A DETERMINATION OF MULTIPLE SCATTERING
FOR A NEGATIVE PION BEAM

by

LARRY JAMES WATTS
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Department of Physics

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date December 21, 1977
The multiple Coulomb scattering of negative pions has significant effects on the dose distributions resulting from pion beams incident on thick targets. The use of negative pions in radiotherapy requires a detailed knowledge of the distribution of dose and biological effect. Thus it is important to have an accurate description for the lateral distributions of pions which result from multiple scattering. It has been proposed by Fowler and Perkins that these lateral distributions are of a Gaussian nature for incident pencil beams. In this study an attempt has been made to determine experimentally and theoretically the appropriate value for the standard deviation of the Gaussian in the pencil beam description.

The experimental determination involved placing medical x-ray films in a homogeneous water phantom, perpendicular to the beam axis of the M8 biomedical channel at TRIUMF. The distributions recorded on film for circularly collimated beams were measured for optical density and compared to calculated distributions in order to extract the pencil beam information. The presence of contaminating electrons and muons as well as the difficulty in achieving a parallel beam complicated the determination of the standard deviation of the Gaussian for pions. The experimental determination at the end of a 20.1 cm range in water is only 7% greater than the preferred theoretical calculation for pions alone.

This calculation is based on the first (Gaussian) term of Molière's theory modified for the Fano correction and energy loss and
yields results 20% lower than those of the "standard reference" of Fowler and Perkins. The agreement between the theory for pions and the experiment for a real beam in water indicates that the theory presented should be adequate for treatment planning calculations.
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1. INTRODUCTION

The use of negative pi-mesons (pions or $\pi^-$) in the radiotherapeutic treatment of cancer was first proposed by Fowler and Perkins (1), in 1961. Since that time, much effort has been expended in the investigation of the advantages of pions over conventional gamma radiation. The two main advantages cited (2,3) are:

(i) better dose concentration at the tumour location compared to surrounding tissue; this is due to the increased energy deposition from charged particles slowing down (Bragg peak effects) and in the case of $\pi^-$, from the additional charged particles produced in the "stars" resulting from the nuclear capture of the stopped pions;

(ii) the reduced dependence on the presence of oxygen [lower Oxygen Enhancement Ratio (OER)], that results from the high Linear Energy Transfer (LET) character of primary and secondary charged particles at the pion stopping location; it has been demonstrated that fully oxygenated cells suffer more damage than anoxic cells for the same doses and since tumours are believed to have anoxic regions, it is anticipated that the differential sensitivity between oxygenated and anoxic cells in the tumour will be reduced at the pion stopping location.

The desire to exploit these advantages has led to the incorporation of radiobiology and radiotherapy projects at the three facilities capable of producing $\pi^-$ beams of sufficient intensity. In fact, some patients have already been treated in preliminary trials (4,5). At the Batho
Biomedical Facility at TRIUMF, patient treatment is still approximately one year from realization, but physical measurements and preliminary radiobiology have been going on since June, 1975 (6,7).

Before treatment can be carried out, it is necessary to know in detail the distribution of dose with position, both in depth and lateral extent. While some initial treatment and biology can be done using dose distributions determined experimentally in a homogeneous medium, the treatment of many patients, with varying tumour locations and volumes, will require a general method of calculating dose distributions in an inhomogeneous medium. Several factors must be accounted for in the calculation of these dose distributions. One factor that is required in either a simple empirical calculation [such as that of Li et al. (8)] or a detailed Monte Carlo treatment [such as those of Armstrong and Chandler (9) or Turner et al. (10,11)] is a knowledge of the multiple scattering of pions.

The concept of a "pencil" beam, i.e. a beam of particles which, initially, has a unique starting direction and is of zero spatial extent, is used in both of the above schemes for calculating dose distributions. The lateral distribution of pions resulting from the multiple scatter of such pencil beams has been described by Fowler and Perkins (1) to be Gaussian, with symmetry about the incident direction. In addition, Fowler and Perkins indicated that the magnitude of the standard deviation, σ, in the Gaussian, was of the order of 1 cm for pions at the end of a 20 cm range in tissue.

For finite size beams, the multiple scatter will affect the dose primarily near the edges [see for example Hamm et al. (12)]; in
the above case within about 2 cm either side of the geometrical edge. This effect is a major reason for the difficulty in confining the stopping pions to the tumour volume, and the magnitude of these effects creates problems for the treatment of small tumour volumes at these depths [see Fowler and Perkins (1)]. Another situation where the effect of scatter is important is in the case of inhomogeneities, such as the presence of bone or air, in tissue. This situation has been discussed by Santoro et al. (13) and Hamm et al. (14,15) for cases where the inhomogeneities are smaller than the pion beam. There is a large effect on the resulting dose distributions due to both the density difference, and the complication of the paths as a result of multiple scattering.

In view of the importance of the multiple scattering of pions on the resulting dose distributions, a study was made to determine experimentally the appropriate value of $\sigma$ to be used in the pencil beam description. In addition, theoretical calculations of this parameter were made for comparison with experiment and to provide a description of the variation of $\sigma$ with depth, suitable for use in treatment planning calculations.

The experimental determination of a parameter, such as the standard deviation, for a pencil beam of pions from a real finite size beam is complicated in our case due to two problems:

(1) the inability to achieve a parallel beam

(2) the presence of contaminating electrons and muons.

The effect of these problems on our determination of $\sigma$ is discussed in Chapter 4.
Medical x-ray film was chosen as our detector in the experiments due to the following requirements:

1. adequate sensitivity at the initially low dose rate
2. good spatial resolution
3. integrating detector system, because of the inevitable fluctuations in the proton beam intensity during the early development of the cyclotron.

The blackening of the film depends on energy above some threshold being deposited in each grain. For particles with a high LET, film responds primarily to the relative number of particles. We have regarded film as a "particle counter" and in Chapter 4 give some experimental justification for this assumption.

In the experiment, films, in a water tank, were placed perpendicular to the beam axis of the M8 biomedical channel at TRIUMF. Measurements were made at two different channel momenta with and without water in the phantom to determine the scattering effect of water. Water was chosen since it is similar to some types of tissue in the pion ranges and dose deposition (since oxygen is a major component in tissue and in water).

For the majority of the experiments circular fields with two different radii were used. In addition, one experiment was carried out using a beam initially defined by a sharp straight edge. Collimation of the beam was achieved by placing a single brass collimator in front of the water tank. The resulting distributions recorded on the film were measured for optical density (defined in Chapter 3) using a scanning densitometer. In order to derive the value of $\sigma$ for the idealized
pencil beam, the measured distributions were compared to distributions calculated for beams of the appropriate shape with the assumption that multiple scatter is of a Gaussian nature.

In the next chapter the theory of multiple scattering will be discussed, initially for the case of pencil beams and thin scattering sections. This theory will then be extended to thick sections and beams of finite size. In Chapter 3, the experimental setup and evaluation of the value of $\sigma$ for the Gaussian pencil beam is presented. Chapter 4 contains a discussion of the results and the assumptions involved, as well as a comparison of the various theoretical values to experiment.
2. MULTIPLE SCATTER THEORY

2.1 Introduction

In this chapter a brief description of the Gaussian theory of multiple scatter, based on a classical treatment, will be given, followed by a more exact theory due to Molière (16,17). Subsequent modifications suggested by Fano (18), Hungerford et al. (19) and Mayes et al. (20) are also discussed. This theory, which applies to a pencil beam with normal incidence on thin foils, will then be applied to the case of a thick scattering section and finally a real finite size beam will be discussed.

2.2 Pencil Beams

2.2.1 Classical Derivation of Gaussian Distribution

The spreading out of an initially collimated beam of particles, incident at a point on a plane scattering medium, is primarily due to lateral scatter attributable to Coulomb interactions with the medium. In the case of a single elastic interaction, this results in the well known Rutherford formula for the distribution of classical particles with angle. The formula is given by Bethe and Ashkin (21) as

\[ \frac{d\sigma}{d\Omega} = \frac{z^2Z^2e^4}{16T^2\sin^4(\theta/2)} \]

where \( z \) and \( Z \) are the charges of the incident particles and target nucleus respectively, \( T \) is the kinetic energy of the incident particles (units of MeV and MeV/c will be used throughout for energy and momentum respectively), and \( d\Omega \) is the element of solid angle at angle \( \theta \). This expression is correct for the case where the incident particle mass is much less than the mass of the nucleus.
In the small angle approximation \((\sin \theta = \theta \text{ and } \cos \theta = 1)\) and substituting \(p v/2\) for \(T\) (where \(p\) is momentum, \(v\) is velocity) this expression reduces to

\[
f(\theta) \, d\theta = \frac{8\pi}{3} \frac{z^2 Z e^4}{(p^2 v^2 \theta^3)} \, d\theta. \tag{1}
\]

Classifying pions as heavy charged particles requires using \(Z^2\) rather than \(Z (Z+1)\) which applies in the case of incident electrons \((17,18)\). This case of small angle scatter with \(\theta\), the polar angle with respect to the incident direction, is shown in figure 1.

For a foil thin enough for the incident particles to suffer several interactions without significant energy loss, the distribution of emerging particles will have azimuthal symmetry about the incident normal direction. As argued by Bethe and Ashkin \((21)\), the distribution with respect to the angle \(\theta\) should be given by a Gaussian distribution, i.e.

\[
P(\theta) \, d\theta = \frac{2\theta}{\langle \theta^2 \rangle} \exp \left(-\frac{\theta^2}{\langle \theta^2 \rangle}\right) \, d\theta. \tag{2}
\]

Here \(\langle \theta^2 \rangle\) is the mean square angle for multiple scatter which can be evaluated by the following formula

\[
\langle \theta^2 \rangle = \int_0^{\theta_{\max}} \int_{\theta_{\min}}^{\Delta x} N \theta^2 \, f(\theta) \, d\theta \, dx \tag{3}
\]

for a foil with thickness \(\Delta x\) and \(N\) scatterers per unit volume. Substituting for \(f(\theta)\) from eq. (1) we get

\[
\langle \theta^2 \rangle = \frac{8\pi N z^2 Z e^4}{(p^2 v^2)} \Delta x \ln \left(\frac{\theta_{\max}}{\theta_{\min}}\right), \tag{4}
\]

where \(\theta_{\max}\) and \(\theta_{\min}\) allow for corrections to the Rutherford formula due to the finite size of the nucleus and the shielding of the nucleus by
FIGURE 1

MULTIPLE SCATTERING GEOMETRY USED

TO DEFINE SCATTERING ANGLE $\theta$

[adapted from Hungerford et al. (19)]
the atomic electrons. The selection of the limits $\theta_{\text{max}}$ and $\theta_{\text{min}}$ has been discussed by Bethe and Ashkin and only the final expression will be quoted,

$$\langle \theta^2 \rangle = \left[ 4\pi e^4 n_0 z^2 z^2 / (A p^2 v^2) \right] t \ln \left[ 4\pi Z^{4/3} z^2 n_0 t \hbar^2 / (A m_e^2 v^2) \right]$$

where $N$ has been replaced by $N_0 \rho/A$,

- $N_0$ is Avogadro's number.
- $A =$ atomic weight of foil in grams.
- $\rho =$ density in $\text{g/cm}^3$.
- $m_e =$ mass of electron.
- $t =$ $\rho \Delta x$ in $\text{g/cm}^2$.

The angle $\chi_c$ is defined (21) such that there is on the average one scattering, in thickness $t$, of angle greater than $\chi_c$.

Preston and Koehler (22) have used equation (5) for the case of protons with $Z^2$ replaced by $Z(Z+1)$ and $Z^{4/3}$ replaced by $Z^{1/3}(Z+1)$. Fano (18) points out that this is only an order of magnitude correction for the effect of inelastic collisions with the atomic electrons, since it assumes that the Rutherford scattering formula is accurate at all angles. He has proposed a modification to Molière's theory which is discussed in section 2.2.3 and adopted in our calculations.

The difficulties in choosing suitable values for $\theta_{\text{max}}$, $\theta_{\text{min}}$ and in taking account of inelastic electron collisions led us to investigate the theory of Molière. This theory, while being more complicated to evaluate, has been demonstrated to give good agreement
with experiment for "thin" foils as shown by Hungerford et al. (19) and Mayes et al. (20). Although we give the complete formula according to Molière, we will use only the first (Gaussian) term in the expansion for calculations, due to the complexities of thick scattering sections. This Gaussian approximation for lateral scatter has also been made in deriving the pencil beam parameter, $\sigma$, from the measured distributions of finite size non-parallel beams, because of the mathematical complexity for more exact descriptions.

2.2.2 Theory of Molière

The theory of small angle multiple scattering by fast charged particles has been reviewed by Scott (17). While there are several theories that can give accurate results, the theory of Molière, described by Bethe (16), is the best known and is relatively simple to evaluate using the functions described in Bethe's article.

The mathematical description of the succession of single scatterings experienced by a fast charged particle traversing a thin scatterer (this assumes no energy loss) can be most easily described using Hankel (Fourier-Bessel) transforms. This allows the convolutions associated with each successive scattering event to be treated by a single multiplication in the transformed domain. Another way of describing the process (16) uses the Boltzmann transport equation, but use is also made of these transforms to simplify the solution in terms of an ordinary differential equation. The details are found in Bethe (16) and Scott (17) and we merely quote the results.

Molière's theory for small angles involves some assumptions as listed by Hungerford et al. (19):
(i) small angle approximation, i.e. \( \sin \theta = \theta \), \( \cos \theta = 1 \) and integrals from 0 to \( \pi \) are replaced by integrals from 0 to \( \infty \)

(ii) Thomas-Fermi screening by atomic electrons

(iii) absence of spin effects

(iv) absence of scattering by atomic electrons

Molière uses a modified form of the Rutherford formula, but as Bethe (16) points out, the strength of Molière's theory lies in the fact that the scattering is described by a single parameter, the screening angle \( \chi_a \) (analogous to \( \theta_{\text{min}} \)). Also the distribution function \( f(\theta) \) is independent of the shape of the single scattering differential cross section, except that it goes over to the Rutherford law at large angles.

The distribution in \( \theta \) is the inverse transform of the multiplication in the Fourier-Bessel domain described earlier and following Bethe is given by

\[
f(\theta) \, \theta \, d\theta = J_0(\lambda y) \exp \left[ \frac{1}{\lambda} y^2 \left( -b + \ln \left[ \frac{1}{\lambda} y^2 \right] \right) \right] \, y \, dy. \tag{6}
\]

Molière expands this integral as a power series to give

\[
f(\theta) \, \theta \, d\theta = \theta_r \, d\theta_r \left[ 2 \exp \left[ -\theta_r^2 \right] + B^{-1} f^{(1)}(\theta_r) + B^{-2} f^{(2)}(\theta_r) + \ldots \right] \tag{7}
\]

where \( f^{(n)}(\theta_r) = (n!)^{-1} \int_0^\infty J_0(\theta_r u) \exp \left[ -\xi u^2 \right] \left( \xi u^2 \ln \left[ \xi u^2 \right] \right)^n \, u \, du \).

