A RE-ASSESSMENT OF HIERARCHICAL COSMOLOGIES

by

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ABSTRACT

The extension of the concepts of Newtonian cosmology to a universe consisting of a hierarchy of metagalaxies is fairly straightforward. However, in general relativistic cosmology, the construction of such a hierarchical universe is a difficult problem. It is the purpose of this work to examine some aspects of hierarchical cosmology in both the Newtonian and general relativistic cases. It is suggested that the metagalaxy may be a black hole or Schwarzschild object, to account for the fact that no objects which could be identified as metagalaxies have been, as yet, observed. Some features of this concept are discussed. Tidal forces exerted on a metagalaxy, due to others distributed around it, are estimated in the Newtonian case. Such tidal forces may or may not be detectable, depending on the distance between metagalaxies. The interior of a metagalaxy is represented by a Friedmann model, with given values of k and Λ. The Friedmann model is matched at the boundary to a Schwarzschild spacetime. The consequences of this and related calculations suggest that in most cases, a metagalaxy may be a black hole for only part of its lifetime, i.e., for other times, it may be optically detectable to an exterior observer.

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INTRODUCTION

The impressive observational evidence about distant galaxies and other distant objects accumulated during the last half century (see e.g. Weinberg 1972) indicates that the ultimate objects accessible to optical- and radio-astronomical methods are confined to within a "metagalaxy" of finite linear dimension $R \sim 10^{10}$ lightyears and of finite mass $M \sim 10^{23}$ solar masses. The very compatibility of these observational data with cosmological models characterized by a finite length R and a finite mass M has persuaded the majority of modern cosmologists to simply identify this "metagalaxy" seen by the astronomers with the "universe" which is the object of their speculations.

Against the background of the development of astronomical distance scales through a sequence of ever-widening horizons, from the triangulations within the solar system carried out by Tycho Brahe, over the first measurement of a stellar parallax by Bessel, to the modern determination of the metagalactic parameter R by Hubble's observation of galactic red-shifts, such definitive categorization of the metagalaxy as the "universe", in the sense of the "all and everything", seems to exclude, in an apparent excess of confidence in currently held views, any further widening of the cosmic horizon in the future.

Historical precedents counsel taking a more cautionary approach toward the cosmological problem, by considering the possibility of the metagalaxy being but an element in a larger hierarchy of things, and reserving the label "universe" for that hierarchy.

The idea of a hierarchical cosmology is not new. In the 18th

century Lambert (1761) conceived the universe as a planetary hierarchy. In the 20th century the concept of the universe as a hierarchy of clusters was promulgated by Charlier (1908, 1922), who considered each cluster of order n as an element in a cluster of order n+1 with no restriction on the value of n. Taking galaxies as elements of order 0 in the hierarchy, there is definite observational evidence for second-order clustering (i.e. for the existence of clusters of clusters of galaxies), and even some indication of third-order clustering within the metagalaxy (see de Vaucouleurs 1970).

These considerations lead one to consider the possibility of the metagalaxy being a cluster of galaxies of order n (n finite), which is but an element in an N-th order cluster of metagalaxies, where N is not necessarily finite, and to view the "universe" as this entire hierarchy of metagalaxies.

If that view is accepted, one is immediately confronted with the question why the astronomers have not reported, to date, objects that could be identified as metagalaxies, other than the one within which we happen to find ourselves, seen from the outside.

An obvious way out of this difficulty is provided if all metagalaxies are black holes. In that case they may elude detection from the outside by the conventional means of astronomy, and an observer within a metagalaxy might wrongly conclude that clustering terminates at the finite order n.

The purpose of this work is

>> to examine the compatibility of the observational data about the

metagalaxy with the hypothesis that it is a black hole seen from the inside,

>> to assess the merits of a hierarchical cosmology with black holes as elements, and

>> to speculate upon the possibility of detecting the presence of other black hole metagalaxies from the outside by whatever means conceivable.

1. THE METAGALAXY AS A NEWTONIAN BLACK HOLE

The term "black hole" has been used in the cosmological literature as a label for a hypothetical region of space, containing gravitationally collapsed matter, from which no light, matter, or signals of any kind can escape (see e.g. Penrose 1972). Its existence was originally predicted by Oppenheimer and Snyder (1939) using the general theory of relativity. However, it is not necessary to invoke general relativity for the concept of a black hole to be meaningful. Something similar to a black hole was discussed as early as the 18th century by Laplace, who suggested that, according to Newtonian theory, a sufficiently massive and compact object would have an escape velocity greater than the speed of light, and thus, would be invisible at large distances.

More specifically, one can define a "Newtonian black hole" as a spherical object of mass M and radius R whose gravitational field (in the Newtonian sense) is sufficient to capture a photon into a circular orbit near its surface. The condition for this orbital capture is that the gravitational acceleration of the photon must be as large as, or larger than its centrifugal acceleration, i.e.,

[1-1] $GM/R^2 \geq c^2/R$

where c is the speed of light, and G is the gravitational constant. This leads to

[1-2] $R/M \leq G/c^2 \simeq 7.4 \times 10^{-28}$ m/kg

for a Newtonian black hole. It is also possible to define a Newtonian black hole as a massive, compact, spherical object with an escape velocity greater than c, in the sense of Laplace; however, this would

only mean replacing G/c^2 in [1-2] by $2G/c^2$, and for the purposes of this section, this change is insignificant. For the sake of convenience the former definition will be used.

Assuming constant density ρ, one can use the equation

[1-3] $M = 4\pi\rho R^3/3$

to express [1-2] equivalently in terms of R and ρ, or M and ρ, and obtain

[1-4] $R^2\rho \geq 3c^2/4\pi G \simeq 3.2 \times 10^{26}$ kg/m

[1-5] $M^2\rho \geq 3c^6/4\pi G^3 \simeq 5.9 \times 10^{80}$ kg^3/m^3 .

One can also write [1-2], [1-4] and [1-5] as follows,

[1-2a] $R \leq GM/c^2$

[1-4a] $R \geq (c/2)(3/\pi G\rho)^{1/2}$

[1-5a] $M \geq (c^3/2)(3/\pi G^3\rho)^{1/2}$.

In order to see what kind of objects can be thought of as Newtonian black holes, one can construct a radius vs. mass diagram. Fig. 1-1 shows such a diagram, in which several common objects are plotted. The straight line represents the equation

[1-6] $R = GM/c^2$

or, on the graph,

[1-7] $\log R = \log(GM/c^2) = -27.1 + \log M$.

The mass of the metagalaxy was obtained from [1-3] where $\rho \simeq 4 \times 10^{-26}$ kg/m^3 and $R \simeq 10^{10}$ light years $= 9.46 \times 10^{25}$ m. The mass of the metagalaxy is then $M \simeq 1.4 \times 10^{53}$ kg. However, these values for ρ and R are only crude order-of-magnitude estimates and the location of the point on the R-M diagram representing the metagalaxy can vary according-

ly. In fact, if high-order galaxy clustering is prevalent in the metagalaxy, one may suspect that the value for ρ may be meaningless, since for an accurate value of the mean density ρ , one must consider a "large enough" volume of space such that the matter in this volume is roughly uniformly distributed, i.e., the volume must be large enough to contain at least several clusters of the highest order of clustering (de Vaucouleurs 1970). Any estimate of ρ using a smaller volume would be meaningless because it neglects the clustering effect.

Ignoring this for the moment, one sees from Fig. 1-1 that one may consistently think of the metagalaxy as a Newtonian black hole. The other objects deviate from the $R = GM/c^2$ line. Given the mass of an object, the radius would have to be much smaller for that object to be a Newtonian black hole.

For the sake of completeness, it should also be mentioned that one should be wary in extrapolating the $R = GM/c^2$ line in Fig. 1-1 to regions of small R and M, where the possible quantum effects of gravitation may play a role. This is indicated on the graph by a dashed line. The values for R and M which separate the quantum region from the classical region are not well-defined but the so-called Planck length ℓ_0 , and the corresponding mass m_0 ,

$$\ell_0 \equiv (G\hbar/c^3)^{1/2} \simeq 1.6 \times 10^{-35} \text{ m}$$

$$m_0 \equiv \ell_0 c^2/G = (\hbar c/G)^{1/2} \simeq 2.2 \times 10^{-8} \text{ kg}$$

provide plausible estimates for these values. Further discussion of the quantum region would be outside the scope of this work.

For the electron in the R-M diagram, the Compton wavelength,

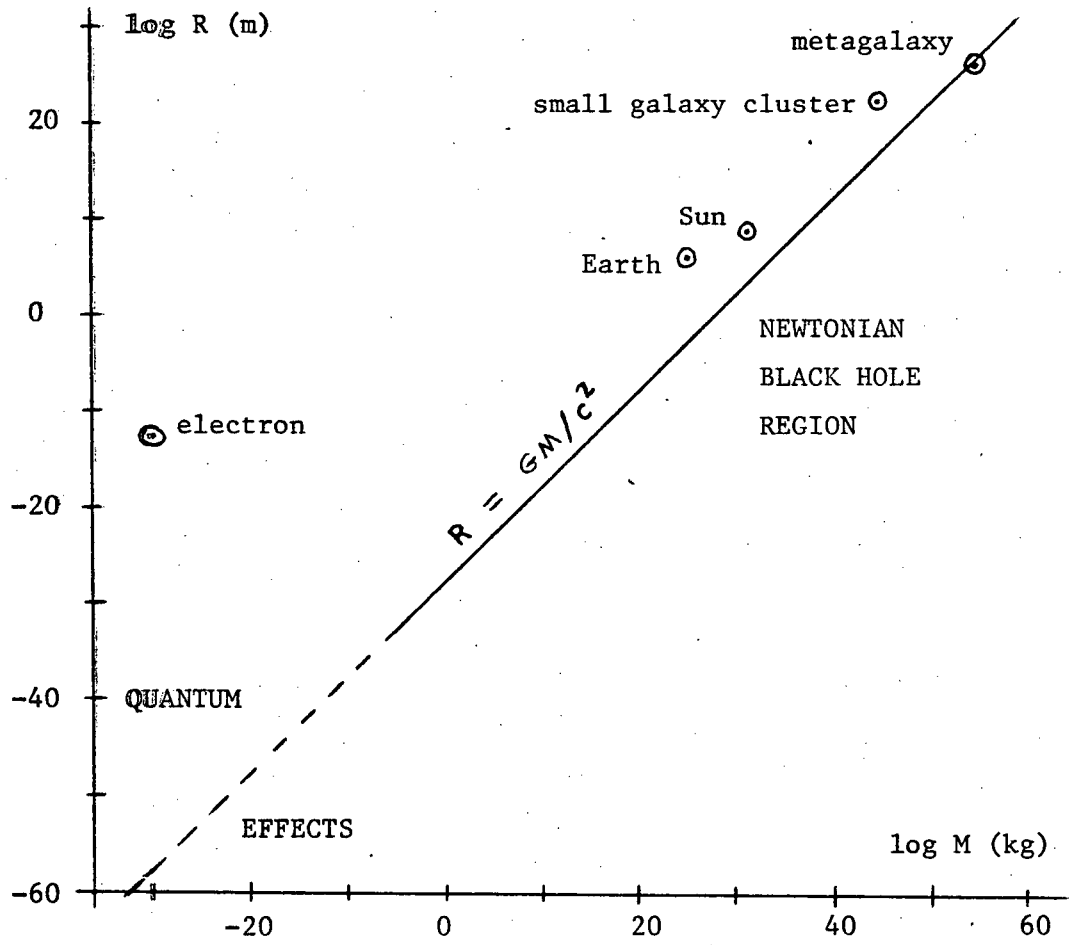

FIG.1-1 . Radius-Mass Diagram .
The Newtonian Black Hole region is defined by $R \leq \frac{GM}{c^2}$.

$\hbar/mc \simeq 3.9 \times 10^{-13}$ m was used as a reasonable estimate of the radius.

Fig.1-2 is merely a blow-up of the upper right-hand section of Fig.1-1, with more objects shown. The points are numbered and the objects they represent are listed in Table 1-1 below. Most of the data for Fig.1-1 and Fig.1-2 was obtained from a recent paper by de Vaucouleurs (1970).

One sees from Fig.1-2 that neutron stars and white dwarfs, whose densities are very high, are situated near the black hole region, as expected.

TABLE 1-1 Mass-Radius Data

Objects	log M (kg)	log R (m)
1. Earth	24.78	6.81
2. Neutron star	30.16	3.93
3. Neutron star	29.54	5.44
4. White dwarf (L930-80)	30.45	6.3
5. White dwarf (vM2)	29.90	7.05
6. Main Sequence star (Sun)	30.30	8.84
7. Supergiant star (M2)	31.7	11.75
8. Compact dwarf elliptical galaxy (N4486-B)	40.4	18.5
9. Giant elliptical galaxy (N4486)	42.5	20.4
10. Dense group of ellipticals (Virgo E, core, or Fornax I)	43.5	21.7
11. Small cluster of galaxies (Virgo E)	44.2	22.3

(cont.)

TABLE 1-1 (cont.)

Objects	log M (kg)	log R (m)
12. Large cluster of ellipticals (Coma)	45.3	22.6
13. Second-order cluster of galaxies (Local)	45.7	23.5
14. Metagalaxy	53.1	26.0

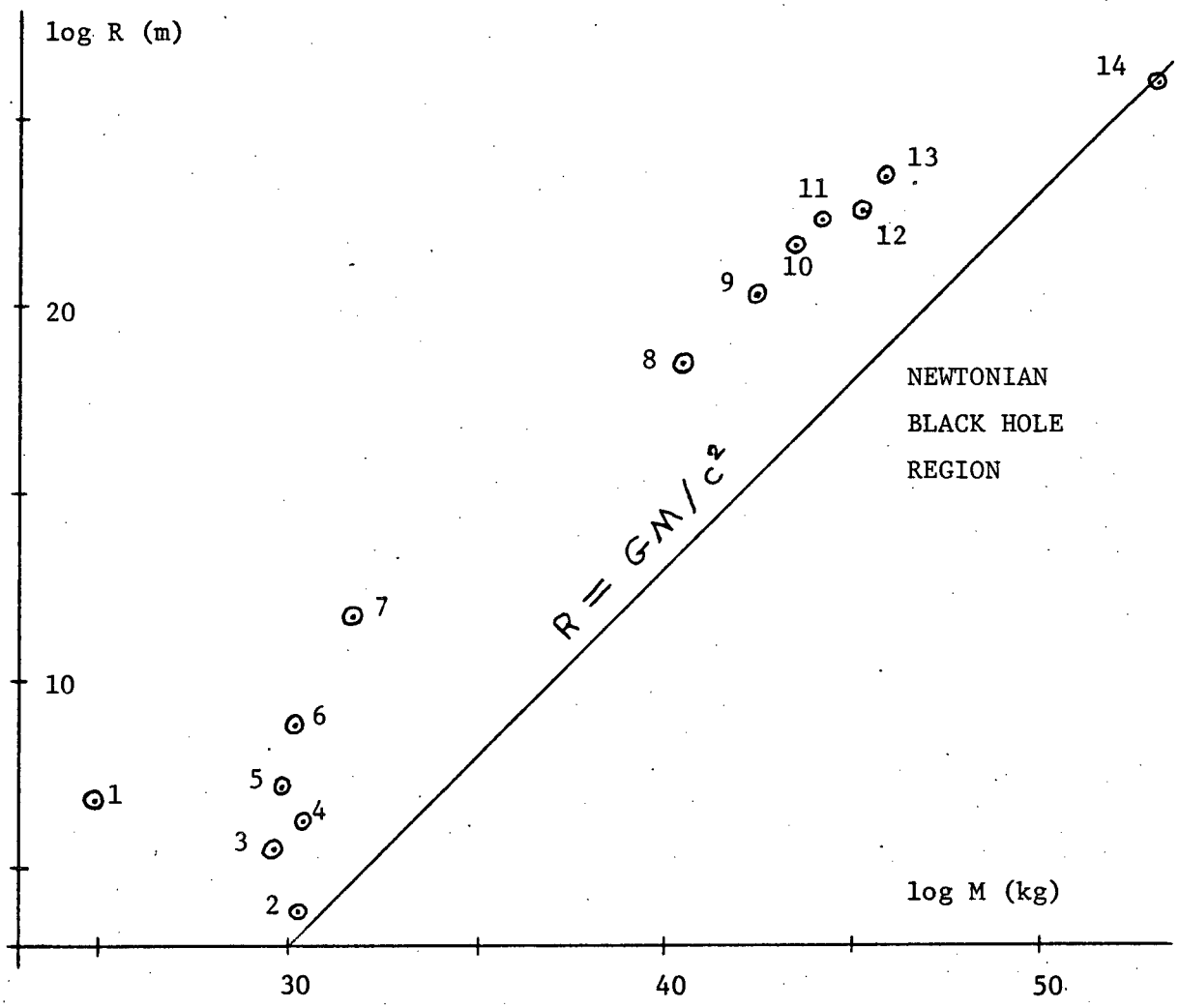

FIG.1-2 . Blow-up of upper right-hand section of Fig.1-1 ($\log M \geq 23$, $\log R \geq 3$). The objects are listed according to number, in Table 1-1 .

