# Twistor Inspired Techniques for Gauge Theory Amplitudes 

by<br>Callum Quigley

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#### Abstract

We examine recent developments in perturbative calculations of gauge theory amplitudes. Motivated by a twistor space analysis, Cachazo, Svrcek and Witten (CSW) formulated a new set of rules for computing scattering amplitudes, which have now been dubbed the CSW rules. We examine the origins of these rules, and apply them to supersymmetric and non-supersymmetric gauge theories. We review many of the recent calculations performed using this new prescription at both the tree and one-loop levels.


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## Chapter 1

## Introduction

Gauge theories possess much structure which is not at all manifest when formulated in conventional terms. For example, scattering amplitudes of perturbative Yang-Mills theory are remarkably simple when expressed in a helicity basis. The simplest non-trivial amplitudes involve two incoming gluons of negative helicity and any number of incoming positive helicity gluons. By crossing symmetry these amplitudes are related to $2 \rightarrow n-2$ processes where all initial and final states have the same helicity and so have maximal violation of helicity conservation. Such processes are called maximally helicity violating (MHV) amplitudes. At tree level, they are given by the simple Parke-Taylor formula which can amazingly be written on a single line. Parke and Taylor first conjectured this solution based on a few simple examples (with small $n$ ) [1]. The all $n$ formula was later proven by Berends and Giele using recursive techniques [2]. That such a simple expression could apply to an infinite set of amplitudes, where the number of external legs is arbitrary, was the first major clue of some hidden structure in Yang-Mills theory. As Yang-Mills theory is effectively supersymmetric at tree-level, it should not be surprising that the Parke-Taylor formula was generalized to the case of $\mathcal{N}=4$ super-Yang-Mills (SYM) [3]. This generalization by Nair also uncovered an unexpected simplicity of the next-to-MHV amplitudes (where three external gluons have negative helicities): they are the product of two MHV amplitudes and $1 / P^{2}$.

Loop amplitudes in Yang-Mills theory are notoriously difficult to calculate. At the oneloop level, these amplitudes are only known for up to five external gluons [4]. In supersymmetric theories, however, the situation has proven to be more tractable. In [5] Bern, Dixon, Dunbar and Kosower (BDDK), demonstrated that a large class of one-loop amplitudes, including all massless supersymmetric gauge theories, can be constructed solely from the knowledge of their four-dimensional unitarity cuts. Amplitudes of this type are called cut-constructible. One-loop $\mathcal{N}=4$ amplitudes for four external gluons were first calculated by Green, Schwarz and Brink as the low energy limit of superstring amplitudes [6]. By applying the power of unitarity, BDDK found general expressions for MHV amplitudes in $\mathcal{N}=4$ SYM [7] and later in $\mathcal{N}=1$ SYM [5] for an arbitrary numbers of external legs. An important feature of cut-constructibility is that the cuts are applied not to individual diagrams, but to the amplitude as a whole, thus avoiding the use of cumbersome Feynman diagrams. This technique was largely known during the sixties under the title $S$-matrix analysis [8]. BBDK capitalized on the new simple tree-level expression, the Parke-Taylor formula and Nair's generalization thereof, and sewed them together using unitarity into loop amplitudes.

More recently, Witten observed that many remarkable features of gauge theories emerge when formulated in twistor space [9]. Specifically, scattering amplitudes must be localized on curves of a specific degree when written in twistor variables. At tree-level, MHV amplitudes lie on degree one curves, NMHV amplitudes lie on curves of degree two, and so on. At loop level, the degree is increased by one, for example MHV loop amplitudes are localized on degree two curves in twistor space. That the amplitudes were so constrained could not
have been deduced simply from their momentum space representations. Witten went on to conjecture a weak-weak duality between $\mathcal{N}=4$ SYM (in Minkowski space) and the so-called B-model of topological strings in twistor space $\left(\mathbb{C P}^{3}\right)$. This duality was further investigated and inspired other twistor string theories $[10,11,12]$.

It was soon realized that the degree $d$ curves which supported scattering amplitudes could equivalently be interpreted as $d$ degree one curves, or some intermediate combination [13]. As each degree one curve corresponds to an MHV amplitude, this led Cachazo, Svrcek and Witten (CSW) to conjecture a new set of rules for perturbative Yang-Mills amplitudes, using MHV amplitudes as vertices [14] and connecting them by scalar propagators. This conjecture generalized Nair's results for NMHV amplitudes. These CSW rules were later extended to include scalars and fermions [15], which passed several tree-level consistency checks [16, 17]. These rules were also used to construct a new recursive technique for tree amplitudes [18] and to compute new sets of explicit tree-level amplitudes in (super)Yang-Mills [19, 20, 21]. Higgs fields and massive vectors were also incorporated at tree-level [22].

At first the twistor space structure of gauge theory loop amplitudes was poorly understood [23]. It was soon realized that the conjectured dual topological string theory would inevitably lead to loop amplitudes for conformal supergravity[24], and it was unclear whether the CSW rules would require modification at the loop level. Nevertheless, Brandhuber, Spence and Travaglini (BST) applied the CSW rules directly to the $\mathcal{N}=4$ SYM MHV loop amplitude and found perfect agreement with BBDK's original computation [25]. This immediately raised the question if the CSW rules held at one-loop in less supersymmetric theories. This author, with Rozali, showed in [26] that the CSW rules work in any supersymmetric gauge theory by computing the MHV one-loop amplitudes. Our results were confirmed by the authors of [27] who went on to show, however, that the CSW rules failed in non-supersymmetric Yang-Mills theory [28].

The original confusion regarding the twistor space structure of loops was traced back to a holomorphic anomaly [29], which has also been used to calculate supersymmetric loop amplitudes [30]. Studies of unknown NMHV loop amplitudes' twistor space support [31] were conducted. Eventually, these amplitudes were computed using generalized unitarity [32] combined with cut-constructibility. Specifically, all NMHV amplitudes in $\mathcal{N}=4$ SYM [33] and many (including all $n \leq 6$ ) NMHV amplitudes with $\mathcal{N}<4$ [34] are now known. Witten's original work on amplitudes in twistor space has also led to new breakthroughs in tree-level calculations [35], non-supersymmetric loop amplitudes [36] and amplitudes in (super)gravity theories [37] including conformal supergravity [38].

The remainder of this thesis is structured as follows. Chapter 2 introduces the essential features of supersymmetry we will need throughout the work. We assume the reader is familiar with quantum field theory, and also has a basic knowledge of group theory. No prior knowledge of supersymmetry is assumed. After reviewing spinors and spinor notation, we discuss Coleman and Mandula's no-go theorem for extending the symmetries of spacetime. By allowing fermionic generators, we by-pass the no-go theorem, and arrive at the supersymmetry algebra. We incorporate $\mathcal{N}=1$ supersymmetry into field theory and discuss the notions of superspace and superfields. We then proceed to write down Lagrangians for $\mathcal{N}=1$ supersymmetric theories. In the last two sections of the chapter we consider larger symmetry groups, such as extended supersymmetry and superconformal groups.

The next three chapters study tree-level amplitudes in gauge theories. Chapter 3 discusses
the Parke-Taylor formula for MHV amplitudes. We analyze the conformal properties of the amplitudes, which motivates us to Fourier transform them to twistor space. We follow Witten's analysis in [9], deriving the fact that scattering amplitudes are localized on curves of specific degrees. Chapter 4 introduces the CSW rules, and explains the steps of calculating MHV diagrams. We derive a general expression for NMHV amplitudes, and check it for a simple 5-point amplitude. We discuss other consistency checks which support the legitimacy of the CSW rules. In Chapter 5, we write down Nair's generalization of the Parke-Taylor amplitude and extend the CSW rules to include fermions and scalars.

The following three chapters of this thesis examine loop amplitudes in supersymmetric gauge theories. Chapter 6 reviews the known results of MHV loop amplitudes in supersymmetric gauge theories, originally found by BBDK. In Chapter 7, we discuss how the MHV diagrams are used in loop calculations. We summarize these steps by giving a review of the BST calculation for the $\mathcal{N}=4 \mathrm{MHV}$ amplitude. Chapter 8 presents a fully-detailed computation of an MHV loop diagram. The calculation we carry out, originally performed by this author and Rozali in [26], is the $\mathcal{N}=1$ chiral multiplet contribution to a one-loop MHV amplitude.

We discuss some recent results which have developed out of these new techniques, including some works in progress, in the final chapter. Our notations and conventions are summarized in the Appendix, where we also present the most general supersymmetry algebra.

## Chapter 2

## Supersymmetry

Though currently unverified by experiment, supersymmetry, or SUSY for short, is the best candidate for physics beyond the Standard Model. By postulating a global symmetry between bosons and fermions, SUSY is able to soften the UV divergences in quantum processes as the contributions to loop amplitudes for each particle type comes with opposite signs. We will see that SUSY links internal symmetries to the external (spacetime) symmetries, thus modifying the Poincaré group into what's called the super-Poincaré group. As any theory which is locally invariant under the Poincare group contains gravity, then any theory locally invariant under the super-Poincaré group will contain supergravity. It is widely believed that incorporating SUSY locally (particularly in higher spacetime dimensions) will help solve the longstanding UV problem of quantum gravity. Of course, no such Bose-Fermi degeneracy has ever been observed, so SUSY must be broken at sufficiently high energy to agree with experiment. Exactly how SUSY is broken remains an important open question, which future collider experiments, such as the Large Hadron Collider, will hopefully give some indication as to its solution.

We will not concern ourselves with many of these exciting ideas here, and will focus only on the much simpler unbroken global SUSY in flat four-dimensional spacetime. This section is mainly a review of the material found in the first seven chapters of [39] and the first two sections of [40], as well as various portions of [41, 42, 43]. For more regarding supergravity, and SUSY in $d>4$ the author suggests [39, 41], and for more on SUSY breaking see [40].

### 2.1 Spinors

Spinors will play a fundamental role in this work, especially in describing supersymmetric theories, and so perhaps a quick review of them is in order. Our conventions and notations for spinors are summarized in the Appendix. In Minkowski space, with signature (,,,+--- ), the (non-compact) Lorentz group is $S O(1,3)$ and is generated by the Lorentz transformations $J_{a b}$. The unbounded actions are the boosts, generated $J_{0 i}=K_{i}$, while the remaining compact rotation group, $S O(3)$, is generated by $J_{i j}=\epsilon_{i j k} J_{k}$, where $i, j=1,2,3$. Because the Lorentz group is not compact, all finite dimensional representations are reducible. To classify these representations we consider the Lorentz group in Euclidean space, $S O(4)$ (which is compact). Locally on the group manifold, $S O(4) \simeq S U(2)_{L} \times S U(2)_{R}$ and the $S U(2)_{L, R}$ are generated by the operators $J_{i} \pm i K_{i}$. We label the reducible representations of the Lorentz group by the (half-)integer pairs $\left(j_{L}, j_{R}\right)$, where $j_{L}$ and $j_{R}$ denote the spins of the corresponding $2 j+1$ dimensional irreducible representations of the associated $S U(2)$.

Lefthanded spinors are those which transform in the ( $\frac{1}{2}, 0$ ) representation and are labeled with spinor indices, for example $\lambda_{\alpha}$. Righthanded spinors transform as $\left(0, \frac{1}{2}\right)$ objects and are denoted by dotted spinor indices, such as $\widetilde{\lambda}_{\dot{\alpha}}$. The dotted and undotted spinor indices run
over $\alpha, \dot{\alpha}=1,2$. We emphasize that these spinors are commutative. Later, when we discuss fermions, we will require spinors that anticommute.

Angular momentum is added in the usual way, separately for each representation, thus

$$
\begin{equation*}
\left(\frac{1}{2}, 0\right) \otimes\left(\frac{1}{2}, 0\right)=(0,0) \oplus(1,0) \tag{2.1}
\end{equation*}
$$

The scalar $(0,0)$ piece comes from the antisymmetric product of two spinors; that is, by contracting via the invariant antisymmetric tensor $\epsilon_{\alpha \beta}$. Spinor indices are raised and lowered by this tensor and its inverse $\epsilon^{\alpha \beta}$, i.e.: $\lambda^{\alpha}=\epsilon^{\alpha \beta} \lambda_{\beta}$, etc. We will often write the scalar product of two (lefthanded) spinors as

$$
\begin{equation*}
\epsilon^{\alpha \beta} \lambda_{\alpha} \mu_{\beta} \equiv\langle\lambda, \mu\rangle \tag{2.2}
\end{equation*}
$$

Note that because of the antisymmetric tensor $\epsilon^{\alpha \beta},\langle\lambda, \mu\rangle=-\langle\mu, \lambda\rangle^{1}$. As one would expect, there exists an identical antisymmetric tensor for righthanded spinors, $\epsilon_{\dot{\alpha} \dot{\beta}}$. For scalar products of righthanded spinors we often use the notation

$$
\begin{equation*}
\epsilon^{\dot{\alpha} \dot{\beta}} \widetilde{\lambda}_{\dot{\alpha}} \widetilde{\mu}_{\dot{\beta}} \equiv[\widetilde{\lambda}, \widetilde{\mu}] . \tag{2.3}
\end{equation*}
$$

The symmetric ( 1,0 ) part in equation (2.1) is not a vector; in general, a Lorentz tensors must contain an equal number of dotted and undotted spinor indices. Rather, it is the selfdual portion of an antisymmetric 2 -form (a ( 0,1 ) object is the anti-self-dual part), which will be discussed briefly in the next paragraph. Forming a vector out of spinors requires one of each chirality, since this transforms as

$$
\begin{equation*}
\left(\frac{1}{2}, 0\right) \otimes\left(0, \frac{1}{2}\right)=\left(\frac{1}{2}, \frac{1}{2}\right) \tag{2.4}
\end{equation*}
$$

which is the correct representation of a vector. Thus, a vector in spinor notation is written $V_{\alpha \dot{\alpha}}$. To translate between spinor and tensor notations, we use the Pauli matrices $\sigma^{a}$ as Clebsh-Gordon coefficients

$$
V_{\alpha \dot{\alpha}} \equiv\left(\sigma^{a}\right)_{\alpha \dot{\alpha}} V_{a}=\left(\begin{array}{cc}
V_{0}+V_{3} & V_{1}-i V_{2}  \tag{2.5}\\
V_{1}+i V_{2} & V_{0}-V_{3}
\end{array}\right)_{\alpha \dot{\alpha}}
$$

Clearly $\operatorname{det}(V)=V_{a} V^{a}$, so a lightlike vector is one with vanishing determinant. This is possible only when

$$
\begin{equation*}
V_{\alpha \dot{\alpha}}=\lambda_{\alpha} \widetilde{\lambda}_{\dot{\alpha}} \tag{2.6}
\end{equation*}
$$

for some $\lambda_{\alpha}$ and $\widetilde{\lambda}_{\dot{\alpha}}$, since these are commuting spinors ${ }^{2}$. Note that while giving $\lambda$ (or equivalently $\widetilde{\lambda}$ ) determines $V$ uniquely, the converse is not true. Given a null vector $V, \lambda$ and $\widetilde{\lambda}$ are only fixed up to the overall scaling

$$
\begin{equation*}
\lambda \rightarrow z \lambda \quad, \quad \tilde{\lambda} \rightarrow \frac{1}{z} \tilde{\lambda} \tag{2.7}
\end{equation*}
$$

[^0]for any non-zero $z \in \mathbb{C}$. For two null vectors $V_{\alpha \dot{\alpha}}=\lambda_{\alpha} \widetilde{\lambda}_{\dot{\alpha}}$ and $W_{\alpha \dot{\alpha}}=\mu_{\alpha} \widetilde{\mu}_{\dot{\alpha}}$, their scalar product is given by
\[

$$
\begin{equation*}
2 V_{a} W^{a}=V_{\alpha \dot{\alpha}} W^{\alpha \dot{\alpha}}=\langle\lambda, \mu\rangle[\widetilde{\lambda}, \widetilde{\mu}] \tag{2.8}
\end{equation*}
$$

\]

We conclude this section with some remarks on rank two tensors. A general rank two tensor may be considered as the product of two Lorentz vectors, which has the Clebsh-Gordon decomposition

$$
\begin{equation*}
\left(\frac{1}{2}, \frac{1}{2}\right) \otimes\left(\frac{1}{2}, \frac{1}{2}\right)=(0,0) \oplus(1,0) \oplus(0,1) \oplus(1,1) . \tag{2.9}
\end{equation*}
$$

This partition makes explicit the division between symmetric and anti-symmetric parts. The symmetric $(0,0) \oplus(1,1)$ portion contains $1 \times 1+3 \times 3=10$ components, while the antisymmetric $(1,0) \oplus(0,1)$ piece has $3 \times 1+1 \times 3=6$, as expected. The symmetric piece further decomposes into a scalar, the trace, and a traceless symmetric tensor. The simplest, most commonly used symmetric rank-two tensor is the flat spacetime metric $\eta_{a b}$, which in spinor notation is written

$$
\begin{equation*}
\eta_{\alpha \dot{\alpha} \beta \dot{\beta}}=\epsilon_{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}} . \tag{2.10}
\end{equation*}
$$

The decomposition (2.9) also illustrates the fact that any anti-symmetric 2 -form can be broken up into its self-dual and anti-self-dual components,

$$
\begin{equation*}
F_{a b}=-\dot{-} F_{b a}=F_{a b}^{+}+F_{a b}^{-} \tag{2.11}
\end{equation*}
$$

which, as noted before, are the $(1,0)$ and $(0,1)$ pieces, respectively. The self-dual, $F^{+}$, and anti-self-dual, $F^{-}$, tensors are defined by

$$
\begin{equation*}
F^{ \pm} \equiv \frac{1}{2}(F \mp i * F), \quad \text { where } \quad * F_{a b} \equiv \frac{1}{2} \epsilon_{a b c d} F^{c d} \tag{2.12}
\end{equation*}
$$

is dual to $F_{a b}$, because then

$$
\begin{equation*}
* F^{ \pm}= \pm i F^{ \pm} \tag{2.13}
\end{equation*}
$$

These names refer to the fact that in Euclidean signature, the eigenvalues are $\pm 1$, instead of $\pm i$. In spinor notation, this decomposition takes the form

$$
\begin{equation*}
F_{\alpha \dot{\alpha} \dot{\beta}}=F_{\alpha \dot{\beta}}^{+} \epsilon_{\dot{\alpha} \dot{\beta}}+\epsilon_{\alpha \beta} F_{\dot{\alpha} \dot{\beta} \dot{\prime}}^{-} \tag{2.14}
\end{equation*}
$$

When the 2-form $F_{a b}$ represents the gauge invariant field strength of some gauge field $A_{a}$, then $F_{a b}^{+}$corresponds to positive helicity particles, while $F_{a b}^{-}$gives the negative helicity states.

### 2.2 Super-Poincaré Algebras

Discussions of SUSY often begin by considering the Coleman-Mandula Theorem [44] from 1967. This rigourously proved theorem tightly constrains the allowed symmetry group $\mathcal{G}$ of the $S$-matrix (whose matrix elements are scattering amplitudes). It states that given the following assumptions:
i) (Lorentz invariance) $\mathcal{G}$ contains the Poincaré group, $\operatorname{ISO}(1,3)$, as a subgroup,
ii) (Particle finiteness) For any finite $M$, there exist a finite number of particle species with mass $<M$,
iii) (Weak elastic analyticity) Scattering amplitudes are analytic functions of the Mandelstam variables $s$ and $t$,
iv) (Occurrence of scattering) In general, any two particles will interact to some degree,
v) (Dependance on Lie algebras) Any element of $\mathcal{G}$ may be obtained by the appropriate exponentiation of its Lie algebra generators,

Then, $\mathcal{G}$ is a direct product of the Poincaré group and an internal symmetry group.
The Coleman-Mandula Theorem does not mention discrete symmetries, so to be precise we should include at least $C P T$. To any physicist, with a basic understanding of particle physics and group theory, the above assumptions are quite reasonable and mild. How then, could one go beyond the symmetry group of the Standard Model whose symmetries are $\mathcal{G}=I S O(1,3) \times S U(3) \times S U(2) \times U(1)$ ? In theories with a completely massless spectrum, the Poincar'e group may be enlarged to the conformal group. However, quantum corrections generically spoil this invariance so we will not consider this possibility now (though we will return to it at the end of this chapter). It was long thought that the only other possible extensions were to enlarge the internal gauge group, for example to $S U(5)$ or $S O(10)$. However, a more profound result arises from weakening the fifth assumption. Though the best known continuous symmetries are described by Lie algebras with their commutation relations, there are mathematical groups called graded Lie algebras which possess anti-commutation relations.

The usual symmetries of a field theory are generated by: Lorentz transformations $J_{a b}$ and spacetime translations $P_{a}$ and satisfy

$$
\begin{gather*}
{\left[J_{a b}, J_{c d}\right]=i\left(\eta_{a d} J_{b c}-\eta_{a c} J_{b d}+\eta_{b c} J_{a d}-\eta_{b d} J_{a c}\right)}  \tag{2.15}\\
{\left[J_{a b}, P_{c}\right]=i\left(\eta_{b c} P_{a}-\eta_{a c} P_{b}\right)}  \tag{2.16}\\
{\left[P_{a}, P_{b}\right]=0} \tag{2.17}
\end{gather*}
$$

as well as any internal (gauge) symmetries each with generators $T_{r}$ such that

$$
\begin{gather*}
{\left[T_{r}, T_{s}\right]=i f_{r s}{ }^{t} T_{t}}  \tag{2.18}\\
{\left[J_{a b}, T_{r}\right]=0=\left[P_{a}, T_{r}\right]} \tag{2.19}
\end{gather*}
$$

where $f_{r s}{ }^{t}$ are the group's structure constants. The conserved quantities associated with these symmetries are: (generalized) angular momentum, 4 -momentum and any relevant quantum numbers (electric charge, isospin, etc.). Angular momentum is not a distinct charge as it is determined by the moment of the momentum vector $J_{a b}=x_{a} P_{b}-x_{b} P_{a}$. A corollary to the Coleman-Mandula Theorem is then, 4 -momentum and various quantum numbers are the only possible (distinct) conserved quantities. Note that (2.16) and (2.19) tell us that these charges transform as in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(0,0)$ representations of the Lorentz group, respectively, i.e.: they are vectors and scalars.

By relaxing the fifth assumption we allow for anticommuting (fermionic) generators which transform in the $\left(\frac{1}{2}, 0\right) \bigoplus\left(0, \frac{1}{2}\right)$ representations of the Lorentz group. That is to say, we
introduce spinor charges $Q_{\alpha}^{A}$, as well as their complex conjugates $\bar{Q}_{\dot{\alpha} A}$, where $A=1, \ldots, \mathcal{N}$. These spinor operators must satisfy the relations

$$
\begin{gather*}
\left\{Q_{\alpha}^{A}, \bar{Q}_{\dot{\alpha} B}\right\}=2\left(\sigma^{a}\right)_{\alpha \dot{\alpha}} P_{a} \delta_{B}^{A}  \tag{2.20}\\
\left\{Q_{\alpha}^{A}, Q_{\beta}^{B}\right\}=\epsilon_{\alpha \beta} Z^{A B}  \tag{2.21}\\
\left\{\bar{Q}_{\dot{\alpha} A}, \bar{Q}_{\dot{\beta} B}\right\}=-\epsilon_{\dot{\alpha} \dot{\beta}} \bar{Z}_{A B} \tag{2.22}
\end{gather*}
$$

The antisymmetric matrices $Z$ and $\bar{Z}$ are called the central charges and they commute with all other charges in the theory including themselves

$$
\begin{equation*}
\left[Z^{A B}, \text { anything }\right]=0=\left[\bar{Z}_{A B}, \text { anything }\right] \tag{2.23}
\end{equation*}
$$

in particular they are Lorentz scalars. There exists an additional internal symmetry, called R-symmetry, which rotates all the $Q_{\alpha}^{A}$ amongst themselves. When all central charges vanish, which we will assume from now on, this extra symmetry is $U(\mathcal{N})_{R}$. Its generators $R_{r}$ obey relations analogous to (2.18) and (2.19), while their commutation relations with the SUSY generators are

$$
\begin{gather*}
{\left[Q_{\alpha}^{A}, R_{r}\right]=\left(U_{r}\right)_{B}^{A} Q_{\alpha}^{B}}  \tag{2.24}\\
{\left[\bar{Q}_{\dot{\alpha} A}, R_{r}\right]=\left(U_{r}^{\dagger}\right)_{A}^{B} \bar{Q}_{\dot{\alpha} B}} \tag{2.25}
\end{gather*}
$$

The commutators of the SUSY charges with the Poincare generators show that they are constant spinors

$$
\begin{gather*}
{\left[Q_{\alpha}^{A}, P_{a}\right]=0=\left[\bar{Q}_{\dot{\alpha} A}, P_{a}\right]}  \tag{2.26}\\
{\left[Q_{\alpha}^{A}, J_{a b}\right]=\left(\sigma_{a b}\right)_{\alpha}^{\beta} Q_{\beta}^{A}}  \tag{2.27}\\
{\left[\bar{Q}_{\dot{\alpha} A}, J_{a b}\right]=\left(\bar{\sigma}_{a b}\right)_{\dot{\alpha}}^{\dot{\beta}} \bar{Q}_{\dot{\beta} A} .} \tag{2.28}
\end{gather*}
$$

Notice that since the SUSY charges are spinors this automatically means that they do not generate a new internal (scalar) symmetry. SUSY modifies/extends the external symmetry group of spacetime itself into the super-Poincaré group.

It should not be surprising that the must satisfy (2.20), with $P_{a}$ on the right hand side, as it is the only possibility. After all, the $Q$ and $\bar{Q}$ charges are conserved, thus their anticommutator should be as well. Since the left hand side transforms in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation so should the right hand side, and the Coleman-Mandula allows for a single such conserved quantity, namely $P_{a}$. Similar, though more involved arguments hold for the remaining relations in the SUSY algebra, and are given in [41]. We may also realize why spin $\frac{3}{2}$, that is $\left(1, \frac{1}{2}\right) \oplus\left(\frac{1}{2}, 1\right)$, generators were not considered: their anticommutators would lead to a spin 3 conserved charge which cannot exist. In 1975, Haag, Lopuszanski and Sohnius proved that the graded Lie algebra given in (2.15)-(2.28) is the unique extension to the Coleman-Mandula case obtainable by allowing fermionic generators [45]. Without convincing physical reasons to further weaken the assumptions of the Coleman-Mandula Theorem, one could claim [41]:

Supersymmetry is the only possible extension of the known spacetime symmetries of particle physics.

## $2.3 \mathcal{N}=1$ Supermultiplets

We will now incorporate SUSY algebras into quantum field theories. Later chapters will only require the understanding of massless SUSY field theories, so the massive cases will not be addressed here (for more on these, consult the references). This section will develop the formalism for $\mathcal{N}=1$ SUSY theories, and a latter section will address extended, $\mathcal{N}>1$, cases.

To classify massless fields with energy $E$, we boost to the lightcone frame where $P_{a}=$ ( $E, 0,0, E)$ so that

$$
\left\{Q_{\alpha}, \bar{Q}_{\dot{\alpha}}\right\}=4 E\left(\begin{array}{ll}
1 & 0  \tag{2.29}\\
0 & 0
\end{array}\right)_{\alpha \dot{\alpha}}
$$

By rescaling the charges we can define the annihilation and creation operators

$$
\begin{equation*}
a_{\alpha} \equiv \frac{1}{2 \sqrt{E}} Q_{\alpha} \quad, \quad \bar{a}_{\dot{\alpha}} \equiv \frac{1}{2 \sqrt{E}} \bar{Q}_{\dot{\alpha}} \tag{2.30}
\end{equation*}
$$

which satisfy

$$
\begin{equation*}
\left\{a_{i}, \bar{a}_{j}\right\}=\delta_{1 i} \delta_{1 j} \quad\left\{a_{i}, a_{j}\right\}=0=\left\{\bar{a}_{i}, \bar{a}_{j}\right\} . \tag{2.31}
\end{equation*}
$$

Thus, $\mathcal{N}=1$ field theories possess a single ladder algebra, $\left\{a_{1}, \bar{a}_{1}\right\}=1$, for building Foch spaces. Suppose $\mid \Omega_{j}>$ is a spin $j$ state which $a_{1}$ annihilates. To be precise, as we are only considering massless states, $\mid \Omega_{j}>$ should be a state of helicity $j$; we will often use the two terms synonymously. Then, its Foch space is 2-dimensional:

$$
\begin{equation*}
\mid \Omega_{j}>\quad \text { and } \quad \bar{a}_{1} \mid \Omega_{j}> \tag{2.32}
\end{equation*}
$$

No other states may be built from $\mid \Omega_{j}>$ since there exists a single anticommuting creation operator. Since $\bar{a}_{\dot{\alpha}}$, like $\bar{Q}_{\dot{\alpha}}$, is spin $\frac{1}{2}$, the state $\bar{a}_{1} \mid \Omega_{j}>$ will have spin $j+\frac{1}{2}$. Thus, we may conclude that an irreducible massless $\mathcal{N}=1$ multiplet contains exactly one boson and one fermion, both of which are massless ${ }^{3}$. For SUSY to hold, we will always require an equal number of bosonic and fermionic degrees of freedom. This must be so if there is to be a Bose-Fermi degeneracy. By CPT, there must also exist a similar pair of massless states with opposite helicities $\left(-j,-j-\frac{1}{2}\right)$. So in general two irreducible massless $\mathcal{N}=1$ multiplets will pair up giving four states with helicities $\left(j, j+\frac{1}{2},-j-\frac{1}{2},-j\right)$.