\( J_0 = \) zero order Bessel function.

\( u = B^{1/2} y. \)

\( \theta_r = \lambda B^{-1/2} = \theta/(\chi_c B^{1/2}). \)

\( B \) is defined by \( B - \ln B = b. \)

\( b = \ln \left( \chi_c^2 / \chi_a^2 \right) - .1544. \)
\[ \chi_c^2 = 4\pi e^4 N_o t z^2 Z^2 / (A p^2 \beta^2), \] described earlier.

\[ \chi_a^2 = \chi_0^2 \cdot f(a^2) = \chi_0^2 (1.13 + 3.76 a^2). \]

\[ \alpha = \frac{\gamma e^2}{(\hbar v)} = \frac{zZ}{(137 \beta)}. \]

\[ \chi_0 = \frac{\hbar}{(pr_0)}. \]

\[ r_0 = 0.468 \times 10^{-8} \, Z^{-1/3} \, \text{cm}, \text{ Thomas-Fermi radius of the atom.} \]

\[ \beta, p = \text{velocity, momentum of incident particle.} \]

Bethe (16) has discussed the significance of the power series expansion, and states that only the first three terms are required to achieve an accuracy of 1%. The first term is Gaussian, the second goes over to the single scattering formula at large angles, while the third is a correction with no simple interpretation.

Using tables given in Bethe (16), \( f(\theta) \) is plotted vs \( \theta \) for two values of \( B \) ranging near the extremes in our case (figure 2). When we substitute numerical values for the constants in the above formula we find,

\[ \chi_c^2 = .1569 \, z^2 Z^2 \, t / (A p^2 \beta^2), \] (8)

where \( t \) is in grams/cm\(^2\), \( A \) is in grams and \( p\beta \) is in MeV,

\[ \chi_c^2/\chi_a^2 = 8838.4 \, z^2 Z^4/3 \, t / (A \beta^2 (1.13 + 3.76 a^2)), \] (9)

\[ b = \ln \left( \frac{\chi_c^2}{\chi_a^2} \right) - .1544 = B - \ln B. \] (10)

Scott (17) gives an approximate formula, accurate to .5% for \( \chi_c^2/\chi_a^2 \) from \( 10^2 \) to \( 10^5 \) (where \( \chi_c^2/\chi_a^2 \) is representative of the mean number of collisions in thickness \( t \)).

\[ B = 1.153 + 2.583 \, \log_{10} \left( \frac{\chi_c^2}{\chi_a^2} \right). \] (11)
FIGURE 2

MOLIÈRE DISTRIBUTION PLOTTED AGAINST REDUCED ANGLE $\theta_r$ FOR TWO VALUES OF $B$ PARAMETER
2.2.3 Modifications Proposed for Molière's Theory

As stated, Molière's original theory neglected the scattering by atomic electrons, but Fano (18) has calculated the contribution of inelastic collisions with the atomic electrons in a form compatible with Molière's theory. He states that it is incorrect to simply replace $Z^2$ with $Z(Z+1)$, because the actual cross sections depart at small angles from the Rutherford formula. Instead of rising to infinity, the cross sections are cut off with different cutoffs for elastic and inelastic collisions. Fano gives a recipe:

(i) if the incident particles are electrons replace $Z^2$ by $Z(Z+1)$ and replace Molière's $b$ by $b + B''$ where

$$B'' = (Z+1)^{-1} \left( \ln \left[ \frac{0.160}{Z^{2/3}} \frac{Z^{1/3}}{1+3.33 \frac{Z}{137g}} \right] - u_{in} \right);$$

(ii) if the incident particles are heavy charged particles leave $Z^2$ unaltered but replace Molière's $b$ by $b + B''$ where

$$B'' = Z^{-1} \left( \ln \left[ \frac{1130}{Z^{-4/3}} \frac{Z^{2}}{(1-Z^2)} \right] - u_{in} - \frac{1}{2} \beta^2 \right). \quad (12)$$

Here $u_{in}$ is the integral over the incoherent scattering function and Fano gives values for $u_{in}$ as found in table I.

**TABLE I**

<table>
<thead>
<tr>
<th>Z</th>
<th>$u_{in}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3.6</td>
</tr>
<tr>
<td>3</td>
<td>-4.6</td>
</tr>
<tr>
<td>8</td>
<td>-5.0</td>
</tr>
<tr>
<td>82</td>
<td>-6.3</td>
</tr>
</tbody>
</table>
We have adopted Fano's suggestion for calculating B and this results in different values for B than those calculated by others, such as Bichsel (23) (details in Appendix A).

Another suggested modification to Molière's theory is given by Hungerford et al. (19), who point out that the Rutherford cross section applies to the centre of mass frame, and as such the values for momentum $p$, velocity $\beta$, and angle $\theta$ are c.m. values. They point out, that in the small angle approximation, the relativistic cross section is, to first order, equal to the Rutherford cross section with c.m. angles and velocities. They derive a new expression for $\chi_c^2$ as

$$\chi_c^2 = 0.157 z^2z^2 t / (A \varepsilon^2 (p_{cm} / \beta_{cm})^2) \text{ (rad}^2)\text{)}.$$ 

They give a complicated expression for $\varepsilon^2$ but it turns out that $\varepsilon = p_{lab} / p_{cm}$ and therefore the corrected expression is given by

$$\chi_c^2 = 0.157 z^2z^2 t / (A (p_{lab} / \beta_{cm})^2).$$

If one ignores the correction to $\beta^2$ in the evaluation of Molière's B (which should be a small effect) then to first order

$$\langle \theta^2 \rangle_{corr} / \langle \theta^2 \rangle = (\chi_c^2)_{corr} B / (\chi_c^2 B)$$

$$= (\beta_{lab} / \beta_{cm})^2,$$

or as shown by Highland (24)

$$\langle \theta^2 \rangle_{corr}^{1/2} = \langle \theta^2 \rangle^{1/2} (1 + M_p^2 / (E_p M_t)),$$

where $M_p$, $E_p$ are the mass and total energy of the incident particle and...
$M_t$ is the mass of the target atom. Calculations indicate, that over the range of pion energies that we are dealing with, in tissue equivalent materials, the corrections amount to only 1% to 1.5% and have been neglected. However, they may be added later if desired.

A more significant modification proposed was that of Mayes et al. (20) who suggested, that on the basis of the experimental results for various particles, targets, and energies, a better fit of the theory to the data could be effected by changing Molière's $f(a^2)$ equation from

$$f(a^2) = 1.13 + 3.76 \ a^2$$

to

$$f(a^2) = 0.59 + 3.44 \ a^2.$$ 

They still, however, have a discrepancy, especially for low Z, that $\langle s^2 \rangle_{\text{exp}} / \langle s^2 \rangle_{\text{th}}$ is approximately 1.06 - 1.08. Since they have not applied Fano's correction, which is particularly important for low Z materials, their proposal should be held in abeyance pending more refined experiments (25). We have calculated values for $B$ in Appendix A on the basis of both $f(a^2)$ equations. In view of the uncertainty of the validity of their modification, we have elected to use only Molière's results.

As mentioned earlier, in view of the complexities involved in the transition to thick scattering sections and finite size beams, only the first (Gaussian) term will be retained, using for the mean square angle

$$\langle s^2 \rangle = \chi_c^2 B. \quad (13)$$

In order to see the size of this approximation we have plotted in figure 3 the Gaussian term and the complete distributions for the two values of $B$. 

FIGURE 3

COMPARISON OF GAUSSIAN TERM WITH NORMALIZED MOLIÈRE DISTRIBUTION

FOR TWO VALUES OF B PARAMETER
used for figure 2. Here we have normalized all distributions for $\theta^r = 0$. It is seen that the Gaussian appears as the limit as the number of collisions goes to infinity. Since in general we will be dealing with a very thick section this is a reasonable approximation.

As mentioned earlier, the theory of Molière, as outlined, holds for thin foils where the energy loss is negligible. In radiotherapy we are dealing with cases where the amount of scattering material is very large. In fact, we have the case for negative pi-mesons where the region of interest extends from their relativistic incidence at the surface to the point where they have lost all their energy and come to rest, causing a nuclear disintegration following capture.

It is appropriate at this point to summarize the modifications to Molière's theory that have been discussed and to note which of them will be used.

(i) Fano's correction for calculation of B—used, discussed in Appendix A.

(ii) Centre of mass corrections—not used.

(iii) $f(a^2)$ equation corrections—calculated but not used.

(iv) Gaussian angular approximation—used with mean square angle of $\chi_c^2 B$.

(v) Energy loss correction—used, discussed in the next section and in detail in Appendix A.
2.2.4 Application to Thick Scatterers

In the case of radiotherapy we are not primarily interested in the angular distributions after multiple scatter from a thin section, but in the radial distribution at some point in the finite range of a pion. We will follow in general the treatment of Fowler and Perkins (1), Preston and Koehler (22,26) and Carlsson and Rosander (27).

Referring to figure 4, the case of interest is the lateral distribution at a plane located at depth \( s \) in the medium, for a mono-energetic beam of particles entering at \( x = 0 \) with a total range \( R_0 \) (no straggling). If the thickness \( dx \) is sufficiently thick to produce a Gaussian angular distribution we will argue that the lateral distribution will also be Gaussian in the small angle approximation.

Since each increment in range results in a Gaussian distribution of angle, then two successive increments also result in a Gaussian, with mean square angle increased by twice the mean square deviation that one increment would give, i.e., the mean square deviations are additive. (Equivalently the convolution of two Gaussians is a Gaussian with \( \langle \theta^2 \rangle_t = \langle \theta^2 \rangle_1 + \langle \theta^2 \rangle_2 \).) The contribution of a thin scattering section, at the surface, to the distribution of lateral displacement at \( s \), would be the projection of this angular distribution magnified by the distance from the surface to the plane, \( s \). This implies a Gaussian distribution in lateral displacement for a Gaussian angular distribution at the surface. For a series of scatterers between the surface and \( s \), each with a Gaussian distribution in angle, we seek a description of the contributions of each to the resultant mean square lateral displacement \( \langle r^2 \rangle \), at a depth \( s \).

In general, for uncorrelated parameters \( a \) and \( b \) with a functional
FIGURE 4

SCHEMATIC DIAGRAM OF SMALL ANGLE SCATTER CONTRIBUTION TO LATERAL DISPLACEMENT FOR THICK SCATTERING MEDIUM
relationship \( y = f(a,b) \) the resultant standard deviation is given by

\[
\sigma_y^2 = (\partial y/\partial a)^2 \sigma_a^2 + (\partial y/\partial b)^2 \sigma_b^2.
\]

In the case of small angle scatter (figure 4)

\[
dr = d\theta (s-x)
\]

i.e. \( r = \theta (s-x) \),

and therefore

\[
\sigma_r^2 = (s-x)^2 \sigma_\theta^2 + (-\theta)^2 \sigma_x^2,
\]

which for Gaussian distributions results in

\[
\langle r^2 \rangle = (s-x)^2 \langle \theta^2 \rangle + \theta^2 \langle x^2 \rangle.
\]

Assuming \( (s-x)^2 \langle \theta^2 \rangle \) to be much greater than \( \theta^2 \langle x^2 \rangle \), which is reasonable since we assume no straggling in \( x \) and each angle \( \theta \) is small, then the contribution to \( \langle r^2 \rangle \) for a thin section \( dx \) at depth \( x \) is given by

\[
d\langle r^2 \rangle = (s-x)^2 d\langle \theta^2 \rangle
\]

and therefore

\[
\langle r^2 \rangle = \int_0^S (s-x)^2 d\langle \theta^2 \rangle.
\]

(14)

The radial distribution is given by a Gaussian

\[
N(r,s) = (\pi \langle r^2 \rangle)^{-1} \exp \left( -r^2/\langle r^2 \rangle \right)
\]

where \( \langle r^2 \rangle \) is a function of \( s \). In order to evaluate \( \langle r^2 \rangle \) we substitute from equations (8) and (13) for \( \langle \theta^2 \rangle \) to give

\[
\langle x^2 \rangle = \int_0^S (s-x)^2 \left[ .157 z^2 Z^2 B / (A p^2 \beta^2) \right] dx
\]

(15)

where \( t \) has been replaced by \( dx \). From figure 4 we see that \( s-x = R-R_p \),
where $R$ is the residual range. Substituting this in equation (15) and changing the limits of integration we get,

$$<r^2> = \int_{R_F}^{R_0} (R-R_F)^2 \left[ .157 z^2 z^2 B / (A p^2 \beta^2) \right] dR$$

$$= K \int_{R_F}^{R_0} (R-R_F)^2 \left[ B / (p^2 \beta^2) \right] dR.$$  \hspace{1cm} (16)

For heavy charged particles $p\beta$ can be well described as a function of the residual range $R$ by a simple power law,

$$p\beta = C_1 R^2$$

with the numerical evaluation of the constants given in Appendix A. Now we must find a suitable expression for $B$. One approach is to average $B$ over the entire range and give it a constant value and then integrate equation (16). This is undesirable in our case since we will be dealing with pions of varying ranges $R_0$. We have found that $B$ can also be well approximated as a simple function of the residual range, as shown in Appendix A, to be

$$\overline{B} = C_3 R^4$$

where $\overline{B}$ indicates we have averaged over the various atomic constituents of the scatterer. Thus equation (16) may be written

$$<r^2> = \int_{R_F}^{R_0} (R-R_F)^2 K \left( C_3 / C_1 \right)^2 R^{-2C_2} + C_4 dR,$$

$$= K' \int_{R_F}^{R_0} (R-R_F)^2 R^{-2c'} dR,$$
and integrated to yield

$$\langle r^2 \rangle = \frac{K'R_0}{3-2c'} \left[ 1 - \frac{(3-2c')}{(1-c')} \left( \frac{R_F}{R_0} \right) + \frac{(3-2c')}{(1-2c')} \left( \frac{R_F}{R_0} \right)^2 \right]$$

$$+ \left( \frac{R_F}{R_0} \right)^{3-2c'} \left( \frac{(3-2c')}{(1-c')} - \frac{(3-2c')}{(1-2c')} - 1 \right) .$$  \hspace{1cm} (17)

This is evaluated numerically in Appendix A.

