RIGOROUS TREATMENT OF $\theta$-STATES

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Abstract

In the last twenty years there has been growing interest in topological effects in Quantum Field Theories (Q.F.T.'s). In particular, the existence of $\theta$-vacuums in Yang-Mills theories has attracted a lot of attention. One problem with these topological effects in Q.F.T.'s is that they are derived using the assumption that field configurations are represented by smooth functions. It is well known that the fields found in (rigorous) Q.F.T.'s are typically distributional and thus the usual topological arguments are not valid. A natural question to ask is: Do $\theta$-vacuums really exist in the rigorous theory of Q.C.D.? Here we construct the rigorous quantum theory of the electromagnetic field on a 3-torus and prove that $\theta$-vacuums exist in this Q.F.T. still exist. Hence, we show the rigorous existence of topological effects in this Q.F.T. and thus provide strong evidence for the existence of $\theta$-vacuums in Q.C.D..
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Chapter 1

Introduction

In recent years there has been growing interest in the role that topology plays in quantum physics (see Balachandran et als.[1]). This study has lead to many interesting results. In particular, the existence of particles with fractional spin and statistics in lower dimensions (see Leinaas and Myrheim [2]), PT violation in molecular physics (see Balachandran, Simoni and Witt [3]) and topological proofs of the spin and statistics theorem which does not require relativity or analyticity (see Balachandran et als. [4]).

The appearance of topological effects in quantum physics dates backs to the work of Dirac on the magnetic monopole in 1931 (see Dirac [5]). Unfortunately at that time the word topology was not part of the vocabulary of most physicists and the topological origin of the the magnetic monopole was not recognized. In was only in the late 1950's that the study of topology in quantum physics emerged. Tony H R Skyrme and David Finkelstein independently recognized the effects that topology can have on quantum states. In particular, they realized how Fermionic states can be built out of Bosonic variables [6] and the existence of topological conserved charges [7].

Another interesting topological effect is the existence of $\theta$-vacuums in Yang-Mills theories. In 1976 R.Jackiw and C.Rebbi [8] and also C.G.Callan, R.F.Dashen and D.J.Gross [9] noticed that the classical vacuum for Yang-Mills theories is degenerate. There exists an infinite number of classical ground states labeled by $n \in \mathbb{Z}$. This implies that the
correct quantum vacuum is equal to the linear superposition of all the classical ground states with weight factor $e^{i\theta n}$ for each classical vacuum $|n\rangle$. Hence, the quantum vacuums are given by $|\theta\rangle = \sum e^{i\theta n}|n\rangle$, where $\theta \in [0, 2\pi)$. Thus, there exists a one parameter family of vacuums for Yang-Mills theories labeled by $\theta$. The parameter $\theta$ has no observables effects in pure Yang-Mills theories but when coupled to matter fields it leads to CP violation in Q.C.D. (see Jackiw [10]).

When one argues the existence of topological effects in quantum physics one uses the assumption that the path integral is equal to the sum over all continuous paths (see Schulman [11] and Laidlaw and DeWitt [12]). One problem with such arguments in Quantum Field Theory (Q.F.T.) is that the path integral is given by the sum over all distributional paths. That is, it is equal to the sum over all distributional valued space-time fields. This is well known for rigorous Q.F.T.'s in lower dimensions (see Glimm and Jaffe [13]). Hence, the usual arguments to show the presence of topological effects in Q.F.T.'s can no longer be applied and thus there is no guarantee a priori that these topological effects are still present once we have rigorously constructed the field theory.

The distributional nature of the fields found in quantum field theories has been known for a long time (see Streater and Wightman [14]). Note that the canonical commutation relations $[\hat{\phi}(x), \hat{\Pi}(x')] = i\delta(x - x')$ only make sense if $\hat{\phi}(x)$ and $\hat{\Pi}(x)$ are smeared with smooth test functions. That is, $\hat{\phi}(x)$ and $\hat{\Pi}(x)$ are replaced by $\hat{\phi}(f)$ and $\hat{\Pi}(g)$, where $f$ and $g$ are smooth functions, and the canonical commutation relations are replaced by $[\hat{\phi}(f), \hat{\Pi}(g)] = i\int dx f(x)g(x)$. Because of the distributional nature of the field operators the classical configuration space must be enlarged in order to be compatible with the quantum theory. Thus, there is no guarantee a priori that the topology of the classical
configuration space will still be present once we have appropriately enlarged the configuration for the (rigorous) quantum theory. This leads to the question: Do $\theta$-states really exits in Yang-Mills theories?

To answer this question we rigorously construct the quantum theory of the electromagnetic field on a 3-torus and explicitly show that $\theta$-states are still present in the quantum theory. The necessary preliminary material and the explicit construction of this quantum system is ordered as follows. In chapter 2 we review the role that topology plays in quantum physics. In chapter 3 the necessary tools to rigorously construct quantum systems are developed and the precise meaning of quantization is given. Chapter 4 is devoted to the rigorous quantization of the harmonic oscillator and a free particle on a 3-torus. We construct the quantum theory of a free massive and massless scalar field on a 3-torus in chapter 5. Finally, in chapter 6 we prove that $\theta$-states exist in the quantum theory of the electromagnetic field on a 3-torus.
Chapter 2

Topology and quantum states

The first appearance of topological effects in quantum physics dates back to the work of Dirac on the magnetic monopole (see Dirac [5]). Dirac realized that wave functions in the presence of a magnetic monopole cannot be described by single valued functions on the configuration space, $Q$. Instead quantum states are represented by multivalued functions on $Q$. It is this observation which has lead physicists to realize the role that topology plays in quantum physics.

2.1 Path integral on multiply connected spaces

In the usual path integral approach to quantum mechanics one computes the propagator, $K(q,0;q',t)$, by summing over all continuous paths which go from $q$ to $q'$. Each of these paths are weighted by the phase factor $\exp(iS/h)$, where $S$ is the action functional. When the configuration space, $Q$, is simply connected, $\Pi_1(Q) = 0$, this prescription suffices to perform this sum, aside the usual technical difficulties. If we are given any two paths, $\phi(t)$ and $\psi(t)$, we cannot add relative phase factors between their contributions to $K(q,0;q',t)$. We cannot have a situation such as

$$K \sim e^{iS[\phi]/\hbar} - e^{iS[\psi]/\hbar}$$

(2.1)

since it is always possible to continuously deform $\phi(t)$ to $\psi(t)$ and thus continuously change the contribution of $\phi(t)$ to that of $\psi(t)$. Hence, any attempt to add relative phases between different paths leads to inconsistencies and thus are not permitted.
When the configuration space is multiply connected, $\Pi_1(Q) \neq 0$, the above argument is no longer valid. There is no reason why we cannot add relative phases between paths which cannot be continuously deformed one to another. Nothing prohibits us from writing the propagator as

$$K(q,0;q',t) = \sum_{\alpha \in \Pi_1(Q)} \chi(\alpha) K^\alpha(q,0;q',t) \quad \chi(\alpha) \in U(1), \quad (2.2)$$

where $\alpha$ labels the different homotopy classes of paths from $q$ to $q'$ and $K^\alpha(q,0;q',t)$ denotes the partial sum over all paths in the $\alpha$ homotopy class of paths. Even though the phase factors can be non-trivial, $\chi(\alpha) \neq 1$, they are not arbitrary. As it was first shown by M. Laidlaw and C. M. DeWitt [12] the phase factors $\chi(\alpha)$, where $\alpha \in \Pi_1(Q)$, form a unitary irreducible representation (U.I.R.) of the fundamental group of the configuration space, $\Pi_1(Q)$.

For example, consider a free particle on a circle, $Q = S^1$. The space $Q = S^1$ is represented by the interval $[0,2\pi]$ with 0 and $2\pi$ identified as a single point. Note that $\Pi_1(Q) = \Pi_1(S^1) = \mathbb{Z}$. The action functional for this system is equal to

$$S[q] = \int_0^t dt \frac{q'^2}{2m}, \quad (2.3)$$

where $q$ denotes the position of the particle on $Q = S^1$ and $m$ its mass. To compute the propagator, $K(q,0;q',t)$, we must sum over all paths from $q$ to $q'$. To accomplish this sum we lift up the system to the universal covering of $S^1$, $\hat{Q} = \mathbb{R}$. This permits us to visualize paths in different homotopy classes, that is, paths which wind around $S^1$ a different number of times. To simplify the sum we choose $q = q'$. Thus, the propagator is given by the sum over all closed loops which start and end at $q$. On the universal covering, $\hat{Q} = \mathbb{R}$, $q$ and $q + 2\pi n$, where $n \in \mathbb{Z}$, correspond to the same point $q$ on $S^1$. 
Thus, all closed loops on $S^1$ are represented by paths on $\mathbb{R}$ which go from $q$ to $q + 2\pi n$ for some $n \in \mathbb{Z}$. Recall that all U.I.R. of $\Pi_1(\mathbb{Q}) = \mathbb{Z}$ are given by $\{e^{i\theta}\}$, where $\theta \in [0, 2\pi)$. The parameter $\theta$ labels the set of all unitarily inequivalent U.I.R. of $\mathbb{Z}$.

It now follows from Eq.(2.2) that the propagator is equal to

$$K(q_0; q, t) = \sum_{n \in \mathbb{Z}} e^{i\theta} K^n(q_0; q, t), \quad (2.4)$$

where

$$K^n(q_0; q, t) = \int_{(q(0) = q)}^{q(t) = q + 2\pi n} Dq(t) e^{iS[q]/\hbar}. \quad (2.5)$$

Hence, the contribution of paths which wind around $S^1$ a different number of times differ by the phase factor $e^{i(n_1 - n_2)\theta}$, where $n_1$ and $n_2$ counts the number of times each loop goes around the circle.

It should be noted that the effects of the phase factor $e^{i\theta}$ can be incorporated into the action. This is accomplished by adding an extra term to the action $S$. We can rewrite Eq.(2.4) as

$$K(q_0; q, t) = \int_{\text{all closed loops}} Dq(t) e^{i(S + S_{\text{topological}})}, \quad (2.6)$$

where

$$S_{\text{topological}} = \int_0^t dt \frac{\theta}{2\pi} \dot{q}, \quad (2.7)$$

since

$$S_{\text{topological}} = \int_0^t dt \frac{\theta}{2\pi} \frac{dq}{dt} = \int_0^t dt \frac{d}{dt} \left( \frac{\theta}{2\pi} q \right) = \frac{\theta}{2\pi} (q(t) - q(0)) = \theta n. \quad (2.8)$$

Note that the integrand in $S_{\text{topological}}$ is a total derivative and thus has no physical effect at the classical level. Hence, the topological effects from $\Pi_1(\mathbb{Q}) = \mathbb{Z}$ causes the appearance of a new term in the action which can only be detected at the quantum level.
2.2 Topological effects and quantum field theories

In the previous section we showed how topological effects can appear in the path integral approach to quantum mechanics. A key element in our argument was the fact that the propagator is equal to the sum over all continuous paths. When one considers quantum field theories the situation is quite different. The propagator is no longer given by the sum over all continuous paths. Instead it is equal to the sum over all distributional paths, that is, to the sum over all distributional valued fields on spacetime. This is well known for rigorous $\lambda\phi^4$ theory in lower dimensions (see Glimm and Jaffe [13]). Thus, the path integral argument for the existence of topological effects is no longer valid. This leads us to the question: Do topological effects exist in rigorous quantum field theories?

Let us show why one must consider non continuous field configurations when dealing with quantum field theories. We follow the exposition of Horowitz and Witt [15]. Consider first quantized free string theory. A natural question to ask is: What string configurations are set of measure zero in the Fock space inner product? To answer this question first recall that the unnormalized ground state for string theory is given by

$$\Psi_0(x^\mu_n) = \exp\left(-\frac{1}{2} \sum_n n x^2_n\right),$$  \hspace{1cm} (2.9)

where we have decomposed the string, $X^\mu(\sigma)$, into Fourier modes,

$$X^\mu(\sigma) = \sum_n x^\mu_n e^{i n \sigma}. \hspace{1cm} (2.10)$$

Since all states in the Fock space are obtained by acting on the ground state with finite polynomials of the Fourier coefficients it suffices that we only consider the vacuum state. The normalization of the ground state to unity,

$$1 = \langle \Psi_0 | \Psi_0 \rangle = \int DX \exp\left(-\sum_n n x^2_n\right),$$  \hspace{1cm} (2.11)
Chapter 2. Topology and quantum states

fixes the explicit form of the measure. We find that

$$\prod_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{db_n}{\sqrt{\pi}} \exp(-b_n^2) = 1,$$

where $b_n = \sqrt{n} x_n$. Now, consider the set of string configurations which satisfy $\sum_n b_n^2 < \infty$. Because of the convergence condition on the sum there exists a constant $L$ such that $|b_n| < L$ for all $n$. We find that the contribution of all these string configurations to the Fock space inner product is equal to

$$\lim_{L \to \infty} \prod_{n=1}^{\infty} \int_{-L}^{L} \frac{db_n}{\sqrt{\pi}} \exp(-b_n^2) = 0,$$

where the zero on the right hand side follows from the fact that the on left hand side we have the infinite product of a number smaller then one. Hence, all such strings are set measure zero in the Fock space inner product. Note that these string configurations, $X^\mu(\sigma)$, correspond to continuous maps from $S^1$ to spacetime, $M$ (see Horowitz and Witt [15]). Thus, one must consider less regular string configurations since continuous strings are set of measure zero in the Fock space inner product. It should be noted that the analogous result in quantum mechanics is that differentiable paths are set of measure zero for the harmonic oscillator path integral (see Coleman [16]).

2.3 Canonical quantization on a multiply connected space

We now describe how to canonically quantize a system with a multiply connected configuration space (see Balachandran et als.[1]). Consider a classical system with configuration space $Q$ such that $\Pi_1(Q) \neq 0$. The first step to quantize is to lift up the classical system from $Q$ to its universal covering, $\hat{Q}$. Recall that $\Pi_1(Q)$ acts freely on $\hat{Q}$ and that $Q = \hat{Q}/\Pi_1(Q)$,

$$\Pi_1(Q) \to \hat{Q} \to Q = \hat{Q}/\Pi_1(Q).$$
We lift up classical observables from $\mathcal{O}$ to $\hat{\mathcal{O}}$ by demanding that they project down to well defined functions on $\mathcal{O}$. This is accomplished by requiring that observables be functions on $\hat{\mathcal{O}}$ which are invariant under the action of $\Pi_1(\mathcal{O})$.

The next step is to lift up wave functions from $\mathcal{O}$ to $\hat{\mathcal{O}}$. Since wave functions need not be single valued functions on $\mathcal{O}$ it follows that wave functions on $\hat{\mathcal{O}}$ don't have to be functions which are invariant under $\Pi_1(\mathcal{O})$. Instead we choose them to be functions on $\hat{\mathcal{O}}$ which transform under some U.I.R., $\Gamma$, of $\Pi_1(\mathcal{O})$, that is,

$$\psi(\hat{q}h) = D(h)\psi(\hat{q}) \quad \hat{q} \in \hat{\mathcal{O}}, \quad (2.15)$$

where $h$ is an element of $\Pi_1(\mathcal{O})$ and $\{D(h)\}$ is a U.I.R. of $\Pi_1(\mathcal{O})$. Finally, we quantize the system as usual.

One should note that for each distinct U.I.R. of $\Pi_1(\mathcal{O})$ there corresponds a unitarily inequivalent quantum theory. For example, if $\mathcal{O} = S^1$ we have that $\Pi_1(\mathcal{O}) = \mathbb{Z}$. Since all the U.I.R. of $\mathbb{Z}$ are given by $\{e^{in\theta}\}$, where $\theta \in [0, 2\pi)$ and $n \in \mathbb{Z}$, it follows that wave functions are functions on $\hat{\mathcal{O}} = \mathbb{R}$ which obey the transformation law

$$\psi_\theta(\hat{q} + 2\pi n) = e^{in\theta} \psi_\theta(\hat{q}). \quad (2.16)$$

Thus, for each value of $\theta$ we have unitarily inequivalent quantum theories. One should note that this canonical approach is consistent with the path integral approach studied in section 1.
2.4 Yang-Mills theories and $\theta$-states

Let us now consider Yang-Mills theories. The configuration space, $\mathcal{Q}$, for Yang-Mills theories is given by

$$\mathcal{Q} = \mathcal{A}/\mathcal{G},$$

(2.17)

where $\mathcal{A}$ is the set of all gauge potentials and $\mathcal{G}$ is the group of all gauge transformations. The gauge group, $\mathcal{G}$, is equal to the set of all continuous mappings from space, $\Sigma^3$, to the little gauge group, $\mathcal{G}$, that is, $\mathcal{G} = \text{Map}(\Sigma^3, \mathcal{G})$. We call a gauge transformation $g_0(x) \in \mathcal{G}$ small if it is connected to the identity, that is, $g_0(x) \in \mathcal{G}_0$, where $\mathcal{G}_0$ denotes the identity component of $\mathcal{G}$. A gauge transformation $g(x) \in \mathcal{G}$ is said to be large if it is not continuously deformable to the identity, that is, $g(x) \notin \mathcal{G}_0$.

Let $\mathcal{G} = SU(n)$ with $n > 1$ and $\Sigma^3 = \mathbb{R}^3$ with the usual boundary conditions at infinity, $\lim_{x \to \infty} g(x) = I$, where $I$ denotes the identity element of $\mathcal{G}$. In order to find all possible unitarily inequivalent quantum theories we need to compute $\pi_1(\mathcal{A}/\mathcal{G})$. Because of the boundary conditions at infinity $\mathcal{G}$ has a free action on $\mathcal{A}$ and thus $\mathcal{A}/\mathcal{G}$ is a fiber bundle (see Spanier [17]). Using the exact sequence of this fiber bundle (see Spanier [17]) and the contractability of $\mathcal{A} \simeq \ast$ it follows that

$$\pi_1(\mathcal{A}/\mathcal{G}) = \pi_0(\mathcal{G}),$$

(2.18)

where $\pi_0(\mathcal{G})$ denotes the set of path components of $\mathcal{G}$. We call $\pi_0(\mathcal{G})$ the group of large gauge transformations. It is well known that (see Jackiw [10])

$$\pi_0(\mathcal{G}) = \pi_3(\mathcal{G}) = \mathbb{Z},$$

(2.19)

where we have used the fact that $\pi_3(\mathcal{G})$ is equal to the group of integers, $\mathbb{Z}$, for all $\mathcal{G} = SU(n)$ with $n > 1$. Thus, just like for the free particle on a circle the set of inequivalent quantizations for this system is labeled by $\theta \in [0, 2\pi)$. Hence, the vacuum states
for Yang-Mills theories are labeled by \( \theta \), one for each unitarily inequivalent quantization. We call these quantum states \( \theta \)-states.

Let us now consider the case of interest to us, electromagnetism on a 3-torus, \( T^3 = S^1 \times S^1 \times S^1 \). In this case \( G = U(1) \) and \( \Sigma^3 = T^3 \). Because of the constant gauge transformations \( g(x) = e^{i\alpha} \), where \( \alpha \) is some constant, the action of \( G \) on \( A \) is not free and thus \( A/G \) is not a fiber bundle. Note that these constant gauge transformations have trivial action on \( A \) since

\[
A'(x) = A + g^{-1}(x) \nabla g(x) = A(x)
\]  

when \( g(x) = e^{i\alpha} \). Hence, \( A/G = A \), where \( G \) denotes the set of constant gauge transformations.

Now notice that the right hand side of

\[
\frac{A}{G} = \frac{A/G}{G/G}
\]  

is a fiber bundle since \( G/G \) act freely on \( A/G = A \). Taking the exact sequence of this fiber bundle and using \( A/G = A \simeq \ast \) it follows that

\[
\Pi_1(A/G) = \Pi_0(G/G).
\]  

Since \( G \) and \( G/G \) have the same path components it follows that \( \Pi_0(G/G) = \Pi_0(G) \). Hence, \( \Pi_1(A/G) = \Pi_0(G) \).

In order to find \( \Pi_1(A/G) \) we must compute \( \Pi_0(G) \). By definition we have that

\[
\Pi_0(G) = [T^3, U(1)],
\]  

(2.23)
Chapter 2. Topology and quantum states

where $[X, Y]$ denotes the homotopy classes of maps from $X$ to $Y$. Since $U(1) = K(Z; 1)$ it follows that (see Spanier [17])

$$[T^3; U(1)] = [T^3; K(Z; 1)] = H^1(T^3; Z),$$

where $H^1(T^3; Z)$ is the first cohomology group of $T^3$ with integer coefficients. Hence,

$$\Pi_1(\mathcal{A}/\mathcal{G}) = Z^3$$

since $H^1(T^3; Z) = Z^3$.

From this formal computation we conclude that the set of inequivalent quantum theories for electromagnetism on a 3-torus is labeled by $\vec{\theta} \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi)$ since all U.I.R. of $Z^3$ are given by $\{e^{i\vec{n} \cdot \vec{\theta}}\}$, where $\vec{\theta} \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi)$ and $\vec{n} \in Z^3$. This calculation is only formal since we have assumed that the set of all gauge transformations are represented by continuous mappings. Typically this is not true in the quantum theory and thus there is no guarantee a priori that we will find the same $\Pi_1(\mathcal{A}/\mathcal{G})$ in the rigorous quantum theory of electromagnetism on a 3-torus.
In this chapter we develop the necessary machinery to rigorously construct quantum systems. In particular, we introduce the notion of self-adjointness which will play a central role in rigorous quantum mechanics and quantum field theories. For a complete exposition see Reed and Simon [18].

3.1 Operators and Domains

A central feature of quantum systems is that physical observables are represented by operators acting on some Hilbert space $\mathcal{H}$. Hence, in order to construct rigorous quantum systems one must understand what is an operator.

First, consider the case when the Hilbert space $\mathcal{H}$ is finite dimensional. In this case $\mathcal{H} = \mathbb{C}^n$, where $n < \infty$, and operators are represented by matrices defined on all of $\mathcal{H}$. When the Hilbert space is infinite dimensional the situation is quite different. Operators are no longer given by matrices and they typically cannot be defined on all of $\mathcal{H}$. To understand this, let us give a precise meaning to the word operator and then introduce the notion of boundedness (see Reed and Simon [18]).

**Definition 1** An operator $A$ is a linear mapping from $\mathcal{D}(A) \subseteq \mathcal{H}$ to $\mathcal{H}$, where $\mathcal{D}(A)$ is a dense\(^1\) subset of the Hilbert space $\mathcal{H}$. We call $\mathcal{D}(A)$ the domain of the operator $A$.

\(^1\)A subset $\mathcal{D}(A) \subseteq \mathcal{H}$ is said to be dense if and only if its closure is equal to $\mathcal{H}$.

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We demand that $\mathcal{D}(A)$ be dense since we want $A$ to have a well defined adjoint $A^\dagger$ (see section 2). We now define boundedness.

**Definition 1** An operator $A$ is said to be bounded if and only if there exists some constant $k \in \mathbb{R}$ such that $||A\psi|| \leq k||\psi||$ for all $\psi \in \mathcal{D}(A)$.

