ANOMALOUS COMMUTATORS AND THE BJL LIMIT

by

DUNCAN M. ROGERS

B.Sc., York University, 1985

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
in
THE FACULTY OF GRADUATE STUDIES
DEPARTMENT OF PHYSICS

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA
August 1987

4 6

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Physics

The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3

Date October 10, 1987

Abstract

The BJL limit is derived and used to calculate the anomalous vector current commutator in QED. It is then shown, by a calculation with the BJL limit for double commutators, that the Jacobi identity fails for two vector currents and one axial vector current in QED.

Contents

List of Figures

Acknowledgement

I would like to thank Dr. Gordon W. Semenoff for helpful discussions.

1. Introduction

Calculating current commutators in field theory is an ambiguous task. The problem of operator ordering in currents containing products of operators gives rise to infinities. It is not easy to define a method which takes an unequal time commutator into an equal time commutator. Fortunately, there exists a procedure which appears to overcome these difficulties. This method calculates the commutators from the well known Green's functions of field theory and lends itself well to perturbative techniques. The procedure is called the Bjorken-Johnson-Low (BJL) limit. But the results obtained with the BJL limit are often different from the values found using canonical commutation relations. Even worse, at times the Jacobi identity actually fails when calculated with the BJL limit. Before we discuss the BJL limit and the anomalies we should look at some results concerning time ordered products and equal time commutators.

2. Schwinger Terms and Seagulls

The equal time commutator of operators A and B can be written as

$$\left[A(t,\vec{x}),B(t,\vec{y})\right]=C(t,\vec{x})\delta^3(\vec{x}-\vec{y})+S^i(t,\vec{x})\frac{\partial}{\partial x^i}\delta^3(\vec{x}-\vec{y}).$$

The terms proportional to the derivative of the delta function do not arise from canonical commutation relations. They were first postulated by Schwinger[1] for current-current commutators in electromagnetism and are called Schwinger terms. Higher order derivatives of the delta function may also be present.

We write the time ordered product of A and B as

$$\begin{aligned}T(x,y)&=TA(x)B(y)\\&=\theta(x_0-y_0)A(x)B(y)+\theta(y_0-x_0)B(y)A(x).\end{aligned}$$

This is not in general Lorentz covariant. We therefore define the covariant time ordered product $T^*(x,y)$ by

$$T^*(x,y)=T(x,y)+\tau(x,y)$$

where $\tau(x, y)$ is the covariantizing seagull. To find the explicit form of the seagull in terms of the Schwinger term we first introduce a unit timelike vector n^μ and the spacelike projection operator

$$P^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu.$$

Then we write the noncovariant expressions in terms of n,

$$\big[A(x), B(y)\big]\, \delta\big((x-y)\cdot n\big) = C(x; n)\delta^4(x-y) + S^\mu P_{\mu\nu}\frac{\partial}{\partial x_\nu}\delta^4(x-y) \tag{1}$$

and

$$T(x, y; n) = \theta\big((x-y)\cdot n\big)\, A(x)\, B(y) + \theta\big((y-x)\cdot n\big) B(y) A(x).$$

The T^* product is, of course, n independent,

$$T^*(x, y) = T(x, y; n) + \tau(x, y; n).$$

We next vary T^* with respect to n, remembering that only spacelike variations are permitted since n^2 is constrained to unity and n is timelike,

$$\begin{aligned} P^{\mu\nu}\frac{\delta}{\delta n^\nu} T^*(x, y) &= P^{\mu\nu}(x-y)_\nu \big[A(x), B(y)\big] + P^{\mu\nu}\frac{\delta}{\delta n^\nu}\tau(x, y; n) \\ &= 0. \end{aligned}$$

Using the commutator of equation (1), this becomes

$$P^{\mu\nu}\frac{\delta}{\delta n^\nu}\tau(x, y; n) = P^{\mu\nu} S_\nu(x; n)\delta^4(x-y).$$

