ENERGY LEVEL BROADENING IN AN N-PARTICLE SYSTEM: A
SOLVABLE MODEL WITH A HIERARCHY OF
INTERACTIONS

by

EUGENE G. LUKAC
B.S., Lowell Technological Institute, 1968
M.S., University of Connecticut, 1970

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

in the Department
of
Physics

We accept this thesis as conforming to the
required standard

THE UNIVERSITY OF BRITISH COLUMBIA
August, 1975
In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the Head of my Department or by his representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Physics

The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date August 20, 1975
ABSTRACT

A one-dimensional chain of particles interacting through harmonic forces is used in a theoretical study of properties of the strength function which are of current interest in nuclear physics. The work was motivated by the observation of intermediate structure in the low-energy $^{12}\text{C}+^{12}\text{C}$ total reaction cross section. A nested hierarchy of harmonic oscillator systems is constructed to parallel the hierarchy of three stages through which the $^{12}\text{C}+^{12}\text{C}$ reaction is envisioned to proceed. The hierarchy consists of a "single particle" system in which one group of 12 particles interacts with another group of 12 particles through an average potential, a "doorway" system in which six groups of 4 particles interact with each other via another average potential, and a "compound" system in which all particles are grouped together.

The construction and usefulness of the model is discussed with reference to the choice of potentials, the elimination of spurious states, and the possibility of actually obtaining the wavefunctions. Overlap integrals of wavefunctions, in terms of which the strength function is defined, are shown to be doable although they are, in general, N-dimensional. For each step of the hierarchy the strength function is determined, and by analyzing the fluctuations and the effect of the density of states its shape is compared to the widely used Lorentzian shape. Group theory is used to study the effect of degeneracy on the overlap integrals. Without the inclusion of quartet clustering in the last step of the hierarchy, the ratio of widths of the strength function in the hierarchy is found not to correspond to the ratio obtained in the $^{12}\text{C}+^{12}\text{C}$ experiments. Attention is drawn to the consequences of using a Lorentzian strength function for cross section extrapolations.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>1-1. Evolution of a Quantum Mechanical State</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>1-2. The Role of Overlaps in Nuclear Physics</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>1-3. Calculation of the Strength Function from Nuclear Properties: Early Attempts</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>1-4. Experimental Status of Intermediate Structure in Heavy Ion Reactions</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>1-5. Stellar Reaction Rates</td>
<td>15</td>
</tr>
<tr>
<td></td>
<td>1-6. The Alpha Particle Doorway States</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>1-7. Statement of the Problem</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>FEATURES OF THE MODEL</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
<td>THE NORMAL COORDINATES</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>3-1. Meaning and Existence of Normal Coordinates</td>
<td>29</td>
</tr>
<tr>
<td></td>
<td>3-2. Linear Chain with Periodic Boundary Conditions</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>3-3. Linear Chain with Free Ends</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>3-4. Remarks</td>
<td>35</td>
</tr>
<tr>
<td>4</td>
<td>PROTOTYPE CALCULATIONS WITH FOUR PARTICLES</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>4-1. Compound System</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>4-2. Single Particle System</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td>4-3. Overlap of the Ground State Wavefunctions</td>
<td>47</td>
</tr>
<tr>
<td></td>
<td>4-4. Comparison of the Energies</td>
<td>53</td>
</tr>
<tr>
<td></td>
<td>4-5. Moments of the Strength Function</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>4-6. The Model Interactions as an Average Over Exact Interactions</td>
<td>60</td>
</tr>
</tbody>
</table>
Chapter 5. MODEL INTERACTION FOR SYSTEMS WITH N PARTICLES .......... 65

Chapter 6. OVERLAP INTEGRALS FOR A SYSTEM WITH 24 PARTICLES .......... 69

6-1. Single Particle System in Its Ground State ............... 69
6-2. Single Particle System in an Excited State ............... 75

Chapter 7. THE DENSITY OF STATES ............................................. 82

Chapter 8. THE SHAPE OF THE STRENGTH FUNCTION ....................... 85

Chapter 9. A HIERARCHY OF SYSTEMS ........................................... 93

9-1. The Hierarchy ......................................................... 93
9-2. Spreading Widths for Excited States .............................. 97

Chapter 10. SUMMARY AND DISCUSSION ...................................... 101

APPENDIX A. NUCLEAR POTENTIALS AND HARMONIC OSCILLATOR POTENTIALS .. 109

A-2. Construction of a Nucleon-Nucleon Potential ..................... 111

APPENDIX B. DIFFICULTIES WITH THE EXCLUSION PRINCIPLE ................. 114

APPENDIX C. PREDICTION OF NULL OVERLAP INTEGRALS ..................... 120

C-1. Structure of the Overlap Integral .................................. 120
C-2. Group Theoretical Prediction of Null Overlap Integrals .......... 123
C-3. The Invariance Groups of the Single Particle and the Doorway Systems ......................................................... 125
C-4. The Representations of the Common Invariance Group \( \tilde{G} \) .... 129
C-5. Classification of the Eigenfunctions .............................. 140

BIBLIOGRAPHY .............................................................................. 160
LIST OF TABLES

Table                                      Page

9-I. The normal mode frequencies of the three systems in
     the harmonic oscillator hierarchy ................. 96

C-I. Traces of the (reducible) representation D(\tilde{H}), and
     some of the irreducible representations d_i(\tilde{H}) ....... 134

C-II. Classification of fundamental wavefunctions according
      to energy and to representation of \tilde{G} ............... 148
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>The Lorentzian shape of $\langle \mathcal{C}^2_{\lambda;cp} \rangle / D$</td>
<td>9</td>
</tr>
<tr>
<td>1-2</td>
<td>A typical absorption cross section, normalized to a black nucleus cross section, as predicted by the optical model, and observed experimentally with low energy resolution</td>
<td>12</td>
</tr>
<tr>
<td>1-3</td>
<td>Typical intermediate structure in a cross section normalized to the black nucleus cross section</td>
<td>14</td>
</tr>
<tr>
<td>1-4</td>
<td>Schematic illustration of the integrand I(E) necessary for the calculation of the reaction rate for carbon burning at $0.75 \times 10^9 , ^\circ K$</td>
<td>16</td>
</tr>
<tr>
<td>1-5</td>
<td>The three steps in the $^{12}C + ^{12}C$ reaction according to the picture of Michaud and Vogt (1972)</td>
<td>19</td>
</tr>
<tr>
<td>1-6</td>
<td>The linear harmonic oscillator arrangements which correspond to the three stages of the $^{12}C + ^{12}C$ reaction illustrated in Figure 1-5 a), b), and c) respectively</td>
<td>24</td>
</tr>
<tr>
<td>4-1</td>
<td>Compound system with 4 particles</td>
<td>38</td>
</tr>
<tr>
<td>4-2</td>
<td>Single particle system with 4 particles</td>
<td>43</td>
</tr>
<tr>
<td>5-1</td>
<td>The maximum overlap, $C_{oo}^2$, between the ground states of the single particle and compound systems, and the corresponding single particle spring constant ratio, $k'/k$, as functions of the number of particles, N</td>
<td>67</td>
</tr>
<tr>
<td>6-1</td>
<td>The number, m, of 2-quanta compound states having non-zero overlap with the single particle ground state as a function of the compound state energy, $E_\lambda$</td>
<td>72</td>
</tr>
</tbody>
</table>
Figure

6-2. The average overlap squared, $\langle C^2_{\lambda}; cp \rangle$, between 2-quanta compound states and the single particle ground state as a function of compound state energy, $E_{\lambda}$ ............................... 72

6-3. The number, $m$, of 4-quanta compound states having non-zero overlap with the single particle ground state as a function of the compound state energy, $E_{\lambda}$ ............................... 73

6-4. The average overlap squared, $\langle C^2_{\lambda}; cp \rangle$, between 4-quanta compound states and the single particle ground state as a function of compound state energy, $E_{\lambda}$ ............................... 73

6-5. The strength, $\sum C^2/\Delta E$, of the single particle ground state in the compound states with 0, 2, and 4 quanta as a function of compound state energy, $E_{\lambda}$ ............................... 74

6-6. The strengths, $\sum C^2/\Delta E$, of the $n_{12}(2)$ single particle state in the compound states with 2, 4, and 6 quanta as a function of compound energy, $E_{\lambda}$ ............................... 78

6-7. Comparison of the strength $S^{(2)}$ of the $n_{12}(2)$ single particle state in 4-quanta compound states with the strength $S^{(2)'}$ of the single particle ground state in 2-quanta compound states ......................................................... 79

8-1. The distribution of $C^2/D$ about 10 local averages ............... 87

8-2. The strengths of Figure 6-6 compared to a Lorentzian strength function with parameters $\Gamma = 0.0918 \hbar \omega$ and $E_0 = 2.493 \hbar \omega$ obtained through a maximum likelihood procedure ......................................................... 90
9-1. The strength, $S_{s-d}$, of the $n_2(4)$ single particle state in 4-quanta doorway states, and the strength, $S_{d-c}$, of the $n_4(4)$ doorway state in 4-quanta compound states ...... 98

9-2. A possible modification of the compound system whereby quartet clustering is reproduced by taking $k >> k'$ .......... 100

B-1. A simple linear chain of particles .............................. 114

C-1. The single particle system ....................................... 126

C-2. The doorway system .................................................. 128
ACKNOWLEDGMENT

It is a real pleasure for me to express my appreciation to my supervisor, Professor Erich W. Vogt, for his guidance and encouragement throughout the course of my work at the University of British Columbia. His energy and enthusiasm have been a contagious inspiration not only in physics, but extending far beyond it into life's other realms.

The cheerful assistance of Dr. Dan Litvin with group theoretical questions greatly facilitated my task. I am indebted also to Robert Esch for many useful discussions and comments during the initial stages of this work.

Special thanks are due to Barbara Mitchell, whose ability to translate scribble into typescript has made the production of this manuscript much less painful than it would have been otherwise. I was fortunate also in obtaining important typing assistance from Diane Boyd.

Perhaps the most important contribution, however, has been made wittingly or unwittingly by those scores of people, known or unknown, who by their attitudes, greetings, words, and smiles, whether for an instant or through an extended period, have made my days pleasanter, my vision higher, my heart warmer. To them I owe my ability to enjoy my work, and to them I wish to dedicate this thesis.
CHAPTER 1
INTRODUCTION

How does the state of a quantum mechanical system evolve when it is disturbed? This question, as old as quantum mechanics itself, has been asked by physicists in almost every branch of physics. In the mid 1950's it confronted nuclear physicists as the key to accounting for the spectacular successes of the optical model for nuclear reactions and the shell model for nuclear structure. Yet, the intractability of a realistic many-nucleon calculation forced the question to be abandoned after some artificial, though elegant, mathematical modeling. Recently, however, the observation of nuclear reactions proceeding in a hierarchy of steps not only provides a new setting for the question, but adds to it a new urgency due to the consequent uncertainty in stellar reaction rates.

In this introductory chapter the opening question will be given a quantitative formulation in Section 1-1. Section 1-2, showing how the question arises in nuclear physics via its role in the justification of the optical model, will be followed by Section 1-3 with a discussion of previously attempted solutions. The new setting for the problem is revealed by reviewing in Section 1-4 the recent experimental observations, indicating in Section 1-5 their importance to astrophysical calculations, and presenting in Section 1-6 a theoretical framework within which the observations may be understood. The foregoing, serving as a historical introduction, leads in Section 1-7, to the statement of the problem to which this thesis addresses itself together with the assumptions adopted for its solution.
1-1 Evolution of a Quantum Mechanical State

Wanted in this section is a quantitative meaning for the question, "How does the state of a quantum mechanical system evolve when it is disturbed?" Begin by considering a system of $N$ particles in an eigenstate $\phi_m$ with energy $E'_m$ of the Hamiltonian $H'$. Now disturb the system by permitting among the $N$ particles a new interaction which will be embodied in a new Hamiltonian $H$. The two Hamiltonians $H'$ and $H$ may, in practice, be different approximations to the true Hamiltonian for the system. The new Hamiltonian $H$ will have eigenstates $\chi_n$ and eigenvalues $E_n$, which will be assumed, for simplicity, to be discrete and non-degenerate. Since the states of an observable form a complete set, one may write, for example, a state $\phi_m$ as an expansion in terms of states $\chi_n$:

$$\phi_m = \sum_n c_{mn} \chi_n$$  \[1.1\]

where the expansion coefficient, $c_{mn} = \int \phi_m^* \chi_n \, d\tau$, is called "the overlap between the state $\phi_m$ and the state $\chi_n"$, or, simply, "the overlap". The square of the overlap, $c_{mn}^2$, can then be interpreted as the probability of finding the system in a particular state $\chi_n$ if the system had been originally prepared in the state $\phi_m$. In this sense the state $\phi_m$ evolves with probability $c_{mn}^2$ into the states $\chi_n$ of the new Hamiltonian. Thus, a plot of $c_{mn}^2$ versus some identifying feature of the state $\chi_n$, for example the energy $E_n$, gives a quantitative description of how a state $\phi_m$ evolves when it is disturbed.

In terms of the energy, one says that the system of $N$ particles initially having total energy $E'_m$ is found, after the Hamiltonian is changed from $H'$ to $H$, to have energy $E_n$ with probability $c_{mn}^2$. (The changing of the Hamiltonian rescinds the obligation to conserve energy.)
Thus, the energy level $E'_m$ is said to spread, or broaden itself, into the range of energies $E_n$ for which $C_{mn}^2$ is non negligible.

1-2 The Role of Overlaps in Nuclear Physics

This section will show how the overlaps defined in Section 1-1 enter into the understanding of nuclear reactions.

In 1954 Feshbach, Porter, and Weisskopf (1954) addressed themselves to the great richness of data for neutron scattering off nuclear targets from across the entire range of atomic weights, $A$. Although, even when averaged over resonances, there was a large variation in cross section across the range of $A$, the variation was small between target nuclei neighboring in $A$. The gradualness of this variation from one nucleus to the next led to the idea that the neutron cross sections did not depend on the detailed interaction between the neutron and the target, but, rather, that it depended, for the most part, on some gross features of the target nucleus – for example the nuclear radius which changes very little between neighboring nuclei. Furthermore, the then recent accomplishments of the shell model in nuclear structure calculations indicated that a nucleon moves independently inside a nucleus, and experiences the presence of the remaining nucleons only through a potential well typifying their average effect. Thus it was proposed by Feshbach, Porter and Weisskopf to represent the interaction of the incoming neutron with the particles of the target nucleus by means of a single, simple potential describing the interaction of the neutron with the nucleus as a whole. They were careful to point out, however, that this simplification of the interaction cannot hope to predict the details of the cross section, but only the overall behavior after one averages over the resonances of the compound nucleus. Thus, one begins with the
Breit-Wigner expression for the absorption cross section near a single s-wave resonance

\[ \sigma_c(\text{abs}) = \frac{\pi}{k_c^2} \frac{\Gamma_{\lambda_c} \Gamma_{\lambda}}{(E_{\lambda} - E)^2 + \Gamma_{\lambda}/4} \]

where \( k_c \) is the wavenumber of the incident particle of energy \( E \), \( E_{\lambda} \) is the resonance energy of the compound nucleus, \( \Gamma_{\lambda_c} \) is interpreted as the width for decay of the compound nucleus level \( \lambda \) into the entrance channel \( c \), and \( \Gamma_{\lambda} = \sum_c \Gamma_{\lambda c} \). Following Vogt (1962), the cross section in Eq. [1.2] is averaged by integrating over its energy dependence, and dividing by the average compound nucleus level spacing \( D_c \) to obtain

\[ \langle \sigma_c(\text{abs}) \rangle = \frac{2\pi}{k_c^2} \frac{\langle \Gamma_{\lambda c} \rangle}{D_c} \]

It is customary to write \( \langle \gamma_{\lambda c}^2 \rangle = 2P_c \langle \gamma_{\lambda c}^2 \rangle \) where the penetration factor \( P_c \) is proportional to the probability that the incident particle enter the interaction region when hindered by the Coulomb and angular momentum barriers, and the reduced width \( \gamma_{\lambda c}^2 \) depends only on the internal nuclear properties. If, in addition, one defines the strength function

\[ \zeta_c = \frac{\langle \gamma_{\lambda c}^2 \rangle}{D_c} \]

then

\[ \langle \sigma_c(\text{abs}) \rangle = \frac{4\pi}{k_c^2} P_c \zeta_c \]

which contains the important result that the giant resonances in the average cross section reflect the energy dependence of the strength
function. This expression for the average absorption cross section, derived here from isolated compound nuclear resonances, can be shown (Lane and Thomas 1958) to provide a good approximation even when these resonances overlap.

The simple potential (the optical potential) proposed by Feshbach, Porter, and Weisskopf (1954) was a complex square well of radius $R_0$, namely,

$$V = U_0 - i W_0 \quad \text{for} \quad r < R_0$$

$$= 0 \quad \text{for} \quad r > R_0$$

[1.6]

where the real part $U_0$ represents the average interaction of the incident particle with the target nucleus, and the imaginary part $W_0$ is responsible for the absorption of particles out of the incident beam into the compound nucleus. This role of $W_0$ can be easily seen by writing the time dependent Schrödinger equation for the incident particle inside the nuclear radius using potential [1.6]. The resulting time dependence of the single particle wavefunction is $\exp[-i(E-iW_0)t/\hbar]$ so that the mean lifetime of the state would be $\tau = \hbar/W_0$, while $E$ is one of the energy levels (discrete or continuous) of the real potential $U_0$. However, the finite lifetime imposes on the energy levels, because of the uncertainty principle, a width $\Gamma = \hbar/\tau = W_0$. This width is in addition to, and, usually, larger than the widths $\Gamma_p$ of the resonances of the real part of the potential. The mean free path of a particle in a state of lifetime $\tau$ is $\nu\Gamma$, which for a low incident energy nucleon incident on a typical well ($U_0 = -50$ MeV, $W_0 = 6.6$ MeV) is about 10 fm, comparable to a nuclear diameter. That a nucleon can traverse a nucleus without encountering the effects of the remaining nucleons was to be expected from the validity of the shell model, that heavy ion scattering will also be describable by
the optical model speaks for the unexpected persistence of clusters inside nuclei.

Calculation of the s-wave absorption cross section due to the potential [1.6] in the vicinity of a single resonance \( p \), again following Vogt (1962), yields,

\[
\sigma(ab) = \frac{4\pi k^2}{m k R_0} \frac{W_0}{(E_p - E)^2 + W_0^2}
\]

where \( m \) and \( k \) are the mass and wave number, respectively, of the incident particle, and \( E_p \) is its resonant energy. It is this cross section that represents the average compound nucleus cross section found in Eq. [1.5]. Comparing the s-wave part of Eq. [1.5] (\( p = kR_c \)) with Eq. [1.7] one obtains for the strength function

\[
S_c = \frac{k^2}{m R_c^2} \frac{1}{\pi} \frac{W_0}{(E_p - E)^2 + W_0^2}
\]

This simple approach, the nuclear optical model, had a surprising success in describing the trends of the neutron cross sections across the periodic table, and to this extent one accepts Eq. [1.8] as providing the connection between the strength function and the optical model parameters. However, it would, at the same time, be desirable to have an understanding in terms of detailed nuclear properties of why the strength function has the shape given in Eq. [1.8]. It was in response to this question that Lane, Thomas, and Wigner wrote their famous paper of 1955.

To understand the work of Lane, Thomas, and Wigner (1955), it is convenient to begin with the following definitions. Let \( \chi_\lambda \) be the eigenstates of the full Hamiltonian, \( H \), for \( N \) particles which obey a boundary condition at the origin and at the nuclear radius \( R \). Let \( \psi_p \) be the
eigenstates of the N particles partitioned into 2 clumps. Their Hamiltonian, $H'$, contains the interaction of the 2 clumps with each other through a potential $V'$ which is the average of the 2-body potentials $V_{ij}$,

$$[1.9] \quad V' = \left\langle \sum_{i=1}^{N'} \sum_{j=N'+1}^{N} V_{ij} \right\rangle_{\alpha \nu}.$$

The $u_\nu$ are the radial eigenstates of this potential satisfying the same boundary conditions as $\chi_\lambda$, and the $\psi_c$ contain all the remaining variables and quantum numbers for the N particles. Since both $\chi_\lambda$ and $\psi_c u_\nu$ form a complete set of states for N particles inside the nuclear radius, one can write

$$[1.10] \quad \chi_{\lambda} = \sum_{c_r} C_{\lambda; c_r} \psi_c u_\nu,$$

where the coefficients $C_{\lambda; c_r}$ are the overlaps described in Section 1-1.

Now, although the reduced widths $\gamma_{\lambda c}^2$ appearing through $\Gamma_{\lambda c}$ in the Breit-Wigner expression [1.2] are treated as parameters in the theory of resonance reactions, the $\gamma_{\lambda c}^2 \mathcal{M}$, in fact, express formally the probability per unit time that the entrance channel particles in the state $\psi_c$ be found on the surface of the compound nucleus whose wavefunction is $\chi_\lambda$, after the effects of the Coulomb and angular momentum barriers have been absorbed into the penetrabilities $P_c$. Thus, the reduced width amplitudes $\gamma_{\lambda c}$ are defined in terms of the integral of $\chi_\lambda \psi_c$ over all the compound nuclear coordinates except the separation of the channel constituents which is kept at $R_c$.

$$[1.11] \quad \gamma_{\lambda c} \equiv \left( \frac{\hbar}{2 M_c R_c} \right)^{1/2} \int \chi_{\lambda} \psi_c \ dS,$$
where $M_c$ is the channel reduced mass.

Putting Eq. [1.10] into Eq. [1.11] gives

$$\gamma_{\lambda c} = \sum_p \left( \frac{\hbar^2}{2M_c R_c} \right)^{1/2} u_x (R_c) C_{\lambda; cp}$$

So far this is an exact result. Lane, Thomas, and Wigner pointed out, however, that the average separation $d$ between the single particle levels $E_p$ is much larger than the average separation $D$ between the compound nucleus levels $E_\lambda$. Further, in contrast to the "strong-coupling", or "black nucleus" model, the optical model predicts (correctly) giant resonances at the single particle energies $E_p$. Thus in the mixture [1.10] of single particle states into compound states each one of the $\gamma_{\lambda}$ can be associated with a particular $u_p$. In other words, the coefficients $C_{\lambda; cp}$ are such that for each value of $\lambda$, there is only one value of the index $p$ for which $C_{\lambda; cp}$ contributes appreciably to the expansion [1.10]. This interpretation of the giant resonances permits the reduced widths, from Eq. [1.12], to be written

$$\gamma_{\lambda c} = \left( \frac{\hbar^2}{2M_c R_c} u_x (R_c) \right)^2 C_{\lambda; cp}$$

where the first factor on the right side is the single particle reduced width, and the $C_{\lambda; cp}^2$ indicates how the coupling of the entrance channel into the compound nucleus reduces the single particle reduced width into the actual reduced width $\gamma_{\lambda c}^2$.

Now, since for the square well $u_p^2(R) = 2/R$, one may compare the average of the reduced widths in Eq. [1.13] with the square well strength function [1.8] to find
\[ [1.14] \quad \frac{\langle C^2_{\lambda;cp} \rangle}{D} = \frac{1}{\pi} \frac{W}{(E_p - E)^2 + W^2} \]

which is the well known Lorentzian shape illustrated in Figure 1-1.

![Fig. 1-1. - The Lorentzian shape of $\langle C^2_{\lambda;cp} \rangle/D$](image)

The generality of the above discussion is not restricted by its illustration in terms of square wells. Various "diffuse edge" potentials have been used successfully (Woods and Saxon 1954), and, furthermore, Michaud, Scherk, and Vogt (1970) have shown that the effects of such potentials may be reproduced by an appropriately constructed square well.

To recapitulate, then, the optical model leads to a strength function which, in general, is a sum of Lorentzians. Lane, Thomas, and Wigner indicated that one of these terms, $\langle C^2_{\lambda;cp} \rangle/D$ with a given $p$, will dominate this sum over $p$. Further, they assigned the Lorentzian shape of the $\langle C^2_{\lambda;cp} \rangle/D$ to the behavior of the $C^2_{\lambda;cp}$ rather than to the effect of the spacing $D$. It is to be noted that the Lorentzian shape in Eq. [1.14] for $\langle C^2 \rangle /D$, which leads to the optical model strength function, arose from Lane, Thomas, and Wigner's assignment of the giant resonance structure to
the behavior of the $C^2_{\lambda;cp}$. To put it differently, it is because the squared overlaps, $C^2_{\lambda;cp}$, have a giant resonance behavior that the optical model had been successful in reproducing the giant resonances in the data. This leads immediately to the question "Can the giant resonance behavior assigned by Lane, Thomas, and Wigner to the $C^2_{\lambda;cp}$ be connected to nuclear properties?", which gave birth to the next generation of research activities. It is these efforts which will be described in Section 1-3.

1-3 Calculation of the Strength Function from Nuclear Properties:

**Early Attempts**

To calculate the overlaps $C_{\lambda;cp}$ between the compound nuclear state $\chi_\lambda$ and the composite nuclear state $\psi_{cp}$ involving $N$ particles each moving in a 3-dimensional world is an extremely difficult problem, and, in fact, impossible given the present day understanding of nuclear forces and the capabilities of present day computers. One would like, ideally, to obtain the compound states $\chi_\lambda$ by solving

$$H \chi_\lambda = E_\lambda \chi_\lambda$$

and to obtain the single particle states $\psi_{cp}$ by solving

$$H' \psi_{cp} u_\rho = E_{cp} \psi_{cp} u_\rho$$

where the Hamiltonians $H$ and $H'$, and the boundary conditions are those described in Section 1-2. The straightforward calculation of $C_{\lambda;cp}$ requires, then, the evaluation of the $3N$-dimensional integral

$$C_{\lambda;cp} = \int \chi_\lambda^* \psi_{cp} u_\rho \, dV.$$
Faced with the unfeasibility of the direct procedure just described, Wigner (1955), (1956), and Bloch (1957) approached the problem by expressing $\langle C_2^\lambda; \alpha \beta \rangle / D$ in terms of the expectation value in the single particle states of the Green's function $G(E) = (E-H)^{-1}$, which may be expanded as a power series in $H_r$, where $H_r$ is the residual interaction ($H_r \equiv H-H'$) given by the difference between the exact interaction in $H$ and the single particle (or average) interaction in $H'$. This series converges provided the matrix of $H_r$ in the single-particle states is a bordered matrix with finite norm. To this end Wigner assumed the levels $E_{\alpha \beta}$ to be equidistant, and the matrix elements $\langle \alpha \beta | H_r | \gamma \delta \rangle$ to have the value $+V$ or $-V$ with equal probability if $|E_{\alpha \beta} - E_{\gamma \delta}| < \epsilon$, and to have the value zero if $|E_{\alpha \beta} - E_{\gamma \delta}| > \epsilon$. The model of Bloch is a slight generalization of these conditions. By keeping only the lowest order terms in $1/\epsilon$ the series may be summed to obtain the Lorentzian behavior of $\langle C_2^\lambda; \alpha \beta \rangle / D$ as in Eq. [1.14].

However, by adopting assumptions on the residual interaction $H_r$ to cater to the mathematical tractability of the problem, one is left with a mathematical artifice that is no longer able to answer the question "What is the physical reason for the Lorentzian shape?" In the careful words of Wigner, "the model which underlies the present calculation shows only a limited similarity to the model which is believed to be correct". At this point, partly because of the difficulty of the problem, and partly because the optical model parameters could always be obtained by fits to the data, the problem was abandoned without a satisfactory reason having been found for the giant resonance behavior of the squared overlaps, $C_2^\lambda; \alpha \beta$.

The next three sections will explore the source of the new impetus for the revival of the problem posed originally by the success of
the optical model; a revival necessitated now by its limitations.

1-4 Experimental Status of Intermediate Structure in Heavy Ion Reactions

The optical model discussed in Section 1-2 predicts the existence of giant resonances at the single particle energies $E_p$. This means that in an experiment involving the collision between nucleus A and nucleus B, which is done with low energy resolution (a spread in the incident energy of a few tens of keV), one will not observe in the absorption cross section the many narrow resonances of the compound nucleus A+B (typically a few keV apart around mass number 24), but a broad resonance, a few MeV in width, which is an average over many of the compound nucleus resonances, as illustrated in Figure 1-2. The centres of the broad resonances are associated with the levels, $E_p$, of the average potential through which particle A interacts with particle B.

![Graph](image)

**Fig. 1-2.** - A typical absorption cross section, normalized to a black nucleus cross section, as predicted by the optical model, and observed experimentally with low energy resolution. The markings on the energy axis indicate the positions of some of the compound nuclear levels which would be observed with much higher resolution.

The years following the introduction of the optical model for neutron scattering saw, as described above, its successive refinement to
include diffuse edge potentials as well as spin-orbit interactions, and its surprisingly successful extension to the analysis of heavy ion
scattering experiments (Shapiro 1962, Hodgson 1967). The first puzzle in the development of the optical model began emerging in 1960 with the observation at low energies (~ 6 MeV in the centre of mass) by Almqvist, Bromley, and Kuehner (1960) of 3 resonances in the \(^{12}\text{C} + ^{12}\text{C}\) reaction having a width and spacing of a few hundred keV. Ten years later the puzzle became consolidated with the discovery, upon extension of the measurements, that the same resonant structure was continued down to the lowest energies (~ 4 MeV) where statistical uncertainties are small enough to make it visible (Patterson, Winkler, and Zaidins 1969, Mazarakis and Stephens 1972).