We have chosen to write the expression for the Gaussian as

$$N(r,x) = \frac{(2\pi(\sigma_{ms}^2)^{-1}}{2} \exp \left[ -r^2/(2(\sigma_{ms}^2)^2) \right],$$  \hspace{1cm} (18)

where $\sigma_{ms} = \sigma(x)$ and $x = R_0 - R_F$. This is normalized in two dimensions and $\sigma_{ms} = \langle r^2 \rangle/2^{1/2}$, corresponding to Fowler and Perkins, $D_{\text{proj}}$.

We have so far been considering pencil beams of particles. In the next section we will extend the Gaussian theory [equation (18)] to finite field sizes, with the aim of the eventual extraction of the pencil beam value for $\sigma$ from the distributions that result for finite fields.

### 2.3 Finite Size Beams

Before dealing with experimental conditions, the description of multiple scattering that we have developed will be extended to some finite fields of simple shape.

In the case of proton scattering, the transition to large fields has been investigated for beam cross sections of regular (22) and arbitrary geometry (27). The methods involve the convolution in two dimensions of the pencil beam (Gaussian) description with the given
field shapes and result in numerical computations for the distributions at depth in the scatterer.

Preston and Koehler (22) considered three primary geometrical shapes: circular, rectangular and infinite half plane. For the first two shapes they primarily dealt with special cases, almost exclusively with the intensity on the axis. Only in the case of the infinite straight edge do they calculate the general intensity off the axis. They do, however, demonstrate the problem of small field sizes and significant multiple scattering and give a formula of interest in our case.

The intensity on the axis at a depth \( x \), with standard deviation \( \sigma_{ms}(x) \), for a circular collimated beam of radius \( r_c \) is given by

\[
I(0,x) = 1 - \exp \left[ -\frac{r_c^2}{(2\sigma_{ms})^2} \right].
\]  

(19)

This decrease in central axis intensity is due to the scattering out of the beam from the axis. This becomes an important consideration for the case of pions at the end of their range for small treatment volumes, as indicated in Chapter 1.

We have settled on a circular beam shape for the greatest part of this study as a result of a number of considerations which include:

(i) symmetry

(ii) ease of machining

(iii) analogy to pencil beams.

The application to an infinite straight edge is outlined in Appendix B.

For the case of an incident uniform and parallel beam, the problem of determining the distribution, at a depth behind a circular
opening, can be regarded as the convolution (in two dimensions) of a Gaussian (multiple scatter) distribution with the shape of the opening, projected to the depth of interest. The geometrical arrangement is presented in figure 5 for the contribution from an element Q on the surface to the distribution at a point P (with position \((r, \theta, x)\) in cylindrical polar coordinates). The distribution at P is the sum of contributions Q from the entire circular cross section at the surface. Mathematically this can be written

\[
f(r, \theta, x) = \int_0^{2\pi} \int_0^\infty A(r', \phi, 0) G(r_p, \theta + \phi, x) \ r' \ dr' \ d\phi, \tag{20}
\]

where for our case \(A(r', \phi, 0) = 1 \ r < r_c\),

and

\[
G(r_p, \theta + \phi, x) = (2\pi(\sigma_{ms})^2)^{-1} \ \exp[-r_p^2 / (2(\sigma_{ms})^2)],
\]

where

\[
r_p^2 = r^2 + r'^2 - 2rr' \cos(\theta + \phi).
\]

Circular symmetry allows us to choose \(\theta = 0\) and simplify the integral to

\[
f(r, x) = \int_0^{r_c} \int_0^{2\pi} (2\pi(\sigma_{ms})^2)^{-1} \ \exp[-r^2 - r'^2 + 2rr' \cos\phi/(2(\sigma_{ms})^2)] \ r' \ dr' \ d\phi \tag{21}
\]

where \(\sigma_{ms} = \sigma_{ms}(x)\) as before.

We have decided, that rather than evaluate \(f(r, x)\) using equation (21), we would follow Molière's example and use the well known fact [see, for example, Bracewell (29)] that convolution in one domain is equivalent to multiplication in the Fourier transformed domain. This technique is frequently used in data processing (e.g. for seismic geophysical data) as a time saving step.
FIGURE 5

SCHEMATIC DIAGRAM OF CONTRIBUTION TO
THE DISTRIBUTION FOR A POINT AT
DEPTH \( x \) FROM AN ELEMENT AT THE SURFACE
The use of a circular collimator and the resultant circular symmetry allows the conventional two-dimensional Fourier transform to reduce to the Fourier-Bessel transform [see Bracewell (29)]

\[ F(s) = 2\pi \int_0^\infty f(r) J_0(2\pi rs) r \, dr, \]  

(22)

and the inverse transform is given by

\[ f(r) = 2\pi \int_0^\infty F(s) J_0(2\pi rs) s \, ds. \]  

(23)

As may be easily shown (29), the transform, \( F_c(s) \), of a circle of radius \( r_c \) is given as

\[ F_c(s) = (r_c/s) J_1(2\pi r_c s), \]

where \( J_1 \) is the first order Bessel function and where

\[ f_c(r) = 1 \quad r < r_c. \]

The transform of a Gaussian such as that of the multiple scattering

\[ g(r) = (2\pi \sigma^2)^{-1} \exp(-r^2/(2\sigma^2)), \]

is given by

\[ G(s) = \exp(-2\pi^2 \sigma^2 s^2). \]

Using these facts, the transform of \( f(r,x) \) denoted by \( F_f \) is given by

\[ F_f = G(s) F_c(s) = (r_c/s) J_1(2\pi r_c s) \exp(-2\pi^2 \sigma^2 s^2). \]

Thus using equation (23), the radial distribution \( f(r,x) \) is given by

\[ f(r,x) = 2\pi r_c \int_0^\infty J_0(2\pi rs) J_1(2\pi r_c s) \exp(-2\pi^2 \sigma^2 s^2) \, ds, \]  

(24)

which involves only a single integration. The computation of this
expression is accomplished using an integration routine (QINF) available on the UBC computer and the Bessel functions are evaluated using polynomial approximations listed in Abramowitz and Stegun (30). The calculation using equation (24) is approximately twice as fast as that using equation (21). The evaluation of equation (24) gives good agreement for \( r = 0 \) with the values calculated using equation (19). It also may be shown to reduce to the appropriate form for \( f(r,0) \) using properties of integrals of Bessel functions found in reference (30), although the numerical computation of equation (24) for small values of \( \sigma \) requires some care. It should be noted that for \( \sigma = 0 \), \( f(r,0) = 1 \) for \( r < r_c \) as expected and the normalization integral is given by

\[
\int_0^\infty f(r,x) 2\pi r dr = \pi r_c^2.
\]

Using this calculation for \( f(r,x) \) we have shown, in figure 6, the resultant distributions that illustrate the effects of multiple scattering in relation to the collimator hole radius. We have plotted \( f(r') \) versus \( r' \) for various values of \( \sigma' \) using the reduced parameters \( \sigma' = \sigma/r_c \) and \( r' = r/r_c \). This reduction in parameters allows the convenient plotting of radial distributions without reference to any particular value of \( \sigma \) or radius \( r_c \).

This section has assumed that the incident beam of particles is parallel and spatially uniform in intensity. As will be discussed in the next chapter, the preliminary analysis of the experimental data had indicated that the beam was not parallel. However, for some restricted conditions discussed in Appendix C, it is possible to make a modification in equation (24) that permits it to be used in these cases also [equation (C-1)].
CALCULATED DISTRIBUTIONS FOR GAUSSIAN SCATTER WITH CIRCULAR APERTURE FOR SELECTED VALUES OF $\sigma'$ (DESCRIBED IN REDUCED PARAMETERS)
3. EXPERIMENTAL CONSIDERATIONS

3.1 Introduction

In this chapter we will discuss the relevant details of the experimental setup for determining the value of $\sigma_{ms}$ in the multiple scattering formalism. We begin by looking at film as a detector, then discuss the characteristics of the M8 biomedical channel at TRIUMF along with the experimental setup and then present the results of the film measurements.

3.2 Film as a Detector

The mention of film for use in quantitative measurements of radiation fields usually evokes feelings such as those described by Dutreix (31) with regard to film dosimetry, "Film dosimetry does not usually raise great enthusiasm among physicists, and conversely, it sometimes generates too much confidence among radiotherapists". However, as she goes on to point out, "Experience shows that film dosimetry is an excellent practical method particularly for high energy photons and electrons". The prime value is in relative, rather than absolute measurements.

In the detection of heavy charged particles the standard technique is to use nuclear emulsions and analyze the resulting particle tracks with high magnification microscopes. This method was used in the early detection of pions and determination of the frequency and energy spectra of particles emitted from pion stars (32). This technique has also been used in microdosimetric studies on pion beams (33), however, we will be using measurements of optical density to determine
the lateral distributions of pions in a water phantom. A detailed
description of film and the photographic process, including sensitometry,
is given by Mees and James (34). We will define only the term optical
density and give a description of characteristic curves in this discussion.

The formation of latent image centres in the silver halide
grains, by the deposition of energy (through excitation and ionization
loss of passing charged particles), and subsequent chemical development
produce grains of metallic silver in the emulsion layer. These blackened
grains absorb and scatter incident light and the macroscopic effect is
to attenuate the light intensity exponentially such that (35),

\[ I_t = I_o \exp(-n\sigma_b x). \]

Here \( I_o \) refers to the incident intensity, \( I_t \) the transmitted intensity,
\( x \) is the emulsion thickness, \( n \) is the number of blackened grains per
unit volume and \( \sigma_b \) is the total cross section for scattering and
absorption of light. The ratio \( I_t/I_o \) is called the transmittance \( T \),
and the optical density \( (D \) or O.D.) is defined by

\[ D = \log_{10} \left( \frac{1}{T} \right). \]

A plot of the net density (total minus background) against the relative
exposure is called the characteristic curve. The curve for x rays and
charged particles is typically a straight line through the origin for
low densities whereas for light it has a characteristic "toe" at low
densities (see figure 7). This has been attributed to the fact that
a single quantum of x-radiation or a single charged particle can render
a grain developable.
FIGURE 7

TYPICAL FILM CHARACTERISTIC CURVES FOR LIGHT AND IONIZING RADIATION (x RAYS OR CHARGED PARTICLES)

[adapted from Mees and James (34)]
The role of film in radiation dosimetry has been reviewed by Dudley (36) who lists some of the advantages of film as

(i) wide dose range (sensitivity)
(ii) large time scale (integrating detector)
(iii) large areas
(iv) high spatial resolution.

He also lists some disadvantages as

(i) dependence on LET
(ii) dependence on dose rate
(iii) dependence on environment and processing.

The dependence on dose rate is not a problem in the range of dose rates at which we are operating. The environment and processing problems can be minimized by careful handling with regard to temperature and humidity and with the use of automated film processors to reduce the nonuniformity of development.

The chief disadvantage appears to be the variation of response with respect to LET as shown by Tochilin et al. (37). The films that they used (with the exception of nuclear emulsion) showed little or no increase in optical density with depth, even when located in the Bragg peak of a charged particle beam, where the specific ionization (as measured with a parallel plate ionization chamber) was increasing rapidly with depth. From these observations they concluded that each particle deposits enough energy to render a film grain developable and any extra energy deposited (due to increased LET) is wasted. Hence for a given dose, as the LET increases the dose sensitivity decreases,
where dose sensitivity is inversely proportional to the exposure required to produce a given density. Tochilin et al. state that at the energies in their study, film response was primarily a record of particle flux. Similar results using more modern emulsions have been quoted by Dutreix (31).

The previous multiple scattering experiments (22,27) employed a diode as the detector. However, for the low dose rates available at the time of our experiments, a silicon diode was shown to be unsuitable. In order to get the spatial resolution required we elected to use medical x-ray film (Kodak XM2). The use of film allowed for very efficient use of available beam time, in that all the information at one depth could be collected on one film. In addition the film holder design allowed the exposure of up to three films for a particular experiment, if desired.

In view of the observation of Tochilin et al. mentioned earlier, we treated the films as "particle counters". The effects of LET dependence should be reduced by placing the films perpendicular to the incident beam direction, so that each point on the film is subject to approximately the same LET spectrum. A benefit in using film as a particle counter was that it allowed direct comparison with the theory, since the theoretical calculations produced number distributions rather than dose distributions.

3.3 Beamline Characteristics and Setup

The M8 biomedical beamline at TRIUMF has been described by Harrison (38,39) and Henkelman et al. (7). It consists of an achromatic system of nine electromagnets comprising two dipoles, five quadrupoles,
PION RADIOThERAPY BEAM LINE AT TRIUMF

FIGURE 8

SCHEMATIC DIAGRAM OF M8

BIOMEDICAL CHANNEL AT TRIUMF
and two sextupoles over a length of 7.5 meters (figure 8). In the middle of the third quadrupole there is a dispersion plane and two movable blades for selecting a spread of momentum (from $\pm 1.5\% \frac{\Delta p}{p}$, at half maximum, to $\pm 6.7\% \frac{\Delta p}{p}$). As in any pion channel there is the inevitable contamination of muons ($\mu$) and electrons ($e$). This contamination has been investigated using a time of flight system (6,40,41). The muons arise from the decay of pions in-flight, whereas the electrons are produced predominantly in the target, due to pair production by the $\gamma$ rays which arise from the decay of neutral pions produced along with the charged pions. The relative contribution of contaminants to the beam flux has been shown to be dependent on target type and channel momentum (40). In addition, it appears that the relative steering of the proton beam on the target has an effect on the relative yield of $\pi^-$ and the $e^-$ contamination. This is thought to be due to greater or lesser thicknesses of target for the secondary beam to traverse and is a significant effect as shown by figure 9.

Another effect of importance is the loss of $\pi^-$ from the beam due to in-flight interactions. This has been measured in a water phantom using integral range curves (42) employing large plastic scintillators and yields a value of 1.62%/cm.

In order to reduce the effects of multiple scatter before the patient treatment area, the beam is transported from the target in a vacuum system that extends up to the last quadrupole. After the vacuum window there is a transmission ionization chamber, then a variable length of air path ($\approx 1$ meter, depending on beam tunes) before the experimental apparatus which is mounted on a motor driven table. The
FIGURE 9

EXAMPLE OF RELATIVE PARTICLE FRACTIONS PLOTTED AGAINST MOMENTUM SHOWING EFFECT OF PROTON BEAM STEERING

[from Poon (40)]
physical layout is depicted in figure 10 and the thicknesses and conversions to equivalent depth in water are listed in table II.

The film-holding apparatus consisted of three Perspex film envelopes, each capable of holding an 8 x 10 inch "ready pack" film. These envelopes were then suspended from a frame attached to the water phantom. The frame allowed adjustment of the films to any depth in the phantom (except for the first 2.0 cm from the front). This setup for the multiple scattering experiments is illustrated in figure 10. The repositioning error for the film envelopes has been estimated to be about ± .1 cm and the reproducibility in setting the depths and lateral position of the frame as ± .2 cm. For simplicity in sharing the water phantom with other experiments a single collimator was located just before the water tank, rather than a two collimator system. The collimator was brass, two inches thick and allowed the use of inserts with hole diameters up to 7/8 inch. The repositioning error of the collimator system was estimated to be ± .3 cm.

