RADIATIVE MUON CAPTURE BY $^3$He

by

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Abstract

The rate of the nuclear reaction $^3\text{He} + \mu^- \rightarrow ^3\text{H} + \nu_\mu + \gamma$ has been calculated by two different methods: the *elementary particle model* (EPM) approach and the *impulse approximation* (IA) approach. The exclusive statistical radiative muon capture (RMC) rate for photon energy greater than 60 MeV, $\Gamma_{\text{RMC}}^{\text{stat}}$, using the elementary particle model approach is found to be 0.2113 s\(^{-1}\) and the ordinary muon capture (OMC) rate is 1503 s\(^{-1}\). Several trinucleon wavefunctions from different types of realistic nucleon potentials are used for the impulse approximation calculation and it is found that the capture rates calculated via the IA exhibit slight model dependences, possibly arising from differences in binding energy predictions, nature of potentials used or partial wave properties. The impulse approximation version of $\Gamma_{\text{RMC}}^{\text{stat}}$ ranges from 0.1313 to 0.1387 s\(^{-1}\) and the corresponding OMC rate ranges from 1260 to 1360 s\(^{-1}\). The difference in reaction rates between IA and EPM is larger in RMC due to some of the extra Adler and Dothan terms. Therefore, a meson exchange current (MEC) calculation of RMC seems necessary to account for this discrepancy.
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To study in spring is treason;
And summer is sleep's best reason;
If winter hurries the fall,
Then stop till next spring season.

A Chinese poem for bibliophobes. Poet unknown.
English translation by Yutang Lin [1].
1. Introduction

1.1 What and why of radiative muon capture by $^3$He?

1.1.1 What?

Radiative muon capture (RMC) by $^3$He is the process\(^1\)

$$^3\text{He} + \mu^- \rightarrow ^3\text{H} + \nu_\mu + \gamma$$

(1.1)

and the corresponding non-radiative process is called the ordinary muon capture (OMC) by $^3$He.

$$^3\text{He} + \mu^- \rightarrow ^3\text{H} + \nu_\mu$$

(1.2)

Here $\mu^-$ stands for the muon that is captured; $\nu_\mu$ and $\gamma$ are respectively the muon neutrino and photon emitted. The capture takes place from a muonic atomic orbital, similar to an electronic state except that it is about 200 times smaller. While it is in this orbit the muon can decay in the normal way with rate of $0.46 \times 10^6 \text{ s}^{-1}$. Thus, as one will see later, that muon capture is a fairly rare occurrence.

Both capture processes (1.1) and (1.2) can be studied in at least two levels: the level of the nucleus and the level of the nucleons\(^2\). At the nucleus level, one takes $^3$He and $^3$H as whole entities and calculates the capture rate based on phenomenological nuclear form factors. This method is called the elementary particle model (EPM) and will be discussed in chapter 2.

One way to visualize processes (1.1) and (1.2) at the nucleon level is to regard the capture taking place on the constituent nucleons. By recognizing

---

\(^1\) The "breakup" reactions: $^3\text{He} + \mu^- \rightarrow d + n + \nu_\mu + \gamma$ and $^3\text{He} + \mu^- \rightarrow p + n + n + \nu_\mu + \gamma$

are not considered here.

\(^2\) not to mention the level of quarks!
1. Introduction

that $^3$He ($^3$H) is some antisymmetric combination of $|ppn\rangle$ ($|pnn\rangle$) and its permutations, one sums the following reaction that happens inside the nucleus to obtain a first order approximation to the capture rate:

$$p + \mu^- \rightarrow n + \nu_\mu + \gamma \quad (1.3)$$

for RMC and

$$p + \mu^- \rightarrow n + \nu_\mu \quad (1.4)$$

for OMC.

This method is called the impulse approximation (IA) and will be discussed in chapter 3. Notice that since one is summing the contributions of the rate from constituent nucleons, the interactions between the nucleons are ignored by this method\(^3\).

1.1.2 Why?

There are several motivations to calculate RMC (and OMC) rates. Below are some major ones:

1. Experiment E592 at TRIUMF [2, 3] is an experiment to measure the photon spectrum of the radiative muon capture by $^3$He. An accurate theoretical prediction seems necessary to interpret the experimental results.

2. With the advent of computing hardware and software and methods of trinucleon wavefunction calculation, a more precise impulse approximation seems overdue. Indeed, the most recent (and the only other) impulse approximation calculation was done about twenty years ago by Klieb and Rood [4, 5] yet the Hamiltonian they were using did not seem to completely satisfy some low energy theorems and they also handled some proton momentum terms in a crude way. The present impulse approximation calculation will hopefully straighten out some of the errors and improve the approximation they made.

---

\(^3\) Please see section 3.1 for justification on ignoring the interactions between the nucleons.
1.2 From ordinary muon capture to radiative muon capture

Since there are many similarities between radiative muon capture and ordinary muon capture\(^4\), it is worthwhile to take a step back and have a glimpse at the simpler problem of ordinary muon capture. The relationship between different form factors of OMC derived from various hypotheses will eventually be applied to the problem of RMC with some modifications.

1.2.1 Ordinary Muon Capture

In ordinary muon capture, the leading order Feynman diagram is shown in figure (1.1)

\[
\begin{align*}
&\mu \rightarrow \nu \rightarrow P_f \\
&\mu \rightarrow P_i
\end{align*}
\]

\textbf{Fig. 1.1:} The Feynman diagram for ordinary muon capture.

In figure (1.1), \(P_i\) stands for the four momentum of initial nucleus or nucleon (i.e. \(^3\)He for the EPM or proton for the IA) and \(P_f\) is the same quantity for the final nucleus or nucleon (i.e. \(^3\)H for the EPM or neutron for the IA); \(\mu\) is the four momentum of the muon and \(\nu\) is that of the neutrino produced. The above diagram follows from the Fermi's original current-current coupling of the weak interaction.

\[
M^{\text{omc}} = \frac{G_F}{\sqrt{2}} V_{ud} J_{\text{leptonic}} J_{\text{hadronic}}^a
\]  

\(^4\)See Mukhopadhyay [6] and Measday [7] for reviews on the topic of ordinary muon capture
where the leptonic current $J_{\alpha}^{\text{leptonic}}$ is
\[ J_{\alpha}^{\text{leptonic}} = \bar{u}(\nu)\{\gamma_{\alpha}(1 - \gamma^5)\}u(\mu) \quad (1.6) \]
and the hadronic current $J_{\alpha}^{\text{hadronic}}$ is
\[ J_{\alpha}^{\text{hadronic}} = \bar{u}(P_f)W\alpha u(P_i) \quad (1.7) \]

The $W\alpha$ is the weak hadronic vertex, which can be parameterized by four form factors.

\[ W\alpha = G^V\gamma^\alpha + G^M i\sigma^{\alpha\beta}\frac{Q_\beta}{2M_n} + G^A\gamma^\alpha \gamma^5 + G^P\gamma^5\frac{Q_\alpha}{m} \quad (1.8) \]

where
\[ \sigma^{\alpha\beta} \equiv \frac{i}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) \]
\[ \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (1.9) \]

All the $G_i$s ($i = V, M, A, P$) are functions of $Q^2$, the square of the momentum transfer at the hadronic vertex. Note that $M_n$ denotes the mass of the nucleus in EPM but the mass of nucleon in IA.

**Relationship between various form factors**

The *Isotriplet Vector Current Hypothesis* (IVC) stipulates that $G_V(Q^2)$ should be of the form,
\[ G_V(Q^2) = e_f G^f_V(Q^2) - e_i G^i_V(Q^2) \quad (1.10) \]
and $G_M(Q^2)$
\[ G_M(Q^2) = \mu_f G^f_M(Q^2) - \mu_i G^i_M(Q^2) \quad (1.11) \]

---

5 Indeed, the form factors also carry the $P_i^2 (P_f^2)$ dependence if the initial (final) hadron is off-shell, but this dependence will, as usual, be neglected.

6 Please see appendix D for definition of $m$ and for all other expressions that are not defined.

7 IVC is a statement saying that the vector current $J^{i\alpha} \equiv \bar{\Psi} W^{i\alpha}(Q)_{\text{vector}}\tau^i\Psi$, $i = +, 0, -, \Psi = (|^{3}\text{He or p}>, |^{3}\text{H or n})^T$, forms an isovector. Since $\tau^i$ already transforms like a vector during isospin rotation, all that is required is $W^{i\alpha}(Q)_{\text{vector}}$ forms an isoscalar. The magnitude of it can be found for the $i = 0$ case.
where $G_V^i$ ($G_V^f$) is the electric form factor of the initial (final) particle and $G_M^i$ ($G_M^f$) is the magnetic form factor of the initial (final) particle. Notations of $e_i$, $\mu_i$ etc stand for charge and anomalous magnetic moment respectively. These form factors are supposed to be found experimentally from electron scattering experiments on the respective particles.

The *Conserved Vector Current Hypothesis* (CVC) states that the vector current is conserved. This statement can be written in equation:

$$Q \cdot J_{\text{hadronic}}^V = 0$$ (1.12)

where $J_{\text{hadronic}}^V$ is the vector part of the hadronic current. Notice that equation (1.12) is automatically satisfied (up to a small isospin breaking) given the form of $W^\alpha$ in equation (1.8).

The *Partial Conservation of Axial Current Hypothesis* (PCAC) relates the divergence of the axial vector current to the pion field. A pictorial representation is shown in figure (1.2).

![Fig. 1.2: The Partial Conservation of Axial Current Hypothesis](image)

Algebraically, it is:

$$Q \cdot J_{\text{hadronic}}^A = \frac{a_\pi m^3_\pi}{m^2_\pi - Q^2} \langle f | j_\pi(Q) | i \rangle$$ (1.13)

\footnote{It is a slight abuse of notation not to differentiate between $J^\alpha$ (a current operator) and $\langle f | J^\alpha | i \rangle$ (a current amplitude) but the context in which the above notations are used will usually clear the confusion.}
where $\phi(Q)$ is the pion field which operates between the initial and final states and $\alpha_\pi$ is the pion decay constant ($\alpha_\pi = 0.9436$). In the EPM, $i$ is $^3$He and $f$ is $^3$H while in the IA, $i$ is the proton and $f$ is the neutron. Note that

$$\langle f \mid j_\pi(Q) \mid i \rangle = -\frac{2M_n}{m_\pi}G_\pi(Q^2)\bar{u}(P_f)\gamma^5u(P_i) \quad (1.14)$$

where $G_\pi(Q^2)$ is the pion-i-f coupling constant. Since,

$$Q \cdot J_{\text{hadronic}}^5 = \bar{u}(P_f)(2M_n G_A + G_P \frac{Q^2}{m})\gamma^5u(P_i) \quad (1.15)$$

equating (1.13) and (1.15) using (1.14) one will obtain a relation between $G_A$ and $G_P$, which is:

$$G_P = \frac{2M_n m}{m_\pi^2 - Q^2}G_A\{1 + \varepsilon(Q^2)\} \quad (1.16)$$

$$\varepsilon(Q^2) = \frac{m_\pi^2}{-Q^2}\{1 - \frac{G_\pi(Q^2)/G_\pi(0)}{G_A(Q^2)/G_A(0)}\} \quad (1.17)$$

Note that $\alpha_\pi$ disappears by virtue of the relation

$$G_A(0) = -\alpha_\pi G_\pi(0) \quad (1.18)$$

This is just the PCAC at $Q = 0$. The $\varepsilon$ part is small and is approximately a constant over the range of $Q^2$ concerned [4, 5]. Therefore the dominant part of $G_P$ looks as if it were caused by a virtual pion exchange (Figure (1.3)). Please do not confuse $\varepsilon$ with $\epsilon$, the latter will be defined as the photon polarization four vector later in the chapter.

### 1.2.2 Radiative Muon Capture

Let us shift the focus to radiative muon capture. There are several differences between OMC and RMC. These include:

1. Unlike the case of OMC where $Q^2$ is a constant, $Q^2$ for RMC varies approximately between $-m^2$ and $m^2$. Moreover, there are two kinds of momentum transfer at the hadronic vertex: $Q^L$, signifying momentum transfer when the muon is radiating, and $Q^H$, the momentum transfer...
1. Introduction

![Virtual Pion Exchange Diagram](image)

Fig. 1.3: The virtual pion exchange diagram, a consequence of the PCAC when one of the hadrons is radiating. Specifically,

$$
Q_L \equiv P_f - P_i = \mu - \kappa - \nu \quad Q_H \equiv (P_f + \kappa) - P_i = \mu - \nu
$$

(1.19)

where $\mu$, $\nu$ and $\kappa$ are the four momenta of the muon, neutrino and photon respectively.

2. Gauge invariance (GI). Let $M = \epsilon \cdot \vec{M}$ be the RMC amplitude ($\epsilon$ is the polarization vector of the emitted photon), gauge invariance means that $\kappa \cdot \vec{M} = 0$ where $\kappa$ is the momentum of the photon. Gauge invariance essentially says that the longitudinal photon polarization is unphysical and would not affect physical observables. Of course, there is no such thing as gauge invariance in the case of OMC since the photon is absent in the first place.

3. Using minimal substitution $Q_\alpha \rightarrow Q_\alpha - eA_\alpha$, CVC and PCAC now read (compare equations (1.12) and (1.13)):

$$
(Q - eA) \cdot J_{\text{hadronic,RMC}}^\nu = 0 \quad (1.20)
$$

$$
(Q - eA) \cdot J_{\text{hadronic,RMC}}^5 = a_\pi m_\pi^3 \phi(Q) \quad (1.21)
$$

Note that $A_\alpha J^\alpha$ literally means "sticking a photon on the vertex of current $J^\alpha"."
1. Introduction

Our aim is to construct an RMC amplitude that is gauge invariant and satisfies both the CVC and PCAC. This method of constructing the RMC amplitude, to be shown below, is due to Adler and Dothan [8]. The method essentially checks the PCAC, CVC and GI of the currents with a photon attached to either the muon, proton ($^3$He), neutron ($^3$H) or the pion and adds in counterterms so that the sum of all the currents (plus counterterms) satisfies GI, CVC and PCAC.

First consider the four external radiating diagrams in figure (1.4) and let the sum of their hadronic currents be $J_{\text{hadronic,RMC}}^{\alpha,\text{ext}}$.

![Fig. 1.4: The external radiating diagrams](image)

Let $M_1$ be the value of the diagram on the upper left hand side of the figure. Enumerating the diagrams clockwise, one has, after suppressing the constant $\frac{G_F}{\sqrt{2}}V_{ud}$ for convenience:

$$
M_1 = \bar{u}(\nu)^{\gamma_\alpha}(1 - \gamma^5)u(\mu)\bar{u}(P_f)W^\alpha(Q^H)S_F(P_f - \kappa)Q_iu(P_i)
$$

$$
M_2 = \bar{u}(\nu)^{\gamma_\alpha}(1 - \gamma^5)u(\mu)\bar{u}(P_f)Q_iS_F(P_f + \kappa)W^\alpha(Q^H)u(P_i)
$$

$$
M_3 = \bar{u}(\nu)^{\gamma_\alpha}(1 - \gamma^5)u(\mu)\bar{u}(P_f)\left\{\frac{-i}{m^2_\pi - (Q^H - \kappa)^2}\right\}
$$
1. Introduction

\[ M_4 = \frac{\bar{u}(\nu)\gamma_\alpha(1 - \gamma^5)S_F(\mu - \kappa)(-ie\ell)u(\mu)}{m} \]

\[ \bar{u}(P_f)W^\alpha(Q^L)u(P_i) \]

where \( S_F \) is the Feynman propagator for spin \( \frac{1}{2} \) particles and \( Q_{i(f)} = ie_{i(f)}\ell + \frac{\mu_{i(f)}}{2M_0} \sigma^{\nu\rho}k_\rho\epsilon_\lambda, \mu_{i(f)} \) being the anomalous magnetic moment of the initial (final) hadron.

Up to one photon emission, \( (Q_\alpha - eA_\alpha)J_{\text{hadronic,RMC}}^{\alpha,\text{ext}} \) can be calculated schematically as shown in figure (1.5).

**Fig. 1.5:** Covariant derivative of the external weak hadronic currents illustrated. The \( A^\alpha \) part, in one photon limit, only affects the non-radiating weak hadronic current (i.e. the muon radiating diagram).
The result of the calculation\(^9\) is:

\[
(Q - eA) \cdot J^\text{ext}_{\text{hadronic,RMC}} = \\
\bar{u}(P_f)\{G^H V Q^H + G^A H Q^H \gamma^5 + G_P^H \frac{Q_H^2}{m} \gamma^5\} S_F(P_f - k) Q_i u(P_i) + \\
\bar{u}(P_f) Q_J S_F(P_f + k)\{G^H V Q^H + G^A H Q^H \gamma^5 + G_P^H \frac{Q_H^2}{m} \gamma^5\} u(P_i) - \\
\bar{u}(P_f)\left\{\frac{(2Q^H - k) \cdot \epsilon}{m^2 - (Q^H - k)^2} \frac{2mM_\Upsilon(1 + \epsilon) Q^2_H}{m} \gamma^5\right\} u(P_i) + \\
\bar{u}(P_f)\{G^L V Q^L + 2M_n G_A^L \gamma^5 + G_P^L \frac{Q_L^2}{m} \gamma^5\} u(P_i) - \\
\frac{1}{\sqrt{2\kappa_V}} \bar{u}(P_f)\{G^L V \bar{\kappa} + G^L_{M \sigma} \bar{\alpha} \beta Q^2_L \frac{2M_n}{\epsilon} \gamma^5\} u(P_i) + \\
\bar{u}(P_f)\{G^L \bar{\sigma} \epsilon + G^L_{M \epsilon} \bar{\alpha} \beta \kappa \beta \gamma^5\} u(P_i) (1.23)
\]

where \(\epsilon\) has been defined in equation (1.17). Equation (1.23) can be reduced to the following form\(^10\):

\[
(Q - eA) \cdot J^\text{ext}_{\text{hadronic,RMC}} = \\
\bar{u}(P_f)\{G^H V Q^H + G^A H Q^H \gamma^5 + G_P^H \frac{Q_H^2}{m} \gamma^5\} S_F(P_f - k) Q_i u(P_i) + \\
\bar{u}(P_f)\{G^L V \bar{\kappa} + G^L_{M \sigma} \bar{\alpha} \beta Q^2_L \frac{2M_n}{\epsilon} \gamma^5\} u(P_i) + \\
\bar{u}(P_f)\left\{\frac{(2Q^H - k) \cdot \epsilon}{m^2 - (Q^H - k)^2} \frac{2mM_\Upsilon(1 + \epsilon) Q^2_H}{m} \gamma^5\right\} u(P_i) + \\
\bar{u}(P_f)\{G^L \bar{\sigma} \epsilon + G^L_{M \epsilon} \bar{\alpha} \beta \kappa \beta \gamma^5\} u(P_i) (1.24)
\]

\(^9\) One calculates \((Q - eA) \cdot J^\text{ext}_{\text{hadronic,RMC}}\) in order to find the counterterms needed to satisfy equations (1.20) and (1.21).

