The Dressed Oscillator Approach and Particle Creation in Two Simple Models of a Friedmann-Robertson-Walker Universe

by

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Abstract

In the First part of this thesis I look at the Algebraic Method which is a very straightforward technique. The idea behind the Algebraic Method is to generate all the states of a quantum system beginning with a well defined base state, generally the lowest energy state, through successive application of a creation operator (also known as a raising operator) which modifies the lowest energy state in such a fashion as to then characterize the rest of the spectrum of the system. The lowest energy state is defined as the state that is annihilated by the annihilation operator (also known as the lowering operator). Several examples of the Algebraic technique are presented including Landau Levels.

In the Second part of this thesis I look at several examples of Unitary Similarity Transformations and how they can be used to simplify Hamiltonians describing quantum systems. Examples of the Similarity Transformation Method discussed in this thesis include a method to determine the ground state eigenfunction using a generating function, Electron-Spin Resonance, the Foldy and Wouthuysen Transformation and an approach first proposed by Wentzel and applied by Schwinger to describe the non-relativistic interaction of an electron with a field. Schwinger used this approach to solve for the Lamb shift of the electron in a central coulombic potential.

In the Third Part of this thesis I look at the Bogoliubov Transformation which can be used for Diagonalizing a Quadratic Bosonic Hamiltonian.

In the Fourth Part I describe the coupling between a non-relativistic system of oscillators coupled linearly to a scalar field in ordinary Euclidean 3-space. From a physical point of view we give a nonperturbative treatment to the oscillator radiation introducing some coordinates that permit us to divide the coupled system into two parts, the "dressed" oscillator and the field. I also look at how one can describe transitions due to a forcing function.

The first four sections of this thesis build up the mathematical tools, namely the Algebraic Method, the Bogoliubov transformation and the "dressed" oscillator approach, for Part Five in which I look at uniform acceleration n Rindler space, particle creation in two simple models of a Friedmann-Robertson-Walker Universe, as well as a hypothesis that Gravity is an Induced Quantum Effect.

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Chapter 1

Introduction

The investigation of most quantum systems leads to the solution of the Schrödinger equation with a well chosen interaction Hamiltonian or potential. Unfortunately, exact solutions for the Schödinger equation are known for a rather restricted set of interaction Hamiltonian or potentials, so the standard problem we are faced with is to find a good approximation in place of an exact solution.

Confronted with a challenge of finding an approximate solution, the goal becomes to obtain a representation $\hat{H} = \hat{H}_0 + \hat{H}_{int}$, in which $\hat{H}_0$ describes a known physical system with characteristics close to $\hat{H}$, and the interaction Hamiltonian $\hat{H}_{int}$ provides a correction to the Hamiltonian $\hat{H}_0$.

To describe a quantum system means choosing a Hilbert space of states on which the canonical variables are defined as operators. In turn, this means that definite representations of the canonical (or anti-) commutation relations has been chosen. For the case of quantum systems with a denumerably finite number of degrees of freedom, all representations are unitary equivalent to each other. This fact may be used to our advantage when attempting to solve the Schrödinger equation for the quantum system.

Many of our descriptions of quantum systems have been influenced by the Quantum Harmonic Oscillator (QHO). For quite a wide range of quantum systems, it is valid to look for an initial approximation in the form of an oscillator basis, that is, a stable quantum system in a well chosen representation can be described by some set of harmonic oscillators with a spectrum of frequencies.

Many systems may be treated as a set of oscillators with a frequency defined by a mass parameter. The interaction does not change the oscillator nature of the underlying quantum field, but only redefine their masses and other physical characteristics.

The use of the Algebraic Method and Unitary Similarity Transformations in quantum mechanics has proven to be useful particularly when dealing with systems with discrete spectrum.