The presence of scattering material, such as scintillators, etc., before the collimator and the wall of the water tank after the collimator, makes the attainment of a parallel beam by collimation clearly impossible in the present setup. A request was made of the beam tuning group for a broad, spatially uniform beam, as nearly parallel as possible. However, when dealing with charged particle beams of finite emittance (43) it is theoretically impossible to achieve a completely parallel beam. In addition, the focussing and defocussing characteristics of magnetic lenses make achieving even a practically parallel beam difficult. It turned out that the beam envelope was essentially parallel in one direction (vertical, y')
FIGURE 10

SCHEMATIC DIAGRAM OF EXPERIMENTAL SETUP IN PLAN VIEW

(MULTIPLE SCATTERING EXPERIMENTS)
TABLE II

CALCULATION OF WATER EQUIVALENT THICKNESS
FOR MATERIALS IN FRONT OF WATER

<table>
<thead>
<tr>
<th>SOURCE</th>
<th>THICKNESS cm</th>
<th>DENSITY g/cm³</th>
<th>THICKNESS g/cm²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum Window</td>
<td>0.024</td>
<td>1.19</td>
<td>0.029</td>
</tr>
<tr>
<td>Transmission Chamber</td>
<td>0.008</td>
<td>1.19</td>
<td>0.010</td>
</tr>
<tr>
<td>T. Chamber Cover</td>
<td>0.015</td>
<td>1.19</td>
<td>0.018</td>
</tr>
<tr>
<td>Scintillators</td>
<td>0.635</td>
<td>1.032</td>
<td>0.655</td>
</tr>
<tr>
<td>Covers for Scint.</td>
<td>0.090</td>
<td>1.19</td>
<td>0.107</td>
</tr>
<tr>
<td>MWPC</td>
<td>0.010</td>
<td>1.19</td>
<td>0.012</td>
</tr>
<tr>
<td>Plastic Tank</td>
<td>1.240</td>
<td>1.19</td>
<td>1.476</td>
</tr>
</tbody>
</table>

Total thickness in g/cm² 2.307
Multiply by $S_{\text{Perspex}}/S_{\text{H}_2\text{O}}$ 0.971
Total water equivalent thickness in g/cm² 2.240
and divergent in the perpendicular (horizontal, x') direction. The divergence, while not negligible, appeared to be small enough to satisfy the conditions of Appendix C.

3.4 Experimental Results

Measurements were carried out at two channel momenta: the first series at 148 MeV/c corresponding to a pion range in water of 13.5 cm and the second series, three months later, at 180 MeV/c corresponding to a pion range in water of 22.3 cm. The beam tune for the 180 MeV/c experiments was derived by scaling the magnet settings for the 148 MeV/c tune by the ratio 180/148. This should give some correspondence between beam tunes, but due to the decrease in electron contamination at higher momentum (22% at 180 MeV/c versus 43% at 148 MeV/c) and the time separation between experiments, it is advisable to regard them as two completely different beam tunes.

This section will be divided into two subsections, first the preliminary and ancillary experiments followed by the multiple scattering experiments.

3.4.1 Preliminary Experiments

Experiments that fall into this category are exposure tests, calibration curves, and films parallel to the beam direction.

Exposure tests were carried out at each midline momentum (with "wide blades", i.e. ± 6.7% \( \Delta p/p \)) by exposing films at different depths in the water phantom for a series of "monitor counts". The "monitor counts" refer to ionization collected in the transmission
ionization chamber, mounted on the last quadrupole and used as a beam monitor. The current was integrated and converted to output pulses, each corresponding to a fixed amount of charge, which were counted by a preset scaler. The accumulation of a preset number of counts automatically terminated the exposure by causing the beam stop to be inserted. The films were developed in a standard medical x-ray auto-processor within one to two days after exposure. It should be pointed out that the film processor was changed between the 148 MeV/c experiment and the 180 MeV/c experiment.

Optical density measurements were obtained using a scanning densitometer (Kipp and Zonen, model DD 691-E) with a light slit approximately .02 x .4 cm in size. The densitometer signal was amplified and recorded on a strip chart recorder (Hewlett-Packard, model 680-M). Densities for all experiments included base and fog densities. Graphs of film density against relative exposure are plotted for the 148 MeV/c and 180 MeV/c runs in figures 11 and 12 respectively. They show the linear curve predicted (34) in the density range we have used. The intercept corresponds to the background density of the film and the difference between the two backgrounds is probably due to emulsion changes and different processing conditions. The different slopes for different depths will be discussed in section 4.3.4. Dutreix (31) gives some figures on the expected accuracy when comparing optical densities and these are listed in table III.
Figure 11

EXPOSURE TEST FOR 148 MEV/c CASE
FIGURE 12

EXPOSURE TEST FOR 180 MEV/c CASE
TABLE III

EXPECTED ACCURACY FOR COMPARING
OPTICAL DENSITIES [Dutreix (31)]

<table>
<thead>
<tr>
<th>Comparison of Optical Density</th>
<th>Expected Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>On same film</td>
<td>2%</td>
</tr>
<tr>
<td>On films processed simultaneously</td>
<td>3%</td>
</tr>
<tr>
<td>On films processed separately (identical processing condition)</td>
<td>5%</td>
</tr>
<tr>
<td>On films of different batches</td>
<td>?</td>
</tr>
</tbody>
</table>

This may explain the variation of background between experiments, and points out the need for care in using film of the same batch when making comparisons between films. Each multiple scattering experiment (next section) used films from the same box so that the variations of optical density from film to film would be expected to be less than 5%. The relative measurements of the distributions on each film could be expected to show less than 2% variation without taking account of densitometer variations.

In order to test the reproducibility of the densitometer we periodically scanned a calibrated test strip (diffuse density) obtained from Kodak. The densities measured on our densitometer were not numerically equal to those given with the test strip due to the differing geometries of light collection (34). However, the requirement of reliability, as shown in figure 13, is well satisfied in the lower density range. These are results taken over a time span of a few months and
FIGURE 13
DENSITOMETER RESPONSE
(TEST OF REPRODUCIBILITY)
the indicated ranges represent the minimum and maximum values measured. As a result of this data, we elected to use an exposure of 2100 monitor counts (measured O.D. approximately 1.2) for our multiple scatter experiments to stay in the linear portion of the densitometer response.

In addition to the exposure tests, we exposed three films parallel to the beam direction. These films were located approximately 3 to 4 cm inside the water phantom and were in fact angled approximately 5° from the central axis of the beam to try to overcome problems due to air gaps surrounding the film. Dutreix (31) has pointed out the need for care in exposing films parallel to 20–25 MeV electron beams and it is assumed that the same problems apply here. In general, the results exhibit a level plateau at the beginning of the film, followed by a rise in O.D. near the pion stopping peak, followed by a decline to a lower level again after the peak. The results are summarized in table IV.

<table>
<thead>
<tr>
<th>Momentum MeV/c</th>
<th>Exposure Monitor Counts</th>
<th>Δp/p %</th>
<th>Optical Density (including base and fog)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Plateau</td>
</tr>
<tr>
<td>148</td>
<td>1500</td>
<td>6.7</td>
<td>0.75</td>
</tr>
<tr>
<td>180</td>
<td>2100</td>
<td>6.7</td>
<td>1.02</td>
</tr>
<tr>
<td>180</td>
<td>2100</td>
<td>2.0</td>
<td>0.85</td>
</tr>
</tbody>
</table>

The increase in O.D. (plateau to peak) ranges from a low of 2% to a high of 13%. In spite of the associated problems mentioned by Dutreix and the
LET dependency, the interpretation of these results indicate the behaviour of the film to be primarily as a "particle counter".

3.4.2 Multiple Scatter Experiments

In the multiple scatter experiments we used the setup illustrated in figure 10. Collimator aperture sizes of .50 cm radius and 1.125 cm radius were used. A number of experiments were performed for both hole sizes and various depths in water (inside tank wall to film plane). These are summarized in table V.

<table>
<thead>
<tr>
<th>Momentum</th>
<th>$\Delta p/p$</th>
<th>Hole Radius</th>
<th>Depths in Water</th>
<th>Number of Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>MeV/c</td>
<td>%</td>
<td>cm</td>
<td>cm</td>
<td>cm</td>
</tr>
<tr>
<td>148</td>
<td>6.7</td>
<td>1.125</td>
<td>2.4</td>
<td>11.3</td>
</tr>
<tr>
<td>180</td>
<td>6.7</td>
<td>1.125</td>
<td>7.55</td>
<td>15.1</td>
</tr>
<tr>
<td>180</td>
<td>6.7</td>
<td>0.50</td>
<td>7.55</td>
<td>15.1</td>
</tr>
<tr>
<td>180</td>
<td>2.0</td>
<td>1.125</td>
<td>7.55</td>
<td>15.1</td>
</tr>
<tr>
<td>180</td>
<td>2.0</td>
<td>0.50</td>
<td>7.55</td>
<td>15.1</td>
</tr>
<tr>
<td>180</td>
<td>6.7</td>
<td>Straight edge</td>
<td>7.55</td>
<td>15.1</td>
</tr>
</tbody>
</table>

For each experiment successive runs were made with and without water in the tank, to determine the scattering effect of the water. After development each film was scanned on the densitometer in two perpendicular directions, coinciding with the $x',y'$ directions of the beam mentioned earlier. This resulted in four densitometer scans for
each depth for each experiment. For the circular apertures listed in table V there were 88 profiles in all to be analyzed.

Rather than digitize each profile, a convenient way of describing the resulting distributions in optical density was sought. The procedure chosen was to take the full width of the profile at the 75%, 50% and 25% levels of the net density distribution (total minus background, determined from each profile). These values were then divided by the diameter of the aperture which resulted in normalized parameters $Y_{75}$, $Y_{50}$ and $Y_{25}$. These parameters were compared with theoretical values for distributions calculated using equation (24) for various values of $\sigma'$. 

Preliminary attempts to analyze the data indicated that it was not correct to assume that the beam was parallel. As mentioned in section 3.3 the beam diverges in one direction, resulting in an approximately elliptical shape for the distributions at depth. A first approximation for this situation (details are given in section 4.3.5 and Appendix C) can be made by assuming a point source divergence. This assumes that each densitometer profile is the result of scatter from a circle with a magnified (or decreased) radius. This magnification at depth is described by an additional parameter, $A$. It should be noted that in this description, the value of $A$ for the two axes of the ellipse on the same film is not expected to be the same.

Using equation (C-1) with parameters $A$ and $\sigma'$, new values for $Y_{75}$, $Y_{50}$, $Y_{25}$ were calculated and in figure 14 these parameters are plotted against $\sigma'$ for three values of $A$. It can be seen that the variation of the $Y$ parameters with $\sigma'$ is complicated but there are at
FIGURE 14

VARIATION OF Y PARAMETERS WITH SCATTER

PARAMETER $\sigma^*$ FOR VALUES OF MAGNIFICATION A
least two regions which can be distinguished. For the region of \( \sigma' < .4 \), the main effect is due to the divergence while in the region for \( \sigma' > .6 \), the increasing lateral scatter (\( \sigma \)) causes the effect of divergence on \( \Lambda \) to be masked. The effects of \( \Lambda \) on \( \sigma' \) are discussed later in Chapter 4.

In order to search the two parameter space in a systematic fashion, a computer program was written based on a subroutine called CURFIT (28), to determine the values of \( \Lambda \) and \( \sigma' \) that gave best agreement with the experiment. The outline of the program is as follows:

(i) Assume \( \Lambda \), \( \sigma' \), calculate \( Y_{75}', Y_{50}', Y_{25}' \).

(ii) Compute chi-square parameter with experimental point.

(iii) Step in a new direction until chi-square decreases by calculating new \( \Lambda \), \( \sigma' \) and hence new \( Y_{75}', Y_{50}', Y_{25}' \).

(iv) When variation in chi-square is less than some number, like 10% of previous value, stop search.

The results of the determinations of \( \sigma_w \), \( \sigma_{wo} \) and \( \sigma_{ms} \) are given in tables VI, VII and VIII. Here \( \sigma_w \) is the value of \( \sigma \) determined for water in the tank, \( \sigma_{wo} \) is the value for no water in the tank and \( \sigma_{ms} \) the resultant multiple scatter value. Note that in the tables \( \sigma' \) has been corrected for the respective hole sizes to give \( \sigma \), e.g. \( \sigma_w = \sigma_w' r_c \).