The spacelike projection operator can be removed if a term proportional to n_ν is introduced. However, the definition of S in equation (1) allows us to absorb these extra terms in S. We therefore find

$$\frac{\delta}{\delta n^\nu}\tau(x, y; n) = S_\nu(x; n)\delta^4(x-y). \tag{2}$$

This can be integrated if S is curlless, that is, if

$$\frac{\delta}{\delta n^{\nu}} S_{\mu}(x;n) = \frac{\delta}{\delta n^{\mu}} S_{\nu}(x;n). \tag{3}$$

Gross and Jackiw[2] have shown that equation (3) is always satisfied if the equal time commutator of $A(x)$ and $B(y)$ is considered to be the limit as $x_0 \rightarrow y_0$ of the unequal time commutator of $A(x)$ and $B(y)$. We will see in the next section that the BJL limit allows us to calculate equal time commutators this way. After integrating equation (2) we finally obtain

$$\tau(x,y;n) = \int_n dn'_{\nu}\, S^{\nu}(x;n')\delta^4(x-y) + \tau_{\circ}(x,y)$$

where $\tau_{\circ}(x,y)$ is an arbitrary Lorentz covariant term. This equation shows that the T product is covariant if there are no Schwinger terms in the equal time commutator of A and B.

It is easy to extend our analysis to the case of three operators. We define the covariant T^* product and the seagull τ by

$$T^*(x,y,z) = TA(x)B(y)C(z) + \tau(x,y,z).$$

Following a procedure similiar to the two operator analysis we obtain

$$\begin{aligned}\frac{\delta}{\delta n_{\nu}}\tau(x,y,z;n) = {} & \delta^4(x-y)TS^{\nu}_{AB}(x;n)C(z) \\ & + \delta^4(x-z)TS^{\nu}_{AC}(x;n)B(y) \\ & + \delta^4(y-z)TS^{\nu}_{BC}(y;n)A(x)\end{aligned}$$

where S^{ν}_{AB} is the Schwinger term in the commutator of A and B. This equation can be integrated if the right hand side is curlless. It has been shown[2] that the curlless condition is satisfied if A, B, and C satisfy the Jacobi identity. But, as we will see later, the Jacobi identity sometimes fails. Before we look at these failures we must discuss the BJL limit.

3. The Bjorken-Johnson-Low Limit

The BJL limit is a relation between equal time commutators and time ordered products and allows us to calculate the commutator from the Feynman diagrams of perturbation theory. Consider the matrix element of the T product of operators A and B between arbitrary states,

$$T(q) = \int d^4x\, e^{iq\cdot x}\langle\alpha|TA(x)B(0)|\beta\rangle.$$

To derive the BJL limit we use the integral representation of the step function

$$\theta(x_0) = \int \frac{du}{2\pi}\frac{e^{-ix_0u}}{u+i\epsilon}$$

and write $T(q)$ in terms of the spectral functions

$$\rho(q) = \int d^4x\, e^{iq\cdot x}\langle\alpha|A(x)B(0)|\beta\rangle$$

and

$$\bar{\rho}(q) = \int d^4x\, e^{iq\cdot x}\langle\alpha|B(0)A(x)|\beta\rangle.$$

This results in

$$T(q) = i\int\frac{dq_0'}{2\pi}\left[\frac{\rho(q_0',\vec{q})}{q_0-q_0'+i\epsilon} - \frac{\bar{\rho}(q_0',\vec{q})}{q_0-q_0'-i\epsilon}\right].$$

Now let $q_0\rightarrow\infty$ for fixed $\vec{q}$ to obtain

$$\begin{aligned}
\lim_{q_0\rightarrow\infty} T(q) &= \frac{i}{q_0}\int\frac{dq_0'}{2\pi}\left[\rho(q_0',\vec{q}) - \bar{\rho}(q_0',\vec{q})\right] \\
&+ \frac{i}{q_0^2}\int\frac{dq_0'}{2\pi}q_0'\left[\rho(q_0',\vec{q}) - \bar{\rho}(q_0',\vec{q})\right] \\
&+ \cdots \\
&= \frac{i}{q_0}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\langle\alpha|\left[A(0,\vec{x}),B(0)\right]|\beta\rangle \\
&- \frac{i}{q_0^2}\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\langle\alpha|\left[\dot{A}(0,\vec{x}),B(0)\right]|\beta\rangle \\
&+ \cdots.
\end{aligned}$$

Johnson and Low[3] have shown that the commutator of A and B is logarithmically divergent if the expansion of $T(q)$ at large q_0 contains terms of the form $\frac{1}{q_0}\log q_0$. Similarly, they have shown that the commutator is quadratically divergent if the expansion contains terms of the form $q_0 \log q_0 = \frac{1}{q_0}q_0^2\log q_0$.

