) and using the resulting delta functions to eliminate two of the integrals, equation (D.7) can be expressed in the laboratory frame as

$$T_L^{(0)}(\underline{k}_{\mathcal{D}'},\underline{m};\ 0,\underline{k};\ M',M)\quad \delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k})$$

$$= \sum_{\ell,\ell'=-1}^{+1} \int d^3\kappa\ \phi_{\ell'}^{M'*}(\underline{\kappa}+\tfrac{1}{2}(\underline{k}-\underline{m}))\ \phi_{\ell}^{M}(\underline{\kappa})\ \delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k})$$

$$\times\ 2 \sum_{\tau_A,\tau_B,\tau_C=-\frac{1}{2}}^{+\frac{1}{2}} \chi_o(\tau_A,\tau_B)\chi_o(\tau_C,\tau_B) \sum_{\sigma_A,\sigma_B,\sigma_C=-\frac{1}{2}}^{+\frac{1}{2}} \chi_{1\ell'}(\sigma_A,\sigma_B)\chi_{1\ell}(\sigma_C,\sigma_B)$$

$$T(\underline{k}_A,\underline{m};\ \underline{k}_C,\underline{k};\ \sigma_A,\sigma_C;\ \tau_A,\tau_m,\tau_C,\tau_k) \qquad \text{(D.8)}$$

where
$$\underline{k}_A = \underline{\kappa} + \underline{k}-\underline{m} \qquad \text{(D.9)}$$
$$\underline{k}_C = \underline{\kappa} \qquad \text{(D.10)}$$

The sums over isospin can now be done and the final result written in the form

$$T_L^{(0)}(\underline{k}_{\mathcal{D}'},\underline{m};\ 0,\underline{k};\ M',M)\ \delta\ (\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k})$$
$$= \sum_{\ell,\ell'=-1}^{+1} \int d^3\kappa\ \phi_{\ell'}^{M'*}(\underline{\kappa}+\tfrac{1}{2}(\underline{k}-\underline{m}))\phi_{\ell}^{M}(\underline{\kappa})\{T_{\ell'\ell}^{\pi p}+T_{\ell'\ell}^{\pi n}\}\delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k}) \qquad \text{(D.11)}$$

where

$$T_{\ell'\ell}^{\pi p} = \sum_{\sigma_A,\sigma_B,\sigma_C=-\frac{1}{2}}^{+\frac{1}{2}} \chi_{1\ell'}(\sigma_A,\sigma_B)\chi_{1\ell}(\sigma_C,\sigma_B)T_L(\underline{k}_A,\underline{m};\underline{k}_C,\underline{k};\sigma_A,\sigma_C;\tau_A=\tau_C=+\tfrac{1}{2},\tau_m=\tau_k) \qquad \text{(D.12)}$$

and where $T_{\ell'\ell}^{\pi n}$ is the same quantity with $\tau_A = \tau_C = -\frac{1}{2}$

It should be noted that the pion-nucleon T-matrix in equation (D.12) refers to the laboratory frame. This can be related to the pion-nucleon T-matrix in the pion-nucleon CM frame using equation (B.7) of McMillan and Landau(1974).

D.3 Double-Scattering

In section 4.2 the pion-deuteron double-scattering T-matrix is written

$$T^{(1)}_{\mathcal{D}'m,\mathcal{D}k} = -\pi i \sum_{A,B,C,D,q} \langle\mathcal{D}_o'|AB)\langle A|[mV]|Cq\rangle\langle B|[qV]|Dk\rangle(CD|\mathcal{D}_o\rangle\delta(E_B+E_q-E_D-E_k) \tag{D.13}$$

Using the same techniques as in section D.2, the above equation can be written

$$T^{(1)}_L\ (\underline{k}_{\mathcal{D}'},\underline{m};\ 0,\underline{k};\ M',M)\quad \delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k}) \tag{D.14}$$
$$= -\pi i \sum_{\ell,\ell'=-1}^{+1} \int d^3\kappa d^3q\ \phi^{M'*}_{\ell'}\ (\underline{\kappa}+q-\tfrac{1}{2}(\underline{k}-\underline{m}))\ \phi^M_\ell(\underline{\kappa})\ T_{\ell'\ell}\ \delta(\underline{k}_{\mathcal{D}'}+\underline{m}-\underline{k})$$

where

$$T_{\ell'\ell} = 2 \sum_{\substack{\tau_A,\tau_B\\ \tau_C,\tau_D=-\frac{1}{2}}}^{+\frac{1}{2}} \chi_o(\tau_A,\tau_B)\chi_o(\tau_C,\tau_D) \sum_{\substack{\sigma_A,\sigma_B\\ \sigma_C,\sigma_D=-\frac{1}{2}}}^{+\frac{1}{2}} \chi_{1\ell'}(\sigma_A,\sigma_B)\ \chi_{1\ell}(\sigma_C,\sigma_D) \tag{D.15}$$
$$\times \sum_{\tau_q=-1}^{+1} T_L(\underline{k}_A,\underline{m};\ \underline{k}_C,\underline{k};\ \sigma_A,\sigma_C;\ \tau_A,\tau_C,\tau_m,\tau_q)\ T_L(\underline{k}_B,\underline{q};\ \underline{k}_D,\underline{k};\ \sigma_B,\sigma_D;\ \tau_B,\tau_D,\tau_q,\tau_k)$$

and

$$\underline{k}_A = \underline{\kappa} + \underline{q} - \underline{m} \tag{D.16}$$
$$\underline{k}_B = -\underline{\kappa} + \underline{k} - \underline{q} \tag{D.17}$$
$$\underline{k}_C = -\underline{k}_D = \underline{\kappa} \tag{D.18}$$

As stated in the previous section, the pion-nucleon T-matrices in the laboratory frame can be related to the T-matrices in the pion-nucleon CM frame using equation (B.7) of McMillan and Landau(1974).