2. SALIENT FEATURES OF THE WORLD MODEL OF LAMBERT AND CHARLIER

In the past, the concept of a hierarchical cosmology was considered by several investigators (e.g., Lambert 1761, Charlier 1908, 1922). Lambert proposed that, just as Jupiter's moons are satellites of Jupiter, and Jupiter is a satellite of the sun, so is the sun a satellite of some other center of force, and this center of force is a satellite of another center of force, and so on. Charlier's model of the universe was basically a revision of the model proposed by Lambert, the change being necessary, because by the time of Charlier, Lambert's views on the structure of the universe were known to be in error. The model due to Charlier is a hierarchy of systems consisting of galaxies (basic elements, or zeroth order clusters C_0), clusters of galaxies (or first order clusters C_1), clusters of clusters of galaxies (or second order clusters C_2), and so on without limit. Actually, Charlier's clusters C_0 were stars not galaxies, but here galaxies are assumed to be the basic elements, to agree with current trends. Each cluster, of any order, is assumed to be spherical, for mathematical simplicity. More quantitatively, one assumes that M_0 and R_0 are the mass and radius of a galaxy, and that

N_1 galaxies form a 1st order cluster C_1 of mass M_1 and radius R_1,
N_2 clusters C_1 form a 2nd order cluster C_2 of mass M_2 and radius R_2,
N_3 clusters C_2 form a 3rd order cluster C_3 of mass M_3 and radius R_3,

and so on, with the order of clustering unbounded (the notation here is slightly different from that of Charlier). The elements C_{i-1} of each cluster C_i ($i = 1,2,3,\ldots$) are assumed to be the same size, and to have

the same mass, and to be evenly distributed within C_i . One notes that

[2-1] $M_i = N_i M_{i-1} = N_i N_{i-1} N_{i-2} \cdots N_2 N_1 M_0$.

The main objections to an infinite universe are those due to Olbers (1823) and Seeliger (1895). Seeliger's objection is that an application of Newton's law of gravity to an infinite universe would lead to difficulties and contradictions since the total mass is infinite. Using Charlier's hierarchical model, one can remove Seeliger's objection, as Charlier has done, by showing that the total gravitational attraction of the infinite universe on some object within it is finite, if the radii R_i are chosen properly (see [2-4] below). One can consider a unit mass lying on the edge of a galaxy C_0 which is on the edge of a cluster C_1, which in turn is on the edge of a cluster C_2, and so on. The total attraction of the universe on this unit mass is given approximately by

[2-2] $F = (GM_0/R_0^2) + (GM_1/R_1^2) + (GM_2/R_2^2) + \ldots$.

This is obviously an overestimate of the actual force but if F is finite, then so is the actual force. Charlier states that the condition for the convergence of the series [2-2] is that

[2-3] $GM_i/R_i^2 < GM_{i-1}/R_{i-1}^2$.

Using [2-1], this leads to

[2-4] $R_i/R_{i-1} > N_i^{1/2}$.

The condition [2-3] is actually not a sufficient condition for convergence in a strict mathematical sense (e.g., the <u>divergent</u> harmonic series 1 + 1/2 + 1/3 + ... also obeys this condition), however it is possible to establish a slightly more rigorous derivation of [2-4] in

the following way: the ratio test for the convergence of an infinite series $\sum_i a_i$ states that the series will converge (absolutely) if

$$\lim_{i\to\infty} |a_i/a_{i-1}| < 1 \quad .$$

This condition will obviously be satisfied if

[2-5] $|a_i/a_{i-1}| \leq 1-2\varepsilon$,

for all i, where ε is an arbitrarily small, but positive real number, i.e., $0 < \varepsilon << 1$. Using this argument for the series [2-2], one gets

$$\begin{aligned} R_i/R_{i-1} &\geq [N_i/(1-2\varepsilon)]^{1/2} \\ &\simeq N_i^{1/2}(1+\varepsilon) \\ &> N_i^{1/2} \quad . \end{aligned}$$

If ε is very small, this is equivalent to [2-4] for all practical purposes. In this way, one sees that the total attraction of the infinite universe on a unit mass is finite if the inequality [2-4] is satisfied. This essentially eliminates Seeliger's objection.

The objection due to Olbers is expressed by the well-known Olbers Paradox, which may be explained as follows: if it is assumed that stars are uniformly distributed throughout an infinite universe, i.e. their average number density is constant throughout space and time, and that the average luminosity of each star is constant throughout space and time, and that the universe is static (no systematic large-scale motions), then it is possible to show that the total luminosity at any point in the universe should be infinite, or in more familiar terms, the night sky should be infinitely bright. More specifically, the

average apparent luminosity of a star at a distance r from some arbitrary point P is ℓ/r^2 ($4\pi\ell$ is the average absolute luminosity, i.e. it is a constant for all stars). The number of stars in the shell between distances r and r+dr is

$$4\pi n r^2 dr$$

where n is the average number density of stars. Hence the total luminosity at P is

$$\int_0^\infty (\ell/r^2) 4\pi n r^2 dr = 4\pi n \ell \int_0^\infty dr = \infty \quad .$$

It is the very distant stars that are responsible for the divergence of the integral, and to avoid this paradox of infinite energy density, Olbers postulated the existence of an interstellar medium which absorbs their light. This was unsatisfactory because in an infinite universe, the energy absorbed by the medium would heat the medium until it emitted as much energy as it absorbed, and hence could not reduce the average energy density of radiation (Weinberg 1972, Chap. 16; Bondi 1960, Chap. III). Other solutions of the paradox have been proposed, such as assuming that space is non-Euclidean, or that the universe is finite, or young, or expanding. One of the most recent proposals is that "the night sky is dark because the time required for the radiation field to reach thermodynamic equilibrium is large compared with all other time scales of interest" (Harrison 1974).

It is shown in Charlier's paper that the Olbers Paradox is also eliminated in the hierarchical model if [2-4] is satisfied. Charlier's argument (with the present notation) basically proceeds as follows: assume that a cluster C_0^* is at the center of a cluster C_1^*, and that

C_1^* is at the center of C_2^*, and so on. The asterisk distinguishes the central cluster C_i^* from its neighbours C_i. Let r be the radial distance from C_0^*. Let

ℓ_1/r_1^2 = average apparent luminosity of a cluster C_0 (within C_1^*) at the distance $r = r_1$

ℓ_2/r_2^2 = average apparent luminosity of a cluster C_1 (within C_2^*) at the distance $r = r_2$, etc.

where r_i is of the order of the distance between two elements of C_i^*, and ℓ_i is a constant for all elements of C_i^*. Also, let

L_i = total luminosity of C_i^* as seen from its center.

The number of clusters C_0 in the shell between r and r+dr (within C_1^*) is

$$(N_1/\tfrac{4}{3}\pi R_1^3)(4\pi r^2 dr) = (3N_1 r^2/R_1^3)dr \quad .$$

The average apparent luminosity of a cluster C_0 (within C_1^*) at a distance r is ℓ_1/r^2, hence,

$$L_1 = \int_0^{R_1} (\ell_1/r^2)(3N_1 r^2/R_1^3)dr = 3N_1\ell_1/R_1^2 \quad .$$

In general, one has

$$L_i = 3N_i\ell_i/R_i^2 \quad , \quad \text{hence,}$$

[2-6] $$L_i/L_{i-1} = (N_i/N_{i-1})(R_{i-1}/R_i)^2(\ell_i/\ell_{i-1}) \quad .$$

One calculates ℓ_i/ℓ_{i-1} as follows. The average apparent luminosity of a cluster C_0 (within C_1^*) at a distance $r = r_2$ from C_0^* is ℓ_1/r_2^2. Hence the average apparent luminosity of C_1^* at $r = r_2$ is

$$N_1\ell_1/r_2^2 = \ell_2/r_2^2 \quad \text{i.e.,}$$

$\ell_2 = N_1 \ell_1$, and in general,

$\ell_i = N_{i-1} \ell_{i-1}$.

Equation [2-6] then gives

$L_i/L_{i-1} = N_i (R_{i-1}/R_i)^2$.

The total luminosity of the universe at r = 0 is

$L = L_1 + L_2 + L_3 + \ldots$.

This is an overestimate, as in [2-2]. L must converge to avoid the Olbers Paradox. Applying the "ε-proof" above yields the convergence condition for L,

[2-4] $R_i/R_{i-1} > N_i^{1/2}$.

The calculation can be done without the above simplifying assumptions (i.e. that C_0^* is at the center of C_1^*, and C_1^* is at the center of C_2^*, etc.). The same result, [2-4], holds (see Charlier 1922).

Another interesting feature of Charlier's infinite hierarchical universe is that the average mass density ρ_i of a cluster C_i vanishes in the limit $i \to \infty$, i.e.

[2-7] $\lim_{i\to\infty} \rho_i = 0$.

This can be shown in the following way. The expression for ρ_i in terms of M_i and R_i is

$\rho_i = M_i / \frac{4}{3}\pi R_i^3$.

Taking the limit $i \to \infty$ yields,

$$\lim_{i\to\infty} \rho_i = (3/4\pi) \lim_{i\to\infty} M_i/R_i^3$$

$$= (3/4\pi)(\lim_{i\to\infty} M_i/R_i^2)(\lim_{i\to\infty} 1/R_i) \quad .$$

Now, using a mathematical theorem which states that $\lim_{i\to\infty} a_i = 0$ if the infinite series $\sum_i a_i$ converges, it is seen that

[2-8] $\lim_{i\to\infty} M_i/R_i^2 = 0$

since the series for F in [2-2] converges (assuming [2-4] is satisfied). Also, since $\lim_{i\to\infty} 1/R_i = 0$, [2-7] is established (one may question whether the number $q \equiv \lim_{i\to\infty} 1/R_i$ is perhaps a positive real number, instead of zero. This would not alter the result [2-7], but for the sake of completeness, one can argue that q is actually zero. To do this, first suppose that q is a positive finite real number. Then $\lim_{i\to\infty} R_i^2$ is also a positive finite real number. If [2-2] converges, then [2-8] is true, hence one must have $\lim_{i\to\infty} M_i = 0$, which is a contradiction. Therefore q must be zero. It can also be seen intuitively that a finite value for $\lim_{i\to\infty} R_i^2$ would essentially mean that the order of clustering is bounded above).

One sees from the above arguments that Charlier's hierarchical cosmology is a plausible model of the universe. It may be argued that this model is oversimplified and unrealistic because of its discrete mathematical nature and because of the assumptions that all clusters are spherical, and that clusters of the same order have the same size and mass. Observation clearly indicates that these assumptions are not true. Galaxies are certainly not spherical, but elliptical in shape, and they vary in size and mass. Also, more convincing solutions to the Olbers Paradox have been provided by modern theoretical and experimental investigations, such as the discovery of the universal red shift and the development of finite world models, as mentioned by de Vaucouleurs

(1970). However, these findings do not disprove the suggestion that the universe may have a hierarchical structure. If one is concerned first mainly with investigating the basic properties of a hierarchical universe, then the simplified nature of the above-mentioned assumptions does not present a serious difficulty. The oversimplification is obviously a mathematical convenience, but it clearly demonstrates some of the basic concepts and ideas involved with a hierarchical structure, and these may indicate, to some extent, the way to a more complex and realistic model, perhaps of a quasi-continuous nature. In any event, cosmological models consisting of a hierarchy of systems should be seriously considered in any investigations of the structure of the universe.

3. HIERARCHIES WITH BLACK HOLES AS ELEMENTS

As mentioned previously, the metagalaxy can be consistently thought of as a Newtonian black hole. Combined with the notions in the preceding sections of a hierarchical cosmic structure, this leads to the concept of a hierarchical universe with black holes as elements. It is interesting to investigate the conditions that must be satisfied by the elements of Charlier's hierarchical universe in order for Charlier's model to be compatible with the notion that the elements or clusters C_i are Newtonian black holes.

Let each cluster C_i be a Newtonian black hole. This means that

[3-1] $R_i/M_i \leq G/c^2$, $i = 0,1,2, \ldots$.

For instance, in the actual case of the universe, this means that one could take our metagalaxy to be a basic element or cluster C_0, and assume that the other clusters C_0, C_1, C_2, etc., are Newtonian black holes.

Multiplying condition [3-1] by M_i/R_{i-1} gives

[3-2] $R_i/R_{i-1} \leq (G/c^2)(M_i/R_{i-1})$

$= (GM_{i-1}/c^2R_{i-1})N_i$, using [2-1]

$= \alpha_{i-1}N_i$

where $\alpha_i \equiv GM_i/c^2R_i$

[3-3] and, from [3-1], $\alpha_i \geq 1$, $i = 0,1,2, \ldots$.

Condition [3-2] must be satisfied by the radii of the clusters C_i in order for each C_i to be a Newtonian black hole.

In the previous section, a different inequality involving R_i/R_{i-1} was derived, namely

[2-4] $R_i/R_{i-1} > N_i^{1/2}$.

This was shown to be a necessary condition in Charlier's infinite hierarchical model for the elimination of the Olbers Paradox and Seeliger's objection. The Olbers Paradox does not apply to a hierarchical universe where each cluster C_i is a Newtonian black hole, however, Seeliger's criticism is still valid, if one assumes that Newtonian black holes, which are simply compact clumps of matter in the Newtonian sense, are subject to static gravitational forces. Hence [2-4] may be invoked to eliminate Seeliger's objection. This then leads to the inequality,

[3-4] $N_i^{1/2} < R_i/R_{i-1} \leq \alpha_{i-1}N_i$.

Due to [3-3], this is a consistent inequality, i.e.

$$N_i^{1/2} < \alpha_{i-1}N_i \quad .$$

Hence, a Charlier-type hierarchical universe, with each C_i being a Newtonian black hole, must satisfy [3-4].

4. OBSERVABLE EFFECTS OF A HIERARCHY ON THE METAGALAXY

The reason for speculating, in the last section, upon a hierarchical universe with black holes as elements is to provide a possible explanation of the fact that no astronomical objects that could be identified with metagalaxies have been detected as yet (other than our own metagalaxy). Elements of a hierarchical universe that are black holes could be undetectable to each other by conventional optical means, thereby hampering the establishment of a complete picture of the universe. However, there still remains the possibility of detection through the effects of static gravitational fields. For instance, the detection of tidal forces on our metagalaxy would suggest the existence of metagalaxies outside our own.

In order to calculate an order-of-magnitude estimate of the tidal forces on some metagalaxy due to others distributed around it, it will be assumed that the distribution of metagalaxies is such that each one is located at the center of a sphere and that the spheres are close-packed. For simplicity it will be assumed that all spheres are the same size, and that all metagalaxies have the same radius R and mass M. There are two ways to pack equivalent spheres as close as possible. One way leads to the hexagonal close-packed lattice structure and the other way leads to the face-centered cubic (cubic close-packed) lattice structure. Both structures are equally close-packed (see e.g. Kittel 1971, P.28). The face-centered cubic (fcc) structure will be used in the following calculation. The conventional unit cell for the fcc lattice is shown in Fig.4-1. Metagalaxies are situated at the fcc lattice

points. Each metagalaxy has twelve nearest neighbours, each at a distance $a/\sqrt{2}$. Eight of the nearest neighbours to lattice point A are shown (shaded). The remaining four are on the sides of the cell directly above the one shown.

The gravitational field (i.e. gravitational acceleration on a unit mass) at some point P in space near the metagalaxy A due to the surrounding distribution of metagalaxies will be calculated. Only the nearest neighbours to A will first be considered. The difference between the gravitational acceleration at some point on the surface of the metagalaxy A due to the surrounding distribution, and that at the center of A, due to the surrounding distribution is a measure of the tidal effect at that point on A. The structure of the metagalaxy A and its nearest neighbours is depicted in Fig.4-2.

The gravitational potential at P due to a mass M at some point Q (see Fig.4-3) is given by

[4-1] $\mathcal{G}_{PQ} = -GM/\ell$.

The points P and Q have spherical coordinates (r,θ,ϕ) and $(\bar{r},\bar{\theta},\bar{\phi})$ respectively. With the help of Fig.4-3, ℓ can easily be expressed in terms of $r,\theta,\phi,\bar{r},\bar{\theta}$ and $\bar{\phi}$:

[4-2] $\ell^2 = \bar{r}^2 + r^2 - 2\bar{r}r[\cos\bar{\theta}\cos\theta + \sin\bar{\theta}\sin\theta\cos(\bar{\phi}-\phi)]$.