In perturbation theory we calculate the covariant T^* product and not the T product. The difference between T and T^* is the seagull and in position space it depends on a delta function and possibly derivatives of delta functions. In momentum space the difference is a polynomial in q_0. Thus, to calculate T from T^* we let $q_0 \to \infty$ at fixed $\vec{q}$ and drop all polynomials in q_0. This technique is illustrated in the next section where we look at a commutator anomaly in QED.

4. The Vector Current Commutator in Quantum Electrodynamics

The time and space components of the vector current $j^\mu = \overline{\psi}\gamma^\mu\psi$ in QED exhibit some interesting commutator properties. If we calculate the commutator by simply using the canonical commutation relations we get the result

$$\left[j^0(t,\vec{x}), j^i(t,\vec{y})\right] = 0. \tag{4}$$

It is easily seen that this result cannot be correct. First, take the vacuum expectation value of equation (4) and differentiate with respect to y^i. Then use current conservation to obtain

$$\langle 0|\left[j^0(t,\vec{x}), \partial_0 j^0(t,\vec{y})\right]|0\rangle = 0.$$

Next, write $\partial_0 j^0$ as $i\left[H, j^0\right]$ and insert a complete set of eigenstates of the energy. Taking the limit $\vec{y} \to \vec{x}$, we finally obtain

$$\sum_n (E_n - E_0)\left|\langle 0|j^0(x)|n\rangle\right|^2 = 0. \tag{5}$$

Since $\langle 0|j^0(x)|n\rangle$ cannot be zero for all excited states, the left hand side of equation (5) must be greater than zero. Canonical reasoning therefore fails in this example.

But the BJL limit allows us to find a different value for the commutator. The T^* product of j^μ and j^ν,

$$T^{*\mu\nu}(q) = \int d^4x\, e^{iq\cdot x}\langle 0|T^* j^\mu(x) j^\nu(0)|0\rangle,$$

is given to one loop order by Fig. 1,

$$T^{*\mu\nu} = -e^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\left[\gamma^\mu \frac{1}{\not p - m + i\epsilon}\gamma^\nu \frac{1}{\not p - \not q - m + i\epsilon}\right].$$

This integral is solved using the Pauli-Villars regularization[4]. The result is

$$T^{*\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\left[-\frac{ie^2}{3\pi}\ln\frac{\Lambda^2}{m^2} - \frac{ie^2}{3\pi}q^2 \int_{4m^2}^{\infty} dr^2 \frac{f(r^2)}{r^2(r^2-q^2)}\right] \tag{6}$$

where

$$f(r^2) = \left(1 + \frac{2m^2}{r^2}\right)\sqrt{1 - \frac{4m^2}{r^2}}$$

and $\Lambda \to \infty$. The terms containing $\ln\frac{\Lambda^2}{m^2}$ are seagulls and can be ignored in our calculation. We handle the terms containing the integral by noting that

$$\lim_{r^2 \to \infty} f(r^2) = 1 + O(r^{-4})$$

and defining

$$g(r^2) = f(r^2) - 1$$

so that

$$\lim_{r^2 \to \infty} g(r^2) = O(r^{-4}).$$

We can now write

$$\begin{aligned} q^2 \int_{4m^2}^{\infty} dr^2 \frac{f(r^2)}{r^2(r^2-q^2)} &= q^2 \int_{4m^2}^{\infty} dr^2 \frac{g(r^2)}{r^2(r^2-q^2)} + q^2 \int_{4m^2}^{\infty} dr^2 \frac{1}{r^2(r^2-q^2)} \\ &= -\int_{4m^2}^{\infty} dr^2 \frac{g(r^2)}{r^2} + \int_{4m^2}^{\infty} dr^2 \frac{g(r^2)}{r^2-q^2} - \ln\left|1 - \frac{q^2}{4m^2}\right| \end{aligned}$$

