OPEN STRINGS IN A BACKGROUND GUAGE FIELD

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B. Sc. University of Victoria, 1998

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

DEPARTMENT OF PHYSICS AND ASTRONOMY

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

August 2000

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Abstract

The vacuum energy for open strings in a constant gauge field background is calculated for the disk and the annulus in a path integral formalism. This same formalism is then used to reproduce and generalize previous results for D-brane non-commutative geometry at string disk level. The contribution to non-commutative geometry from string annulus diagrams is calculated, and shown to be proportional to the divergence of the open string tachyon.

Table of Contents

Abstract **ii**

Table of Contents **iii**

Acknowledgments **vi**

1 Introduction to the Free String **1**

1.1 Introduction 1

1.2 Boundary Conditions and Mode Expansion 2

1.3 Commutation Relations 6

1.4 Virasoro Generators and Algebra 7

1.5 Ghosts and Vacuum Energy 10

2 Vacuum Energy by Path Integral Calculations **13**

2.1 Introduction 13

2.2 The Annulus 15

2.2.1 The Bosonic Propagator 15

2.2.2 Bosonic Vacuum Energy 16

2.2.3 The Fermionic Propagator 20

2.2.4 Fermionic Vacuum Energy 24

2.2.5 Superstring Vacuum Energy 28

2.3 The Disk 31

2.3.1 Boson Vacuum Energy 31

2.3.2 Fermion Vacuum Energy 33
2.4 The Bosonic Mobius Strip 34

3 Non-Commutative Geometry **36**

3.1 Introduction 36
3.2 Disk Calculations 38
3.3 One Loop Correction to Non-Commutative Geometry 40
3.3.1 The Charged String 43
3.3.2 The Neutral String 44
3.3.3 Distinct Branes 46
3.4 Non-Commutative Geometry from the Mobius Strip 47

4 Conclusion **49**

A Alternate Calculation of World-sheet Propagators **51**

A.1 The Bosonic Propagator 51
A.1.1 Equivalence to Published Expressions 52
A.2 The Fermion Propagator 55

B A Lemma **58**

C Canonical Quantization of the Open String in a $U(1)$ Background **61**

C.1 Motivation 61
C.2 Bosonic Degrees of Freedom 62
C.2.1 Boundary Conditions and Mode Expansion 62
C.2.2 Commutation Relations and the Virasoro Algebra 64
C.3 Fermionic Degrees of Freedom 68
C.3.1 Boundary Conditions and Mode Expansion 68

C.3.2 Commutation Relations and the Virasoro Algebra 71

Bibliography **74**

Acknowledgments

I gratefully acknowledge the encouragement, assistance and constructive criticism from Dr. G. Semenoff, and the patient ears and ready minds of Kirk Buckley, Joel Erickson, Todd Fugleberg, Matthew Horton and Michel Olivier who helped clarify the problem for me many times. I would like to thank my wife for her support and encouragement. This work is supported by NSERC.

Chapter 1

Introduction to the Free String

1.1 Introduction

The purpose of this text is to calculate in a well motivated manner the vacuum energy of a superstring in a background gauge field, and to show how one can recover non-commutative geometry from the study of stringy degrees of freedom. This is a problem that is of historical interest [1] [2] [3] [4], as the effective action for this background field, which can be obtained by integrating out the string degrees of freedom, is the Born-Infeld action, an non-linear generalization of Maxwell's equation. This problem is also of interest in current research as open string which end on D-branes are used in such diverse areas as the AdS/CFT correspondence and non-commutative geometry.

A non-trivial question to ask is whether the question is well posed, since the photons that make up a background $F_{\mu\nu}$ field are in principle described by stringy excitations themselves, would it not be more useful to fully understand the interaction of two open strings and leave it at that? Two considerations make the approach taken reasonable, the first is that string phenomenology is not yet fully understood, so that the determination of what excitations constitute a photon is a difficult task in and of itself. The second is that in the limit of low energies the spectrum of the string is dominated by the massless excitations anyways, and it could be appropriate to regard the background field as some condensate of strings.

To the end of better understanding of the subject, this chapter is intended to review the development of the open string. While certainly not exhaustive, as the subject is covered in detail in the literature [5] [6] this has two motivations. The first is to make this text as self contained as possible, and the second is to highlight some of the differences between a string that interacts with a background gauge field and the free case.

1.2 Boundary Conditions and Mode Expansion

When discussing any field theory, a useful starting place is the action, and string theory can be characterized as a free field theory on a two dimensional manifold with some arbitrary topology, with interactions being described by computations on higher genus surfaces. In the case of open string theory interactions only take place at the edge of the strings, in the sense that interactions involve a string breaking off and propagating freely, and this changes the topology of the surface.

The action for a free Neveu-Schwarz-Ramond open string is given [5] as

$$S = \frac{-1}{4\pi\alpha'} \int d\sigma d\tau \left(\partial_a X^\mu \partial^a X_\mu + \bar{\psi}^\mu \left(-i\gamma^a \partial_a\right) \psi_\mu\right) \tag{1.1}$$

where μ is the index on the target space, a is an index on the world-sheet, which is parameterized by $\sigma \in [0, \pi]$ and τ and the target space metric is taken to be minkowskian, which eliminates the terms such as $\sqrt{-g}$ as they become trivial. In addition, the two dimensional γ matrices are given by $\gamma^\tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $\gamma^\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ which satisfy the Clifford algebra $\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}$, where η^{ab} is the world sheet metric. The X fields represent bosonic degrees of freedom on the world-sheet, and commute, while the ψ fields are world-sheet fermions and hence anticommuting, but they transform under target space Lorentz transforms as vectors. In a sense they can be regarded as auxiliary degrees of freedom, since they do not have the attractive physical interpretation of target space position that the X fields do, however, they contribute the necessary number of degrees of freedom to have a critical theory in 10 dimensional target space.

A classical method for calculations in string theory is to expand the fields in terms of orthogonal modes, and then to use the canonical commutation relations of the theory to deduce the commutators of the mode coefficients which are then interpreted as raising and lowering operators. In addition, the constraints on the theory which are given by conditions on the energy momentum tensor are conveniently expressed in terms of these operators, and can be calculated. The first step in finding the orthogonal modes is to find boundary conditions that the fields must satisfy, and this is best accomplished by using variational arguments. Using the above and letting the X field vary by δX it is clear that

$$\begin{aligned}\delta S &= \frac{-1}{4\pi\alpha'}\int d\sigma d\tau 2\partial_a \delta X^\mu \partial^a X_\mu \\ &= \frac{-1}{4\pi\alpha'}\int d\sigma d\tau \left[\partial^a \left(2\delta X^\mu \partial_a X_\mu\right) - 2\delta X^\mu \partial^a \partial_a X_\mu\right].\end{aligned} \tag{1.2}$$

This gives the equation of motion for the boson,

$$\partial^a \partial_a X_\mu = 0. \tag{1.3}$$

Imposing that the X fields either vanish at $\tau = \pm\infty$ or are periodic in compact τ, depending on the topology of the string, this leaves the boundary condition

$$\int d\tau \delta X^\mu \partial_\sigma X_\mu |_{\sigma=0}^{\sigma=\pi} = 0. \tag{1.4}$$

There are numerous boundary conditions that satisfy this, however, the literature singles out two cases, Neumann boundary conditions, where the normal derivative vanishes at the ends of the string, and Dirichlet boundary conditions, where the position of the string end is held fixed. Dirichlet boundary conditions, which give rise to D-branes, explicitly break Lorentz invariance, while Neumann boundary conditions are chosen because they preserve the maximum possible symmetry for open strings. The boundary conditions and demands for orthogonality only apply to the σ dependence of X and the τ dependence is undetermined. As given in [5] the mode expansion satisfies

$$X^\mu = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma) . \tag{1.5}$$

so the modes are

$$\phi_m = \frac{1}{m} e^{-im\tau} \cos(m\sigma) . \tag{1.6}$$

These modes satisfy the orthogonality relation [2]

$$\int \frac{d\sigma}{\pi} \left[\bar{\phi}_m i \partial_\tau \phi_n - \phi_n i \partial_\tau \bar{\phi}_m \right] = \frac{1}{m} \delta_{mn} \tag{1.7}$$

and redefining the α oscillators to be the conventional raising and lowering operators eliminates the factor of $\frac{1}{m}$.

In a similar manner it is necessary to determine mode expansions for the world sheet fermions. Writing the fermion as a spinor in two dimensions, $\psi^\mu = \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix}$, and varying the above action results in the change in the action

$$\begin{aligned} \delta S &= \frac{-1}{4\pi\alpha'} \int d\sigma d\tau \left[\psi_-^{*\mu} (\partial_\tau + \partial_\sigma) \delta\psi_{\mu-} + \psi_+^{*\mu} (\partial_\tau - \partial_\sigma) \delta\psi_{\mu+} \right. \\ &\quad \left. + \delta\psi_-^{*\mu} (\partial_\tau + \partial_\sigma) \psi_{\mu-} + \delta\psi_+^{*\mu} (\partial_\tau - \partial_\sigma) \psi_{\mu+} \right] . \end{aligned} \tag{1.8}$$

Imposing that the fermion fields are majorana gives, using the fact that the ψs anticommute,

$$\begin{aligned} &= \frac{-1}{4\pi\alpha'} \int d\sigma d\tau \left[2\delta\psi_-^{*\mu} (\partial_\tau + \partial_\sigma) \psi_{\mu-} + 2\delta\psi_+^{*\mu} (\partial_\tau - \partial_\sigma) \psi_{\mu+} \right. \\ &\quad \left. + (\partial_\tau + \partial_\sigma) (\psi_-^{*\mu} \delta\psi_{\mu-}) + (\partial_\tau - \partial_\sigma) (\psi_+^{*\mu} \delta\psi_{\mu+}) \right] . \end{aligned} \tag{1.9}$$

This gives the standard equations of motion,

$$\begin{aligned} (\partial_\tau + \partial_\sigma) \psi_- &= 0 \\ (\partial_\tau - \partial_\sigma) \psi_+ &= 0 \end{aligned} \tag{1.10}$$

for the fermion fields, which can be summarized as $i\gamma^a \partial_a \psi = 0$ using the matrices listed above. The boundary terms are of interest, and assuming, as in the case of the bosons, either a periodicity in τ or that the fields vanish at $\tau = \pm\infty$ the conditions are

$$\int d\tau \left[\psi_-^\mu \delta\psi_{\mu-} - \psi_+^\mu \delta\psi_{\mu+}\right]_{\sigma=0}^{\sigma=\pi} = 0. \tag{1.11}$$

Naively, an ansatz $\psi_{\mu-} = M_\mu^\nu \psi_{\nu+}$ at $\sigma = 0, \pi$ will solve this for any M that satisfies $M^\dagger M = 1$, and M may be different at $\sigma = 0$ and $\sigma = \pi$. This causes the surface terms to vanish at both ends of the string independently. However, there is another consideration which restricts the choice of boundary conditions. The theory ought to be invariant under Lorentz transforms on the indices μ, since the X^μ are regarded as target space coordinates. The only boundary conditions that preserve this symmetry are $M_\mu^\nu = \pm\delta_\mu^\nu$. Due to the fact that there is ambiguity in the choice of boundary conditions for the fermions, there are two inequivalent choices of boundary conditions. Since the overall sign of ψ_- is a normalization convention one can unambiguously set $\psi_{\mu-} = \psi_{\mu+}$ at $\sigma = 0$. The most general mode expansion of ψ_+ that satisfies the condition that $(\partial_\tau - \partial_\sigma)\,\psi_+ = 0$ is $\psi_+ = \sum_m d_m e^{im(\tau+\sigma)}$, and similarly ψ_- can only depend on $\tau - \sigma$. In the case where $\psi_{\mu-} = \psi_{\mu+}$ at $\sigma = \pi$, the Ramond sector of the theory, the expansion for ψ is

$$\psi_{R\mu} = \sum_m d_{m\mu} \begin{pmatrix} e^{-im(\tau-\sigma)} \\ e^{-im(\tau+\sigma)} \end{pmatrix} \tag{1.12}$$

where m is an integer, whereas for the case $\psi_{\mu-} = -\psi_{\mu+}$ at $\sigma = \pi$ the expression in the Neveu-Schwarz sector becomes

$$\psi_{NS\mu} = \sum_m b_{m\mu} \begin{pmatrix} e^{-im(\tau-\sigma)} \\ e^{-im(\tau+\sigma)} \end{pmatrix} \tag{1.13}$$

where m is half-integral. The Ramond sector gives rise to target space fermions because of the non-commuting zero modes, while the Neveu-Schwarz sector gives rise to target-space bosons.

The fermion modes are also orthogonal in the following sense,

$$\begin{aligned}\int_0^\pi \frac{d\sigma}{\pi}\psi_m^\dagger\psi_n &= \int_0^\pi \frac{d\sigma}{\pi}\left(e^{im(\tau-\sigma)}, e^{im(\tau+\sigma)}\right)\begin{pmatrix} e^{-in(\tau-\sigma)} \\ e^{-in(\tau+\sigma)}\end{pmatrix} \\ &= \int_0^\pi \frac{d\sigma}{\pi} e^{i(m-n)\tau}\cos(m-n)\sigma = \delta_{mn}. \end{aligned} \tag{1.14}$$

1.3 Commutation Relations

The commutation relations for the modes must be determined, and the easiest way is to express canonically conjugate variables in terms of raising and lowering operators, impose commutation relations, and fourier analyze the results.

For the bosonic variables X^μ and $\partial_\tau X^\mu$ the commutation relation is

$$[\partial_\tau X^\mu(\sigma), X^\nu(\sigma')] = -i\delta(\sigma-\sigma')\eta^{\mu\nu}, \tag{1.15}$$

and substitution of the expansion 1.5 gives

$$\begin{aligned}-i\delta(\sigma-\sigma')\eta^{\mu\nu} &= \left[p^\mu + \sum_{n\neq 0}\alpha_n^\mu e^{in\tau}\cos(n\sigma), x^\nu + p^\nu\tau + i\sum_{m\neq 0}\frac{1}{m}\alpha_m^\nu e^{-im\tau}\cos(m\tau)\right] \\ &= [p^\mu, x^\nu] + \sum_{n\neq 0}\sum_{m\neq 0}\frac{i}{m}\delta_{m+n}\cos(n\sigma)\cos(m\sigma')\left[\alpha_n^\mu, \alpha_m^\nu\right] \end{aligned} \tag{1.16}$$

using the fact that the commutator must be independent of τ. Fourier analysis of the above immediately gives the standard commutation relationships for the modes,

$$\begin{aligned}[x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [\alpha_n^\mu, \alpha_m^\nu] &= n\delta_{m+n}\eta^{\mu\nu}. \end{aligned} \tag{1.17}$$

For the fermions the momentum canonically conjugate to $\psi_\pm$ is $\psi_\pm$, and so the anti-commutation relationship reads

$$\{\psi_a^\mu(\sigma), \psi_b^\nu(\sigma')\} = \pi\delta(\sigma-\sigma')\eta^{\mu\nu}\delta_{ab} \tag{1.18}$$

where $a, b = \pm$. Applying the expansion for the fermions given above gives for the case of $a = b = -$

$$= \left\{ \sum_m d_m^\mu e^{-im(\tau-\sigma)}, \sum_n d_n^\nu e^{-in(\tau-\sigma')} \right\}, \tag{1.19}$$

which again is forced to give the standard relationship since there must be no τ dependence in the anticommutator, giving

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n} \tag{1.20}$$

with an identical result for $a = b = +$, while the cross terms vanish, giving the proper orthogonality.

1.4 Virasoro Generators and Algebra

From the free string action, it is possible to determine the two dimensional energy-momentum tensor by varying the action with respect to that metric. As given above the action is

$$S = \frac{-1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-g} g_{ab}\partial^a X^\mu \partial^b X_\mu - \frac{i}{2}\sqrt{-g}g^{ab}\bar{\psi}^\mu\gamma^a\partial^b\psi_\mu - \frac{i}{2}\sqrt{-g}g^{ab}\bar{\psi}^\mu\gamma^b\partial^a\psi_\mu \tag{1.21}$$

where $g = \det(g_{ab})$. Taking the variation of this action with respect to g_{ab} is simple with the identity that $\det(g_{ab}) = \exp(Tr \ln g_{ab})$. Then for a change in the metric of δg_{ab} the change in the action is

$$\begin{aligned} \delta S &= \frac{-1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-g}\delta g_{ab}\partial^a X^\mu \partial^b X_\mu - \frac{i}{2}\sqrt{-g}\delta g^{ab}\bar{\psi}^\mu\gamma^a\partial^b\psi_\mu \\ &\quad - \frac{i}{2}\sqrt{-g}\delta g^{ab}\bar{\psi}^\mu\gamma^b\partial^a\psi_\mu - \frac{1}{2}\delta g_{ab}S. \end{aligned} \tag{1.22}$$

Then, because the energy-momentum tensor is given by varying with respect to the two dimensional metric,

$$T^{ab} = \frac{\delta S}{\delta g_{ab}} = \partial^a X^\mu \partial^b X_\mu + \frac{i}{4}\bar{\psi}^\mu\gamma^a\partial^b\psi_\mu + \frac{i}{4}\bar{\psi}^\mu\gamma^b\partial^a\psi_\mu - \frac{1}{2}g^{ab}\left(\partial_c X^\mu \partial^c X_\mu + \frac{i}{2}\bar{\psi}^\mu\gamma^c\partial_c\psi_\mu\right) \tag{1.23}$$

as in [5]. Since the left and right moving degrees of freedom on the string are mixed, the independent fourier components of this are defined as

$$L_m = \frac{1}{\pi} \int_0^{\pi} d\sigma \left(e^{im\sigma} T_{++} + e^{-im\sigma} T_{--} \right). \tag{1.24}$$

This expression can be explicitly evaluated for the modes as developed above, and it is found that

$$L_m = \frac{1}{2} \sum_n \alpha_{-n} \cdot \alpha_{m+n} + \left(n + \frac{1}{2} m \right) d_{-n} \cdot d_{m+n}. \tag{1.25}$$

This is unambiguous because of the diagonal commutation relationships except for the case of $m = 0$ in which case the operators are defined to be normal ordered, and there is some constant addition to L_0 which reflects this. In the sum above the sum is implicitly over half-odd integers for the case of anti-periodic fermions.

Now, the condition which arises as an equation of motion, that $\frac{\delta S}{\delta g^{ab}} = 0$ arises as a constraint, which implies that all components of T_{ab} vanish. This condition is very restrictive, and is relaxed so only the positive fourier components of this will annihilate any physical states, so that

$$L_n | \phi >= 0, \quad n > 0 \tag{1.26}$$

and

$$L_0 - a | \phi >= 0 \tag{1.27}$$

for some value of a. It is also possible to calculate the commutators of the Ls, and explicitly one finds that

$$\begin{aligned} [L_a, L_b] &= \left[\frac{1}{2} \sum_n \alpha_{-n} \cdot \alpha_{a+n} + \left(n + \frac{1}{2} a \right) d_{-n} \cdot d_{a+n}, \right. \\ &\qquad \left. \frac{1}{2} \sum_m \alpha_{-m} \cdot \alpha_{b+m} + \left(m + \frac{1}{2} b \right) d_{-m} \cdot d_{b+m} \right] \\ &= \frac{1}{4} \sum_{mn} [\alpha_{-n} \cdot \alpha_{a+n}, \alpha_{-m} \cdot \alpha_{b+m}] \end{aligned}$$

$$
\begin{aligned}
&+\left(n+\frac{1}{2}a\right)\left(m+\frac{1}{2}b\right)[d_{-n}\cdot d_{a+n}, d_{-m}\cdot d_{b+m}] \\
=\;& (a-b)\,L_{a+b}+A(a)\delta_{a+b}. \qquad (1.28)
\end{aligned}
$$

The term $A(a)$ is the central extension of the algebra, present because of the normal ordering ambiguity in L_0. To determine the normal ordering constant, one takes the expectation value of commutators [5], between two states in the perturbative vacuum which have zero momentum. It can be shown that because of the Jacobi identity for commutators the central extension can be determined by only evaluating the expectation values of $[L_1, L_{-1}]$ and $[L_2, L_{-2}]$.

For the Ramond sector the calculation is

$$
\begin{aligned}
\langle 0|\,[L_1, L_{-1}]\,|0\rangle &= \langle 0|\,\frac{1}{4}\,[d_0\cdot d_1, d_{-1}\cdot d_0]\,|0\rangle \\
&= \langle 0|\,\frac{1}{4}d_0^\mu d_{1\mu} d_{-1\nu} d_0^\nu\,|0\rangle \\
&= \langle 0|\,\frac{1}{4}d_0^\mu d_0^\nu \eta_{\mu\nu}\,|0\rangle = \frac{1}{8}\eta_{\mu\nu}\eta^{\mu\nu} = \frac{1}{8}D. \qquad (1.29)
\end{aligned}
$$

This makes use of the facts that $\alpha_0\,|0\rangle = 0$ on a zero momentum state and the normal ordering of the terms in the fermionic part of L_1. The expectation for L_2 gives

$$
\begin{aligned}
\langle 0|\,[L_2, L_{-2}]\,|0\rangle &= \frac{1}{4}\,\langle 0|\,[\alpha_1\cdot\alpha_1, \alpha_{-1}\cdot\alpha_{-1}] + [2d_0\cdot d_2, 2d_{-2}\cdot d_0]\,|0\rangle \\
&= \frac{1}{4}\,\langle 0|\,2\alpha_1^\mu\alpha_{-1}^\nu\eta_{\mu\nu} + 4d_0^\mu d_{2\mu} d_{-2\nu} d_0^\nu\,|0\rangle \\
&= \frac{1}{4}\left(2\eta_{\mu\nu}\eta^{\mu\nu} + 4\frac{1}{2}\eta_{\mu\nu}\eta^{\mu\nu}\right) = D \qquad (1.30)
\end{aligned}
$$

which confirms that for the Ramond sector the central extension is

$$
A(m) = \frac{1}{8}m^3 D \qquad (1.31)
$$

where D is the dimension of the target space.