\(^{10}\) The photon normalization factor \(\frac{1}{\sqrt{2\kappa_V}}\) will be dropped for convenience as it will always appear in the phase space factor when one calculates the capture rate.
The two tricks involved in the above calculation are the observations that

\[
(P_i - \kappa)S_F(P_i - \kappa) = (P_i - \kappa)\frac{i}{P_i - \kappa - m_n} = i + m_nS_F(P_i - \kappa)
\]

\[
S_F(P_f + \kappa)(P_f + \kappa) = i + m_nS_F(P_f + \kappa)
\]

and the Dirac equation for the initial and final hadrons. Notice that the second last term of equation (1.24) has already been worked out for the case of ordinary muon capture. In other words, equations (1.20) and (1.21) imply that,

\[
\bar{u}(P_f)(G^H - G^L)\not\!\!\!\!F \not\!\!\!\!F + (G^H - G^L)\not\!\!\!\!F \gamma^5)u(P_i) - \bar{u}(P_f)(G^L i\sigma^{\alpha\beta}\frac{Q^L}{2m_n}\epsilon_{\alpha})
\]

\[
+ G^L P \cdot \epsilon \gamma^5 u(P_i) + \bar{u}(P_f)\{2m_nG^H_A + G^H P \cdot \frac{Q^2_H}{m}\}S_F(P_i - \kappa)Q_i u(P_i) +
\]

\[
\bar{u}(P_f)Q_F S_F(P_f + \kappa)\{2m_nG^H_A + G^H P \cdot \frac{Q^2_H}{m}\}S_F(P_i - \kappa)Q_i u(P_i) +
\]

\[
\frac{2mM_nG^H_A(1 + \epsilon)\frac{Q^2_H}{m}}{m^2 - Q^2_H} \not\!\!\!\!F \gamma^5 u(P_i) + \bar{u}(P_f)\{G^H P \cdot \frac{Q^H}{2m_n}\}S_F(P_i - \kappa)Q_i u(P_i) +
\]

\[
(Q - e\lambda) \cdot \Delta J_{\text{hadronic,RMC}} = a_{\pi}m^2_\pi \langle f\gamma | \phi(Q) | i \rangle
\]

provided the relationship between \(G_A\) and \(G_P\) is enforced by equation (1.16). In the above equation \(\Delta J_{\text{hadronic,RMC}}\) is the extra piece of RMC hadronic current whose terms are to be derived from the Adler and Dothan procedure.

\(\langle f\gamma | \phi(Q) | i \rangle\) is related to the pion photoproduction amplitude. The Born terms of \(\langle f\gamma | \phi(Q) | i \rangle\) are given by:

\[
\langle f\gamma | j_\pi(Q) | i \rangle = \bar{u}(P_f)\{-\frac{2M_n}{m_\pi}G^H_\pi \gamma^5\}S_F(P_i - \kappa)Q_i u(P_i) +
\]

\[
\bar{u}(P_f)Q_F S_F(P_f + \kappa)\{-\frac{2M_n}{m_\pi}G^H_\pi \gamma^5\}u(P_i) +
\]

\[
\bar{u}(P_f)\{-\frac{2M_n}{m_\pi}G^L_\pi \frac{-i}{m^2_\pi - (Q^H - \kappa)^2}
\]

\[
(-ie(2Q^L - \kappa)) \cdot \epsilon \gamma^5 u(P_i) -
\]

\[
\bar{u}(P_f)\{-\frac{2M_n}{m_\pi}G^L_\pi (2Q^H \cdot \epsilon) \gamma^5\}u(P_i)
\]

(1.27)
In the above equation, \( G^L_\pi \equiv \frac{d G_\pi(Q^2)}{dQ^2} |_{Q^2=Q_L^2} \). The first three terms correspond to the three external radiation diagrams and the last term is the gauge term to ensure the amplitude is gauge invariant when the pion is on shell. A look at equation (1.26) will reveal that the third, fourth and fifth terms almost fit the first, second and third term of the above expression respectively by virtue of equation (1.16) yet there are still two problematic spots: missing a \(-\bar{u}(P_f)\frac{2M_n G^H_H}{m^2 - Q_L^2} (2Q^H \cdot \epsilon)\gamma^5 u(P_i)\) in equation (1.26) and the mismatching of momentum transfer between \( G^L_\pi \) and \( G^H_\pi \) there. The first problem can be solved by some rearrangement of terms in equation (1.26) and for the second one, one can expand \( G^L_\pi \) as \( G^L_\pi = G^H_\pi + G^H_H (Q_L^2 - Q_H^2) + \mathcal{O}(Q^4) \). After all the work, equation (1.26) can be written as\(^{11}\):

\[
\bar{u}(P_f) \{ (G^H_H - G^L_H) \cdot \epsilon + (G^A_H - G^A_L) \cdot \kappa \gamma^5 \} u(P_i) - \bar{u}(P_f) \{ G^L_\pi \gamma^5 \} u(P_i) - \bar{u}(P_f) \{ G^L_\pi \sigma^{\alpha \beta} \frac{Q^L_\beta}{2M_n} \epsilon_{\alpha} \}
- G^L_\pi \frac{Q^L_\pi \cdot \epsilon}{m} \gamma^5 u(P_i) + \bar{u}(P_f) \{ 2M_n G^H_H + G^H_H \frac{Q^2_H}{m} \} \gamma^5 S_F(\not P_i - \not\kappa) Q_\pi u(P_i) + \bar{u}(P_f) Q_F S_F(\not P_f + \not\kappa) \{ 2M_n G^H_H + G^H_H \frac{Q^2_H}{m} \} \gamma^5 u(P_i) - \bar{u}(P_f) \{ \frac{2M_n G^H_H}{m^2 - Q^2_L} \} \gamma^5 u(P_i) + \frac{1}{m^2 - (Q^H \cdot \kappa)^2} \frac{2m M_n G^H_H (1 + \epsilon) Q^2_H}{m^2 - Q^2_H} (2Q^H \cdot \epsilon) \gamma^5 u(P_i) + \bar{u}(P_f) \frac{2M_n G^H_H (2Q^H \cdot \kappa)}{m^2 - Q^2_L} (2Q^H \cdot \epsilon) \gamma^5 u(P_i) - \bar{u}(P_f) \{ \frac{\epsilon G^H_H}{m} (2Q^H \cdot \epsilon) \gamma^5 \} u(P_i) + \bar{u}(P_f) \{ e G^H_H (Q^H \cdot \epsilon) \gamma^5 \} u(P_i) + \mathcal{O}(\epsilon^2) \Delta J_{\text{hadronic,RMC}}
\]

\[
\langle f \gamma | j_\pi(Q) | i \rangle
= a_\pi m_\pi^2 \langle f \gamma | \phi(Q) | i \rangle
\]

\(\text{and } \langle f \gamma | j_\pi(Q) | i \rangle \text{ as}\):

\[
\langle f \gamma | j_\pi(Q) | i \rangle = \bar{u}(P_f) \{ -\frac{2M}{m_\pi^2} G^H_H \gamma^5 \} S_F(\not P_i - \not\kappa) Q_\pi u(P_i) + \bar{u}(P_f) Q_F S_F(\not P_f + \not\kappa) \{ -\frac{2M}{m_\pi^2} G^H_H \gamma^5 \} u(P_i) + \mathcal{O}(\epsilon^2) \Delta J_{\text{hadronic,RMC}}
\]

\(^{11}\) Notice the gauge condition of the photon is chosen to be such that \( \kappa \cdot \epsilon = 0 \); therefore, \( Q^L \cdot \epsilon = Q^H \cdot \epsilon \).
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\[
\bar{u}(P_f)\left\{-\frac{2M_n}{m_\pi} G^H_\pi \frac{m_\pi^2}{m_\pi^2 - (Q^H - \kappa)^2} \right.
\]
\[
\left. (-ie(2Q^L - \kappa)) \cdot \gamma^5 \right\} u(P_i) -
\]
\[
\bar{u}(P_f)\left\{-\frac{2M_n}{m_\pi} G'^L_\pi (2Q^H \cdot \epsilon) \gamma^5 \right\} u(P_i) +
\]
\[
\bar{u}(P_f)\left\{-\frac{2M_n}{m_\pi} G^H_\pi \frac{2Q^H \cdot \kappa}{m_\pi^2 - (Q^H - \kappa)^2} (2Q^L - \kappa) \cdot \epsilon \gamma^5 \right\} u(P_i)
\]
(1.29)

Assuming the linearity of \( G_\pi(Q^2) \) over the range of \( Q^2 \) concerned, the last two terms of equation (1.29) can be combined to form

\[
-\frac{m_\pi^2 - Q^H L}{m_\pi^2 - Q^L L} \bar{u}(P_f)\left\{-\frac{2M_n}{m_\pi} G^H_\pi (2Q^H \cdot \epsilon) \gamma^5 \right\} u(P_i)
\]

That is to say the CVC and PCAC conditions for RMC are:

\[
\bar{u}(P_f)\{(G^H_V - G^H_W) \kappa + (G^H_A - G^L_A) \kappa \gamma^5 \} u(P_i) - \bar{u}(P_f)\{G^L_M \sigma^{\alpha \beta} \frac{Q^L_P}{2M_n} \epsilon_{\beta}\}
\]
\[
-\frac{G^L_P \epsilon \gamma^5}{m} u(P_i) + \bar{u}(P_f)\left\{2M_n G^L_A (2Q^H \cdot \kappa) \right.\}
\]
\[
\left. \frac{m_\pi^2 - Q^H L}{m_\pi^2 - Q^L L} (2Q^H \cdot \epsilon) \gamma^5 u(P_i) - \bar{u}(P_f)\left\{\frac{Q^H_L}{m} \gamma^5 \right\} u(P_i) + \bar{u}(P_f)\{G^H_W \mu_f - \mu_i \sigma^{\alpha \beta} \kappa_{\beta} \epsilon_{\alpha}\} u(P_i) +
\]
\[
(Q - eA) \cdot \Delta J_{\text{hadronic,RMC}} \gamma^5 u(P_i)
\]
(1.30)

provided equation (1.16) holds true.

Now turn to gauge invariance, let \( M_i = \epsilon \cdot \tilde{M}_i \) and if one then calculates \( \kappa \cdot \tilde{M}_i \) the results of calculation would be:

\[
\kappa \cdot \tilde{M}_1 = e_i \bar{u}(\nu) \gamma_\alpha (1 - \gamma^5) u(\mu) \bar{u}(P_f) W^\alpha(Q^H) u(P_i)
\]
\[
\kappa \cdot \tilde{M}_2 = -e_f \bar{u}(\nu) \gamma_\alpha (1 - \gamma^5) u(\mu) \bar{u}(P_f) W^\alpha(Q^H) u(P_i)
\]
\[
\kappa \cdot \tilde{M}_3 = \bar{u}(\nu) \gamma_\alpha (1 - \gamma^5) u(\mu) \bar{u}(P_f) \left\{-\frac{i}{m_\pi^2 - (Q^H - \kappa)^2} \right.\}
\]
\[
\left. (-ie(2Q^H \cdot \kappa) G^H_P \frac{Q^H_P}{m} \gamma^5 u(P_i) \right.
\]
\[
\kappa \cdot \tilde{M}_4 = -\bar{u}(\nu) \gamma_\alpha (1 - \gamma^5) u(\mu) \bar{u}(P_f) W^\alpha(Q^L) u(P_i)
\]
(1.31)

The tricks used in the above calculation involve the following:
1. Introduction

1. Using the antisymmetry of $\sigma^{\beta\lambda}$ with respect to switching indices to conclude that $\sigma^{\beta\lambda}\kappa_{\beta}\kappa_{\lambda} = 0$.

2. Using equations like

\[
S_{\gamma}(P_{i} - \kappa) i \kappa u(P_{i}) = i \frac{P_{i} - \kappa + M_{n}}{(P_{i} - \kappa)^2 - M_{n}^2} i \kappa u(P_{i})
\]

\[= -\frac{P_{i} \kappa + M_{n} \kappa}{-2P_{i} \cdot \kappa} u(P_{i})
\]

\[= -\frac{2P_{i} \cdot \kappa - \kappa P_{i} + M_{n} \kappa}{-2P_{i} \cdot \kappa} u(P_{i})
\]

\[= u(P_{i})
\]

(1.32)

to simplify expressions.

Since $e_{i} - e_{f} = e$, it is easy to show that,

\[
\sum_{i=1}^{4} \kappa \cdot \vec{M}_{i} = \bar{u}(v)\gamma_{\alpha}(1 - \gamma^{5})u(\mu)\bar{u}(P_{f})\left\{(G_{V}^{H} - G_{V}^{L})\gamma^{\alpha} + (G_{A}^{H} - G_{A}^{L})\gamma^{\alpha}\gamma^{5} + \frac{2mM_{n}G_{A}^{H}(1 + \varepsilon)}{m_{n}^{2} - Q_{L}^{2}}\gamma^{5}\right\}u(P_{i})
\]

(1.33)

To satisfy gauge invariance up to $O(\kappa^{2})$, one needs $\Delta \vec{M}$ such that

\[
\kappa \cdot \left(\sum_{i=1}^{4} \vec{M}_{i} + \Delta \vec{M}\right) = \kappa \cdot \vec{M}
\]

\[= 0 + O(\kappa^{2})
\]

(1.34)

Postulate\(^{12}\)

\[
\Delta J_{\text{hadronic, RMC}}^{\alpha} = -2\bar{u}(P_{f})\left\{G_{V}^{H}\gamma_{\mu} + G_{A}^{H}\gamma_{\mu}\gamma^{5}\right\}\left\{g^{\mu\alpha}Q_{H}\epsilon + \kappa^{\alpha}\epsilon^{\mu} - \kappa^{\mu}\epsilon^{\alpha}\right\}u(P_{i}) - \bar{u}(P_{f})\left\{G_{M}^{H}\delta_{\alpha}^{\gamma_{c}}\epsilon^{c}_{\mu} + G_{P}^{L}\frac{G_{A}^{H}\gamma^{5} + G_{V}^{H}(\mu_{f} - \mu_{i})2m_{n}^{2}\sigma^{c\beta}\kappa_{\beta}\epsilon_{\beta}}{2m_{n}^{2} - Q_{L}^{2}}\right\}u(P_{i})
\]

\(^{12}\)Klieb and Rood did not have\(-\bar{u}(P_{f})\left\{\frac{2mM_{n}G_{V}^{H}(1+\epsilon)}{m_{n}^{2} - Q_{L}^{2}}\epsilon\gamma^{5} + G_{V}^{H}(\mu_{f} - \mu_{i})2m_{n}^{2}\sigma^{c\beta}\kappa_{\beta}\epsilon_{\beta}\right\}u(P_{i})\) in their RMC calculation.
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\[ -\bar{u}(P_f) \frac{2mM_nG'_A(1+\varepsilon)}{m^2 - Q_L^2} \frac{2Q^H \cdot \varepsilon}{m} Q_H^\alpha \gamma^5 u(P_i) - \]

\[ \bar{u}(P_f) \{ G'_M i\sigma^{\alpha\beta} \frac{Q_H^\beta}{2M_n} (2Q^H \cdot \varepsilon) + \]

\[ G'_V (\mu_f - \mu_i) \frac{2\kappa^\alpha}{2M_n} - i\sigma^{\beta\gamma\kappa\epsilon_\beta} u(P_i) \] (1.35)

will satisfy both equations (1.30) and (1.34) up to, but excluding, terms of \( O(\kappa Q) \). \(^{13}\)

**Proof**

Let \( \Delta M \equiv J_{\text{leptonic}} - \Delta J_{\text{hadronic,RMC}}. \) The fulfilment of equation (1.34) of \( \Delta M \) can immediately be read off once one expands the form factors of equation (1.33) to first derivatives. Note that all the pole terms are treated exactly.