and take the limit $q_0 \to \infty$,

$$\begin{aligned} \lim_{q_0 \to \infty} q^2 \int_{4m^2}^{\infty} dr^2 \frac{f(r^2)}{r^2(r^2-q^2)} = &-\int_{4m^2}^{\infty} dr^2 \frac{g(r^2)}{r^2} - \frac{1}{q_0^2}\int_{4m^2}^{\infty} dr^2 g(r^2) \\ &- \ln\frac{q_0^2}{4m^2} + \frac{4m^2}{q_0^2} + \frac{\vec{q}^{\,2}}{q_0^2} + O(q_0^{-4}). \end{aligned}$$

Inserting this into equation (6) and dropping all seagull terms results in

$$\begin{aligned} \lim_{q_0 \to \infty} q_0 T^{0i}(q) = &\frac{ie^2}{3\pi}\left[4m^2 - \int_{4m^2}^{\infty} dr^2 g(r^2)\right] q^i \\ &- \frac{ie^2}{3\pi}\lim_{q_0 \to \infty} q_0^2 \ln\frac{q_0^2}{4m^2}\, q^i \\ &+ \frac{ie^2}{3\pi} q^i \vec{q}^{\,2}. \end{aligned}$$

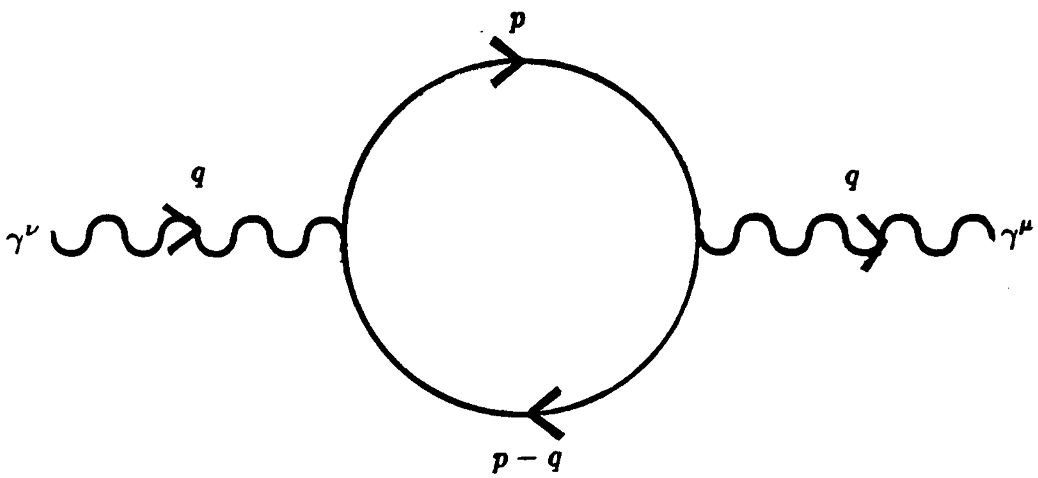

Fig. 1 One loop diagram for $T^{*\mu\nu}$.

From this we finally obtain

$$\langle 0|[j^0(0,\vec{x}), j^i(0)]|0\rangle = -\frac{ie^2}{3\pi}\left[4m^2 - \int_{4m^2}^{\infty} dr^2 g(r^2)\right]\partial_i\delta^3(\vec{x})$$
$$+\frac{ie^2}{3\pi}\lim_{q_0\to\infty} q_0^2 \ln\frac{q_0^2}{4m^2}\,\partial_i\delta^3(\vec{x})$$
$$+\frac{ie^2}{3\pi}\partial_i\vec{\nabla}^2\delta^3(\vec{x}).$$

The commutator calculated with the BJL limit contains a single derivative of a delta function whose coefficients are a quadratic divergence and finite terms. There also is a triple derivative of a delta function with a finite coefficient. The difference between the BJL limit value and the canonical value is one kind of commutator anomaly. The discussion of our other kind of commutator anomaly, the Jacobi identity failure, will have to wait while we look at the BJL limit for double commutators.