In the Neveu-Schwarz sector where the fermionic excitations are half-integrally moded a similar calculation can be performed and one obtains

$$\langle 0| [L_1, L_{-1}] |0\rangle = 0 \tag{1.32}$$

because the bosonic part gives zero, and the fermionic part has normal ordered creation and annihilation operators, which will annihilate the vacuum. However,

$$\begin{aligned}\langle 0| [L_2, L_{-2}] |0\rangle &= \frac{1}{4} \langle 0| [\alpha_1 \cdot \alpha_1, \alpha_{-1} \cdot \alpha_{-1}] + \left[d_{1/2} \cdot d_{3/2}, d_{-3/2} d_{-1/2}\right] |0\rangle \\ &= \frac{1}{4} \langle 0| 2\alpha_1^\mu \alpha_{-1}^\nu \eta_{\mu\nu} + d_{1/2}^\mu d_{-1/2}^\nu \eta_{\mu\nu} |0\rangle \\ &= \frac{3}{4} D \end{aligned} \tag{1.33}$$

which shows that the central extension for the NS sector is

$$A(m) = \frac{1}{8}\left(m^3 - m\right) D. \tag{1.34}$$

1.5 Ghosts and Vacuum Energy

For the free string in a Minkowski target space there is an additional consideration, which is that some of the states created by naive application of the raising and lowering operators for world-sheet bosons and fermions have negative norm. These negative norm states are known as ghosts, and they do not correspond to physical excitations. There are several approaches [5] [6] that will eliminate them, such as quantizing in light cone coordinates or proving that they decouple in critical dimensions. It is shown in the literature that string theory is critical, that is that the negative norm states decouple from the physical spectrum, in the case of $D = 10$ and $a = 1$, where a is as defined above in 1.27. The approach that generalizes best to the case of an interacting string is to introduce additional non-physical degrees of freedom to the system which have statistics that will cancel the undesired excitations. This is analogous to the field-theory formalism of adding 'Fadeev-Popov ghosts' in QED to cancel non-physical excitations. In addition, the perturbative spectrum must be truncated by the GSO projection. The reason for this is that there are naively too many degrees of freedom for the preservation of target-space supersymmetry, which would require that the number of space-time bosons and fermions should be equal. This condition is such that for fermions the trace becomes the operator $1 + (-1)^f$ where f is the fermion number operator.

It only remains to calculate the vacuum energy for the free string. Since the general form of the partition function for any field theory is

$$Z = \int [Df] \exp \left[\int dx f \frac{1}{propagator} f \right] = const(\det propagator)^{-1/2}. \tag{1.35}$$

This is in general infinite, with the constant proportional to the volume of the space that the calculation is done in. On the critical dimensions the propagator for the string is $L_0 - 1$. [5] Using an identity for determinants we find that

$$Z = V \exp \left[-\frac{1}{2} Tr \ln(L_0 - 1) \right]. \tag{1.36}$$

This implies that

$$\begin{aligned} E_v &= \frac{1}{2} Tr \ln(L_0 - 1) \\ &= -\frac{1}{2} \int \frac{dt}{t} Tr e^{-t\pi(L_0 - 1)}. \end{aligned} \tag{1.37}$$

To evaluate $Tre^{-t\pi(L_0-1)}$ note both $L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n\geq 1} \alpha_{-n}\alpha_n$ and the convention that $\alpha_0^\mu = p^\mu$. This, together with the mass shell condition that $\alpha' M^2 = \sum_{n\geq 1} \alpha_{-n}\alpha_n - 1$, gives the vacuum energy as

$$E_v = \int \frac{dt}{t} \int \frac{d^D x d^D p}{(2\pi)^D} e^{-\pi t \frac{p^2}{2}} \sum_{phys} e^{-\pi t \alpha' M^2}. \tag{1.38}$$

In critical theory with $\alpha' = \frac{1}{2}$ this becomes

$$E_v = \int \frac{dt}{t} V_D \sqrt{\frac{1}{(2\pi^2 t)}}^D \sum_{phys} e^{-\frac{\pi t}{2} M_{phys}^2}. \tag{1.39}$$

The sum over physical states is nothing but the product of geometric series, since excitations at a given level have a unique polarization tensor. The sum over physical states then becomes $\sum_{phys} e^{-\frac{\pi t}{2} M^2_{phys}} = e^{\pi t} \prod_{n \geq 1} (1 - e^{-\pi t n})^{-8}$. This then gives the contribution from the bosonic components to be

$$E_v = \int \frac{dt}{t} V_D \sqrt{\frac{1}{2\pi^2 t}}^{D} \eta^{-8} \left(\frac{it}{2}\right) \tag{1.40}$$

agreeing with [4]. However, the contribution from the fermions is multiplicative, since the fermion and boson oscillators commute, and it vanishes by virtue of the GSO projection, as is appropriate for a theory which has space-time supersymmetry.

Chapter 2

Vacuum Energy by Path Integral Calculations

2.1 Introduction

The previous development was exclusively for free strings that do not interact with any form of background. The generalization that is now being considered is that of a string that interacts at its endpoints with a background antisymmetric field, and the goal is to compute the vacuum energy of these strings. The original motivation for this type of calculation was to produce a 'stringy' nonlinear generalization of Maxwell's equations, since the procedure can equally well be thought of as integrating out all string degrees of freedom, and interpreting the resulting vacuum energy as an action for the background fields.

This chapter presents a method which stands in contrast to the canonical quantization method of the previous chapter, and that of the treatment for the background dependent string vacuum energy presented in the appendix. The philosophy which is embraced is to take the fields that live on the string world-sheet and calculate a free Greens function for them for a given topology, which is in this case the annulus or the disk. Once the propagator has been determined the interaction with the background field is treated as a perturbation, and the vacuum energy of the string is calculated by simply summing all the possible world-sheet vacuum diagrams for a particular field.

This method, unlike canonical quantization can be seen to generalize much more readily. In the case of a non-constant field the interaction at the end of the string gives rise to higher order vertices on the world-sheet [2]. These, in turn will give higher loop corrections to the theory that exists on the world-sheet. In general these vertices need renormalization, and that provides a check on the effective action for the external fields. In addition, the method obviously extends to interactions between the world-sheet fields and the background in the bulk of the world-sheet. In the case of canonical quantization, this would present a conceptual challenge, whereas in this method, it adds computational rather than conceptual difficulties. Another strength of particular interest for current research is that the coupling to antisymmetric fields which themselves are confined to D-branes is immediately accounted for in this methodology. This is because there is no reliance on a particular form of the background than for convenience in the presentation of a compact final answer.

The point of this exercise is to compute the vacuum energy of a superstring in a background gauge field, and the calculation is performed explicitly for both the fermion and boson degrees of freedom.

The starting place for the calculation is the string action as given in [4] [7] for the string coupled to a background $F_{\mu\nu}$ field, with the world-sheet topology of the annulus

$$\begin{aligned} S &= \frac{-1}{4\pi\alpha'}\int d\sigma d\tau \left(\sqrt{-g}g_{ab}\partial^a X^\mu \partial^b X_\mu - i\bar{\psi}^\mu\gamma^a\partial_a\psi_\mu\right) \\ &\quad + \frac{1}{2}e_1\int d\tau F_{\mu\nu}\left(X^\nu\partial_\tau X^\mu - \frac{i}{2}\bar{\psi}^\nu\gamma^\tau\psi^\mu\right)|_{\sigma=0} \\ &\quad - \frac{1}{2}e_2\int d\tau F_{\mu\nu}\left(X^\nu\partial_\tau X^\mu - \frac{i}{2}\bar{\psi}^\nu\gamma^\tau\psi^\mu\right)|_{\sigma=a}. \end{aligned} \tag{2.1}$$

The generalization to an arbitrary topology coupled to the background is accomplished by the addition of a Wilson line along each boundary [8]. The addition of a bulk $B_{\mu\nu}$ field is trivial, since it is a total derivative, and is treated explicitly in the next chapter.

2.2 The Annulus

2.2.1 The Bosonic Propagator

The first task is to calculate the Greens function for the Bose degrees of freedom. While this calculation can be performed in several different ways (see Appendix A), the most transparent is to perform a conformal transformation of the Euclidean annulus of inner radius a and outer radius 1 to a cylinder of length $\ln a$. If the annulus is parameterized by the complex coordinate $z = re^{i\theta}$ then this transformation is accomplished by the identification of the cylinder coordinates ρ, ϕ as $\rho = \ln r$, $\phi = \theta$.

The equation that must be solved to determine the Greens function is

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial\phi^2}\right) G(\rho, \phi, \rho', \phi') = 2\pi\alpha'\bar{\delta}(\phi - \phi', \rho - \rho'), \tag{2.2}$$

where $\bar{\delta}$ is a modified δ-function with the constant mode subtracted so that $\bar{\delta} = \delta - \frac{1}{2\pi \ln a}$, and the δ-function in ϕ understood to have a periodicity 2π. Obviously G will have to be periodic in ϕ, and Neumann boundary conditions are imposed. The correct boundary conditions are then

$$\begin{aligned} \left.\frac{\partial}{\partial\rho} G(\phi, \rho; \phi', \rho')\right|_{\rho=0} &= 0 \\ \left.\frac{\partial}{\partial\rho} G(\phi, \rho; \phi', \rho')\right|_{\rho=\ln a} &= 0. \end{aligned} \tag{2.3}$$

Similar conditions apply for ρ' also.

A properly normalized eigenfunction of the Laplacian is

$$\psi_\lambda(\rho, \phi) = \frac{1}{\sqrt{2\pi \ln a}} e^{im\phi} \cos\left(\frac{n\pi}{\ln a}\rho\right), \tag{2.4}$$

with eigenvalue $\lambda = m^2 + \left(\frac{n\pi}{\ln a}\right)^2$, and so using the relation

$$G(\rho, \phi, \rho', \phi') = -\sum_\lambda \frac{\psi_\lambda(rho, \phi)\psi^*_\lambda(\rho', \phi')}{\lambda} \tag{2.5}$$

and taking into account the $2\pi\alpha'$ multiplying the δ function, we obtain the Greens function

$$\begin{aligned} G(\rho, \phi, \rho', \phi') &= \frac{\alpha'}{\ln a} \sum_{m\neq 0, m=-\infty}^{\infty} e^{i(\phi-\phi')m} \frac{1}{m^2} + \\ &+ \frac{2\alpha'}{\ln a} \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} e^{i(\phi-\phi')m} \cos\frac{n\pi}{\ln a}\rho \cos\frac{n\pi}{\ln a}\rho' \frac{1}{m^2 + \left(\frac{n\pi}{\ln a}\right)^2}. \end{aligned} \tag{2.6}$$

This can be re-written to obtain

$$\begin{aligned} G(\rho, \phi, \rho', \phi') &= \frac{\alpha'}{\ln a} \sum_{m\neq 0, m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{i(\phi-\phi')m} \cos\frac{n\pi}{\ln a}\rho \cos\frac{n\pi}{\ln a}\rho' \frac{1}{m^2 + \left(\frac{n\pi}{\ln a}\right)^2} \\ &+ 2\pi\alpha' \sum_{n=1}^{\infty} \cos\frac{n\pi}{\ln a}\rho \cos\frac{n\pi}{\ln a}\rho' \left(\frac{\ln a}{n\pi}\right)^2. \end{aligned} \tag{2.7}$$

This expression turns out to be a useful form of the bosonic propagator, and in the appendices, this is shown to be equivalent to other similarly derived expressions which are found in the literature.

2.2.2 Bosonic Vacuum Energy

The bosonic part of the action given in equation 2.1 is

$$S = \frac{-1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{-g} g_{ab} \partial^a X^\mu \partial^b X_\mu + \frac{1}{2} e_1 \int d\tau F_{\mu\nu} X^\nu \partial_\tau X^\mu |_{\sigma=0} - \frac{1}{2} e_2 \int d\tau F_{\mu\nu} X^\nu \partial_\tau X^\mu |_{\sigma=a}. \tag{2.8}$$

This means that the interaction Lagrangian takes the form

$$L_{int} = \frac{1}{2} e_1 F_{\mu\nu} X^\nu \partial_\tau X^\mu \delta(\sigma) - \frac{1}{2} e_2 F_{\mu\nu} X^\nu \partial_\tau X^\mu \delta(\sigma - \ln a), \tag{2.9}$$

which is identical to the statement that the X field can interact at either of the strings ends with the background field. To determine the vacuum energy in field theory one procedure is to take all the vacuum Feynman diagrams, sum them, and exponentiate the result. Now the possible elements in these diagrams can be enumerated. Due to the fact that this is a vacuum diagram there can be no external X propagators, and since the only interactions take place at $\rho = 0, \ln a$ the possibilities are that the field either experiences two subsequent interactions on the same edge of the cylinder, or on opposite edges, for a total of four possibilities.

It is shown in Appendix B that the appropriate thing to do in this situation is to generalize the propagator from a number to a matrix, sacrificing the ρ dependence, which has the following structure:

$$G(\rho,\phi,\rho',\phi') \to G(\phi,\phi') = \begin{pmatrix} A(\phi,\phi') & B(\phi,\phi') \\ B'(\phi,\phi') & A'(\phi,\phi') \end{pmatrix}. \tag{2.10}$$

In this $A(\phi,\phi') = G(0,\phi,0,\phi')$ and $B(\phi,\phi') = G(\ln a,\phi,0,\phi')$ with A' and B' being the obvious generalizations, and in the appendix this is shown to be a special case of a more general result. In this language, the vertex can be re-written as [1]

$$Interaction = \int d\phi \Omega \frac{F_{\mu\nu}}{2} X^{\nu}\partial_{\phi}X^{\mu}, \quad \Omega = \begin{pmatrix} e_1 & 0 \\ 0 & -e_2 \end{pmatrix}. \tag{2.11}$$

The rational for this is in direct analogy with the re-expression of the propagator, and the reason that this matrix is diagonal is that the interaction does not mix the ends of the string. An interesting feature of this is that since the interactions of the strings bosonic fields are only at the boundary this has converted a $1+1$ dimensional scalar theory into a $0+1$ dimensional theory described by matrices.

The calculation of $G(\phi,\phi')$ must be done carefully. The starting point is the propagator calculated above,

$$\begin{aligned} G(\rho,\phi,\rho',\phi') &= \frac{\alpha'}{\ln a}\sum_{m\neq 0, m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{i(\phi-\phi')m}\cos\frac{n\pi}{\ln a}\rho\cos\frac{n\pi}{\ln a}\rho'\frac{1}{m^2+\left(\frac{n\pi}{\ln a}\right)^2} \\ &\quad + 2\pi\alpha'\sum_{n=1}^{\infty}\cos\frac{n\pi}{\ln a}\rho\cos\frac{n\pi}{\ln a}\rho'\left(\frac{\ln a}{n\pi}\right)^2, \end{aligned} \tag{2.12}$$

and so it is immediately clear that

$$A(\phi,\phi') = G(0,\phi,0,\phi')$$

$$\begin{aligned}
&= \frac{\alpha'}{\ln a} \sum_{m\neq 0, m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{i(\phi-\phi')m} \frac{1}{m^2 + \left(\frac{n\pi}{\ln a}\right)^2} \\
&\quad + 2\pi\alpha' \sum_{n=1}^{\infty} \left(\frac{\ln a}{n\pi}\right)^2 \\
&= \frac{\alpha'}{\ln a} \sum_{m\neq 0, m=-\infty}^{\infty} \ln a \frac{e^{i(\phi-\phi')m}}{m} coth(m \ln a) + \pi\alpha' \frac{\ln^2 a}{3} \\
&= \pi\alpha' \frac{\ln^2 a}{3} + 2\alpha' \sum_{m>0} \frac{\cos(m(\phi-\phi'))}{m} coth(m \ln a). \qquad (2.13)
\end{aligned}$$

Clearly the same result holds for $A'(\phi,\phi') = G(\ln a, \phi, \ln a, \phi')$, since the only change in the calculation would be that the sum over n would now be multiplied by $\cos^2(n\pi)$ which is identically 1. Similarly, for $B(\phi, \phi')$ the sum is multiplied by $\cos(n\pi) = (-1)^n$, and the identity $\sum_n \frac{(-1)^n}{m^2+n^2} = \frac{\pi}{m} csch(m\pi)$ immediately gives

$$B(\phi,\phi') = B'(\phi,\phi') = \pi\alpha' \frac{\ln^2 a}{6} + 2\alpha' \sum_{m>0} \frac{\cos(m(\phi-\phi'))}{m} csch(m \ln a). \qquad (2.14)$$

Thus the Greens function can be compactly expressed as

$$G(\phi,\phi') = 2\alpha' \sum_{m>0} \frac{\cos(m(\phi-\phi'))}{m} G_m, \quad G_m = \begin{pmatrix} coth(m \ln a) & csch(m \ln a) \\ csch(m \ln a) & coth(m \ln a) \end{pmatrix}. \qquad (2.15)$$

It is also possible to obtain similar expressions for arbitrary ρ and ρ', however, that is unnecessary in this analysis.

Now, armed with both a propagator and an interaction vertex it is possible to calculate any Feynman diagram that would arise in the theory. Because the coupling to the external field has only two X legs, there is a unique world-sheet diagram at each order in n, and it can be explicitly calculated as

$$\begin{aligned}
n^{th} &= \frac{1}{n!}(n-1)! Tr(F^n)(2\alpha')^n \int d\phi_1 \ldots d\phi_n \sum_{m_1 \ldots m_n > 0} \partial_{\phi_1} \frac{\cos(m_1(\phi_1 - \phi_2))}{m_1} \times \\
&\quad \ldots \partial_{\phi_n} \frac{\cos(m_n(\phi_n - \phi_1)}{m_n} Tr\left[G_{m_1}\Omega \ldots G_{m_n}\Omega\right]. \qquad (2.16)
\end{aligned}$$

The factor of $\frac{1}{n!}$ is due to the order of the diagram, $(n-1)!$ comes from the inequivalent orderings of the vertices, 2^n is a result of the fact that there are two propagators on each vertex that can be connected either way, and $Tr(F^n)$ together with $\frac{1}{2^n}$ come from the interaction Lagrangian, and the factors of 2^n cancel. The reason that the derivatives are shown operating on distinct propagators is that if two operate on the same one, the expression can be converted into the above form by integration by parts and using the antisymmetry of $F_{\mu\nu}$, noting that the surface terms vanish because the world-sheet is compact in the ϕ direction. The ϕ integrals enforce the equality of the m_i, so the final expression for the diagram is

$$n^{th} = \frac{1}{n}(-2\pi\alpha')^n Tr(F^n) \sum_{m>0} Tr\left[G_m\Omega\right]^n . \tag{2.17}$$

The matrix $G_m\Omega$ can be diagonalized, and will generically have two eigenvalues, λ_{m+} and λ_{m-} which can be determined explicitly. This results in the simplification

$$n^{th} = \frac{1}{n}(-2\pi\alpha')^n Tr(F^n) \sum_{m>0} \left(\lambda_{m+}^n + \lambda_{m-}^n\right) \tag{2.18}$$

Now that there is an expression for the n^{th} diagram in the set of all vacuum Feynman diagrams, it is possible to sum explicitly,

$$\begin{aligned}
\sum connected &= \sum_{n\geq 1} \frac{(-2\pi\alpha')^n}{n} Tr(F^n) \sum_{m\geq 1} \left(\lambda_{m+}^n + \lambda_{m-}^n\right) \\
&= -\sum_{m\geq 1} Tr\left[\ln\left(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\lambda_{m+}\right) + \ln\left(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\lambda m-\right)\right] \\
&= -\sum_{m\geq 1} Tr \ln\left(\delta_{\mu\nu} + 2\pi\alpha' A_m(e_1 - e_2)F_{\mu\nu} + (2\pi\alpha')^2 F_{\mu\nu}^2(-e_1 e_2)\right) ,
\end{aligned} \tag{2.19}$$

where the last manipulation is obtained by evaluating the secular equation for $G_m\Omega$ and having set $A_m = \frac{1+a^{2m}}{1-a^{2m}}$. Thus, the exponential of the connected diagrams is

$$e^{\sum connected} = \prod_{m\geq 1} \exp -Tr \ln\left(\delta_{\mu\nu} + 2\pi\alpha' A_m(e_1 - e_2)F_{\mu\nu} + (2\pi\alpha')^2 F_{\mu\nu}^2(-e_1 e_2)\right)$$

$$= \prod_{m\geq 1} \left(det\left[\delta_{\mu\nu} + 2\pi\alpha' A_m(e_1 - e_2)F_{\mu\nu} + (2\pi\alpha')^2 F^2_{\mu\nu}(-e_1e_2)\right]\right)^{-1}, \quad (2.20)$$

which is the expression for the vacuum energy of the bosonic degrees of freedom of the string, and agrees with [1] [4] [9]. Note that in the case of the neutral string ($e_1 = e_2$) this is exactly the square of the Born Infeld action for the background field $F_{\mu\nu}$. This is because of the contribution of the different ends with opposite charges, each giving rise to a copy of the Born Infeld action.