What was the nature of this puzzle? It was precisely the width and spacing of these resonances that made them so intriguing. An order of magnitude larger than the compound nucleus resonances, their widths and spacings were yet one order of magnitude smaller than those one had learned to expect from the optical model. Specifically, the spacing of the single particle resonances, determined primarily by the reduced mass and radii of the interacting nuclei, is in this case approximately 8 MeV; the width, determined by the imaginary part of the optical potential, is 1 - 2 MeV. In contrast, the new resonances had a spacing of 0.3 MeV and a width of 0.1 - 0.2 MeV. On the other hand, the compound nucleus levels available in \(^{24}\text{Mg}\), though overlapping at the 20 MeV excitation energy probed by the \(^{12}\text{C} + ^{12}\text{C}\) experiment, can be estimated to be separated by 0.02 MeV and to have a width of about 0.12 MeV (Almqvist et al. 1964, Vogt et al. 1964, Voit et al. 1974). The new resonances were, thus, given the name "intermediate resonances" or "intermediate structure". In Figure 1-3 they can be seen encompassing many compound nucleus levels, and, in turn,
being encompassed and modulated by the giant resonance of the optical model.

\[ \sigma(\omega) \]

\[ E \]

Fig. 1-3. - Typical intermediate structure in a cross section normalized to the black nucleus cross section. The markings on the energy axis are some of the compound nucleus levels. The dashed line is the optical model cross section, which is, in fact, an average over the intermediate structure.

Although the procrustean exercise of adapting the optical model to fit the first three observed intermediate resonances was imaginatively attempted, it eventually became clear (Michaud and Vogt 1972) that no reasonable optical model potential for the $^{12}\text{C} + ^{12}\text{C}$ system could give rise to all the observed structure. The data, in particular the spacings of the intermediate resonances, suggested that they were due to the excitation of some simple internal degrees of freedom. It will be seen in Section 1-6 how this remark leads to a natural interpretation of intermediate resonances in terms of a hierarchy of structures for the 24 nucleons involved in the $^{12}\text{C} + ^{12}\text{C}$ reaction.

Since the first appearance of intermediate structure in the $^{12}\text{C} + ^{12}\text{C}$ system it has been sought systematically in other heavy ion
reactions (Hanson et al. 1974) though, at present, only $^{12}\text{C}^{16}\text{O}$ and 
$\alpha + ^{20}\text{Ne}$ besides $^{12}\text{C}^{12}\text{C}$ (Voit et al. 1974) exhibit the necessary correlation
of resonances in all exit channels that distinguish intermediate structure
from statistical fluctuations. These reactions, however, are just the ones
where one would expect to see intermediate structure according to the
framework of Michaud and Vogt, to be described in Section 1-6, which was
first advanced to account for it in the $^{12}\text{C}^{12}\text{C}$ system.

The description of the framework of Michaud and Vogt will be
postponed in order to first underscore in Section 1-5 the importance of a
detailed understanding of the $^{12}\text{C}^{12}\text{C}$ reaction for astrophysical calcula-
tions.

1-5 Stellar Reaction Rates

After the exhaustion of its helium supply in the production of
$^{12}\text{C}$, $^{16}\text{O}$, $^{20}\text{Ne}$, and some $^{24}\text{Mg}$, a star may achieve, either quiescently or
explosively, temperatures favorable for the heavy ion reaction $^{12}\text{C}^{12}\text{C}$.
Lack of detailed models for hydrostatic carbon-burning stars leads to
estimates (Baudet and Salpeter 1969, Arnett and Truran 1969, Arnett 1969,
Barnes 1971) in the range $0.3 \leq T_9 \leq 1.2$ for the most likely temperature at
which this occurs ($T_9 = 10^{-9} \text{TK}$). To calculate the reaction rate, $\langle \sigma v \rangle$
per cm$^3$ per sec, one averages the product of the cross section $\sigma(E)$ and
the relative speed $v$ using the Maxwell-Boltzman distribution $N_T(v)$ for the
number of pairs per cm$^3$ having relative speed $v$ at the temperature $T$ to
obtain (Reeves 1968)

$$\langle \sigma v \rangle = \int_0^\infty N(E) v(E) \sigma(E) dE$$

[1.18]
For the case of the $^{12}\text{C} + ^{12}\text{C}$ reaction at $0.75 \times 10^9$ °K the nature of the integrand, $I(E) = N(E) \nu(E) \sigma(E)$, of Eq. [1.18] is illustrated in Figure 1-4.

![Figure 1-4](image)

**Fig. 1-4.** - Schematic illustration of the integrand $I(E)$ necessary for the calculation of the reaction rate for carbon burning at $0.75 \times 10^9$ °K. After Reeves (1968)

It is seen from Figure 1-4 that the integrand $I(E)$ has a maximum, the "Gamow peak", which is centered at 2 MeV, and has a width of 0.8 MeV. Thus the Gamow peak occurs in the tail of the Maxwell distribution, and in a region where the Coulomb barrier has already caused the cross section to plummet from $\sim 10^3$ mb at 10 MeV to $\sim 10^{-5}$ mb at 2.5 MeV, the lowest energy at which the cross section has been measured.

Thus, the calculation of astrophysical reaction rates requires knowledge of the cross section at energies where it is too small to be accessible to laboratory measurements. It is here that the optical model must rise from its descriptive role to the position of a tool for extrapolation. Although it is possible to extrapolate the cross section to lower energies by obtaining a phenomenological fit to the experimental data at higher energies, such a process is sensitive to the assumptions made regarding the dominant physical features below 2.5 MeV, leading, at
present, to an uncertainty factor of 10 in the reaction rate (Michaud and Vogt 1972).

This uncertainty demands a revival of the search (last taken up by Bloch (1957) as described in Section 1-3) for those physical features in a nuclear reaction responsible for the giant resonances in the cross section. Further, since, as explained in Section 1-4, a giant resonance represents an average over the intermediate resonances, it is necessary, in turn, to expose as well the physical features giving rise to the intermediate structure. It is thus that Wigner and Bloch's endeavor to clarify the origin of the giant resonances acquires, thanks to the richness of the $^{12}\text{C} + ^{12}\text{C}$ system, the dual task of understanding both the giant resonances and the intermediate resonances. It was in this context that in 1972 Michaud and Vogt opened a door to future approaches by introducing an $\alpha$-particle framework which permits a qualitative understanding of the intermediate structure in the $^{12}\text{C} + ^{12}\text{C}$ reaction. An outline of this description will now be given in Section 1-6.

1-6 The Alpha-Particle Doorway States

Intermediate structure in nuclear cross sections is an observed phenomenon. Its interpretation centres on the concept of a "doorway state". This idea was already implicitly contained in a qualitative discussion of nuclear reactions given by Weisskopf (1961). The expression "doorway state" was coined by Block and Feshbach (1963) who first used it to interpret the systematic deviations of the neutron strength function around the optical model predictions. A doorway state refers to a simple nuclear excitation ($2$ particle $- 1$ hole in the treatment of Block and Feshbach) conceived as the only way the entrance channel ($1$ particle $- 0$ hole) can begin to excite the compound nucleus by means of $2$-body forces.
Later work by Feshbach and collaborators (Feshbach, Kerman, and Lemmer 1967, Kennedy and Shrills 1968, Feshbach 1973a, Feshbach 1973b) extended the concept to permit more general types of doorway states.

Michaud and Vogt (1972) were the first to conceive of $\alpha$-particle doorway states as appropriate for the $^{12}\text{C}+^{12}\text{C}$ reaction as follows. After the carbon nuclei enter the region of nuclear interaction the reaction is envisioned as occurring in three steps. In the first step the two carbon nuclei interact through some average $C-C$ potential, $V_{ij}^{(C-C)}$, whose resonances give rise to the large "single particle" structure in the cross section. The large width indicates that, as one would expect, the carbon nuclei are not able to maintain their identity for very long in the presence of the residual interaction which is due to the forces the three alpha particles in one carbon exert on those of the other. These $\alpha-\alpha$ interactions occurring through an average potential $V_{ij}^{(\alpha-\alpha)}$ soon break up each of the 2 carbons into 3 alpha particles. (It is easier to remove an $\alpha$-particle than a nucleon from carbon - the former costs 7.4 MeV, the latter 18.8 MeV). The increased number of degrees of freedom gives rise to the narrower spacing of the intermediate resonances, and the high stability of alpha particles suggests its role in determining their narrower width compared to single particle resonances. Thus the doorway states give rise to the intermediate structure as the resonances of a system of 6 alpha particles interacting through the potential $V_{ij}^{(\alpha-\alpha)}$. Eventually, however, even the alpha particles will succumb to the nuclear forces since they are not yet eigenstates of the full Hamiltonian for 24 nucleons interacting pair wise through potentials $V_{ij}^{(N-N)}$. The residual interaction due to $V_{ij}^{(N-N)}$ acting between members of different alpha particles tears them apart to form the compound nuclear system $^{24}\text{Mg}$ whose excited states are the closely spaced compound nucleus resonances. These
three steps in the reaction are illustrated pictorially in Figure 1-5.

![Diagram showing three steps in the reaction](image)

Fig. 1-5. - The three steps in the $^{12}\text{C}+^{12}\text{C}$ reaction according to the picture of Michaud and Vogt (1972). The type of potential which dominates the behavior at each stage is indicated. The depiction of $^{12}\text{C}$ as three $\alpha$-particles derives from the known clusterization properties of $^{12}\text{C}$ (Igo et al. 1963) as well as the success of $\alpha$-particle modeling for its ground state (Ikeda et al. 1972, Mendez-Moreno et al. 1974).

The above interpretation of the $^{12}\text{C}+^{12}\text{C}$ reaction proceeding into the compound nucleus $^{24}\text{Mg}$ through $\alpha$-particle doorway states received striking confirmation from the experiments of Voit et al. (1973) who found that quartet states in $^{20}\text{Ne}$ are preferentially populated in the reaction $^{12}\text{C}(^{12}\text{C},\alpha)^{20}\text{Ne}$ at bombarding energies corresponding to maxima in the
intermediate structure. This same inconsistency with statistical model predictions for this reaction had been noticed earlier by Middleton et al. (1971) at slightly higher energies.

It is possible, as Michaud and Vogt have done, to envisage their picture of the $^{12}\text{C}+^{12}\text{C}$ reaction as proceeding through a natural nested hierarchy of systems - the single particle system, the alpha particle doorway system, and the compound nucleus system - and to express the states of each system as the eigenstates of a corresponding hierarchy of nested Hamiltonians. These three Hamiltonians are given, respectively, by

\[
\begin{align*}
H'' &= -\frac{\hbar^2}{2m_C} \sum_{i=1}^{2} \nabla_i^2 + \sum_{i<j}^{2} V''_{ij}(\mathbf{c}-\mathbf{c}) \\
H' &= -\frac{\hbar^2}{2m_\alpha} \sum_{i=1}^{4} \nabla_i^2 + \sum_{i<j}^{4} V'_{ij}(\mathbf{\alpha}-\mathbf{\alpha}) \\
H &= -\frac{\hbar^2}{2m_N} \sum_{i=1}^{8} \nabla_i^2 + \sum_{i<j}^{8} V_{ij}(\mathbf{N}-\mathbf{N})
\end{align*}
\]

where $m_C$ is the mass of a carbon nucleus, $m_\alpha$ is the mass of an alpha particle, and $m_N$ is the mass of a nucleon. At each stage an average potential is expressible as an average over the next most exact one. Thus,

\[
V'_{ij}(\mathbf{\alpha}-\mathbf{\alpha}) \equiv \langle \phi_{i}^{\alpha} \phi_{j}^{\alpha} \mid \sum_{\nu=1}^{8} \sum_{k=5}^{8} V_{\nu k}(\mathbf{N}-\mathbf{N}) \mid \phi_{i}^{\alpha} \phi_{j}^{\alpha} \rangle
\]
where $\phi_{\alpha_i}^j$ is the ground state wavefunction of the $i$th alpha particle (with nucleons 1-4) and $\phi_{\alpha_j}^j$ is the ground state wavefunction of the $j$th alpha particle (with nucleons 5-8); and

$$\sqrt{c_i''} (c - c) \equiv \langle \phi_{c_i} \phi_{c_i}^j \mid \sum_{k=1}^3 \sum_{l=5}^6 \sqrt{c_{k_l}'} (\alpha - \alpha) \mid \phi_{c_i} \phi_{c_i}^j \rangle$$

where $\phi_{c_i}^j$ is the ground state wavefunction of the $i$th carbon nucleus (with $\alpha$'s 1-3) and $\phi_{c_j}^j$ is the ground state wavefunction of the $j$th carbon nucleus (with $\alpha$'s 4-6). The residual interaction, $H_r'$, spreading single particle states into doorway states is

$$H_r' = H' - H''$$

and the residual interaction, $H_r$, spreading doorway states into compound states is

$$H_r = H - H'$$

This framework accounts qualitatively not only for all the characteristics of the low energy $^{12}C + ^{12}C$ reaction (giant, intermediate, and compound resonances, their widths and spacings, and the branching ratios in the exit channel), but also for all the other cases where intermediate structure has been seen in heavy ion reactions (Voit et al. 1974). Its only shortcoming seems to be the present impossibility of quantifying the description of the systems specified by Hamiltonians [1.19] through [1.21] using nuclear forces for 24 particles in 3 dimensions. However the success of this scheme, which is based simply on the
hierarchial grouping of the 24 nucleons involved, compels one to believe that a treatment which is inattentive to some of the complexities of real nuclei, but which preserves the distinctive groupings of the model of Michaud and Vogt, might be capable of quantitatively reproducing its features as well as providing a connection with the Lorentzian shape of the resonances. It is this belief which motivated the work which is reported in this thesis.

1-7 Statement of the Problem

The aim of this thesis is two-fold. On the one hand the intent is to reopen the question asked by Wigner and Bloch (see Section 1-3) regarding the Lorentzian shape of the strength function by using a more physical model than the contrived random matrices employed previously. On the other hand it is sought to determine whether clustering alone as embodied in a hierarchy of interactions can reproduce the intermediate structure seen in low energy $^{12}\text{C}+^{12}\text{C}$ scattering. These two aspects will be folded into each other by calculating the spread of a single particle state into doorway states, and, in turn, calculating the spread of doorway states into the compound states. In each case the shape of the energy dependence of the average overlap coefficients squared, $\langle c^2 \rangle$, divided by the average level spacing, $D$, will be compared to the Lorentzian shape of the optical model (see Eq. [1.14]).

Motivated by the desire for some physical realism, and, yet, restricted by the requirement of mathematical solubility, two initial assumptions for the N-body nuclear system will be made which, though drastic, will not interfere with the questions addressed. The first is to confine the N nucleons to a one-dimensional space thus disregarding the possible effect of the ability of the particles to get around each other
on the shape of the strength function. This assumption invites itself not only because of the experimentally observed one-dimensional chains of a particles in states of $^{12}_C$ (Ikeda et al. 1972), but also because of the repeated usefulness of one-dimensional prototypes in physics (Lieb and Mattis 1966). The second is to approximate the nucleon-nucleon force by a harmonic force chosen to best simulate its effects. The extent to which this can be done will have to be examined, but, more importantly, it will be shown that the problem can be cast in a form which makes the results insensitive to the particular choice made. The ensuing system of $N$ one-dimensional particles with harmonic interactions is soluble: the eigenfunctions of any Hamiltonian (single particle, doorway, compound) can be found. Further, the overlap integral, $C$, between any two wavefunctions, even though it is $N$-dimensional, can always be evaluated, and thus one can directly study the behavior of the average overlaps, $<C^2>$.

In particular, to each of the three steps of the $^{12}_C+^{12}_C$ reaction as depicted in Figure 1-5 there corresponds an arrangement of harmonic oscillators as illustrated in Figure 1-6.

It is to be emphasized that one does not pretend that the linear chain of Figure 1-6c is a replica of the nucleus $^{24}_Mg$. The intention is simply that, to the extent that the various structures in the cross section for $^{12}_C+^{12}_C$ are due to the clusterings of Figure 1-5, the relationship between the systems of Figures 1-6a, b, and c will duplicate the relationship between the systems of Figures 1-5a, b, and c. That is, for the purposes of studying the successive spreading of simpler states into more complex ones leading to giant resonances, intermediate resonances, etc, as well as for obtaining the first quantitative evaluation of a strength function, one may avail oneself of the simplicity of harmonic
Fig. 1-6. - The linear harmonic oscillator arrangements which correspond to the three stages of the $^{12}\text{C}+^{12}\text{C}$ reaction illustrated in Figure 1-5 a), b), and c) respectively. The Hooke's constants for each type of spring are indicated.
oscillators while preserving those physical features under investigation. This thesis constitutes the actualization of this tenet.

With this intent, Chapter 2 discusses certain features of the model. The normal coordinates are presented in Chapter 3. Chapter 4 collects the prototype calculations which were performed first with four particles. Model interactions for N particles, treated in Chapter 5, are followed by overlap calculations in Chapter 6, and by a study of the density of states in Chapter 7, leading to their incorporation, in Chapter 8, into the evaluation of the strength function. The hierarchial system of Figure 1-6 finally makes its appearance in Chapter 9 where the experience of the previous four chapters is applied to calculate the wavefunctions, their overlaps from model to model, and the corresponding strength functions. Lastly, in Chapter 10, the results are summarized to permit a discussion in the spirit of the questions raised by this introduction.
CHAPTER 2
FEATURES OF THE MODEL

In constructing a physically relevant harmonic oscillator representation of nuclei attention must be paid to several features. This chapter describes briefly how some of these are handled. Further details will be found in the appendixes.

One asks first what choice for the parameters $A$ and $B$ in the harmonic oscillator potential

$$V_H(r) = A + \frac{1}{2} (r-r_0)^2 B$$

will give the best approximation to a nuclear potential? It will be seen in what follows in this thesis, however, that the results are independent of this choice, and, thus, the question need not be pursued here. A discussion of this point is nevertheless given in Appendix A where it is shown that it is difficult for a harmonic oscillator potential to reproduce the effect of a nuclear potential with a stiff core, but that one can construct a nuclear potential that is amenable to a harmonic oscillator approximation.

Secondly, before one can confidently calculate the wavefunctions for a system of particles subject to harmonic oscillator forces, it is necessary to ascertain that no spurious states will be introduced due to the motion of the centre of mass of the system. In the nuclear shell model, where wavefunctions are referred to a fixed external origin, there are introduced three extra degrees of freedom which correspond to the motion of the nucleus as a whole. The shell model wavefunctions will thus contain more states than the wavefunctions of the internal motion.
Elliott and Skyrme (1955) have shown how to deal with the spurious states in this case. For the case of one-dimensional harmonic oscillator interactions the problem is considerably simpler as will be now shown.

Consider for illustration two particles of mass $m$ in a harmonic oscillator potential of spring constant $k$ fixed at the origin. In units where \( \hbar = m = k = 1 \), the Hamiltonian of the system is

\[ H_0 = -\frac{1}{2} \gamma r_1^2 - \frac{1}{2} \gamma r_2^2 + \frac{1}{2} r_1^2 + \frac{1}{2} r_2^2 \]

Now, if the two particles are in a harmonic oscillator potential fixed at their centre of mass, then, in the same units, the Hamiltonian describing the (internal) motion of the system is

\[ H_{\text{int}} = -\frac{1}{2} \gamma r_1^2 - \frac{1}{2} \gamma r_2^2 + \frac{1}{2} (r_1 - R)^2 + \frac{1}{2} (r_2 - R)^2 \]

where \( R = (r_1 + r_2)/2 \) is the coordinate of the centre of mass. Therefore, one may write

\[ H_0 = H_{\text{int}} + R^2 \]

which reveals that the difference between the two Hamiltonians in question is a term describing the force on the centre of mass. Elliott and Skyrme (1955) give a prescription for obtaining the eigenfunctions of \( H_{\text{int}} \) in terms of the eigenfunctions of \( H_0 \) which are the shell model eigenfunctions.

However, by noticing that \( H_{\text{int}} \) may be written as

\[ H_{\text{int}} = -\frac{1}{2} \frac{\partial^2}{\partial r_1^2} - \frac{1}{2} \frac{\partial^2}{\partial r_2^2} + \frac{1}{4} (r_1 - r_2)^2 \]
and by remarking that the essential feature of $H_{\text{int}}$ is that the potential energy term depends only on the relative coordinates, one may, for harmonic oscillator interactions, write the Hamiltonian directly in the form [2.5], and directly obtain the internal eigenfunctions. The effectiveness of this procedure will be explicitly portrayed in Chapter 3 which deals with normal coordinates, and in Chapter 4 where prototype calculations are performed using a system of four particles.

The Pauli exclusion principle is a feature which one would like to take into consideration in the model. However, it is argued in Appendix B that this added realism would make the model completely unmanageable, and thus the rest of this work restricts itself to the treatment of unsymmetrized wavefunctions. Often in physics (see, for example, Lane, Thomas, and Wigner, 1955) the neglect of the Pauli principle has resulted in a useful first approach. Already the present model has averted the artificiality of the models described in Section 1-3, and, in fact, concentrating on the arrangement of the particles in various linear chain configurations while disregarding their fermion character is not out of keeping with the intent of exploring the simplest physics which might be responsible for the intermediate structure in heavy ion reactions described in the introductory chapter.
CHAPTER 3

THE NORMAL COORDINATES

This chapter deals with a set of coordinates called normal coordinates which can be obtained by a transformation on the particle coordinates, and in terms of which the classical or quantum mechanical solution of a system with harmonic forces may be readily obtained. Section 3-1 defines these new coordinates and establishes their existence for a wide class of systems. Certain simple cases can be worked out explicitly in closed form and this is done in Section 3-2 and 3-3. A few speculative remarks regarding normal coordinates in Section 3-4 close this chapter.

3-1 Meaning and Existence of Normal Coordinates

The Hamiltonian appropriate to a system of N particles coupled to nearest neighbors by harmonic forces is

\[ H = \frac{1}{2} \sum_{i=1}^{N} \left[ m \dot{r}_i^2 + \kappa (r_i - r_{i+1})^2 \right] \]

where all the particles have mass m, and the \( i \)th particle is displaced by \( r_i \) from its equilibrium position. Dotted quantities have been differentiated with respect to time. With \( r_{n+1} = r_n \) the Hamiltonian in Eq. [3.1] serves for a chain with free ends, and with \( r_{n+1} = r_1 \) it serves for a chain with periodic boundary conditions (which can be visualized as placing the particles on the circumference of a circle).

The Hamiltonian in Eq. [3.1] which is designed to, and does, couple the particles to each other, is for the same reason not in the form most conducive to a solution. For a system of harmonic oscillators
the solutions, both classical and quantum mechanical, are straightforward if the Hamiltonian can be written as a sum of Hamiltonians each one describing the behavior of an isolated entity subject only to a single harmonic force. In the quantum mechanical case the wavefunction for the whole system is then given by the product of wavefunctions associated with each of the Hamiltonians being added. To effect this separation of the Hamiltonian one transforms from the particle coordinates $\mathbf{r}_1$, which may be written as a column matrix $\mathbf{r}$, to a new set of coordinates $\mathbf{x}$ via a matrix $A$ such that $\mathbf{r} = A \mathbf{x}$. The new coordinates $\mathbf{x}$ are called the "normal coordinates" for the system.

Do such "normal coordinates", $\mathbf{x}$, actually exist? In other words, under what conditions can one find a transformation $A$ connecting the particle coordinates to the normal coordinates? Consider the Hamiltonian in Eq. [3.1], and notice that it may be written as

\[ H = \frac{m}{2} \mathbf{r}^T \mathbf{1} \mathbf{r} + \frac{k}{2} \mathbf{r}^T \mathbf{U} \mathbf{r} \]

where $\mathbf{r}^T$ is the transpose of the column matrix $\mathbf{r}$, $\mathbf{1}$ is the $N \times N$ unit matrix, and the matrix $\mathbf{U}$ is, for example for a chain with free ends, given by

\[ \mathbf{U} = \begin{bmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 1 & -1 & \cdots & 0 \\
0 & -1 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 1
\end{bmatrix}_{N \times N} \]

It can now be clearly seen that the normal coordinates are those in which
the Hamiltonian $H$ in Eq. [3.2] is "diagonal", specifically in which the matrix $U$ is diagonal. Moreover, it should be noticed that the matrix $U$ is always a real symmetric matrix not only for springs between nearest neighbors, but for any arbitrary set of springs among $N$ particles. Recalling (see, for eg., Holm 1964) that if $U$ is a real symmetric matrix, there exists an orthogonal matrix $A$ such that $A^T U A$ is diagonal, the existence of normal coordinates has been established. Indeed, the Hamiltonian in Eq. [3.2] becomes

$$H = \frac{m}{2} \dot{x}^T \frac{1}{m} \dot{x} + \frac{k}{2} x^r (A^T U A) x$$

Obtaining matrix $A$ from any given $U$ may be done by standard analytical or numerical methods. There exist cases, however, where $A$ may be expressed in a closed form. The next sections examine some of these.

3-2 Linear Chain with Periodic Boundary Conditions

For a simple linear chain of $N$ particles interacting only with nearest neighbors through a harmonic force the Hamiltonian is, from Eq. [3.1], given by

$$H = \frac{1}{2} \sum_{n=1}^{N} \left[ m \dot{r}_{n}^2 + k (r_{n} - r_{n-1})^2 \right]$$

For periodic boundary conditions one takes $r_{n+1} = r_1$. Wanted now is a new set of coordinates $x$ given by a matrix transformation $r = A x$, such that the $x$'s are normal coordinates as defined in Section 5-1. Following Henley and Thirring (1962) one attempts a transformation in the form
Note that this expression satisfies the periodic boundary condition $r_{n+1} \equiv r_1$. Now, by substitution of transformation [3.6] into the Hamiltonian [3.5], and by using the formula

$$[3.7] \quad \sum_{n=1}^{N} \exp \left( \frac{2\pi i n (l-l')}{N} \right) = N \delta_{l,l'}$$

one obtains

$$[3.8] \quad H = \frac{1}{2} \sum_{k=1}^{N} m k^2 \dot{x}_k^2 + k \left( 4 \sin^2 \frac{\pi l}{N} \right) x_k x_k^*$$

where the superscript * indicates complex conjugation. This Hamiltonian, [3.8], now describes a system of $N$ independent harmonic oscillators, each one of frequency

$$[3.9] \quad \omega_k = \left( 2 \sin \frac{\pi l}{N} \right) \sqrt{\frac{k}{m}}$$

where $l(= 1, 2, \ldots N)$ labels a particular "normal mode frequency". Thus the coordinates $r_n$ defined by Eq. [3.6] are indeed normal coordinates for this problem.

One notes from Eq. [3.9], however, that $\omega_k = \omega_{N-l}$, so that the normal mode frequencies, except for $l = N$, and for $l = N/2$ (which exists only if $N$ is even), are doubly degenerate. Further, the normal coordinates $r_n$ are complex so that it is not possible at a glance to visualize, from the transformation matrix $A$, the motion of the particles. Nonetheless it is possible to determine the real motion of the particles by taking
linear combinations of two coordinates $r_n$ corresponding to the same frequency. These two complications do not arise in a linear chain with free ends as will now be seen in Section 3-3.

### 3-3 Linear Chain with Free Ends

For a simple linear chain with free ends consisting of $N$ particles interacting only with nearest neighbors through a harmonic force it is easier, for the purposes of this section, to deal with the [classical] equations of motion, rather than with the Hamiltonian. The reason for this is that in this case the closure relations which would be needed for uncoupling the Hamiltonian, such as Eq. [3.7], are not simple. Further, the equations of motion are linear in the coordinates, whereas the Hamiltonian is quadratic.

If the particles all have mass $m$ and the springs all have force constant $k$, the equation of motion for the $n$th particle is

$$[3.10] \quad m \ddot{r}_n = -k (r_n - r_{n+1}) - k (r_n - r_{n-1})$$

or, letting $\omega^2 = k/m$,

$$[3.11] \quad \ddot{r}_n = \omega^2 (r_{n+1} - 2r_n + r_{n-1})$$

For free ends one lets $r_0 = r_1$ and $r_{N+1} = r_N$. Wanted now is a transformation $r_n = \sum_{\ell=1}^{N} a_{n\ell} x_{\ell}$ such that the $x_{\ell}$ obey $\ddot{x}_{\ell} = -\omega_{\ell}^2 x_{\ell}$; that is they are normal coordinates. Putting these two expressions in Eq. [3.11], and using the fact that the $x_{\ell}$'s are independent, one obtains
as the condition which is to be satisfied by the elements \(a_{nl}\) of the 
transformation matrix \(A\). Following Weinstock (1971), a solution of 
Eq. [3.12] is given by

\[
[3.13] \quad a_{nl} = C_{l} \frac{\sin n \phi_{l} - \sin (n-1) \phi_{l}}{\sin \phi_{l}}
\]

where \(\phi_{l} \equiv \cos^{-1}[1 - \omega_{l}^{2}/2\omega^{2}]\). One may now use the boundary conditions for 
free ends which may be expressed as \(a_{N+1,l} = a_{N,l}\) to obtain the allowable 
values of \(\phi_{l}\). Thus putting the expression [3.13] into this boundary con­
dition one obtains

\[
[3.14] \quad \sin N \phi_{l} \left( \cos \phi_{l} - 1 \right) = 0
\]

from which it follows either that \(\cos \phi_{l} = 1\) so that \(\omega_{l} = 0\) for all \(l\) 
(which is the physically uninteresting N-fold degenerate case in which all 
modes are pure translation), or \(\sin(N\phi_{l}) = 0\) so that

\[
[3.15] \quad \omega_{l}^2 = 2 \omega^2 \left( 1 - \cos \frac{\pi}{N} (l-1) \right)
\]

for \(l = 1, 2, \ldots, N\). Eq. [3.15] gives N distinct normal mode frequencies 
\(\omega_{l}\), including the pure translation \(\omega_{1}\).

Although, in general it is not possible to give a closed form 
for the matrix \(A\) as was done in this and the previous sections, it is 
always possible, as mentioned in Section 3-1, to diagonalize the potential 
energy matrix \(U\) for any one-dimensional arrangement of particles and
springs.

3-4 Remarks

To put all the nucleons of a nucleus into a common potential well is a useful beginning for many nuclear problems. This standard practice is the nuclear shell model. The linear chain model being used in this thesis offers a different handle on certain nuclear problems. It is interesting, though not necessary for the present purposes, to speculate about the relationship between the two models. This section, then, appends to Chapter 3 a brief comment regarding the two models.

In the linear chain the normal coordinates are those which have an oscillatory motion independent of one another. In the shell model the particles are put in a common potential well independent of one another. This leaves room for conjecturing that the normal modes of the linear chain are the entities which correspond to the particles in the shell model. In both cases it is through excitation of these to higher states that the system absorbs energy. However, whereas in the shell model a ground state nucleus is kept from having all its particles in the energy level of the lowest energy particle by the Pauli principle, in the linear chain a ground state nucleus is kept from having all its normal modes at the energy level of the lowest energy normal mode by the fact that each one is subject to a different effective spring constant.