The total gravitational potential at P due to the twelve masses M in Fig.4-2 is

[4-3] $\mathcal{G} = -GM\sum_{i=1}^{12} (1/\ell_i)$

where ℓ_i is the distance from P to the point i in Fig.4-2. Since the nearest neighbour distance $\bar{r}_i$ is the same for all i, it will be denoted

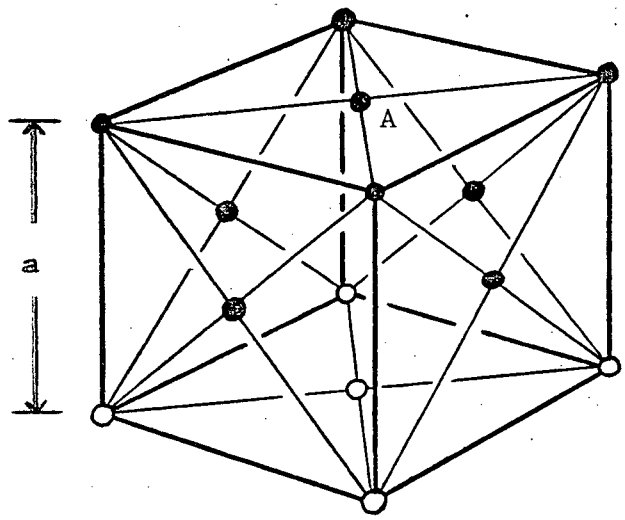

FIG.4-1 . Conventional unit cell for fcc lattice.

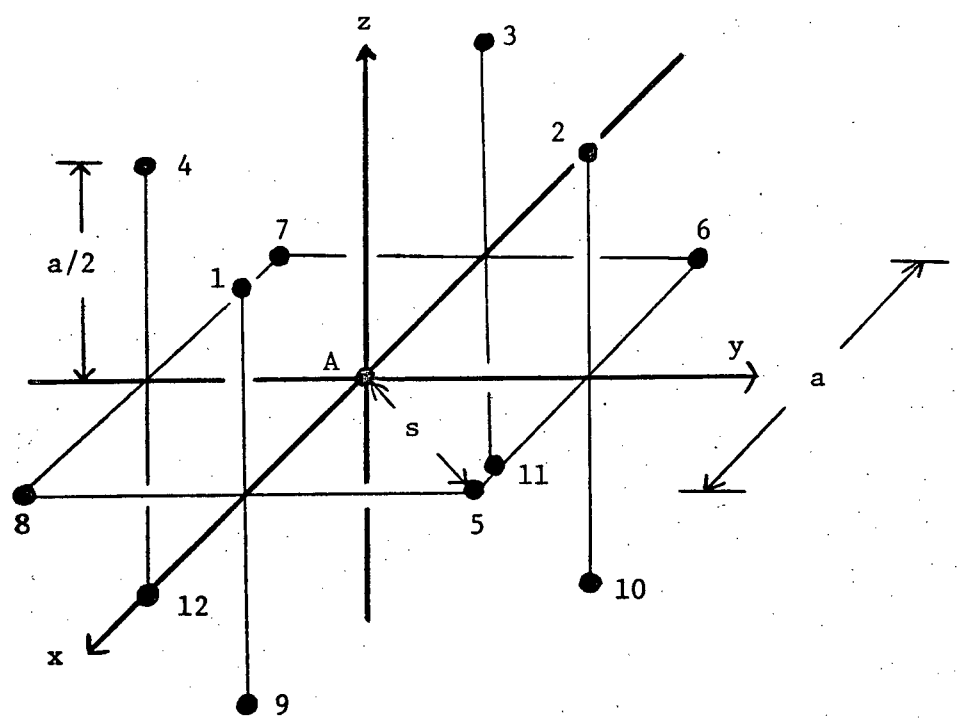

FIG.4-2 . Metagalaxy A and nearest neighbours, each at a distance $a/\sqrt{2} \equiv s$.

by $\bar{r}_i \equiv s$. From [4-2], $(1/\ell_i)$ can be written as

[4-4] $$(1/\ell_i) = \frac{1}{s}(1 - 2q_i w + w^2)^{-1/2} = \frac{1}{s}\sum_{n=0}^{\infty} P_n(q_i)w^n$$

[4-5] where $w \equiv r/\bar{r}_i = r/s$

[4-6] and $q_i \equiv \cos\bar{\theta}_i\cos\theta + \sin\bar{\theta}_i\sin\theta\cos(\bar{\phi}_i-\phi)$

and P_n is the nth order Legendre polynomial. Hence [4-3] becomes

[4-7] $$\mathcal{G} = -\frac{GM}{s}\sum_{n=0}^{\infty}\left[\sum_{i=1}^{12} P_n(q_i)\right]w^n \quad .$$

The coordinates $\bar{\theta}_i$, $\bar{\phi}_i$ for the twelve metagalaxies in Fig.4-2 are given in Table 4-1.

TABLE 4-1. Coordinates of nearest neighbours

i	1	2	3	4	5	6	7	8	9	10	11	12
$\bar{\theta}_i$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	$\frac{3\pi}{4}$
$\bar{\phi}_i$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{5\pi}{4}$	$\frac{7\pi}{4}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

Using Table 4-1 and equation [4-6], one has

[4-8]
$$\begin{aligned}
q_1 &= (\cos\theta + \sin\theta\cos\phi)/\sqrt{2} = -q_{11}\\
q_2 &= (\cos\theta + \sin\theta\sin\phi)/\sqrt{2} = -q_{12}\\
q_3 &= (\cos\theta - \sin\theta\cos\phi)/\sqrt{2} = -q_9\\
q_4 &= (\cos\theta - \sin\theta\sin\phi)/\sqrt{2} = -q_{10}\\
q_5 &= \sin\theta(\sin\phi + \cos\phi)/\sqrt{2} = -q_7\\
q_6 &= \sin\theta(\sin\phi - \cos\phi)/\sqrt{2} = -q_8 \quad .
\end{aligned}$$

This means that

[4-9] $$\sum_{i=1}^{12} q_i^{2m+1} = 0 \qquad \text{and}$$

$$[4\text{-}10] \quad \sum_{i=1}^{12} q_i^{2m} = 2 \sum_{i=1}^{6} q_i^{2m}$$

where m is an integer. Hence

$$[4\text{-}11] \quad \sum_{i=1}^{12} P_n(q_i) = \begin{Bmatrix} 0 & n \text{ odd} \\ 2 \sum_{i=1}^{6} P_n(q_i) & n \text{ even} \end{Bmatrix} .$$

since $P_n(q_i)$ is an odd or even polynomial in q_i if n is odd or even respectively. Therefore, [4-7] becomes

$$[4\text{-}12] \quad \mathcal{G} = -\frac{2GM}{s} \sum_{n=0}^{\infty} \left[\sum_{i=1}^{6} P_{2n}(q_i) \right] w^{2n} .$$

From [4-8], one can easily calculate

$$[4\text{-}13] \quad \sum_{i=1}^{6} q_i^2 = 2 \quad \text{and}$$

$$\sum_{i=1}^{6} q_i^4 = 3\cos^2\theta - 2\cos^4\theta + \sin^4\theta(3\cos^2\phi - 2\cos^4\phi + \sin^4\phi)$$
$$\equiv f(\theta,\phi) .$$

Using [4-13], one has then

$$[4\text{-}14] \quad \sum_{i=1}^{6} P_0(q_i) = \sum_{i=1}^{6} 1 = 6$$

$$\sum_{i=1}^{6} P_2(q_i) = (1/2)\sum_{i=1}^{6} (3q_i^2-1) = 0$$

$$\sum_{i=1}^{6} P_4(q_i) = \frac{1}{8}\sum_{i=1}^{6} (35q_i^4-30q_i^2+3) = \frac{1}{8}[35f(\theta,\phi) - 42] .$$

Hence [4-12] is

$$[4\text{-}15] \quad \mathcal{G} = -\frac{2GM}{s}[6 + \frac{1}{8}(35f - 42)w^4 + O(w^6)] .$$

The gravitational field at P due to the twelve surrounding masses is

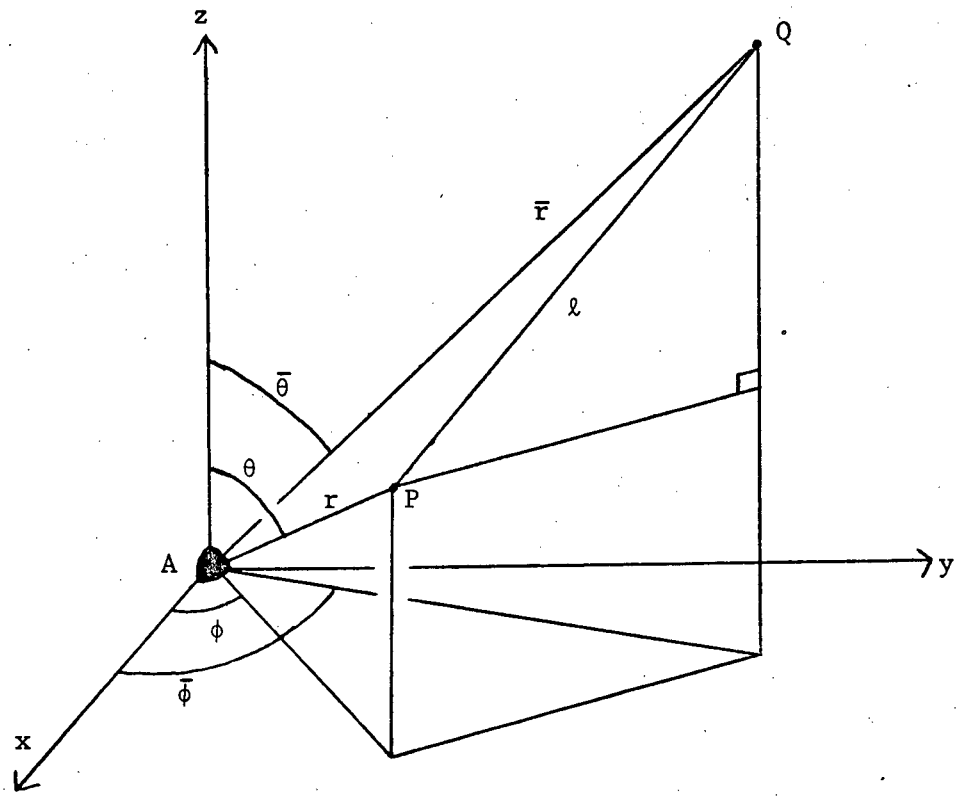

FIG.4-3 . For use in potential calculation.

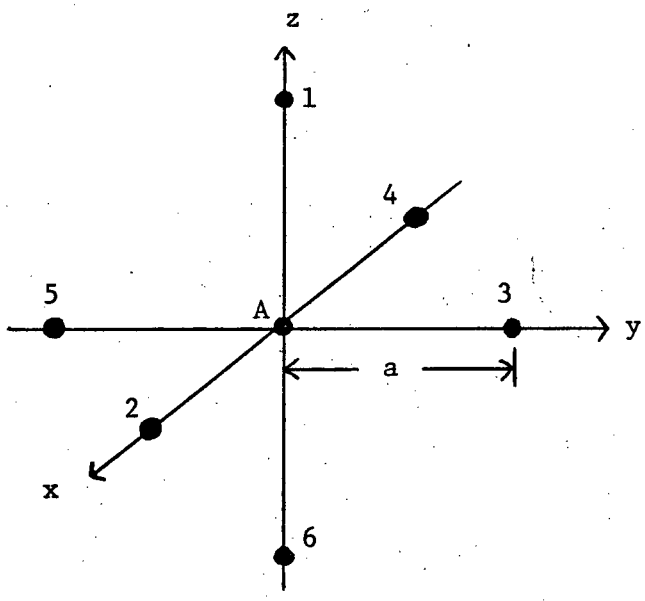

FIG.4-4 . Metagalaxy A and second nearest neighbours, each at a distance $a = \sqrt{2}\,s$.

[4-16] $\vec{\mathcal{F}} = -\nabla\mathcal{G} = -\frac{\partial\mathcal{G}}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial\mathcal{G}}{\partial\theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial\mathcal{G}}{\partial\phi}\hat{\phi}$

where $\hat{r},\hat{\theta},\hat{\phi}$ are unit vectors in the r, θ and ϕ directions. This gives

[4-17] $\mathcal{F}_r = -\frac{\partial\mathcal{G}}{\partial r} = \frac{GMr^3}{s^5}(35f-42)$

[4-18] $\mathcal{F}_\theta = -\frac{1}{r}\frac{\partial\mathcal{G}}{\partial\theta} = \frac{35}{4}\left(\frac{GMr^3}{s^5}\right)\frac{\partial f}{\partial\theta}$

[4-19] $\mathcal{F}_\phi = -\frac{1}{r\sin\theta}\frac{\partial\mathcal{G}}{\partial\phi} = \frac{35}{4}\left(\frac{GMr^3}{s^5}\right)\frac{(\partial f/\partial\phi)}{\sin\theta}$

where it has been assumed that

[4-20] $w^5 = (r/s)^5 << 1$,

i.e. only terms up to the fourth order in w have been kept in $\vec{\mathcal{F}}$ (the coefficients of w^4 in $\vec{\mathcal{F}}$ are all zero, of course).

The gravitational field at the center of A due to the twelve surrounding metagalaxies is obviously zero because of symmetry. Hence $\vec{\mathcal{F}}$ is the tidal acceleration at some point r, θ, ϕ. To get an estimate of the tidal effect on the metagalaxy A due to the twelve masses, consider points on the surface of A (r=R) at which the tidal acceleration is radial i.e. $\mathcal{F}_\theta = \mathcal{F}_\phi = 0$. At other points, the tidal acceleration will be a combination of radial and angular accelerations.

After some calculation, $\partial f/\partial\theta$ and $\partial f/\partial\phi$ are given by

[4-21] $\frac{\partial f}{\partial\theta} = -\sin 2\theta[3 - 4\cos^2\theta - 2\sin^2\theta(3\cos^2\phi - 2\cos^4\phi + \sin^4\phi)]$

[4-22] $\frac{\partial f}{\partial\phi} = \frac{1}{2}\sin^4\theta\sin 4\phi$.

The conditions $\mathcal{F}_\theta = 0$, $\mathcal{F}_\phi = 0$ require that

[4-23] $\frac{\partial f}{\partial\theta} = 0$ and $\frac{1}{\sin\theta}\frac{\partial f}{\partial\phi} = \frac{1}{2}\sin^3\theta\sin 4\phi = 0$

which is equivalent to

[4-24] $\partial f/\partial\theta = 0$ and $\partial f/\partial\phi = 0$

since $\sin^3\theta$ and $\sin^4\theta$ have the same zeros. Equations [4-24] and [4-17] show that points on a surface of constant r at which $\mathcal{F}_\theta = \mathcal{F}_\phi = 0$ are also the extremum points of $\mathcal{F}_r$. It is easy to show that the points in the first quadrant ($0 \le \phi \le \pi/2$, $0 \le \theta \le \pi/2$) on a surface of constant r at which [4-24] holds are :

[4-25] $(\theta,\phi) = (0,\phi),\ (\pi/4,0),\ (\pi/2,0),\ (\cos^{-1}\frac{1}{\sqrt{3}},\pi/4),\ (\pi/2,\pi/4),$
$(\pi/4,\pi/2),\ (\pi/2,\pi/2)$.

Other quadrants have symmetrically corresponding points. Using [4-17] and the definition of f in [4-13], gives $\vec{\mathcal{F}}$ at these points :

[4-26] $(0,\phi)$: $\vec{\mathcal{F}}/\mathcal{F}_0 = (\mathcal{F}_r/\mathcal{F}_0)\hat{r} = -7\hat{r}$

(where $\mathcal{F}_0 \equiv GMr^3/s^5$)

$(\pi/4,0)$: $\vec{\mathcal{F}}/\mathcal{F}_0 = +(7/4)\hat{r}$

$(\pi/2,0)$: $\vec{\mathcal{F}}/\mathcal{F}_0 = -7\hat{r}$

$(\cos^{-1}\frac{1}{\sqrt{3}},\pi/4)$: $\vec{\mathcal{F}}/\mathcal{F}_0 = +(14/3)\hat{r}$

$(\pi/2,\pi/4)$: $\vec{\mathcal{F}}/\mathcal{F}_0 = +(7/4)\hat{r}$

$(\pi/4,\pi/2)$: $\vec{\mathcal{F}}/\mathcal{F}_0 = +(7/4)\hat{r}$

$(\pi/2,\pi/2)$: $\vec{\mathcal{F}}/\mathcal{F}_0 = -7\hat{r}$.

Hence the radial tidal acceleration of largest magnitude is

[4-27] $|\mathcal{F}_r|_{max} = 7GMr^3/s^5$.

Comparing this with the acceleration, α_R, at the surface r=R of the metagalaxy A due to A itself, gives

[4-28] $|\mathcal{F}_{r=R}|_{max}/\alpha_R = (7GMR^3/s^5)/(GM/R^2) = 7(R/s)^5$.