2.2.3 The Fermionic Propagator

We wish to analogously determine the fermion propagator, and use an identical technique to determine the contribution to the partition function from fermions. Since the topology under consideration is a cylinder with a Euclidean metric, and following the convention [5] for the two dimensional γ matrices we have

$$\gamma_\phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_\rho = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (2.21)$$

Then the eigenvalue problem for the Dirac operator $\gamma^a \partial_a \phi = \lambda\phi$ is expanded as

$$\begin{pmatrix} 0 & \frac{\partial}{\partial\rho} + i\frac{\partial}{\partial\phi} \\ -\frac{\partial}{\partial\rho} + i\frac{\partial}{\partial\phi} & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \lambda \begin{pmatrix} \psi_r \\ \tilde{\psi} \end{pmatrix}. \quad (2.22)$$

Taking as an ansatz for ψ

$$\begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} = \sum_n e^{in\phi} \begin{pmatrix} \psi_n \\ \tilde{\psi}_n \end{pmatrix} \quad (2.23)$$

and noting that $e^{in\phi}$ is a complete set on $\phi \in [0, 2\pi]$ then for each mode the eigenvalue problem becomes

$$\begin{pmatrix} 0 & \frac{\partial}{\partial\rho} - n \\ -\frac{\partial}{\partial\rho} - n & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \tilde{\psi}_n \end{pmatrix} = \lambda_n \begin{pmatrix} \psi_n \\ \tilde{\psi}_n \end{pmatrix} \quad (2.24)$$

which is equivalent to

$$\left(-\frac{\partial^2}{\partial\rho^2} + n^2\right)\psi_n = \lambda_n^2 \psi_n$$

$$\tilde{\psi}_n = \frac{-\frac{\partial}{\partial\rho} - n}{\lambda_n}\psi_{rn}. \tag{2.25}$$

Now, with the definition $k^2 = \lambda_n^2 - n^2$ it is clear that the most general solution to the first of those equations is

$$\psi_n = Ae^{ik\rho} + Be^{-ik\rho}. \tag{2.26}$$

This implies in turn that

$$\tilde{\psi}_n = \frac{-ik - n}{\lambda}Ae^{ik\rho} + \frac{ik - n}{\lambda}Be^{-ik\rho}. \tag{2.27}$$

Now it is necessary to impose the boundary conditions, and since these are free fermions either Ramond or Neveu-Schwarz boundary conditions will suffice. The condition that $\psi_l = \psi_r$ at $\rho = 0$ immediately implies that $B = -A\frac{\lambda+n+ik}{\lambda+n-ik}$ which can be simplified using the definition of k to obtain $B = -A\frac{n+ik}{\lambda}$. This means that the condition $\tilde{\psi} = \pm\psi$ at $\rho = \ln a$ is converted to

$$e^{ik\ln a} - \frac{n+ik}{\lambda}e^{-ik\ln a} = \pm\left(\frac{-ik-n}{\lambda}e^{ik\ln a} + e^{-ik\ln a}\right), \tag{2.28}$$

which has the effect of quantizing k as

$$k = \frac{\pi m}{\ln a}. \tag{2.29}$$

where m is either an integer or half integer depending on whether the + or − sign was chosen in the boundary conditions. Choosing the appropriate normalization constant $A = \sqrt{\frac{1}{4\pi\ln a}}$ the final expression for the eigenstate of the Dirac operator is

$$\psi_{mn} = e^{in\psi}\frac{1}{\sqrt{4\pi\ln a}}\begin{pmatrix} e^{ik\rho} - \frac{n+ik}{\lambda}e^{-ik\rho} \\ -\frac{n+ik}{\lambda}e^{ik\rho} + e^{-ik\rho} \end{pmatrix}. \tag{2.30}$$

Noting that the sign of λ is not chosen by the definition means that, in addition to the value of k and n, the sign of λ is also a choice. In analogy with the approach taken by

[5] in quantizing the Fermionic modes of the open string, note that the above expression for ψ implies that

$$\tilde{\psi}_{mn\pm}(\rho,\phi) = \psi_{mn\pm}(-\rho,\phi). \tag{2.31}$$

This means that the $\tilde{\psi}$ and ψ fields on the domain $0 \le \rho \le \ln a$ can be combined into a single field which lives on the interval $-\ln a \le \rho \le \ln a$ and which has either periodic (Ramond) or anti-periodic (Neveu-Schwarz) boundary conditions

$$\psi(\phi, \rho + 2\ln a) = \pm\psi(\phi,\rho), \tag{2.32}$$

and the eigenvalue equation for the Dirac operator is trivially rewritten in the form

$$\left(-\frac{\partial}{\partial\rho} + i\frac{\partial}{\partial\phi}\right)\psi(\phi,\rho) = \lambda\psi(\phi,-\rho), \tag{2.33}$$

which explains the mixing of left- and right-movers in the expression for ψ, which stands in contrast to the free string case.

The inner product of the eigenfunctions is defined as

$$\langle\psi_{mn\pm}\psi_{m'n'\pm'}\rangle = \int_0^{\ln a} d\rho \int_0^{2\pi} d\phi \psi^\dagger_{mn\pm}\psi_{m'n'\pm'}. \tag{2.34}$$

This is in direct analogy with the orthogonality condition of the previous chapter. This can calculated explicitly as follows,

$$\begin{aligned}
\int_0^{\ln a} d\rho \int_0^{2\pi} d\phi\psi^\dagger_{mn\pm}\psi_{m'n'\pm'} &= \int_0^{\ln a} d\rho \int_0^{2\pi} d\phi \left[\psi^*_{mn}\psi_{m'n'} + \tilde{\psi}^*_{mn}\tilde{\psi}_{m'n'}\right] \\
&= \frac{1}{4\pi\ln a}\int d\rho d\phi \left[e^{-i(n-n')\phi}\left(e^{-ik\rho} - \frac{n-ik}{\lambda}e^{ik\rho}\right)\times\right. \\
&\quad \left(e^{ik'\rho} - \frac{n'+ik'}{\lambda'}e^{-ik'\rho}\right) \\
&\quad \left. + e^{-i(n-n')\phi}\left(-\frac{n-ik}{\lambda}e^{-ik\rho} + e^{ik\rho}\right)\left(-\frac{n'+ik'}{\lambda'}e^{ik'\rho} + e^{-ik'\rho}\right)\right] \\
&= \frac{1}{\ln a}\delta_{nn'}\int_0^{\ln a}\left[\left(1 + \frac{n-ik}{\lambda}\frac{n'+ik'}{\lambda'}\right)\cos(k-k')\rho\right.
\end{aligned}$$

$$
\begin{aligned}
& \left. - \left(\frac{n-ik}{\lambda} + \frac{n'+ik'}{\lambda'} \right) \cos(k+k')\rho \right] \\
= \; & \delta_{nn'}\delta_{k-k'} \left(1 + \frac{n-ik}{\lambda}\frac{n'+ik'}{\lambda'} \right) - \delta_{nn'}\delta_{k+k'} \left(\frac{n-ik}{\lambda} + \frac{n'+ik'}{\lambda'} \right) \\
= \; & \delta_{mm'}\delta_{nn'}\delta_{\pm\pm'}, \qquad (2.35)
\end{aligned}
$$

upon imposition of the condition that k is exclusively positive. This condition can also be deduced from the observation that $\psi_{(-m)n} = (-1)^{\frac{n-ik}{\lambda}}\psi_{mn}$ which means that those degrees of freedom are not linearly independent, and is reminiscent of the condition for the bosons. Now the Greens function can be calculated, and it is simply

$$
G(\rho, \psi, \rho', \psi') = \sum_{m>0,n,\pm} \frac{2\pi\alpha'}{\lambda}\psi_{mn\pm}(\rho, \psi)\psi^{\dagger}_{mn\pm}, \qquad (2.36)
$$

which is a 2×2 matrix, with the factor of $2\pi\alpha'$ accounting for the normalization of the Dirac operator in the action (2.1). Also, the sum is restricted to being over positive m by the observation above that the terms for negative m are not linearly independent. The elements of this matrix can be explicitly evaluated, and are

$$
\begin{aligned}
G_{11} &= \sum_{m>0,n,\pm} \frac{2\pi\alpha'}{\lambda}\psi_{r,mn\pm}(\rho,\phi)\psi^{*}_{r,mn\pm}(\rho',\phi') \\
&= \frac{\alpha'}{2\ln a} \sum_{m>0,n,\pm} \frac{1}{\lambda} e^{in(\phi-\phi')} \left(e^{ik\rho} - \frac{n+ik}{\lambda}e^{-ik\rho} \right) \left(e^{-ik\rho'} - \frac{n-ik}{\lambda}e^{ik\rho'} \right). \qquad (2.37)
\end{aligned}
$$

The sum over the sign of λ has the effect of eliminating the odd power terms, and what is left can be written as

$$
= \frac{\alpha'}{\ln a} \sum_{m>0,n} \frac{1}{\lambda^2} e^{in(\phi-\phi')} \left[-2n(e^{ik(\rho+\rho')} + e^{-ik(\rho+\rho')}) + 2ik(e^{ik(\rho+\rho')} - e^{-ik(\rho+\rho')}) \right]. \qquad (2.38)
$$

Similar calculations for the other components reveal that the structure of the Greens function is

$$
\begin{aligned}
G(\rho, \psi, \rho', \psi') = \; & \frac{4\alpha'}{\ln a} \sum_{m>0,n} \frac{1}{\lambda^2} e^{in(\phi-\phi')} \left[\begin{pmatrix} -n\cos k(\rho+\rho') & -n\cos k(\rho-\rho') \\ -n\cos k(\rho-\rho') & -n\cos k(\rho+\rho') \end{pmatrix} \right. \\
& \left. + \begin{pmatrix} -k\sin k(\rho+\rho') & +k\sin k(\rho-\rho') \\ -k\sin k(\rho-\rho') & +k\sin k(\rho+\rho') \end{pmatrix} \right]. \qquad (2.39)
\end{aligned}
$$

2.2.4 Fermionic Vacuum Energy

The fermionic part of the action (2.1) is

$$S = \frac{-1}{4\pi\alpha'}\int d\sigma d\tau \left(-i\bar{\psi}^{\mu}\gamma^{a}\partial_{a}\psi_{\mu}\right)+\frac{1}{2}e_1\int d\tau F_{\mu\nu}\frac{-i}{2}\bar{\psi}^{\nu}\gamma^{\tau}\psi^{\mu}|_{\sigma=0}-\frac{1}{2}e_2\int d\tau F_{\mu\nu}\frac{-i}{2}\bar{\psi}^{\nu}\gamma^{\tau}\psi^{\mu}|_{\sigma=\ln a}. \tag{2.40}$$

The interaction Lagrangian can be written as

$$L_{int} = \frac{1}{2}e_1 F_{\mu\nu}\frac{-i}{2}\bar{\psi}^{\nu}\gamma^{\tau}\psi^{\mu}\delta(\rho) - \frac{1}{2}e_2 F_{\mu\nu}\frac{-i}{2}\bar{\psi}^{\nu}\gamma^{\tau}\psi^{\mu}\delta(\ln a-\rho), \tag{2.41}$$

which means just as in the case of the bosons that the fermions only interact at the ends of the string. Exactly as in the case of the bosons the fermion propagator can be restricted to the edges and made into a larger matrix of the form

$$G(\rho,\phi,\rho',\phi') \to G(\phi,\phi') = \begin{pmatrix} A(\phi,\phi') & B(\phi,\phi') \\ B'(\phi,\phi') & A(\phi,\phi') \end{pmatrix}, \tag{2.42}$$

with the same definitions for A and B as in the bosonic case.

Now it is necessary to calculate the explicit form of A, A', B, B' for both Ramond and Neveu-Schwarz boundary conditions. The case of periodic fermions is straightforward, and explicitly

$$\begin{aligned} A_R = G(0,\phi,0,\phi') &= \frac{4\alpha'}{\ln a}\sum_{m>0,n}\frac{1}{\lambda^2}e^{in(\phi-\phi')}\begin{pmatrix} -n & -n \\ -n & -n \end{pmatrix} \\ &= \frac{4\alpha'}{\ln a}\sum_{m>0,n}\frac{1}{n^2+\frac{m\pi}{\ln a}}e^{in(\phi-\phi')}\begin{pmatrix} -n & -n \\ -n & -n \end{pmatrix} \\ &= 4\pi\alpha'\sum_{n}e^{in(\phi-\phi')}\begin{pmatrix} coth(n\ln a) & coth(n\ln a) \\ coth(n\ln a) & coth(n\ln a) \end{pmatrix}. \end{aligned} \tag{2.43}$$

The calculation of A'_R is identical, because m is quantized as an integer, so $\cos(2\pi m) = \cos(0) = 1$ while $\sin(m\pi)$ is vanishing, and because there should be no physical difference between the two ends of the string, the two values are expected to be equal. In particular, the vacuum energy is not changed by the exchange of e_1 and $-e_2$. Similarly, B_R can be calculated as

$$\begin{aligned} B_R = G(0,\phi,\ln a,\phi') &= \frac{4\alpha'}{\ln a}\sum_{m>0,n}\frac{1}{\lambda^2}e^{in(\phi-\phi')}\begin{pmatrix}-n(-1)^m & -n(-1)^m\\ -n(-1)^m & -n(-1)^m\end{pmatrix}\\ &= 4\pi\alpha'\sum_n e^{in(\phi-\phi')}\begin{pmatrix}csch(n\ln a) & csch(n\ln a)\\ csch(n\ln a) & csch(n\ln a)\end{pmatrix}, \end{aligned} \tag{2.44}$$

and the calculation is identical for B'_R.

For the Neveu-Schwarz sector the calculation of A_{NS} and B_{NS} requires more care, due to the fact that m is half integer instead of integer. The calculation for A_{NS} can be done as

$$\begin{aligned} A_{NS} = G(0,\psi,0,\psi') &= \frac{4\alpha'}{\ln a}\sum_{m>0,n}\frac{1}{\lambda^2}e^{in(\phi-\phi')}\begin{pmatrix}-n & -n\\ -n & -n\end{pmatrix}\\ &= +4\pi\alpha'\sum_n e^{in(\phi-\phi')}\begin{pmatrix}tanh(n\ln a) & tanh(n\ln a)\\ tanh(n\ln a) & tanh(n\ln a)\end{pmatrix}. \end{aligned} \tag{2.45}$$

The calculation for A'_{NS} is analogous, resulting in

$$A'_{NS} = +4\pi\alpha'\sum_n e^{in(\phi-\phi')}\begin{pmatrix}-tanh(n\ln a) & tanh(n\ln a)\\ tanh(n\ln a) & -tanh(n\ln a)\end{pmatrix}. \tag{2.46}$$

The subtlety comes from the evaluation of

$$\begin{aligned} B_{NS} = G(0,\psi,\ln a,\psi') &= \frac{4\alpha'}{\ln a}\sum_{m>0,n}\frac{1}{\lambda^2}e^{in(\phi-\phi')}\begin{pmatrix}-n\cos k(+\ln a) & -n\cos k(-\ln a)\\ -n\cos k(-\ln a) & -n\cos k(+\ln a)\end{pmatrix}\\ &\quad+\begin{pmatrix}-k\sin k(+\ln a) & +k\sin k(-\ln a)\\ -k\sin k(-\ln a) & +k\sin k(+\ln a)\end{pmatrix}\\ &= \frac{4\alpha'}{\ln a}\sum_{m>0,n}\frac{1}{\lambda^2}e^{in(\phi-\phi')}\begin{pmatrix}-\frac{m\pi}{\ln a}\sin(m\pi) & -\frac{m\pi}{\ln a}\sin(m\pi)\\ \frac{m\pi}{\ln a}\sin(m\pi) & \frac{m\pi}{\ln a}\sin(m\pi)\end{pmatrix}\\ &= 4\pi\alpha'\sum_n e^{in(\phi-\phi')}\begin{pmatrix}sech(n\ln a) & sech(n\ln a)\\ -sech(n\ln a) & -sech(n\ln a)\end{pmatrix}, \end{aligned} \tag{2.47}$$

and the subtle point is that when evaluating B'_{NS} there is a difference compared with the above calculation. The signs of the off-diagonal elements change, because they depend on $\rho - \rho'$, while ρ and ρ' are interchanged. The result is that

$$B'_{NS} = 4\pi\alpha' \sum_n e^{in(\phi-\phi')} \begin{pmatrix} sech(n\ln a) & -sech(n\ln a) \\ sech(n\ln a) & -sech(n\ln a) \end{pmatrix}. \tag{2.48}$$

From the interaction Lagrangian it is clear that we can re-write it as

$$L_{int} = i\int d\phi F_{\mu\nu} \begin{pmatrix} \frac{e_1}{4}(-i\gamma^{\phi}) & 0 \\ 0 & \frac{-e_2}{4}(-i\gamma^{\phi}) \end{pmatrix}. \tag{2.49}$$

Given this interaction term, it is possible to write down the contribution from the unique diagram at order n. This takes the form

$$\begin{aligned} n^{th} &= \frac{1}{n!}(n-1!)\, Tr(F^n_{\mu\nu})\alpha'^n \int d\phi_1 \dots d\phi_n \sum_{m_1\dots m_n} e^{im_1(\phi_2-\phi_1)} \\ &\quad \dots e^{im_n(\phi_1-\phi_n)} \times Tr\left[G_{m_1}\Omega \dots G_{m_n}\Omega\right], \end{aligned} \tag{2.50}$$

where Ω is the matrix component of the interaction term, and G_m is the matrix portion of the fermion propagator. The term $\frac{1}{n!}$ comes from the order of the diagram, and $(n-1!)$ counts the number of ways of ordering the vertices. The integration over the ϕs is trivial and the result is

$$n^{th} = \frac{1}{n} i^n\, Tr(F^n_{\mu\nu})\,(\alpha'\pi)^n \sum_m Tr\,(G_m\Omega)^n. \tag{2.51}$$

Now, using the fact that the matrix can be diagonalized, and denoting the four eigenvalues of $G_m\Omega$ by $\lambda_{1m}\dots\lambda_{4m}$ this can be re-expressed as

$$n^{th} = \frac{1}{n} i^n\, Tr(F^n_{\mu\nu})\,(\alpha'\pi)^n \sum_m (\lambda^n_{1m} + \lambda^n_{2m} + \lambda^n_{3m} + \lambda^n_{4m}). \tag{2.52}$$

To obtain the vacuum energy it is necessary to sum all the diagrams so

$$\begin{aligned} \sum connected &= \sum_{mnj} \frac{1}{n} Tr(F^n_{\mu\nu})\,(\alpha'\pi)^n \lambda^n_{jm} \\ &= \sum_{mj} \log Tr\left(\delta_{\mu\nu} + (\alpha'\pi)\, F_{\mu\nu}\lambda_{jm}\right), \end{aligned} \tag{2.53}$$

so

$$e^{\sum connected} = \prod_{mj} \det\left(\delta_{\mu\nu} - \left(\frac{\alpha'\pi}{2}\right) F_{\mu\nu}\lambda_{jm}\right). \tag{2.54}$$

and the sum in the determinant depends on the eigenvalues of $G_m\Omega$.

The task is then to determine the eigenvalues of $G_m\Omega$ for both the case of Ramond, and Neveu-Schwarz boundary conditions. In the periodic sector the matrix

$$\begin{aligned} G_{mR}\Omega &= \begin{pmatrix} A_R & B_R \\ B'_R & A'_R \end{pmatrix} \begin{pmatrix} e_1\gamma^\phi & 0 \\ 0 & -e_2\gamma^\phi \end{pmatrix} \\ &= \begin{pmatrix} coth(m\ln a) & coth(m\ln a) & csch(m\ln a) & csch(m\ln a) \\ coth(m\ln a) & coth(m\ln a) & csch(m\ln a) & csch(m\ln a) \\ csch(m\ln a) & csch(m\ln a) & coth(m\ln a) & coth(m\ln a) \\ csch(m\ln a) & csch(m\ln a) & coth(m\ln a) & coth(m\ln a) \end{pmatrix} \times \\ &\quad \begin{pmatrix} 0 & e_1 & 0 & 0 \\ e_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_2 \\ 0 & 0 & -e_2 & 0 \end{pmatrix} \end{aligned} \tag{2.55}$$

had two non-zero eigenvalues λ_{1R}, λ_{2R} that satisfy

$$\begin{aligned} \lambda_{1R} + \lambda_{2R} &= 2(e_1 - e_2)coth(m\ln a) \\ \lambda_{1R}\lambda_{2R} &= -4e_1e_2. \end{aligned} \tag{2.56}$$

In the antiperiodic sector the result is different, but analogous. Evaluating

$$G_{mNS}\Omega = \begin{pmatrix} tanh(m\ln a) & tanh(m\ln a) & sech(m\ln a) & sech(m\ln a) \\ tanh(m\ln a) & tanh(m\ln a) & -sech(m\ln a) & -sech(m\ln a) \\ sech(m\ln a) & -sech(m\ln a) & -tanh(m\ln a) & tanh(m\ln a) \\ sech(m\ln a) & -sech(m\ln a) & tanh(m\ln a) & -tanh(m\ln a) \end{pmatrix} \times$$

$$\begin{pmatrix} 0 & e_1 & 0 & 0 \\ e_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_2 \\ 0 & 0 & -e_2 & 0 \end{pmatrix} \tag{2.57}$$

Again, two non-zero eigenvalues are found, which satisfy

$$\begin{aligned} \lambda_{1NS} + \lambda_{2NS} &= 2(e_1 - e_2)tanh(m \ln a) \\ \lambda_{1NS}\lambda_{2NS} &= -4e_1e_2. \end{aligned} \tag{2.58}$$

With this information the contribution to the vacuum energy in both the Ramond and Neveu Schwarz sectors can be written down explicitly. The result is

$$\begin{aligned} e_R^{\sum connected} &= \prod_m \det\left(\delta_{\mu\nu} + 2\alpha'\pi(e_1 - e_2)coth(m\ln a)F_{\mu\nu} + (2\alpha'\pi)^2(-e_1e_2)F_{\mu\nu}^2\right) \\ e_{NS}^{\sum connected} &= \prod_m \det\left(\delta_{\mu\nu} + 2\alpha'\pi(e_1 - e_2)tanh(m\ln a)F_{\mu\nu} + (2\alpha'\pi)^2(-e_1e_2)F_{\mu\nu}^2\right), \end{aligned} \tag{2.59}$$

in agreement with the results of [4].