In conclusion, then, it has been seen in this chapter how normal coordinates may be treated in general. It should be reiterated that the normal coordinates by uncoupling the equations of motion or the Hamiltonian apply generally as an aid to both classical and quantum mechanical solutions. The next chapter while illustrating with a four-particle chain the techniques to be used in this work will deal explicitly
with the correspondence between classical and quantum mechanical systems.
CHAPTER 4

PROTOTYPE CALCULATIONS WITH FOUR PARTICLES

In order to incorporate the discussion to this point into a concrete example, and in order to provide a visualizable illustration for the techniques to be used henceforth, this chapter contains calculations using a system of four particles. The four particles move in one dimension and are subject to harmonic forces which in Section 4-1 are arranged to correspond to a compound nucleus, and in section 4-2 to correspond to a single particle state in nuclear reactions. The overlap of the ground state wavefunctions of the "compound" system and the "single particle" system is calculated in Section 4-3. The energies of the two systems are compared in Section 4-4. Section 4-5 examines the residual interaction between the two systems, and its moments in the ground state of the single particle system. These are shown to be related to the moments of the strength function. Finally, an alternative method for obtaining the single particle interaction as an average interaction is shown, in Section 4-6, to lead again to harmonic oscillator interactions.

It will turn out that, although the four particle system already contains much of the essential physics of interest, the small number of particles and low density of states results in one state of the compound system acquiring 97% of the strength of the single particle system. This precludes the possibility of a meaningful calculation of the strength function. The methods learned, though, can be applied to larger, richer systems.
4-1 Compound System

The compound system is characterized by having all the participating particles grouped together into one system. In the case of a linear chain with free ends having four particles and harmonic interactions between nearest neighbors it can be depicted as in Figure 4-1.

Fig. 4-1. - Compound system with 4 particles. k indicates the spring constant.

Let \( r_1 \) be the displacement to the right from equilibrium of the \( i \)th particle. If the particles all have mass \( m \) the [classical] Hamiltonian describing the system is

\[
H = \frac{m}{2} \left( r_1^2 + r_2^2 + r_3^2 + r_4^2 \right) + \frac{k}{2} \left[ (r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_3 - r_4)^2 \right].
\]

Using the results of Section 3-3 one transforms to a new set of coordinates \( \mathbf{x} \) using a matrix transformation \( \mathbf{A} \) such that \( \mathbf{r} = \mathbf{A} \mathbf{x} \). From Eq. [3.13] and Eq. [3.15] the elements of \( \mathbf{A} \) are given by

\[
\mathbf{a}_{jk} = C_k \frac{\sin j(k+1)\pi/N - \sin (j-i)(k+1)\pi/N}{\sin (k+1)\pi/N}
\]

where \( C_k \) is to be chosen such that \( \sum_j a_{jk}^2 = 1 \). Explicitly, then, one may write
and it is readily verified that $A A^T = A^T A = I$. Inserting this transformation, $[4.3]$, into the Hamiltonian $[4.1]$ gives

$$H = \frac{m}{2} \left( \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 + \dot{x}_4^2 \right) + \frac{k}{2} \left( 0.50 x_1^2 + 2.0 x_3^2 + 3.414 x_4^2 \right).$$

One notices that this Hamiltonian does not couple the $x$'s to each other, and, thus, the $x$'s are indeed normal coordinates for this system. Further, letting $\omega^2 = \sqrt{k/m}$ one sees from the Hamiltonian $[4.4]$ that the [classical] normal mode frequencies for this system are

$$\omega_1 = 0$$
$$\omega_2 = \sqrt{0.50} \omega = 0.766 \omega$$
$$\omega_3 = \sqrt{2.0} \omega = 1.414 \omega$$
$$\omega_4 = \sqrt{3.414} \omega = 1.846 \omega$$

which agree with the predictions which would be made using the general expression $[3.15]$ for normal mode frequencies.

Having written the classical Hamiltonian in terms of the normal coordinates it is necessary now to obtain from it the quantum mechanical Hamiltonian. This transition from a classical Hamiltonian is obtained by requiring that the generalized coordinates $q_n$ and the generalized momenta $p_n$ satisfy the commutation relations
It can be shown that the transformation [4.3] from particle coordinates to normal coordinates preserve the commutation relations [4.6], and thus in the present problem one can take $x_n$ as the generalized coordinates and $m \dot{x}_n$ as the generalized momenta from which it follows that the quantum mechanical Hamiltonian corresponding to Eq. [4.4] is

$$[4.7] \quad q_n = q_n \quad \quad p_n = -i \hbar \frac{\partial}{\partial q_n}$$

It can be shown that the transformation [4.3] from particle coordinates to normal coordinates preserve the commutation relations [4.6], and thus in the present problem one can take $x_n$ as the generalized coordinates and $m \dot{x}_n$ as the generalized momenta from which it follows that the quantum mechanical Hamiltonian corresponding to Eq. [4.4] is

$$[4.8] \quad H = H_{cm} + H_{inf}$$

where

$$[4.9] \quad H_{cm} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2}$$

$$[4.10] \quad H_{inf} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \frac{0.586 \hbar}{2} x_i^2 \right) + \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + \frac{2.0 \hbar}{2} x_j^2 \right) + \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_k^2} + \frac{3.414 \hbar}{2} x_k^2 \right)$$

It is well to point out that the coordinate $R$ of the centre of mass of the system is given by
so using transformation [4.3] one may write

\[ x_i = \frac{1}{2} \left( r_i + r_{i+1} + r_{i+2} + r_{i+3} \right) = 2 \cdot \mathcal{R} \]

Therefore the expression in Eq. [4.9] may be written

\[ H_{\text{cm}} = \frac{-\hbar^2}{2 (\gamma m)} \frac{\partial^2}{\partial \mathcal{R}^2} \]

and is seen to truly represent the motion of the centre of mass. Thus, as promised in Chapter 2, the problem has very effortlessly separated itself into the motion of the centre of mass and the internal motion. By considering only the states of the Hamiltonian \( H_{\text{int}} \) of Eq. [4.10] which does not contain \( x_1 \) one is sure not to include any spurious centre of mass states.

The Hamiltonian \( H_{\text{int}} \) in Eq. [4.10] describes three independent one-dimensional harmonic oscillators. The solution of the time-independent Schrödinger equation with \( H_{\text{int}} \) is the product of the three wavefunctions associated with each of the harmonic oscillators in \( H_{\text{int}} \). These wavefunctions are well known (see, for example, Merzbacher 1961) so that the total wavefunction may be immediately written down as
where the \( \omega_i \) are the classical normal mode frequencies given in Eq. [4.5], \( n_i \) is the number of quanta in the \( i \)th normal mode, and \( H_n \) is the Hermite polynomial of order \( n \). The energy eigenvalues corresponding to this wavefunction are

\[
E_{n_1 n_2 n_3} = (n_1 + \frac{1}{2}) \hbar \omega_1 + (n_2 + \frac{1}{2}) \hbar \omega_2 + (n_3 + \frac{1}{2}) \hbar \omega_3
\]

and it can be easily verified that the wavefunction is properly normalized so that

\[
\int_{-\infty}^{\infty} \chi^*_{n_1 n_2 n_3} \chi_{n'_1 n'_2 n'_3} \, dx_1 \, dx_2 \, dx_3 = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \delta_{n_3 n'_3}.
\]
The single particle system is characterized by having the participating particles grouped, actually, into two clumps. Each clump experiences the average field of the other as if it were a single particle—hence the name. If the two clumps interact via only a central force then, of course, the problem may be reduced to an equivalent one-body problem in terms of the reduced mass and the relative coordinates (Goldstein 1950).

The division of the particles into two clumps may ascribe any number of the available particles to one of the clumps. In the present case of four particles each clump will be chosen to contain half the particles since the physical problem eventually being aimed at is the interaction of $^{12}$C with $^{12}$C. These two clumps will be made to interact with each other through a harmonic force coupling their centres of mass. To allow for the fact that the strength of the interaction between the two clumps could be different from that of the interaction between two particles, a spring of constant $k'$ will be associated with it. The value of $k'$ will have to be discussed later. The single particle system can, thus, be depicted as in Figure 4-2.

![Diagram of single particle system with 4 particles](image)

Fig. 4-2. - Single particle system with 4 particles.

Bearing in mind that these are the same four particles belonging to the compound system of Section 4-1 but arranged differently here, one may
use the same notation and write the Hamiltonian describing this system as

\[ H' = \frac{m}{2} \left( \dot{\mathbf{r}}^2 + \dot{\mathbf{r}}_{2}^2 + \dot{\mathbf{r}}_{3}^2 + \dot{\mathbf{r}}_{4}^2 \right) + \frac{\hbar}{2} \left[ (\mathbf{r} \cdot \mathbf{r})^T + (\mathbf{r} \cdot \mathbf{r}) \right] + \frac{\hbar}{2} \left( \frac{\mathbf{r}_{1} \cdot \mathbf{r}_{2}}{2} - \frac{\mathbf{r}_{3} \cdot \mathbf{r}_{4}}{2} \right). \]

In matrix notation this may be written

\[ H' = \frac{m}{2} \mathbf{r}^T \mathbf{M}^{-1} \mathbf{r} + \frac{\hbar}{2} \mathbf{r}^T \mathbf{U} \mathbf{r}, \]

where

\[ \mathbf{U} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \frac{k'}{k} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}. \]

The normal coordinates, \( \mathbf{y} \), needed to uncouple \( H' \) are those in which \( \mathbf{U} \) is diagonal. Since \( \mathbf{U} \) is a real symmetric matrix it is diagonalizable by an orthogonal transformation, \( \mathbf{R} \), such that \( \mathbf{r} = \mathbf{R} \mathbf{y} \). Since for this arrangement of harmonic oscillators (see Figure 4-2) there exists no known transformation in closed form as for the compound system of Figure 4-1, the task requires the diagonalization of a 4x4 matrix for which there are standard methods. This algebraic procedure, however, is extremely tedious, and since it is possible in the case of four particles to guess at the solution one is led to try the transformation
It is readily verified that $B^T B = B^T B = 1$, and that inserting this transformation into the Hamiltonian $H'$ of Eq. [4.17] gives

$$[4.21] \quad H' = \frac{\hbar}{2} \left( \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2 + \dot{\gamma}_4^2 \right) + \frac{\hbar}{2} \gamma_1^2 + \frac{\hbar}{2} \gamma_2^2 + \frac{\hbar}{2} \gamma_3^2 + \frac{\hbar}{2} \gamma_4^2,$$

so that indeed the $y$'s are normal coordinates for this system. One sees also that the normal mode frequencies are

$$[4.22] \quad \omega_1 = 0 \quad \omega_2 = \sqrt{\frac{\mu}{m}} \approx \omega' \quad \omega_3 = \sqrt{2} \omega = 1.414 \omega \quad \omega_4 = \sqrt{2} \omega = 1.414 \omega \; .$$

As in Section 4-1, the quantum mechanical Hamiltonian corresponding to $H'$ in Eq. [4.21] is

$$[4.23] \quad H' = H'_{cm} + H'_{int} \; ,$$

where

$$[4.24] \quad H'_{cm} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \gamma_i^2} \; .$$
Again, the transformation [4.20] shows that \( H'_{\text{CM}} = H'_{\text{int}} \) and that it is the proper description of the centre of mass of the system. Thus the states of interest are the eigenfunctions of \( H'_{\text{int}} \), which describes three independent harmonic oscillators, so that the complete wave function associated with \( H'_{\text{int}} \) can be immediately written down as

\[
\psi_{n_x, n_y, n_z} = \left(2^{n_x} n_x!\right)^{1/2} \left(\frac{m \omega_x'}{\hbar}\right)^{1/2} H_n_x \left(\frac{m \omega_x'}{\hbar}\gamma_x\right) \exp\left(-\frac{m \omega_x'}{2 \hbar} \gamma_x^2\right)
\]

\[
\times \left(2^{n_y} n_y!\right)^{1/2} \left(\frac{m \omega_y'}{\hbar}\right)^{1/2} H_n_y \left(\frac{m \omega_y'}{\hbar}\gamma_y\right) \exp\left(-\frac{m \omega_y'}{2 \hbar} \gamma_y^2\right)
\]

\[
\times \left(2^{n_z} n_z!\right)^{1/2} \left(\frac{m \omega_z'}{\hbar}\right)^{1/2} H_n_z \left(\frac{m \omega_z'}{\hbar}\gamma_z\right) \exp\left(-\frac{m \omega_z'}{2 \hbar} \gamma_z^2\right)
\]

where the \( \omega_i' \) are the normal frequencies of Eq. [4.22] and the rest of the notation is as in the compound system solutions [4.14]. The energy eigenvalues corresponding to this wavefunction are

\[
E'_{n_x, n_y, n_z} = \left(n_x + \frac{1}{2}\right) \hbar \omega_x' + \left(n_y + \frac{1}{2}\right) \hbar \omega_y' + \left(n_z + \frac{1}{2}\right) \hbar \omega_z'
\]

and it can be easily verified that the wavefunction is properly normalized so that

\[
\int \cdots \int \psi_{n_x, n_y, n_z}^{*} \psi_{n_x', n_y', n_z'} \, d\gamma_x \, d\gamma_y \, d\gamma_z = \delta_{n_x, n_x'} \delta_{n_y, n_y'} \delta_{n_z, n_z'}
\]
Now that both the single particle solutions and the compound system solutions have been obtained, the next section will be devoted to calculating the overlap of the two ground state wavefunctions.

4-3 Overlap of Ground State Wavefunctions

In the context of this chapter one may restate the aim of this thesis as expressed in Section 1-7 by asking how good a model for the compound system of Figure 4-1 is provided by the single particle system of Figure 4-2. The question needs to be asked for each state of each system. If for each single particle state there is one compound state with which its overlap is close to unity, then the model is very good. If all the overlaps of the single particle state with compound states are close to zero, then the model is very bad. The behaviour with energy of the average overlap squared per average energy interval is the strength function (up to a constant) and it characterizes how a single state of the single particle system behaves when the particles enter a compound system arrangement. As explained in the introductory chapter, this is the quantity of interest in nuclear physics. Its evaluation requires the calculation of a great many overlap integrals. This section will show how one of these, namely the overlap of the ground states, is to be calculated.

Wanted, then, is the overlap

\[ C_{00} = \int_{-\infty}^{\infty} \psi_0^*(x) \chi_0(x) d\tau \]

where \( \chi_0(x) \) is the ground state of the compound system obtained from Eq. (4.14) by setting \( n_2 = n_3 = n_4 = 0 \), and \( \psi_0(x) \) is the ground state
of the single particle system obtained from Eq. [4.26] by setting \(n_2 = n_3 = n_4 = 0\). In order to evaluate the integral in Eq. [4.29] both functions in the integrand must be written in terms of the same variables. This is straightforward since, it will be recalled, \(\mathbf{r} = \mathbf{A} \mathbf{x}\) and \(\mathbf{r} = \mathbf{B} \mathbf{y}\) so that

\[
[4.30] \quad \mathbf{x} = \mathbf{A}^T \mathbf{B} \mathbf{y} \quad \text{and} \quad \mathbf{y} = \mathbf{B}^T \mathbf{A} \mathbf{x}.
\]

Letting \(\mathbf{C} = \mathbf{B}^T \mathbf{A}\), one has, explicitly,

\[
[4.31] \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .92 & 0 & -.38 \\ 0 & .27 & .70 & .64 \\ 0 & .27 & -.70 & .64 \end{bmatrix}
\]

One sees that \(\mathbf{C}\), which is the product of two orthogonal matrices, is, as expected, also orthogonal, and that the Jacobian of the transformation from \((x_2, x_3, x_4)\) to \((y_2, y_3, y_4)\) is 1 and thus preserves the normalization of the wavefunctions.

The integrand in the overlap consists then of the product of

\[
\Psi_0 (\chi) = \Psi_0 (\mathbf{C} \chi) = \left(\frac{m \omega'}{\hbar \pi} \right)^{1/4} \exp \left[ -\frac{m \omega'}{2 \hbar} (0.92 x_2 - 0.38 x_4)^2 \right]
\]

\[
[4.32] \quad \times \left(\frac{m \omega_3}{\hbar \pi} \right)^{1/4} \exp \left[ -\frac{m \omega_3}{2 \hbar} (0.27 x_3 + 0.70 x_3 + 0.4 x_4)^2 \right]
\]

\[
\times \left(\frac{m \omega_4}{\hbar \pi} \right)^{1/4} \exp \left[ -\frac{m \omega_4}{2 \hbar} (0.27 x_4 - 0.70 x_4 + 0.4 x_4)^2 \right]
\]
The overlap of these two wavefunctions will be a three-dimensional integral. In general, for N particles the overlap integral will be (N-1)-dimensional (since in the internal wavefunctions being considered the pure translational mode has been excluded). One notices, however, that the product \( \psi_0 \chi_0 \) contains the exponential of a general quadratic function of the \( x \)'s. The coefficients of the \( x \)'s in the quadratic contain the normal mode frequencies \( \omega_2, \omega_3, \omega_4, \omega_3, \omega_4 \), which are given in terms of \( \omega (\equiv \sqrt{k/m}) \) in Eq. [4.5] and Eq. [4.22]. Further, if \( k' = ak \), and one knows the number \( a \), then \( \omega_2 (\equiv \sqrt{k'/m}) \) can also be expressed in terms of \( \omega \) so that

\[
\psi_0 \chi_0 \propto \exp \left( -\frac{m \omega}{2k} x^T M x \right)
\]

where \( M \) is a real symmetric matrix containing all the numerical coefficients. Explicitly, \( M \) is given by

\[
M = \mathcal{C}^T \mathcal{W}' \mathcal{C} + \mathcal{W}
\]

where \( \mathcal{W}' \) is the diagonal matrix containing the ratios \( \omega_1/\omega \) along the diagonal, and \( \mathcal{W} \) is the diagonal matrix containing the ratios \( \omega_1/\omega \) along the diagonal. (In doing numerical work a trial value for the number \( \alpha (\equiv k'/k) \) must be inserted at this stage). Again, a real
symmetric matrix can always be diagonalized with an orthogonal transformation, and thus there exist a set of coordinates \( x \) obtained from \( \bar{x} \) by an orthogonal transformation \( \bar{x} = X z \) such that the overlap integrand can be written

\[
\tag{4.36}
\psi \chi \propto \exp \left[ -\frac{m \omega}{2k} z^T (X^T M X) z \right]
\]

The matrix \( (X^T M X) \) is diagonal. \( X \) is formed as usual from the eigenvectors of \( M \). Finally, since Eq. [4.36] permits the integrand to be written in the form

\[
\tag{4.37}
\psi \chi \propto e^{-\alpha z_1^2} e^{-b z_2^2} \cdots e^{-n z_n^2}
\]

the three-dimensional overlap integral becomes a product of three one-dimensional integrals.

This useful separation can be carried out for the overlap integral of any two linear harmonic oscillator states of \( N \) particles. Indeed, having obtained the matrix \( X \) from the eigenvectors of \( M \) in Eq. [4.35] one has

\[
\tag{4.38}
x = X z \quad \text{and} \quad y = Y z
\]

where \( Y = C X \). Inserting the transformations [4.38] respectively into any wavefunction \( \chi(x) \) and any wavefunction \( \psi(y) \), any overlap integral

\[
\int \cdots \int \psi(y) \chi(z) \, dz
\]

can be written as a product (or, at worst, a sum of such products) of integrals of the form

\[
\int f(z_i) e^{-\alpha z_i^2} \, dz_i
\]
Integrals of this type can be found in tables of integrals. Thus, one of the most important features of the present one-dimensional harmonic oscillator model is that it is a solvable problem for N bodies. That is, the N-dimensional overlap integrals can be done analytically with no more computer assistance than the diagonalization of NxN matrices.

Returning now to the four particle illustrative example of this chapter, one remarks that even diagonalization of a 4x4 matrix requires considerable effort if done by hand. Further, this cannot be done without committing oneself to specifying the ratio k'/k. For small numbers of particles, however, these difficulties can be bypassed using an alternative method as follows. Consider the overlap

\[ C_{oo} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_0(x_1) \Psi_0(x_2) \Psi_0(x_3) \Psi_0(x_4) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \]

with the wavefunctions as given in Eq. [4.32] and Eq. [4.33]. By noticing from Eq. [4.22] that \( \omega_1 \) = \( \omega'_1 \), the overlap can be written in the form

\[ C_{oo} \propto \int_{-\infty}^{\infty} dx_1 \, e^{-a x_1^2} \int_{-\infty}^{\infty} dx_4 \, e^{-b x_4^2} \int_{-\infty}^{\infty} dx_2 \, e^{-c x_2^2} \int_{-\infty}^{\infty} dx_3 \, e^{-d x_3^2} \]

The integral over \( x_3 \) is a standard form and with the aid of tables may be done immediately. An expression for the integral over \( x_2 \) is given Bierens de Haan (1867, Table 28, #1). The answer, in terms of \( x_4 \), is incorporated into the integral over \( x_4 \) which then becomes of the same form as the integral over \( x_3 \) and is done immediately. This procedure, using the \( \omega_1 ' \)s and \( \omega_1 '' \)s from Eq. [4.5] and Eq. [4.22] respectively, gives
At this point, to get a numerical answer, one has to fix the value of \( k' \) with respect to \( k \) (or, equivalently, \( \omega' \) with respect to \( \omega \)). What is being sought here is the strength function which serves as an indicator of how good a model for the compound system is provided by the single particle system. Therefore one would like to begin with that single particle system which is the best possible model for the compound system. One way to obtain this - other ways will be described in the following sections - is to require that the choice of \( k' \) maximize the overlap \( C_{oo} \) between the ground states of the two systems. Setting the derivative of \( C_{oo} \) in Eq. 4.41 with respect to \( \omega' \) equal to zero gives

\[ 0.42 \quad \omega' = 0.877 \omega \]

or, equivalently,

\[ 0.43 \quad k' = 0.769 k \]

With this value for \( k' \), the constant of the spring joining the two clumps in the single particle system, the overlap squared of the ground states is

\[ 0.44 \quad C_{oo}^2 = 0.973 \]
This high value indicates that the single particle system is a very good model indeed for the ground state of the compound system, and at the same time indicates that the strength function will resemble a spike. Since the sum of all the overlaps squared must add up to 1.0, overlaps other than $C_{oo}$ will be very small. The four particle system is in this sense too discrete to show the shape of the strength function. This is not the case when the number of particles is increased as will be shown in the next chapter.

Notice, however, that once having fixed the relationship between $k'$ and $k$ as in Eq. [4.43] one can obtain the numerical value of overlaps. In particular, as promised in Chapter 2, it is not necessary to have a numerical value for $k$ - the overlaps, and hence the strength function, are independent of $k$. This can also be seen using a dimensionality argument: since the overlaps are dimensionless numbers they can only contain the spring constants $k$ and $k'$ in ratios. There are, however, other ways besides the maximization of the ground state overlap to fix the value of $k'$ with respect to $k$. The next sections consider these alternatives.

4-4 Comparison of the Energies

It was pointed out in Section 4-3 that one would like the single particle spring constant $k'$ to be such that the single particle system is the best model possible for the compound system. This was done by requiring that the overlap between the ground states of the two systems be maximized. One may, alternatively, use the ground state
energies as a criterion, and require, as in this section, that the ground state energies for the two systems match.

From Eq. [4.15] and Eq. [4.5] the energy levels of the compound system are

\[ E_{n_{1}, n_{2}, n_{3}} = \frac{1}{\hbar} \omega \left[ 0.765 \left( n_{1}^{+} \right) + 1.414 \left( n_{2}^{+} \right) + 1.846 \left( n_{3}^{+} \right) \right] \]

From Eq. [4.27] and Eq. [4.22] the energy levels of the single particle system are

\[ E'_{n_{1}', n_{2}', n_{3}'} = \frac{1}{\hbar} \omega' \left( n'_{1}^{+} \right) + \frac{1}{\hbar} \omega \left[ 1.414 \left( n'_{2}^{+} \right) + 1.414 \left( n'_{3}^{+} \right) \right] \]

Setting the ground state energies equal to each other, \( E_{000} = E'_{000} \), gives

\[ \omega' = 1.199 \omega \]

and, consequently,

\[ k' = 1.437 \, k \]

This value for \( k' \) is different from the value for \( k' \) in Eq. [4.43] found by maximizing the overlap. One needs to ask how sensitive are the energy and the overlap to these changes in \( k' \)? Putting \( k' = 1.437k \) found in Eq. [4.48] by matching the energies into the expression [4.41] for the overlap gives
so that

$$C_{oo}(k' = 0.769 K) - C_{oo}(k' = 1.437 K) \quad C_{oo}(k' = 0.769 K) = 0.61 \%$$

On the other hand putting $k' = 0.769K$, found in Eq. [4.43] by maximizing the overlap, into the expression $E'_{ooo}$ in Eq. [4.46] for the ground state of the single particle system gives

$$\frac{E_{ooo} - E'_{ooo}}{E_{ooo}} = 8.0 \%$$

This reveals that both the energy and the overlap are remarkably insensitive to the choice of $k'$. In particular, the overlap changes by only 0.6% when $k'$ is changed by 100%. Thus, as is desired, it is the clustering arrangement of the particles which dominate the physics - the choice of interaction constants are playing a minor role.

The way the four particles are clustered in the two systems leads to two different Hamiltonians. Their difference is called the "residual interaction". The next section examines the role of the residual interaction in determining the shape of the strength function.

4-5 Moments of the Strength Function

One of the main concerns of this thesis is the question of the shape of the nuclear strength function. In the introductory chapter the strength function was defined in Eq. [1.4] as

$$C_{oo} = 0.962$$
\[ S_c \equiv \frac{\langle \gamma^2 \rangle}{D} \]

where in the giant resonance interpretation of Lane, Thomas and Wigner

\[ \gamma^2 = (\gamma_{cp}^{sp})^2 C_{\lambda;cp}^2 \]

The \((\gamma_{cp}^{sp})^2\) are the single particle reduced widths of Eq. [1.13]. In this interpretation the energy dependence of the strength function is contained in the energy dependence of the overlaps \(C_{\lambda;cp}^2\). The present approach using linear chains of harmonic oscillators permits, as seen in Section 4-3, the direct calculation of the \(C_{\lambda;cp}^2\) as a function of the compound system energy \(E_\lambda\), giving directly the shape of the strength function.

Information on the shape of the strength function is also contained in its moments with respect to the single particle energy levels \(E_{cp}\). It will now be shown how these moments can be expressed in terms of the moments of the residual interaction in the single particle wavefunctions \(\psi_{cp}\). Using the notation of the this chapter for single particle and compound system quantities, and letting \(V = H - H'\) be the residual interaction, one begins with the definition of the \(n\)th moment, \(M_n\), of the strength function

\[ M_n \equiv \int (E_\lambda - E_{cp})^n S_c (E_\lambda) \, dE_\lambda \]

\[ = \sum_{\lambda} (E_\lambda - E_{cp})^n \gamma^2_{\lambda c} \]
Therefore

\[ M_1 \propto (\Psi_{cp}, V \Psi_{cp}) \]

\[ M_2 \propto (\Psi_{cp}, V^2 \Psi_{cp}) \]

\[ M_3 \propto (\Psi_{cp}, V (V + H' - E_{cp}) V \Psi_{cp}) \]

In the present four-particle linear chain problem the residual interaction \( V \) is the difference between the compound system Hamiltonian \( H \) of Eq. [4.1] and the single particle system Hamiltonian \( H' \) of Eq. [4.17],
namely

\[ V = \frac{\hbar}{2} \left( r_r^1 - r_r^1 \right)^2 - \frac{\hbar'}{2} \left( \frac{r_r^r + r_r^r}{2} - \frac{r_r^r + r_r^r}{2} \right)^2 \]

where the \( r_r^1 \) are the particle coordinates. The single particle ground state wavefunction over which the moments are to be calculated is, from Eq. [4.26] and Eq. [4.22],

\[ \psi_o = \left( \frac{m \omega}{\hbar} \right)^{1/4} \left( \frac{1 + i \mu m \omega}{\hbar} \right)^{1/4} e^{-\frac{m \omega}{2 \hbar} y_1^2} e^{-\frac{1 + i \mu m \omega}{2 \hbar} y_2^2} e^{-\frac{1 + i \mu m \omega}{2 \hbar} y_3^2} \]

where the \( y_r^1 \) are the normal coordinates of this system.

To calculate the integral in the first moment

\[ M_1 = \left( \gamma_{sp}^{c_p} \right)^2 \int_{-\infty}^{\infty} \psi_o^* V \psi_o \ dy \]

it is necessary to express the residual interaction \( V \) in terms of the normal coordinates \( y \) by using the explicit form of the transformation \( \mathbf{r} = \mathbf{B} y \) given in Eq. [4.20] whereupon the integral, since the exponential does not couple different \( y_r^1 \)'s, is doable, yielding

\[ M_1 = \left( \gamma_{sp}^{c_p} \right)^2 \left[ \frac{\hbar}{4} \left( \frac{\omega}{\omega'} - \omega' \right) + \frac{\hbar \omega}{4 \sqrt{2}} \right] \]

For now it is sufficient to notice that the value of \( \omega' = 0.877 \omega \) obtained in Eq. [4.42] by maximizing the overlap of the wavefunctions does not make \( M_1 \) vanish, giving, in fact, \( M_1 = \left( \gamma_{sp}^{c_p} \right)^2 0.242 \hbar \omega \). This means that the strength function of the single particle system in the compound states is not centered at the single particle energy. This result was anticipated in Section 4-4 where it was shown, Eq. [4.51].
that with the value of \( \omega' \) which maximizes the overlap the ground state energies differ by 8%. This question of the centering of the strength function will arise again in Section 4-6 where reference will be made to the first moment \( M_1 \) of Eq. [4.70].