## - Superspace and Superfields

A convenient method of packaging $\mathcal{N}=1$ multiplets is achieved through the use of superspace. This technique adapts well for $\mathcal{N}=2$ but not for more supersymmetries, nor is it useful in $d>4$. Superspace, or more precisely $\mathcal{N}=1$ rigid superspace (rigid since we are considering global SUSY), is the fermionic extension of four-dimensional spacetime. To the usual four bosonic dimensions of spacetime, $x^{a}$, we add four fermionic dimensions $\theta^{\alpha}$ and $\bar{\theta}_{\dot{\alpha}}$,

$$
\begin{equation*}
x^{a} \longrightarrow\left(x^{a}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}\right) \tag{2.33}
\end{equation*}
$$

Being fermionic, the new Grassman coordinates anticommute

$$
\begin{equation*}
\left\{\theta^{\alpha}, \theta^{\beta}\right\}=\left\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\right\}=\left\{\theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}\right\}=0 . \tag{2.34}
\end{equation*}
$$

[^1]However, the combinations $\theta^{\alpha} Q_{\alpha}$ and $\bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$ are bosonic and so may be exponentiated to obtain a finite translation in superspace. Analogous to how momentum operators generate spacetime translations, SUSY generators lead to superspace translations. A general translation would be given by

$$
\begin{equation*}
G\left(x^{a}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}\right)=e^{i\left(-x^{a} P_{a}+\theta^{\alpha} Q_{\alpha}+\bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right)} \tag{2.35}
\end{equation*}
$$

or infinitesimally as

$$
\begin{equation*}
\left(x^{a}, \theta^{\alpha}, \bar{\theta}_{\dot{\alpha}}\right) \longrightarrow\left(x^{a}+\epsilon^{a}+i \theta \sigma^{a} \bar{\xi}-i \xi \sigma^{a} \bar{\theta}, \theta^{\alpha}+\xi^{\alpha}, \bar{\theta}_{\dot{\alpha}}+\bar{\xi}_{\dot{\alpha}}\right) . \tag{2.36}
\end{equation*}
$$

Notice that an infinitesimal translation by $\xi^{\alpha}$ in the fermionic directions induces a change in the bosonic spacetime. This implies that the differential operators for the SUSY charges are not $\partial_{\alpha}$ and $\partial_{\dot{\alpha}}$ but rather

$$
\begin{align*}
Q_{\alpha} & =\partial_{\alpha}-i\left(\sigma^{a} \bar{\theta}\right)_{\alpha} \partial_{a}  \tag{2.37}\\
\bar{Q}_{\dot{\alpha}} & =-\partial_{\dot{\alpha}}+i\left(\theta \sigma^{a}\right)_{\dot{\alpha}} \partial_{a} \tag{2.38}
\end{align*}
$$

In fact, the "extra" terms in the differential operator expressions for $Q, \bar{Q}$ are essential to ensure that the relation $\{Q, \bar{Q}\} \sim P$ holds true. So, while the rigid superspace we've been dealing with has zero curvature there is a non-zero torsion present. To compensate for this, we introduce covariant derivatives

$$
\begin{align*}
D_{\alpha} & =\partial_{\alpha}+i\left(\sigma^{a} \bar{\theta}\right)_{\alpha} \partial_{a}  \tag{2.39}\\
\bar{D}_{\dot{\alpha}} & =-\partial_{\dot{\alpha}}-i\left(\theta \sigma^{a}\right)_{\dot{\alpha}} \partial_{a} \tag{2.40}
\end{align*}
$$

which only differ from the SUSY generators by a relative sign. One could also check that the covariant derivatives anticommute with the SUSY charges.

$$
\begin{gather*}
\left\{D_{\alpha}, Q_{\beta}\right\}=\left\{D_{\alpha}, \bar{Q}_{\dot{\alpha}}\right\}=0  \tag{2.41}\\
\left\{\bar{D}_{\dot{\alpha}}, Q_{\alpha}\right\}=\left\{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\}=0 \tag{2.42}
\end{gather*}
$$

which will be important later. Their own anticommutation relations are the same (up to a sign) as the SUSY charges'

$$
\begin{gather*}
\left\{D_{\alpha}, D_{\beta}\right\}=0=\left\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right\}  \tag{2.43}\\
\left\{D_{\alpha}, \bar{D}_{\dot{\alpha}}\right\}=-2\left(\sigma^{a}\right)_{\alpha \dot{\alpha}} P_{a} \tag{2.44}
\end{gather*}
$$

A general superfield is a function written over all of superspace, and so by definition is a SUSY invariant object. The superfields of interest to physicists are the ones which transform irreducibly under the SUSY algebra. Since the Grassman variables of superspace anticommute, a general function on superspace $\Phi(x, \theta, \bar{\theta})$ can be Taylor expanded in the fermionic variables into a sum which necessarily terminates:

$$
\begin{align*}
\Phi(x, \theta, \bar{\theta}) & =\phi(x)+\theta \psi(x)+\bar{\theta} \bar{\chi}+\theta^{2} F(x)+\bar{\theta}^{2} G(x)  \tag{2.45}\\
& +\theta \sigma^{a} \bar{\theta} A_{a}(x)+\theta^{2} \overline{\theta \lambda}(x)+\bar{\theta}^{2} \theta \rho(x)+\theta^{2} \bar{\theta}^{2} D(x) \tag{2.46}
\end{align*}
$$

where the summation over spinor indices is left implicit. This most general (scalar) function on superspace, however, is not an irreducible representation of SUSY. To see this, note that it
contains: complex scalars $\phi, F, G, D$, lefthanded fermions $\psi_{\alpha}, \rho_{\alpha}$, righthanded fermions $\chi^{\dot{\alpha}}, \lambda^{\dot{\alpha}}$ and a vector $A_{a}$. As we've already discussed, the massless SUSY irreducible representations have fields of spin $\left(j, j+\frac{1}{2}\right)$ which is smaller then what we found for the general scalar superfield $\Phi$. To obtain irreducible representations, we must impose constraints on the superfields. These constraints must (anti)commute with the SUSY charges to ensure the constrained superfields are SUSY invariant as well.

## - Chiral Superfields

The first constraint we will impose is the chiral superfield ( $\chi \mathrm{sf}$ ) constraint:

$$
\begin{equation*}
\bar{D}_{\dot{\alpha}} \Phi=0 . \tag{2.47}
\end{equation*}
$$

As we already noted, $\bar{D}_{\dot{\alpha}}$ anticommutes with the SUSY charges, so this is an allowed constraint. It is easy to check that $\theta$ and $y^{a}=x^{a}+i \theta \sigma^{a} \bar{\theta}$ are both annihilated by $\bar{D}_{\dot{\alpha}}$, so any function $\Phi=\Phi(y, \theta)$ will be too. The general $\chi \mathrm{sf}$ is

$$
\begin{align*}
\Phi(y, \theta) & =\phi(y)+\sqrt{2} \theta \psi(y)+\theta^{2} F(y) \\
& =\phi+i \theta \sigma^{a} \bar{\theta} \partial_{a} \phi+\frac{1}{4} \theta^{2} \bar{\theta}^{2} \partial^{2} \phi+\sqrt{2} \theta \psi-\frac{i}{\sqrt{2}} \theta^{2} \partial_{a} \psi \sigma^{a} \bar{\theta}+\bar{\theta}^{2} F \tag{2.48}
\end{align*}
$$

Under an infinitesimal superspace translation $\xi^{\alpha}$ the component fields transform as

$$
\begin{align*}
\delta_{\xi} \phi & =\sqrt{2} \xi \psi \\
\delta_{\xi} \psi_{\alpha} & =\sqrt{2} \xi_{\alpha} F+i \sqrt{2}\left(\sigma^{a} \bar{\xi}\right)_{\alpha} \partial_{a} \phi  \tag{2.49}\\
\delta_{\xi} F & =-i \sqrt{2} \partial_{a} \psi \sigma^{a} \bar{\xi}
\end{align*}
$$

So $\chi$ sf do indeed transform into themselves under SUSY.
A $\chi$ sf only contains the complex scalars $\phi, F$ and the lefthanded Weyl fermion $\psi_{\alpha}$, and therefore only spins $j=0, \frac{1}{2}$, consistent with the restrictions on irreducible representations. However, we noted earlier that massless $\mathcal{N}=1$ irreducible multiplets contained only one boson and one fermion. The reason for this was that when we boosted to the particle's lightcone frame we were assuming the fields were on-shell. What we have derived is the off-shell $\chi \mathrm{sf}$ multiplet. Off shell, it contains 4 (real) degrees of freedom for both bosons and fermions. It will always happen that the F field is auxiliary (non-propagating). On shell, the multiplet reduces to $\left\{\phi, \psi_{\alpha}\right\}$, each with 2 degrees of freedom (as we expect for complex scalars and massless (on-shell) fermions). Though the F field is not physical, it is necessary to preserve SUSY off-shell.

One can analogously define anti-chiral superfields, $\bar{\chi}$ sf, subject to

$$
\begin{equation*}
D_{\alpha} \Phi=0 . \tag{2.50}
\end{equation*}
$$

These must be functions of $\bar{\theta}$ and $\bar{y}^{a}=x^{a}-i \theta \sigma^{a} \bar{\theta}$. Since (in Minkowski space) $D^{*}=\bar{D}_{\dot{\alpha}}$, if $\Phi$ is a $\chi \mathrm{sf}$, then $\bar{\Phi}$ is an $\bar{\chi} \mathrm{sf}$. Note that for $\chi \mathrm{sfs} \Phi^{i}, \Phi^{i}+\Phi^{j}$ and $\Phi^{i} \Phi^{j}$ are also $\chi \mathrm{sfs}$, but $\Phi^{i}+\bar{\Phi}^{i}$ and $\Phi^{i} \bar{\Phi}^{j}$ are not. $\chi$ sf and $\bar{\chi}$ sf are the SUSY analogues of matter and anti-matter fields, and by CPT they must come in chiral/anti-chiral pairs. The fermionic fields are thought of as the standard matter fields (quarks, electrons, etc.), while their superpartner scalars have been dubbed sfermions and have names like squarks and selectrons.

## - Abelian Vector Superfields

The next superfield we shall consider is the vector superfield (Vsf) $V$, which obeys the reality condition

$$
\begin{equation*}
V=\bar{V} . \tag{2.51}
\end{equation*}
$$

The most general superfield which obeys this constraint is

$$
\begin{align*}
V & =B+\theta \chi+\bar{\theta} \bar{\chi}+\theta^{2} C+\bar{\theta}^{2} \bar{C}-\theta \sigma^{a} \bar{\theta} A_{a}+ \\
& +i \theta^{2} \bar{\theta}\left(\bar{\lambda}+\frac{1}{2} \bar{\sigma}^{a} \partial_{a} \chi\right)-i \bar{\theta}^{2} \theta\left(\lambda-\frac{1}{2} \sigma^{a} \partial_{a} \bar{\chi}\right)+\frac{1}{2} \theta^{2} \bar{\theta}^{2}\left(D+\partial^{2} B\right) \tag{2.52}
\end{align*}
$$

where $B, D, A_{a}$ are real and $C$ is complex. Under a translation in superspace by $\xi^{\alpha}$, the Vsf's components vary by

$$
\begin{align*}
\delta_{\xi} B & =\xi \chi+\bar{\xi} \bar{\chi} \\
\delta_{\xi} \chi_{\alpha} & =2 \xi_{\alpha} C+\left(\sigma^{a} \bar{\xi}\right)_{\alpha}\left(i \partial_{a} B+A_{a}\right) \\
\delta_{\xi} C & =\overline{\xi \lambda}-\frac{i}{2} \partial_{a} \chi \sigma^{a} \bar{\xi} \\
\delta_{\xi} A_{a} & =\xi \sigma_{a} \bar{\lambda}+\lambda \sigma_{a} \bar{\xi}+\frac{i}{2}\left(\xi \partial_{a} \chi-\partial_{a} \bar{\chi} \bar{\xi}\right)  \tag{2.53}\\
\delta_{\xi} \lambda_{\alpha} & =2 \xi_{\alpha} D-\frac{i}{2} \xi_{\alpha} \partial^{a} A_{a}+i\left(s^{a} \bar{\xi}\right)_{\alpha} \partial_{a} C \\
\delta_{\xi} D & =\frac{i}{2} \partial_{a}\left(\lambda \sigma^{a} \bar{\xi}+\xi \sigma^{a} \bar{\lambda}\right),
\end{align*}
$$

so Vsfs are indeed scalars under SUSY as well.
The presence of the vector $A_{a}$ suggests that Vsfs should possess a gauge symmetry. The supersymmetric generalization of a regular gauge transformation is

$$
\begin{equation*}
V \longrightarrow V+i(\hat{\Lambda}-\overline{\hat{\Lambda}}) \tag{2.54}
\end{equation*}
$$

where $\hat{\Lambda}$ is an arbitrary $\chi$ sf

$$
\begin{equation*}
\hat{\Lambda}=\Lambda+\sqrt{2} \theta \psi_{\Lambda}+\theta^{2} F_{\Lambda}+i \theta \sigma^{a} \bar{\theta} \partial_{a} \Lambda+\frac{i}{\sqrt{2}} \theta^{2} \bar{\sigma}^{a} \partial_{a} \psi_{\Lambda}+\frac{1}{4} \theta^{2} \bar{\theta}^{2} \partial^{2} F_{\Lambda} \tag{2.55}
\end{equation*}
$$

The Vsf's components transform under this gauge transformation as

$$
\begin{align*}
\delta_{\Lambda} B & =i(\Lambda-\bar{\Lambda}) \\
\delta_{\Lambda} \chi & =i \sqrt{2} \psi_{\Lambda} \\
\delta_{\Lambda} C & =i F_{\Lambda} \\
\delta_{\Lambda} A_{a} & =\partial_{a}(\Lambda+\bar{\Lambda})  \tag{2.56}\\
\delta_{\Lambda} \lambda & =0 \\
\delta_{\Lambda} D & =0 .
\end{align*}
$$

In particular, the vector field $A_{a}$ transforms correctly with respect to the gauge parameter $\operatorname{Re}(\Lambda)$. This supersymmetric gauge invariance is, however, larger than standard ones as it
possesses the additional gauge parameter $\operatorname{Im}(\Lambda)$. In this case, with only a single gauge field, the full gauge group is $U(1)_{\mathbb{C}}$, rather the usual $U(1)_{\mathbb{R}}$. In general, the gauge group $G$ of $A_{a}$ is complexified to $G_{\mathbb{C}}$ for the corresponding Vsf.

One can determine by inspection that the $B, \chi$ and $C$ fields are all gauge artifacts and may be gauged away. This is attained by fixing $\operatorname{Im}(\Lambda), \psi_{\Lambda}$ and $F_{\Lambda}$ to cancel them, and is called the Wess-Zumino gauge. The WZ gauge fixed Vsf is written

$$
\begin{equation*}
V_{W Z}=-\theta \sigma^{a} \bar{\theta} A_{a}+i \theta^{2} \overline{\theta \lambda}-i \bar{\theta}^{2} \theta \lambda+\frac{1}{2} \theta^{2} \bar{\theta}^{2} D \tag{2.57}
\end{equation*}
$$

though from now on we shall not include the WZ subscript. In addition to reducing the number of fields, the WZ gauge also has the advantage that

$$
\begin{equation*}
e^{V}=1+V-\frac{1}{4} \theta^{2} \bar{\theta}^{2} A^{2} \tag{2.58}
\end{equation*}
$$

as $V^{3}=0$ in this gauge. The finite form of the gauge transformation (2.54) is then

$$
\begin{equation*}
e^{V} \longrightarrow e^{-i \bar{\Lambda}} e^{V} e^{i \bar{\Lambda}} \tag{2.59}
\end{equation*}
$$

Notice that $\operatorname{Re}(\Lambda)$ is left unaffected by fixing to the WZ gauge, thus the vector field still has its usual gauge freedom. In effect, the WZ gauge breaks $G_{\mathbb{C}}$ down to the standard gauge group $G$. The disadvantage is that SUSY is no longer manifest in the WZ gauge. Also note that (2.56) says that the $\lambda$ and $D$ fields are gauge invariant, though they are not both physical. As before, there is an auxiliary scalar field, $D$, which is necessary to continue the Vsf off-shell. In this case, the extra scalar field is real and when combined with the massive vector particle produce an equal number of degrees of freedom as the massive fermion. Onshell, the physical degrees of freedom of the Vsf are the massless fermion $\lambda_{\alpha}$ and the gauge boson $A_{a}$. The on-shell Vsf has fields of helicities $\left(-1,-\frac{1}{2}, \frac{1}{2}, 1\right)$ and so is already a CPT singlet. The vector fields are interpreted as standard gauge fields, in this simple example $A_{a}$ is the photon, while their fermionic superpartners are called gauginos, in this case $\lambda$ is the photino.

One would like the analogue of a gauge invariant field strength for the Vsfs. This is another irreducible representation of the SUSY algebra, called the field strength multiplet, and it has the same field content as the WZ gauge fixed Vsf. It is given by the field strength superfields

$$
\begin{equation*}
W_{\alpha}=-\frac{1}{4} \bar{D}^{2} D_{\alpha} V, \quad \bar{W}_{\dot{\alpha}}=-\frac{1}{4} D^{2} \bar{D}_{\dot{\alpha}} V \tag{2.60}
\end{equation*}
$$

which are indeed invariant under the supergauge transformation (2.54). Let's check this for $W_{\alpha}$ :

$$
\begin{align*}
\delta_{\Lambda} W_{\alpha} & \propto \bar{D}^{2} D_{\alpha}(\hat{\Lambda}-\overline{\hat{\Lambda}})=\bar{D}^{2} D_{\alpha} \hat{\Lambda}=\bar{D}_{\dot{\alpha}}\left\{\bar{D}^{\dot{\alpha}}, D_{\alpha}\right\} \hat{\Lambda}  \tag{2.61}\\
& \propto \bar{D}_{\dot{\alpha}}\left(\sigma^{a}\right)_{\alpha}^{\dot{\alpha}} P_{a} \hat{\Lambda}=\left(\sigma^{a}\right)_{\alpha}^{\dot{\alpha}} P_{a} \bar{D}_{\dot{\alpha}} \hat{\Lambda}=0,
\end{align*}
$$

where we've used the facts that the the $D \overline{\hat{\Lambda}}=\bar{D} \hat{\Lambda}=0,\{D, \bar{D}\} \sim P_{a}$ and $[\bar{D}, P]=0$. The superfields $W, \bar{W}$ are also $\chi$ sf and $\bar{\chi}$ sf, respectively:

$$
\begin{equation*}
\bar{D}_{\dot{\alpha}} W_{\alpha}=-\frac{1}{4} \bar{D}_{\dot{\alpha}}(\overline{D D}) D_{\alpha} V=0 \tag{2.62}
\end{equation*}
$$

since the $\bar{D}$ anticommute and have only two components, thus $\bar{D}^{3}=0$. They are not, however, general (anti) $\chi$ sfs, as they also satisfy the Bianchi identity

$$
\begin{equation*}
D^{\alpha} W_{\alpha}=\bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \tag{2.63}
\end{equation*}
$$

We can still expand them as functions of $y$ or $\bar{y}$ and, by taking $V$ in WZ gauge, the definitions (2.60) imply

$$
\begin{align*}
& W_{\alpha}=-i \lambda_{\alpha}(y)+\theta_{\alpha} D(y)-\frac{i}{2}\left(\sigma^{a} \bar{\sigma}^{b} \theta\right)_{\alpha} F_{a b}(y)+\theta^{2}\left(\sigma^{a} \partial_{a} \bar{\lambda}(y)\right)_{\alpha}  \tag{2.64}\\
& \bar{W}_{\dot{\alpha}}=i \bar{\lambda}_{\dot{\alpha}}(\bar{y})+\bar{\theta}_{\dot{\alpha}} D(\bar{y})=\frac{i}{2}\left(\bar{\sigma}^{a} \sigma^{b} \bar{\theta}\right)_{\dot{\alpha}} F_{a b}(\bar{y})+\bar{\theta}^{2}\left(\bar{\sigma}^{a} \partial_{a} \lambda(\bar{y})\right)_{\dot{\alpha}} \tag{2.65}
\end{align*}
$$

where $F_{a b}=\partial_{a} A_{b}-\partial_{b} A_{a}$ is the gauge field strength. Notice that $\sigma^{a} \bar{\sigma}^{b} F_{a b}$ projects $F$ onto $F^{+}$, its self-dual $(1,0)$ portion. Thus, $W_{\alpha}$ contains the self-dual field strength, while $\bar{W}_{\dot{\alpha}}$ contains the anti-self-dual piece.

## - Non-Abelian Vector Superfields

Everything we have discussed so far generalizes to non-Abelian gauge fields. Many of the details are rather complicated, so we will suppress them here, however the final results are quite similar to the Abelian case above.

First, we define the non-Abelian Vsf and gauge parameter $\chi$ sf as

$$
\begin{align*}
& V=T_{r} V^{r}  \tag{2.66}\\
& \hat{\Lambda}=T_{r} \hat{\Lambda}^{r}, \tag{2.67}
\end{align*}
$$

where the matrices $T_{r}$ form the Lie algebra of some gauge group $G$ and obey the commutation relations (2.18). The finite gauge transformation of the Vsf is the same as in the Abelian case

$$
\begin{equation*}
e^{V} \longrightarrow e^{-i \overline{\hat{\Lambda}}} e^{V} e^{i \hat{\Lambda}} \tag{2.68}
\end{equation*}
$$

however the infinitesimal form is more complicated, as one might expect for non-commuting fields. Fortunately, a WZ gauge exists for these fields which, again, breaks the supersymmetric gauge group from $G_{\mathbb{C}}$ down to $G$, however, the remaining gauge parameter is no longer $\operatorname{Re}(\Lambda)$ but rather

$$
\begin{equation*}
\hat{\Lambda}_{W Z}=\left(1+i \theta \sigma^{a} \bar{\theta} \partial_{a}+\frac{1}{4} \theta^{2} \bar{\theta}^{2} \partial^{2}\right) \operatorname{Re}(\Lambda) \tag{2.69}
\end{equation*}
$$

In WZ gauge, we again find

$$
\begin{equation*}
V_{W Z}=-\theta \sigma^{a} \bar{\theta} A_{a}+i \theta^{2} \overline{\theta \lambda}-i \bar{\theta}^{2} \theta \lambda+\frac{1}{2} \theta^{2} \bar{\theta}^{2} D \tag{2.70}
\end{equation*}
$$

though now the fields are matrix valued and transform in the adjoint representation. Having fixed WZ gauge, the remaining non-Abelian infinitesimal gauge transformation is

$$
\begin{equation*}
V \longrightarrow V+i(\hat{\Lambda}-\overline{\hat{\Lambda}})-\frac{i}{2}[(\hat{\Lambda}+\overline{\hat{\Lambda}}), V] \tag{2.71}
\end{equation*}
$$

where we have once again dropped the WZ subscripts. Componentwise, this transformation reads

$$
\begin{align*}
\delta_{\Lambda} A_{a}^{r} & =\partial_{a}(\Lambda+\bar{\Lambda})^{r}+f_{s t}^{r} A_{a}^{s}(\Lambda+\bar{\Lambda})^{t} \\
\delta_{\Lambda} \lambda_{\alpha}^{r} & =f_{s t}^{r} \lambda_{\alpha}^{s}(\Lambda+\bar{\Lambda})^{t}  \tag{2.72}\\
\delta_{\Lambda} D^{r} & =f_{s t}^{r} D^{s}(\Lambda+\bar{\Lambda})^{t}
\end{align*}
$$

The gauge invariant non-Abelian field strength $\chi$ sfs are defined as:

$$
\begin{equation*}
W_{\alpha}=-\frac{1}{4} \bar{D}^{2} e^{-V} D_{\alpha} e^{V}, \quad \bar{W}_{\dot{\alpha}}=-\frac{1}{4} D^{2} e^{V} \bar{D}_{\dot{\alpha}} e^{-V} . \tag{2.73}
\end{equation*}
$$

Note that, for Abelian gauge fields this definition is equivalent to (2.60). Under a finite gauge transformation, they transform in the correct covariant manner

$$
\begin{equation*}
W_{\alpha} \longrightarrow e^{-i \overline{\hat{\Lambda}}} W_{\alpha} e^{i \hat{\Lambda}} \tag{2.74}
\end{equation*}
$$

and similarly for $\bar{W}_{\dot{\alpha}}$. Expanded into functions of $y$, the field strength superfields are

$$
\begin{equation*}
W_{\alpha}=-\lambda_{\alpha}(y)+\theta_{\alpha} D(y)-\left(\sigma^{a b} \theta\right)_{\alpha} F_{a b}+\theta^{2}\left(\sigma^{a} \nabla_{a} \bar{\lambda}(y)\right)_{\alpha} \tag{2.75}
\end{equation*}
$$

where $F_{a b}=\partial_{a} A_{b}-\partial_{b} A_{a}+i\left[A_{a}, A_{b}\right]$ is the Yang-Mills field strength and $\nabla_{a}=\partial_{a}+i\left[A_{a}, \bullet\right]$ is the Yang-Mills covariant derivative. As always, $\bar{W}_{\dot{\alpha}}$ is obtained by taking the hermitian conjugate of $W_{\alpha}$.

## $2.4 \mathcal{N}=1$ Supersymmetric Actions

Actions for superfields must be invariant under all the symmetries of the theory: Poincaré, gauge, CPT and of course SUSY. To keep matters simple, we will also impose the constraint that our theories be renormalizable in four-dimensional spacetime. Satisfying the symmetry requirements is quite straightforward: we build Lagrangian densities from gauge-invariant scalar objects made of superfields, always pairing $\chi$ sfs with $\bar{\chi}$ sfs, and integrate over superspacetime.

Using only $\Phi^{i}$ and $\bar{\Phi}^{i}$, the simplest Lagrangian one can construct is

$$
\begin{equation*}
\mathcal{L}_{K}=\int d^{4} x d^{4} \theta \bar{\Phi}_{i} \Phi^{i} \tag{2.76}
\end{equation*}
$$

where $d^{4} \theta=d^{2} \theta d^{2} \bar{\theta}$. The subscript $K$ here stands for Kähler, and in its most general form the integrand is an arbitrary function $K\left(\bar{\Phi}^{i}, \Phi^{i}\right)$ called the Kähler potential. Recall that Grassman integration is the same as differentiation, so the fermionic integral picks out the $\theta^{2} \bar{\theta}^{2}$ component. With some straightforward manipulations we find,

$$
\begin{equation*}
\mathcal{L}_{K}=\bar{F}_{i} F^{i}-\partial_{a} \bar{\phi}_{i} \partial^{a} \phi^{i}-i \bar{\psi}_{i} \bar{\sigma}^{a} \partial_{a} \psi^{i} \tag{2.77}
\end{equation*}
$$

As promised, the $F$ fields have no propagators so their Euler-Lagrange equations read $F^{i}=0$. This simple Lagrangian describes a set of identical free massless complex scalars and Weyl fermions, which we know to be mixed under SUSY transformations.

We can add interactions to this by including a superpotential $\mathcal{W}\left(\Phi^{i}\right)+\overline{\mathcal{W}}\left(\bar{\Phi}^{i}\right)$. Being (anti)holomorphic functions (functions only of $\Phi^{i}$ or $\bar{\Phi}^{i}$ ), superpotentials must be integrated over the appropriate half of superspace to give non-zero contributions. The most general renormalizable superpotential is that in the Wess-Zumino model, given by

$$
\begin{align*}
\mathcal{L}_{\mathcal{W}} & =\int d^{2} \theta\left(\nu_{i} \Phi^{i}+m_{i j} \Phi^{i} \Phi^{j}+g_{i j k} \Phi^{i} \Phi^{j} \Phi^{k}\right)+c . c .  \tag{2.78}\\
& =\left(\nu_{i}+m_{i j} \phi^{j}+g_{i j k} \phi^{j} \phi^{k}\right) F^{i}-\frac{1}{2} m_{i j} \psi^{i} \psi^{j}-\frac{1}{2} g_{i j k} \phi^{i} \psi^{j} \psi^{k}+c . c .
\end{align*}
$$

One can again solve the $F$ fields' equations of motion (including the Kähler term) and substitute the solutions into the above superpotential to obtain

$$
\begin{equation*}
\mathcal{L}_{\mathcal{W}}=V(\phi, \bar{\phi})-\frac{1}{2} m_{i j} \psi^{i} \psi^{j}-\frac{1}{2} g_{i j k} \phi^{i} \psi^{j} \psi^{k}+c . c . \tag{2.79}
\end{equation*}
$$

where the scalar potential is

$$
\begin{equation*}
V(\phi, \bar{\phi})=\left|\frac{\partial \mathcal{W}(\phi)}{\partial \phi^{i}}\right|=\left|\nu_{i}+m_{i j} \phi^{j}+g_{i j k} \phi^{j} \phi^{k}\right|^{2} \tag{2.80}
\end{equation*}
$$

A remarkable result about superpotentials, whose derivation will take us too far off course here, is that they receive no perturbative quantum corrections. That is to say, the superpotential is exact and un-renormalized to all orders in perturbation theory!

The Kähler potential in (2.76) is too simple for us, as it is not a gauge invariant quantity. Under a finite gauge transformation, $\chi \mathrm{sfs}$ in the representation $R$ of the gauge group $G$ transform as

$$
\begin{equation*}
\Phi_{R}^{i} \rightarrow\left(e^{-i \hat{\Lambda}_{R}}\right)_{j}^{i} \Phi_{R}^{j} \quad \bar{\Phi}_{\bar{R} i} \rightarrow \bar{\Phi}_{\bar{R} j}\left(e^{i \bar{\Lambda}_{\bar{R}}}\right)_{i}^{j} \tag{2.81}
\end{equation*}
$$

where the gauge parameter $\left(\hat{\Lambda}_{R}\right)_{j}^{i}=\Lambda^{r}\left(T_{r}\right)_{j}^{i}$. We also write $V_{j}^{i}=V^{r}\left(T_{r}\right)_{j}^{i}$, with the generators $\left(T_{r}\right)_{j}^{i}$ in the same representation $R$. Then according to (2.68) the quantity

$$
\begin{equation*}
K^{\prime}=\bar{\Phi}_{i}\left(e^{V}\right)_{j}^{i} \Phi^{j} \tag{2.82}
\end{equation*}
$$

is gauge invariant and the kinetic terms for the $\chi \mathrm{ffs}$ is now

$$
\begin{align*}
\mathcal{L}_{K^{\prime}} & =\int d^{4} \theta \bar{\Phi}_{i}\left(e^{V}\right)_{j}^{i} \Phi^{j}  \tag{2.83}\\
& =\left|F^{i}\right|^{2}-\left|\left(\nabla_{a}\right)_{j}^{i} \phi^{j}\right|-i \overline{\psi_{i}} \bar{\sigma}^{a}\left(\nabla_{a}\right)_{j}^{i} \psi^{j}-\frac{i}{\sqrt{2}}\left(\phi^{i} \lambda_{i}^{j} \bar{\psi}_{j}-\bar{\phi}_{i} \lambda_{j}^{i} \psi^{i}\right)+\frac{1}{2} \bar{\phi}_{i} D_{j}^{i} \phi^{j}
\end{align*}
$$

where the Yang-Mills covariant derivative in a general representation is

$$
\begin{equation*}
\left(\nabla_{a}\right)_{j}^{i}=\partial_{a} \delta_{j}^{i}-i A_{a}^{r}\left(T_{r}\right)_{j}^{i} \tag{2.84}
\end{equation*}
$$

Of course, the Vsf requires its own kinetic terms and they are given by the super-Yang-Mills Lagrangian

$$
\begin{equation*}
\mathcal{L}_{S Y M}=\tau \int d^{2} \theta \operatorname{Tr} W^{\alpha} W_{\alpha}+c . c \tag{2.85}
\end{equation*}
$$

where the complex gauge coupling $\tau$ contains both the Yang-Mills coupling $g$ and the thetaangle $\vartheta$ :

$$
\begin{equation*}
\tau=\frac{1}{4 g^{2}}-\frac{i \vartheta}{32 \pi^{2}} \tag{2.86}
\end{equation*}
$$

After integrating over superspace the field strength Lagrangian becomes

$$
\begin{equation*}
\mathcal{L}_{S Y M}=\frac{1}{g^{2}} \operatorname{Tr}\left(-\frac{1}{4} F_{a b} F^{a b}-i \lambda \sigma^{a} \nabla_{a} \bar{\lambda}+\frac{1}{2} D^{2}\right)+\frac{\vartheta}{32 \pi^{2}} \operatorname{Tr} F_{a b} * F^{a b} . \tag{2.87}
\end{equation*}
$$

We can again solve for the auxiliary fields $D$, appearing in both the gauge invariant Kähler and field strength Lagrangians, and find a new term in the scalar scalar potential

$$
\begin{equation*}
V^{\prime}(\phi, \bar{\phi})=\left|\frac{\partial \mathcal{W}(\phi)}{\partial \phi^{i}}\right|^{2}+\frac{g^{2}}{8}\left(\bar{\phi}_{i}\left(T_{r}\right)_{j}^{i} \phi^{j}\right)^{2} . \tag{2.88}
\end{equation*}
$$

Putting all this together, the most general renormalizable $\mathcal{N}=1$ supersymmetric Lagrangian is the Yang-Mills-Wess-Zumino Lagrangian

$$
\begin{align*}
S & =S_{K^{\prime}}+S_{\mathcal{W}}+S_{S Y M} \\
& =\int d^{4} \theta \bar{\Phi}_{i}\left(e^{V}\right)_{j}^{i} \Phi^{j}+\left(\int d^{2} \theta\left[\nu_{i} \Phi^{i}+m_{i j} \Phi^{i} \Phi^{j}+g_{i j k} \Phi^{i} \Phi^{j} \Phi^{k}+\tau \operatorname{Tr} W^{\alpha} W_{\alpha}\right]+c . c\right) \\
& =\frac{1}{g^{2}} \operatorname{Tr}\left(-\frac{1}{4} F_{a b} F^{a b}-i \lambda \sigma^{a} \nabla_{a} \bar{\lambda}\right)+\frac{\vartheta}{32 \pi^{2}} \operatorname{Tr} F_{a b} * F^{a b}-\left|\left(\nabla_{a}\right)_{j}^{i} \phi^{j}\right|-i \bar{\psi}_{i} \bar{\sigma}^{a}\left(\nabla_{a}\right)_{j}^{i} \psi^{j} \\
& +\left(\frac{i}{\sqrt{2}} \bar{\phi}_{i} \lambda_{j}^{i} \psi^{j}+m_{i j} \psi^{i} \psi^{j}+g_{i j k} \phi^{i} \psi^{j} \psi^{k}+c . c .\right)  \tag{2.89}\\
& +\left|\nu_{i}+m_{i j} \phi^{j}+g_{i j k} \phi^{j} \phi^{k}\right|^{2}+\frac{g^{2}}{8}\left(\bar{\phi}_{i}\left(T_{r}\right)_{j}^{i} \phi^{j}\right)^{2} .
\end{align*}
$$

A remark should be made that the couplings are gauge invariant only if [43]

$$
\begin{align*}
m_{i j}\left(T_{r}\right)_{k}^{i}+m_{i j}\left(T_{r}\right)_{k}^{j} & =0 \\
g_{i j \ell}\left(T_{r}\right)_{k}^{\ell}+g_{i \ell k}\left(T_{r}\right)_{j}^{\ell}+g_{\ell j k}\left(T_{r}\right)_{i}^{\ell} & =0  \tag{2.90}\\
\nu_{i}\left(T_{r}\right)_{j}^{i} & =0 .
\end{align*}
$$

The first of these constraints implies $m_{i j} \neq 0 \Leftrightarrow R_{i}=\bar{R}_{j}$, the second that $g_{i j k} \neq 0 \Leftrightarrow$ $R_{i} \times R_{j} \times R_{k}$ contains the singlet, the last equation requires that the coupling $\nu_{i} \neq 0 \Leftrightarrow R_{i}$ is the singlet.