5. The Bjorken-Johnson-Low Limit for Double Commutators

The BJL limit for double commutators is derived by an extension of the method for the single commutator BJL limit. The result is

$$\lim_{p_0\to\infty}\lim_{q_0\to\infty} q_0 p_0 T(q,p) = -\int d^3x\, d^3y\, e^{i\vec{q}\cdot\vec{x}} e^{i\vec{p}\cdot\vec{y}}\Big\langle\alpha\Big|\Big[B(0,\vec{y}),\, [A(0,\vec{x}), C(0)]\Big]\Big|\beta\Big\rangle$$

where

$$T(q,p) = \int d^4x\, d^4y\, e^{iq\cdot x} e^{ip\cdot y}\langle\alpha|TA(x)B(y)C(0)|\beta\rangle.$$

The order of the limits is important. If the order is reversed the BJL limit becomes

$$\lim_{q_0\to\infty}\lim_{p_0\to\infty} q_0 p_0 T(q,p) = -\int d^3x\, d^3y\, e^{-i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}}\Big\langle\alpha\Big|\Big[A(0,\vec{x}),\, [B(0,\vec{y}), C(0)]\Big]\Big|\beta\Big\rangle.$$

As in the single commutator case the T product is obtained from the T^* product calculated in perturbation theory by dropping all polynomials in p_0 and q_0. With this formula in hand, we can now look at the Jacobi identity failure.

6. The Jacobi Identity Failure

To test the validity of the Jacobi identity for two vector currents $j^\mu = \overline{\psi}\gamma^\mu\psi$ and one axial vector current $j_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi$ in QED, we must calculate

$$J^{\mu\nu\alpha}(q,p) = \int d^3x\, d^3y\, e^{-i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \Big\langle 0\Big| \Big[j^\mu(0,\vec{x}), \big[j^\nu(0,\vec{y}), j_5^\alpha(0) \big] \Big] + \Big[j^\nu(0,\vec{y}), \big[j_5^\alpha(0), j^\mu(0,\vec{x}) \big] \Big] + \Big[j_5^\alpha(0), \big[j^\mu(0,\vec{x}), j^\nu(0,\vec{y}) \big] \Big] \Big| 0\Big\rangle$$

and see if it is zero or some other value. Each of the terms of $J^{\mu\nu\alpha}$ is obtained from the double BJL limit;

$$\int d^3x\, d^3y\, e^{-i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \Big\langle 0\Big| \Big[j^\mu(0,\vec{x}), \big[j^\nu(0,\vec{y}), j_5^\alpha(0) \big] \Big] \Big| 0\Big\rangle = - \lim_{q_0\to\infty} \lim_{p_0\to\infty} q_0 p_0 T^{\mu\nu\alpha}(q,p), \quad (7)$$

$$\int d^3x\, d^3y\, e^{-i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \Big\langle 0\Big| \Big[j^\nu(0,\vec{y}), \big[j_5^\alpha(0), j^\mu(0,\vec{x}) \big] \Big] \Big| 0\Big\rangle = \lim_{p_0\to\infty} \lim_{q_0\to\infty} q_0 p_0 T^{\mu\nu\alpha}(q,p), \quad (8)$$

and

$$\int d^3x\, d^3y\, e^{-i\vec{q}\cdot\vec{x}} e^{-i\vec{p}\cdot\vec{y}} \Big\langle 0\Big| \Big[j_5^\alpha(0), \big[j^\mu(0,\vec{x}), j^\nu(0,\vec{y}) \big] \Big] \Big| 0\Big\rangle = - \lim_{k_0\to\infty} \lim_{q_0\to\infty} q_0 k_0 T^{\mu\nu\alpha}(q,-q-k) \quad (9)$$

where

$$T^{\mu\nu\alpha}(q,p) = \int d^4x\, d^4y\, e^{iq\cdot x} e^{ip\cdot y} \langle 0| T j^\mu(x) j^\nu(y) j_5^\alpha(0) |0\rangle$$

and $k = -(q+p)$.

