Kalb-Ramond Solitons in Bosonic String Theory

by

Benjamin Jacob Sussman

B.Sc., University of Waterloo, 2000

A THESIS SUBMITTED IN PARTIAL FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Master of Science

in

The Faculty of Graduate Studies

Department of Physics and Astronomy

We accept this thesis as conforming to the required standard

The University of British Columbia

January 2002

In presenting this thesis in partial fulfillment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Physics and Astronomy

The University of British Columbia
Vancouver, Canada

Date Jan 25, 2002

Abstract

A soliton of the Kalb-Ramond field in closed bosonic string theory is introduced. Under appropriate configurations the cosmological constant becomes a periodic function of the Kalb-Ramond field. This vacuum degeneracy permits the formation of sine-Gordon solitons. The energy and length scale of the soliton is inversely proportional to the string coupling constant g_s. The stability of the the solitons is discussed and it is shown that these objects are stable.

Table of Contents

List of Figures

Acknowledgements

I wish to gratefully acknowledge Prof. Gordon Semenoff for his patient supervision and support. Tom Davis and Mark Laidlaw could always be counted on for friendly and engaging discussions. I would also like to thank my friends and family, without whom, this thesis would not have been possible. This work was supported in part by the Natural Sciences and Engineering Council of Canada.

1: Introduction

String theory purports to be a 'theory of everything' and it is currently the lone serious candidate for the unification of gravity and quantum mechanics. Decades of intense theoretical work have explored only a few avenues of the large landscape that is string theory, but to determine if string theory really is a 'theory of everything', we must continue to investigate its various aspects. A particularly interesting area of research involves looking at certain topological features of string theory.

Herein we discuss one topological feature that occurs in closed bosonic string theory: the Kalb-Ramond soliton. One of the massless states of closed bosonic string theory gives rise to a field known as the Kalb-Ramond field, which is often called simply the B-field. Under appropriate compactifications, the B-field can couple to closed strings. In this situation, the B-field shifts the momentum, much like the shift in momentum caused by the application of an electromagnetic field to an electron. The shift in momentum drastically changes the value of the vacuum energy density. In fact, the cosmological constant becomes a function that is periodic in the B-field. This degeneracy of the vacuum can produce domain walls. To an approximation, the system is equivalent to the well known Sine-Gordon system. The energy, size, and structure of the domain wall depends on the compactification radius and may be observable in certain configurations.

We begin what follows with a brief review of bosonic string theory. Chapter 2 provides the necessary prerequisites for a discussion of Kalb-Ramond solitons. Chapter 3 contains a discussion of the covariant path integral formalism. The key point here is the idea of modular invariance. The influence of the Kalb-Ramond field on closed strings is analyzed in chapter 4. It is shown that the Kalb-Ramond field can shift the canonical momenta string modes. Chapter 5 details the calculation of the vacuum energy with and without toroidal compactification. The calculation is then consider for the configuration of a constant Kalb-Ramond field. The finding is that that vacuum energy is a periodic function of the Kalb-Ramond field. After a brief discussion of the tachyon divergence in chapter 6, the string theory effective action is introduced in chapter 7. The cosmological constant is added to the effective action and dimensional reduction is considered. Chapter 8 shows that under appropriate circumstances the Kalb-Ramond field can have sine-Gordon solitons, the details of which are discussed in chapter 9. The remainder of the thesis discuss the stability of these solitons. Chapter 10 begins with an explanation of why one might at first expect the solitons to be stable and concludes by explaining that the soliton may decay to the true vacuum through its coupling to massive modes. Chapter 11 estimates the rate of decay of the soliton through nucleation. Chapter 12 concludes the thesis with a summary.

2: Review of Bosonic String Theory

2.1 History and Motivation

Einstein's general theory of relativity is the currently accepted theory of gravitation. However, it is a classical theory. In order to describe gravitational effects in extreme circumstances a quantum version of general relativity is required. After many years of effort, though, gravity has prooven impossible to quantize. The efforts to do so have persisted since the early days of quantum mechanics, but all conventional attempts have failed [1].

During the 1960s and 1970s, the so called dual resonance model was explored as a possible theory of the hadrons. Ultimately it failed and was replaced by quantum chromodynamics. One of its failures was that it predicted a spin-2 particle that was not observed in hadronic reactions. This failure was, however, was somewhat fortuitous. For as the dual resonance model was abandoned, some realized that the spin-2 particle may in fact be a candidate for the graviton - the quanta of the gravitational field. That is, the dual resonance model should not be taken as a theory of hadronic interactions, but as a quantum theory of everything. In modern times, string theory has become the leading candidate for a consistent theory of quantum gravity. Thorough monographs have been written on string theory [2, 3, 4, 5] as well as excellent review articles [6, 7, 8, 9].

Herein we shall give a brief review of the elements of string theory that are immediately applicable to the discussion B-field solitons.

2.2 From Particles to Strings

2.2.1 Particles

In special relativity, the action of a particle is given by the integral of the invariant length $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ times an arbitrary constant m:

$$S = -m \int ds = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \tag{2.1}$$

The connection with non-relativistic physics is obtained by parameterizing $\tau = X^0$ and taking the following limit:

$$S = -m \int dX^0 \sqrt{1 - \dot{X^i}^2} = \int dX^0 \left\{ \frac{m}{2} \dot{X^i}^2 - m + \mathcal{O}(\dot{X^i}^4) \right\} \tag{2.2}$$

We therefore conclude that m is the particle mass.

2.2.2 Strings

To move to string theory, we note that a one dimensional string sweeps out a two dimensional sheet as it moves through spacetime. Cover the surface with a two dimensional grid parameterized by $\{c^a | a = 1, 2\}$ and metric h_{ab}. The element of length for any Riemannian (or pseudo-Riemmanian) surface is given by $ds^2 = h_{ab}dc^a dc^b$. Thus the length of a small coordinate displacement is given by $||\mathbf{dc}^a|| = \sqrt{h_{aa}}dc^a$ (no sum). Therefore the area of a particular (parallelogram) element of the grid is given by:

$$\begin{aligned}
\Delta A &= \left\|\mathbf{dc^0} \times \mathbf{dc^1}\right\| && (2.3)\\
&= \left\|\mathbf{dc}^0\right\| \left\|\mathbf{dc}^1\right\| \sin(\theta) && (2.4)\\
&= \left\|\mathbf{dc}^0\right\| \left\|\mathbf{dc}^1\right\| (1-\cos^2(\theta)) && (2.5)\\
&= \sqrt{\left\|\mathbf{dc}^0\right\|^2 \left\|\mathbf{dc}^1\right\|^2 - \left\|\mathbf{dc}^0 \cdot \mathbf{dc}^1\right\|^2} && (2.6)\\
&= \sqrt{h_{00}h_{11} - (h_{01})^2} dc^0 dc^1 && (2.7)\\
&= \sqrt{-h} dc^0 dc^1 && (2.8)
\end{aligned}$$

where $h = \det(h_{ab})$. Writing $\tau = c^0, \sigma = c^1$ we then write the so-called Nambu-Goto action as the total area of the surface times arbitrary constant T of dimension $(length)^{-2}$:

$$S = -T \int d\sigma d\tau \sqrt{-h} \tag{2.9}$$

To determine the induced metric h_{ab} in terms of the spacetime metric note that the line element as calculate in both frames should be the same:

$$ds^2 = h_{ab} dc^a dc^b = G_{\mu\nu} dX^\mu dX^\nu \tag{2.10}$$

$$h_{ab} = G_{\mu\nu} \frac{\partial X^\mu}{\partial c^a} \frac{\partial X^\nu}{\partial c^b} \tag{2.11}$$

To make the connection with non-relativistic physics, we parameterize $\tau = X^0$. We then parametrize σ as the proper length perpendicular to τ: $(\frac{\partial}{\partial\sigma} X^i)^2 = 1$, $\frac{\partial}{\partial\sigma} X^0 = 0$. Substituting these values into (2.11) and subsequently into (2.9) we obtain:

$$S = -T \int d\tau d\sigma \sqrt{1 - (\frac{\partial}{\partial\tau} X^i)^2 + (\frac{\partial}{\partial\tau} X^i \frac{\partial}{\partial\sigma} X^i)^2} \tag{2.12}$$

Writing $v^2 = (\frac{\partial}{\partial\tau}X^i)^2 - (\frac{\partial}{\partial\tau}X^i\frac{\partial}{\partial\sigma}X^i)^2$ and expanding in small v, we obtain

$$S = T\int d\tau d\sigma\left(\frac{1}{2}v^2 - 1 + \mathcal{O}(v^4)\right). \tag{2.13}$$

Thus we may interpret v as the transverse velocity of the string and the second term as a potential energy term. I.e., T is the classical tension of the string. T is often written as $T = \frac{1}{2\pi\alpha'}$

We will find that an alternative, but equivalent version of the string action (2.9) will be easier to work with. Introduce a Lorentzian $(-,+)$ metric γ^{ab} for the so-called Polyakov action:

$$S_P = \frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu\partial_b X_\mu \tag{2.14}$$

The equation of motion for γ^{ab} following from this action can be used to eliminate itself from this equation and recover the Nambu-Goto action (2.9). We may also define the energy-momentum tensor via the equation of motion for γ_{ab} :

$$T^{ab} = -4\pi\frac{1}{\sqrt{-\gamma}}\frac{\delta}{\delta\gamma^{ab}}S_P = -\frac{1}{\alpha'}(\partial^a X^\mu\partial^b X_\mu - \frac{1}{2}\gamma^{ab}\partial_a X^\mu\partial^a X_\mu) = 0. \tag{2.15}$$

The equation of motion for X^μ is

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) - \sqrt{-\gamma}\partial^2 X^\mu = 0 \tag{2.16}$$

2.3 Symmetries

The concept of symmetry plays an important role in string theory. Demanding certain symmetries puts extreme constraints on string theory.

First we note that the action S_p should be independent of intrinsic properties of the world-sheet. That is, physics should be independent of parameterization of the world-sheet. This is called diffeomorphism invariance under:

$$X'^{\mu}(\tau', \sigma') = X^{\mu}(\tau, \sigma) \tag{2.17}$$

$$\frac{\partial \sigma'^{c}}{\partial \sigma'^{a}} \frac{\partial \sigma'^{d}}{\partial \sigma'^{b}} \gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma). \tag{2.18}$$

An additional symmetry satisfied by S_P is that of D-dimensional Poincaré invariance:

$$X'^{\mu}(\tau', \sigma') = \Lambda^{\mu}_{\nu} X^{\nu}(\tau, \sigma) + a^{\mu} \tag{2.19}$$

$$\gamma'_{ab}(\tau, \sigma) = \gamma_{ab}(\tau, \sigma). \tag{2.20}$$

A third symmetry is unique to that fact that the world-sheet is two dimensional and is called Weyl Symmetry:

$$X'^{\mu}(\tau', \sigma') = X^{\mu}(\tau, \sigma) \tag{2.21}$$

$$\gamma'_{ab}(\tau, \sigma) = e^{2\omega(\tau,\sigma)} \gamma_{ab}(\tau, \sigma). \tag{2.22}$$

An arbitrary symmetric matrix has $D(D+1)/2$ free components. Diffeomorphism invariance implies that we can fix D components. Therefore there is $D(D-1)/2$ degrees of freedom. In particular, in $D = 2$ we have 1 degree of freedom which means that we can write the world-sheet metric as:

$$\gamma_{ab} = e^{2\omega(\tau,\sigma)} \eta_{ab}(\tau, \sigma) \tag{2.23}$$

where η_{ab} is the flat Lorentzian two-dimensional metric. Subsequently, Weyl invariance implies that that the action is independent of the final degree of freedom.

Of particular importance are conformal transformations: There are coordinate changes that scale the metric which can subsequently be undone via a Weyl transformation. These compound transformations are called conformal transformations.