In the First part of this thesis I look at the Algebraic Method which is a very straightforward technique. The idea behind the Algebraic Method is to generate all the states of the system beginning with a well defined base state, generally the lowest energy state, through successive application of a creation operator (also known as a raising operator) which modifies the lowest energy state in such a fashion as to then characterize the rest of the spectrum of the system. The lowest energy state is defined as the state that is annihilated by the annihilation operator (also known as the lowering operator). The methods outlined in this part of the thesis are by no means the only methods to characterize the eigenfunctions of quantum systems. Other techniques such as the *Factorization Method* of Hull and Infeld [1] may be used, as well as a number of more specialized techniques.

In the Second part of this thesis I look at Unitary Similarity Transformations and how they can be used to simplify Hamiltonians describing quantum systems. Examples of the Similarity Transformation Method discussed in this thesis include a method to determine the ground state eigenfunction using a generating function, Electron-Spin Resonance , the Foldy and Wouthuysen Transformation and an approach first proposed by Wentzel and applied by Schwinger to describe the non-relativistic interaction of an electron with a field. Schwinger used this approach to solve for the Lamb shift of the electron in a central coulombic potential.

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Part I

The Algebraic Method

Chapter 2

Introduction

The idea behind the Algebraic Method is to generate all the states of the system beginning with a well defined base state, generally the lowest energy state, through successive application of a creation operator (also known as a raising operator) which modifies the lowest energy state in such a fashion as to then characterize the rest of the system. The lowest energy state is defined as the state that is annihilated by the annihilation (lowering) operator.

Key to the application of the Algebraic Method is the proper formulation of the creation operator, the operator algebra and the formulation of the fundamental state. The Algebraic Method amounts essentially to the replacement of a second-order differential equation by equivalent products of first-order equations from which the appropriate creation operators are identified.

The archetypical discrete system that is built up using the Algebraic Method is the Quantum Harmonic Oscillator (QHO) which was first proposed by P.A.M. Dirac early in the development of Quantum Mechanics. [2]

The Algebraic Method has also been applied to other discrete quantum systems. The example of a Charged Particle in a Magnetic Field and how one arrives at the Landau Levels is given as an example of the Algebraic Method applied to a degenerate quantum system.

Chapter 3

Building up an Operator using Creation and Annihilation Operators

Let $\hat{P}_n$ be a Hermitian operator in Hilbert space with a discrete spectrum and let $\mid n >$ be a denumerable set of eigenvectors so that $(n = 0, 1, 2, 3, \dots)$.

$$\hat{P} \mid n >= p_n \mid n > \tag{3.1}$$

Construct a creation operator $\eta^\dagger$ and an annihilation operator η associated with $\hat{\mathrm{P}}$. [3] The creation operator will have the form

$$\eta^\dagger = \sum_n C_n \ \mid n+1 >< n \mid \tag{3.2}$$

and the annihilation operator will have the form

$$\eta = \sum_n C^*_{n-1} \ \mid n-1 >< n \mid \tag{3.3}$$

Applying these operators to an eigenstate of the operator $\hat{\mathrm{P}}$ yields

$$\eta^\dagger \mid k >= C_k \mid k+1 > \tag{3.4}$$

and

$$\eta \mid k >= C^*_{k-1} \mid k-1 > \tag{3.5}$$

It is possible then to define the relationship between the coefficients and the creation and annihilation operators in the following fashion, namely

$$\eta\eta^{\dagger} \mid k >=\mid C_k \mid^2 \mid k > \tag{3.6}$$

and similarly

$$\eta^{\dagger}\eta \mid k >=\mid C_{k-1} \mid^2 \mid k > \tag{3.7}$$

Assume now that the spectrum for $\hat{P}$ has a lower bound corresponding to $n = 0$, so that $C_{-1} = 0$. This defines a zeroth state $\mid 0 >$ in such a way that $\eta \mid 0 >= 0$. If there exists an upper bound at state N then $C_N = 0$. Not all systems have an upper bound.