An operator $A$ is said to be unbounded if it is not bounded. From the linearity of operators it follows that bounded operators are continuous mappings and unbounded are discontinuous. Consider the Laplacian, $-\nabla^2$, on the 2-sphere. The Laplacian acts on $\mathcal{H} = L^2(S^2)$, the space of square summable functions. Since not all $\psi \in L^2(S^2)$ are differentiable we must pick the domain of the Laplacian, $\mathcal{D}(-\nabla^2)$, to be a proper subset of $L^2(S^2)$. We choose it to be $\mathcal{D}(-\nabla^2) = C^\infty(S^2)$, the set of smooth functions on $S^2$. Solving the eigenvalue equation $-\nabla^2\psi = \lambda^2\psi$ we find that $-\nabla^2$ has eigenvalues $l(l+1)$ with $l = 0, 1, 2, \ldots$. If the Laplacian was a bounded operator then there would exist a constant $k \in \mathbb{R}$ such that $||-\nabla^2\psi|| \leq k||\psi||$ for all $\psi \in \mathcal{D}(-\nabla^2) = C^\infty(S^2)$. But since we can always choose a large enough $l$ such that $l(l+1) > k$ it follows that the Laplacian is an unbounded operator.

A natural question to ask is: Is there a connection between boundedness and domains? The answer to this question is found in the following theorem (see Reed and Simon [18]).

**Theorem 1 (Hellinger-Toeplitz)** If a Hermitian operator $A$ is defined on all the Hilbert space, $\mathcal{D}(A) = \mathcal{H}$, then $A$ is a bounded operator.

If we now take a moment to recall the operators that are encountered in elementary quantum mechanics (Hamiltonian, momentum, angular momentum,...), we quickly come to the realization that almost all\(^2\) these operators are unbounded. Then the above theorem implies that we must pick domains for most of our usual quantum observables.

\(^2\)An exception is spin operators since they act on finite dimensional Hilbert spaces.
3.2 Why Hermiticity is not enough

It is typically stated in quantum mechanics text books (see Messiah [20]) that physical observables are represented by Hermitian operators. This statement is simplistic or at least a sloppy use of the term Hermitian. A Hermitian operator can be plagued with an array of problems. The most serious is that they do not necessarily possess a spectrum or exponentiate to a well defined unitary operator. Thus, the use of Hermitian operators can cause illnesses which can render the quantum system inconsistent. The reason why these problems don’t appear in elementary quantum mechanics is that, in simple situations, one explicitly finds eigenvectors and eigenvalues which corresponds to the eigenstates and eigenvalues of a self-adjoint extension of the Hermitian that one considers.

Let us now give precise meaning to the word Hermitian. We first introduce adjoint operators.

**Definition 1** Let $A$ be some operator on the Hilbert space $\mathcal{H}$. Let $\mathcal{D}(A^\dagger)$ be the set of $\phi \in \mathcal{H}$ for which there is an $\eta \in \mathcal{H}$ such that

$$ (A\psi, \phi) = (\psi, \eta) \quad \text{for all } \psi \in \mathcal{D}(A). $$

(3.26)

For each such $\phi \in \mathcal{D}(A^\dagger)$, we define $A^\dagger \phi = \eta$. $A^\dagger$ is called the adjoint of $A$.

Note that if $\mathcal{D}(A)$ is not a dense subset then $A^\dagger$ is not well defined. This follows by first noting that if $\mathcal{D}(A)$ is not dense then there exists a $\chi \in \mathcal{H}$ such that $\chi$ is orthogonal to $\mathcal{D}(A)$, $(\chi, \psi) = 0$ for all $\psi \in \mathcal{D}(A)$. By noting that $(A\phi, \psi) = (\phi, \eta) = (\psi, \eta + \chi)$ it follows that $A^\dagger$ is not well defined if $\mathcal{D}(A)$ is not dense. It should also be noted that $A^\dagger$ may not even be an operator since there is no guarantee a priori that $\mathcal{D}(A^\dagger)$ is dense. In fact $A^\dagger$ can have an empty domain, $\mathcal{D}(A^\dagger) = \emptyset$ (see Reed and Simon [18]). We now define Hermiticity.
Chapter 3. Operators, Domains and Quantization

**Definition 1** An operator $A$ is said to be Hermitian if and only if $(A\phi, \psi) = (\phi, A\psi)$ for all $\psi, \phi \in D(A)$.

This definition simply states that the restriction of $A^\dagger$ to $D(A)$ is equal to $A$, $A^\dagger|_{D(A)} = A$, which is not equivalent to $A^\dagger = A$. For example, the momentum operator $p = -i\frac{d}{dx}$ on the interval $[0, \pi]$ with domain $D(p) = \{\phi \in C^\infty([0, \pi]) | \phi(0) = \phi(\pi) = 0\} \subset L^2([0, \pi])$ does not coincide with its adjoint, $p \neq p^\dagger$. This follows by first noting that for any $\psi \in D(p)$ and $\phi \in C^1([0, \pi])$

$$
(p\psi, \phi) = \int_0^\pi dx (-i\frac{d}{dx}\psi)^* \phi = \int_0^\pi dx \psi^*(-i\frac{d}{dx}\phi),
$$

where we have integrated by parts and used $\psi(\pi) = \psi(0) = 0$. Hence, $C^1([0, \pi]) \subset D(p^\dagger)$ and thus $D(p)$ is a proper subset of $D(p^\dagger)$ since it is a proper subset of $C^1([0, \pi])$. It now follows that $p \neq p^\dagger$ since $D(p) \neq D(p^\dagger)$. The statement $A^\dagger = A$ is known as self-adjointness. We will return to this later. Let us now show why Hermitian operators are not suited as quantum observables.

Consider a free particle on the interval $I = [0, \pi]$. The Hamiltonian for this system is equal to $H = -\frac{1}{2m}\frac{d^2}{dx^2}$, where $m$ is the mass of the particle. We choose $D = C_0^\infty((0, \pi))$, the set of smooth functions which vanish on some compact set in the interior of the interval $I = [0, \pi]$, as domain for $H$. Note that all $\psi \in D$ and all their derivatives vanish at the end points 0 and $\pi$. A simple calculation shows that $-\frac{1}{2m}\frac{d^2}{dx^2}$ is Hermitian on $D$. A natural question to ask is: What is the spectrum of the Hamiltonian? To find the spectrum of $H$ we first solve the eigenvalue equation

$$
-\frac{1}{2m}\frac{d^2}{dx^2}\psi_\lambda = \frac{\lambda^2}{2m}\psi_\lambda.
$$
We find the general solution

\[ \psi_{\lambda} = a + bx + ce^{\lambda x} + de^{-\lambda x}, \tag{3.30} \]

where \( a, b, c, d \) and \( \lambda \) are all arbitrary complex numbers. Next, we pick out the \( \psi_{\lambda} \) which are contained in \( \mathcal{D} = \mathcal{C}^\infty((0, \pi)) \). It is easy to see that the only solution of Eq.(3.29) contained in \( \mathcal{C}^\infty((0, \pi)) \) is the trivial solution, \( \psi_{\lambda} = 0 \). Thus, the Hamiltonian with domain has no spectrum. Hence, one cannot assume \textit{a priori} that a Hermitian operator has a non-trivial spectrum.

It should be noted that one can extend the domain of \( H \) in order to obtain a non-trivial spectrum, but one problem with this option is that there is no unique way to do this. For example, if we choose \( \mathcal{D}_\delta = \{ \phi \in \mathcal{C}^\infty([0, \pi]) \mid \phi(\pi) = e^{i2\pi\delta}\phi(0) \} \supset \mathcal{C}^\infty((0, \pi)) \), where \( 0 \leq \delta < 1 \), as our extension we find that the unnormalized eigenfunctions are given by

\[ \exp(i2(n + \delta)x) \quad n \in \mathbb{Z} \tag{3.31} \]

with eigenvalues \( \frac{2}{m}(n + \delta)^2 \). Hence, for each value of \( \delta \in [0, 1) \) we find a distinct spectrum. Thus, Hermitian operators don’t always posses a well defined spectrum.

We now turn to the exponentiation of Hermitian operators and the notion of commutation. We say that two operators \( A \) and \( B \) commute iff they generate one parameter families of unitary transformations defined on all the Hilbert space \( \mathcal{H} \), \( e^{isA} \) and \( e^{itB} \), such that \([e^{isA}, e^{itB}]\phi = 0\) for all \( \phi \in \mathcal{H} \). This is not equivalent to the statement that \( A \) and \( B \) commute on a dense subset \( \mathcal{D} \subset \mathcal{H} \), \([A, B]\phi = 0\) for all \( \phi \in \mathcal{D} \). The proof that \( A \) and \( B \) commute on a dense subset of \( \mathcal{H} \) does not imply that \( e^{isA} \) and \( e^{itB} \) commute on all of \( \mathcal{H} \) is known in literature as Nelson’s example (see Reed and Simon [18]). We will review Nelson’s example in section 4. It should be noted that the operators \( A \) and \( B \) in Nelson’s
example are essentially self-adjoint operators which is a much more restrictive class of operators than Hermitian operators. Note that this result implies that the naive notion of commutation for Hermitian operators is ill-defined. Moreover, this result implies that the formal power series expression for the exponential, $e^{iA} = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!}$, is ill-defined since if it was well defined it would contradict Nelson's example. Hence, exponentiation for Hermitian operators can be ill-defined.

Since Hermitian operators are plagued with an array of problems (ill-defined notion of commutation, cannot exponentiate, empty spectrum,...) we must find a new type of operator which is suited to represent physical observables for quantum systems.

### 3.3 Self-Adjointness

In the previous section we showed why Hermitian operators cannot represent physical observables. Let us now introduce a new type of operator which can. These are known as self-adjoint operators.

**Definition 1** An operator $A$ is said to be self-adjoint iff it is Hermitian and $\mathcal{D}(A^\dagger) = \mathcal{D}(A)$.

At this point the reader might be a bit perplexed since Hermiticity and self-adjointness only differ by an apparently small condition on the domain of $A^\dagger$. But it is this subtlety which is at the root of all the problems which plague Hermitian operators. It should be noted that given a Hermitian operator there can exist many self-adjoint extensions. Another way to say this is that for a given Hermitian operator there can exist many choices of boundary conditions, that is, domains of self-adjointness.
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The reason why self-adjoint operators is the correct choice of operator to representing physical observables is summarized in a fundamental result from functional analysis known as the spectral theorem (see Reed and Simon [18]). I will not state the theorem in its exact form because of its technical nature. Instead I will state the theorem in the particular case when the self-adjoint operator has discrete spectrum and point out what must be modified when spectrum is continuous.

**Theorem 1 (Spectral)** We can always decompose any self-adjoint operators as follows:

\[ A\psi = \sum_n \lambda_n (\psi, \phi_n) \phi_n \quad \lambda_n \in \mathbb{R} \quad (3.32) \]

where \( \lambda_n \) are the eigenvalues of \( A \) and \( \phi_n \) are its eigenfunctions.

An immediate consequence of the theorem is that all self-adjoint operators posses a well defined spectrum, which is crucial for these operators to represent physical observables. Moreover, the spectral theorem also provides the necessary tools for defining exponentiation. The exponential of a self-adjoint operator \( A \) is defined by

\[ e^{itA}\psi = \sum_n e^{it\lambda_n} (\psi, \phi_n) \phi_n, \quad (3.33) \]

which is a well defined (strongly continuous) unitary operator on \( \mathcal{H} \) (see Reed and Simon [18]). Thus, self-adjointness guarantees unitary evolution and unitary implementation of symmetries for quantum systems. Hence, two more key elements for quantum systems are satisfied with self-adjoint operators.

In the case where the self-adjoint operator has a continuous spectrum one must use projection valued measures (see Reed and Simon [18]) to accomplish the decomposition of Eq.(3.32). For example, consider the Hamiltonian for a free particle on the real line \( \mathbb{R} \),
\[ H = -\frac{1}{2m} \frac{d^2}{dx^2} \]. It is well known that, with a suitable choice of domain, the eigenfunctions of \( H \) are given by plane waves \( \frac{e^{ikx}}{\sqrt{2\pi}} \) which are not elements in \( L^2(\mathbb{R}) \). In this situation we use a projection valued measure to decompose \( H \).

It should be noted that since a Hermitian operator can have many self-adjoint extensions (see Reed and Simon [18]) it can be made to exponentiate to different unitary operators. Hence, if one wants to make sense of the formal power series expansion for the exponential one needs information about the domain of self-adjointness. For example, the Hamiltonian \( H = -\frac{d^2}{dx^2} \) on the interval \([0, \pi]\) with domain \( \mathcal{D}(H) = C_0^\infty((0, \pi)) \) has a 2 complex parameter family of self-adjoint extensions (see Reed and Simon [18]). If one chooses the self-adjoint extension where wave functions vanish at 0 and \( \pi \), \( \psi(0) = 0 \) and \( \psi(\pi) = 0 \), then the evolution operator, \( e^{itH} \), is given by

\[
U(t)\psi = \sum_{n=1}^{\infty} e^{in^2t} \left( \psi, \sin(nx) \right) \sin(nx).
\]  (3.34)

If instead one picks the self-adjoint extension where wave functions obey the boundary conditions \( \frac{d}{dx}\psi(0) = 0 \) and \( \psi(\pi) = 0 \), then the evolution operator is equal to

\[
U(t)\psi = \sum_{l=0}^{\infty} e^{i(l+1/2)^2t} \left( \psi, \cos((l + 1/2)x) \right) \cos((l + 1/2)x).
\]  (3.35)

Thus, making sense of the power series expression of the exponential requires more work than one could naively believe.

### 3.4 Essential Self-Adjointness

One major drawback with self-adjoint operators is that they are extremely difficult to find; given the analytical expression of an operator it is often difficult to find a domain on which the operator is self-adjoint. One way to solve this problem is to use essentially
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self-adjoint operators. These will help us find self-adjoint operators.

Let us define essential self-adjointness. We start by stating a few definitions. We say that an operator \( A \) is closed if its image in \( \mathcal{H} \) is closed. An operator \( A \) is said to be an extension of \( A_0 \) if \( \mathcal{D}(A_0) \subseteq \mathcal{D}(A) \) and \( A\phi = A_0\phi \) for all \( \phi \in \mathcal{D}(A_0) \). We now define the closure of an operator.

**Definition 1** An operator \( A \) is closable if it has a closed extension. Every closable operator has a smallest closed extension which we call the closure and denote by \( \bar{A} \).

We can now define essential self-adjointness.

**Definition 2** A Hermitian operator \( A \) is said to be essentially self-adjoint if its closure \( \bar{A} \) is self-adjoint.

By definition each essentially self-adjoint operator corresponds to a unique self-adjoint operator obtained by taking its closure. It should be noted that given a Hermitian operator \( A \) there can exit many self-adjoint extensions. In fact a Hermitian operator can have an uncountable number of self-adjoint extensions (see Reed and Simon [18]).

There is no guarantee *a priori* that these operators are any easier to find then self-adjoint operators. The following result from functional analysis provides us with the necessary tools to detect essential self-adjointness.

**Theorem 1** Let \( A \) be a Hermitian operator on some Hilbert space. Then \( A \) is essentially self-adjoint iff \( \text{Ker}(A^\dagger \pm i) = \{0\} \).

Thus, if we are given a Hermitian operator the above criteria provides us with a computational method to see if the given operator is essentially self-adjoint or not. This will
be a key tool for constructing quantum theories.

Consider the Hamiltonian $H = -\frac{d^2}{dx^2}$ on the real line with domain $\mathcal{D}(H) = C_0^\infty(\mathbb{R})$. This operator is essentially self-adjoint on $\mathcal{D}(H)$.

**Theorem 2** The operator $H = -\frac{d^2}{dx^2}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$.

**Proof:** First, the domain of $H$ is dense in $L^2(\mathbb{R})$ since $L^2(\mathbb{R})$ is the Cauchy completion of $C_0^\infty(\mathbb{R})$ with respect to the usual inner product

$$ (\phi, \psi) = \int_{\mathbb{R}} dx \phi^*(x) \psi(x). \tag{3.36} $$

Second, $H$ is Hermitian on $\mathcal{D}(H)$. This follows from

$$ \int_{\mathbb{R}} dx \phi^*(x) \left( -\frac{d^2 \psi(x)}{dx^2} \right) = \int_{\mathbb{R}} dx \left( -\frac{d^2 \phi(x)}{dx^2} \right)^* \psi(x) \tag{3.37} $$

for all $\phi, \psi \in \mathcal{D}(H)$. Note that all boundary terms from integration by parts vanish because all functions in $\mathcal{D}(H)$ along with their derivatives vanish outside a compact set.

Now, suppose that $H$ is not essentially self-adjoint. Then from theorem (1) there exists a function or functions in $\mathcal{D}(H^*) \subset L^2(\mathbb{R})$ which satisfy the equation $H^* \phi \pm i \phi = 0$. This equation has no solution in $L^2(\mathbb{R})$ because all solutions of this differential equation are of the form $\phi(x) = Ae^{ik_x x} + Be^{-ik_x x}$, where $k_x^2 = \mp i$ and $A, B$ are constants. Therefore, $H$ is essentially self-adjoint.

One should still be careful when manipulating essentially self-adjoint operators since there is no Spectral theorem for these operators and thus no guarantee that these operators always have a well defined spectrum or exponentiate to unitary operators. Consider the following example.
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Example 1 (Nelson) Let $\mathcal{M}$ be the Riemann surface for $\sqrt{z}$ and $\mathcal{H} = L^2(\mathcal{M})$ with the usual Lebesgue measure. Let $A = -i \frac{\partial}{\partial x}$ and $B = -i \frac{\partial}{\partial y}$ with domain $\mathcal{D}$ which consist of all smooth functions with compact support not containing the origin. Then

(a) $A$ and $B$ are essentially self-adjoint on $\mathcal{D}$ \hspace{1cm} (3.38)

(b) $A : \mathcal{D} \to \mathcal{D}$, $B : \mathcal{D} \to \mathcal{D}$ \hspace{1cm} (3.39)

(c) $[A, B] \phi = 0$ for all $\phi \in \mathcal{D}$ \hspace{1cm} (3.40)

(d) $e^{itA}$ and $e^{isB}$ do not commute. \hspace{1cm} (3.41)

The proofs of (b) and (c) are obvious. To prove (a), first observe that integration by parts shows that $A$ and $B$ are Hermitian. Let $\mathcal{D}_x \subset \mathcal{D}$ be the functions in $\mathcal{D}$ whose support does not contain the $x$ axis on either sheet. $\mathcal{D}_x$ is also dense in $L^2(\mathcal{M})$. On $\mathcal{D}_x$ define $(U(t)\phi)(x, y) = \phi(x + t, y)$, the formal exponential of $A$. Then $U(t)$ is a norm-preserving map with dense range and so extends to a unitary operator on $L^2(\mathcal{M})$ (see Reed and Simon [18]). Since $U(t)$ is strongly continuous on $\mathcal{D}_x$, $U(t)$ is strongly continuous on $L^2(\mathcal{M})$. Now, on $\mathcal{D}_x$, $U(t)$ is strongly differentiable and $i^{-1}$ times its strong derivative is $A$. Thus, $A$ is essentially self-adjoint on $\mathcal{D}_x$ (see Reed and Simon [18]) and therefore on $\mathcal{D}$ and its closure generates $U(t)$. A similar proof shows that the closure of $B$ is the infinitesimal generator of $V(s)$ defined by $(V(s)\phi)(x, y) = (x, y + s)$. This proves (a).

Note that $V(s)$ is the formal exponential of $B$.

To prove (d), let $\phi$ be an infinitely differentiable function with support contained in a small circle about the point $(-\frac{1}{2}, \frac{1}{2})$ on the first sheet. Then

$$U(1)V(1)\phi \neq V(1)U(1)\phi$$ \hspace{1cm} (3.42)

since the functions will have their support around $(\frac{1}{2}, \frac{1}{2})$ on different sheets! Hence, essentially self-adjoint operators don’t always exponentiate and thus are not suited to represent quantum observables.
3.5 Quantization

In the previous sections we have developed the necessary tools required to rigorously construct quantum systems. Let us now give a precise meaning to the word quantization.

We showed in the previous sections that Hermitian operators cannot represent physical observables since they don't always possess a non-trivial spectrum or exponentiate to a well defined unitary operator. Moreover, it follows from Nelson's example that essentially self-adjoint operators are also not suited to represent physical observables since they don't always exponentiate. It is the spectral theorem for self-adjoint operators which guarantees that self-adjoint operators poses a non-trivial spectrum and exponentiate to a well defined unitary operator. Hence, self-adjoint operators is the natural choice of operators to represent physical observables for quantum systems.

It is often stated that a quantization is a mapping from classical observables to operators acting on some Hilbert space $\mathcal{H}$ such that these operators form a Hilbert space representation of the classical algebra. One problem with this definition is that such a mapping does not exist. One cannot map all classical observables to operators such that the classical algebra is satisfied (see Abraham and Marsden [19]). This observation was first made by Grönewald (1946) and then rigorously proved by van Hove (1951). Hence, we must pick a proper subset of the classical observables which we map to operators. A natural choice is the canonical phase space variables and the Hamiltonian. By canonical phase space variables we mean any linear combination of $p_i$ and $q_j$. It should be noted that the canonical variables don't have to be mapped to self-adjoint operators. This follows from the fact that if one chooses the raising and lowering operators, $a$ and $a^\dagger$, as canonical phase space variables one cannot implement these as self-adjoint operators.
since they are not even Hermitian.

We define a quantization as a mapping of the canonical phase space variables and the Hamiltonian to operators on some Hilbert space $\mathcal{H}$ such that the operators corresponding to the canonical variables form a Hilbert space representation of the classical algebra and the Hamiltonian is implemented as a self-adjoint operator on $\mathcal{H}$. Self-adjointness of the Hamiltonian guarantees that the system has a well defined energy spectrum and evolves unitarily. This is our working definition of quantization.
In this chapter we will construct the rigorous quantum theory of two systems: first a one-dimensional harmonic oscillator and second a free particle on a 3-torus. The latter will provide an example of a rigorous quantum system with non-trivial topological effects. The rigorous quantization of these systems will be of use to us in later chapters, when we will construct quantum field theories. It should be noted that the quantum mechanical analog of a free quantum field theory is provided by the harmonic oscillator, studied in elementary quantum mechanics. Some of the features encountered in this simple system are very similar to those found in free field theories.