### 2.5 Extended Supersymmetry

Recall that the $\mathcal{N}=1$ (on-shell) massless multiplets contained two fields with spins ( $j, j+\frac{1}{2}$ ). With $\mathcal{N}$ SUSY charges, we have more raising operators so we may build a total of $2^{\mathcal{N}}$ states of the form $\bar{a}_{1 A_{1}} \bar{a}_{1 A_{2}} \ldots \bar{a}_{1 A_{n}} \mid \Omega_{j}>$, where $n=0,1, \ldots, \mathcal{N}$. Thus, fields of helicity $j+\frac{n}{2}$ will be $\binom{\mathcal{N}}{n}$-fold degenerate.

An important feature of extended SUSY is the enlarged $R$ symmetry. The mixing of the SUSY charges under $U(\mathcal{N})_{R}$ implies that the components with common helicities must transform in some representation of $S U(\mathcal{N})$. Essentially, the extra $U(1)_{R}$ factor continues to act as in the $\mathcal{N}=1$ case, by assigning an " $R$-charge" to the components and rotating each component into itself. This $U(1)_{R}$ must be present as every extended SUSY group contains $\mathcal{N}=1$ subgroups ${ }^{4}$. Because of this structure, every supermultiplet can be decomposed into its $\mathcal{N}=1$ components which must mix correctly under $S U(\mathcal{N})_{R}$. We use this property to classify extended SUSY multiplets below.

In principle, one may construct SUSY algebras for any $\mathcal{N}$, however physical constraints place an upper bound on this number. On general grounds, massless particles of $\operatorname{spin} j \geq 1$ must couple to conserved charges of spin $j-1$. Thus, spin- 1 gauge fields couple to scalar charges; the spin- $\frac{3}{2}$ gravitino couples to the SUSY charges $Q_{\alpha}^{A}$; and the spin- 2 graviton couples to the momentum vector $P_{a}$. Since angular momentum is given by the moment of $P_{a}, J_{a b}=x_{a} P_{b}-x_{b} P_{a}$, no additional (higher spin) fields are required to couple to it. A corollary of the Coleman-Mandula Theorem is then: there are no interacting massless fields of spin> 2. These observations imply that $\mathcal{N}=8$ is the largest number of SUSY charges physically possible. An $\mathcal{N}=8$ multiplet will contain states with all helicities from -2 to +2 . For theories without (super)gravity $\mathcal{N}=4$ is the maximal amount of SUSY possible, as the gravitino must always appear in a multiplet with the graviton, and the helicities will then range from -1 to +1 .

## - $\mathcal{N}=2$ SUSY

We begin by examining the simplest extended supersymmetric theory which contains two supercharges. In $\mathcal{N}=2$ SUSY, there are two types of multiplets possible: vector multiplets and hypermultiplets, whose lowest spin states are are $j=0$ (or -1 ) and $j=-\frac{1}{2}$, respectively. The $\mathcal{N}=2$ Yang-Mills vector multiplet is composed of an $\mathcal{N}=1 \mathrm{Vsf} V=V^{r} T_{r}$ and an $\mathcal{N}=1$ $\chi$ sf $\Phi=\Phi^{r} T_{r}$ in the adjoint representation of the gauge group $G$. The on-shell states of this multiplet are: gauge fields $A_{a}^{r}$ and complex scalars $\phi^{r}$, which are $S U(2)$ singlets, and an $S U(2)$ doublet of fermions ( $\lambda_{\alpha}^{r}, \psi_{\alpha}^{r}$ ), along with their CPT conjugate partners. The relations between the various components is displayed in Figure 2.1. This multiplet is governed by the Lagrangian (2.90) with a vanishing superpotential ( $\nu_{i}=m_{i j}=g_{i j k}=0$ ) as required by the $(\lambda, \psi) S U(2)$ symmetry. In particular, the possible interactions involve gauge fields through covariant derivatives $\operatorname{Tr}\left(\left[A_{a}, \bullet\right]\right)$, and scalars through the Yukawa couplings $\operatorname{Tr}(\bar{\phi} \cdot[\lambda, \psi])$ and quartic self-interactions $\operatorname{Tr}([\phi, \bar{\phi}])^{2}$.
$\mathcal{N}=2$ theories may also contain a matter sector and this is given in terms of the hypermultiplets. Each of these multiplets are built from a $\chi$ sf $\Phi^{\prime i}$ and a distinct $\bar{\chi}$ sf $\bar{\Phi}^{\prime \prime i}$ (different from $\bar{\Phi}^{\prime i}$ ). We use the primes to distinguish these superfields from the $\mathcal{N}=2$ vector components $\Phi^{r}$. The on-shell field content is: two $S U(2)$ fermion singlets $\psi_{\alpha}^{\prime i}, \psi_{\alpha}^{\prime \prime i}$, and an $S U(2)$ doublet of complex scalars ( $\phi^{\prime i}, \phi^{\prime \prime i}$ ), together with their CPT conjugates. The realtionship among fields is also summarized in Figure 2.1. When coupled to $\mathcal{N}=2$ gauge fields, the hypermultiplets may appear in any representation of the gauge group $G$ provided $R^{\prime}=\bar{R}^{\prime \prime}$. The only superspotential which respects the $S U(2)_{R}$ symmetry is

$$
\begin{equation*}
\mathcal{W}\left(\Phi^{\prime i}, \Phi^{\prime \prime i}, \Phi^{r}\right)=\Phi^{\prime i}\left(\Phi^{r}\right)_{i j} \Phi^{\prime \prime j}+m_{i j} \Phi^{\prime i} \Phi^{\prime \prime j} \tag{2.91}
\end{equation*}
$$

[^2]


Figure 2.1: The relations among the components of the $\mathcal{N}=2$ vector multiplet (left) and hypermultiplet (right)

## - $\mathcal{N}=4$ SUSY

Our main concern with extended SUSY theories will be the $\mathcal{N}=4$ case. As mentioned previously, this is the maximally supersymmetric field theory possible (without gravity) and it contains a unique multiplet. $\mathrm{N}=4$ super-Yang-Mills (SYM) can be considered as a sector $\mathcal{N}=2$ theory, with an $\mathcal{N}=2$ vector multiplet coupled to a hypermultiplet in the adjoint representation. In terms of $\mathcal{N}=1$ components, the $\mathcal{N}=4$ vector contains a Vsf and $3 \chi$ sfs. Its on-shell spectrum consists of a gauge field $A_{a}, 4$ left-handed fermions $\psi_{\alpha}^{A}$, and 3 complex (or 6 real) scalars $\phi^{A B}$ plus their $C P T$ conjugates. These fields transform in the singlet, fundamental (4), and anti-symmetric tensor (6) representations of $S U(4)_{R}$, respectively. Symmetries dictate that right-handed fermions $\bar{\psi}_{A \dot{\alpha}}$ transform in the anti-fundamental representation ( $\overline{4}$ ), and the scalars obey the reality condition $\overline{\phi^{A B}}=* \phi^{A B}=\frac{1}{2} \varepsilon^{A B C D} \phi_{C D}$. The $S U(4)$ symmetry of the fermions $\psi^{A}$ forbids a mass term in the $\mathcal{N}=2$ superpotential and it imposes the equality of all couplings (except $\vartheta$ ). Furthermore all fields are matrix valued and transform in the adjoint of $G$. Thus, symmetries fix the Lagrangian to be

$$
\begin{align*}
\mathcal{L}_{\mathcal{N}=4}=\frac{1}{g^{2}} \operatorname{Tr}\left(-\frac{1}{4} F_{a b} F^{a b}\right. & +\frac{g^{2} \vartheta}{32 \pi^{2}} F_{a b} * F^{a b}-i \psi^{A} \sigma^{a} \nabla_{a} \bar{\psi}_{A}-\left|\nabla_{a} \phi^{A B}\right|^{2} \\
& \left.-\sqrt{2} \operatorname{Re}\left(\bar{\phi}_{A B} \cdot\left[\psi^{A}, \psi^{B}\right]\right)+\frac{1}{8}\left|\left[\phi^{A B}, \phi^{C D}\right]\right|^{2}\right) . \tag{2.92}
\end{align*}
$$

We have not mentioned $\mathcal{N}=3$ in our discussion so far for a simple reason. In nongravitational theories, $\mathcal{N}=3$ SUSY is the same as $\mathcal{N}=4$. Consider what an $\mathcal{N}=3$ multiplet might look like, a quick count reveals there would be: a positive helicity gauge field $A_{a}^{+}, 3$ righthanded fermions $\psi_{\alpha}^{A+}, 3$ complex scalars $\phi^{A B}$ and a lefthanded fermion $\bar{\psi}_{\dot{\alpha}}$ (which must be distinct from the other $\psi^{A}$ ). When combined with their CPT conjugates this multiplet is exactly that which appeared in the $\mathcal{N}=4$ case. Further analysis would reveal that the two theories are in fact equivalent. Of course, there are $\mathcal{N}=3$ supergravity theories which are distinct from their $\mathcal{N}=4$ cousins. However in acquires the fourth one "for free".

### 2.6 Superconformal Algebras

In a massless theory, with dimensionless coupling, the spacetime symmetry group must be extended to include conformal transformations (at least classically). This enlarged symmetry
group includes, in addition to the Poincaré group, dilations $D: x^{a} \rightarrow t x^{a}$ and special conformal transformations $K_{a}: x^{a} \rightarrow \frac{x^{a}+b^{a} x^{2}}{1+2 x^{a} b_{a}+b^{2} x^{2}}$. The transformation properties of $K_{a}$ may seem peculiar, however they insure that inversions $x^{a} \rightarrow x^{a} / x^{2}$ are included in this group. The indices on the new generators $D$ and $K_{a}$ suggest that they are scalar and vector operators, respectively, and indeed they are. Their behaviour under Lorentz transformations are identical to (2.16) and (2.19) with $T_{r} \leftrightarrow D$ and $P_{a} \leftrightarrow K_{a}$. Their remaining relations are

$$
\begin{gather*}
{\left[D, K_{a}\right]=-i K_{a},\left[D, P_{a}\right]=i P_{a}}  \tag{2.93}\\
{\left[P_{a}, K_{b}\right]=2 i J_{a b}-2 i \eta_{a b} D}  \tag{2.94}\\
{\left[K_{a}, K_{b}\right]=0=[D, D] .} \tag{2.95}
\end{gather*}
$$

We refer to the eigenvalues of the $D$ commutation relations as ( $i$ times) the scaling dimension. Thus, the dimensions of $D, P_{a}, K_{a}$ and $J_{a b}$ are $0,+1,-1$ and 0 , respectively.

The number of fermionic generators must also be increased when passing from the superPoincaré to the superconformal group. We must include the superconformal generators $S_{\alpha}^{A}$, which are in a sense complimentary to the $Q_{\alpha}^{A}$ in a similar manner to how $K_{a}$ is complimentary to $P_{a}$. In minimal SUSY algebras we have $\{Q, \bar{Q}\} \sim P$, the $S_{\alpha}^{A}$ obey the analogous relation

$$
\begin{equation*}
\left\{S_{\alpha}^{A}, \bar{S}_{\dot{\alpha} B}\right\}=-2\left(\sigma^{a}\right)_{\alpha \dot{\alpha}} K_{a} \delta_{B}^{A} \tag{2.96}
\end{equation*}
$$

As $P_{a}$ and $K_{a}$ have opposite scaling dimensions, so too do $Q_{\alpha}^{A}$ and $S_{\alpha}^{A}$ :

$$
\begin{equation*}
\left[D, Q_{\alpha}^{A}\right]=\frac{i}{2} Q_{\alpha}^{A} \quad, \quad\left[D, S_{\alpha}^{A}\right]=-\frac{i}{2} S_{\alpha}^{A} \tag{2.97}
\end{equation*}
$$

For completeness, we list the remaining (anti)commutators of the superconformal group in the Appendix.

Usually, conformal symmetries are only valid classically since so-called trace anomalies tend introduce a scale dependance at the loop level. However, a most remarkable fact about $\mathrm{N}=4 \mathrm{SYM}$ is that it is an exactly conformal theory, even at the quantum level. To see this, note that in general the strength of the gauge coupling constant is governed by the renormalization group coefficient

$$
\begin{equation*}
\beta(g)=-\frac{g^{3}}{4 \pi^{2}}\left(\frac{11}{12} C_{1}(G)-\frac{1}{6} \sum_{i} C_{2}^{f}\left(R_{i}\right)-\frac{1}{12} \sum_{a} C_{2}^{s}\left(R_{a}\right)\right) \tag{2.98}
\end{equation*}
$$

where the sums are taken over fermions in representations $R_{i}$ and complex scalars in $R_{a}$. The $C_{i}$ are the Casimir operators for the representations $R_{i}, R_{a}$ and $G$ (adjoint). Since the 4 fermions and 3 complex scalars of $\mathcal{N}=4 \mathrm{SYM}$ all transform in the adjoint, then $C_{2}^{f}=C_{2}^{s}=$ $C_{1}(G)$. The one-loop $\beta$-function is then

$$
\begin{equation*}
\beta(g) \propto-\frac{g^{3}}{4 \pi^{2}}\left(\frac{11}{12}-\frac{4}{6}-\frac{3}{12}\right)=0 . \tag{2.99}
\end{equation*}
$$

Although this is only a one-loop result there are theorems, similar those asserting the nonrenormalization of the superpotential, which protect the coupling constant from further perturbative corrections beyond one-loop. Thus, the $\mathcal{N}=4$ theory is (super)conformal and the gauge coupling does not vary with momentum scales.

## Chapter 3

## MHV Amplitudes and Twistors

### 3.1 MHV Amplitudes at Tree Level

Consider a general Yang-Mills theory with an unbroken $S U(N)$ gauge group in four dimensions, with or without supersymmetry, and a coupling constant $g_{Y M}$. If all the fields are massless, then $n$-particle scattering amplitudes are functions of: colour labels, $a_{i}$, external momenta, $p_{i}$ and helicities, $h_{i}$, for $i=1, \ldots, n$. The full amplitude, $\mathcal{A}_{n}$, can be decomposed as a sum of kinematic partial amplitudes, $A_{n}$, multiplied by an appropriate colour factor, $T_{n}$, with an overall momentum conservation,

$$
\begin{equation*}
\mathcal{A}_{n}\left(a_{i}, p_{i}, h_{i}\right)=i g_{Y M}^{n-2}(2 \pi)^{4} \delta^{(4)}\left(\sum_{i=1}^{n} p_{i}\right) \sum_{\sigma \epsilon S_{n} / Z_{n}} T_{n}\left(a_{\sigma(i)}\right) A_{n}\left(p_{\sigma(i)}, h_{\sigma(i)}\right) . \tag{3.1}
\end{equation*}
$$

To ensure Bose symmetry, the sum is taken over all non-cyclic permutations of the $n$ external particles $S_{n} / Z_{n}$, as the partial amplitudes themselves are cyclicly invariant. The colour factors are easily determined, for example, at tree level with all particles in the adjoint representation its simply a single trace of generators: $T_{n}=\operatorname{Tr}\left(T_{a_{1}} \ldots T_{a_{n}}\right)$. The difficulty lies in computing the partial amplitudes.

We will focus on tree level amplitudes, with all particles defined to be incoming. We use the notation $g_{i}^{ \pm}$to denote the $i^{\text {th }}$ gauge boson (henceforth referred to as gluon) with on-shell momentum $p_{\alpha \dot{\alpha}}^{i}=\lambda_{\alpha}^{i} \widetilde{\lambda}_{\dot{\alpha}}^{i}$ and helicity $h_{i}= \pm 1$. When all or all but one particles have the same helicity, then the amplitude is identically zero,

$$
\begin{align*}
& A_{n}\left(g_{1}^{+}, g_{2}^{+}, \ldots, g_{n}^{+}\right)=A_{n}\left(g_{1}^{-}, g_{2}^{-}, \ldots, g_{n}^{-}\right)=0 \\
& A_{n}\left(g_{1}^{-}, g_{2}^{+}, \ldots, g_{n}^{+}\right)=A_{n}\left(g_{1}^{+}, g_{2}^{-}, \ldots, g_{n}^{-}\right)=0 . \tag{3.2}
\end{align*}
$$

The simplest non-trivial (partial) amplitudes are the Parke-Taylor amplitudes [1] where all but two particles have the same helicity, hence their alternate name - maximally helicity violating (MHV) amplitudes. Writing the momentum vectors in the spinor basis outlined in Section 2.1, the MHV amplitudes take an exceptionally simple form. To further simplify notation, we write spinor products

$$
\begin{equation*}
\left\langle\lambda_{i}, \lambda_{j}\right\rangle \equiv\langle i j\rangle \quad \text { and } \quad\left[\tilde{\lambda}_{k}, \tilde{\lambda}_{\ell}\right] \equiv[k \ell] . \tag{3.3}
\end{equation*}
$$

With this shorthand, the MHV amplitudes for the "mostly plus" case are

$$
\begin{equation*}
A_{n}\left(g_{r}^{-}, g_{s}^{-}\right)=\frac{\langle r s\rangle^{4}}{\prod_{\ell=1}^{n}\langle\ell \ell+1\rangle}, \tag{3.4}
\end{equation*}
$$

where the gluons in the $r$ and $s$ positions ( $1 \leq r, s \leq n$ ) carry negative helicity, while the remaining $n-2$ (which are suppressed on the left hand side) carry positive helicity. Also,
we cyclicly identify $\lambda_{n+1} \simeq \lambda_{1}$ throughout this thesis. The "mostly minus" Parke-Taylor amplitudes shall be referred to as MHV, though they are also called googly, amplitudes. They may be obtained from the MHVs by the simultaneous parity transformations $+\leftrightarrow-$ and complex conjugation $\langle i j\rangle \leftrightarrow[i j]$, yielding

$$
\begin{equation*}
A_{n}\left(g_{r}^{+}, g_{s}^{+}\right)=\frac{[r s]^{4}}{\prod_{\ell=1}^{n}[\ell \ell+1]} \tag{3.5}
\end{equation*}
$$

(where now the negative helicity gluons are suppressed on the left). Henceforth, positive helicity gluons will usually be left implicit when denoting "mostly plus" amplitudes, and similarly for the "mostly minus" cases.

The case of $n=3$ is slightly exceptional as the MHV amplitudes only contain one gluon of opposite helicity. These amplitudes vanish when on-shell, as expected. For instance, consider the amplitude

$$
\begin{equation*}
A_{3}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{+}\right)=\frac{\langle 12\rangle^{3}}{\langle 23\rangle\langle 31\rangle} \tag{3.6}
\end{equation*}
$$

To demonstrate the vanishing of this amplitude, suppose only the first two gluons are massless, then by momentum conservation

$$
p_{3}^{2}=\left(p_{1}+p_{2}\right)^{2}=2 p_{1} \cdot p_{2}=\left\langle\begin{array}{ll}
1 & 2
\end{array}\right]\left[\begin{array}{ll}
1 & 2 \tag{3.7}
\end{array}\right]=|\langle 12\rangle|^{2},
$$

since for real momenta in Lorentzian signature, $\widetilde{\lambda}_{i}= \pm \bar{\lambda}_{i}$. So,

$$
\begin{equation*}
p_{3}^{2}=0 \Longleftrightarrow\langle 12\rangle=0, \tag{3.8}
\end{equation*}
$$

thus, in the limit where all three particles go on-shell

$$
\begin{equation*}
A_{3}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{+}\right) \xrightarrow[p_{i}^{2} \longrightarrow 0]{\longrightarrow} 0 \tag{3.9}
\end{equation*}
$$

Similar arguments hold for helicity configurations. One issue that deserves attention is the fact that all factors $p_{i} \cdot p_{j} \propto\langle i j\rangle$ vanish for on-shell 3 -point functions. Fortunately, the numerator of such amplitudes is of higher order than the denominator, so no singularities arise. Off-shell continuations of 3-point (and higher) MHV amplitudes will be used considerably in the coming sections.

### 3.2 Conformal Invariance and Twistor Space

The CSW rules were motivated by [9], where Witten first noticed that Yang-Mills amplitudes are supported on algebraic curves in Penrose's twistor space. Along with suggesting a new approach to perturbative calculations, this work reveals a deeper structure of gauge theory, and so is doubly deserving of our attention. Here, we'll review the relevant points of Sections 2 and 3 of [9], readers interested in the details and its interpretation via topological string theory are referred to the original paper.

A renormalizable (unbroken) $S U(N)$ gauge theory contains only dimensionless parameters at tree level, and so should be invariant under all conformal transformations. For a
massless field, we may write the generators of the conformal group in bi-spinor notation:

$$
\begin{array}{cc}
J_{\alpha \dot{\alpha} \dot{\beta} \dot{\beta}}=\frac{i}{2}\left(\lambda_{\alpha} \frac{\partial}{\partial \lambda^{\beta}}+\lambda_{\beta} \frac{\partial}{\partial \lambda^{\alpha}}\right) \epsilon_{\dot{\alpha} \dot{\beta} \dot{\beta}} \quad, \quad \widetilde{J}_{\dot{\alpha} \dot{\beta}}=\frac{i}{2} \epsilon_{\alpha \beta}\left(\widetilde{\lambda}_{\dot{\alpha}} \frac{\partial}{\partial \widetilde{\lambda}_{\dot{\beta}}}+\widetilde{\lambda}_{\dot{\beta}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{\alpha}}}\right) \\
P_{\alpha \dot{\alpha}}=\lambda_{\alpha} \widetilde{\lambda}_{\dot{\alpha}} \quad, \quad K_{\alpha \dot{\alpha}}=\frac{\partial^{2}}{\partial \alpha^{\alpha} \partial \widetilde{\lambda}^{\dot{\alpha}}}  \tag{3.10}\\
D=\frac{i}{2}\left(\lambda^{\alpha} \frac{\partial}{\partial \lambda^{\alpha}}+\widetilde{\lambda}^{\dot{\alpha}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{\alpha}}}+2\right) .
\end{array}
$$

For $n$ massless particles, we sum all $n$ generators, so $P_{\alpha \dot{\alpha}}=\sum_{i} P_{\alpha \dot{\alpha}}^{i}=\sum_{i} \lambda_{\alpha}^{i} \widetilde{\lambda}_{\dot{\alpha}}^{i}$, etc. Given that $\lambda, \widetilde{\lambda}$ have scaling dimension $\frac{1}{2}$, (i.e.: $-i[D, \lambda]=-i[D, \widetilde{\lambda}]=\frac{1}{2}$ ), we see that $J, P, K, D$ all have the proper scaling dimensions of $0,+1,-1,0$, respectively. The inhomogeneous factor of 2 in the definition of $D$ is necessary to produce the correct commutation relation $\left[P_{\alpha \dot{\alpha}}, K_{\beta \dot{\beta}}\right]=i\left(J_{\alpha \dot{\alpha} \dot{\beta}}-\widetilde{J}_{\alpha \dot{\alpha} \beta \dot{\beta}}+\epsilon_{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}} D\right)$.

Physical quantities in conformal theories (such as classical Yang-Mills) must be annihilated by the above generators. As an explicit example, we will follow Witten's demonstration [9] that MHV amplitudes

$$
\begin{equation*}
\mathcal{A}^{\mathcal{M H \mathcal { V }}}=i g_{Y M}^{n-2}(2 \pi)^{4} \delta^{4}\left(\sum_{i} \lambda_{\alpha}^{i} \widetilde{\lambda}_{\dot{\alpha}}^{i}\right) \frac{\left\langle\lambda_{r}, \widetilde{\lambda}_{s}\right\rangle^{4}}{\prod_{1=1}^{n}\left(\lambda_{i}, \lambda_{i+1}\right\rangle} \tag{3.11}
\end{equation*}
$$

are in fact annihilated by all the generators of the conformal group. Note that Poincaré invariance is evident as these amplitudes contain a momentum conserving delta function and the remaining terms $\left\langle\lambda, \lambda^{\prime}\right\rangle$ are Lorentz scalars. Thus the non-trivial checks are the annihilation by $D$ and $K_{\alpha \dot{\alpha}}$.

First, we consider the action of $D$. The delta-function has scaling dimension -4, which precisely cancels the +4 contribution from $\langle r s\rangle^{4}$. So the numerator is annihilated by $D$. The denominator is as well since it is homogeneous in each $\lambda_{i}$ of degree -2 which cancels the +2 inhomogeneity in the definition of each $D^{i}$. So MHV amplitudes are invariant under dilations.

We will now show that $K_{\alpha \dot{\alpha}} \mathcal{A}^{M H V}=0$. Write $\mathcal{A}^{M H V}=\delta^{4}\left(P_{\alpha \dot{\alpha}}\right) A\left(\lambda_{i}\right)$ and recall $P_{\alpha \dot{\alpha}}=$ $\sum_{i} \lambda_{\alpha}^{i} \widetilde{\lambda}_{\dot{\alpha}}^{i}$. Since $\partial A / \partial \widetilde{\lambda}=0$, then by using the chain rule we find

$$
\begin{align*}
K_{\alpha \dot{\alpha}} \mathcal{A} & =\sum_{i} \frac{\partial^{2}}{\partial \lambda_{i}^{\alpha} \partial \widetilde{\lambda}_{i}^{\dot{\alpha}}} \mathcal{A}  \tag{3.12}\\
& =\left(\left(n \frac{\partial}{\partial P_{\alpha \dot{\alpha}}}+P_{\beta \dot{\beta}} \frac{\partial^{2}}{\partial P_{\alpha \dot{\beta}} \partial P_{\dot{\beta} \alpha}}\right) \delta^{4}\left(P_{\alpha \dot{\alpha}}\right)\right) A\left(\lambda_{i}\right)+\left(\frac{\partial}{\partial P_{\beta \dot{\alpha}}} \delta^{4}\left(P_{\alpha \dot{\alpha}}\right)\right) \sum_{i} \lambda_{\beta}^{i} \frac{\partial A\left(\lambda_{i}\right)}{\partial \lambda_{\alpha}^{i}} .
\end{align*}
$$

Using the Lorentz invariance of $A$, we can replace

$$
\begin{equation*}
\sum_{i} \lambda_{\beta}^{i} \frac{\partial A}{\partial \lambda_{\alpha}^{i}} \longrightarrow \frac{1}{2} \delta_{\beta}^{\alpha} \sum_{i} \lambda_{\gamma}^{i} \frac{\partial A}{\partial \lambda_{\gamma}^{i}}=-(n-4) \delta_{\beta}^{\alpha} A . \tag{3.13}
\end{equation*}
$$

We may also replace

$$
\begin{equation*}
P_{\beta \dot{\beta}} \frac{\partial^{2}}{\partial P_{\alpha \dot{\beta}} \partial P_{\dot{\beta} \alpha}} \delta^{4}\left(P_{\alpha \dot{\alpha}}\right) \longrightarrow-4 \frac{\partial}{\partial P_{\alpha \dot{\alpha}}} \delta^{4}\left(P_{\alpha \dot{\alpha}}\right), \tag{3.14}
\end{equation*}
$$

as the two distributions are equal (this is easy to check by integrating some test function by parts). Upon making these two substitutions, we realize that the right hand side of (3.12) vanishes, as required.

Minkowski space is a redundant setting in which to formulate conformal field theories because both the background and the field theory are scale invariant. It seems natural to look for a space where this scale invariance has been divided out, leaving a more appropriate background for the theory. Mathematicians have long studied such spaces and collectively call them projective spaces. We shall be particularly interested in two of these spaces: $\mathbb{R}^{3}$ and its complex cousin $\mathbb{C P}^{3}$, which Penrose named twistor space [49], or more precisely (complex) projective twistor space, denoted $\mathbb{P T}$.

For those unfamiliar with projective spaces, consider regular 3-space with one point (the origin) removed, $\mathbb{R}^{3} \backslash\{0\}$. Next, "squash" every point radially onto the unit sphere, i.e. map $\mathbb{R}^{3} \backslash\{0\} \rightarrow S^{2}$ via

$$
\begin{equation*}
\vec{x} \rightarrow \vec{x}^{\prime}=\frac{\vec{x}}{|\vec{x}|} \tag{3.15}
\end{equation*}
$$

Finally, identify antipodal points with each other, $\vec{x}^{\prime} \simeq-\vec{x}^{\prime}$; this new space is $\mathbb{R} \mathbb{P}^{2}$. To construct other projective spaces, like $\mathbb{R}^{\mathbb{P}^{n}}$ or $\mathbb{C P}^{n}$, follow the same procedure beginning instead with $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$ (and mod out an overall phase as well in the complex case, instead of just $\pm 1$ ). Equivalently, $\mathbb{R}^{\left(P^{n}\right.}\left(\mathbb{C P}^{n}\right)$ is defined as the space of (complex) lines in $\mathbb{R}^{n+1}\left(\right.$ math $\left.b b C^{n}\right)$ which pass through the origin.

While MHV amplitudes are holomorphic functions, that is functions of the $\lambda_{i}$ and not their conjugates $\widetilde{\lambda}_{i}$, a generic amplitude is not. Thus, an arbitrary $n$-point amplitude is parameterized by $n$ spinor pairs ( $\lambda_{\alpha}, \widetilde{\lambda}_{\dot{\alpha}}$ ). Naïvely, one might suspect that to encode these functions in twistor space, which is $\mathbb{C P}^{3}$, one simply takes the four (possibly complex) values in each spinor pair as the coordinates of a point in $\mathbb{P T}$, so $n$-point amplitudes are defined on $\mathbb{P T}^{n}$. The process is, however, slightly more subtle. As in [9], first we must Fourier transform half the coordinates

$$
\begin{equation*}
\tilde{\lambda}_{\dot{\alpha}} \rightarrow i \frac{\partial}{\partial \mu^{\dot{\alpha}}} \quad-i \frac{\partial}{\partial \widetilde{\lambda^{\dot{\alpha}}}} \rightarrow \mu_{\dot{\alpha}} \tag{3.16}
\end{equation*}
$$

This transformation, the so-called twistor transformation, leads to a more natural representation of the conformal group than (3.10). Instead of the generators being a mix of multiplication operators, and differential operators of degree one and two, the twistor transformation convert all the generators into first-order homogeneous operators. In particular, the two vector operators, $P_{\alpha \dot{\alpha}}$ and $K_{\alpha \dot{\alpha}}$, now appear on equal footing

$$
\begin{equation*}
P_{\alpha \dot{\alpha}}=i \lambda_{\alpha} \frac{\partial}{\partial \mu^{\dot{\alpha}}} \quad K_{\alpha \dot{\alpha}}=i \mu_{\dot{\alpha}} \frac{\partial}{\partial \lambda^{\alpha}} \tag{3.17}
\end{equation*}
$$

and the inhomogeneous term in $D$ is no longer necessary

$$
\begin{equation*}
D=\frac{i}{2}\left(\lambda_{\alpha} \frac{\partial}{\partial \lambda_{\alpha}}-\mu^{\dot{\alpha}} \frac{\partial}{\partial \mu^{\dot{\alpha}}}\right) . \tag{3.18}
\end{equation*}
$$

The Lorentz generators are unaltered in form: $J(\lambda) \rightarrow J(\lambda), \widetilde{J}(\widetilde{\lambda}) \rightarrow \widetilde{J}(\mu)$.
A (null) twistor, then, is a point in $\mathbb{P T}$ with coordinates $Z^{A}=\left(\lambda^{\alpha}, \mu_{\dot{\alpha}}\right)$, which corresponds to a null vector in Minkowski space $V_{\alpha \dot{\alpha}}=\lambda_{\alpha} \widetilde{\lambda}_{\dot{\alpha}}$. Since the null vector $V$ is invariant under
the simultaneous rescaling

$$
\begin{equation*}
\lambda \rightarrow z \lambda \quad, \quad \widetilde{\lambda} \rightarrow \frac{1}{z} \widetilde{\lambda}, \tag{3.19}
\end{equation*}
$$

for $z \in \mathbb{C}^{*}$, then the twistor variables $\lambda, \mu$ will have the symmetry

$$
\begin{equation*}
\left(\lambda^{\alpha}, \mu_{\dot{\alpha}}\right) \sim z\left(\lambda^{\alpha}, \mu_{\dot{\alpha}}\right) \tag{3.20}
\end{equation*}
$$

We recognize this as the symmetry of $\mathbb{C P}^{3}$, so we are justified in associating twistor space with this projective space. For an introduction to twistor theory see [50], or the recent review [51], for a more comprehensive examination of the subject see [52].