2.2.5 Superstring Vacuum Energy

For the bosons, the energy associated with an annulus of inner radius a and charges e_1 and $-e_2$ at the ends was determined to be

$$E = \prod_{m\geq 1}\left(det\left[\delta_{\mu\nu} + 2\pi\alpha' A_m(e_1 - e_2)F_{\mu\nu} + (2\pi\alpha')^2F_{\mu\nu}^2(-e_1e_2)\right]\right)^{-1}, \tag{2.60}$$

with A_m defined as above. Since the external field can be block diagonalized, it is only necessary to determine the effect of one of those blocks on the energy, and the rest will be identical calculations. If the field strength is f in this pair of directions, with the identifications

$$\beta_1 = 2\pi\alpha' e_1 f, \quad \beta_2 = -2\pi\alpha' e_2 f \tag{2.61}$$

then for the two dimensions in question, the energy is

$$
\begin{aligned}
&= \prod_{m\geq 1} det \begin{pmatrix} 1-\beta_1\beta_2 & -A_m\beta_1+\beta_2 \\ A_m\beta_1+\beta_2 & 1-\beta_1\beta_2 \end{pmatrix}^{-1} \\
&= \left(\prod_{m\geq 1} (1-\beta_1\beta_2)^2 + (A_m(\beta_1+\beta_2))^2\right)^{-1} \\
&= \left(\prod_{m\geq 1} \left[1-2\beta_1\beta_2+\beta_1^2\beta_2^2+\frac{1+2a^{2m}+a^{4m}}{1-2a^{2m}+a^{4m}}\left(\beta_1^2+\beta_2^2+2\beta_1\beta_2\right)\right]\right)^{-1} \\
&= \left(\prod_{m\geq 1}\frac{1}{(1-a^{2m})^2}\prod_{n\geq 1}(1+\beta_1^2)(1+\beta_2^2)\,\times\right. \\
&\qquad \left.\prod_{m\geq 1}\left(1-2\frac{1-\beta_1^2-\beta_2^2-4\beta_1\beta_2+\beta_1^2\beta_2^2}{(1+\beta_1^2)(1+\beta_2^2)}a^{2m}+a^{4m}\right)\right)^{-1}.
\end{aligned}
\tag{2.62}
$$

Now, making the identification

$$\nu = \frac{1}{2\pi}cos^{-1}\frac{1-\beta_1^2-\beta_2^2-4\beta_1\beta_2+\beta_1^2\beta_2^2}{(1+\beta_1^2)(1+\beta_2^2)}, \tag{2.63}$$

the energy can be re-written in terms of a Jacobi θ function as

$$\bar{Z} = \left(\prod_{n\geq 1}(1+\beta_1^2)(1+\beta_2^2)\left(\frac{1}{a^{1/12}}\prod_{m\geq 1}\frac{1}{1-a^{2m}}\right)^3\frac{1}{2\sin\pi\nu}\theta_1\left(\nu|\frac{lna}{i\pi}\right)\right)^{-1}, \tag{2.64}$$

which can be re-cast with a Dedekind η function as

$$
\begin{aligned}
&= \left(\prod_{n\geq 1}(1+\beta_1^2)(1+\beta_2^2)\frac{1}{2\sin\pi\nu}\frac{\theta_1\left(\nu|\frac{lna}{i\pi}\right)}{\eta^3\left(\frac{lna}{i\pi}\right)}\right)^{-1} \\
&= \left(\left((1+\beta_1^2)(1+\beta_2^2)\right)^{-1/2}\frac{1}{2}\sqrt{\frac{(1+\beta_1^2)(1+\beta_2^2)}{(\beta_1+\beta_2)^2}}\frac{\theta_1\left(\nu|\frac{lna}{i\pi}\right)}{\eta^3\left(\frac{lna}{i\pi}\right)}\right)^{-1} \\
&= 2(\beta_1+\beta_2)\frac{\eta^3\left(\frac{lna}{i\pi}\right)}{\theta_1\left(\nu|\frac{lna}{i\pi}\right)}.
\end{aligned}
\tag{2.65}
$$

The second step makes use of the fact that $\sin(\frac{1}{2}\cos^{-1}k) = \sqrt{\frac{1-k}{2}}$ and also ζ function regularization. This gives the effect of one independent block of $F_{\mu\nu}$ on the vacuum energy of the bosons.

The above must be integrated over the Teichmuller parameter, and the obvious generalization is that the vacuum energy in 10 dimensions is

$$Z = \int \frac{da}{a} \prod_{n=1}^{5} 2(\beta_{1n} + \beta_{2n}) \frac{\eta^3\left(\frac{lna}{i\pi}\right)}{\theta_1\left(\nu_n | \frac{lna}{i\pi}\right)} \prod_{k \geq 1} \left(1 - e^{-\pi kt}\right)^2, \tag{2.66}$$

where as before, $f_1, \ldots f_5$ are the independent components of the background $F_{\mu\nu}$ field, equation 2.61 is extended to read

$$\beta_{1n} = 2\pi\alpha' e_1 f_n, \quad \beta_{2n} = -2\pi\alpha' e_2 f_2, \tag{2.67}$$

and the generalization of 2.63 is

$$\nu_n = \frac{1}{2\pi} cos^{-1} \frac{1 - \beta_{1n}^2 - \beta_{2n}^2 - 4\beta_{1n}\beta_{2n} + \beta_{1n}^2\beta_{2n}^2}{(1 + \beta_{1n}^2)(1 + \beta_{2n}^2)}. \tag{2.68}$$

The contribution from the ghosts is simply the vacuum energy in the limit of zero field for a pair of the coordinates, but in the numerator, as is appropriate for the degrees of freedom which reduce the number of bosons. The integration has an imaginary part for a background electric field, which indicates, as discussed in [4] that there is pair production of strings from the vacuum.

The contribution from the fermions is more subtle. They are multiplicative with the bosonic contribution, but must be summed over four spin structures [10] [11] which is equivalent to performing the GSO projection on the string. These spin structures correspond to periodicity or antiperiodicity in both the ρ and ϕ directions on the annulus, and have their origin in the fact that the annulus is also a tree level closed string diagram. The contribution of the two terms that have periodic ϕ are

$$Z_{R,P} = \frac{1}{2(\beta_1 + \beta_2)} \frac{\theta_1\left(\nu | \frac{\ln a}{i\pi}\right)}{\eta^3\left(\frac{\ln a}{i\pi}\right)} \tag{2.69}$$

$$Z_{NS,P} = \frac{1}{2(1 - \beta_1\beta_2)} \frac{\theta_2\left(\nu | \frac{\ln a}{i\pi}\right)}{\eta\left(\frac{\ln a}{i\pi}\right)} a^{-1/6} \prod_{m>0} \left(1 + a^{2m}\right)^{-2}, \tag{2.70}$$

while the terms antiperiodic in ϕ are

$$Z_{R,A} = \frac{1}{\sqrt{(1+\beta_1^2)(1+\beta_2^2)}} \frac{\theta_4\left(\nu|\frac{\ln a}{i\pi}\right)}{\eta\left(\frac{\ln a}{i\pi}\right)} a^{-1/12} \prod_{m>0} \left(1-a^{2m-1}\right)^{-2} \tag{2.71}$$

$$Z_{NS,A} = \frac{1}{\sqrt{(1+\beta_1^2)(1+\beta_2^2)}} \frac{\theta_3\left(\nu|\frac{\ln a}{i\pi}\right)}{\eta\left(\frac{\ln a}{i\pi}\right)} a^{-1/12} \prod_{m>0} \left(1+a^{2m-1}\right)^{-2}. \tag{2.72}$$

As in [11] the sum over spin structures is given as

$$Z_{R,A} + Z_{NS,P} - Z_{R,P} - Z_{NS,A} \tag{2.73}$$

and as noted in [4] the zero field limit has this exactly vanishing, as expected for a model with unbroken supersymmetry.

2.3 The Disk

2.3.1 Boson Vacuum Energy

The amplitude for the disk is a simpler calculation than that for the annulus, and can be obtained as a limit of the annulus calculation. The calculation starts in the same place as the others, with the action for a set of bosonic fields on the string that are only coupled at the boundary to the external field. This action is [1]

$$S = \frac{1}{2}\int d\sigma d\tau \partial_a X^\mu \partial^a X_\mu + \frac{i2\pi\alpha' e}{2}\int d\tau F_{\mu\nu} X^\nu \partial_\tau X^\mu, \tag{2.74}$$

where the X fields have been rescaled by a factor of $\frac{1}{\sqrt{2\pi\alpha'}}$ compared with the previous section, for ease of computation.

Reading off from this the term associated with each vertex is

$$-e\pi\alpha' F_{\mu\nu}\int d\tau X^\nu \partial_\tau X^\mu, \tag{2.75}$$

and the analogy with the annulus is that because there is only one boundary $\Omega = 1$ here. A convenient way to do calculations on the disk is to parameterize it as the unit disk in the complex plane, because the Greens function satisfying Neumann boundary conditions is well known there and is [12]

$$G(z, z') = -\frac{1}{2\pi} \ln \left[|z - z'| \left| z - \bar{z}'^{-1} \right| \right]. \tag{2.76}$$

Using the parameterization $z = re^{i\tau}$, $z' = r'e^{i\tau'}$ and denoting $\zeta = \tau - \tau'$, because there is only one boundary in analogy with the calculations done for the annulus it is necessary to calculate the propagator restricted to the disk's edge. With $r = r' = 1$ the propagator immediately becomes

$$G(\tau, \tau') = -\frac{1}{2\pi} \ln (2 - 2\cos\zeta), \tag{2.77}$$

and using the identity

$$\ln \left(1 + b^2 - 2b\cos\zeta \right) = -2 \sum_{n=1}^{\infty} \frac{b^m}{m} \cos m\zeta, \tag{2.78}$$

then

$$G(\tau, \tau') = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\cos m\zeta}{m}. \tag{2.79}$$

This expression also has an analogy in terms of the annulus, where the matrix $A_m = 1$ in this case. This considerable simplification makes the closed form for the disk simpler than that for the annulus.

Now it is necessary to calculate an arbitrary Feynman diagram, and exactly as in the case of the annulus, the n^{th} diagram is

$$n^{th} = \frac{1}{n!} (-e\pi\alpha')^n \, Tr(F^n) \, (n-1)! 2^n \int d\tau_1 \ldots d\tau_n \frac{\partial}{\partial\tau_1} G(\tau_1, \tau_2) \Omega \ldots \frac{\partial}{\partial\tau_n} G(\tau_n, \tau_1) \Omega. \tag{2.80}$$

The term 2^n comes from the observation that each vertex has two equivalent bosonic legs, and no matter how they are connected the diagram can be restored to the form above by and integration by parts. Using the above expression for the propagator this can be re-expressed as

$$\begin{aligned}
&= \frac{1}{n}\left(-2e\pi\alpha'\right)^n Tr(F^n) \sum_{m_1\ldots m_n} \frac{1}{\pi^n}\int d\tau_1\ldots d\tau_n \sin m_1(\tau_1-\tau_2)\times \\
&\quad \ldots \sin m_n(\tau_n-\tau_1) A_{m_1}\Omega\ldots A_{m_n}\Omega \\
&= \frac{1}{n}\left(-2e\pi\alpha'\right)^n Tr(F^n) \sum_{m>0} \left(A_m\Omega\right)^n . \qquad (2.81)
\end{aligned}$$

So the sum of the connected vacuum diagrams is

$$\begin{aligned}
\sum connected &= \sum_{n\geq 1}\sum_{m\geq 1} \frac{1}{n}\left(-2e\pi\alpha'\right)^n Tr(F^n)\left(A_m\Omega\right)^n \\
&= \sum_{m\geq 1} -Tr\ln\left(\delta_{\mu\nu}+2e\pi\alpha' F_{\mu\nu}A_m\Omega\right). \qquad (2.82)
\end{aligned}$$

So that

$$\begin{aligned}
e^{\sum connected} &= \prod_{m\geq 1}\exp\left(-Tr\ln\left(\delta_{\mu\nu}+2e\pi\alpha' F_{\mu\nu}A_m\Omega\right)\right) \\
&= \prod_{m\geq 1}\frac{1}{\det\left(\delta_{\mu\nu}+2e\pi\alpha' F_{\mu\nu}A_m\Omega\right)}. \qquad (2.83)
\end{aligned}$$

But since the factors A_m and Ω are trivial, this becomes

$$= \sqrt{\det\left(\delta_{\mu\nu}+2e\pi\alpha' F_{\mu\nu}\right)} \qquad (2.84)$$

by ζ function regularization, and this is exactly the Born-Infeld action.

It is worth noting that the expression for free energy of the annulus exactly reproduces this in the limit where $a \to 0$ and $e_2 \to 0$, which corresponds to the disk amplitude.

2.3.2 Fermion Vacuum Energy

It is more difficult to place fermions conventionally on the disk, because while the topology had a Greens function that satisfied the two dimensional Laplacian, the mapping to appropriate boundary conditions is non-trivial for the fermions. However, since the question has already been solved for the annulus, and since for the bosons one can get the disk amplitude from the annulus amplitude by taking the limits $a \to 0$ and $e_2 \to 0$, it is reasonable to suppose that the same procedure will work for the fermions. In this limit the energy for the fermions from the Ramond sector (2.59) is

$$e_R^{\sum connected} = \prod_m \det\left(\delta_{\mu\nu} + 2\alpha'\pi e_1 F_{\mu\nu}\right), \tag{2.85}$$

and equally the sum in the Neveu-Schwarz sector is

$$e_{NS}^{\sum connected} = \prod_m \det\left(\delta_{\mu\nu} + 2\alpha'\pi e_1 F_{\mu\nu}\right), \tag{2.86}$$

so the contribution from either sector is equal, which confirms the naive expectation that there should only be one way to include fermions on such a simple topology. After ζ function regularization the fermion contribution becomes

$$= \frac{1}{\sqrt{\det\left(\delta_{\mu\nu} + 2e\pi\alpha' F_{\mu\nu}\right)}}. \tag{2.87}$$

This means that for superstrings in a background $U(1)$ field the disk amplitude becomes trivial, and the tachyonic divergences are cancelled.

2.4 The Bosonic Mobius Strip

The mobius strip is very similar to the annulus, and the propagator and vacuum energy can be calculated similarly. The fields must obey the same Neumann boundary conditions as on the annulus, with the only significant difference coming from the topological boundary condition

$$G(\rho, \phi, \rho', \phi') = G(\ln a - \rho, \phi + 2\pi, \rho', \phi'). \tag{2.88}$$

This has the effect of suppressing the odd modes in ρ and ρ', which leaves the Greens function in the form

$$G(\rho, \phi, \rho', \phi') = \frac{\alpha'}{\ln a} \sum_{m,k \neq 0,0} \frac{1}{m^2 + \left(\frac{2\pi k}{\ln a}\right)^2} e^{im(\phi-\phi')} \cos\frac{2\pi k}{\ln a}\rho \cos\frac{2\pi k}{\ln a}\rho', \tag{2.89}$$

which is exactly the same form as the propagator for the annulus, with the multiplicative change in the constant multiplying k.

When restricted to the edge of the strip, of which there is only one, we find that

$$\begin{aligned} G(0,\phi,0,\phi') &= G(\ln a,\phi,0,\phi') = G(0,\phi,\ln a,\phi') = G(\ln a,\phi,\ln a,\phi') \\ &= 4\alpha' \sum_{m>0} \frac{\cos m(\phi-\phi')}{m} coth \frac{m \ln a}{2}, \end{aligned} \tag{2.90}$$

and so, following the exact same steps as in the case of the disk we find that the partition function, given by

$$\begin{aligned} Z = e^{\sum conn} &= \prod_{m>0} \exp\left(-Tr\ln(\delta_{\mu\nu} + 2\cdot 2\pi\alpha' e F_{\mu\nu} coth \frac{m\ln a}{2})\right) \\ &= \prod_{m>0} \left(det\left[\delta_{\mu\nu} + 2\cdot 2\pi\alpha' e F_{\mu\nu} coth \frac{m\ln a}{2}\right]\right)^{-1}. \end{aligned} \tag{2.91}$$

Several things can be observed. In the limit $a \to 0$ this reproduces the form of the Born-Infeld action for the field $F_{\mu\nu}$. The additional 2 multiplying the field strength can be thought of as encoding the information that the mobius strip has an edge twice as long as that of the disk, and the lack of a F^2 term indicates that there is one and only one edge, in contrast to the annulus where the "squaring" of the Born-Infeld action is precisely because of the existence of two edges.

Chapter 3

Non-Commutative Geometry

3.1 Introduction

Non-commutative geometry is a subject that has received a great deal of attention in the past few years [13] [14] [15] [16] [17] [18] [19]. It is interesting for two particular reasons. The first is that field theory on non-commutative spaces are currently being studied because of interesting properties such as UV/IR mixing, and the second is that if the X^μ fields on the string world-sheet are regarded as the coordinates of the string embedded into a larger space it can be shown that they are themselves non-commutative.

The general form of the non-commutativity among the coordinates is encoded in the non-commutativity parameter $\theta^{\mu\nu}$ which is defined by the relation

$$[X^\mu, X^\nu] \;=\; i\theta^{\mu\nu}, \tag{3.1}$$

and it is taken to be a constant. The algebra of the functions on the non-commutative space is governed by the Moyal product, defined as

$$f * g(x) \;=\; e^{\frac{i}{2}\theta^{\mu\nu}\partial_{\xi^\mu}\partial_{\zeta^\nu}} f(x+\xi)g(x+\zeta)|_{\xi=\zeta=0}, \tag{3.2}$$

which is associative and non-commutative to order θ. The way that non-commutative geometry appears is by considering the end-points of open strings, for which it is possible to derive an exact propagator in the presence of a antisymmetric background. Non-commutativity appears in the guise of coordinate dependent coefficients in the OPE.

The purpose of this section is to derive the non-commutativity of space time coordinates in the perturbative framework developed previously for the calculation of the partition function for strings in an electro-magnetic field background.

The methodology is as follows, given the action for a string ending on a Dp-brane with a constant gauge field confined to the brane and an antisymmetric $B_{\mu\nu}$ field in the bulk of target space, the two fields appear in a gauge invariant combination, which is treated as a perturbation which gives rise to an interaction term in the two dimensional field theory of the bosons on the string world sheet exactly as in the previous section. The modification to the X field propagator due to an arbitrary number of such interactions is calculated, and using a technique due to [14] the non-commutative nature of the Dp-brane is revealed both at the disk and annulus levels. This is generalized for the case of a string ending on two separate branes.

It is not surprising that there should be some modification to the standard commutation relations given an antisymmetric background. The well known problem of the motion of a charged particle in a constant magnetic field background gives rise to a non-trivial commutation relation with the anticommutativity parameter depending on the magnetic field strength. Indeed, it is noted in [14] the $\alpha' \to 0$ limit of string theory, which is equivalent to that of a large ($\mathcal{F} \gg 1$) background field, has an action which is exactly that describing the motion of electrons in a large background field. It is shown [14] [20] that the dynamics of the X fields is that of electrons in the lowest Landau level, and the commutation relations, derived later, reduce to

$$[X^\mu, X^\nu] = \theta^{\mu\nu} = \left(\frac{1}{\mathcal{F}}\right)^{\mu\nu} \tag{3.3}$$

in the large field limit. Correspondingly, the noncommutativity parameter,

$$\theta^{\mu\nu} = \left(\frac{\mathcal{F}}{1-\mathcal{F}^2}\right)^{\mu\nu} \tag{3.4}$$

goes in the limit of small fields ($\mathcal{F} \ll 1$) to $\theta = \mathcal{F}$ which illustrates the fact that commutativity should be restored in the case with no background field.

3.2 Disk Calculations

It is well known that the action for a fundamental string ending on a Dp-brane in the presence of a B-field, with a gauge field confined to the brane is [21] [22]

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \left[g^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu + \pi\alpha'\epsilon^{\alpha\beta}B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu\right] + \int d\tau A_\mu \partial_\tau X^\mu, \tag{3.5}$$

and since the term involving the B field is a total derivative, and in the approximation of constant B and F fields the last two terms can be combined in the form

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma g^{\alpha\beta}G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + \frac{1}{2}\int d\tau \mathcal{F}_{\mu\nu}X^\nu\partial_\tau X^\mu, \tag{3.6}$$

with

$$\mathcal{F}_{\mu\nu} = B_{\mu\nu} - F_{\mu\nu} \tag{3.7}$$

as the gauge invariant field strength that appears in the Born-Infeld action. Using the gauge transformations

$$\begin{gathered} A \to A + d\lambda, \quad B \to B \\ A \to A + \lambda, \quad B \to B + d\lambda \end{gathered} \tag{3.8}$$

it is possible to leave only components of $\mathcal{F}$ which are on the brane.