4.1 Harmonic oscillator

The standard approach to quantize the harmonic oscillator is as follows. One starts with the classical Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2)$$  \hspace{1cm} (4.43)

and Poisson brackets (obtained from the usual symplectic structure $w = dq \wedge dp$)

$$\{q, q\}_{P.B.} = 0 \quad \{p, p\}_{P.B.} = 0$$  \hspace{1cm} (4.44)

and

$$\{q, p\}_{P.B.} = 1.$$  \hspace{1cm} (4.45)

Quantization, is formally accomplished by mapping the classical variables $q$ and $p$ to
quantum operators $\hat{q}$ and $\hat{p}$, which satisfy the canonical commutation relations

$$[\hat{q}, \hat{q}] = 0, \quad [\hat{p}, \hat{p}] = 0$$  \hspace{1cm} (4.46)

and

$$[\hat{q}, \hat{p}] = i \quad (\hbar = 1).$$  \hspace{1cm} (4.47)

One way of solving this system is to map $\hat{q}$ and $\hat{p}$ to the raising and lowering operators, $a$ and $a^\dagger$, defined by

$$a = \frac{1}{\sqrt{2\omega}}(\omega \hat{q} + i \hat{p}) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2\omega}}(\omega \hat{q} - i \hat{p}),$$  \hspace{1cm} (4.48)

or equivalently by

$$\hat{q} = \frac{1}{\sqrt{2\omega}}(a^\dagger + a) \quad \text{and} \quad \hat{p} = i \sqrt{\frac{\omega}{2}}(a^\dagger - a).$$  \hspace{1cm} (4.49)

In these new variables the Hamiltonian becomes

$$\hat{H} = \omega(a^\dagger a + \frac{1}{2})$$  \hspace{1cm} (4.50)

and the commutation relations

$$[a, a] = 0 \quad [a^\dagger, a^\dagger] = 0,$$  \hspace{1cm} (4.51)

$$[a, a^\dagger] = 1.$$  \hspace{1cm} (4.52)

It is now a basic exercise in quantum mechanics to find the eigenvectors and eigenvalues of $\hat{H}$. To refresh our memory let us go through the details of the construction following the exposition of Messiah [20]. It is important to note that the following construction is only formal since we assume that we have well defined operators on some Hilbert space but we have not specified this Hilbert space or picked domains for these operators.

We first note that if $\phi_\alpha$ is a normalized eigenvector of the Hermitian operator $a^\dagger a$,

$$a^\dagger a \phi_\alpha = \alpha \phi_\alpha,$$  \hspace{1cm} (4.53)
then

$$\alpha = (\phi_\alpha, a^\dagger a \phi_\alpha) = (a \phi_\alpha, a \phi_\alpha) = ||a \phi_\alpha||^2 \geq 0. \tag{4.54}$$

Hence, the eigenvalues are all real and nonnegative. Using the identity \([AB, C] = A[B, C] + [A, C]B\), we find that

$$[a^\dagger a, a] = [a^\dagger, a]a = -a, \tag{4.55}$$

$$[a^\dagger a, a^\dagger] = a^\dagger[a, a^\dagger] = a^\dagger; \tag{4.56}$$

or, equivalently,

$$(a^\dagger a)a = a(a^\dagger a - 1), \tag{4.57}$$

$$(a^\dagger a)a^\dagger = a^\dagger(a^\dagger a + 1). \tag{4.58}$$

From these equations, it follows that for an eigenvector \(\phi_\alpha\)

$$(a^\dagger a)\phi_\alpha = a(a^\dagger a - 1)\phi_\alpha = a(\alpha - 1)\phi_\alpha = (\alpha - 1)a \phi_\alpha. \tag{4.59}$$

Thus, \(a \phi_\alpha\) is an eigenvector with eigenvalue \(\alpha - 1\), unless \(a \phi_\alpha = 0\). Similarly \(a^\dagger \phi_\alpha\) is an eigenvector with eigenvalues \(\alpha + 1\), unless \(a^\dagger \phi_\alpha = 0\). The norm of \(a \phi_\alpha\) is obtained from

$$||a \phi_\alpha||^2 = (a \phi_\alpha, a \phi_\alpha) = (\phi_\alpha, a^\dagger a \phi_\alpha) = \alpha(\phi_\alpha, \phi_\alpha) = \alpha, \tag{4.60}$$

or

$$||a \phi_\alpha|| = \sqrt{\alpha}. \tag{4.61}$$

Similarly,

$$||a^\dagger \phi_\alpha|| = \sqrt{\alpha + 1}. \tag{4.62}$$

Now, suppose that \(a^n \phi_\alpha \neq 0\) for all \(n\). Then by repeated application of Eq.(4.59), \(a^n \phi_\alpha\) is an eigenvector of \(a^\dagger a\) with eigenvalue \(\alpha - n\). This contradicts Eq.(4.54), since \(\alpha - n < 0\) for a sufficiently large \(n\). Thus we must have

$$a^n \phi_\alpha \neq 0 \text{ but } a^{n+1} \phi_\alpha = 0 \tag{4.63}$$
for some nonnegative integer \( n \).

Let \( \phi_{\alpha-n} = a^n \phi_\alpha/||a^n \phi_\alpha|| \), so that \( \phi_{\alpha-n} \) is a normalized eigenvector with eigenvalues \( \alpha - n \). Then from Eqs.(4.61) and (4.63),

\[ \sqrt{\alpha - n} = ||a\phi_{\alpha-n}|| = 0, \tag{4.64} \]

and therefore \( \alpha = n \). Hence, the eigenvalues of \( a^\dagger a \) are nonnegative integers, \( n = 0, 1, ... \), and there is a ground state \( \phi_0 \) such that

\[ a\phi_0 = 0. \tag{4.65} \]

By repeatedly applying \( a^\dagger \) to the ground state we find that

\[ \phi_n = \frac{1}{\sqrt{n!}}(a^\dagger)^n\phi_0 \tag{4.66} \]

are eigenvectors of \( H = \omega(a^\dagger a + 1/2) \) with eigenvalues \( \omega(n + 1/2) \) where \( n = 0, 1, ... \). Defined in this way, \( \phi_n \) are orthonormal (see Messiah [20]) and satisfy

\[ a^\dagger \phi_n = \sqrt{n+1}\phi_{n+1} \tag{4.67} \]

\[ a\phi_n = \sqrt{n}\phi_{n-1} \tag{4.68} \]

\[ a^\dagger a\phi_n = n\phi_n. \tag{4.69} \]

Equations Eqs.(4.66)-(4.69), with the Hamiltonian \( H = \omega(a^\dagger a + 1/2) \), form a solution to the harmonic oscillator problem.

The operators \( a^\dagger \) and \( a \) are often referred to as the raising and lowering operators respectively, since they raise and lower the eigenvalues of \( a^\dagger a \). In later applications operators similar to \( a^\dagger a \) will be interpreted as the operator which counts the number of particles of a certain type. In this case \( a^\dagger \) and \( a \) are called the creation and annihilation
operators.

One problem with this construction is that it is only formal. For example, Eq.(4.60) uses the definition of adjoint but no inner product has been defined. We have not specified the Hilbert space on which these operators act on or picked a domain on which the Hamiltonian is self-adjoint. Recalling Nelson's example we quickly realize that we have no guarantee that the Hamiltonian exponentiates to a well defined unitary operator. Hence, we still do not know if the system evolves unitarily.

Let us now give a rigorous construction of the quantum theory of the harmonic oscillator. The first step is to find a Hilbert space representation of the operator algebra Eqs.(4.51) and (4.52). That is, to find a Hilbert space on which we can implement the creation and annihilation operators.

A natural choice of Hilbert space is $\mathcal{H} = l^2(\mathbb{C})$, the space of square summable sequences of complex numbers, since $H$ formally has discrete spectrum. $l^2(\mathbb{C})$ is defined by

$$
\psi_a = (a_0, a_1, ..., a_n, ...) \quad \text{and} \quad \psi_b = (b_0, b_1, ..., b_n, ...) \quad a_i, b_i \in \mathbb{C} \quad i = 0, 1, ...
$$

such that,

$$
(\psi_a, \psi_b) = \sum_{i=0}^{\infty} a_i^* b_i < \infty.
$$

This Hilbert space has an obvious orthonormal basis which we denote by

$$
\phi_n = (0, 0, ..., 0, 1, 0, ..)
$$

where the non-zero entry occurs in the $n + 1$ slot.
We can easily implement the raising and lowering operators on this Hilbert space. The action of the creation operator $a^\dagger$ on the basis vectors is defined by Eq. (4.67). Using this equation we find
\[
(\phi_n, a^\dagger \phi_n) = \sqrt{n + 1} \delta_{m,n+1}.
\] (4.73)

The domain of $a^\dagger$ is chosen to be the set of all finite linear combinations of basis vectors, that is,
\[
D(a^\dagger) = \{\psi \in \mathcal{H} | \psi = \sum_n c_n \phi_n \text{ where only finitely many } c_n \neq 0\}.
\] (4.74)

It is easy to see that this domain is dense in $l^2(\mathbb{C})$. Given any $\psi \in l^2(\mathbb{C})$ and $\epsilon$ there is always a sequence $\psi_\epsilon$ with only finitely many non-zero entries such that $||\psi - \psi_\epsilon|| < \epsilon$. This is true since we can choose $\psi_\epsilon$ to have an arbitrary large number of non-zero entries (see Lang [22]). Thus, $D(a^\dagger)$ is dense in $l^2(\mathbb{C})$. Since $D(a^\dagger)$ is dense $a^\dagger$ has a well defined adjoint $(a^\dagger)^\dagger$, which we denote by $a \equiv (a^\dagger)^\dagger$. Using the definition of adjoint and interchanging the indices $n$ and $m$ in Eq. (4.73) we find,
\[
(\phi_m, a \phi_n) = \sqrt{n} \delta_{m,n-1}.
\] (4.75)

It now follows that Eq. (4.68) is satisfied. Moreover, since $(\psi, a^\dagger \phi) = (a \psi, \phi)$ is well defined for all $\psi, \phi \in D(a^\dagger)$, we see that $D(a^\dagger) \subseteq D(a)$. It is easy to check that the commutation relations of the raising and lowering operators are satisfied on $D(a^\dagger)$. It should be emphasised that we need the commutation relation to be satisfied on $D(a^\dagger)$ since they were used to obtain the Hamiltonian Eq. (4.50). Thus, we have a Hilbert space representation of the operator algebra Eqs. (4.51) and (4.52).

Now that we have accomplished the first step in our rigorous construction, we can proceed to the second part: implementation of the Hamiltonian as a self-adjoint operator.
To accomplish this we do the following: we find a domain on which $H$ is essentially self-adjoint and then take its closure $\tilde{H}$ as our Hamiltonian. A natural candidate is $\mathcal{D}(a^\dagger)$ since $H = \omega(a^\dagger a + \frac{1}{2})$ has a well defined action on these states. This follows from the observation that $a$ is well defined on $\mathcal{D}(a^\dagger)$ and maps these states to other finite particle states. Since $a^\dagger$ is defined on $\mathcal{D}(a^\dagger)$ it follows that $H$ is well defined on $\mathcal{D}(a^\dagger)$. In fact the Hamiltonian is essentially self-adjoint on $\mathcal{D}(a^\dagger)$.

**Theorem 1** The Hamiltonian $H = \omega(a^\dagger a + \frac{1}{2})$ is essentially self-adjoint on $\mathcal{D}(a^\dagger) \subset l^2(\mathbb{C})$.

*Proof:* First recall that the orthonormal basis vectors $\phi_n$ are eigenstates of the $H = \omega(a^\dagger a + \frac{1}{2})$ with eigenvalues $\omega(n + \frac{1}{2})$, where $n \in \mathbb{N}$. Suppose that $H$ is not essentially self-adjoint. Then by theorem (1) from chapter 3 section 4 there exist a $\psi \in \mathcal{D}(H^\dagger)$ such that $H^\dagger \psi = i\psi$ or $H^\dagger \psi = -i\psi$. This implies

\begin{equation}
0 = (\langle H^\dagger \pm i \rangle \psi, \phi_n) \quad (4.76)
\end{equation}

\begin{equation}
= (\psi, (H \mp i) \phi_n), \quad (4.77)
\end{equation}

where we have used the fact that $\phi_n \in \mathcal{D}(a^\dagger) = \mathcal{D}(H) \subset \mathcal{D}(H^\dagger)$. From Eq.(4.69) we find

\begin{equation}
0 = (\psi, (\omega(n + \frac{1}{2}) \mp i) \phi_n), \quad (4.78)
\end{equation}

which implies $(\psi, \phi_n) = 0$ for all $\phi_n$. Since $\phi_n$ is a complete basis for $\mathcal{H} = l^2(\mathbb{C})$, we must have $\psi = 0$. If $\psi \neq 0$, it would define a new basis vector orthogonal to all $\phi_n$ and thus orthogonal to $\mathcal{D}(a^\dagger)$, $\psi \perp \mathcal{D}(a^\dagger)$. This contradicts the fact that $\mathcal{D}(a^\dagger)$ is a dense subset of $l^2(\mathbb{C})$. Hence, there exists no non-trivial solution to $H^\dagger \psi = \pm i\psi$. Thus, the Hamiltonian is essentially self-adjoint.
Now that we have an essentially self-adjoint Hamiltonian $H$, we take its closure $\bar{H}$ and obtain a self-adjoint operator, which is our choice for Hamiltonian. We can now claim that we have a well defined quantum theory. It is worth emphasizing that self-adjointness is crucial to the construction of a rigorous quantum system. It is only with a self-adjoint Hamiltonian that unitary evolution and existence of a unique non-trivial energy spectrum is guaranteed.

Having a solution to the system, we can easily give the explicit expression for the evolution operator, the exponential of the Hamiltonian $\bar{H}$, as defined in chapter 3. First note that $\bar{H}$ has the same eigenvectors and eigenvalues as $H$. This follows by first noting that the eigenvectors of $H$ are also eigenvectors of $\bar{H}$ since $\mathcal{D}(H) \subseteq \mathcal{D}(\bar{H})$. Now suppose there exists an eigenvector, $\psi$, of $\bar{H}$ which is not an eigenvector of $H$. Then $\psi$ would be orthogonal to all the eigenvectors of $H$, $(\psi, \phi_n) = 0$ for all $\phi_n$. But since $\phi_n$ is a complete basis this implies that $\psi = 0$. Hence, $H$ and $\bar{H}$ have the same eigenvectors and eigenvalues. By the spectral theorem (see Reed and Simon [18]), the exponential of the Hamiltonian, defined by,

$$U(t)\psi = \sum_{n=0}^{\infty} e^{i\omega(n+1/2)t} (\psi, \phi_n) \phi_n$$

(4.79)

is a well defined (strongly continuous) unitary operator. An alternate way to obtain the evolution operator is to use the power series expression for the exponential,

$$e^{i\bar{H}t} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\bar{H}t)^n.$$ 

(4.80)

First note that $e^{i\bar{H}t}$ is a well defined operator on $\mathcal{D}(a^\dagger)$. This follows by noting that $e^{i\bar{H}t}$ is well defined on individual basis states since

$$e^{i\bar{H}t} \phi_n = \sum_{n=0}^{\infty} \frac{1}{n!} (i\bar{H}t)^n \phi_n = \sum_{n=0}^{\infty} \frac{1}{n!} (i\omega(n + 1/2)t)^n \phi_n = e^{i\omega(n+1/2)t} \phi_n,$$ 

(4.81)
and thus $e^{i\hat{H}t}$ is well defined on all finite linear combinations of the basis states. Note that $e^{i\hat{H}t}$ is unitary on $\mathcal{D}(a^\dagger)$. Since $e^{i\hat{H}t}$ is a well defined unitary operator on a dense domain in $l^2(\mathbb{C})$ it uniquely extends to a unitary operator on all of $l^2(\mathbb{C})$ (see Reed and Simon [18]). It should be noted that since $U(t)$ and $e^{i\hat{H}t}$ coincide on a dense subset, $\mathcal{D}(a^\dagger)$, it follows that they define the same unitary operator on $l^2(\mathbb{C})$.

4.1.1 Position representation

The reader might wonder how is this abstract representation related to the usual coordinate representation. Let us now show how we can construct an isomorphism between this abstract representation and the usual coordinate representation. In the latter wave functions are square summable functions of the position, $q$. The creation and annihilation operators, obtained from Eq.(4.50), are formally given by,

$$ a^\dagger = \frac{1}{\sqrt{2}}(q - \frac{d}{dq}) $$

and

$$ a = \frac{1}{\sqrt{2}}(q + \frac{d}{dq}), $$

where we have set $\omega = 1$ for simplicity. From Eq.(4.65) we have,

$$ 0 = a \phi_0(q) $$

$$ = \frac{1}{\sqrt{2}}(q + \frac{d}{dq})\phi_0(q), $$

which we can formally solve. The general solution of this differential equation is

$$ \phi_0(q) = Ae^{-1/2q^2}, $$

where $A$ is a normalization constant. It is fixed by the condition,}

$$ 1 = (\phi_0(q), \phi_0(q)) = |A|^2 \sqrt{\pi}, $$
which implies,

$$A = (1/\pi)^{1/4},$$  \hspace{1cm} (4.88)

where we have set the arbitrary phase factor to unity. Thus, formally the ground state in the position representation is

$$\phi_0(q) = (1/\pi)^{1/4} e^{-1/2q^2}.\hspace{1cm} (4.89)$$

Using Eq.(4.66) we find that

$$\phi_n(q) = \frac{1}{\sqrt{n!}} (a^\dagger)^n \phi_0(q) \hspace{1cm} (4.90)$$

$$= \frac{1}{\sqrt{n!}} (1/2)^{2/n} (q - \frac{d}{dq})^n \phi_0(q) \hspace{1cm} (4.91)$$

$$= \frac{1}{\sqrt{n!}} (1/\pi)^{1/4} (1/2)^{2/n} (q - \frac{d}{dq})^n e^{-1/2q^2}.\hspace{1cm} (4.92)$$

It should be noted that these are the Hermite polynomials (up to a weight factor $xe^{1/2x^2}$), which are well known to be a complete orthonormal basis vectors for $L^2(\mathbb{R})$ (see Reed and Simon [18]).

We can now construct an isomorphism between representations. We define a mapping

$$l^2(\mathbb{C}) \rightarrow L^2(\mathbb{R}),$$  \hspace{1cm} (4.93)

from one Hilbert space to another, by

$$\phi_n \rightarrow \phi_n(q).$$  \hspace{1cm} (4.94)

We can now use the solution in the $l^2(\mathbb{C})$ representation to rigorously solve the harmonic oscillator problem in the position representation.
Using Eq.(4.94), we map \( \mathcal{D}(a^\dagger) \in l^2(\mathbb{C}) \) to \( \mathcal{D}_\varphi(a^\dagger) \in L^2(\mathbb{R}) \), where
\[
\mathcal{D}_\varphi(a^\dagger) = \{ \psi \in L^2(\mathbb{R}) \mid \psi = \sum_n c_n \phi_n(q) \text{ where finitely many } c_n \neq 0 \}.
\] (4.95)
This maps the operator \( a^\dagger \) from one Hilbert to another. It is now easy to see that the mapping from the abstract representation to the usual representation can be used to carry all the operators from one representation to another. In particular the self-adjoint Hamiltonian \( \tilde{H} \) is mapped to a self-adjoint operator on \( L^2(\mathbb{R}) \). Thus, we can map the solution of the system from the abstract representation to the coordinate representation.

The point to remember is that once we have rigorously solved the system in one representation we can construct isomorphisms and solve the system in other representations. The solution from the old representation can be carried to the new one.

### 4.2 Free particle on a 3-torus

In this section we consider the problem of a free particle on a 3-torus, or equivalently, as it will be shown, a particle in a lattice with zero potential, \( V=0 \). We will first review Bloch's theorem and then rigorously quantize a free particle on a 3-torus. Recall that a particle on a perfect lattice can be viewed as a particle on a 3-torus with twisted boundary conditions on the wave functions. This will provide an example of rigorous quantum system with non-trivial topological effects.

#### 4.2.1 Bloch's theorem

We briefly review Bloch's theorems. For a complete exposition see Ashcroft and Mermin [23].
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Theorem 1 (Bloch's) The eigenstates $\psi$ of the one-electron Hamiltonian $H = -\nabla^2/2m + U(\vec{r})$, where $U(\vec{r}) = U(\vec{r} + \vec{R})$ for all $\vec{R}$ in a lattice, can be chosen to have the form of a plane wave times a function with periodicity of the lattice:

$$\psi_k(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} u_k(\vec{r}), \quad (4.96)$$

where

$$u_k(\vec{r} + \vec{R}) = u_k(\vec{r}), \quad (4.97)$$

for all $\vec{R}$ in the lattice. Note that Eqs. (4.96) and (4.97) imply that

$$\psi_k(\vec{r} + \vec{R}) = e^{i\vec{k} \cdot \vec{R}} \psi_k(\vec{r}), \quad (4.98)$$

for every $\vec{R}$ in the lattice. In fact Eq. (4.98) is an alternate formulation of Bloch's theorem.

Let us now prove Bloch's theorem. For each lattice vector $\vec{R}$ we define a translation operator $T_{\vec{R}}$ which, when operating on any function $f(\vec{r})$, shifts the argument by $\vec{R}$:

$$T_{\vec{R}} f(\vec{r}) = f(\vec{r} + \vec{R}). \quad (4.99)$$

Since the Hamiltonian is periodic, we have

$$T_{\vec{R}} H \psi = H(\vec{r} + \vec{R}) \psi(\vec{r} + \vec{R}) = H(\vec{r}) \psi(\vec{r} + \vec{R}) = HT_{\vec{R}} \psi. \quad (4.100)$$

Because (4.100) holds identically for any function $\psi$, we have the operator identity

$$T_{\vec{R}} H = HT_{\vec{R}}. \quad (4.101)$$

In addition, the result of applying two successive translations does not depend on the order in which they are applied, since for any $\psi(\vec{r})$

$$T_{\vec{R}} T_{\vec{R}'} \psi(\vec{r}) = \psi(\vec{r} + \vec{R} + \vec{R}') = T_{\vec{R}'} T_{\vec{R}} \psi(\vec{r}). \quad (4.102)$$
Therefore

\[ T_{\mathbf{R}}T_{\mathbf{R}'} = T_{\mathbf{R}}T_{\mathbf{R}} = T_{\mathbf{R}+\mathbf{R}'.} \] (4.103)

Equations (4.101) and (4.103) assert that \( T_{\mathbf{R}} \) for all lattice vectors and the Hamiltonian form a set of commuting operators. It follows from the fundamental theorem of quantum mechanics that the eigenstates of \( H \) can therefore be chosen to be simultaneously eigenstates of all the \( T_{\mathbf{R}} \):

\[ H\psi = E\psi \] (4.104)

\[ T_{\mathbf{R}}\psi = c(\mathbf{R})\psi. \] (4.105)

The eigenvalues \( c(\mathbf{R}) \) of the translation operators are related because on the one hand

\[ T_{\mathbf{R}}T_{\mathbf{R}}\psi = c(\mathbf{R})T_{\mathbf{R}}\psi = c(\mathbf{R})c(\mathbf{R}')\psi, \] (4.106)

while, according to (4.103),

\[ T_{\mathbf{R}}T_{\mathbf{R}}T_{\mathbf{R}'} = T_{\mathbf{R}+\mathbf{R}'}\psi = c(\mathbf{R}+\mathbf{R}')\psi. \] (4.107)

Now let \( \mathbf{a}_i \) be three primitive vectors for the lattice (see Ashcroft and Mermin [23]). We can always write the \( c(\mathbf{a}_i) \) in the form

\[ c(\mathbf{a}_i) = e^{i2\pi x_i}, \] (4.108)

by a suitable choice of \( x_i \). It then follows by successive applications of (4.106) that if \( \mathbf{R} \) is a general lattice vector given by

\[ \mathbf{R} = n_1\mathbf{a}_1 + n_2\mathbf{a}_2 + n_3\mathbf{a}_3, \] (4.109)

then

\[ c(\mathbf{R}) = c(\mathbf{a}_1)^{n_1}c(\mathbf{a}_2)^{n_2}c(\mathbf{a}_3)^{n_3}. \] (4.110)
But this is precisely equivalent to

\[ c(\vec{R}) = e^{i\vec{k} \cdot \vec{R}}, \]

(4.111)

where

\[ \vec{k} = x_1 \vec{b}_1 + x_2 \vec{b}_2 + x_3 \vec{b}_3 \]

(4.112)

and the \( \vec{b}_i \) are defined by \( \vec{b}_i \cdot \vec{a}_j = 2\pi \delta_{ij} \).