Introducing $\sigma^{\pm} = \tau \pm \sigma$ we note that in the conformal gauge $-\det(\gamma_{ab}) = e^{4\omega}$ and $\gamma^{ab} = e^{-2\omega}\eta^{ab}$. The action becomes

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu \tag{2.24}$$

Thus, the equation of motion is the simple wave equation:

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0. \tag{2.25}$$

we may then factorize the differential operator yielding:

$$\left(\frac{\partial}{\partial\sigma^+}\frac{\partial}{\partial\sigma^-}\right) X^\mu = 0. \tag{2.26}$$

The general solution is then the sum of a holomorphic and anti-holomorphic (right- and left-moving, respectively) part:

$$X^\mu(\sigma,\tau) = X_L(\sigma^+) + X_R(\sigma^-) \tag{2.27}$$

$$\begin{aligned} X_R^\mu(\sigma^-) &= \tfrac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + \sum_{n\neq 0} \tfrac{\alpha_n^\mu}{n} e^{-2in\sigma^-} \\ X_L^\mu(\sigma^+) &= \tfrac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + \sum_{n\neq 0} \tfrac{\tilde{\alpha}_n^\mu}{n} e^{-2in\sigma^+} \end{aligned} \tag{2.28}$$

We may now quantize in the usual way. In the conformal gauge we have

$$S = \frac{T}{2} \int d^2\sigma \partial^a X^\mu \partial_a X_\mu \tag{2.29}$$

and thus

$$P^{\mu} = \frac{\partial}{\partial \dot{X}^{\mu}} L = T\dot{X}^{\mu}. \tag{2.30}$$

To quantize the system, we introduce the equal time canonical commutators

$$[X^{\mu}(\sigma,\tau), P^{\nu}(\sigma',\tau)] = i\delta(\sigma-\sigma')\eta^{\mu\nu}. \tag{2.31}$$

We then arrive at

$$[x^{\mu}, p^{\nu}] = i\eta^{\mu\nu} \tag{2.32}$$

$$[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = \delta_{m+n}\eta^{\mu\nu} \tag{2.33}$$

$$[\tilde{\alpha}^{\mu}_{m}, \tilde{\alpha}^{\nu}_{n}] = \delta_{m+n}\eta^{\mu\nu} \tag{2.34}$$

$$[\alpha^{\mu}_{m}, \tilde{\alpha}^{\nu}_{n}] = 0. \tag{2.35}$$

We recognize these as the harmonic oscillator raising and lower operators if we write

$$\alpha^{\mu}_{m} = \sqrt{m}a^{\mu}_{m} \qquad m > 0 \tag{2.36}$$

$$\alpha^{\mu}_{-m} = \sqrt{m}a^{\mu\dagger}_{m} \qquad m < 0. \tag{2.37}$$

One then constructs the various string states by using the raising operators on the vacuum state $|0;p^{\mu}\rangle$. It will turn out that the higher states are increasingly more massive. However, not all states constructed in this fashion are physical. The equation of motion for γ in (2.15) yielded the energy-momentum tensor which may be written as

$$T_{\pm\pm} = \frac{\partial X_\mu}{\partial \sigma^\pm}\frac{\partial X^\mu}{\partial \sigma^\pm} = 0 \tag{2.38}$$

In order to facilitate the application of this constraint we take it's Fourier modes. For example:

$$L_m = \frac{1}{4\pi\alpha'}\int_0^\pi e^{-2im\sigma}T_{--}d\sigma = \frac{1}{4\pi\alpha'}\int_0^\pi e^{-2im\sigma}\dot{X}_R^2, \tag{2.39}$$

$$\tilde{L}_m = \frac{1}{4\pi\alpha'}\int_0^\pi e^{-2im\sigma}T_{++}d\sigma = \frac{1}{4\pi\alpha'}\int_0^\pi e^{-2im\sigma}\dot{X}_L^2. \tag{2.40}$$

After substituting in the mode expansion we find that for the right modes

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n = 0 \qquad m \neq 0 \tag{2.41}$$

however, since $[\alpha_{-n}, \alpha_n] \neq 0$, after quantization there is an operator ordering ambiguity in the $m = 0$ case

$$L_0 = -\alpha' p^2 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = a. \tag{2.42}$$

That is, this normal ordered expression is ambiguous up to a constant a. The constant must be determined. For closed strings we find $a = 1$ [4]. Similar relations hold for the left modes.

If we define mass as $m^2 = -p_\mu p^\mu$ then the Virasoro generators give the important level matching and mass-shell condition. Taking the sum and the difference of the zero-mode Virasoro constraints, we find

$$N - \tilde{N} = 0 \tag{2.43}$$

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2). \tag{2.44}$$

3: The Covariant Path Integral and Modular Invariance

3.1 The Covariant Path Integral

The Polyakov path integral formalism is essential for a proper understanding of string theory [10]. Here we investigate the the closed bosonic string path integral by analytically continuing the matrix γ_{ab} to the Euclidean signature metric g_{ab}. The path integral we are interested is given by

$$\int DXDge^{-S} \tag{3.1}$$

where S is the sum of the action and the topologically invariant Euler character

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}\int d^2\sigma\sqrt{g}R. \tag{3.2}$$

This is the only local functional consistent with Weyl invariance, two-dimensional general covariance and Poincare invariance. The Euler character is often denoted by χ and and is related to the number of handles (genus) of the surface in question:

$$\chi = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R = 2 - 2\gamma. \tag{3.3}$$

In addition, however, we also need to add a counter term of the form $\mu^2 \int d^2\sigma\sqrt{G}$ to (3.2). This is required to compensate the regulator that is used to define path integrals of products of X^μ at the same point.

In $D = 26$ all anomalies cancel and the action is invariant under two-dimensional coordinate transformations and Weyl transformations. We must therefore divide out the group volumes for the general coordinate transforms (diff-invariance) and Weyl invariance: $V = V_{GC}V_W$. This is in fact an oversimplified notation since the volumes are not quite independent. The proper path integral is then given by

$$Z = \int \frac{DX^\mu Dg_{ab}}{V_{GC}V_W} e^{-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu - \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R - \int d^2\sigma\sqrt{g}\mu^2}. \tag{3.4}$$

This integral was first evaluated by operator methods [11] but has also been determined via the path integral method [12] for the torus amplitude. For the torus amplitude it was determined to be

$$Z_{torus} = T^{13}V_{26}\int_F \frac{d^2\tau}{4\pi\tau_2^2} e^{4\pi\tau_2}(2\pi\tau_2)^{-12}\left|f(e^{2\pi i\tau})\right|^{-48} \tag{3.5}$$

where $\tau \equiv \tau_1 + i\tau_2$, $\partial^2\tau = d\tau_1 d\tau_2$, $f(q) \equiv \prod_{n=1}^{\infty}(1-q^n)$, T is the string tension, and V_{26} is the volume of space time. The one loop cosmological constant is just $\frac{Z_{torus}}{-V_{26}}$. Of critical importance here is the fact that the integral is performed over the fundamental domain F defined as

$$-\frac{1}{2} < \tau_1 < \frac{1}{2}, \quad \tau_2 > 0, \quad |\tau| > 1. \tag{3.6}$$

3.2 Modular Invariance and the Torus Amplitude

The restriction to the fundamental domain can be understood by recalling the definition of a torus. Consider a parallelogram in the complex plane with sides represented by the complex parameters λ_1 and λ_2 (that behave as vectors in the complex plane). A torus is obtained by identifying opposite edges of the parallelogram. If the plane is tiled with these parallelograms then we may identify equivalent points on the tori:

$$z \sim z + n_1\lambda_1 + n_2\lambda_2 \tag{3.7}$$

for $n_i = 0, \pm1, \pm2, \cdots$.

There is, however, a certain amount of redundancy here. Consider

$$n_1\lambda_1 + n_2\lambda_2 = n_1'\lambda_1' + n_2'\lambda_2'. \tag{3.8}$$

If, for a given λ and for arbitrary $\mathbf{n}$ we can find always find a λ' that satisfies this equation for a particular choice of $\mathbf{n}'$, then λ and λ' describes the same torus.

To be more precise all λ that are related by

$$\lambda' = \mathbf{M}\lambda \tag{3.9}$$

$$M = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \tag{3.10}$$

where M is an SL(2,Z) matrix are equivalent tori. The world sheet action is conformally invariant so, in fact, the absolute size of the tori is irrelevant, only the ratio of its sides is important. We thus define $\tau = \frac{\lambda_2}{\lambda_1}$ Equation (3.9) then implies that all the relationships between all equivalent tori are given by

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{3.11}$$

All of these transformations can be generated by the two simple transformations

$$S: \ \tau \to -\frac{1}{\tau} \tag{3.12}$$

and

$$T: \ \tau \to -\tau + 1. \tag{3.13}$$

If we wish to parametrize the path integral in terms of unique tori then we must restrict τ to a region where it is unique. One such region is the fundamental domain

$$-\frac{1}{2} < \tau_1 < \frac{1}{2}, \quad \tau_2 > 0, \quad |\tau| > 1. \tag{3.14}$$

The remaining infinite number of unique regions lie within the semi-circle in the strip of the upper half plane

$$-\frac{1}{2} < \tau_1 < \frac{1}{2}, \quad \tau_2 > 0, \quad |\tau| < 1. \tag{3.15}$$

These regions are all equivalent and can be mapped to each other via (3.11).

4: Strings in a Kalb-Ramond Field

4.1 The Massless States of The Closed Bosonic String

The massless states of the closed bosonic string are the graviton $G_{\mu\nu}$, the dilaton ϕ, and the Kalb-Ramond field (or antisymmetric field) $B_{\mu\nu}$. To be explicit, the massless states of momentum p in the light-cone gauge are given as

$$e_{ij}\,\alpha^{i}_{-1}\tilde{\alpha}^{j}_{-1}\,|0;p\rangle \tag{4.1}$$

where $j = 1, 2, \ldots (D-2)$ and e_{ij} is the transverse polarization tensor. The states transform as a two index tensor under SO(D-2). We may reduce this representation into a symmetric traceless tensor, an antisymmetric tensor, and a scalar that do not mix under SO(D-2) transformations. Any tensor e^{ij} can be decomposed as:

$$[\frac{1}{2}(e_{ij}+e_{ji}) - \frac{1}{D-2}\delta_{ij}tre] + [\frac{1}{2}(e_{ij}-e_{ji})] + [\frac{1}{D-2}\delta_{ij}tre] \tag{4.2}$$

The three terms in brackets correspond, respectively, as:

$$[g_{ij}] + [B_{ij}] + [\delta_{ij}\Phi] \tag{4.3}$$

which represent the graviton, the Kalb-Ramond field and the dilaton, respectively. Herein, we shall be mostly interested in B.

The world sheet action for strings is the Polyakov action

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \tag{4.4}$$

In the presence of background fields corresponding to coherent superpositions of the massless states, the action is modified to

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}[(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X))\partial_a X^\mu \partial_b X^\nu + \alpha' R\phi(X)]. \tag{4.5}$$

This equation will be justified in 7.1

4.2 Mass-Shell Condition and Level Matching

We wish to consider the allowed momenta for a string in only a background B field where space time is toroidally compactified on a 26-torus of radius R. In Euclidean space we have

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma[(g^{ab}G_{\mu\nu}(X) + \epsilon^{ab}B_{\mu\nu}(X))\partial_a X^\mu \partial_b X^\nu] \tag{4.6}$$

If $B_{\mu\nu}$ is constant, the second term can be written as a total divergence $\partial_a(\epsilon^{ab}B_{\mu\nu}X^\mu\partial_b X^\nu)$. This term's contribution to the canonical momentum is zero except when X^μ is non-periodic. Therefore, to determine the allowed momenta we consider only the non-periodic parts of X^μ and write

$$X^\mu = x^\mu(\tau) + w^\mu R\sigma + oscillators \tag{4.7}$$

where w^μ is an integer that represents the number of times the string wraps around the X^μ direction. The action then becomes

$$S = \int d\tau[\frac{1}{2\alpha'}G_{\mu\nu}(\dot{x}^\mu\dot{x}^\nu + w^\mu w^\nu R^2) + \frac{1}{\alpha'}B_{\mu\nu}\dot{x}^\mu w^\nu R]. \tag{4.8}$$

The Lagrangian L for the system is just the integrand. We may form the canonical momenta

$$P_\mu = \frac{1}{\alpha'}(G_{\mu\nu}\dot{x}^\nu + B_{\mu\nu}w^\nu R). \tag{4.9}$$

We recognize this momentum as a sum of the conventional momentum and and a field momentum contributed from B. Under quantization it is the canonical momentum that must be quantized and we thus require $P_\mu = \frac{n_\mu}{R}$. We may now write the mode expansion as

$$X^\mu = x^\mu + 2\sigma R w^\mu + 2\tau(G^{\mu\nu}\alpha'\frac{n_\nu}{R} - B^\mu{}_\nu w^\nu R) + oscillators \tag{4.10}$$

Since we have

$$X^\mu(\sigma,\tau) = X_L(\sigma^+) + X_R(\sigma^-) \tag{4.11}$$

with

$$\begin{aligned} X^\mu_R(\sigma^-) &= \tfrac{1}{2}x^\mu + \alpha' p^\mu_R \sigma^- + \sum_{n\neq 0} \tfrac{\alpha^\mu_n}{n} e^{-2in\sigma^-} \\ X^\mu_L(\sigma^+) &= \tfrac{1}{2}x^\mu + \alpha' p^\mu_L \sigma^+ + \sum_{n\neq 0} \tfrac{\tilde{\alpha}^\mu_n}{n} e^{-2in\sigma^+} \end{aligned} \tag{4.12}$$

We may determine the right and left momenta as

$$\alpha' p^\mu_L = w^\mu R + (G^{\mu\nu}\alpha'\frac{n_\nu}{R} - B^\mu{}_\nu w^\nu R) \tag{4.13}$$

and

$$\alpha' p^\mu_R = -w^\mu R + (G^{\mu\nu}\alpha'\frac{n_\nu}{R} - B^\mu{}_\nu w^\nu R) \tag{4.14}$$

For illustrative purposes, we consider a special case for non-zero $B = B_{12}$ with X^1 toroidally compactified. The momentum conjugate to X^μ for flat spacetime ($G^{\mu\nu} =$

$\eta^{\mu\nu}$) is

$$P^{\mu} = \frac{d}{dX_{\mu}}L = \frac{1}{2\pi\alpha'}\int_0^{\pi}[\partial_{\tau}X^{\mu} + B^{\mu}_{\nu}\partial_{\sigma}X^{\nu}]d\sigma. \tag{4.15}$$

We write $P^{\mu} = p^{\mu} + p^{\mu}_{B}$. The first term here is the usual kinetic momentum p^{μ} that multiplies the term linear in τ in the mode expansion of X^{μ}.

In the X^1 direction the left and right moving parts are modified to:

$$\begin{aligned} X^1_R(\sigma^-) &= \tfrac{1}{2}x^1 + (-wR + \alpha'\tfrac{n}{R} - BwR)\sigma^- + \sum_{n\neq 0}\tfrac{\alpha^1_n}{n}e^{-2in\sigma^-} \\ X^1_L(\sigma^+) &= \tfrac{1}{2}x^1 + (wR + \alpha'\tfrac{n}{R} - BwR)\sigma^+ + \sum_{n\neq 0}\tfrac{\tilde{\alpha}^1_n}{n}e^{-2in\sigma^+} \end{aligned} \tag{4.16}$$

The second term in (4.15) is the field momentum p^{μ}_{B}. This term only contributes when X^1 is compactified on a torus with radius R then we have $X^1(\tau, \sigma + \pi) \equiv X^1(\tau,\sigma) + 2\pi Rw$. In this case

$$\int_0^{\pi}\partial_{\sigma}X^1 d\sigma = 2\pi Rw \tag{4.17}$$

and hence the field momentum is non-zero.