Since the operators $\eta\eta^{\dagger}$ and $\eta^{\dagger}\eta$ have the same eigenvectors as $\hat{\mathrm{P}}$, it is possible to formulate an expression for $\hat{\mathrm{P}}$ in terms of $\eta\eta^{\dagger}$ and $\eta^{\dagger}\eta$.

Without losing any generality we write $\hat{P}$ as an ordered function,

$$\hat{P} = \sum_{m,n} a_{mn}(\eta^{\dagger}\eta)^n \ (\eta\eta^{\dagger})^m \tag{3.8}$$

It is now possible, through a wise choice of coefficients C_k to reduce an operator such as a quadratic Hamiltonian into an ordered function of a simple linear form, of the creation and annihilation operators,

$$\hat{P} = a_{00} + a_{10}\eta\eta^{\dagger} + a_{01}\eta^{\dagger}\eta \tag{3.9}$$

that is when operating on $\mid n >$

$$\hat{P} = a_{00} + a_{10} \mid C_n \mid^2 + a_{01} \mid C_{n-1} \mid^2 \tag{3.10}$$

It is convenient for the purpose of analysis to separate out the anti-symmetric and symmetric combinations of the creation and annihilation operators. Define the anti-symmetric $\hat{A}$ operator by

$$\hat{A} = \eta\eta^{\dagger} - \eta^{\dagger}\eta \tag{3.11}$$

and the symmetric operator $\hat{S}$ by

$$\hat{S} = \eta\eta^{\dagger} + \eta^{\dagger}\eta \tag{3.12}$$

So then we can express $\hat{P}$ in terms of the anti-symmetric and symmetric operators

$$\hat{P} = q_0 + q_a\hat{A} + q_S\hat{S} \tag{3.13}$$

Let a_k and s_k be the eigenvalues of the operators $\hat{A}$ and $\hat{S}$, respectively, so then

$$\hat{A} \mid k >= a_k \mid k >= (\mid C_k \mid^2 - \mid C_{k-1} \mid^2) \mid k > \tag{3.14}$$

and

$$\hat{S} \mid k >= s_k \mid k >= (\mid C_k \mid^2 + \mid C_{k-1} \mid^2) \mid k > \tag{3.15}$$

yielding for the eigenvalues

$$a_k = \mid C_k \mid^2 - \mid C_{k-1} \mid^2 \tag{3.16}$$

and

$$s_k = \mid C_k \mid^2 + \mid C_{k-1} \mid^2 \tag{3.17}$$

From these two equations it follows that

$$\mid C_k \mid^2 = \frac{1}{2}(s_k + a_k) = \frac{1}{2}(s_{k+1} - a_{k+1}) \tag{3.18}$$

From these results it is possible to set up a series of relations known as consistency relations which prove valuable in the analysis of specific systems. The consistency relations are the following:

$$s_k \geq 0 \tag{3.19}$$

$$s_k + a_k \geq 0 \tag{3.20}$$

$$s_k - a_k \geq 0 \tag{3.21}$$

From the equation for $\mid C_k \mid^2$ we also have

$$s_k + a_k = s_{k+1} - a_{k+1} \tag{3.22}$$

which fixes the spectrum of the operator $\hat{P}$, which along with the lower bound term $C_{-1} = 0$ and the upper bound term $C_N = 0$ provides the final pair of consistency relations

$$s_0 = a_0 \tag{3.23}$$

and if an upper bound indeed exists

$$s_N = -a_N \tag{3.24}$$

This set of consistency relations contains all the information about the spectrum of the operator $\hat{P}$. These relations are compact and easier to use then other techniques, provided one can formulate the operators $\hat{A}$ and $\hat{S}$ for a specific system.