### 3.3 Twistor Transformed Amplitudes

Before going further, let's pause and examine this twistor transformation. Note that we have (arbitrarily) chosen to transform the righthanded spinors over the lefthanded ones, apparently breaking parity symmetry. The repercussions of this choice are that amplitudes with fewer negative helicity gluons will be simpler to compute, while their parity conjugates (like $\overline{\text { MHV }}$ amplitudes) will be more cumbersome. Also, recalling that for real momenta in Minkowski space $\widetilde{\lambda}= \pm \bar{\lambda}$, one may wonder exactly how to Fourier transform only the $\widetilde{\lambda}$ without affecting the $\lambda$. The simplest solution is to Wick rotate to the signature (,,,++-- ) where $\lambda$ and $\widetilde{\lambda}$ are both real and independent. This is the signature used whenever Fourier transforms to twistor space are discussed in this work.

The claim of [9] is that amplitudes involving $q$ negative helicity gluons with $\ell$ loops are supported on an algebraic curves in twistor space of degree $d$ and genus $g$, where

$$
\begin{equation*}
d=q-1+\ell \quad \text { and } \quad g \leq \ell . \tag{3.21}
\end{equation*}
$$

This conjecture offers a new explanation for the vanishing of tree-level amplitudes in (??). There are no algebraic curves with $d=-1$, and when $d=0$ the curve is just a point, so $\lambda_{i}=\lambda_{j}$ for all $i, j=1, \ldots, n$, this implies $2 p_{i} \cdot p_{j}=\langle i j\rangle[i j]=0$, so the amplitudes must vanish. The simplest non-trivial example is a curve with $d=1$; MHV tree amplitudes should be of this type, as they possess $q=2$ and $\ell=0$. They can be written simply as

$$
\begin{equation*}
\mathcal{A}^{\mathcal{M} \mathcal{H} \mathcal{V}}\left(\lambda_{i}, \widetilde{\lambda}_{i}\right)=\int d^{4} x \exp \left(i x_{\alpha \dot{\alpha}} \sum_{i=1}^{n} \lambda_{i}^{\alpha} \widetilde{\lambda}_{i}^{\dot{\alpha}}\right) A\left(\lambda_{i}\right), \tag{3.22}
\end{equation*}
$$

where the momentum conserving delta function is written as an integral over $x$-space, and the holomorphic function $A\left(\lambda_{i}\right)$ is the Park-Taylor formula given by (3.4), other constant factors are irrelevant here. Fourier transforming the $\widetilde{\lambda}$ is quite simple, yielding

$$
\begin{align*}
\widetilde{\mathcal{A}}^{\mathcal{M} \mathcal{H} \mathcal{V}}\left(\lambda_{i}, \mu_{i}\right) & =\int d^{4} x\left(\prod_{i=1}^{n} \frac{d^{2} \widetilde{\lambda}_{i}}{(2 \pi)^{2}}\right) \exp \left(i \sum_{i=1}^{n} \mu_{i \dot{\alpha}} \widetilde{\lambda}_{i}^{\dot{\alpha}}\right) \exp \left(i x_{\alpha \dot{\alpha}} \sum_{i=1}^{n} \lambda_{i}^{\alpha} \widetilde{\lambda}_{i}^{\dot{\alpha}}\right) A\left(\lambda_{i}\right) \\
& =\int d^{4} x\left(\prod_{i=1}^{n} \delta^{(2)}\left(\mu_{i \dot{\alpha}}+x_{\alpha \dot{\alpha}} \lambda_{i}^{\alpha}\right)\right) A\left(\lambda_{i}\right) . \tag{3.23}
\end{align*}
$$

Evidently, the amplitude vanishes unless all $n$ twistors $Z_{i}^{A}=\left(\lambda_{i}^{\alpha}, \mu_{i \dot{\alpha}}\right)$ lie on the curve in $\mathbb{P T}$ defined by the equations

$$
\begin{equation*}
\mu_{\dot{\alpha}}+x_{\alpha \dot{\alpha}} \lambda^{\alpha}=0, \dot{\alpha}=1,2 . \tag{3.24}
\end{equation*}
$$

For such a simple curve, called a "complete intersection", its degree is defined by $d=d_{1} d_{2}$, where $d_{1}$ and $d_{2}$ are the degrees of the defining polynomial equations. In this case, we indeed find $d=1$ as predicted, since both polynomials on the left hand side of (3.24) are linear. The real variables $x_{\alpha \dot{\alpha}}$ parameterizes the moduli space: degree one, genus zero curves in twistor space.

To make this more intuitive, consider the space $\mathbb{R} \mathbb{P}^{3} \backslash\left\{\lambda_{1}=0\right\}$. We may describe this space by coordinates

$$
\begin{equation*}
\left(x_{1}, x_{2}, x_{3}\right)=\left(\frac{\lambda_{2}}{\lambda_{1}}, \frac{\mu_{1}}{\lambda_{1}}, \frac{\mu_{2}}{\lambda_{1}}\right), \tag{3.25}
\end{equation*}
$$

which is in effect nothing more than $\mathbb{R}^{3}$. In this representation, the non-vanishing of the amplitude requires all $n$ points to lie on a straight line through $\mathbb{R}^{3}$.

Continuing the study of tree-level amplitudes, the next case to examine is $q=3$, or next-to-MHV (NMHV), which should lie on curves of degree two. Using the representation (3.25) of the last paragraph, these degree two curves may be realized as conic sections. Since $2=d=d_{1} d_{2}=1 \cdot 2$ is the only possible integer factorization, the twistor points must lie on the solutions of the linear and quadratic equations

$$
\begin{equation*}
\sum_{A=1}^{4} a_{A} Z^{A}=0 \quad \text { and } \quad \sum_{A, B=1}^{4} b_{A B} Z^{A} Z^{B}=0 \tag{3.26}
\end{equation*}
$$

for some real coefficients $a_{A}$ and $b_{A B}$. Verifying this is, however, more difficult than the previous case. The simplest example, $n=5$ (an $\overline{M H V}$ ), requires Fourier transforming (3.5) which is rather challenging indeed. Witten offers the following alternate method. Instead of transforming the amplitudes to twistor space, convert the twistor coordinates back to momentum space operators

$$
\begin{equation*}
Z^{A}=\left(\lambda^{\alpha}, \mu_{\dot{\alpha}}\right) \rightarrow\left(\lambda^{\alpha},-i \partial / \partial \widetilde{\lambda}^{\dot{\alpha}}\right) . \tag{3.27}
\end{equation*}
$$

The amplitudes are supported on conics if they satisfy certain differential equations, which is confirmed for $n=5,6$ and to one loop for $n=5$. Furthermore, Roiban et. al., using Witten's topological string interpretation, demonstrated in [10] that all $\overline{\text { MHV }}$ tree amplitudes lie on curves of degree $d=n-3$, and in general an amplitude supported on a degree $d$ curve is related to its parity conjugate on a curve of degree $d^{\prime}=n-d-2$, as predicted.

The CSW prescription was motivated by the realization in [9] that in addition to solving the differential equations required for conic support, the $n=5 \overline{\mathrm{MHV}}$ amplitudes also satisfy the requirements to lie on two disjoint straight lines. One possible configuration, depicted in Figure 3.1, involves three gluons with helicities,,++- attached to one twistor line while the remaining two negative helicity particles are attached to the other. Some type of propagating internal field connects the two lines, with opposite helicities at either end (since particles are defined as incoming, internal lines must flip sign). The internal helicity values are chosen to ensure each line contains exactly $q=2$, as required.


Figure 3.1: a)Twistor graph depicted in $\mathbb{R}^{3}$, b) The corresponding MHV diagram.

## Chapter 4

## Tree Level Amplitudes from Scalar Diagrams

The observation at the end of the last chapter, that an amplitude supported on a degree two curve is also supported on two skew lines, was originally difficult to interpret. Which moduli space is integrated over, as in (3.23): curves of degree one or two? Are there contributions coming from each integral, or should a unique one be used? Is this situation just a coincidence, or does this happen generally? Gukov, Motl and Neitzke proved that the integrals over each space are equivalent [13], and in general, an amplitude supported on a degree $d$ curve also has support on $n$ curves of degree $d_{i}$, so long as $\sum_{i=1}^{n} d_{i}=d$. Integrating over one degree $d$ curve is beneficial in some situations, e.g. proving parity symmetry [10], [11]. While the other extreme, $d$ disconnected degree one curves, produces the CSW method. This has the advantage of generating a novel diagrammatic expansion for all tree amplitudes in terms of MHV amplitudes.

### 4.1 Constructing MHV Diagrams

The CSW formalism says to interpret MHV amplitudes (degree one curves) as interaction vertex points. This interpretation seems natural in twistor theory since, according to Penrose [49], lines in $\mathbb{P T}$ correspond points in Minkowski space and visa-versa. These vertices are then connected by scalar propagators to produce an MHV diagram. Finally, summing over all possible diagrams consistent with a given cyclic order gives the partial amplitude $A_{n}$.

There is one subtlety involved in using the CSW rules, since the internal gluons are offshell and the MHV amplitudes require each leg to be massless. The CSW prescription is to define the spinor associated with a massive momentum $P_{\alpha \dot{\alpha}}$ as

$$
\begin{equation*}
\lambda_{P \alpha} \equiv P_{\alpha \dot{\alpha}} \eta^{\dot{\alpha}}, \tag{4.1}
\end{equation*}
$$

for some arbitrary, but fixed, anti-chiral spinor $\eta$. While the on-shell amplitudes are Lorentz invariant, by lifting them off-shell and introducing $\eta$ this symmetry is broken. This can equivalently be viewed as introducing a gauge dependance (where the choice of $\eta$ is a choice of lightcone gauge), or a breaking of Lorentz invariance (by choosing some preferred vector $\eta \bar{\eta}$ ), into the vertex. In [14], CSW demonstrated that contracted vertices are independent of $\eta$ and therefore gauge (or Lorentz) invariant.

Writing the sum of the massless momenta ( $p_{i}, p_{i+1}, \ldots, p_{j}$ ) as

$$
\begin{equation*}
P_{i j}=p_{i}+p_{i+1}+\ldots+p_{j}, \tag{4.2}
\end{equation*}
$$

the CSW rules are as follows:

- Spinors for off-shell particles are defined by $\lambda_{P \alpha}=P_{\alpha \dot{\alpha}} \eta^{\dot{\alpha}}$, for some fixed reference spinor $\eta^{\dot{\alpha}}$
- All graphs contain MHV vertices of the form

- All vertices are connected by scalar propagators

- All possible diagrams which preserve the the colour ordering must be summed over
- Any undetermined variables must be integrated over

At tree level, the only potentially undetermined variable is the internal momentum, however the momentum-conserving $\delta$-function renders this integration trivial. This last rule will, however, be relevant when including lower spin fields and in loop processes.

Since each vertex corresponds to an MHV amplitude, which is a degree one curve curve in twistor space, the number of vertices required, $v$, should equal the amplitudes total degree $d$. This new framework provides a natural explanation for (3.21), which was only conjectured in [9], and can be rewritten

$$
\begin{equation*}
v=q-1+\ell . \tag{4.3}
\end{equation*}
$$

Each vertex, being MHV, has two negative helicity legs. Each internal line connects a negative helicity leg to a positive one. As usual, the number of internal lines is $v-1+\ell$. The number of leftover (external) negative helicity lines is then

$$
\begin{equation*}
q=2 v-(v-1+\ell)=v+1-\ell \tag{4.4}
\end{equation*}
$$

confirming Witten's suggested selection rule. Of course, no derivation of the CSW prescription is yet known, so this "proof" has only traded one conjecture for another. It is, however, more appealing to derive selection rules from a theory rather than the other way round. Also, this diagrammatic interpretation offers yet another explanation for the vanishing of amplitudes with $q<2$ : they contain no vertices.

The use of a scalar propagator may seem puzzling at first, since these diagrams are supposed to describe vector gauge fields. Recall, however, that $S$-matrix elements are always contracted with external wavefunctions, which in this case are the polarization vectors $\epsilon_{\mu}^{ \pm}$ for $h= \pm 1$. Consider the NMHV amplitude. If we factor the polarization vectors out of the
vertex amplitudes and include them in the propagator instead, it can be rewritten as the massless. Feynmann gauge propagator

$$
\begin{align*}
A_{\mathcal{M H V}} \frac{1}{q^{2}} A_{\mathcal{M H V}}^{\prime} & =A_{\mathcal{M H V}}^{\mu} \frac{\epsilon_{\mu}^{+} \epsilon_{\nu}^{-}}{q^{2}} A_{\mathcal{M H V}}^{\prime \nu} \\
& =A_{\mathcal{M H V}}^{\mu} \frac{-g_{\mu \nu}}{q^{2}} A_{\mathcal{M H V}}^{\prime \nu} \tag{4.5}
\end{align*}
$$

where we have used the facts that internal lines must connect opposite helicity states and summing over polarizations gives $-g_{\mu \nu}$. Since scalars are generally simpler to manipulate than tensors, the ingredients on the left hand side (the CSW building blocks) will prove especially convenient.

At this point it is quite simple to write down an expression for the general NMHV tree level amplitude. Such an all $n$ result was unknown before the advent of the CSW method. Using the the cyclic invariance of the partial amplitudes, we may always choose one the of the negative helicity gluons to be $g_{1}^{-}$, the other two are in some arbitrary positions $s$ and $t$. Following the rules given above, we can draw all the diagrams in Figure 4.1 and calculate


Figure 4.1: The MHV Diagrams contributing to the Next-to-MHV amplitude the general NMHV amplitude:

$$
\begin{align*}
A_{n}\left(g_{1}^{-}, g_{s}^{-}, g_{t}^{-}\right)=\prod_{\ell=1}^{n} \frac{1}{\langle\ell \ell+1\rangle}\left[\sum_{i=2}^{s-1} \sum_{j=s}^{t-1} \frac{\langle 1 t\rangle^{4}\left\langle s P_{i j}\right\rangle^{4}}{D_{i j}}\right. & +\sum_{i=2}^{s-1} \sum_{j=t}^{n} \frac{\left\langle 1 P_{i j}\right\rangle^{4}\langle s t\rangle^{4}}{D_{i j}}+ \\
& \left.+\sum_{i=s}^{t-1} \sum_{j=t}^{n} \frac{\langle 1 s\rangle^{4}\left\langle t P_{i j}\right\rangle^{4}}{D_{i j}}\right] \tag{4.6}
\end{align*}
$$

where the common denominator is

$$
\begin{equation*}
D_{i j}=\frac{\left\langle i P_{i j}\right\rangle\left\langle P_{i j} j+1\right\rangle P_{i j}^{2}\left\langle P_{i j} i+1\right\rangle\left\langle j P_{i j}\right\rangle}{\langle i i+1\rangle\langle j j+1\rangle} \tag{4.7}
\end{equation*}
$$

One might worry that the above expression will produce poles by appropriate choices of the reference spinor $\eta$. Also, the explicit dependance of $\eta$ seems to destroy Lorentz invariance by singling out some preferred direction in momentum space. Kosower showed in [19], however, that this expression is indeed independent of $\eta$, and derived an equivalent form which depends only on the external momenta. The exact expression, which requires half a page to write, need not concern us here. Suffice it to say, such an expression exists and Lorentz invariance is ultimately preserved.

### 4.2 An Example: The Amplitude $A_{5}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{-}, g_{4}^{+}, g_{5}^{+}\right)$

Before calculating an explicit amplitude, let us derive some useful identities. First, write momentum conservation in spinor notation,

$$
\begin{equation*}
p_{1}+p_{2}+\ldots+p_{n}=0 \quad \Rightarrow \quad \lambda_{1} \widetilde{\lambda}_{1}+\lambda_{2} \widetilde{\lambda}_{2}+\ldots+\lambda_{n} \widetilde{\lambda}_{n}=0 \tag{4.8}
\end{equation*}
$$

then contract with $\lambda_{i}^{\alpha}$ and $\tilde{\lambda}_{j}^{\dot{\alpha}}$, for some $i, j=1, \ldots, n$, to obtain the equivalent statement

$$
\begin{equation*}
\sum_{k=1}^{n}\langle i k\rangle[k j]=0 \quad \forall i, j=1, \ldots, n . \tag{4.9}
\end{equation*}
$$

Another spinor manipulation, which is used ubiquitously throughout this work, is the Schouten identity,

$$
\begin{equation*}
\langle a b\rangle\langle c d\rangle+\langle a c\rangle\langle d b\rangle+\langle a d\rangle\langle b c\rangle=0 . \tag{4.10}
\end{equation*}
$$

This statement is probably better known through the equivalent relation $\epsilon_{\alpha \beta} \epsilon^{\gamma \delta}=-\delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta}+$ $\delta_{\alpha}^{\delta} \delta_{\beta}^{\gamma}$, which lies at the heart of the Fierz identities. We now turn to the $n=5 \overline{\text { MHV }}$ amplitude $A_{5}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{-}, g_{4}^{+}, g_{5}^{+}\right)$. It is calculated by summing over the four appropriate diagrams involving a 3 -point and a 4 -point MHV vertex with one leg from each contracted by a scalar propagator, shown in Figure 4.2.

Using the general formula (4.6) for NMHV amplitudes, we find only the third term survives, as $s=2=t-1$. This leaves:

$$
\begin{align*}
A_{5}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{-}\right)=\prod_{\ell=1}^{5} \frac{1}{\langle\ell \ell+1\rangle} \sum_{j=3}^{5} & {\left[\frac{\left\langle 1 P_{2 j}\right\rangle^{3}\langle 23\rangle^{4}\langle 12\rangle\langle j j+1\rangle}{\left\langle 2 P_{2 j}\right\rangle P_{2 j}^{2}\left\langle j P_{2 j}\right\rangle\left\langle j+1 P_{2 j}\right\rangle}+\right.} \\
& \left.+\frac{\langle 12\rangle^{4}\left\langle 3 P_{3 j}\right\rangle^{3}\langle 23\rangle\langle j j+1\rangle}{\left\langle 2 P_{3 j}\right\rangle P_{3 j}^{2}\left\langle j P_{3 j}\right\rangle\left\langle j+1 P_{3 j}\right\rangle}\right] . \tag{4.11}
\end{align*}
$$

Note that, of the six terms in the sum above, two terms will vanish (when $j=5,\left\langle 1 P_{25}\right\rangle=$ $-\left\langle 1 P_{11}\right\rangle=0$, and for $j=3,\left\langle 3 P_{33}\right\rangle=0$ ). This leaves the expected four terms, whose sum gives:

$$
\begin{align*}
& A_{5}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{-}\right)=  \tag{4.12}\\
& \frac{\left\langle 1 P_{23}\right\rangle^{3}}{\langle 45\rangle\langle 51\rangle\left\langle P_{23} 4\right\rangle} \frac{1}{\langle 23\rangle[32]} \frac{\langle 23\rangle^{3}}{\left\langle P_{23} 2\right\rangle\left\langle 3 P_{23}\right\rangle}+\frac{\left\langle 1 P_{24}\right\rangle^{3}}{\langle 51\rangle\left\langle P_{24} 5\right\rangle} \frac{1}{\langle 51\rangle[15]} \frac{\langle 23\rangle^{3}}{\langle 34\rangle\left\langle 4 P_{24}\right\rangle\left\langle P_{24} 2\right\rangle} \\
& +\frac{\langle 12\rangle^{3}}{\left\langle 2 P_{12}\right\rangle\left\langle P_{12} 1\right\rangle} \frac{1}{\langle 12\rangle[21]} \frac{\left\langle 3 P_{35}\right\rangle^{3}}{\langle 34\rangle\langle 45\rangle\left\langle 5 P_{35}\right\rangle}+\frac{\langle 12\rangle^{3}}{\langle 51\rangle\left\langle 2 P_{34}\right\rangle\left\langle P_{34} 5\right\rangle} \frac{1}{\langle 34\rangle[43]} \frac{\left\langle P_{34} 3\right\rangle^{3}}{\langle 34\rangle\left\langle 4 P_{34}\right\rangle}
\end{align*}
$$



Figure 4.2: The four MHV diagrams contributing to the amplitude $A_{5}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{-}, g_{4}^{+}, g_{5}^{+}\right)$
Using the facts that $P_{i j}=-P_{j+1 i-1},\left\langle a P_{i j}\right\rangle=\sum_{k=i}^{j}\langle a k\rangle[k \eta]$, and recalling that spinor products are anti-symmetric, the complexity of this expression is easily reduced. Also, the clever choice of the arbitrary reference anti-chiral spinor $\eta=\widetilde{\lambda}_{4}$ further simplifies the sum, in fact the last term is canceled altogether. This leaves

$$
\begin{aligned}
& +\frac{\langle 15\rangle^{2}\left[\begin{array}{ll}
4 & 5
\end{array}{ }^{3}\right.}{\langle 45\rangle\left[\begin{array}{ll}
4 & 4]\left[\begin{array}{ll}
3 & 3
\end{array}\right][4]
\end{array}\right]\left(4 P_{23}\right\rangle}
\end{aligned}
$$

As we know, this amplitude corresponds to an $\overline{\mathrm{MHV}}$, so the terms in parenthesis should equal unity. Repeated applications of 4.9) and (4.10), and roughly a page of algebra, show that this is indeed the case. Thus, the CSW rules give

$$
\begin{equation*}
A_{5}\left(g_{1}^{-}, g_{2}^{-}, g_{3}^{-}, g_{4}^{+}, g_{5}^{+}\right)=\frac{[45]^{3}}{[12][23][34][51]} \tag{4.14}
\end{equation*}
$$

as expected.

### 4.3 Some Consistency Checks: Mostly Minus Amplitudes

As remarked above, the required number of MHV vertices,

$$
\begin{equation*}
v=q-1+\ell, \tag{4.15}
\end{equation*}
$$

forces amplitudes with $q<2$ to vanish. We can examine the conjugates to these cases, that is $q>n-2$, using MHV diagrams and find they too vanish, as they should.

The amplitudes where $q=n$ will vanish at tree level. It is impossible to construct an MHV diagram where all external legs carry negative helicity, since MHV vertices contain at least one positive helicity leg and internal lines flip helicity between vertices. Choosing this external configuration will force an internal vertex to consist of all negative helicities, which is not allowed.

Though valid diagrams exist when $q=n-1$, it is easy to demonstrate that their amplitudes also vanish. Begin by noting that the only available graphs must only contain trivalent vertices. Higher point vertices must necessarily contain a greater number of positive helicity legs, as the number of negatives is fixed at two. Since internal helicities flip signs between sites, additional positive helicities will always propagate to the boundary of the graph. Having established that all graphs with $q=n-1$ contain trivalent vertices only, we recognize that there are two possible cases:
i) the positive helicity gluon $g_{p}^{+}$shares a vertex with a single internal line
ii) the positive helicity gluon $g_{p}^{+}$shares a vertex with two internal lines.

In case ii), the diagram is naturally split by $g_{p}^{+}$into two groups $\{p+1, p+2, \ldots, k\}$ and $\{k+1, k+2, \ldots, p-1\}$, for some gluon $g_{k}^{-}$.
i)



Figure 4.3: $M H V$ diagrams for $q=n-1$
In each case the vertices containing $g_{p}^{+}$contribute the following factors:

$$
\begin{align*}
& \text { i) } \frac{\left\langle p \pm 1 P_{p p+1}\right\rangle^{3}}{\langle p p \pm 1\rangle\left\langle p P_{p p \pm 1}\right\rangle}=\frac{\langle p \pm 1 p\rangle^{3}[\eta p]^{3}}{\langle p p \pm 1\rangle\langle p p \pm 1\rangle[\eta p \pm 1]}=\frac{\langle p r \pm 1\rangle[\eta p]^{3}}{[\eta p \pm 1]}  \tag{4.16}\\
& \text { ii) } \frac{\left\langle P_{p+1 k} P_{p k}\right\rangle^{3}}{\left\langle p P_{p+1 k}\right\rangle\left\langle p P_{p k}\right\rangle}=\frac{\left(\left\langle p P_{p+1 k}\right\rangle[\eta p]+\left\langle P_{p+1 k} P_{p+1 k}\right\rangle\right)^{3}}{\left\langle p P_{p+1 k}\right\rangle\left(\langle p p\rangle[\eta p]+\left\langle p P_{p+1 k}\right\rangle\right)}=\left\langle p P_{p+1 k}\right\rangle[\eta p]^{3} \tag{4.17}
\end{align*}
$$

Irrespective of the remaining terms in the diagrams, we can always choose $\eta=\widetilde{\lambda}_{p}$ and these vertices will vanish. This confirms the expected result that $q>n-2$ amplitudes do indeed vanish in the CSW formalism.

The case of $q=n-2$ correspond to $\overline{\text { MHV }}$ amplitudes, and we have already seen an example of how the CSW method produces the correct result for $n=5$ in the previous section. In fact, Zhu showed that this result holds for general $n$ [16]. Following the arguments given above, we conclude that an MHV amplitude will contain a single four-point vertex and the rest must be trivalent. Also, by choosing the reference spinor $\eta$ to correspond to one of the two positive helicity gluons, say $g_{p}^{+}$, non-vanishing contributions will come only from diagrams where $g_{p}^{+}$lies on the four-point vertex. Thus, amplitudes where the only positive helicity gluons are $g_{p}^{+}$and $g_{q}^{+}$, are determined by the sum of the diagrams shown in Figure 4.4.


Figure 4.4: The googly amplitude is the sum of the above diagrams
Next, Zhu showed by induction that the "blobs" in the diagrams, which represent the sums of only three-point vertices, are given by the formulas

$$
\begin{equation*}
V\left(g_{i}^{-}, g_{i+1}^{-}, \ldots, g_{j}^{-}, g_{P_{i j}}^{+}\right)=\frac{P_{i j}^{2}}{[i \eta][j \eta]} \frac{1}{[i i+1] \ldots[j-1 j]} \tag{4.18}
\end{equation*}
$$

$$
\begin{equation*}
V\left(g_{i}^{-}, \ldots, g_{q}^{+}, \ldots, g_{j}^{-}, g_{P_{i j}}^{+}\right)=\frac{P_{i j}^{2}[q \eta]^{4}}{[i \eta][j \eta]} \frac{1}{[i i+1] \ldots[j-1 j]} . \tag{4.19}
\end{equation*}
$$

This result will not be proven here, though the reader may readily verify that the case of a single off-shell three-point vertex is reproduced by the above formulae. Also, notice that these sums vanish when the off-shell gluon, $g_{P_{i j}}^{+}$, is taken on shell, confirming the above result regarding $q=n-1$.

Completing the proof for the $\overline{\mathrm{MHV}}_{\mathrm{s}}$ requires summing all possible diagrams in Figure 4.4, using the effective "blob" vertices where appropriate. First, note that the $1 / P_{i j}$ propagators which connect to the four-point vertex will cancel those terms in the "blob" vertices. Also, while choosing $\eta=\widetilde{\lambda}_{p}$ simplifies the calculation, it also leads to the correct numerator for the $\overline{\mathrm{MHV}}:[q p]^{4}$. Finally, many of the anti-chiral spinor products, $[\widetilde{\lambda}, \widetilde{\mu}]$, which appear in $\overline{\mathrm{MHV}}$ s are already present in the "blobs". This results in the sum

$$
\begin{array}{r}
A_{n}\left(g_{p}^{+}, g_{q}^{+}\right)=\frac{[q p]^{4}}{\prod_{\ell=1}^{n}[\ell+1]}\left(\sum_{k=q}^{p-2} C_{k}^{+}\left\langle p+1 P_{p k}\right\rangle^{4}+\sum_{k=p+2}^{q-1} C_{k}^{+}\left\langle p+1 P_{p+2 k}\right\rangle^{4}+\right.  \tag{4.20}\\
\left.\quad+\sum_{k=q}^{p-3} C_{k}^{-}\left\langle p-1 P_{p-1 k}\right\rangle^{4}+\sum_{k=p+1}^{q-1} C_{k}^{-}\left\langle p-1 P_{p+1 k}\right\rangle^{4}\right)
\end{array}
$$

where

$$
\begin{align*}
& C_{k}^{+}=\frac{[p p+1][p+1 p+2][k k+1]}{[p p+2][k p][p k+1]\langle p p+1\rangle\left\langle p+1 P_{p+2 k}\right\rangle\left\langle P_{p+2 k} P_{p k}\right\rangle\left\langle P_{p k} p\right\rangle}  \tag{4.21}\\
& C_{k}^{-}=\frac{[p p-1][p-1 p-2][k k+1]}{[p p-2][k p][p k+1]\langle p p-1\rangle\left\langle p-1 P_{p+1 k}\right\rangle\left\langle P_{p+1 k} P_{p-1 k}\right\rangle\left\langle P_{p-1 k} p\right\rangle} . \tag{4.22}
\end{align*}
$$

The fact that the bracketed terms in (4.20) sums to unity is a highly non-trivial check of the CSW rules' validity, which was carried out in [16] by taking $p_{\mu}=(p, 0,0, p)$, so that we can scale $\eta=\widetilde{\lambda}_{p}=(1,0)$.

Thus far, we have used the CSW method to derive formulas for $n$ gluon scattering amplitudes with with up to 3 negative helicity particles. Using the conjugate set of rules (that is using $\overline{\text { MHV }}$ vertices), one can equally determine all amplitudes with up to 3 positive helicity gluons. With these results alone, one now knows all tree level gluon scattering amplitudes with up to 7 external states. To obtain all amplitudes with $n \leq 7$ using traditional techniques would require summing a total of tens of thousands of Feynman diagrams. This new technique dramatically simplifies the process requiring only a few dozen diagrams to produce all $n \leq 7$ amplitudes! In fact, the result (4.6) for general NMHV amplitudes, being an infinite series of solutions, would indeed require summing an infinite number of Feynman diagrams, which are summarized above by just three (albeit infinite sets of) MHV diagrams. This method's remarkable simplicity cannot be emphasized enough.

## Chapter 5

## Amplitudes with Fermions and Scalars

We have so far considered tree level amplitudes for purely gluonic interactions. However, the CSW rules are easily modified to include particles of spin $<1$. In fact, we will see later that including fermions and scalars is necessary at the 1 -loop level, even in pure Yang-Mills theories (i.e. theories of gauge bosons only).