The zero-mode Virasoro constraints are implemented as usual:

$$L_0 = \frac{T}{2}\int_0^{\pi}\dot{X_R}^2 d\sigma = 1 \tag{4.18}$$

$$\tilde{L}_0 = \frac{T}{2}\int_0^{\pi}\dot{X_L}^2 d\sigma = 1. \tag{4.19}$$

Inserting (5.26) and (4.16) into these constraints we obtain the mass-shell and level matching rules:

$$\sum_{\mu\neq 1}p^{\mu}p_{\mu} = (p^1)^2 + (\frac{wR}{\alpha'})^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{4.20}$$

and

$$N - \tilde{N} = wn. \tag{4.21}$$

Since the canonical momentum is the generator of translations, it must be discrete in the X^1 direction: $P^1 = \frac{n}{R}$. Also, we may replace the kinetic momentum with the canonical one which changes the mass-shell condition to [13]:

$$(P^2 - 2\pi BRw)^2 + \sum_{\mu\neq 1,2} p^\mu p_\mu = (\frac{n}{R})^2 + (\frac{wR}{\alpha'})^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{4.22}$$

This contrasts the the free-field string case in that there is now a shift in the mass.

5: The Vacuum Energy

The vacuum energy can be calculated in essentially two different ways: via the operator formalism, or via the path integral formalism. Application of either of these methods will reveal that for an appropriate configuration the vacuum energy is a periodic function of B. The configuration we shall choose to investigate is two toroidally compactified dimensions with a nonzero B field in the component of those directions. The operator formalism is instructive, but the more correct method in this case is the path integral formalism. However, as discussed above, the operator formalism can be corrected by performing the restriction to the fundamental domain. We will proceed in this manner.

5.1 Vacuum energy of the uncompactified string

We may determine the vacuum energy by evaluating the vacuum persistence amplitude in the euclidean formalism.

$$Z = \langle 0| \, e^{-HT} \, |0\rangle = e^{-E_0 T}. \tag{5.1}$$

For the scalar field ϕ in D non-compact dimensions we have

$$Z = \int D\phi e^{-S} = \int D\phi e^{-\frac{1}{2}\int d^D x \; \phi(-\partial^2 + m^2)\phi} = e^{-\frac{1}{2}Tr\ln(-\partial^2+m^2)} \tag{5.2}$$

We may evaluate the trace as the sum of the eigenfunctions and obtain the vacuum energy

$$E_0T = \frac{1}{2}VT\int \frac{d^Dp}{(2\pi)^D}\ln(p^2+m^2) \tag{5.3}$$

where V is the spatial volume of $D-1$ dimensions and T is the temporal length or time.

A convenient representation of the logarithm for an arbitrary operator $\hat{O}$ is

$$\ln(\hat{O}) = -\lim_{\epsilon\to 0}\left(\int_\epsilon^\infty \frac{d\lambda}{\lambda}e^{-\lambda\hat{O}} - \int_\epsilon^\infty \frac{d\lambda}{\lambda}e^{-\lambda}\right). \tag{5.4}$$

This can be verified by repeated differentiation and by observing that $\ln(1) = 0$.We see that the divergent part is independent of $\hat{O}$ and can neglected. Defining the vacuum energy density as $\Lambda = \frac{E_0}{V}$ we find that

$$\Lambda = -\frac{1}{2}\int_0^\infty \frac{d\lambda}{\lambda}\int \frac{d^Dp}{(2\pi)^D}e^{-\lambda(p^2+m^2)} = -\frac{1}{2}\int_0^\infty \frac{d\lambda}{\lambda}e^{-\lambda m^2}(4\pi\lambda)^{-D/2}. \tag{5.5}$$

This is the vacuum energy for scalar field on an uncompactified flat space. For bosonic strings, we should sum over the masses of the spectrum while satisfying the level matching condition (4.21). That is, the vacuum energy becomes

$$\Lambda = -\frac{1}{2}\sum_m \int_0^\infty \frac{d\lambda}{\lambda}e^{-\lambda m^2}(4\pi\lambda)^{-D/2}. \tag{5.6}$$

with $m^2 = \frac{2}{\alpha'}(N+\tilde{N}-2)$. We may enforce the level matching condition (4.21) with a delta function.

$$\delta_{N-\tilde{N}} = \int_{-1/2}^{1/2} d\tau_1 e^{2\pi i(N-\tilde{N})\tau_1} \tag{5.7}$$

and then trace over N and $\tilde{N}$:

$$\Lambda = -\frac{1}{2}\left(\frac{1}{4\pi^2\alpha'}\right)^{13}\int d\tau_1 d\tau_2 \tau_2^{-14}e^{4\pi\tau_2}tr(e^{2\pi i\tau N - 2\pi i\bar{\tau}\tilde{N}}) \tag{5.8}$$

where we have defined $\tau_2 = \lambda/\alpha'\pi$ $\tau = \tau_1 + i\tau_2$.

For the evaluation of the trace we recall the relations (2.36) and (2.37) and define $q = e^{2\pi i\tau}$. Now we note

$$tr(q^{\sum_{m=1}^{\infty} m\mathbf{a}_m^{\dagger}\cdot\mathbf{a}_m}) = \left(\prod_{m=1}^{\infty}\frac{1}{1-q^m}\right)^{24}. \tag{5.9}$$

The factor of 24 comes from performing the trace over the transverse states. We now define the Dedekind eta function as

$$\eta(q) \equiv q^{1/24}\prod_{m=1}^{\infty}(1-q^m). \tag{5.10}$$

In particular, then, we have

$$\sum_m e^{-\tau_2 m^2} = \int_{-1/2}^{1/2} d\tau_1 \left|\eta(e^{2\pi i\tau})\right|^{-48} \tag{5.11}$$

and finally:

$$\Lambda = -\frac{1}{2}\left(\frac{1}{4\pi^2\alpha'}\right)^{13}\int\frac{d\tau_1 d\tau_2}{\tau_2^{14}}\left|\eta(e^{2\pi i\tau})\right|^{-48}. \tag{5.12}$$

The integral here ranges from $-1/2 \leq \tau_1 \leq 1/2$ and $0 < \tau_2 < \infty$. However as we have learned from the path integral formalism we should in fact truncate the integral to the fundamental region $-1/2 \leq \tau_1 \leq 1/2$, $\tau_2 > 0$ and $abs(\tau) > 1$. The physical reason for this is that the above is effectively a quantum field theory calculation and that we are over counting states. From the path integral perspective, we are calculating the torus amplitude. The τ_i correspond to the Teichmuller parameters that characterize an arbitrary torus. However, some different τ_i correspond to the same torus and we must only count each unique torus once. Hence we only integrate over the fundamental domain.

5.2 Vacuum Energy with one compact direction

In a configuration with one toroidally compact direction X^1 we make the identification

$$X^1 \sim X^1 + 2\pi R \tag{5.13}$$

Because the momentum is the generator of spacetime translation, the momentum P^1 must be discrete

$$P^1 = \frac{n}{R} \qquad n = 0, \pm 1, \pm 2, \cdots . \tag{5.14}$$

Additionally, there exists the winding modes, so we must include modes that wrap around the compact direction. Denoting the number of wraps by w, we write the mode expansion as

$$X^1 = x^1 + 2\sigma R w + 2\alpha' \frac{n}{R}\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha_n^1 e^{-2in(\tau - \sigma)} + \tilde{\alpha}_n^1 e^{-2in(\tau + \sigma)} \right) . \tag{5.15}$$

Evaluating the zero mode Virasoro generators $L_0 = \tilde{L}_0 = 1$ we obtain the mass-shell (for the 25- and 26-dimensional mass - see section 7.2 for their definition) and level matching conditions

$$m_{25}^2 = \frac{\alpha'}{2}(N + \tilde{N} - 2) + \frac{n^2}{R^2} + \frac{R^2 w^2}{(\alpha')^2} \tag{5.16}$$

$$m_{26}^2 = \frac{\alpha'}{2}(N + \tilde{N} - 2) + \frac{R^2 w^2}{(\alpha')^2} \tag{5.17}$$

$$N - \tilde{N} = nw \tag{5.18}$$

with

$$N = \sum_{n=1}^{\infty} \left(\alpha^{\mu}_{-n} \alpha_{n\mu} + \alpha^{1}_{-n} \alpha_{n1} \right) \tag{5.19}$$

$$\tilde{N} = \sum_{n=1}^{\infty} \left(\tilde{\alpha}^{\mu}_{-n} \tilde{\alpha}_{n\mu} + \tilde{\alpha}^{1}_{-n} \tilde{\alpha}_{n1} \right) \tag{5.20}$$

where μ here excludes the compactified dimensions X^1.

To convert (5.12) into the correct form 26-dimensional vacuum energy for a single toroidal compactification, we replace the integration $\int dp_1$ with a sum $\frac{2\pi}{R} \sum_n$ and also include the winding modes in the mass by replacing m with m_{26}. The vacuum energy is then modified to

$$\Lambda_{M^{25} \times T^1} = -\frac{1}{2} \frac{(\alpha')^{(1-D)/2}}{(2\pi)^D} \sum_{w,n} \frac{1}{R} \int_F d\tau_1 d\tau_2 \, \tau_2^{-(D+1)/2} e^{-\alpha' \tau_2 \pi \left((\frac{n}{R})^2 + (\frac{wR}{\alpha'})^2 \right) - 2\pi i w n \tau_1} \left| \eta(\tau) \right|^{-48} \tag{5.21}$$

where $D = 26$.

5.3 Vacuum Energy with many compact directions

We now consider the case of $M^d \times T^k$. In this case use label index m for the compact directions and write the mode expansion as

$$X^m = x^m + 2\sigma L^m + 2\alpha' p^m \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha^m_n e^{-2in(\tau - \sigma)} + \tilde{\alpha}^m_n e^{-2in(\tau + \sigma)} \right). \tag{5.22}$$

It turns out that the are various ways one can consistently choose compactified momenta. To make this manifest, we use the left and right momenta.

$$\alpha' p^m_L = L^m + \alpha' p^m \tag{5.23}$$

$$\alpha' p_R^m = L^m - \alpha' p^m \tag{5.24}$$

with

$$X^\mu(\sigma,\tau) = X_L(\sigma^+) + X_R(\sigma^-) \tag{5.25}$$

$$\begin{aligned} X_R^\mu(\sigma^-) &= \tfrac{1}{2}x^\mu + \alpha' p_R^\mu \sigma^- + \sum_{n\neq 0} \tfrac{\alpha_n^\mu}{n} e^{-2in\sigma^-} \\ X_L^\mu(\sigma^+) &= \tfrac{1}{2}x^\mu + \alpha' p_L^\mu \sigma^+ + \sum_{n\neq 0} \tfrac{\tilde{\alpha}_n^\mu}{n} e^{-2in\sigma^+} \end{aligned} \tag{5.26}$$

The zero-mode Virasoro constraints now give

$$N - \tilde{N} = \frac{\alpha'}{4}\left(\mathbf{p}_L^2 - \mathbf{p}_R^2\right). \tag{5.27}$$

where the boldface indicates a vector over the indices m. The left hand side of this equation is an integer and therefore we must demand that the right hand side of this equation be integers. The modification here contrasts the T^1 compactification because the periodic coordinate identifications we make need not be at right angles. For example, we may identify points as

$$\mathbf{X} \equiv \mathbf{X} + 2\pi\sqrt{\alpha'}\mathbf{E}_m \tag{5.28}$$

Where the basis vectors $\mathbf{E_m}$ are complete over T^k. Thus we may write

$$\mathbf{L} = \sqrt{\alpha'} w_m \mathbf{E}_m. \tag{5.29}$$

In order to make sure that the momenta are the generators of translations we write

$$\mathbf{p} = \sqrt{\alpha'} n_m \mathbf{e}_m \tag{5.30}$$

where we have formed the reciprocal lattice vectors $\mathbf{e_i}$

$$\mathbf{e}_p \cdot \mathbf{E}_q = \delta_{pq}. \tag{5.31}$$

In this notation we may write the mass shell and level matching condition as

$$m_{D-d}^2 = \frac{\alpha'}{2}(N + \tilde{N} - 2) + \mathbf{p}^2 + \frac{\mathbf{L}^2}{(\alpha')^2} \tag{5.32}$$

$$N - \tilde{N} = \mathbf{n} \cdot \mathbf{w} \tag{5.33}$$

We will typically be interested in the case where the basis vectors $\mathbf{E}_i$ are all orthogonal and equal to R in length. In this case we have simply

$$m_{D-d}^2 = \frac{\alpha'}{2}(N + \tilde{N} - 2) + \frac{\mathbf{n}^2}{R^2} + \frac{R^2\mathbf{w}^2}{(\alpha')^2}. \tag{5.34}$$

Now the cosmological constant is adjusted to

$$\begin{aligned} \Lambda_{M^{D-k}\times T^k} &= -\frac{1}{2}\frac{(\alpha')^{(k-D)/2}}{(2\pi)^D}\sum_{\mathbf{w},\mathbf{n}} \\ &\frac{1}{R^k}\int_F d\tau_1 d\tau_2\ \tau_2^{-(D+2-k)/2} e^{-\alpha'\tau_2\pi\left((\frac{\mathbf{n}}{R})^2+(\frac{\mathbf{w}R}{\alpha'})^2\right)-2\pi i \mathbf{w}\cdot\mathbf{n}\tau_1}\, |\eta(q)|^{-48}\,. \end{aligned} \tag{5.35}$$

5.4 Vacuum Energy with 2 compact directions and a B field

In the configuration where two spatial dimensions are compactified (say X^1 and X^2) and a non-zero (and constant) B-field is permitted only in the $B_{12} = -B_{21}$ component, an unusual feature appears: the vacuum energy becomes a periodic function of B. This configuration will be the central focus of what follows, as it can give rise to a unique topological structure.