For several systems the operators $\hat{A}$ and $\hat{S}$ may be constructed by simple inspection. To facilitate this task we express the creation and annihilation operators in terms of two Hermitian Operators α and β, namely

$$\eta^\dagger = \frac{1}{\sqrt{2}}(\alpha - i\beta) \tag{3.25}$$

and

$$\eta = \frac{1}{\sqrt{2}}(\alpha + i\beta) \tag{3.26}$$

so then the operators $\hat{A}$ and $\hat{S}$ are given by

$$\hat{A} = -i[\alpha, \beta] \tag{3.27}$$

and

$$\hat{S} = \alpha^2 + \beta^2 \tag{3.28}$$

which allows one to write the operator $\hat{P}$ as

$$\hat{P} = q_0 - iq_a[\alpha, \beta] + q_s(\alpha^2 + \beta^2) \tag{3.29}$$

The usefulness of this formulation is that it makes the form of the operators more self evident when studying specific systems.

One can solve for the eigenstates of a system by defining the lower bound state $\mid 0 >$ or the upper bound state $\mid N >$ of a system and then solving the equation for $\eta \mid 0 >$ for the lower bound state or $\eta^{\dagger} \mid N >$, whichever is appropriate.

Chapter 4

The Quantum Harmonic Oscillator

Consider the Hamiltonian for the Quantum Harmonic Oscillator (QHO) given by

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{4.1}$$

Define two operators α and β so that

$$\alpha = \omega\hat{x}\ \sqrt(\frac{m}{2}) \tag{4.2}$$

and

$$\beta = \frac{\hat{p}}{\sqrt{2m}} \tag{4.3}$$

Construct now the Antisymmetric operator $\hat{A}$ and the Symmetric operator $\hat{S}$ from α and β, namely

$$\hat{A} = -i[\alpha, \beta] = -i\frac{\omega}{2}[x, p] = \frac{1}{2}\hbar\omega \tag{4.4}$$

and

$$\hat{S} = \alpha^2 + \beta^2 = \frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m} \tag{4.5}$$

By inspection we see that the eigenvalues are

$$s_k = E_k \tag{4.6}$$

and

$$a_k = \frac{1}{2}\hbar\omega \tag{4.7}$$

We see that

$$E_k \geq \frac{1}{2}\hbar\omega \tag{4.8}$$

and

$$E_{k+1} - E_k = \hbar\omega \tag{4.9}$$

Iterating this result and using $E_0 = \frac{1}{2}\hbar\omega$ as the lower bound energy we arrive at

$$E_k = E_0 + k\hbar\omega = (k + \frac{1}{2})\hbar\omega \tag{4.10}$$

where k = 0,1,2,3, ... In this system no upper bound exists.

Using α and β we can construct the creation operator $\eta^\dagger$ and annihilation operator η

$$\eta^\dagger = \frac{1}{\sqrt{2}}\ (\alpha - i\ \beta) = \ \frac{\omega\sqrt{m}}{2}(x - \frac{ip}{m\omega}) \tag{4.11}$$

$$\eta = \frac{1}{\sqrt{2}}\ (\alpha + i\ \beta) = \ \frac{\omega\sqrt{m}}{2}(x + \frac{ip}{m\omega}) \tag{4.12}$$

A coefficient C_k is defined by

$$\mid C_k \mid^2 = (k+1) \tag{4.13}$$

The creation and annihilations operators for the QHO can now be formulated in a compact manner,

$$\eta^\dagger = \sum_n \ C_n \mid n+1 >< n \mid = \sum_n \ \sqrt{[(n+1)]} \mid n+1 >< n \mid \tag{4.14}$$

and the annihilation operator will have the form

$$\eta = \sum_n C_{n-1} \mid n-1 >< n \mid = \sum_n \sqrt{[n]} \ \mid n-1 >< n \mid \tag{4.15}$$

As one can see then

$$\eta^\dagger \ \eta = \mid C_{n-1} \mid^2 \mid n >< n \mid = \ n \ \mid n >< n \mid \tag{4.16}$$