### 5.1 MHV Amplitudes in $\mathcal{N}=4$ SYM

At tree level, pure Yang-Mills theory is effectively supersymmetric. This is because a gluon may only couple to a pair of fermions or scalars. So, if all external fields are gluons and a pair of matter fields are produced in a scattering process, then the pair must necessarily form a closed loop. Thus at tree level, gluons do not "know" if they live in a supersymmetric theory or not. Because of this, we may treat the gauge particles as members of a theory with maximal SUSY which is $\mathcal{N}=4$ SYM, so long as we restrict ourselves to the classical level.

This hidden symmetry of gluons may be responsible for the extreme simplicity of the MHV amplitudes. One would not expect such a simple expression unless the theory possed an incredible amount of symmetry. We might ask whether the MHV amplitudes may be generalized to include all the fields of the $\mathcal{N}=4$ multiplet. Indeed, such a generalization exists and was first written down by Nair in [3]. Before giving the $\mathcal{N}=4$ MHV amplitude, we must introduce some additional notation.

As described in Section 2.5, $\mathcal{N}=4$ SYM contains fields of helicities -1 to +1 . To package this additional information in scattering amplitudes, we introduce scalar Grassman (anticommuting) variables $\eta^{A}, A=1, \ldots, 4$, which transform in the 4 of $S U(4)_{R}$. Similar to the manner in which $\bar{Q}_{\dot{\alpha} A}$ raises helicity by $\frac{1}{2}, \eta^{A}$ lowers the helicity by $\frac{1}{2}$. This is not a perfect analogy as $\bar{Q}_{A}$ is an operator, and $\eta^{A}$ is only a variable, but it is a useful mnemonic. We measure a particle's helicity with the operator

$$
\begin{equation*}
h=1-\sum_{A} \frac{1}{2} \eta^{A} \frac{\partial}{\partial \eta^{A}}, \tag{5.1}
\end{equation*}
$$

so a term in an amplitude with the factor $(\eta)^{k}$ will correspond to a particle of helicity $h=1-\frac{k}{2}$. This implies the following associations:

$$
\begin{array}{ccc}
1 & \sim & g^{+}  \tag{5.2}\\
\eta^{A} & \sim & \psi^{A} \\
\eta^{A} \eta^{B} & \sim & \phi^{A B} \\
\eta^{A} \eta^{B} \eta^{C} & \sim \varepsilon^{A B C D} \bar{\psi}_{D} \\
\eta^{A} \eta^{B} \eta^{C} \eta^{D} & \sim \varepsilon^{A B C D} g^{-}
\end{array}
$$

Nair's generalization of the Park-Taylor MHV formula is given by the partial amplitude

$$
\begin{equation*}
A_{n}\left(\lambda_{\alpha}^{i}, \eta_{i}^{A}\right)=\delta^{(8)}\left(\sum_{i=1}^{n} \lambda_{\alpha}^{i} \eta_{i}^{A}\right) \prod_{\ell=1}^{n} \frac{1}{\langle\ell \ell+1\rangle} . \tag{5.3}
\end{equation*}
$$

Recall that Grassman $\delta$-functions are given by $\delta(\theta)=\theta$. Because of the anticommuting variables, the $\delta$-function can be Taylor expanded into a finite sum

$$
\begin{align*}
\delta^{(8)}\left(\sum_{i} \lambda_{\alpha}^{i} \eta_{i}^{A}\right) & =\prod_{A=1}^{4} \delta^{2}\left(\sum_{i} \lambda_{\alpha}^{i} \eta_{i}^{A}\right)=\prod_{A=1}^{4}\left(\sum_{i} \lambda_{\alpha}^{i} \eta_{i}^{A}\right)\left(\sum_{j} \lambda_{j}^{\alpha} \eta_{j}^{A}\right)  \tag{5.4}\\
& =\prod_{A=1}^{4}\left(\sum_{i j}\langle i j\rangle \eta_{i}^{A} \eta_{j}^{A}\right) \\
& =\sum_{i j} \sum_{k l} \sum_{m n} \sum_{r s}\langle i j\rangle\langle k \ell\rangle\langle m n\rangle\langle r s\rangle \eta_{i}^{1} \eta_{j}^{1} \eta_{k}^{2} \eta_{\ell}^{2} \eta_{m}^{3} \eta_{\eta}^{3} \eta_{r}^{4} \eta_{s}^{4}
\end{align*}
$$

By (5.2), the term proportional to $\left(\eta_{r} \eta_{s}\right)^{4}$ should correspond to the scattering of two gluons of negative helicity and $n-2$ of positive helicity. Indeed, in this case we recover the expected numerator $\langle r s\rangle^{4}$. Thus, we find the following $\mathcal{N}=4$ SYM amplitudes,

$$
\begin{align*}
& A_{n}\left(g^{-}, g^{-}\right), A_{n}\left(g^{-}, \bar{\psi}_{A}, \psi^{A}\right), A_{n}\left(\bar{\psi}_{A}, \bar{\psi}_{B}, \psi^{A}, \psi^{B}\right) \\
& A_{n}\left(g^{-}, \psi^{1}, \psi^{2}, \psi^{3}, \psi^{4}\right), A_{n}\left(\bar{\psi}_{A}, \psi^{A}, \psi^{1}, \psi^{2}, \psi^{3}, \psi^{4}\right) \\
& A_{n}\left(\psi^{1}, \psi^{2}, \psi^{3}, \psi^{4}, \psi^{1}, \psi^{2}, \psi^{3}, \psi^{4}\right), A_{n}\left(\bar{\phi}_{A B}, \psi^{A}, \psi^{B}, \psi^{1}, \psi^{2}, \psi^{3}, \psi^{4}\right) \\
& A_{n}\left(g^{-}, \bar{\phi}_{A B}, \phi^{A B}\right), A_{n}\left(g^{-}, \bar{\phi}_{A B}, \psi^{A}, \psi^{B}\right), A_{n}\left(\bar{\psi}_{A}, \bar{\psi}_{B}, \phi^{A B}\right)  \tag{5.5}\\
& A_{n}\left(\bar{\psi}_{A}, \phi^{A B}, \bar{\phi}_{B C}, \psi^{C}\right), A_{n}\left(\bar{\psi}_{A}, \bar{\phi}_{B C}, \psi^{A}, \psi^{B}, \psi^{C}\right), A_{n}\left(\bar{\phi}_{A B}, \phi^{A B}, \bar{\phi}_{C D}, \phi^{C D}\right), \\
& A_{n}\left(\bar{\phi}_{A B}, \phi^{A B}, \bar{\phi}_{C D}, \psi^{C}, \psi^{D}\right), A_{n}\left(\bar{\phi}_{A B}, \bar{\phi}_{C D}, \psi^{A}, \psi^{B}, \psi^{C}, \psi^{D}\right),
\end{align*}
$$

are all coefficients in the expansion of (5.3) and given by the formula

$$
\begin{equation*}
A_{n}= \pm \frac{\langle i j\rangle\langle k \ell\rangle\langle m n\rangle\langle r s\rangle}{\prod_{\ell=1}^{n}\langle\ell \ell+1\rangle} \tag{5.6}
\end{equation*}
$$

where the values of $i, j, k, \ell, m, n, r, s$ are determined by matching the appropriate factors of $\eta^{A}$ in (5.2) to $\eta_{i}^{1} \eta_{j}^{1} \eta_{k}^{2} \eta_{\ell}^{2} \eta_{m}^{3} \eta_{n}^{3} \eta_{r}^{4} \eta_{s}^{4}$. The overall sign may arise from anticommuting the $\eta^{A}$ into the correct order. We could also determine the $\overline{\mathrm{MHV}}$ amplitudes by switching $\langle,\rangle \leftrightarrow[$,$] .$ Since all the fields transform in the adjoint, the colour factor which multiplies the partial amplitude is the same as in pure Yang-Mills $T_{n}\left(a_{i}\right)=\operatorname{Tr}\left(T_{a_{1}} T_{a_{2}} \ldots T_{a_{n}}\right)$.

Notice that not all the amplitudes above are what we would normally call MHV, that is, with two particles of different helicity then the rest (by an "opposite helicity scalar" we mean its complex conjugate). In particular the second and third lines are not of this type. The common feature is that they all possess $8 \eta^{A}$ in the expansion of (5.3). Nevertheless, for simplicity we shall continue to refer to any amplitude in (5.5) as MHV.

As in Sections 3.2 and 3.3, we could perform an analysis of the $\mathcal{N}=4 \mathrm{MHV}$ amplitudes similar to the one carried out in the non-supersymmetric case. We will summarize the results of this analysis found in [9]. As $\mathcal{N}=4$ SYM is a superconformal theory, we would find that
the amplitudes are indeed annihilated by the superconformal algebra's generators. In bispinor notation, these generators are also a mix of multiplication operators, and differential operators of degree one and two. The Fourier transform

$$
\begin{array}{cl}
\widetilde{\lambda}_{\dot{\alpha}} \rightarrow i \frac{\partial}{\partial \mu^{\dot{\alpha}}} & , \quad-i \frac{\partial}{\partial \widetilde{\lambda}_{\dot{\alpha}}} \rightarrow \mu_{\dot{\alpha}} \\
\eta^{A} \rightarrow i \frac{\partial}{\partial \chi_{A}} \quad, \quad-i \frac{\partial}{\partial \eta^{A}} \rightarrow \chi_{A} \tag{5.8}
\end{array}
$$

takes us to super-twistor space $\mathbb{C} \mathbb{P}^{3 / 4}$ parameterized by $\left(\lambda^{\alpha}, \mu_{\dot{\alpha}}, \chi_{A}\right)$ subject to the equivalence

$$
\begin{equation*}
\left(\lambda^{\alpha}, \mu_{\dot{\alpha}}, \chi_{A}\right) \sim z\left(\lambda^{\alpha}, \mu_{\dot{\alpha}}, \chi_{A}\right), \forall z \in \mathbb{C}^{*} \tag{5.9}
\end{equation*}
$$

In super-twistor space the $\mathcal{N}=4 \mathrm{MHV}$ amplitudes are localized on $\mathbb{C P}^{1}$ "curves" which simultaneously satisfy the six equations

$$
\begin{equation*}
\mu_{\dot{\alpha}}+x_{\alpha \dot{\alpha}} \lambda^{\alpha}=0 \quad, \quad \chi_{A}+\theta_{A \alpha} \lambda^{\alpha}=0 \tag{5.10}
\end{equation*}
$$

These $\mathbb{C P}^{1}$ curves are parameterized by $\lambda_{\alpha} \sim z \lambda_{\alpha}$ and have the modulus $x_{\alpha \dot{\alpha}}$ as in the bosonic case, however now the additional fermionic modulus $\theta_{A \alpha}$ appears.

### 5.2 MHV Amplitudes with $\mathcal{N}<4$

We might inquire whether some version of the above results carry over to theories with less SUSY. After all the $\mathcal{N}=4$ theory is just a particular type of $\mathcal{N}=2$ theory, which in turn is a particular type of $\mathcal{N}=1$ theory. The main difference between the various SYM theories is in the fermion and scalar content. As pointed out in [15], if we restrict the external states to lie in appropriate multiplets, then the above amplitudes are valid in theories with less SUSY. For $\mathcal{N}=2$ SYM, we require that $A, B, \ldots=1,2$. This completely eliminates the amplitudes on the second and third lines which involve all four fermion fields. By further restricting $A$ to take only a single value, we limit ourselves to $\mathcal{N}=1 \mathrm{SYM}$. In addition to the excluded amplitudes for the $\mathcal{N}=2$ theory, this restriction forbids any of the amplitudes with external scalars (as $\phi^{A B}=-\phi^{B A}$ ), which is encouraging because the $\mathcal{N}=1$ version of SYM contains no such fields.

It is possible to use these MHV amplitudes for SYM with $\mathcal{N}<4$ coupled to (massless) matter fields. To do this we allow the indices $A$ to run over additional values which corresponds to the matter multiplets. Though we may consider theories with an arbitrary number of matter fields, these amplitudes will only apply to processes where a limited number of those multiplets interact. Specifically, the Yang-Mills multiplet may interact with 1 hypermultiplet for $\mathcal{N}=2$ and up to $3 \chi$ sfs in $\mathcal{N}=1$. Any more matter would exceed the field content of the $\mathcal{N}=4$ theory and spoil the agreement. The kinematic partial amplitudes $A_{n}$ will be the same, however the colour factors $T_{n}\left(a_{i}\right)$ must be altered in an appropriate manner depending on the representation of the matter fields [46].

A particularly simple example of this actually has applications in the real (nonsupersymmetric) world. Consider $\mathcal{N}=1$ SYM coupled to $n_{f} \chi$ sfs in the fundamental representation. We restrict $A$ to a single value so only the first line of amplitudes in (5.5), with
$m=0,1,2$ fermion pairs and $\ell=n-2 m$ gluons, will survive. The external fermions may either be of the adjoint (gluino) type or the fundamental (quark) type. At tree-level, the only internal states which can appear are the same as the ones we started with, $g^{ \pm}, \psi^{A}, \bar{\psi}_{A}$ for fixed $A$. Thus, by fixing the external fermions to be quarks (of the same flavour) the remaining fields (gluinos and squarks) decouple from these amplitudes. We are left with an effective non-supersymmetric theory of gluons and quarks - that is to say QCD with $S U(N)$ gauge group and $n_{f}$ massless quarks. Thus we have deduced the non-supersymmetric MHV amplitudes

$$
\begin{equation*}
A_{\ell}\left(g^{-}, g^{-}\right), A_{\ell+2}\left(g^{-}, \bar{q}_{f}, q_{f}\right), A_{\ell+4}\left(\bar{q}_{f}, q_{f}, \bar{q}_{f}, q\right) \tag{5.11}
\end{equation*}
$$

from $\mathcal{N}=4$ SYM. As we mentioned in the last paragraph, we must adjust the colour factor in front of the amplitude to account for the presence of fundamental matter. For $\ell$ gluons and $m>0$ quark/antiquark pairs, the exact tree level colour factor is [15, 46],

$$
\begin{equation*}
T_{\ell+2 m}=\frac{(-1)^{p}}{N^{p}}\left(T_{a_{1}} \ldots T_{a_{\ell_{1}}}\right)_{i_{1} \alpha_{1}}\left(T_{a_{\ell_{1}+1}} \ldots T_{a_{\ell_{2}}}\right)_{i_{2} \alpha_{2}} \ldots\left(T_{a_{\ell_{m-1}+1}} \ldots T_{a_{\ell}}\right)_{i_{m} \alpha_{m}} \tag{5.12}
\end{equation*}
$$

The $\ell_{1} \ldots \ell_{m}$ are an arbitrary permutation of the $\ell$ gluon indices; $i_{1} \ldots i_{m}$ and $\alpha_{1} \ldots \alpha_{m}$ are the quark and antiquark colour indices, respectively. When quark $i_{k}$ is connected by a fermion line to the antiquark $\alpha_{k}$, we set $i_{k}=\bar{\alpha}_{k}$. Finally, the power $p$ is the number of times $i_{k}=\bar{\alpha}_{k}$ minus one, thus $p \in\{0, \ldots, m-1\}$. This introduces the correct multi-trace colour factors with $1 / N^{p}$ suppression, as required for fundamental matter. Amazingly, the partial amplitudes $A_{\ell+2 m}$ are identical to the $\mathcal{N}=4$ theory, and given by (5.6).

### 5.3 Generalized MHV Diagrams

The generalization of the CSW rules is straightforward, and following the discussion of the preceding two sections probably quite obvious. This extension was first employed at tree level in [15] and later in [17, 20, 21] We simply replace the gluonic MHV vertex with the $\mathcal{N}=4$ MHV supervertex:


Each vertex has 8 powers of $\eta^{A}$ associated to it, which are distributed amongst anywhere from two to eight external legs. We continue to connect vertices by scalar propagators, though the propagating field may have helicity $0, \pm \frac{1}{2}, \pm 1$. Since propagators flip helicities between vertices, internal lines have $\left(g^{+}, g^{-}\right)$, $\left(\psi^{A}, \bar{\psi}_{A}\right)$ or ( $\phi^{A B}, \bar{\phi}_{A B}$ ) endpoints. A quick count reveals that each propagator uses up 4 of the $\eta^{A}$ associated with the whole diagram. Once again, we must sum over all possible diagrams which agree with the colour ordering. For theories with less SUSY, we must constrain the vertices as in the previous section.

By now, we are used to using scalar propagators to connect gluonic MHV diagrams, so perhaps we are not so surprised that the same scalar propagator works for fermions as well.

As in the bosonic case, recall that the MHV amplitudes have all the external wavefunctions incorporated into them already, including those which we take off shell to connect to other vertices. Thus, by extracting the propagators' wavefunctions from the vertices and including them with the $1 / P^{2}$ terms we reproduce the fermionic propagators. For this to work, it is once again essential that propagators flip helicities between vertices. For example, consider the NMHV amplitude with an internal fermion line (such an amplitude will necessarily have an identical fermion and its CPT conjugate on different MHV vertices), then

$$
\begin{align*}
A_{M H V} \frac{1}{P^{2}} A_{M H V}^{\prime} & =A_{M H V}^{\alpha} \frac{\lambda_{P \alpha} \widetilde{\lambda}_{P \dot{\alpha}}}{P^{2}} A_{M H V}^{\prime \dot{\alpha}} \\
& =A_{M H V}^{\alpha} \frac{\left(\sigma^{\alpha}\right)_{\alpha \dot{\alpha}} P_{a}}{P^{2}} A_{M H V}^{\prime \dot{\alpha}}  \tag{5.13}\\
& =A_{M H V}^{\alpha} \frac{P_{\alpha \dot{\alpha}}}{P^{2}} A_{M H V}^{\prime \dot{\alpha}}
\end{align*}
$$

Using these modified CSW rules, we can easily construct the general NMHV amplitude in $\mathcal{N}=4$ SYM, first presented in [20]. Of course, not all the amplitudes will be NMHV (just as not all the amplitudes in Nair's formula were MHV), however we use this name nonetheless.


Figure 5.1: General diagram contributing to NHMV amplitude
The general diagram in depicted in Figure 5.1. It has two vertices connected by a single propagator. We leave the specific helicities of the external particles arbitrary, only requiring that the total amplitude have 12 powers of $\eta^{A}$ associated to it. This is simply because there are two vertices with $8 \eta^{A}$ each, with the internal line using up 4 of them, as explained above. Also, we orient the diagram so that the positive helicity end of the propagator is attached to the left vertex. The division of the $4 \eta^{A}$ between left and right endpoints of an internal line is arbitrary unless the external helicities are specified, so we must integrate over these variables.

As before, we define the off-shell spinor $\lambda_{P \alpha}=P_{\alpha \dot{\alpha}} \eta^{\dot{\alpha}}$ for some arbitrary fixed spinor $\eta^{\dot{\alpha}}$ (not to be confused with the Grassman variable $\eta^{A}$ ). The momentum transfer vector is still $P_{i j}=p_{i}+\ldots+p_{j}=-P_{(j-1)(i-1)}$, which we will sometimes abbreviate to just $P$. With all
this in mind, we can write down the general NMHV amplitude in $\mathcal{N}=4$ SYM:

$$
\begin{equation*}
A_{n}=\frac{1}{\prod_{\ell=1}^{n}\langle\ell \ell+1\rangle} \sum_{i j} \frac{1}{D_{i j}} \int d^{4} \eta_{P}^{A} \delta^{(8)}\left(-\lambda_{\alpha}^{P} \eta_{P}^{A}+\sum_{k_{2}=i}^{j} \lambda_{\alpha}^{k_{2}} \eta_{k_{2}}^{A}\right) \delta^{(8)}\left(\lambda_{\alpha}^{P} \eta_{P}^{A}+\sum_{k_{1}=j+1}^{i-1} \lambda_{\alpha}^{k_{1}} \eta_{k_{1}}^{A}\right) \tag{5.14}
\end{equation*}
$$

where the denominator $D_{i j}$ is the same as in (4.7):

$$
\begin{equation*}
D_{i j}=\frac{\left\langle i P_{i j}\right\rangle\left\langle P_{i j} j+1\right\rangle P_{i j}^{2}\left\langle P_{i j} i+1\right\rangle\left\langle j P_{i j}\right\rangle}{\langle i i+1\rangle\langle j j+1\rangle} . \tag{5.15}
\end{equation*}
$$

The sum over $i, j$ must be arranged to ensure that $8 \eta^{A}$ are associated to each vertex and the diagram's orientation, discussed above, is preserved. Notice that we take one of the $\lambda_{P}$ to be negative, this is a reflection of the momentum flow of $P_{i j}$.

We make use of the fact that $\int \delta\left(f_{2}\right) \delta\left(f_{1}\right)=\int \delta\left(f_{1}+f_{2}\right) \delta\left(f_{1}\right)$ to simplify the $\eta^{A}$ integration,

$$
\begin{align*}
& \delta^{(8)}\left(-\lambda_{\alpha}^{P} \eta_{P}^{A}+\sum_{k_{2}=i}^{j} \lambda_{\alpha}^{k_{2}} \eta_{k_{2}}^{A}\right) \delta^{(8)}\left(\lambda_{\alpha}^{P} \eta_{P}^{A}+\sum_{k_{1}=j+1}^{i-1} \lambda_{\alpha}^{k_{1}} \eta_{k_{1}}^{A}\right) \\
= & \delta^{(8)}\left(\sum_{k=1}^{n} \lambda_{\alpha}^{k} \eta_{k}^{A}\right) \delta^{(8)}\left(\lambda_{\alpha}^{P} \eta_{P}^{A}+\sum_{k_{1}=j+1}^{i-1} \lambda_{\alpha}^{k_{1}} \eta_{k_{1}}^{A}\right)  \tag{5.16}\\
= & \delta^{(8)}\left(\sum_{k=1}^{n} \lambda_{\alpha}^{k} \eta_{k}^{A}\right) \prod_{A=1}^{4} \delta^{2}\left(\lambda_{\alpha}^{P} \eta_{P}^{A}+\sum_{k_{1}=j+1}^{i-1} \lambda_{\alpha}^{k_{1}} \eta_{k_{1}}^{A}\right) .
\end{align*}
$$

Now the $\eta^{A}$ integration is trivial, and in doing it the $\lambda_{P}$ get paired with the $\lambda_{k_{1}}$. The final result is [20]

$$
\begin{equation*}
A_{n}=\frac{1}{\prod_{\ell=1}^{n}\langle\ell \ell+1\rangle} \delta^{(8)}\left(\sum_{k=1}^{n} \lambda_{\alpha}^{k} \eta_{k}^{A}\right) \sum_{i j} \frac{1}{D_{i j}} \prod_{A=1}^{4}\left(\sum_{m=j+1}^{i-1}\left\langle P_{i j} m\right\rangle \eta_{m}^{A}\right) \tag{5.17}
\end{equation*}
$$

As expected, these amplitudes have 12 powers of $\eta^{A}$ associated with them. We also might have expected they would be proportional to $\lambda_{P}^{4}$, as the internal line uses up 4 of the original $16 \eta^{A}$, and indeed it is. Taylor expanding (5.17) in $\eta^{A}$ yields all the $\mathcal{N}=4$ SYM amplitudes involving 12 factors of $\eta^{A}$ as determined by the map between $\eta^{A}$ and helicity states (5.2).

## Chapter 6

## One-Loop MHV Amplitudes

### 6.1 Loop Amplitudes in Supersymmetric Gauge Theories

As discussed in Section 2.5, extended SUSY multiplets can always be decomposed into $\mathcal{N}=1$ components. In calculating loop amplitudes we can perform a similar decomposition. By knowing the contributions arising from $\mathcal{N}=1$ vector and (adjoint) chiral superfields propagating around the loop, we can combine these results into the one-loop contribution of any SUSY multiplet. Rather then calculating the $\mathcal{N}=1$ Vsf contribution directly, we utilize the following linear combination

$$
\begin{equation*}
\mathcal{A}^{\mathcal{N}=1 V}=\mathcal{A}^{\mathcal{N}=4}-3 \mathcal{A}^{\mathcal{N}=1 \chi} . \tag{6.1}
\end{equation*}
$$

The advantage of this is that the $\mathcal{N}=4$ theory possesses greater amounts of symmetry and so proves simpler to compute. So, when calculating one-loop amplitudes in any supersymmetric gauge theory, we can use the contributions from the $\mathcal{N}=4$ multiplet and $\mathcal{N}=1 \chi$ sf as a basis for all amplitudes.

Unitarity plays an essential role in determining supersymmetric loop amplitudes. Recall that the amplitudes which are completely fixed by their unitarity cuts were dubbed cutconstructible by Bern, Dixon, Dunbar and Kosower (BBDK) in the mid-nineties [5]. The essential feature of cut-constructible amplitudes in that they are can be written as a linear combination of a set of well-know basis functions. These functions are the solutions to scalar loop integrals with up to four external legs. By comparing the cuts of the basis functions to the cuts in the amplitude, they were able to determine the correct coefficients in basis expansion.

So for cut-constructible theories, one really only needs the information of tree-level amplitudes to determine one-loop results. By applying the power of unitarity, BDDK found general expressions for MHV amplitudes in $\mathcal{N}=4$ SYM [7] and later in $\mathcal{N}=1$ SYM [5] for an arbitrary numbers of external legs. We will present these amplitudes in the next two sections.

To obtain a pure Yang-Mills amplitude, the decomposition (6.1) needs an additional contribution arising from an internal scalar field in the loop

$$
\begin{equation*}
\mathcal{A}^{Y M}=\mathcal{A}^{\mathcal{N}=4}-4 \mathcal{A}^{\mathcal{N}=1 \chi}+\mathcal{A}^{\phi} . \tag{6.2}
\end{equation*}
$$

A lone fermion could also be found with knowledge of the scalar contribution, subtracting it from the chiral multiplet. The scalar is always the most difficult of the three basis amplitudes to compute, however it is far simpler then solving for the gluon or fermion circulating in
the loop directly. In particular, the scalar loop (being non-supersymmetric) is not cutconstructible [5]. This is the main obstruction to performing loop calculations in pure YangMills theory.

At the one-loop level, an additional complication arises concerning the colour decomposition. We no longer have

$$
\begin{equation*}
\mathcal{A}_{n}\left(a_{i}, p_{i}, h_{i}\right)=i g_{Y M}^{n-2}(2 \pi)^{4} \delta^{(4)}\left(\sum_{i=1}^{n} p_{i}\right) \sum_{\sigma \in S_{n} / Z_{n}} T_{n}\left(a_{\sigma(i)}\right) A_{n}\left(p_{\sigma(i)}, h_{\sigma(i)}\right) \tag{6.3}
\end{equation*}
$$

as in (3.1). Instead, assuming an $S U(N)$ gauge group with all particles transforming in the adjoint, the colour decomposition is [7]

$$
\begin{equation*}
\mathcal{A}_{n}\left(a_{i}, p_{i}, h_{i}\right)=i g_{Y M}^{n-2}(2 \pi)^{4} \delta^{(4)}\left(\sum_{i=1}^{n} p_{i}\right) \sum_{J} n_{J} \sum_{c=1}^{\lfloor n / 2\rfloor+1} \sum_{S_{n} / S_{n ; c}} \operatorname{Gr}_{n ; c}(\sigma) A_{n ; c}^{[J]}(\sigma), \tag{6.4}
\end{equation*}
$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to $x$ and $n_{J}$ is the number particles of spin $J$. The leading colour-structure factor

$$
\begin{equation*}
\operatorname{Gr}(1)=N \operatorname{Tr}\left(T_{a_{1}} \ldots T_{a_{n}}\right) \tag{6.5}
\end{equation*}
$$

is just $N$ times the tree-level colour factor, and the sub-leading colour structures are given by

$$
\begin{equation*}
\operatorname{Gr}_{n ; c}(1)=\operatorname{Tr}\left(T_{a_{1}} \ldots T_{a_{c-1}}\right) \operatorname{Tr}\left(T_{a_{c}} \ldots T_{a_{n}}\right) \tag{6.6}
\end{equation*}
$$

$S_{n}$ is still the set of all permutations of $n$ objects, while $S_{n ; c}$ is the subset leaving $\operatorname{Gr}_{n ; c}$ invariant. The leading contributions to scattering amplitudes, for large $N$, comes from $A_{n ; 1}^{[J]}$, while the sub-leading corrections, down by a factor of $1 / N$, are given by $A_{n ; c}^{[J]}$ for $c>1$. Fortunately, we need not calculate all of the sub-leading terms as they can be determined algebraically from the leading contribution [7]. It is therefore sufficient to consider only the leading term $A_{n ; 1}^{[J]}$, which we will do in what follows.

In any supersymmetric gauge theory, amplitudes in which all or all but one external gluons have the same helicity vanish [47],

$$
\begin{equation*}
A_{n}^{1-l o o p}\left(g^{ \pm}, g^{+}, \ldots, g^{+}\right)=0 \tag{6.7}
\end{equation*}
$$

exactly as in the tree-level case. This is due to a supersymmetric Ward identity, however we may also see this by appealing to cut-constructibility, which we will now explain. To calculate a unitarity cut we use the Cutkosky cutting rules [48]. For a one-loop amplitude, we replace two of the internal propagators with $\delta$-functions. This separates the loop amplitude into a product of two tree-level amplitudes and two $\delta$-functions, which we then integrate using an appropriate Lorentz-invariant phase-space measure. This calculates the cut in one channel, we repeat this process for all possible channels and sum all the cuts. Any state which crosses the cut must have different helicities in the two tree amplitudes, as we always consider amplitudes to have all particles incoming. In one-loop amplitudes of the form $A_{n}\left(g^{ \pm}, g^{+}, \ldots, g^{+}\right)$(or their parity conjugates), we can easily see that there is no possible assignment of internal helicities states which prevents a tree amplitude of the form
$A_{n}^{\text {tree }}\left(g^{ \pm}, g^{+}, \ldots, g^{+}\right)$from forming on one side of the cut. We know these amplitudes vanish at tree level, from Section 3.1, so we conclude that the unitarity cuts vanish in all channels for this external helicity configuration. As this theory is cut-constructible this implies the amplitude is identically zero at one loop.

The first non-vanishing amplitudes at one loop in supersymmetric gauge theories involve two particles with opposite helicities from the rest - that is, the MHV amplitudes.