In the free field case, two compactified dimensions have a cosmological constant of

$$\Lambda = -\frac{1}{2}\frac{(\alpha')^{(2-D)/2}}{(2\pi)^D}\sum_{\mathbf{w},\mathbf{n}}\frac{1}{R^2}\int_F d\tau_1 d\tau_2\ \tau_2^{-D/2} e^{-\alpha'\tau_2\pi\left((\frac{\mathbf{n}}{R})^2+(\frac{\mathbf{w}R}{\alpha'})^2\right)-2\pi i\mathbf{w}\cdot\mathbf{n}\tau_1}\,|\eta(q)|^{-48}\,. \tag{5.36}$$

where $D = 26$. However, if we turn on $B \equiv B_{12} = -B_{21}$ we must now incorporate the momenta shifts as discussed in section 4.2. The mode expansions for the directions that are modified display the momenta shift (the coefficient of $2\tau\alpha'$):

$$X^1 = x^1 + 2\sigma R w^1 + 2\tau(\alpha'\frac{n^1}{R} - Bw^2R) + oscillators \tag{5.37}$$

$$X^2 = x^2 + 2\sigma R w^2 + 2\tau(\alpha'\frac{n^2}{R} + Bw^1R) + oscillators \tag{5.38}$$

Thus the cosmological constant is modified to

$$\begin{aligned}\Lambda = & - \frac{1}{2}\frac{(\alpha')^{(2-D)/2}}{(2\pi)^D}\sum_{\mathbf{w},\mathbf{n}}\frac{1}{R^2}\int_F d\tau_1 d\tau_2\ \tau_2^{-D/2} \\ & \cdot\ e^{-\alpha'\tau_2\pi\left(\frac{1}{R^2}(n_1-\Phi w_2)^2+\frac{1}{R^2}(n_2+\Phi w_1)^2+(\frac{\mathbf{w}R}{\alpha'})^2\right)-2\pi i\mathbf{w}\cdot\mathbf{n}\tau_1}\,|\eta(\tau)|^{-48}\,.\end{aligned} \tag{5.39}$$

We have defined the flux as $\Phi \equiv 2\pi\frac{R^2}{\alpha'}B$. Note that the level matching condition is left unchanged after modifying the right a left momenta. While it is not immediately apparent, (5.39) is periodic in Φ. To make the periodicity manifest we must perform some rearrangement. Consider the component

$$Q \equiv \sum_{\mathbf{n}} e^{-\alpha'\tau_2\pi\left(\frac{1}{R^2}(n_1-\Phi w_2)^2+\frac{1}{R^2}(n_2+\Phi w_1)^2+(\frac{\mathbf{w}R}{\alpha'})^2\right)-2\pi i\mathbf{w}\cdot\mathbf{n}\tau_1}. \tag{5.40}$$

We shall now perform a Poisson re-sum:

$$Q = \int_{-\infty}^{\infty} d^2p \sum_{\mathbf{k}} e^{2\pi i \mathbf{k}\cdot\mathbf{p}} e^{-\alpha' \tau_2 \pi \left(\frac{1}{R^2}(p_1 - \Phi w_2)^2 + \frac{1}{R^2}(p_2 + \Phi w_1)^2 + (\frac{\mathbf{w}R}{\alpha'})^2\right) - 2\pi i \mathbf{w}\cdot\mathbf{p}\tau_1}. \tag{5.41}$$

Define the dual of $\mathbf{w}$ as $\hat{\mathbf{w}} = (-w_2, w_1)$ and shift $\mathbf{p} \to \mathbf{p} + \hat{\mathbf{w}}\Phi$ and perform the integration over $\mathbf{p}$:

$$Q = \sum_{\mathbf{k}} \frac{R^2}{\alpha' \tau_2} e^{-\frac{\pi R^2}{\alpha' \tau_2}(k_1 - w_1\tau_1)^2 - \frac{\pi R^2}{\alpha' \tau_2}(k_2 - w_2 \tau_1)^2 - \alpha' \tau_2 \pi (\frac{\mathbf{w}R}{\alpha'})^2} e^{2\pi i \mathbf{k}\cdot\hat{\mathbf{w}}\Phi} \tag{5.42}$$

and insert this back into (5.39) and we obtain

$$\Lambda = -\frac{1}{2}\frac{(\alpha')^{-D/2}}{(2\pi)^D} \int d\tau_1 \frac{d\tau_2}{\tau_2^{D/2+1}} \sum_{\ell} e^{2\pi i \Phi \ell} g_\ell \, |\eta(q)|^{-48} \tag{5.43}$$

$$g_\ell = \sum_{\mathbf{w},\mathbf{k}} e^{-\frac{\pi R^2}{\alpha' \tau_2}\left[(k_1 - w_1 \tau_1)^2 + (k_2 - w_2 \tau_1)^2\right] - \tau_2 \pi \frac{(R\mathbf{w})^2}{\alpha'}} \delta(\ell, \mathbf{k}\cdot\hat{\mathbf{w}}) \tag{5.44}$$

Immediately the periodicity in Φ is manifest. Further, if g_ℓ is a rapidly decreasing function for large negative and positive ℓ we find that the most important contribution to the periodic function is the first harmonic. We thus write the cosmological constant as

$$\Lambda_1 \cos(2\pi \frac{R^2}{\alpha'} B). \tag{5.45}$$

In summary, we have found that the cosmological constant for $T^2 \times M^{24}$ immersed in a constant B-field is periodic in B. Obtaining the value of Λ_1 is not trivial. On dimensional grounds Λ_1 must scale as $(\alpha')^{-D/2}$. We also expect Λ_1 to tend to zero as $R \to 0$ (this is the no compactification limit and there should be no periodic term). Further, since Λ_1 arises from the torus amplitude, we expect there to be no g_s dependence. A dimensional estimation then yields

$$\Lambda_1 \sim (\alpha')^{-D/2} \frac{\alpha'}{R^2}. \tag{5.46}$$

A more careful analysis involving the method of steepest descent modifies this by adding an exponential suppression. Consider the term in the exponent found in the cosmological constant

$$-\pi \frac{R^2}{\alpha'} U(\tau_1, \tau_2) \equiv -\frac{\pi R^2}{\alpha' \tau_2} \left[(k_1 - w_1 \tau_1)^2 + (k_2 - w_2 \tau_1)^2 \right] - \tau_2 \pi \frac{(R\omega)^{\mathbf{2}}}{\alpha'} + 2\pi i \mathbf{k} \cdot \hat{\mathbf{w}} \Phi. \tag{5.47}$$

By using 'Polchinski's Trick' [14] we may eliminate w_2 in favour of integrating over the full strip, and not just the fundamental domain. Then we may trade integrating over $\tau_1 = [-1/2, 1/2]$ to $\tau_1 = [-\infty, \infty]$ but summing only over $k_1 < w_1$. Minimizing U with respect to τ_1 and τ_2 we obtain the extrema, respectively

$$t_1 = \frac{k_1}{w_1} \tag{5.48}$$

$$t_2 = \left| \frac{k_2}{w_1} \right|. \tag{5.49}$$

The exponent at the extremum now becomes

$$-2\pi \frac{R^2}{\alpha'} |k_2 w_1| + 2\pi i k_2 w_1 \Phi. \tag{5.50}$$

Now we may proceed to write

$$U(\tau_1, \tau_2) = U(t_1, t_2) + \frac{1}{2} U_{1,1} (\tau_1 - t_1)^2 + \frac{1}{2} U_{2,2} (\tau_2 - t_2)^2 + U_{1,2} (\tau_1 - t_1)(\tau_2 - t_2) + \cdots \tag{5.51}$$

and perform the integration over the upper-half plane. Noting

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-a(x-x_0)^2-b(y-y_0)^2-c(x-x_0)(y-y_0)}dxdy = \frac{2\pi}{\sqrt{4ab-c}} \tag{5.52}$$

We find that the dominant periodic term behaves as

$$\Lambda_1 \cos(2\pi\frac{R^2}{\alpha'}) \tag{5.53}$$

where

$$\Lambda_1 = (\alpha')^{-D/2}\frac{\alpha'}{R^2}e^{-\frac{R^2}{\alpha'}}. \tag{5.54}$$

6: Elimination of the Tachyon Contribution

While the discovery of a periodic cosmological constant is rather interesting, it is mired by the fact that the constant which multiplies the periodic term is in fact divergent. We must eliminate this divergence to make a physically reasonable interpretation of the vacuum energies. We may proceed in two ways.

6.1 Spectrum Truncation

The tachyon contribution comes from the $N = \tilde{N} = 0$ modes. In evaluating the vacuum energy we traced over

$$e^{-2\pi i\tau\tilde{N}+2\pi i\bar{\tau}N} = |f(q)|^{-48}\,. \tag{6.1}$$

This causes the integral to diverge when $N = \tilde{N} = 0$. One way to remove the tachyon contribution, is to simply subtract off the $N = \tilde{N} = 0$ contribution, which amounts to replacing

$$|f(q)|^{-48} \to |f(q)|^{-48} - 1 \tag{6.2}$$

or equivalently

$$|\eta(q)|^{-48} \to |\eta(q)|^{-48} - e^{4\pi\tau_2} \tag{6.3}$$

Note however, that the modular invariance of the vacuum energy is now spoiled since $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$ is no longer a symmetry. The value of the cosmological constant has been carried out numerically for this scheme [15].

6.2 Relative Subtraction

Perhaps a more reasonable (i.e., modular invariant) way is to look at the change in vacuum energy with respect to the non-compactified, field free vacuum energy.We define the renormalized cosmological constant as

$$\Lambda_r = \Lambda(B) - \Lambda(B=0). \tag{6.4}$$

We should note, that there is some ambiguity in the choice of regularization. In reality, any contribution to the vacuum energy should be observable due to its coupling to gravity. Bosonic string theory is pathological in this sense and cannot be remedied. However, for exploratory purposes we must choose a method of regularization so that we may have tenable results. Further, the contribution to Λ_1 is finite, so we may proceed by considering this term only.

7: String Theory Effective Action

The path integral formalism is a general prescription for quantizing theories with classical actions. The key element of this method is the vacuum amplitude Z. The vacuum amplitude is formed by taking the weighted sum over all possible classical configurations. For strings, the different configurations are the different worldsheets. The weight is given as e^{-S} where S is the action for the worldsheet. S is the area of the surface multiplied by the dimensional parameter T which produces the dimensionless action plus some additional terms. One of the additional terms (as discussed in section 3.1) is the Euler character. The value of this term is set by the number of handles in the world sheet and thus becomes the coupling constant g_s (the string Feynman diagrams are weighted such that each additional loop provides a weight of g_s):

$$Z = \sum_{genus\ h=0}^{\infty} g_s^{2h-2} \int DxDge^{-S_P[x,g]}. \tag{7.1}$$

Consider the worldsheet depicted in figure 7.1. Here we have a closed string that propagates to another closed string. Consider, now, the more complicated worldsheet shown in figure 7.2. Here we have a closed string that propagates, splits into two strings, and recombines in to one string. This diagram is analogous to a one-loop Feynman diagram.

The relative weighting of these diagrams is set by the expectation value of the dilaton though the Euler character. There is no way to guess at a value for the coupling constant and we must therefore introduce it as an arbitrary string coupling

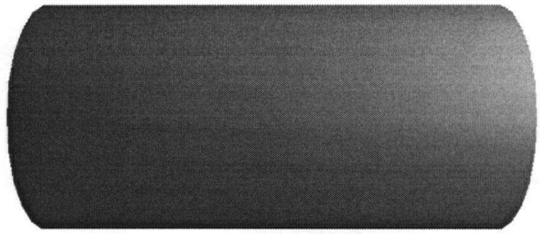

Figure 7.1: Zero loop string diagram

constant that fixes their relative weights of the two diagrams. More precisely we see that $\Phi \to \Phi \cdot const.$ sends $g_s \to e^{const.} g_s$. However, while the relative weighting of two diagrams is arbitrary, once we choose a value, the relative weight with respect to a third diagram is fixed.

We shall be interested in topological features of the massless Kalb-Ramond field. The most convenient way to deal with these features is through the use of the effective action. The effective action method transforms the world-sheet perspective to the spacetime perspective. The effective action is the spacetime action for string theory. By this, we mean that is should generate the spacetime S-matrix elements of interest.

The ambiguity of the relative weighting of string diagrams is also apparent in thee effective action. It is invariant under $g_s \to e^{const.} g_s$, $\Phi \to \Phi - 2const.$. We see that the constant part of the dilaton shifts g_s and in a sense, g_s is set by the vacuum expectation value of the dilaton.