Under the assumption [4-20], it is seen that tidal effects on a metagalaxy may be too small to be noticeable. Because of the fifth power of R/s in [4-28], it is seen that metagalaxies would have to be relatively close together before Roche's limit comes into effect, i.e. before tidal forces would be great enough to make each metagalaxy fly apart. For instance if

[4-29] $(R/s)^5 \sim 1/32$

then [4-20] is adequately satisfied to make the previous tidal calculations valid. However $|\mathcal{F}_{r=R}|_{max}/\alpha_R$ is then $\sim 7/32 \sim 0.22$ which is not much less than unity i.e. Roche's limit is being approached. Equation [4-29] gives $s \sim 2R$, i.e. metagalaxies may have to be "touching" or "overlapping" before they cause each other to fly apart. Of course, if metagalaxies vary widely in mass, then this argument may not be valid since [4-28] would depend on mass ratios.

Note that [4-17]-[4-19] show a third-order (in w, or r) tidal effect. A first or second order calculation would have yielded $\vec{\mathcal{F}} = 0$. This is because of the high spherical symmetry and the relatively large number of masses. If the twelve masses would be spread out uniformly in a shell of constant density, then $\vec{\mathcal{F}}$ would be exactly zero since the gravitational field inside a uniform spherical shell is exactly zero. Equations [4-17]-[4-19] show the deviations from a uniform spherical shell distribution.

It has been assumed that contributions from second nearest neighbours are negligible. To get an estimate of their effect, one can do the same calculations as above. The metagalaxy A has six second neighbours each at a distance of $a = \sqrt{2}s$. The structure of A and its second

neighbours is depicted in Fig.4-4. The coordinate system is the same as that of Fig.4-2. As before,

[4-30] $$\mathcal{G} = -\frac{GM}{a}\sum_{n=0}^{\infty}\left[\sum_{i=1}^{6} P_n(u_i)\right]v^n$$

[4-31] with $v = r/a = r/\sqrt{2}\,s$

[4-32] and $u_i = \cos\bar{\theta}_i\cos\theta + \sin\bar{\theta}_i\sin\theta\cos(\bar{\phi}_i-\phi)$.

From Fig.4-4 it is easy to show that

[4-33] $$u_1 = \cos\theta = -u_6$$
$$u_2 = \sin\theta\cos\phi = -u_4$$
$$u_3 = \sin\theta\sin\phi = -u_5 \quad .$$

Hence, as before, this reduces $\mathcal{G}$ to

[4-34] $$\mathcal{G} = -\frac{2GM}{a}\sum_{n=0}^{\infty}\left[\sum_{i=1}^{3} P_{2n}(u_i)\right]v^{2n} \quad .$$

It is easy to show that

[4-35] $$\sum_{i=1}^{3} u_i^2 = 1 \quad \text{and}$$

$$\sum_{i=1}^{3} u_i^4 = \cos^4\theta + \sin^4\theta(\cos^4\phi + \sin^4\phi) \equiv j(\theta,\phi) \quad .$$

This yields

[4-36] $$\sum_{i=1}^{3} P_0(u_i) = \sum_{i=1}^{3} 1 = 3$$

$$\sum_{i=1}^{3} P_2(u_i) = \frac{1}{2}\sum_{i=1}^{3}(3u_i^2-1) = 0$$

$$\sum_{i=1}^{3} P_4(u_i) = \frac{1}{8}\sum_{i=1}^{3}(35u_i^4-30u_i^2+3) = \frac{1}{8}[35j(\theta,\phi) - 21]$$

[4-37] hence $\mathcal{Y} = -\frac{2GM}{a}[3 + \frac{1}{8}(35j - 21)v^4 + O(v^6)]$.

Using [4-16] and $a = \sqrt{2}\, s$, this gives

[4-38] $\mathcal{F}_r = \frac{1}{4\sqrt{2}}\left(\frac{GMr^3}{s^5}\right)(35j - 21)$

[4-39] $\mathcal{F}_\theta = \frac{35}{16\sqrt{2}}\left(\frac{GMr^3}{s^5}\right)\frac{\partial j}{\partial \theta}$.

[4-40] $\mathcal{F}_\phi = \frac{35}{16\sqrt{2}}\left(\frac{GMr^3}{s^5}\right)\frac{(\partial j/\partial \phi)}{\sin\theta}$

where it has been assumed that

[4-41] $v^5 = (r/a)^5 << 1$.

One can compare $\mathcal{F}_r$ with the nearest neighbour values of $\mathcal{F}_r$ at some of the points in [4-26]. Denoting the second neighbour acceleration by ${}^2\mathcal{F}_r$ and the nearest neighbour acceleration by ${}^1\mathcal{F}_r$, one has :

[4-42] $(\pi/4,0)$: ${}^2\mathcal{F}_r/\mathcal{F}_0 = -7/8\sqrt{2}$ which gives

$|{}^2\mathcal{F}_r/{}^1\mathcal{F}_r| = (7/8\sqrt{2})/(7/4) = 1/2\sqrt{2} \sim 0.36$

$(\cos^{-1}\frac{1}{\sqrt{3}}, \pi/4)$: ${}^2\mathcal{F}_r/\mathcal{F}_0 = -7/3\sqrt{2}$ which gives

$|{}^2\mathcal{F}_r/{}^1\mathcal{F}_r| = (7/3\sqrt{2})/(14/3) = 1/2\sqrt{2}$

$(0,\phi)$: ${}^2\mathcal{F}_r/\mathcal{F}_0 = +7/2\sqrt{2}$ which gives

$|{}^2\mathcal{F}_r/{}^1\mathcal{F}_r| = (7/2\sqrt{2})/7 = 1/2\sqrt{2}$.

Calculation of $\partial j/\partial\phi$ gives

[4-43] $\partial j/\partial\phi = -\sin^4\theta\sin 4\phi = -2\partial f/\partial\phi$

[4-44] which gives $|{}^2\mathcal{F}_\phi/{}^1\mathcal{F}_\phi| = 1/2\sqrt{2}$ at all points.

It appears that nearest neighbour tidal effects are roughly three

times greater than second neighbour effects, i.e. the major parts of the tidal forces on a metagalaxy are due to its nearest neighbours.

5. THE METAGALAXY AS A SCHWARZSCHILD OBJECT

In the previous sections, hierarchical cosmology has been discussed in Newtonian terms. The concepts of the metagalaxy being a Newtonian black hole, and a hierarchical universe with black holes as elements, were proposed, and their features were examined. In order to obtain a more modern point of view, it would be interesting to treat the problem using the general theory of relativity.

The classical notion presented previously that the metagalaxy is simply an element in a hierarchy of metagalaxies, i.e. that it is one of a number of clumps of matter grouped together, leads to the general relativistic concept of the metagalaxy being one of a number of Schwarzschild objects grouped together. Also, just as the metagalaxy may be consistently thought of as a Newtonian black hole in the classical case, so it is conceivable that the metagalaxy may be a black hole in the general relativistic sense. It is the purpose of this section to examine the nature of this concept.

The spacetime comprising the interior of the metagalaxy may be described by a metric of the Robertson-Walker type

[5-1] $$ds_F^2 = dt^2 - R^2(t)[(1 - kr^2)^{-1}dr^2 + r^2 d\Omega^2]$$

with $$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

(see e.g. Weinberg 1972, p. 412). The constant k takes the values -1,0, or +1 depending on whether metagalactic space is infinite or finite respectively. Here, θ and ϕ are spherical angle variables. Units in which c = speed of light = 1 have been used. The metric ds_F^2 describes a spherically symmetric, isotropic and homogeneous spacetime.

The spatial coordinates r, θ and ϕ of a point comoving with the "cosmic fluid" do not change their values, i.e. they are comoving coordinates.

Using [5-1] in Einstein's field equations, with the cosmic constant Λ unequal to zero, one obtains the following differential equation for R(t) :

[5-2] $\dot{R}^2 + k = 8\pi G\rho R^2/3 + \Lambda R^2/3$

where $\dot{R} \equiv dR/dt$ (see e.g. Weinberg 1972, p.613-614) and $\rho = \rho(t)$ is the energy density. The equation for the conservation of energy is

[5-3] $d(\rho R^3)/dR = -3pR^2$

where p is the pressure of cosmic matter (based on Weinberg 1972, p.472, p.614; and, Adler, Bazin, Schiffer 1965, p.358). Given an equation of state $p = p(\rho)$, equation [5-3] will determine ρ as a function of R, which can then be used in [5-2] to determine R as a function of t. The values assigned to k and Λ will also determine the behaviour of R as a function of t. Cosmological models in which R(t) is derived in this way comprise the well-known Friedmann models, hence the subscript "F" on the line element ds in equation [5-1]. For the case k = +1, R is sometimes called the "radius" of the metagalaxy, since it is the radius of a four-dimensional Euclidean hypersphere. For the cases k = -1,0 , R is simply a scale factor i.e. it sets the scale of the geometry of space (Weinberg 1972, p.412).

In pressureless cosmological models (p = 0) one has from [5-3],

[5-4] $\rho R^3 = \text{constant}$.

In this case [5-2] can be written as

[5-5] $\dot{R}^2 + k = \mu/R + \Lambda R^2/3$

with $\mu \equiv 8\pi G\rho R^3/3 = \text{constant}$.

If the metagalaxy may be thought of as a Schwarzschild object, then the spacetime exterior to the metagalaxy is described by the Schwarzschild line element for empty space,

[5-6] $ds_S^2 = (1 - 2m/\bar{r} - \Lambda\bar{r}^2/3)d\bar{t}^2 - (1 - 2m/\bar{r} - \Lambda\bar{r}^2/3)^{-1}d\bar{r}^2 - \bar{r}^2 d\bar{\Omega}^2$

where $d\bar{\Omega}^2 = d\bar{\theta}^2 + \sin^2\bar{\theta}d\bar{\phi}^2$

and $m \equiv GM = G \times \text{mass of Schwarzschild object}$

(see e.g. Adler, Bazin, Schiffer 1965, p.335-336).

The two metrics ds_F^2 and ds_S^2 must be matched at the boundary dividing the interior and exterior regions. Both metrics describe spherically symmetric spacetimes, hence, in the matching process, the two coordinate systems may be "centered" with respect to the angle variables by taking

[5-7] $\bar{\theta} = \theta$ and $\bar{\phi} = \phi$

i.e. $d\bar{\Omega} = d\Omega$.

Also, for the sake of generality, it will first be assumed that the Friedmann and Schwarzschild regions have different values of the cosmic constant, i.e. the cosmic constant is Λ_F for the Friedmann region and Λ_S for the Schwarzschild region. It will be shown that this assumption leads to an undesirable result, and it will then be dropped. The two metrics are

[5-8] $ds_F^2 = dt^2 - R^2(t)[(1 - kr^2)^{-1}dr^2 + r^2 d\Omega^2]$

[5-9] with $\dot{R}^2 + k = \mu/R + \Lambda_F R^2/3$

assuming that pressure is negligible, i.e. $p \sim 0$, and

[5-10] $ds_S^2 = (1 - 2m/\bar{r} - \Lambda_S\bar{r}^2/3)d\bar{t}^2 - (1 - 2m/\bar{r} - \Lambda_S\bar{r}^2/3)^{-1}d\bar{r}^2 - \bar{r}^2 d\Omega^2.$

In order to match them at the boundary, the metric ds_F^2 will be expressed in terms of the standard coordinates (or Schwarzschild coordinates) $\bar{r},\theta,\phi,\bar{t}$. The most general form for a spherically symmetric, isotropic metric in these coordinates is the standard form

[5-11] $ds_F^2 = B(\bar{r},\bar{t})d\bar{t}^2 - A(\bar{r},\bar{t})d\bar{r}^2 - \bar{r}^2 d\Omega^2$

(see e.g. Weinberg 1972, equation 11.7.1, p.336). The following calculation has been adapted from other calculations for matching spacetimes (see e.g. Einstein and Strauss 1945; Oppenheimer and Snyder 1939; see also Weinberg 1972, p.342-346). One must find a transformation

[5-12a] $\bar{t} = \bar{t}(r,t)$

[5-12b] $\bar{r} = \bar{r}(r,t)$

which transforms ds_F into standard form. Taking differentials and solving for dr and dt gives

[5-13] $dr = (\bar{t}_t d\bar{r} - \bar{r}_t d\bar{t})/\eta$ and $dt = (\bar{r}_r d\bar{t} - \bar{t}_r d\bar{r})/\eta$

where $\eta \equiv \bar{t}_t\bar{r}_r - \bar{r}_t\bar{t}_r$

and $\bar{t}_t \equiv \partial\bar{t}/\partial t$, etc. .

Using [5-13] in [5-8] yields

[5-14] $ds_F^2 = (\bar{r}_r^2 - \sigma\bar{r}_t^2)\eta^{-2}d\bar{t}^2 - (\sigma\bar{t}_t^2 - \bar{t}_r^2)\eta^{-2}d\bar{r}^2$

$-2(\bar{r}_r\bar{t}_r - \sigma\bar{r}_t\bar{t}_t)\eta^{-2}d\bar{r}d\bar{t} - r^2R^2d\Omega^2$

[5-15] where $\sigma \equiv R^2(t)(1 - kr^2)^{-1}$.

Comparing with [5-11], this yields the transformation [5-12b]

[5-16] $\bar{r} = rR(t)$

(see e.g. Weinberg 1972, p. 345) and also

[5-17a] $B(\bar{r},\bar{t}) = (\bar{r}_r^2 - \sigma\bar{r}_t^2)\eta^{-2}$

[5-17b] $A(\bar{r},\bar{t}) = (\sigma\bar{t}_t^2 - \bar{t}_r^2)\eta^{-2}$ and

[5-17c] $\bar{r}_r\bar{t}_r = \sigma\bar{r}_t\bar{t}_t$.

It is understood that r and t in [5-17a] and [5-17b] are expressed as functions of $\bar{r}$ and $\bar{t}$. Equation [5-17c] can be used to eliminate $\bar{r}_r$ from [5-17a] and η. This gives

[5-18a] $B^{-1} = \bar{t}_t^2[1 - \sigma^{-1}(\bar{t}_r/\bar{t}_t)^2]$

[5-18b] $A^{-1} = \bar{r}_t^2[\sigma(\bar{t}_t/\bar{t}_r)^2 - 1]$.

Using [5-15] and [5-16], equation [5-17c] gives

[5-19] $(\bar{t}_r/\bar{t}_t) = \sigma\bar{r}_t/\bar{r}_r = r R\dot{R}(1 - kr^2)^{-1}$.

Then, using [5-9], [5-15], [5-16] and [5-19] in [5-18], B and A become

[5-20a] $B = (1/\bar{t}_t^2)(1 - kr^2)(1 - \mu r^2/R - \Lambda_F r^2R^2/3)^{-1}$

[5-20b] $A = (1 - \mu r^2/R - \Lambda_F r^2R^2/3)^{-1}$.

A is now known, but B is still undetermined due to the factor $(1/\bar{t}_t^2)$. A must agree with the coefficient of $d\bar{r}^2$ in the Schwarzschild metric [5-6] at the boundary. B will agree with the coefficient of $d\bar{t}^2$ in [5-6] at the boundary if $\bar{t}_t$ is chosen appropriately at the boundary. The boundary may be described by

[5-21] $r = b$

where b is a constant since r is a comoving coordinate. Matching A at r = b yields

[5-22] $1 - \mu b^2/R - \Lambda_F b^2R^2/3 = 1 - 2m/bR - \Lambda_S b^2R^2/3$

where [5-16] has been used. This is equivalent to

[5-23] $2m = b^3[\mu + (\Lambda_F - \Lambda_S)R^3/3]$.

This means that in order for ds_F^2 to be expressible in standard form,

the Schwarzschild geometrized mass m must be time-dependent. This is unrealistic, and so the cosmic constant Λ will henceforth be assumed to have one universal value, i.e.,

$$\Lambda_F = \Lambda_S = \Lambda \quad .$$

In this case, one has

[5-24] $m = \frac{1}{2}\mu b^3$

or, using the definition of μ in [5-5],

[5-25] $M = \frac{4}{3}\pi\rho(bR)^3$,

a familiar result. Matching B at r = b gives

[5-26] $\bar{t}_t^{-2}(b,t)(1 - kb^2)(1 - \mu b^2/R - \Lambda b^2R^2/3)^{-1} = 1 - \mu b^2/R - \Lambda b^2R^2/3$

where [5-24] is understood. This gives the expression for $\bar{t}_t$ at the boundary,

[5-27] $\bar{t}_t(b,t) = \pm(1 - kb^2)^{1/2}(1 - \mu b^2/R - \Lambda b^2R^2/3)^{-1}$

where the sign is arbitrary. The general solution to the differential equation of [5-19], combined with the integral of [5-27] gives the required transformation for $\bar{t}$ in [5-12a]. The quantity $\bar{t}_t(r,t)$ can then be calculated to give the expression for B in [5-20a].