Summarizing, we have shown that we can choose the eigenstates \( \psi \) of \( H \) so that for every lattice vector \( \vec{R} \),

\[ T_{\vec{R}} \psi = \psi(\vec{r} + \vec{R}) = c(\vec{R}) \psi = e^{i\vec{k} \cdot \vec{R}} \psi(\vec{r}). \]

(4.113)

This is precisely Bloch's theorem, Eq.(4.98).

Notice that since the lattice is perfectly periodic, the configuration space (the space of all physically inequivalent configurations) is obtained by identifying any two points of \( \mathbb{R}^3 \) which differ by a lattice translation, that is, by an element in \( \mathbb{Z}^3 \) the lattice group. Hence, the configuration space is equal to

\[ Q = \mathbb{R}^3 / \mathbb{Z}^3, \]

(4.114)

which is topologically a 3-torus.

### 4.2.2 Topological method and Bloch's theorem

Let us now rigorously quantize a free particle on a 3-torus. The system is classically described by the free Hamiltonian,

\[ H = \frac{1}{2m} \vec{p}^2, \]

(4.115)
and the symplectic structure,
\[ \omega = \sum_{i=1}^{3} dx_i \wedge dp_i . \] (4.116)

From Eq.(4.116) we deduce the usual Poisson brackets,
\[ \{ x_i, x_j \}_B.P. = 0, \quad \{ p_i, p_j \}_B.P. = 0 \] (4.117)
and
\[ \{ x_i, p_j \}_B.P. = \delta_{i,j} . \] (4.118)

Let us recall the procedure for quantizing a system with multiply connected configuration space, \( \Pi_1(Q) \neq 0 \). The first step is to lift up classical observables and wave functions to the universal covering, \( \hat{Q} \). Observables are functions on \( \hat{Q} \) which are invariant under the action of the fundamental group. Wave functions are functions on \( \hat{Q} \) which transform under some U.I.R., \( \Gamma \), of \( \Pi_1(Q) \). One then quantizes the system as usual.

The fundamental group of \( Q = \mathbb{R}^3 / \mathbb{Z}^3 \) is
\[ \Pi_1(Q) = \Pi_1(\mathbb{R}^3 / \mathbb{Z}^3) = \mathbb{Z}^3, \] (4.119)
where \( \mathbb{Z}^3 \) is the group of lattice translations. The universal covering of \( Q = \mathbb{R}^3 / \mathbb{Z}^3 \) is
\[ \hat{Q} = \mathbb{R}^3. \] (4.120)

Observables, like the Hamiltonian, are functions on \( \mathbb{R}^3 \) which are invariant under \( \mathbb{Z}^3 \),
\[ H(\vec{r} + \vec{R}) = H(\vec{r}), \] (4.121)
where \( \vec{R} \in \mathbb{Z}^3 \). Wave functions, \( \psi \), are functions on \( \hat{Q} = \mathbb{R}^3 \), \( \psi \in L^2(\mathbb{R}^3) \), which transform under a U.I.R., \( \Gamma \), of \( \Pi_1(Q) = \mathbb{Z}^3 \). Let us refresh the memory of the reader about U.I.R. of \( \mathbb{Z}^3 \). First, note that the mapping
\[ \Gamma_{\vec{\delta}} : \mathbb{Z}^3 \rightarrow U(1) \] (4.122)
\[ : \vec{R} \rightarrow e^{i\vec{\delta} \cdot \vec{R}} \] (4.123)
is a unitary representation of $\mathbb{Z}^3$. It is irreducible since the representation is one dimensional. Since $\vec{\theta}$ and $\vec{\theta} + 2\pi(n_x, n_y, n_z)$, where $n_x, n_y$ and $n_z \in \mathbb{Z}^3$, define the same mapping and thus the same representation, we restrict $\vec{\theta}$ to live in $\vec{\theta} \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi)$. For each $\vec{\theta} \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi)$ the $\Gamma_{\vec{\theta}}$'s define unitarily inequivalent representations. This follows by noting that if two representations $\Gamma_{\vec{\theta}}$ and $\Gamma_{\vec{\theta}'}$ are unitarily equivalent then there exists $e^{i\alpha} \in U(1)$ such that

$$e^{i\vec{\theta} \cdot \vec{R}} = e^{-i\alpha} e^{i\vec{\theta}' \cdot \vec{R}} e^{i\alpha} = e^{i\vec{\theta} \cdot \vec{R}}.$$  (4.124)

This implies that $\vec{\theta}' = \vec{\theta}$ and thus each $\vec{\theta} \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi)$ defines a unitarily inequivalent representation.

Since the action of $\Pi_1(\mathcal{Q}) = \mathbb{Z}^3$, the lattice group, on $\hat{\mathcal{Q}} = \mathbb{R}^3$ is equal to

$$\Pi_1(\mathcal{Q}) : \hat{\mathcal{Q}} \to \hat{\mathcal{Q}}$$

$$: \vec{r} \to \vec{r} + \vec{R}, \quad \vec{R} \in \mathbb{Z}^3$$  (4.126)

it follows that wave functions satisfy the transformation law

$$U_{\vec{R}} \psi_{\vec{\theta}}(\vec{r}) = \psi_{\vec{\theta}}(\vec{r} + \vec{R}) = e^{i\vec{\theta} \cdot \vec{R}} \psi_{\vec{\theta}}(\vec{r}),$$  (4.127)

where $U_{\vec{R}}$ is a unitary operator on $L^2(\mathbb{R}^3)$ which implements translations by $\vec{R}$. Unitarity of $U_{\vec{R}}$ follows from

$$(U_{\vec{R}} \psi, U_{\vec{R}} \psi) = \int_{\mathbb{R}^3} d^3 \vec{r} \psi(\vec{r} + \vec{R})^* \psi(\vec{r} + \vec{R}) = \int_{\mathbb{R}^3} d^3 \vec{r}' \psi(\vec{r}')^* \psi(\vec{r}') = (\psi, \psi),$$  (4.128)

where we changed variables $\vec{r} \to \vec{r}' = \vec{r} + \vec{R}$ and used the fact that the Jacobian for this transformation is unity. Moreover, $U_{\vec{R}}$ is well defined on all of $L^2(\mathbb{R}^3)$ since it is well defined on all square summable functions. Let $L^2_{\vec{\theta}}(\mathbb{R}^3)$ denote the subspace of $L^2(\mathbb{R}^3)$ which consists of all wave functions that have transformation law Eq.(4.127). With the induced inner product from $L^2(\mathbb{R}^3)$, $L^2_{\vec{\theta}}(\mathbb{R}^3)$ is a Hilbert space, which we choose as the
Hilbert space for the system.

The next step to rigorously quantize the system is to find a self-adjoint Hamiltonian. Let \( \mathcal{D}_g = \{ \text{smooth functions in } L^2_g(\mathbb{R}^3) \} \). Since all elements in \( \mathcal{D}_g \) are smooth it follows that the Hamiltonian is well defined on \( \mathcal{D}_g \). Moreover, since the set of all smooth functions in \( L^2(\mathbb{R}^3) \) is dense it follows that \( \mathcal{D}_g \) is dense in \( L^2_g(\mathbb{R}^3) \). We pick \( \mathcal{D}_g \) as domain for the Hamiltonian. Let us now prove that with this choice of domain the Hamiltonian is essentially self-adjoint.

**Theorem 1** The Hamiltonian \( H = \frac{-1}{2m} \vec{\nabla}^2 \) is essentially self-adjoint on \( \mathcal{D}_g \).

**Proof:** First, note that the generalized eigenfunctions of \( H \),
\[
\phi_K = \frac{1}{(2\pi)^{3/2}} e^{i(K+\vec{\theta}) \cdot \vec{r}} \text{ for } \vec{K} \in \mathbb{Z}^3, \tag{4.129}
\]
form a complete basis for \( L^2_g(\mathbb{R}^3) \), that is, for any \( \psi_g \in L^2_g(\mathbb{R}^3) \) we have that
\[
\psi_g = \sum_{\vec{K} \in \mathbb{Z}^3} c_{\vec{K}} \phi_{\vec{K}}. \tag{4.130}
\]
This follows from Fourier analysis. Suppose that \( H \) is not essentially self-adjoint. Then from theorem (1) in chapter 3 section 4 there exists a \( \psi \in \mathcal{D}(H^\dagger) \) such that,
\[
(-\frac{1}{2m} \vec{\nabla}^2 - i)\psi = 0 \text{ or } (-\frac{1}{2m} \vec{\nabla}^2 + i)\psi = 0, \tag{4.131}
\]
where we have used the fact that \( H^\dagger = -\frac{1}{2m} \vec{\nabla}^2 \) on \( \mathcal{D}(H^\dagger) \). Using Eq.(4.129) we find that Eq.(4.131) implies
\[
\sum_{\vec{K} \in \mathbb{Z}^3} \frac{1}{2m}(\vec{K} + \vec{\theta})^2 \pm i)c_{\vec{K}} = 0, \tag{4.132}
\]
where we have used the fact that \( \psi \in \mathcal{D}(H^\dagger) \) has at least two generalized derivatives (see Reed and Simon [18]). Since \( \phi_{\vec{K}} \) is an orthogonal basis it follows that
\[
\frac{1}{2m}(\vec{K} + \vec{\theta})^2 \pm i)c_{\vec{K}} = 0 \text{ for all } \vec{K} \in \mathbb{Z}^3. \tag{4.133}
\]
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It now follows that $c_{\vec{K}} = 0$ for all $\vec{K} \in \mathbb{Z}^3$ since $\frac{1}{2m}(\vec{K} + \vec{\theta})^2 \pm i \neq 0$ for all $\vec{K} \in \mathbb{Z}^3$. Thus $\psi = 0$. Hence, the Hamiltonian $H$ is essentially self-adjoint.

Having an essentially self-adjoint operator $H$, we take its closure $\tilde{H}$ and obtain a self-adjoint Hamiltonian. We now have a well defined quantum theory. It is worth emphasising that self-adjointness is crucial for rigorous quantum systems. It is only with a self-adjoint Hamiltonian that unitary evolution and existence of a unique energy spectrum is guaranteed.

Having an explicit solution of the system we can write down the expression of the evolution operator,

$$U_{\vec{\theta}}(t)\psi = \sum_{\vec{K} \in \mathbb{Z}^3} e^{i\frac{1}{2m}(\vec{K}+\vec{\theta})^2t} (\psi, \phi_{\vec{K}}) \phi_{\vec{K}}, \quad (4.134)$$

where we have used the fact that $H$ and its closure $\tilde{H}$ have the same eigenvectors and eigenvalues. This follows from the same arguments that were used for the harmonic oscillator. Eq.(4.134) shows that the evolution of the system depends on the $\vec{\theta}$ parameter.

Having quantized the system on the universal covering $\mathbb{R}^3$, let us now project the system back down to the 3-torus. The projection is accomplished in two ways:

1) Wave functions, $\psi_{\vec{\theta}}$, are projected to the 3-torus by taking their restriction to the unit cell $E = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$, $\psi_{\vec{\theta}} \rightarrow \psi_{\vec{\theta}}|_E$. One should recall that a 3-torus can be obtained by taking the unit cell and identifying opposite sides, that is, imposing periodic boundary conditions. Now, since $\psi_{\vec{\theta}}$ is not periodic, except for $\vec{\theta} = (0,0,0)$, it follows that wave functions are not single valued on the 3-torus. They are multivalued functions on the 3-torus or equivalently sections of a $U(1)$-bundle over the 3-torus (see Husemoller [24]). Hence, topological effects forces wave functions to be multivalued on $Q = \mathbb{R}^3/\mathbb{Z}^3$. 


2) We demand that wave functions be single valued on the 3-torus and find the corresponding Hamiltonian. First note that functions with transformation law Eq. (4.127) can be written as \( \psi_\theta = e^{i\theta} u(\vec{r}) \), where \( u(\vec{r}) \) is a periodic function. Substituting this equation into

\[
H \psi_\theta = -\frac{1}{2m} \vec{\nabla}^2 \psi_\theta
\]

we find

\[
H_\theta u(\vec{r}) = \frac{1}{2m} (-i \vec{\nabla} + \vec{\theta})^2 u(\vec{r}).
\]

We now project wave functions down to the 3-torus by \( u(\vec{r}) \rightarrow u(\vec{r})|_E \). Hence, the topological effects now appear in the Hamiltonian.

A natural question to ask is: Does \( \vec{\theta} \) have observables physical effects? In the case of a particle in a lattice with zero potential, \( V = 0 \), we have that \( \vec{\theta} = \vec{k} \), where \( \vec{k} \) is the lattice momentum (see Ashcroft and Mermin [23]). It is well known that when interactions are turned on \( \vec{k} \) can be observed via scattering experiments. Since the mathematical structure of a free particle on a 3-torus is the same as a Bloch electron with \( V = 0 \) it follows that \( \vec{\theta} \) can be detected when interaction are included in the system. Hence, these topological effects do have observables effects.
In this chapter we rigorously quantize a free massive and massless Bosonic scalar field on a 3-torus. This construction will be of use to us when constructing the quantum theory of the electromagnetic field on a 3-torus. Moreover, it is interesting to see how these quantum field theories, which most graduate students encounter, can be rigorously constructed.

5.1 Massive scalar field theory

In this section we consider the quantum theory of a free massive scalar field on a 3-torus, that is, in a box with periodic boundary conditions. The selected box is a cube with sides of a length of $2\pi$. We first formally quantize the system and then give a rigorous construction.

5.1.1 Formal quantization

We start with the classical Hamiltonian,

$$H = \int d^3x \, \frac{1}{2}(\Pi^2 + m^2 \phi^2 + (\vec{\nabla}\phi)^2),$$

(5.137)

and Poisson brackets,

$$\{\phi(\bar{x}), \phi(\bar{y})\}_{P.B.} = 0 \quad \{\Pi(\bar{x}), \Pi(\bar{y})\}_{P.B.} = 0,$$

(5.138)

$$\{\phi(\bar{x}), \Pi(\bar{y})\}_{P.B.} = \delta(\bar{x} - \bar{y}).$$

(5.139)
Quantization is formally accomplished by mapping $\phi(\vec{x})$ and $\Pi(\vec{x})$ to operators $\hat{\phi}(\vec{x})$ and $\hat{\Pi}(\vec{x})$, which satisfy the canonical commutations relations,

$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = 0, \quad [\hat{\Pi}(\vec{x}), \hat{\Pi}(\vec{y})] = 0$$

and

$$[\hat{\phi}(\vec{x}), \hat{\Pi}(\vec{y})] = i\delta(\vec{x} - \vec{y}).$$ (5.141)

The first step to formally solve the system is to decompose $\hat{\phi}(\vec{x})$ and $\hat{\Pi}(\vec{x})$ into Fourier modes,

$$\hat{\phi}(x) = \sum_{k \in \mathbb{Z}^3} \hat{\phi}_k \frac{e^{i\vec{k} \cdot \vec{x}}}{\sqrt{V}}$$

and

$$\hat{\Pi}(x) = \sum_{k \in \mathbb{Z}^3} \hat{\Pi}_k \frac{e^{i\vec{k} \cdot \vec{x}}}{\sqrt{V}},$$ (5.143)

where $V = (2\pi)^3$ is the volume of the box. To this decomposition we add the conditions

$$\hat{\phi}_k^* = \hat{\phi}_{-\vec{k}} \quad \text{and} \quad \hat{\Pi}_k^* = \hat{\Pi}_{-\vec{k}}$$ (5.144)

to ensure that $\hat{\phi}(\vec{x})$ and $\hat{\Pi}(\vec{x})$ are real.

The Hamiltonian in terms of the Fourier modes is obtained by substituting Eqs.(5.142) and (5.143) into Eq.(5.137), we find

$$H = \sum_{k,k'} \int d^3x \frac{1}{2} \left( \hat{\Pi}_{k} \hat{\Pi}_{k'} + (m^2 - \vec{k} \cdot \vec{k'}) \hat{\phi}_{k} \hat{\phi}_{k'} \right) \frac{e^{i(\vec{k} + \vec{k'}) \cdot \vec{x}}}{V}. \quad (5.145)$$

Using the identity

$$\int d^3x \frac{e^{i(\vec{k} + \vec{k'}) \cdot \vec{x}}}{V} = \delta_{\vec{k}, -\vec{k'}},$$ (5.146)

Eq.(5.145) becomes,

$$H = \sum_{k,k'} \frac{1}{2} \left( \hat{\Pi}_{k} \hat{\Pi}_{k'} + (m^2 - \vec{k} \cdot \vec{k'}) \hat{\phi}_{k} \hat{\phi}_{k'} \right) \delta_{\vec{k}, -\vec{k'}}. \quad (5.147)$$
Thus,

\[ H = \sum_{\kappa \in \mathbb{Z}^3} \frac{1}{2} (\hat{\Pi}_\kappa \hat{\Pi}_{-\kappa} + \omega_\kappa^2 \hat{\phi}_\kappa \hat{\phi}_{-\kappa}), \tag{5.148} \]

where \( \omega_\kappa^2 = \omega_{-\kappa}^2 = m^2 + \vec{k}^2. \)

The commutation relations for the Fourier modes are obtained as follows. From the Fourier inversion formula we have

\[ \hat{\phi}_\kappa = \int d^3x \ \hat{\phi}(\vec{x}) \frac{e^{-i \kappa \cdot \vec{x}}}{\sqrt{V}} \tag{5.149} \]

and

\[ \hat{\Pi}_\kappa = \int d^3x \ \hat{\Pi}(\vec{x}) \frac{e^{-i \kappa \cdot \vec{x}}}{\sqrt{V}}. \tag{5.150} \]

Using Eqs. (5.140)-(5.141) and (5.149)-(5.150) we find

\[ [\hat{\phi}_\kappa, \hat{\phi}_{\vec{p}}] = 0, \quad [\hat{\Pi}_\kappa, \hat{\Pi}_{\vec{p}}] = 0 \tag{5.151} \]

and

\[ [\hat{\phi}_\kappa, \hat{\Pi}_{\vec{p}}] = \int d^3x \int d^3x' [\hat{\phi}(x), \hat{\Pi}(x')] \frac{e^{-i \kappa \cdot \vec{x}} e^{-i \vec{p} \cdot \vec{x}'}}{\sqrt{V} \sqrt{V}} \tag{5.152} \]

\[ = \int d^3x \int d^3x' i \delta(\vec{x} - \vec{x}') \frac{e^{-i(\kappa + \vec{p}) \cdot \vec{x}'}}{V} \tag{5.153} \]

\[ = i \int d^3x \frac{e^{-i(\kappa + \vec{p}) \cdot \vec{x}}}{V} \tag{5.154} \]

\[ = i \delta_{\vec{k}, -\vec{p}}. \tag{5.155} \]

We now map \( \hat{\phi}_\kappa \) and \( \hat{\Pi}_\kappa \) to the creation and annihilation operators, \( a_\kappa^\dagger \) and \( a_\kappa \), by

\[ a_\kappa = \frac{1}{\sqrt{2 \omega_\kappa}} (\omega_\kappa \hat{\phi}_\kappa + i \hat{\Pi}_\kappa), \tag{5.156} \]

and

\[ a_\kappa^\dagger = \frac{1}{\sqrt{2 \omega_\kappa}} (\omega_\kappa \hat{\phi}_{-\kappa} - i \hat{\Pi}_{-\kappa}) \tag{5.157} \]
or, equivalently by
\[ \dot{\phi}_k = \frac{1}{\sqrt{2\omega_k}} (a_k + a_k^*) \] (5.158)

and
\[ \dot{\Pi}_k = i\frac{\sqrt{\omega_k}}{2} (a_{-k} - a_k^*). \] (5.159)

In these new variables the Hamiltonian becomes
\[ H = \sum_{k \in \mathbb{Z}^3} \frac{1}{2} \omega_k(a_k^* a_k + a_k a_k^*). \] (5.160)

and the commutation relations
\[ [a_k, a_k^*] = 0, \quad [a_k^*, a_k^+] = 0 \] (5.161)

and
\[ [a_k, a_{k'}^+] = \delta_{k,k'}. \] (5.162)

The Hamiltonian, \( H \), at this point is not an operator, it is only a quadratic form (see Reed and Simon [18]). In order to obtain an operator from Eq.(5.160) we must regularize \( H \) (see Itzikson and Zuber [26]). This is accomplished by using the formal commutation relations, Eqs.(5.161) and (5.162), and then subtracting an infinite constant which represents the ground state energy. Once this is done we obtain the regularized Hamiltonian
\[ H = \sum_{k \in \mathbb{Z}^3} \omega_k a_k^* a_k. \] (5.163)

It is this expression which will concern us from now on.

Having regularized the Hamiltonian we can formally find the eigenstates of \( H \) by repeating the exact same arguments that were used for the harmonic oscillator (see chapter 4). The only difference is that in this case we have an infinite number of harmonic
oscillators and not just one. We omit the details of this formal construction since they are found in the next section where we rigorously construct this quantum system.

5.1.2 Bosonic Fock space

We now rigorously construct this quantum field theory. The first step is to build a Hilbert space $\mathcal{H}$ on which we can implement the creation and annihilation operators. To accomplish this we introduce the notion of tensor product of Hilbert spaces (see Reed and Simon [18]).

Given any $\phi \in \mathcal{H}_1$ and $\psi \in \mathcal{H}_2$, we define a bilinear form on $\mathcal{H}_1 \times \mathcal{H}_2$ by

$$\phi \otimes \psi(\psi_1, \psi_2) = (\psi_1, \phi)_{\mathcal{H}_1}(\psi_2, \psi)_{\mathcal{H}_2},$$

(5.164)

where $\psi_1 \in \mathcal{H}_1$ and $\psi_2 \in \mathcal{H}_2$. Let $\mathcal{E}$ denote the set of all finite linear combinations of these bilinear forms and introduce on $\mathcal{E}$ the inner product defined by

$$(\phi \otimes \psi, \tilde{\phi} \otimes \tilde{\psi})_\mathcal{E} = (\phi, \tilde{\phi})_{\mathcal{H}_1}(\psi, \tilde{\psi})_{\mathcal{H}_2}.$$  

(5.165)

The tensor product of $\mathcal{H}_1$ with $\mathcal{H}_2$, denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$, is defined as the Cauchy completion of $\mathcal{E}$ with this inner product, that is, the space obtained by adding the limit points for all Cauchy sequences. This process is well defined and leads to a unique Hilbert space (see Reed and Simon [18]). We can easily extend the construction to build the tensor product of $n$ Hilbert spaces $\mathcal{H}_1, ..., \mathcal{H}_n$, $\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_n$.