Let us define the energy of a state $\mid n >$ so that

$$\hat{H} = \ \epsilon_n \ \mid n >< n \mid \tag{4.17}$$

where $E\epsilon_n$ is the energy eigenvalue for state $\mid n >$. For the QHO, $E_n = n\hbar\,\omega$ which means that expressed in terms of the creation operator $\eta^\dagger$ and the annihilation operator η, the energy of a state $\mid n >$ of the Quantum Harmonic Oscillator is

$$\hat{H} \ \mid n > = \ (n + \frac{1}{2}) \ \hbar\omega \ \mid n > = \ (\eta^\dagger \ \eta \ + \frac{1}{2}) \ \hbar\omega \ \mid n > \tag{4.18}$$

Chapter 5

Solving for the $\mid 0 >$ state for the Quantum Harmonic Oscillator

Let us now solve for the lower bound state $\mid 0 >$ for the Quantum Harmonic Oscillator using the Algebraic Method first proposed by P.A.M. Dirac. [2]

Define the lower bound state $\mid 0 >$ by the condition $\eta \ \mid 0 > = 0$. This means that

$$\eta \ \mid 0 >= \frac{\omega\sqrt{m}}{2}(x + \frac{ip}{m\omega}) \ \mid 0 >= 0 \tag{5.1}$$

Reformulate this defining equation we arrive at

$$(\frac{d}{dx} + \frac{m\omega x}{\hbar}) \ \mid 0 >= (\hat{D} + \frac{m\omega x}{\hbar}) \ \mid 0 >= 0 \tag{5.2}$$

where D is the differential Operator. The solution to this differential equation is

$$\mid 0 >= C \ exp \ (\frac{-m\omega x^2}{2\hbar}) \tag{5.3}$$

where C is a normalization constant which is found to be

$$C = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \tag{5.4}$$

so that the lower bound state $\mid 0 >$ for the harmonic oscillator is given by

$$| 0 >= (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} \; exp \; (\frac{-m\omega x^2}{2\hbar}) \tag{5.5}$$

Having found the lower bound state $| \; 0 >$ the next state is arrived at by application of the creation operator $\eta^\dagger$ to the lower bound state, namely

$$| 1 > = \eta^\dagger \;\; | 0 > \tag{5.6}$$

From the expression for $\eta^\dagger$ we see that $| \; n >$ and $| \; n-1 >$ are related by

$$| n >= \frac{\eta^\dagger}{\sqrt{n}} \;\; | n-1 > \tag{5.7}$$

In terms of the lower bound state $| \; 0 >$ one can see by inspection that any other state of the QHO is given by

$$| n >= \frac{(\eta^\dagger)^n}{\prod_{m=0}^{n-1} C_m} \;\; | 0 > \tag{5.8}$$

However by inspection we see that

$$\prod_{m=0}^{n-1} C_m = \sqrt{n!} \tag{5.9}$$

so then the state $| \; n >$ expressed in terms of the lower bound state $| \; 0 >$ is given by

$$| n >= \frac{1}{\sqrt{n!}} \; (\eta^\dagger)^n \;\; | 0 > \tag{5.10}$$

This description of the Quantum Harmonic Oscillator (QHO) using creation and annihilation operators (also know as raising and lowering operators) is well known.

Chapter 6

Angular Momentum

Consider the commutation relationship given by [3]

$$\hat{J}_3 = -i[\hat{J}_1, \hat{J}_2] \tag{6.1}$$

Let

$$\begin{aligned} \alpha &= \hat{J}_2 \\ \beta &= \hat{J}_1 \end{aligned} \tag{6.2}$$

so then $q_0 = q_s = 0$ and $q_a = -1$ then

$$\hat{A} = -i[\alpha, \beta] = i[\hat{J}_1, \hat{J}_2] \tag{6.3}$$

and

$$\hat{S} = \alpha^2 + \beta^2 = \hat{J}_1^2 + \hat{J}_2^2 = \hat{J}^2 - \hat{J}_3^2 \tag{6.4}$$