The diagrams in Fig. 2 are the lowest order contributions to the covariant time ordered product. The sum of the two contributions is

$$T^{*\mu\nu\alpha}(q,p) = -2ie^2 \int \frac{d^4r}{(2\pi)^4} \mathrm{tr}\left[\frac{1}{\not{r} - \not{q} - m + i\epsilon}\gamma^\mu \frac{1}{\not{r} - m + i\epsilon}\gamma^\nu \frac{1}{\not{r} + \not{p} - m + i\epsilon}\gamma^\alpha\gamma_5 \right].$$

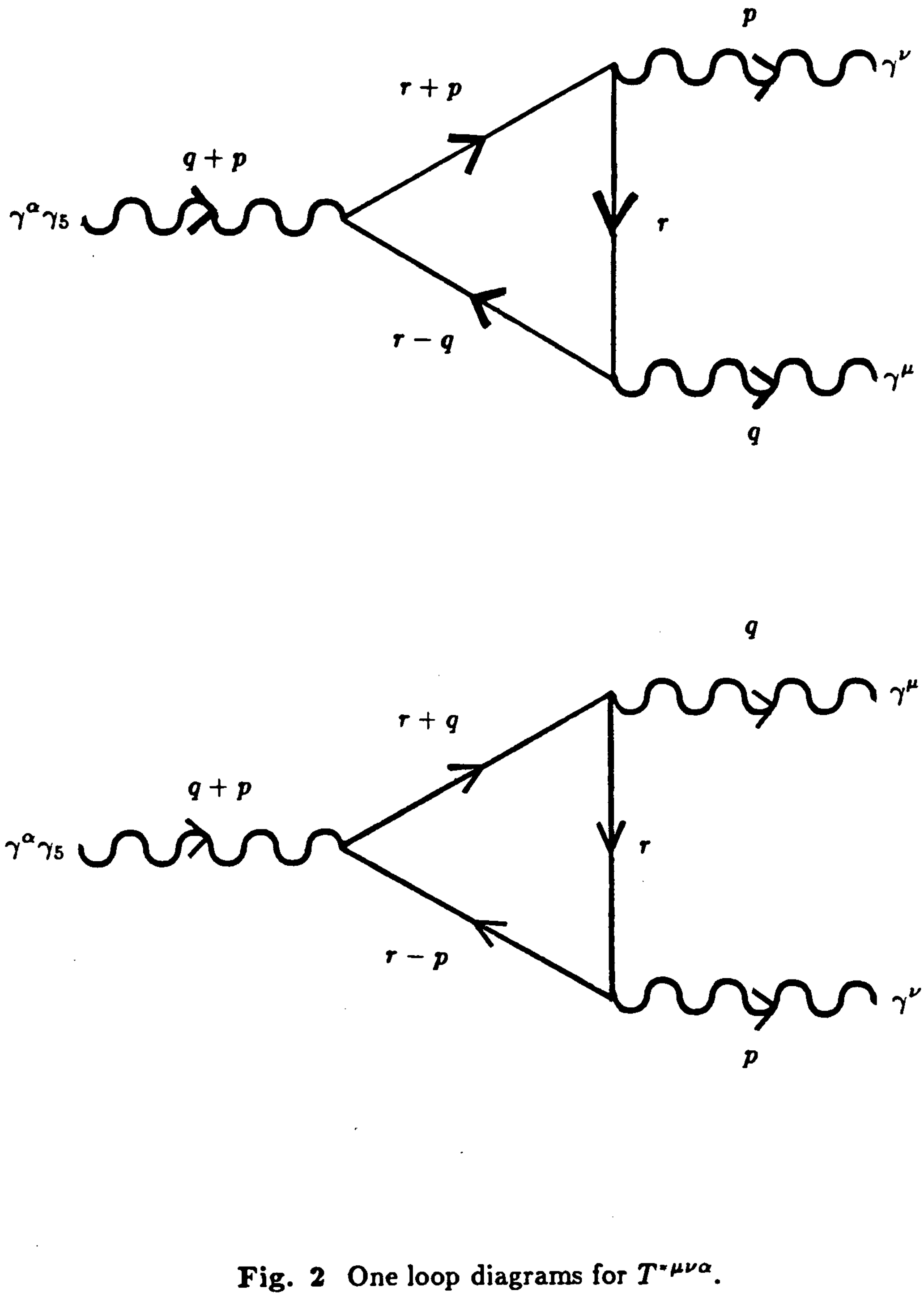

Fig. 2 One loop diagrams for $T^{*\mu\nu\alpha}$.