Before defining the Hilbert space of interest to us, the Bosonic Fock space, let us take a moment to fill up our bag of tricks for future needs. Let $A$ and $B$ be operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ respectively. We define the tensor product of domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$, which we denote by $\mathcal{D}(A) \otimes \mathcal{D}(B)$, as the set of finite linear
combinations of vectors of the form \( \psi_1 \otimes \psi_2 \), where \( \psi_1 \in \mathcal{D}(A) \) and \( \psi_2 \in \mathcal{D}(B) \). Note that \( \mathcal{D}(A) \otimes \mathcal{D}(B) \) is dense in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) iff \( \mathcal{D}(A) \) and \( \mathcal{D}(B) \) are dense in \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively (see Reed and Simon [18]). The tensor product of operators \( A \) and \( B \), denoted by \( A \otimes B \), is then defined by

\[
A \otimes B (\psi_1 \otimes \psi_2) = (A \psi_1) \otimes (B \psi_2)
\]

on states of the form \( \psi_1 \otimes \psi_2 \), where \( \psi_1 \in \mathcal{H}_1 \) and \( \psi_2 \in \mathcal{H}_2 \), and extended by linearity to all of \( \mathcal{D}(A) \otimes \mathcal{D}(B) \).

Having these definitions in hand, we are ready to ask the following question: Given two essentially self-adjoint operators \( A \) and \( B \), with domains \( \mathcal{D}(A) \) and \( \mathcal{D}(B) \), is \( A + B \) (\( = A \otimes I + I \otimes B \)) essentially self-adjoint on \( \mathcal{D}(A) \otimes \mathcal{D}(B) \)? The answer is yes! (see Reed and Simon [18]).

**Theorem 1** Let \( A_i, i = 1, ..., n \) be a set of essentially self-adjoint operators on \( \mathcal{H}_i, i = 1, ..., n \), with domains of essential self-adjointness \( \mathcal{D}(A_i) \) \( i = 1, ..., n \). Then the operator \( \sum_{i=1}^{n} A_i \) is essentially self-adjoint on \( \otimes_{i=1}^{n} \mathcal{D}(A_i) \).

For example, if \( A \) is some essentially self-adjoint operator and \( B \) is the identity operator, \( B = I \), then it is clear that \( A \otimes I \) is essentially self-adjoint on \( \mathcal{D}(A) \otimes \mathcal{D}(B) \). It then follows that the theorem is true since \( A \otimes I \) and \( I \otimes B \) are both essentially self-adjoint and their sum \( A \otimes I + I \otimes B \) acts as two independent operators, each acting non-trivially on different Hilbert spaces. This result will be very useful when constructing the quantum theory of systems which split up into two (or more) independent systems.

Before defining the Bosonic Fock space, let us first recall the definition of the usual
Fock space. Given an Hilbert space $\mathcal{H}$, the Fock space for $\mathcal{H}$ is defined by

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n,$$

(5.167)

where $\mathcal{H}^n = \mathcal{H} \otimes ... \otimes \mathcal{H}$ for $n > 0$ and $\mathcal{H}^0 = \mathbb{C}$. For example, if $\mathcal{H} = L^2(\mathbb{R})$, then an element $\psi \in \mathcal{F}(\mathcal{H})$ is a sequence of functions,

$$\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_3(x_1, x_2, x_3), ...\},$$

(5.168)

such that,

$$|\psi_0|^2 + \sum_{n=1}^{\infty} \int_{\mathbb{R}^n} |\psi_n(x_1, x_2, ..., x_n)|^2 dx_1 dx_2 ... dx_n < \infty.$$

(5.169)

It should be noted that $\mathcal{F}(\mathcal{H})$ is separable iff $\mathcal{H}$ is (see Reed and Simon [18]).

We now define the Bosonic Fock space. Let $S_n$ be the permutation group of $n$ elements and let $\{\phi_k\}$ be a basis for $\mathcal{H}$. For each $\sigma \in S_n$, we define an operator on $\mathcal{H}^n$ by

$$\sigma(\phi_{k_1} \otimes \phi_{k_2} \otimes ... \otimes \phi_{k_n}) = \phi_{k_{\sigma(1)}} \otimes \phi_{k_{\sigma(2)}} \otimes ... \otimes \phi_{k_{\sigma(n)}}$$

(5.170)

and extend $\sigma$ by linearity to all of $\mathcal{H}^n$. We define the symmetrizer operator $S_n$ on $\mathcal{H}^n$ by $S_n = \sum_{\sigma \in S_n} \sigma$. The range of $S_n$, denoted by $S_n \mathcal{H}^n$, is a well defined Hilbert space with the induced inner product from $\mathcal{H}^n$ (see Reed and Simon [18]). In the case where $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{H}^n = L^2(\mathbb{R}) \otimes ... \otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^n)$, $S_n \mathcal{H}^n$ is just the subspace of $L^2(\mathbb{R}^n)$ which consists of all functions invariant under any permutation of its variables.

We define the Bosonic Fock space, for a Hilbert space $\mathcal{H}$, by

$$\mathcal{F}_b(\mathcal{H}) = \bigoplus_{n=0}^{\infty} S_n \mathcal{H}^n.$$  

(5.171)

Note that since $\mathcal{F}_b(\mathcal{H})$ can be viewed as a subspace of $\mathcal{F}(\mathcal{H})$ it follows that $\mathcal{F}_b(\mathcal{H})$ is separable if $\mathcal{H}$ is. Given the resemblance between the Hamiltonian of Eq.(5.163) and
that of the harmonic oscillator, $\mathcal{F}_b(\mathcal{H})$ is a natural choice for Hilbert space since creation and annihilation of particle excitations are easily implemented on $\mathcal{F}_b(\mathcal{H})$. Since we are interested in the quantum theory of a scalar field on a 3-torus, we take $\mathcal{H} = L^2(T^3)$, where $T^3$ denotes the 3-torus. This follows from the fact that $a_k^\dagger$ creates particle excitations which live in $L^2(T^3)$. Thus, our choice of Hilbert space is $\mathcal{F}_b(L^2(T^3))$. It should be noted that states in $\mathcal{F}_b(L^2(T^3))$ are of the form Eq.(5.168) with

$$\psi_n(x_1, x_2, ..., x_n) = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \psi_\sigma(1) \otimes \psi_\sigma(2) \otimes ... \otimes \psi_\sigma(n),$$

(5.172)

where $\psi_i \in L^2(T^3)$ with $i = 1, ..., n$.

Before rigorously quantizing this system, let us construct a complete basis for $\mathcal{F}_b(L^2(T^3))$. We will use Feynman’s notation [25]. Since $\{\phi_{\vec{k}} = \frac{e^{ik \cdot x}}{\sqrt{V}}\}$ is a basis for one particle states, that is, a basis for $L^2(T^3)$, it follows that

$$\frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \phi_{\vec{k}_\sigma(1)} \otimes \phi_{\vec{k}_\sigma(2)} \otimes ... \otimes \phi_{\vec{k}_\sigma(n)} \text{ where } \vec{k}_i \in \mathbb{Z}^3$$

(5.173)

is a basis for n-particle states. It should be noted that $\phi_{\vec{k}} \in L^2(T^3)$, that is, the basis vectors are elements in the Hilbert space. This is contrasts working on $\mathbb{R}^3$ where the plane wave form a basis for $L^2(\mathbb{R}^3)$ but are not in the Hilbert space, $\frac{e^{ik \cdot x}}{(2\pi)^{2/3}} \notin L^2(\mathbb{R}^3)$. Note that the set of all finite particle states, states of the form

$$\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), ..., \psi_n(x_1, ..., x_n)\},$$

(5.174)

is a dense subset of $\mathcal{F}_b(\mathcal{H})$. This follows from the same arguments that were used to show that $\mathcal{D}(a_k^\dagger)$ is dense in $l^2(\mathbb{C})$ (see chapter 4). It now follows that Eq.(5.173) defines a complete basis for $\mathcal{F}_b(L^2(T^3))$. 

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To simplify the notation of the basis vectors, Eq. (5.173), we set

\[ |\vec{k}_1, \ldots, \vec{k}_n \rangle = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \phi_{k_\sigma(1)} \otimes \ldots \otimes \phi_{k_\sigma(n)}. \]  

It should be noted that \( |\vec{k}_1, \ldots, \vec{k}_n \rangle \) are orthonormal vectors since

\[ (|q_1, \ldots, q_n \rangle, |k_1, \ldots, k_m \rangle) = \delta_{nm} \]  

\[ = \delta_{n,m} \left( \frac{1}{\sqrt{n!}} \sum_{\sigma' \in S_n} \phi_{q_{\sigma'(1)}} \otimes \ldots \otimes \phi_{q_{\sigma'(n)}} \right) \left( \frac{1}{\sqrt{n!}} \sum_{\sigma'' \in S_n} \phi_{k_{\sigma''(1)}} \otimes \ldots \otimes \phi_{k_{\sigma''(n)}} \right) \]

\[ = \delta_{n,m} \frac{1}{n!} \sum_{\sigma' \in S_n} \sum_{\sigma'' \in S_n} \delta_{q_{\sigma'(1)}, k_{\sigma''(1)}} \ldots \delta_{q_{\sigma'(n)}, k_{\sigma''(n)}} \]  

(5.177)

(5.178)

(where we have used \( (\phi_q, \phi_k) = \delta_{q,k} \))

\[ = \delta_{n,m} \frac{1}{n!} \sum_{\sigma' \in S_n} \sum_{\sigma'' \in S_n} \delta_{q_{\sigma'(1)}, k_{\sigma''(1)}} \ldots \delta_{q_{\sigma'(n)}, k_{\sigma''(n)}} \]  

(5.179)

(where we have permuted the factors by \( \sigma^{-1} \))

\[ = \delta_{n,m} \left( \frac{1}{n!} \sum_{\sigma'' \in S_n} \sum_{\sigma''' \in S_n} \delta_{q_{\sigma''(1)}, k_{\sigma'''(1)}} \ldots \delta_{q_{\sigma''(n)}, k_{\sigma'''(n)}} \right) \]  

(5.180)

(where we have set \( \sigma''' = \sigma' \sigma^{-1} \))

\[ = \delta_{n,m} \sum_{\sigma \in S_n} \delta_{q_{\sigma(1)}, k_{\sigma(n)}} \]  

(5.181)

where we set \( \sigma = \sigma''' \). It should also be noted that \( \vec{k}_i \) do not have to be distinct. For example, we can have states of the form

\[ |\vec{k}, \vec{k} \rangle ; \ |\vec{k}, \vec{k}, \vec{k}' \rangle ; \ |\vec{k}, \vec{k}, \vec{k}, \vec{k}' \rangle. \]  

(5.182)

### 5.1.3 Rigorous Q.F.T.

We now construct the rigorous quantum theory of a free massive scalar field on a 3-torus. This will be accomplished in two steps: first we implement the operator algebra of the
creation and annihilation operators, Eqs.(5.161) and (5.162), on the Bosonic Fock space and second we find a self-adjoint Hamiltonian.

The creation operators $a^\dagger(\phi)$, where $\phi \in L^2(T^3)$, are mappings from $n$-particle states to $(n+1)$-particle states,

$$a^\dagger(\phi): \mathcal{H}^n \rightarrow \mathcal{H}^{n+1}. \quad (5.183)$$

They are defined by

$$a^\dagger(\phi)\psi_n(x_1, x_2, \ldots, x_n) = \frac{1}{\sqrt{(n+1)!}} \sum_{\sigma \in \mathcal{S}_{n+1}} \psi_{\sigma(1)} \otimes \psi_{\sigma(2)} \otimes \cdots \otimes \psi_{\sigma(n+1)}, \quad (5.184)$$

where we have set $\phi = \psi_{n+1}$. Hence, $a^\dagger(\phi)$ creates an excitation $\phi$. We set $a^\dagger(\phi_k) = a^\dagger_k$. The action of $a^\dagger_k$ on basis states is then given by

$$a^\dagger_k|\vec{k}_1, \ldots, \vec{k}_n> = |\vec{k}, \vec{k}_1, \ldots, \vec{k}_n>, \quad (5.185)$$

where we have used Eqs.(5.175) and (5.184). We pick the domain of $a^\dagger_k$ to be the set of all finite linear combinations of basis states,

$$\mathcal{D}(a^\dagger_k) = \{ |\psi > = \sum_{n=0}^{\infty} \sum_{\vec{k}_1, \ldots, \vec{k}_n \in \mathbb{Z}^3} \lambda_{\vec{k}_1, \ldots, \vec{k}_n} |\vec{k}_1, \ldots, \vec{k}_n > \text{ only finitely many } \lambda_{\vec{k}_1, \ldots, \vec{k}_n} \neq 0 \}. \quad (5.186)$$

By Eq.(5.185) we see that $a^\dagger_k$ are well defined on individual basis states, $|\vec{k}_1, \ldots, \vec{k}_n>$. From this it follows that $a^\dagger_k$ are well defined on all finite linear combinations of basis states. Hence, $a^\dagger_k$ are well defined on $\mathcal{D}(a^\dagger_k)$. It should be noted that $a^\dagger_k$ are not defined the same way the raising operator was defined for the one dimensional harmonic oscillator.

The proof that $\mathcal{D}(a^\dagger_k)$ is dense in $\mathcal{F}_b$ is as follows. First, recall that $|\vec{k}_1, \ldots, \vec{k}_n>$ is a complete basis for $\mathcal{F}_b$. Next, given any countable basis for a separable Hilbert space $\mathcal{H}$, the set of all finite linear combinations of these vectors form a dense subset of $\mathcal{H}$ (see Lang
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[22]). This statement follows from the proof that $\mathcal{D}(a^\dagger)$ is dense in $l^2(\mathbb{C})$ (see chapter 4). Notice that $\mathcal{F}_b(L^2(\mathbb{R}^3))$ is separable since $L^2(\mathbb{R}^3)$ is separable. Since $\mathcal{F}_b(L^2(\mathbb{R}^3))$ is separable and $\mathcal{D}(a^\dagger_k)$ is the set of all finite linear combinations of basis vectors we conclude that $\mathcal{D}(a^\dagger_k)$ is dense in $\mathcal{F}_b$.

Since $\mathcal{D}(a^\dagger_k)$ is a dense domain the creation operators, $a^\dagger_k$, have well defined adjoints, $(a^\dagger_k)^\dagger$, which we denote by $a_k = (a^\dagger_k)^\dagger$. To find the action of $a_k$ on basis states we note the following:

\begin{align}
(|q_1, \ldots, q_m >, a^\dagger_k |k_1, \ldots, k_n >) &= (|q_1, \ldots, q_m >, |k, k_1, \ldots, k_n >) \\
&= \delta_{m,n+1} \sum_{\sigma \in S_m} \delta_{q_1, k_\sigma(1)} \cdots \delta_{q_m, k_\sigma(m)}
\end{align}

(5.187)

(5.188)

(where we have used Eqs.(5.181), (5.185) and set $k = k_m$)

\begin{align}
&= \delta_{m-1,n} \sum_{i=1}^n \delta_{q_i, k_m} \sum_{\sigma \in S_n} \delta_{q_1, k_\sigma(1)} \cdots \delta_{q_m, k_\sigma(n)} \\
&= \delta_{m-1,n} \sum_{i=1}^n \delta_{q_i, k_m} (|q_1, \ldots, \hat{q}_i, \ldots, q_m >, |k_1, \ldots, k_n >) \\
&= (a_k |q_1, \ldots, q_m >, |k_1, \ldots, k_n >),
\end{align}

(5.189)

(5.190)

(5.191)

where the hat signifies an omitted slot. Then from the definition of adjoint we find,

\begin{equation}
a_k |\vec{k}_1, \ldots, \vec{k}_n >= \sum_{i=1}^n \delta_{\vec{k}_i, \vec{k}_m} |\vec{k}_1, \ldots, \hat{\vec{k}}_i, \ldots, \vec{k}_n >
\end{equation}

(5.192)

and

\begin{equation}
a_k |0 >= 0.
\end{equation}

(5.193)

It follows from Eqs.(5.187)-(5.191) that $\mathcal{D}(a^\dagger_k) \subseteq \mathcal{D}(a_k)$ and thus the annihilation operators, $a_k$, are also well defined on $\mathcal{D}(a^\dagger_k)$. It should be noted that $a_k$ maps $\mathcal{D}(a^\dagger_k)$ to itself since its action is to kill a one particle excitation, Eqs.(5.192) and (5.193).
Using the definition of $a^\dagger_k$ and of its adjoint, $a_k$, it is easy to verify that the commutation relations are satisfied on $\mathcal{D}(a^\dagger_k)$. First note that

$$a^\dagger_k a^\dagger_{k'} |\vec{k}_1, ..., \vec{k}_n > = a^\dagger_k |\vec{k}_1', \vec{k}_1, ..., \vec{k}_n >$$

$$= \delta_{k, k'} |\vec{k}_1, ..., \vec{k}_n > + \sum_{i=1}^{n} \delta_{\vec{k}, \vec{k}_i} |\vec{k}_1', \vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_n >. \quad (5.195)$$

Now, since

$$a^\dagger_k a_k |\vec{k}_1, ..., \vec{k}_n > = \sum_{i=1}^{n} \delta_{\vec{k}, \vec{k}_i} |\vec{k}_1', \vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_n >,$$

it follows that

$$[a_k, a^\dagger_k] = \delta_{\vec{k}, \vec{k}'}$$

is satisfied for all states in $\mathcal{D}(a^\dagger_k)$. It is worth emphasising that we have used the rigorous definition of adjoint to show that the commutation relations of the creation and annihilation operators are satisfied on a dense domain.

The next step to rigorously quantize the system is to find a self-adjoint Hamiltonian. To accomplish this we first find an essentially self-adjoint Hamiltonian $H$ and then take its closure $\bar{H}$. Note that

$$H |\vec{k}_1, ..., \vec{k}_n > = \sum_{\vec{k} \in \mathbb{Z}^3} \omega_\vec{k} a^\dagger_\vec{k} a_\vec{k} |\vec{k}_1, ..., \vec{k}_n >$$

$$= \sum_{\vec{k} \in \mathbb{Z}^3} \omega_\vec{k} a^\dagger_\vec{k} \sum_{i=1}^{n} \delta_{\vec{k}, \vec{k}_i} |\vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_n >$$

$$= \sum_{\vec{k} \in \mathbb{Z}^3} \omega_\vec{k} \sum_{i=1}^{n} \delta_{\vec{k}, \vec{k}_i} |\vec{k}, \vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_n >$$

$$= \sum_{i=1}^{n} \omega_{\vec{k}_i} |\vec{k}_i, \vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_n >$$

$$= \sum_{i=1}^{n} \omega_{\vec{k}_i} |\vec{k}_1, ..., \vec{k}_i, ..., \vec{k}_n >. \quad (5.202)$$

where we have used Eqs.(5.185) and (5.192). Thus the Hamiltonian is well defined on individual basis vectors $|\vec{k}_1, ..., \vec{k}_n >$. Hence, $H$ is well defined on $\mathcal{D}(a^\dagger_k)$ since $\mathcal{D}(a^\dagger_k)$ is
the set of all finite linear combinations of basis states \( |\vec{k}_1, ..., \vec{k}_n> \). Let us now show that the Hamiltonian is essentially self-adjoint on \( D(a^\dagger_k) \).

**Theorem 1** The Hamiltonian for a free massive scalar field on a 3-torus is essentially self-adjoint on \( D(a^\dagger_k) \).

**Proof:** Suppose that \( H \) is not essentially self-adjoint. Then by theorem (1) from chapter 3 section 4 there exists a \( |\chi \rangle \in D(H^\dagger) \) such that \( H^\dagger |\chi \rangle = i|\chi \rangle \) or \( H^\dagger |\chi \rangle = -i|\chi \rangle \). This implies \( 0 = ((H^\dagger \pm i)|\chi \rangle, |\vec{k}_1, ..., \vec{k}_n> ) \)

\[
0 = ((H^\dagger \pm i)|\chi \rangle, |\vec{k}_1, ..., \vec{k}_n> )
\]

(5.203)

\[
= (|\chi \rangle, (H \mp i)|\vec{k}_1, ..., \vec{k}_n> ),
\]

(5.204)

where we have used the fact that \( |\vec{k}_1, ..., \vec{k}_n> \in D(a^\dagger_k) = D(H) \subset D(H^\dagger) \). Using Eq.(5.202) we find that

\[
0 = (|\chi \rangle, \sum_{i=1}^{n} \omega_{\vec{k}_i} \mp i)|\vec{k}_1, ..., \vec{k}_n> ).
\]

Hence,

\[
(|\chi \rangle, |\vec{k}_1, ..., \vec{k}_n> ) = 0
\]

(5.205)

(5.206)

for all basis vectors \( |\vec{k}_1, ..., \vec{k}_n> \). Since \( |\vec{k}_1, ..., \vec{k}_n> \) is a complete basis we conclude that \( |\psi \rangle = 0 \). Thus, the Hamiltonian is essentially self-adjoint on \( D(a^\dagger_k) \).

Now that we have an essentially self-adjoint operator \( H \), we take its closure \( \bar{H} \) and obtain a self-adjoint Hamiltonian. We can now claim that we have a well defined quantum field theory. Self-adjointness of \( \bar{H} \) guarantees that the system has a well defined energy spectrum and evolves unitarily. It should be noted that we can construct an isomorphism between this representation and other representations, just as it was done
for the harmonic oscillator. Hence, we can rigorously quantize this field theory in other representations.

Having a solution for the system, we can give an explicit expression for the evolution operator. By repeating the arguments that were used for the harmonic oscillator (see chapter 4) we conclude that the eigenstates and eigenvalues of $\tilde{H}$ coincides with those of $H$. Then by the spectral theorem (see Reed and Simon [18]), the exponential of $\tilde{H}$, defined by

$$U(t)|\psi> = \sum_{n=0}^{\infty} \sum_{\vec{k}, \vec{k}_n \in \mathbb{Z}^3} e^{i(\sum_{n=1}^{\infty} \omega_{\vec{k}_n})t} (|\psi>, |\vec{k}_1, ..., \vec{k}_n>) |\vec{k}_1, ..., \vec{k}_n>,$$

is a well defined (strongly continuous) unitary operator. It should be noted that the formal power series expression for the exponential can be made to coincide with the above expression as it was done for the harmonic oscillator (see chapter 4).

5.2 Massless scalar field theory

In the previous section we rigorously quantized a free massive scalar field on a 3-torus. We now consider the case where the scalar field is massless. We will first formally quantize the system and then give a rigorous construction for this quantum field theory.

5.2.1 Formal quantization

The formal quantum theory of a free massless scalar field is described by the Hamiltonian

$$H = \int d^3x \frac{1}{2}(\vec{\nabla}^2 + (\vec{\nabla}\phi)^2)$$

and the commutation relations Eqs.(5.140) and (5.141).
In terms of modes, Eqs.(5.142) and (5.143), the Hamiltonian becomes

$$ H = \sum_{\vec{k} \in \mathbb{Z}^3} \left( \hat{\Pi}_{\vec{k}} \hat{\phi}_{\vec{k}}^* + \vec{k}^2 \hat{\phi}_{\vec{k}} \hat{\phi}_{\vec{k}}^* \right), \tag{5.209} $$

where we have used Eq.(5.144). As for the commutation relations they are given by Eqs.(5.151)-(5.155).