The general solution to [5-19] is

[5-28] $\bar{t} = F(x)$

[5-29] where $x = J(1 - kr^2)^{-\gamma/2k}e^{\gamma I(R)}$

[5-30] with $I(R) = \int^R (\mu + \Lambda R^3/3 - kR)^{-1}dR$

where [5-5] has been used. J and γ are arbitrary constants but it is convenient to choose them as

$J = (1 - kb^2)^{-1/2}$ and $\gamma = -k$, hence,

$$[5\text{-}31] \quad x = \left(\frac{1 - kr^2}{1 - kb^2}\right)^{1/2} e^{-kI(R)} \quad .$$

F is an arbitrary function of x. Equation [5-27] gives

$$[5\text{-}32] \quad \bar{t}(b,t) = \pm\sqrt{1-kb^2}\int^{t} (1 - \mu b^2/R - \Lambda b^2R^2/3)^{-1}dt$$

$$= \pm\varepsilon\sqrt{1-kb^2}\int^{R} \frac{dR}{(1 - \mu b^2/R - \Lambda b^2R^2/3)}\left(\frac{R}{\mu + \Lambda R^3/3 - kR}\right)^{1/2}$$

where [5-5] has been used, and ε is the sign of R. Equation [5-32] can also be written as

$$[5\text{-}33] \quad \bar{t}(b,t) = \pm\varepsilon\sqrt{1-kb^2}\int^{f(x_b)} \frac{dR}{(1 - \mu b^2/R - \Lambda b^2R^2/3)}\left(\frac{R}{\mu + \Lambda R^3/3 - kR}\right)^{1/2}$$

$$[5\text{-}34] \quad \text{where} \quad x_b \equiv x(r=b) = e^{-kI(R)}$$

and f(z) is a solution of

$$[5\text{-}35] \quad z = e^{-kI(f(z))} \quad .$$

Hence, for any r, an expression for $\bar{t}$ is obtained by replacing x_b by x, i.e.,

$$[5\text{-}36] \quad \bar{t}(r,t) = \pm\varepsilon\sqrt{1-kb^2}\int_{f(x_0)}^{f(x)} \frac{dR}{(1 - \mu b^2/R - \Lambda b^2R^2/3)}\left(\frac{R}{\mu + \Lambda R^3/3 - kR}\right)^{1/2}$$

$$[5\text{-}37] \quad \text{where} \quad f(x_0) = f(x(t=0)) = f\left(\left[\frac{1 - kr^2}{1 - kb^2}\right]^{1/2} e^{-kI(R_0)}\right)$$

$$\text{and} \quad R_0 = R(t=0) \quad .$$

The lower limit to the integral has been conveniently chosen, so that one has

$$[5\text{-}38] \quad \bar{t}(r,0) = 0 \quad .$$

Equation [5-36] is the required transformation [5-12a].

To obtain B, one must first calculate $\bar{t}_t$ from [5-36]. The result is

$$[5\text{-}39] \quad \bar{t}_t(r,t) = \pm\sqrt{1-kb^2}\sqrt{\frac{f(x)}{R}}\,\frac{\sqrt{\mu + \frac{1}{3}\Lambda f^3(x) - kf(x)}}{[1 - \frac{\mu b^2}{f(x)} - \frac{1}{3}\Lambda b^2 f^2(x)]\sqrt{\mu + \frac{1}{3}\Lambda R^3 - kR}} \; .$$

Using this in [5-20a] gives

$$[5\text{-}40] \quad B = \left[\frac{R}{f(x)}\right]\frac{[1 - \frac{\mu b^2}{f(x)} - \frac{1}{3}\Lambda b^2 f^2(x)]^2}{[1 - \frac{\mu r^2}{R} - \frac{1}{3}\Lambda r^2 R^2]}\left[\frac{1-kr^2}{1-kb^2}\right]\left[\frac{\mu + \frac{1}{3}\Lambda R^3 - kR}{\mu + \frac{1}{3}\Lambda f^3(x) - kf(x)}\right] .$$

The expression for A is repeated here for convenience,

$$[5\text{-}41] \quad A = (1 - \frac{\mu r^2}{R} - \frac{1}{3}\Lambda r^2 R^2)^{-1} \; .$$

Summarizing, the Friedmann metric is given in standard form by [5-11] with A and B given by [5-40] and [5-41]. The transformations connecting $(\bar{r},\bar{t})$ to (r,t) are given by equations [5-16] and [5-36]. The Schwarzschild metric is expressed in standard form by equation [5-10] (with $\Lambda_S = \Lambda$, of course).

Since, at the boundary $r = b$, one has

$$x = x_b = e^{-kI(R)}$$

$$\text{and} \quad f(x) = f(x_b) = R$$

equation [5-40] reduces to

$$[5\text{-}42] \quad B = 1 - \mu b^2/R - \Lambda b^2 R^2/3$$

at $r = b$, as required.

In the case $\Lambda = 0$, one has

$$[5\text{-}43] \quad I(R) = -k^{-1}\ln(\mu - kR)$$

$$[5\text{-}44] \quad \text{hence} \quad x = \left(\frac{1 - kr^2}{1 - kb^2}\right)^{1/2}(\mu - kR)$$

[5-45] and $x_b = \mu - kR$

[5-46] and $x_0 = \left(\frac{1 - kr^2}{1 - kb^2}\right)^{1/2} (\mu - kR_0)$

[5-47] hence $f(x) = (\mu - x)/k$

[5-48] and $f(x_0) = (\mu - x_0)/k$.

Therefore B and A reduce to

[5-49a] $B = \left[\frac{R}{f(x)}\right] \frac{[1 - \mu b^2/f(x)]^2}{[1 - \mu r^2/R]} \left[\frac{1 - kr^2}{1 - kb^2}\right]^{1/2} \qquad (\Lambda = 0)$

[5-49b] $A = (1 - \mu r^2/R)^{-1} \qquad (\Lambda = 0)$.

An easier way to match metrics at the boundary is simply to equate ds_S^2 and ds_F^2 at $r = b$. However, this method yields only equations [5-16] at the boundary $r = b$, and [5-32], i.e. the transformations for $\bar{r} = \bar{r}(r,t)$ and $\bar{t} = \bar{t}(r,t)$ are revealed only for $r = b$, not for any r. B and A are also known only at the boundary $r = b$ by this method. The more rigorous matching process has been used above for completeness. The simpler method goes as follows :

Repeating, the two metrics are

[5-50] $ds_F^2 = dt^2 - R^2(t)[(1 - kr^2)^{-1}dr^2 + r^2 d\Omega^2]$, and

[5-51] $ds_S^2 = (1 - 2m/\bar{r} - \Lambda\bar{r}^2/3)d\bar{t}^2 - (1 - 2m/\bar{r} - \Lambda\bar{r}^2/3)^{-1}d\bar{r}^2 - \bar{r}^2 d\Omega^2$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$.

At the boundary $r = b$, one has

[5-52] $ds_F^2 = dt^2 - R^2 b^2 d\Omega^2 = ds_S^2$

$$= (1 - 2m/\bar{r}_b - \Lambda\bar{r}_b^2/3)d\bar{t}_b^2 - (1 - 2m/\bar{r}_b - \Lambda\bar{r}_b^2/3)^{-1}d\bar{r}_b^2 - \bar{r}_b^2 d\Omega^2 .$$

This yields

[5-53] $\bar{r}_b = bR(t)$

[5-54] hence $d\bar{r}_b = bdR = b\dot{R}dt$.

With the help of [5-53] and [5-54], [5-52] becomes

[5-55] $dt^2 = (1 - 2m/bR - \Lambda b^2R^2/3)d\bar{t}_b^2 - (1 - 2m/bR - \Lambda b^2R^2/3)^{-1}b^2\dot{R}^2dt^2.$

Using equations [5-2] for $\dot{R}$, and assuming the validity of equation [5-24] or [5-25], one can solve [5-55] for $d\bar{t}_b/dt$ to give

[5-56] $\frac{d\bar{t}_b}{dt} = \frac{d\bar{t}}{dt}(b,t) = \frac{\partial\bar{t}}{\partial t}(b,t) = \pm\sqrt{1-kb^2}\,(1 - \mu b^2/R - \Lambda b^2R^2/3)^{-1}$

which yields [5-32] when integrated (for similar expressions, see e.g. Kantowski 1969).

The foregoing methods have been used to embed a Friedmann spacetime at the center of a Schwarzschild spacetime. The exact same methods can obviously be used to embed Schwarzschild objects in a Friedmann spacetime. This has been done in the so-called "Swiss cheese" models of the metagalaxy (see e.g. Kantowski 1969). The results of these methods will be used in later parts of this work in the consideration of a hierarchy of metagalaxies represented as Schwarzschild objects embedded in a Friedmann spacetime.

Various dynamical models of the metagalaxy are obtained by assigning different values to the constants k and Λ. These models are described by the scale factor R as a function of t, which is a solution of [5-2], or of [5-5] if pressure is negligible i.e. if [5-4] is assumed. The various solutions of [5-5] are depicted graphically in Figs.5-1 to 5-3. These diagrams have been adapted from "Cosmology"

(Bondi 1960, Chap. IX). In each of the possible models, R(t) behaves in one of the following ways :

I. R(t) is a constant for all t (Fig.5-3(ii)(a)).

II. R(t) expands monotonically ($R(t)\to\infty$ as $t\to\infty$) starting at a definite time from R = 0 (Figs.5-1(i), 5-1(ii), 5-2(i), 5-2(ii), 5-3(i)).

III. R(t) expands monotonically from a finite value of $\dot{R}$ at $t = -\infty$ ($R(t)\to\infty$ as $t\to\infty$) (Fig.5-3(ii)(c)).

IV. R(t) expands monotonically starting at a finite time from R = 0 and tends to a limit as $t\to\infty$ (Fig.5-3(ii)(b)).

V. R(t) oscillates between R = 0 and a finite value of R (Figs. 5-1(iii), 5-2(iii), 5-3(iii)(a), 5-3(iv)).

VI. R(t) contracts from infinity to a finite value of R and then expands again to infinity (Fig.5-3(iii)(b)).

If the metagalaxy is a black hole, then it would be undetectable by optical means from the outside. Electromagnetic signals could be received from the outside; however, if the outside consists of nothing but other black hole-metagalaxies, then there would be no connection at all via optical means. Again, only tidal forces on some metagalaxy due to others around it might be noticeable according to an argument similar to the one presented in section 4. However, if metagalaxies are Schwarzschild objects but not black holes, then optical detection may be possible.

To clarify this, consider the following simple example: two metagalaxies are represented as Schwarzschild objects embedded in a Friedmann spacetime. For simplicity, assume $\Lambda = 0$. Since the Schwarzschild $\bar{r}$ coordinate of the boundary of a metagalaxy is given by [5-53], i.e.

[5-53] $\bar{r}_b \equiv \bar{r}(b,t) = bR(t)$

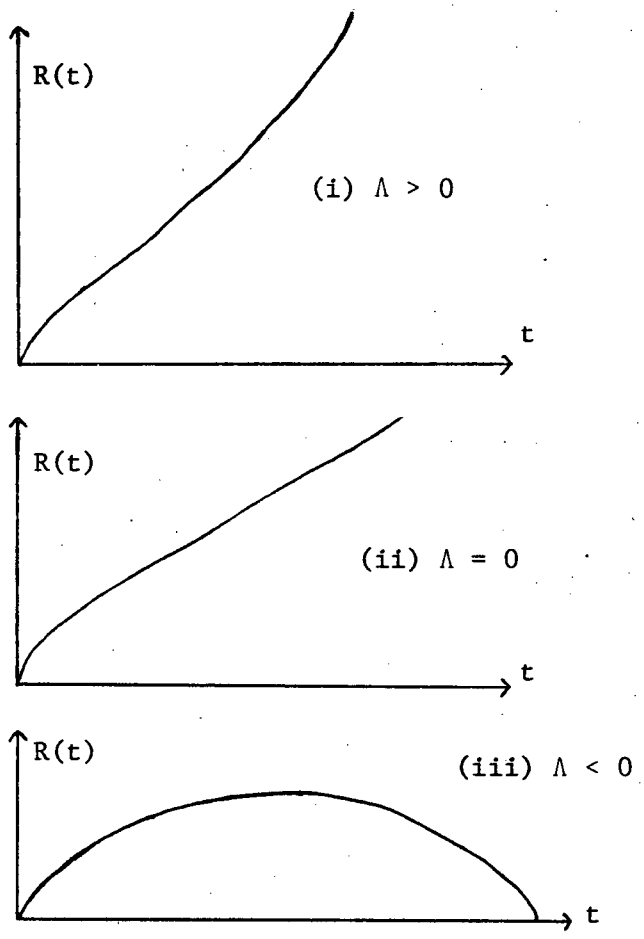

FIG.5-1 . Metagalactic models for k = -1.

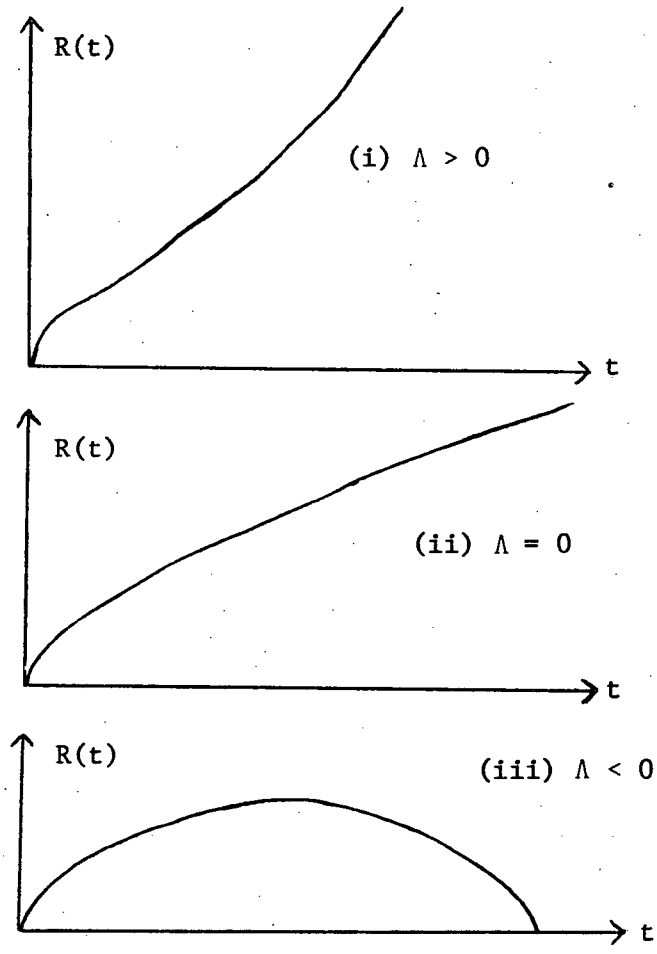

FIG.5-2 . Metagalactic models for k = 0.