### 6.2 MHV Loops in $\mathcal{N}=4$ SYM

In the case of $\mathcal{N}=4 \mathrm{SYM}$, the MHV amplitude is particularly simple and given by [7]

$$
\begin{equation*}
A_{n}^{\mathcal{N}=4}=c_{\Gamma} A_{n}^{\text {tree }} V_{n} \tag{6.8}
\end{equation*}
$$

where $A_{n}^{\text {tree }}$ is the regular tree level MHV amplitude, either the pure gluon Parke-Taylor formula (3.4) or Nair's generalized version (5.3), and $V_{n}$ is a universal one-loop function which only depends on the number of external legs, in particular $V_{n}$ is independent of the helicity ordering. The prefactor is

$$
\begin{equation*}
c_{\Gamma}=\frac{(4 \pi)^{\epsilon}}{16 \pi^{2}} \frac{\Gamma(1+\epsilon) \Gamma^{2}(1-\epsilon)}{\Gamma(1-2 \epsilon)}, \tag{6.9}
\end{equation*}
$$

where $\epsilon=(4-D) / 2$ the dimensional regularization parameter.
$V_{n}$ is most naturally written as a sum over pairs of external states in the set of $n$. We take these distinguished particles to be in the positions $r, s$ and write their momenta as $k_{r}, k_{s}$. The remaining $n-2$ external momenta then naturally combine into two sets ( $k_{s+1}, \ldots, k_{r-1}$ ) and ( $k_{r+1}, \ldots, k_{s-1}$ ). We write the sums of these sets as

$$
\begin{equation*}
P=k_{s+1}+k_{s+2}+\ldots+k_{r-1} \quad, \quad Q=k_{r+1}+k_{r+2}+\ldots+k_{s-1} \tag{6.10}
\end{equation*}
$$

With this notation, momentum conservation reads $k_{r}+Q+k_{s}+P=0$ and the universal function is written

$$
\begin{equation*}
V_{n}=\sum_{r=1}^{n} \sum_{s=r+2}^{r+\lfloor n / 2\rfloor}\left(1-\frac{1}{2} \delta_{r+\frac{n}{2}, s}\right) F\left(k_{r}, Q, k_{s}, P\right) \tag{6.11}
\end{equation*}
$$

where $F\left(k_{r}, Q, k_{s}, P\right)$ is the so-called (two-mass easy) box function, which we explain momentarily. First, we find it convenient to define the momentum invariants

$$
\begin{equation*}
s=\left(P+k_{r}\right)^{2} \quad, \quad t=\left(P+k_{s}\right)^{2}, \tag{6.12}
\end{equation*}
$$

for reasons which will some become clear ${ }^{1}$. With these definitions in place, we can write the box function compactly as

[^3]\[

$$
\begin{align*}
F\left(k_{r}, Q, k_{s}, P\right) & =-\frac{1}{\epsilon^{2}}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}-\left(-P^{2}\right)^{-\epsilon}-\left(-Q^{2}\right)^{-\epsilon}\right] \\
& +\mathrm{Li}_{2}\left(1-\frac{P^{2}}{s}\right)+\mathrm{Li}_{2}\left(1-\frac{P^{2}}{t}\right)+\mathrm{Li}_{2}\left(1-\frac{Q^{2}}{s}\right)+\mathrm{Li}_{2}\left(1-\frac{Q^{2}}{t}\right) \\
& -\mathrm{Li}_{2}\left(1-\frac{P^{2} Q^{2}}{s t}\right)+\frac{1}{2} \log ^{2}\left(\frac{s}{t}\right) . \tag{6.13}
\end{align*}
$$
\]

Although the $\mathcal{N}=4$ theory is conformal, implying that the Yang-Mills coupling constant does not diverge on any scale, IR divergences will occur in the soft or collinear limit. While these infinities may always be avoided at tree-level through the appropriate choice of external momenta, they will inevitably arise in loop amplitudes. As we see here, this IR behaviour is contained in the $1 / \epsilon^{2}$ terms.

The peculiar name "two-mass easy" box function is derived from the fact that it is proportional to the "two-mass easy" scalar box integral,

$$
\begin{equation*}
F\left(k_{r}, Q, k_{s}, P\right)=\frac{i}{c_{\Gamma}} \frac{s t-P^{2} Q^{2}}{2} I_{4: r ; s}^{2 m e} \tag{6.14}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{4: r ; s}^{2 m e}=\int \frac{d^{D} \ell}{(2 \pi)^{D}} \frac{1}{\ell^{2}\left(\ell+k_{r}\right)^{2}\left(\ell+k_{r}+Q\right)^{2}(\ell-P)^{2}} \tag{6.15}
\end{equation*}
$$

which we depict in Figure 6.1.


Figure 6.1: The loop diagram for the scalar box integral
As we can see, the above 4-point scalar loop integral contains two massive legs $P, Q$, hence the name "two-mass", the "easy" part refers to the fact that the massive legs separated by a null leg. There is also a "two-mass hard" scalar box integral where the massive legs are adjacent, however, it does not appear until computations of NMHV one-loop amplitudes and is not needed for the simpler MHV case.

### 6.3 MHV Loops in $\mathcal{N}<4$ SYM

As explained in Section 6.1, any supersymmetric gauge theory amplitude can be decomposed into a linear combination of contributions from an $\mathcal{N}=4$ multiplet and $\mathcal{N}=1$ chiral superfields ( $\chi \mathrm{sf}$ ). The $\mathcal{N}=1 \chi$ sf contribution to one-loop MHV amplitudes was discovered in [5]. As in the $\mathcal{N}=4$ case, the tree-level MHV amplitude factors out, however the remaining factor $V_{n}$ is no longer universal as it depends explicitly on the locations of the negative helicity states. We label the negative helicity particles as $p, q$. Using the same notation as in the previous section, the $\mathcal{N}=1 \chi$ sf contribution is

$$
\begin{equation*}
A_{n}^{\mathcal{N}=1 \chi}=\frac{c_{\Gamma}}{2} A_{n}^{\text {tree }} V_{n}^{p q} \tag{6.16}
\end{equation*}
$$

where the ordering dependant factor is

$$
\begin{array}{r}
V_{n}^{p q}=\sum_{r=p+1}^{q-1} \sum_{s=q+1}^{p-1} b_{r s}^{p q} B\left(k_{r}, Q, k_{s}, P\right)+\sum_{r=p+1}^{q-1} \sum_{s=q}^{p-1} c_{r s}^{p q} T\left(k_{r}, P, \widetilde{Q}\right)+ \\
+\sum_{r=p}^{q-1} \sum_{s=q+1}^{p-1} c_{s r}^{p q} T\left(k_{s}, Q, \widetilde{P}\right)+\mathcal{A}_{I R} . \tag{6.17}
\end{array}
$$

Notice the ranges of summation over $r, s$ in (6.17) is always such that $k_{p}$ belongs to the set of momenta in $P=k_{s+1}+\ldots+k_{r-1}$, and likewise $k_{q}$ is one of the momenta in $Q=k_{r+1}+\ldots+k_{s-1}$. In particular the massless momenta $k_{r}, k_{s}$ always have positive helicity. The first term in $V_{n}^{p q}$ is related the scalar box function $F$, then there are two terms coming from' (two-mass) triangles functions and finally the last part $\mathcal{A}_{I R}$ comes from exceptional, boundary terms. We explain each these terms below.

First, the functions $B\left(k_{r}, Q, k_{s}, P\right)$ are the finite parts of the box functions (6.13), which appeared in the $\mathcal{N}=4$ amplitude. More explicitly,

$$
\begin{align*}
B\left(k_{r}, Q, k_{s}, P\right) & =F\left(k_{r}, Q, k_{s}, P\right)+\frac{1}{\epsilon^{2}}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}-\left(-P^{2}\right)^{-\epsilon}-\left(-Q^{2}\right)^{-\epsilon}\right] \\
& =\mathrm{Li}_{2}\left(1-\frac{P^{2}}{s}\right)+\mathrm{Li}_{2}\left(1-\frac{P^{2}}{t}\right)+\mathrm{Li}_{2}\left(1-\frac{Q^{2}}{s}\right)+\mathrm{Li}_{2}\left(1-\frac{Q^{2}}{t}\right) \\
& -\mathrm{Li}_{2}\left(1-\frac{P^{2} Q^{2}}{s t}\right)+\frac{1}{2} \log ^{2}\left(\frac{s}{t}\right) \tag{6.18}
\end{align*}
$$

The triangle functions $T$ depend on only one massless momentum $k_{r, s}$, and two massive ones: $P, \widetilde{Q} \equiv Q+k_{s}$ or $Q, \widetilde{P} \equiv P+k_{r}$, each of which contains a single negative helicity gluon $p$ or $q$. The two triangle functions are identical in form, in general

$$
\begin{equation*}
T(k, P, Q)=\frac{\log \left(P^{2}\right)-\log \left(Q^{2}\right)}{P^{2}-Q^{2}} \tag{6.19}
\end{equation*}
$$

We use the name triangle functions as they are proportional to a scalar 3-point integral

$$
\begin{equation*}
T\left(k_{r}, P, \widetilde{Q}\right)=\frac{1}{c_{\Gamma}}\left(\frac{\epsilon}{1-2 \epsilon}\right) I_{3: r ; s}^{2 m} \tag{6.20}
\end{equation*}
$$

where

$$
\begin{equation*}
I_{3: r ; s}^{2 m}=i \int \frac{d^{D} \ell}{(2 \pi)^{D}} \frac{1}{\ell^{2}(\ell-\widetilde{Q})^{2}\left(\ell+k_{r}\right)^{2}} . \tag{6.21}
\end{equation*}
$$

We point out that the scalar integral $I_{3: r ; s}^{2 m}$ has a $1 / \epsilon$ divergence [5], implying that the function $T$ remains finite as $\epsilon \rightarrow 0$. The set of diagrams contributing to triangle functions $T\left(k_{r}, P, \widetilde{Q}\right)$ is drawn in Figure 6.2, the others follow the same pattern. Though it was not explicit in the


Figure 6.2: Diagrams contributing to triangle functions
given ranges of summation for the triangles, we require $|r-s|>1$ and $|r-s-1|>1$. This constraint ensures each of the triangles have two massive legs.

The representation (6.20) of the function $T$ is not unique. When we calculate this amplitude using the CSW rules we will need to know the follow equivalent forms of representing this function [5]:

$$
\begin{equation*}
c_{\Gamma} T\left(k_{r}, P, \widetilde{Q}\right)=\left(\frac{\epsilon}{1-2 \epsilon}\right) I_{3: r ; s}^{2 m}=I_{3: r ; s}^{2 m}\left[x_{2}\right]=\frac{1}{s-Q^{2}}\left(I_{2: r ; s}-I_{2: r+1 ; s}\right) \tag{6.22}
\end{equation*}
$$

The integral $I_{3: r ; s}^{2 m}\left[x_{2}\right]$ is the same as the triangle integral (6.21) except the Feynman parameter $x_{2}$ appears in the numerator. We will explain this representation more explicitly when it arises in the calculations of Section 8. We have also introduced the scalar 2-point integral

$$
\begin{equation*}
I_{2: r ; s}=-i \int \frac{d^{D} \ell}{(2 \pi)^{D}} \frac{1}{\ell^{2}(\ell-Q)^{2}} \tag{6.23}
\end{equation*}
$$

whose diagrams have momenta $k_{s}+P+k_{r}$ on one side of the loop and $Q$ on the other.
The last term in (6.17), $\mathcal{A}_{I R}$, is the only source of infrared divergences in these amplitudes. IR divergences arise from the degeneration of triangle diagrams for which one of the massive momenta become massless. That is, when it contains only a single external momentum, which is then necessarily a negative helicity gluon. There are four such degenerations, for
which $p=P, \widetilde{P}$ or $q=Q, \widetilde{Q}$. These cases are drawn in Figure 6.3 , they give rise to the following 4 terms:

$$
\begin{equation*}
\mathcal{A}_{I R}=c_{p+1, p-1}^{p q} \frac{\left(-\left(k_{p+1}+k_{p-1}\right)^{2}\right)^{-1-\epsilon}}{\epsilon(1-2 \epsilon)}+c_{p-1, p}^{p q} \frac{\left(-\left(k_{p-1}+k_{p}\right)^{2}\right)^{-1-\epsilon}}{\epsilon(1-2 \epsilon)}+(p \leftrightarrow q) . \tag{6.24}
\end{equation*}
$$



Figure 6.3: Two of the degenerate triangle diagrams
Finally, the coefficients appearing in (6.17) are as follows. For the box functions

$$
\begin{equation*}
b_{r s}^{p q}=2 \frac{\langle p r\rangle\langle p s\rangle\langle q r\rangle\langle q s\rangle}{\langle r s\rangle^{2}\langle p q\rangle^{2}} \tag{6.25}
\end{equation*}
$$

whereas for the triangles (and the boundary terms) one has

$$
\begin{equation*}
\left.\left.\left.c_{r s}^{p q}=\frac{\langle p r\rangle\langle r q\rangle}{\langle p q\rangle^{2}} \frac{\langle s, s+1\rangle}{\langle s r\rangle\langle s, r+1\rangle}(\langle q r\rangle\langle p| P \mid r]+\langle p r\rangle\langle q| P \right\rvert\, r\right]\right), \tag{6.26}
\end{equation*}
$$

where we have introduced the spinor product

$$
\begin{equation*}
\left.\langle\lambda| P_{i j} \mid \widetilde{\mu}\right]=\sum_{k=i}^{j}\left\langle\lambda \lambda_{k}\right\rangle\left[\tilde{\lambda}_{k} \widetilde{\mu}\right] . \tag{6.27}
\end{equation*}
$$

Notice that in the coefficients $c_{r s}^{p q}$ the null leg is $k_{r}$. In $c_{s r}^{p q}$ the null leg is $k_{s}$ and we must change $P=k_{s+1}+\ldots+k_{r-1}$ to $Q=k_{r+1}+\ldots+k_{s-1}$.

## Chapter 7

## One-Loop MHV Diagrams

In applying the CSW formalism to one-loop amplitudes, we begin by noticing that the number of MHV vertices must equal the number of negative helicity particles, as

$$
\begin{equation*}
v=q-1+\ell=q . \tag{7.1}
\end{equation*}
$$

We immediately conclude that the first non-trivial one-loop amplitudes are MHV with two external negative helicity states. This agrees with the statement made in Section 6.1 that $A_{n}^{1-l o o p}\left(g^{ \pm}, g^{+}, \ldots, g^{+}\right)=0$. Unfortunately, these amplitudes are non-vanishing in pure Yang-Mills theory, so the CSW rules must be modified somehow for non-supersymmetric theories at the loop level. What modifications must be made is still an open question, although some speculations are presented in [23].

The first application of the CSW rules to loop amplitudes was conducted in [25] by Brandhuber, Spence and Travaglini (BST). They calculated the simplest set of one-loop amplitudes, which are the $\mathcal{N}=4$ MHV ones, and found perfect agreement with the BDDK calculation. Later, this author together with Rozali performed an analogous calculation [26] involving the $\mathcal{N}=1$ chiral multiplet and also found perfect agreement with BDDK. Our result was simultaneously confirmed by BST and Bedford [27]. When applied to the scalar loop contribution, however, the MHV diagrams did not reproduce the full amplitude [28]. In particular, the scalar amplitude contains terms which are not cut-constructible and these terms are invisible to the MHV diagrams. This might have been anticipated, as we have already noticed that the CSW rules require modification at the loop level in non-supersymmetric theories.

Although the origins of the BBDK and BST approaches are far removed from each other (unitarity cuts vs. CSW rules), after a point the two calculations follow almost identical paths. The reason is the following. In performing the unitarity cuts, BDDK found that the two tree-level amplitudes to be integrated over Lorentz invariant phase space (LIPS) were necessarily both MHV [7]. In the BST calculation, as we shall soon see, the LIPS measure naturally arises in the loop integral. With some rearrangement of terms, the sum over MHV diagrams becomes equivalent to the sum over the cuts in all possible channels. It is quite surprising that such disparate methods would produce such parallel computations. It may simply be a coincidence, however this author believes it is a hint of some deeper hidden structure in field theory.

### 7.1 The BST Measure

The crucial step in the BST calculation was rewriting the measure $d^{4} L$ in terms of spinor variables and ultimately obtaining the phase space measure $d$ LIPS. To do this, we begin
with a slight modification to the CSW prescription for off-shell spinors. As in $[18,19]$, we can always decompose a massive (loop) momentum vector $L_{\alpha \dot{\alpha}}$ into two null vectors

$$
\begin{equation*}
L_{\alpha \dot{\alpha}}=\ell_{\alpha \dot{\alpha}}+z \eta_{\alpha \dot{\alpha}} \tag{7.2}
\end{equation*}
$$

where $\ell^{2}=0=\eta^{2}, \eta$ is a fixed (but arbitrary) null vector and $z \in \mathbb{R}$. Essentially, $z$ measures how far the momentum $L$ is from being null. The choice of $\eta$ amounts to choosing a lightcone frame. Writing the null vectors as bi-spinors, $\ell_{\alpha \dot{\alpha}}=\lambda_{\alpha} \widetilde{\lambda}_{\dot{\alpha}}$ and $\eta_{\alpha \dot{\alpha}}=\eta_{\alpha} \widetilde{\eta}_{\dot{\alpha}}$, it follows from contracting (7.2) with $\widetilde{\eta}^{\dot{\alpha}}$ that

$$
\begin{equation*}
\lambda_{\alpha}=\frac{P_{\alpha \dot{\alpha}} \widetilde{\eta}^{\dot{\alpha}}}{[\widetilde{\lambda} \tilde{\eta}]} \quad, \quad \widetilde{\lambda}_{\dot{\alpha}}=\frac{\eta^{\alpha} L_{\alpha \dot{\alpha}}}{\langle\lambda \eta\rangle} . \tag{7.3}
\end{equation*}
$$

The expressions we will construct using the CSW rules will always be homogeneous in the off-shell spinors, and so the $[\widetilde{\lambda} \widetilde{\eta}]$ or $\langle\lambda \eta\rangle$ factors will always cancel. Because of this, these factors may be neglected and this prescription becomes equivalent to CSW's. The advantage of this approach is that we now know

$$
\begin{equation*}
L^{2}=2 z\langle\lambda \eta\rangle[\tilde{\lambda} \tilde{\eta}] \tag{7.4}
\end{equation*}
$$

Writing out the loop measure in spinor variables, we are led to the result that

$$
\begin{equation*}
d^{4} L=2 d z d \mathcal{N}(\ell)\langle\lambda \eta\rangle[\widetilde{\lambda} \tilde{\eta}], \tag{7.5}
\end{equation*}
$$

where the Nair measure [3] is (proportional to) the LIPS measure of a single particle with null momentum $\ell$ :

$$
\begin{equation*}
d \mathcal{N}(\ell)=\langle\lambda d \lambda\rangle d^{2} \tilde{\lambda}-[\tilde{\lambda} d \widetilde{\lambda}] d^{2} \lambda=2 i d^{4} \ell \delta^{(+)}\left(\ell^{2}\right), \tag{7.6}
\end{equation*}
$$

where $\delta^{(+)}\left(\ell^{2}\right)$ restricts $\ell_{0}>0$. The important observation to make now is that the combination which arises naturally when using MHV diagrams

$$
\begin{equation*}
\frac{d^{4} L}{L^{2}}=\frac{d z}{z} d \mathcal{N}(\ell) \tag{7.7}
\end{equation*}
$$

is independent of our choice of reference vector $\eta$.
For one-loop integrals, there will be two off-shell momenta $L_{1}, L_{2}$ related through momentum conservation $\delta^{(4)}\left(L_{1}-L_{2}+P\right)$, where $P$ is the total external momentum flowing into one side of the loop. We decompose the massive vectors as above,

$$
\begin{equation*}
L_{i \alpha \dot{\alpha}}=\lambda_{i \alpha} \widetilde{\lambda}_{i \dot{\alpha}}+z_{i} \eta_{\alpha} \widetilde{\eta}_{\dot{\alpha}}, \quad i=1,2, \tag{7.8}
\end{equation*}
$$

using the same reference null vector $\eta$ for both $L_{i}$. The combinations $d^{4} L_{i} / L_{i}^{2}$ are as described above, while the quantity appearing in the $\delta$-function is

$$
\begin{equation*}
L_{1}-L_{2}+P=\ell_{1}-\ell_{2}+P-\left(z_{2}-z_{1}\right) \eta \tag{7.9}
\end{equation*}
$$

Defining the quantities

$$
\begin{array}{r}
z=z_{2}-z_{1} \\
P_{z}=P-z \eta, \tag{7.11}
\end{array}
$$

then we have

$$
\begin{array}{r}
\frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{2}}{L_{2}^{2}} \delta^{(4)}\left(L_{1}-L_{2}+P\right)=\frac{d z_{1}}{z_{1}} \frac{d z_{2}}{z_{2}} d \mathcal{N}\left(\ell_{1}\right) d \mathcal{N}\left(\ell_{2}\right) d^{(4)}\left(\ell_{1}-\ell_{2}+P_{z}\right) \\
\quad=-4 \frac{d z_{1}}{z_{1}} \frac{d z_{2}}{z_{2}} \cdot\left[d^{4} \ell_{1} \delta(+)\left(\ell_{1}^{2}\right) d^{4} \ell_{2} \delta(+)\left(\ell_{2}^{2}\right) \delta^{(4)}\left(\ell_{1}-\ell_{2}+P_{z}\right)\right] . \tag{7.12}
\end{array}
$$

Notice that the quantity in brackets in precisely the LIPS measure for two massless particles whose momenta differ by $P_{z}$

$$
\begin{equation*}
d \operatorname{LIPS}\left(\ell_{1},-\ell_{2} ; P_{z}\right)=d^{4} \ell_{1} \delta(+)\left(\ell_{1}^{2}\right) d^{4} \ell_{2} \delta(+)\left(\ell_{2}^{2}\right) \delta^{(4)}\left(\ell_{1}-\ell_{2}+P_{z}\right) \tag{7.13}
\end{equation*}
$$

For $z=0$ this is precisely the LIPS measure we would require to apply unitarity cuts. We therefore conclude that

$$
\begin{equation*}
\frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{2}}{L_{2}^{2}} \delta^{(4)}\left(L_{1}-L_{2}+P\right)=-4 \frac{d z_{1}}{z_{1}} \frac{d z_{2}}{z_{2}} d \operatorname{LIPS}\left(\ell_{1},-\ell_{2} ; P_{z}\right) \tag{7.14}
\end{equation*}
$$

We call the measure on the righthand side the BST measure; it differs from the regular LIPS measure used for unitarity cuts in two important ways. First, the momentum flowing into the loop has been shifted by $z$ dependent terms $P \rightarrow P_{z}=P-z \eta$. Second, there are additional dispersion integrals $d z_{i} / z_{i}$ which integrate over the shifts in $P_{z}$.

The dispersion integrals over $z_{i}$ have an extremely elegant role in CSW/BST formalism. Recall that BBDK's unitarity approach required comparing an amplitude's cuts to the cuts of the known basis functions. By invoking a uniqueness theorem for the basis' cuts, they could match each term in the amplitude to the appropriate basis function, thereby fixing the amplitude uniquely. In using the BST measure, however, no such analysis is required. The $d$ LIPS calculates the cuts in each channel (whose momentum-invariant quantity contains $z$ dependent terms). Then while integrating out the $z$ dependance, the dispersion integrals reconstruct the full amplitude exactly. The mechanics of this process will be more transparent when we present an explicit one-loop computation in Chapter 8.

## $7.2 \mathcal{N}=4 \mathrm{MHV}$ Loop Diagram

We summarize here the procedure used in [25] to calculate the $\mathcal{N}=4$ MHV amplitude at oneloop using the CSW rules for MHV diagrams. We will be brief, omitting many of the details, as an analogous calculation is presented in full detail in the next chapter. Nevertheless, it will be useful to have an overview of how such a computation is carried out, before diving into the calculation.

## - Computing the Diagram

We begin by drawing the typical MHV loop diagram, shown in Figure 7.1, which has total momentum $P_{L}=k_{m_{1}}+k_{m_{1}+1}+\ldots+k_{m_{2}}$ on the left and $P_{R}=k_{m_{2}+1}+k_{m_{2}+2}+\ldots+k_{m_{1}-1}$ on the right. The total amplitude will require summing over all possible pairs $\left(m_{1}, m_{2}\right)$. For the $\mathcal{N}=4$ loop, we may use the generalized MHV vertex with arbitrary external helicity states. This diagram leads to the following integral

$$
\begin{equation*}
\mathcal{A}_{n}^{\mathcal{N}=4}=i(2 \pi)^{4} \delta^{(4)}\left(P_{L}+P_{R}\right) \int \frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{2}}{L_{2}^{2}} \delta^{(4)}\left(L_{1}-L_{2}+P_{L}\right) \int d^{8} \eta_{L_{1}} d^{8} \eta_{L_{2}} A_{L} A_{R} \tag{7.15}
\end{equation*}
$$



Figure 7.1: Typical one-loop MHV diagram
where $A_{L}, A_{R}$ are the vertices given by

$$
\begin{align*}
A_{L} & =\delta^{(8)}\left(\Theta_{L}\right) \frac{1}{\lambda_{2}\left\langle m_{2}\right\rangle\left\langle\lambda_{1} m_{1}\right\rangle\left\langle-\lambda_{2} \lambda_{1}\right\rangle} \prod_{i=m_{1}}^{m_{2}} \frac{1}{\langle i i+1\rangle}  \tag{7.16}\\
A_{R} & =\delta^{(8)}\left(\Theta_{R}\right) \frac{1}{\left\langle\lambda_{1} m_{1}-1\right\rangle\left\langle-\lambda_{2} m_{2}+1\right\rangle\left\langle-\lambda_{1} \lambda_{2}\right\rangle} \prod_{j=m_{2}+1}^{m_{1}-1} \frac{1}{\langle j j+1\rangle},
\end{align*}
$$

and

$$
\begin{align*}
& \left(\Theta_{L}\right)_{\alpha}^{A}=\sum_{i=m_{1}}^{m_{2}} \lambda_{\alpha}^{i} \eta_{i}^{A}-\lambda_{1 \alpha} \eta_{L_{1}}^{A}+\lambda_{2 \alpha} \eta_{L_{2}}^{A},  \tag{7.17}\\
& \left(\Theta_{R}\right)_{\alpha}^{A}=\sum_{j=m_{2}+1}^{m_{1}-1} \lambda_{\alpha}^{j} \eta_{j}^{A}+\lambda_{1 \alpha} \eta_{L_{1}}^{A}-\lambda_{2 \alpha} \eta_{L_{2}}^{A} .
\end{align*}
$$

As we have left the helicity assignments arbitrary we must integrate over the $\eta^{A}$ variables. This step is carried out [25] using an elegant trick of Fourier transforming the $\delta^{(8)}(\Theta)$ into an exponential in superspace and integrating out the $\eta^{A}$. The result is that the tree-level amplitude factors out,

$$
\begin{equation*}
\mathcal{A}_{n}^{\mathcal{N}=4}=\mathcal{A}_{n}^{\text {tree }} \int \frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{2}}{L_{2}^{2}} \delta^{(4)}\left(L_{1}-L_{2}+P_{L}\right) R \tag{7.18}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{A}_{n}^{\text {tree }}=i(2 \pi)^{4} \delta^{(4)}\left(P_{L}+P_{R}\right) \delta^{(8)}\left(\sum_{i=1}^{n} \lambda_{\alpha}^{i} \eta_{i}^{A}\right) \prod_{i=1}^{n} \frac{1}{\langle i i+1\rangle}, \tag{7.19}
\end{equation*}
$$

and the integrand is

$$
\begin{equation*}
R=\frac{\left\langle m_{1}-1 m_{1}\right\rangle\left\langle\lambda_{1} \lambda_{2}\right\rangle}{\left\langle m_{1}-1 \lambda_{1}\right\rangle\left\langle-\lambda_{1} m_{1}\right\rangle} \frac{\left\langle m_{2} m_{2}+1\right\rangle\left\langle\lambda_{2} \lambda_{1}\right\rangle}{\left\langle m_{2} \lambda_{2}\right\rangle\left\langle-\lambda_{2} m_{2}+1\right\rangle} . \tag{7.20}
\end{equation*}
$$

## - Reducing the Integrand

Applying the Schouten identity to the numerators in each half of $R$ leads to a sum of four similar terms

$$
\begin{equation*}
R=R\left(m_{1}, m_{2}+1\right)-R\left(m_{1}, m_{2}\right)+R\left(m_{1}-1, m_{2}\right)+R\left(m_{1}-1, m_{2}+1\right) \tag{7.21}
\end{equation*}
$$

where

$$
\begin{equation*}
R(r, s)=\frac{\left\langle r \lambda_{2}\right\rangle}{\left\langle r \lambda_{1}\right\rangle} \frac{\left\langle s \lambda_{1}\right\rangle}{\left\langle s \lambda_{2}\right\rangle} . \tag{7.22}
\end{equation*}
$$

Next, we use the fact that

$$
\begin{equation*}
\left\langle\bullet \lambda_{i}\right\rangle=\frac{\left\langle\bullet \lambda_{i}\right\rangle\left[\lambda_{i} \bullet\right]}{\left[\lambda_{i} \bullet\right]}=\frac{\left(\ell_{i}+k_{\bullet}\right)^{2}}{\left[\lambda_{i} \bullet\right]}, \tag{7.23}
\end{equation*}
$$

since $\left(\ell_{i}+k_{\bullet}\right)^{2}=2\left(\lambda_{i} \cdot k_{\bullet}\right)$, to convert all the spinor variables into the null vectors $\ell_{i}$ which appear in $d$ LIPS. After some manipulations, we obtain

$$
\begin{equation*}
R(r, s)=\frac{\frac{1}{2}\left(s_{z} t_{z}-P_{z}^{2} Q_{z}^{2}\right)}{\left(\ell_{2}-k_{r}\right)^{2}\left(\ell_{1}+k_{s}\right)^{2}} \tag{7.24}
\end{equation*}
$$

where we have introduced the shifted momentum invariants

$$
\begin{array}{rll}
s_{z}=\left(P_{L ; z}\right)^{2} & , & P_{z}^{2}=\left(P_{L ; z}-k_{r}\right)^{2}  \tag{7.25}\\
t_{z}=\left(P_{L ; z}-k_{r}+k_{s}\right)^{2} & , & Q_{z}^{2}=\left(P_{L ; z}+k_{s}\right)^{2}
\end{array}
$$

When combined with the $1 / L_{i}^{2}$ factors, this integral at $z=0$ precisely corresponds to the scalar box function (6.14).

## - Reorganizing the Sum

Next, we replace the measure over loop variables $L_{i}$ with the BST measure (7.14). Integrating (7.24) with respect to $d$ LIPS will calculate a cut in the $P_{L ; z}$ channel of the scalar box function. For fixed $m_{1}, m_{2}$, each term $R(r, s)$ corresponds to the same channel, $P_{L ; z}$, but different values of $r, s$ correspond to cutting different propagators in the box diagram (Figure 6.1), and hence different channels of the box $s_{z}, t_{z}, P_{z}^{2}, Q_{z}^{2}$. Thus, each MHV diagram gives. a cut in one channel of four different boxes. The sum of all MHV diagrams gives the sum of all four (non-vanishing) cuts of all the boxes. It is convenient, then, to organize the sum of diagrams into a sum over boxes, where each box is the sum of its four cuts.