It is equally well know that for the disk amplitude the world-sheet of the string can be conformally transformed into a unit disk on the complex plane, and using the propagator from the previous section

$$G(z,z') = -\frac{1}{2\pi}\ln|z-z'|\left|z-\bar{z}'^{-1}\right|. \tag{3.9}$$

When this is restricted to the boundaries $|z| = |z'| = 1$ it becomes

$$G(e^{i\phi}, e^{i\phi'}) = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\cos m(\phi - \phi')}{m}, \tag{3.10}$$

so using the standard Feynman rules for the now interacting field theory the correction to the propagator from N interactions with the external field is

$$\begin{aligned} G_N(\rho e^{i\phi}, \rho' e^{i\phi'}) &= \int d\theta_1 \ldots d\theta_N G(\rho e^{i\phi}, e^{i\theta_1}) \partial_{\theta_1} G(e^{i\theta_1}, e^{i\theta_2}) \ldots \partial_{\theta_N} G(e^{i\theta_N}, \rho' e^{i\phi'}) \\ &= (-2\pi\alpha' e)^N \left(\mathcal{F}^N\right)^{\mu\nu} \int d\theta_1 \ldots d\theta_N \frac{1}{\pi^{N+1}} \sum_{m m_1 \ldots m_N} \frac{\rho^m}{m} \cos m(\phi - \theta_1) \\ &\quad \times \sin m_1(\theta_1 - \theta_2) \sin m_2(\theta_2 - \theta_3) \ldots \frac{\rho'^{m_N} + \rho'^{-m_N}}{2} \sin m_N(\theta_N - \phi') \\ &= (-2\pi\alpha' e)^N \left(\mathcal{F}^N\right)^{\mu\nu} \frac{1}{\pi} \sum_m \frac{\rho^m}{m} \frac{\rho'^m + \rho'^{-m}}{2} \left\{ \begin{array}{ll} \cos m(\phi - \phi') & N even \\ \sin m(\phi - \phi') & N odd \end{array} \right\}. \end{aligned} \tag{3.11}$$

Then summing the contributions from all the values of N gives the propagator as

$$\begin{aligned} G_{int}(\rho e^{i\phi}, \rho' e^{i\phi'}) &= \left(\frac{1}{1 - (2\pi\alpha' e)^2 \mathcal{F}^2}\right)^{\mu\nu} \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{(\rho\rho')^m + (\rho/\rho')^m}{m} \cos m(\phi - \phi') N \ even \\ &\quad + \left(\frac{-2\pi\alpha' e \mathcal{F}}{1 - (2\pi\alpha' e)^2 \mathcal{F}^2}\right)^{\mu\nu} \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{(\rho\rho')^m + (\rho/\rho')^m}{m} \sin m(\phi - \phi') N \ odd. \end{aligned} \tag{3.12}$$

The only thing that remains is to calculate the equal time commutator of the X fields, and to do that using the technique due to Witten's paper,

$$[X^\mu(\tau), X^\nu(\tau)] = \lim_{\epsilon \to 0} \mathcal{T} \left(X^\mu(\tau) X^\nu(\tau - \epsilon) - X^\mu(\tau) X^\nu(\tau + \epsilon) \right), \tag{3.13}$$

where $\mathcal{T}$ represents time ordering. In the case $\rho = \rho' = 1$ the sine series above is nothing but a step function in the limit $\epsilon \to 0$, while for either ρ or ρ' not on the boundary this converges to something proportional to $\tan^{-1}\left(\frac{2\rho\rho' \sin\epsilon}{1-(\rho\rho')^2}\right)$, which vanishes in the limit

$\epsilon \to 0$. This shows that for the end-points of the open string at the disk amplitude there is a non-trivial commutation relation, which is the signature of non-commutative geometry on the brane, and this relation is, after rescaling the Xs by $\sqrt{2\pi\alpha'}$ to restore the fields to a form comparable to those used in analyzing the annulus,

$$[X^\mu(\tau), X^\nu(\tau)] = 2\pi\alpha' \left(\frac{-2\pi\alpha' e\mathcal{F}}{1-(2\pi\alpha' e)^2\mathcal{F}^2}\right)^{\mu\nu}. \tag{3.14}$$

This reproduces the non-commutative geometry on the brane [14] [21], and it also demonstrates that the non-commutative geometry is in fact a perturbative effect, and not due to instantons, or other non-perturbative effects.

3.3 One Loop Correction to Non-Commutative Geometry

Since the foundation of the non-commutative geometry is given in tree level diagrams, it is interesting to consider the corrections that come from one loop string diagrams. In this section one loop diagrams are discussed with two main thrusts, the first being to examine the effect of an open string stretched between two branes, and the second, to give the one loop correction to the commutator found at the disk level for the X^μ fields.

As is developed elsewhere, the free propagator for the X fields on the annulus with Teichmuller parameter $\ln a$, which has been conformally transformed to a cylinder is (2.7)

$$\begin{aligned} G(\rho,\phi,\rho',\phi') &= \frac{2\alpha'}{\ln a}\left(\sum_{m>0}\sum_n \cos m(\phi-\phi')\cos\frac{n\pi\rho}{\ln a}\cos\frac{n\pi\rho'}{\ln a}\frac{1}{m^2+\left(\frac{n\pi}{\ln a}\right)^2}\right. \\ &\qquad \left. +2\pi\alpha'\sum_{n>0}\cos\frac{n\pi\rho}{\ln a}\cos\frac{n\pi\rho'}{\ln a}\left(\frac{\ln a}{n\pi}\right)^2\right), \end{aligned} \tag{3.15}$$

and as discussed in the calculation of the partition function of the string, this propagator can be re-cast as a matrix valued propagator of only one variable when the interaction happens only at the edges. The Lagrangian for the string coupled to the two branes is the same as that given before, with the $\mathcal{F}_i$ being the gauge invariant field strength on the i^{th} brane. In this language, the N^{th} diagram for the propagator on the annulus $\langle X^i(\rho,\tau)X^j(\rho',\tau')\rangle$ is

$$\begin{aligned}
G_N(\rho,\tau,\rho',\tau') &= \frac{1}{N!}N!\int d\theta_1\ldots d\theta_N\,(G(\rho,\tau,0,\theta_1),G(\rho,\tau,\ln a,\theta_1)) \\
&\quad\times\begin{pmatrix} e_1\mathcal{F}_1^{ik_1} & 0 \\ 0 & -e_2\mathcal{F}_2^{ik_1}\end{pmatrix}\partial_{\theta_1}M(\theta_1,\theta_2)\times\ldots\times\begin{pmatrix} e_1\mathcal{F}^j_{1k_{N-1}} & 0 \\ 0 & -e_2\mathcal{F}^j_{2k_N}\end{pmatrix} \\
&\quad\times\partial_{\theta_N}\begin{pmatrix} G(0,\theta_N,\rho',\tau') \\ G(\ln a,\theta_N,\rho',\tau')\end{pmatrix}, \qquad (3.16)
\end{aligned}$$

where

$$M(\theta,\theta')=\sum_{m>0}\frac{\cos m(\theta-\theta')}{m}\begin{pmatrix} cothm\ln a & cschm\ln a \\ cschm\ln a & cothm\ln a\end{pmatrix}. \qquad (3.17)$$

The integration is trivial and gives

$$\begin{aligned}
G_N(\rho,\tau,\rho',\tau') &= (2\alpha')^{N+1}(-\pi)^N\frac{1}{\ln^2 a}\sum_{m>0}\begin{Bmatrix} \sin m(\tau-\tau') & Nodd \\ \cos m(\tau-\tau') & Neven\end{Bmatrix} \\
&\quad\times\sum_n\left(\cos\frac{n\pi\rho}{\ln a}\frac{1}{m^2+\left(\frac{n\pi}{\ln a}\right)^2},\cos\frac{n\pi\rho}{\ln a}\frac{(-1)^n}{m^2+\left(\frac{n\pi}{\ln a}\right)^2}\right)\begin{pmatrix} e_1\mathcal{F}_1^{ik_1} & 0 \\ 0 & -e_2\mathcal{F}_2^{ik_1}\end{pmatrix} \\
&\quad\times\begin{pmatrix} cothm\ln a & cschm\ln a \\ cschm\ln a & cothm\ln a\end{pmatrix}\ldots\begin{pmatrix} cothm\ln a & cschm\ln a \\ cschm\ln a & cothm\ln a\end{pmatrix} \\
&\quad\times\begin{pmatrix} e_1\mathcal{F}^j_{1k_{N-1}} & 0 \\ 0 & -e_2\mathcal{F}^j_{2k_N}\end{pmatrix}\sum_{n'}\begin{pmatrix} \cos\frac{n'\pi\rho'}{\ln a}\frac{m}{m^2+\left(\frac{n'\pi}{\ln a}\right)^2} \\ \cos\frac{n'\pi\rho'}{\ln a}\frac{m(-1)^{n'}}{m^2+\left(\frac{n'\pi}{\ln a}\right)^2}\end{pmatrix}. \qquad (3.18)
\end{aligned}$$

Note that this expression reproduces and generalizes the non-commutativity relation found by Chu [22] for strings between two branes, namely that

$$\begin{aligned}
[X^\mu,X^\nu] &= \left(\frac{\pi\alpha' 2\pi\alpha' e_1\mathcal{F}_1}{1-(2\pi\alpha' e_1\mathcal{F})_1^2}\right)^{\mu\nu} \quad \rho=\rho'=0 \\
&= \left(\frac{\pi\alpha' 2\pi\alpha' e_2\mathcal{F}_2}{1-(2\pi\alpha' e_2\mathcal{F})_2^2}\right)^{\mu\nu} \quad \rho=\rho'=\ln a \\
&= 0 \ \ otherwise. \qquad (3.19)
\end{aligned}$$

The generalization is that there is no requirement that the fields on the two branes are aligned in any way with each other, which is no surprise, but reassuring to discover. The limit in which the above expression reproduces Chu's result is in the limit $a \to 0$. The way to understand this is to note that this limit corresponds to the disk limit, since a is the Teichmuller parameter for the annulus. Chu's results rely upon a time average to calculate commutators, and this average is taken over non-compact time (on the string world-sheet) which corresponds to an infinite string world-sheet extending between the branes in question. This is nothing but a disk amplitude.

The last thing to note is that in the most general case, with two branes with independent $\mathcal{F}$ fields on them the commutator of the X fields will generally depend on both the $\mathcal{F}$s in a non-trivial way. This means that in a Randall-Sundrum type scenario in which there are two branes, with us living on one, subtle deviations in the non-commutativity parameter would enable the measurement of the $\mathcal{F}$ field on the other brane, which is an exciting possibility.

Now, consider a string beginning and ending on the same D-brane. This brane has a gauge field with field strength F on it, and there is a B field in the background. The commutation relation for the X fields will be determined exclusively by the values of the string endpoint charges. The dependence on the background fields occurs in the combination $\mathcal{F} = B - F$ which is gauge invariant. The action is re-written so that the X and $\mathcal{F}$ fields have a derivative interaction at the edge of the string world-sheet. This process in itself suggests that the string coupling to the $\mathcal{F}$ should be the same strength at each end because the bulk interaction with the B field is a total derivative, and appears with equal coefficients when rewritten as a surface term on the world-sheet. This is in direct analogy with the procedure used previously to determine the string partition function.

3.3.1 The Charged String

For a charged string with equal coupling at each end, we have $e_1 = -e_2 = e$ and the above expression (3.18) can be modified. With equal charges and $\rho = \rho' = 0$ analysis identical to that for the partition function that we've discussed before gives that

$$\begin{aligned}\langle X^i(0,\phi)X^j(0,\phi')\rangle_N &= 2\alpha'(2\pi\alpha')^N \left(e^N\mathcal{F}^N\right)^{ij}\sum_m \frac{1}{m}\left\{\begin{array}{ll}\sin m(\phi-\phi') & N\ odd\\ \cos m(\phi-\phi') & N\ even\end{array}\right\}\\ &\times\frac{1}{2}\left(\frac{1}{1-a^{2m}}\right)^{N+1}\left(1+a^{2m},2a^m\right)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &\times\begin{pmatrix}(1+2a^m+a^{2m})^{N-1} & 0\\ 0 & (1-2a^m+a^{2m})^{N-1}\end{pmatrix}\\ &\times\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\left(1+a^{2m},2a^m\right),\end{aligned} \tag{3.20}$$

and so

$$\begin{aligned}\langle X^i(0,\phi)X^j(0,\phi')\rangle_N &= 2\alpha'(2\pi\alpha')^N \left(e^N\mathcal{F}^N\right)^{ij}\sum_m \frac{1}{m}\left\{\begin{array}{ll}\sin m(\phi-\phi') & N\ odd\\ \cos m(\phi-\phi') & N\ even\end{array}\right\}\frac{1}{2}\left(\frac{1}{1-a^{2m}}\right)^{N+1}\\ &\left((1+2a^m+a^{2m})^{N+1}+(1-2a^m+a^{2m})^{N+1}\right).\end{aligned} \tag{3.21}$$

The same calculation for $\rho = \rho' = \ln a$ gives an identical result. The results for $\rho = 0, \rho' = \ln a$ and $\rho = \ln a, \rho' = 0$ are equal to each other with

$$\begin{aligned}\langle X^i(0,\phi)X^j(\ln a,\phi')\rangle_N &= 2\alpha'(2\pi\alpha')^N \left(e^N\mathcal{F}^N\right)^{ij}\sum_m \frac{1}{m}\left\{\begin{array}{ll}\sin m(\phi-\phi') & N\ odd\\ \cos m(\phi-\phi') & N\ even\end{array}\right\}\\ &\times\frac{1}{2}\left(\frac{1}{1-a^{2m}}\right)^{N+1}\left((1+2a^m+a^{2m})^{N+1}-(1-2a^m+a^{2m})^{N+1}\right).\end{aligned} \tag{3.22}$$

Thus, the sum of these four contributions, which must be taken to evaluate the commutator gives

$$\langle X^i(\phi)X^j(\phi')\rangle_N = 2\alpha'(2\pi\alpha')^N \left(e^N\mathcal{F}^N\right)^{ij}\sum_m \frac{1}{m}\left\{\begin{array}{ll}\sin m(\phi-\phi') & N\ odd\\ \cos m(\phi-\phi') & N\ even\end{array}\right\}2\left(\frac{1+a^m}{1-a^m}\right)^{N+1}.$$

(3.23)

Since in the end, a commutator is going to be evaluated, it is obvious that the terms which have even N will cancel, while the odd N terms add. Summing over all odd N and integrating over the Teichmuller parameter one obtains

$$\left[X^i, X^j\right] = \lim_{\epsilon \to 0_+} \int \frac{da}{a} \sum_m \frac{1}{m} \sin m(\epsilon) \times 4\alpha' \frac{fk^2}{1 - f^2k^2}, \tag{3.24}$$

where $f = 2\pi\alpha' e\mathcal{F}$, and $k = \frac{1+a^m}{1-a^m}$. The sum of sine terms is the fourier series for a sawtooth wave, and so the limit $\epsilon \to 0$ can be evaluated as

$$\left[X^i, X^j\right] = \int \frac{da}{a} 2\pi\alpha' \frac{fk^2}{1 - f^2k^2}. \tag{3.25}$$

This integral has two poles, one as $a \to 0$, and one near $a = 1$. The second is a simple pole, and the contributions of the integral on either side of it cancel, and the pole at $a = 0$ can be regulated by introducing a small cutoff, and then up to constant terms

$$\begin{aligned} \left[X^i, X^j\right] &= \int_\epsilon^1 \frac{da}{a} 2\pi\alpha' \frac{fk^2}{1 - f^2k^2} \\ &= -2\pi\alpha' \frac{f}{1 - f^2} \ln \epsilon, \end{aligned} \tag{3.26}$$

Thus the contribution at one loop level from a charged string to the non-commutative geometry is a renormalization effect. The divergence is the same encountered in the free energy for the free annulus, which is interpreted in [6] as originating from the open string tachyon.

3.3.2 The Neutral String

For the neutral string, a very similar process can be followed. In particular, taking the initial form of the N^{th} interaction, and substituting $e_1 = e_2 = e$, which is a neutral string, one obtains

$$\begin{aligned}
\langle X^i(\rho,\phi)X^j(\rho',\phi')\rangle_N &= 2\alpha'(2\pi\alpha' e)^N \left(\mathcal{F}^N\right)^{ij} \sum_m \frac{1}{m} \begin{Bmatrix} \sin m(\phi-\phi') & N\ odd \\ \cos m(\phi-\phi') & N\ even \end{Bmatrix} \\
&\times \left(f_m(\rho,0), f_m(\rho,\ln a)\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & a^m \\ -a^m & -1 \end{pmatrix} \\
&\begin{pmatrix} (1-a^{2m})^{N-1} & 0 \\ 0 & (-1)^{N-1}(1-a^{2m})^{N-1} \end{pmatrix} \begin{pmatrix} 1 & a^m \\ -a^m & -1 \end{pmatrix} \begin{pmatrix} f_m(0,\rho') \\ f_m(\ln a,\rho') \end{pmatrix} \\
&\times \left(\frac{1}{1-a^{2m}}\right)^{N-1} \left(\frac{1}{1-a^{2m}}\right). \qquad (3.27)
\end{aligned}$$

Since the properties which determine whether a particular N survives the difference in the commutator depends only on f and sin, and from previous experience we know that only the odd N survive, in the following we only consider odd N. For the case of $\rho = \rho' = 0$ the expression simplifies to

$$\begin{aligned}
\langle X^i(0,\phi)X^j(0,\phi')\rangle_N &= 2\alpha'(2\pi\alpha' e)^N \left(\mathcal{F}^N\right)^{ij} \sum_m \frac{1}{m} \sin m(\phi-\phi') \left(\frac{1}{1-a^{2m}}\right)^3 \\
&\times \left(1+a^{2m}, 2a^m\right) \begin{pmatrix} 1-a^{2m} & 0 \\ 0 & -(1-a^{2m}) \end{pmatrix} \begin{pmatrix} 1+a^{2m} \\ 2a^m \end{pmatrix} \\
&= 2\alpha' 2\pi\alpha' e)^N \left(\mathcal{F}^N\right)^{ij} \sum_m \frac{1}{m} \sin m(\phi-\phi') \left(\frac{1}{1-a^{2m}}\right)^2 \\
&\times \left(\left(1+a^{2m}\right)^2 - 4a^{2m}\right). \qquad (3.28)
\end{aligned}$$

Following similar logic, we find that for $\rho = \rho' = \ln a$,

$$\begin{aligned}
\langle X^i(\ln a,\phi)X^j(\ln a,\phi')\rangle_N &= 2\alpha'(2\pi\alpha' e)^N \left(\mathcal{F}^N\right)^{ij} \sum_m \frac{1}{m} \sin m(\phi-\phi') \left(\frac{1}{1-a^{2m}}\right)^2 \\
&\left(-\left(1+a^{2m}\right)^2 + 4a^{2m}\right), \qquad (3.29)
\end{aligned}$$

and for $\rho = 0, \rho' = \ln a$, the result is

$$\langle X^i(0,\phi)X^j(\ln a,\phi')\rangle_N = 2\alpha'(2\pi\alpha' e)^N \left(\mathcal{F}^N\right)^{ij} \sum_m \frac{1}{m} \sin m(\phi-\phi') \left(\frac{1}{1-a^{2m}}\right)^2$$

$$\begin{aligned} & \times\left(\left(1+a^{2m}\right)2a^m - 2a^m\left(1+a^{2m}\right)\right) \\ = \; & 0. \end{aligned} \tag{3.30}$$

So since the four contributions must be summed to obtain the annulus correction we see explicitly that the contribution at each order in perturbation theory from the neutral string to the commutation relations vanishes.

3.3.3 Distinct Branes

As noted above for an open string stretched between two branes, in the limit $a \to 0$ the non-commutativity parameter is $\theta = \frac{\mathcal{F}}{1-\mathcal{F}^2}$ where $\mathcal{F}$ is the gauge invariant field strength appropriate to the brane. This will, of course, be modified when not in the disk amplitude limit. Since the field strengths on the two branes are not *a priori* related to each other, it is unreasonable to expect that they have similar decomposition or block diagonal forms. Here, we examine the limit $\mathcal{F}_2 \ll \mathcal{F}_1$, which gives leading order corrections to the non-commutativity parameter.

With $\mathcal{F}_2 \ll \mathcal{F}_1$ we find that the contribution of the N interactions with either background field gives

$$\begin{aligned} G_N^{ij}(\phi,\phi') \;=\; & 2\alpha'\left(2\pi\alpha'\right)^N \sum_m \frac{1}{m}\left\{\begin{array}{ll} \sin m(\phi-\phi') & N\ odd \\ \cos m(\phi-\phi') & N\ even \end{array}\right\} \\ & \times coth^{N-1} m \ln a \left[coth^2 m \ln a \left(\mathcal{F}_1^N\right)^{ij}\right. \\ & \left. + csch^2 m \ln a \left(\mathcal{F}_2\mathcal{F}_1^{N-1} + \mathcal{F}_1\mathcal{F}_2\mathcal{F}_1^{N-2}\ldots\mathcal{F}_1^{N-1}\mathcal{F}_2\right)^{ij}\right]. \end{aligned} \tag{3.31}$$

Since the expression multiplying $csch^2 m \ln a$ is manifestly odd for odd N, and even for even N it is clear that following the same procedure as above

$$\begin{aligned} \left[X^\mu, X^\nu\right]_N \;=\; & \pi\alpha'\left(2\pi\alpha'\right)^N coth^{N-1} m \ln a \left[coth^2 m \ln a \left(\mathcal{F}_1^N\right)^{\mu\nu}\right. \\ & \left. + csch^2 m \ln a \left(\mathcal{F}_2\mathcal{F}_1^{N-1} + \mathcal{F}_1\mathcal{F}_2\mathcal{F}_1^{N-2}\ldots\mathcal{F}_1^{N-1}\mathcal{F}_2\right)^{\mu\nu}\right]. \end{aligned} \tag{3.32}$$

Now, making the assumption that $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same block diagonal form we obtain.

$$\begin{aligned}[X^\mu, X^\nu] &= \int \frac{da}{a} \pi\alpha' \left[\left(\frac{2\pi\alpha' \mathcal{F}_1 coth^2 m \ln a}{1 - (2\pi\alpha'\mathcal{F}_1)^2 coth^2 m \ln a} \right)^{\mu\nu} \right. \\ &\quad \left. + \left(\frac{1 + (2\pi\alpha'\mathcal{F}_1)^2 coth^2 m \ln a}{1 - (2\pi\alpha'\mathcal{F}_1)^2 coth^2 m \ln a} csch^2 m \ln a \mathcal{F}_2 \right)^{\mu\nu} \right]. \end{aligned} \tag{3.33}$$

Using the same techniques as before we can see that upon integration over the Teichmuller parameter, there will be the same divergent structure for the full commutator as in the case of the charged string.