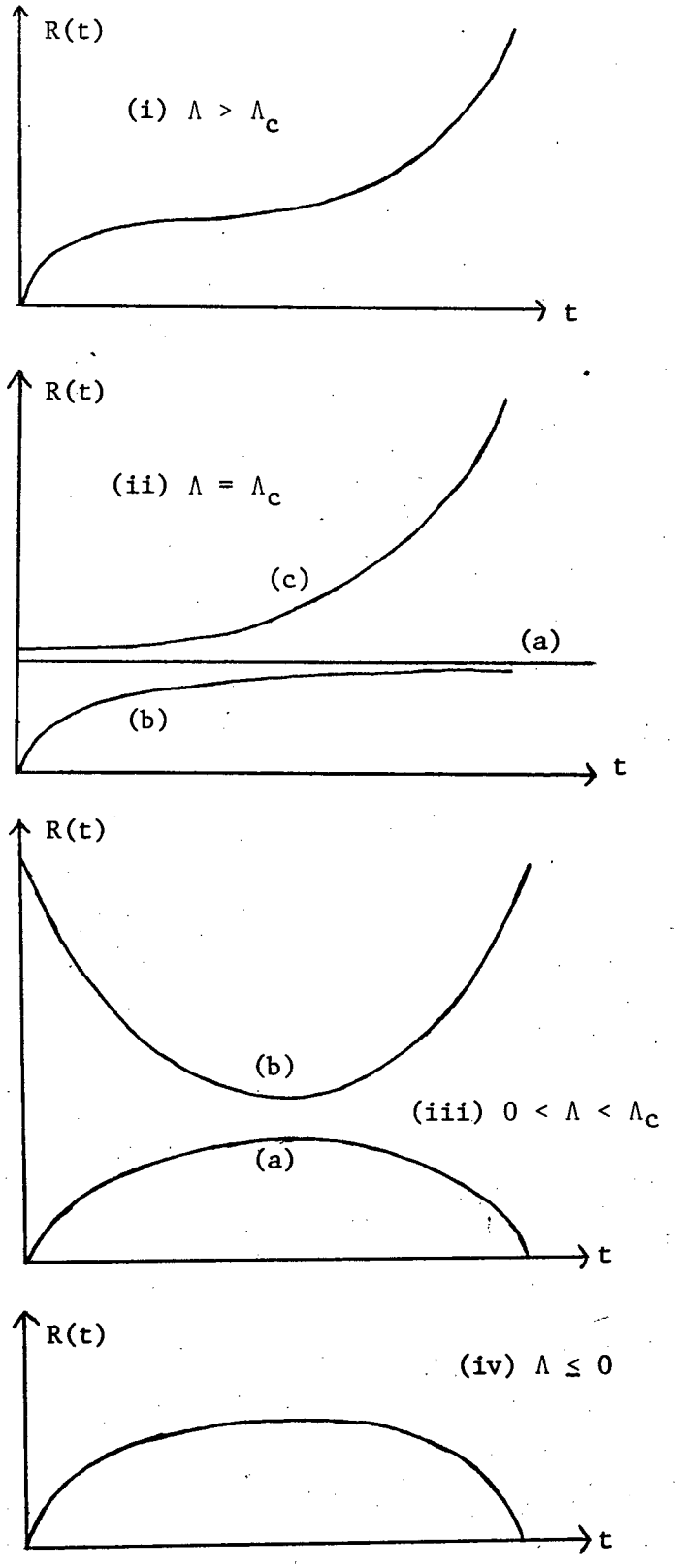

FIG.5-3 . Metagalactic models for k = +1. $\Lambda_c = 4/9\mu^2$.

one may denote the gravitational radius (or Schwarzschild radius) of the metagalaxy by $\bar{r}_{bg}$, where $\bar{r}_{bg} = 2m$. The Friedmann spacetime is connected to the two Schwarzschild spacetimes at regions exterior to their gravitational radii i.e. at regions where $\bar{r} > \bar{r}_{bg} = 2m$. In order for an optical connection to exist between the two metagalaxies it is necessary that the boundary of at least one of them be beyond its gravitational radius $\bar{r}_{bg}$, so to speak. For instance, if the metagalaxy is expanding starting from R = 0, i.e. if it is of type II, then the boundary of this metagalaxy would have to expand with t until t reached t_g, such that

[5-57] $\bar{r}_{bg} \equiv \bar{r}(b,t_g) = 2m = bR(t_g) \equiv bR_g$.

For $t > t_g$, the Schwarzschild coordinate $\bar{r}_b$ of the boundary would be greater than $\bar{r}_{bg}$, hence the metagalaxy would no longer be a black hole, and would be able to transmit optical signals to the other metagalaxy.

The behaviour of $\bar{t}_b \equiv \bar{t}(b,t)$ near $t = t_g$ can be determined by using [5-27]. Since $\bar{r}_{bg}$ is defined by

[5-58] $1 - 2m/\bar{r}_{bg} - \Lambda\bar{r}_{bg}^2/3 = 0$ i.e.,

[5-59] $1 - \mu b^2/R_g - \Lambda b^2 R_g^2/3 = 0$

(using [5-24]), equation [5-27] shows that

[5-60] $\bar{t}_t(b,t_g) = \pm\infty$

[5-61] $\Rightarrow \quad \bar{t}_b \to \pm\infty \quad \text{as} \quad t \to t_g$.

This shows that it would take an infinite Schwarzschild time $\bar{t}_b$ for the boundary to reach the gravitational radius.

It has been assumed in the last paragraph that R_g exists, however,

since R_g is defined by [5-59], it is seen that R_g may not exist, i.e. [5-59] may not have any real solutions for R_g (such that $R_g > 0$), if Λ is positive and large enough. This can be shown by rewriting [5-59] as

[5-62] $\frac{1}{3}\Lambda R^3 = \frac{R}{b^2} - \mu \qquad \text{at } R = R_g$

and plotting both sides simultaneously as functions of R as in Fig.5-4. If $\Lambda > \Lambda^*$, then R_g does not exist. Note that for $\Lambda < \Lambda^*$, two gravitational radii $\bar{r}_{bg}$ can exist for one metagalaxy. The critical value Λ^* can be derived by noting that at $R = R_g^*$, the slopes of $\Lambda^* R^3/3$ and $R/b^2 - \mu$ are equal, i.e.

[5-63] $\Lambda^*(R_g^*)^2 = 1/b^2$

[5-64] $\Rightarrow \qquad \Lambda^* = 1/(bR_g^*)^2$.

Using this in the equation

[5-65] $\frac{1}{3}\Lambda^*(R_g^*)^3 = \frac{1}{b^2}R_g^* - \mu$

[5-66] gives $\qquad R_g^* = 3\mu b^2/2 = 3m/b$

[5-67] which means $\qquad \Lambda^* = 4/9\mu^2 b^6 = 1/9m^2$.

For the metagalaxy,

[5-68] $m = GM \sim R \qquad (c = 1)$

according to Fig.1-1. Hence

[5-69] $\Lambda^* \sim 1/9R^2 \sim 10^{-53}\ \text{meter}^{-2}$

assuming $R \sim 10^{10}$ light years $\sim 10^{26}$ meters. Hence if

[5-70] $\Lambda > \Lambda^* \sim 1/9R^2$,

then a gravitational radius does not exist for the metagalaxy as a Schwarzschild object.

In such a case, in which there is no gravitational radius, the

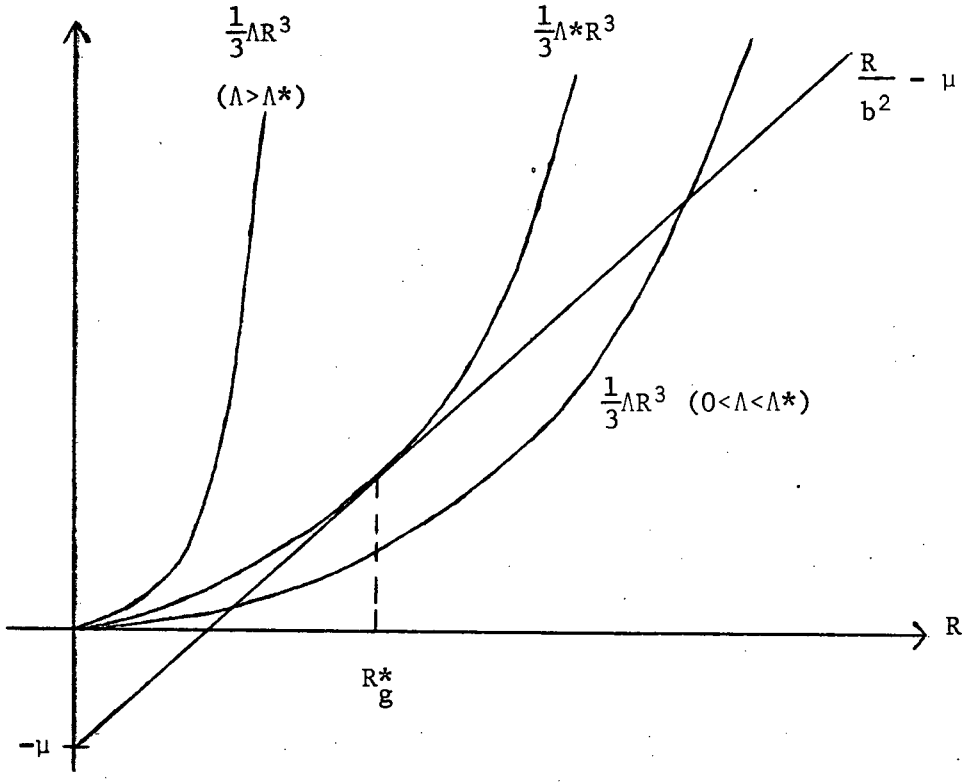

FIG.5-4 . Plot of [5-62]. If $\Lambda > \Lambda^*$, then R_g does not exist. One has $0 \leq b < \infty$ for $k = -1,0$ and $0 \leq b < 1$ for $k = +1$. For $0 < \Lambda < \Lambda^*$, there are two values of R_g .

following inequality holds :

[5-71] $1 - \mu b^2/R - \Lambda b^2 R^2/3 < 0 \qquad (r = b)$

for all values of R(t), or equivalently,

[5-72] $1 - 2m/\bar{r}_b - \Lambda \bar{r}_b^2/3 < 0$

for all values of $\bar{r}_b$.

With reference to Figs. 5-1 to 5-3, it is seen that there are only three metagalactic models that could possibly belong to the category $\Lambda > \Lambda^*$. They are

(1) the case $k = -1$, $\Lambda > 0$ i.e. Fig.5-1(i)

(2) the case $k = 0$, $\Lambda > 0$ i.e. Fig.5-2(i)

(3) the case $k = +1$, $\Lambda > \Lambda_c$ i.e. Fig.5-3(i) .

These three cases do not have to have $\Lambda > \Lambda^*$ — it is only a possibility. They are all models of type II. The remaining models do not belong in the category $\Lambda > \Lambda^*$ for the following reasons: some of them have $\Lambda \leq 0$ and so obviously do not belong. Others have $0 < \Lambda \leq \Lambda_c$, but in these cases, $k = +1$, hence b can only take values in the range $0 \leq b < 1$. This means that

[5-73] $\Lambda_c = 4/9\mu^2 < \Lambda^* = 4/9\mu^2 b^6$

[5-74] hence $\Lambda < \Lambda^*$.

Therefore cases (1)-(3) above are the only ones that do not necessarily have a gravitational radius.

If $\Lambda > \Lambda^*$, cases (1)-(3) would represent monotonically expanding metagalaxies (type II) which satisfy [5-71] for all values of R(t). Examination of [5-27] shows that $\bar{t}_b$ would be finite for all values of t, and that $\bar{t}_b$ would be a monotonically increasing or decreasing

function of t (depending on the sign of $\bar{t}_t(b,t)$). According to [5-72], the coefficient of $d\bar{t}_b^2$ in the Schwarzschild metric is negative for all values of $\bar{r}_b$, and the coefficient of $d\bar{r}_b^2$ is positive for all values of $\bar{r}_b$. This means that $\bar{r}_b$ is a timelike coordinate and $\bar{t}_b$ is a spacelike coordinate. The increase of $\bar{r}_b$ in these models represents the passage of Schwarzschild time, and the increase or decrease of $\bar{t}_b$ represents the passage of Schwarzschild space (see e.g. Misner, Thorne and Wheeler 1973, p.823). Such a region of Schwarzschild spacetime, in which the spacelike and timelike roles of $\bar{r}$ and $\bar{t}$ are reversed, will henceforth be referred to as a "negative region of Schwarzschild spacetime" since $1 - 2m/\bar{r} - \Lambda\bar{r}^2/3$ is negative. The region of Schwarzschild spacetime in which $\bar{r}$ and $\bar{t}$ play the familiar spacelike and timelike roles, respectively, will henceforth be referred to as the "positive region of Schwarzschild spacetime", since $1 - 2m/\bar{r} - \Lambda\bar{r}^2/3$ is positive. If $\Lambda > \Lambda^*$, cases (1)-(3) would be type II metagalaxies whose boundaries are always situated in a negative region of Schwarzschild spacetime.

Consider now the cases in which a gravitational radius (or radii) does exist for a metagalaxy, i.e. cases in which $\Lambda < \Lambda^*$. It would be interesting to find an oscillating metagalactic model, i.e. a model of type V, in which the maximum value of R(t) coincides with the gravitational radius (or the gravitational radius of smallest value if there are two). In other words it would be interesting to have the case

[5-75] $R_m = R_g$

where $R_m \equiv R(t_m)$ is the maximum value of R, and $t = t_m$ is the time at which this maximum is reached. This means that this metagalaxy would

be optically undetectable from regions beyond the gravitational radius for all values of t.

One might also consider the case

[5-76] $R_m < R_g$

however this case would be unphysical because in this region where $\bar{r}_b < \bar{r}_{bg}$ i.e. $R(t) < R_g$, the coordinate $\bar{r}_b$ is a timelike coordinate and $\bar{t}_b$ is a spacelike one. A reversal of R(t), from expansion to contraction, within this negative region means a reversal of $\bar{r}_b$ which would signify a reversal of the direction of the flow of Schwarzschild time. However, it will be seen that this case does not occur anyway.

The possibility of the case [5-75] will now be investigated. The equation that R_g must satisfy is given by [5-59] which can be written as

[5-77] $\frac{1}{3}\Lambda R^3 = \frac{R}{b^2} - \mu \qquad \text{at } R = R_g$.

In models of type V, the maximum value R_m must satisfy

[5-78] $\frac{1}{3}\Lambda R^3 = kR - \mu \qquad \text{at } R = R_m$

which is obtained by setting $\dot{R} = 0$ in [5-5] (pressure is assumed negligible). To determine whether or not the case $R_m = R_g$ is possible, equations [5-77] and [5-78] can be discussed graphically by plotting both sides of both equations simultaneously as functions of R, for each of the four models of type V, as in Figs.5-5 to 5-7. These four models will now be discussed :

<u>$k = -1$, $\Lambda < 0$ (Fig.5-1(iii))</u>: In this case, one has $0 \leq b < \infty$. One sees from Fig.5-5 that $R_m > R_g$ for all possible values of Λ and b.

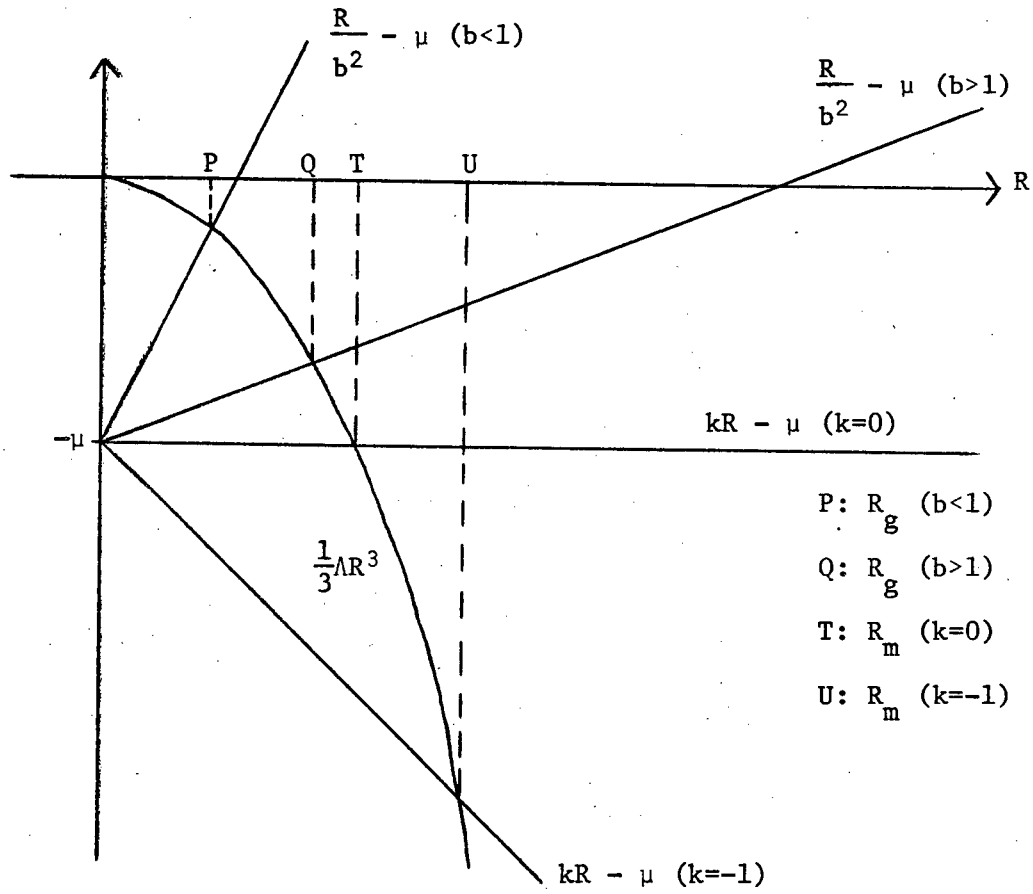

FIG.5-5 . Plot of [5-77] and [5-78] for models of type V with $k = -1,0$ and $\Lambda < 0$.

<u>$k = 0$, $\Lambda < 0$ (Fig.5-2(iii))</u>: In this case, one has $0 \leq b < \infty$. One sees from Fig.5-5 that $R_m > R_g$ for all possible values of Λ and b.

<u>$k = +1$, $0 < \Lambda < \Lambda_c$ (Fig.5-3(iii)(a))</u>: In this case one has $0 \leq b < 1$. One sees from Fig.5-6 that there are two values of R_g and R_m. One has $R_{m1} > R_{g1}$. For this model, R(t) does not exceed R_{m1} and hence does not reach R_{g2}. R_{m2} actually represents the <u>minimum</u> value of R for the model of type VI — it will be discussed later. Note that for $R_{m1} < R < R_{m2}$ in Fig.5-6, one has $\Lambda R^3/3 < kR - \mu$ which means that $R\dot{R}^2 < 0$ which is unphysical. Similar regions exist in Figs. 5-5 and 5-7.