Calculation of the second moment

\[
M_2 = \langle Y_{r'}^r \rangle^2 \int \int \int \psi_0^* \psi_0 \, d\gamma
\]

requires squaring the residual interaction \( V \) of Eq. [4.67] and transforming it from the particle coordinates \( \mathbf{r} \) to the normal coordinates \( \mathbf{y} \). However, the integral is, for the reason expressed in the previous paragraph, still doable, and gives

\[
M_2 = \langle Y_{r'}^r \rangle^2 \frac{k^2 \omega^2}{8} \left[ -\frac{9}{4} + \frac{\omega}{\omega'} \frac{3}{4\alpha} - \frac{\omega'}{\omega} \frac{3}{4\alpha} + \frac{\omega^2}{\omega'^2} \frac{3}{2} + \frac{\omega'^2}{\omega^2} \frac{3}{2} \right].
\]

This provides yet a third alternative for determining \( \omega' \). The requirement that the single particle system provide a good model for the compound system implies that the value of the strength function will be appreciable only in a narrow range of energies around the single particle energy. This condition is optimized by requiring \( M_2 \) to be minimized with respect to \( \omega' \) which gives

\[
\omega' = 1.233 \, \omega
\]

\[
k' = 1.520 \, k
\]

comparable to the values obtained in Section 4-4 by matching the energies.
In the case of the second moment it is to be noticed that the value \( \omega' (= 0.877 \omega) \) obtained in Eq. [4.42] by maximizing the overlap does not make \( M_2 \) infinite, giving, in fact, \( M_2 = (\gamma_{cp}^{sp})^2 0.332 \hbar^2 \omega^2 \). Lane, Thomas, and Wigner (1955) have shown that any local complex potential will lead to a strength function with an infinite second moment. The present work has dealt only with local real potentials. Moreover, Vogt (1957) showed that the total cross sections are largely insensitive to a change in the form of the strength function provided the full width at half maximum, \( W \), is kept constant, and, further, that the second moment provides an upper limit on \( W \),

\[ [4.75] \quad \frac{1}{4} J_{M_2} \geq W. \]

This means that the width of the distribution of the \( C_{\lambda;cp}^2 \) for the four particle chain will be less than 2.3 \( \hbar \omega \). This is indeed the case. Since the maximum of the \( C_{\lambda;cp}^2 \) occurs at \( C_{oo}^2 = 0.973 \), the sum rule on the squared overlaps does not permit an overlap to exist for which \( C_{\lambda;cp}^2 = C_{oo}^2 / 2 \). Thus the width in this case is effectively zero.

The next section, 4-6, presents an alternative way of obtaining the interaction between the two clumps in the single particle system. The new method to be presented, closer to the spirit of the optical model, will turn out to lead to results much the same as those which were obtained in the previous sections by simpler methods.

4-6 The Model Interaction as an Average Over Exact Interactions

The single particle system is here being conceived as a model for the compound system which can be called the "exact system". When the particles forming the compound system are torn apart into the
two clumps of the single particle system, individual interactions between particles which now find themselves in different clumps are sundered. In their places one has a single interaction between one clump and the other clump as a whole. It is physically sensible that the interaction between the clumps - the model interaction - be taken as the average over the individual interactions dissolved in the clumping process, i.e. as an average over exact interactions.

This is, in fact, the prescription suggested by Michaud and Vogt (1972) for obtaining the model interactions at each stage of the hierarchy of models described in Chapter 1. It is equivalent to Hartree's self-consistent method, although Hartree's method leads, in addition, to a renormalization of the individual interactions within each clump. While this method of obtaining model interactions has been prescribed for a long time (see, for e.g. Lane, Thomas, and Wigner 1955) it has not been well illustrated in the literature. It is not difficult to understand why this should be so: realistic nuclear wavefunctions and potentials are too complicated to make the calculation feasible. Here, again, the linear harmonic oscillator systems come to exhibit their main virtue: calculability.

In the present problem, then, the exact system is represented by

\[ V(s_{12}) V(s_{34}) V(s_{14}) \]

where the interparticle distance \( s_{ij} = r_i - r_j \), and the interparticle interaction \( V(s_{ij}) = \frac{1}{i} k s_{ij}^2 \). The model system is represented by
where the distance between the centres of mass of the two clumps is

\[ s_{12,34} = \frac{r_1 + r_2}{2} - \frac{r_3 + r_4}{2} = s \]

and \( V(s_{12,34}) \equiv V(s) \) is the interaction to be determined. In the previous sections of this chapter \( V(s) \) was simply assigned a harmonic oscillator behaviour with spring constant \( k' \). In this section \( V(s) \) is obtained as the average of the interactions it replaces - in this case \( V(s_{23}) \). This averaging is to take place over the internal motion of each of the two clumps, that is

\[ V(s) = \int \int V(s_{13}) u^2_o(s_{1i}) u^2_o(s_{3j}) ds_{1i} ds_{3j} \]

where the internal wavefunction of a clump, \( u_o(s_{1j}) \), is the ground state wavefunction of particles \( i \) and \( j \) interacting through \( V(s_{1j}) \). That is,

\[ u_o(s_{ij}) = \left( \frac{\sqrt{2m}}{\sqrt{\hbar}} \right)^{\frac{1}{2}} \exp \left( - \frac{\sqrt{2m}}{\sqrt{\hbar}} s_{ij} \right) \]

The task now is to calculate \( V(s) \) using Eq. (4.77). It will not necessarily turn out to be a harmonic oscillator potential as has been assumed heretofore.

It is to be noted that in the integral [4.77] the coordinate
\( s_{ij} \) is not independent of \( s_{12} \) and \( s_{34} \), but they may all conveniently be expressed in terms of the normal coordinates \( \gamma \) using the transformation [4.20] which gives

\[
\begin{align*}
\gamma_3 & = \sqrt{2} \gamma_1 \\
\gamma_4 & = \gamma_2 - \frac{1}{\sqrt{2}} \gamma_1 - \frac{1}{\sqrt{2}} \gamma_4 \\
\gamma_4 & = \sqrt{2} \gamma_4 \\
\gamma & = \sqrt{2} \gamma_4
\end{align*}
\]

[4.79]

With these coordinates the integration is straightforward and gives

\[
\sqrt{s} = \frac{1}{2} k s^2 + \frac{\hbar \omega}{4\sqrt{2}}
\]

[4.80]

One, first of all, notices immediately that the resultant interaction is indeed a harmonic oscillator interaction. This gives support to the assumption of harmonic oscillator interactions made in the previous sections. Secondly, one notices that the model spring constant \( k' = k \), a value intermediate between \( k' = 0.769 k \) obtained in Eq. [4.43] by maximizing the ground state overlap and \( k' = 1.437 k \) obtained in Eq. [4.48] matching the energies. Since it was seen in Section 4-4 that the overlap is hardly affected by this change in \( k' \), the overlap can still be considered to be maximized with the choice \( k' = k \). Thirdly, the interaction [4.80] has an added constant energy \( \hbar \omega / 4\sqrt{2} \) which is precisely the value needed to insure that the strength function is centered around the single particle energy. This is apparent by considering the first moment \( M_1 \) obtained in Eq. [4.70]. Setting \( \omega' = \omega \), one sees that the quantity \( \hbar \omega / 4\sqrt{2} \) in the single particle Hamiltonian, \( H' \), will reduce the residual interaction, \( V = H - H' \), by just
the right amount to make $M_1$ vanish. The vanishing of the first moment of the strength function has been shown by Lane, Thomas and Wigner (1955) to be a general property of model interactions obtained by the averaging procedure of this section. The present example has the ability to show this property in action.

The present chapter has shown by illustration how the linear harmonic oscillator chains are adapted, because of their analytic simplicity, to test the calculational tools of nuclear physics. However, not much more than what has been done in this chapter with the four particle system can be done without resorting to computer assistance for handling the matrices. The rest of this work will consist of computer assisted calculations which begin in the next chapter with an examination of the effect of increasing the number of particles, N.
CHAPTER 5
MODEL INTERACTION FOR SYSTEMS WITH N PARTICLES

The previous chapter showed the method by which overlaps are to be calculated, and illustrated it by calculating the overlap of the ground states of two systems of four particles. The best model wavefunction is that single particle wavefunction which has the largest overlap with the compound system wavefunction. It was found in Chapter 4 that, with four particles, the best single particle wavefunction is a very good model for the compound system wavefunction - in fact, too good: the strength of the single particle states resides 97% in a single compound state, and not enough strength is left in other compound states to permit a determination of the shape of the strength function. The present chapter inquires whether increasing the number of particles has any effect in diluting the strength of the single particle system.

Let the total number of particles be N (N = 2, 4, 6, ..., 24). For each value of N a compound system is formed and solutions obtained in the manner indicated in Section 4-1. Next, as in Section 4-2, the single particle system is formed by dividing the particles into two clumps with N/2 particles each. The two clumps are allowed to interact through a harmonic force with spring constant k' in terms of which the solutions are found. Following the procedure of Section 4-3 the overlap of the two ground state wavefunctions is found. This requires that both the normal coordinates x of the compound system and the normal coordinates y of the single particle system be expressed in terms of the integration coordinates z, described in Section 4-3, which permit the (N-1)-dimensional overlap integral to be written as a product of (N-1) one-dimensional integrals. The overlap depends on the ratio of spring constants k'/k,
and it is maximized numerically with respect to this ratio. The results are shown in Figure 5-1 which displays, for each value of N, the maximum ground state overlap and the corresponding value of the ratio k'/k.

It is seen from Figure 5-1 that although the maximum overlap decreases gradually as the number of particles, N, is increased, even for N=24 it has only decreased to $C_\infty^2 = 0.829$ (corresponding to $k' = 0.195$ k). However, although the strength of the single particle ground state seems to still be rather concentrated on one compound state (there being only 17% of strength left to be distributed among other states), with higher N the other states appear much closer together. In particular, for N=24 the density of states is higher and the excited states are closer to the ground state than for N=4. This is apparent from the expression for the total energy of a linear harmonic oscillator chain.

\[ E = \sum_{\lambda=1}^{N} \left( n_\lambda \cdot \omega_\lambda \right) \frac{k}{m} \omega_\lambda \]

where the normal mode frequencies for a compound system are given in Eq. [3.9] as

\[ \omega_\lambda = \left( 2 \sin \frac{\pi \lambda}{N} \right) \sqrt{\frac{k}{m}} \]

The higher density of states suggests that it is possible that the distribution of the remaining 17% of the strength give an indication of the shape of the overlaps $C_{\lambda;cp}^2$ as a function of energy.

Further, if one considers an excited state of the single particle system, its higher energy will locate it in the midst of a larger density of compound states (the density of states increases as the energy goes up). At these higher energies there is no reason to
Fig. 5-1. - The maximum overlap, $C_{oo}^2$, between the ground states of the single particle and compound systems, and the corresponding single particle spring constant ratio, $k'/k$, as functions of the number of particles, $N$. 
expect that there exists a compound state which will monopolize most of the single particle strength as in the case of the ground state. This should permit the calculation of enough closely spaced overlaps $C_{\lambda;\text{cp}}^2$ to give a fair indication of the shape of their distribution.

To correspond most closely to the number of particles involved in the $^{12}\text{C}+^{12}\text{C}$ reaction of interest, the remaining chapters will restrict themselves to systems with $N=24$. The next chapter will report the calculations of overlaps for $N=24$ which this chapter suggested as worthwhile: overlaps of the ground state of the single particle system with excited states of the compound system; and overlaps of an excited state of the single particle system with excited states of the compound system.
CHAPTER 6

OVERLAP INTEGRALS FOR A SYSTEM WITH 24 PARTICLES

As indicated in Chapter 1, in the giant resonance model of Lane, Thomas, and Wigner (1955), the strength function is proportional to
\[
\frac{C^2_{\lambda;cp}}{D}
\]
where \( C^2_{\lambda;cp} \) is the average overlap obtained by averaging over a few neighboring values of \( \lambda \), and \( D \) is the average spacing between the energy levels \( E_\lambda \). The quantity of interest is thus

\[
[6.1] \quad \frac{\langle C^2_{\lambda;cp} \rangle}{D} = \frac{\sum \frac{C^2_{\lambda;cp}}{\Delta E}}{\Delta E / \#} = \frac{\sum C^2_{\lambda;cp}}{\Delta E}
\]

where \( \# \) indicates the number of states \( \lambda \) included in the sum, i.e. it is the number of states over which the average is done; and \( \Delta E \) is the averaging interval which must be large enough to average out local fluctuations, and yet not so large that it obscures the shape of the strength function. This chapter determines \( \frac{\sum C^2_{\lambda;cp}}{\Delta E} \) as a function of compound system energy \( E_\lambda \) for two different values of \( cp \). In Section 6-1 \( \psi_{cp} \) is taken as the ground state of the single particle system. In Section 6-2 \( \psi_{cp} \) is taken as the excited state of the single particle system having 2 quanta of excitation in the 12th normal mode.

6-1 Single Particle System in its Ground State

It is well to remember that the integral over all space of the product of an even function with an odd function vanishes. A harmonic oscillator wavefunction is the product of Hermite polynomials with an exponential. The exponential is an even function of the coordinates. A
Hermite polynomial is an even or odd function of the coordinates depending on whether it is of even or odd order respectively. The order of the Hermite polynomial is equal to the number of quanta in the normal mode with which the Hermite polynomial is associated. Thus the ground state wavefunction will be an even function of the coordinates and will have zero overlap with any odd wavefunction. In particular, the ground state of the single particle system will have zero overlap with all compound system states containing a total of 1, 3, 5 \ldots quanta of excitation.

To proceed systematically, then, one begins by calculating the overlap of the single particle ground state with the compound ground state. This was actually done in Chapter 5 giving $C^2 = 0.829$. Next come the compound states with 1 quantum, which are odd functions, and, thus, all their overlaps with the single particle ground state vanish. Thirdly, one considers compound states with 2 quanta.

Since a linear system of 24 particles has 23 internal normal modes there are 23 states having 2 quanta in the same normal mode, and there are $23 \times \frac{22}{2!} = 253$ states having 1 quantum in each of two different normal modes. These two types of states with 2 quanta must be kept separate for the purposes of calculating the overlap integral. The reason for this rests in the transformation to the integration coordinates $z$ which are described in Chapter 4. If 2 quanta are both in the $i$th normal mode, the Hermite polynomial involved is

\[ [6.2] \quad H_z \left( \int \frac{m \omega_i}{\kappa} x_i \right) = -2 + 4 \left( \frac{m \omega_i}{\kappa} \right) \left( \sum_j x_{ij} \ z_j \right)^2, \]

but if one quantum is in the $i$th mode and another is in the $j$th mode, then
[6.3] \[ H_i \left( \sqrt{\frac{m \omega_i}{\hbar}} x_i \right) \quad H_i \left( \sqrt{\frac{m \omega_i}{\hbar}} x_i \right) = 2 \sqrt{\frac{m \omega_i}{\hbar}} \left( \sum_k X_{ik} z_k \right) \]

In the first case the factors contributing to the overlap are \( X_{ij}^2 \), in the second case the contribution to the integral comes from \( X_{ik} X_{jk} \). Thus a different computer program is needed in each case to handle the elements of the matrix \( X \) together with the proper constants in the Hermite polynomials. Carrying out these computations it turns out that out of a possible 276 overlaps only 78 do not vanish (the reasons for this will be explored in Appendix C). The results are gathered in Figures 6-1, 6-2 and 6-5.

Next in complexity are the compound states with 3 quanta. These, as explained above, are odd functions of the coordinates and all their overlaps with the single particle ground state will vanish.

Adding one more quantum to the compound system gives the 4 quanta states. In this case there are five types of states (equal to the number of partitions of 4) depending on whether the 4 quanta are all in the same mode or distributed among two, three, or four modes. Since there are 23 modes it is easy to show that there are 14,950 compound states having 4 quanta. However, only 1,365 of these will have a non-vanishing overlap with the ground state of the single particle system (see Appendix C). Carrying out each of the five types of integrals separately gives the results gathered in Figures 6-3, 6-4, and 6-5.

The compound system states having 5 quanta will, again, since they are odd functions of the coordinates, have no overlap with the single particle ground state. However, adding up at this point the
Fig. 6-1. - The number, \( m \), of 2-quanta compound states having non-zero overlap with the single particle ground state as a function of the compound state energy, \( E^\lambda \). The energy \( E^\lambda \) is in units of \( \hbar \omega \) above the compound ground state energy.

Fig. 6-2. - The average overlap squared, \( \langle C_{\lambda,\Gamma P}^2 \rangle \), between 2-quanta compound states and the single particle ground state as a function of compound state energy, \( E^\lambda \). The squared overlaps have been multiplied by \( 10^4 \). The energy units are the same as in Figure 6-1.
Fig. 6-3. - The number, \( m \), of 4-quanta compound states having non-zero overlap with the single particle ground state as a function of the compound state energy, \( E_\lambda \). Conventions are the same as in Figure 6-1.

Fig. 6-4. - The average overlap squared, \( \langle \xi^2 \rangle \), between 4-quanta compound states and the single particle ground state as a function of compound state energy, \( E_\lambda \). Conventions are the same as in Figure 6-2.
Fig. 6-5. - The strength, $\overline{C^2}/\Delta E$, of the single particle ground state in the compound states with 0, 2, and 4 quanta as a function of compound state energy, $E^\lambda$. The energy $E^\lambda$ is in units of $\hbar \omega$ above the compound ground state. $\Delta E$ is the energy interval over which the overlaps $C^2$ are summed.
distribution of strength of the single particle ground state among com-
pound states shows that 82.9% went to the 0 quanta compound state
13.3% went to the 2 quanta compound states
2.9% went to the 4 quanta compound states
which accounts already for 99% of the strength. Therefore no more effort
will be spent in pursuing higher excited states of the compound system -
their effect on the strength function will be small.

Looking at the results in Figures 6-1 through 6-5, it is
apparent that there are two distinct factors contributing to the shape of
the strength function: the density of states and the average overlap.
Figure 6-5 seems to indicate that although the gross structure of the
strength function is, as considered by Lane, Thomas, and Wigner (1955), a
reflection of the behaviour of the overlaps $C_A^{2;cp}$, the density of states
associated with different number of quanta might be responsible for intro-
ducing some systematic variations into the shape. The present case (where
the single particle system is in its ground state) does not allow
stronger conclusions for two reasons: a disproportionate amount of the
strength resides in a single compound state, and all the available com-
pound states are of higher energy than the single particle state being
considered which makes visible only the high energy side of the strength
function. To overcome these limitations Section 6-2 considers the
spread of an excited single particle state into compound states.

6-2 Single Particle System in an Excited State

The strength of the single particle ground state in the compound
states resides 83% in one compound state; the remaining strength being
distributed among higher energy compound states. Since a more uniform
distribution of the strength would serve to better show off the shape of the strength function one ought to begin with a higher energy single particle state which because of its placement among a higher density of compound states can be expected to distribute its strength more equitably.

A preliminary calculation of the overlaps of the 1-quantum single particle states with compound states shows that 50% of the strength is still concentrated on one compound state. However, the trend is promising, and one is led to examine next the strength of a 2-quanta single particle state. To choose this state so that it does not occur near either end of the distribution of 2-quanta compound states, one places the 2-quanta in the 12th normal mode of the single particle system. This state will be called the $n_{12}(2)$ single particle state, and this section examines the distribution of its strength among compound system states.

Again, one can, at the outset eliminate from consideration the compound states with an odd number of quanta since these are odd functions of the coordinates and will have zero overlap with the $n_{12}(2)$ single particle state which is an even function of the coordinates. To proceed systematically, then, one begins by taking the overlap of the $n_{12}(2)$ single particle state with the compound ground state which gives $C^2 = 0.0000218$.

Next one considers the compound states with 2 quanta. As mentioned in Section 6-1, there are 276 of these and there are two types of overlap integrals depending on whether both quanta are in the same mode or in different modes. However, only 91 of these have non zero overlap with the $n_{12}(2)$ single particle state. The largest overlap gives $C^2 = 21\%$ which shows that indeed, as expected, the strength is not con-
centrated mostly on one state. The rest of the 2-quanta compound states account for an additional 61.5% of the strength.

It will be convenient to call the strength of an n-quanta single particle state in the m-quanta compound states $S^{m-n}$. Thus the strength of the $n_{12}(2)$ single particle state in the 2-quanta compound states is $S^{(0)}$. The strength $S^{(0)}$ is shown in Figure 6-6.

The 4-quanta compound states number 14,950 and involve five different types of overlap integrals as will be recalled from Section 6-1. In the present case only 1,807 states have non-zero overlap with the $n_{12}(2)$ single particle state, and they contribute 15.6% to the strength. The strength $S^{(2)}$ is also shown in Figure 6-6.

The 6-quanta compound states can be shown to number 376,740, and to involve eleven types of overlap integrals (equal to the partitions of the number 6). However, it is perhaps not necessary to do these since 98% of the strength has already been exhausted, and thus the contribution of $S^{(4)}$ to the shape of the strength function will be almost negligible. Nevertheless it is possible to obtain an indication of the shape of $S^{(4)}$ as follows. Plotting, as in Figure 6-7, $S^{(2)}$, the strength of the $n_{12}(2)$ single particle state in the 4-quanta compound states (from Figure 6-6), and $S^{(2)}'$, the strength of the single particle ground state in the 2-quanta compound states (from Figure 6-5), one sees that they are very similar to each other. Although the number of states involved in each case is quite different - 1807 and 78, respectively - their contributions to the total strength are not dissimilar - 15.6% and 13.3% respectively. Likewise, the strength of the $n_{12}(2)$ single particle state in the 2-quanta compound states is 82.5% of the total, while the strength of the single particle ground state in the compound ground state is 82.9%
Fig. 6-6. - The strengths, $\sum c^2/\Delta E$, of the $n_{12}(2)$ single particle state in the compound states with 2, 4, and 6 quanta as a function of compound energy, $E_\Lambda$. The three types of strength are called $S(0)$, $S(2)$, and $S(4)$ respectively. $S(4)$ has not been calculated but obtained by approximating it with the strength of the single particle ground state in the 4-quanta compound states. $E_\Lambda$ and $\Delta E$ are as described in Figure 6-5. The energy of the $n_{12}(2)$ single particle state is 2.435 MeV above the ground state.
Fig. 6-7. - Comparison of the strength $S^{(2)}$ of the $n_{12}(2)$ single particle state in 4-quanta compound states with the strength $S^{(2)'}$ of the single particle ground state in 2-quanta compound states. $S^{(2)'}$ is obtained from Figure 6-5. $S^{(2)}$ is obtained from Figure 6-6 but has been shifted back along the energy axis by $2.435 \hbar \omega$ - the energy of the $n_{12}(2)$ single particle state above the ground state.
of the total. Thus, one can consider the strength function \( S^{(n)} \) to be approximately independent of the state of the single particle system. In particular, one can get a good indication of the shape of \( S^{(4)} \), the strength of the \( n_{12}(2) \) single particle state in the 6-quanta compound states, by looking at \( S^{(4)}' \), the strength of the single particle ground state in the 4-quanta compound states. Thus, Figure 6-6 also contains \( S^{(4)} \) obtained in this way.

One of the most intriguing features of Figure 6-6 is its terraced shape which indicates that the contributions to the strength come from different populations of states. In a given energy interval one of the strengths \( S^{(0)} \), \( S^{(2)} \), or \( S^{(4)} \) dominates the others. To see whether this competition between two or more populations has been seen in nuclear data, the literature on neutron cross sections was scanned for any evidence of deviations from the expected Porter-Thomas distribution (see Chapter 8) of the level width fluctuations. Rohr and Friedland (1967) reported just such a discrepancy in the neutron scattering on \(^{51}\)V. Some enumeration of shell model states in the compound nucleus was carried out to see if alpha-like quartet excitations, which compete favorably in energy with single particle excitations (Arima, Gillet, and Ginocchio, 1970), would yield a second population of states accounting for the observed width fluctuations. A further search of the literature, carried out at the same time, revealed, however, that Stieglitz, Hockenbury, and Block (1971) had repeated the experiment and shown that proper accounting of p-wave resonances removes any discrepancy with the Porter-Thomas distribution. This question, which is only incidental to the main theme of this thesis, was not pursued further, but remains a subject for future work.
Figure 6-6 provides, in histogram form, a distribution of the average of the $C^2/D$. As described in Chapter 1, this must be compared to a Lorentzian shape. Having done the overlaps, the $C^2$ are now known. Thus, the next step in the procedure is to determine the function $D(E)$, the average spacing between states. This forms the subject of Chapter 7.
CHAPTER 7
THE DENSITY OF STATES

In Section 6-2 the strength of the n_{12}(2) single particle state in compound states was computed by evaluating the overlaps $C_{\lambda;cp}$ and summing them in an energy interval $\Delta E$ to obtain $\sum_{\lambda} C_{\lambda;cp}^2 / \Delta E$ as a function of compound system energy $E_{\lambda}$. In the limit where the summing interval contains only one non-zero overlap the quantity $\sum_{\lambda} C_{\lambda;cp}^2 / \Delta E$ becomes $C_{\lambda;cp}^2 / D$, where $D$ is the average spacing between compound states $\chi_{\lambda}$ having non-zero overlap with the single particle state $\psi_{cp}$. A curve fitted to the quantities $C_{\lambda;cp}^2 / D$ (as a function of $E_{\lambda}$) will average out the fluctuations in $C_{\lambda;cp}^2 / D$, and, if it is a good fit, will reproduce the shape of the histogram of Figure 6-6 which provides such an average directly.

Before this strength function can be obtained by fitting a curve to the quantities $C_{\lambda;cp}^2 / D$ (which will be done in the next chapter) each one of the overlaps $C_{\lambda;cp}^2 (E_{\lambda})$, which was evaluated in the previous chapter, must be divided by the corresponding average spacing $D(E_{\lambda})$. This chapter concerns itself with obtaining $D(E_{\lambda})$.

As a first approximation one could take

$$[7.1] \quad D (E_{\lambda}) = \frac{E_{\lambda+1} - E_{\lambda-1}}{2}$$

However, an average taken in this way would be a very local average, and the resulting spacing $D$ would transmit its local fluctuation to the strength $C_{\lambda;cp}^2 / D$. At the other extreme one could take the total energy interval containing non-zero overlaps and divide by the total number of overlaps to obtain
which, while certainly providing an overall average, would give a constant spacing insensitive to its dependence on \( E^\lambda \). Of course, one may also take an intermediate position and average over an energy interval large enough to contain a few but not all of the overlaps. This has, in fact, already been illustrated in Figures 6-1 and 6-3 which indicate that not only both of the above objections are still applicable, but, in addition, one would be introducing a certain amount of unwanted structure into the strength function due to the discontinuous nature of the histograms.

All the above shortcomings may be circumvented by approximating the distribution of states with a continuous function. The shape of the histograms 6-1 and 6-3 themselves suggest that the functional dependence of the density of states could be Gaussian. In fact, Wong and French (1972) have shown that two-body interactions do lead to a Gaussian partial level density, and Wong and Wong (1973) have shown that this agrees with shell model calculations. Thus for the present case the Gaussian form

\[
D^{-1}(E) = \phi E \exp \left[ -\frac{(E - \bar{E})^2}{\bar{\phi}} \right] \left( 1 + p_3 E \right)
\]

will be adopted for the level density. \( p_3 \) is a parameter which permits the distribution to be skewed in order to accommodate the effect of a small sample size.

Using a least squares procedure to fit the density \( D^{-1}(E) \) of Eq. [7.3] to the distribution of the 91 levels of the 2-quanta compound
system having non-zero overlap with the $n_{12}(2)$ single particle state
gives

\begin{align*}
[7.4] 
   p_1 &= 2.80 \\
   p_2 &= 1.11 \\
   p_3 &= 0.023 \\
   p_4 &= 63.8
\end{align*}

for the parameters to be used in Eq. [7.3]. This, then, is all the
information which is needed to tabulate the quantities $C^2/D$ as a function of
energy which permits, in the next chapter, a Lorentzian curve to be
fitted to them.
CHAPTER 8

THE SHAPE OF THE STRENGTH FUNCTION

The Gaussian shape adopted in Chapter 7 for the density of states \( D^{-1}(E) \) permits the strength \( C_{λ;cp}^2/D \) to be obtained as a function of energy. In the giant resonance interpretation of Lane, Thomas, and Wigner the average of this function has, as discussed in Section 1-2, a Lorentzian shape. In order to put in evidence to what extent \( \langle C_{λ;cp}^2/D \rangle \) is indeed Lorentzian, this chapter will fit a Lorentzian curve to the strength of the \( n_{12}(2) \) single particle state in the 2-quanta compound states. These states account for 82.5% of the strength and are represented in Figure 6-6 as \( S^{(o)} \). Figure 6-6 serves, as well, to show that only \( S^{(o)} \) (and not \( S^{(o)} + S^{(2)} \) nor \( S^{(o)} + S^{(2)} + S^{(4)} \)) will be suitably represented by a Lorentzian.

The Lorentzian curve fitted to \( S^{(o)} \) will be an average over the fluctuations of the quantities \( C^2/D \). In order that this average be indeed a representative average the nature of the fluctuations of \( C^2/D \) about their (local) mean must be known beforehand. Porter and Thomas (1956) have argued that the fluctuations in the reduced widths \( γ_{λc}^2 \) (which are proportional to \( C_{λ;cp}^2 \) - see Eq. [1.13]) obey a chi-squared distribution with one degree of freedom. That is, they are described by a probability density function

\[
[8.1] \quad f(x) = \frac{1}{\sqrt{2πx}} \ x^{-\lambda} e^{-x/\lambda}
\]

where \( x \equiv γ_{λc}^2/⟨γ_{λc}^2⟩ \). In nuclear physics the distribution [8.1] is called a Porter-Thomas distribution.
To determine whether in fact the fluctuations of the $C^2/D$ obey a Porter-Thomas distribution reference is made to the results collected in Figure 6-6. Each of the $C^2/D$ is divided by the local average in each of the ten boxes in the histogram $S^{(o)}$ of Figure 6-6. Calling these quantities $x$, one plots in Figure 8-1 their frequency distribution together with a Porter-Thomas distribution. Comparison of the two distributions in Figure 8-1 indicates that the Porter-Thomas distribution does represent the fluctuations of the $C^2/D$.