Introduce the eigenstates $\mid \lambda, \mu >$ of the operators $\hat{J}^2$ and $\hat{J}_3^2$ so that

$$\hat{J}^2 \mid \lambda, \mu >= \lambda \mid \lambda, \mu > \tag{6.5}$$

and

$$\hat{J}_3^2 \mid \lambda, \mu >= \mu \mid \lambda, \mu > \tag{6.6}$$

Consider the raising and lowering operators that act on the eigenvalue μ for a given λ. By inspection we see that

$$a_k = -\mu_k \tag{6.7}$$

and

$$s_k = \lambda - \mu_k^2 \tag{6.8}$$

We see then that $\lambda \geq \mu_k^2$ and that $\lambda \geq \mu_k(\mu_k \pm 1)$ and that

$$\lambda - \mu_{k+1}^2 - \lambda + \mu_k^2 = -\mu_{k+1} - \mu_k \tag{6.9}$$

which means that $\mu_{k+1} = \mu_k + 1$. Iterating this result we get $\mu_k = \mu_0 + k$ where $k = 0, 1, 2, \ldots$.

From the remaining consistency equations we see that

$$\begin{aligned} \lambda &= \mu_0(\mu_0 - 1) \\ \lambda &= \mu_N(\mu_N - 1) \end{aligned} \tag{6.10}$$

It is evident that μ_0 and μ_N are related by

$$\mu_N = \mu_0 + N \tag{6.11}$$

where N is the integral number of steps needed to go from the lower bound state to the upper bound state.

Let $N = 2j$. Solving for λ

$$\lambda = j(j+1) \tag{6.12}$$

where j is an integer or a half-integer. We see therefore that $\mu_0 = -j$ and $\mu_N = j$ as expected.

It is straighforward to formulate the creation and annihilation operators $\eta^\dagger$ and η in terms of the Euler angles θ and ϕ, namely

$$\eta^\dagger = -\frac{i}{\sqrt{2}}(\hat{J}_1 + i\hat{J}_2) = -\frac{i}{\sqrt{2}}\ \hat{J}_+ = -\frac{i\hbar}{\sqrt{2}}\ e^{i\phi}(\frac{\partial}{\partial\theta} + i\cot\theta\ \frac{\partial}{\partial\phi}) \tag{6.13}$$

and

$$\eta = \frac{i}{\sqrt{2}}(\hat{J}_1 - i\hat{J}_2) = \frac{i}{\sqrt{2}}\ \hat{J}_- = \frac{i\hbar}{\sqrt{2}}\ e^{-i\phi}(\frac{\partial}{\partial\theta} - i\cot\theta\ \frac{\partial}{\partial\phi}) \tag{6.14}$$

The coefficients C_k are given by

$$C_k =< k+1 \mid \eta^\dagger \mid k >= -\frac{i}{\sqrt{2}} < j,\mu \mid \hat{J}_+ \mid j,\mu > \tag{6.15}$$

so that

$$\mid C_k \mid^2 = \frac{1}{2}[j(j+1) - \mu(\mu+1)] = \frac{1}{2}(j+\mu+1)(j-\mu) \tag{6.16}$$

Let us solve for the upper bound state $\mid \lambda, \lambda >$ for this system. Define state $\eta^\dagger \mid \lambda, \lambda >$ by

$$\eta^\dagger \ \mid \lambda, \lambda >= 0 \tag{6.17}$$

This means that

$$\eta^\dagger \mid \lambda, \lambda >= \frac{i\hbar}{\sqrt{2}} \ e^{-i\phi} \ (-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}) \ \mid \lambda, \lambda >= 0 \tag{6.18}$$

Use separation of variables and define

$$\mid \lambda, \lambda >= \Theta_{\lambda,\lambda}(\theta) \ \ e^{-i\lambda\phi} \tag{6.19}$$