<u>$k = +1$, $\Lambda \leq 0$ (Fig.5-3(iv))</u>: In this case one has $0 \leq b < 1$. One sees from Fig.5-7 that $R_m > R_g$ for all possible values of Λ and b.

Hence in all models of type V, one has

[5-79] $R_m > R_g$

i.e. the case $R_m = R_g$ does not occur.[1] In these models, the boundary expands from R = 0 in the negative region, crosses the gravitational radius into the positive region, reaches a maximum, contracts, crosses the gravitational radius into the negative region, and collapses to R = 0. This cycle is indefinitely repeated. There are two values of t_g at which the boundary crosses the gravitational radius, and at both of these values one has

[5-80] $\bar{t}_{bg} = \bar{t}(b,t_g) = \pm\infty$.

For clarity, a specific example will now be discussed more quantitatively. Consider the case of Fig.5-3(iv), with k = +1 and $\Lambda = 0$. For this case, the solution to [5-5] is the well-known cycloid,

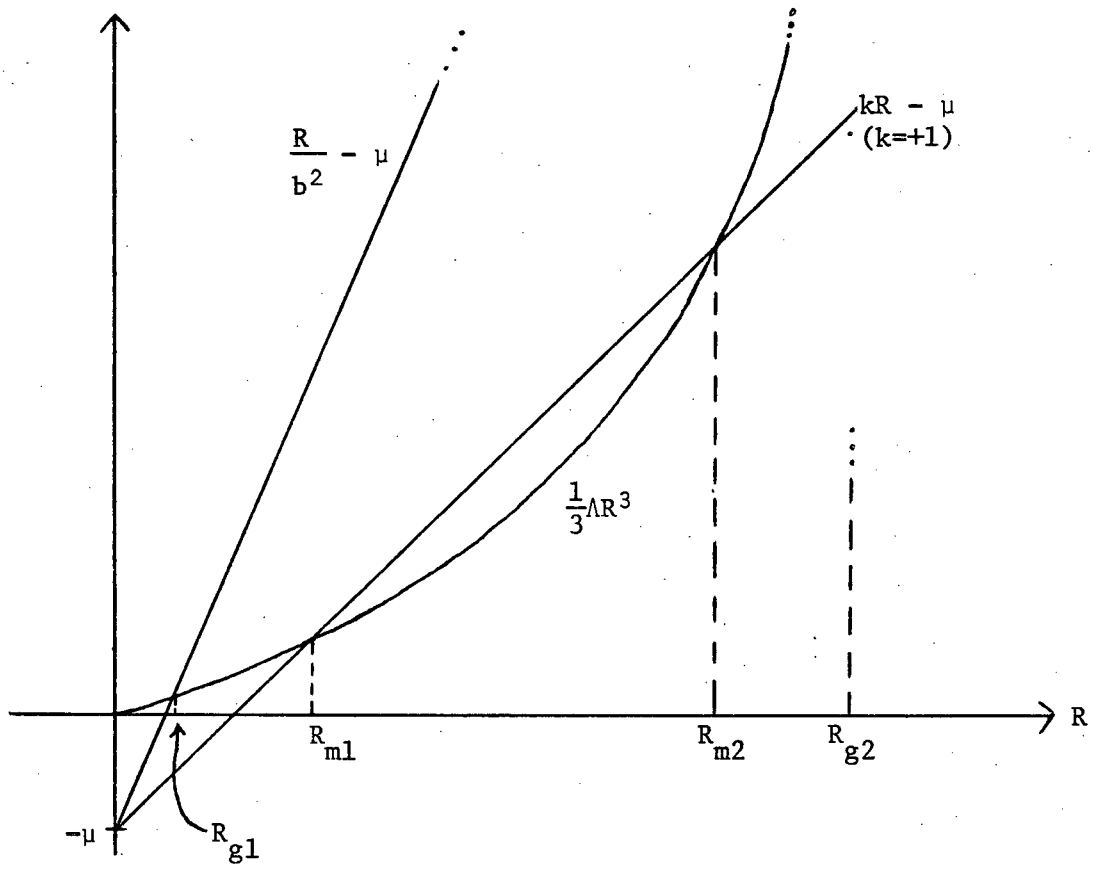

FIG.5-6 . Plot of [5-77] and [5-78] for the models of types V and VI with $k = +1$, $0 < \Lambda < \Lambda_c$.

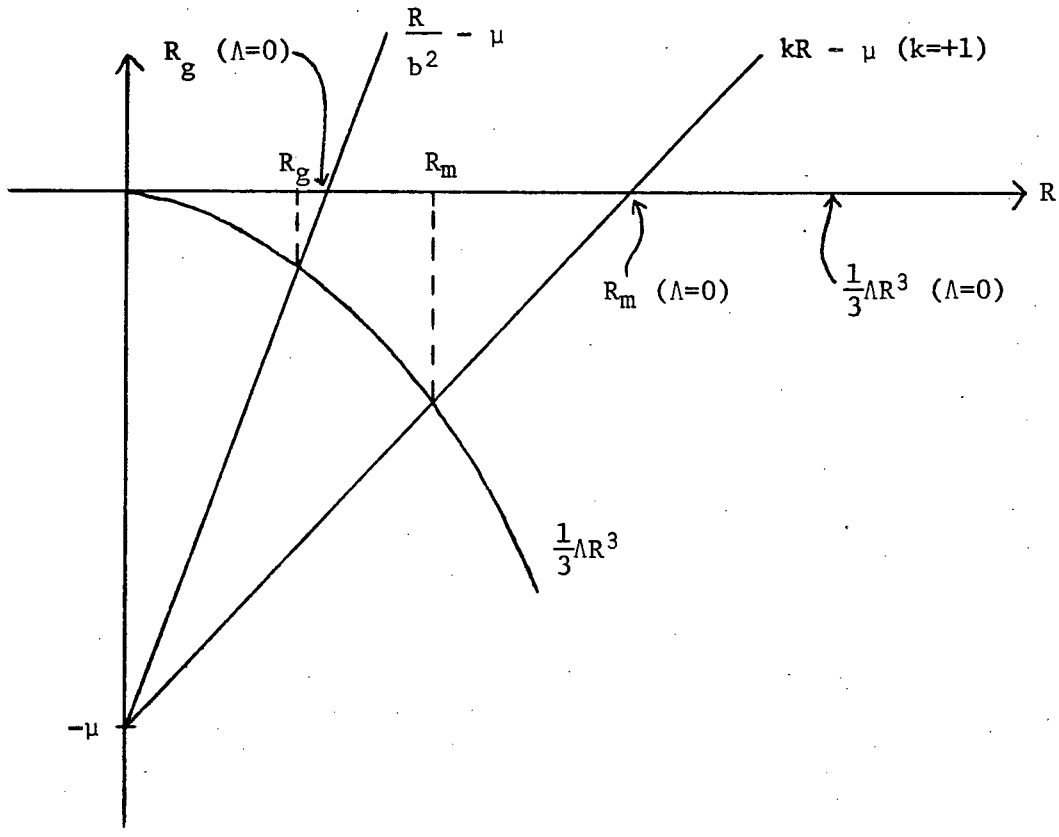

FIG.5-7 . Plot of [5-77] and [5-78] for the model of type V with $k = +1$, $\Lambda \leq 0$.

[5-81] $R = \frac{1}{2}\mu(1 - \cos\psi)$

[5-82] $t = \frac{1}{2}\mu(\psi - \sin\psi)$

(see e.g. Weinberg 1972, p.482-483). Equation [5-27] yields

[5-83] $$\frac{\partial\bar{t}}{\partial t}(b,t) = \frac{\sqrt{1-b^2}}{(1 - \mu b^2/R)} = \frac{\sqrt{1-b^2}\,R}{R - \mu b^2}$$

where the "+" sign has been chosen arbitrarily for $\bar{t}_t(b,t)$. Hence,

[5-84] $$\bar{t}(b,t) = \bar{t}_b = \sqrt{1-b^2}\int^t \frac{R\,dt}{R - \mu b^2} + \text{constant} \quad .$$

Using [5-81] and [5-82], this becomes

[5-85] $$\bar{t}(b,t) = \bar{t}_b = \frac{1}{2}\mu\sqrt{1-b^2}\int^\psi \frac{(1 - \cos\psi)^2 d\psi}{1 - 2b^2 - \cos\psi} + \text{constant} \quad .$$

Upon integration, this becomes

[5-86] $$\bar{t}_b = \frac{1}{2}\mu\sqrt{1-b^2}\left[(1+2b^2)\psi - \sin\psi - \frac{2b^3}{\sqrt{1-b^2}}\ln\left|\frac{b + \sqrt{1-b^2}\tan(\psi/2)}{b - \sqrt{1-b^2}\tan(\psi/2)}\right|\right]$$

where ψ is understood to be a function of t through [5-82]. The constant of integration is chosen to be zero so that $\bar{t}$ is zero when t is zero. The equation in [5-86] is plotted as $\bar{t}_b$ vs. t , with the help of [5-82], in Fig.5-8, for one cycle i.e. $0 \leq \psi \leq 2\pi$. The two values of t_g are denoted by t_{g1} and t_{g2} . In the negative region A the boundary of the metagalaxy is expanding. As $t \to t_{g1}$, the Schwarzschild time $\bar{t}_b \to -\infty$. In other words, it would take an infinite Schwarzschild time $\bar{t}_b$ for the boundary to cross the gravitational radius. An observer in the Schwarzschild spacetime would never see the metagalaxy emerge from the negative region. In the positive region B the metagalaxy has crossed the gravitational radius and is not a black hole.

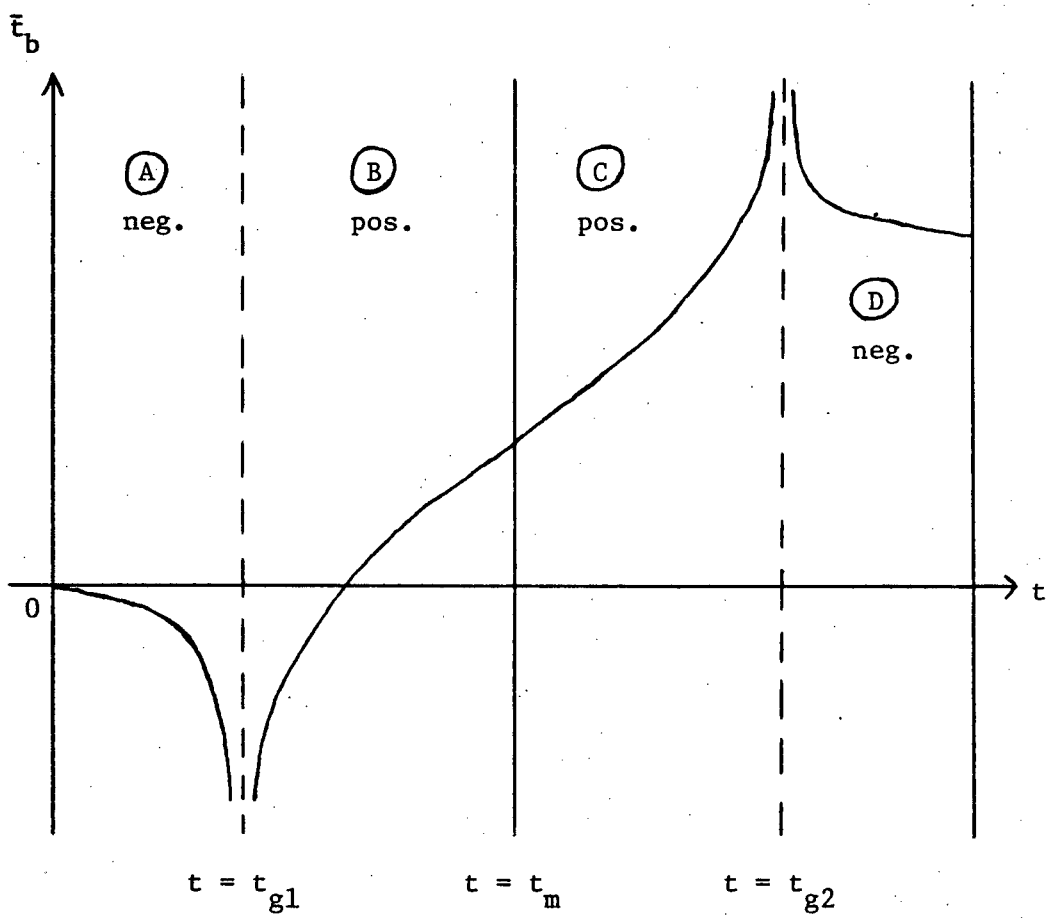

FIG.5-8 . Plot of [5-86]. The two values of t_g are denoted by t_{g1} and t_{g2} .

The boundary is expanding towards the maximum value of R which it reaches at $t = t_m$. It then enters region C in which it begins to contract towards the gravitational radius. In the regions B and C, the metagalaxy may be optically detectable to an observer in the Schwarzschild spacetime. One sees that the metagalaxy emerged from the negative region at $\bar{t}_b = -\infty$, and that it will enter it again at $\bar{t}_b = +\infty$. To a Schwarzschild observer, the boundary of the metagalaxy has always been, and will always be outside the negative region. In region D the metagalaxy has crossed the gravitational radius again, and is a black hole.

The case $R_m = R_g$ does not occur for type V metagalaxies; they are optically detectable to positive region observers in a certain part of their lifetime. If our metagalaxy is of type V, and if the universe consists of a hierarchy of type V metagalaxies, then it is possible that we may be able to detect some of these other metagalaxies.

Turning to other models, one may similarly ask whether or not the constant value of R in the model of type I is equal to R_g, and whether or not the upper limit of R in the model of type IV is equal to R_g. In the type I model ($k = +1$, $\Lambda = \Lambda_c$) the constant value of R, R_c, satisfies [5-78] at $R = R_c$, i.e.

$$[5\text{-}87] \quad \frac{1}{3}\Lambda_c R^3 = kR - \mu \qquad \text{at } R = R_c$$

which is obtained by setting $\dot{R} = 0$ in [5-5], as in [5-78]. Similarly, the upper limit of R, R_ℓ, in the type IV model ($k = +1$, $\Lambda = \Lambda_c$) satisfies

$$[5\text{-}88] \quad \frac{1}{3}\Lambda_c R^3 = kR - \mu \qquad \text{at } R = R_\ell \quad .$$

One sees from Fig.5-3(ii) that

[5-89] $R_\ell = R_c$.

As usual, R_g must satisfy [5-77]. Discussing these models graphically as before, one has:

<u>$k = +1$, $\Lambda = \Lambda_c$ (Fig.5-3(ii)(a) and 5-3(ii)(b))</u>: In these cases, one has $0 \leq b < 1$. One sees from Fig.5-9 that $R_\ell = R_c > R_g$ for all possible values of b. Note that there are two values of R_g, however R(t) does not exceed $R_\ell = R_c$ and hence does not reach the larger value of R_g.

Hence the possibility of $R_\ell = R_c = R_g$ is eliminated. The model of type I is a static metagalaxy. It is a Schwarzschild object whose boundary is situated in the positive region of Schwarzschild spacetime hence it would be optically detectable to an observer in the positive region. In the model of type IV, the boundary expands from R = 0, crosses the gravitational radius and continues expanding towards its upper limit. As usual, at $t = t_g$ one has $\bar{t}_{bg} = \pm\infty$. For $t > t_g$, this metagalaxy is also optically detectable to a positive region observer, and it is optically undetectable to a positive region observer for $t < t_g$.

The remaining models to be considered are of types II, III, and VI. Consider models of type II. They have no values of R at which $\dot{R} = 0$. They can be discussed graphically by plotting both sides of [5-77] as functions of R. There are five models of type II :

<u>$k = -1$, $0 < \Lambda < \Lambda^*$ (Fig.5-1(i))</u>: In this case, $0 \leq b < \infty$. One sees from Fig.5-10 that there are two values of R_g.