Now that the distribution of the fluctuations of the $C^2/D$ are known, one can proceed to fit a curve through their mean. A standard method for fitting a curve is the maximum likelihood method. The likelihood for obtaining a set of points is defined as the product of the probability densities evaluated at the given points. The curve which best estimates the given points is that for which the likelihood (or, equivalently, the logarithm of the likelihood) is maximized. It turns out that in connection with a Porter-Thomas probability density, as in Eq. [8.1], the best estimate of a fit obtained by the maximum likelihood method coincides with a good visual fit, i.e. if there is no scatter the points all lie on the curve. This is not trivial but it is a characteristic of the chi-squared distributions of which the Porter-Thomas is a special case. The proof is as follows.

The distribution [8.1] used by Porter and Thomas belongs to the class of chi-squared distributions

\[ f(x) \, dx = \frac{1}{\Gamma(n/2)} \left( \frac{n}{2} \right)^{n/2} x^{n/2 - 1} e^{-n x/2} \, dx \]

where $n$ is the "number of degrees of freedom", and $x$ is the ratio of a
Fig. 8-1. - The distribution of $C^2/D$ about 10 local averages. The quantity $x$ is defined as the ratio of $C^2/D$ to its local average. The $C^2/D$ are in this case the strengths of the $n_{12}(2)$ single particle state in the 2-quanta compound states. The dashed lines represent a Porter-Thomas distribution.
a quantity to its average (say, $x = y/\bar{y}$). Thus

$$f'(y) \, dy = \frac{(\pi/2)^{n/2}}{\Gamma(n/2)} \bar{y}^{n/2} \, e^{-\frac{\pi}{2} \frac{y}{\bar{y}}} \, dy.$$  

Now, given for $y$ a specific value $y^0$, what is the best estimate of $\bar{y}$?

The likelihood $L(\bar{y})$ that a particular choice of $\bar{y}$ leads to the given value $y^0$ is defined by

$$L(\bar{y}) \equiv f'(y^0).$$

Obtaining the best estimate of $\bar{y}$ by maximizing the logarithm of $L(\bar{y})$ with respect to $\bar{y}$ gives

$$\bar{y} = y^0$$

which is independent of the number of degrees of freedom $n$. This says that a curve describing the behavior of $\bar{y}$ will pass through the [single] point $y^0$ as one would expect from a good visual fit.

Having established the appropriateness of the maximum likelihood method for the present case, one now wants the parameters $\Gamma$ and $E_0$ of the Lorentzian curve

$$\bar{y}(E) = \frac{N}{\pi} \frac{\Gamma/2}{(E - E_0)^2 + \Gamma^2/4}$$

that describes the average behaviour of the strengths $C^2/D$ of the $n_{12}(2)$ single particle state in the 2-quanta compound states. The quantities $C^2/D$ were found in Chapter 7 (there were 91 of them), and here they are
given the symbol \( y^O(E_\lambda) \). The normalization constant \( N \) in Eq. [8.6] is
designed to allow for the fact that \( S^{(0)} \) contains only 82.5% of the
strength and to minimize the effect of variations in the average level
spacing \( D \). It is chosen such that

\[
\sum_{\lambda=1}^{91} \bar{y}(E_\lambda) = \sum_{\lambda=1}^{91} y^O(E_\lambda)
\]

Now, the likelihood \( L_\lambda \) for obtaining the observed value \( y^O(E_\lambda) \) given the
predicted or mean value \( \bar{y}(E_\lambda) \) is, according to the Porter–Thomas distri­
bution [8.1], given by

\[
L_\lambda = \left[ 2\pi y^O(E_\lambda) \bar{y}(E_\lambda) \exp \left( y^O(E_\lambda)/\bar{y}(E_\lambda) \right) \right]^{-\frac{1}{2}}
\]

If there are 91 points, [numerical] maximization of

\[
\omega = \sum_{\lambda=1}^{91} \ln L_\lambda
\]

with respect to \( \Gamma \) and \( E_0 \) gives

\[
\Gamma = 0.0918
\]

\[
E_0 = 2.493
\]

This, then, is the result of the maximum likelihood method. In Figure
8–2 the Lorentzian strength function with the parameters [8.10] is com­
pared to the histogram for the average strengths previously shown in
Figure 6–6.

Several features are apparent from Figure 8–2. Although the
Fig. 8-2. - The strengths of Figure 6-6 compared to a Lorentzian strength function with parameters $\Gamma = 0.0918$ and $E_0 = 2.493$ obtained through a maximum likelihood procedure.
Lorentzian fit is adequate near the central \((E_0 \approx 2.5 \hbar \omega)\) region of the strength, in regions farther from the centre than about \(1 \hbar \omega\) on either side the fit is noticeably deficient. However, the next lowest \((n_{12}(1))\) and next highest \((n_{12}(3))\) single particle states occur at \(1.2 \hbar \omega\) and \(3.7 \hbar \omega\), respectively, so the Lorentzian does describe the strength in the region where one would expect it (see Section 1-2) to be applicable. Interestingly, the deviations of the strength function from the Lorentzian differ on the high and low energy sides: in the higher energy region of the curve the Lorentzian underestimates the strength, while on the lower energy region the Lorentzian overestimates the strength. The reason for this is that although a single particle state will preferentially populate a certain type of compound state \((S^o)\) carries 82.5% of the strength), the strength is in principle spread among all possible compound states (subject, of course, to the conservation laws), and there are many more states at higher energies than at lower energies. Further, since in actual nuclei, as in harmonic oscillator systems, the total density of states increases with increasing energy, one should be prepared, when extrapolating cross sections with Lorentzian strength functions down to lower energies, to overestimate the cross section. This will be particularly significant in the region below the lowest single particle resonance.

The fitting in this chapter of the Lorentzian curve to the strength function completes, together with the discussions of Chapters 2 to 7, the machinery needed for the discussion of strength functions proposed in the introduction. The last four chapters have dealt with its application to the spread of a state of the single particle system into states of the compound system. Its application to
the spreading of states in a hierarchy of systems will now be considered in Chapter 9.
CHAPTER 9
A HIERARCHY OF SYSTEMS

The promise made in the introduction - to study the behaviour of the strength function in the hierarchy of harmonic oscillator systems which are the counterparts of the three stages of the $^{12}\text{C}+^{12}\text{C}$ reaction - has yet to be fulfilled. However, the method to be used is that which has been developed after the introduction through the chapters which preceded the present. This chapter, then, will not re-elaborate on the method, but will indicate how it is to be extended to encompass the hierarchy, and will succinctly quote the results obtained therefrom. The hierarchy itself and the choice of spring constants is presented in Section 9-1. Section 9-2 contains the evaluation of the strength functions. As remarked in Section 4-3, because of the properties of harmonic oscillator wavefunctions, no numerical integration is required, but computer assistance was employed in diagonalizing matrices, gathering results, and fitting curves. At the speed of the IBM 370 a total of 1.2 hours of central processing unit time were required for these tasks. A single overlap can take up to about 40 seconds, but subsequent overlaps of the same type need only a few additional seconds each.

9-1 The Hierarchy

The three stages through which Michaud and Vogt (1972) have depicted the $^{12}\text{C}+^{12}\text{C}$ reaction were illustrated in Figure 1-5. The three harmonic oscillator systems which correspond to the three stages were illustrated in Figure 1-6, and called the single particle system, the doorway system, and the compound system, respectively. It should be remarked that the single particle system of the present chapter is not the
same as the single particle system used in Chapters 5 to 8. The single particle system used then was simply a division of the compound system into two sections of 12 particles each. The single particle system used in this chapter reflects the clustering properties of $^{12}$C described in the introduction. With this observation one sees that the systems used in this chapter form a natural hierarchy in the sense that, as one proceeds from the single particle system through the doorway system to the compound system, at each step an average interaction is replaced by a more exact interaction — one closer to the compound system. The idea, then, is to study how a single particle state is spread among doorway states, and how one of these doorway states is in turn spread among compound states. The quantity which reflects this spreading is the strength function, and it will have to be determined, as was done in Chapters 6, 7, and 8, for each step of the hierarchy.

One begins by solving each system one at a time. Thus, first, using trial values for $k'$, the spring constant of the interaction between quartets, and for $k''$, the spring constant of the interaction between dodecaplets, the classical Hamiltonians are written down. Next, they are diagonalized and written in terms of the normal coordinates which provides at the same time the normal mode frequencies. These frequencies depend on the choice of $k'$ or $k''$. Having obtained the classical Hamiltonians in terms of the normal coordinates, the quantum mechanical Hamiltonians follow immediately together with their solutions which consist of products of 23 harmonic oscillator wavefunctions similar to those in Eq. [4.14] and Eq. [4.26]. The next step is to determine $k'$ and $k''$ by maximizing the overlap $C_{00}$ between the ground states. This is done numerically by varying the trial values of $k'$ and $k''$, and doing the integration using the integration coordinates $z$ described in Section
4-3. The result for the maximum overlap between the ground state of the compound system with the ground state of the doorway system is

\[ C_{d-c}^2 = 0.6978 \]

corresponding to a value \( k' = 0.4797 \). Similarly, the maximum overlap between the ground state of the doorway system and the ground state of the single particle is

\[ C_{s-d}^2 = 0.9474 \]

corresponding to a value \( k'' = 0.2890 \). These values of \( k' \) and \( k'' \) permit the normal mode frequencies to be written down as is done in Table 9-1. The Hamiltonians, which will not be used further in this chapter, are, to avoid repetition, written down explicitly in Appendix C where their symmetry properties are investigated.

A comparison of the ground state overlaps [9.1] and [9.2] already reveals something about the hierarchy. They make it clear that as far as the ground states are concerned the single particle system is 95\% like the doorway system, while the doorway system is only 70\% like the compound system. It will be recalled from Chapter 1 that in the \(^{12}\text{C} + ^{12}\text{C}\) reaction the intermediate structure in the cross section has widths an order of magnitude smaller than the single particle resonances. This is interpreted to mean that the doorway structure lasts longer than the single particle structure because, in terms of the residual interactions (see Eq. [1.24] and Eq. [1.25]), the doorway is more like the compound system than like the single particle system. This is in contradistinction
TABLE 9-1

THE NORMAL MODE FREQUENCIES OF THE THREE SYSTEMS IN THE HARMONIC OSCILLATOR HIERARCHY

[All frequencies are in units of $\omega (=\sqrt{k/m})$ ]

<table>
<thead>
<tr>
<th>Normal Mode Number</th>
<th>Compound System</th>
<th>Doorway System</th>
<th>Single Particle System</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>0.131</td>
<td>0.179</td>
<td>0.219</td>
</tr>
<tr>
<td>3</td>
<td>0.261</td>
<td>0.346</td>
<td>0.346</td>
</tr>
<tr>
<td>4</td>
<td>0.390</td>
<td>0.490</td>
<td>0.346</td>
</tr>
<tr>
<td>5</td>
<td>0.518</td>
<td>0.600</td>
<td>0.600</td>
</tr>
<tr>
<td>6</td>
<td>0.643</td>
<td>0.669</td>
<td>0.600</td>
</tr>
<tr>
<td>7</td>
<td>0.765</td>
<td>0.765</td>
<td>0.765</td>
</tr>
<tr>
<td>8</td>
<td>0.885</td>
<td>0.765</td>
<td>0.765</td>
</tr>
<tr>
<td>9</td>
<td>1.000</td>
<td>0.765</td>
<td>0.765</td>
</tr>
<tr>
<td>10</td>
<td>1.111</td>
<td>0.765</td>
<td>0.765</td>
</tr>
<tr>
<td>11</td>
<td>1.218</td>
<td>0.765</td>
<td>0.765</td>
</tr>
<tr>
<td>12</td>
<td>1.319</td>
<td>0.765</td>
<td>0.765</td>
</tr>
<tr>
<td>13</td>
<td>1.414</td>
<td>1.414</td>
<td>1.414</td>
</tr>
<tr>
<td>14</td>
<td>1.504</td>
<td>1.414</td>
<td>1.414</td>
</tr>
<tr>
<td>15</td>
<td>1.587</td>
<td>1.414</td>
<td>1.414</td>
</tr>
<tr>
<td>16</td>
<td>1.663</td>
<td>1.414</td>
<td>1.414</td>
</tr>
<tr>
<td>17</td>
<td>1.732</td>
<td>1.414</td>
<td>1.414</td>
</tr>
<tr>
<td>18</td>
<td>1.794</td>
<td>1.414</td>
<td>1.414</td>
</tr>
<tr>
<td>19</td>
<td>1.848</td>
<td>1.848</td>
<td>1.848</td>
</tr>
<tr>
<td>20</td>
<td>1.894</td>
<td>1.848</td>
<td>1.848</td>
</tr>
<tr>
<td>21</td>
<td>1.932</td>
<td>1.848</td>
<td>1.848</td>
</tr>
<tr>
<td>22</td>
<td>1.962</td>
<td>1.848</td>
<td>1.848</td>
</tr>
<tr>
<td>23</td>
<td>1.983</td>
<td>1.848</td>
<td>1.848</td>
</tr>
<tr>
<td>24</td>
<td>1.996</td>
<td>1.848</td>
<td>1.848</td>
</tr>
</tbody>
</table>
to what has just been found for the ground states of the harmonic oscillator hierarchy. Will this behaviour of the ground states carry over into the strength function for excited states? Section 9-2 examines, thus, the strength function beginning with an excited state of the single particle system.

9-2 Spreading Widths for Excited States

Wanted in this section are two spreading widths: first the spread of an excited single particle state into doorway states will be calculated, and then the spread of one of these doorway states into compound states will be calculated. In each case the spreading width will be given, as in Chapter 8, by the width of the Lorentzian fit to the strength function.

Putting 2 quanta in the second normal mode of the single particle system creates the \( n_2(2) \) state. When this state is spread among 2-quanta doorway states it is found that the \( n_2(2) \) doorway state acquires 58% of the strength of the \( n_2(2) \) single particle state. As was seen in Section 6-1 this disproportionate concentration of the strength does not show off the strength function to best advantage. Therefore one must begin with a higher excited state.

Consider the \( n_2(4) \) state of the single particle system which contains 4-quanta in the second normal mode. As will be seen in Appendix C, because of the degeneracies evident in Table 9-I, not all the 14,950 4-quanta doorway states will have a non-zero overlap with the \( n_2(4) \) single particle state. The results in histogram form can be seen in Figure 9-1.

To fit this strength function with a Lorentzian one follows the
Fig. 9-1. - The strength, $S_{S-d}$, of the $n_2(4)$ single particle state in 4-quanta doorway states, and the strength, $S_{d-c}$, of the $n_d(4)$ doorway state in 4-quanta compound states. In each case the best Lorentzian fit has been superimposed on the histograms obtained from the calculated overlaps.
procedure of Chapter 8: the density of states is described by a Gaussian function, each overlap is divided by the value of the average spacing at the corresponding energy; and the resulting $C^2/D$ is fitted with a Lorentzian using the maximum likelihood method. The best fit is obtained with the Lorentzian parameters

\[ 9.3 \quad \Gamma = 0.301 \hbar \omega \]

\[ 9.4 \quad E_o = 1.073 \hbar \omega \]

In Figure 9-1 the resulting curve has been superimposed on the histogram.

The next step is to choose for investigation one of the doorway states into which the $n_2(4)$ single particle state deposited its strength. The $n_4(4)$ doorway state occurring at an energy of $1.958 \hbar \omega$ above the ground state lies part way between the centre and the tail of the strength function just calculated. It received 0.01% of the strength which is neither disproportionately high nor low, and thus as a typical state is a good candidate for the present purposes. Therefore the task now is to repeat all the above calculations beginning with the $n_4(4)$ doorway state to study its spread into compound states. The resulting histogrammed strength function together with the best Lorentzian fit to it are also shown in Figure 9-1. The parameters of the Lorentzian in this case are

\[ 9.5 \quad \Gamma = 0.974 \hbar \omega \]

\[ 9.6 \quad E_o = 2.509 \hbar \omega \]
Comparison of Eq. [9.3] with Eq. [9.5] shows that the width of the strength function for the $n_4(4)$ doorway state into compound states is three times larger than the width of the $n_2(4)$ single particle state in doorway states. This confirms what was anticipated by the ground state overlaps in Section 9-1: the doorway is a better model for the single particle system than for the compound system. This is also almost apparent by looking at the normal mode frequencies in Table 9-1. The quartet clustering in the doorway and single particle systems introduce degeneracies that are not present in the compound system. It would thus seem that the harmonic oscillator hierarchy can be more closely aligned with its nuclear counterpart by adjusting the spring constants within the compound system such as to mimic the strong internal binding of alpha particles. The ensuing compound system might then look like Figure 9-2 where the spring constant $k$ between particles in the same quartet is much larger than the spring constant $k'$ between particles in different quartets.

Fig. 9-2. - A possible modification of the compound system whereby quartet clustering is reproduced by taking $k \gg k'$. 
CHAPTER 10
SUMMARY AND DISCUSSION

The context within which this work has been performed was defined in the introductory chapter. The key question raised was "How does the state of a quantum mechanical system evolve when it is disturbed?" In Section 1-1 this question was put on a quantitative basis in terms of the overlap integral between two quantum mechanical states. In Section 1-2 it was shown that the giant resonance interpretation of Lane, Thomas, and Wigner for the nuclear optical model centres around the average behaviour of certain overlap integrals. This behaviour with energy serves to define the nuclear strength function, which in this case is expected to have a Lorentzian shape. Why is this shape Lorentzian? Section 1-3 described some early attempts to answer this question. The models used, though elegant, provided little connection with the physics. Section 1-6 described some heavy ion reactions whose cross sections have recently revealed some resonance structure intermediate between the widely spaced single particle resonances and the narrowly spaced compound nucleus resonances. The role of this intermediate structure in the calculation of stellar reaction rates was discussed in Section 1-5. Section 1-6 argued how the alpha particle doorway states conceived by Michaud and Vogt could be responsible for these intermediate resonances. Section 1-7 proposed a hierarchy of one-dimensional harmonic oscillator systems in terms of which one might not only revive the question of the Lorentzian shape of the strength function, but also investigate quantitatively how quartet doorways give rise to intermediate structure.

Chapter 2 discussed several features of the model. One would have liked to begin by choosing for the harmonic oscillator parameters
those values which give the best approximation to a realistic nuclear potential. It turned out, however, that the study of the strength function is independent of the numerical values of these parameters. Harmonic oscillator potentials have been used extensively in nuclear physics, most importantly in the shell model where special care has to be taken to eliminate spurious states due to motion of the centre of mass of the system. Will these kinds of states affect the present calculation of the strength function? It was shown that since the Hamiltonian of a linear chain can be written as a sum of a term describing the centre of mass motion and a term describing the internal motion, the eigenfunctions of this second term will be the desired internal wavefunctions. Thus, in the present problem, in contrast to the shell model, the elimination of spurious states was straightforward.

Both classical and quantum mechanical systems of coupled harmonic oscillators are most easily solved when described in terms of a set of coordinates called normal coordinates which effectively uncouple the oscillators. The normal coordinates are obtained from the particle coordinates by a linear transformation and Section 3-1 showed that this transformation exists for any possible arrangement of the connecting springs in a linear chain. Sections 3-2 and 3-3 illustrated this by giving in closed form the transformation to normal coordinates for two special cases.

With a small number of particles it is possible to do many calculations by hand. Chapter 4 treated chains with four particles in order to develop the techniques which were later generalized to systems of N particles. Sections 4-1 and 4-2 obtained the wavefunctions for two different arrangements of the four particles. The procedure illustrated
how the quantum mechanical solutions are obtained beginning with the classical Hamiltonian, and also how the centre of mass motion gets separated out. Section 4-3 showed how overlap integrals are to be done using a particularly appropriate set of integration coordinates. Oscillator parameters, other than those connecting individual particles, were determined by maximizing the overlap of the ground states, by matching the ground state energies, and by minimizing the second moment of the strength function. It was shown how the moments of the strength function can be expressed in terms of the moments of the residual interaction. Although this result is widely known, the proof is not spelled out in the literature. Finally, in Section 4-6 it was shown how a model interaction can be obtained as an average over exact interactions. In the case of harmonic oscillator exact interactions, the model interaction obtained in this way is also harmonic, and contains in addition a constant term which has the effect of bringing the ground state energies into closer alignment.

Chapter 5 was used to generalize the treatment of Sections 4-1, 4-3, and 4-3 to N particles. It was shown that as the number of particles increased from 2 to 24 the maximum squared overlap between the ground states of the "single particle" arrangement and the "compound" arrangement decreased gradually from 1.0 to 0.83. It was concluded that the more particles involved, the more the strength will be distributed, and hence the more conspicuous will be the shape of the strength function. In the remaining chapters the linear chains consisted of 24 particles.

Chapter 6 continued the work of chapter 5 by calculating with the appropriate overlap integrals, the strength of the single particle ground state in compound states, and the strength of the $n_1^2(2)$ single particle state in compound states. It was found that the strength for spreading into states with the same number of quanta of excitation
dominated the strength for states differing by 2 quanta which, in turn, were larger than the strength for states differing by 4 quanta from the initial state. See Figure 6-6. It is clear that in a given energy interval certain types of states are preferentially populated over certain others. The preferred ones are those whose excitation is most like the initial state; that is, most like the entrance channel. Using the conjecture of Chapter 3 one may put in evidence the nuclear counterpart to this undemocratic behaviour. It is to be reiterated that it was not certain normal modes that were found to be preferentially populated, but certain types of excitations (for example, 2 quanta distributed in all possible ways among all the normal modes). Correspondingly, in the nuclear problem, one would not expect the preferential excitation of certain particles, but, perhaps, of certain types of excitations (for example, 8-particle - 4-hole), which are most similar to the entrance channel. Preferential excitation of certain types of states has in fact been observed in heavy ion scattering experiments by Middleton et al (1971), Voit et al (1973), and Ballini et al (1974). This constitutes some of the most compelling evidence for the interpretation of intermediate structure in $^{24}$Mg in terms of alpha-cluster configurations.

The strength function depends not only on the overlap integrals but also on the density of states. Chapter 7 showed that the Gaussian shape used by Wong and French for the nuclear partial level density is also appropriate for the present problem. This enabled a tabulation to be made of the quantities $C^2/D$ - the squared overlap divided by the corresponding average spacing.

In Chapter 8 a Lorentzian curve was fitted to the tabulated $C^2/D$. It was shown that the fluctuations of the $C^2/D$ obey a Porter-Thomas
distribution, and that the maximum likelihood method of curve fitting is appropriate for all chi-squared distributions. The Porter-Thomas distribution is a special case of these. Using the maximum likelihood method a Lorentzian curve was fitted to the strength function of the \( n_{12}(2) \) single particle state in the 2-quanta compound states. The results were plotted in Figure 8-2 and are discussed below.

The hierarchy of systems (single particle, quartet doorway, and compound) proposed in the introduction was studied in Chapter 9 which required the extension and application of the method developed in Chapters 2 to 8. It was found that the Lorentzian shape fitted to the strength function of a single particle state in doorway states had a width of \( \Gamma_s = 0.30 \) (in units of \( \hbar \omega \)), whereas the one fitted to the strength function of a doorway state in the compound states had a width of \( \Gamma_d = 0.97 \). These results, though at odds with the expectations from the alpha-particle model for the \( ^{12}\text{C}+^{12}\text{C} \) reaction, are not, retrospectively, difficult to understand; they simply expose the fact that, with the harmonic oscillators, the single particle system is closer to the doorway system than the doorway system is to the compound system. Indeed the single particle system differs from the doorway system by the placing of one spring, while the doorway system differs from the compound system by the placing of five springs (see Figure 1-6). Thus the spreading of the doorway into compound states was wider because the residual interaction was larger, i.e. the doorway Hamiltonian left out more terms from the compound Hamiltonian than the single particle Hamiltonian left out from the doorway Hamiltonian. Further, the results had been anticipated by the overlap of the ground states: single particle with doorway - 95%; doorway with compound - 70%.
The widths obtained in Chapter 9 must be contrasted with the experimental broad single particle widths and the narrower doorway widths of the $^{12}\text{C} + ^{12}\text{C}$ reaction. However, one remarks that while the known clustering properties of the $^{12}\text{C}$ nucleus have been expressly designed into the corresponding harmonic oscillator single particle system, no cognizance was taken of possible clustering effects in the structure of $^{24}\text{Mg}$ in the arrangement of the compound harmonic oscillator system. This is what made the doorway model so different (in the sense of having a large spreading width) from the compound model. Though, in $^{24}\text{Mg}$ the ground state has very little quartet clustering (Basu 1972), Arima, Gillet, and Ginocchio (1970) have shown that, energetically, quartet excitations can be expected among low lying shell model states. Had the compound harmonic oscillator system been made to include quartet clusters with strong internal binding as in Figure 9-2, the spread of the doorway state might have been much narrower. One of the implications of the present work is, then, the suggestion that the narrowness of the intermediate structure in the $^{12}\text{C} + ^{12}\text{C}$ reaction is due to the existence of quartet clustering in the compound nucleus $^{24}\text{Mg}$ at the excitation energies being probed.

In Chapter 8 a single particle state was spread among compound states. In Chapter 9 the single particle state was spread among doorway states, and one of these was spread among compound states. In order to examine the applicability of the commonly used Lorentzian shape for the energy dependence of $\langle C^2 \rangle/D$, a Lorentzian curve was fitted in each of the three cases to the calculated $\langle C^2 \rangle/D$. Considering for the moment only the strength of the initial state in states of the same number of quanta, it can be seen from Figures 8-2 and 9-1 that the distribution of $\langle C^2 \rangle/D$ falls off in energy more rapidly
than a Lorentzian. This was not entirely unexpected. A Lorentzian curve has an infinite second moment, but any calculated distribution will have finite moments. In the central regions, that is near the resonant energy, the distribution is not unlike a Lorentzian while a Gaussian curve, on the other hand, would fall off much too rapidly. More importantly, however, adding to the strength function the contribution due to states of different quanta, it was seen that the Lorentzian curve overestimated the strength in the low energy side and underestimated it in the high energy side. The latter effect was directly ascribable to the level spacing $D$. Although the average overlap $<C^2>$ for the states of different quanta is much smaller than for the states of same number of quanta as the initial state, many more states can be formed with a larger number of quanta, hence increasing the level density. Therefore, although the assignment of Lane, Thomas and Wigner (1955) of the giant resonances to the behavior of the $<C^2>$ is corroborated in this model, the decreasing average level spacing $D$ will lead to a skewing of the Lorentzian shape of the resonances.

The other physical consequence that was implied in these results is that an optical model extrapolation to lower energies from a measured resonance will tend to overestimate the absorption cross section, provided that there is no new physics below the lowest resonance. It is not possible, however, to declare the optical model cross section as an upper limit on the cross section since the harmonic oscillator model cannot, nor was it designed to, treat effects such as "absorption under the barrier" (Michaud, Scherk, and Vogt, 1970) which might influence the cross section at low energies. Such effects will require the examination of individual nuclei.

In short, the harmonic oscillator because of its remarkable
solvability is a useful testing ground for many of the tools of theoretical physics (see also, for example, Moshinsky, 1969). The linear chains used in this work have served to make manifest the preferential population of certain types of states, the non-Lorentzian shape of the strength function, the second moment as a measure of the width of the strength function, and the importance for heavy ion reactions of cluster states in the compound nucleus. All these features may survive the harmonic oscillator approximation and they should certainly colour the way in which the data is examined.

It would be interesting in the future to include spin and isospin in the model and to investigate the emergence of tightly bound $\alpha$-like clusters while examining the special role, if any, of 4-body forces. Actually, for springs of zero length the harmonic oscillator systems are solvable exactly in spaces of any dimension. A study of the formalism needed to incorporate finite length springs in more than one dimension could provide an insight into suitable mechanisms for nuclear matter calculations with repulsive core potentials. Another area where this model could be useful is a study of the connection between free particle interactions and effective interactions in nuclei: the system could be solved both exactly and in a shell-model-like approximation. In conclusion it can be said that the full usefulness of the harmonic oscillator model has by no means been exhausted and that one can look forward to seeing it in fruitful service for many years.
APPENDIX A

NUCLEAR POTENTIALS AND HARMONIC OSCILLATOR POTENTIALS

It was not necessary in this work to employ particular numerical values for the parameters of the harmonic oscillator potential used to represent the nucleon-nucleon interaction. Nevertheless, this appendix discusses how the parameters of the harmonic oscillator would be chosen to best approximate a nuclear potential. In Section A-1 this is done by using one of the nuclear potentials from the literature. A better approximation is obtained in Section A-2 by constructing an N-N potential that is better suited to a harmonic oscillator approximation than those in common use.

A-1 Approximation of a Potential by a Harmonic Oscillator Potential

In order to approximate an arbitrary potential \( V(r) \) by a harmonic oscillator potential

\[
[V_{H.O.}(r) = A + \frac{1}{2} (r - r_o)^2 B
\]

one can expand \( V(r) \) in a Taylor series about \( r_o \) obtaining

\[
V(r) = V(r_o) + (r - r_o) \frac{dV}{dr} \bigg|_{r_o} + \frac{(r - r_o)^2}{2!} \frac{d^2 V}{dr^2} \bigg|_{r_o} + \cdots + \frac{(r - r_o)^n}{n!} \frac{d^n V}{dr^n} \bigg|_{r_o} + \cdots
\]

where \( r_o \) is chosen such that

\[
\frac{dV}{dr} \bigg|_{r_o} = 0
\]

and require that
for all \( n > 2 \). The range of \( r \) for which condition (A.4) holds will define the range of applicability of a harmonic oscillator potential approximation to the potential \( V(r) \). This procedure will now be illustrated by applying it to an analytically very manageable phenomenological three Gaussian potential (3GRS) developed by Tamagaki (1968) as a substitute for more realistic potentials.