Let us now formally solve the system in the field representation. State vectors are functionals of the modes $\phi_{\vec{k}}$, $\Psi = \Psi[\phi_{\vec{k}}]$, and the operators $\hat{\Pi}_{\vec{k}}$ and $\hat{\phi}_{\vec{k}}$ are formally given by

$$ \hat{\Pi}_{\vec{k}} = -i \frac{\partial}{\partial \phi_{\vec{k}}^*} \quad \text{and} \quad \hat{\phi}_{\vec{k}} = \phi_{\vec{k}}, \tag{5.210} $$

where we have used Eqs.(5.144) and (5.155).

Notice that the Hamiltonian is an infinite sum of (complex) harmonic oscillators, one for each $\vec{k} \in \mathbb{Z}^3$ with $\vec{k} \neq \vec{0}$, plus a purely kinetic term for the zero mode sector,

$$ H = \sum_{\vec{k} \neq \vec{0}} H_k + \frac{1}{2} \hat{\Pi}_{\vec{0}}^2, \tag{5.211} $$

where

$$ H_k = \frac{1}{2} \left( \hat{\Pi}_{\vec{k}} \hat{\phi}_{\vec{k}}^* + \vec{k}^2 \hat{\phi}_{\vec{k}} \hat{\phi}_{\vec{k}}^* \right). \tag{5.212} $$

From Eq.(5.211) it follows that the eigenstates of $H$ are of the form $\Psi = \psi_0 \Psi_n$, where $\Psi_n$ are eigenstates of $\sum H_k$ and $\psi_0$ are eigenstates of $\frac{1}{2} \hat{\Pi}_{\vec{0}}^2$.

Let us find the eigenstates of $\sum H_k$. For each $H_k$ we can formally find its eigenvectors and eigenvalues. This is accomplished by introducing the creation and annihilation operators, $a_{\vec{k}}^\dagger$ and $a_{\vec{k}}$, defined by Eqs.(5.156) and (5.157). In terms of these variables the regularized $H_\vec{k}$ is given by

$$ H_\vec{k} = |\vec{k}| a_{\vec{k}}^\dagger a_{\vec{k}}, \tag{5.213} $$
where we has subtracted the ground state energy $\frac{1}{2}|\vec{k}|$. It should be noted that this extraction of the ground state energy for each $H_k$ regularizes the Hamiltonian Eq.(5.211).

The eigenstates of $H_k$ can be constructed as it was done for the harmonic oscillator in the coordinate representation (see chapter 4).

The ground state is formally given by

$$a_{\vec{k}} \psi_0(\phi_{\vec{k}}) = 0. \quad (5.214)$$

Using Eqs.(5.156) and (5.210) we find the differential equation

$$(|\vec{k}| \phi_{\vec{k}} + \frac{\partial}{\partial \phi_{\vec{k}}}) \psi_0(\phi_{\vec{k}}) = 0, \quad (5.215)$$

which has solution

$$\psi_0(\phi_{\vec{k}}) = e^{-|\vec{k}| \phi_{\vec{k}}^{\phi_{\vec{k}}^*}}, \quad (5.216)$$

where we have set the normalization factor equal to unity for convenience. All other eigenstates of $H_k$ are obtained by acting on the ground state $\psi_0$ with the creation operator $a_{\vec{k}}^\dagger$, that is, they are given by

$$\psi_n(\phi_{\vec{k}}) = (a_{\vec{k}}^\dagger)^n \psi_0(\phi_{\vec{k}}), \quad (5.217)$$

up to a normalization factor. Having formally found the eigenstates for each $H_k$ it follows that the eigenstates of

$$H_1 = \sum_{\vec{k} \neq 0} |\vec{k}| a_{\vec{k}}^{\dagger} a_{\vec{k}} \quad (5.218)$$

are given by

$$\Psi_n = a_{\vec{k}_1}^{\dagger} ... a_{\vec{k}_n}^{\dagger} \psi_0 \quad \text{where} \quad \vec{k}_i \in \mathbb{Z}^3 \quad \text{and} \quad \vec{k}_i \neq 0 \quad (5.219)$$

and where

$$\psi_0 = \prod_{\vec{k} \neq 0} e^{-|\vec{k}| \phi_{\vec{k}}^{\phi_{\vec{k}}^*}} \quad (5.220)$$
is the unnormalized ground state of $H_1$. It should be noted that we impose no conditions on $\vec{k}_i$ except $\vec{k}_i \neq 0$ for all $i = 1, ..., n$.

We now find the eigenstates for the zero mode sector of the Hamiltonian. Note that the zero mode contribution to the Hamiltonian is equal to

$$H_2 = \frac{1}{2} \hat{\Pi}_0^2,$$

where $\hat{\Pi}_0$ is given by Eq.(5.210). The eigenstates of $H_2$ are plane waves given by

$$\psi_\theta(\phi_0) = \frac{e^{i\theta \phi_0}}{\sqrt{2\pi}} \text{ with } \theta \in \mathbb{R}.$$

Note that even though $\psi_\theta$ are not normalizable states they still form a basis for $L^2(\mathbb{R})$. It should also be noted that since the spectrum of $H_2$ is continuous it follows that the spectrum of $H = H_1 + H_2$ is also continuous. This contrasts the quantum theory of a massive scalar field since in that case the spectrum is discrete.

It now follows that the unnormalized eigenstates of $H = H_1 + H_2$ are given by

$$\Psi_n[\phi_\vec{k}] = a_{\vec{k}_1}^\dagger ... a_{\vec{k}_n}^\dagger \Psi_\theta[\phi_\vec{k}],$$

where

$$\Psi_\theta[\phi_\vec{k}] = e^{i\theta \phi_0} \prod_{\vec{k} \neq 0} e^{-|\vec{k}|^2|\phi_0|_2^2}$$

is the unnormalized ground state. By ground state we mean the state with no particle excitations. Hence, there exists a one parameter family of ground states for the quantum theory of a massless scalar field. This contrasts the massive case where there is a unique vacuum. It should be noted that $\Psi_\theta$ is formally the tensor product of $\psi_\theta$ with $\Psi_0$. One should also note that a general state is of the form

$$\Psi = \sum_{n=0}^{\infty} \sum_{\vec{k}_1, ..., \vec{k}_n \neq 0} \int d\theta f(\theta) \lambda_{\vec{k}_1, ..., \vec{k}_n} a_{\vec{k}_1}^\dagger ... a_{\vec{k}_n}^\dagger \Psi_\theta,$$
where $f(\theta)$ and $\lambda_{\vec{k}_1, ..., \vec{k}_n}$ are weight factors.

One should note that this construction is only formal since we have not specified the Hilbert space in which the states given by Eq.(5.227) live in or picked domains for our operators. Moreover, we have not shown that the Hamiltonian is self-adjoint. Thus, we have no guarantee that the system evolves unitarily.

### 5.2.2 Rigorous construction

We now rigorously construct the quantum theory of a free massless scalar field on a 3-torus. The first step is to build a Hilbert space on which we can implement the operator algebra and find a self-adjoint Hamiltonian.

The Hilbert space for the system is obtained by taking the tensor product of $L^2(\mathbb{R})$ with the Hilbert space generated by the basis vectors $|\theta; \vec{k}_1, ..., \vec{k}_n \rangle$. Let $|\theta; 0 \rangle$ represent symbolically the ground state of the system, $\Psi_\theta$. By $|\theta; 0 \rangle$ we mean the tensor product of $|\theta \rangle$ with $|0 \rangle$, $|\theta \rangle \otimes |0 \rangle$, where $|0 \rangle = e^{i\varphi_0} / \sqrt{2\pi} \in L^2(\mathbb{R})$ and $|0 \rangle$ represents the ground state $\Psi_0$. We define the basis vectors $|\theta; \vec{k}_1, ..., \vec{k}_n \rangle$ symbolically by

$$a_{\vec{k}_1}^\dagger ... a_{\vec{k}_n}^\dagger |\theta; 0 \rangle = |\theta; \vec{k}_1, ..., \vec{k}_n \rangle \quad \text{where} \quad \vec{k}_i \in \mathbb{Z}^3 \quad \text{and} \quad \vec{k}_i \neq 0. \quad \text{(5.227)}$$

One should note that the basis vectors are by definition invariant under permutations since

$$|\theta; \vec{k}, \vec{k}' \rangle = a_{\vec{k}}^\dagger a_{\vec{k}'}^\dagger |\theta; 0 \rangle = a_{\vec{k}'}^\dagger a_{\vec{k}}^\dagger |\theta; 0 \rangle = |\theta; \vec{k}', \vec{k} \rangle, \quad \text{(5.228)}$$
where \( \vec{k} \neq 0 \) and \( \vec{k}' \neq 0 \).

Let \( \Lambda \) be the set of all finite linear combinations of the basis vectors \( |\theta; \vec{k}_1, ..., \vec{k}_n> \) for a fixed value of \( \theta \). Introduce on \( \Lambda \) an inner product defined by

\[
(|\theta; \vec{q}_1, ..., \vec{q}_m>, |\theta; \vec{k}_1, ..., \vec{k}_n>)_{\Lambda} = \delta_{n,m} \sum_{\sigma \in S_n} \delta_{\vec{q}_1, \vec{k}_{\sigma(1)}} ... \delta_{\vec{q}_n, \vec{k}_{\sigma(n)}}
\]

(5.229)
on the basis vectors and extend by linearity to all of \( \Lambda \). One should note the resemblance between Eq.(5.229) and (5.181). We define the Hilbert space \( \bar{\Lambda} \) as the Cauchy completion of \( \Lambda \) with this inner product. One should recall that the Cauchy completion of \( \Lambda \) is the Hilbert space obtained by adding all the limit points of all the Cauchy sequences.

The Hilbert space for the quantum theory is the tensor product of \( L^2(\mathbb{R}) \) with \( \bar{\Lambda} \), that is,

\[
\mathcal{H} = L^2(\mathbb{R}) \otimes \bar{\Lambda}.
\]

(5.230)

A general element in \( \mathcal{H} \) is of the from

\[
|f; \Psi> = \sum_{n=0}^{\infty} \sum_{\vec{k}_1, ..., \vec{k}_n \neq 0} \int d\theta \ f(\theta) \lambda_{\vec{k}_1, ..., \vec{k}_n} |\theta; \vec{k}_1, ..., \vec{k}_n>.
\]

(5.231)

where \( f(\theta) \) is square summable,

\[
\int d\theta |f(\theta)|^2 < \infty,
\]

(5.232)

and \( \lambda_{\vec{k}_1, ..., \vec{k}_n} \) satisfies

\[
\sum_{n=0}^{\infty} \sum_{\vec{k}_1, ..., \vec{k}_n \neq 0} |\lambda_{\vec{k}_1, ..., \vec{k}_n}|^2 < \infty.
\]

(5.233)

Having found the Hilbert space for the system we now implement the operator algebra, Eqs.(5.161)-(5.162) and (5.151)-(5.155), on \( \mathcal{H} \). We define the creation operators on \( \mathcal{H} \) by \( I \otimes a^\dagger_k \), where \( I \) is the identity operator on \( L^2(\mathbb{R}) \) and \( a^\dagger_k \) is defined by Eq.(5.227).
We pick $\mathcal{D} = \mathcal{C}_0^\infty(\mathbb{R}) \otimes \Lambda$ as domain for the creation operators. It should be noted that $\mathcal{D}$ provides a common domain for all the operators in the algebra. Note that $\mathcal{D}$ is dense in $\mathcal{H}$ since $\Lambda$ is dense in $\overline{\Lambda}$ and $\mathcal{C}_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$. From Eq.(5.227) it follows that $a_k^\dagger$ are well defined on $\Lambda$ since they are well defined on individual basis vectors and thus are well defined on all finite linear combinations of basis states. Since $I$ is well defined on all of $L^2(\mathbb{R})$ it follows that the creation operators $I \otimes a_k^\dagger$ are well defined on $\mathcal{D}$. Note that $I = I^\dagger$, that is, the identity operator is self-adjoint on $L^2(\mathbb{R})$. Using $I = I^\dagger$ and recalling Eqs.(5.187)-(5.197) and (5.227) it follows that the adjoint of $I \otimes a_k^\dagger$ is given by $I \otimes a_k^\dagger$, where $a_k$ is defined by Eqs.(5.192) and (5.193). Recalling Eqs.(5.187)-(5.197) it follows that the commutation relations, Eqs.(5.161) and (5.162), are satisfied on $\mathcal{D}$.

The operators $\hat{\Pi}_6$ and $\hat{\phi}_6$, which correspond to the zero modes, are defined by $\hat{\Pi}_6 \otimes I$ and $\hat{\phi}_6 \otimes I$, where $I$ is the identity operator on $\Lambda$ and where $\hat{\Pi}_6$ and $\hat{\phi}_6$ are defined by

$$\begin{align*}
\hat{\Pi}_6 \int d\theta f(\theta)|\theta > &= \int d\theta \theta f(\theta)|\theta > \quad (5.234) \\
\hat{\phi}_6 \int d\theta f(\theta)|\theta > &= i \int d\theta f'(\theta)|\theta > . \quad (5.235)
\end{align*}$$

It should be noted that this is the usual representation of the canonical commutation relations in the Fourier space. We pick $\mathcal{D} = \mathcal{C}_0^\infty(\mathbb{R}) \otimes \Lambda$ as their domain. Recall that $\mathcal{D}$ is dense in $\mathcal{H}$. Since $\hat{\Pi}_6$ and $\hat{\phi}_6$, as given by Eqs.(5.234) and (5.235), are well defined on $\mathcal{C}_0^\infty(\mathbb{R})$ it follows that $\hat{\Pi}_6 \otimes I$ and $\hat{\phi}_6 \otimes I$ are well defined on $\mathcal{D}$. Moreover, from Eqs.(5.234) and (5.235) it follows that the commutation relations, Eqs.(5.151) and (5.155), are satisfied on $\mathcal{D}$. Hence, the operator algebra for the system is satisfied on $\mathcal{D}$ a dense subset of $\mathcal{H}$.

The next step is to find a self-adjoint Hamiltonian. This is accomplished by first finding an essentially self-adjoint Hamiltonian and then taking its closure. recalling
Eqs. (5.198)-(5.202) it follows that $I \otimes H_1$, where $H_1$ is defined by Eq. (5.218), is well defined on $\mathcal{D}$. Since $\hat{\Pi}_0^2$ is well defined on $C_0^\infty(\mathbb{R})$ it follows that $H_2 \otimes I$, where $H_2$ is defined by Eq. (5.221), is well defined on $\mathcal{D}$. Hence, $H = I \otimes H_1 + H_2 \otimes I$ is well defined on $\mathcal{D}$. Let us now prove that $H$ is essentially self-adjoint on this domain.

**Theorem 1** The Hamiltonian for a free massless scalar field on a 3-torus is essentially self-adjoint on $\mathcal{D}$.

*Proof.* We show that $H_1$ and $H_2$ are essentially self-adjoint on $\Lambda$ and $C_0^\infty(\mathbb{R})$ respectively and then use theorem (1) from section 1.2 to conclude that $H$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}) \otimes \Lambda$. Repeating the proof of theorem (1) from section 1.3 it follows that $H_1$ is essentially self-adjoint on $\Lambda$. Recalling theorem (2) from chapter 3 section 4 we conclude that $H_2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R})$. Hence, the Hamiltonian $H = I \otimes H_1 + H_2 \otimes I$ is essentially self-adjoint on $\mathcal{D} = C_0^\infty(\mathbb{R}) \otimes \Lambda$.

Now that we have an essentially self-adjoint operator, $H = H_1 + H_2$, we take its closure, $\tilde{H} = \tilde{H}_1 + \tilde{H}_2$, and obtain a self-adjoint Hamiltonian. We can now conclude that we have a well defined quantum theory for a free massless scalar field. Self-adjointness of $\tilde{H}$ guarantees that the system has a well defined energy spectrum and evolves unitarily. Repeating the arguments that were used for the harmonic oscillator it follows that $H_1$ and $\tilde{H}_1$ have the same eigenvalues and eigenvectors. Moreover, $H_2$ and $\tilde{H}_2$ also have the same eigenvalues and eigenvectors (see Reed and Simon [18]). It now follows that $H$ and $\tilde{H}$ have the same eigenvectors and eigenvalues. It should be noted that since there are many inequivalent representation of the operator algebra for quantum field theories (see Reed and Simon [18]) it follows that there is no guarantee a priori that this representation is the usual physical representation.
We now give the expression for the evolution operator. Recall the greens function representation of the propagator for a free particle on the real line is given by (see Reed and Simon [18]),

\[(e^{itH_2} f)(\phi_0) = (-4\pi it)^{-1/2} \int_{\mathbb{R}} e^{-i\phi_0 - i^2 t/4t} f(y) dy, \tag{5.236}\]

where \(H_2\) denotes the coordinate representation of \(\frac{1}{2} \Pi_0^2\). It then follows that the evolution operator in the Fourier transform space is equal to

\[U_2(t)|f\rangle = e^{itH_2} \int d\theta f(\theta)|\theta\rangle = \int d\theta e^{it\theta^2/2} f(\theta)|\theta\rangle, \tag{5.237}\]

Note that Eq.(5.237) can also be deduced from Eq.(5.234).

Now, since \(H_1\) and \(H_2\) act on different Hilbert spaces it follows that the exponential of \(H = I \otimes H_1 + H_2 \otimes I\) is equal to \(U(t) = U_1(t) \otimes U_2(t)\), where \(U_1(t)\) and \(U_2(t)\) are the exponentials of \(H_1\) and \(H_2\) respectively. It now follows that

\[U(t)|f; \Psi\rangle = U(t)(|f\rangle \otimes |\Psi\rangle) \tag{5.238}\]

\[= (U_2(t)|f\rangle) \otimes (U_1(t)|\Psi\rangle), \tag{5.239}\]

where \(U_2(t)\) is given by Eq.(5.237) and \(U_1(t)\) by Eq.(5.207). It should be noted that the zero mode sector is responsible for the differences between the quantum theory of a massive and massless scalar field.

### 5.3 Configuration space

Let us now argue the existence of distributional field configurations in these quantum field theories. We first consider the quantum theory of the massive scalar field on a 3-torus. Recall that since we have an explicit solution of the system we can construct
an isomorphism from the abstract representation to the coordinate representation as it was done for the harmonic oscillator (see chapter 4). The isomorphism then allows us to work in the field representation where the distributional nature of the fields is transparent. Since all states are constructed by acting on the vacuum state, \( |\psi\rangle \), with the creation operators, \( a_k^\dagger \), it follows that we can restrict our attention to the unnormalized ground state for a massive scalar field theory is equal to

\[
\Psi_0[\phi_K] = \prod_{k \in \mathbb{Z}^3} e^{-\omega_k^2 \phi_k^2} = \exp\left(-\sum_{k \in \mathbb{Z}^3} \omega_k^2 \phi_K^2 \phi_K^* \right),
\]

(5.240)

where \( \omega_k^2 = k^2 + m^2 \). Naively one might believe that since \( \Psi_0[\phi_K] = 0 \) for any field configuration which does not satisfy \( \sum |\phi_K|^2 < \infty \) then one can restrict oneself to such fields. In fact, just the opposite is the case (see Horowitz and Witt [15]). As we have already argued in chapter 2 section 2, when we normalize \( \Psi_0 \) to unity we find that such configurations are set of measure zero in the Fock space inner product. Such a result could have been predicted if one noticed that the normalization constant for the vacuum is infinite and thus the normalized ground state only makes sense if the sum \( \sum |\phi_K|^2 \) diverges.

Now, consider the field configuration which correspond to a delta function, \( \phi(x) = \delta(x) \). The Fourier coefficients of \( \delta(x) \) are all equal to unity, \( \phi_k = 1 \). It then follows from Eq.(2.12) from chapter 2 section 2 that such a field configuration gives a non-trivial contribution to the Fock space inner product. Thus, such a field configuration must be included in the quantum theory. Hence, we have distributional field configurations in this Q.F.T..

We can apply the same arguments to the quantum theory of a free massless scalar field and conclude that distributional fields also exist in that field theory. It should be
noted that such results are well known. For example, the configuration space for two-dimensional field theories can be taken to be the space of all distributions (see Glimm and Jaffe [13]). Note that all these distributional spaces are contractable and thus simply connected. Hence, there are no topological effects in these systems as expected.
Chapter 6

Electromagnetism on a 3-torus

We now consider the quantum theory of the electromagnetic field on a 3-torus. As it was already shown in chapter 2 this system formally has \( \theta \)-states. We will now prove that the rigorous quantum theory of electromagnetism on a 3-torus still has \( \theta \)-states.

6.1 Classical theory of electromagnetism

Our starting point is the Lagrangian formulation of electromagnetism. In this form the equations of motion are obtained from the least action principle applied to the action functional

\[
S = -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu},
\]

where

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

(6.241)

It is readily seen that the action, \( S \), is invariant under the transformations

\[
A_\mu(x) \rightarrow \tilde{A}_\mu(x) = A_\mu(x) - ig^{-1}(x)\partial_\mu g(x),
\]

(6.242)

where \( g(x) \) are mappings from spacetime \( \mathcal{M} = \mathbb{R} \times T^3 \) to \( U(1) \). We call these gauge transformations.

Once we have performed the Legendre transformation and imposed the temporal gauge we are left with a reduced system described by (see Ramond [27]) the Hamiltonian

\[
H = \int d^3x \frac{1}{2} [\tilde{\Pi}^2 + (\tilde{\nabla} \times \tilde{A})^2]
\]

(6.244)
and subject to the constraint
\[ \nabla \cdot \Pi = 0, \quad (6.245) \]
where \( \Pi = -E \). Hence, Eq.(6.245) is simply Gauss' law. One should note that the constraint generates gauge transformations
\[ \delta A(x) = \{ A(x), \int d^3 y \chi(y) \nabla \cdot \Pi(y) \}_P.B. \]
\[ = \nabla \chi(y). \quad (6.246) \]
\[ \delta A(x) = \{ A(x), \int d^3 y \chi(y) \nabla \cdot \Pi(y) \}_P.B. \]
\[ = \nabla \chi(y). \quad (6.247) \]
These are small static gauge transformations.

6.2 Quantization of a constrained system

The method that we will use to quantize the system is a reduced phase space quantization. In this approach we reduce the system to its classically inequivalent degrees of freedom. For the system of interest to us we reduce the system to the set of inequivalent gauge potential, \( A/G \), and then quantize using the prescription described in chapter 2.

Let us describe the procedure to obtain the physical configuration space, \( Q = A/G \). We start by expressing the phase space, \( P \), in term of the Fourier coefficients of \( \tilde{A}(x) \) and \( \Pi(x) \). We then impose the constraint \( \nabla \cdot \Pi = 0 \) and identify any two points of \( P \) which differ by a gauge transformation connected to the identity. Hence, we find the partially reduced phase space, \( P_c = (P|\nabla \cdot \Pi = 0)/G_0 \). This then gives us the partially reduced configuration space, \( Q_c = A/G_0 \). We still don't have the true configuration space since there exists large gauge transformations. To obtain the physical configuration space, \( Q = A/G \), we must find the set of large gauge transformations in terms of modes and then identify any two points in \( Q_c = A/G_0 \) which are related by one of these transformations. Thus, we find that \( Q = Q_c/\sim \), where \( \sim \) denotes the action of the
large gauge transformations. This will be our configuration space for the quantum theory.