The averages and total uncertainties are calculated using formulas given by Bevington (28)

\[
\bar{\sigma} = \frac{\sum (\sigma_i / \nu_i^2)}{\sum (1/\nu_i^2)}
\]

\[
\Delta \sigma^2 = (\sum (1/\nu_i^2))^{-1}
\]

This has the effect of giving more weight to the values with the smaller
<table>
<thead>
<tr>
<th>RADIUS cm</th>
<th>DEPTH cm</th>
<th>PROFILE</th>
<th>$\sigma_w$ cm</th>
<th>$\sigma_{wo}$ cm</th>
<th>$\sigma_{ms}$ cm</th>
<th>AVERAGE $\sigma_{ms}$ cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.125</td>
<td>2.4</td>
<td>x'</td>
<td>.133±.004</td>
<td>.124±.004</td>
<td>.048±.015</td>
<td>.052±.011</td>
</tr>
<tr>
<td></td>
<td></td>
<td>y'</td>
<td>.172±.003</td>
<td>.163±.004</td>
<td>.055±.015</td>
<td></td>
</tr>
<tr>
<td>1.125</td>
<td>11.3</td>
<td>x'</td>
<td>.542±.013</td>
<td>.340±.005</td>
<td>.422±.017</td>
<td>.430±.016</td>
</tr>
<tr>
<td></td>
<td></td>
<td>y'</td>
<td>.630±.040</td>
<td>.396±.006</td>
<td>.490±.052</td>
<td></td>
</tr>
<tr>
<td>RADIUS cm</td>
<td>DEPTH cm</td>
<td>PROFILE</td>
<td>$\sigma_w$ cm</td>
<td>$\sigma_{wo}$ cm</td>
<td>$\sigma_{ms}$ cm</td>
<td>AVERAGE $\sigma_{ms}$ cm</td>
</tr>
<tr>
<td>------------</td>
<td>----------</td>
<td>---------</td>
<td>---------------</td>
<td>-------------------</td>
<td>------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>1.125</td>
<td>7.55</td>
<td>$x'$</td>
<td>0.306±.004</td>
<td>0.205±.003</td>
<td>0.227±.006</td>
<td>0.220±.005</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>0.341±.004</td>
<td>0.270±.004</td>
<td>0.208±.008</td>
<td></td>
</tr>
<tr>
<td>15.1</td>
<td></td>
<td>$x'$</td>
<td>0.680±.068</td>
<td>0.362±.006</td>
<td>0.576±.080</td>
<td>0.582±.059</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>0.727±.069</td>
<td>0.427±.009</td>
<td>0.588±.086</td>
<td></td>
</tr>
<tr>
<td>20.1</td>
<td></td>
<td>$x'$</td>
<td>1.075±.129</td>
<td>0.431±.005</td>
<td>0.985±.141</td>
<td>0.977±.127</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>1.096±.253</td>
<td>0.562±.012</td>
<td>0.941±.295</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>7.55</td>
<td>$x'$</td>
<td>0.303±.024</td>
<td>0.214±.004</td>
<td>0.215±.034</td>
<td>0.228±.028</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>0.352±.036</td>
<td>0.241±.007</td>
<td>0.257±.050</td>
<td></td>
</tr>
<tr>
<td>15.1</td>
<td></td>
<td>$x'$</td>
<td>0.756±.134</td>
<td>0.412±.004</td>
<td>0.634±.160</td>
<td>0.632±.141</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>0.761±.241</td>
<td>0.434±.040</td>
<td>0.625±.295</td>
<td></td>
</tr>
<tr>
<td>20.1</td>
<td></td>
<td>$x'$</td>
<td>0.984±.111</td>
<td>0.497±.050</td>
<td>0.849±.132</td>
<td>0.843±.117</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>0.997±.209</td>
<td>0.574±.033</td>
<td>0.819±.256</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE VIII

**EXPERIMENTAL VALUES OF $\sigma$ FOR 180 MEV/c**

**NARROW BLADE CASE (1 EXPERIMENT)**

<table>
<thead>
<tr>
<th>RADIUS cm</th>
<th>DEPTH cm</th>
<th>PROFILE</th>
<th>$\sigma_w$ cm</th>
<th>$\sigma_{wo}$ cm</th>
<th>$\sigma_{ms}$ cm</th>
<th>AVERAGE $\sigma_{ms}$ cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.125</td>
<td>7.55</td>
<td>$x'$</td>
<td>.308±.006</td>
<td>.206±.005</td>
<td>.229±.009</td>
<td>.218±.007</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>.324±.005</td>
<td>.253±.006</td>
<td>.202±.011</td>
<td></td>
</tr>
<tr>
<td>15.1</td>
<td></td>
<td>$x'$</td>
<td>.713±.096</td>
<td>.378±.009</td>
<td>.605±.113</td>
<td>.606±.099</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>.758±.163</td>
<td>.450±.020</td>
<td>.610±.203</td>
<td></td>
</tr>
<tr>
<td>20.1</td>
<td></td>
<td>$x'$</td>
<td>1.109±.179</td>
<td>.459±.011</td>
<td>1.010±.197</td>
<td>.957±.135</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>1.067±.156</td>
<td>.555±.033</td>
<td>.911±.184</td>
<td></td>
</tr>
<tr>
<td>0.50</td>
<td>7.55</td>
<td>$x'$</td>
<td>.266±.037</td>
<td>.203±.004</td>
<td>.170±.058</td>
<td>.225±.015</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>.340±.005</td>
<td>.251±.015</td>
<td>.229±.015</td>
<td></td>
</tr>
<tr>
<td>15.1</td>
<td></td>
<td>$x'$</td>
<td>.658±.011</td>
<td>.384±.086</td>
<td>.534±.063</td>
<td>.572±.048</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>.758±.008</td>
<td>.430±.107</td>
<td>.624±.074</td>
<td></td>
</tr>
<tr>
<td>20.1</td>
<td></td>
<td>$x'$</td>
<td>.995±.124</td>
<td>.448±.186</td>
<td>.888±.167</td>
<td>.866±.112</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$y'$</td>
<td>1.008±.120</td>
<td>.546±.080</td>
<td>.847±.152</td>
<td></td>
</tr>
</tbody>
</table>
uncertainties, $v_i$. Here the uncertainties, $v_i$, are the values returned by the program. These values of $v_i$ reflect how steep the chi-square surface is in the vicinity of the minimum. This depends to some extent on the value of the cutoff chosen in searching the two parameter space as well as the uncertainties of the $Y$ parameters input to the program. Some of the effects on the determination of $\sigma$ will be discussed in Chapter 4. $\sigma_{ms}$ is calculated by

$$\sigma_{ms} = [\sigma_w^2 - (\sigma_{wo})^2]^{1/2},$$

and the uncertainty by $\Delta\sigma_{ms} = \sigma_{ms}^{-1} \left[ (\Delta\sigma_w \sigma_w)^2 + (\Delta\sigma_{wo} \sigma_{wo})^2 \right]^{1/2}$.

In addition to the circular field data, one experiment using a straight edge aligned along the $y'$ direction was carried out. The results of this experiment, using equation (B-4) for the determination of $\sigma$, are given in table IX. An overall summary of the $\sigma_{ms}$ values for all field sizes is given in table X.

A discussion of the assumptions involved in the analysis will be given in the next chapter, however, it is well to note that the results at 180 MeV/c for the three field sizes are in substantial agreement with the exception of the deepest depth where there appears to be a systematic increase in $\sigma_{ms}$ with field size.
TABLE IX

EXPERIMENTAL VALUES OF $\sigma$ FOR 180 MEV/c

WIDE BLADE CASE (1 EXPERIMENT) STRAIGHT EDGE

<table>
<thead>
<tr>
<th>DEPTH cm</th>
<th>$\sigma_w$ cm</th>
<th>$\sigma_{wo}$ cm</th>
<th>$\sigma_{ms}$ cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.55</td>
<td>.302 ± .017</td>
<td>.195 ± .018</td>
<td>.231 ± .027</td>
</tr>
<tr>
<td>15.1</td>
<td>.724 ± .035</td>
<td>.344 ± .036</td>
<td>.637 ± .044</td>
</tr>
<tr>
<td>20.1</td>
<td>1.178 ± .053</td>
<td>.510 ± .036</td>
<td>1.062 ± .061</td>
</tr>
</tbody>
</table>

TABLE X

SUMMARY OF $\sigma_{ms}$ FOR ALL EXPERIMENTS

<table>
<thead>
<tr>
<th>MOMENTUM MeV/c</th>
<th>RANGE IN TANK cm</th>
<th>DEPTH cm</th>
<th>$\sigma_{ms}$ cm</th>
<th>$r_c$ = 0.50 cm</th>
<th>$r_c$ = 1.125 cm</th>
<th>Straight Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>148</td>
<td>11.3</td>
<td>2.4</td>
<td></td>
<td>0.052 ± .011</td>
<td>0.430 ± .016</td>
<td>0.231 ± .027</td>
</tr>
<tr>
<td></td>
<td></td>
<td>11.3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>20.1</td>
<td>7.55</td>
<td>0.226 ± .013</td>
<td>0.219 ± .010</td>
<td>0.588 ± .051</td>
<td>0.637 ± .044</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15.1</td>
<td>0.578 ± .045</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>20.1</td>
<td>0.855 ± .081</td>
<td>0.968 ± .093</td>
<td></td>
<td>1.062 ± .061</td>
</tr>
</tbody>
</table>
4. DISCUSSION

4.1 Introduction

In this chapter the assumptions made in the analysis of the experiments are presented, followed by a discussion of the experimental factors involved. These experimental factors affect both the analysis and the overall significance of the experiments and consist of the following:

1. effect of momentum spread,
2. effect of contaminating electrons and muons,
3. effect of background scatter,
4. film as a particle counter,
5. effect of non-parallel nature of beam.

These factors are discussed point by point and then the theory presented in Chapter 2 and Appendix A is compared to the experimental results. Finally, the values of $\sigma$ for both theory and experiment are compared to theories presented in the literature.

4.2 Assumptions in the Data Analysis

As mentioned in Chapter 3, the procedure for analyzing the optical density profiles of the films was based on a comparison of the three experimental points $Y_{75}$, $Y_{50}$, $Y_{25}$ on the profile, with those from calculated distributions based on equation (C-1) for values of the two parameters, $A$ and $\sigma'$. The following assumptions are involved in using the calculated distributions to analyze the experimental data.
(1) In both cases with and without water in the tank, the distributions are assumed to result from a simple Gaussian convolved with a circular aperture. This implies that the value of \( \sigma \) is the same at the three levels (i.e. 75%, 50%, 25%) of the resulting distribution. This assumption is violated by the effects of momentum spread and contamination which involve sums of Gaussian terms (discussed in sections 4.3.1, 4.3.2). In addition, the assumption is made that the background scatter (from before and after the collimator) is also Gaussian (or may be approximated by a Gaussian). Also, as mentioned in Chapter 2, the multiple scatter is assumed to be Gaussian in order that \( \sigma_{ms}^2 \) may be calculated as \( \left[ \sigma_w^2 - (\sigma_{wo})^{2\gamma} \right]^{1/2} \) (discussed in section 4.3.3).

(2) The calculated distributions are number distributions and the assumption is made that the film chosen behaves as a particle counter, particularly in the direction perpendicular to the beam axis (discussed in section 4.3.4).

(3) It is assumed that any non-parallel behaviour of the beam can be accounted for in the magnification parameter, \( A \). This assumes that it is possible to separate the effect of finite emittance into a point source term (zero emittance) described by the parameter \( A \) and a Gaussian term included in the background scatter (discussed in section 4.3.3). Finite emittance is defined as the area in phase space occupied by the beam, but in our case the effects are regarded as due only to a finite source size. The observed divergence in one direction means that an elliptical, rather than circular shape results
at depth, which involves some error in the evaluation of A (discussed in section 4.3.5). The last assumption made is that the point source is far enough from the collimator for the divergence to be small in angle, to allow the simple correction derived in Appendix C.

The extent to which the experimental factors influence these assumptions is given in the next section as each factor is discussed in detail.

4.3 Experimental Factors

4.3.1 Effect of Momentum Spread

This effect includes the effect of range straggling, but since the momentum spread is much larger only the case of momentum spread will be discussed.

The experimental data for the 180 MeV/c experiments with large momentum spread (wide blades, table VII) does not show any significant difference from that for a small momentum spread (narrow blades, table VIII). In order to see if this was reasonable, calculations were made to see what effects could be expected for two shapes of the momentum spectrum: (1) Gaussian and (2) Rectangular. These spectra \( N(p) \) were chosen to have equal areas and the same widths, i.e. \( \pm 7\% \frac{Δp}{p} \) at half maximum. The calculations used equation (17) [numerically (A-5)] for finding \( \sigma \) as a function of depth \( x \) for the ranges \( R_0(p) \) in the 180 MeV/c case. The resultant distribution for each of the two spectra was compared with that for a monoenergetic beam with the range 20.1 cm at three points; where the monoenergetic beam was predicted to have values of 75%, 50%, and 25% of the peak. As expected, the resultant
distribution, which was the sum of Gaussian terms, was not Gaussian. However, at each level it was possible to assign a value of $\sigma$ to give agreement. In doing this, the resulting deviation of $\sigma$ from the average value was less than 1% and the average values themselves were within 1% of the single momentum case, except for the depth corresponding to the mean range of 20.1 cm.

At this depth the spread of momentum leads to a calculated reduction in the value of $\sigma$ by 10% for the assumed Gaussian and 8% for the rectangular momentum spectrum. One possible reason for not observing this difference was that the narrow blade case was not truly mono-energetic, having a momentum spread of approximately $\pm$ 2%. However, calculations for this case indicate that there should still be a 7% difference between the wide and narrow blade Gaussian cases. This 7% spread is within the limits of the estimated error (tables VII and VIII), so it is not surprising that no systematic difference is observed between the two cases.

4.3.2 Effect of Contaminating Electrons and Muons

The contaminating electrons and muons, due to their independent scattering, should affect the resulting scattering distributions, particularly since it is assumed that the different particle types contribute to film blackening on the basis of number. Since it was impossible to separate the scattering effects of each particle in our experiments, a calculation was made to try to estimate the effects of the different particle types.

Due to the fact that electrons are such light particles,
calculations of electron penetration in a thick scattering medium are very difficult; the quantitative assessment of the stopping distributions involves Monte Carlo treatments or solutions of the transport equation, both beyond the scope of this discussion. Thus, we have arrived at a procedure for inferring the values of \( \sigma \) for the electrons. The calculations assumed the case of a single momentum with each particle type represented by a Gaussian scattering distribution weighted according to the fraction by number determined experimentally for the beamline (40). The procedure involved calculating the values of \( \sigma \) for the pions and muons at each depth using equations of the form of equation (17). The difference between the experimental values of \( \sigma_{\text{ms}} \) determined for the \( r_c = 1.125 \text{ cm} \) case and the pion and muon values calculated were attributed entirely to the scattering of the electrons. The values of \( \sigma \) for the electrons determined at each of the three positions (at the 75%, 50%, 25% levels, calculated using \( \sigma_{\text{ms}} \)) were different since again a sum of Gaussian terms was involved. The variation about the average \( \sigma \) for the three positions was typically 3% to 10% and this is larger than for the case of momentum spread.

The values required for the standard deviation, \( \sigma \), of the electron scattering are generally larger than the calculated pion values. Simple calculations for electrons [based on the Continuous Slowing Down Approximation (CSDA) data in reference (44)] yield values too small to account for the whole difference between experiment and the pion and muon calculations. Since accurate values of \( \sigma \) for electrons are difficult to calculate, we have not attempted to separate their contribution from the total measured value of \( \sigma \).
4.3.3 Effects of Background Scatter

This includes all factors which are common to both the "water in" and "water out" distributions. These are primarily scattering from structures before and after the collimator but also included is the finite emittance part of the beam as discussed in section 4.2.

It may be easily shown [especially using the convolution-multiplication properties of Fourier transforms (29)] that the convolution of two Gaussians, with standard deviations \( \sigma_1 \) and \( \sigma_2 \), is simply another Gaussian with \( \sigma_t^2 = \sigma_1^2 + \sigma_2^2 \). Thus if the no water and multiple scattering distributions are both Gaussian then \( \sigma_w^2 = (\sigma_{ms})^2 + (\sigma_{wo})^2 \) or as we have used it \( (\sigma_{ms})^2 = \sigma_w^2 - (\sigma_{wo})^2 \). We assume that the no water distribution accounts for all effects except the desired measurement. Our assumption is that this no water distribution is given by the convolutions of effects that are Gaussian or may be approximated by a Gaussian and hence the total distribution is Gaussian.