7.1 The Effective Action as the Spacetime Action

The Polyakov action (4.5) is the worldsheet action for describing a string moving in a gravitational, Kalb-Ramond, and dilaton field. There is, however, an alternative way to examine strings propagating in background fields: the string effective action. The effective action formalism in string theory provides and spacetime perspective for

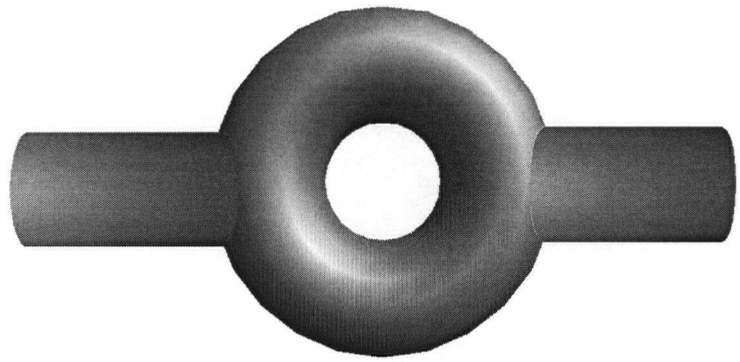

Figure 7.2: One loop string diagram

strings in background fields. The complete derivation of the string spacetime action is straightforward, but somewhat involved [16]. However, the motivation is brief.

The equations of motion for the internal metric (2.15) is traceless when the world sheet is two-dimensional:

$$T_a^a = \frac{1}{\alpha'} G_{\mu\nu}(\partial_a X^\mu \partial^a X^\nu - \frac{1}{2}\delta_a^a \partial^c X^\mu \partial_c X^n) \sim (1 - \frac{q}{2}) = 0 \tag{7.2}$$

where $q = 2$ is the dimension of the worldsheet. This relationship is anomalous. That is, after quantization this condition only holds in special circumstances. The vanishing of the trace corresponds to maintaining conformal invariance in the quantized theory. It turns out that in order to guarantee the disappearance of the anomalies three differential equations must be satisfied:

$$0 = \beta_{\mu\nu}^G = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} H_\nu{}^{\lambda\rho} + \mathcal{O}(\alpha'^2) \tag{7.3}$$

$$0 = \beta_{\mu\nu}^B = -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha'(\nabla^\lambda \Phi) H_{\lambda\mu\nu} + \mathcal{O}(\alpha'^2) \tag{7.4}$$

$$0 = \beta^\Phi = -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha'(\nabla\Phi)^2 - \frac{1}{24}\alpha' H^2 + \mathcal{O}(\alpha'^2) \tag{7.5}$$

where ∇_μ is the covariant derivative. These three equations are linear combinations of the equations of motion which can be derived from the action

$$S = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[\frac{2(D-26)}{3\alpha'} + R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 2\partial_\mu \Phi \partial^\mu \Phi + \mathcal{O}(\alpha') \right]. \tag{7.6}$$

Specifically, we have

$$\delta S = -\frac{1}{2\kappa_0^2 \alpha'} \int d^D X \sqrt{-G} e^{-2\Phi} \left[\delta G_{\mu\nu} \left(\beta^{G\mu\nu} - \frac{1}{2} G^{\mu\nu} (\beta^{G\rho}_{\rho} - 4\beta^{\Phi}) \right) + \delta B_{\mu\nu} \left(\beta^{B\mu\nu} \right) + \delta\Phi \left(2\beta^{G\rho}_{\rho} - 8\beta^{\Phi} \right) \right] = 0. \tag{7.7}$$

The action (8.5) is the effective action for the low energy spacetime fields. It can be shown that using this action to generate S-matrix elements is equivalent to generating S-matrix elements from the standard operator formalism (for one-loop closed strings coupling to massless fields). Equation (8.5) will be of central focus in what follows.

7.2 Kaluza-Klein Theory

Kaluza-Klein theory is a classical attempt to unify electromagnetism with gravity [17]. The original idea was to consider gravity on a five dimensional manifold where one of the directions is periodic. In this section we let the compact direction be specified by y and the non-compact directions by x^μ. The manifold is $M^4 \times S^1$ with $0 \le y \le 2\pi R$ where R is the compactification radius. The 5-D Einstein action is

$$S_5 = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g_5} R_5 \tag{7.8}$$

We may parametrize the metric any way we wish and we choose the following:

$$g_{5\mu\nu} = e^{\frac{\phi}{\sqrt{3}}} \begin{pmatrix} g_{\mu\nu} + e^{-\sqrt{3}\phi} A_\mu A_\nu & e^{-\sqrt{3}\phi} A_\mu \\ e^{-\sqrt{3}\phi} A_\nu & e^{-\sqrt{3}\phi} \end{pmatrix} \tag{7.9}$$

Here, ϕ is a scalar, A_μ is a vector, and $g_{\mu\nu}$ is to be the usual 4-D spacetime metric. Owing to the periodicity of the fifth dimension we may expand ϕ as a Fourier series

$$\phi(x,y) = \sum_{n=-\infty}^{\infty} \phi_n(x) e^{iny/r} \tag{7.10}$$

and similar expansions for the other fields. If we substitute the Fourier expansions and the parametrization of $g_{5\mu\nu}$ into (7.8) and integrate over y only keeping the $n = 0$ terms ($\phi_0 = \phi$, etc.) we obtain (after some algebra, the details of which are discussed in the appendix)

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4} e^{-\sqrt{3}\phi} F_{\mu\nu} F^{\mu\nu}) \tag{7.11}$$

where the field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\kappa^2 = \kappa_5^2/2\pi R$. We have discovered that gravity on $M^4 \times S^1$ is equivalent to gravity on M^4 plus electromagnetism plus a massless scalar field. The presence of the dilaton ϕ was troublesome at first, but in modern times it has become an important theoretical element.

Kaluza-Klein theory also has a mechanism for constructing massive fields out of massless ones. Consider the free massless scalar propagating on $R^4 \times S^1$:

$$\partial_5^2 \phi(x,y) = 0. \tag{7.12}$$

If we plug the Fourier expansion for ϕ in we obtain an infinite number of uncoupled equations of the form

$$(\partial_\mu \partial^\mu - m_n^2)\phi_n = 0 \tag{7.13}$$

where $m_n^2 \equiv \frac{n^2}{R^2}$. To a four dimensional observer (who defines mass as $m_4^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2$) these modes appear as massive scalar fields. In regards to these modes, one speaks of the Kaluza-Klein tower of states. Of course, the five dimensional observer still defines mass as $m_5^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 - p_4^2$.

7.3 Compactification of the Effective Action

The effective action discussed in the previous section was for $D = 26$. We wish to analyze the effective action on the space $M^{24} \times T^2$. We do this by way of Kaluza-Klein compactification.

Parametrize the metric as

$$ds^2 = G^D_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + G_{mn}(dx^m + A^m_\mu dx^\mu)(dx^n + A^n_\mu dx^\mu) \tag{7.14}$$

Then, the effective action can be *dimensionally reduced* to

$$\begin{aligned} S_{0,k} = \frac{(2\pi R)^k}{2\kappa_0^2} \int d^d x \sqrt{-G_d} e^{-2\Phi_d} [R_d + 4\partial_\mu \Phi_d \partial^u \Phi_d \\ - \tfrac{1}{4} G^{mn} G^{pq} (\partial_\mu G_{mp} \partial^u G_{nq} + \partial_\mu B_{mp} \partial^u B_{nq}) \\ - \tfrac{1}{4} G_{mn} F^{(1)m}_{\mu\nu} F^{(1)n\mu\nu} - \tfrac{1}{4} G^{mn} H_{m\mu\nu} H^{\mu\nu}_n - \tfrac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda}] \end{aligned} \tag{7.15}$$

where the following definitions have been employed:

$$\Phi_d \equiv \Phi - \frac{1}{4} - \det(G_{mn}) \tag{7.16}$$

$$F^{(1)m}_{\mu\nu} = \partial_\mu A^{(1)m}_\nu - \partial_\nu A^{(1)m}_\mu \tag{7.17}$$

$$H_{\mu\nu m} = F^{(2)}_{\mu\nu m} - B_{mn} F^{(1)n}_{\mu\nu} \tag{7.18}$$

$$F^{(2)}_{\mu\nu m} = \partial_\mu A^{(2)}_{\nu m} - \partial_\nu A^{(2)}_{\mu m} \tag{7.19}$$

$$A^{(2)}_{\mu m} = \hat{B}_{\mu m} + B_{mn} A^{(1)n}_{\mu} \tag{7.20}$$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{2}\left(A^{(1)m}_{\mu}\ F^{(2)}_{\nu\lambda m} + A^{(2)}_{\mu m} F^{(1)m}_{\nu\lambda}\right) + \text{cyclic perms.} \tag{7.21}$$

$$B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{1}{2} A^{(1)m}_{\mu} A^{(2)}_{\nu m} - \frac{1}{2} A^{(1)m}_{\nu} A^{(2)}_{\mu m} - A^{(1)m}_{\mu} B_{mn} A^{(1)n}_{\nu} \tag{7.22}$$

the details of which are explained in the appendix.

7.4 Adding the Cosmological Term

The general form of the partition function is:

$$Z = \sum_{genus\ h=0}^{\infty} g_s^{2h-2} \int DxDg e^{-S_P[x,g]}. \tag{7.23}$$

Tree level amplitudes correspond $h = 0$ and the torus amplitude is $h = 1$. Alternatively, we take the effective action as the spacetime action. The effective action will then generate the string theory S-matrix. The effective action is invariant under

$$\Phi \to \Phi - c$$

$$g_s \to g_s e^{2c}$$

for constant c. Hence we can always absorb the string constant into the dilaton. We expect the same of the effective action (which must reproduce the same S-matrix). Again, the torus amplitude goes as g_s^0, so we expect the following form for the effective action with the inclusion of the vacuum energy:

$$S = S_{0,k} + \int d^D x \sqrt{-G} \Lambda(B) \tag{7.24}$$

where we may replace $2k_0^2 = (\alpha')^{12} g_s^2$ in $S_{0,k}$. Any uncertainty in the relative size of Λ term from the rest has been absorbed in g_s

We should note that the vacuum energy calculation was performed for a constant B. However, here, we assume B is not constant. We must be in a regime where treating B as non-constant is applicable.

8: Equations of Motion

8.1 Equivalence of String Frame and Einstein Frame

To touch basis with conventional gravity we shall now demonstrate the equivalence of (8.5) and the Einstein-Hilbert action [2]. Consider the action

$$S = \frac{1}{2\kappa^2}\int d^D x\sqrt{-G}e^{-2\Phi}\left[R - \frac{4}{D-2}\partial_\mu\Phi\partial^\mu\Phi\right]. \tag{8.1}$$

This is the Einstein-Hilbert action coupled to a scalar field Φ. In $D = 4$ we identify κ with Newton's gravitational constant G_N: $\kappa^2 = 8\pi G_N$. We shall show that this is equivalent to the string effective action and vice-versa.

Under Weyl transformations of the metric

$$G_{\mu\nu} \to e^{2\omega(x)}G_{\mu\nu} \tag{8.2}$$

the Ricci Scalar Transforms as [18]

$$R \to e^{-2\omega(x)}\left(R - 2(D-1) - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega\right) \tag{8.3}$$

and

$$G \to e^{2D\omega(x)}G. \tag{8.4}$$

If, for constant Φ_0, we define $\kappa = \kappa_0 e^{\Phi_0}$ and choose $\omega = 2\dfrac{\Phi_0 - \Phi}{D-2}$, this transformation turns the action () into the string action

$$S_0 = \frac{1}{2\kappa_0^2} \int d^D x \sqrt{-G} e^{-2\Phi} \left[R + 4\partial_\mu \Phi \partial^\mu \Phi \right]. \tag{8.5}$$

This form of the action not as conveinient for determining the equations of motion as (8.1). We have learnt that string theory predicts general relativity coupled to a scalar field.

8.2 Variation off the Action

We wish to find the equations of motion for the doubly compactified action

$$\begin{aligned} S_{0,2} = & \tfrac{(2\pi R)^2}{(\alpha')^{12} g_s^2} \int d^d x \sqrt{-G_d} e^{-2\Phi_d} \left[R_d + 4\partial_\mu \Phi_d \partial^u \Phi_d \right. \\ & - \tfrac{1}{4} G^{mn} G^{pq} \left(\partial_\mu G_{mp} \partial^u G_{nq} + \partial_\mu B_{mp} \partial^u B_{nq} \right) \\ & \left. - \tfrac{1}{4} G_{mn} F^{(1)m}_{\mu\nu} F^{(1)n\mu\nu} - \tfrac{1}{4} G^{mn} H_{m\mu\nu} H^{\mu\nu}_n - \tfrac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right] \\ & + \int d^D x \sqrt{G} \Lambda. \end{aligned} \tag{8.6}$$