Let us now explicitly find the configuration space, $\mathcal{Q} = \mathcal{A}/G$. The first step is to express the vector potential, $\vec{A}(x)$, and its conjugate momentum, $\vec{\Pi}(x)$, in terms of Fourier modes. We have that

$$\vec{A}(x) = \sum_{\vec{k} \in \mathbb{Z}^3} \sum_{\sigma = 1, 2, 3} A_{\vec{k}}^\sigma \hat{u}_{\vec{k}}^\sigma e^{i\vec{k} \cdot \vec{x}} \sqrt{V}$$

and

$$\vec{\Pi}(x) = \sum_{\vec{k} \in \mathbb{Z}^3} \sum_{\sigma = 1, 2, 3} \Pi_{\vec{k}}^\sigma \hat{u}_{\vec{k}}^\sigma e^{i\vec{k} \cdot \vec{x}} \sqrt{V},$$

where $V = (2\pi)^3$ is the volume of space and $\hat{u}_{\vec{k}}^\sigma$ are real ($\hat{u}_{\vec{k}}^\sigma = (\hat{u}_{\vec{k}}^\sigma)^*$) orthonormal basis vectors on the 3-torus. These are defined by the following relations

$$\hat{u}_{\vec{k}}^T \cdot \vec{k} = 0 \quad \text{where} \quad \vec{k} \in \mathbb{Z}^3, \vec{k} \neq 0 \text{ and } T = 1, 2$$

and

$$\hat{u}_{\vec{k}}^L \times \vec{k} = 0 \quad \text{where} \quad \vec{k} \in \mathbb{Z}^3, \vec{k} \neq 0 \text{ and } L = 3.$$

As for the basis vectors for the zero modes, $\hat{u}_{\vec{k}}^0$, they are given by $\hat{x}$, $\hat{y}$ and $\hat{z}$ for $\sigma = 1, 2$ and 3 respectively. To the decomposition, Eqs.(6.248) and (6.249), we add the conditions

$$A_{\vec{k}}^{\sigma \ast} = A_{-\vec{k}}^\sigma, \quad \Pi_{\vec{k}}^{\sigma \ast} = \Pi_{-\vec{k}}^\sigma,$$

and

$$(\hat{u}_{\vec{k}}^\sigma)^* = \hat{u}_{\vec{k}}^\sigma = \hat{u}_{-\vec{k}}^\sigma,$$

in order to ensure that $\vec{A}(x)$ and $\vec{\Pi}(x)$ are real fields. The physical meaning of this decomposition is as follows. The modes $A_{\vec{k}}^T$ with $T = 1, 2$ and $\vec{k} \neq 0$ are the transverse modes which represents the two possible polarization of the photon field. The modes $A_{\vec{k}}^L$ with $\vec{k} \neq 0$ and $L = 3$ represents the longitudinal part of the electromagnetic field which
is purely gauge. The final piece is the zero mode sector, $A^\sigma_0$ with $\sigma = 1, 2$ and 3. It is this piece which carries all the topology of the system.

We now express Gauss' law in terms of modes. Eq.(6.245) in terms of modes is given by

$$\sum_{\vec{k} \in \mathbb{Z}^3} \sum_{\sigma = 1, 2, 3} \Pi_{\vec{k}}^\sigma \vec{k} \cdot \vec{u}_{\vec{k}}^\sigma e^{i\vec{k} \cdot \vec{x}} = 0,$$

which implies

$$\sum_{\vec{k} \neq 0} \Pi_{\vec{k}}^3 \vec{u}_{\vec{k}}^3 e^{i\vec{k} \cdot \vec{x}} = 0,$$

where we have used Eq.(6.250). It then follows that

$$\Pi_{\vec{k}}^L = 0 \quad \text{for all } \vec{k} \in \mathbb{Z}^3, \vec{k} \neq 0 \text{ and } L = 3.$$  \hspace{1cm} (6.256)

Hence, Gauss' law implies that the longitudinal modes of $\bar{\Pi}(x)$ vanish.

Let us now find the transformations generated by the constraints, Eq.(6.256), on the phase space. To accomplish this we must first express the Poisson Brackets in terms of modes. We have that

$$\{A_{\vec{k}}^\sigma, \Pi_{\vec{k}'\rho}'\}^B = \int d^3 x \int d^3 x' \, (\vec{u}_{\vec{k}}^\sigma)_{i} \, (\vec{u}_{\vec{k}'}^\rho)_{j} \, \{A_{i}(x), \Pi_{j}(x')\}^B \frac{e^{i\vec{k} \cdot \vec{x}} e^{i\vec{k}' \cdot \vec{x}'} V}{\sqrt{V} \sqrt{V}}.$$  \hspace{1cm} (6.257)

Using the canonical brackets,

$$\{A_{i}(\vec{x}), \Pi_{j}(\vec{x}')\}^B = \delta_{ij} \delta(\vec{x} - \vec{x}')$$  \hspace{1cm} (6.258)

we find that

$$\{A_{\vec{k}}^\sigma, \Pi_{\vec{k}'\rho}'\}^B = \int d^3 x \, \vec{u}_{\vec{k}}^\sigma \cdot \vec{u}_{\vec{k}'}^\rho \, \frac{e^{i(\vec{k} + \vec{k}') \cdot \vec{x}}}{V}$$  \hspace{1cm} (6.259)

$$= \delta_{\vec{k}, \vec{k}'} \delta_{\rho, \rho'} \delta_{\sigma, \sigma'}.$$  \hspace{1cm} (6.260)
where we have used Eq. (6.253) and the fact that $\hat{u}_k^T$ are orthogonal. We now find that

$$\delta A_k^L = \{A_k^L, \epsilon \Pi_{-k}^L\} = \epsilon,$$

(6.262)

where $\epsilon$ is an arbitrary complex number $\epsilon \in \mathbb{C}$ and $L = 3$. Hence, the constraints, Eq. (6.256), generate arbitrary translations along the longitudinal modes, $A_k^L$ where $\vec{k} \in \mathbb{Z}^3$, $\vec{k} \neq 0$ and $L = 3$.

From Eq. (6.256) and (6.262) we find that the partially reduced phase space, $\mathcal{P}_c$, is given by

$$\mathcal{P}_c = (\mathcal{P}_{|\Pi_{\vec{k}}^T = 0})/\mathcal{G}_0 \quad \text{where } \vec{k} \in \mathbb{Z}^3, \; \vec{k} \neq 0 \; \text{and} \; L = 3$$

(6.263)

and where $\mathcal{G}_0$ is the group generated by the constraints. We can simplify the expression for $\mathcal{P}_c$ by picking a gauge. From Eq. (6.262) it follows that $A_k^L = 0$ for all $\vec{k} \neq 0$ and $L = 3$ is a good gauge condition. It eliminates the unphysical longitudinal part of the electromagnetic field. One should note that this gauge choice is simply Coulomb’s gauge, $\nabla \cdot \vec{A} = 0$. Using this gauge fixing we find

$$\mathcal{P}_c = (\mathcal{P}_{|\Pi_{\vec{k}}^T = 0})/\mathcal{G}_0$$

(6.264)

$$= \mathcal{P}_{|\Pi_{\vec{k}}^T = 0, A_k^L = 0}$$

(6.265)

$$= (A_k^T, \vec{A}_0, \Pi_k^T, \Pi_0)$$

(6.266)

where $T = 1, 2$ and where $\vec{A}_0$ and $\Pi_0$ denote the zero modes. It now follows that the partially reduces configuration space, $\mathcal{Q}_c$, is equal to

$$\mathcal{Q}_c = (A_k^T, \vec{A}_0),$$

(6.267)

where $T = 1, 2$. We must now find the large gauge transformations in order to obtain the physical configuration space, $\mathcal{Q} = \mathcal{A}/\mathcal{G} = \mathcal{Q}_c/\sim$. 

First, notice that we can express any gauge transformation, Eq.(6.246), by

$$\vec{A}'(x) = \vec{A}(x) + \vec{\omega}(x),$$

(6.268)

where $\vec{\omega}$ satisfies

$$\vec{\nabla} \times \vec{\omega} = 0.$$  

(6.269)

One should note that Eq.(6.269) simply states that $\vec{\omega}$ produces a trivial magnetic field, $\vec{B} = 0$. In terms of modes Eq.(6.269) becomes

$$0 = \sum_{k \in \mathbb{Z}^3} \sum_{\sigma = 1,2,3} (i\vec{k} \times \hat{u}^\sigma_k) \omega_k^\sigma \frac{e^{i\vec{k} \cdot \vec{x}}}{V}$$

(6.270)

$$= \sum_{k \neq 0} \sum_{T = 1,2} (i\vec{k} \times \hat{u}^T_k) \omega_k^T \frac{e^{i\vec{k} \cdot \vec{x}}}{\sqrt{V}},$$

(6.271)

where we have used Eq.(6.251) and where $\omega_k^\sigma$ denotes the Fourier coefficients of $\vec{\omega}(x)$. This implies

$$\omega_k^T = 0 \quad \text{for all } \vec{k} \in \mathbb{Z}^3, \; \vec{k} \neq 0 \text{ and } T = 1,2.$$  

(6.272)

Thus, all gauge transformations leave the transverse part of the gauge field invariant.

Let us now express the generator of all gauge transformations, which we denote by $Q$, in terms of modes. We have that $Q$ is equal to

$$Q = \int d^3 x \; \vec{\omega}(x) \cdot \vec{\Pi}(x),$$

(6.273)

where $\vec{\omega}$ satisfies Eq.(6.269). In terms of modes $Q$ becomes

$$Q = \int d^3 x \; \vec{\omega}(x) \cdot \vec{\Pi}(x)$$

(6.274)

$$= \sum_{k,k'} \sum_{\sigma,\sigma'} \int d^3 x \; \hat{u}^\sigma_k \cdot \hat{u}^{\sigma'}_{k'} \; \omega_{k'}^\sigma \Pi_k^{\sigma} \frac{e^{i(\vec{k}+\vec{k}') \cdot \vec{x}}}{V}$$

(6.275)

$$= \sum_{k \in \mathbb{Z}^3} \sum_{\sigma = 1,2,3} \omega_{-k}^\sigma \Pi_k^\sigma$$

(6.276)
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\[ = \sum_{k \neq 0} \omega_{k}^L \Pi_k^L + \sum_{\sigma=1,2,3} \omega_0^\sigma \Pi_0^\sigma \] (6.277)
\[ = \sum_{k \neq 0} \omega_{k}^L \Pi_k^L + \tilde{\omega}_0^L \cdot \tilde{\Pi}_0 \] (6.278)

where we have used Eq.(6.272) and denoted the zero modes of \(\tilde{\omega}(x)\) by \(\tilde{\omega}_0\). If we define \(\chi(x)\) by

\[ \chi(x) = \sum_{k \neq 0} \frac{\omega_{-k}^L}{|k|} e^{ik \cdot x} \] (6.279)

we can rewrite \(Q\) as

\[ Q = \int d^3x \tilde{\nabla} \chi \cdot \tilde{\Pi} + \tilde{\omega}_0^L \cdot \tilde{\Pi}_0. \] (6.280)

Thus, the generator of all gauge transformations splits up into two pieces. The first term is the usual expression for the generator of small gauge transformations. As for the second piece it generates the set of large gauge transformations since these transformations are not generated by Gauss’ law. Note that

\[ \delta \tilde{A}_0 = \{ \tilde{A}_0, Q \}_P.R. = \tilde{\omega}_0. \] (6.281)

Thus, large gauge transformations are given by translations along the zero modes.

We still have not finished the job since \(\tilde{\omega}(x)\) are not arbitrary curl free vector fields and thus \(\tilde{\omega}_0\) are not arbitrary vectors. The vector fields \(\tilde{\omega}(x)\) must be of the form \(\tilde{\omega}(x) = -ig^{-1}(x)\tilde{\nabla} g(x)\), where \(g(x) = e^{i\chi(x)}\) is a single valued function from space, \(T^3\), to \(U(1)\). It is important to note that even though \(g(x) = e^{i\chi(x)}\) is single valued \(\chi(x)\) need not be single valued. The only constraints that \(\chi(x)\) must satisfy are

\[ \chi(0, y, z) = \chi(2\pi, y, z) + n_y \quad \text{where} \quad 2\pi n_y \in \mathbb{Z}, \] (6.282)
\[ \chi(x, 0, z) = \chi(x, 2\pi, z) + 2\pi n_y \quad \text{where} \quad n_y \in \mathbb{Z} \] (6.283)

and

\[ \chi(x, y, 0) = \chi(x, y, 2\pi) + 2\pi n_z \quad \text{where} \quad n_z \in \mathbb{Z}. \] (6.284)
These conditions guarantee that $g(x) = e^{ix(x)}$ is a well defined function from $T^3$ to $U(1)$. Let us now find the allowed values for $\chi(x)$. Since we are working with Coulomb’s gauge we must have that $\tilde{\omega} = -ie^{-ix(x)}\tilde{\nabla}e^{ix(x)}$ satisfies

$$0 = \tilde{\nabla} \cdot \tilde{\omega}(x)$$

$$= \tilde{\nabla} \cdot (-ie^{-ix(x)}\tilde{\nabla}e^{ix(x)})$$

$$= \tilde{\nabla} \cdot \tilde{\nabla} \chi(x),$$

where the last equation only makes sense locally. The general solution of Eq.(6.287) is

$$\chi(x) = n \cdot x + b,$$

where $n$ is a constant vector and $b$ is a constant which we set to zero since it produces only trivial gauge transformation since $\vec{A} \to \vec{A} + \vec{\nabla}b = \vec{A}$. It follows from Eqs.(6.282)-(6.284) that $\vec{n} \in \mathbb{Z}^3$. Hence, large gauge transformations are translations by $\vec{n} \in \mathbb{Z}^3$ on the zero modes, $\vec{A}_0 \to \vec{A}_0 + \vec{n}$. Since $\vec{A}(x)$ is real it follows that $\vec{A}_0 \in \mathbb{R}^3$. Thus, the zero mode sector in the totally reduced configuration space, $\mathcal{A}/\mathcal{G}$, is topologically a 3-torus, $\mathbb{R}^3/\mathbb{Z}^3$. Hence, the zero modes describe a “particle” on a 3-torus (see chapter 4). It should be noted that this torus is equal to the unit cube $I = [0,1] \times [0,1] \times [0,1]$ with opposite sides identified.

It now follows that the true configuration space is equal to

$$\mathcal{Q} = \mathcal{A}/\mathcal{G} = \mathcal{Q}_c/\sim = \mathcal{Q}_c/\mathbb{Z}^3 = (\mathbb{A}_k^T, \vec{A}_0)/\mathbb{Z}^3,$$

where $\mathbb{Z}^3$ is the group of large gauge transformations which is given by the group of lattice translations on the zero modes. Having explicitly found the configuration space we can now compute $\Pi_1(\mathcal{Q})$ and compare the result with the formal computation found in chapter 2. Since all the configuration space, except the zero modes, is contractable it
follows that

\[
\Pi_1(Q) = \Pi_1(A/G) = \Pi_1(A_6/Z^3) = \Pi_1(R^3/Z^3) = Z^3,
\]

where \( Z^3 \) is the group of large gauge transformations. Hence, we find that the same result as the formal calculation where the fields were assumed to be continuous. It should be noted that since we do not assume a strong convergence condition on the Fourier coefficients the fields in the configuration space don't have to be continuous.

Before formally quantizing the system let us take a moment to find the Hamiltonian for the reduced system. We start with the unconstrained Hamiltonian which is given by

\[
H = \int d^3x \frac{1}{2} (\Pi^2 + (\nabla \times \vec{A})^2).
\]

The kinetic energy expressed in terms of modes is given by

\[
\frac{1}{2} \int d^3x \Pi^2 = \frac{1}{2} \sum_{k,k',\sigma,\sigma'} \Pi^\sigma_k \Pi'^\sigma_{k'} \hat{u}^\sigma_k \cdot \hat{u}^{\sigma'}_{k'} \int d^3x \frac{e^{i(k \cdot \vec{x})}}{V}
\]

As for the potential energy of \( H \) it becomes

\[
\frac{1}{2} \int d^3x (\vec{\nabla} \times \vec{A})^2 = \frac{1}{2} \sum_{k,k',\sigma,\sigma'} A^\sigma_k A'^\sigma_{k'} (i\vec{k} \times \hat{u}^\sigma_k) \cdot (i\vec{k}' \times \hat{u}^{\sigma'}_{k'}) \int d^3x \frac{e^{i(k \cdot \vec{x})}}{V}
\]
Using the identity
\[(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})\] (6.300)
we find that
\[
\frac{1}{2} \int d^3x \ (\vec{\nabla} \times \vec{A})^2 = \frac{1}{2} \sum_{\kappa} \sum_{\sigma, \sigma'} A^\sigma_{-\kappa} A^{\sigma'}_\kappa \{ (\vec{k} \cdot \vec{\kappa})(\dot{\vec{u}}_{-\kappa}^{\sigma'} \cdot \dot{\vec{u}}_\kappa^{\sigma}) - (\vec{k} \cdot \dot{\vec{u}}_\kappa^{\sigma})(\vec{k} \cdot \dot{\vec{u}}_{-\kappa}^{\sigma'}) \} \tag{6.301}
\]
which implies
\[
\frac{1}{2} \int d^3x \ (\vec{\nabla} \times \vec{A})^2 = \frac{1}{2} \sum_{\kappa} \sum_{\sigma, \sigma'} A^\sigma_{-\kappa} A^{\sigma'}_\kappa (\vec{k}^2 - \vec{\kappa}^2 \delta_{\sigma, 3}) \tag{6.302}
= \frac{1}{2} \sum_{\kappa \neq 0} \sum_{T=1, 2} \vec{k}^2 A^T_{-\kappa} A^T_\kappa. \tag{6.303}
\]
Hence, only the transverse part of the photon field has non-vanishing potential energy.

Using Eqs.(6.297), (6.303) and imposing the constraints Eq.(6.256) we find that the Hamiltonian for the reduced system is given by
\[
H = \sum_{\kappa \neq 0} \sum_{T=1, 2} \frac{1}{2} (\Pi^T_{-\kappa} \Pi^T_\kappa + \vec{k}^2 A^T_{-\kappa} A^T_\kappa) + \frac{1}{2} \vec{\Pi}_0^2. \tag{6.304}
\]
Hence, the reduced Hamiltonian describes two massless scalar fields and three zero modes.

6.3 Formal quantization

We now formally quantize the electromagnetic field on a 3-torus. The 3-torus is given by the cube \( I = [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \) with opposite sides identified. Let us first recall the procedure for quantizing a system with a multiply connected configuration space, \( \Pi_1(Q) \neq 0 \). The first step is to lift up observables and wave functions to the universal covering, \( \hat{Q} \). Observables are functions on \( \hat{Q} \) which are invariant under the action of \( \Pi_1(Q) \). Wave functions are functions on \( \hat{Q} \) which transform under some U.I.R., \( \Gamma \), of
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\( \Pi_1(\mathcal{Q}) \). One then quantizes the system as usual.

In the previous section we showed that \( \Pi_1(\mathcal{Q}) = \mathbb{Z}^3 \), where \( \mathbb{Z}^3 \) is the group of lattice translations acting on the zero modes, \( \vec{A}_0 \). Since

\[
\mathcal{Q} = (A_k^T, \vec{A}_0)/\mathbb{Z}^3,
\]

where \( \mathbb{Z}^3 \) acts freely, it follows that the universal covering of \( \mathcal{Q} \) is equal to

\[
\hat{\mathcal{Q}} = (\vec{A}_0, A_k^T).
\]

One should note that \( \hat{\mathcal{Q}} = \mathcal{Q}_c \), that is, the universal covering of \( \mathcal{Q} \) is equal to the partially reduced configuration space. Observables like the Hamiltonian are functions on \( \hat{\mathcal{Q}} \) which are invariant under large gauge transformations. State vectors are functionals on \( \hat{\mathcal{Q}} \), \( \Psi = \Psi[A_k^T, \vec{A}_0] \), which transform under some U.I.R., \( \Gamma \), of \( \Pi_1(\mathcal{Q}) = \mathbb{Z}^3 \) (see chapter 4). The action of \( \Pi_1(\mathbb{Z}^3) \) on \( \hat{\mathcal{Q}} \) is given by

\[
\Pi_1(\mathcal{Q}) : \hat{\mathcal{Q}} \to \hat{\mathcal{Q}} \quad (A_k^T, \vec{A}_0) \to (A_k^T, \vec{A}_0 + \vec{n}) \quad \text{where} \quad \vec{n} \in \mathbb{Z}^3.
\]

Thus, wave functions satisfy the transformation law

\[
U_{\vec{n}} \Psi[\bar{A}_k^T, \vec{A}_0] = e^{i\vec{\theta} \cdot \vec{n}} \Psi[\bar{A}_k^T, \vec{A}_0 + \vec{n}],
\]

where \( \vec{\theta} \in [0, 2\pi] \times [0, 2\pi] \times [0, 2\pi] \) and where \( U_{\vec{n}} \) is the formal unitary operator which implements the large gauge transformations.

Quantization is formally accomplished by mapping the classical variables \( \Pi_k^T, A_k^T, \vec{A}_0 \) and \( \vec{A}_0 \) to operators which are formally given by

\[
\hat{\Pi}_k^T = -i \frac{\partial}{\partial A_k^T}, \quad \hat{A}_k^T = A_k^T
\]
and
\[
\hat{\Pi}_0 = -i \frac{\partial}{\partial \hat{A}_0}, \quad \hat{A}_0 = \tilde{A}_0,
\] (6.311)
where we have used Eqs. (6.252) and (6.261).

Notice that the Hamiltonian is equal to two infinite sums of (complex) harmonic oscillators plus a purely kinetic term for the zero modes,
\[
H = \sum_{T=1,2} \sum_{k \neq 0} H_k^T + \frac{1}{2} \hat{\Pi}_0^2,
\] (6.312)
where
\[
H_k^T = \frac{1}{2}(\hat{\Pi}_k^T \hat{\Pi}_k^T + \tilde{k}^2 \hat{A}_k^T \hat{A}_k^T \ast). \quad (6.313)
\]
Note that \(H\) is similar to the Hamiltonian for a free massless scalar field on a 3-torus.

Because of the form of the Hamiltonian, Eq. (6.312), the eigenstates of \(H\) are of the form
\[
\Psi[A_k^T, \hat{A}_0] = \psi_\theta(\hat{A}_0) \Psi_n[A_k^T],
\] (6.314)
where \(\psi_\theta\) is an eigenstate of \(\frac{1}{2} \hat{\Pi}_0^2\) and \(\Psi_n\) is an eigenstate of \(\sum H_k^T\).