The three range Gaussian potential used by Tamagaki (1968) for the \(^1S\) state is

\[
V_{3G}(r) = -5 e^{-r^2/10^{-15}} - 270 e^{-(r^2/542)^2} + 630 e^{-(r^2/62)^2}
\]

where \( r \) is in units of \( 10^{-15} \) m and \( V \) is in units of MeV. This potential can be found to have a minimum at \( r = r_0 = 1.03 \) fm. The range of applicability of a harmonic oscillator approximation to Eq. (A.5) is then given by condition (A.4) as

\[
(1.62 \text{ fm}) \quad r \quad (0.44 \text{ fm})
\]

In this region \( V_{3G} \) can be approximated by

\[
V_{HO}(r) = V_0 + \frac{1}{2} k (r - 1.03)^2
\]

where \( V_0 = -53 \) MeV and \( k = 653 \) MeV/fm\(^2\). However, the ground state energy of the potential \( V_{HO} \) in Eq. (A.7) is \( E_0 = +63.542 \) MeV. When
this value is compared to the $^1S$ system of two nucleons, which is unbound by about 0.5 MeV, one obtains a clear indication that the harmonic oscillator potential [A.7] is too steep: the value of the spring constant $k$ is too large. This is because the harmonic oscillator is being forced to approximate the relatively hard core of the 3GRS potential. This is not surprising. The N-N potentials which are usually constructed are not designed to be approximated by a harmonic oscillator potential and thus they do not resemble a parabolic well. The next section will aim at constructing a new N-N potential explicitly designed to lend itself easily to a harmonic oscillator approximation.

A-2 Construction of a Nucleon-Nucleon Potential

Wanted in this section is a nucleon-nucleon potential which reproduces N-N scattering data fairly well and which most easily lends itself to a harmonic oscillator approximation. The most straightforward of such potentials is a harmonic oscillator well matched on to a one pion exchange tail. The matching radius has to be chosen to make full use of the known validity of the one pion exchange potential (OPEP) for large distances. Breit (1962) reviewed this question and concluded that for $r > 2.9$ fm the OPEP is dominant. Later work (Tamagaki 1967) has indicated that the OPEP can probably be trusted down to 2.0 fm, and, in this section, this value will be used as the matching radius. The centre of the harmonic oscillator well will be chosen to coincide with the minimum of the 3GRS potential as found in Section A-1; namely, $r_o = 1.0$ fm. Thus, one seeks a potential of the form

$$[A.8a] \quad V = -V_o + \frac{1}{2} k (r - r_o)^2 \quad r < 2.0 \text{ fm}$$
where \( \mu = m \pi / \hbar \) and \( f^2 / 4\pi = 0.08 \). The potential in Eq. [A.8a] is the harmonic oscillator (HO) potential, and Eq. [A.8b] is the OPEP. The task is to determine the two parameters \( V_o \) and \( k \). Two methods will be used -

a) Matching the value and slope of the HO potential to the OPEP at \( r = 2.0 \) fm gives \( V_o = 3.14 \) MeV and \( k = 2.35 \) MeV/fm\(^2\). This value of \( V_o \) makes the potential of Eq. [A.8] extremely shallow compared to typical N-N depths of \( \sim 50 \) MeV. The ground state energy of this potential, however, is \( +3.82 \) MeV, which is of the same order of magnitude as the virtual state of the so-called "singlet deuteron". It is necessary to ask now whether this potential can be improved by relinquishing the requirement of a smooth slope at \( r = 2.0 \) fm.

b) Matching the values of expressions [A.8a] and [A.8b] at \( r = 2.0 \) fm gives the relationship

\[
[A.9] \quad k = 2 \left( V_o - 1.97 \right) \text{MeV/fm}^2
\]

Now by matching the scattering length from the potential [A.8] to the experimental value (either \( a_{nn}^s = -17 \) fm or \( a_{np}^s = -23.7 \) fm; see Moyes 1972), one should be able to determine the value of \( V_o \), the remaining parameter. In practice one chooses a trial value for \( V_o \) and calculates the scattering length numerically. A choice of \( V_o = 30 \) MeV gives \( a = -21 \) fm. Making \( V_o \) smaller by about 1 MeV will increase the scattering length to the singlet neutron-neutron value. Making \( V_o \) larger by about 1 MeV will decrease the scattering length to the singlet neutron-proton.
value. Thus, $V_0 = 30$ MeV is a good order-of-magnitude estimate. The
value of the spring constant $k$ is then determined from Eq. [A.9], which
gives $k = 56$ MeV/fm$^2$. A harmonic oscillator well with these parameters has
a ground state energy of +4 MeV, which, again, is of the same order of
magnitude as the virtual state of the singlet deuteron.

This appendix has served to display the feasibility of a
harmonic oscillator approximation to realistic nuclear potentials. It
is not necessary to pursue further the question regarding the best
choice of parameters. Their numerical values are only incidental to
the developments in this thesis, which addresses the question of the
spread of a state of one system into states of another system. The
treatment which was followed uses harmonic oscillator forces with
arbitrary parameters. The results are independent of their numerical
values.

It is appropriate to point out here that non-nearest-neighbor
forces in a system consisting of many particles may also be treated
successfully with harmonic oscillator forces. One simply mimics the
weakening of the nuclear force with increasing distance by using
successively weaker springs (smaller $k$) to connect successively more
distant neighbors. Although the problem of linear chains considered
in this thesis would still be solvable using non-nearest-neighbor forces,
these would constitute an added complication. A preliminary calculation
indicated that the spring constants associated with next-to-nearest-
neighbor forces are only 1% of those associated with nearest neighbors.
Since the physical effects that are studied here do not depend on the
inclusion or non-inclusion of these added forces (whose effects, in
any case, are small), use of these is avoided, and the calculations
employ nearest neighbor forces only.
APPENDIX B

DIFFICULTIES WITH THE EXCLUSION PRINCIPLE

The linear chains of nucleons which are used in this thesis are made up of fermions. It is the purpose of this appendix to study the place of the Pauli Exclusion Principle in the formulation of the problem. For the purposes of illustration, the discussion will be presented in reference to a three-particle chain – the simplest non-trivial example. See Figure B-1.

![Diagram of a three-particle chain](image)

Fig. B-1. – A simple linear chain of particles. The labels $\alpha$, $\beta$, $\gamma$ denote positions on the chain; they do not label the particles.

If the three particles $1$, $2$ and $3$ are identical fermions, the wavefunction $\Psi$ describing the system must be totally antisymmetric under the exchange of any two particles, and it may be written

$$\Psi = \Psi(1,2,3) - \Psi(1,3,2) + \Psi(2,3,1) - \Psi(2,1,3) + \Psi(3,1,2) - \Psi(3,2,1)$$

$$\equiv A \Psi(1,2,3)$$

where $1$, $2$, $3$ in the argument of $\Psi$ are to be understood as representing all the coordinates of each of these particles respectively, the functions $\Psi(i,j,k)$ are the exchange degenerate eigenfunctions for the same Hamiltonian and same energy as $\Psi$, and $A$ is an antisymmetrizing operator. The wavefunction $\Psi$ is then the unique antisymmetric combination of the $\Psi(i,j,k)$. 
First of all, however, it is necessary to establish that all the particles are identical. This means that any one of the particles 1, 2, or 3 may appear with equal probability in any one of the positions $\alpha$, $\beta$, or $\gamma$ on the linear chain. The identity of the particles must also be reflected in the Hamiltonian describing the system which must remain unchanged whenever any two particles are substituted for each other. Indeed, it is possible (Messiah 1961) to define particles as identical whenever the Hamiltonian has this property.

An appropriate Hamiltonian describing the system of Figure B-1 with harmonic oscillator interactions of spring constant $k$ between particles in adjacent position is

\[ H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + V_{\alpha} + V_{\beta} \]

where $p_i$ and $m$ are, respectively, the momentum and mass of the $i$th particle. The interaction $V_{\alpha\beta}$ between a particle at site $\alpha$ and a particle at site $\beta$ is

\[ V_{\alpha\beta} = \sum_{\substack{i,k,l}} \frac{k}{2} (r_i - r_k)^2 Q_{\alpha}(i) Q_{\beta}(k) \]

where $r_i$ is the displacement from equilibrium of the $i$th particle, and $Q_{\alpha}(i)$ is a projection operator which acting on a wavefunction gives zero unless the $i$th particle is in the position $\alpha$. Using a convention on the ordering of the arguments of $\psi(j,k,\ell)$ such that $(j,k,\ell)$ indicates that $i$th particle at $\alpha$, the $k$th particle at $\beta$, and the $\ell$th particle at $\gamma$, one may write concisely
Clearly now, the Hamiltonian of Eq. [B.2] is symmetric under exchange of any two particles, and produces a harmonic interaction between adjacent particles and no interaction between non-adjacent particles. It acts on the total antisymmetric wavefunction to give

\[ H \Psi = E \Psi, \]

and on the exchange degenerate wavefunctions to give

\[ H \Psi (j, k, \ell) = E \Psi (j, k, \ell). \]

It is now possible using the explicit effect [B.4] of the operators \(Q_{\alpha}(i)\) on the wavefunctions \(\psi(j, k, \ell)\) to simplify the Schrödinger equation [B.6] as follows.

\[ V_{a\beta} \psi(\hat{j}, k, \ell) = \sum_{m, n} \frac{k}{2} (r_m - r_n)^2 Q_{\alpha}(m) Q_{\beta}(n) \psi(\hat{j}, k, \ell) \]

\[ = \frac{k}{2} (r_j - r_k)^2 \psi(\hat{j}, k, \ell) \]

so that the Schrödinger equation [B.6] becomes
where \( j \) (and \( k \), and \( \ell \)) = 1, 2, 3. Finally, since \( \Psi = \Delta \psi(1,2,3) \), it is necessary to solve only

\[
[B.9] \sum_{i=1}^{3} \frac{\rho_i}{2m} \psi(1,2,3) + \frac{k}{2} (r_i - r_e)^2 \psi(1,2,3) + \frac{k}{2} (r_e - r_i)^2 \psi(1,2,3) = E \psi(1,2,3)
\]

That is, one need only consider particle 1 fixed in the first position, particle 2 fixed in the second position, etc., and obtain \( \Psi \), at the end, by antisymmetrizing \( \psi(1,2,3) \) with respect to the particle coordinates.

This last step, extremely simple in principle, and manageable with only three particles, becomes prohibitive with twenty four particles. The reason is the following. If equation [B.9] did not couple the coordinates of the different particles, then one would be able to write \( \psi(1,2,3) \) as a product function

\[
[B.10] \psi(1,2,3) = \phi_1(x) \phi_2(y) \phi_3(z)
\]

and \( \Psi \) would be given by the 3 x 3 Slater determinant formed with the \( \phi \)'s (or an \( N \times N \) determinant for \( N \) particles). However, Eq. [B.9] does couple the coordinates of different particles, and requires, for its solution, transformation to a new set of (uncoupled) coordinates \( x \).
which are called the "normal coordinates". (A single underline denotes a column matrix, a double underline a square matrix) Then

$$
\psi(r_1, r_2, r_3) = \psi'[x_1(r_1, r_2, r_3), x_2(r_1, r_2, r_3), x_3(r_1, r_2, r_3)]
$$

In the case of 24 particles a single antisymmetrized wavefunction $\psi$ would consist of $24!$ (= Avogadro's number) terms, each term consisting of a product of 24 functions, and each of these functions depending on all 24 particle coordinates with respect to which the antisymmetrization is to be carried out. The fact that these wavefunctions are used in overlap integrals does not significantly reduce the complexity of the problem as can be appreciated even by considering chains of two particles. Let $\psi$ and $\phi$ be two antisymmetrized wavefunctions each describing two particles. Writing

$$
\begin{align*}
\Psi &= \psi(1, 2) - \psi(2, 1) \\
\Phi &= \phi(1, 2) - \phi(2, 1)
\end{align*}
$$

enables their overlap integral to be written
where the last step is obtained by relabeling the variables of integration. Thus, out of 4 possible integrals only 2 have to be evaluated. For the case of three particles the same argument shows that out of a possible 36 terms in the overlap integral 6 have to be evaluated. In general, for N particles the antisymmetrized wavefunctions have N! terms, and the overlap integral of two of them has \((N!)^2\) terms out of which only N! are distinct. This is the difficulty that led to the restriction to unsymmetrized wavefunctions in the body of this work.

The simple harmonic oscillator is often used to illustrate the elegance of the number representation and the role of creation and annihilation operators together with their commutation relations. However, the operators which create or annihilate states of the harmonic oscillator obey boson commutation relations, and the states of a set of uncoupled harmonic oscillators can be identified with the states of an assembly of bosons. When written in terms of normal coordinates the Hamiltonian of a linear chain is indeed uncoupled, but the antisymmetrization has to be carried out with respect to the particle coordinates - not the normal coordinates. Even in the number representation, then, it would be necessary to transform from the normal labels to the particle labels and to carry out the antisymmetrization as above with no gain in economy of labour. For this reason the coordinate representation has been used throughout this work.
APPENDIX C

PREDICTION OF NULL OVERLAP INTEGRALS

In determining a strength function by the method used in this work, a great many overlap integrals have to be calculated. One wonders, naturally, if a considerable amount of labour may be saved by examination of the structure of the integrals, or of the structure of the Hamiltonians themselves. In this appendix the spreading of a single particle state into doorway states will be considered by examining in Section C-1 the overlap integral between the single particle state and a doorway state, and by using group theory in Sections C-2 to C-5 to exploit the symmetry of the Hamiltonians into yielding predictions about the overlaps of their wavefunctions. In Section C-2 the orthogonality relation between two irreducible representations of a group is used to show that two wavefunctions will have zero overlap unless they are both basis for the same representation of the group. The common invariance group $\tilde{G}$ of the two Hamiltonians is found in Section C-3. In Section C-4 the representation of the group $\tilde{G}$, found in Section C-3, is decomposed into the irreducible representations of the group. Finally, in Section C-5 the eigenfunctions are classified according to the representations of $\tilde{G}$ and use is made of the theorem of Section C-2 to obtain predictions about the overlap integrals. This appendix is not a contribution to the methods of group theory, but consists of an application of standard techniques to the overlap problem.

C-1 Structure of the Overlap Integral

Consider a single particle state with $m_2$ quanta in the second normal mode and no quanta in all other modes. Its wavefunction is
Consider an arbitrary doorway state

\[ \Psi_m (\gamma) = \mathcal{N}_m \ H_{m_{\alpha}} (\beta_{\alpha} \gamma_{\alpha}) \ \exp \left[ - \sum \frac{\omega_{\alpha}}{2} \gamma_{\alpha}^2 \right] \ . \]

The overlap integral of interest in the determination of the strength function is

\[ \chi_n (x) = \mathcal{N}_n \ H_{n_{\alpha}} (\beta_{\alpha} x_{\alpha}) \ H_{n_{\beta}} (\beta_{\beta} x_{\beta}) \ldots \ H_{n_{\mu}} (\beta_{\mu} x_{\mu}) \ \exp \left[ - \sum \frac{\omega_{\alpha}}{2} x_{\alpha}^2 \right] \ . \]

The overlap integral of interest in the determination of the strength function is

\[ I_{mn} = \int \psi_m (\gamma) \ \chi_n (x) \ d^\uparrow \]

where \( d^\uparrow \) is an element of volume in the space of the \( y \)'s or \( x \)'s. It is appropriate to note that both the coordinates \( \gamma \) and the coordinates \( x \) are related to the particle coordinates by an orthogonal transformation; thus they span the same space and are related to each other by an orthogonal transformation \( \mathcal{C} \).

\[ \mathcal{C} = \mathcal{C} \mathcal{C} = x \]

It is, further, advantageous to note that the matrix \( \mathcal{C} \) has the block diagonal form

\[ \mathcal{C} = \begin{array}{cc}
\begin{matrix}
\begin{array}{c}
\end{array}
\end{matrix}
\end{array} \]

which permits the overlap \([C.3]\) to be written as
In Eq. [C.6] it is understood that $y = y(x)$ according to Eq. [C.4], and since the transformation $\mathbb{C}$ is orthogonal, each of the sub-matrices in the block diagonal form will also be orthogonal, and in each case the magnitude of the Jacobian will be +1.

Now, finally, observing in Table 9-1 that

[C.7a] $\omega_7 = \omega_8 = \omega_9 = \omega_{10} = \omega_{11} = \omega_1'$ = $\omega_9' = \omega_1 = \omega_1'$

[C.7b] $\omega_{12} = \omega_{13} = \omega_{14} = \omega_{15} = \omega_{16} = \omega_1'$ = $\omega_1'$ = $\omega_1'$

[C.7c] $\omega_1' = \omega_{20} = \omega_{21} = \omega_{22} = \omega_{23} = \omega_{24} = \omega_{25} = \omega'_{26} = \omega_{27} = \omega_{28} = \omega_{29} = \omega_{210} = \omega_{211} = \omega_{212}$

one may write

[C.8] $I_{lm} \propto \int H_{n_l} (\beta, x_1) \cdots H_{n_{10}} (\beta, x_{10}) e^{-\sum_{i=1}^{10} \frac{1}{2} \omega_i x_i^2} \, dx_1 \cdots dx_{10}$

$\times \int H_{n_{11}} (\beta_1, x_{11}) \cdots H_{n_{14}} (\beta_4, x_{14}) e^{-\sum_{i=11}^{14} \frac{1}{2} \omega_i x_i^2} \, dx_{11} \cdots dx_{14}$

$\times \int H_{n_{15}} (\beta_4, x_{15}) \cdots H_{n_{20}} (\beta_{20}, x_{20}) e^{-\sum_{i=21}^{20} \frac{1}{2} \omega_i x_i^2} \, dx_{15} \cdots dx_{20}$
Hence, using the orthogonality relations of the Hermite polynomials one finds

[C.9] \[ I_{nm} \propto \delta_{n_1,0} \delta_{n_2,0} \cdots \delta_{n_4,0} \]

For the special case where 4 quanta are available to the doorway system, these can be distributed among the 23 vibrational modes to give 14,950 possible doorway states. But according to Eq. [C.9] a doorway state having any quanta in any of the modes 7 through 24 will have zero overlap with the single particle state \( m_2 \neq 0 \). Thus for the purposes of calculating these overlaps, the 4 quanta in the doorway system need only be distributed among modes 2 through 6 to give only 70 possible non-zero overlaps.

The conclusion of this section may be summarized by remarking that the orthogonality properties of the transformations and of the Hermite polynomials, together with the fortuitous equality of many of the eigenfrequencies of the two systems, permit the prediction of a great many zero overlaps. The remaining sections of this appendix will examine how many of these zero overlap integrals are non-fortuitous, and may be ascribed, instead, to the common invariance properties of the Hamiltonians belonging to the overlapping wavefunctions. In the example just discussed, for which only 0.47% of the 14,950 overlaps are non-zero, it will turn out that group theory removes only 50% of the overlaps to be calculated. Nonetheless, the analysis which follows could be a more useful tool in other cases.

C-2 Group Theoretical Prediction of Null Overlap Integrals

Using the notation of this thesis this section will serve as a
reminder that if two wavefunctions $\chi_n(r)$ and $\psi_m(r)$ are basis functions for two inequivalent irreducible representations of a group $\tilde{G}$, their overlap integral $I_{nm}$ is equal to zero.

The overlap integral $I_{nm}$ is defined by

$$I_{nm} = \int \chi_n(r) \psi_m(r) \, dr$$

Now let

$$r' = G \cdot r$$

where $G$ is an element of the group $\tilde{G}$ of matrix transformations on the $r$. Then

$$I_{nm} = \int \chi_n(G^{-1}r') \psi_m(G^{-1}r') \, dr' |J|$$

where the Jacobian $J$ is the determinant of $G^{-1}$, so that $|J|=1$ for orthogonal transformations. Now relabel the coordinate of integration by simply dropping the prime to obtain

$$I_{nm} = \int \chi_n(G^{-1}r) \psi_m(G^{-1}r) \, dr$$

Here $\chi_n(G^{-1}r)$ is a new function of $r$. Call it $P_G \chi_n(r)$, i.e.

$$P_G \chi_n(r) = \chi_n(G^{-1}r)$$

If $G$ is an invariance group of the Hamiltonian of $\chi_n(r)$, then this new function of $r$ may be expressed as a linear combination of
functions $\chi_n(r)$ belonging to the same eigenvalue of energy as $\chi_n(G^{-1}r)$:

$$[C.15] \quad P_G \chi_n(r) = \sum_{n'} D^\alpha(G)_{nn'} \chi_{n'}(r)$$

The matrix $D^\alpha(G)$ is a representation of the element $G$ of $\tilde{G}$. This representation has been labeled by the superscript $\alpha$.

A similar treatment of $\psi_m(G^{-1}r)$ leads to the representation $D^\beta(G)$, and permits the overlap $[C.13]$ to be written as

$$[C.16] \quad I_{nm} = \sum_{n',n''} D^\alpha(G)_{nn'} D^\beta(G)_{m'n''} \int \chi_{n'}(r) \psi_{n''}(r) \, dr .$$

Since Eq. [C.16] holds for every element $G$ of the group $\tilde{G}$, one may sum over all the elements $G$ and divide by the number of elements $|\tilde{G}|$ in the group $\tilde{G}$ to obtain

$$[C.17] \quad I_{nm} = \frac{1}{|\tilde{G}|} \sum_G D^\alpha(G)_{nn'} D^\beta(G)_{m'n'} \int \chi_{n'}(r) \psi_{n''}(r) \, dr .$$

But the well-known (see for example Wigner (1959) p. 83) orthogonality relation between matrices of two irreducible representations states that

$$[C.18] \quad \sum_G D^\alpha(G)_{nn'} D^\beta(G)_{m'm'} = 0 \quad \text{if} \quad \alpha \neq \beta .$$

Therefore, unless $\chi_n(r)$ and $\psi_m(r)$ are both basis functions for the same irreducible representation of $\tilde{G}$, their overlap, $I_{nm}$, will be zero.

C-3 The invariance Groups of the Single Particle and the Doorway Systems

In the previous section, Section C-2, it was shown that two wavefunctions may have non-zero overlap only if they each form part of a
basis for the same irreducible representation of the invariance group
which is common to their two Hamiltonians. In this section one asks just
what are the invariance groups of the two Hamiltonians at hand, namely,
the single particle Hamiltonian and the doorway system Hamiltonian.

An invariance group of a Hamiltonian is a group of coordinate
transformations every one of whose elements commutes with the Hamiltonian.
Such a group is now sought for the Hamiltonian of the single particle
system.

Consider the following single particle system

![Diagram of single particle system]

Fig. C-1. - The single particle system

where the coupling between individual particles is by means of a spring
of constant \( k \), the coupling between "a particles" is by means of a spring
of constant \( k' \), and the coupling between "\(^{12}\)C particles" is by means of a
spring of constant \( k'' \). Let \( r_i \) be the displacement to the right from
equilibrium of the \( i \)th particle. Each of the 24 particles has mass \( m \).
The Hamiltonian of this system is then

\[
H_s = \frac{m}{2} \dot{r}^T \mathbf{I} \dot{r} + \frac{k}{2} r^T \mathbf{V}_1 r + \frac{k'}{2} \frac{r^T \mathbf{U}_2 r}{2 \times 16} + \frac{k''}{2 \times 144} r^T \mathbf{U}_3^T r
\]

where

\[
\mathbf{V}_1 = \begin{pmatrix} A & A & A & A \\ A & A & A & A \end{pmatrix} \\
\mathbf{U}_2 = \begin{pmatrix} 0 & -6 & -6 & -6 \\ -6 & 0 & -6 & -6 \\ -6 & -6 & 0 & -6 \\ -6 & -6 & -6 & 0 \end{pmatrix} \\
\mathbf{U}_3 = \begin{pmatrix} 0 & -6 & -6 & -6 \\ -6 & 0 & -6 & -6 \\ -6 & -6 & 0 & -6 \\ -6 & -6 & -6 & 0 \end{pmatrix}
\]
where

\[ A = \begin{pmatrix} 1 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \theta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \]

By looking at Figure C-1 it is apparent that the system is invariant under certain coordinate transformations,

\[ r \rightarrow r' = S \cdot r \]

and, in fact, it can be verified that the Hamiltonian \[ C.19 \] is invariant if \( S \) is any of the following matrices \( g_i \) (\( i = 1, 2, 3, 4, 5, 6, 7, 8, 9 \)) or their products.

\[ g_i = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad g_j = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad g_k = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]

\[ g_l = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad g_m = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad g_n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]
These matrices \( g_i \) (\( i = 1,2, \ldots, 9 \)) are the generators of a 512 element (non-abelian) group \( \tilde{G}' \). This group is a subgroup of the symmetric group on 24 objects, i.e. the group of all possible permutations on 24 symbols. \( \tilde{G}' \) is an invariance group of the Hamiltonian [C.19].

Now consider the following doorway system

where the coupling between individual particles is by means of a spring of constant \( k \), and the coupling between "a particles" is by means of a spring of constant \( k' \). Then in the same notation used in the previous paragraph for the single particle system, the Hamiltonian for the doorway system may be written

\[
H_d = \frac{m}{2} \mathbf{r}^T \mathbf{1} \mathbf{r} + \frac{k}{2} \mathbf{r}^T \mathbf{V}\mathbf{1} \mathbf{r} + \frac{k'}{2 \times 16} \mathbf{r}' \mathbf{V}^2 \mathbf{r}'
\]

where
Again, by looking at Figure C-2 one sees that the system is invariant under certain coordinate transformations

\[ r' \rightarrow r'' = S r \]

which leave the Hamiltonian \[ C.25 \] invariant, where \( S \), in this case, can be any of the previously defined \( g_i \) (\( i = 1, 2, \ldots, 7 \)) or any of their products. These matrices \( g_i \) (\( i = 1, 2, \ldots, 7 \)) are the generators of a 128 element (non-abelian) group \( \tilde{G} \) which is a subgroup of the group \( \tilde{G}' \). \( \tilde{G} \) is an invariance group of the Hamiltonian \[ C.25 \]. Since it is also an invariance group of the single particle Hamiltonian \[ C.19 \], \( \tilde{G} \) is the common invariance group of both Hamiltonians under consideration.

C-4 The Representations of the Common Invariance Group \( \tilde{G} \)

In Section C-3 the invariance group \( \tilde{G} \), common to both the single particle Hamiltonian, and the doorway system Hamiltonian, was found. The elements of \( \tilde{G} \) were matrices having 24 x 24 entries, and these, in fact, constitute a 24-dimensional representation of \( \tilde{G} \), \( D(\tilde{G}) \). This representation \( D(\tilde{G}) \) will, in general, be reducible, which, in this case, means that it can be written as a direct sum of irreducible representations of the group \( \tilde{G} \). Wanted, then, in this section, are the irreducible representations of \( \tilde{G} \) and the decomposition of \( D(\tilde{G}) \) in terms of them.

It is convenient to begin by noting the abstract properties of the group \( \tilde{G} \). The group \( \tilde{G} \) has 7 generators \( g_i \), \( i = 1, 2, \ldots, 7 \), which have the following relations
The irreducible representations of \( \hat{G} \) can be found most easily by finding, first, the irreducible representations of a subgroup \( \tilde{H} \) of \( \hat{G} \), and, then, using the theory of induced representations to find the representations of \( \hat{G} \). Let \( \tilde{H} \) be the subgroup of \( \hat{G} \) with generators \( g_i (i = 1, 2, 3, 4, 5, 6) \). \( \tilde{H} \) will then be an abelian group of order 64, i.e. it is of index 2 in \( \hat{G} \) (which has 128 elements), and one may write

\[
\tilde{G} = \tilde{H} + q_\gamma \tilde{H}.
\]

Since a conjugate class is the set of \( H' \) obtained by \( H' = H_k H H^{-1}_k \), an abelian group will have as many conjugate classes as elements; in this case, 64 conjugate classes. For each conjugate class of \( \tilde{H} \) there exists an irreducible representation of \( \tilde{H} \). If \( d^i \) is the dimension of the \( \gamma \)-th representation, and \( N \) is the number of irreducible representations, then

\[
\sum_{i=1}^{N} d_i = \sum_{i=1}^{64} d_i = \text{order of the group} = 64
\]

which can only be satisfied if \( d_i = 1 \) for all \( i \). In other words, each one of the 64 irreducible representations of \( \tilde{H} \) is one dimensional. Note
that since for each \( H \in \tilde{H} \), \( H^2 = E \), a one dimensional representation \( d(H) \) will have each \( d(H) = \pm 1 \), and the 64 representations are each obtained from one of the \( 2^6 \) ways that +1 and -1 can be assigned to the 6 generators of \( \tilde{H} \).

Now that the representations of \( \tilde{H} \) have been found, it is necessary to consider the group \( \tilde{G} \) (of which \( \tilde{H} \) is a subgroup), and to recall the following definition: an element \( G \) of \( \tilde{G} \) belongs to the orbit \( \tilde{L}[d_j(H_i)] \) if \( d_j(H_i) \sim d_j(G H_i G^{-1}) \). One remarks that, in the present case, all elements \( H \) of \( \tilde{H} \) belong to \( \tilde{L}[d_j(H_i)] \) because \( \tilde{H} \) is abelian, and thus \( H H_i H^{-1} = H_i \), so that trivially \( d_j(H_i) \sim d_j(H H_i H^{-1}) \). If, in addition, \( d_j(H_i) \sim d_j(g_7 H_i g_7^{-1}) \), then \( g_7 \) and all the elements of \( \tilde{G} \) belong to \( \tilde{L}[d_j(H_i)] \), so that \( \tilde{L}[d_j(H_i)] = \tilde{G} \). If, however, \( d_j(H_i) \not\sim d_j(g_7 H_i g_7^{-1}) \), then \( g_7 \) and all \( g_7 \tilde{H} \) elements of \( \tilde{G} \) do not belong to \( \tilde{L}[d_j(H_i)] \), so that \( \tilde{L}[d_j(H_i)] = \tilde{H} \).

The orbit of \( \tilde{H} \) in \( \tilde{G} \), as defined in the previous paragraph, plays a role in inducing a representation \( D(\tilde{G}) \) from a representation \( d(\tilde{H}) \) according to the following theorem:

a) If \( \tilde{L}[d_j(H_i)] = \tilde{G} \), then one obtains 2 irreducible representations of \( \tilde{G} \):

\[
[C.30] \quad \begin{align*}
D(H_i) &= d(H_i) = D(q, H_i), \\
\text{and}
\end{align*}
\]

b) If \( \tilde{L}[d_j(H_i)] = \tilde{H} \), then one obtains only 1 irreducible representation of \( \tilde{G} \):

\[
[C.31] \quad \begin{align*}
D(H_i) &= d(H_i) = -D(q, H_i).
\end{align*}
\]
The proof of this theorem can be found in many textbooks dealing with induced representations, for example Jansen (1967). The symbol ** represents expressions involving \( d_j(H_i) \) and \( d_j(g_7 H_i g_7^{-1}) \) which will not be needed in subsequent calculations.