To demonstrate that the \( \Delta J_{\text{hadronic,RMC}}^\alpha \) in equation (1.35) satisfies equation (1.30), one can calculate \( Q \cdot \Delta J_{\text{hadronic,RMC}}^\alpha. \) The \( -\varepsilon A_\alpha \) part does not enter since it is higher order than required. The result of the calculation is:

\[ Q \cdot \Delta J_{\text{hadronic,RMC}}^\alpha = -\bar{u}(P_f) \{ G'_V \gamma^5 + G'_A \gamma^5 \} (Q_L^2 - Q_H^2) u(P_i) \]

\[ -\bar{u}(P_f) \{ 2mM_nG'_A(2Q^H \cdot \varepsilon) \gamma^5 \} u(P_i) \]

\[ -\bar{u}(P_f) \{ 2mM_nG'_B(1+\varepsilon) 2Q^H \cdot \varepsilon^{2M_n} - Q_H^2 \gamma^5 \} u(P_i) \]

\[ -\bar{u}(P_f) \{ G'_M i\sigma^{\alpha\beta} \frac{Q_H^\beta}{2M_n} + G'_L \frac{Q_H^L \cdot \varepsilon^{2M_n}}{m} \gamma^5 \} u(P_i) - \]

\[ u(P_f) \{ G'_V (\mu_f - \mu_i) \frac{2Q^L \cdot \kappa}{2M_n} - i\sigma^{\beta\gamma\kappa\epsilon_\beta} u(P_i) \} \] (1.36)

Putting it back into the left hand side of equation (1.30), one would get for the left hand side:

\[ -\bar{u}(P_f) \{ 2M_nG'_A(2Q^H \cdot \varepsilon) \gamma^5 \} u(P_i) + \bar{u}(P_f) \{ G'_V \frac{\mu_f - \mu_i}{2M_n} i\sigma^{\beta\gamma\kappa\epsilon_\beta} u(P_i) + \]

\[ \bar{u}(P_f) \{ \frac{2mM_nG'_A(2Q^H \cdot \kappa - (1+\varepsilon)Q_H^2)}{m^2 - Q_L^2} (2Q^H \cdot \varepsilon) \gamma^5 - \]

\[ \frac{\varepsilon G'_P}{m} (2Q^H \cdot \varepsilon) \gamma^5 \} u(P_i) \] (1.37)

\(^{13}\) Equation (1.30) determines terms up to \( O(Q^0\kappa^n) \) and equation (1.34) determines terms up to \( O(\kappa Q^m) \). Here \( \kappa^0 \) means "photon four momentum up to order zero" and \( n, m \) are arbitrary positive integers.
which equals

\[-\frac{m_\pi^2}{m_\pi^2 - Q_L^2} \bar{u}(P_f)2M_nG'_{A}(2Q^H \cdot \epsilon)\gamma^5 u(P_i) + \bar{u}(P_f)\{G'_{V} \frac{\mu_f - \mu_i}{2M_n}\}

\[i\sigma^{\beta\kappa}\epsilon_\beta\}u(P_i) - \bar{u}(P_f)\{\frac{\epsilon 2mM_nG'_{A}2Q^H \cdot \epsilon}{m} Q_H^2 \gamma^5 + \frac{\epsilon G'_{V}}{m} (2Q^H \cdot \epsilon)\gamma^5\} u(P_i)\] (1.38)

Using

\[\epsilon = m_\pi^2 \left\{ \frac{G'_{\pi}}{G_{\pi}(0)} - \frac{G'_{A}}{G_{A}(0)} \right\} + O(Q^2)\] (1.39)

to eliminate $G'_{A}$ in favour of $G'_{\pi}$ and $\epsilon$ in expression (1.38) one would end up having,

\[-\frac{a_\pi m_\pi^2}{m_\pi^2 - Q_L^2} \bar{u}(P_f)\{-\frac{2M_n}{m} G'_{\pi}(2Q^H \cdot \epsilon)\gamma^5\} u(P_i) - \epsilon G_{A}(0)\left(\frac{G'_{\pi}}{G_{\pi}(0)}Q_H^2 + \frac{G'_{A}}{G_{A}(0)}Q_L^2\right)\bar{u}(P_f)\{\frac{2M_n}{m(m_\pi^2 - Q_L^2)}(2Q^H \cdot \epsilon)\gamma^5\} u(P_i) + \bar{u}(P_f)\{G'_{V} \frac{\mu_f - \mu_i}{2M_n}\}

\[i\sigma^{\beta\kappa}\epsilon_\beta\}u(P_i) + O(\epsilon^2)\] (1.40)

The first term is just the right hand side of equation (1.30). The magnitude of the second term can be approximated as follows:

\[\epsilon G_{A}(0)\left(\frac{G'_{\pi}}{G_{\pi}(0)}Q_H^2 + \frac{G'_{A}}{G_{A}(0)}Q_L^2\right) \sim \epsilon G_{A}(0)\left(\frac{G'_{\pi}}{G_{\pi}(0)} + \frac{G'_{A}}{G_{A}(0)}\right)Q^2

\[\sim m_\pi^2 G_{A}(0)\left(\frac{G'_{\pi}^2}{G_{\pi}(0)^2} - \frac{G'_{A}^2}{G_{A}(0)^2}\right)Q^2

\[\sim 2K G'' Q^4\] (1.41)

keeping in mind that $a_\pi \sim 1$ (0.9436, to be exact); in the above equation, $G''$, $|G''| << 1$ is the small second derivative of either $G_{A}$ or $G_{\pi}$ and $K$ is some constant with magnitude $|K| \sim 1$. The third term is annoying but the Adler and Dothan procedure is not sufficient to determine uniquely counterterms to get rid of it. To see this point, notice that a term in $\Delta J_{\text{hadronic, RMC}}^a$ with the format

\[-\bar{u}(P_f)\{G'_{V}Y^\alpha \frac{\mu_f - \mu_i}{2M_n}, i\sigma^{\beta\kappa}\epsilon_\beta\}u(P_i)\] (1.42)
with \( Y^\alpha = \frac{\kappa^\alpha}{Q^4} \) or \( \frac{\partial Y^\alpha}{\partial Q^4} \) (and probably among many others) can both do the trick but both have some undesirable pole behaviours. Notice that the latter is formally a term of \( \mathcal{O}(\kappa Q) \) but it is exactly these terms that the Adler and Dothan procedure cannot determine uniquely. A more detailed study of these structure terms should probably involve a model that incorporates the exchange currents of mesons. Please see section 3.7.2 for possible implications of these undeterminable terms on the resulting capture rates.

\( \mathcal{O}(\kappa Q) \) terms aside, the Adler and Dothan procedure does indeed generate terms of \( \Delta J_{\text{hadronic}}^\alpha \) as in equation (1.35) that satisfy the gauge invariant requirement up to \( \mathcal{O}(\kappa^0) \) and PCAC up to terms of \( \mathcal{O}(\varepsilon G^\mu) \) and \( \mathcal{O}(Q^0) \).

1.3 Summary

After a brief introduction of the process of ordinary muon capture by \(^5\text{He}\) and its various hypotheses involving the OMC hadronic current, the corresponding radiative process is compared against the ordinary one. For the RMC amplitude to be gauge invariant and its hadronic currents to satisfy the CVC and the PCAC, an extra current \( \Delta J_{\text{hadronic,RMC}}^\alpha \) must be added in addition to a naive insertion of photons to external particles. The Adler and Dothan procedure is employed to generate those extra terms as far as possible. This procedure guarantees the uniqueness of those extra terms up to \( \mathcal{O}(\kappa^0) \) by gauge invariant requirement and \( \mathcal{O}(Q^0) \) by the CVC and the PCAC except the pion pole terms which are treated exactly. Terms of \( \mathcal{O}(\kappa Q) \), however, are not determined uniquely by the procedure and these terms can only be revealed by a detailed examination of the structure of the weak hadronic vertex.
2. The elementary particle model

2.1 Introduction

The elementary particle model is probably the simplest method which can be used in the calculation of the RMC rate. It was first used by Kim and Primakoff [9, 10] in calculating the beta decay of complex nuclei. As its name suggests, the central idea of the method is to completely ignore the internal structure of both the $^3$He and $^3$H and regard them as "elementary particles". Since both $^3$He and $^3$H have spin $\frac{1}{2}$, and isospin $\frac{1}{2}$ up to a small isospin breaking, the calculation is relatively simple compared with other methods. The drawback is one still has to rely on experiments done on these nuclei for the phenomenological form factors parameterizing the reaction. For example, one can perform electron scattering experiments for determining the vector current form factor and beta decay experiments for the axial current form factor.

Previous EPM calculations of RMC by $^3$He include Fearing [11], Klieb and Rood [4, 5] and Hwang and Primakoff [12] but only the calculation of Klieb and Rood included some Adler and Dothan terms. Moreover, some physical quantities were obtained by Klieb and Rood via a non-relativistically reduced amplitude. The present calculation differs from those previous ones in that the full Adler and Dothan terms are included and the resultant amplitude is not non-relativistically approximated in the hope of getting a more accurate reaction rate.

2.2 Kinematics and nuclear form factors

For the elementary particle model description of RMC by $^3$He, the degrees of freedom are the four momenta of $^3$He ($P_{^3\text{He}}$), $^3$H ($P_{^3\text{H}}$), photon ($\kappa$), neutrino ($\nu$) and muon ($\mu$), together with their respective spinors. The four-
momentum conservation is written as:

\[ P_{3\text{He}} + \nu + \kappa = P_{3\text{He}} + \mu \]  

(2.1)

The quantity that is of interest is the *differential capture rate*, which is given by

\[ \frac{d\Gamma}{d\kappa} = \sum_{\text{photon polarization}} \int \frac{d(\text{phase space})}{d\kappa} \text{Tr}(\rho M^\dagger M) \]  

(2.2)

for RMC, and the *total capture rate*

\[ \Gamma_{\text{omc}} = \int d(\text{phase space})_{\text{omc}} \text{Tr}(\rho M_{\text{omc}}^\dagger M_{\text{omc}}) \]  

(2.3)

for OMC. Here \( M (M_{\text{omc}}) \) is the RMC (OMC) amplitude that was discussed in chapter 1. One just needs to recognize the following name changes

- \( u(P_i) \rightarrow u(P_{3\text{He}}) \)
- \( \bar{u}(P_f) \rightarrow \bar{u}(P_{3\text{He}}) \)
- \( M_n \rightarrow M_i \) on the hadronic vertex
- \( G_i \rightarrow F_i, \ i = V, M, A, P \)  

(2.4)

for \( M \) (denote it as \( M(P_{3\text{He}}, P_{3\text{He}}, M_i) \)). The name change from \( G \rightarrow F \) is merely a change in accordance with the usual convention.

\( \rho \) is the density matrix which describes the initial spin configurations of the muonic atom. The detailed implementation of \( \rho \) will be discussed in the next section.

\( d(\text{phase space}) \) (\( d(\text{phase space})_{\text{omc}} \)) is the differential phase space factor of the emitted particles for RMC (OMC). The detailed calculation of these will appear in the next section. Also note the integral sign of equation (2.2) does not apply to \( \kappa \) since what one needs is the differential rate.

### 2.2.1 Nuclear form factors

One can conveniently parameterize the nuclear electromagnetic form factors linearly in terms of \( Q^2 \) when \( Q^2 \) is small, as in the case of RMC by \(^3\)He.

\[ F^{3\text{He}}_V(Q^2) = 1 + \frac{1}{6} \langle r^2 \rangle_{3\text{He}, \text{electric}} Q^2 + \mathcal{O}(Q^4) \]  

(2.5)

\[ F^{3\text{He}}_M(Q^2) = 1 + \frac{1}{6} \langle r^2 \rangle_{3\text{He}, \text{magnetic}} Q^2 + \mathcal{O}(Q^4) \]  

(2.6)

\(^1\) Despite its notation, \( \frac{d(\text{phase space})}{d\kappa} \) is actually a differential of angular variables.

\(^2\) One integrates \( \kappa \) in equation (2.2) to get the *total capture rate* of RMC.
and analogous equations also hold true for the case of $^3$H. The above equations relate the electromagnetic form factors to $<r^2>$ where $<r^2>^{\frac{1}{2}}$ is some form of root mean square charge or magnetic radius of the nucleus which are supposed to be found from experiments (see, for example, refs. [13, 14, 15, 16] for related experiments).

<table>
<thead>
<tr>
<th>$&lt;r^2&gt;^{\frac{1}{2}}$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^3$He, electric</td>
</tr>
<tr>
<td>$1.976 \pm 0.015$</td>
</tr>
<tr>
<td>$^3$He, magnetic</td>
</tr>
<tr>
<td>$1.99 \pm 0.06$</td>
</tr>
<tr>
<td>$^3$H, electric</td>
</tr>
<tr>
<td>$1.68 \pm 0.03$</td>
</tr>
<tr>
<td>$^3$H, magnetic</td>
</tr>
<tr>
<td>$1.72 \pm 0.06$</td>
</tr>
</tbody>
</table>

**Tab. 2.1:** A set of $<r^2>^{\frac{1}{2}}$ values. The $<r^2>$ notation indicates that the above quantities are root mean square values of nuclear radii. Experimental data from Ottermann et al. [14] ($^3$He) and Beck et al. [13] ($^3$H).

By the IVC discussed in chapter 1, one has

$$F_V = 2F_{V^3\text{He}} - F_{V^3\text{H}}$$
$$F_M = \mu_\text{He}F_{M^3\text{He}} - \mu_\text{H}F_{M^3\text{H}}$$  \(2.7\)

where $\mu_\text{He}$ is the anomalous magnetic moment of $^3$He with a similar definition for $^3$H. Their values are

$$\mu_\text{He} = -8.3689 \text{ n.m}$$
$$\mu_\text{H} = 7.9173 \text{ n.m}$$  \(2.8\)

Equations (2.7) imply that

$$F_V(0) = 1$$
$$F_M(0) = \mu_\text{He} - \mu_\text{H}$$  \(2.9\)

The functional form of $F_V$ and $F_M$ are chosen to be

$$F_V = 1 + \frac{1}{6}R_V^2Q^2$$
$$F_M = (\mu_\text{He} - \mu_\text{H})(1 + \frac{1}{6}R_M^2Q^2)$$  \(2.10\)
2. The elementary particle model

with \( R_V = 1.94 \text{ fm} \) and \( R_M = 1.72 \text{ fm} \). These values are taken from a grand average of various experiments and (with \( R_A \)) they produce "good" numerical values for OMC. The function \( F_A(Q^2) \) is not so clear cut since there does not seem to have any direct experimental determination of that form factor. One has to resort to the impulse approximation to estimate the dependence of \( F_A \) on \( Q^2 \) over the range of \( Q^2 \) concerned. Delorme [17], using impulse approximation, expressed the nuclear form factors in terms of nucleon form factors. This method is also employed by Klieb and Rood [4, 5] and Congleton and Fearing [18, 19] in their EPM calculations.

\[
F_V(Q^2) = g_V(Q^2)[1]^0 \\
F_M(Q^2) = 3\{g_V(Q^2) + g_M(Q^2)\}[\bar{\sigma}] - g_V(Q^2)[1]^0 - \\
\frac{3\sqrt{2}}{2|Q|}g_V(Q^2)[i\bar{P}]_{1,1} \\
F_A(Q^2) = g_A(Q^2)[\bar{\sigma}] \\
F_P(Q^2) = 9g_P(Q^2) + 6\sqrt{2}\frac{M_t^2}{Q^2}g_A(Q^2)[\bar{\sigma}]^{2,1}
\]

where \([1]^0, [\bar{\sigma}]^+, [\bar{\sigma}]^-, [\bar{\sigma}]^{2,1} \) and \([i\bar{P}]_{1,1} \) are the impulse approximation reduced matrix elements resulting from integration of internal momenta. They will be defined in section 3.6.1. The \( Q^2 \) dependence of \( F_A \) is given by

\[
\frac{F_A(Q^2)}{F_A(0)} = \frac{g_A(Q^2)[\bar{\sigma}]^-(Q^2)}{g_A(0)[\bar{\sigma}]^-(0)}
\]

Notice that

\[
F_V(Q^2) + F_M(Q^2) \rightarrow 3\{g_V(Q^2) + g_M(Q^2)\}[\bar{\sigma}]^-(Q^2)
\]

when \([i\bar{P}]_{1,1} \rightarrow 0\). Since \([i\bar{P}]_{1,1} \) is indeed small, one can use the above limiting equation as an approximation and put it back in equation (2.15). The result is:

\[
\frac{F_A(Q^2)}{F_A(0)} = \left(1 - \frac{Q^2/M_V^2}{1 - \frac{Q^2}{M_A^2}}\right)^2 \left\{\frac{F_V(Q^2) + F_M(Q^2)}{F_V(0) + F_M(0)}\right\}
\]

where \( M_V^2 = 0.710 \text{ GeV}^2 \) and \( M_A^2 = 1.08 \pm 0.04 \text{ GeV}^2 \) [18]. This follows from the expansion \( g_i = g_i(0)/(1 - Q^2/M_i^2)^2 \) \( i = V, A \). Upon expansion to \( O(Q^2) \),
2. The elementary particle model

$F_A$ can be written as

$$ F_A = F_A(0)(1 + \frac{1}{6} R_A^2 Q^2) \quad (2.18) $$

$$ F_A(0) = 1.212 \pm 0.004 [18] \quad (2.19) $$

where $R_A^2 = \frac{F_V(0) R_L^2 + F_M(0) R_M^2}{F_V(0) + F_M(0)} + \frac{1}{3} \left( \frac{1}{M_A^2} - \frac{1}{M_P^2} \right)$ which makes $R_A = 1.703$ fm.

The PCAC hypothesis discussed in chapter 1 relates $F_P$ to $F_A$. The relationship is shown as equation (1.16) with $G_A$ replaced by $F_A$ and $G_P$ replaced by $F_P$.