3.4 Non-Commutative Geometry from the Mobius Strip

The mobius strip is the simplest non-orientable surface that can be incorporated in string theory, and as such it is interesting to study its effects in the context of non commutative geometry, since it will give the contribution of non-planar diagrams. As was derived previously, the propagator for bosons on the annulus is

$$G(\rho, \phi, \rho', \phi') = \frac{\alpha'}{\ln a} \sum_{m,k \neq 0,0} \frac{1}{m^2 + \left(\frac{2\pi k}{\ln a}\right)^2} e^{im(\phi - \phi')} \cos \frac{2\pi k}{\ln a} \rho \cos \frac{2\pi k}{\ln a} \rho', \tag{3.34}$$

and following the procedure given for the annulus, it can be shown that the correction to this due to N interactions with the background field can be written as

$$\begin{aligned}\langle X^i X^j \rangle_N &= 2\alpha' \left(-2\pi\alpha' e\right)^N \left(\mathcal{F}^N\right)^{ij} \sum_{mkk'} \left(\frac{1}{\ln a}\right)^2 \begin{Bmatrix} \sin m(\phi - \phi') N \ odd \\ \cos m(\phi - \phi') N \ even \end{Bmatrix} \left(2coth \frac{m \ln a}{2}\right)^{N-1} \\ &\quad \times m \frac{\cos \frac{2\pi k}{\ln a}\rho}{m^2 + \left(\frac{2\pi k}{\ln a}\right)^2} \frac{\cos \frac{2\pi k'}{\ln a}\rho}{m^2 + \left(\frac{2\pi k'}{\ln a}\right)^2}, \end{aligned} \tag{3.35}$$

and similarly to the case for the disk, for $\rho, \rho' \neq 0, \ln a$ this will identically vanish for $\phi \rightarrow \phi'$.

Now, calculating the commutator of the fields on the endpoint, we find that after summing the contributions of all possible interactions with the background field for a particular value of a,

$$\left[X^i, X^j\right] = \lim_{\epsilon \to 0} \sum_m \frac{\sin m\epsilon}{m} 2\alpha' \left(\frac{fk^2}{1 - f^2k^2} \right)^{ij}, \tag{3.36}$$

where $k = 2coth\frac{m \ln a}{2}$ and $f = 2\pi\alpha' e\mathcal{F}$. This is clearly of the same form as the case of the annulus, and so when integrating over the parameter a, we see a logarithmic divergence near $a = 0$ and the pole for the critical value of a near $a = 1$ does not contribute any infinite pieces to the integral, because it is a simple pole. Thus, we see that the contribution to non-commutativity from the mobius strip is the same as that of the charged annulus.

Chapter 4

Conclusion

There are two major themes that are presented in this thesis. The first is the verification of previous work on the vacuum energy of strings. The contributions from both the disk and the annulus for the bosonic degrees of freedom were analyzed and found to be in exact agreement with the results presented by Fradkin and Tseytlin [1]. The fermions on the world-sheet were similarly analyzed and their contribution to the vacuum energy were also in agreement with those of Bachas and Porrati [4] for the disk and annulus, however the method used to obtain these results are distinct from those used in the original paper and thus add confidence in the final results. Finally, the bosonic vacuum energy was found for the Mobius strip, and while it had the same form as that for the disk, namely the Born-Infeld action for the background field, the coefficient multiplying the field-strength encoded the fact that the boundary of the nonorientable surface is twice the length of the boundary of the disk.

Using the mechanism and propagator developed for the calculation of vacuum energies several results pertaining to the non-commutativity of D-brane world-sheets were obtained. It was possible to reproduce the classic disk contribution to non-commutative geometry, and also more recent calculation for non-commutativity from strings stretched between two separate branes. The new results obtained from this analysis were the determination that the uncharged string makes no contribution to the non-commutativity at the annulus level, while for the charged string there is a logarithmic divergence which has as its coefficient the standard non-commutativity parameter, and the origin relations for the Mobius strip are shown to have the same structure as those for the annulus. Perhaps the most significant result is the determination of corrections to non-commutativity on one brane due to a weak independent antisymmetric field on another brane.

There are several opportunities for further research presented by these results. One is the calculation of tachyon and vector particle scattering amplitudes in the presence of an antisymmetric background, using the techniques applied above. Another is the exciting possibility of combining the results for non-commutativity within a Randall-Sundrum framework, which could provide a cosmological explanation for non-commutativity in our world.

Appendix A

Alternate Calculation of World-sheet Propagators

A.1 The Bosonic Propagator

Another method of calculating the bosons green function can be obtained from a path integral formalism on the string world sheet. The action for the boson is written

$$\begin{aligned} S_{boson} &= \frac{-1}{4\pi\alpha'}\int d^2\sigma\partial_a X^\mu \partial^a X_\mu \\ &= \frac{-1}{4\pi\alpha'}\int d\rho d\phi(\partial_\phi X^\mu \partial_\phi X_\mu + \partial_\rho X^\mu \partial_\rho X_\mu) \end{aligned} \tag{A.1}$$

in Euclidean space. Since the boson satisfies Neumann boundary conditions an ansatz is

$$X^\mu = \sum_{n>0,m} f^\mu_{mn} e^{im\phi} \cos\frac{n\pi\rho}{\ln a}. \tag{A.2}$$

Applying this it is clear that

$$\begin{aligned} S &= \frac{-1}{4\pi\alpha'}\int d\rho d\phi \sum_{mnm'n'}\left(-mm'F^\mu_{mn}F_{m'n'\mu}e^{im\phi}\cos\frac{n\pi\rho}{\ln a}e^{im'\phi}\cos\frac{n'\pi\rho}{\ln a}+\right. \\ &\quad \left.\frac{n\pi}{\ln a}\frac{n'\pi}{\ln a}F^\mu_{mn}F_{m'n'\mu}e^{im\phi}\sin\frac{n\pi\rho}{\ln a}e^{im'\phi}\sin\frac{n'\pi\rho}{\ln a}\right) \\ &= \frac{-1}{4}\frac{\ln a}{\alpha'}\sum mnF^\mu_{mn}F_{-mn\mu}\left(m^2+\left(\frac{n\pi}{\ln a}\right)^2\right). \end{aligned} \tag{A.3}$$

Now the fact that X is a real valued field implies that $F_{-mn} = F^*_{mn}$ so

$$S = \frac{-1}{4}\frac{\ln a}{\alpha'}\sum mn\left|F^\mu_{mn}\right|^2\left(m^2+\left(\frac{n\pi}{\ln a}\right)^2\right). \tag{A.4}$$

Now consider the path integral

$$\int [dF]\, e^{-S} = \int [dReF]\,[dImF]\exp\left(\frac{-1}{4}\frac{\ln a}{\alpha'}\sum mn\left|F^\mu_{mn}\right|^2\left(m^2+\left(\frac{n\pi}{\ln a}\right)^2\right)\right), \tag{A.5}$$

with the assumption that the target space metric is Euclidean

$$= \prod_{m>0} \prod_{n>0} \sqrt{\frac{2\alpha'\pi}{\ln a \left(m^2 + \left(\frac{n\pi}{\ln a}\right)^2\right)}}^{-2d}. \tag{A.6}$$

The power of 2 in for the result of the Gaussian integral is due to the degrees of freedom of the real and imaginary parts of F and the d is from the number of free bosons, the dimension of the target space. Now consider the integral

$$\begin{aligned} \int [dF]\, e^{-S} X^\mu(\phi,\rho) X^\nu(\phi',\rho') &= \int [dF] \exp\left(\frac{-1}{4}\frac{\ln a}{\alpha'} \sum mn \left|F^\mu_{mn}\right|^2 \left(m^2 + \left(\frac{n\pi}{\ln a}\right)^2\right)\right) \\ &\quad \times \sum_{ab} F^\mu_{ab} e^{ia\phi} \cos\frac{b\pi\rho}{\ln a} \sum_{cd} F^\nu_{cd} e^{ic\phi} \cos\frac{d\pi\rho}{\ln a}. \end{aligned} \tag{A.7}$$

The only non-vanishing contributions to the integral come from $c = -a$, $b = d$ and $\mu = \nu$, so this becomes

$$= \sum_{mn} e^{im(\phi-\phi')} \cos\frac{n\pi\rho}{\ln a} \cos\frac{n\pi\rho'}{\ln a} \frac{2\alpha'}{\ln a} \frac{1}{m^2 + \left(\frac{n\pi}{\ln a}\right)^2}, \tag{A.8}$$

which is equivalent to the propagator calculated previously.

A.1.1 Equivalence to Published Expressions

The bosonic Greens function was determined above in two different ways, resulting in identical expressions. However, both of these methods are for the cylinder, and it is instructive to verify that they correspond to the the Greens function on the annulus, which can be obtained from complex analysis.

The free propagator on the annulus with Neumann boundary conditions is [1] [2]

$$G(z-z') = \frac{-1}{2\pi}\left[\ln|z-z'|\left|z-\bar{z}'^{-1}\right| + \sum_{n\geq 1} \ln\left[|1-a^{2n}\frac{z}{z'}||1-a^{2n}\frac{z'}{z}||1-a^{2n}z\bar{z}'||1-\frac{a^{2n}}{z\bar{z}'}|\right]\right], \tag{A.9}$$

and it is desired to compare this with the previous result for the Greens function, which, appropriately scaled, is (2.7)

$$\begin{aligned} G(\rho,\phi,\rho',\phi') &= \frac{1}{2\pi\ln a}\sum_{m\neq 0,m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} e^{i(\phi-\phi')m}\cos\frac{n\pi}{\ln a}\rho\cos\frac{n\pi}{\ln a}\rho'\frac{1}{m^2+\left(\frac{n\pi}{\ln a}\right)^2} \\ &\quad +\sum_{n=1}^{\infty}\cos\frac{n\pi}{\ln a}\rho\cos\frac{n\pi}{\ln a}\rho'\left(\frac{\ln a}{n\pi}\right)^2 . \end{aligned} \tag{A.10}$$

This expression has to be converted into a form that facilitates comparison. The first thing to note is that the second term is formally divergent, and further is eliminated from consideration by the fact that the X^μ fields have a derivative coupling to the external field. The above expression, without the $m=0$ modes, is even in n, and can be rewritten as a contour integral:

$$G(\rho,\phi,\rho',\phi') = \frac{1}{2}\sum_m\int dne^{i(\phi-\phi')m}\cos\frac{n\pi}{\ln a}\rho\cos\frac{n\pi}{\ln a}\rho'\frac{1}{\pi\ln a}\times\frac{1}{m^2+\left(\frac{n\pi}{\ln a}\right)^2}\frac{-i}{2}\cot\pi n. \tag{A.11}$$

In the preceding the factor $\frac{1}{2}$ came from using the evenness in n to expand the sum over all integers, and the factor of $\frac{-i}{2}$ is to ensure that the coefficient of the residue at each integer is 1, and the contour is an infinitesimal circle around each element in the sum. The contour can then be continuously deformed to a pair of semi-circles in the upper and lower half plane respectively, that do not overlap the real line. This integral can then be evaluated at the poles $n=\pm\frac{im\ln a}{\pi}$, and the objection to the procedure that the cosine terms diverge at $n=\pm i\infty$ is taken care of by the fact that the cotangent in the expression gives an additional sign change, so the curves at infinity cancel. This implies

$$G(\rho,\phi,\rho',\phi') = \sum_{m>0}\cos m(\phi-phi')\frac{\cosh m\rho\cosh m\rho'}{m\pi\ln a}\coth m\ln a \tag{A.12}$$

is an equivalent expression.

Now explicitly expand the above in terms of ϕ and ρ to obtain

$$=\sum_{m>0}\frac{e^{im(\phi-\phi')}+e^{-im(\phi-\phi')}}{m\pi\ln a}\frac{e^{m\rho}+e^{-m\rho}}{2}\frac{e^{m\rho'}+e^{-m\rho'}}{2}\frac{a^m+a^{-m}}{a^m-a^{-m}}, \tag{A.13}$$

and using the identification $z = e^{\rho + i\phi}$ this becomes

$$\begin{aligned}
&= \sum_{m>0} \frac{1}{m\pi 8 \ln a} \left[(z\bar{z}')^m + (\frac{z}{z'})^m + (\frac{\bar{z}'}{\bar{z}})^m + (\frac{1}{z'\bar{z}})^m + (\bar{z}z')^m \right. \\
&\quad \left. + (\frac{\bar{z}}{\bar{z}'})^m + (\frac{z'}{\bar{z}})^m + (\frac{1}{z\bar{z}'})^m \right] \frac{a^m + a^{-m}}{a^m - a^{-m}} \\
&= \sum_{m>0, n\geq 0} \frac{1}{m\pi 8 \ln a} \left[(z\bar{z}')^m + (\frac{z}{z'})^m + (\frac{\bar{z}'}{\bar{z}})^m + (\frac{1}{z'\bar{z}})^m + (\bar{z}z')^m \right. \\
&\quad \left. + (\frac{\bar{z}}{\bar{z}'})^m + (\frac{z'}{z})^m + (\frac{1}{z\bar{z}'})^m \right] (1 + a^{2m}) a^{2mn}.
\end{aligned} \tag{A.14}$$

Now, doing the sum carefully this becomes

$$\begin{aligned}
&= \sum_{m>0} \frac{1}{m\pi 8 \ln a} \left[(z\bar{z}')^m + (\frac{z}{z'})^m + (\frac{\bar{z}'}{\bar{z}})^m + (\frac{1}{z'\bar{z}})^m + (\bar{z}z')^m \right. \\
&\quad \left. + (\frac{\bar{z}}{\bar{z}'})^m + (\frac{z'}{z})^m + (\frac{1}{z\bar{z}'})^m \right] \\
&\quad + 2 \sum_{m>0, n>0} \frac{1}{m\pi 8 \ln a} \left[(z\bar{z}')^m + (\frac{z}{z'})^m + (\frac{\bar{z}'}{\bar{z}})^m + (\frac{1}{z'\bar{z}})^m + (\bar{z}z')^m \right. \\
&\quad \left. + (\frac{\bar{z}}{\bar{z}'})^m + (\frac{z'}{z})^m + (\frac{1}{z\bar{z}'})^m \right] a^{2mn}.
\end{aligned} \tag{A.15}$$

This expression simplifies to

$$\begin{aligned}
&= \frac{1}{m\pi 8 \ln a} \left[\ln \left(|1 - z\bar{z}'| \left|1 - \frac{z}{z'}\right| \left|1 - \frac{\bar{z}'}{\bar{z}}\right| \left|1 - \frac{1}{z'\bar{z}}\right| \right) \right. \\
&\quad \left. + 2 \sum_{n>0} \ln \left(\left|1 - a^{2n} z\bar{z}'\right| \left|1 - a^{2n}\frac{z}{z'}\right| \left|1 - a^{2n}\frac{\bar{z}'}{\bar{z}}\right| \left|1 - a^{2n}\frac{1}{z'\bar{z}}\right| \right) \right] \\
&= \frac{1}{m\pi 8 \ln a} \left[2 \ln \left(|z - z'| \left|z - \frac{1}{\bar{z}'}\right| \right) - 2 \ln |z| \right. \\
&\quad \left. + 2 \sum_{n>0} \ln \left(\left|1 - a^{2n} z\bar{z}'\right| \left|1 - a^{2n}\frac{z}{z'}\right| \left|1 - a^{2n}\frac{\bar{z}'}{\bar{z}}\right| \left|1 - a^{2n}\frac{1}{z'\bar{z}}\right| \right) \right].
\end{aligned} \tag{A.16}$$

Now, this can be seen to be equal to the expression from [1] apart from the $\ln |z|$ term, which is clearly harmonic in the region. Due to a general property of differential equations any function f that satisfies $\nabla f = 0$ and does not affect the boundary conditions can be added to the Greens function and it is still a Greens function. This demonstrates the equivalence of the two different representations of the Greens function.

A.2 The Fermion Propagator

In close analogy with the case of the boson, a path integral calculation of the fermion propagator can also be done. Again, the starting point for the calculation is the free fermion action, which can be written as

$$S = \frac{-1}{4\pi\alpha'} \int d^2\sigma \left(-i\bar{\psi}^{\mu}\gamma^{a}\partial_a\psi_{\mu} \right). \tag{A.17}$$

In this case the mode expansion for the fermions is

$$\psi(\rho,\phi) = \sum_{mn} \psi_{mn} \begin{pmatrix} e^{in\frac{\pi\rho}{\ln a}} \\ e^{-in\frac{\pi\rho}{\ln a}} \end{pmatrix} e^{-im\phi}. \tag{A.18}$$

This expansion clearly satisfies Neveu-Schwarz or Ramond boundary conditions depending on whether n is integer or half integer. Using the convention for the γ matrices

$$\gamma^{\phi} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^{\rho} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{A.19}$$

In this case the Dirac operator $-i\gamma^{\alpha}\partial_{\alpha}$ becomes $\begin{pmatrix} 0 & \partial_\phi - \partial_\rho \\ \partial_\phi + \partial_\rho & \end{pmatrix}$. Making the assumption that the target space metric is Euclidean, and recalling that the Dirac conjugate of a spinor is $\bar{\psi} = \psi^{\dagger}\gamma^{\phi}$ the action can be rewritten as

$$\begin{aligned} S &= \frac{-1}{4\pi\alpha'} \int d^2\sigma \sum_{mn}\sum_{m'n'} \psi^{\dagger}_{mn}\psi_{m'n'} e^{im\phi} e^{-im'\phi} \begin{pmatrix} e^{im\phi} \\ e^{-im'\phi} \end{pmatrix}^{\dagger} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \times \\ &\quad \begin{pmatrix} 0 & \partial_\phi - \partial_\rho \\ \partial_\phi + \partial_\rho & \end{pmatrix} \begin{pmatrix} e^{in'\frac{\pi\rho}{\ln a}} \\ e^{-in'\frac{\pi\rho}{\ln a}} \end{pmatrix} \\ &= \frac{-i}{4\pi\alpha'} \int d^2\sigma \sum_{mn}\sum_{m'n'} \psi^{\dagger}_{mn}\psi_{m'n'} e^{im\phi} e^{-im'\phi} \times \end{aligned}$$

$$
\begin{aligned}
&\left(e^{in\frac{\pi\rho}{\ln a}}\ ,\ e^{-in\frac{\pi\rho}{\ln a}}\right)\begin{pmatrix} 0 & \partial_\phi - \partial_\rho \\ \partial_\phi + \partial_\rho & \end{pmatrix}\begin{pmatrix} e^{in'\frac{\pi\rho}{\ln a}} \\ e^{-in'\frac{\pi\rho}{\ln a}} \end{pmatrix} \\
= \ & \frac{-i}{4\pi\alpha'}\int d^2\sigma \sum_{mn}\sum_{m'n'} \psi^\dagger_{mn}\psi_{m'n'}e^{im\phi}e^{-im'\phi} \times \\
&\left(e^{in\frac{\pi\rho}{\ln a}}\ ,\ e^{-in\frac{\pi\rho}{\ln a}}\right)\left(-im' + i\frac{n'\pi}{\ln a}\right)\begin{pmatrix} e^{-in'\frac{\pi\rho}{\ln a}} \\ e^{in'\frac{\pi\rho}{\ln a}} \end{pmatrix} \\
= \ & \frac{1}{4\pi\alpha'}\int d\rho \sum_{mnm'n'} \psi^\dagger_{mn}\psi_{m'n'}\delta^{mm'}2\pi\left(-m' + \frac{n'\pi}{\ln a}\right)\left(e^{i(n-n')\frac{\pi\rho}{\ln a}} + e^{-i(n-n')\frac{\pi\rho}{\ln a}}\right) \\
= \ & \frac{\ln a}{\alpha'}\sum_{mn}\psi^\dagger_{mn}\psi_{mn}\left(-m' + \frac{n'\pi}{\ln a}\right) \qquad \text{(A.20)}
\end{aligned}
$$

for each of the d fermion fields. Now consider the free vacuum energy

$$
\int [d\psi]\, e^{iS} = \int [d\psi]\left[d\psi^\dagger\right]\exp\left(\frac{i\ln a}{\alpha'}\sum_{mn}\psi^\dagger_{mn}\psi_{mn}\left(-m + \frac{n\pi}{\ln a}\right)\right). \qquad \text{(A.21)}
$$

This exponential can be Taylor expanded very simply since the coefficients ψ_{mn} are Grassman parameters, so

$$
\begin{aligned}
= \ & \int [d\psi]\left[d\psi^\dagger\right]\prod_{mn}\left(1 + \frac{i\ln a}{\alpha'}\psi^\dagger_{mn}\psi_{mn}\left(-m + \frac{n\pi}{\ln a}\right)\right) \\
= \ & \prod_{mn}\frac{i\ln a}{\alpha'}\left(-m + \frac{n\pi}{\ln a}\right). \qquad \text{(A.22)}
\end{aligned}
$$

Now, to calculate the Greens function for the fermions, it is necessary to similarly consider the quantity

$$
\int [d\psi]\, e^{iS}\psi(\rho,\phi)\psi^\dagger(\rho',\phi'). \qquad \text{(A.23)}
$$