The two contributions, beam emittance and scatter from before the collimator, can be treated together since, as Banford (43) points out, scatter changes a zero emittance beam to a finite emittance beam. Under some specialized assumptions for a finite emittance beam (uniform density of elliptical shape in phase space) the beam profile in one direction can be described by \( 1 - x^2/a^2 \) for \( y = 0 \). This can be fitted by a Gaussian with some adjustment in the value of \( \sigma \) at the three measurement levels. The variation is approximately 10% from the average value of \( \sigma \). The effect of this variation on \( \sigma_{ms} \) will be presented after the following discussion.
The last consideration is that of scatter from the water tank wall after the collimator. If the thickness of the Perspex wall is replaced by the equivalent thickness in water (table II), i.e. 1.43 cm, then the effect of this scatter can be calculated for the various film depths by modifying equation (16). If one changes the limits to \( R_0 - \delta \) and \( R_0 \) for the lower and upper limits respectively, then one obtains on integration

\[
\langle r^2 \rangle = \frac{K^* R_0^3 \cdot 2c'}{(3-2c')} \left[ 1 - \left(1 - \frac{\delta}{R_0}\right)^{3-2c'} + 2 \left(\frac{2c'}{3-2c'}\right) \left(\frac{R_F}{R_0}\right) \left[ 1 - \left(1 - \frac{\delta}{R_0}\right)^{2-2c'} \right] - 1 \right]
\]

\[+ \left(\frac{3-2c'}{1-2c'}\right) \left(\frac{R_F}{R_0}\right)^2 \left[ 1 - \left(1 - \frac{\delta}{R_0}\right)^{1-2c'} \right]. \tag{25}\]

This is evaluated, using values of \( K^* \) and \( c' \) given in Appendix A, for \( R_0 = 21.5 \text{ cm} \) (20.1 + 1.4) and \( \delta = 1.4 \text{ cm} \) for the various depths (+ 1.4 cm) of the experiment in table XI. Here \( \sigma_{\text{wt}} = \langle \langle r^2 \rangle / 2 \rangle^{1/2} \), and \( \sigma_{\text{wt}} \) is the value for scatter from the water tank only.

**TABLE XI**

**COMPARISON OF \( \sigma_{\text{wt}} \) WITH CALCULATED CONTRIBUTIONS FROM THE WATER TANK WALL FOR \( R_0 = 21.5 \) AND \( \delta = 1.4 \)**

<table>
<thead>
<tr>
<th>Depth (+1.4) cm</th>
<th>( \sigma_{\text{wt}} ) (Eq. 25)</th>
<th>( \sigma_{\text{wo}} ) (minimum-maximum)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.95</td>
<td>.136</td>
<td>.203 - .270</td>
</tr>
<tr>
<td>16.5</td>
<td>.261</td>
<td>.362 - .450</td>
</tr>
<tr>
<td>21.5</td>
<td>.343</td>
<td>.43 - .58</td>
</tr>
</tbody>
</table>
The values for \( \sigma_{wt} \) and \( \sigma_{wo} \) from table XI can be used in conjunction with the values of \( \sigma_{ms} \) (table X) to determine the effect of the 10% variation in \( \sigma_{fe} \) (finite emittance) on \( \sigma_{ms} \). Considering only the variation in \( \sigma_{fe} \), it may easily be shown that if \( (\sigma_{wo})^2 = (\sigma_{fe})^2 + (\sigma_{wt})^2 \) then

\[
\frac{\Delta \sigma_{ms}}{\sigma_{ms}} = \frac{(\sigma_{wo})^2 - (\sigma_{wt})^2}{(\sigma_{ms})^2} \frac{\Delta \sigma_{fe}}{\sigma_{fe}}.
\]

This results in variations for \( \sigma_{ms} \) generally less than the 10% variation proposed for \( \sigma_{fe} \), especially at the two largest depths in the experiment. It thus appears that the Gaussian approximation is well established and little error should result in determining \( \sigma_{ms} \) using \( [\sigma_w^2 - (\sigma_{wo})^2]^{1/2} \).

4.3.4 Film as a Particle Counter

It was stated in section 3.2 that on the basis of the data of Tochilin et al. (37) film would be considered to be a particle counter. In our experiments we are interested in the radial distributions measured from films placed perpendicular to the beam axis. There are situations, such as that for a pure monoenergetic beam, where the radial distributions at any depth (except where \( \pi^- \) star products are important) should not depend on whether film behaves as a particle counter, since approximately the same LET spectrum should be present at any radial position. The case where the effects of the particle counter assumption on the radial distributions should be most noticeable is for a beam with a large momentum spread and having contaminants with significantly different scattering from the pions. The effect of momentum spread, as indicated in section 4.3.1 for the 180 MeV/c experiments, does not seem to cause any experimentally detectable difference in the value of \( \sigma_{ms} \) determined
for large or small momentum spreads. In this case, while it is difficult to determine the scattering of the electrons (section 4.3.2), the derived values of σ for the electrons are larger than those calculated for the pions. Thus, it appears that in this case where the effects of the particle counter assumption should be most noticeable, if there is any non-compliance with the assumption, it does not greatly affect the radial distribution. However, we would still like to see how closely the film behaves as a particle counter.

We have three sets of experimental data that lend support for the assumption of film as behaving as a particle counter, on the basis of the overall film darkening with depth. They are:

(i) parallel films
(ii) exposure tests
(iii) axial density from multiple scatter data.

The films parallel to the beam axis were mainly simple initial attempts to measure the variation in film blackening with depth, and no special precautions were taken to eliminate air gaps. In spite of the problems mentioned by Dutreix (31), which make using the numbers for the film densities of little value, the generally uniform response with depth lends support for considering the film to behave as a particle counter, rather than as a dosimeter.

In the exposure test films, which were done without collimation, there appears to be some inconsistency in the film darkening with depth. In the 148 MeV/c case (figure 11) the O.D. at the peak (x = 11.3 cm) is greater than the O.D. at the front (x = 2.4 cm). In the 180 MeV/c test
(figure 12) the O.D. decreases steadily as depth increases. The lack of collimation and non-uniform intensity, which causes different densities at different areas on the film, may account for some of this discrepancy due to the scattering out from the higher density areas into the lower density areas to give a more uniform density at depth. This explanation appears to be likely since the results based on the collimated multiple scatter data (discussed next) are consistent for both momenta.

As mentioned, the multiple scattering data can also be analyzed to see how closely film behaves like a particle counter. If equation (19) is used, which relates the intensity on the axis to the radius of the aperture, \( r_c \), and the standard deviation of the scatter, \( \sigma \), and corrections are made to the net O.D. (total minus background) for the loss of particles due to in-flight interactions, it is possible to calculate a value for \( \sigma \) based on the optical density on the axis only. Data for the 148 MeV/c case (including a run of \( r_c = 0.50 \) cm not analyzed for lateral distribution) with the corrections made is given in table XII. This calculation has also been done for the 180 MeV/c case and is given in table XIII. The shallowest depth and largest field size is used as the point of reference and normalization in each case. Comparison with the values of \( \sigma_w \) in tables VI, VII and VIII shows that the values for \( \sigma \) are, within 10% to 15%, in agreement for the two methods.

From the results of the variation of film darkening with depth it would appear that the film may be assumed, within 10% to 15%, to be a particle counter. The effect that this has on the radial
distribution is certainly much less than this limit, as witnessed by the good agreement for the wide and narrow blade cases discussed earlier. Thus, our determination of $\sigma$ from the radial distributions should not be restricted by this 15% limit on the validity of the particle counter assumption.

**TABLE XII**

**CALCULATION OF $\sigma$ FROM AXIAL DENSITY**

**AFTER CORRECTION FOR IN-FLIGHT INTERACTIONS, 148 MEV/c**

<table>
<thead>
<tr>
<th>Depth (cm)</th>
<th>Net Density</th>
<th>Corrected, Normalized Net Density</th>
<th>$\sigma$ [Equation (19)] (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_c = 1.125$</td>
<td>$r_c = 0.50$</td>
<td>$r_c = 1.125$</td>
<td>$r_c = 0.50$</td>
</tr>
<tr>
<td>2.4</td>
<td>0.99</td>
<td>0.90</td>
<td>1.00</td>
</tr>
<tr>
<td>11.3</td>
<td>0.72</td>
<td>0.27</td>
<td>0.85</td>
</tr>
</tbody>
</table>

**TABLE XIII**

**CALCULATION OF $\sigma$ FROM AXIAL DENSITY**

**AFTER CORRECTION FOR IN-FLIGHT INTERACTIONS, 180 MEV/c**

<table>
<thead>
<tr>
<th>Depth (cm)</th>
<th>Net Density</th>
<th>Corrected, Normalized Net Density</th>
<th>$\sigma$ [Equation (19)] (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_c = 1.125$</td>
<td>$r_c = 0.50$</td>
<td>$r_c = 1.125$</td>
<td>$r_c = 0.50$</td>
</tr>
<tr>
<td>7.55</td>
<td>0.92</td>
<td>0.62</td>
<td>1.00</td>
</tr>
<tr>
<td>15.1</td>
<td>0.62</td>
<td>0.20</td>
<td>0.767</td>
</tr>
<tr>
<td>20.1</td>
<td>0.36</td>
<td>0.10</td>
<td>0.491</td>
</tr>
</tbody>
</table>
4.3.5 Effect of Non-parallel Beam

In this section only the point source approximation is discussed since the assumption of the separation of the finite emittance from the divergence has been discussed in section 4.2 and 4.3.3.

As mentioned in Chapter 3, the beam envelope is observed to diverge in the $x'$ direction and remain approximately parallel in the $y'$ direction, although the focus seems to be far enough from the film to satisfy the small angle criterion. At points downstream from the collimator, the resulting beam will be somewhat elliptical in shape. It is hard to estimate quantitatively the effect this may have in determining $\sigma'$ but as may be seen in tables VI, VII, and VIII there is a systematic tendency for values of $\sigma_w$ and $\sigma_{wo}$ to be lower for the $x'$ profile (major axis of ellipse) than for the $y'$ profile (minor axis of ellipse). The explanation of this observation involves consideration of the effect of assuming circular symmetry in the fitting procedure and the effect of variations in the magnification parameter $A$, on the scattering parameter $\sigma'$.

Along the major axis of the elliptical distribution there is less scatter contribution from off the axis than for a circular distribution of the same radius. Thus, if one assumes a circular shape in the fitting procedure, the value of the magnification parameter $A$ determined will be an underestimate of the actual value. Conversely, along the minor axis of the ellipse the magnification ($A$) will be overestimated. The effect of these errors in $A$ for estimating $\sigma'$ can be seen from figure 14 to depend on the region of $\sigma'$ that is involved. Generally for $\sigma'$ below about .5, the effect of underestimating $A$ is to
underestimate $\sigma'$ since both the $Y_{75}$ and $Y_{50}$ tend in this direction, although the $Y_{25}$ goes in the opposite direction. Similarly $\sigma'$ is overestimated when $A$ is overestimated. In the region of $\sigma'$ greater than .5, the effect of underestimating $A$ is for $\sigma'$ to be overestimated and vice versa for $A$ being overestimated. Within the limits of the estimated error for $\sigma'$, this argument tends to explain the observations of $\sigma_w$ and $\sigma_{wo}$ for the $x'$, $y'$ data given in tables VI, VII and VIII. The effect on $\sigma_{ms}$ is difficult to assess, although it is possible to conceive of situations which reverse the systematic difference between $x'$, $y'$ directions. Averaging the values of $\sigma_{ms}$ for the two directions as has been done should result in a better estimate of $\sigma_{ms}$.

4.4 Comparison with Theory

The theoretical calculations of $\sigma_{ms}$ for pions, based on the values for $K'$ and $\epsilon'$ given in Appendix A for the two $f(\alpha^2)$ equations, are evaluated using equation (17). These values, as well as the results of the experiments, are given in table XIV.

Comparison of the values in table XIV shows that, with the exception of the peak value of the 148 MeV/c case and the peak value for $r_{c/0.50}$ cm in the 180 MeV/c case, the experimental values are greater than the theoretical values. This is what has been observed by Mayes et al. (20), but even with their suggested correction the situation is not altered significantly. Clearly the uncertainty in the experiment is such that their proposal cannot be commended over that of Molière's original theory. In order to better show the variation of $\sigma_{ms}$ with depth, the theoretical curves [Molière's $f(\alpha^2)$] are plotted against depth in figure 15. Also shown are the experimental values,
FIGURE 15
COMPARISON OF $\sigma_{ms}$ FOR THEORY (CURVES) AND EXPERIMENT (POINTS) AGAINST DEPTH IN WATER
with error bars for the $r_c = 1.125$ cm case only. An alternate form of presentation due to Preston and Koehler (22), which incorporates all the data, is shown in figure 16. Here $\sigma/\sigma_{R_0}$ is plotted against $x/R_0$, where $\sigma_{R_0}$ is the value given by Molière's $f(\alpha^2)$ equation when $x = R_0$, again only the error bars for the $r_c = 1.125$ cm case are shown.

**TABLE XIV**

<table>
<thead>
<tr>
<th>Momentum MeV/c</th>
<th>Range in Tank cm</th>
<th>Depth cm</th>
<th>$\sigma_{ms}$ (Theory) cm</th>
<th>$\sigma_{ms}$ (Experiment) cm</th>
<th>$r_c = 0.50$ cm</th>
<th>$r_c = 1.125$ cm</th>
<th>Straight Edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>148</td>
<td>11.3</td>
<td>2.4</td>
<td>.042</td>
<td>.043</td>
<td>.052±.011</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>11.3</td>
<td>.508</td>
<td>.523</td>
<td>.430±.016</td>
<td></td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>20.1</td>
<td>7.55</td>
<td>.179</td>
<td>.184</td>
<td>.226±.013</td>
<td>.219±.010</td>
<td>.231±.027</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15.1</td>
<td>.544</td>
<td>.560</td>
<td>.578±.045</td>
<td>.588±.051</td>
<td>.637±.044</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20.1</td>
<td>.904</td>
<td>.930</td>
<td>.855±.081</td>
<td>.968±.093</td>
<td>1.062±.061</td>
</tr>
</tbody>
</table>

The agreement is not too bad, when one considers that the theoretical value is for pions only and the experiment includes contributions due to electrons and muons, as well as star products at those depths where pions are stopping. When calculating the lateral distribution of pions, for purposes of dose distributions in treatment planning, it thus seems reasonable to assume that the lateral distributions of the electrons and muons are the same as for the pions as Li et al. (8) have done. Indeed, when one is calculating the dose distributions, the contributions of electrons and muons to the dose are a much smaller fraction of the total than their fractions by numbers. (See for example
COMPARISON OF SCATTER PARAMETER FOR THEORY (CURVE) AND EXPERIMENT (POINTS) AS A FUNCTION OF RANGE RELATIVE TO END OF RANGE VALUES.
Turner et al. (10) where the assumed relative numbers are comparable to the 180 MeV/c case.)