### 2.3 Results

When all the relevant Feynman diagrams are evaluated and the gauge freedom is exploited so that the photon is made transverse (i.e. $\vec{\kappa} \cdot \vec{e} = 0$ and $\vec{e}^0 = 0$), the relativistic amplitude $M (M = \sum_{i=1}^{4} M_i + j^{\text{leptonic}} \cdot \Delta J_{\text{hadronic,RMC}}$ in equation (1.35)) can be written in the following non-relativistic like form, which operates on the product space (see ref. [20], chapters 3 and 4 for information on direct products of matrices) of the hadronic spin (via $\vec{\sigma}$) and leptonic spin (via $\vec{\sigma}_L$):

$$ M(P_{\Phi}, P_{\Phi e}, M_t) = N' \frac{G_F}{\sqrt{2}} V_{ud} (1 - \vec{\sigma}_L \cdot \vec{\nu}) \{ f_1 \vec{\sigma}_L \cdot \vec{\epsilon}_\lambda + f_2 \vec{\sigma} \cdot \vec{\epsilon}_\lambda + $$

$$ i f_3 \vec{\epsilon}_\lambda \times \vec{\sigma}_L \cdot \vec{\epsilon}_\lambda + \frac{f_4}{2m} \vec{\sigma}_L \cdot \vec{\epsilon}_\lambda \cdot \vec{s} + \frac{f_5}{2m} \vec{\nu} \cdot \vec{\epsilon}_\lambda + $$

$$ i \frac{f_6}{2m} \vec{s} \times \vec{\epsilon}_\lambda \cdot \vec{\sigma} + \frac{f_7}{2m} \vec{\sigma}_L \cdot \vec{\epsilon}_\lambda \cdot \vec{s} + \frac{f_8}{2m} \vec{\sigma}_L \cdot \vec{\sigma} \cdot \vec{\epsilon}_\lambda + $$

$$ \frac{f_9}{4m^2} \vec{\sigma} \cdot \vec{\nu} \cdot \vec{\epsilon}_\lambda + i \frac{f_{10}}{4m^2} \vec{s} \times \vec{\nu} \cdot \vec{\sigma}_L \cdot \vec{\epsilon}_\lambda + $$

$$ \frac{f_{11}}{4m^2} \vec{\sigma} \cdot \vec{\nu} \cdot \vec{\epsilon}_\lambda + \frac{f_{12}}{2m} \vec{\sigma} \cdot \vec{\nu} \cdot \vec{\epsilon}_\lambda + $$

$$ i \frac{f_{13}}{4m^2} \vec{s} \times \vec{\epsilon}_\lambda \cdot \vec{\sigma}_L \cdot \vec{s} + i \frac{f_{14}}{8m^3} \vec{s} \times \vec{\nu} \cdot \vec{\sigma} \cdot \vec{\nu} \cdot \vec{\epsilon}_\lambda \} \quad (2.20) $$

where $\vec{s} \equiv \vec{\nu} + \vec{\kappa}$ and $N'$ is the normalization factor for the particle spinors,

$$ N' = \frac{1}{2} \left( \frac{P_{\Phi}^0 + M_t}{2P_{\Phi}^0} \right) \quad (2.21) $$
The coefficient of $\frac{1}{2}$ in $N'$ comes from the normalizations of the photon and the neutrino. The photon polarization unit vector $\vec{\epsilon}_\lambda$, $\lambda = 1, -1$ is defined as,

$$\vec{\kappa} \times \vec{\epsilon}_\lambda = -i\lambda\kappa\vec{\epsilon}_\lambda \quad (2.22)$$

That means the $\sum_{\text{photon polarization}}$ in equation (2.2) should be replaced by $\sum_{\lambda=-1,+1}$. $f_i, i = 1 - 15$ are some very complicated scalar functions independent of $\bar{\sigma}, \bar{\sigma}_L$ and $\vec{\epsilon}_\lambda$ and will not be denoted.

Define $d(\text{phase space})$ to be the differential phase space factor which has the form

$$d(\text{phase space}) = C |\phi_\mu(0)|^2 \frac{d^3\vec{p}_\mu}{(2\pi)^3} \frac{d^3\vec{p}'_\mu}{(2\pi)^3} \frac{1}{\kappa} \frac{d^3\kappa}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(\mu + P_{\text{He}} - p_{\text{He}} - \kappa - \nu) \quad (2.23)$$

where $\phi_\mu(0)$ is the muon wavefunction at the origin and $C$ (close to 1, chosen to be 0.9788 [18]) is the correction factor which accounts for the non-pointlike nature of the nucleus (an averaging of the muon wavefunction near the origin over the small but finite size of the nucleus). Some standard manipulations of equation (2.23) lead to:

$$d(\text{phase space}) = C |\phi_\mu(0)|^2 \frac{1}{(2\pi)^5} \frac{2P_\mu^0}{\kappa} |2\kappa(1 - \cos(\theta)) - 2(m + M_i)|^{-1} \nu^2(\kappa, \theta) d\nu d\kappa d\kappa \quad (2.24)$$

where $\cos(\theta) = \hat{\nu} \cdot \hat{\kappa}$ and

$$\nu(\kappa, \theta) = \frac{2\kappa(m + M_i) - m^2 - 2mM_i}{2\kappa(1 - \cos(\theta)) - 2(m + M_i)} \quad (2.25)$$

expressing the four-momentum conservation of the $\delta$ function.

Define $\rho_{ij}$ as the elements of the density matrix which takes care of the initial spin of the muonic atom. Since $\rho$ is a density matrix, it has the property $\text{Tr}(\rho) = 1$. $\rho$ is chosen to be diagonal in the $| f, f_z \rangle$ (coupled lepton ($\vec{s}_\mu$) spin and nuclear ($\vec{s}_\text{nuclear}$) spin state; $\vec{f} \equiv \vec{s}_\mu + \vec{s}_\text{nuclear}$) basis rather than the $| s_\mu, s_\text{nuclear}; s'_\mu, s'_\text{nuclear} \rangle$ because of the hyperfine splitting of the muon in the muonic $1S$ orbit. In the special case when the initial spin mixtures
are equal, \( \rho_{ij} = \frac{1}{4} \delta_{ij} \) and is basis independent. In that case, equation (2.2) reduces to

\[
\frac{d\Gamma_{\text{stat}}}{d\kappa} = \sum_{\lambda=-1,+1} \int \left\{ \frac{d(\text{phase space})}{d\kappa} \right\} \frac{1}{4} \text{Tr}(M(P_{\text{He}}, P_{\text{He}}, M_{i})^\dagger M(P_{\text{He}}, P_{\text{He}}, M_{i})) \}
\]

(2.26)

The quantity \( \frac{d\Gamma_{\text{stat}}}{d\kappa} \) is commonly called the differential statistical rate and the corresponding photon spectrum is plotted in figure (2.1).

The total RMC rate is obtained by \( \int \frac{d\Gamma_{\text{stat}}}{d\kappa} d\kappa \). The EPM description of RMC statistical rate for photon momentum higher than 5 MeV \(^3\) is

\[
\Gamma_{\text{stat}}^{\text{RMC}}(\kappa > 5 \text{ MeV}) = 0.8189 \text{ s}^{-1}
\]

(2.27)

when \( F_P \) is set at the PCAC value. Figure (2.2) shows the variation of \( \Gamma_{\text{stat}}^{\text{RMC}} \) with respect to changes in \( F_P \).

Perhaps a more important quantity is the RMC statistical rate for photon momentum higher than 60 MeV because experiments cannot (yet) measure photons resulting from RMC with momentum lower than 53 MeV because of bremsstrahlung\(^4\). Its “PCAC” value is:

\[
\Gamma_{\text{stat}}^{\text{RMC}}(\kappa > 60 \text{ MeV}) = 0.2113 \text{ s}^{-1}
\]

(2.28)

Figure (2.3) shows different photon polarization\(^5\) \( P_\gamma(\kappa) \) predictions vs. \( F_P \).

### 2.3.1 Ordinary Muon Capture

The calculation of ordinary muon capture (OMC) is also done for comparison. Since RMC is the major topic of this work, the presentation below will be brief.

Let \( M_{\text{omc}} \) be the amplitude for the ordinary muon capture as shown in section 1.2.1. The momentum transfer squared at the hadronic vertex

---

\(^3\) The \( \kappa \) integration cannot go to zero because of infrared divergence.

\(^4\) The two step process \( \mu^- \rightarrow e^- + \nu_e + \bar{\nu}_\mu \) followed by \( e^- + ^3\text{He} \rightarrow e^- + ^3\text{He} + \gamma \) produces photons up to 53 MeV.

\(^5\) To obtain \( \frac{d\Gamma_{\text{stat}}}{d\kappa} \), simply replace \( \sum_{\text{polarization}} \) in equation (2.2) by \( \sum_{\lambda=-1,+1} (-1)^{\lambda+1}/2 \).
Fig. 2.1: The EPM description RMC photon spectra $\frac{d\sigma}{dx}$ using the full Adler Dothan amplitude for various values of $F_P$. The "Klieb and Rood" values are taken from the relativistic calculation of ref. [5] which are not shown in ref. [4]

$Q^2 = \left(-\frac{m^2 + 2mM_t}{2(m + M_t)}\right)^2 = -0.2763$ fm$^{-2}$. Upon expanding $M^{\text{omec}}(P_H, P_{\text{He}}, M_t)$ and rewriting in a non-relativistic like format, one has:

$$M^{\text{omec}}(P_H, P_{\text{He}}, M_t) = N' \frac{G_F}{\sqrt{2}} V_{ud}(1 - \vec{\sigma}_L \cdot \vec{\nu})\left\{g_1 + g_2 \vec{\nu} \cdot \vec{\nu} + g_3 \vec{\sigma}_L \cdot \vec{\sigma}\right\}$$

(2.29)

where

$$g_1 = F_V \left(1 + \frac{\nu}{M_t + P_{\text{He}}^0}\right) - F_M \frac{\nu}{2M_t(M_t + P_{\text{He}}^0)}$$

(2.30)

$$g_2 = (F_P - F_A - F_V - F_M) \frac{\nu}{M_t + P_{\text{He}}^0}$$

(2.31)

$$g_3 = F_A - (F_V + F_M) \frac{\nu}{M_t + P_{\text{He}}^0}$$

(2.32)
Fig. 2.2: Sensitivity of $\Gamma_{\text{omc}}(\kappa > 60 \text{ MeV})$ using EPM with respect to $F_P$.

\[
N' = \sqrt{\frac{1}{2} \frac{M_t + P_{\text{He}}^0}{2 P_{\text{He}}^0}} \tag{2.33}
\]

The differential phase space factor for OMC $d(\text{phase space})_{\text{omc}}$ is:

\[
d(\text{phase space})_{\text{omc}} = C|\phi_{\mu}(0)|^2 q^3 F_{\text{He}}^0 \frac{d\tilde{\sigma}}{(2\pi)^3} (2\pi)^4 \delta(4)(P_{\text{He}} + \nu - P_{\text{He}} - \mu) \tag{2.34}
\]

\[
= C|\phi_{\mu}(0)|^2 \frac{1}{(2\pi)^2} \frac{P_{\text{He}}^0 \nu^2}{m + M_t} d\nu
\]

where $\nu$ is evaluated at, due to the $\delta$ function

\[
\nu = \frac{m^2 + 2mM_t}{2(m + M_t)} \tag{2.35}
\]

The value of $P_{\text{He}}^0$ for EPM description of OMC is therefore:

\[
P_{\text{He}}^0 = \sqrt{M_t^2 + \nu^2} \tag{2.36}
\]

using the value of $\nu$ from equation (2.35).

The statistical rate for OMC is obtained via equation (2.3) with $\rho_{ij} = \frac{1}{4} \delta_{ij}$.

The result, with $F_P$ set at the PCAC value, is:

\[
\Gamma_{\text{omc}} = 1503 \text{ s}^{-1} \tag{2.37}
\]
2. The elementary particle model

2.3 The elementary particle model

Fig. 2.3: Photon polarization for various values of $F_P$.

with

\[
F_V(-0.2763 \text{ fm}^{-2}) = 0.8267
\]
\[
F_M(-0.2763 \text{ fm}^{-2}) = -14.067
\]
\[
F_A(-0.2763 \text{ fm}^{-2}) = 1.050
\]

which should be compared with the most recent calculation by Congleton and Fearing [19] who obtained a value of $1497 \text{ s}^{-1}$. The closeness of the above result to theirs is not so surprising given that the values of the formfactors (i.e. $F_V$, $F_M$ and $F_A$) at the $Q^2$ for OMC are approximately the same as the ones they were using (which also gives a reason why a linear parameterization of formfactors in section 2.2.1 is sufficient). Also interesting to mention is the most recent experimental result of statistical OMC rate of $1496 \pm 4 \text{ s}^{-1}$ [21].

2.3.2 Discussion

Below are some points to note with regards to the EPM description of RMC. Please refer to section 3.7.2 discussions on the comparison between EPM and IA.
1. The photon spectrum (figure (2.1)) and total RMC rate agree with the results of Klieb and Rood [4, 5] who obtained a total rate of 0.814 s\(^{-1}\) via a non-relativistically approximated amplitude (also see figure (2.2) for variation of RMC statistical rate vs. \(F_P\)), showing that the extra terms not included by Klieb and Rood do not contribute significantly to the photon spectrum.

2. The photon polarization \(P_7(K)\) obtained in this work does differ from that of Klieb and Rood quite significantly (see figure 2.3). The reason behind that is while a non-relativistically reduced amplitude used by Klieb and Rood produces the correct spectrum within a few percent, it cannot produce \(P_7(K)\) accurately. Fearing [22] noted that the first order contribution of \(P_7(K)\) actually comes from \(O(\frac{1}{M^2})\) terms in the squared Hamiltonian but Klieb and Rood compromised \(P_7(K)\)'s accuracy by truncating many \(O(\frac{1}{M^2})\) terms when they squared the already non-relativistically reduced amplitude (see section 3.7.2).

3. The Adler and Dothan terms have a great influence on the resulting photon spectrum. They contribute about 10% of the spectrum at the high photon energy range \((K > 60\ \text{MeV})\). The large size of \(G'_v\) and \(G'_A\) because of the large root mean square nuclear radii of \(^3\text{He}\) and \(^3\text{H}\) compared to proton or neutron is primarily responsible for the contribution.

### 2.4 Summary

In the framework of the EPM description of RMC by \(^3\text{He}\), the "PCAC" value of the exclusive statistical capture rate, \(\Gamma_{\text{stat}}^{\text{exc}}(\kappa > 5\ \text{MeV})\), via a fully Adler and Dothan amplitude, is 0.8189 s\(^{-1}\) and the OMC statistical rate is 1503 s\(^{-1}\). While these results agree with Klieb and Rood [4, 5] and Congleton and Fearing [18, 19] respectively, Klieb and Rood's photon polarization \(P_7(\kappa)\) disagrees with the result of this work, possibly due to the fact that they threw away some terms in the squared amplitude which contribute significantly to \(P_7(\kappa)\).
3. The impulse approximation

3.1 Introduction

In this chapter, the method of the impulse approximation will be used to investigate the RMC by \(^{3}\)He. Even though the elementary particle model is a convenient approach to the reaction, it does not give the detailed mechanism of "what is happening inside". Indeed, each of the \(^{3}\)He and \(^{3}\)H nuclei is a three-nucleon bound state. Perhaps it would be a good idea to look at the nuclear reaction using a nucleon perspective. The idea is, using a trinucleon wavefunction, to integrate out all the degrees of freedom relevant to the internal coordinates of the trinucleon system and to extract quantities that are dependent on the EPM degrees of freedom. In other words, one wants an expression similar to \(M(P_{^{3}\text{He}}, P_{^{3}\text{H}}, M_t)\) in the EPM but one with all the information regarding the internal degrees of freedom hidden in some formfactors resulting from internal integration\(^1\).

The first step is to regard the three bound state nucleons as free nucleons and the Hamiltonian of the nuclear reaction as the sum of the Hamiltonian of its nucleon counterparts. This procedure is called the impulse approximation. Naively speaking, the approximation of bound state nucleons by free nucleons in this case is a good one since the binding energy (~ 8 MeV) is about 0.3% of the mass of the nucleus. However, the impulse approximation deliberately neglects the contribution to the RMC from meson exchange currents between the nucleons, which may be large. Figure (3.1) shows the impulse approximation for OMC by \(^{3}\)He.

\(^1\) The method of finding this expression will be discussed in section 3.6
3. The impulse approximation

![Diagram of the impulse approximation for OMC by $^3$He](image)

Fig. 3.1: Impulse approximation for OMC by $^3$He illustrated. The blobs represent the strong force that binds the nucleons. Please note that one has to sum all the contributions of the above process from every nucleon in order to get the complete rate.

3.2 "Internal" and "external" degrees of freedom

In describing any non-relativistic three-particle system, there exists a coordinate transformation which separates the centre of mass coordinates and the internal coordinates of the three particles. This enables one to treat the three-body problem as an effective two-body problem.

Let $\vec{k}_i$ be the (three) momentum of the of the $i^{th}$ nucleon and define:

$$\vec{P} = \vec{k}_\alpha + \vec{k}_\beta + \vec{k}_\gamma,$$
$$\vec{q}_\alpha = \frac{2}{3}\vec{k}_\alpha - \frac{1}{3}\vec{k}_\beta - \frac{1}{3}\vec{k}_\gamma,$$
$$\vec{p}_\alpha = \frac{1}{2}\vec{k}_\beta - \frac{1}{2}\vec{k}_\gamma,$$

where $(\alpha, \beta, \gamma)$ are any set of numbers obtained from the cyclic permutation of $(1, 2, 3)$.\(^2\)

\(^2\) From now on, when no confusion arises, $\vec{q}_\alpha$ and $\vec{q}_\alpha'$ will sometimes be denoted as $\vec{q}$ and $\vec{q}'$. 


3. The impulse approximation

It is now obvious that $\vec{P}$ is the centre of mass momentum vector and $\vec{q}_a$ is the momentum of particle $\alpha$ (spectator) with respect to the centre of mass momentum of the other two particles (subsystem), while $\vec{p}_\alpha$ is the momentum of particle $\beta$ with respect to particle $\gamma$ (or vice versa). Also note that the Jacobian of the transformation $\left[ \frac{\partial(k_\alpha,k_\beta,k_\gamma)}{\partial(\vec{P},\vec{q}_\alpha,\vec{p}_\alpha)} \right]$ is 1.

3.3 Wavefunction

The tri-nucleon momentum space wavefunction, provided by Schadow [23, 24, 25], is obtained by solving the Faddeev equation (see, for example, ref. [26]) with various model potentials. It can be written as a sum of channel wavefunctions where each channel represents a specific angular momentum and spin eigenstate.

$$ | \Psi \rangle = \sum_{i_c} \psi_{i_c}(p,q) | i_c \rangle | \vec{P} \rangle $$

(3.2)

The $| \vec{P} \rangle$ represents the centre of mass momentum of the tri-nucleon system and is independent of the internal interactions of the nucleons and so it can be safely omitted for convenience in later calculations.