Now, using the expression for ψ from above it is possible to directly substitute into the above expression and find

$$
\begin{aligned}
= \ & \int [d\psi]\left[d\psi^\dagger\right]\prod_{mn}\left(1 + \frac{i\ln a}{\alpha'}\psi^\dagger_{mn}\psi_{mn}\left(-m + \frac{n\pi}{\ln a}\right)\right)\sum_{ab}\psi_{ab}e^{-ia\phi}\begin{pmatrix} e^{ib\frac{\pi\rho}{\ln a}} \\ e^{-ib\frac{\pi\rho}{\ln a}} \end{pmatrix} \times \\
& i\sum_{cd}\psi^\dagger_{cd}e^{ic\phi'}\left(e^{id\frac{\pi\rho'}{\ln a}}\ ,\ e^{-id\frac{\pi\rho'}{\ln a}}\right). \qquad \text{(A.24)}
\end{aligned}
$$

It is immediately obvious that the Berezin integral forces the conditions $a = c$, $b = d$, and noting that the mode coefficients anticommute,

$$= \int [d\psi]\left[d\psi^\dagger\right]\prod_{mn}\left(1+\frac{i\ln a}{\alpha'}\psi^\dagger_{mn}\psi_{mn}\left(-m+\frac{n\pi}{\ln a}\right)\right)(-i)\sum_{ab}\psi^\dagger_{ab}\psi_{ab}e^{ia(\phi'-\phi)}\times$$
$$\begin{pmatrix} e^{ib(\rho+\rho')\frac{\pi}{\ln a}} & e^{ib(\rho-\rho')\frac{\pi}{\ln a}} \\ e^{-ib(\rho-\rho')\frac{\pi}{\ln a}} & e^{-ib(\rho+\rho')\frac{\pi}{\ln a}} \end{pmatrix}$$
$$= \sum_{ab}(-i)e^{ia(\phi'-\phi)}\begin{pmatrix} e^{ib(\rho+\rho')\frac{\pi}{\ln a}} & e^{ib(\rho-\rho')\frac{\pi}{\ln a}} \\ e^{-ib(\rho-\rho')\frac{\pi}{\ln a}} & e^{-ib(\rho+\rho')\frac{\pi}{\ln a}} \end{pmatrix}\prod_{m,n\neq a,b}\frac{i\ln a}{\alpha'}\left(-m+\frac{n\pi}{\ln a}\right). \quad \text{(A.25)}$$

From the above result it is immediately apparent that for the fermions the propagator can be written as

$$G(\rho,\phi,\rho',\phi') = \sum_{mn}\frac{1}{m-\frac{n\pi}{\ln a}}\frac{\alpha'}{\ln a}e^{im(\phi'-\phi)}\begin{pmatrix} e^{in(\rho+\rho')\frac{\pi}{\ln a}} & e^{in(\rho-\rho')\frac{\pi}{\ln a}} \\ e^{-in(\rho-\rho')\frac{\pi}{\ln a}} & e^{-in(\rho+\rho')\frac{\pi}{\ln a}} \end{pmatrix}, \quad \text{(A.26)}$$

which can be trivially re-expressed as

$$= \sum_{mn}\frac{m+\frac{n\pi}{\ln a}}{m^2-\left(\frac{n\pi}{\ln a}\right)^2}\frac{\alpha'}{\ln a}e^{im(\phi'-\phi)}\begin{pmatrix} e^{in(\rho+\rho')\frac{\pi}{\ln a}} & e^{in(\rho-\rho')\frac{\pi}{\ln a}} \\ e^{-in(\rho-\rho')\frac{\pi}{\ln a}} & e^{-in(\rho+\rho')\frac{\pi}{\ln a}} \end{pmatrix}. \quad \text{(A.27)}$$

which can be seen to be of the same form as the propagator calculated from the eigenfunctions of the Dirac operator, with the observation that because this calculation was performed with a Minkowski world-sheet metric it is necessary to let $n \to in$ for the purposed of comparison.

Appendix B

A Lemma

The purpose of this lemma is to offer a proof of the validity of the technique used above to promote the bosonic and fermionic propagators to matrices.

Consider a field $X(x_1 \ldots x_n)$ which is defined on a manifold of dimension n with p boundaries all of dimension $n-1$, and interacts with some external potential F only on these boundaries. Further, assume that there exists some choice of coordinates such that each of the p boundaries corresponds to a unique value of x_n. A Greens function can be determined for the field on the manifold, and in general it will have the form $G(x_1, \ldots x_n, x'_1, \ldots x'_n)$.

The statement to be proved is:

'The vacuum Feynman diagrams of a field which only interacts at the boundaries of a manifold can be evaluated by integrating over all the coordinates transverse to the boundary and replacing the propagator by the matrix constructed in the following way

$$G(x_1, \ldots x_n, x'_1, \ldots x'_n) \rightarrow G(x_1 \ldots x_{n-1}, x'_1 \ldots x'_{n-1}) = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1p} \\ A_{21} & A_{22} & \ldots & A_{2p} \\ \vdots & & & \vdots \\ A_{p1} & A_{p2} & \ldots & A_{pp} \end{pmatrix}, \quad \text{(B.1)}$$

where

$$A_{ab} = G(x_1, \ldots, x_{n-1}, x_n^a, x'_1, \ldots x'_{n-1}, x_n^b), \quad \text{(B.2)}$$

with x_n^a as the coordinate of the a^{th} boundary. The interaction vertex is given by a diagonal matrix whose entries are the coupling to the field at each boundary with

$$\Omega = diag(g_1, g_2, \dots g_p)F \tag{B.3}$$

and taking the trace of the result.'

The proof is by induction, and the case of 0 interactions with the external potential the statement is trivial. In the case of one interaction it must be shown that the two methods of calculating the vacuum diagrams are the same. In the case of p boundaries, there are p integrals of the form

$$\int dx_1 \dots dx_n G(x_1, \dots, x_n, x_1, \dots x_n) g_a F \delta(x_n - x_n^a) \tag{B.4}$$

and the sum of them is

$$\int dx_1 \dots dx_{n-1} \sum_{a=1}^{p} (G(x_1, \dots, x_{n-1}, x_n^a, x_1, \dots, x_{n-1}, x_n^a) F g_a)). \tag{B.5}$$

Now, for the matrix method one evaluates

$$\int dx_1 \dots dx_{n-1} Tr \left[\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2p} \\ \vdots & & & \vdots \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{pmatrix} \begin{pmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & g_p \end{pmatrix} F \right] \tag{B.6}$$

with $\bar{x} = \bar{x}'$ which can be trivially seen to be identical to the expression above.

Now, assume that the expression above is valid for the interactions of order m, the task is now to show that it is valid for order $m+1$. Consider a general Feynman diagram for the field X interacting with the potential F. There is a propagator for X between the m^{th} point (which is on boundary a_m) and the 1^{st} point (which is on boundary a_1). To construct a diagram of order $m+1$ that propagator will be replaced by two propagators and a vertex. There are a total of p possibilities for which boundary the inserted vertex is on, and the modification to the Feynman diagrams will be as follows Prior to the insertion the additional vertex, the Feynman diagram would be evaluated as

$$\int d\bar{x}^1 \ldots d\bar{x}^m G(\bar{x}^1, x_n^{a_1}, \bar{x}^2, x_n^{a_2}) g_{a_1} \ldots G(\bar{x}^m, x_n^{a_m}, \bar{x}^1, x_n^{a_1}) g_{a_m} \tag{B.7}$$

where $\bar{x}^i$ is the i^{th} set of coordinates $x_1^i \ldots x_{n-1}^i$. Summing the p possible insertions gives

$$\sum_{b=1}^{p} \int d\bar{x}^1 \ldots d\bar{x}^m d\bar{x}^{m+1} G(\bar{x}^1, x_n^{a_1}, \bar{x}^2, x_n^{a_2}) g_{a_1} \ldots G(\bar{x}^m, x_n^{a_m}, \bar{x}^{m+1}, x_n^p) g_{a_m} G(\bar{x}^{m+1}, x_n^p, \bar{x}^1, x_n^{a_1}) g_p. \tag{B.8}$$

The analogous operation with the matrices would be to replace

$$G(\bar{x}^m, \bar{x}^1) \rightarrow \int d\bar{x}^{m+1} G(\bar{x}^m, \bar{x}^{m+1}) \Omega G(\bar{x}^{m+1}, \bar{x}^1) \tag{B.9}$$

which is equivalent to the above expression.

This justifies the approach used in the calculations using the propagators.

Appendix C

Canonical Quantization of the Open String in a $U(1)$ Background

C.1 Motivation

The interacting open string is different in several respects from the free string. The most notable difference is that the world-sheet fields, which were free apart from boundary conditions, have those boundary conditions changed. In a sense the external gauge field acts as a source on the boundary for the degree of freedom on the string. Another difference is that spacetime supersymmetry is broken by the presence of background field which chooses particular directions in space by its components. The consequence of this breaking of supersymmetry is the absence of certain zero modes that are present in the spectrum of the open string.

The method used throughout this chapter can be summarized as follows: First, using variational arguments the world-sheet equations of motion are reproduced, and coincide with those of the free string. Second, the surface terms are analyzed and appropriate boundary conditions which involve the external $F_{\mu\nu}$ are determined. The boundary conditions and the equations of motion determine the mode expansion, and care is taken to produce modes that are orthogonal with respect to the inner product that is natural given the action. Using canonical commutation relationships between the fields, commutation relations for the coefficients of the modes are determined, and those coefficients are promoted to operators in the usual way. The Virasoro generators and central extension are calculated.

This method is useful, since it follows very closely from the development of the free string which is well understood. It has limitations however. In particular, it relies on the fact that the external field is constant. If it were not, there would be higher derivative interactions at the edge of the string which would complicate the boundary conditions. In addition, a non-constant external field could not necessarily be block diagonalized consistently at both ends, that is to say different rotations would be necessary to obtain an identical block diagonal form, which would mean that there would be a non-trivial interaction between the different fields on the string world-sheet, which would complicate quantization. This same objection makes this an inappropriate vehicle for the discussion of open string coupled to an antisymmetric field living on a stack of D-branes.

C.2 Bosonic Degrees of Freedom

C.2.1 Boundary Conditions and Mode Expansion

To use the operator formalism, it is necessary to construct orthogonal modes for use in the expansion of the fields, which account for the background $F_{\mu\nu}$ field. As in the case of free strings, the first step is to determine appropriate boundary conditions from a variational method. Re-writing the bosonic part of the superstring action gives

$$S = \frac{-1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-g}g_{ab}\partial^a X^\mu\partial^b X_\mu + \frac{1}{2}e_1\int d\tau F_\mu\nu X^\nu\partial_\tau X^\mu|_{\sigma=0} - \frac{1}{2}e_2\int d\tau F_\mu\nu X^\nu\partial_\tau X^\mu|_{\sigma=a}, \tag{C.1}$$

so varying the X^μ field gives

$$\begin{aligned}\delta S &= \frac{-1}{4\pi\alpha'}\int d\sigma d\tau\sqrt{-g}g_{ab}2\partial^a X^\mu\partial^b\delta X_\mu + \frac{1}{2}e_1\int d\tau F_{\mu\nu}\left(X^\nu\partial_\tau\delta X^\mu + \delta X^\nu\partial_\tau X^\mu\right)|_{\sigma=0} \\ &\quad - \frac{1}{2}e_2\int d\tau F_{\mu\nu}\left(X^\nu\partial_\tau\delta X^\mu + \delta X^\nu\partial_\tau X^\mu\right)|_{\sigma=a}. \end{aligned}\tag{C.2}$$

Imposing that the string world-sheet metric is Euclidean, this becomes

$$\delta S = \frac{-1}{2\pi\alpha'}\int d\sigma d\tau\left[\delta X_\mu\left(-\partial_\tau^2 - \partial_\sigma^2\right)X^\mu + \partial_\tau\left(\delta X_\mu\partial_\tau X^\mu\right) + \partial_\sigma\left(\delta X_\mu\partial_\sigma X^\mu\right)\right]$$

$$
\begin{aligned}
&+\frac{1}{2}e_1\int d\tau F_\mu\nu\left(\partial_\tau\left(X^\nu\delta X^\mu\right)+2\delta X^\nu\partial_\tau X^\mu\right)|_{\sigma=0}\\
&-\frac{1}{2}e_2\int d\tau F_\mu\nu\left(\partial_\tau\left(X^\nu\delta X^\mu\right)+2\delta X^\nu\partial_\tau X^\mu\right)|_{\sigma=a}.
\end{aligned}
\tag{C.3}
$$

Integrating over the total derivatives gives, after imposing the periodicity of X on the annulus,

$$
\begin{aligned}
=\ &\frac{-1}{2\pi\alpha'}\int d\sigma d\tau\delta X_\mu\left(-\partial_\tau^2-\partial_\sigma^2\right)X^\mu+\int d\tau\left(\frac{-1}{2\pi\alpha'}\delta X_\mu\partial_\sigma X^\mu-\frac{1}{2}e_2F_{\mu\nu}2\delta X^\nu\partial_\tau X^\mu\right)|_{\sigma=a}\\
&+\int d\tau\left(\frac{-1}{2\pi\alpha'}-\delta X_\mu\partial_\sigma X^\mu+\frac{1}{2}e_1F_{\mu\nu}2\delta X^\nu\partial_\tau X^\mu\right)|_{\sigma=0}.
\end{aligned}
\tag{C.4}
$$

This gives the equations of motion

$$\left(-\partial_\tau^2-\partial_\sigma^2\right)X^\mu=0, \tag{C.5}$$

which is the same as those for the free string, and the boundary conditions

$$
\begin{aligned}
\partial_\sigma X^\mu &= -2\pi\alpha' e_2F^\mu_\nu\partial_\tau X_\nu|_{\sigma=a}\\
\partial_\sigma X^\mu &= -2\pi\alpha' e_1F^\mu_\nu\partial_\tau X_\nu|_{\sigma=0}.
\end{aligned}
\tag{C.6}
$$

The mode expansion for the bosons must satisfy both the boundary conditions and the wave equation, and also be orthogonal. One of the principal differences between the case of the free string and the charged string in a $U(1)$ gauge field is that while the boundary conditions for the free string are diagonal in the target space indices, the boundary conditions for the charged string mix them. The boundary conditions can be made diagonal however by use of the fact that $F_{\mu\nu}$ can be re-cast in a block diagonal form by a rotation of target space coordinates, so

$$
F_{\mu\nu}=\begin{pmatrix}
0 & f_1 & & & \dots\\
-f_1 & 0 & & & \dots\\
 & & 0 & f_2 & \dots\\
 & & -f_2 & 0 & \dots\\
\vdots & & \vdots & &
\end{pmatrix}.
\tag{C.7}
$$

Now, this can be diagonalized for any individual block by a change of basis

$$X_{+n} = \frac{1}{\sqrt{2}}\left(X_{2n-1} + iX_{2n}\right), \quad X_{-n} = \frac{1}{\sqrt{2}}\left(X_{2n-1} - iX_{2n}\right), \tag{C.8}$$

as noted in [2] this has the effect of changing the boundary conditions to

$$\begin{aligned} \partial_\sigma X_{+n} &= i\alpha_n \partial_\tau X_{+n}|_{\sigma=0} \\ \partial_\sigma X_{+n} &= i\beta_n \partial_\tau X_{+n}|_{\sigma=a}, \end{aligned} \tag{C.9}$$

where $\alpha_n = 2\pi\alpha' e_1 f_n$ and $\beta_n = 2\pi\alpha' e_2 f_n$, and since $X_{-n} = X_{+n}^\dagger$ the sign of the boundary conditions changes for X_{-n}.

Restricting attention for a moment to one of the n independent components of the field strength and, for instance the $+$ component, the normalized mode that satisfies the boundary conditions and the wave equation is

$$X_m = \frac{1}{|n-\epsilon|^{1/2}} \cos\left[(m-\epsilon)\,\sigma + \gamma\right] e^{-i(n-\epsilon)\tau} \tag{C.10}$$

with

$$\epsilon = \frac{1}{\pi}\left(\gamma' - \gamma\right), \quad \gamma = \tan^{-1}\alpha, \quad \gamma' = \tan^{-1}\beta, \tag{C.11}$$

where α and β are the appropriate constants as defined above for the mode in question. The particularly interesting thing about this mode expansion is that the right and left moving terms that were in the free mode expansion are mixed by the boundary conditions. The orthogonality condition for the modes can be expressed as

$$\int_0^\pi \frac{d\sigma}{\pi}\left[\bar{X}_n i\partial_\tau X_m - X_m i\partial_\tau \bar{X}_n + \alpha\delta(\sigma)\bar{X}_n X_m - \beta\delta(\sigma-\pi)\bar{X}_n X_m\right] = \delta_{mn} sign(n-\epsilon). \tag{C.12}$$

C.2.2 Commutation Relations and the Virasoro Algebra

In terms of the modes given above, the expansion for X_+ is [2] [4]

$$X_+ = x_+ - ib_0^\dagger X_0 + i\sum_{n=1}^{\infty}\left[a_n X_n - b_n^\dagger X_{-n}\right], \tag{C.13}$$

and similarly for X_-

$$X_- = x_- + ib_0 X_0 + i\sum_{n=1}^{\infty}\left[b_n X_n - a_n^\dagger X_{-n}\right]. \tag{C.14}$$

An interesting point to note is that unlike the case of the open string in this mode expansion there is no term linear in $\tau \pm \sigma$. This is simply due to the fact that there can be no such term that satisfies the boundary conditions.

Using the orthogonality relation above, the oscillators can be obtained by taking the inner product defined above with X_n in the following manner,

$$a_n = \int \frac{d\sigma}{\pi}\bar{X}_n\left(i\bar{\partial}_\tau + \alpha\delta(\sigma) - \beta\delta(\sigma-\pi)\right)X_+ \tag{C.15}$$

where $X\bar{\partial}_\tau Y = X\partial_\tau Y - Y\partial_\tau X$ given that X and Y commute. Similar expressions hold for $a_n^\dagger$, b_n, and $b_n^\dagger$. This can be written in an easier way for the purpose of calculating commutation relations. The momentum conjugate to X_+ satisfies

$$\pi P_+ = \partial_\tau X_- + \frac{i}{2}X_-\left[\alpha\delta(\sigma) - \beta\delta(\sigma-\pi)\right], \tag{C.16}$$

and the conjugate of this for P_-, since X_+ and X_- are complex conjugates of each other. Given the expressions for P, the expression for a_n can be re-expressed as

$$a_n = \int \frac{d\sigma}{\pi}\bar{X}_n\left[\pi P_- - i\left((n-\epsilon) + \frac{1}{2}\alpha\delta(\sigma) - \frac{1}{2}\beta\delta(\pi-\sigma)\right)X_+\right], \tag{C.17}$$

and so the calculation of commutators between the a operators becomes

$$\begin{aligned}
\left[a_n, a_m^\dagger\right] &= \left[\int d\sigma\bar{X}_n\left(P_- - \frac{i}{\pi}\left((n-\epsilon) + \frac{1}{2}\alpha\delta(\sigma) - \frac{1}{2}\beta\delta(\pi-\sigma)\right)X_+\right),\right. \\
&\qquad \left.\int d\sigma'\bar{X}_m^\dagger\left(P_+ + \frac{i}{\pi}\left((m-\epsilon) + \alpha\delta(\sigma') - \frac{1}{2}\beta\delta(\pi-\sigma')\right)X_-\right)\right] \\
&= \int d\sigma d\sigma'\bar{X}_n(\sigma)X_m(\sigma')\left(\frac{-i}{\pi}\left((n-\epsilon) + \frac{1}{2}\alpha\delta(\sigma) - \frac{1}{2}\beta\delta(\pi-\sigma)\right)\right)\left[X_+(\sigma), P_+(\sigma')\right] \\
&\qquad + \left(\frac{i}{\pi}\left((m-\epsilon) + \frac{1}{2}\alpha\delta(\sigma) - \frac{1}{2}\beta\delta(\pi-\sigma)\right)\right)\left[P_-(\sigma), X_-(\sigma')\right] \\
&= \int d\sigma\bar{X}_n(\sigma)X_m(\sigma)\left((n-\epsilon) + (m-\epsilon) + \alpha\delta(\sigma) - \beta\delta(\pi-\sigma)\right) \\
&= \delta_{mn}
\end{aligned} \tag{C.18}$$

after using the commutation relations between X and P and the orthogonality of the modes. Similar calculations for the other commutators give

$$\begin{aligned}
\left[a_m, a_n^\dagger\right] &= \delta_{mn} \\
[a_m, a_n] &= \left[a_m^\dagger, a_n^\dagger\right] = 0 \\
\left[b_m, b_n^\dagger\right] &= \delta_{mn} \\
[b_m, b_n] &= \left[b_m^\dagger, b_n^\dagger\right] = 0 \\
[x_+, x_-] &= \frac{\pi}{\alpha+\beta}.
\end{aligned} \tag{C.19}$$

The conditions for x_+ and x_- appear unusual, but since there is no term linear in τ it is natural to expect that these will be conjugate to each other.