Write $S = S_{GD} + S_{BFG} + S_\Lambda$, where S_{GD} refers to the terms on the first line, S_{BFG} refers to the terms on the second and third line, and S_Λ refers to the final line. The cosmological constant term is constructed from

$$\Lambda = \Lambda_1 (1 - cos(AB)). \tag{8.7}$$

We now perform the variation of the action to obtain the equations of motion for $g_{\mu\nu}$:

$$\frac{\delta S_{GD}}{\delta G^{\mu\nu}} = \frac{\sqrt{-G_d} e^{-2\Phi_d}}{(\alpha')^1 2 g_s^2} \left[R_{d\mu\nu} + 2D_\mu D_\nu - \frac{1}{2} G_{\mu\nu} (R_d + 4D_\mu \partial^\mu \Phi - 4\partial_\mu \Phi \partial^\mu \Phi) \right] \tag{8.8}$$

$$
\begin{aligned}
&\frac{\delta S_{BFG}}{\delta G^{\mu\nu}} = \\
&-\frac{1}{2}\frac{\sqrt{-G_d}}{(\alpha')^{12}g_s^2}e^{-2\Phi_d}G_{\mu\nu}\left[-\frac{1}{4}G^{mn}G^{pq}\left(\partial_\rho G_{mp}\partial^\rho G_{nq}+\partial_\rho B_{mp}\partial^\rho B_{nq}\right)\right. \\
&\left.-\frac{1}{4}G_{mn}F^m{}_{\rho\sigma}F^{n\rho\sigma}-\frac{1}{4}G^{mn}H_{m\rho\sigma}H_n{}^{\rho\sigma}-\frac{1}{12}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}\right] \\
&+\frac{\sqrt{-G_d}}{(\alpha')^{12}g_s^2}e^{-2\Phi_d}\left[-\frac{1}{4}G^{mn}G^{pq}\left(\partial_\mu G_{mp}\partial_\nu G_{nq}+\partial_\mu B_{mp}\partial_\nu B_{nq}\right)\right. \\
&\left.-\frac{1}{2}G_{mn}F^m{}_{\mu\sigma}F^n_\nu{}^\sigma-\frac{1}{2}G^{mn}H_{m\mu\lambda}H^\lambda_{n\nu}-\frac{1}{4}H_{\mu\rho\sigma}H_\nu{}^{\rho\sigma}\right]
\end{aligned}
\tag{8.9}
$$

$$
\frac{\delta S_\Lambda}{\delta G^{\mu\nu}} = -\frac{1}{2}\sqrt{-G_d}G_{\mu\nu}\sqrt{G_{D-d}}\Lambda. \tag{8.10}
$$

The first of these three terms is obtained most easily from the Einstein-Hilbert form of the action (8.1). In this case the variation of the Ricci scalar can be carried out in the usual way [19]. The equation of motion for $G_{\mu\nu}$ is then

$$
\frac{\delta}{\delta G^{\mu\nu}}(S_{GD}+S_{GBF}+S_\Lambda)=0. \tag{8.11}
$$

For G_{mn}:

$$
\frac{\delta S_{GD}}{\delta G^{mn}} = 0 \tag{8.12}
$$

$$
\begin{aligned}
&\frac{\delta S_{BFG}}{\delta G^{mn}} = \\
&\frac{\sqrt{-G_d}}{(\alpha')^{12}g_s^2}e^{-2\Phi_d}\left[-\frac{1}{2}G^{pq}\left(\partial_\rho G_{mp}\partial^\rho G_{nq}+\partial_\rho B_{mp}\partial^\rho B_{nq}\right)\right. \\
&-\left(\partial_\mu\Phi_d G^{ol}G^{pq}\partial^\mu G_{nq}-\frac{1}{2}\nabla_\mu\left(G^{ol}G^{pq}\partial^\mu G_{nq}\right)\right)G_{lp}G_{mo} \\
&\left.+\frac{1}{4}G_{nl}G_{om}F^o_{\rho\sigma}F^{l\rho\sigma}-\frac{1}{4}H_{m\rho\sigma}H_n^{\rho\sigma}\right]
\end{aligned}
\tag{8.13}
$$

$$
\frac{\delta S_\Lambda}{\delta G^{mn}} = -\frac{1}{2}\sqrt{-G_d}\sqrt{-G_{D-d}}G_{mn}\Lambda. \tag{8.14}
$$

The equation of motion for G_{mn} is then

$$\frac{\delta}{\delta G^{mn}}(S_{GD} + S_{GBF} + S_{\Lambda}) = 0. \tag{8.15}$$

For Φ_d

$$\frac{\delta S_{GD}}{\delta \Phi_d} = \frac{\sqrt{-G_d}e^{-2\Phi_d}}{(\alpha')^{12}g_s^2}\left[-2R_d + 8\partial_\mu \Phi_d \partial^\mu \Phi_d - 8D_\mu \partial^\mu \Phi_d\right] \tag{8.16}$$

$$\begin{aligned} \frac{\delta S_{BFG}}{\delta \Phi_d} &= \frac{\sqrt{-G_d}e^{-2\Phi_d}}{(\alpha')^{12}g_s^2}(-2) \\ &\left[-\tfrac{1}{4}G^{mn}G^{pq}\left(\partial_\mu G_{mp}\partial^u G_{nq} + \partial_\mu B_{mp}\partial^u B_{nq}\right)\right. \\ &\left. -\tfrac{1}{4}G_{mn}F^{(1)m}_{\mu\nu}F^{(1)n\mu\nu} - \tfrac{1}{4}G^{mn}H_{m\mu\nu}H^{\mu\nu}_{n} - \tfrac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \end{aligned} \tag{8.17}$$

$$\frac{\delta S_{\Lambda}}{\delta \Phi_d} = 0. \tag{8.18}$$

The equation of motion for Φ_d is then

$$\frac{\delta}{\delta \Phi_d}(S_{GD} + S_{GBF} + S_{\Lambda}) = 0. \tag{8.19}$$

For B

$$\frac{\delta S_{GD}}{\delta B^{mn}} = 0$$

$$\frac{\delta S_{BFG}}{\delta B^{mn}} = \frac{1}{2}\frac{\sqrt{-G_d}}{(\alpha')^{12}g_s^2}\nabla_\mu\left(e^{-2\Phi_d}G^{lo}G^{pq}\partial^\mu B_{oq}\right)G_{lm}G_{pn}$$

$$\frac{\delta S_{\Lambda}}{\delta B^{mn}} = \begin{cases} 0 & \{m,n\} \neq \{1,2\} \\ A\Lambda_0 \sin(BA)\sqrt{-G_d}\sqrt{G_{D-d}} & \{m,n\} = \{1,2\} \end{cases}$$

The equation of motion for B_{mn} is then

$$\frac{\delta}{\delta B^{mn}}(S_{GD} + S_{GBF} + S_{\Lambda}) = 0 \tag{8.20}$$

We are interested in static solutions about flat space with all fields flat except for $B \equiv B_{12}(X^3)$ and therefore the additional equations of motion do not couple in this regime.

The equation of motion for B_{mn} then leads us to the equation of motion for the celebrated sine-Gordon equation:

$$\frac{1}{(\alpha')^{12} g_s^2} \partial_3^2 B(X^3) - A\Lambda_1 \sin(B(X^3)A) = 0 \tag{8.21}$$

Unfortunately, allowing non-zero B means we must consider the other equations of motion that B couples to. We will resolve this shortly, but to do this we must see how (8.21) scales with g_s

Equation (8.21) admits a first integral. We multiply by $\partial_3 B$ and integrate to find

$$\frac{1}{2(\alpha')^{12} g_s^2}(B')^2 + \Lambda_1 \cos(BA) = c \tag{8.22}$$

The constant is determined by requiring B'=0 and $BA = 2\pi n$ at spatial infinity: $c = \Lambda_1$. And thus

$$(B')^2 + 2(\alpha')^{12} g_s^2 \Lambda_1(\cos(BA) - 1) = 0. \tag{8.23}$$

This equation can be solved (as will be done in what follows), however, the most important feature is that $(B')^2 \sim g_s^2$. Since equations of motion for the effective action satisfy the flat solution, we may consider the sine-Gordon solution as pertubative solution in g_s. Since $(B')^2 \sim g_s^2$, to lowest order, we may ignore the coupling of B in (8.11, 8.20) and consider (8.21) alone.

9: Sine-Gordon Solitons

9.1 Two Dimensional Sine-Gordon Solutions

The sine-Gordon model is well known from field theory and many other fields [20]. It is one of few important example where various analytic results can be obtained. The sine-Gordon model exhibits various interesting solitonic solutions. We examine the model here in $(1+1)$ dimensions with a $(+-)$ metric. The Lagrangian density is constructed from a kinetic term and a periodic potential

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{m^4}{\lambda}\left(\cos(\frac{\sqrt{\lambda}}{m}\phi) - 1\right) \tag{9.1}$$

The action is obtained by integrating over the two spacetime dimensions:

$$S = \int dtdx\mathcal{L} \tag{9.2}$$

The equation of motion for this Lagrangian follow as the extremum of the action:

$$\partial^2\phi + \frac{m^3}{\sqrt{\lambda}}\sin(\frac{\sqrt{\lambda}}{m}\phi) = 0. \tag{9.3}$$

The naming of the constants m and λ is clear if we expand out the cosine in the Lagrangian to reveal

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \cdots \tag{9.4}$$

We recognize m as a mass term (of dimension one in D=2) and λ as a coupling constant (of dimension 2 in D=2).

We may obtain a solution by multiplying the time-independent version of the equation of motion 9.3 by $\frac{\partial}{\partial x}\phi$ and integrating. Choosing boundary conditions such that at spacetime infinity $\partial_x\phi$=0 and $\frac{\sqrt{\lambda}}{m}\phi = 2\pi n$ for integer n we obtain the so called soliton solution

$$\phi(x) = \frac{4m}{\sqrt{\lambda}} \arctan(e^{\pm mx}). \tag{9.5}$$

the length scale of the soliton is thus given by $\ell = 1/m$. To determine the energy of this solution we appeal to the canonical formalism by forming the the Hamiltonian density

$$\mathcal{H} = p\partial_t\phi - \mathcal{H} \tag{9.6}$$

$$p \equiv \frac{\partial}{\partial_0\phi}\mathcal{L} \tag{9.7}$$

which yields

$$E = \int dx\mathcal{H} = \int dx\frac{1}{2}(\partial\phi)^2 - \frac{m^4}{\lambda}\left(\cos(\frac{\sqrt{\lambda}}{m}\phi) - 1\right). \tag{9.8}$$

Applying this formula to our solution (9.5) we obtain the energy of the soliton:

$$E = 8\frac{m^3}{\lambda}. \tag{9.9}$$

The sine-Gordon model also admits a time-dependent version of (9.5) which may be obtained by performing a Lorentz boost on x. Likewise, its energy is increased by the Lorentz factor $1/\sqrt{1-v^2}$. There are also various superposition solutions. For example, the so-called soliton-antisoliton:

$$\phi(x) = \frac{4m}{\sqrt{\lambda}} \arctan\left(\frac{\sinh(vmt/\sqrt{1-v^2})}{v\cosh(mx/\sqrt{1-v^2})}\right). \tag{9.10}$$

9.2 B-field Solitons

We are considering the configuration where we have two compactified directions X^1, X^2 and a non zero B field in that various only in a third direction $B = B_{12}(x^3) = -B_{21}(X^3)$. To relevant order in g_s we may then write the action as

$$S = \frac{1}{(\alpha')^{12} g_s^2} \int d^{26}X \left\{ -\frac{1}{4}(\partial_3 B)^2 - Y(1 - \cos(AB)) \right\} \tag{9.11}$$

where we have defined $Y = (\alpha')^{12} g_s^2 \Lambda_1$. We my proceed as in the previous section to obtain the energy an size of the domain wall:

$$B = \frac{4}{A} \arctan(e^{\pm A\sqrt{Y}x}) \tag{9.12}$$

$$\ell = \frac{1}{A\sqrt{Y}} \tag{9.13}$$

$$E = \frac{\sqrt{2}16\pi^2}{(\alpha')^5 g_s} \sqrt{\Lambda_1} \int d^{22}X \tag{9.14}$$

where the integral is over $d^{22}X = dX^4 dX^5 \cdots dX^{25}$. Using the estimation for Λ_1 from (5.54) we obtain

$$\ell = \frac{\alpha'}{2\pi R g_s} e^{(\frac{R^2}{\alpha'})} \tag{9.15}$$

$$E = \frac{\sqrt{2}16\pi^2}{(\alpha')^{11} R g_s} e^{-(\frac{R^2}{2\alpha'})} \int d^{22}X. \tag{9.16}$$

Since we have assumed $\frac{R^2}{\alpha'} \gg 1$ we find that ℓ is large and E is small. The dependance of E on g_s is inverse and that is somewhat peculiar. Note the similarity to the physical D-brane tension, which has the same g_s dependance [2]

$$\frac{\sqrt{\pi}}{16\alpha'^6 g_s}(2\pi\sqrt{\alpha'})^{11-p} \tag{9.17}$$

for a p dimensional brane.

10: Topological Indices and Vacuum Manifold

10.1 Topological Charge

We now take a brief digression into topology. Topological defects have become of considerable interest in recent times in field theory, in string theory, and in cosmology [20] [21] [22]

Consider a (1+1) scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi) \tag{10.1}$$

where the potential is given by

$$U = \frac{m^4}{\lambda}\left(\cos(\frac{\sqrt{\lambda}}{m}\phi) - 1\right). \tag{10.2}$$

Classically there are many ground states corresponding to the to minima of the potential

$$\phi_0 = 2\pi n \frac{m}{\sqrt{\lambda}}. \tag{10.3}$$

However, the equation of motion also admits another solution

$$\phi(x) = \frac{4m}{\sqrt{\lambda}} \arctan(e^{\pm m(x-x_0)}). \tag{10.4}$$

which has finite energy. For all finite energy solutions we expect the asymptotic values of ϕ to take on the values

$$\phi(x) = \begin{cases} 2\pi n_1 \dfrac{m}{\sqrt{\lambda}} & x = -\infty \\ 2\pi n_2 \dfrac{m}{\sqrt{\lambda}} & x = +\infty \end{cases} . \tag{10.5}$$

Consider the following integral:

$$\frac{1}{2\pi}\frac{\sqrt{\lambda}}{m}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}\phi dx = n_1 - n_2 \equiv \Delta n \tag{10.6}$$

where $\Delta n = 0$ corresponds to the true vacuum and $\Delta n = \pm 1$ corresponds to the soliton and anti-soliton. We see what appears to be a charge that characterizes the solutions. This motivates us to define the topological charge

$$j_\mu = \frac{1}{2\pi}\frac{\sqrt{\lambda}}{m}\epsilon_{\mu\nu}\partial^\nu\phi \tag{10.7}$$

This current is conserved identically (i.e., independent of the field equations) $\partial_\mu j^\mu \equiv 0$ and its zeroth component is the topological charge

$$Q \equiv \int j_0 dx = \Delta n. \tag{10.8}$$

We should note that Q does not generate a continuous symmetry transformation, i.e., it is not a Noether charge. Regardless, it is a charge that is conserved and this indicates that the soliton is stable.