The coupling scheme of the channels is:

$$ | i_c \rangle = | (\langle 0L_\alpha|0l_\alpha) \rangle L_\alpha, (S_\alpha s_\alpha) S_\alpha J \rangle | (I_\alpha i_\alpha) I \rangle $$

$$ \equiv | i_c(J) \rangle | i_c(I) \rangle $$

(3.3)

$| (\langle 0L_\alpha|0l_\alpha) \rangle L_\alpha, (S_\alpha s_\alpha) S_\alpha J \rangle$ is the spin and angular momentum part: $L_\alpha$ ($S_\alpha$) is the angular momentum (spin) of subsystem ($\beta \gamma$), $l_\alpha$ ($s_\alpha$) is the angular momentum (spin) of particle $\alpha$ (the spectator particle) which are coupled to form $L_\alpha$ ($S_\alpha$) and then to $J = \frac{1}{2}$. The $| (I_\alpha i_\alpha) I \rangle$ is the isospin part: $I_\alpha$, being the subsystem isospin which couples with $i_\alpha$, the spectator isospin, to form $I = \frac{1}{2}$. Note that the angular momentum part and the isospin part are not coupled together. The zeros in $| i_c \rangle$ are “dud” couplings. They are merely used to facilitate computation.

Indeed, the LS coupling scheme that is used here is not suitable for calculating the wavefunction, the $jj$ or the channel coupling scheme was used to solve the Faddeev equation and the resulting channel wavefunctions were then transformed to LS coupling scheme.
3.3.1 Antisymmetrization and normalization of wavefunctions

The full wavefunction $| \psi \rangle$ is antisymmetrized with respect to the labelling of nucleons. That is,

$$| \psi \rangle = | \psi_{\text{Fad}} \rangle + P | \psi_{\text{Fad}} \rangle \tag{3.4}$$

where $| \psi_{\text{Fad}} \rangle$ is the "Faddeev component" of the wavefunction obtained by solving the wave equation with one particular choice of nucleon labelling scheme. and $P$ is the necessary permutation operator to make the full wavefunction antisymmetric. The normalization is defined as:

$$\langle \psi | \psi \rangle = 3 \langle \psi_{\text{Fad}} | \psi \rangle = 1 \tag{3.5}$$

The fact that the wavefunction is antisymmetric with respect to nucleon labellings makes the second expression of the above equation legitimate. It is also numerically possible since it consists of a finite number of terms. The first expression is, however, numerically impossible in general since permuting the nucleon label ($P | \psi_{\text{Fad}} \rangle$) will project the Faddeev component onto a set of eigenstates which is usually much larger, if not infinitely larger, than the original (finite) set of eigenstates comprising the Faddeev components. It is therefore necessary in practice to restrict the set of projected states to one with a finite number of elements. Suppose one defines

$$\tilde{P} | \psi_{\text{Fad}} \rangle$$

to be the permutation projected on a restricted set of states, then the numerical normalization of the wavefunction

$$\langle \psi | \psi \rangle_{\text{num}} = \left( \langle \psi_{\text{Fad}} | + \langle \psi_{\text{Fad}} | \tilde{P} \rangle \right) \left( | \psi_{\text{Fad}} \rangle + \bar{P} | \psi_{\text{Fad}} \rangle \right) \tag{3.6}$$

will not be 1 because of this restriction. For example, a 8-8 channel wavefunction (i.e. $| \psi_{\text{Fad}} \rangle$ has 8 channels and the permutation is projected on the same 8 channel) has a normalization of 0.93 and a 22-22 channel wavefunction typically has a numerical normalization around 0.99 (see appendix C).
3.4 Hamiltonian

The effective Hamiltonian is defined by:

$$\mathcal{H} = \sum_{i=1}^{3} \tau_{i}^{-} H_{i}$$  \hspace{1cm} (3.7)

where $H_{i}$ and $\tau_{i}^{-}$ are the weak interaction Hamiltonian and the isospin lowering operator acting on $i^{th}$ nucleon of the trinucleon system. The relevant Feynman diagrams are exact duplicates of the EPM ones except that the $^{3}\text{He}$ is replaced by a proton and the $^{3}\text{H}$ is replaced by a neutron. Since the wavefunction is fully antisymmetric with respect to interchange of particle labels, one can write equation (3.7) as

$$\mathcal{H} = \sum_{i=1}^{3} \tau_{i}^{-} H_{i} = 3 \tau_{j}^{-} H_{j}$$  \hspace{1cm} (3.8)

where $j$ is 1 or 2 or 3, provided $\mathcal{H}$ is acting on the above wavefunction.

3.5 Nucleon form factors

The relation between different nucleon form factors is an exact duplicate of the nuclear ones. In other words, all the CVC, PCAC, IVC hold for the nucleon case once one replaces $^{3}\text{He}$ with a proton and $^{3}\text{H}$ with a neutron. One has,

$$g_{i}(Q^{2}) = g_{i}(0)(1 + \frac{1}{6}r_{i}^{2}Q^{2})$$  \hspace{1cm} (3.9)

where $i = V, M, A$ and

$$g_{V}(0) = 1$$
$$g_{M}(0) = \mu_{p} - \mu_{n}$$
$$g_{A}(0) = -1.267 \pm 0.0035 [27]$$  \hspace{1cm} (3.10)

with $r_{V}^{2} = 0.576 \text{ fm}^{2}$, $r_{M}^{2} = 0.771 \text{ fm}^{2}$ and $r_{A}^{2} = 0.433 \text{ fm}^{2}$ [18].
3. The impulse approximation

3.6 IA ↔ EPM translation

The aim of this section is to find the correspondence between the EPM amplitude and the IA amplitude.

Let $M_{ia}$ be the RMC transition amplitude obtained by the impulse approximation. This means $M_{ia}$ is the IA equivalent of $M(P_{He}, P_{Hi}, M_t)$ in the EPM. More explicitly, one would use the following formula in order to have the quantity $\frac{d\Gamma}{d\kappa}$ (compare with equation (2.2)):

$$\frac{d\Gamma}{d\kappa} = \sum_{\lambda=-1,+1} \int \frac{d\text{(phase space)}}{d\kappa} \text{Tr}(\rho M_{ia}^{\dagger} M_{ia})$$

(3.11)

and

$$\Gamma_{\text{omc}} = \int d\text{(phase space)}_{\text{omc}} \text{Tr}(\rho M_{ia}^{\text{omc}} M_{ia}^{\text{omc}})$$

(3.12)

for OMC.

Define $M_{epm}$ as:

$$M_{epm} \equiv (2\pi)^4 \delta^{(4)}(P_{He} + (\nu + \kappa - \mu) - P_{Hi})M(P_{He}, P_{Hi}, M_t)$$

(3.13)

It then follows that the IA equivalent of $M_{epm}$, $M_{ia}$, would be:

$$M_{ia} = 3 \int (2\pi)^4 \delta^{(4)}(k'_{\alpha} + (\nu + \kappa - \mu) - k_{\alpha})(2\pi)^4 \delta^{(4)}(k'_{\beta} - k_{\beta})(2\pi)^4 \delta^{(4)}(k'_{\gamma} - k_{\gamma})$$

$$\psi_{He}^*(k'_{\alpha}, k_{\beta}, k_{\gamma}) \psi_{He}(k_{\alpha}, k_{\beta}, k_{\gamma}) \omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k_{\cdot 0})M(k_{\alpha}, k_{\beta}, k_{\gamma}, M_p)$$

$$\prod_{i=\beta,\gamma} \frac{d^4k'_i}{(2\pi)^4} \frac{d^4k_i}{(2\pi)^4}$$

(3.14)

In words, the above equation merely says the capture takes place on (spectator, in the sense of equations (3.1)) nucleon $\alpha$ and so one integrates out the degrees of freedom of the two subsystem nucleons. The 3 comes from the antisymmetrization of the wavefunction.

$\omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k_{\cdot 0}, k_{\gamma})$, loosely speaking, specifies the “dispersion relation” of the subsystem particles in the phase space. There is a degree of arbitrariness in choosing the functional form of $\omega$ since all the nucleons are

---

4 The $\delta$ function merely enforces the four momentum conservation of the reaction.

5 That is why $M(k'_{\cdot}, k_{\alpha}, M_p)$ appears.
off-shell and therefore the non-relativistic wavefunction cannot enforce the energy-momentum relation. Suppose one were to integrate $M_{epm}$ over the phase-space of $P_H$ and $P_{He}$, that is, one does the following:

$$
\int M_{epm} \frac{d^4P_H}{(2\pi)^4} \frac{d^4P_{He}}{(2\pi)^4}
$$

(3.15)

The corresponding thing in IA is:

$$
3 \int M_{ia} \frac{d^4k_{a}}{(2\pi)^4} \frac{d^4k'_{a}}{(2\pi)^4}
$$

(3.16)

(3.16) can be written as:

$$
(3.16) = 3 \int (2\pi)^3 \delta^{(3)}(\vec{k}'_{a} - \vec{k}_{a} + \vec{v} + \vec{k} - \vec{\mu})(2\pi)\delta(k'_{a}^0 - k_{a}^0 + \nu + \kappa - \mu^0) \\
(2\pi)^3 \delta^{(3)}(\vec{k}'_{\beta} - \vec{k}_{\beta})(2\pi)\delta(k'_{\beta}^0 - k_{\beta}^0)(2\pi)^3 \delta^{(3)}(\vec{k}'_{\gamma} - \vec{k}_{\gamma})(2\pi)\delta(k'_{\gamma}^0 - k_{\gamma}^0) \\
\psi_{H}^{*}(\vec{k}'_{a}, \vec{k}'_{\beta}, \vec{k}'_{\gamma})\psi_{He}(\vec{k}_{a}, \vec{k}_{\beta}, \vec{k}_{\gamma})\omega(M(k'_{a}, k_{a}, \mu_{p}) \\
\prod_{i=\alpha,\beta,\gamma} \frac{d^3\vec{k}'_{i}}{(2\pi)^3} \frac{d^3\vec{k}_{i}}{(2\pi)^3} \frac{dk_{i}^0}{(2\pi)^3} \frac{dk'_{i}^0}{(2\pi)^3}
$$

(3.17)

Using the inverse of equation set (3.1) to transform variables from $\{\vec{k}_{(\alpha,\beta,\gamma)}\}$ to $\{\vec{P}, \vec{p}_{a}, \vec{q}_{a}\}$ and from $k_{a}^0$ to $k_{\beta}^0 + k_{\gamma}^0$, one has:

$$
(3.16) = 3 \int (2\pi)^3 \delta^{(3)}(\frac{1}{3}(\vec{P}' - \vec{P}) + (\vec{q}'_{a} - \vec{q}_{a} + \vec{v} + \vec{k} - \vec{\mu})) \\
(2\pi)\delta(k'_{a}^0 - k_{a}^0 + \nu + \kappa - \mu^0) \\
(2\pi)^3 \delta^{(3)}(\frac{1}{3}(\vec{P}' - \vec{P}) - \frac{1}{2}(\vec{q}'_{a} - \vec{q}_{a} + (\vec{p}'_{a} - \vec{p}_{a})))\omega(M(k'_{a}, k_{a}, \mu_{p}) \\
\prod_{i=\alpha,\beta,\gamma} \frac{d^3\vec{P}'_{i}}{(2\pi)^3} \frac{d^3\vec{q}'_{i}}{(2\pi)^3} \frac{dk_{i}^0}{(2\pi)^3} \frac{dk'_{i}^0}{(2\pi)^3}
$$

(3.18)

One does this integration to get a relativistically invariant quantity so that a direct comparison of the same physical quantity obtained by the IA and the EPM is possible.
Again, using the multivariable version of one of the basic properties of $\delta$ function:

$$\delta^{(n)}(x) = |\det(T)| \delta^{(n)}(T\bar{x})$$  \hspace{1cm} (3.19)

where $n$ is any positive integer and $T$ is any non-singular linear transformation and $|\det(T)|$ is the absolute value of the determinant of $T$. We have,

$$\begin{align*}
(3.16) &= 3 \int (2\pi)^3 \delta^{(3)}(\bar{P}' - \bar{P} + \bar{v} + \bar{k} - \bar{\mu})(2\pi)^3 \delta(E' - E + \nu + \kappa - \mu^0) \\
&\quad \times (2\pi)^3 \delta^{(3)}(\bar{q}'_{\alpha} - \bar{q}_{\alpha})(2\pi)^3 \delta^{(3)}(\bar{q}'_{\beta} - \bar{q}_{\beta} + \frac{2}{3}(\bar{v} + \bar{k} - \bar{\mu})) \\
&\quad \times (2\pi)^3 \delta(k'_{\gamma} - k'_{0})(2\pi)^3 \delta(k'_{0} - k_{0})\psi_{He}(\bar{P}', \bar{P}_{\alpha}, \bar{q}_{\alpha})M(k', k_{\alpha}, M_{p}) \\
&\quad \omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k'_{0}, k'_{0}) \\
&\quad \frac{d^3 \bar{P}'}{(2\pi)^3} \frac{d^3 \bar{q}'_{\alpha}}{(2\pi)^3} \frac{d^3 \bar{P}_{\alpha}}{(2\pi)^3} \frac{d^3 \bar{q}_{\alpha}}{(2\pi)^3} \frac{dE'}{(2\pi)^3} \frac{dE}{(2\pi)^3} \prod_{i=\beta, \gamma} \frac{dk'_{0}}{2\pi} \frac{dk_{0}}{2\pi} \\
&= \omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k'_{0}, k'_{0}) \\
&= (2k'_{0})(2k_{0})(2\pi)^3 \delta(k'_{0} - M_{p}^2)(2\pi)^3 \delta(k_{0}^2 - M_{p}^2) \hspace{1cm} (3.21)
\end{align*}$$

The assumption of impulse approximation is that the two subsystem nucleons are both on-shell; therefore, $\omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k'_{0}, k'_{0})$ can be written as:

$$\omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k'_{0}, k'_{0}) = (2k'_{0})(2k_{0})(2\pi)^3 \delta(k'_{0} - M_{p}^2)(2\pi)^3 \delta(k_{0}^2 - M_{p}^2) \hspace{1cm} (3.21)$$

Using the above functional form of $\omega$ and integrating equation (3.20), we have:

$$\begin{align*}
(3.16) &= 3 \int (2\pi)^3 \delta^{(3)}(\bar{P}' - \bar{P} + \bar{v} + \bar{k} - \bar{\mu})(2\pi)^3 \delta(E' - E + \nu + \kappa - \mu^0) \\
&\quad \times (2\pi)^3 \delta^{(3)}(\bar{q}'_{\alpha} - \bar{q}_{\alpha})(2\pi)^3 \delta^{(3)}(\bar{q}'_{\beta} - \bar{q}_{\beta} + \frac{2}{3}(\bar{v} + \bar{k} - \bar{\mu})) \\
&\quad \times (2\pi)^3 \delta(k'_{\gamma} - k'_{0})(2\pi)^3 \delta(k'_{0} - k_{0})\psi_{He}(\bar{P}', \bar{P}_{\alpha}, \bar{q}_{\alpha})M(k', k_{\alpha}, M_{p}) \\
&\quad \omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k'_{0}, k'_{0}) \\
&\quad \frac{d^3 \bar{P}'}{(2\pi)^3} \frac{d^3 \bar{q}'_{\alpha}}{(2\pi)^3} \frac{d^3 \bar{P}_{\alpha}}{(2\pi)^3} \frac{d^3 \bar{q}_{\alpha}}{(2\pi)^3} \frac{dE'}{(2\pi)^3} \frac{dE}{(2\pi)^3} \prod_{i=\beta, \gamma} \frac{dk'_{0}}{2\pi} \frac{dk_{0}}{2\pi} \\
&= \omega(k_{\alpha}, k_{\beta}, k_{\gamma}, k'_{0}, k'_{0}) \\
&= (2k'_{0})(2k_{0})(2\pi)^3 \delta(k'_{0} - M_{p}^2)(2\pi)^3 \delta(k_{0}^2 - M_{p}^2) \hspace{1cm} (3.22)
\end{align*}$$

It is tempting to associate the centre of mass momentum of the three nucleons $\bar{P}(\bar{P}')$ of IA with $\bar{P}_{He}(\bar{P}_{He})$ of EPM and $E = k_{\alpha}^0 + k_{\beta}^0 + k_{\gamma}^0$ with $P_{He}$ of EPM and $E' = k_{\alpha}'^0 + k_{\beta}'^0 + k_{\gamma}'^0$ with $P_{He}'$ of EPM; however, there are still two problems which need to be resolved:

1. What is the dispersion relation of the struck nucleon?
2. Do \((E,\vec{P})\) and \((E',\vec{P}')\) satisfy the “on-shell” condition?

It turns out that if one puts the struck nucleon on-shell, the answer for question (2) is negative; one could have adjusted the energy-momentum relation of both the struck incident and final nucleon in a way such that \((E,\vec{P})\) and \((E',\vec{P}')\) are on-shell but the latter approach has the disadvantage that the off-shell nucleon will no longer satisfy the Dirac equation, the equation that has been used extensively to obtain the Adler and Dothan terms in the IA amplitude.

Let us see the extent the on-shell condition is not satisfied by \((E,\vec{P})\) and \((E',\vec{P}')\) if we put the struck nucleon on-shell. One wants to know whether \(P^2 = P_{\text{He}}^2 = M_t^2 \) and \(P'^2 = P_{3\text{H}}^2 = M_t^2\). Now,

\[
P^2 = \sum_{i,j=\alpha,\beta,\gamma} k_i \cdot k_j
= 3M_p^2 + \sum_{i,j\neq j} k_i \cdot k_j
= 3M_p^2 + \sum_{i,j\neq j} (M_p^2 + \frac{1}{2}(k_i - k_j)^2) + O(k^4)
\]

(3.23)

Except for the difference in binding energy, the error in assuming that \(P^2 = M_t^2\) is of the order of the square of the relative momenta of the initial nucleons. Even for the case of \(^3\text{H}\), the error is at most \(O\left(\frac{m}{M_p}\right)^2\) which is about 1%.