Rosenberg[5] has solved this integral. The result is

$$
\begin{aligned}
\frac{i\pi^2}{e^2} T^{*\mu\nu\alpha}(q,p) = {} & q_\beta \varepsilon^{\beta\mu\nu\alpha}\Big[q\cdot p I_{11}(q,p) - p^2\big[I_{20}(q,p) - I_{10}(q,p)\big]\Big] \\
& - p_\beta \varepsilon^{\beta\mu\nu\alpha}\Big[q\cdot p I_{11}(p,q) - q^2\big[I_{20}(p,q) - I_{10}(p,q)\big]\Big] \\
& - p^\nu q_\rho p_\beta \varepsilon^{\rho\beta\mu\alpha}\big[I_{20}(q,p) - I_{10}(q,p)\big] \\
& + q^\mu q_\rho p_\beta \varepsilon^{\rho\beta\nu\alpha}\big[I_{20}(p,q) - I_{10}(p,q)\big] \\
& + q^\nu q_\rho p_\beta \varepsilon^{\rho\beta\mu\alpha} I_{11}(q,p) \\
& - p^\mu q_\rho p_\beta \varepsilon^{\rho\beta\nu\alpha} I_{11}(p,q)
\end{aligned}
\tag{10}
$$

where

$$
I_{st}(q,p) = \int_0^1 dx \int_0^{1-x} dy \frac{x^s y^t}{y(1-y)q^2 + x(1-x)p^2 + 2xyq\cdot p - m^2}.
$$

To calculate the right hand side of equation (8), we first do the y integration in I_{st} and then expand the remaining integrand in powers of q_0, dropping any seagull terms as we proceed. For $I_{s0}(q,p)$ this results in

$$
I_{s0}(q,p) = \int_0^1 dx\, x^s \left[-\frac{1}{q_0^2} + 2x\frac{p_0}{q_0^3} + O(q_0^{-4})\right] \ln\left|\frac{x(1-x)p^2 - m^2}{q_0^2}\right|. \tag{11}
$$

The logarithm in the integrand can be written as

$$
-\ln\frac{q_0^2}{m^2} + \ln\frac{p_0^2}{m^2} + \ln\left|x(1-x) - \frac{m^2}{p_0^2} - x(1-x)\frac{\vec{p}^{\,2}}{p_0^2}\right|.
$$

We can now expand the third logarithm in the above expression in powers of p_0 and after dropping a few more seagull terms we obtain

$$
I_{s0}(q,p) = \int_0^1 dx\, x^s \left[-\frac{1}{q_0^2} + 2x\frac{p_0}{q_0^3} + O(q_0^{-4})\right] \left[\ln\frac{p_0^2}{m^2} - \frac{m^2}{p_0^2}\frac{1}{x(1-x)} - \frac{\vec{p}^{\,2}}{p_0^2} + O(p_0^{-4})\right].
$$

Upon examining equation (10) and our expansion of $I_{s0}(q,p)$ we see that those terms in equation (10) proportional to $q_\beta\varepsilon^{\beta\mu\nu\alpha}p^2$ are either seagull terms and do not contribute to $T^{\mu\nu\alpha}$ or are zero when the limits in equation (8) are taken. There is, however, a non-zero result for some of the terms proportional to $p^\nu q_\rho p_\beta \varepsilon^{\rho\beta\mu\alpha}$. When the other I_{st} integrals are expanded in a similiar manner we find that none of them give a non-zero value for the right hand side of equation (8). The result we find for the right hand side of equation (8) is

$$\frac{ie^2}{\pi^2} g^{\nu 0} \varepsilon^{i\mu 0\alpha} p_i \left[\frac{1}{6} \lim_{p_0 \to \infty} p_0^2 \ln \frac{p_0^2}{m^2} - m^2 - \frac{1}{6} \vec{p}^{\,2} \right].$$