## - Calculating the Cuts

We consider a single term $R(r, s)$ and assume it corresponds to the $s_{z}$-channel cut of some box. Its integral over the LIPS is given by

$$
\begin{equation*}
\mathcal{I}\left(s_{z}\right)=\int d^{D} \operatorname{LIPS}\left(\ell_{1},-\ell_{2} ; P_{L ; z}\right) \frac{s_{z} t_{z}-P_{z}^{2} Q_{z}^{2}}{\left(\ell_{2}-k_{r}\right)^{2}\left(\ell_{1}+k_{s}\right)^{2}} \tag{7.26}
\end{equation*}
$$

This integral is most easily solved in the rest frame of $\ell_{1}-\ell_{2}=P_{L ; z}$ with $k_{r}$ oriented along the $x^{D}$ axis, and $k_{s}$ in the $x^{1} x^{D}$ plane:

$$
\begin{align*}
\ell_{1}=\frac{1}{2}\left|P_{L ; z}\right|(1, \mathbf{v}) \quad, \quad \ell_{2} & =\frac{1}{2}\left|P_{L ; z}\right|(-1, \mathbf{v})  \tag{7.27}\\
k_{r}=\left(k_{r}, 0, \ldots, 0, k_{r}\right) \quad, \quad k_{s} & =(A, B, 0, \ldots, 0, C),
\end{align*}
$$

where $|\mathrm{v}|=1$ and $A^{2}=B^{2}+C^{2}$. After a long calculation and some help from Appendix B of [53], BST find

$$
\begin{equation*}
\mathcal{I}\left(s_{z}\right)=\frac{\pi^{\frac{3}{2}-\epsilon}}{\Gamma\left(\frac{1}{2}-\epsilon\right)}\left(\frac{1}{\epsilon}\right)\left|\frac{s_{z}}{4}\right|\left(1-a s_{z}\right)^{\epsilon}\left[1+\epsilon^{2} \operatorname{Li}_{2}\left(\frac{-a s_{z}}{1-a s_{z}}\right)+\mathcal{O}\left(\epsilon^{3}\right)\right] \tag{7.28}
\end{equation*}
$$

where the quantity

$$
\begin{equation*}
a=\frac{P^{2}+Q^{2}-s-t}{P^{2} Q^{2}-s t} . \tag{7.29}
\end{equation*}
$$

## - Performing the Dispersion Integrals

The final step in the calculation is the dispersion integrals over $z_{1}, z_{2}$. First, we change variables to $z=z_{2}-z_{1}$ and $z^{\prime}=z_{2}+z_{1}$

$$
\begin{equation*}
\frac{d z_{1}}{z_{1}} \frac{d z_{2}}{z_{2}}=-\frac{1}{2} \frac{d z d z^{\prime}}{\left(z^{\prime}-z\right)\left(z^{\prime}+z\right)} . \tag{7.30}
\end{equation*}
$$

The $z^{\prime}$ integration is trivial, as the rest of the integrand is independent of it. The remaining $z$ integration can be recast as an integration over the shifted channel's momentum invariant, in this case $s_{z}$ :

$$
\begin{equation*}
\frac{d z}{z}=\frac{d s_{z}}{s_{z}-s} . \tag{7.31}
\end{equation*}
$$

They found that the combination

$$
\begin{equation*}
\int_{0}^{\infty} \frac{d s_{z}}{s_{z}-s} \mathcal{I}\left(s_{z}\right)-\int_{0}^{\infty} \frac{d Q_{z}^{2}}{Q_{z}^{2}-Q^{2}} \mathcal{I}\left(Q_{z}^{2}\right)=-\frac{1}{\epsilon^{2}}\left[(-s)^{-\epsilon}-\left(-Q^{2}\right)^{-\epsilon}\right]+\mathrm{Li}_{2}\left(1-a Q^{2}\right)-\mathrm{Li}_{2}(1-a s) \tag{7.32}
\end{equation*}
$$

Combining this with the remaining $t$ and $P^{2}$ channels, they obtained the result

$$
\begin{align*}
F\left(k_{r}, Q, k_{s}, P\right) & =-\frac{1}{\epsilon^{2}}\left[(-s)^{-\epsilon}+(-t)^{-\epsilon}-\left(-Q^{2}\right)^{-\epsilon}-\left(-P^{2}\right)^{-\epsilon}\right]  \tag{7.33}\\
& +\operatorname{Li}_{2}\left(1-a Q^{2}\right)+\mathrm{Li}_{2}\left(1-a P^{2}\right)-\mathrm{Li}_{2}(1-a s)-\mathrm{Li}_{2}(1-a t)
\end{align*}
$$

This is not the form in which the box function was originally presented. Amazingly, BST went on to prove the following non-trivial identity involving nine dilogarithms [25]:

$$
\begin{align*}
& \mathrm{Li}_{2}\left(1-a Q^{2}\right)+\mathrm{Li}_{2}\left(1-a P^{2}\right)-\mathrm{Li}_{2}(1-a s)-\mathrm{Li}_{2}(1-a t)=\frac{1}{2} \log ^{2}\left(\frac{s}{t}\right)  \tag{7.34}\\
& +\mathrm{Li}_{2}\left(1-\frac{P^{2}}{s}\right)+\mathrm{Li}_{2}\left(1-\frac{P^{2}}{t}\right)+\mathrm{Li}_{2}\left(1-\frac{Q^{2}}{s}\right)+\mathrm{Li}_{2}\left(1-\frac{Q^{2}}{t}\right)-\mathrm{Li}_{2}\left(1-\frac{P^{2} Q^{2}}{s t}\right)
\end{align*}
$$

and so their result does indeed match that of BDDK. While the CSW rules still lack any formal derivation, there is strong evidence to support their legitimacy. The fact that they would be invalid if the above identity were not true is rather surprising. But the fact that such a non-trivial identity is true is perhaps the strongest piece of evidence to date.

The question of using MHV diagrams to compute sub-leading amplitudes $A_{n ; c}$ was examined in [54]. It was indeed confirmed that the CSW rules fix the sub-leading terms of $\mathcal{N}=4$ MHV amplitudes as linear combinations of the leading terms, thus the technique applies for finite $N$ and not just in the large $N$ limit.

## Chapter 8

## The $\mathcal{N}=1$ MHV Loop Diagram

Here we present an explicit computation of an MHV loop diagram ${ }^{1}$. The amplitude we are interested in calculating the contribution of $N=1$ chiral multiplet to one-loop MHV amplitude. We consider the case of external gluons only, but many other diagrams with external fermions or scalar are related to this amplitude by supersymmetry.

### 8.1 Computing the Diagram

The typical one-loop MHV diagram of interest is the same as Figure 7.1, however we must impose some constraints. We cannot use the general supervertex as we require the $\mathcal{N}=1$ chiral multiplet to circulate in the loop. Also, we must have one negative helicity gluon on each side of the diagram, as there is no possible helicity assignment for the intermediate states if both negative helicity gluons are on the same side of the diagram. We label the momenta on the left side as $k_{m_{1}}, \ldots, k_{m_{2}}$, one of which is negative helicity, denoted by $p$. The momenta on the right side as $k_{m_{2}+1}, \ldots, k_{m_{1}-1}$, the negative helicity momentum labeled $q$. As always, all momentum labels are cyclically ordered. When calculating the complete amplitude one has to sum over all such MHV diagram. All loop momenta are evaluated using dimensional regularization, in $D$ dimensions, with $D=4-2 \epsilon$.

Now the amplitude for this constrained diagram is given by

$$
\begin{equation*}
A_{n}^{\mathcal{N}=1 \chi}=i(2 \pi)^{4} \delta\left(P_{L}+P_{R}\right) \int \frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{2}}{L_{2}^{2}} \delta^{(4)}\left(L_{1}-L_{2}+P_{L}\right)\left(A_{L}^{F} A_{R}^{\tilde{F}}+A_{L}^{\tilde{F}} A_{R}^{F}+2 A_{L}^{S} A_{R}^{S}\right) \tag{8.1}
\end{equation*}
$$

where $P_{L}, P_{R}$ are the momenta flowing into the diagrams from the left and right correspondingly. Each vertex $A_{L}, A_{R}$ is obtained from the appropriate coefficient of the supervertex by ensuring the two internal lines are members of a chiral multiplet, including a fermion (of two helicities, resulting in vertices $A^{F}$ and $A^{\tilde{F}}$ ) and a complex scalar (resulting in a vertex $A^{S}$ ).

As reviewed above, each of the off-shell momenta $L_{i}, i=1,2$ has an associated null momentum $\ell_{i}$ and the corresponding spinors $\ell_{i}=\lambda_{i} \bar{\lambda}_{i}$, specifically $L_{i}=\lambda_{i} \bar{\lambda}_{i}+z_{i} \eta \bar{\eta}$.

Factoring out the tree (Parke-Taylor) amplitude results ${ }^{2}$ in:

$$
\begin{align*}
A_{n}^{\mathcal{N}=1 \chi}=A_{n}^{\text {tree }} \int \frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{2}}{L_{2}^{2}} \delta^{(4)}\left(L_{1}-L_{2}\right. & \left.+P_{L}\right) \frac{1}{\left\langle\lambda_{1} \lambda_{2}\right\rangle\left\langle\lambda_{2} \lambda_{1}\right\rangle}\left(2 I^{S}+I^{F}+I^{\tilde{F}}\right) \times \\
& \times \frac{\left\langle m_{2} m_{2}+1\right\rangle\left\langle m_{1}-1 m_{1}\right\rangle}{\left\langle\lambda_{1} m_{1}\right\rangle\left\langle m_{2} \lambda_{2}\right\rangle\left\langle\lambda_{2}, m_{2}+1\right\rangle\left\langle m_{1}-1, \lambda_{1}\right\rangle} \tag{8.2}
\end{align*}
$$

[^4]where
\[

$$
\begin{align*}
I^{S} & =\frac{\left\langle\lambda_{1} p\right\rangle^{2}\left\langle\lambda_{2} p\right\rangle^{2}\left\langle\lambda_{1} q\right\rangle^{2}\left\langle\lambda_{2} q\right\rangle^{2}}{\langle p q\rangle^{4}} \\
I^{F} & =-I^{S} \frac{\left\langle\lambda_{2} q\right\rangle\left\langle\lambda_{1} p\right\rangle}{\left\langle\lambda_{2} p\right\rangle\left\langle\lambda_{1} q\right\rangle} \\
I^{\tilde{F}} & =-I^{S} \frac{\left\langle\lambda_{2} p\right\rangle\left\langle\lambda_{1} q\right\rangle}{\left\langle\lambda_{2} p\right\rangle\left\langle\lambda_{1} q\right\rangle} \tag{8.3}
\end{align*}
$$
\]

To sum the 3 terms in (8.2) one uses the Schouten identity

$$
\begin{equation*}
\langle a b\rangle\langle c d\rangle=\langle a d\rangle\langle c b\rangle+\langle a c\rangle\langle b d\rangle \tag{8.4}
\end{equation*}
$$

which will be repeatedly used below. This leads to the following expression for the chiral multiplet contribution to the one loop gluon MHV amplitude

$$
\begin{equation*}
A_{n}^{\mathcal{N}=1 \chi}=A_{n}^{\text {tree }} \int \frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{2}}{L_{2}^{2}} \delta^{(4)}\left(L_{1}-L_{2}+P_{L}\right) R \tag{8.5}
\end{equation*}
$$

with

$$
\begin{equation*}
R=\frac{\left\langle m_{1}-1, m_{1}\right\rangle\left\langle m_{2}, m_{2}+1\right\rangle\left\langle\lambda_{1} q\right\rangle\left\langle\lambda_{2} q\right\rangle\left\langle\lambda_{1} p\right\rangle\left\langle\lambda_{2} p\right\rangle}{\langle p q\rangle^{2}\left\langle m_{1}-1, \lambda_{1}\right\rangle\left\langle\lambda_{1} m_{1}\right\rangle\left\langle m_{2} \lambda_{2}\right\rangle\left\langle\lambda_{2}, m_{2}+1\right\rangle} \tag{8.6}
\end{equation*}
$$

### 8.2 Simplifying the Integrand

Our next step is to split the spinor expression $R$ into 4 terms of identical structure. Using the pair of Schouten identities:

$$
\begin{align*}
& \left\langle m_{1}-1, m_{1}\right\rangle\left\langle\lambda_{1} q\right\rangle=\left\langle m_{1}-1, q\right\rangle\left\langle\lambda_{1} m_{1}\right\rangle+\left\langle m_{1}-1, \lambda_{1}\right\rangle\left\langle m_{1} q\right\rangle \\
& \left\langle m_{2}, m_{2}+1\right\rangle\left\langle\lambda_{2} p\right\rangle=\left\langle m_{2} p\right\rangle\left\langle\lambda_{2}, m_{2}+1\right\rangle+\left\langle m_{2} \lambda_{2}\right\rangle\left\langle m_{2}+1, p\right\rangle \tag{8.7}
\end{align*}
$$

we get the following sum:

$$
\begin{equation*}
R=R\left(m_{2}, m_{1}-1\right)-R\left(m_{2}+1, m_{1}-1\right)-R\left(m_{2}, m_{1}\right)+R\left(m_{2}+1, m_{1}\right) \tag{8.8}
\end{equation*}
$$

where

$$
\begin{equation*}
R(r, s)=\frac{\left\langle\lambda_{1} p\right\rangle\left\langle\lambda_{2} q\right\rangle}{\langle p q\rangle^{2}} \frac{\langle s q\rangle\langle r p\rangle}{\left\langle s \lambda_{1}\right\rangle\left\langle r \lambda_{2}\right\rangle} \tag{8.9}
\end{equation*}
$$

Let us simplify $R(r, s)$ : once again we use Schouten identities to split $R(r, s)$ to 4 term, which (when integrated) give rise to tensor box, vector triangle and scalar bubble diagrams ${ }^{3}$. The 4 terms are:

$$
\begin{align*}
R^{A}(r, s) & =\frac{\langle s q\rangle\langle r p\rangle\langle r q\rangle\langle p s\rangle}{\langle p q\rangle^{2}\langle r s\rangle^{2}} \frac{\left\langle\lambda_{2} s\right\rangle\left\langle\lambda_{1} r\right\rangle}{\left\langle\lambda_{1} s\right\rangle\left\langle\lambda_{2} r\right\rangle} \\
R^{B}(r, s) & =\frac{\langle s q\rangle\langle r p\rangle^{2}\langle r q\rangle}{\langle p q\rangle^{2}\langle r s\rangle^{2}} \frac{\left\langle\lambda_{2} s\right\rangle}{\left\langle\lambda_{2} r\right\rangle} \\
R^{C}(r, s) & =\frac{\langle s q\rangle^{2}\langle r p\rangle\langle s p\rangle}{\langle p q\rangle^{2}\langle r s\rangle^{2}} \frac{\left\langle\lambda_{1} r\right\rangle}{\left\langle\lambda_{1} s\right\rangle} \\
R^{D}(r, s) & =\frac{\langle s q\rangle^{2}\langle r p\rangle^{2}}{\langle p q\rangle^{2}\langle r s\rangle^{2}} \tag{8.10}
\end{align*}
$$

[^5]Let us simplify these expressions one at a time:

## - Simplifying $R^{A}$

The $\lambda_{i}$ independent factors in $R^{A}$ give the box coefficient $b_{r s}^{p q}$ while the remaining factors give the tensor box function. We decompose this, in [5, 25], into scalar components by expanding

$$
\begin{align*}
& \frac{\left\langle s \lambda_{2}\right\rangle\left\langle r \lambda_{1}\right\rangle}{\left\langle r \lambda_{2}\right\rangle\left\langle s \lambda_{1}\right\rangle}=\frac{\left[\lambda_{2} r\right]\left\langle r \lambda_{1}\right\rangle\left[\lambda_{1} s\right]\left\langle s \lambda_{2}\right\rangle}{\left\langle r \lambda_{2}\right\rangle\left[\lambda_{2} r\right]\left\langle s \lambda_{1}\right\rangle\left[\lambda_{1} s\right]}=-\frac{\operatorname{tr}\left(\frac{1}{2}\left(1-\gamma^{5}\right) \not \ell_{2} \not \not{ }_{r} \ell_{1} \not \not{ }_{s}\right)}{\left(\ell_{2}-k_{r}\right)^{2}\left(\ell_{1}+k_{s}\right)^{2}}=  \tag{8.11}\\
& =\frac{-2\left\{\left(\ell_{2} \cdot k_{r}\right)\left(\ell_{1} \cdot k_{s}\right)+\left(\ell_{2} \cdot k_{s}\right)\left(\ell_{1} \cdot k_{r}\right)-\left(k_{r} \cdot k_{s}\right)\left(\ell_{1} \cdot \ell_{2}\right)+i \varepsilon_{a b c d} \ell_{2}^{a} k_{r}^{b} \ell_{1}^{c} k_{s}^{d}\right\}}{\left(\ell_{2}-k_{r}\right)^{2}\left(\ell_{1}+k_{s}\right)^{2}}
\end{align*}
$$

The term proportional to the $\varepsilon$-tensor vanishes upon integration. As before, we define $P_{L ; z}=\ell_{1}-\ell_{2}=P_{L}-\left(z_{1}-z_{2}\right) \eta \bar{\eta}$, then the rest of the numerator may be rewritten as

$$
\begin{align*}
\left(2\left(P_{L ; z} \cdot k_{r}\right)\left(P_{L ; z} \cdot k_{s}\right)-\left(k_{r} \cdot k_{s}\right) P_{L ; z}^{2}\right) & -\left(\ell_{1}+k_{s}\right)^{2}\left(P_{L ; z} \cdot k_{r}\right)- \\
\left(\ell_{2}-k_{r}\right)^{2}\left(P_{L ; z} \cdot k_{s}\right) & +\left(\ell_{2}-k_{r}\right)^{2}\left(\ell_{1}+k_{s}\right)^{2} \tag{8.12}
\end{align*}
$$

The terms collected in the first brackets contribute to a scalar box integral, while the next two terms each contain a factor which cancels one of the propagators in the denominator, leaving scalar triangles. The last term reduces to a scalar bubble, since both propagators cancel. Next, we make use of the identity

$$
\begin{equation*}
4(P \cdot i)(P \cdot j)-2 P^{2}(i \cdot j)=(P+i)^{2}(P+j)^{2}-P^{2}(P+i+j)^{2} \tag{8.13}
\end{equation*}
$$

valid for any momentum $P$ and null momenta $i$ and $j$, to rewrite the box's coefficient in terms the shifted momentum invariants, (recall their definitions; $s_{z}=\left(P_{L ; z}\right)^{2}, P_{z}^{2}=\left(P_{L ; z}-k_{r}\right)^{2}, t_{z}=$ $\left(P_{L ; z}-k_{r}+k_{s}\right)^{2}$, and $\left.Q_{z}^{2}=\left(P_{L ; z}+k_{s}\right)^{2}\right)$ :

$$
\begin{equation*}
2\left(P_{L ; z} \cdot k_{r}\right)\left(P_{L ; z} \cdot k_{s}\right)-\left(k_{r} \cdot k_{s}\right) P_{L ; z}^{2}=\frac{1}{2}\left(P_{z}^{2} Q_{z}^{2}-s_{z} t_{z}\right) \tag{8.14}
\end{equation*}
$$

Thus, the result of the tensor box's decomposition is

$$
\begin{equation*}
\frac{\left\langle s \lambda_{2}\right\rangle\left\langle r \lambda_{1}\right\rangle}{\left\langle r \lambda_{2}\right\rangle\left\langle s \lambda_{1}\right\rangle}=\left\{\frac{\frac{1}{2}\left(P_{z}^{2} Q_{z}^{2}-s_{z} t_{z}\right)}{\left(\ell_{2}-k_{r}\right)^{2}\left(\ell_{1}+k_{s}\right)^{2}}-\frac{P_{L ; z} \cdot k_{r}}{\left(\ell_{2}-k_{r}\right)^{2}}-\frac{P_{L ; z} \cdot k_{s}}{\left(\ell_{1}+k_{s}\right)^{2}}\right\}+1 \tag{8.15}
\end{equation*}
$$

The terms collected in the bracket lead to the finite portion of the scalar box function, complete with the correct coefficient $b_{r s}^{p q}$, as in equation (6.17). The second term contributes to scalar bubbles, which cancel against other contributions. We demonstrate this cancelation below.

## -Simplifying $R^{B}$ and $R^{C}$

We now turn to the linear triangle terms $R^{B}(r, s)$ and $R^{C}(r, s)$. First, we write the loop momentum dependant part of $R^{B}(r, s)$ as

$$
\begin{equation*}
\left.\left.\frac{\left\langle s \lambda_{2}\right\rangle}{\left\langle\lambda_{2} r\right\rangle}=-\frac{\left\langle s \lambda_{2}\right\rangle\left[\lambda_{2} r\right]}{\left\langle r \lambda_{2}\right\rangle\left[\lambda_{2} r\right]}=\frac{\left.\langle s| \ell_{2} \mid r\right]}{\left(\ell_{2}-k_{r}\right)^{2}}=\langle s| \gamma_{a} \right\rvert\, r\right] \frac{\ell_{2}^{a}}{\left(\ell_{2}-k_{r}\right)^{2}} . \tag{8.16}
\end{equation*}
$$

So, $R^{B}$ is the integrand of the (cut) vector two-mass triangle integral $I_{3: r ; m_{1}}^{2 m}\left[\ell_{2}^{a}\right]$, defined as in (6.21), except with the numerator $\ell_{2}^{a}$. Next, we introduce Feynman parameters $x_{i}$ into the vector triangle integral, and shift the integration variable:

$$
\begin{align*}
I_{3: r ; m_{1}}^{2 m}\left[\ell^{a}\right] & =i \int \frac{d^{D} \ell}{(2 \pi)^{D}} \frac{\ell^{a}}{\ell^{2}(\ell+\widetilde{Q})^{2}\left(\ell-k_{r}\right)^{2}}  \tag{8.17}\\
& =i \int \frac{d^{D} \ell^{\prime}}{(2 \pi)^{D}} d^{3} x_{i} \delta\left(\sum_{i=1}^{3} x_{i}-1\right) \frac{2\left(\ell^{\prime}-x_{2} \widetilde{Q}+x_{3} k_{r}\right)^{a}}{\left(\ell^{\prime 2}-\Delta\right)^{3}}
\end{align*}
$$

The term linear in $\ell^{\prime a}$ vanishes upon integration, so we are left with

$$
\begin{equation*}
I_{3: r ; m_{1}}^{2 m}\left[\ell^{a}\right]=-\widetilde{Q}_{z}^{a} I_{3: r ; m_{1}}^{2 m}\left[x_{2}\right]+k_{r}^{a} I_{3: r ; m_{1}}^{2 m}\left[x_{3}\right], \tag{8.18}
\end{equation*}
$$

where the arguments in square brackets are the numerators in the integrals, $\widetilde{Q}_{z}$ is the momentum of one massive leg (shifted by $z$ dependent terms, as defined above) and $k_{r}$ is the momentum of the massless leg, as drawn in Figure 6.2. Since $\widetilde{Q}_{z}=P_{z}+k_{r}$ and $[r r]=0$, we can write

$$
\begin{equation*}
\left.\langle s| \gamma_{a}|r| I_{3: r ; m_{1}}^{2 m}\left[\ell_{2}^{a}\right]=-\langle s| P_{z} \mid r\right] I_{3: r ; m_{1}}^{2 m}\left[x_{2}\right] . \tag{8.19}
\end{equation*}
$$

Now, the full coefficient of $R^{B}$ is

$$
\begin{equation*}
-\frac{\langle p r\rangle\langle r q\rangle\langle p r\rangle\langle s q\rangle\left\langle s P_{z}\right\rangle\left[P_{z} r\right]}{\langle p q\rangle^{2}\langle r s\rangle^{2}} . \tag{8.20}
\end{equation*}
$$

Applying the Schouten identity to the terms $\langle p r\rangle\left\langle s P_{z}\right\rangle$ and $\langle r q\rangle\left\langle s P_{z}\right\rangle$, then averaging over the two gives

$$
\begin{align*}
& -\frac{\langle p r\rangle\langle s q\rangle}{\langle p q\rangle^{2}\langle r s\rangle}\left(\frac{\langle p r\rangle\left\langle q P_{z}\right\rangle+\langle q r\rangle\left\langle p P_{z}\right\rangle}{2}\right)\left[P_{z} r\right] \\
& -\frac{\langle p r\rangle\langle s q\rangle}{\langle p q\rangle^{2}\langle r s\rangle^{2}}\left(\frac{\langle p s\rangle\langle r q\rangle+\langle p r\rangle\langle s q\rangle}{2}\right) 2\left(P_{z} \cdot k_{r}\right) \tag{8.21}
\end{align*}
$$

We use the Schouten identity again, on the first term of the first pair only.

$$
\begin{equation*}
\frac{\langle p r\rangle\langle s q\rangle}{\langle r s\rangle}=\frac{\langle p s\rangle\langle r q\rangle}{\langle r s\rangle}+\langle p q\rangle \tag{8.22}
\end{equation*}
$$

Note that the piece containing $\langle p q\rangle$ is independent of $s$, so it will vanish when summing over $s=\left\{m_{1}-1, m_{1}\right\}$, as that sum has alternating signs. Now the first pair of terms in equation (8.21) reads

$$
\begin{equation*}
\frac{\langle p r\rangle\langle r q\rangle}{\langle r s\rangle\langle p q\rangle^{2}}\left(\frac{\langle s q\rangle\left\langle p P_{z}\right\rangle+\langle s p\rangle\left\langle q P_{z}\right\rangle}{2}\right)\left[P_{z} r\right] \tag{8.23}
\end{equation*}
$$

A similar analysis of $R^{C}(r, s)$ shows that the coefficient of the integral function $I_{3: s ; m_{2}+1}^{2 m}\left[x_{2}\right]$ is

$$
\begin{align*}
& \frac{\langle p s\rangle\langle s q\rangle}{\langle r s\rangle\langle p q\rangle^{2}}\left(\frac{\langle r p\rangle\left\langle q Q_{z}\right\rangle+\langle r q\rangle\left\langle p Q_{z}\right\rangle}{2}\right)\left[Q_{z} s\right] \\
+ & \frac{\langle p r\rangle\langle s q\rangle}{\langle r s\rangle^{2}\langle p q\rangle^{2}}\left(\frac{\langle p s\rangle\langle r q\rangle+\langle p r\rangle\langle s q\rangle}{2}\right) 2\left(Q_{z} \cdot k_{s}\right) \tag{8.24}
\end{align*}
$$

where $Q_{z}$ is the shifted momentum transfer defined above. In this decomposition the first term contributes to the coefficient of the scalar triangle function, and the second one will be used below to cancel the bubble diagrams.

## -Simplifying $R^{D}$

First, the scalar bubble in $R^{D}$ can be combined with the one that arose from decomposing $R^{A}$, giving a single bubble with coefficient

$$
\begin{equation*}
\frac{\langle p r\rangle\langle s q\rangle}{\langle r s\rangle\langle p q\rangle^{2}}(\langle p s\rangle\langle r q\rangle+\langle p r\rangle\langle s q\rangle) \tag{8.25}
\end{equation*}
$$

Now, to cancel the bubbles notice that they possess the same coefficient as the last pair of terms in (8.21) and (8.24). These integrals combine into

$$
\begin{array}{r}
\frac{\langle p r\rangle\langle s q\rangle}{\langle r s\rangle^{2}\langle p q\rangle^{2}}\left(\frac{\langle p s\rangle\langle r q\rangle+\langle p r\rangle\langle s q\rangle}{2}\right) \times  \tag{8.26}\\
\times\left(2 I_{2: r ; s}-2\left(P_{z} \cdot k_{r}\right) I_{3: r ; m_{1}}^{2 m}\left[x_{2}\right]+2\left(Q_{z} \cdot k_{s}\right) I_{3: s ; m_{2}+1}^{2 m}\left[x_{2}\right]\right)
\end{array}
$$

which vanishes in each channel of each cut because of the relation (6.22)

$$
\begin{equation*}
\left(s-Q^{2}\right) I_{3: r ; s}^{2 m}\left[x_{2}\right]=I_{2: r ; s}-I_{2: r+1 ; s} \tag{8.27}
\end{equation*}
$$

In summary, the net result of this decomposition is then

$$
\begin{align*}
R(r, s) & =\frac{\langle p r\rangle\langle r q\rangle\langle p s\rangle\langle s q\rangle}{\langle r s\rangle^{2}\langle p q\rangle^{2}}\left\{\frac{\frac{1}{2}\left(P_{z}^{2} Q_{z}^{2}-s_{z} t_{z}\right)}{\left(\ell_{2}-k_{r}\right)^{2}\left(\ell_{1}+k_{s}\right)^{2}}-\frac{P_{L ; z} \cdot k_{r}}{\left(\ell_{2}-k_{r}\right)^{2}}-\frac{P_{L ; z} \cdot k_{s}}{\left(\ell_{1}+k_{s}\right)^{2}}\right\} \\
& +\frac{\langle p r\rangle\langle r q\rangle}{\langle r s\rangle\langle p q\rangle^{2}}\left\{\frac{\left.\left.\langle s q\rangle\langle p| P_{z} \mid r\right]+\langle s p\rangle\langle q| P_{z} \mid r\right]}{2}\right\} \frac{\left(\frac{\epsilon}{1-2 \epsilon}\right)}{\left(\ell_{2}-k_{r}\right)^{2}}  \tag{8.28}\\
& +\frac{\langle p s\rangle\langle s q\rangle}{\langle r s\rangle\langle p q\rangle^{2}}\left\{\frac{\left.\left.\langle r p\rangle\langle q| Q_{z} \mid s\right]+\langle r q\rangle\langle p| Q_{z} \mid s\right]}{2}\right\} \frac{\left(\frac{\epsilon}{1-2 \epsilon}\right)}{\left(\ell_{1}+k_{s}\right)^{2}}
\end{align*}
$$

where we have again used (6.22)

$$
\begin{equation*}
I_{3: r ; s-1}^{2 m}\left[x_{2}\right]=\left(\frac{\epsilon}{1-2 \epsilon}\right) I_{3: r ; s-1}^{2 m} \tag{8.29}
\end{equation*}
$$

to conveniently express the triangles' integrands.
The first coefficient above is easily recognizable as $\frac{1}{2} b_{r s}^{p q}$ from equation (6.25), but to get the remaining two into the correct form requires an additional step. Consider the second line of each of the four $R(r, s)$ terms. Those with a common value for $r$ differ only in the $s$ dependance of their coefficients. So when we add $R\left(r, m_{1}-1\right)-R\left(r, m_{1}\right)$, the only change is

$$
\begin{equation*}
\sum_{s} \frac{\langle s \bullet\rangle}{\langle r s\rangle}=\frac{\left\langle m_{1}-1 \bullet\right\rangle}{\left\langle r m_{1}-1\right\rangle}-\frac{\left\langle m_{1} \bullet\right\rangle}{\left\langle r m_{1}\right\rangle}=\frac{\langle\bullet r\rangle\left\langle m_{1}-1 m_{1}\right\rangle}{\left\langle m_{1}-1 r\right\rangle\left\langle r m_{1}\right\rangle} \tag{8.30}
\end{equation*}
$$

where we used the Schouten identity to combine the two terms. Now the coefficient of the second line is $\frac{1}{2} c_{r\left(m_{1}-1\right)}^{p q}$. An analogous treatment of the third line produces the coefficient $\frac{1}{2}{ }_{2} p_{s m_{2}}$.

### 8.3 Reorganizing the Sum

We have decomposed the integrand of each one of the MHV diagrams into a sum (8.28) which should now be compared with the sum occurring in the exact result (6.17). The crucial point is the BST measure [25]:

$$
\begin{equation*}
\frac{d^{4} L_{1}}{L_{1}^{2}} \frac{d^{4} L_{1}}{L_{1}^{2}} \delta^{(4)}\left(L_{1}-L_{2}+P_{L}\right)=-4 \frac{d z_{1}}{z_{1}} \frac{d z_{2}}{z_{2}} d \operatorname{LIPS}\left(\ell_{2},-\ell_{1}, P_{L, z}\right) \tag{8.31}
\end{equation*}
$$

where dLIPS is the Lorenz invariant phase space measure appearing in the cut rules. For fixed $z_{1}, z_{2}$, after performing the integration over $l_{1}, l_{2}$ we have a sum over cuts of Feynman graphs (at shifted values of the momentum invariant). The claim is that this sum, at $z=0$, coincides exactly with the cut of the exact result (6.17). This is not true diagram by diagram, rather there is some re-arrangement of the cuts which we now demonstrate.

Having completed the decomposition of $R=\sum_{r, s} R(r, s)$ in the previous section, we find it contains eight distinct terms which are: 4 (cut) finite box functions ( 1 for each pair of null legs $k_{r}$ and $k_{s}$ ), and 4 modified (cut) two-mass triangles ( 1 for each case where $k_{r}$ or $k_{s}$ is the null leg). When we cut the loop in the MHV diagram, this is equivalent to cutting the boxes and triangles, as shown in figure 5, so as to keep $\left\{k_{m_{1}}, \ldots, k_{m_{2}}\right\}$ on the same side of the cut. Clearly, which lines get cut depends completely on where $k_{m_{1}}, k_{m_{2}}$ are in relation to $k_{r}, k_{s}$. We stress that all these cuts are in the same channel, $s=P_{L}^{2}$. Alternatively, we could combine the contributions from different MHV diagrams (with different $m_{1}, m_{2}$ ) which have the same null legs $k_{r}, k_{s}$ and therefore must produce the same boxes and triangles. Different MHV diagrams will lead to different cuts. In this manner, we may combine: the 4 boxes with common $k_{r}$ and $k_{s}$, with cuts in the channels $s, t, P^{2}, Q^{2}$, the 2 triangles with common $k_{r}$, with cuts in the channels $s=\widetilde{Q}^{2}$ and $P^{2}$, and the 2 triangles with common $k_{s}$, with cuts in the channels $s=\widetilde{P}^{2}$ and $Q^{2}$, for all values of $r, s$.

In the exceptional cases where one of the triangles massive legs becomes massless, then this diagram has the single non-trivial cut which isolates the remaining massive leg, as the trivalent vertex vanishes on-shell. We will show below that each of these terms are reconstructed from their single cut.