Since the X fields are expressed in terms of raising and lowering operators, the Virasoro operators can be calculated. Just as in the case of the free string, the bosonic contribution to the energy momentum tensor is

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} \partial_\gamma X^\mu \partial^\gamma X_\mu \tag{C.20}$$

because there is no dependence on the metric in the surface interaction terms. The Virasoro generators will be determined in the same way as for the free string, by taking the fourier modes of $T_{\alpha\beta}$, so that

$$L_k = \frac{1}{\pi}\int_0^\pi d\sigma \left(e^{ik\sigma}\partial_+ X^\mu(\tau,\sigma)\partial_+ X_\mu(\tau,\sigma) + e^{-ik\sigma}\partial_- X^\mu(\tau,\sigma)\partial_- X_\mu(\tau,\sigma)\right), \tag{C.21}$$

where as for the free string $x^+ = \tau + \sigma$ and $x^- = \tau - \sigma$. Using the mode expansion for X^μ and noting that $X^\mu X_\mu = X_+ X_-$ for the pair of coordinates under consideration, the expression becomes

$$L_k = \frac{1}{\pi}\int_0^\pi d\sigma \left(\sum_{n>0}\sum_{m>0} a_n a_m^\dagger \left(e^{ik\sigma}\partial_+ X_n \partial_+ \bar{X}_m + e^{-ik\sigma}\partial_- X_n \partial_- \bar{X}_m\right)\right.$$

$$-\sum_{n>0}\sum_{m\geq 0} a_n b_m \left(e^{ik\sigma}\partial_+ X_n \partial_+ \bar{X}_{-m} e^{-ik\sigma}\partial_- X_n \partial_- \bar{X}_{-m}\right)$$
$$-\sum_{n\geq 0}\sum_{m>0} b_n^\dagger a_m^\dagger \left(e^{ik\sigma}\partial_+ X_{-n} \partial_+ \bar{X}_{m} e^{-ik\sigma}\partial_- X_{-n} \partial_- \bar{X}_{m}\right)$$
$$\left.+\sum_{n\geq 0}\sum_{m\geq 0} b_n^\dagger b_m \left(e^{ik\sigma}\partial_+ X_{-n} \partial_+ \bar{X}_{-m} e^{-ik\sigma}\partial_- X_{-n} \partial_- \bar{X}_{-m}\right)\right), \tag{C.22}$$

and from the definition of X_n it can be verified that

$$\begin{aligned} \partial_+ X_n &= -\partial_+ \bar{X}_n = \frac{n-\epsilon}{|n-\epsilon|^{1/2}}(-i)e^{-i(n-\epsilon)\tau}e^{-i((n-\epsilon)\sigma+\gamma)} \\ \partial_- X_n &= -\partial_- \bar{X}_n = \frac{n-\epsilon}{|n-\epsilon|^{1/2}}(-i)e^{-i(n-\epsilon)\tau}e^{i((n-\epsilon)\sigma+\gamma)}. \end{aligned} \tag{C.23}$$

Upon re-insertion of these terms into the above, the integration over σ becomes trivial and the result for the Virasoro generator in the direction of the field is as in [4]

$$\begin{aligned} L_n &= \sum_{m>0}\sqrt{(m-\epsilon)(n+m-\epsilon)}a_m^\dagger a_{n+m} + \sum_{m\geq 0}\sqrt{(m+\epsilon)(n+m+\epsilon)}b_m^\dagger b_{n+m} \\ &\quad + \sum_{0}^{n-1}\sqrt{(m+\epsilon)(n-m-\epsilon)}b_m a_{m-n}, \end{aligned} \tag{C.24}$$

for $n \geq 0$ and

$$L_{-n} = L_n^\dagger \tag{C.25}$$

as can be seen from direct evaluation of the above expression for L_k.

The L_ms obey the standard Virasoro algebra, as can be verified easily. The only possible change comes from the central extension of the algebra, and this can be calculated in the same way as was done in the case of a free open string. Because the Virasoro algebra is the same up to the central extension, the central extension can be uniquely determined by taking the expectation value both $[L_1, L_{-1}]$, and $[L_2, L_{-2}]$. In the first case, taking $|0>$ to be a state with zero momentum,

$$<0|\left[L_1, L_{-1}\right]|0> = <0|\left[\sqrt{\epsilon(1-\epsilon)}b_0 a_1, \sqrt{\epsilon(1-\epsilon)}b_0^\dagger a_1^\dagger\right]|0> \tag{C.26}$$

because of normal ordering

$$=< 0|\epsilon(1-\epsilon)b_0 b_0^\dagger a_1 a_1^\dagger|0 >= \epsilon(1-\epsilon). \tag{C.27}$$

Similarly for L_2,

$$\begin{aligned} < 0|\left[L_2, L_{-2}\right]|0 > &= < 0|\left[\sqrt{\epsilon(2-\epsilon)}b_0 a_2 + \sqrt{(1+\epsilon)(1-\epsilon)}b_1 a_1,\right. \\ &\qquad \left.\sqrt{\epsilon(2-\epsilon)}b_0^\dagger a_2^\dagger + \sqrt{(1+\epsilon)(1-\epsilon)}b_1^\dagger a_1^\dagger\right]|0 > \\ &= \epsilon(2-\epsilon) + (1+\epsilon)(1-\epsilon), \end{aligned} \tag{C.28}$$

which confirms that the central extension of the algebra for bosons degrees of freedom is

$$A_n = \frac{2}{12}\left(n^3 - n\right) + n\epsilon(1-\epsilon). \tag{C.29}$$

C.3 Fermionic Degrees of Freedom

C.3.1 Boundary Conditions and Mode Expansion

To use an analogous technique to those in [5] it is necessary to find a mode expansion for the fermions that is orthogonal and that account for the background $U(1)$ gauge field. The first task then is to determine the boundary conditions for the fermionic fields, which can be found by the usual variational arguments, in analogy with what is done for the bosons.

The starting point is the fermionic part of the action on a Euclidean world sheet,

$$S = \frac{-1}{4\pi\alpha'}\int d\sigma d\tau \bar{\psi}^\mu \left(-i\partial_a \gamma^a\right)\psi^\nu g_{\mu\nu} + \frac{1}{4}e_1 \int d\tau F_{\mu\nu}\bar{\psi}^\nu \gamma^\tau \psi^\mu|_{\sigma=0} - \frac{1}{4}e_2 \int d\tau F_{\mu\nu}\bar{\psi}^\nu \gamma^\tau \psi^\mu|_{\sigma=\ln a}. \tag{C.30}$$

If the ψ field is written as $\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}$, then, with the convention that for a Euclidean world-sheet,

$$\gamma^\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{C.31}$$

the action can be re-expressed as

$$
\begin{aligned}
S &= \frac{-1}{4\pi\alpha'}\int d\sigma d\tau\left[-i\psi_-^{*\mu}\partial_\tau\psi_-^\nu - i\psi_+^{*\mu}\partial_\tau\psi_+^\nu - \psi_-^{*\mu}\partial_\sigma\psi_-^\nu + \psi_+^{*\mu}\partial_\sigma\psi_+^\nu\right]g_{\mu\nu}\\
&\quad + \frac{1}{4}e_1\int d\tau F_{\mu\nu}\left[\psi_-^{*\nu}\psi_-^\mu + \psi_+^{*\nu}\psi_+^\mu\right]|_{\sigma=0}\\
&\quad - \frac{1}{4}e_2\int d\tau F_{\mu\nu}\left[\psi_-^{*\nu}\psi_-^\mu + \psi_+^{*\nu}\psi_+^\mu\right]|_{\sigma=\ln a}.
\end{aligned} \tag{C.32}
$$

Now, varying the action according to $\delta\psi = \begin{pmatrix}\delta\psi_-\\ \delta\psi_+\end{pmatrix}$ the change in the action is, ignoring the bulk term that gives the equation of motion, and ignoring the total derivative in τ which is eliminated by the periodicity around the cylinder,

$$
\begin{aligned}
\delta S &= \frac{-1}{4\pi\alpha'}\int d\sigma d\tau\partial_\sigma\left[-\psi_-^{*\mu}\delta\psi_-^\nu + \psi_+^{*\mu}\delta\psi_+^\nu\right]g_{\mu\nu}\\
&\quad + \frac{1}{4}e_1\int d\tau F_{\mu\nu}\left[\psi_-^\nu\delta\psi_-^\mu + \delta\psi_-^\nu\psi_-^\mu + \psi_+^\nu\delta\psi_+^\mu + \delta\psi_+^\nu\psi_+^\mu\right]_{\sigma=0}\\
&\quad - \frac{1}{4}e_2 d\tau F_{\mu\nu}\left[\psi_-^\nu\delta\psi_-^\mu + \delta\psi_-^\nu\psi_-^\mu + \psi_+^\nu\delta\psi_+^\mu + \delta\psi_+^\nu\psi_+^\mu\right]_{\sigma=\ln a}.
\end{aligned} \tag{C.33}
$$

Now, impose that these are pure Majorana spinors, which as in the case of the free string imposes constraints on the raising and lowering operators for the fermions. This also has the effect of simplifying the above to

$$
\begin{aligned}
\delta S &= \frac{-1}{4\pi\alpha'}\int d\tau g_{\mu\nu}\left[-\psi_-^\mu\delta\psi_-^\nu + \psi_+^{*\mu}\delta\psi_+^\nu\right]_{\sigma=\ln a} - g_{\mu\nu}\left[-\psi_-^\mu\delta\psi_-^\nu + \psi_+^\mu\delta\psi_+^\nu\right]_{\sigma=0}\\
&\quad + \frac{1}{2}e_1\int d\tau F_{\mu\nu}\left(\psi_-^\nu\delta\psi_-^\mu + \psi_+^\nu\delta\psi_+^\mu\right)_{\sigma=0} - \frac{1}{2}e_2\int d\tau F_{\mu\nu}\left(\psi_-^\nu\delta\psi_-^\mu + \psi_+^\nu\delta\psi_+^\mu\right)_{\sigma=\ln a}
\end{aligned} \tag{C.34}
$$

Imposing that this must vanish gives the boundary conditions

$$
\begin{aligned}
\frac{-1}{4\pi\alpha'}g_{\mu\nu}\left(\psi_-^\mu\delta\psi_-^\nu - \psi_+^\mu\delta\psi_+^\nu\right) + \frac{1}{2}e_1F_{\mu\nu}\left(\psi_-^\nu\delta\psi_-^\mu + \psi_+^\nu\delta\psi_+^\mu\right) &= 0, \quad \sigma = 0\\
\frac{-1}{4\pi\alpha'}g_{\mu\nu}\left(\psi_-^\mu\delta\psi_-^\nu - \psi_+^\mu\delta\psi_+^\nu\right) + \frac{1}{2}e_2F_{\mu\nu}\left(\psi_-^\nu\delta\psi_-^\mu + \psi_+^\nu\delta\psi_+^\mu\right) &= 0, \quad \sigma = \ln a.
\end{aligned} \tag{C.35}
$$

These boundary conditions are solved first by rotating the $F_{\mu\nu}$ field into its canonical block diagonal form, and then going to coordinates that are an exact analog of the change of bosonic coordinates introduced above, that is

$$\psi_a = \psi_1 + i\psi_2, \quad \psi_b = \psi_1 - i\psi_2, \tag{C.36}$$

and it is understood that similar definitions occur in every pair of coordinates, which correspond to the independent components of F. Writing for simplicity $f_1 = 2\pi\alpha' e_1 F$ where F is the component of $F_{\mu\nu}$ for the pair of fields in question, the the boundary conditions can be re-cast as

$$\psi_{a-} = \sqrt{\frac{1+if_1}{1-if_1}}\psi_{a+}|_{\sigma=0}, \quad \psi_{a-} = \sqrt{\frac{1+if_2}{1-if_2}}\psi_{a+}|_{\sigma=\ln a}, \tag{C.37}$$

with the appropriate modification for ψ_b, which is the complex conjugate of ψ_a. It is important to note that these boundary conditions are distinct from the ones found in the literature [4]. The difference is due to the fact that the variation of the fields must obey the same boundary conditions that the fields themselves obey. Now defining

$$\gamma_1 = \frac{1}{2}\ln\sqrt{\frac{1+if_1}{1-if_1}}, \quad \gamma_2 = \frac{1}{2}\ln\sqrt{\frac{1+if_2}{1-if_2}}, \tag{C.38}$$

and

$$\epsilon = \frac{1}{\pi}\left(\gamma_2 - \gamma_1\right), \tag{C.39}$$

the mode expansion for the fermions becomes

$$\begin{aligned}
\psi_- &= \sum_n d_n e^{-i(n+i\epsilon)(\tau - \frac{\sigma\pi}{\ln a}) + \gamma_1} \\
\psi_+ &= \sum_n d_n e^{-i(n+i\epsilon)(\tau + \frac{\sigma\pi}{\ln a}) - \gamma_1},
\end{aligned} \tag{C.40}$$

where as in the case of free fermions the index n can be either integer or half integer to correspond to the analog of Ramond or Neveu-Schwarz boundary conditions. The inner product that these are orthogonal with respect to is a generalization of the one for the the free case

$$\int d\sigma \psi_n^\dagger \left(1 + 2\pi\alpha' e_1 f \gamma^\tau \delta(\sigma) - 2\pi\alpha' e_2 f \gamma^\tau \delta(\sigma - \pi)\right) \psi_m = \delta_{mn}. \tag{C.41}$$

C.3.2 Commutation Relations and the Virasoro Algebra

The action for the interacting string, given above (C.30) gives rise to the same cannonicaly conjugate momentum for the fermion fields as it does in the case of a free string, the reason being that the interaction at the string ends is not derivative with respect to the coordinate τ. Since there is no change from the case of the free string and since the modes of the expansion are orthogonal, the commutation relationships are identical to those for the free string,

$$\{d_n^\mu, d_m^\nu\} = \eta^{\mu\nu}\delta_{m+n}, \tag{C.42}$$

which is in agreement with the statement in [4] It is important to note that the coordinate system which diagonalized the boundary conditions has the same effect as the light cone coordinates of making the spacetime metric off diagonal.

As in the case of the free string the Virasoro generators are given by

$$L_k = \frac{1}{\ln a}\int d\sigma \frac{i}{2}\left(\psi_+^{\mu *}\partial_+\psi_{+\mu}e^{ik\sigma\frac{\pi}{\ln a}} + e^{-ik\sigma\frac{\pi}{\ln a}}\psi_-^{\mu *}\partial_-\psi_{-\mu}\right), \tag{C.43}$$

and examining only the contribution from some particular coordinate, a, this becomes, since $\psi_a = i\psi^b$,

$$\begin{aligned} L_k &= \frac{1}{\ln a}\int d\sigma \frac{-1}{2}\left(\psi_+^{a*}\partial_+\psi_+^b e^{ik\sigma\frac{\pi}{\ln a}} + e^{-ik\sigma\frac{\pi}{\ln a}}\psi_-^{a*}\partial_-\psi_-^b\right) \\ &= \frac{1}{\ln a}\int d\sigma \frac{-1}{2}\sum_m\sum_n d_m^{a*}d_n^b\left[e^{ik\sigma\frac{\pi}{\ln a}}\left(e^{-i(m+i\epsilon)(\tau+\sigma\frac{\pi}{\ln a})-\gamma_1}\right)^*\partial_+ e^{-i(n+i\epsilon)(\tau+\sigma\frac{\pi}{\ln a})-\gamma_1}\right. \\ &\quad \left. + e^{-ik\sigma\frac{\pi}{\ln a}}\left(e^{-i(m+i\epsilon)(\tau-\sigma\frac{\pi}{\ln a})+\gamma_1}\right)^*\partial_- e^{-i(n+i\epsilon)(\tau-\sigma\frac{\pi}{\ln a})+\gamma_1}\right]. \end{aligned} \tag{C.44}$$

It is important to note that since ϵ and γ are logarithms and the effect of conjugation is the same as inverting the external field, the field dependent parts of the exponential cancel, and what remains is

$$\begin{aligned} &= \frac{1}{\ln a}\int d\sigma\frac{-1}{2}\sum_m\sum_n d_{-m}^a d_n^b\,(n+i\epsilon)\left(1+\frac{\pi}{\ln a}\right)\left[e^{i(m+k-n)\sigma\frac{\pi}{\ln a}} + e^{-i(m+k-n)\sigma\frac{\pi}{\ln a}}\right] \\ &= \left(1+\frac{\pi}{\ln a}\right)\sum_m d_{-m}^a d_{m+k}^b\,(m+k+i\epsilon). \end{aligned} \tag{C.45}$$

This can be converted into the standard form for the fermionic part of the Virasoro algebra except for the field dependent part, and that is what distinguishes the case of a non-zero external field.

To complete the analogy with the free string, it remains to show that the Virasoro algebra is satisfied, and also to determine the central extensions. The commutator of two Ls, with the expression given above is

$$\begin{aligned}
[L_i, L_j] &= \left(1+\frac{\pi}{\ln a}\right)^2 \sum_m \sum_n \left[d^a_{-m} d^b_{m+i}, d^a_{-n} d^b_{n+j}\right](m+i+i\epsilon)(n+j+i\epsilon) \\
&= \left(1+\frac{\pi}{\ln a}\right)^2 \sum_m \sum_n (m+i+i\epsilon)(n+j+i\epsilon)\left[d^a_{-n} d^a_{-m}\left\{d^b_{m+i}, d^b_{n+j}\right\}\right. \\
&\qquad \left. -d^a_{-n} d^b_{m+i}\left\{d^a_{-m}, d^b_{n+j}\right\} - d^a_{-m} d^b_{n+j}\left\{d^b_{m+i}, d^a_{-n}\right\} + d^b_{m+i} d^b_{n+j}\left\{d^a_{-m}, d^a_{-n}\right\}\right] \\
&= \left(1+\frac{\pi}{\ln a}\right)^2 \sum_n \left(d^a_{-n} d^b_{m+i+j}(n+i+j+i\epsilon)(n+b+i\epsilon)(-1)\right. \\
&\qquad \left. +d^a_{-n+i} d^b_{n+j}(n+i\epsilon)(n+j+i\epsilon)\right) \\
&= (i-j)L_{i+j}
\end{aligned} \tag{C.46}$$

after re-indexing.

As for the bosons, expectation values for L commutators must be calculated to determine the central extension. In the sector where the fermions are integrally moded

$$\begin{aligned}
\langle 0|\,[L_1, L_{-1}]\,|0\rangle &= \langle 0|\left[d^a_0 d^b_1(1+i\epsilon) + d^a_1 d^b_0 i\epsilon, d^a_{-1} d^b_0 i\epsilon + d^a_0 d^b_{-1}(-1+i\epsilon)\right]|0\rangle \\
&= \langle 0|\,(1+i\epsilon) i\epsilon d^a_0 d^b_1 d^a_{-1} d^b_0 + (-1+i\epsilon) i\epsilon d^a_1 d^b_0 d^a_0 d^b_{-1}\,|0\rangle \\
&= (1+i\epsilon) i\epsilon,
\end{aligned} \tag{C.47}$$

and

$$\begin{aligned}
\langle 0|\,[L_2, L_{-2}]\,|0\rangle &= \langle 0|\,(2+i\epsilon) d^a_0 d^b_2 + (1+i\epsilon) d^a_1 d^b_1 + i\epsilon d^a_2 d^b_0, (-2+i\epsilon) d^a_0 d^b_{-2} \\
&\quad +(-1+i\epsilon) d^a_{-1} d^b_{-1} + i\epsilon d^a_{-2} d^b_0\,|0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle 0| \, (2+i\epsilon) d_0^a d_2^b i\epsilon d_{-2}^a d_0^b + (1+i\epsilon) d_1^a d_1^b (-1+i\epsilon) d_{-1}^a d_{-1}^b \\
&\quad + i\epsilon d_2^a d_0^b (-2+i\epsilon) d_0^a d_{-2}^b \, |0\rangle \\
&= \langle 0| \, (2+i\epsilon) i\epsilon d_0^a d_0^b + i\epsilon(-2+i\epsilon) d_0^b d_0^a \, |0\rangle + (1+i\epsilon)(-1+i\epsilon) \\
&= (2+i\epsilon) i\epsilon + (1+i\epsilon)(-1+i\epsilon). \qquad \text{(C.48)}
\end{aligned}
$$

So for the Ramond sector the central extension is

$$
A(m) = \frac{2}{12}\left(m^3 - m\right) + m(1+i\epsilon)i\epsilon \qquad \text{(C.49)}
$$

which exactly analogous to the contribution in the bosonic sector. By contrast, in the case of Neveu-Schwarz boundary conditions one has

$$
\begin{aligned}
\langle 0| \, [L_1, L_{-1}] \, |0\rangle &= \langle 0| \left[d_{1/2}^a d_{1/2}^b (\frac{1}{2} + i\epsilon), d_{-1/2}^a d_{-1/2}^b (-\frac{1}{2} + i\epsilon) \right] |0\rangle \\
&= \langle 0| \left[d_{1/2}^a d_{1/2}^b (\frac{1}{2} + i\epsilon) d_{-1/2}^a d_{-1/2}^b (-\frac{1}{2} + i\epsilon) \right] |0\rangle \\
&= (\frac{1}{2} + i\epsilon)(-\frac{1}{2} + i\epsilon), \qquad \text{(C.50)}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle 0| \, [L_2, L_{-2}] \, |0\rangle &= \langle 0| \left[(\frac{3}{2} + i\epsilon) d_{1/2}^a d_{3/2}^b + (\frac{1}{2} + i\epsilon) d_{3/2}^a d_{1/2}^b, \right. \\
&\quad \left. (-\frac{3}{2} + i\epsilon) d_{-1/2}^a d_{-3/2}^b + (-\frac{1}{2} + i\epsilon) d_{-3/2}^a d_{-1/2}^b \right] |0\rangle \\
&= \langle 0| \, (2+i\epsilon)(-2+i\epsilon) d_{1/2}^a d_{3/2}^b d_{-3/2}^a d_{-1/2}^b \\
&= (2+i\epsilon)(-2+i\epsilon). \qquad \text{(C.51)}
\end{aligned}
$$

So for the Neveu-Schwarz sector the central extension is

$$
A(m) = \frac{m^3}{8} + m\epsilon^2, \qquad \text{(C.52)}
$$

which is in agreement with the results of [4].

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