The approach that is taken is to regard all the nucleons as free nucleons and assume (to a considerable accuracy) that \((E, \vec{P})\) and \((E', \vec{P}')\) is a four-vector satisfying the on-shell particle criterion. Then, noting that \(\psi_{3\text{He}}(\vec{P}, \vec{P}', \vec{q}_a, \vec{q}) = \psi_{3\text{He}}(\vec{P}, \vec{P}', \vec{q}_a, \vec{q})\), one can associate

\[
M(P_{\text{He}}, P_{3\text{H}}, M_t) \leftrightarrow 3 \int \left(2\pi\right)^3 \delta^{(3)}(\vec{p}_\alpha - \vec{p}_a)(2\pi)^3 \delta^{(3)}(\vec{q}_a - \vec{q}_a + \frac{2}{3}(\vec{\nu} + \vec{\kappa} - \vec{\mu}))
\psi_{3\text{He}}(\vec{P}, \vec{q}_a)\psi_{3\text{He}}(\vec{P}, \vec{q}_a)
M(k'_\alpha, k_\alpha, M_p) \frac{d^3 q'_a}{(2\pi)^3} \frac{d^3 P'_a}{(2\pi)^3} \frac{d^3 q_a}{(2\pi)^3} \frac{d^3 P_a}{(2\pi)^3}
\]

(3.24)

The right hand side of the above expression is just \(M_{3\alpha}\) defined previously. We have successfully separated the centre of mass coordinates from the internal coordinates of the IA amplitude. In order to actually do the calculation, one also needs to expand those \(\delta\) functions in (3.24) in terms of angular momentum eigenstates. To this end, one has (after setting \(\vec{\mu} = 0\) and denoting
The impulse approximation

\[ \vec{v} + \vec{k} = \vec{\varepsilon} \]

\[ (2\pi)^3 \delta^{(3)}(\vec{p}_\alpha' - \vec{p}_\alpha) = \frac{(2\pi)^3 \delta(p_\alpha' - p_\alpha)}{p_\alpha^2} \sum_i (-1)^i \sqrt{2l + 1} Y_{li}^{00}(\hat{p}_\alpha', \hat{p}_\alpha) \] (3.25)

\[ (2\pi)^3 \delta^{(3)}(\vec{q}_\alpha' - \vec{q}_\alpha + \frac{2}{3} \vec{\varepsilon}) = \sum_{l_1, l_2, l_3} \left[ \frac{(4\pi)^5 (2l_1 + 1)(2l_2 + 1)}{(2l_3 + 1)} \right] \int j_{l_1}(r) j_{l_2}(r) j_{l_3}(\frac{2}{3} \varepsilon) r^2 dr \] (3.26)

\[ \langle \vec{p}_\alpha, \vec{q}_\alpha | \psi \rangle = \sum_i \psi_i(p, q) \langle \hat{p}, \hat{q} | i_c \rangle \] where \( | i_c \rangle \) is a particular LS coupled state; therefore, one can separate the angular and radial components and use angular momentum algebra to evaluate (3.24) by picking up terms which give non-zero results. The radial integral is a little bit difficult to deal with, as it involves a product of three (highly oscillating) spherical Bessel functions (see ref. [20], chapter 11). Analytical methods, if any, to handle those spherical Bessel functions are not feasible since the radial channel wavefunctions are in numerical form. One numerical method to get around this is to handle these three functions one by one (see appendix A) to avoid this highly singular integral (see also appendix D for definitions of expressions).

### 3.6.1 Operators

Upon the expansion of those \( \delta \) functions into their respective angular momentum eigenfunctions, the next thing to do is to manipulate \( M(k'_\alpha, k_\alpha, M_p) \) in equation (3.24) into a form that is suitable to couple with those \( \delta \) functions, keeping in mind that the aim of doing all this is to let \( M_{ia} \) have a similar format to \( M(P_{H}, P_{He}, M_i) \) in the EPM except that all the non-EPM degrees of freedom are encapsulated in some formfactors or reduced matrix elements. The method used is to expand the expression non-relativistically in powers of momentum of the struck nucleon \( \vec{k}_\alpha \) (which equals \( \vec{q}_i \) upon setting the initial centre of mass momentum of the trinucleon zero) and then couple all the spin and angular momentum operators into tensors of rank 0 or 1. Since the total angular momentum of both initial and final states are \( \frac{1}{2} \), there is no need to couple the operators into tensors of other ranks. There is also no need to couple operators into odd parity as both the initial and final states are of even parity. Recognizing the total angular momentum of the trinucleon system in the IA as that of spin in the EPM, one can easily see that any
operators of rank 1 in the IA corresponds to (within a factor) $\sigma$ matrices in the EPM and operators of rank 0 in the IA corresponds to identity hadronic operators in the EPM.

All the coefficients are expanded to $O\left(\frac{k_p}{M_p}\right)$ except that of $g_P$ since one of the kinematic endpoints of RMC is quite close to the pole of $g_P$ and might make the value of $g_P$ large at those places. Therefore, coefficients of $g_P$ are expanded to $O\left(\left(\frac{k_p}{M_p}\right)^2\right)$.

The correspondence between the IA and the EPM for all forms of operators up to first order in momentum are shown below. Please refer to appendix B for operators of second order of momentum. Please note that the [...]s are actually reduced matrix elements between the initial and final states (i.e. results of integration of "internal" degrees of freedom) and the numbers inside denote some specific spin and angular momentum combination. They will be defined in equation (3.33). For now, it is sufficient to note that the first digit of [...] is about the nucleon momentum $\vec{q}$ and the second digit comes from the spherical harmonics of the $\delta$ function. These two terms couple together to have angular momentum of value represented by the third digit. The fourth digit is about the hadronic spin matrix and the subscript is the rank of the whole reduced matrix element. 1 (or sometimes denoted as $1_0$) is defined as the hadronic identity matrix element in both the IA and the EPM, please do not confuse it with $[1]^0$ although they are related.

\[
\text{IA after coupling and reexpressing in EPM format}
\]

\[
\begin{align*}
1 & \leftrightarrow [(0,0)0 \otimes 0]_0 1 \\
\vec{\sigma} \cdot \vec{v} & \leftrightarrow \frac{3}{\sqrt{2}}\{(0,2)2 \otimes 1)_{11}\vec{\sigma} \cdot \vec{s}\vec{v} \cdot \hat{s} + \\
& \quad \left\{ \frac{1}{\sqrt{2}} \left[ ((0,2)2 \otimes 1)_{11} + ((0,0)0 \otimes 1)_{11} \right]\vec{\sigma} \cdot \vec{v} \right. \\
\vec{\sigma} \cdot \vec{q} & \leftrightarrow \left\{ -\sqrt{\frac{5}{3}} \left[ ((1,1)2 \otimes 1)_{11} - \frac{1}{2}((1,1)1 \otimes 1)_{11} \right] - \\
& \quad \frac{1}{\sqrt{3}}\left[ ((1,1)0 \otimes 1)_{11}\vec{\sigma} \cdot \vec{s} \right. \\
\vec{q} \cdot \vec{v} & \leftrightarrow -\frac{1}{\sqrt{3}}\left[ ((1,1)0 \otimes 0)_{0}\vec{v} \cdot \hat{s} - \frac{i}{\sqrt{2}} \left[ ((1,1)1 \otimes 0)_{1}\vec{\sigma} \cdot \vec{v} \times \hat{s} \right. \\
& \quad \left. \left(3.28\right) \right. \\
& \quad \left. \left(3.29\right) \right. \\
& \quad \left. \left(3.30\right) \right. \\
\end{align*}
\]
3. The impulse approximation

\[ \vec{\sigma} \times \vec{q} \cdot \vec{v} \leftrightarrow \left\{ -\frac{\sqrt{5}}{12} \left[ \langle (1,1)1 \otimes 1 \rangle_{1} + \frac{1}{2} \langle (1,1)1 \otimes 1 \rangle_{1} \right] + \frac{1}{3} \langle (1,1)0 \otimes 1 \rangle_{1} \right\} \vec{v} \times \vec{\sigma} + i \frac{2}{3} \langle (1,1)1 \otimes 1 \rangle_{0} \vec{v} \cdot \vec{\sigma} \]

(3.31)

\[ \vec{q} \cdot \vec{v} \cdot \vec{\sigma} \cdot \vec{u} \leftrightarrow \left\{ \sqrt{\frac{15}{2}} \langle (1,1)2 \otimes 1 \rangle_{1} \right\} - \left\{ \frac{1}{3} \langle (1,1)0 \otimes 1 \rangle_{1} \right\} - \frac{1}{2} \sqrt{\frac{10}{5}} \langle (1,3)2 \otimes 1 \rangle_{1} \vec{\sigma} \cdot \vec{v} \vec{u} \cdot \vec{\sigma} + \frac{1}{2} \sqrt{\frac{10}{5}} \langle (1,3)2 \otimes 1 \rangle_{1} \vec{\sigma} \cdot \vec{u} \vec{v} \cdot \vec{\sigma} + \frac{1}{2} \sqrt{\frac{10}{5}} \langle (1,3)2 \otimes 1 \rangle_{1} \vec{\sigma} \cdot \vec{u} \vec{v} \cdot \vec{\sigma} + \frac{1}{2} \sqrt{\frac{10}{5}} \langle (1,3)2 \otimes 1 \rangle_{1} \vec{\sigma} \cdot \vec{v} \vec{u} \cdot \vec{\sigma} + \frac{1}{2} \sqrt{\frac{10}{5}} \langle (1,3)2 \otimes 1 \rangle_{1} \vec{\sigma} \cdot \vec{u} \vec{v} \cdot \vec{\sigma} + \frac{1}{2} \sqrt{\frac{10}{5}} \langle (1,3)2 \otimes 1 \rangle_{1} \vec{\sigma} \cdot \vec{v} \vec{u} \cdot \vec{\sigma} + i \frac{1}{6} \langle (1,1)1 \otimes 1 \rangle_{0} \vec{u} \cdot \vec{v} \times \vec{\sigma} \]

(3.32)

Note that while \( \vec{\sigma} \) on the left hand side acts on the spin of the spectator nucleon, \( \vec{\sigma} \) on the right acts on the entire nucleus; \( \vec{u} \) and \( \vec{v} \) are mutually commuting vectors that are not concerned with the internal momentum (i.e. not \( \vec{p} \) nor \( \vec{q} \)) and commute with \( \vec{\sigma} \). In this notation, \( [\mathbf{1}]_{0} = \langle (0,0)0 \otimes 0 \rangle_{0} \); \( [\vec{\sigma}]^{0,1} = \langle (0,0)0 \otimes 1 \rangle_{1} \), \( [\vec{\sigma}]^{2,1} = -\langle (0,2)2 \otimes 1 \rangle_{1} \), \( [\vec{\sigma}]^{+} = [\vec{\sigma}]^{0,1} + \sqrt{2}[\vec{\sigma}]^{2,1} \) and \( [\vec{\sigma}]^{-} = [\vec{\sigma}]^{0,1} - \frac{1}{\sqrt{2}}[\vec{\sigma}]^{2,1} \). The precise relationship between \( [i\vec{P}]^{1,1} \) and the reduced matrix elements defined here is unclear but it has a magnitude to the order of \( \langle (1,1)1 \otimes 0 \rangle_{1} \) or \( \langle (1,1)1 \otimes 1 \rangle_{1} \). As one will see later, these two matrix elements are very small. The definition of \( \langle (a,b)c[\vec{a}] \otimes d \rangle_{e} \) (a function of \( s \equiv ||\vec{v} + \vec{\kappa}|| \)) is:

\[ \langle (a,b)c[\vec{a}] \otimes d \rangle_{e} = 3 \frac{1}{2\pi^{5}} \left( \frac{1}{2} \langle ||\vec{e}|| \rangle_{2} \right)^{1} \left( \frac{1}{2} \langle ||\vec{\kappa}|| \rangle_{2} \right)^{-1} \sum_{i_{e},i_{c},i_{1},i_{2},L_{1}} (-1)^{L_{1}} i_{1}^{-1} i_{2}^{-1} b \]
3. The impulse approximation

\[
\int p^2 dp r^2 dr j_b \left( \frac{2}{3} sr \right) \{ \psi_L^e(p, q') j_r(q' r) q^2 dq' \} \{ \psi_L^e(p, q) j_r(q r) q^2 dq \}
\]

\[
\left( \begin{array}{cc}
   b & l_1 \\
   0 & l_2
\end{array} \right) F(l_1, l_2; a, b, c; \iota_c, i_c) \sqrt{ \frac{(2l_1 + 1)(2l_2 + 1)(2l_1 + 1)}{2b + 1} }
\]

\[
\langle \iota_c'(J) \| \{ (Y^0_{L_1, L_1}(\hat{p}', \hat{p}) \otimes Y^0_{J_1}(\hat{q}', \hat{q})) \otimes (1_0 \otimes T_d) \} \| i_c(J) \rangle
\]

\[
\langle \iota_c'(I) \| (1_0 \otimes \tau) \| i_c(I) \rangle
\] (3.33)

Notice \( \tilde{a} \) specifies the dimension of the matrix element. When \( \tilde{a} \) is not shown explicitly on a reduced matrix element, \( \tilde{a} = a \); that is \( \{(a, b)c \otimes d\}_e \equiv \{(a, b)c[a] \otimes d\}_e \). \( F(l_1, l_2; a, b, c; \iota_c, i_c) \) is defined as:

\[
F(l_1, l_2; a, b, c; \iota_c, i_c) = \begin{pmatrix}
   c & a & b \\
   m_c & m_a & m_b
\end{pmatrix}^{-1}
\]

\[
\int Y^c_{l', l} (\hat{v}, \hat{\chi})^* Y^{a, m_a}_{0, a} (\hat{v}, \hat{\chi}) Y^{b, m_b}_{l_1, l_2} (\hat{v}, \hat{\chi}) d\hat{v} d\hat{\chi}
\]

\[
= \frac{1}{4\pi (-1)^{l_1 + l_2 + b}} \sum_{l_1, l_2, a, c, l} \begin{pmatrix}
   l_1 & l_2 & b \\
   a & c & l
\end{pmatrix} \sqrt{ \frac{(2l_2 + 1)(2a + 1)}{2b + 1} } \delta_{l_1 l_2} (3.34)
\]

\( \hat{v}, \hat{\chi} \) being some dummy angular variables. The notations used here are the same as Brink and Satchler [28] (also see ref. [29]) except that the Clebsch-Gordon coefficients are denoted by \( \begin{pmatrix}
   J & J_1 & J_2 \\
   M & M_1 & M_2
\end{pmatrix} \) as opposed to \( \langle JM \| J_1 J_2 M_1 M_2 \rangle \). Please note that the definition of reduced matrix element is:

\[
(-1)^{2J''} \begin{pmatrix}
   J & J' & J'' \\
   M & M' & M''
\end{pmatrix} \langle J \| T_{J''} \| J' \rangle = \langle JM \| T_{J'' M''} \| J' M' \rangle
\] (3.35)

\( \langle \iota_c'(J) \| \{ (Y^0_{L_1, L_1}(\hat{p}', \hat{p}) \otimes Y^0_{J_1}(\hat{q}', \hat{q})) \otimes (1_0 \otimes T_d) \} \| i_c(J) \rangle \) is the spin and angular momentum part of the reduced matrix element between the helion and triton channels. Its calculation is tedious but standard.

\[
\langle \iota_c'(J) \| \{ (Y^0_{L_1, L_1}(\hat{p}', \hat{p}) \otimes Y^0_{J_1}(\hat{q}', \hat{q})) \otimes (1_0 \otimes T_d) \} \| i_c(J) \rangle =
\]
3. The impulse approximation

\[ 4\pi(-1)^{\ell'+2l+c+L'+S'+\frac{1}{2}+S'+d}\{(2\ell+1)(2\ell'+1)(2S+1)(2S'+1) \]
\( \times (2c+1)(2e+1)\}\left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & e \\ \ell' & \ell & c \end{array} \right\} \left\{ \begin{array}{ccc} L & L' & c \\ S' & S & d \end{array} \right\} \delta_{SS'} \]
\( \langle \frac{1}{2}||T_d||\frac{1}{2}\rangle\langle L' \ 0||Y^0_{L_1,L_1}(\hat{p}',\hat{p})||0 \ L \rangle \] (3.36)

where \( T_d = 1_0 \) for \( d = 0 \) and \( \tilde{\sigma} \) for \( d = 1 \); also note that

\[ \sum_{L_1}(-1)^{L_1}\sqrt{2L_1+1}\langle L' \ 0||Y^0_{L_1,L_1}(\hat{p}',\hat{p})||0 \ L \rangle = \frac{1}{4\pi}\delta_{LL'} \] (3.37)

\( \langle \hat{t}_c(I)||1_0 \otimes \tilde{t}||\hat{t}_c(I) \rangle \) is the isospin contribution of the reduced matrix element, which equals

\[ 2(-1)^{I'}\left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & I' \end{array} \right\} \delta_{II'}\langle \frac{1}{2}||\tilde{t}||\frac{1}{2}\rangle \] (3.38)

By doing all the procedures mentioned in this section, one can match the IA amplitude piece by piece with its EPM counterparts and thus a direct comparison between each piece would become possible.

3.7 Results

Upon doing all the appropriate couplings and integrating the internal coordinates, one would obtain the right hand side of equation (3.24) which is \( M_{ia} \).

One then obtains \( \frac{d\sigma}{d\omega} \) via equation (3.11).

Figure (3.2) shows the statistical IA RMC photon spectra from wavefunctions of various model potentials. Figure (3.3) shows the photon spectra at different values of \( g_P \) and figure (3.4) plots the statistical RMC rate vs. \( g_P \).

3.7.1 Ordinary Muon Capture

Since the IA treatment of OMC is less complicated than RMC, all the nucleon momentum (\( \hat{q} \)) terms in the IA OMC amplitude are expanded to \( O(\frac{\hat{q}^4}{M^2}) \).