10.2 Modularity of B

Note that if ϕ is a periodic variable then only Q is relevant, not n_1, n_2. For example, since $B_{\mu\nu}$ is a gauge field

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\chi_\nu - \partial_\nu\chi_\mu. \tag{10.9}$$

We may change gauge with $\chi_\mu = \frac{c}{2}(\delta^2_\mu X^1 - \delta^1_\mu X^2)$. This shifts $B_{\mu\nu}$ by

$$\delta B_{\mu\nu} = c(\delta^2_\nu \delta^1_\mu - \delta^1_\nu \delta^2_\mu). \tag{10.10}$$

That is we can shift the field in the 1-2 direction and leave the other directions unchanged and it is the same physical state. This will have important consequences for the B-field solitons. Consider the term in the Polyakov action associated with the B-field

$$S_B = \frac{i}{4\pi\alpha'} \int d^2\sigma[\epsilon^{ab} B_{\mu\nu}(X))\partial_a X^\mu \partial_b X^\nu]. \tag{10.11}$$

For the path integral to be invariant under gauge transformations, this term must change by $2\pi n i$, for $n \in \mathbb{Z}$. We must be careful about the fact that we are on a compactified manifold. Under the gauge transformation (10.9) S_B changes by

$$\begin{aligned}
\Delta S_B &= \frac{i}{2\pi\alpha'} \int d^2\sigma[\epsilon^{ab}\partial_\mu \chi_\nu \partial_a X^\mu \partial_b X^\nu] && (10.12)\\
&= \frac{i}{2\pi\alpha'} \int d^2\sigma[\partial_\mu \chi_\nu(\partial_1 X^\mu \partial_2 X^\nu - \partial_2 X^\mu \partial_1 X^\nu)] && (10.13)\\
&= \frac{i}{2\pi\alpha'} \int d^2\sigma[\partial_\mu \chi_\nu(\partial_1(X^\mu \partial_2 X^\nu) - \partial_2(X^\mu \partial_1 X^\nu))] && (10.14)
\end{aligned}$$

since $\partial_\mu \chi_\nu$ is a constant we may use Green's theorem to turn this integral into a line integral

$$\Delta S_P = \frac{ic}{2\pi\alpha'} \oint X^1 dX^2 - X^2 dX^1 \tag{10.15}$$

Since the string must wrap around the compact directions a discrete number of times and the momentum modes are also discrete, we find the equivalence between physical states

$$B \sim B + \frac{\alpha'}{R^2}\mathbb{Z}. \tag{10.16}$$

10.3 Unstable Topological Configurations

Ordinarily one would expect the sine-Gordon soliton to be stable: It has a topological charge that commutes with the Hamiltonian [20]. However, we have been dealing with an effective field theory where other modes have been 'integrated out'. These modes can change the topology of the configuration space and permit a decay [23, 24, 25]. Figure illustrates two different homotopy classes corresponding to the vacuum ($Q = 0$) and the soliton ($Q = 1$). The vacuum cannot be continuously transformed into the soliton (and vice versa) without passing through the forbidden region. In reality, though, the problem is not two dimensional. Through coupling to other modes the problem introduces a third dimension perpendicular to the plane. In this case, the forbidden region becomes a hump barrier. Loops around the hump can be deformed to $Q = 0$ loops by dragging them over the hump. This of course has some energy cost. The hump will be modeled in the next chapter.

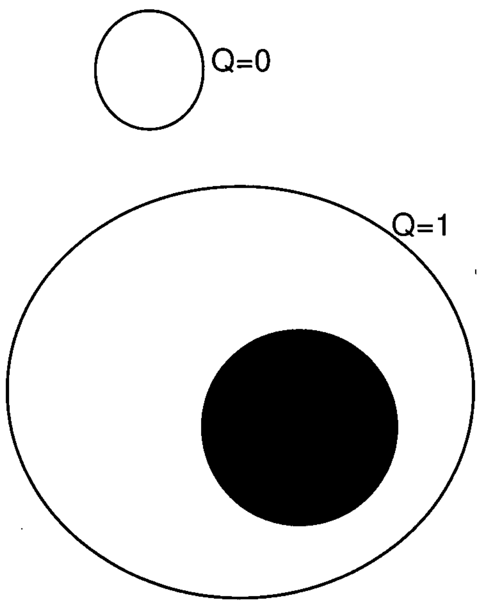

Figure 10.1: Homotopy classes. The black region is forbidden. The two loops correspond to the vacuum ($Q = 0$) and the soliton ($Q = 1$).

11: Stability

11.1 The Bounce

The decay of false vacuum states has been discussed extensively in the context of homogeneous states [26, 27, 28] as well as in the application to domain walls [24, 25].

Consider a (2+1) scalar field theory with action

$$S = \int d^3x \{ \frac{1}{2}(\partial_\mu \phi)^2 - U(\phi) \} \tag{11.1}$$

where $U(\phi)$ is a potential with two uniform (constant field) extrema: one local minima at ϕ_+ and a global minima at ϕ_-. We wish to consider the probability for decay from ϕ_+ to ϕ_-.

The relevance of false vacuum decay to cosmology is important. In the early universe the energy density was very high and was not nearly a vacuum. The universe may have settled into a false vacuum ϕ_+ and may decay toward the true vacuum ϕ_-. There are also more terrestrial issues where false vacuum decay is important. For example, the nucleation process in a supercooled fluid as occurs in cold clouds.

Physically the process is facilitated by quantum fluctuations. A fluctuation bubble of ϕ_- is formed in the ϕ_+ state. If the energy decrease from the volume of the bubble exceeds the energy increase due to the wall of the bubble, the bubble will rapidly grow, consuming the ϕ_- solution.

In the semi-classical approximation, the decay rate per unit volume per unit time for a tunneling particle process is given as [27]

$$\frac{\Gamma}{V} = Ae^{-B/\hbar}(1 + \mathcal{O}(\hbar)). \tag{11.2}$$

where

$$S_E[\phi] = \int d^3x \{\frac{1}{2}(\partial_\mu \phi)^2 + U(\phi)\} \tag{11.3}$$

$$B = S_E(\phi) - S_E(\phi_+) \tag{11.4}$$

and ϕ here is the bounce solution. The bounce solution is so called because after forming the euclidean action, the sign of the potential is reversed. Hence the solution corresponding to tunneling through a barrier is actually a solution that slips down the inverted barrier, stops at the other end, and returns to its starting point. If the bounce has $O(3)$ symmetry then we may convert the action integral into a radial integral. We introduce the D dimensional solid angle $\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$ and write

$$S_E = \Omega_3 \int_0^\infty dr r^2 \{\frac{1}{2}(\frac{\partial}{\partial r}\phi)^2 + U\}. \tag{11.5}$$

The equation of motion is

$$2\phi' + r\phi'' - r\frac{\partial}{\partial \phi}U = 0. \tag{11.6}$$

We wish to investigate the system when the energy difference between the two vacuo ϕ_+ and ϕ_- is small

$$\rho \equiv U(\phi_+) - U(\phi_-) \tag{11.7}$$

and think of ρ as a perturbative parameter that turns on the asymmetry of the energy levels in in U. We thus write

$$U(\phi) = U_0(\phi) + \rho U_1(\phi) + \cdots . \tag{11.8}$$

where $U_0(\phi_+) = U_0(\phi_-)$ and $\frac{\partial}{\partial \phi} U_0|_{\phi_\pm} = 0$. An example for U_0 is

$$U_0 = \frac{\lambda}{8}(\phi^2 - \frac{\mu}{\lambda})^2 \tag{11.9}$$

We are looking for bubble solutions to the equation of motion: the bubble has ϕ_- in its interior and at some large $r = r_0$ it quickly grows to ϕ_+. We can therefore ignore the $\frac{\partial}{\partial r}\phi$ in the equation of motion as it is either small, or small with respect to r. Our approximate equation of motion is then

$$r\phi'' - r\frac{\partial}{\partial \phi} U_0 = 0. \tag{11.10}$$

Multiply by $\frac{\partial}{\partial r}\phi$ and integrate using the boundary condition $\phi(\infty) = \phi_+$:

$$\frac{\partial}{\partial r}\phi = \sqrt{2}(U_0(\phi) - U_0(\phi_+))^{1/2}. \tag{11.11}$$

Letting r be the radius at which ϕ is midway between ϕ_+ and ϕ_- we find

$$r = r_0 + \int_{(\phi_+ + \phi_-)/2}^{\phi} \frac{\partial}{\partial \phi} r \, d\phi \tag{11.12}$$

For (11.9) we find

$$\phi = \frac{\mu}{\sqrt{\lambda}} \tanh(\frac{1}{2}\mu(r - r_0)) \tag{11.13}$$

Now that we have explored the nature of the bubble solutions, we my evaluate B by treating the bubble as having a thin wall. B is given to lowest order by an integral over all space

$$B = 4\pi \int_0^\infty dr\ r^2[\frac{1}{2}((\phi')^2 - (\phi'_+)^2) + U_0(\phi) - U_0(\phi_+)] \tag{11.14}$$

This integral may separated over three significant regions: the region inside r_0, the region about r_0, and the region outside r_0.

$$B = B_{in} + B_{on} + B_{out} \tag{11.15}$$

$$B_{in} = 4\pi \int_0^{r_0} dr\ r^2[U_0(\phi) - U_0(\phi_+)] = -\frac{4\pi}{3} r_0^3 \rho \tag{11.16}$$

this is the volume of a 3-sphere times the density ρ.

$$B_{on} = 4\pi r_0^2 \int_{on} dr[\frac{1}{2}(\phi')^2 + U_0(\phi) - U_0(\phi_+)] = 4\pi r_0^2 \sigma \tag{11.17}$$

This is the the surface area of a 3-sphere times the surface density $\sigma = \int_{on} dr[\frac{1}{2}(\phi')^2 + U_0(\phi) - U_0(\phi_+)] = \int_{\phi_-}^{\phi_+} d\phi\, (2U_0(\phi) - 2U_0(\phi_+))^{1/2}$. The right hand sides follows from the equation of motion. For (11.9) we find

$$\sigma = \frac{2\mu^3}{3\lambda} \tag{11.18}$$

Outside the bubble $\phi = \phi_+$ so we find

$$B_{out} = 0. \tag{11.19}$$

In total the bounce is given by

$$B = -\frac{4\pi}{3} r_0^3 \rho + 4\pi r_0^2 \sigma. \tag{11.20}$$

The extremum of B occurs when $r_0 = \frac{2\sigma}{\rho}$ which gives

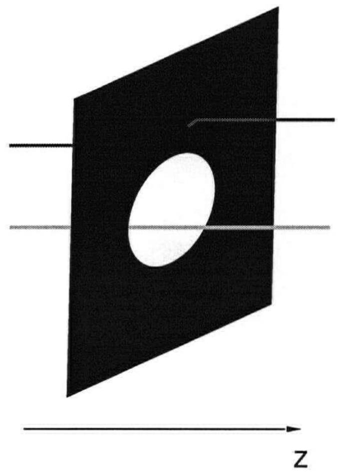

Figure 11.1: The lines represent the value of the Kalb-Ramond flux Φ. The light line passes through the bubble and travels from $\Phi = 0 \to 0$ as $z = -\infty \to \infty$. The dark line avoids the bubble and travels from $\Phi = 0 \to 2\pi$ as $z = -\infty \to \infty$.

$$B = \frac{16\pi}{3}\frac{\sigma^3}{\rho^2}. \tag{11.21}$$

If $B \gg 1$ we find the decay process is strongly suppressed via the exponential in (11.2).

So far, we have been discussing a bubble of ϕ_- forming in a uniform ϕ_+ world. However, this can also be applied to the case of a domain wall that separates two physically equivalent vacuua in arbitrary dimension. For example, the above (2+1) theory is equivalent to a (3+1) theory where a 3-dimensional bubble forms in the wall. In this case we write ϕ_+ for the domain wall solution and ϕ_- for the empty vacuum. Figure 11.1 illustrates a bubble forming in a domain wall.

For bubbles of dimensions d we have

$$B = -\Omega_d \frac{r^d}{d}\rho + \Omega_d r^{d-1}\sigma \tag{11.22}$$

where ρ and σ are the d-dimensional volume and surface density respectively. The extremum of B with respect to r gives the critical radius r_0

$$r_0 = (d-1)\frac{\sigma}{\rho}. \tag{11.23}$$

Then, the d-dimensional bounce is given as

$$B = \Omega_d \frac{(d-1)^{d-1}}{d} \frac{\sigma^d}{\rho^{d-1}}. \tag{11.24}$$

For large d we may use Sterling's formula for $\Gamma(x)$

$$\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-1/2} \tag{11.25}$$

and expand

$$(d-1)^{d-1} \sim e^{-1} d^{d-1}. \tag{11.26}$$

For large d we find the asymptotic bounce:

$$B \approx 2^{d/2} \pi^{(d-1)/2} e^{(d-2)/2} d^{(d-3)/2} \frac{\sigma^d}{\rho^{d-1}} \tag{11.27}$$

For our domain walls the energy is given by (9.14)

$$E = \frac{\sqrt{2} 16\pi^2}{(\alpha')^5 g_s} \sqrt{\Lambda_1} \int d^{22}X \tag{11.28}$$

and hence

$$\rho = \frac{\sqrt{2} 4}{(\alpha')^5 g_s} \sqrt{\Lambda_1} \tag{11.29}$$

Evaluating the bounce for our domain wall requires $d = 23$. In this case we find

$$B = \frac{(\alpha')^{110} (g_s')^{22}}{\Lambda_1^{11}} \sigma^{23} \tag{11.30}$$

where we have written g_s' to absorb the unnecessary pure number.