By enumeration of the 64 representations for the generators of \( \tilde{H} \), it is easy to see that there are 8 cases for which

\[
[C.33] \quad d_i(H_i) \sim d_j(g_7 H_i g_7^{-1})
\]

and thus, according to part (a) of the above theorem, these lead to 16 one-dimensional irreducible representations for \( \tilde{G} \). The 56 remaining representations of \( \tilde{H} \) pair up so that

\[
[C.34] \quad d_j(H_i) \sim d_k(g_7 H_i g_7^{-1}) \quad , \quad j \neq k
\]

and thus, according to part (b) of the theorem, one obtains 28 two-dimensional irreducible representations for \( \tilde{G} \). A useful check is provided by expression \([C.29]\):

\[
[C.29] \quad \sum_{i=1}^{N} d_i^2 = \text{order of the group}
\]

which in this case
verifies that all the irreducible representations of $\tilde{G}$ have been identified.

The groundwork has now been laid for asking the question, what irreducible representations of $\tilde{G}$ are contained in the 24-dimensional representation $D(\tilde{G})$? In other words, wanted is an explicit expression for

$$D_{\phi} \sim \sum \mu_i D_i(\phi)$$

where $\mu_i$ is the number of times that the irreducible representation $D_i(\tilde{G})$ is contained in $D(\tilde{G})$, and the sum is a direct sum. Taking the trace of both sides of Eq. [C.36] and using the orthogonality relation between irreducible representations of a group, one obtains

$$\mu_i = \frac{1}{|\mathbb{Z}|} \sum_{\phi} \text{Tr} [D(\phi)] \text{Tr} [D_i(\phi)]$$

which will be used repeatedly in what follows. The application of Eq. [C.37] is simplified by noting that

$$\text{Tr} [D(\phi, h)] = 0$$

so that

$$\mu_i = \frac{1}{128} \sum_h \text{Tr} [D(h)] \text{Tr} [D_i(h)]$$

The required traces are listed in Table C-I.
TABLE C-I

TRACES OF THE (REDUCIBLE) REPRESENTATION $D(\tilde{H})$, AND
OF SOME OF THE IRREDUCIBLE REPRESENTATIONS $d(\tilde{H})$

Notation:
The 6 generators of $H$ are labeled $H_j$ with $j = 1, 2, 3, 4, 5, 6$.

$$H_k H_\ell = H_k \ell$$

<table>
<thead>
<tr>
<th>Element k</th>
<th>$Tr(D(H_k))$</th>
<th>$Tr(d_1(H_k))$</th>
<th>$Tr(d_2(H_k))$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>24</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td></td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td></td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>16</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td></td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td></td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td></td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
### TABLE C-I (Continued...)

<table>
<thead>
<tr>
<th>Element k</th>
<th>$\text{Tr}(D(H_k))$</th>
<th>$\text{Tr}(d_1(H_k))$</th>
<th>$\text{Tr}(d_2(H_k))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>45</td>
<td>16</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>46</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>56</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>123</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>124</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>125</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>126</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>134</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>135</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>136</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>145</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>146</td>
<td>12</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>156</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>234</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>235</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>236</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>245</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>246</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>256</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>345</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>346</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>356</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>456</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
TABLE C-I (Continued...)

<table>
<thead>
<tr>
<th>Element k</th>
<th>$\text{Tr}(D(H_k))$</th>
<th>$\text{Tr}(d_1(H_k))$</th>
<th>$\text{Tr}(d_2(H_k))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1234</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1235</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1236</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1245</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1246</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1256</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>1345</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1346</td>
<td></td>
<td>8</td>
<td>-1</td>
</tr>
<tr>
<td>1356</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>1456</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>2345</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2346</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2356</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2456</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>3456</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>12345</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>12346</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>12356</td>
<td></td>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>12456</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>13456</td>
<td></td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>23456</td>
<td></td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>123456</td>
<td></td>
<td>-1</td>
<td>-1</td>
</tr>
</tbody>
</table>

-136-
The identity representation $d^G_1(\bar{H})$ induces 2 one-dimensional representations of $\bar{G}$:

$D_1(\bar{G})$: $D_1(G) = 1$ (the identity representation) and,

$D_{II}(\bar{G})$: $D_{II}(H) = 1$ and $D_{II}(g_H) = -1$.

Let $\mu_I$ be the number of times $D_1(\bar{G})$ is contained in $D(\bar{G})$. Then,

$$[C.40] \quad \mu_I = \frac{1}{128} \sum_H Tr[D(H)] \quad Tr[D_1(H)] = 6 .$$

Similarly,

$$[C.41] \quad \mu_{II} = \frac{1}{128} \sum_H Tr[D(H)] \quad Tr[D_{II}(H)] = 6 .$$

Thus, so far, one may write

$$[C.42] \quad D(\bar{G}) \sim 6 D_1(\bar{G}) \oplus 6 D_{II}(\bar{G}) \oplus \text{others} .$$

It is useful to note now that any representation of $\bar{G}$ when restricted to $\bar{H}$ is a representation of $\bar{H}$, and to ask which irreducible representations of $\bar{H}$ are contained in the 24 dimensional representations $D(\bar{H})$. Let $\nu_I$ be the number of times that the irreducible representation $d_1(\bar{H})$ is contained in $D(\bar{H})$. Then,

$$[C.43] \quad \nu_I = \frac{1}{64} \sum_H Tr[D(H)] \quad Tr[d_1(H)] .$$
Now, let $d_i(\tilde{H})$ be the identity representation. Let $d_i(\tilde{H})$ for $i = 1, 2, 3, 4, 5, 6$, be the representation of $\tilde{H}$ for which the $i$th generator $g_i$ is represented by $-1$ and all other generators are represented by $+1$. Then, it is easy to apply expression [C.43] using the traces tabulated in Table C-I to find that

\begin{align*}
\text{[C.44]} & \quad \gamma_1 = 12 \\
\text{[C.45]} & \quad \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = 2
\end{align*}

Therefore

\begin{align*}
\text{[C.46]} & \quad D(\tilde{H}) \sim 12 d_1(\tilde{H}) \oplus 2 d_1(\tilde{H}) \oplus 2 d_3(\tilde{H}) \oplus 2 d_4(\tilde{H}) \oplus 2 d_5(\tilde{H}) \\
& \quad \oplus 2 d_3(\tilde{H}) \oplus 2 d_4(\tilde{H})
\end{align*}

Note that in Eq. [C.46] the representations on the Right Hand Side exhaust the dimensionality (24) of $D(\tilde{H})$, and, thus, only these irreducible representations and no others are contained in $D(\tilde{H})$. Now it is necessary to ask, which representations of $G$ will, when restricted to $\tilde{H}$, result in the representations of Eq. [C.46]? Using the symbol $\downarrow$ to mean "restricted to", it is already clear that

\begin{align*}
\text{[C.47]} & \quad D(G) \downarrow \tilde{H} = D(\tilde{H}) \\
\text{[C.48]} & \quad D_1(G) \downarrow \tilde{H} = d_1(\tilde{H}) \\
\text{[C.49]} & \quad D_2(G) \downarrow \tilde{H} = d_1(\tilde{H})
\end{align*}
Further, by making use of the group properties [C.27] one observes that

\[ q_1 = q_1 q_4 q_7^{-1} \]

so that

\[ \rho_1(H) \sim \rho_4(q_7 H q_7^{-1}) \]

Thus the theorem on inducing representations quoted earlier in this chapter indicates the existence of a representation \( \rho_1(\tilde{G}) \) such that

\[ \rho_1(H) = \begin{array}{c}
\rho(H) & 0 \\
0 & \rho_4(q_7 H q_7^{-1})
\end{array} \]

\[ \rho_1(q_7 H) = \begin{array}{c}
0 & \cdots \\
\cdots & 0
\end{array} \]

This irreducible representation of \( \tilde{G} \) is, when restricted to \( \tilde{H} \), a reducible representation of \( \tilde{H} \). In fact, using Eq. [C.51], one may write

\[ \rho_1(\mathcal{E}) \downarrow \tilde{H} = \rho_1(\tilde{H}) \oplus \rho_4(\tilde{H}) \]

Similarly, and with analogous notation,

\[ \rho_2(\mathcal{E}) \downarrow \tilde{H} = \rho_2(\tilde{H}) \oplus \rho_5(\tilde{H}) \]

\[ \rho_3(\mathcal{E}) \downarrow \tilde{H} = \rho_3(\tilde{H}) \oplus \rho_4(\tilde{H}) \]
It is necessary here to observe carefully that since $d_1(\tilde{H})$ and $d_6(\tilde{H})$ induce only $D_1(\tilde{G})$, only $D_1(\tilde{G})$ will, when restricted to $\tilde{H}$, yield $d_1(\tilde{H})$ and $d_6(\tilde{H})$. This observation permits one to write, finally,

\[ D(\tilde{G}) \sim \mathbb{D}_1(\tilde{G}) \oplus \mathbb{D}_6(\tilde{G}) \oplus \mathbb{D}_2(\tilde{G}) \oplus \mathbb{D}_3(\tilde{G}) \]

as the only decomposition of $D(\tilde{G})$ in terms of the irreducible representations of $\tilde{G}$ which is consistent with Eq. [C.46]. Expression [C.56] is therefore the decomposition of $D(\tilde{G})$ sought for at the beginning of this section.

**C-5 Classification of the Eigenfunctions**

As was shown in Section C-2, the prediction of a null overlap integral between two wavefunctions depends on whether or not the two wavefunctions are each part of a basis for the same representation of a group $\tilde{G}$. One of the wavefunctions being considered is the single particle wavefunction $\psi_m(y)$ which is an eigenstate of the Hamiltonian $H_s$ of Eq. [C.19]. The other is the doorway system wavefunction $\chi_n(x)$ which is an eigenstate of the Hamiltonian $H_d$ of Eq. [C.25]. The group $\tilde{G}$ is the common invariance group of the two Hamiltonians. It is necessary, then, in this section, to classify the states $\psi_m(y)$ by specifying for each set of quantum numbers $m$ the representation of $\tilde{G}$ for which $\psi_m(y)$ serves as a basis function. A similar classification must also be made for the states $\chi_n(x)$. Since the wavefunctions are known functions of the coordinates, it is appropriate to begin by considering the transformation properties of the coordinates.
C-5(a) Transformation Properties of the Coordinates

The representation $D(G)$ has the particle coordinates $\mathbf{r}$ as its basis functions; i.e. the representation $D(G)$ consists of matrices of size $24 \times 24$ which act on the 24-dimensional vector $\mathbf{r}$. The wavefunction $\chi_n(\mathbf{r})$ is, however, written in terms of the normal coordinates

\[\chi^* = \mathbf{A}^T \mathbf{r}\]

where $\mathbf{A}^T$ is a real, orthogonal, $24 \times 24$ matrix. The effect of the representation $D(G)$ on the coordinates $\mathbf{x}$ can be found by performing the similarity transformation

\[D^x(G) = \mathbf{A}^T D(G) \mathbf{A}\]

where the representation $D^x(G)$ is a representation equivalent to $D(G)$ and having the coordinates $\mathbf{x}$ as basis functions. Carrying out the transformation $[C.58]$ for the 7 generators $g_i$ of the group $\tilde{G}$ one obtains

\[D^x(g_i) = \begin{bmatrix} M_1 & M_{i2} & M_{i3} \\ M_{i2} & M_{1} & \ldots \\ M_{i3} & \ldots & \ldots \end{bmatrix}_{24 \times 24} \quad D^y(g_4) = \begin{bmatrix} M_1 & M_5 & M_6 \\ M_5 & M_1 & \ldots \\ M_6 & \ldots & \ldots \end{bmatrix}_{24 \times 24}

where $i = 1, 2, 3, 4, 5, 6,$
and $M_1, M_3, M_5, M_6, M_7$ are, each, non-zero $6 \times 6$ matrices.

(1) One sees immediately, by inspection, that

$x_1$ is a basis for $D_1(\tilde{G})$,

$x_2$ is a basis for $D_{II}(\tilde{G})$,

$x_3$ is a basis for $D_1(\tilde{G})$,

$x_4$ is a basis for $D_{II}(\tilde{G})$,

$x_5$ is a basis for $D_1(\tilde{G})$,

$x_6$ is a basis for $D_{II}(\tilde{G})$.

(2) Again, by inspection, one sees that $x_13, x_14, x_15, x_16, x_17$ and $x_18$ are each a basis for $d_1(\tilde{H})$. $d_1(\tilde{H})$ induces $D_1(\tilde{G})$ and $D_{II}(\tilde{G})$.

However, after the previous paragraph, (1), and because of the decomposition [C.56] of $D(\tilde{G})$ in terms of irreducible representations of $\tilde{G}$, there are only 3 $D_1(\tilde{G})$ and 3 $D_{II}(\tilde{G})$ which have not yet been assigned basis functions. Thus, there exist 6 linear combinations of $x_13, x_14, \ldots, x_18$ such that each of 3 of them forms a basis for $D_1(\tilde{G})$, and each of the other 3 forms a basis for $D_{II}(\tilde{G})$. Nevertheless, if one wishes to adhere to the normal coordinates as given, it suffices to state that

$\{x_13, x_14, \ldots, x_18\}$ is a basis for 3 $D_1(\tilde{G}) \oplus 3 D_{II}(\tilde{G})$. 
(3) It remains to assign bases to the six 2-dimensional representations $2D_1(\tilde{G})$, $2D_2(\tilde{G})$, and $2D_3(\tilde{G})$ out of $x_7, x_8, \ldots, x_{12}$ and $x_{19}, x_{20}, \ldots, x_{24}$. Consider $\{x_7, \ldots, x_{12}\}$. It must form a basis for a 6-dimensional representation of $\tilde{G}$ which is equivalent to a direct sum of 3 of the 6 irreducible representations above. This 6-dimensional representation is in fact given in [C.59] by $D^\chi(G)_{ij}$, where $i,j = 7,8, \ldots, 12$. Recalling that two equivalent representations will have the same trace, it is possible to identify which 3 irreducible representations, when summed directly, are equivalent to the 6-dimensional representation by using the following compilation of traces.

$$1 = 1 \ 2 \ 3 \ 4 \ 5 \ 6$$

$$\text{Tr}[D_1(g_1)] = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix} \equiv [1]$$
$$\text{Tr}[D_2(g_1)] = \begin{bmatrix} 2 & 0 & 2 & 2 & 0 & 2 \end{bmatrix} \equiv [2]$$
$$\text{Tr}[D_3(g_1)] = \begin{bmatrix} 2 & 2 & 0 & 0 & 2 & 2 \end{bmatrix} \equiv [3]$$

so that

$$[1 \otimes 1 \otimes 2] = \begin{bmatrix} 2 & 4 & 6 & 6 & 4 & 2 \end{bmatrix}$$
$$[1 \otimes 1 \otimes 3] = \begin{bmatrix} 2 & 6 & 4 & 4 & 6 & 2 \end{bmatrix}$$
$$[1 \otimes 2 \otimes 2] = \begin{bmatrix} 4 & 2 & 6 & 6 & 2 & 4 \end{bmatrix}$$
$$[1 \otimes 2 \otimes 3] = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 & 2 \end{bmatrix}$$
$$[1 \otimes 3 \otimes 3] = \begin{bmatrix} 4 & 6 & 2 & 2 & 6 & 4 \end{bmatrix}$$
$$[2 \otimes 2 \otimes 3] = \begin{bmatrix} 6 & 2 & 4 & 4 & 2 & 6 \end{bmatrix}$$
$$[2 \otimes 3 \otimes 3] = \begin{bmatrix} 6 & 4 & 2 & 2 & 4 & 6 \end{bmatrix}$$
These must be compared with what is obtained from direct calculation on the matrices of Eq. [C.59] which yield

\[ \sum_{j=1}^{n} D^X(q_i)_{jj} = 4 \quad \text{for } i = 1, 2, \ldots, 6 \]

This permits an immediate and unique assignment of \( \{x_7, \ldots, x_{12}\} \) as the basis for a representation equivalent to \( D_1(\tilde{G}) \oplus D_2(\tilde{G}) \oplus D_3(\tilde{G}) \).

(4) By elimination, \( \{x_{19}, \ldots, x_{24}\} \) is a basis for a 6-dimensional representation equivalent to \( D_1(\tilde{G}) \oplus D_2(\tilde{G}) \oplus D_3(\tilde{G}) \).

The above 4-step reasoning can be repeated for the normal coordinates \( y \) of the single particle system in order to find their transformation properties. Let \( D^Y(\tilde{G}) \) be a representation equivalent to \( D(\tilde{G}) \) but having the coordinates \( y \) as basis functions. By straightforward calculation its matrices are found to have a structure similar to the matrices \( D^X(G) \) given in Eq. [C.59] with the exception that

\[ M_y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \]

(1) By inspection one sees that

- \( y_1 \) is a basis for \( D_I(\tilde{G}) \)
- \( y_2 \) is a basis for \( D_{II}(\tilde{G}) \)
\{y_3, y_4\} is a basis for \(D_I(\tilde{G}) \otimes D_{II}(\tilde{G})\)
\{y_5, y_6\} is a basis for \(D_I(\tilde{G}) \otimes D_{II}(\tilde{G})\)

\[(2)\] By inspection one sees that \(y_{13}, y_{14}, \ldots, y_{18}\) are each a basis for \(d_I(\tilde{H})\). \(d_I(\tilde{H})\) induces \(D_I(\tilde{G})\) and \(D_{II}(\tilde{G})\) of which there are now only 3 of each to be assigned bases. Thus, \(\{y_{13}, y_{14}, \ldots, y_{18}\}\) is a basis for \(3 \times D_I(\tilde{G}) \otimes 3 \times D_{II}(\tilde{G})\).

\[(3)\] Consider \(y_7, \ldots, y_{12}\). They transform among themselves under the group \(\tilde{G}\) according to \(D^y(\tilde{G})_{ij}\), where \(i, j = 7, 8, \ldots, 12\). The trace of this representation can be found by direct calculation to be

\[
[C.63] \sum_{j=7}^{12} D^y(q_{ij})_{ij} = 4 \quad \text{for} \quad i = 1, 2, \ldots, 6 \]

Comparing this with the table of traces in paragraph (3) above, one arrives, similarly, at a unique assignment of \(\{y_7, \ldots, y_{12}\}\) as the basis for a representation equivalent to \(D_I(\tilde{G}) \otimes D_2(\tilde{G}) \otimes D_3(\tilde{G})\).

\[(4)\] By elimination, \(\{y_{19}, \ldots, y_{24}\}\) is a basis for a 6-dimensional representation equivalent to \(D_I(\tilde{G}) \otimes D_2(\tilde{G}) \otimes D_3(\tilde{G})\).

C-5(b) Transformation Properties of the Exponential

The wavefunction of a system with harmonic interactions, like the doorway system under consideration, consists of a product of Hermite polynomials times the exponential factor

\[
[C.64] e^{-\frac{i}{\hbar} \sum_{i=1}^{N} \omega_i x_i^4} .
\]
Wanted are the transformation properties of the wavefunctions. Having just obtained in Section C-5(a) the transformation properties of the coordinates, one now proceeds to examine the transformation properties of the exponential part of the wavefunction.

Since several modes of the doorway system share the same frequency, it is possible to write the exponential in Eq. [C.64] as

\[ e^{-\frac{i}{\hbar} \sum_j \omega_j \sum_{k_j} x_{k_j}} \]

where all the modes labeled by \( k_j \) have the same frequency \( \omega_j \). It is important to remark now that for each value of \( j \) the \( x_{k_j} \) form a basis for a representation of \( \tilde{G} \) as described in Section C-5(a). With this remark in mind, it is useful to recall the following theorem:

If \{\( F_1, \ldots, F_n \)\} is a set of basis functions for a representation of a group \( \tilde{G} \), then \( \sum_{i=1}^{n} F_i^2 \) is invariant under all elements of \( \tilde{G} \).

Proof:

\[ P_g \sum_i F_i F_i = \sum_i \sum_{j,k} D^*(g)_{ij} D^{\ast}(g)_{ik} F_j F_k \]

\[ = \sum_{j,k} \sum_i D^*(g^{-1})_{ij} D^{\ast}(g)_{ik} F_j F_k \]

\[ = \sum_{j,k} D^{\ast}(E)_{jk} F_j F_k \]

\[ = \sum_i F_i F_i \quad Q.E.D. \]
Hence the factor $\sum_{k,l} x^2_{kj}$ in Eq. [C.65] is invariant under all elements of the group $\tilde{G}$, and hence the entire exponential is invariant under any element of $\tilde{G}$. Thus the transformation properties of the wavefunctions are determined exclusively by the transformation properties of the Hermite polynomials they contain.

C-5(c) Transformation Properties of the Fundamental Wavefunctions

Having seen that the exponential in the wavefunction is invariant under all elements of the group $\tilde{G}$, it follows trivially that the ground state which contains only the zeroth Hermite polynomial,

$$[C.70] \quad H_0 = 1$$

is also invariant under all elements of the group. The wavefunctions containing one quantum of excitation in one mode are called the fundamental wavefunctions. Since the fundamental wavefunctions contain only the first Hermite polynomial

$$[C.71] \quad H_1(\xi) = 2\xi$$

they will transform in the same way that the coordinates transform. That is, a state with one quantum in the $i$th mode has the same transformation properties as the $i$th normal coordinate. Further, having found the transformation properties of the normal coordinates in Section C-5(a), it is straightforward to classify all the fundamental wavefunctions according to energy and to representation of $\tilde{G}$. This is done in Table C-II where a subscript $n_0 = 1$ on a wavefunction indicates that there is one quantum in
the \( i \)th mode, and no quanta in any of the other modes.

### TABLE C-II

CLASSIFICATION OF FUNDAMENTAL WAVEFUNCTIONS ACCORDING TO ENERGY AND TO REPRESENTATION OF \( \hat{G} \)

<table>
<thead>
<tr>
<th>Single Particle System</th>
<th>Doorway System</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi(y), \ldots, \psi(y) )</td>
<td>( \chi(z), \ldots, \chi(z) )</td>
</tr>
<tr>
<td>( m_{i1}^{} )</td>
<td>( D_{i} \otimes D_{2} \otimes D_{3} )</td>
</tr>
<tr>
<td>( m_{v1}^{} )</td>
<td>( n_{a1}^{} )</td>
</tr>
<tr>
<td>( n_{x1}^{} )</td>
<td></td>
</tr>
<tr>
<td>( n_{y1}^{} )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\psi(y), \psi(y) & \quad D_{1} \otimes D_{2} \otimes D_{3} \\
\psi(y), \psi(y) & \quad D_{1} \otimes D_{2} \otimes D_{3} \\
\psi(y) & \quad D_{1} \otimes D_{2} \otimes D_{3} \\
\psi(y) & \quad D_{1} \otimes D_{2} \otimes D_{3} \\
\psi(y) & \quad D_{1} \otimes D_{2} \otimes D_{3} \\
\psi(y) & \quad D_{1} \otimes D_{2} \otimes D_{3} \\
\end{align*}
\]

This table may be used directly to determine when an overlap integral between a single particle state and a doorway state will be zero by
making use of the theorem of Section C-2. For example

\[ [C.72] \int \psi(y) \chi_{n_i \tilde{n}_i} \, d\mathbf{r} = 0 \]

because \( \psi(y) \) is a basis function for \( \mathbb{D}_2(G) \), but \( \chi(x) \) is a basis function for \( \mathbb{D}_1(G) \otimes \mathbb{D}_2(G) \otimes \mathbb{D}_3(G) \).

C-5(d) Transformation Properties of the Overtone Levels

An overtone state is defined as having more than one quantum of excitation in one frequency, and no quanta in other frequencies. Note that more than one mode may share the same frequency. An overtone wavefunction will consist of a product of Hermite polynomials times an exponential factor, and since the exponential factor was shown in Section C-5(b) to be invariant under the action of the group, the symmetry species of the wavefunction is that of its Hermite polynomial(s).

Suppose that there are \( d \) normal coordinates \( r_1, r_2, \ldots, r_d \) with equal frequencies \( \omega_1 = \omega_2 = \ldots = \omega_d = \omega \). Suppose that \( n \) quanta are to be distributed into modes with frequency \( \omega \). This will give rise to an \( f \)-fold degenerate overtone level with \( f = f(d,n) \). A wavefunction for this level will be

\[ [C.73] \phi \propto \mathcal{H}_{n_1}(\sqrt{\omega} \, r_1) \mathcal{H}_{n_2}(\sqrt{\omega} \, r_2) \ldots \mathcal{H}_{n_d}(\sqrt{\omega} \, r_d) \]

where \( n_1 + n_2 + \ldots + n_d = n \). Now, since the Hermite polynomial of order \( i \), \( H_i(\xi) \), has \( \xi^i \) as the highest power of \( \xi \), the wavefunction [C.73] may be written
Hence, it is apparent that the degree $f$ of degeneracy of the level is equal to the number of ways one may write

$$\phi \propto (r_1^{n_1} r_2^{n_2} \ldots r_d^{n_d} + \text{smaller powers of the } r's).$$

with $n_1 + n_2 + \ldots + n_d = n$. Now it is convenient to recall the following definition:

If $\Gamma(G)$ is a representation of $\tilde{G}$ transforming $r_1, r_2, \ldots, r_d$ among themselves, then $[\Gamma]^n(G)$ is a representation of $\tilde{G}$ transforming the products $(r_1^{n_1} r_2^{n_2} \ldots r_d^{n_d}), (r_1^{n'_1} r_2^{n'_2} \ldots r_d^{n'_d}), \ldots$ among themselves. Here $n_1 + n_2 + \ldots + n_d = n_1' + n_2' + \ldots + n_d' = n$.

$[\Gamma]^n(G)$ is called the symmetric $n$th power representation.

Now, the $f$ wavefunctions $\phi$ will form a basis for a representation of $\tilde{G}$, $\Delta(\tilde{G})$. The $f$ products of the type $(r_1^{n_1} r_2^{n_2} \ldots r_d^{n_d})$ will also form a basis for a representation of $\tilde{G}$, namely, $[\Gamma]^n(\tilde{G})$. It follows, therefore, that these 2 representations must have the same dimension. Further, since the transformation $\Delta(G)$ is linear, i.e.

$$P_\phi \phi_i = \sum_j \Delta(\tilde{G})_{ij} \phi_j$$

it will only mix together equal powers of the coordinates. However, this transformation of the powers of the coordinates among themselves is ac-
complished, as defined above, by the symmetric \( n \)th power representation. Hence,

\[ \Delta(\tilde{G}) \sim [\Gamma]^n(\tilde{G}) \]

In other words, the wavefunction of the state with \( n \) quanta distributed among modes which have the same frequency will be a basis function for the [reducible] \( n \)th power representation \([\Gamma]^n(\tilde{G})\).

C-5(e) Transformation Properties of Combination Levels

A combination state is one in which two or more distinct normal frequencies are excited with at least one quantum each. Let \( \phi_{(ij)} \) be the wavefunction of such a state. Then, if for example, 2 modes are excited,

\[ \phi_{(ij)} \propto H_{n_1}(\overline{\omega}_1 r_1) H_{n_2}(\overline{\omega}_2 r_2) \]

\[ \propto \phi_i' \phi_j'' \]

where \( \phi_i' \) is the state having \( n_1 \) quanta in the mode with normal coordinate \( r_1 \), and \( \phi_j'' \) is the state having \( n_2 \) quanta in the mode with normal coordinate \( r_2 \). Then the action of an element \( G \) of the group \( \tilde{G} \) on the wavefunction \( \phi_{(ij)} \) of Eq. [C.78] may be written

\[ \rho_G \phi_{(ij)} \propto \sum_k \Delta'(\tilde{G})_{ik} \phi_k' \sum_l \Delta''(\tilde{G})_{jl} \phi_l'' \]

\[ = \sum_{k, l} \Delta'(\tilde{G})_{ik} \Delta''(\tilde{G})_{jl} \phi_k' \phi_l'' \]
where \( \Delta'(G) \) and \( \Delta''(G) \) are the representations of \( \tilde{G} \) for which \( \phi'_{i} \) and \( \phi''_{j} \), respectively, are basis functions. Recall, however, that the direct product \( \Delta \) of two matrices \( \Delta' \) and \( \Delta'' \), written \( \Delta = \Delta' \otimes \Delta'' \), is given by

\[
\Delta_{ij, kl} = \Delta'_{ik} \Delta''_{jl}
\]

Therefore,

\[
\rho_{\phi} \phi_{(ij)} \propto \sum_{k,l} \Delta_{ik, jl} \phi_{(kl)}
\]

In other words, the combination level \( \phi_{(ij)} \) transforms like the direct product of the representations transforming \( \phi'_{i} \) and \( \phi''_{j} \).

C-5(f) Transformation Properties of the Most General Wavefunction

The foregoing may be applied to determine the transformation properties of, for example, the most general doorway state. The doorway system has only 8 distinct frequencies. Its wavefunction can, thus, be written

\[
\chi_{n} = \chi_{n_{1}, n_{2}, n_{3}, n_{4}, n_{5}, n_{6}, n_{7}, n_{8}}
\]

Therefore, bearing in mind the transformation properties of the fundamental doorway wavefunctions as listed in the Table C-II in Section C-5(c), \( \chi_{n} \) will transform under the representation

\[
\Delta_{a}(\tilde{G}) = [D_{1}]^{n_{1}} \otimes [D_{1}]^{n_{3}} \otimes [D_{1}]^{n_{5}} \otimes [D_{1}]^{n_{6}} \otimes [D_{1}]^{n_{7}} \otimes [D_{1}]^{n_{9}} \]
\[
[D_{1} \oplus D_{2} \oplus D_{3}]^{n_{10}} \otimes [3 D_{2} \oplus 3 D_{1}]^{n_{12}} \otimes [D_{1} \oplus D_{2} \oplus D_{3}]^{n_{14}}
\]
A similar expression applies, of course, to the most general single particle state $\psi_m$. The two wavefunctions $\chi_n$ and $\psi_m$ may have a non zero overlap only if the transformations for which they are basis functions contain, when reduced, at least one common irreducible representation of the group $\tilde{G}$.