Note the four-momentum transfer at the hadronic vertex \( Q^2 \) for OMC is

\[ Q^2 = -\left(\frac{m^2+2mM}{2m+M}\right)^2 \simeq -0.964m^2 \]

\[ M^{omc}(k'_\alpha, k_\alpha, M_p) = \frac{G_F V_{ud} \left(1 - \tilde{\sigma}_L \cdot \hat{\nu} \right)}{\sqrt{2}} \left\{ \tilde{g}_1 + \tilde{g}_2 \tilde{\sigma} \cdot \hat{\nu} + \tilde{g}_3 \tilde{\sigma}_L \cdot \tilde{\sigma} + \frac{1}{2m} (\tilde{g}_4 \tilde{\sigma} \cdot \hat{q} + \right. \]

\[ \left. \right. \]
3. The impulse approximation

Fig. 3.2: RMC photon spectra from two EPM calculations (one with the full Adler and Dothan amplitude and the other with only gauge invariant terms) and from IA calculations using various model potentials. All wavefunctions used have 22 Faddeev components and the permutation is projected on the same set of states. The infrared divergent part is not shown.

\[ \bar{g}_5 \bar{\sigma}_L \cdot \bar{q} + \bar{g}_6 \bar{\sigma} \cdot \bar{q} + \bar{g}_7 \bar{\sigma}_L \cdot \bar{\sigma} \cdot \bar{q} + \bar{g}_8 \bar{\sigma}_L \cdot \bar{\sigma} \cdot \bar{q} \cdot \bar{\sigma} + \]
\[ \bar{g}_9 \bar{\sigma} \cdot \bar{q} \cdot \bar{\sigma} + i \bar{g}_{10} \bar{\sigma} \times \bar{\sigma} \cdot \bar{q} + \frac{1}{4m^2} \{ \bar{g}_{11} \bar{\sigma} \cdot \bar{q} \bar{\sigma}_L \cdot \bar{q} + \]
\[ \bar{g}_{12} |\bar{q}|^2 \bar{\sigma}_L \cdot \bar{\sigma} + \bar{g}_{13} \bar{\sigma} \cdot \bar{q} \bar{\sigma} \cdot \bar{q} + \bar{g}_{14} |\bar{q}|^2 \bar{\sigma} \cdot \bar{\sigma} \} \]  \tag{3.39}

where

\[ \bar{g}_1 = g \nu (1 + \frac{2M_p + \frac{\nu^2}{8M_p^2}}{g_M \frac{m \nu}{4M_p^2}} - \]
\[ \bar{g}_2 = g_P (\frac{\nu}{2M_p} - \frac{3\nu^3}{16M_p^3}) - g \nu \frac{2M_p}{g_M \frac{\nu}{2M_p}} - g \frac{\nu}{2M_p} + \]
\[ \bar{g} \]
3. The impulse approximation

Fig. 3.3: Photon spectra from IA calculations for various values of $g_p$. Wavefunction derived from Bonn-A potential is used for calculation.

\begin{align*}
    g_3 &= g_M \left( -\frac{\nu}{2M_p} + \frac{m\nu}{4M_p^2} - \frac{\nu^2}{4M_p^2} \right) \\
    \bar{g}_3 &= -g_M \left( \frac{\nu}{2M_p} + g_A (1 - \frac{\nu^2}{8M_p^2}) \right) + \\
    \bar{g}_3 &= g_M \left( -\frac{\nu}{2M_p} + \frac{m\nu}{4M_p^2} - \frac{\nu^2}{4M_p^2} \right) \\
    \bar{g}_4 &= g_p \frac{m\nu^2}{4M_p^3} + g_A \left( \frac{2m}{M_p} + \frac{m\nu}{2M_p^2} \right) \\
    \bar{g}_5 &= -g_A \frac{m\nu^2}{2M_p^2} + g_M \frac{2m}{M_p} \\
    \bar{g}_6 &= -g_A \frac{m\nu}{2M_p^2}
\end{align*}
Fig. 3.4: Same as figure (2.2) but for IA calculation. Using Bonn-A potential, $\Gamma_{\text{staf}}(\kappa > 60 \text{ MeV}) = 0.1387 \text{ s}^{-1}$.

\begin{align}
\bar{g}_7 &= g_M \left( \frac{m^2}{M_p^2} - \frac{m\nu}{M_p^2} \right) - g_A \frac{m\nu}{2M_p^2} \\
\bar{g}_8 &= -g_M \left( \frac{m^2}{M_p^2} - \frac{m\nu}{M_p^2} \right) + g_A \frac{m\nu}{M_p^2} \\
\bar{g}_9 &= g_P \frac{3m\nu^2}{4M_p^3} \\
\bar{g}_{10} &= g_{\nu} \frac{m\nu}{2M_p^2} + g_M \frac{m^2}{M_p^2} \\
\bar{g}_{11} &= g_A \frac{2m^2}{M_p^2} \\
\bar{g}_{12} &= -g_A \frac{2m^2}{M_p^2} \\
\bar{g}_{13} &= -g_P \frac{m^2\nu}{M_p^3} \\
\bar{g}_{14} &= -g_P \frac{m^2\nu}{M_p^3} \\
\end{align}

(3.40)
3. The impulse approximation

To perform the internal integration, one uses equation (3.28) to (3.32) and the formulae of appendix B to couple the internal operators with the appropriate spherical harmonics from the δ functions. The final product after internal integrations will then be a amplitude that, similar to the RMC one, depends on the external EPM degrees of freedom. The only traces left by internal integration are the "formfactors" that are functions of s. In the case of OMC, \( \delta = \nu \) and therefore \( s = |\nu| \). Use equation (3.12) to integrate externally with the phase space and get the IA OMC rate.

The statistical rates for both the ordinary and radiative muon capture are shown on table (3.1):

<table>
<thead>
<tr>
<th>Potentials</th>
<th>OMC rate (statistical) (s(^{-1}))</th>
<th>( \Gamma_{stat}^{\nu mc} ) (s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bonn-A</td>
<td>1368.6 1368.1 1367.8 1357.9</td>
<td>0.6114 0.1387</td>
</tr>
<tr>
<td>Bonn-B</td>
<td>1341.3 1340.8 1340.4 1330.7</td>
<td>0.6033 0.1366</td>
</tr>
<tr>
<td>CD-Bonn</td>
<td>1336.1 1335.7 1335.3 1326.1</td>
<td>0.6022 0.1364</td>
</tr>
<tr>
<td>Nijmegen</td>
<td>1298.0 1297.5 1297.1 1288.2</td>
<td>0.5894 0.1334</td>
</tr>
<tr>
<td>Paris</td>
<td>1270.9 1270.4 1270.1 1260.3</td>
<td>0.5800 0.1313</td>
</tr>
<tr>
<td>AV14</td>
<td>1271.0 1270.6 1270.2 1260.0</td>
<td>0.5796 0.1313</td>
</tr>
<tr>
<td>EPM</td>
<td>1503</td>
<td>0.8189 0.2113</td>
</tr>
</tbody>
</table>

Tab. 3.1: Various OMC and RMC statistical rates when \( g_P \) or \( F_P \) is at the PCAC value. The numbers in the second column are the values obtained by using the "\( \frac{g}{3} \)" prescription up to \( O(\frac{g}{M_p}) \) terms. The numbers in the third column are obtained via the correct approach up to \( O(\frac{g}{M_p}) \) terms. The fourth column has values of the OMC rates using "\( \frac{g}{3} \)" prescription up to \( O((\frac{g}{M_p})^2) \) terms. Numbers in the fifth column are the values obtained by the correct approach up to \( O((\frac{g}{M_p})^2) \) terms; the sixth and rightmost columns contain \( \Gamma_{stat}^{\nu mc}(\kappa > 5 \text{ MeV}) \) and \( \Gamma_{stat}^{\nu mc}(\kappa > 60 \text{ MeV}) \) respectively.

3.7.2 Discussion

There are several points to note:
1. Using the correct approach to handle the momentum terms instead of the traditional \( \frac{q}{3} \) approach decreases the OMC statistical rate by a mere 0.5 s\(^{-1}\) in all IA calculations up to first order in nucleon momentum terms. In the prescription that is used here, there are 24 reduced matrix elements instead of 3 in the \( \frac{q}{3} \) prescription. The decrease in capture rate is due to the non-zero value of those extra “internal formfactors” which can be attributed to the non-vanishing amplitude of the non-S wave components of the tri-nucleon wavefunction since the \( \frac{q}{3} \) prescription is exact for S waves. The smallness of the effect is primarily due to the minute contribution of the P-state wavefunction to the trinucleon wavefunction. Please note that the effect is a genuine effect and not a effect caused by numerical calculation. To prove this, a two channel Yamaguchi wavefunction consisting solely of S-waves is used to gauge the numerical uncertainty in wavefunction integration. There is an increase of the rate (due only to numerical integration) by 0.1 s\(^{-1}\) for the correct approach. This difference is much smaller than the difference (stemming from errors in both the numerical integration and the \( \frac{q}{3} \) approach) of the calculations of other regular 22-channel wavefunctions. Even though the \( \frac{q}{3} \) approach is exact for S waves up to first order, it is no longer valid at second order and thus a correct prescription for second order nucleon momentum terms is essential, as can be seen by a 0.6% decrease in capture rate when the correct prescription is used for second order nucleon momentum terms.

2. The determinable Adler and Dothan terms have a significant influence on the EPM calculation of RMC rates but their effect on the IA calculation is much smaller. They make the ratio of the IA RMC total capture rate to the EPM RMC total capture rate about 10% smaller than the OMC counterparts. For OMC, the ratio of the IA calculation to the EPM calculation ranges from 85 to 90% but in RMC the same ratio is just around 75%. If only the gauge terms were added

---

7 The \( \frac{q}{3} \) prescription, first used by Peterson in his OMC calculation, replaces all nucleon momentum terms \( q \) by \( \frac{q}{3} \).

8 Notice that the influence of those extra Adler and Dothan terms other than those GI terms on the IA amplitude is minimal. One can barely distinguish the two IA photon spectra (one with the full Adler and Dothan amplitude and the other with only GI terms) visually when they are plotted on figure (3.2)

9 The gauge terms are the terms in \( \Delta J_{\text{hadronic RMC}}^a \) (equation (1.35)) that are explicitly
but not the full Adler and Dothan terms, the ratio would be about 83% (see figure (3.2)). One might blame the discrepancy on the failure of the IA in describing RMC by $^3$He but that seems unlikely given its good predictions for OMC. Indeed, the Adler and Dothan procedure seems suspicious when it is applied to both the EPM and IA description of the $^3$He $\rightarrow$ $^3$H transition, in particular the EPM one when the derivatives of formfactors are large$^{10}$. Since the magnitude of the Adler and Dothan terms are large in EPM due primarily to the large size of the derivatives of the nuclear form factors, it is important that all the extra terms (to a sufficient high order) should be determined in order to have the full effect of those terms on the resulting EPM capture rate. However, this does not happen: the terms of $O(\kappa Q)$ are not determinable (or at least cannot be determined uniquely) by the Adler and Dothan procedure. Some of those undetermined terms of $O(\kappa Q)$ carry coefficients of magnitude of $(\mu_{^3\text{He}} - \mu_{^3\text{H}})R_M^2$ and these terms are actually first order terms disguised as second order and might have a considerable influence on the resulting spectrum. As a result, a full meson exchange current (MEC) calculation seems necessary to tell whether the inadequacy, if there is any, lies with the IA or the EPM or both. Congleton and Truhlik $^{31}$ calculated the MEC contribution to the simpler problem of OMC by $^3$He and found that IA+MEC prediction of the exclusive OMC statistical rate agrees with both the EPM calculation and experiment $^{21}$.

3. The EPM calculation of both the OMC and RMC rate and the IA calculation of OMC rate using wavefunctions of non-Bonn potentials agree with that of Klieb and Rood. However, the IA RMC spectrum is about 4% lower$^{11}$ than theirs. At first it looks a bit contradictory that a more or less the same IA OMC rate but a slightly lower RMC spectrum is obtained in this work but a closer look reveals that Klieb and Rood used a lot of approximations evaluating the reduced matrix elements for the RMC spectrum which they did not use for OMC.

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3. The impulse approximation

(a) Instead of evaluating the reduced matrix elements directly in terms of \( s \), they expressed \( s \) in terms of an infinite sum of spherical harmonics of \( \hat{u} \) and \( \hat{k} \) and an artificial cutoff was imposed on them.

(b) They did not fully square the resulting matrix element: only products of any two most dominant terms and products of one dominant and one small term were considered.

(c) In expanding the plane wave \( \exp(i\vec{s} \cdot \vec{r}) \), they only included the term having \( j_0(\nu r)j_0(kr) \) and they used this approximation as the premise in deriving several relationships between various reduced matrix elements for RMC.

None of these approximations is used here.

Some terms in \( \Delta J^\alpha_{\text{hadronic, RMC}} \) that are present here but not included by Klieb and Rood also tend to decrease the resulting photon spectrum.

4. The Bonn type potentials seem to give a higher (and perhaps better) RMC and OMC results than the other potentials. To analyze this properly, let us take a look at the three dominant reduced matrix elements: \( [1]^0 \), \( [\sigma]^{0,1} \) and \( [\sigma]^{2,1} \). All the curves produced by the non-Bonn potentials seem to be a bit separated from the curves of the Bonn type potentials, especially in the region when \( s \) is large. For \( [1]^0 \) the problem may be partly associated with the numerical normalization since \( [1]^0(0) \) should be one in principle. However, even though one takes this into account (say, for example, scale the Paris potential curve so that it agrees with all others at \( s = 0 \)) the value of \( [1]^0 \) is still smaller than that of the Bonn type potentials, as can be easily seen from the fact that the fractional deviation is larger at large value of \( s \). This seems to have to do with the binding energy differences by the Bonn and non-Bonn potentials: the non-Bonn potentials generally underbind the trinucleon by about 1 MeV. Congleton and Truhlik [31] explained since \( [1]^0 \sim 1 - \text{const}(\langle r^2 \rangle s^2) \) (To see this, expand \( j_0(\frac{3}{2}sr) \) in polynomials, note \( \text{const} > 0 \)) and \( \langle r^2 \rangle \) scales like the inverse of binding energy so one can expect a lower value for \( [1]^0 \) when a wavefunction with a lower binding energy is used. For \( [\sigma]^{0,1} \), they argued that because of the weak tensor force of the Bonn type potentials, one-body currents will be favoured \( [\sigma]^{0,1} \), which is already dominated by contributions from one-body currents. If their analysis is correct, this would potentially explain why
the Bonn type potentials consistently give higher results in both OMC and RMC IA calculations. The quadratic like curve of $[\hat{\sigma}]^{2,1}$ is obvious if one notices $j_2(x) \sim \frac{x^2}{15}$ for $x \ll 1$. Since $[\hat{\sigma}]^{2,1}$ is the result of S-D mixing, a tri-nucleon wavefunction with a larger component of D wave would probably have a larger magnitude of $[\hat{\sigma}]^{2,1}$ which seems to be the case for the generally higher D wave component of the non-Bonn potentials. Recently Lahiff and Afnan [32] suggested that the Bonn-type potentials might be a better choice than the Nijmegen potential because the energy dependence of propagators is treated exactly (during the evaluation of the potential via time-ordered perturbation theory) in the Bonn type potential but the Nijmegen group removes this energy dependence.

In a nutshell, the reasons that the Bonn type potentials give a higher result for both RMC and OMC are primarily due to its higher binding energy and possibly its weak tensor force. The higher D wave components of those non-Bonn potentials may give a boost to $[\hat{\sigma}]^{2,1}$ but the smallness of $[\hat{\sigma}]^{2,1}$ as compared to the other two would make its effect on the IA capture rates small. Figures (3.5) to (3.7) show the variation of $[1]^0, [\hat{\sigma}]^0,1$ and $[\hat{\sigma}]^{2,1}$ for various model potentials. Ordinary muon capture occurs at $s = 0.52574 \text{ fm}^{-1}$ and the kinematic range for radiative muon capture is $0 < s < 0.52574 \text{ fm}^{-1}$.

### 3.8 Summary and conclusion

An impulse approximation calculation of RMC by $^3\text{He}$ has been done using various trinucleon wavefunctions. Unlike most, if not all, other calculations, the calculation is done with the momentum space wavefunctions instead of the traditional position space ones and the nucleon momentum terms are treated exactly without using the common $\frac{f_3}{3}$ approach. Moreover, the non-relativistic reduction of the Hamiltonian is better in the present work than that of Klieb and Rood's [4, 5] since all terms having coefficient $g_P$ are expanded to second order in nucleon momentum terms here. There are slight variations in the resulting photon spectra from the IA calculations for different wavefunctions and these variations can possibly be accounted for by some general characteristics of tri-nucleon wavefunctions such as binding energy and partial wave probabilities. The difference between the impulse approximation and the EPM description of RMC by $^3\text{He}$ is larger than in the
case of OMC since the extra Adler and Dothan terms contribute a lot more in the EPM calculation. It is not possible to say whether the EPM or the IA is a better description of the nuclear reaction as all the experimental results are preliminary [2, 3]. A MEC calculation seems necessary because it might remedy the inadequacy of the Adler and Dothan procedure by providing terms of $\mathcal{O}(\kappa Q)$, thus leading to a satisfactory resolution of the discrepancy between the IA and EPM RMC calculation.

![Plot of $|1|^0$ vs. $s$ for different nuclear potentials.](image)

**Fig. 3.5:** Plot of $|1|^0$ vs. $s$ for different nuclear potentials.
3. The impulse approximation

Fig. 3.6: Plot of $|\sigma|^{0,1}$ vs. $s$.

Fig. 3.7: Plot of $|\sigma|^{2,1}$ vs. $s$. 
A. Handling of radial wavefunction integrations in the IA

A.1 Introduction

Since the radial trinucleon wavefunction employed in this calculation is numerical and the spherical Bessel functions are highly oscillating, this appendix explains the special procedures needed when doing evaluations like that in equation (3.33). These special procedures are indeed "brute force" approaches: partition each zero of the Bessel function and perform a gaussian integration on each of the partitions. There is no sure method of determining the accuracy of this procedure but the smoothness of IA form factors as a function of $s$ obtained after integrating out the internal degrees of freedom, among others, suggests that a high level of accuracy is achieved by this procedure.

A.2 Integration of type:

$$\zeta(p, r) = \int \psi(p, q)j_1(qr)q^{2+\bar{a}}dq$$

A set of about 70 mesh points on $r$ is chosen in advance. To increase accuracy, this set of mesh points contains more points when $\zeta(p, r)$ has more structure.\(^1\) Of course, one does not know where $\zeta$ has more structure until after doing the integration but it is almost always the case that for $r$ small, $\zeta$ would tend to have more structure.\(^2\)

---

\(^1\) A loose definition of structure can be related to second derivatives: the smaller the second partial derivatives of a function, the less the structure of it.