Equations (7) and (9) are calculated in a similiar manner and we finally obtain

$$\begin{aligned} J^{\mu\nu\alpha}(q,p) &= \frac{ie^2}{\pi^2} g^{\mu 0} \varepsilon^{i0\nu\alpha} q_i \left[\frac{1}{6} \lim_{q_0 \to \infty} q_0^2 \ln \frac{q_0^2}{m^2} - m^2 - \frac{1}{6} \vec{q}^{\,2} \right] \\ &+ \frac{ie^2}{\pi^2} g^{\nu 0} \varepsilon^{i\mu 0\alpha} p_i \left[\frac{1}{6} \lim_{p_0 \to \infty} p_0^2 \ln \frac{p_0^2}{m^2} - m^2 - \frac{1}{6} \vec{p}^{\,2} \right] \\ &+ \frac{ie^2}{\pi^2} g^{\alpha 0} \varepsilon^{i\mu\nu 0} k_i \left[\frac{1}{6} \lim_{k_0 \to \infty} k_0^2 \ln \frac{k_0^2}{m^2} - m^2 - \frac{1}{6} \vec{k}^{\,2} \right]. \end{aligned}$$

The Jacobi identity failure for the massless theory has been analysed by Levy[6]. In his analysis he encounters some ambiguous terms. To see how these ambiguities arise we expand the logarithm in equation (11) for $m = 0$,

$$\ln \left| x(1-x) \frac{p^2}{q_0^2} \right| = \ln x(1-x) + \ln \frac{p_0^2}{q_0^2} - \ln \left| 1 - \frac{\vec{p}^{\,2}}{m^2} \right|.$$

The $\ln \frac{p_0^2}{q_0^2}$ term can be interpreted in several different ways depending on how we choose to define the limits. For the appropriate definition of the limits it can even be made to disappear. This ambiguous term becomes in our theory a quadratic divergence.

The Jacobi identity fails if one of μ, ν and α is zero and the other two components are spatial and unequal. When this condition is satisfied, $J^{\mu\nu\alpha}$ in position space contains a single derivative of a delta-function whose coefficients are a quadratic divergence and a term proportional to the square of the mass. It also contains a triple derivative of a delta function with a finite coefficient. We have seen that the Jacobi identity does fail. In the conclusion we will attempt to answer the question—why?

7. Conclusion

Are the anomalies we have discussed a part of the theory itself or are they due solely to our method of calculating the commutators? It is possible that the anomalies occur because we worked to lowest order in perturbation theory; if the entire perturbation expansion were summed then perhaps the anomalies would disappear. Perhaps the failure lies with the BJL limit itself. But we showed without the BJL limit, from elementary considerations, that the vector current commutator in QED does not maintain its canonical value; it cannot be zero. Maybe the answer is that the BJL limit value and the canonical value are completely unrelated, that they are entirely different objects. If this were the case the BJL limit would still be useful since it provides, through the sum rules, an experimental means of measuring commutators. A connection between the anomaly in the axial vector Ward identity and the non-canonical value of the vector-axial vector commutator calculated with the BJL limit has been found[7]. It is the Schwinger term in the commutator which causes the anomaly in the Ward identity. No connection has been found for the vector-vector commutator nor for our Jacobi identity failure, but future work may help us discover a connection and then the theory and the BJL limit might someday be consistent.

References

[1] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).

[2] D. J. Gross and R. Jackiw, Nucl. Phys. **B14**, 269 (1969).

[3] K. Johnson and F. E. Low, Prog. Theoret. Phys. (Kyoto), Suppl. **37-38**, 74 (1966).

[4] C. Itzykson and J. Zuber, *Quantum Field Theory*, (McGraw-Hill, New York, 1980).

[5] L. Rosenberg, Phys. Rev. **129**, 2786 (1963).

[6] D. Levy, Imperial College preprint, (Feb. 1986).

[7] S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory*, ed. S. Deser, M. Grisaru, and H. Pendleton, (MIT Press, Cambridge, Mass., 1970);R. Jackiw, in *Lectures on Current Algebra and Its Applications*,ed. S. Treiman, R. Jackiw, and D. Gross, (Princeton University Press, Princeton, NJ, 1972).