Thus for the analysis of any given overlap it will be necessary to reduce a representation of the form [C.85] for the doorway system, and a similar representation for the single particle system. It is to be noticed that, although $A_a(\tilde{G})$ in Eq. [C.85] is written in terms of irreducible representations of $\tilde{G}$, it is not written in reduced form; i.e. it is not written as a direct sum of irreducible representations. To carry out this reduction one makes repeated use of the expression [C.37], namely,

\[
\mu_i = \frac{1}{|\tilde{G}|} \sum_\alpha T_r[\Delta_a(\tilde{G})] \cdot T_r[D_i(\tilde{G})]
\]

where $\mu_i$ is the number of times that the representation $\Delta_a$ contains the irreducible representation $D_i$. For this purpose one needs the traces of the representation $\Delta_a$ which can be obtained with the help of the expressions given in Eq. [C.87] to Eq. [C.92] below.

A general expression for the trace $[\chi]^n(\tilde{G})$ of the symmetric $n$th power representation $[\Gamma]^n(\tilde{G})$ is given by Lyubarski (1960), from which

\[
[C.87] \quad [\chi]^2(\tilde{G}) = \frac{1}{2} \chi^2(\tilde{G}) + \frac{1}{2} \chi(\phi^2)
\]

\[
[C.88] \quad [\chi]^3(\tilde{G}) = \frac{1}{3} \chi(\phi^3) + \frac{1}{2} \chi(\phi^2) \chi(\phi) + \frac{1}{6} \chi^3(\phi)
\]
The symmetrized $n$th product representation of a direct sum of 2 representations is given in reduced form by Watson (1972) as follows.

\begin{align*}
\text{[C.89]} \quad [\chi]^n \phi (G) &= \frac{1}{4} \chi (G^4) + \frac{1}{4} \chi (G^3) \chi (G) + \frac{1}{4} \chi (G^2) \\
&\quad + \frac{1}{4} \chi (G) \chi^2 (G) + \frac{1}{4} \chi^4 (G) .
\end{align*}

The reduction of representations such as $[\Gamma, \Gamma_2 \Gamma_3 + \ldots]^n$ is easily obtained from the immediately preceding three expressions by iteration.

Further, since some of the irreducible representations of $\tilde{G}$ are one-dimensional, it is convenient to gather here some results for the case of one-dimensional representations. Let $\Gamma(\tilde{G})$ be a one-dimensional representation, i.e. it has one basis function $r$; i.e.

\begin{align*}
\text{[C.93]} \quad \rho_r \phi r &= \Gamma(\tilde{G}) r .
\end{align*}

Therefore

\begin{align*}
\text{[C.94]} \quad \rho_r \Gamma^n &= \Gamma(\tilde{G}) \odot \Gamma(\tilde{G}) \odot \ldots \odot r^n .
\end{align*}
Thus it follows from the definition of the symmetric $n$th power representation given in Section C-5(d) that

\[[C.95] \quad [\Gamma]^n(\rho) = \rho(\rho) \otimes \rho(\rho) \otimes \cdots \text{n times},\]

i.e. for a one-dimensional representation the symmetric $n$th power representation is the same as the direct product representation. Now let $\mu_1$ be the number of times that the identity representation, $\Gamma_1(\rho)$ is contained in the direct product representation $\Gamma(\rho) \otimes \Gamma(\rho)$. Then, from Eq. [C.86],

\[[C.96] \quad \mu_1 = \frac{1}{|G|} \sum_\rho \rho(\rho) \Gamma(\rho) \Gamma_1(\rho) = 1\]

since $\Gamma_1(\rho) = 1$ for all $\rho \in \mathbb{G}$ and $\Gamma(\rho)$ contains $\Gamma(\rho)$ once. However, $\Gamma(\rho) \otimes \Gamma(\rho)$ is one dimensional. $\Gamma_1(\rho)$ exhausts its dimensionality. Hence $\Gamma(\rho) \otimes \Gamma(\rho) = \Gamma_1(\rho)$, and in general,

\[[C.97a] \quad [\Gamma]^n(\tilde{\rho}) = \Gamma_1(\tilde{\rho}) \quad \text{for even } n\]

\[[C.97b] \quad [\Gamma]^n(\tilde{\rho}) = \Gamma(\tilde{\rho}) \quad \text{for odd } n.\]

C-5(g) Prediction of Null Overlaps - An Example

The foregoing machinery permits the prediction of whether or not the overlap integral $I_{nm}$ of Eq. [C.3] between any single particle wavefunction $\psi_m$ and any doorway system wavefunction $\chi_n$ must equal zero. As an example of the procedure to be followed, the special case treated in Section C-1 from a different point of view, will be reanalyzed here for comparison.
Consider a single particle state with 4 quanta in the second normal mode. Since the corresponding fundamental wavefunction transforms like the 1-dimensional representation $D_{11}(\tilde{G})$, the 4 quanta wavefunction $\psi_m$ will transform like the identity representation $D_1(\tilde{G})$. Consider all doorway system states with 4 quanta. As given in Eq. [C.85], their wavefunctions transform under the representation

$$\Delta_{\alpha}(\tilde{G}) = \left[ D_1 \right]^{n_2} \otimes \left[ D_2 \right]^{n_3} \otimes \left[ D_2 \right]^{n_4} \otimes \left[ D_2 \right]^{n_5} \otimes \left[ D_3 \right]^{n_6} \otimes \left[ D_3 \right]^{n_7} \otimes \left[ D_2 \right]^{n_8} \otimes \left[ D_3 \right]^{n_9}$$

with $n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 = 4$. Thus, following the theorem of Section C-2, the overlap integral $I_{nm}$ between the two wavefunctions will be zero if $\Delta_{\alpha}(\tilde{G})$ does not contain $D_1(\tilde{G})$.

Let $\mu$ be the number of times the representation $\Delta_{\alpha}(\tilde{G})$ contains the representation $D_1(\tilde{G})$. Then, from Eq. [C.86], it follows that

$$\mu = \frac{1}{128} \sum_\alpha \text{Tr} \left[ \Delta_{\alpha}(\tilde{G}) \right]$$

In the next 5 paragraphs Eq. [C.99] is applied in turn to each of the 5 possible partitions of 4 quanta with the aid of the expressions [C.87] to [C.92]. Note that the possible values of the index $i$, indicating the normal frequency having $n_i$ quanta, are 2, 3, 4, 5, 6, 7, 13, and 19.

(1) The case $n_i = 4$. It is easily shown that each of the representations $[D_{11}]^4$, $[D_2]_4$, $[D_1 \otimes D_2 \otimes D_3]^b$, and $[3D_1 \otimes 3D_{11}]^4$ contain $D_1$. Therefore, any doorway state with 4 quanta in any one frequency may have a non-zero overlap with $m_2=4$ single particle state.
(2) The case \( n_i = 3, n_j = 1 \), with \( i \neq j \). Again, by direct
calculation using Eq. [C.99] one can show that the representations
\[
[D_{\text{II}}]^3 \otimes [D_{\text{II}}]^1, [D_{\text{I}}]^3 \otimes [D_{\text{I}}]^1, [D_{\text{I}}]^3 \otimes [3D_{\text{I}} \otimes 3D_{\text{II}}]^1, [D_{\text{II}}]^3 \otimes
[3D_{\text{I}} \otimes 3D_{\text{II}}]^1, [3D_{\text{I}} \otimes 3D_{\text{II}}]^3 \otimes [D_{\text{I}}]^1, [3D_{\text{I}} \otimes 3D_{\text{II}}]^3 \otimes [D_{\text{II}}]^1,
\]
and
\[
[D_{\text{I}} \otimes D_{\text{II}} \otimes D_{\text{III}}]^3 \otimes [D_{\text{I}} \otimes D_{\text{II}} \otimes D_{\text{III}}]^1
\]
are, in this case, the only representa-
tions containing \( D_{\text{I}} \). Thus, the following numbers of overlaps between the
\( m_2=4 \) single particle state, and the doorway state with 3 quanta in \( \omega_i \) and
1 quantum in \( \omega_j \) are predicted to be zero.

\[
\begin{array}{cccccccccccccccc}
\# & 1 & 1 & 6 & 6 & 1 & 1 & 6 & 6 & 1 & 1 & 6 & 6 & 1 & 1 & 6 & 6 \\
i = 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 \\
j = 3 & 5 & 7 & 19 & 2 & 4 & 6 & 7 & 19 & 3 & 5 & 7 & 19 & 2 & 4 & 6 & 7 & 19 & 3 & 5 & 7 & 19 \\
\end{array}
\]

This adds up to 1976 overlaps of this type predicted to be zero.

(3) The case \( n_i = 2, n_j = 2 \), with \( i \neq j \). In this case, it can
be shown that each of the representations \([D_{\text{I}}]^2, [D_{\text{II}}]^2, [3D_{\text{I}} \otimes 3D_{\text{II}}]^2\),
and \([D_{\text{I}} \otimes D_{\text{II}} \otimes D_{\text{III}}]^2\) contain \( D_{\text{I}} \). Therefore, if a state of the doorway
system has 2 quanta in one frequency and 2 quanta in another frequency,
none of it overlaps with the \( m_2=4 \) single particle state have to be zero.

(4) The case \( n_i = 2, n_j = 1, n_k = 1 \), with \( i \neq j \neq k \). In this case,
it can be shown that the representations
\( [D_I]^2 \otimes [D_1 \otimes D_2 \otimes D_3]^1 \otimes [D_1 \otimes D_2 \otimes D_3]^1, [D_{II}]^2 \otimes [D_1 \otimes D_2 \otimes D_3]^1 \otimes [D_1 \otimes D_2 \otimes D_3]^1, [D_I]^2 \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}], [D_{II}] \otimes [D_{II}] \otimes [D_1 \otimes D_2 \otimes D_3]^1, [D_{II}]^2 \otimes [D_{II}] \otimes [D_{II}], [D_I] \otimes [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}], [D_{II}]^2 \otimes [D_{II}] \otimes [D_{II}], [D_I] \otimes [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_{II}] \otimes [D_{II}], [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_I] \otimes [D_{II}], [D_{II}] \otimes [D_{II}], \text{and } [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}] \otimes [D_I] \otimes [D_{II}], \text{ and [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_{II}] \otimes [3D_I \otimes 3D_{II}]\text{ are the only representations of this type which contain } D_I. Therefore, the following numbers of overlaps between the } m_2=4 \text{ single particle state, and the doorway state with 2 quanta in } \omega_i \text{ and 1 quantum in } \omega_j \text{ and in } \omega_k \text{ are predicted to be zero.}

\begin{array}{ccc}
i = 2,3,4,5,6 & 2,3,4,5,6,13 & 7,19 \\
j = 2,4,6 & 2,3,4,5,6,13 & 7,19 \\
k = 3,5 & 7,19 & 2,3,4,5,6,13 \\
\# = 18 & 1860 & 2772
\end{array}

This makes 4650 overlaps of this type predicted to be zero.

(5) The case } n_i = n_j = n_k = n_\ell = 1, \text{ with } i \neq j \neq k \neq \ell. \text{ The representations } [D_I] \otimes [D_I] \otimes [D_{II}] \otimes [D_{II}], [D_I] \otimes [D_{II}] \otimes [D_{II}], [D_{II}] \otimes [D_I] \otimes [D_{II}], [D_{II}] \otimes [D_{II}] \otimes [D_{II}], [D_I] \otimes [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_{II}] \otimes [D_{II}] \otimes [D_{II}], [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_{II}] \otimes [D_I] \otimes [D_{II}], \text{ and [D_1 \otimes D_2 \otimes D_3]^2 \otimes [D_{II}] \otimes [3D_I \otimes 3D_{II}]\text{ are the only representations containing } D_I. \text{ can be shown to be the only representations containing } D_I. \text{ can be shown to be the only representations containing } D_I.
Therefore the following numbers of overlaps between the $m_2=4$ single particle state, and the doorway state with 1 quantum in each of $\omega_i$, $\omega_j$, $\omega_k$, and $\omega_\ell$ are predicted to be zero.

\[
\begin{array}{ccc}
    i = 3,5 & 7,19 & 7,19 \\
    j = 2 & 2,3,4,5,6 & 13 \\
    h = 4 & 2,3,4,5,6 & 2,3,4,5,6 \\
    \ell = 6 & 2,3,4,5,6 & 2,3,4,5,6 \\
    \# = 2 & 120 & 720 \\
\end{array}
\]

These are 842 overlaps of this type predicted to be zero.

In conclusion, a straightforward group theoretical analysis of the overlaps between the single particle state $m_2=4$ and all the possible doorway states with 4 quanta has led to the prediction of a total of 7468 null overlaps. This is to be contrasted with the 14,880 null overlaps predicted in Section C-1 by means of an examination of the structure of the integrals themselves. Thus, a group theoretical analysis is only of limited usefulness in the present study of strength functions. The reason for some of the limitation of this analysis was pointed out in paragraph (2) of Section C-5(a). In effect the wavefunctions which in Table C-II are classified according to representations of the group $\tilde{G}$ are not basis for irreducible representations of $\tilde{G}$. Although it would have been possible to search for a transformation of the wavefunctions among themselves such that the new wavefunctions being basis for irreducible representations of $\tilde{G}$ could have resulted in a larger number of zero predictions, it would have been difficult to connect in a detailed manner the null overlaps of the new wavefunctions with the overlaps of the wavefunctions actually used.
BIBLIOGRAPHY

Report of the first three intermediate resonances in the $^{12}\text{C} + ^{12}\text{C}$ reaction.

The first application of Ericson fluctuations (due to overlapping compound nucleus levels) is made using $^{12}\text{C}(^{12}\text{C}, \alpha)^{20}\text{Ne}$. Compound level widths and spacings are discussed.

By calculating the energies of quartet structures in even-even N=Z nuclei from $^{12}\text{C}$ to $^{52}\text{Fe}$ it is shown that these can compete favorably in energy with single nucleon excitations.

Four-nucleon direct transfer reactions such as $^{16}\text{O}(^{16}\text{O}, ^{12}\text{C})^{20}\text{Ne}$ are discussed by considering $^{16}\text{O} = ^{12}\text{C} + \alpha$ and $^{20}\text{Ne} = ^{16}\text{O} + \alpha$.

A treatment of explosive $^{12}\text{C}$ burning in supernovae. At the onset, however, hydrostatic carbon burning may occur near $T_9 \approx 0.3$ (see p. 190).

Using new estimates for the reaction rate the estimated temperature range for carbon burning is increased to $T = (0.8 - 1.2) \times 10^9$ K.

It is found that $^{20}\text{Ne} + \alpha$ has intermediate resonances at the same energies as $^{12}\text{C} + ^{12}\text{C}$. The $^{23}\text{Na} + \text{p}$ reaction does not. This supports the $\alpha$-cluster doorway interpretation.

A study of $\alpha$ clustering in the ground states of light nuclei. Clustering is almost complete in the ground state of $^{12}\text{C}$, but is much less so for $^{24}\text{Mg}$ and heavier nuclei.

A discussion of the conditions (masses and temperatures) necessary for hydrostatic carbon burning in stars.

Bethe, H. A., Intermediate Quantum Mechanics, Benjamin, New York (1964)
The Hartree-Fock self-consistent method is described in Chapter 6.
Biarens de Haan, D., Nouvelles tables d'intégrales définies, Amsterdam (1867)
A useful compendium of definite integrals. Table 28, #1, permits the evaluation of integrals of the exponential function of a general binomial expression without the use of a principle axis transformation.

The strength function and its moments are discussed in terms of a simple model that gives a Lorentzian strength function shape.

____, Nucl. Phys. 4, 503 (1957)
A unified formulation of the theory of nuclear reactions based on writing the scattering matrix as the Green function for the Schrödinger equation. Special cases lead to the Wigner resonance model, to the optical model, and to direct interactions for inelastic scattering.

A review of statistical nuclear theory covering level densities, statistics of level spacings and partial widths, average cross sections, and cross section fluctuations.

Block, B., and Feshbach, H., Ann. Phys. 23, 47 (1963)
The first use of the concept "doorway state" to interpret the systematic deviations of the neutron strength function around the optical model prediction.

Breit, G., Rev. Mod. Phys. 34, 766 (1962)
Reviews some aspects of N-N scattering including the validity of the OPEP. Concludes (p. 800) that the OPEP is the main interaction for r > 2.9 fm.

They show that a Hamada-Johnston potential with a repulsive square well core of height ~ 670 MeV yields the best fit to date for the phase shifts.

De Benedetti, S., Nuclear Interactions, Wiley, New York (1964)
Excellent introduction to all aspects of nuclear interactions "containing a description of the basic facts and a clear presentation of the theory".

They sort out the centre of mass motion from the internal motion of the system in a shell model with a harmonic oscillator potential. Spurious states are those that do not refer to the internal motion.

A review of nuclear energy levels for A = 21-44. In particular, there is a complete treatment of $^{24}_{5}Mg$, its energy levels, decays, and resonances where $^{24}_{5}Mg$ is the compound nucleus. Extensive up-to-date references.
One of a series of papers from Duke U. on s- and p-wave neutron spectroscopy for $A = 40-96$. "Together the Wigner and Porter-Thomas distributions explain the data very well".

Feshbach, H., in Reaction Dynamics ed. by E. W. Montroll et al., Gordon and Breach, New York (1973)
1968 Summer School in Mexico. Nuclear reactions are dealt with in 45 pages covering scattering theory, transition amplitude, resonance theory, optical model, energy averaging, and intermediate structure.

The introduction of the optical model as a tool for the description of average neutron cross sections.

A thorough and detailed introduction to the subject of intermediate structure and doorway states.

Contributions by Wigner, Feshbach, Lane, Mahaux, etc. on spacings, correlations, strength functions, level densities, intermediate structure, etc.

In page 152 can be found a clear account of the principles of thermonuclear reactions including Coulomb barrier penetration and reaction rates.

Goldstein, H., Classical Mechanics, Addison-Wesley, Reading (1950)
This well-known text gives a clear and elegant presentation of many topics in classical mechanics. Chapter 10 deals with small oscillations and the principle axis transformation to normal coordinates.

A review of the existence and non-existence of intermediate structure in heavy ion reactions. The level density of the compound nucleus is important but the details of nuclear structure must also be invoked in some (unknown) way.

Harrington, D. R., Phys. Rev. 147, 685 (1966)
The ground state of $^{12}$C is calculated quite successfully using s-wave interactions among 3 rigid alpha particles.
Quantum Field Theory is treated non-relativistically. The first chapters deal with a vibrating line of atoms, the harmonic oscillator, and coupled oscillators. Periodic boundary conditions are used.

Short note on non-exponential decay.

A brief review of the optical model for the nucleon-nucleus interaction up to 1967.

A very elementary but complete treatment of linear algebra. Very readable. All concepts are clearly defined. Nice exposition of orthogonal transformations, eigenvalue problems, etc.

As obtained from quasielastic (α, 2α) scattering at 910 MeV, it is claimed that "alpha clusterization in carbon is nearly 100% complete".

The entire issue entitled Alpha-Like Four Body Correlations and Molecular Aspects in Nuclei contains 7 articles devoted to topics such as Nuclear Forces and Clusterization, 8Be, etc.

Calculations on the trinucleon bound state using Reid's soft core potential and a harmonic oscillator basis.

Jansen, L., and Boon, M., Theory of Finite Groups. Applications in Physics, North-Holland, Amsterdam (1967)
Useful theorems of the theory of induced representations are presented. Page 161 treats the special case of a subgroup of index 2.

June 1966 lecture series on some of the early ideas on the topic. Contributions by R. Lemmer, L. S. Rodberg, J. E. Young, J. J. Griffin, A. Lande.

A very useful and readable text discussing many distribution functions and the relations between them.
Lane, A. M., Rev. Mod. Phys. 32, 519 (1960)
A careful discussion of the concept of reduced width and a compilation of the experimental reduced widths for particle and photon channels in light nuclei.

________, Thomas, R. G., Wigner, E. P., Phys. Rev. 98, 693 (1955)
A giant resonance model is introduced to account for the success of the optical model in nucleon-nucleus scattering. The nuclear strength function and the moments of the residual interaction are discussed and calculated.

________, and Thomas, R. G., Rev. Mod. Phys. 30, 257 (1958)
A comprehensive review of the R-Matrix theory of nuclear reactions. It is also shown (pp. 305, 306) that the average cross section can be expressed in terms of the strength function whether or not the compound nucleus levels overlap.

A simple model is solved exactly, and also using the Hartree-Fock and the Maximum Overlap methods. The two approximation methods are compared. Which method is better depends on the problems in which one is interested.

A collection of reprints with introductory text illustrating the vast usefulness of one-dimensional prototypes in a wide range of physical problems.

Alpha particle transfer reactions can populate (sd) \(^3\) (fp) states in \(^{32}\)S and \(^{36}\)A\(_N\) with strength comparable to (sd) \(^4\) states.

Expressions for symmetrized multiple products of representations are found in p. 75. The symmetry species of excited states of an oscillatory system are considered in p. 278.

A discussion of reduced widths and spectroscopic factors. Information from stripping reactions is used to study the structure of light nuclei.

A brief study of the status of intermediate structure in nuclear reactions.

Mathews, J., and Walker, R. L., Mathematical Methods of Physics, Benjamin, New York, 1964
A large number of useful techniques are explained. Ch. 6 deals with coordinate transformations and diagonalization of matrices. Ch. 14 discusses distribution functions, moments, etc.
The $^{12}\text{C} + ^{12}\text{C}$ reaction is measured down to 2.54 MeV in the c.m.
Intermediate structure is seen as low as ~ 4 MeV; at lower energies
large error bars preclude any conclusions re intermediate structure.

Conference on Nuclear Physics, Munich, 1973, ed. by J. de Boer
The level density is a global property of nuclear spectra, while
the level spacing distributions are local properties. Level
 spacings for a particular J$^\pi$ have a Wigner distribution. If all
J$^\pi$ values are allowed the distribution is Poisson.

Mendez-Moreno, R. M., Moreno, M., and Seligman, T. H., Nucl. Phys. A221,
381 (1974)
Using the best available phenomenological $\alpha$-$\alpha$ potentials the
spectra and form factors of $^{12}\text{C}$ and $^{16}\text{O}$ are calculated in the
$\alpha$-particle model.

Merzbacher, E., Quantum Mechanics, Wiley, New York (1961)
A textbook in elementary quantum mechanics. In Chapter 5 the
Schrödinger equation is solved for a harmonic oscillator potential.
The orthogonal eigenfunctions of the harmonic oscillator are given.

Messiah, A., Quantum Mechanics, North-Holland, Amsterdam (1961)
A well-known, pedagogically written textbook. Appendix B of
Vol. 1 is a useful treatment of special functions (Coulomb,
spherical Bessel, etc) and associated formulae. Chap. XVIII
discusses the Hartree-Fock method.

Michaud, G., Astrophys. J. 175, 751 (1972)
A description of the status of the $^{12}\text{C} + ^{12}\text{C}, ^{12}\text{C} + ^{16}\text{O}$, and $^{16}\text{O} + ^{16}\text{O}$
stellar reaction rates. New rates are calculated with a soft
core optical potential giving a factor of 10 increase over previous
calculations.

An "equivalent square well" is constructed for diffuse edge nuclear
potentials such that nuclear reaction calculations depending on
wave properties are simplified.

The $^{12}\text{C} + ^{12}\text{C}$ reaction is analysed phenomenologically using an optical
potential to describe the gross energy dependence of the cross
section. A framework is given in which to understand the inter­
mediate structure of the cross section in terms of alpha-particle
doorway states.

Middleton, R., Garret, J. D., and Fortune, H. T., Phys. Rev. Lett. 27,
950 (1971)
The $^{12}\text{C}(^{12}\text{C}, \alpha)^{20}\text{Ne}$ reaction is shown, at one angle, to preferentially
populate quartet states in $^{20}\text{Ne}$ compared to other states of the
same spin and parity. The quartet states consist of two sd-shell
$\alpha$ particles outside a $^{12}\text{C}$ core.
An exploration of the application of harmonic oscillator states to a variety of many-body problems.

A discussion of the decay law of unstable states. It is shown that at large times there are deviations from the exponential decay law.

A review of the N-N effective range parameters, $a_{np} \neq a_{nn}$ because the scattering length is too sensitive to higher order failures of charge independence.

A collection of important reprints together with an introduction by Porter which includes an application of statistics to prime numbers. Contains work by Wigner, Porter, Mehta, Dyson, Ericson.

The usefulness of a particular chi-squared distribution in describing the fluctuations of reduced widths is pointed out. The number of "degrees of freedom" is a lower bound on the number of open channels.

Very clear and didactic source book. Extensive treatment of N-N interactions. Sect. 18-3 deals with the strength function. Sect. 17-2 deals with the statistical distribution of reaction widths.

A brief, very readable book, suitable as an introduction to the subject.

Excellent review of N-N potentials. Lots of references. Also constructs his own potentials to fit latest phase shifts. Often quoted. His cores, however, are still too hard for a harmonic oscillator approximation.

Deviations from exponential decay due to the dependence of the decay rate on the amplitude at earlier times is discussed. Non-exponential decay implies non-Lorentzian line shapes.

The distribution of neutron widths in $^{51}$V from 5 to 16Q keV is found not to obey a Porter-Thomas distribution.
A concise description of how single-particle effects are to be handled with an application to alpha decay. Discussion of widths, reduced widths, spectroscopic factors.

Schiff, L. I., Phys. Rev. B133, 802 (1964)
The electromagnetic form factors of $^3$H and $^3$He are calculated in terms of the properties of the ground state wavefunction. The choice of wavefunctions is discussed in terms of the isospin formalism and the Pauli principle. Gaussian wavefunctions are given.

Quantum Mechanics, McGraw-Hill, New York (1968)
A well-known textbook. Chapter 10 presents a particularly clear treatment of identical particles.

Shapiro, I. S., Sov. Phys.-Usp. 4, 674 (1962)
Review of the optical model for nucleon and heavier particle scattering. The unexpectedness of its success for composite particles is pointed out.

Neutron strength functions of nuclei in the deformed region. The introduction gives a nice account of the meaning and significance of the concept of neutron strength function.

Taking into account the contribution of p-waves to the neutron cross section on $^{51}$V eliminates the disagreement with the Porter-Thomas distribution found by Rohr and Friedland.

The discovery of heavy ion reactions leading to compound nuclei of level densities similar to $^{24}$Mg and not possessing intermediate structure was a puzzle to the understanding of intermediate structure.

Tamagaki, R., Rev. Mod. Phys. 39, 629 (1967)
Combines meson-theoretical predictions at large distances (> 1.5 fm) with phenomenological description at small distances to obtain a soft core N-N potential.

Contains parameters for various types of potentials in various states including a very useful (for calculations) phenomenological three-range Gaussian potential for the N-N interaction.
They report the successes of a super soft core potential in fitting two-body data and nuclear matter properties.

Vogt, E. W., Rev. Mod. Phys. 34, 732 (1962)
A very didactic presentation of the theory of low energy nuclear reactions.

An early description of the alpha particle doorway states for the intermediate structure in the $^{12}\text{C}+^{12}\text{C}$ reaction was given in the context of a review of nuclear models.

, Nuclear Astrophysics unpublished lecture notes (1971)
An introduction to the basic nuclear physics of astrophysics including level widths, barrier penetration, strength function, nuclear models, reaction rates, etc.

, and Lascoux, J., Phys. Rev. 107, 1028 (1957)
The role of the moments and the widths of the strength function and of the overlap coefficients is discussed. A relationship is derived between the second moment and the width of the strength function.

Analysis of the $^{12}\text{C}(^{12}\text{C}, a)^{20}\text{Ne}$ reaction provides a thorough test for the statistical compound nucleus theory. Level spacings and widths for the compound nucleus $^{24}\text{Mg}$ are discussed.

Experimental evidence for alpha-particle doorway states in the $^{12}\text{C}+^{12}\text{C}$ reaction is presented lending support to the framework of Michaud and Vogt.

A review of correlated intermediate resonances shows they occur only for $^{12}\text{C}+^{12}\text{C}$, $^{12}\text{C}+^{16}\text{O}$, and $^{4}\text{He}+^{20}\text{Ne}$. Two empirical conditions for their observation are i) low level density in the compound nucleus and small number of open channels, and ii) both nuclei must be a nuclei.

Watson, J. K. G., J. Mol. Spectrosc. 41, 229 (1972)
Expressions for $[\Gamma_1 \otimes \Gamma_2 \otimes \ldots]_N$ are given in terms of the symmetric multiple power representations of $\Gamma_1, \Gamma_2 \ldots$.

Exact solutions to linear chains of harmonic oscillators are found with all possible boundary conditions. Chains are both uniform and with single mass defect.
Weisskopf, V., Physics Today 14, 18 (July 1961)
A qualitative discussion of nuclear reactions containing implicitly
the concept of doorway state later cemented by Block and Feshbach.


Motivated by the giant resonance model of Lane, Thomas, and
Wigner, strength functions for eigenvectors of certain kinds of
random matrices are calculated.

A well-known textbook containing many important theorems of group
theory - particularly those useful in quantum mechanics.

Wilson, E. B., Decius, J. C., and Cross, P. C., Molecular Vibrations,
McGraw Hill, Toronto (1955)
A very readable brief treatise on vibrations dealing with classical
vibrations, quantum mechanical vibrations, symmetries, selection
rules.

The level density is Gaussian if there are only 2-body interactions
and has a Wigner semi-circular shape if there are many-body
(> 7) interactions.

Conference on Nuclear Physics, Munich, 1973, ed. by J. de Boer and
Comparison of a Gaussian energy level distribution with that
obtained from shell model calculations give favorable results.

Lett. 28, 1656 (1972)
Latest (1972) description of experiment and analysis giving \(^1\Sigma_0^+
neutron-neutron effective range parameters.