<u>$k = -1$, $\Lambda = 0$ (Fig.5-1(ii))</u>: In this case, $0 \leq b < \infty$. One sees from Fig.5-10 that there is one value of R_g. The expression $\Lambda R^3/3$ is the

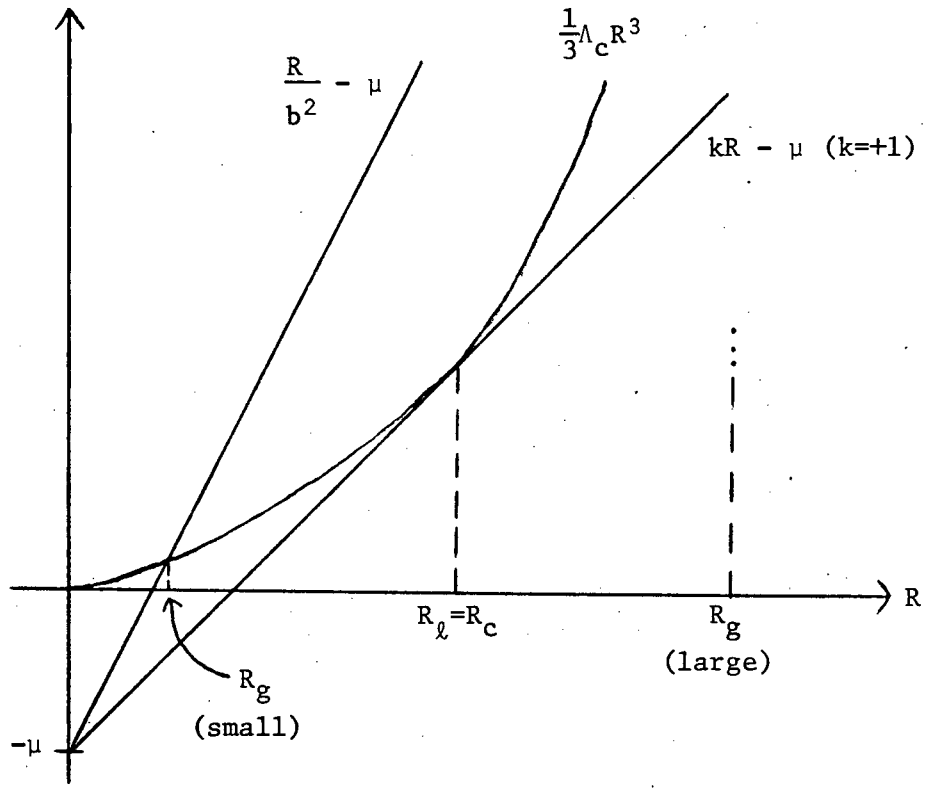

FIG.5-9 . Plot of [5-77], [5-87] and [5-88] for models of types I, III, and IV with $k = +1$, $\Lambda = \Lambda_c$.

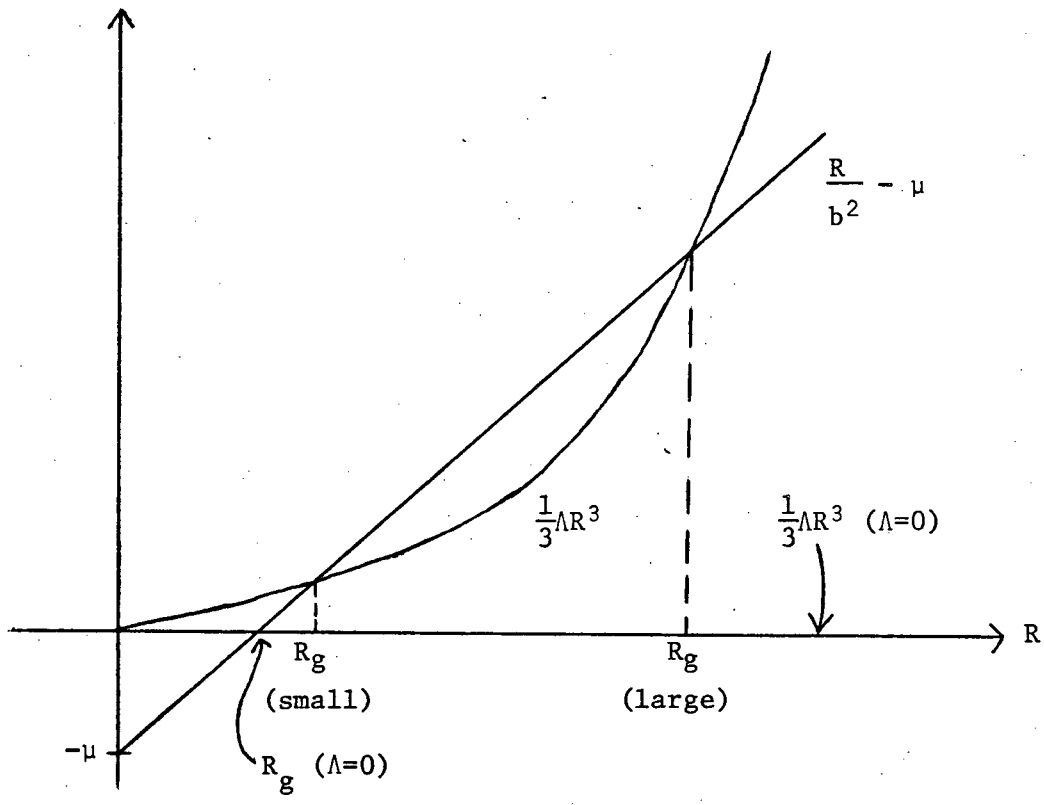

FIG.5-10 . Plot of [5-77] for type II models with $k = -1$, $\Lambda = 0$; $k = -1$, $0 < \Lambda < \Lambda^*$; $k = 0$, $\Lambda = 0$; $k = 0$, $0 < \Lambda < \Lambda^*$; and $k = +1$, $\Lambda_c < \Lambda < \Lambda^*$.

horizontal axis for $\Lambda = 0$.

<u>$k = 0$, $0 < \Lambda < \Lambda^*$ (Fig.5-2(i))</u>: In this case, $0 \leq b < \infty$. This is similar to the first case above, $k = -1$, $0 < \Lambda < \Lambda^*$. There are two values of R_g, as seen in Fig.5-10 .

<u>$k = 0$, $\Lambda = 0$ (Fig.5-2(ii))</u>: In this case $0 \leq b < \infty$. This is similar to the second case above, $k = -1$, $\Lambda = 0$. There is one value of R_g, as seen in Fig.5-10 .

<u>$k = +1$, $\Lambda_c < \Lambda < \Lambda^*$ (Fig.5-3(i))</u>: In this case, $0 \leq b < 1$. This is similar to the first and third cases above. There are two values of R_g, as seen in Fig.5-10 .

In the above models in which there are two values of R_g, the boundary of the metagalaxy initially expands from $R = 0$ in a negative region of Schwarzschild spacetime, crosses the first gravitational radius into a positive region, continues expanding, and crosses the second gravitational radius into a negative region, and continues expanding from there on. One has, as usual, $\bar{t}_{bg} = \pm \infty$ at both values of $t = t_g$. When the boundary is in the positive region between the two gravitational radii, the metagalaxy may be optically detectable to a positive region observer. When the boundary is in the first negative region, the metagalaxy is optically undetectable to a positive region observer.

In the above type II models with $\Lambda = 0$, the boundary begins expanding at $R = 0$, crosses the gravitational radius and continues expanding. After it has crossed the gravitational radius into the positive region, it may be optically detectable to an observer in this region. Also $\bar{t}_{bg} = \pm \infty$ at $t = t_g$.

Now consider the model of type III :

$k = +1$, $\Lambda = \Lambda_c$ (Fig.5-3(ii)(c)): This case has the same values for k and Λ as the models of type I and IV discussed above. One has $0 \leq b < 1$. Fig.5-9 shows there are two values of R_g. R(t) has a minimum value at $R = R_c$ i.e. [5-87] is satisfied. The boundary of this metagalactic model begins expanding at $R = R_c$ at $t = -\infty$, i.e. outside the first gravitational radius in a positive region. It continues expanding in this region (in which it may be optically detectable to an observer in this region) and then crosses the second gravitational radius into a negative region. One has $\bar{t}_{bg} = \pm\infty$ at $t = t_g$.

Finally, consider the model of type VI :

$k = +1$, $0 < \Lambda < \Lambda_c$ (Fig.5-3(iii)(b)): This model has the same values of k and Λ as the type V model of Fig.5-3(iii)(a) discussed earlier. One has $0 \leq b < 1$. R(t) has a minimum value at $R = R_{m2}$. Equations [5-77] and [5-78] (where R_m is the minimum value of R, R_{m2}) can be discussed graphically in Fig.5-6. The boundary of this metagalaxy contracts from $R = \infty$ in a negative region, crosses $R = R_{g2}$ into a positive region, reaches the minimum, expands, crosses R_{g2} into the initial negative region, and expands to infinity. While in the positive region, it may be detectable to a positive region observer. Again $\bar{t}_{bg} = \pm\infty$ at $t = t_g$.

All of the models have been considered now, and it is seen that there exist no models in which the boundary of the metagalaxy is always confined to negative regions, except for those models with $\Lambda > \Lambda^*$ in which no gravitational radius exists.

6. SOME CONSIDERATIONS ABOUT HIERARCHIES WITH SCHWARZSCHILD OBJECTS AS ELEMENTS

The basic features of the concept of the metagalaxy being a Schwarzschild object have been discussed in section 5. These features help to determine the behaviour of a hierarchy of metagalactic Schwarzschild objects. It has been shown how it is possible to embed a Friedmann spacetime at the center of a Schwarzschild spacetime. The same methods can obviously be used to embed Schwarzschild objects in a Friedmann spacetime. These Schwarzschild objects consist of metagalaxies matched to Schwarzschild spacetimes, and these Schwarzschild spacetimes have to be matched to the Friedmann spacetime in which they are embedded. In other words, one can have zeroth order metagalaxies embedded in first order metagalaxies, which may be similarly embedded in second order metagalaxies, and so on. This is analogous to the Newtonian case of zeroth order clusters contained in first order clusters, which may be contained in second order clusters, and so on.

Since there are quite a few metagalactic models that can describe a given metagalaxy, and since there is no obvious reason why all metagalaxies in the hierarchy should be described by the same model, the universe may be a system displaying complicated dynamical motions. An ith order metagalaxy may be contracting while its (i-1)th order elements are expanding, possibly resulting in blue shifts (if the boundaries are in positive Schwarzschild regions), rates of expansion and contraction differ from one metagalaxy to another, and so on.

Consider the interior of our metagalaxy to be a Friedmann spacetime, F_0 , of order zero. To construct a hierarchy of metagalactic

Schwarzschild objects, zeroth order metagalaxies must be embedded in first order metagalaxies, which must be embedded in second order metagalaxies, and so on. Each Friedmann spacetime F_0 must be connected to a Schwarzschild spacetime S_0 and each of these Schwarzschild spacetimes S_0 must be connected to a common first order Friedmann spacetime F_1. Similarly, these Friedmann spacetimes F_1 must be connected to Schwarzschild spacetimes S_1, each of which must be connected to a common Friedmann spacetime F_2, and so on.

Consider the connection between F_0 and S_0. Let r_0 be the comoving r-coordinate for F_0, and let

[6-1] $r_0 = b_0 = \text{constant}$

be the equation for the boundary separating F_0 and S_0. Then equation [5-24] gives

[6-2] $m_0 = \frac{1}{2}\mu_0 b_0^3$.

The subscripts "0" pertain to the Friedmann spacetime F_0. For simplicity, assume that all zeroth order metagalaxies have the same values of m_0, b_0, μ_0 etc. The same goes for higher-order metagalaxies.

Next consider the connection between S_0 and F_1. Let r_1 be the comoving r-coordinate for F_1, and let

[6-3] $r_1 = a_1 = \text{constant}$

be the equation for the boundary separating F_1 and a given S_0. Note that [6-3] describes the S_0-F_1 boundary only for a particular S_0. For the other S_0-F_1 boundaries, different r_1-coordinates, r_1', r_1'', etc., must be used in order for an equation like [6-3] to be valid, i.e. for some other given S_0, the S_0-F_1 boundary may be described by

[6-4] $r_1' = a_1$

assuming that a_1 is the same for all the S_0-F_1 boundaries. To clarify this by analogy, note that only a certain class of straight lines on a two-dimensional x-y plane can be described by x = constant or y = constant. The remaining ones must be described by equations like y = mx + c , or by equations like x′ = constant or y′ = constant where the x′-y′ coordinate system is rotated appropriately with respect to the x-y system. However, the result of this calculation will not involve these different r-coordinates, so they need receive no further mention. Continuing, for the S_0-F_1 boundary described by [6-3], equation [5-24] gives

[6-5] $m_0 = \frac{1}{2}\mu_1 a_1^3$.

Consider now the F_1-S_1 boundary, described similarly by

[6-6] $r_1'' = b_1 =$ constant

where r_1'' is the appropriate r_1-coordinate. Equation [5-24] gives

[6-7] $m_1 = \frac{1}{2}\mu_1 b_1^3$.

Note that a given ith order Friedmann spacetime F_i is connected to S_i by an equation like $r_i = b_i$, and F_i is connected to the Schwarzschild spacetimes S_{i-1} it contains by an equation like $r_i = a_i$. One can easily obtain an expression for b_i/a_i. Dividing [6-7] by [6-5] gives

[6-8] $m_1/m_0 = M_1/M_0 = (b_1/a_1)^3$

[6-9] i.e. $b_1/a_1 = (M_1/M_0)^{1/3}$.

In general,

[6-10] $b_i/a_i = (M_i/M_{i-1})^{1/3}$.

This equation shows a comparison of the "sizes" of the two types of Schwarzschild-Friedmann boundaries of a metagalaxy of order i.

An objection to the concept of the metagalaxy being a Schwarzschild object is that by "cutting off" the Friedmann spacetime that describes the metagalaxy at some boundary r = b, spatial isotropy at all points, on a metagalactic scale, is apparently lost — an observer near the "edge", r = b, of the metagalaxy would not see the same thing in all directions. However, if the universe consists of an infinite hierarchy of metagalaxies of all orders, then spatial isotropy at all points is valid, although on a grander scale than metagalactic. In other words, the cosmological principle, which states that the universe presents the same aspect from every point except for local irregularities (Bondi 1960, p.11), is valid for a hierarchical universe (see Bondi 1960, p.14). The fact that an observer near the edge, r = b, of a metagalaxy sees different things in different directions, would be classified as a "local irregularity" .

As always,observational evidence will have to determine the correctness of the above concepts. Optical detection of other metagalaxies, if they exist, may have to await a future date.

FOOTNOTES

1. (p.54) It has recently been brought to my attention that an earlier attempt to consider the metagalaxy as a black hole, such that $R_m = R_g$, has been made by Pathria(1972). He concludes that the case $R_m = R_g$ can, in fact, occur. However, I believe that his conclusion may be wrong. For further explanation, see appendix A.

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APPENDIX A : ON PATHRIA'S PAPER

Pathria(1972) assumes the interior of the metagalaxy to be described by the Robertson-Walker metric and the exterior by the Schwarzschild metric (with $\Lambda = 0$). However, he fails, in his analysis, to consider the conditions imposed when the two metrics are matched at the boundary separating them. In the present work, it is shown (p.52-54) that the case $R_m = R_g$ does not occur. For $k = +1$ (which is the case Pathria concentrates on) it is seen (p.54) that this is due to the condition $0 \leq b < 1$, i.e. $b \neq 1$. If $b = 1$ was possible, then Figs. 5-6 and 5-7 would show that R_m and R_g could coincide, in agreement with Pathria's results. However, the transformation $\bar{t} = \bar{t}(r,t)$ derived by matching the two metrics at the boundary $r = b$, becomes singular at $b = 1$, as seen from equations [5-27], [5-36] or [5-39]. For instance, consider the case $k = +1$, $\Lambda = 0$ (for this case $\bar{t}_b$ is worked out explicitly on p.57-58). This is one of the cases of Pathria's paper in which he claims $R_m = R_g$ is possible. To agree with Pathria's results, one must have $b = 1$ in equation [5-85], which gives $\bar{t}_b$ = constant for any value of t. This is an unphysical result, hence it seems one must have $b \neq 1$ and $R_m > R_g$.

The singularity in $\bar{t}$ at $b = 1$ is apparently not avoided by transforming the Robertson-Walker metric to another coordinate system. For instance, by transforming to a system with $r = \sin\omega$ ($k = +1$) gives

$$[A\text{-}1] \quad ds_F^2 = dt^2 - R^2(t)[d\omega^2 + \sin^2\omega \, d\Omega^2]$$

which shows that $r = 1$ is only an apparent singularity in the Robertson-Walker metric. However, in this ω,θ,ϕ,t coordinate system, the

factor $\sqrt{1-b^2}$ in $\bar{t}$ becomes

[A-2] $\sqrt{1-b^2} \rightarrow \sqrt{1-\sin^2\omega_b} = \pm\cos\omega_b$

which, again, makes $\bar{t}$ singular at $\omega_b = \pi/2$.

It is interesting to question the physical significance of the condition $0 \leq b < 1$. If one is dealing with only the Robertson-Walker spacetime, then b can certainly assume the value 1. However, when one matches it to a Schwarzschild spacetime, one must apparently have $b \neq 1$. It seems that $b \neq 1$ is necessary to allow for the possibility of connecting the Robertson-Walker spacetime to a Schwarzschild spacetime, i.e. one must "leave room" so to speak, for a smooth connection between the two spacetimes (from remarks made by Prof. G. Fahlman in a conversation).