We now formally find the eigenvectors and eigenvalues of \(H_k^T\). This will be accomplished exactly the same way it was done for the free field theories (see chapter 5). We introduce the creation and annihilation operators, \(a_k^T\) and \(a_k^{T\dagger}\). They are defined by
\[
a_k^T = \frac{1}{\sqrt{2|\vec{k}|}} (|\vec{k}| A_k^T + i \Pi_k^T), \quad \vec{k} \in \mathbb{Z}^3, \ \vec{k} \neq 0 \text{ and } T = 1, 2,
\] (6.315)
and
\[
a_k^{T\dagger} = \frac{1}{\sqrt{2|\vec{k}|}} (|\vec{k}| A_{-k}^T - i \Pi_{-k}^T), \quad \vec{k} \in \mathbb{Z}^3, \ \vec{k} \neq 0 \text{ and } T = 1, 2.
\] (6.316)
In terms of these new variables the regularized Hamiltonian (see chapter 5) is given by

\[ H = \sum_{T=1,2} \sum_{\vec{k} \neq 0} H_{\vec{k}}^T + \frac{1}{2} \hat{\Pi}_0^2, \]

with

\[ H_{\vec{k}}^T = |\vec{k}|a_{\vec{k}}^T \dagger a_{\vec{k}}^T. \]

As for the commutation relations they become

\[ [a_{\vec{k}}^T, a_{\vec{k}'}^T \dagger] = 0, \quad [a_{\vec{k}}^T, a_{\vec{k}'}^T] = 0 \]

and

\[ [a_{\vec{k}}^T, a_{\vec{k}'}^T \dagger] = \delta_{\vec{k},\vec{k}'} \delta_{T,T'}. \]

Repeating the construction of the eigenstates for the harmonic oscillator in the coordinate representation we obtain the eigenstates of \( H_{\vec{k}}^T \). The ground state is formally given by

\[ a_{\vec{k}}^T \psi_0(A_{\vec{k}}^T) = 0. \]

Using Eqs.(6.310) and (6.315) we obtain the differential equation

\[ (|\vec{k}|A_{\vec{k}}^T + \frac{\partial}{\partial A_{\vec{k}}^T})\psi_0(A_{\vec{k}}^T) = 0, \]

which has solution

\[ \psi_0(A_{\vec{k}}^T) = e^{-|\vec{k}|A_{\vec{k}}^T A_{\vec{k}}^T \dagger}. \]

We have set the normalization factor in Eq.(6.323) to one for convenience. The other eigenstates of \( H_{\vec{k}}^T \) are obtained by acting on \( \psi_0 \) with the creation operator, \( a_{\vec{k}}^T \dagger \). We find that the unnormalized eigenstates of \( H_{\vec{k}}^T \) are

\[ \psi_n(A_{\vec{k}}^T) = (a_{\vec{k}}^T \dagger)^n \psi_0(A_{\vec{k}}^T). \]
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It now follows that the unnormalized eigenstates of

\[ H^T = \sum_{\vec{k} \neq 0} H^T_{\vec{k}} \]  

(6.325)

are given by

\[ \Psi_n^T = a_{\vec{k}_1}^T \ldots a_{\vec{k}_n}^T \Psi_0^T, \]  

(6.326)

where

\[ \Psi_0^T = \prod_{\vec{k} \neq 0} e^{-i|\vec{k}|^2 A_{\vec{k}}^T A_{\vec{k}}^T}. \]  

(6.327)

is the unnormalized ground state of \( H^T \). It should be noted that there are no restrictions on \( \vec{k}_i \) except that \( \vec{k}_i \neq 0 \) for all \( i = 1, \ldots, n \).

We now find the eigenstates for the zero mode sector of the Hamiltonian. The zero mode contribution to the Hamiltonian is

\[ H_0 = \frac{1}{2} \vec{\Pi}_0^2, \]  

(6.328)

where \( \vec{\Pi}_0 \) is defined by Eq.(6.311). Thus, \( H_0 \) describes a free particle on a 3-torus since \( \vec{A}_0 \in \mathbb{R}^3/\mathbb{Z}^3 \). Let us recall the analysis from chapter 4. Wave functions are functions on \( \mathbb{R}^3 \), the universal covering of \( \mathbb{R}^3/\mathbb{Z}^3 \), which transform under some U.I.R., \( \Gamma_{\vec{\theta}} \), of \( \mathbb{Z}^3 \). Hence, wave functions satisfy the transformation law

\[ \mathcal{U}_{\vec{\theta}} \psi_{\vec{\theta}}(\vec{A}_0) = \psi_{\vec{\theta}}(\vec{A}_0 + \vec{n}) = e^{i\vec{\theta} \cdot \vec{n}} \psi_{\vec{\theta}}(\vec{A}_0). \]  

(6.329)

From this it follows that the Hilbert space for this system is equal to \( L^2_\theta(\mathbb{R}^3) \), the space of square integrable functions which have transformation law Eq.(6.329). Since the Hamiltonian for the zero modes is given by \( H_0 = -\frac{1}{2} \frac{\partial^2}{\partial A_0^2} \) it follows that the eigenstates are given by

\[ \phi_{\vec{K}}(\vec{A}_0) = \frac{e^{i(\vec{\theta} + \vec{K}) \cdot \vec{A}_0}}{(2\pi)^{2/3}} \quad \text{where} \quad \vec{K} \in \mathbb{Z}^3, \]  

(6.330)
and have eigenvalues $\frac{1}{2}(\tilde{\theta} + \tilde{K})^2$. Note that the eigenfunctions $\phi_{\tilde{K}}$ are not normalizable states but still form a basis for $L^2_{\tilde{\theta}}(\mathbb{R}^3)$. It should be noted that we cannot take linear superpositions of wave functions which transform under different U.I.R. of $\mathbb{Z}^3$. Hence, wave functions with different values of $\tilde{\theta}$ cannot be superposed. This contrasts the case of a massless scalar field where we can take linear superpositions of states with different values of $\theta$.

It now follows that the unnormalized eigenstates of $H$ are given by

$$\Psi_{n,m,\tilde{K}} = a_{\tilde{k}_1}^{\dagger} \ldots a_{\tilde{k}_n}^{\dagger} a_{\tilde{m}_1}^{\dagger} \ldots a_{\tilde{m}_m}^{\dagger} e^{i\tilde{K} \cdot \tilde{A}_5} \Psi_{\tilde{\theta}},$$

(6.331)

where

$$\Psi_{\tilde{\theta}} = e^{i\tilde{\theta} \cdot \tilde{A}_5} \prod_{T=1,2} \prod_{\tilde{k} \neq 0} e^{-i|\tilde{k}| A^T_{\tilde{k}} A^T_{\tilde{k}}},$$

(6.332)

is the unnormalized ground state. By ground state we mean state with no particle excitations. For each values of $\tilde{\theta} \in [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi)$ we have distinct vacuums.

It should be noted that $\Psi_{\tilde{\theta}}$ is formally a tensor product of states.

### 6.4 Rigorous quantization

We now rigorously construct the quantum theory of electromagnetism on a 3-torus. The first step is the build a Hilbert space on which we can implement the operator algebra and find a self-adjoint Hamiltonian.

The Hilbert space for the system is the tensor product of $L^2_{\tilde{\theta}}(\mathbb{R}^3)$ (see chapter 4) with two Hilbert spaces generated by the basis vectors $|\tilde{k}_1, ..., \tilde{k}_n >_1$ and $|\tilde{k}_1, ..., \tilde{k}_m >_2$ (see chapter 5). The basis states $|\tilde{k}_1, ..., \tilde{k}_n >_T$ are symbolically defined by

$$a_{\tilde{k}_1}^{T\dagger} \ldots a_{\tilde{k}_n}^{T\dagger} |0 >_T = |\tilde{k}_1, ..., \tilde{k}_n >_T,$$

(6.333)
where $|0\rangle_T$ represents the ground state. One should note that the basis vectors are by definition invariant under permutations.

We define $\Lambda_T$ exactly the same way $\Lambda$ was defined for the massless scalar field theory (see chapter 5). Let $\Lambda_T$ be the set of all finite linear combinations of the basis states $|\vec{k}_1, ..., \vec{k}_n\rangle_T$. Define an inner product on $\Lambda_T$ by

$$
(|\vec{q}_1, ..., \vec{q}_n\rangle_T, |\vec{k}_1, ..., \vec{k}_m\rangle_T)_{\Lambda_T} = \delta_{n,m} \sum_{\sigma \in S_n} \delta_{\vec{q}_1, \vec{k}_{\sigma(1)}} \cdots \delta_{\vec{q}_n, \vec{k}_{\sigma(n)}}
$$

(6.334)
on the basis vectors and extend by linearity to all of $\Lambda_T$. We define the Hilbert space $\overline{\Lambda}_T$ as the Cauchy completion of $\Lambda_T$ with this inner product.

The Hilbert space for the system is the tensor product of $L^2_\theta(\mathbb{R}^3)$ with $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$, that is,

$$
\mathcal{H} = L^2_\theta(\mathbb{R}^3) \otimes \overline{\Lambda}_1 \otimes \overline{\Lambda}_2.
$$

(6.335)

A general element in $\mathcal{H}$ is of the form

$$
\psi_\theta \otimes |\Psi\rangle_1 \otimes |\Psi\rangle_2,
$$

(6.336)

where

$$
\psi_\theta = \sum_{\vec{K} \in \mathbb{Z}^3} c_{\vec{K}} e^{i(\theta + \vec{K}) \cdot \vec{A}_\theta} (2\pi)^{2/3}
$$

(6.337)

with

$$
\sum_{\vec{K} \in \mathbb{Z}^3} |c_{\vec{K}}|^2 < \infty,
$$

(6.338)

and where

$$
|\Psi\rangle_T = \sum_{n=0}^{\infty} \sum_{\vec{k}_1, ..., \vec{k}_n \neq 0} \lambda_{\vec{k}_1, ..., \vec{k}_n}^T |\vec{k}_1, ..., \vec{k}_n\rangle_T
$$

(6.339)

with

$$
\sum_{n=0}^{\infty} \sum_{\vec{k}_1, ..., \vec{k}_n \neq 0} |\lambda_{\vec{k}_1, ..., \vec{k}_n}^T|^2 < \infty \quad T = 1, 2.
$$

(6.340)
Having found the Hilbert space for the system we now implement the operator algebra, Eqs.(6.319)-(6.320) and (6.261), as it was done for the quantum theory of a free massless scalar field (see chapter 5). We define the creation operators on $\mathcal{H}$ by $I \otimes a^T_k$, where $I$ is the identity operator on the tensor product of $L^2_\phi(\mathbb{R}^3)$ with one of the copies of $\Lambda_T$ and where $a^T_k$ is defined by Eq.(6.333) on the other copy of $\Lambda_T$. We choose $\mathcal{D} = \mathcal{D}_\phi \otimes \Lambda_1 \otimes \Lambda_2$ as their domain, where $\mathcal{D}_\phi$ is defined in chapter 4. It follows from the analysis from chapter 5 that $\mathcal{D}$ is dense in $\mathcal{H}$ and that the adjoint of $I \otimes a^T_k$ is equal to $I \otimes a^T_k$, where $a^T_k$ is defined by Eqs.(5.192) and (5.193) from chapter 5. It now follows that the operator algebra for the creation and annihilation operators is satisfied on $\mathcal{D}$. We define the operators $\hat{\Pi}_\phi$ and $\hat{A}_\phi$ on $\mathcal{H}$ by $\hat{\Pi}_\phi \otimes I$ and $\hat{A}_\phi \otimes I$, where $\hat{\Pi}_\phi$ and $\hat{A}_\phi$ are defined by Eq.(6.311) and where $I$ is the identity operator on $\Lambda_1 \otimes \Lambda_2$. We choose $\mathcal{D}$ as their domain. It follows from Eq.(6.311) that $\hat{\Pi}_\phi \otimes I$ and $\hat{A}_\phi \otimes I$ are well defined on $\mathcal{D}$ and satisfy the canonical commutation relations. Hence, the operator algebra is satisfied on $\mathcal{D}$.

The next step is to find a self-adjoint Hamiltonian. This is accomplished by first finding an essentially self-adjoint Hamiltonian and then taking its closure. Notice that the Hamiltonian, $H = \sum_{T=1,2} H^T + H_\phi$, is well defined on $\mathcal{D}$ since $H_\phi$ is well defined on $\mathcal{D}_\phi$ and $H^T$ is well defined on $\Lambda_T$. Let us now show that the Hamiltonian is essentially self-adjoint on $\mathcal{D}$.

**Theorem 1** The Hamiltonian for the electromagnetic field on a 3-torus is essentially self-adjoint on $\mathcal{D}$.

**Proof:** To prove the theorem we repeat the arguments of theorem (1) from chapter 5 section 2.2. We first show that $H_\phi$ and $H^T$ are essentially self-adjoint on $\mathcal{D}_\phi$ and $\Lambda_T$ respectively and then use theorem (1) from chapter 5 section 1.2 to conclude that $H = H^1 + H^2 + H_\phi$ is essentially self-adjoint on $\mathcal{D} = \mathcal{D}_\phi \otimes \Lambda_1 \otimes \Lambda_2$. Recalling theorem
(1) from chapter 4 section 3.1 it follows that \( H_\delta \) is essentially self-adjoint on \( D_\delta \). Repeating the proof of theorem (1) from chapter 5 section 1.3 it follows that \( H^T \) is essentially self-adjoint on \( \Lambda_T \). Thus, the Hamiltonian \( H \) is essentially self-adjoint on \( \mathcal{D} \).

Now that we have an essentially self-adjoint operator \( H = \sum_{T=1,2} H^T + H_\delta \), we take its closure, \( \bar{H} = \sum_{T=1,2} \bar{H}^T + \bar{H}_\delta \) and obtain a self-adjoint Hamiltonian. We can now conclude that we have a well defined quantum theory for the electromagnetic field on a 3-torus. Self-adjointness of \( \bar{H} \) guarantees that the system has a well defined energy spectrum and evolves unitarily. Repeating the arguments used for the harmonic oscillator it follows that the eigenvectors and eigenvalues of \( \bar{H} \) and \( H \) are the same.

We now give the expression for the evolution operator. Repeating the arguments given for the massless scalar field theory it follows that \( \mathcal{U}(t) = \mathcal{U}_\delta(t) \otimes \mathcal{U}_1(t) \otimes \mathcal{U}_2(t) \), where \( \mathcal{U}_\delta(t), \mathcal{U}_1(t) \) and \( \mathcal{U}_2(t) \) are the exponentials of \( H_\delta, H^1 \) and \( H^2 \) respectively. Hence,

\[
\mathcal{U}(t) \left( \psi_\delta \otimes |\Psi_1 \rangle \otimes |\Psi_2 \rangle \right) = \left( \mathcal{U}_\delta(t) \psi_\delta \right) \otimes \left( \mathcal{U}_1(t) |\Psi_1 \rangle \right) \otimes \left( \mathcal{U}_2(t) |\Psi_2 \rangle \right),
\]

where \( \mathcal{U}_\delta \) is defined by Eq.(4.134) from chapter 4 and \( \mathcal{U}_T(t) \) is defined by Eq.(5.207) from chapter 5. It should be noted that the evolution operator depends explicitly on \( \vec{\theta} \).

As a final step in our rigorous construction we define \( \mathcal{U}_\eta \), the unitary operator which implements the large gauge transformations, by \( \mathcal{U}_\eta \otimes I \), where \( I \) is the identity operator on \( \Lambda_1 \otimes \Lambda_2 \) and where \( \mathcal{U}_\eta \) is defined by

\[
\mathcal{U}_\eta \psi_\delta(A_\delta) = \psi_\delta(A_\delta + \vec{n}) \quad \vec{n} \in \mathbb{Z}^3
\]

for all \( \psi_\delta \in L^2_\delta(\mathbb{R}^3) \). Recalling the arguments for the free particle on a 3-torus (see chapter 4) it follows that \( \mathcal{U}_\eta \) is a well defined unitary operator on \( L^2_\delta(\mathbb{R}^3) \). Hence, \( \mathcal{U}_\eta \otimes I \) is a
well defined unitary operator on $\mathcal{H}$. Thus, we have unitary implementation of the group of large gauge transformations.

It should be noted that we can incorporate the effects of $\bar{\theta}$ into the Hamiltonian as it was done for the free particle on a 3-torus (see chapter 4). We find that

$$H_{\bar{\theta}} = \sum_{T=1,2} \sum_{k \neq 0} |k| a_k^T \dagger a_k^T + \frac{1}{2} (\hat{\Pi}_0 + \bar{\theta})^2. \quad (6.343)$$

Using Eqs.(6.315) and (6.316) we obtain

$$H_{\bar{\theta}} = \sum_{k \neq 0} \sum_{\sigma=1,2,3} \frac{1}{2} (\Pi_0^\sigma \Pi_k^\sigma + k^2 A_0^\sigma A_k^\sigma) + \sum_{\sigma=1,2,3} \frac{1}{2} (\Pi_0^\sigma + (\bar{\theta})^\sigma)^2 \quad (6.344)$$

after reinstating the longitudinal modes and undoing the normal ordering. This last expression can be rewritten in terms of $\bar{\Pi}$ and $\bar{A}$. Using Eqs.(6.248)-(6.253) and the fact that $\bar{\theta}$ is a constant vector it follows that

$$H_{\bar{\theta}} = \int d^3x \frac{1}{2} (\bar{\Pi} + \bar{\theta})^2 + \frac{1}{2} (\bar{\nabla} \times \bar{A})^2. \quad (6.345)$$

This expression is analogous to that found in Yang-Mills theories on flat space $\mathbb{R}^3$. In that case the modified Hamiltonian is (see Jackiw [8])

$$H_{\theta} = \int d^3x \frac{1}{2} (\Pi_a - \frac{\theta g^2}{8\pi^2} \nabla \times A_a)^2 + \frac{1}{2} (\nabla \times A_a)^2. \quad (6.346)$$

It should be noted that, even though the effects of $\bar{\theta}$ is to change wave functions by a phase factor, when interactions are turned on the effects of $\bar{\theta}$ can be observed. This is analogous to the effects due to the crystal momentum (see chapter 4).

### 6.5 Topological term

Let us now find how the classical action is effected by $\bar{\theta}$. We expect that the effects of $\bar{\theta}$ appear in the Lagrangian by the addition of a new term which is (at least locally) a
total derivative since these effects are purely quantum mechanical. In the usual physics jargon this term is called the topological term. It should be noted that this term need not be a topological invariant of the spatial manifold, that is, invariant under metric perturbations. It only needs to be a total derivative. In the case of Yang-Mills theories on $\mathbb{R}^3$ the additional term is a topological invariant.

It follows from the analysis found in chapter 2 section 1 that the topological term for a free particle on a unit circle, $S^1 = [0, 1]$ with the end points identified, is equal to

$$S_{\text{topological}} = \int dt \, \theta \dot{q}. \quad (6.347)$$

This implies that the topological term for our “particle” on a 3-torus, $\mathbb{R}^3/\mathbb{Z}^3 = S^1 \times S^1 \times S^1$, is equal to

$$S_{\text{topological}} = \int dt \bar{\theta} \cdot \vec{A}_{\bar{\theta}}. \quad (6.348)$$

Note that we can rewrite $S_{\text{topological}}$ as

$$S_{\text{topological}} = -\bar{\theta} \cdot \int d^4 x \vec{E}(x). \quad (6.349)$$

This follows from

$$-\bar{\theta} \cdot \int d^4 x \vec{E}(x) = \int dt \int d^3 x \, \bar{\theta} \cdot \frac{d\vec{A}}{dt}(x) \quad (6.350)$$

$$= \int dt \frac{d}{dt} \left( \int d^3 x \, \bar{\theta} \cdot \vec{A}(x) \right) \quad (6.351)$$

$$= \int dt \frac{d}{dt} (\bar{\theta} \cdot \vec{A}_{\bar{\theta}}) \quad (6.352)$$

$$= \int dt \, \bar{\theta} \cdot \vec{A}_{\bar{\theta}}. \quad (6.353)$$

It should be noted that since the mathematical structure of the topological effects is the same as a free particle on a 3-torus it follows that $\bar{\theta}$ has observable effects when coupled to other fields, that is, when interactions are turned on. Note that for Yang-Mills theories $\theta$ has physical effects when coupled to Fermions (see Jackiw [8]).
Chapter 7

Conclusion

In chapters 2 through 5 we developed the necessary tools to construct the rigorous quantum theory of the electromagnetic field on a 3-torus. This system, which was studied in chapter 6, is an example where \( \theta \)-vacuums survive rigorous quantization and hence provides strong evidence that \( \theta \)-vacuums exist in Yang-Mills theories. Similar results have already been found in Yang-Mills theories in 1+1-dimensions (see Chandra and Ercolessi [28]). But in that case once the system is reduced to its true degrees of freedom one is left with a finite dimensional system. Hence, the system is no longer a field theory, it is only a quantum mechanical system. Thus, this does not answer the question: Do topological effects exist in quantum field theories? Since the distributional nature of the fields is lost in the reduction. The system that we have considered is a field theory where distributional field configurations do exist. Hence, our work provides an example where rigorous topological effects exist in a quantum field theory.

Even though the existence of \( \theta \)-vacuums in electromagnetism provides evidence of the presence of \( \theta \)-vacuums in the quantum theory of Yang-Mills systems there is no guarantee \textit{a priori} that this will be the case. One argument that supports the claim that \( \theta \)-vacuums do survive rigorous quantization is as follows. In all known interacting field theories the ground state of the interacting theory coincides with the vacuum of the free theory (see Glimm and Jaffe [13]). Since Yang-Mills theories with interactions turned off is described by several copies of a \( U(1) \) gauge theories (electromagnetism), one for each generator of
the gauge group $G$, it then follows that the $\theta$-state structure, which does exist in the rigorous quantum theory of electromagnetism, can provide $\theta$-vacuums for the interacting Yang-Mills theory. Hence, our example of rigorous $\theta$-states is a strong indicator that $\theta$-vacuums are present in Q.C.D..

There are still many questions which remain unanswered. We showed using a reduced phase space quantization that $\theta$-states survive rigorous quantization. It was the domains of our operators which carried the topology of the system in the quantum theory. If one performs a Dirac quantization a natural question to ask is: Do the domains of the operators still carry the topology of the system? Recall that the Hilbert spaces for Dirac quantization and reduced phase space quantization are different and thus the domains of operators in these different approaches don't coincide. Hence, it is not obvious that $\theta$-states exist in a rigorous Dirac quantization. The same question arises if one uses a Gupta-Bleuler quantization or B.R.S.T. quantization. One should recall that different quantization procedures can lead to different answers (see Schleich [21] and Hetrick[29]). Thus, there is no guarantee that all these different approaches will produce the same $\theta$-state structure.

Another interesting problem is to rigorously prove that the partition function for a topological field theory is a topological invariant of the underlying spacetime manifold (see Witten [30]). Recall that continuous field configurations are typically set of measure zero in the path integral measure. Thus, the fields which contribute to the path integral are at least discontinuous and thus do not directly detect the topology of the spacetime manifold. A natural question to ask is: How can the partition function carry information about the topology of the spacetime manifold? From our work we have learned that
the topology of the system is carried in the domains of the operators. Since the rigorous construction of the path integral representation starts with the rigorous canonical quantization of the system it then follows that our work provides a starting point for the rigorous construction of topological invariants via topological field theories.

In conclusion we hope to address some of these questions in future research and report our progress in future publications.
Bibliography