As observed in Chapter 3, there appears to be a systematic increase in $\sigma_{\text{ms}}$ with field size for the peak depth in the 180 MeV/c case (table X or XIV). In order to decide if there is a reasonable explanation for this difference, we re-assess the data analysis for the various fields.

Consideration of the case of circular fields shows that for large values of $\sigma'$, the central axis intensity is considerably reduced (see for example figure 6). In the case of $r_c = 0.50$ cm at the peak depth ($x = 20.1$ cm) the value of $\sigma'$ is on the order of 2 which results in the distribution being very spread out. This means that the determination of the $Y$ parameters, especially $Y_{25}$, is very sensitive to variations in the background film density. A means of checking the reliability of the determination of $\sigma'$ is to examine the consistency of the values of $A$ resulting from the fitting procedure. One way of doing this is to assume that the value of the magnification, $A$, is correct for the shallowest depth (smaller $\sigma'$, sharp edge), then calculate the value of the focal distance $f$ (Appendix C). Using this value for $f$, values of $A$ for the succeeding depths are calculated for the same experimental conditions. Comparing the calculated values with the values given by the fitting program indicated that for $r_c = 0.5$ cm the $A$ values for the two methods do not agree well, whereas for $r_c = 1.125$ cm the agreement was much better. Thus it would appear that the values for $r_c = 1.125$ cm, (and $r_c = 0.50$ cm at shallow depths) because they have smaller values of $\sigma'$, resulting in steeper distributions, are less sensitive to variations in the background film density and provide a more reliable
value for $A$ and thus $\sigma_{ms}$. Matching the aperture size to the value of $\sigma (\sigma' = 1)$ should be expected to give sufficient information without overly reducing the signal to noise ratio at any of the levels chosen for analysis.

In the case of the straight edge experiment we have used a different method of analysis. There is no term to correct for divergence so that it has been assumed that there is either no divergence or that the projection of the edge at the film remains a straight edge, except for scatter effects. This is not expected to be a particularly good assumption since there will be some spread at depth due to the observed divergence. One possible correction that can be made is to assume that the value of the magnification, $A$, for the $r_c = 1.125$ cm case is valid for the straight edge and divide the straight edge value of $\sigma_{ms}$ by $A$. Using a value of $A = 1.2$, which is typical for the wide blade case at $x = 20.1$ cm depth, this reduces $\sigma_{ms}$ from 1.062 cm to 0.885 cm, which is in substantial agreement with the values for the circular apertures. Another problem arises due to the difficulty in achieving a uniform intensity distribution for a beam that is uncollimated and of large spatial extent. It is therefore argued that the values for the $r_c = 1.125$ cm case are more reliable and should be considered to represent the best experimental values. However, all values will be included in the comparison with the published values.

4.5 Comparison to Values in the Literature

Most references to calculations of pion scattering have been to the work of Fowler and Perkins (1), whose values were used in both dose calculating schemes (8,11,12) mentioned in Chapter 1. Curtis and
Raju (45) have also given a formula using different numerical coefficients than Fowler and Perkins. These two formulas for the standard deviation in the multiple scattering Gaussian, for $x = R_0$, are given in table XV, as well as that of our theoretical derivation presented in Chapter 2. When comparison is made to the experimental data it would appear that the values of Curtis and Raju are too large and that those of Fowler and Perkins are probably too large also, although there is the single point for the straight edge experiment which would overlap their value of $\sigma_{ms}$.

**TABLE XV**

**COMPARISON OF CALCULATED VALUES**

<table>
<thead>
<tr>
<th>Source</th>
<th>$K''$</th>
<th>$c''$</th>
<th>Calculated Values of $\sigma_{ms} = K''R_0c''$ ($R_0 = 20.1$ cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fowler and Perkins (1)</td>
<td>0.07</td>
<td>0.92</td>
<td>1.11 cm</td>
</tr>
<tr>
<td>Curtis and Raju (45)</td>
<td>0.0763</td>
<td>0.95</td>
<td>1.32 cm</td>
</tr>
<tr>
<td>Equation (A-5)$^a$</td>
<td>0.045</td>
<td>1.00</td>
<td>0.90 cm</td>
</tr>
</tbody>
</table>

$^a\sigma_{ms} = \langle x^2 \rangle/2^{1/2}$ evaluated for $R_p = 0$

The theory presented in Chapter 2 gives better agreement with the experiment than either of the other theories.

In order to demonstrate whether the differences between the theories have practical significance, we have calculated the radial distributions for two circular fields of radii, 1.0 cm and 5.0 cm at a depth of 20.1 cm, and these are plotted in figure 17. These are the resultant distributions for a parallel beam of uniform intensity over
FIGURE 17

CALCULATED DISTRIBUTIONS FOR TWO CIRCULAR RADII COMPARING VALUES OF $\sigma_{ms}$ AT $R_0 = 20.1$ cm
the area of each field at zero depth. The differences are significant and since the theory presented in this work gives better agreement with experiment, it is suggested that this theory be adopted at least until more experimental work is done. In any event, the results indicate that multiple scattering will greatly affect the shape of the dose distribution.

When treating patients, it has been proposed to use a fixed channel momentum in the range of 160 MeV/c to 200 MeV/c to reduce the electron contamination and increase the pion flux. It will be necessary in this approach to add bolus (absorber) to adjust the stopping peak to the tumour location. Assuming a mean momentum of 180 MeV/c or a range of 22.3 cm of water, this would imply that field sizes less than 5 cm radius will begin to be strongly affected by the multiple scattering of the pions. Thus, while one of the hoped for advantages of pions is better dose concentration at the peak, it appears that multiple scatter will severely restrict the size of tumours for which this applies. If it is desired to treat small field sizes at shallow depths using a fixed midline momentum, it will be necessary to collimate the beam as close to the end of the bolus as possible.
5. CONCLUSIONS

In this study the theory of multiple scatter has been examined and measurements have been made using medical x-ray film for a beam of negative pions incident on a thick water phantom. Calculations have been presented for the multiple scatter based on the Gaussian term of Molière's theory, with modification for the Fano correction as well as energy loss. This calculation gives values for the standard deviation of the Gaussian lateral distribution which are 20% lower than values from the previous calculation of Fowler and Perkins (1).

The contamination of the pion beam by electrons and muons makes it difficult to derive the pion scattering from the experiments, since it is assumed that the contaminants contribute to the films by number. Since the magnitude of electron scattering for thick sections is difficult to calculate, the separation of the pion scattering from the total is uncertain, and this has not been done. However, when the best value of $\sigma$ for the pencil beam [derived from the finite size beam measurements ($r_c = 1.125 \text{ cm}$)] is compared with that of our theory for pions alone, the experimental value is greater than the theoretical by only 7%. It is possible to correct the theory for centre of mass effects amounting to about 1% and an additional correction of 3% has been proposed by Mayes et al. (20) but the validity of this is uncertain at the present time.

The agreement between the measured and calculated values for depths less than the final range would indicate that equation (A-5) should be adequate for treatment planning. This equation is derived from the theory presented in Chapter 2 [equation (17)] with values for
the constants K' and c' determined in Appendix A and is reproduced below:

\[ \langle r^2 \rangle = 0.004064 R_0^{1.99855} \left[ 1 - 4.002904 \left( \frac{R_F}{R_0} \right) - 1378.370276 \left( \frac{R_F}{R_0} \right)^2 \right] + 1381.373180 \left( \frac{R_F}{R_0} \right)^{1.99855} \]

where \( \sigma = \left[ \frac{\langle r^2 \rangle}{2} \right]^{1/2} \) with \( R_F = R_0 - x \). A simplification of the above formula arises in the case of water as pointed out in Appendix A, such that the expression for \( \langle r^2 \rangle \) becomes

\[ \langle r^2 \rangle = .0041 R_0^2 \left[ 1 - 4 \left( \frac{R_F}{R_0} \right) + \left( \frac{R_F}{R_0} \right)^2 \left( 3 - 2 \ln \left( \frac{R_F}{R_0} \right) \right) \right] \]

which gives results of sufficient accuracy. In addition, the agreement between theory and experiment indicates that it should be valid to assume that the contaminating electrons and muons scatter the same as pions when calculating dose distributions.

Due to the uncertainty of the LET dependency of the film, the electron contamination, as well as a possible field size effect, it is recommended that measurements be made with some other detector (such as a silicon diode). There are problems with detectors which behave more like dosimeters than particle counters, since at depths near the pion stopping peak it will be necessary to account for the broadening effect to the dose distribution by the pion star products. However, alternate measurements may allow resolution of the apparent field size effect and comparison of the results with those of film may give further indication of the accuracy of film. This may allow the use of film in relative
measurements, particularly in the investigation of the effects of inhomogeneities.

In the application of pions for radiotherapy, our results indicate that the stopping distributions should be less dispersed than previously predicted. However, it has also been demonstrated that the effects of multiple scattering are significant. The size of these effects indicates that if absorber is used to shift the depth of the stopping peak then collimation of the pion beam should be done after the absorber if possible.

In summary, the results of the experiment are a first attempt to determine the multiple scattering of pions from the complicated case of a finite size, non-parallel beam, with additional problems due to star products and contaminating electrons and muons. The agreement with the theory presented appears to be sufficient to allow the use of this theory in calculations for radiotherapy, at least until more experimental determinations are completed.


41. H. APPEL, V. BÖHMER, G. BÜCHE, W. KLUGE and H. MATTÄY, "\(\pi^-


APPENDIX A

NUMERICAL EVALUATION OF CONSTANTS FOR MEAN SQUARE LATERAL DISPLACEMENT \( <r^2> \) [Equation (17)]

A.1 Determinations of Constants in Equation Relating Momentum and Residual Range \( (p\beta = C_1R^{C_2}) \)

The data used for this determination is given in table VI of Henry (46). This is range-energy data for pions in water scaled from proton values given by Bichsel (23). Table AI contains the values of \( R, T, p\beta \) and \( \beta^2 \) over the ranges of interest. The values of \( p\beta \) and \( \beta^2 \) are calculated using the following formulas

\[
p\beta = mc^2 \left[ \gamma - \frac{1}{\gamma} \right],
\]

where

\[
mc^2 = \text{rest mass of pion} = 139.6 \text{ MeV},
\]

\[
\gamma = \frac{T}{(mc^2)} + 1,
\]

and

\[
\beta^2 = 1 - \left[ \frac{1}{\gamma^2} \right].
\]

The data was fitted to a power curve and the values determined for the coefficients \( C_1 \) and \( C_2 \) are given below. The correlation coefficient was 0.99997 indicating a good fit for the range of 0.6 cm to 30 cm of water (since \( \rho = 1 \text{ g/cm}^2 \) for water, distances are given in cm). Thus for pions in water

\[
p\beta = C_1R^{C_2}
\]

\[
C_1 = 26.45527,
\]

\[
C_2 = 0.5411887.
\]
### TABLE AI

VALUES FOR ENERGY RELATED PARAMETERS AS A FUNCTION OF RANGE FOR PIONS IN WATER

<table>
<thead>
<tr>
<th>RANGE cm</th>
<th>ENERGY MeV</th>
<th>pβ MeV</th>
<th>$\beta^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.6</td>
<td>10.468</td>
<td>20.206</td>
<td>.1346</td>
</tr>
<tr>
<td>1.0</td>
<td>13.898</td>
<td>26.538</td>
<td>.1729</td>
</tr>
<tr>
<td>1.4</td>
<td>16.754</td>
<td>31.713</td>
<td>.2028</td>
</tr>
<tr>
<td>2.4</td>
<td>22.81</td>
<td>42.416</td>
<td>.2612</td>
</tr>
<tr>
<td>3.4</td>
<td>27.92</td>
<td>51.186</td>
<td>.3056</td>
</tr>
<tr>
<td>5.5</td>
<td>36.98</td>
<td>66.215</td>
<td>.3750</td>
</tr>
<tr>
<td>7.6</td>
<td>44.82</td>
<td>78.746</td>
<td>.4270</td>
</tr>
<tr>
<td>10.8</td>
<td>55.57</td>
<td>95.317</td>
<td>.4884</td>
</tr>
<tr>
<td>14.0</td>
<td>65.39</td>
<td>109.92</td>
<td>.5362</td>
</tr>
<tr>
<td>18.0</td>
<td>76.79</td>
<td>126.33</td>
<td>.5838</td>
</tr>
<tr>
<td>22.0</td>
<td>87.5</td>
<td>141.29</td>
<td>.6221</td>
</tr>
<tr>
<td>26.0</td>
<td>97.69</td>
<td>155.16</td>
<td>.6539</td>
</tr>
<tr>
<td>30.0</td>
<td>107.5</td>
<td>168.23</td>
<td>.6808</td>
</tr>
</tbody>
</table>
A.2 Determination of Constants in Equation Relating Molière's B and Residual Range \((B = C_3 R^{4/3})\)

One can calculate the value of Molière's B parameter quite easily using equation (9) and either equation (10) or (11). The value of \(B\) is seen to depend on \(Z, A\) and \(t\) of the medium and \(z, \beta^2\) of the incident particle. Making the substitution of \(Z(Z+1)\) for \(Z^2\) and \(Z^{1/3} (Z+1)\) for \(Z^{4/3}\), one can reproduce the values of \(B\) tabulated by Bichsel (23). (\(A\) in grams, \(t\) in g/cm\(^2\).)

Molière's theory is valid for thin foils or no energy loss. However, we are interested in thick scattering sections of compounds such as water and tissue. It is possible to find suitable expressions for \(\beta^2\) as a function of residual range \(R\), but one is still left with the problem of what value of the thickness \(t\) should be used. Our solution to this problem is to find the thickness \(t\) at each range such that the value of \(\beta^2\) changes by only 1% in traversing thickness \(t\).

The procedure was to first find an expression for \(\beta^2\) as a function of \(R\). We decided on the form

\[
\ln (\beta^2) = X_1 + X_2 \ln R + X_3 (\ln R)^2. \quad (A-1)
\]

Using the values in table AI for pions in water we find

\[
X_1 = -1.752155,
\]
\[
X_2 = 0.498911,
\]
\[
X_3 = -0.027780.
\]

These values gave good agreement since \(\Sigma(\beta_{i}^2 - \beta_{i}^{2'})^2\) was less than
7 x 10^{-5} where \( \hat{\beta}^2 \) is the value of \( \beta^2 \) calculated using equation (A-1) for each value of \( R \) in table Al.