\(^2\) One "reason" for the above argument is $\zeta(p, r)$, with all its partial derivatives to all orders, is expected to go to zero "sufficiently fast" when $r \to \infty$ that would make the $r$ integral finite.
A. Handling of radial wavefunction integrations in the IA

For each of the mesh points in \( r \) space, the zeros of \( j_i(qr) \) (in \( q \) space) are evaluated. A gaussian integration on \( q \) is then performed on the integral for each interval between two adjacent zeros. Usually, 25-35 gaussian points per interval would suffice but there are cases (when the interval is large) which need as high as 70 gaussian points. Since the number of integration points used in \( q \) space integration is much larger than the number of mesh points, some techniques should be used for interpolation when the integration points do not coincide with the mesh points in \( q \) space on which the wavefunction is defined. A one-dimensional cubic spline (see ref. [33], chapter 3) with the "natural" boundary conditions\(^\text{3}\) is used, this spline interpolation is seemingly good on inspection. This procedure will minimize the error of the integral due to the highly oscillatory nature of the spherical Bessel functions.

A.3 Integration of type:

\[
\int \zeta(p, r)\zeta'(p, r)j_i(sr)p^2dp r^2dr
\]

Since the \( p \) integration is not affected by spherical Bessel functions, it is straightforward. The method used is to integrate using the coefficients provided with the wavefunction (remember that nothing has been done on \( p \) space yet). The technique used in the \( r \) integration is essentially the same as that in the previously discussed \( q \) integration, except that one needs two one-dimensional cubic splines (for \( \zeta \) and \( \zeta' \)) to interpolate the \( r \) space for each integration point in \( p \) space (the number of integration points in \( p \) space varies from 40 to 50). There are also other possible methods, say using a two-dimensional spline for \( \zeta \), but the current method is seemingly more economical in terms of both programming and computing resources.

A.4 "Proofs" of the accuracy of spline interpolation

There is no solid proof that the spline used here is accurate but one can estimate the accuracy of the interpolation by some indirect methods. There are at least three hints showing that the spline interpolation in this work is fairly accurate. They are:

\(^3\) It means setting the second derivative of both endpoints to zero.
1. Calculation of normalization of the trinucleon wavefunctions using spline interpolation and regular gaussian mesh points agrees to the 0.01% level with that of the true normalization calculated using weighting coefficients provided\(^4\), if one uses at least the same number of integration points for integration using interpolation as there are mesh points.

2. \([1]^0\)(0) is in principle the numerical normalization of the wavefunction, the agreement between that and the numerical normalization is at the worst 0.15% for all wavefunctions.

3. The smoothness of the internal "formfactors" suggests that the integration is numerically stable with respect to the variation in \(s\), at least in the region of interest.

### A.5 Integration cutoffs

The integration cutoffs for various spaces are as follows:

1. \(p\) space: 45 fm\(^{-1}\).
2. \(q\) space: 30 fm\(^{-1}\).
3. \(r\) space: 75 fm.

The cutoffs for the \(p\) and \(q\) space are determined by the domain of the wavefunction provided. The 75 fm cutoff of the \(r\) space seems to be a good one since this distance is over 35 times the nuclear radius. As a test, almost no changes in the "formfactors" are observed when the cutoff is reduced to 70 fm.

---

\(^4\) The set of "natural" mesh points (i.e. mesh points with weighting coefficients provided) for the wavefunction is a totally different set than the set of mesh points used for testing the normalization.
B. Second order terms for IA $\leftrightarrow$ EPM translation

The second order terms of the IA amplitude after coupling with the $\delta$ functions are shown below. Note $\eta_{a'}^{(1,1)}$ is defined as

$$\eta_{a'}^{(1,1)} = \frac{3}{\sqrt{2a+1}} \begin{pmatrix} a & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad (B.1)$$

IA after coupling and reexpressing in EPM format

$$\bar{q} \times \bar{\sigma} \cdot \bar{v} \bar{q} \cdot \bar{w} \leftrightarrow \begin{cases} \frac{1}{3} \sqrt{7} \eta_{2}^{(1,1)}[[((2, 2)2 \otimes 1)_{1}] - \frac{1}{\sqrt{6}} \eta_{2}^{(1,1)}[((2, 2)1 \otimes 1)_{1}] - \\
\frac{\sqrt{6}}{9} \eta_{2}^{(1,1)}[[((2, 2)0 \otimes 1)_{1}] + \frac{2\sqrt{6}}{9} \eta_{0}^{(1,1)}[((0, 2)2[2] \otimes 1)_{1}] + \\
\frac{\sqrt{15}}{9} \eta_{2}^{(1,1)}[((2, 2)0 \otimes 1)_{1}] - \\
\frac{2}{3\sqrt{3}} \eta_{0}^{(1,1)}[((0, 2)[2] \otimes 1)_{1}] } \bar{v} \times \bar{w} \cdot \bar{\sigma} + \\
\frac{-\sqrt{7}}{6} \eta_{2}^{(1,1)}[((2, 2)2 \otimes 1)_{1}] - \frac{1}{\sqrt{6}} \eta_{2}^{(1,1)}[((2, 2)1 \otimes 1)_{1}] + \\
\frac{2}{3} \eta_{0}^{(1,1)}[((0, 2)[2] \otimes 1)_{1}] \bar{w} \times \bar{s} \cdot \bar{\sigma} \bar{\bar{v}} \cdot \bar{s} + \\
\frac{\sqrt{2}}{3} \eta_{2}^{(1,1)}[((2, 2)1 \otimes 1)_{1}] + \eta_{2}^{(1,1)}[((2, 2)0 \otimes 1)_{1}] - \\
\eta_{0}^{(1,1)}[((0, 2)[2] \otimes 1)_{1}] } \bar{v} \times \bar{s} \cdot \bar{\sigma} \bar{w} \cdot \bar{s} - \\
\frac{i}{3} \eta_{2}^{(1,1)}[((2, 2)1 \otimes 1)_{0}] \bar{w} \cdot \bar{v} + \\
\frac{i}{3} \eta_{2}^{(1,1)}[((2, 2)1 \otimes 1)_{0}] \bar{v} \cdot \bar{s} \bar{w} \cdot \bar{s} \quad (B.2) \end{cases}$$
\[ q \cdot \bar{q} \cdot \bar{w} \cdot \bar{u} \leftrightarrow \ \begin{align*}
- \frac{5}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] + \sqrt{\frac{3}{14}} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] - \\
\frac{1}{\sqrt{6}} \eta_{2}^{(1,1)} & \left[ ((2,2)1 \otimes 1)_{1} \right] \bar{\sigma} \cdot \bar{w} \bar{u} \cdot \bar{v} \cdot \hat{s} + \\
- \frac{1}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] - \sqrt{\frac{2}{21}} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] - \\
\frac{1}{\sqrt{15}} \eta_{2}^{(1,1)} & \left[ ((2,0)2 \otimes 1)_{1} \right] \bar{\sigma} \cdot \bar{w} \bar{u} \cdot \bar{v} + . \\
\frac{5}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] - 2\sqrt{\frac{2}{21}} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] + \\
\sqrt{\frac{2}{3}} \eta_{2}^{(1,1)} & \left[ ((2,2)0 \otimes 1)_{1} \right] \bar{\sigma} \cdot \bar{w} \bar{u} \cdot \bar{v} + \bar{s} + \\
- \frac{1}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] + \frac{4}{63} \sqrt{42} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] - \\
\frac{\sqrt{6}}{9} \eta_{0}^{(1,1)} & \left[ ((0,2)2[2] \otimes 1)_{1} \right] + \frac{2}{9} \sqrt{\frac{3}{5}} \eta_{2}^{(1,1)} & \left[ ((0,0)2 \otimes 1)_{1} \right] - \\
\frac{2}{9} \sqrt{3} \eta_{0}^{(1,1)} & \left[ ((2,0)0[2] \otimes 1)_{1} \right] - \\
\frac{\sqrt{6}}{9} \eta_{2}^{(1,1)} & \left[ ((2,2)0 \otimes 1)_{1} \right] \bar{\sigma} \cdot \bar{u} \bar{w} \cdot \bar{v} - \\
- \frac{1}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] + \sqrt{\frac{2}{21}} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] + \\
\frac{1}{\sqrt{15}} \eta_{2}^{(1,1)} & \left[ ((2,0)2 \otimes 1)_{1} \right] \bar{\sigma} \cdot \bar{v} \bar{u} \cdot \bar{w} + . \\
\frac{5}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] + \sqrt{\frac{3}{14}} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] + \\
\frac{1}{\sqrt{6}} \eta_{2}^{(1,1)} & \left[ ((2,2)1 \otimes 1)_{1} \right] \bar{\sigma} \cdot \bar{s} \bar{u} \cdot \bar{w} \bar{v} + \bar{s} + \\
\frac{5}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] + \sqrt{\frac{3}{14}} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] + \\
\frac{1}{\sqrt{6}} \eta_{2}^{(1,1)} & \left[ ((2,2)1 \otimes 1)_{1} \right] \bar{\sigma} \cdot \bar{s} \bar{u} \cdot \bar{w} \bar{v} + \bar{s} + \\
\frac{5}{\sqrt{210}} \eta_{2}^{(1,1)} & \left[ ((2,4)2 \otimes 1)_{1} \right] - \sqrt{\frac{8}{21}} \eta_{2}^{(1,1)} & \left[ ((2,2)2 \otimes 1)_{1} \right] + \end{align*} \]
\[
\frac{\sqrt{6}}{3} \eta^{(1,1)}_0 \left[(0, 2) \otimes 1_1\right] \hat{\sigma} \cdot \hat{s} \hat{w} \cdot \hat{v} \hat{u} \cdot \hat{s} + \\
\left\{-\frac{5}{\sqrt{2}} \eta^{(1,1)}_2 \left[(2, 4) \otimes 1_1\right] + \frac{3}{14} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right]\right\} - \\
\frac{1}{\sqrt{6}} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right] \hat{\sigma} \cdot \hat{v} \hat{u} \cdot \hat{s} \hat{w} \cdot \hat{s} + \\
i \frac{3}{3} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right] \hat{w} \times \hat{u} \cdot \hat{s} \hat{w} \cdot \hat{s} + \\
i \frac{3}{3} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right] \hat{w} \times \hat{u} \cdot \hat{s} \hat{v} \cdot \hat{s} - \\
\sqrt{\frac{35}{6}} \eta^{(1,1)}_2 \left[(2, 4) \otimes 1_1\right] \hat{\sigma} \cdot \hat{s} \hat{w} \cdot \hat{s} \hat{u} \cdot \hat{s} \hat{v} \cdot \hat{s} \\
(B.3)
\]

\[
\hat{q} \cdot \hat{v} \hat{\sigma} \cdot \hat{q} \leftrightarrow \left\{-\frac{1}{3} \sqrt{7} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right] - \frac{1}{\sqrt{6}} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right]\right\} - \\
\frac{\sqrt{6}}{9} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right] - \frac{\sqrt{6}}{9} \eta^{(1,1)}_0 \left[(0, 2) \otimes 1_1\right] - \\
\frac{2}{9} \sqrt{15} \eta^{(1,1)}_2 \left[(2, 0) \otimes 1_1\right] - \\
\frac{2}{9} \sqrt{3} \eta^{(1,1)}_0 \left[(0, 0) \otimes 1_1\right] \hat{\sigma} \cdot \hat{v} + \\
\left\{\sqrt{\frac{7}{6}} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right] + \sqrt{\frac{3}{2}} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right]\right\} + \\
\sqrt{\frac{2}{3}} \eta^{(1,1)}_2 \left[(2, 2) \otimes 1_1\right] + \\
\sqrt{\frac{2}{3}} \eta^{(1,1)}_0 \left[(0, 2) \otimes 1_1\right] \hat{\sigma} \cdot \hat{s} \hat{v} \cdot \hat{s} \\
(B.4)
\]

\[
\hat{q} \cdot \hat{v} \hat{q} \cdot \hat{w} \leftrightarrow \frac{i}{\sqrt{3}} \eta^{(1,1)}_2 \left[(2, 2) \otimes 0_1\right] \hat{v} \times \hat{s} \cdot \hat{s} \hat{w} \cdot \hat{s} + \\
i \frac{i}{\sqrt{3}} \eta^{(1,1)}_2 \left[(2, 2) \otimes 0_1\right] \hat{w} \times \hat{s} \cdot \hat{s} \hat{v} \cdot \hat{s} + \\
\left\{-\frac{\sqrt{6}}{9} \eta^{(1,1)}_2 \left[(2, 2) \otimes 0_0\right]\right\} - \\
\frac{2}{9} \sqrt{3} \eta^{(1,1)}_0 \left[(0, 0) \otimes 0_1\right] \hat{v} \cdot \hat{w} + 
\]
B. Second order terms for IA ↔ EPM translation

\[
\frac{\sqrt{6}}{3} \eta_2^{(1,1)} (((2, 2) 0 \otimes 0)_0) [\vec{u} \cdot \vec{w}] \cdot \vec{s}
\]  
(B.5)
C. Wavefunction characteristics

C.1 Channel specifications

All the 22-channel wavefunctions [25] have the following channel specifications. These 22 channels consist of all possible states of the trinucleon system up to and including $J = 2$, where $J$ is the total angular momentum of the subsystem particle.

<table>
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<th>Channel</th>
<th>$L$</th>
<th>$l$</th>
<th>$L'$</th>
<th>$S$</th>
<th>$I$</th>
<th>Channel</th>
<th>$L$</th>
<th>$l$</th>
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<th>$S$</th>
<th>$I$</th>
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<td>1</td>
<td>3/2</td>
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<td>1/2</td>
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<td>3</td>
<td>2</td>
<td>1</td>
<td>3/2</td>
</tr>
</tbody>
</table>

Tab. C.1: Specifications for 22-channel wavefunctions

C.2 Model dependent quantities of wavefunctions

Table (C.2) lists several important quantities that are dependent on the potential of the wavefunction. They are respectively the binding energy given
by the wavefunctions, various partial wave probabilities and the numerical normalization $\langle \psi | \psi \rangle_{\text{num}}$ of the wavefunctions. The experimental binding energy $E_b$ of $^3\text{He}$ is 7.72 MeV and $^3\text{H}$ is 8.48 MeV [34].

| Potential    | $E_b$ | $P(S)$ | $P(S')$ | $P(P)$ | $P(D)$ | $\langle \psi | \psi \rangle_{\text{num}}$ |
|--------------|-------|--------|---------|--------|--------|----------------------------------|
| Bonn A       | 8.29  | 92.59% | 1.23%   | 0.030% | 6.14%  | 0.994                            |
| Bonn B       | 8.10  | 91.61% | 1.19%   | 0.044% | 7.16%  | 0.993                            |
| CD Bonn ($^3\text{He}$) | 7.91  | 91.61% | 1.35%   | 0.041% | 7.01%  | 0.993                            |
| CD Bonn ($^3\text{H}$) | 7.93  | 91.63% | 1.31%   | 0.041% | 7.01%  | 0.993                            |
| Nijmegen     | 7.66  | 90.31% | 1.29%   | 0.065% | 8.34%  | 0.990                            |
| Paris        | 7.38  | 90.11% | 1.40%   | 0.069% | 8.42%  | 0.988                            |
| AV14         | 7.58  | 89.86% | 1.15%   | 0.082% | 8.90%  | 0.987                            |

**Tab. C.2:** Some important quantities of trinucleon wavefunctions. Binding energy $E_b$ in MeV. $P(S)$ denotes the probability of S-wave and so on.
D. Notations and conventions

Below are some notations and conventions that are often used in this work:

1. Units: Natural units are used. That is, \( \hbar = c = 1 \).

2. Four vectors: Let \( U \) and \( V \) be two four vectors, their four "dot product" is defined as

\[
U \cdot V \equiv U^\alpha V_\alpha \equiv U^0 V_0 - \vec{U} \cdot \vec{V} \tag{D.1}
\]

Note that \( U^0 = U_0 \).

Sometimes, \( U \) is used to denote \( |\vec{U}| \) but the context will always clear the confusion.

3. \( \gamma \) matrices: The Dirac representation of \( \gamma \) matrices (the representation of ref. [35]) is used whenever an explicit representation is required.

4. Spherical harmonics: The bipolar spherical harmonic \( Y_{l_1,l_2}^{i_3,m_3}(\hat{x},\hat{y}) \) is defined as

\[
Y_{l_1,l_2}^{i_3,m_3}(\hat{x},\hat{y}) \equiv \sum_{m_1,m_2} \left( \begin{array}{c|cc} l_3 & l_1 & l_2 \\ m_3 & m_1 & m_2 \end{array} \right) Y_{l_1}^{m_1}(\hat{x}) Y_{l_2}^{m_2}(\hat{y}) \tag{D.2}
\]

5. \( e \): \( e = +\sqrt{4\pi\alpha} \) where \( \alpha \) is the fine structure constant whenever \( e \) is used to denote electric charge quantities.

6. \( G_F \): \( G_F \) is the Fermi coupling constant. \( G_F = 1.16639 \times 10^{-11} \) MeV\(^{-2} \).

7. \( V_{ud} \): \( V_{ud} \) is the CKM matrix element connecting up and down quarks. \(|V_{ud}| = 0.9735 \pm 0.0008 \) [27].

8. \( m \): \( m \) is used to denote the mass of a muon \( = 105.6583568 \pm 5.2 \times 10^{-6} \) MeV [27] throughout.
9. $m_\pi$: $m_\pi$ is the mass of a charged pion $= 139.57018 \pm 0.00035$ MeV [27].

10. $M_t$: $M_t$ is the average value of masses of $^3$He and $^3$H. $M_t = 2808.66$ MeV. The difference between the two masses is $\sim 0.6$ MeV.

11. $M_p$: $M_p$ is the average value of masses of a proton and a neutron.
Bibliography