One might worry that not all the cuts exist in all channels for non-degenerate cases. A priori, we must sum over all MHV diagrams with $q+1 \leq m_{1} \leq p$ and $p \leq m_{2} \leq q-1$, but when $m_{2}=p$ or $m_{1}-1=q$ the corresponding boxes and triangles may not be defined. Fortunately, the coefficients

$$
\begin{equation*}
b_{p s}^{p q}=b_{r q}^{p q}=c_{p s}^{p q}=c_{q r}^{p q}=0 \tag{8.32}
\end{equation*}
$$

all vanish. So, we may restrict the sums over $m_{2} \equiv r, m_{1} \equiv s+1$ to the ranges given in Section 6.3, plus the degenerate triangle terms.

So, in summary, we have found that the decomposition of the sum of MHV diagram is simply related to the result (6.16). For any channel $X=s, t, P^{2}, Q^{2}$, of any function $F=B, T, \mathcal{A}_{I R}$ in (6.17), we find a term in our sum of the form $\Delta_{X} F\left(X_{z}\right)$, where $\Delta_{X}$ denotes the cut in the $X$-channel, and $X_{z}$ is $X$ shifted by $z$-dependent terms.


Figure 8.1: The cuts produced by one MHV diagram

### 8.4 Calculating the Cuts

In the last section, we noted that the loop integrations factor into two parts: dispersion integrals over the $z_{i}$ and an integral over $\operatorname{dLIPS}\left(\ell_{2},-\ell_{1}, P_{L ; z}\right)$ which computes the cuts in the diagrams. The cut box integrals were computed in [25] and these results were summarized in Section 7.2, so the only new ingredients are the cut triangles. We will now evaluate these integrals for when $k_{r}$ is the null leg, the other case follows by switching $r \leftrightarrow s$ and $\ell_{2} \leftrightarrow-\ell_{1}$. Also, we focus on the $s$-channel cuts; other channels are treated analogously. The integrals we wish to solve are in dimension $D=4-2 \epsilon$ and of the form

$$
\begin{equation*}
\mathcal{I}\left(s_{z}\right)=\int d^{D} \operatorname{LIPS}\left(\ell_{2},-\ell_{1}, P_{L ; z}\right) \frac{N\left(P_{L ; z}\right)}{\left(\ell_{2}-k_{r}\right)^{2}}, \tag{8.33}
\end{equation*}
$$

where the numerator $N\left(P_{L ; z}\right)$ only depends on $P_{L ; z}$ and external momenta ${ }^{4}$. By boosting to the rest frame of $\ell_{1}-\ell_{2}$, then rotating $k_{r}$ into the $x_{D}$ direction, we have

$$
\begin{equation*}
\ell_{1}=\frac{1}{2}\left|P_{L ; z}\right|(1, \mathbf{v}) ; \quad \ell_{2}=\frac{1}{2}\left|P_{L ; z}\right|(-1, \mathbf{v}) ; \quad k_{r}=\left(k_{r}, 0, \ldots, 0, k_{r}\right), \tag{8.34}
\end{equation*}
$$

where the unit vector $\mathbf{v}$ is such that $\mathbf{v} \cdot \widehat{x}_{D}=\cos \left(\theta_{1}\right)$. This allows us to re-write our phasespace measure as in [25]

$$
\begin{equation*}
d^{D} \operatorname{LIPS}\left(\ell_{2},-\ell_{1}, P_{L ; z}\right)=\frac{\pi^{\frac{1}{2}-\epsilon}}{4 \Gamma\left(\frac{1}{2}-\epsilon\right)}\left|\frac{P_{L ; z}^{2}}{4}\right|^{-\epsilon} d \theta_{1} d \theta_{2}\left(\sin \theta_{1}\right)^{1-2 \epsilon}\left(\sin \theta_{2}\right)^{-2 \epsilon} \tag{8.35}
\end{equation*}
$$

and the integrand's denominator becomes

$$
\begin{equation*}
\left(\ell_{2}-k_{r}\right)^{2}=-2 \ell_{2} \cdot k_{r}=k_{r}\left|P_{L ; z}\right|\left(1-\cos \theta_{1}\right) \tag{8.36}
\end{equation*}
$$

Performing the integral (8.33) is now a simple task (for a computer), with the result

$$
\begin{align*}
\mathcal{I}\left(s_{z}\right) & =\frac{4^{\epsilon} \pi^{\frac{3}{2}-\epsilon}}{2 \Gamma\left(\frac{1}{2}-\epsilon\right)}\left|\frac{s_{z}}{4}\right|^{-\epsilon} \frac{N\left(P_{L ; z}\right)}{k_{r}\left|P_{L ; z}\right|} \frac{\Gamma(-\epsilon)}{\Gamma(1-\epsilon)} \\
& \longrightarrow-\frac{1}{\epsilon} \frac{\pi}{2} \frac{N\left(P_{L ; z}\right)}{k_{r}\left|P_{L ; z}\right|} s_{z}^{-\epsilon} . \tag{8.37}
\end{align*}
$$

Now, for any channel of any function $F(X)$ appearing in the result (6.17), we are left with an integral of the form $\int \frac{d z_{1} d z_{2}}{z_{1} z_{2}} \Delta_{X} F\left(X_{z}\right)$, where the cuts of the triangle graphs are exhibited in (8.37). Furthermore, we have shown that $\Delta_{X} F\left(X_{z=0}\right)$ is precisely the cut of the exact result (6.17). Appealing to cut constructibility, we can anticipate that our dispersion integration will reproduce the correct answer as long as the functions $\Delta_{X} F\left(X_{z}\right)$ are cut free on the integration contour of the $z$-integration. As the cuts (8.37) do include non-analytic functions of $X_{z}$, the correct contour ${ }^{5}$ is $X_{z} \geq 0$. Choosing this contour, it is a simple matter to perform the dispersion integration directly to verify that we get the correct answer, and we turn to that integration now.

[^6]
### 8.5 Performing the Dispersion Integrals

We now show that the final integrations over $z_{1}, z_{2}$ reproduce the result in (6.16). Recall that in Section 8.3, we demonstrated that the sum over MHV diagrams is equivalent to the sum of cuts in all possible channels of the box and triangle diagrams. Thus, it remains to show that a given box (triangle) is reconstructed from the sum of its 4 (2) integrated cuts.

We change our integration variables to $z=z_{1}-z_{2}, z^{\prime}=z_{1}+z_{2}$ and note that for any function $f(z)$ independent of $z^{\prime}$

$$
\begin{equation*}
\int \frac{d z_{1}}{z_{1}} \frac{d z_{2}}{z_{2}} f\left(z_{1}-z_{2}\right)=2(2 \pi i) \int \frac{d z}{z} f(z) \tag{8.38}
\end{equation*}
$$

Next, we use the fact that $s_{z}=s-2 z \eta \cdot P_{L}$ to write

$$
\begin{equation*}
\frac{d z}{z}=-\frac{d s_{z}}{s-s_{z}} \tag{8.39}
\end{equation*}
$$

with a corresponding change of variables in the other channels. Now, we must show that

$$
\begin{align*}
B\left(k_{r}, Q, k_{s}, P\right) & =\int \frac{d s_{z}}{s-s_{z}} \Delta_{s} B\left(s_{z}\right)+\int \frac{d t_{z}}{t-t_{z}} \Delta_{t} B\left(t_{z}\right) \\
& -\int \frac{d P_{z}^{2}}{P^{2}-P_{z}^{2}} \Delta_{P^{2}} B\left(P_{z}^{2}\right)-\int \frac{d Q_{z}^{2}}{Q^{2}-Q_{z}^{2}} \Delta_{Q^{2}} B\left(Q_{z}^{2}\right) \tag{8.40}
\end{align*}
$$

and

$$
\begin{equation*}
T(k, P, Q)=\int \frac{d P_{z}^{2}}{P^{2}-P_{z}^{2}} \Delta_{P^{2}} T\left(P_{z}^{2}\right)-\int \frac{d Q_{z}^{2}}{Q^{2}-Q_{z}^{2}} \Delta_{Q^{2}} T\left(Q_{z}^{2}\right) \tag{8.41}
\end{equation*}
$$

Again, we will consider the $s$-channel only, the other channels follow immediately. As discussed above, we must restrict the integration to $s_{z}>0$, where the expression (8.37) has no cuts.

First, we will reconstruct the divergence free box functions. They possess three types of terms, given in the first line of (8.28). The first of these was calculated in [25], and we quoted earlier that the $s$ dependant terms are

$$
\begin{equation*}
-\frac{1}{\epsilon^{2}}(-s)^{-\epsilon}-\operatorname{Li}_{2}(1-a s) \tag{8.42}
\end{equation*}
$$

The next term has the cut $\mathcal{I}\left(s_{z}\right)$ from the previous section, with numerator $N\left(P_{L ; z}\right)=$ $-P_{L ; z} \cdot k_{r}$. Up to a sign, this numerator is precisely the denominator in our working reference frame (8.34). The dispersion integral is then

$$
\begin{equation*}
-\frac{1}{2 \epsilon} \int_{0}^{\infty} \frac{d s_{z}}{s-s_{z}} s_{z}^{-\epsilon}=\frac{1}{2} \frac{\pi \csc (\pi \epsilon)}{\epsilon}(-s)^{-\epsilon} \longrightarrow \frac{1}{2 \epsilon^{2}}(-s)^{-\epsilon} \tag{8.43}
\end{equation*}
$$

The next term in the divergence free box gives an identical contribution. Summing the three contributions, we find

$$
\begin{equation*}
\int \frac{d s_{z}}{s-s_{z}} \Delta_{s} B\left(s_{z}\right)=-\operatorname{Li}_{2}(1-a s) \tag{8.44}
\end{equation*}
$$

exactly what is required to reproduce (6.18). Treating the other channels similarly proves the equality of (8.40) and (6.18), provided we use BST's representation of the box function (7.34).

Moving on to the triangles, we will consider those where $k_{r}$ is the null leg. These also have cuts of the form $\mathcal{I}\left(s_{z}\right)$, in the reference frame (8.34) the numerator is

$$
\begin{equation*}
\left.\left.\left.N\left(P_{L ; z}\right)=\langle x| P_{z} \mid r\right]=\langle x| P_{L ; z} \mid r\right]=\langle x| \gamma^{0} \mid r\right\}\left|P_{L ; z}\right| \tag{8.45}
\end{equation*}
$$

times $\left(\frac{\epsilon}{1-2 \epsilon}\right)$ and $x=p, q$. The dispersion integral is nearly identical to (8.43):

$$
\begin{equation*}
\frac{1}{2 \epsilon} \int_{0}^{\infty} \frac{d s_{z}}{s-s_{z}} \frac{\epsilon}{1-2 \epsilon} \frac{\left.\langle x| \gamma^{0} \mid r\right]}{k_{r}} s_{z}^{-\epsilon}=\frac{1}{\epsilon(1-2 \epsilon)} \frac{\left.\langle x| \gamma^{0} \mid r\right]}{2 k_{r}}(-s)^{-\epsilon} \tag{8.46}
\end{equation*}
$$

Multiplying the top and bottom by $|P| \equiv P^{0}$, then re-expressing this result in a covariant fashion gives the coefficient

$$
\begin{equation*}
\frac{\left.\langle x| \gamma^{0} \mid r\right]}{2 k_{r}}=\frac{\langle x| P \mid r]}{2 k_{r} \cdot P}=\frac{\langle x| P \mid r]}{\widetilde{Q}^{2}-P^{2}} \tag{8.47}
\end{equation*}
$$

(recall that $s \equiv \widetilde{Q}^{2}$ ). An analogous result holds in the $P^{2}$ channel. Taking the difference of the two, and expanding $\left(-\widetilde{Q}^{2}\right)^{-\epsilon},\left(-P^{2}\right)^{-\epsilon}$ in $\epsilon$ yeilds the desired result:

$$
\begin{equation*}
\frac{1}{\epsilon(1-2 \epsilon)} \frac{\left(-\widetilde{Q}^{2}\right)^{-\epsilon}-\left(-P^{2}\right)^{-\epsilon}}{\widetilde{Q}^{2}-P^{2}}=\frac{\log \left(\widetilde{Q}^{2}\right)-\log \left(P^{2}\right)}{\widetilde{Q}^{2}-P^{2}} \tag{8.48}
\end{equation*}
$$

In the case of the one-mass triangles, the result is even simpler. Consider the case $(r, s)=(p+1, p-1)$, then $P^{2}=p^{2}=0$ and the dispersion integral gives

$$
\begin{equation*}
\frac{1}{\epsilon(1-2 \epsilon)} \frac{\left(-\widetilde{Q}^{2}\right)^{-\epsilon}}{\widetilde{Q}^{2}} \tag{8.49}
\end{equation*}
$$

as desired. We conclude therefore that all triangle terms are reconstructed once we perform the final dispersion integration.

Thus, we have shown by explicit calculation that the MHV diagram formalism is valid for the calculation of the one-loop contribution of the $\mathcal{N}=1$ chiral multiplet to the MHV amplitude. Together with the result of [25] this establishes the validity of the MHV-diagram technique for this helicity configuration in any massless supersymmetric theory.

## Chapter 9

## Conclusions and Future Outlooks

We have presented a wealth of evidence confirming that perturbative gauge theory amplitudes are much simpler than anyone would naïvely suspect. Conventional approaches to such calculations become exponentially cumbersome as the complexity (the number of external legs or the number of loops) in the amplitude increases. The simplicity of the Parke-Taylor formula indicates that standard approaches are overly redundant and are in need of simplification. The first step in this program was factoring out the colour information to reduce the problem to a kinematic one which further simplifies when formulated in a helicity basis. The CSW carry this line-of-thought further by building amplitudes out of blocks where many, many Feynman diagrams have already been summed.

While the CSW rules lack any derivation, we have demonstrated that they pass several non-trivial checks at the tree and one-loop levels. It would interesting to push this further and examine higher loop amplitudes. There is however, the unsettling fact that the CSW rules do not seem to apply to non-supersymmetric loops. Understanding how to obtain the full scalar loop contributions from the CSW rules would be a tremendous boost to the program. It is essential that we develop some method of computing these contributions if we are to push the calculation frontier of pure Yang-Mills loop amplitudes past five external gluons. The six-point one-loop amplitude, for example, will be crucial for next-to-leading order analyses of four-jet events at future collider experiments such as the LHC.

Recent investigations into supersymmetric gauge theories has been more fruitful. The approach of generalized unitarity, where more than two propagators are put on shell, has proven most effective for computing loop amplitudes. This has led to the discovery of all NMHV one-loop amplitudes in $\mathcal{N}=4$ SYM [33], and certain sets of loop amplitudes for the $\mathcal{N}=1$ case [34]. This success is largely due to the fact that such theories are cut-constructible. Since cut-constructible theories may be expressed as a linear combination of known scalar loop integrals, the only remaining task is to compute the coefficients.

The study of $\mathcal{N}=4$ one-loop NHMV amplitudes yielded a surprising result: the coefficients of one of the functions (the three-mass box) determined all the remaining coefficients algebraically! This author is currently investigating whether this holds for more complicated loop amplitudes. Specifically, we are examining the next-to-next-to-MHV amplitudes (with four negative helicities) in $\mathcal{N}=4$ SYM to see whether all the coefficients are determined from a single set (namely the four-mass box coefficients). So far, we have found general expressions for the four-mass box integrals' coefficients and are now determining their relationships to the others.

The $\mathcal{N}=1$ theory, having less symmetry, has proven more challenging. Only limited sets of amplitudes have been uncovered so far. In particular, the set of NMHV $\mathcal{N}=1$ amplitudes of the form $A_{n}\left(g^{-}, g^{-}, g^{-}, g^{+}, g^{+}, g^{+}, \ldots, g^{+}\right)$are now known, as well as all the helicity configurations of the 6 -point amplitude. We are working on an expression for the general NMHV loop amplitude. Our hope is that the coefficients of each basis function will be fixed
in terms of the coefficients of a small subset (from one to three) of them.
Our final line of current investigation is in non-supersymmetric amplitudes. We aim to determine all cut-constructible terms in the contributions from internal scalars to one-loop 6point amplitudes. As our technique relies heavily on unitarity, we do not anticipate producing the full amplitude. Nevertheless, recent parallel developments have produced a recursive formula for one-loop amplitudes of the form $A_{n}\left(g^{ \pm}, g^{+}, g^{+}, \ldots, g^{+}\right)$for non-supersymmetric gauge theories [36]. There is a growing belief that eventually non-supersymmetric theories will be solved through a combination of unitarity (to determine the cut-constructible terms) and analogous recursion relations (which fix the remaining cut-free terms). With this in mind, we feel that determining all cut-constructible terms in non-supersymmetric amplitudes will be an important result for future calculations.

There have also been many interesting new developments in tree-level computations leading to even more compact expression then the CSW prescription does [35]. Undoubtedly, these new compact formulae (when combined with unitarity) will lead to dramatic simplifications in loop amplitudes. Better control over loop calculations will be essential for probing new physics at future colliders. Though the CSW rules may not be suitable at the loop level, or at least require some modification, for non-supersymmetric theories, they are nonetheless an important step towards a more efficient paradigm for calculating loop amplitudes. Their more successful application to supersymmetric theories, however, should prove useful in uncovering hidden structure in gauge theories.

## Appendix A

## Conventions and Notations

Here we summarize our conventions and notations.

## -Spinors

We use the signature $(+,-,-,-)$ for the spacetime metric $\eta_{a b}$. Spacetime indices are denoted $a, b, \ldots=0,1,2,3$. Lefthanded spinor indices are denoted by $\alpha, \beta, \ldots=1,2$, while righthanded spinors follow the same convention with a dotted index. Spinor indices are raised and lowered with the 2-dimensional Levi-Civita symbol

$$
\begin{align*}
& \epsilon_{\alpha \beta}=\epsilon_{\dot{\alpha} \dot{\beta}}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)  \tag{A.1}\\
& \epsilon^{\alpha \beta}=\epsilon^{\dot{\alpha} \dot{\beta}}=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right), \tag{A.2}
\end{align*}
$$

which satisfy

$$
\begin{align*}
\epsilon_{\alpha \epsilon} \epsilon^{\gamma \beta} & =\delta_{\alpha}^{\beta}  \tag{A.3}\\
\epsilon_{\alpha \beta} \epsilon^{\gamma \delta} & =-\delta_{\alpha}^{\gamma}+\delta_{\beta}^{\delta} \tag{A.4}
\end{align*}
$$

and similarly for the dotted symbols.
We convert between tensor and spinor indices using the Pauli matrices

$$
\begin{equation*}
V^{a b \ldots}=\left(\sigma^{a}\right)_{\alpha \dot{\alpha}}\left(\sigma^{b}\right)_{\beta \dot{\beta}} \ldots V^{\alpha \dot{\alpha} \beta \dot{\beta} \ldots} \tag{A.5}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma^{a}=\left(\mathbf{1}, \sigma^{i}\right)=\bar{\sigma}_{a} \quad \text { and } \quad \bar{\sigma}^{a}=\left(\mathbf{1},-\sigma^{i}\right)=\sigma_{a}, \tag{A.6}
\end{equation*}
$$

which are related by the Levi-Civita symbols

$$
\begin{equation*}
\left(\bar{\sigma}^{a}\right)^{\dot{\alpha} \alpha}=\epsilon^{\dot{\alpha} \dot{\beta}} \epsilon^{\alpha \beta}\left(\sigma^{a}\right)_{\beta \dot{\beta}} \quad ; \quad\left(\sigma^{a}\right)_{\alpha \dot{\alpha}}=\epsilon_{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}}\left(\bar{\sigma}^{a}\right)^{\dot{\beta} \beta} \tag{A.7}
\end{equation*}
$$

and satisfy the completeness relations

$$
\begin{align*}
\operatorname{tr} \sigma^{a} \bar{\sigma}^{b} & =2 \eta^{a b}  \tag{A.8}\\
\left(\sigma^{a}\right)_{\alpha \dot{\alpha}}\left(\bar{\sigma}_{a}\right)^{\dot{\beta} \beta} & =2 \delta_{\alpha}^{\beta} \delta_{\dot{\alpha}}^{\dot{\beta}} \tag{A.9}
\end{align*}
$$

The $S L(2, \mathbb{C})$ generators are defined

$$
\begin{equation*}
\sigma^{a b}=\frac{i}{4}\left[\sigma^{a} \bar{\sigma}^{b}-\bar{\sigma}^{a} \sigma^{b}\right] \quad ; \quad \bar{\sigma}^{a b}=\frac{i}{4}\left[\bar{\sigma}^{a} \sigma^{b}-\sigma^{a} \bar{\sigma}^{b}\right] \tag{A.10}
\end{equation*}
$$

which implies they are respectively (imaginary) self-dual and (imaginary) anti-self-dual

$$
\begin{equation*}
* \sigma^{a b}=i \sigma^{a b} \quad ; \quad * \bar{\sigma}^{a b}=-i \bar{\sigma}^{a b} . \tag{A.11}
\end{equation*}
$$

For our purposes, the Hodge star always maps $p$ forms to $4-p$ forms

$$
\begin{equation*}
* \omega_{a_{1} \ldots a_{p}}=\frac{1}{p!} \epsilon_{a_{1} a_{2} a_{3} a_{4}} \omega^{a_{p+1} \ldots a_{4}} . \tag{A.12}
\end{equation*}
$$

We often write (antisymmetric) spinor products as

$$
\begin{align*}
& \langle\lambda, \mu\rangle=\lambda^{\alpha} \mu_{\alpha}=\epsilon^{\alpha \beta} \lambda_{\alpha} \mu_{\beta}=\epsilon^{\alpha \beta} \mu_{\beta} \lambda_{\alpha}=-\epsilon^{\beta \alpha} \mu_{\beta} \lambda_{\alpha}=-\langle\mu, \lambda\rangle  \tag{A.13}\\
& {[\widetilde{\lambda}, \widetilde{\mu}]=\widetilde{\lambda}^{\dot{\alpha}} \widetilde{\mu}_{\dot{\alpha}}=\epsilon^{\dot{\alpha} \dot{\beta}} \widetilde{\lambda}_{\dot{\alpha}} \widetilde{\mu}_{\dot{\beta}}=\epsilon^{\dot{\alpha} \dot{\beta}} \widetilde{\mu}_{\dot{\beta}} \widetilde{\lambda}_{\dot{\alpha}}=-\epsilon^{\dot{\beta} \alpha} \widetilde{\mu}_{\dot{\beta}} \widetilde{\lambda}_{\dot{\alpha}}=-[\widetilde{\mu}, \widetilde{\lambda}] .} \tag{A.14}
\end{align*}
$$

Notice that for fermionic spinors the product is symmetric:

$$
\begin{gather*}
\chi \psi \equiv \epsilon^{\alpha \beta} \chi_{\alpha} \psi_{\beta}=-\epsilon^{\alpha \beta} \psi_{\beta} \chi_{\alpha}=+\epsilon^{\beta \alpha} \psi_{\beta} \chi_{\alpha}=\psi \chi  \tag{A.15}\\
\bar{\chi} \bar{\psi} \equiv \epsilon^{\dot{\alpha} \dot{\beta}} \bar{\chi}_{\dot{\alpha}} \bar{\psi}_{\dot{\beta}}=-\epsilon^{\dot{\alpha} \dot{\beta}} \bar{\psi}_{\dot{\beta}} \bar{\chi}_{\dot{\alpha}}=+\epsilon^{\dot{\beta} \dot{\alpha}} \bar{\psi}_{\dot{\beta}} \bar{\chi}_{\dot{\alpha}}=\bar{\psi} \bar{\chi} . \tag{A.16}
\end{gather*}
$$

In loop amplitudes the product

$$
\begin{equation*}
\langle\lambda| k \mid \widetilde{\mu}]=\left\langle\lambda, \lambda_{k}\right\rangle\left[\widetilde{\lambda_{k}}, \widetilde{\mu}\right]=\lambda^{\alpha} \lambda_{k \alpha} \widetilde{\lambda}_{k \dot{\alpha}} \widetilde{\mu}^{\dot{\alpha}}=\lambda^{\alpha}\left(\sigma^{a} k_{a}\right)_{\alpha \dot{\alpha}} \widetilde{\mu}^{\dot{\alpha}} \tag{A.17}
\end{equation*}
$$

appears quite often.

## -Grassman variables

Grassman variables anticommute:

$$
\begin{equation*}
\theta \eta=-\eta \theta \tag{A.18}
\end{equation*}
$$

in particular,

$$
\begin{equation*}
\theta \theta=0 \tag{A.19}
\end{equation*}
$$

A Grassman function of a single variable has a simple Taylor expansion:

$$
\begin{equation*}
f(\theta)=a+b \theta, \tag{A.20}
\end{equation*}
$$

since $\theta \theta=0$. The Berezian integral of an anticommuting variable is defined :

$$
\begin{align*}
\int d \theta \theta & =1  \tag{A.21}\\
\int d \theta 1 & =0  \tag{A.22}\\
\int d \theta f(\theta) & =b \tag{A.23}
\end{align*}
$$

This definition implies the following properties:

- Berezian integration is translationally invariant

$$
\begin{gather*}
\int d \theta f(\theta+\eta)=\int d \theta f(\theta)  \tag{A.24}\\
\int d \theta \cdot \frac{d}{d \theta} f(\theta)=0 \tag{A.25}
\end{gather*}
$$

- Berezian integration is equivalent to differentiation

$$
\begin{equation*}
\frac{d}{d \theta} f(\theta)=b=\int d \theta f(\theta) \tag{A.26}
\end{equation*}
$$

- We can define a delta function by

$$
\begin{equation*}
\delta(\theta)=\theta \tag{A.27}
\end{equation*}
$$

For the superspace coordinates $\theta^{\alpha}$, we write

$$
\begin{equation*}
\partial_{\alpha}=\frac{\partial}{\partial \theta^{\alpha}} \quad \text { and } \quad \theta^{2}=\frac{1}{2} \epsilon_{\alpha \beta} \theta^{\alpha} \theta^{\beta} . \tag{A.28}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\partial_{\alpha} \theta^{\beta} & =\delta_{\alpha}^{\beta}  \tag{A.29}\\
\partial_{\alpha} \theta^{\beta} \theta^{\gamma} & =\delta_{\alpha}^{\beta} \theta^{\gamma}-\delta_{\alpha}^{\gamma} \theta^{\beta}  \tag{A.30}\\
\partial_{\alpha} \theta^{2} & =\theta_{\alpha}  \tag{A.31}\\
\partial^{2} \theta^{2} & =1 \tag{A.32}
\end{align*}
$$

and similarly for the $\bar{\theta}_{\dot{\alpha}}$. The important features for superspace integration are

$$
\begin{gather*}
\int d^{2} \theta \theta^{2}=1=\int d^{2} \bar{\theta} \bar{\theta}^{2}  \tag{A.33}\\
\int d^{4} \theta \theta^{2} \bar{\theta}^{2}=1 \tag{A.34}
\end{gather*}
$$

where $d^{4} \theta=d^{2} \theta d^{2} \bar{\theta}$.

## - Superconformal Algebras

The most general group we consider in this work is the superconformal group. To emphasize the various sub-groups of this large symmetry group, we divide the commutation relations into parts.

Poincaré: Generators $J_{a b}, P_{a}$

$$
\begin{gather*}
{\left[J_{a b}, J_{c d}\right]=i\left(\eta_{a d} J_{b c}-\eta_{a c} J_{b d}+\eta_{b c} J_{a d}-\eta_{b d} J_{a c}\right)}  \tag{A.35}\\
{\left[J_{a b}, P_{c}\right]=i\left(\eta_{b c} P_{a}-\eta_{a c} P_{b}\right)} \tag{A.36}
\end{gather*}
$$

Supersymmetry: Additional generators $Q_{\alpha}^{A}, \bar{Q}_{\dot{\alpha} A}, R_{r}$

$$
\begin{gather*}
\left\{Q_{\alpha}^{A}, \bar{Q}_{\dot{\alpha} B}\right\}=2\left(\sigma^{a}\right)_{\alpha \dot{\alpha}} P_{a} \delta_{B}^{A}  \tag{A.37}\\
\left\{Q_{\alpha}^{A}, Q_{\beta}^{B}\right\}=\epsilon_{\alpha \beta} Z^{A B} \tag{A.38}
\end{gather*} ; \quad\left\{\bar{Q}_{\dot{\alpha} A}, \bar{Q}_{\dot{\beta} B}\right\}=-\epsilon_{\dot{\alpha} \dot{\beta}} \bar{Z}_{A B}, ~\left(\bar{Q}_{\dot{\alpha} A}, J_{a b}\right]=\left(\bar{\sigma}_{a b} b_{\dot{\alpha}}^{\dot{\beta}} \bar{Q}_{\dot{\beta} A},\right.
$$

$$
\begin{equation*}
\left[R_{r}, R_{s}\right]=i f_{r s}{ }^{t} R_{t} \tag{A.41}
\end{equation*}
$$

Conformal: Generators $K_{a}, D$ in addition to $J_{a b}, P_{a}$

$$
\begin{gather*}
{\left[J_{a b}, K_{c}\right]=i\left(\eta_{b c} K_{a}-\eta_{a c} K_{b}\right)}  \tag{A.42}\\
{\left[P_{a}, K_{b}\right]=2 i J_{a b}-2 i \eta_{a b} D}  \tag{A.43}\\
{\left[D, K_{a}\right]=-i K_{a} \quad ; \quad\left[D, P_{a}\right]=i P_{a}} \tag{A.44}
\end{gather*}
$$

Superconformal: All of the above generators as well as $S_{\alpha}^{A}$

$$
\left.\left.\begin{array}{rl}
\left\{S_{\alpha}^{A}, \bar{S}_{\dot{\alpha} B}\right\}=-2\left(\sigma^{a}\right)_{\alpha \dot{\alpha}} K_{a} \delta_{B}^{A} \\
{\left[S_{\alpha}^{A}, J_{a b}\right]=\left(\sigma_{a b}\right)_{\alpha}^{\beta} S_{\beta}^{A}} & ;
\end{array}\right]\left[\bar{S}_{\dot{\alpha} A}, J_{a b}\right]=\left(\bar{\sigma}_{a b}\right)_{\dot{\alpha}}^{\dot{\beta}} \bar{S}_{\dot{\beta} A}\right)
$$

and all other (anti)commutators vanish. Here $R$ generates the additional $U(1)_{R}$ contained in the full $U(\mathcal{N})_{R}$. Also, in the superconformal algebra, the central charges $Z^{A B}$ must vanish.

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[^0]:    ${ }^{1}$ This is assuming commuting spinors; the scalar product of fermionic spinors is symmetric, for more on this see the Appendix
    ${ }^{2}$ For real $V, \widetilde{\lambda}= \pm \bar{\lambda}$, depending on the sign of $V_{0}$.

[^1]:    ${ }^{3} \mathrm{Had}$ we considered massive fields, we would have found $\left\{a_{i}, \bar{a}_{j}\right\}=\delta_{i j}$, thus, there would exist two independent ladder algebras and therefore twice as many states in the Foch space.

[^2]:    ${ }^{4}$ The case of $\mathcal{N}=4$ is exceptional, as the action of the extra $U(1)_{R}$ is in fact trivial

[^3]:    ${ }^{1}$ We apologize to the reader if the proliferation of variables labeled $s$ is confusing, however it will be quite clear from the context whether we mean the position $s \in(1, \ldots, n)$ or the Lorentz invariant quantity $s=\left(k_{r}+P\right)^{2}$.

[^4]:    ${ }^{1}$ A version of this chapter has been accepted for publication. Quigley, C. and Rozali, M. One-Loop MHV Amplitudes in Supersymmetric Gauge Theories. JHEP 01(2005)053.
    ${ }^{2}$ We do not keep track of the overall sign, which can be fixed at the end of the calculation.

[^5]:    ${ }^{3}$ Tensor, vector and scalar refer to the degree of loop momenta appearing in the numerator.

[^6]:    ${ }^{4}$ Since $[\mathrm{r} \mathrm{r}]=0$, we can always write $\left.\left.\left.\langle\bullet| P_{z} \mid r\right]=\langle\bullet|\left(P_{z}+k_{r}\right) \mid r\right]=\langle\bullet| P_{L ; z} \mid r\right]$.
    ${ }^{5}$ We note that the integration contour then is channel-dependent, as in [25].