11.2 Modelling the Surface Tension

Our final task then is to estimate σ. Unfortunately we know very little about the detailed structure $U_0(\phi)$ for this case. We must proceed cautiously with an estimation. Qualitatively we know that ϕ_+ and ϕ_- must correspond to the minima of U_0 and there must be some barrier between them. We shall model U_0 with (11.9), but to analyze the problem it is more convenient to write it in terms of its width between minima w and its barrier height h.

$$U_0 = \frac{16h}{w^4}\left(\phi^2 - w^2/4\right)^2 \tag{11.31}$$

where the momentum dimensions of the constants are: $[h] = 23$ and $[w] = 21/2$. We obtain these dimensions by modeling the bubble in the wall (excluding the X^3 direction) as a 25-dimensional scalar field theory. That is, we choose an action like

$$S = \int d^{23}x(\frac{1}{2}(\partial_\mu\phi)^2 + U(\phi)). \tag{11.32}$$

This is consistent with obtaining the bounce (11.30).

We should expect that the height of the barrier goes as the string energy scale scale. Therefore we dimensionally estimate $h = (\alpha')^{-23/2}$ In the free field limit, coupling to the massive modes is not present and the sine-Gordon soliton must be stable. Therefore we dimensionally estimate $w = \Lambda_1^{21/52}\frac{1}{g_s^n}$, where n is some positive integer. The g_s dependence must be inverse because we must have $B = \infty$ for $g_s = 0$. The power n can not be obtained easily but is likely some small integer power. Thus the surface tension is

$$\sigma = \frac{\sqrt{32}}{6}w\sqrt{h} = \frac{\sqrt{32}}{6}\Lambda_1^{21/52}(\alpha')^{-23/4}\frac{1}{g_s^n} \tag{11.33}$$

and the bounce is

$$B = c(\alpha')^{-89/4} g_s^{22-23n} \Lambda_1^{-89/52} \tag{11.34}$$

where c is a constant that can be absorbed into g_s. Note that B scales inversely as g_s since n is a positive integer. We may also insert our estimation for Λ_1 from (5.54):

$$B = \left(\frac{R^2}{\alpha'}\right)^{89/52} g_s^{22-23n} e^{\frac{89}{52}\pi\frac{R^2}{\alpha'}} \tag{11.35}$$

where we have dropped c (which is equivalent to absorbing it into g_s). Since we require $\frac{R^2}{\alpha'} \gg 1$ and we expect g_s to be small, we find that the decay is exponentially suppressed: the Kalb-Ramond soliton is stable.

12: Conclusion

After a brief introduction to the relevant parts of string theory, a soliton of the Kalb-Ramond field in closed bosonic string theory was introduced. The soliton discussed forms under a double toroidal compactification. With a constant Kalb-Ramond field permitted in a components of the compact directions, the vacuum energy is found to be a periodic function of the field. This vacuum degeneracy permits the formation of domain walls, as in ferromagnets. We examined the configuration where two halves of the universe are in separate domains. The region in between is approximated by a sine-Gordon soliton whose length and energy are given as

$$\ell = \frac{\alpha'}{2\pi R g_s} e^{(2\pi \frac{R^2}{\alpha'})} \tag{12.1}$$

$$E = \frac{\sqrt{2}16\pi^2}{(\alpha')^{11} R g_s} e^{-(2\pi \frac{R^2}{2\alpha'})} \int d^{22}X \tag{12.2}$$

where the integral is over the spatial directions not including the compactified directions and the direction in which the field varies.

The stability of this structure is threatened by decay through nucleation. However, the decay process is exponentially suppressed by inverse powers of g_s.

The are many other structures and features that may be investigated. Alternative compacitifactions may form other topological structures. As well, instanton gasses may occur. It is also important to investigate how particles would scatter of these objects and permit their observational detection.

Bosonic string theory is pathological and is not the correct model for nature. However, type II string theory is more realistic and should have the same solitonic structures. This is worth investigating in future work.

References

[1] S. Carlip. Quantum gravity: A progress report. *Rept. Prog. Phys.*, 64:885, 2001.

[2] J. Polchinski. String theory. vol. 1: An introduction to the bosonic string. Cambridge, UK: Univ. Pr. (1998) 402 p.

[3] J. Polchinski. String theory. vol. 2: Superstring theory and beyond. Cambridge, UK: Univ. Pr. (1998) 531 p.

[4] M. B. Green, J. H. Schwarz, and Edward Witten. Superstring theory. vol. 1: Introduction. Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics).

[5] B. Hatfield. Quantum field theory of point particles and strings. Redwood City, USA: Addison-Wesley (1992) 734 p. (Frontiers in physics, 75).

[6] Hirosi Ooguri and Zheng Yin. Tasi lectures on perturbative string theories. 1996. http://arXiv.org/abs/hep-th/9612254/.

[7] U. Danielsson. Introduction to string theory. *Rept. Prog. Phys.*, 64:51–96, 2001.

[8] Ashoke Sen. An introduction to non-perturbative string theory. 1998. http://arXiv.org/abs/hep-th/9802051.

[9] Stefan Forste. Strings, branes and extra dimensions. 2001. http://arXiv.org/abs/hep-th/0110055/.

[10] Steven Weinberg. Covariant path integral approach to string theory. Lectures given at 3rd Jerusalem Winter School of Theoretical Physics, Jerusalem, Israel, Dec 30, 1986 - Jan 8, 1987.

[11] J. A. Shapiro. Loop graph in the dual-tube model. *Phys. Rev.*, D5:1945–1948, 1972.

[12] Joseph Polchinski. Evaluation of the one loop string path integral. *Commun. Math. Phys.*, 104:37, 1986.

[13] Gianluca Grignani, Marta Orselli, and Gordon W. Semenoff. Matrix strings in a b-field. *JHEP*, 07:004, 2001.

[14] Gordon W. Semenoff. Dlcq strings and branched covers of torii. 2001. http://arXiv.org/abs/hep-th/0112043/.

[15] N. Sakai and I. Senda. Vacuum energies of string compactified on torus. *Prog. Theor. Phys.*, 75:692, 1986.

[16] Jr. Callan, Curtis G. and L. Thorlacius. Sigma models and string theory. In *Providence 1988, Proceedings, Particles, strings and supernovae, vol. 2* 795-878.

[17] M. J. Duff. Kaluza-klein theory in perspective. 1994. http://arXiv.org/abs/hep-th/9410046/.

[18] R. M. Wald. General relativity. Chicago, Usa: Univ. Pr. (1984) 491p.

[19] Steven Weinberg. General relativity. Toronto, Canada: Wiley (1984) 657p.

[20] R. Rajaraman. Solitons and instantons. an introduction to solitons and instantons in quantum field theory. Amsterdam, Netherlands: North-holland (1982) 409p.

[21] Alejandro Gangui. Topological defects in cosmology. 2001. http://arXiv.org/abs/astro-ph/0110285/.

[22] Mirjam Cvetic and Harald H. Soleng. Supergravity domain walls. *Phys. Rept.*, 282:159–223, 1997.

[23] John Preskill and Alexander Vilenkin. Decay of metastable topological defects. *Phys. Rev.*, D47:2324–2342, 1993.

[24] Michael McNeil Forbes and Ariel R. Zhitnitsky. Domain walls in qcd. *JHEP*, 10:013, 2001.

[25] T. W. B. Kibble, G. Lazarides, and Q. Shafi. Walls bounded by strings. *Phys. Rev.*, D26:435, 1982.

[26] Jr. Callan, Curtis G. and Sidney R. Coleman. The fate of the false vacuum. 2. first quantum corrections. *Phys. Rev.*, D16:1762–1768, 1977.

[27] Sidney R. Coleman. The fate of the false vacuum. 1. semiclassical theory. *Phys. Rev.*, D15:2929–2936, 1977.

[28] Sidney R. Coleman and Frank De Luccia. Gravitational effects on and of vacuum decay. *Phys. Rev.*, D21:3305, 1980.

[29] Jnanadeva Maharana and John H. Schwarz. Noncompact symmetries in string theory. *Nucl. Phys.*, B390:3–32, 1993.

A: Dimensional Reduction

Here we summarize the procedure of dimensional reduction [29]. We choose to parametrize the D-dimensional spacetime metric G^D_{MN} as

$$ds^2 = G_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + G_{mn}(dx^m + A^m_\mu dx^\mu)(dx^n + A^n_\mu dx^\mu). \qquad \text{(A.1)}$$

where $M, N = 0, 1, \cdots 25$ and μ and ν index the non-periodic dimensions and m and n index the periodic dimensions. In matrix form, we may write the metric and its inverse as

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} + G_{mn}A^m_\mu A^n_\nu & G_{mn}A^m_\mu \\ G_{mn}A^n_\nu & G_{mn} \end{pmatrix} \qquad \text{(A.2)}$$

$$G^{MN} = \begin{pmatrix} G^{\mu\nu}+ & -A^{\nu m} \\ -A^{\mu n} & G^{mn} + A^{\rho m}A^n_\rho \end{pmatrix} \qquad \text{(A.3)}$$

This parametrization has a convenient property:

$$\sqrt{-|G_{MN}|} = \sqrt{-|G_{\mu\nu}|}\sqrt{-|G_{mn}|}. \qquad \text{(A.4)}$$

Using the above relations we find that

$$\frac{1}{2\kappa_0^2}\int d^Dx\sqrt{-|G_{MN}|}e^{-2\Phi}\left[R_D + 2\partial_M\Phi\partial^M\Phi\right] \qquad \text{(A.5)}$$

is equivalent to

$$\begin{aligned}\frac{(2\pi R)^k}{2\kappa_0^2}\int d^dx\sqrt{-G_d}e^{-2\Phi_d}\,[R_d &+ 4\partial_\mu\Phi_d\partial^u\Phi_d \\ &- \tfrac{1}{4}G^{mn}G^{pq}\left(\partial_\mu G_{mp}\partial^u G_{nq}\right) \\ &- \tfrac{1}{4}G_{mn}F^{(1)m}_{\mu\nu}F^{(1)n\mu\nu}]\end{aligned} \tag{A.6}$$

with

$$F^{(1)m}_{\mu\nu} = \partial_\mu A^{(1)m}_\nu - \partial_\nu A^{(1)m}_\mu \tag{A.7}$$

$$\Phi_d \equiv \Phi - \frac{1}{4}\det(G_{mn}) \tag{A.8}$$

for k periodic dimensions with $d = D - k$. Typically, we will write $A\mu^{(1)n} = A\mu^n$. The (1) is make a distinction between another tensor that will be defined shortly.

Now we consider the action associated with Kalb-Ramond field

$$\frac{1}{2\kappa_0^2}\int d^Dx\sqrt{-G}e^{-2\Phi}\left[-\frac{1}{12}H_{MNL}H^{MNL}\right] \tag{A.9}$$

The metric parametrization allows us to separate the terms and write

$$-\frac{(2\pi R)^k}{2\kappa_0^2}\int d^dx\sqrt{-G_d}e^{-2\Phi_d}[\frac{1}{12}H_{mnl}H^{mnl} + \frac{1}{4}H_{\mu nl}H^{\mu nl} + \frac{1}{4}H_{\mu\nu l}H^{\mu\nu l} + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}] \tag{A.10}$$

where the following definitions have been employed:

$$H_{\mu mn} = \partial_\mu B_{mn} \tag{A.11}$$

$$H_{\mu\nu m} = F^{(2)}_{\mu\nu m} - B_{mn}F^{(1)n}_{\mu\nu} \tag{A.12}$$

$$F^{(2)}_{\mu\nu m} = \partial_\mu A^{(2)}_{\nu m} - \partial_\nu A^{(2)}_{\mu m} \tag{A.13}$$

$$A^{(2)}_{\mu m} = \hat{B}_{\mu m} + B_{mn} A^{(1)n}_\mu \tag{A.14}$$

$$H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \frac{1}{2}\left(A^{(1)m}_\mu \; F^{(2)}_{\nu\lambda m} + A^{(2)}_{\mu m} F^{(1)m}_{\nu\lambda}\right) + \text{cyclic perms.} \tag{A.15}$$

$$B_{\mu\nu} = \hat{B}_{\mu\nu} + \frac{1}{2} A^{(1)m}_\mu A^{(2)}_{\nu m} - \frac{1}{2} A^{(1)m}_\nu A^{(2)}_{\mu m} - A^{(1)m}_\mu B_{mn} A^{(1)n}_\nu \tag{A.16}$$

Adding all the components together, we find that the effective action can be *dimensionally reduced* to

$$\begin{aligned} S_{0,k} = \frac{(2\pi R)^k}{2\kappa_0^2} \int d^d x \sqrt{-G_d} e^{-2\Phi_d} \, [R_d + 4\partial_\mu \Phi_d \partial^u \Phi_d \\ - \tfrac{1}{4} G^{mn} G^{pq} \left(\partial_\mu G_{mp} \partial^u G_{nq} + \partial_\mu B_{mp} \partial^u B_{nq}\right) \\ - \tfrac{1}{4} G_{mn} F^{(1)m}_{\mu\nu} F^{(1)n\mu\nu} - \tfrac{1}{4} G^{mn} H_{m\mu\nu} H^{\mu\nu}_n - \tfrac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda}] \end{aligned} \tag{A.17}$$