The next step was to find the value of \( t \) at each value of \( R \) in table Al. To do this we differentiated equation (A-1) to give

\[
\frac{1}{\beta^2} \frac{d\beta^2}{dR} = \frac{X_2}{R} + \left( \frac{2X_3}{R} \right) \ln R,
\]

(A-2)
equating \( d\beta^2 \) with \( \Delta \beta^2 \) and \( dR \) with \( \Delta R = t \) we solved for \( t \) such that

\[
\Delta \beta^2/\beta^2 = .01 \quad \text{and arrived at}
\]

\[
t = \Delta R = \left( \frac{.01 R}{X_2 + 2X_3 \ln R} \right).
\]

(A-3)

Using equation (A-3) and the values of \( R \) in table Al we were able to tabulate \( t \) as a function of \( R \) (table All). (\( R \), \( t \) in cm since \( \rho = 1 \) g/cm^2.)

Since we also wanted to incorporate Fano's correction [equation (12)] using values of \( u_{in} \) in table I we then had to solve

\[
B - \ln B = \ln \left[ \frac{\chi_{c}^2}{(\chi_{0}^2 f(x^2))} \right] - .1544 + B''
\]

\[
= \ln \left[ \frac{8838.4 \frac{Z^4/3 t}{A(1.13 \beta^2 + 3.76 \frac{Z^2}{(137)^2})}}{Z^{-1}(\ln [1130 Z^{-4/3} (\beta^2/(1-\beta^2))] - u_{in} - \frac{3}{2} \beta^2} \right] - .1544
\]

(A-4)

When dealing with compounds such as water we then evaluated equation (A-4) for each element at each residual range \( R \) making the substitution of \( t_1 = \varepsilon_1 t \) where \( \varepsilon_1 \) is the fraction of the element in the compound by weight. When we had values for \( B_i \) at each \( R \) we then formed the quantity

\[
J = \sum \varepsilon_i \left( \frac{Z_i^2}{A_i} \right) B_i \quad \text{and divided by } \langle Z^2/A \rangle \quad \text{where}
\]
\[ \langle Z^2/A \rangle = \sum \epsilon_i \langle Z_i^2/A_i \rangle, \]

which gives for each value of \( R \)

\[ \bar{B} = J / \langle Z^2/A \rangle . \]

The next step was then to approximate \( \bar{B} \) as a function of the residual range \( R \). Again a power law was found to give good agreement, with a correlation coefficient of 0.99995 using Molière's \( f(a^2) \) equation. In the case of water the coefficients in the equation

\[ \bar{B} = C_3 R^{C_4}, \]

are given by

\[ C_3 = 9.888340, \]
\[ C_4 = 0.080927. \]

An example for water is given in table AIII.

A.3 Numerical Evaluation of Equation (17)

An evaluation of equation (17) based on these constants, using Molière's \( f(a^2) \) equation where we replace \( Z^2/A \) in \( K \) by \( \langle Z^2/A \rangle \) (since a compound) yields \( K' = 0.008122 \) and \( c' = 0.500725 \) and thus

\[ \langle r^2 \rangle = 0.004064 R_0^{1.99855} \left[ 1 - 4.002904 \left( \frac{R_F}{R_0} \right) - 1378.370276 \left( \frac{R_F}{R_0} \right)^2 \right. \]
\[ + 1381.373180 \left( \frac{R_F}{R_0} \right)^{1.99855} \left], \right. \quad (A-5) \]

and again

\[ \sigma = (\langle r^2 \rangle/2)^{1/2}. \]
### TABLE AIII

**SAMPLE CALCULATION OF \( \bar{B} = C_3 R^{C_4} \) FOR WATER USING MOLIÈRE'S \( f(\alpha^2) \) EQUATION**

<table>
<thead>
<tr>
<th>RANGE (cm)</th>
<th>t (cm)</th>
<th>( t_H )</th>
<th>( \chi_c^2 / \chi_o^2 f(\alpha^2) )</th>
<th>B( '' )</th>
<th>B( _H )</th>
<th>t( _0 )</th>
<th>( \chi_c^2 / \chi_o^2 f(\alpha^2) )</th>
<th>B( '' )</th>
<th>B( _0 )</th>
<th>J</th>
<th>( \bar{B} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>.0114</td>
<td>.00128</td>
<td>73.4156</td>
<td>8.7018</td>
<td>15.5901</td>
<td>.01012</td>
<td>542.2527</td>
<td>.9161</td>
<td>9.2860</td>
<td>34.7203</td>
<td>9.4770</td>
</tr>
<tr>
<td>1.0</td>
<td>.0200</td>
<td>.00224</td>
<td>100.2708</td>
<td>8.9783</td>
<td>16.2179</td>
<td>.01776</td>
<td>753.5909</td>
<td>.9507</td>
<td>9.6926</td>
<td>36.2345</td>
<td>9.8903</td>
</tr>
<tr>
<td>2.4</td>
<td>.0533</td>
<td>.00596</td>
<td>176.9393</td>
<td>9.4596</td>
<td>17.3336</td>
<td>.04734</td>
<td>1357.6684</td>
<td>1.0109</td>
<td>10.4131</td>
<td>38.9180</td>
<td>10.6228</td>
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<td>.00883</td>
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<td>9.6564</td>
<td>17.7919</td>
<td>.07007</td>
<td>1728.2041</td>
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<td>10.7068</td>
<td>40.0123</td>
<td>10.9215</td>
</tr>
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<td>5.5</td>
<td>.1361</td>
<td>.01523</td>
<td>314.7878</td>
<td>9.9316</td>
<td>18.4438</td>
<td>.12087</td>
<td>2445.5927</td>
<td>1.0699</td>
<td>11.1269</td>
<td>41.5772</td>
<td>11.3486</td>
</tr>
<tr>
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<td>.02202</td>
<td>399.7635</td>
<td>10.1224</td>
<td>18.8978</td>
<td>.17478</td>
<td>3116.8202</td>
<td>1.0937</td>
<td>11.4192</td>
<td>42.6660</td>
<td>11.6458</td>
</tr>
<tr>
<td>10.8</td>
<td>.2945</td>
<td>.03296</td>
<td>523.0492</td>
<td>10.3394</td>
<td>19.4104</td>
<td>.26155</td>
<td>4091.0911</td>
<td>1.1208</td>
<td>11.7466</td>
<td>43.8860</td>
<td>11.9788</td>
</tr>
<tr>
<td>14.0</td>
<td>.3974</td>
<td>.04447</td>
<td>642.8971</td>
<td>10.5069</td>
<td>19.8044</td>
<td>.35293</td>
<td>5038.6160</td>
<td>1.1418</td>
<td>11.9970</td>
<td>44.8193</td>
<td>12.2336</td>
</tr>
<tr>
<td>26.0</td>
<td>.8179</td>
<td>.09152</td>
<td>1085.0648</td>
<td>10.9392</td>
<td>20.8096</td>
<td>.72638</td>
<td>8535.3611</td>
<td>1.1958</td>
<td>12.6294</td>
<td>47.1776</td>
<td>12.8773</td>
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<td>30.0</td>
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<td>.10831</td>
<td>1233.3389</td>
<td>11.0470</td>
<td>21.0573</td>
<td>.85959</td>
<td>9708.1559</td>
<td>1.2093</td>
<td>12.7838</td>
<td>47.7536</td>
<td>13.0345</td>
</tr>
</tbody>
</table>
The evaluation of $K'$ and $c'$ using the suggested $f(a^2)$ equation of Mayes et al. (20) gives values

$$K' = 0.008675,$$
$$c' = 0.502792.$$  

These values when inserted in equation (17) allow calculation of $\sigma_{\text{ms}}$ for comparison with values using equation (A-5) as shown in table XIV.

We decided to use only Molière's $f(a^2)$ equation and hence have only presented equation (A-5) for calculating $\sigma_{\text{ms}}$. In the case of water using Molière's $f(a^2)$ equation, the value of $c'$ is close enough to $0.5000$ that a simple approximation may be made before the integration which yields equation (17). This allows an alternate form that is sufficiently accurate for our purposes,

$$\langle r^2 \rangle = 0.0041 \, R_0^2 \left[ 1 - 4 \left( \frac{R_F}{R_0} \right)^2 + \left( \frac{R_F}{R_0} \right)^2 \left[ 3 - 2 \ln \left( \frac{R_F}{R_0} \right) \right] \right]. \quad (A-6)$$

This converges to $0.0041 \, R_0^2$ as $R_F$ goes to zero as is expected. It should be noted that this equation applies only for pions in water and for other particles or materials it is necessary to follow the above procedure for the determination of $K'$ and $c'$ and subsequent substitution in equation (17).

In addition, since we have been dealing with the special case of water with $\rho = 1 \, \text{g/cm}^2$, both $R$ and $t$ have been given in cm resulting in $\sigma$ being given in cm. Obviously if $\rho \neq 1 \, \text{g/cm}^2$ then with $R$ and $t$ measured in g/cm$^2$, $\sigma$ is given in g/cm$^2$. 
APPENDIX B

EXTENSION OF PENCIL BEAM SCATTER TO INFINITE STRAIGHT EDGE

We would like to calculate the scatter contribution at a point 
\((x,y)\) from a point at \((x',y')\), where the multiple scattering distribution 
is given by a Gaussian,

\[ G(x-x', y-y') = (2\pi\sigma^2)^{-1} \exp\left[-\frac{(x-x')^2}{(2\sigma^2)} - \frac{(y-y')^2}{(2\sigma^2)}\right]. \]

For an incident parallel beam the lateral distribution at a point 
\((x,y,z)\) is given by the sum of all contributions in the beam profile, i.e.,

\[ f(x,y,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} N(x',y') G(x-x',y-y') \, dx'dy'. \]  

(\text{B-1})

Here \(\sigma = \sigma(z)\).

If we collimate the beam by blocking out the left half plane, i.e. a 
collimator is aligned along the \(y'\) axis we define

\[ N(x',y') = 1 \quad x' > 0. \]

Therefore equation (B-1) becomes

\[ f(x,y,z) = \int_{-\infty}^{\infty} dy' (2\pi\sigma^2)^{-1} \exp\left[-\frac{(x-x')^2}{(2\sigma^2)}\right] \exp\left[-\frac{(y-y')^2}{(2\sigma^2)}\right] dx'. \]

\[ = \int_{0}^{\infty} (2\pi)^{-1/2} \sigma^{-1} \exp\left[-\frac{(x-x')^2}{(2\sigma^2)}\right] dx' \int_{-\infty}^{\infty} (2\pi)^{-1/2} \sigma^{-1} \exp\left[-\frac{(y-y')^2}{(2\sigma^2)}\right] dy', \]

\[ = \int_{0}^{\infty} (2\pi)^{-1/2} \sigma^{-1} \exp\left[-\frac{(x-x')^2}{(2\sigma^2)}\right] dx' \cdot 1, \]

\[ = (\pi)^{-1/2} \int_{-\infty}^{x/\sigma\sqrt{2}} \exp(-v^2) \, dv, \]  

(\text{B-2})

where we have made the substitution \(v^2 = \frac{(x-x')^2}{(2\sigma^2)}\).
Therefore \( f(x,z) = \frac{1}{2} \pm \pi^{-\frac{1}{4}} \int_{0}^{\frac{x}{\sigma \sqrt{2}}} \exp(-v^2) \, dv \). \hfill (B-3)

Here the plus sign is for \( x>0 \) and minus for \( x<0 \) and obviously for \( x=0 \), \( f(x,z) = 0.5 \).

For comparison with experiment we are interested in the cases where

\[
\begin{align*}
  f(x,z) &= 0.75, \\
  f(x,z) &= 0.25.
\end{align*}
\]

Using formulas given in reference (30) for these types of integrals we find for

\[
\begin{align*}
  f(x,z) &= 0.75, \quad x/\sigma = 0.6742 \\
  f(x,z) &= 0.25, \quad x/\sigma = -0.6742
\end{align*}
\]

Thus \( x_{75} - x_{25} = 1.348 \sigma \). \hfill (B-4)
APPENDIX C

MODIFICATION TO ALLOW FOR DIVERGENCE

In experiments with charged particle beams it is difficult, if not impossible, to get parallel beams. Our preliminary analysis showed that this was true in our beam also, so we undertook a first order approximation to take account of divergence or convergence. Referring to figure C1 we see the case of a converging beam, in fact, convergent to a point a distance $f$ from the location of a collimator of radius $r_c$.

In the case of a parallel beam we visualize the lateral distribution at depth as the sum of pencil beams, where each pencil beam spreads out due to multiple scattering given by a Gaussian. For a parallel beam the angle $\alpha = 0$ and the summation at depth $f$ takes place in the plane perpendicular to the beam.

In the case of a beam convergent to a point at depth $f$, each pencil has a unique angle $\alpha' < \alpha$ and each pencil has a Gaussian distribution perpendicular to this angle, say $f(r')$. If we measure, with a film say, distributions in a plane perpendicular to $\alpha = 0$, we will measure contributions from each pencil beam $f(r')$ at the radial distance $r = r' \cos \alpha'$. The maximum angle is $\alpha$ and thus for small enough values of $\alpha$ the distribution measured perpendicular to $\alpha = 0$ will show little smearing due to summing contributions over a range of $r'$ for a given $r$. We set a limit on $\alpha$ such that $\cos \alpha = .99$ or $\alpha$ to be about $8^\circ$.

For a depth, $x$, less than $f$, the beam has radius $r_c' = r_c (1-x/f)$ and effectively we are dealing with a beam of smaller radius. However,
FIGURE C1

SCHEMATIC DIAGRAM AND PARAMETERS FOR A BEAM CONVERGING TO A POINT

FIGURE C2

PHASE SPACE PLOTS FOR ZERO EMITTANCE BEAMS AT DEPTH $x = 0$
the summation procedure is the same as if we had a parallel beam. If we define

\[ A = 1 - \frac{x}{f} \]

then if we replace \( r_c \) by \( Ar_c \) we should have the proper calculation in equation (24), except that the number per unit area has increased. If we assume a uniform density across the cross sectional area then the number of particles remains the same but the area has decreased so we need to multiply by the factor \( \frac{1}{A^2} \). Thus to correct equation (24) for a converging beam under these assumptions, we replace \( r_c \) by \( Ar_c \) and multiply by \( A^{-2} \), i.e.

\[
f(r,x) = \left( \frac{2\pi Ar_c}{A^2} \right) \int_0^\infty J_0(2\pi rs) J_1(2\pi Ar_c s) \exp[-2\pi^2 \sigma^2 s^2] \, ds. \tag{C-1}
\]

Obviously this holds for a diverging beam where \( f = -f \) and hence \( A > 1 \).

The point source requirement means we are still dealing with a zero emittance beam instead of a real finite emittance beam, but the phase space diagram is altered as shown in figure C2. The physical interpretation of \( A \) is as the magnification, identical with point source light optics.