Reformulate this defining equation we arrive at

$$\hbar \ e^{i(\lambda+1)\phi} \ (\frac{\partial}{\partial\theta} - \lambda\cot\theta) \ \Theta_{\lambda,\lambda}(\theta) = 0 \tag{6.20}$$

the solution to this equation being

$$\Theta_{\lambda,\lambda}(\theta) = (\sin\theta)^\lambda \tag{6.21}$$

so then the upper bound state becomes

$$\mid \lambda, \lambda >= (\sin\theta)^\lambda \ e^{i\lambda\phi} \tag{6.22}$$

The next state down becomes

$$| \lambda, \lambda - 1 >= C(\lambda, \lambda - 1)\ \eta\ | \lambda, \lambda > \tag{6.23}$$

where $C(\lambda, \lambda - 1)$is a normalization constant. Any other state then becomes

$$| \lambda, \mu >= C(\lambda, \mu)\ \eta^{\lambda-\mu}\ \ | \lambda, \lambda > \tag{6.24}$$

which in terms of Legendre Polynomials becomes

$$| \lambda, \mu >= (-1)^{\mu}[\ \frac{2\lambda + 1}{4\pi}\ \frac{(\lambda - \mu)!}{(\lambda + \mu)!}\ ]^{\frac{1}{2}}\ P_{\lambda}^{\mu}(\cos\theta)\ e^{i\mu\phi} \tag{6.25}$$

Chapter 7

The Rigid Rotator

Consider the Hamiltonian for the three-dimensional rigid rotator for the radial wave function $u = rR(r)$, [3]

$$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l^2}{2mr^2} + \frac{\hbar^2 l}{2mr^2} \tag{7.1}$$

where we have expanded the term $l(l+1)$ which by inspection allows us to set α and β in the following fashion

$$\alpha = \frac{\hbar l}{r\sqrt{2m}} \tag{7.2}$$

and

$$\beta = \frac{i\hbar}{\sqrt{2m}}\frac{d}{dr} \tag{7.3}$$

It is them evident that the anti-symmetric and symmetric operators are given by

$$\hat{A} = -i[\alpha, \beta] = -\frac{\hbar^2 l}{2m}[\frac{d}{dr}, \frac{1}{r}] \tag{7.4}$$

and

$$\hat{S} = \alpha^2 + \beta^2 = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l^2}{2mr^2} = \hat{H} - \hat{A} \tag{7.5}$$

and that $q_a = q_s = 1$ and $q_0 = 0$.

The eigenvalues of $\hat{A}$ and $\hat{S}$ are given by

$$a_l = \frac{\hbar^2}{2I}\, l \tag{7.6}$$

$$s_l = E_l - \frac{\hbar^2}{2I}\, l \tag{7.7}$$

where $I = ma^2$ is the moment of inertia of the rotator. The consistency relations means that

$$E_l \geq 2a_l = \frac{l\hbar^2}{I} \tag{7.8}$$

and

$$E_{l+1} - a_{l+1} - E_l + a_l = a_{l+1} + a_l \tag{7.9}$$

or

$$E_{l+1} - E_l = 2a_{l+1} = \frac{\hbar^2}{I}(l+1) \tag{7.10}$$

from which we iterate to get

$$E_l = E_0 + \frac{\hbar^2}{I}\sum_{k=1}^{l} = E_0 + \frac{\hbar^2}{2I}\, l(l+1) \tag{7.11}$$

Now since $E_0 = 0$ the eigenvalues of the energy are

$$E_l = \frac{\hbar^2}{2I}\, l(l+1) \tag{7.12}$$

No upper bound exists for this system.

The creation and annihilation operators $\eta^\dagger$ and η are given by

$$\eta^\dagger = \frac{\hbar}{2\sqrt{m}}(\frac{l}{r} + \frac{d}{dr}) \tag{7.13}$$

and

$$\eta = \frac{\hbar}{2\sqrt{m}}(\frac{l}{r} - \frac{d}{dr}) \tag{7.14}$$

The coefficient C_l are given by

$$\mid C_l \mid^2 = \frac{1}{2}E_l = \frac{\hbar^2}{4I}\ l(l+1) \tag{7.15}$$

So then with $I = mr^2$ the moment of inertia of the rigid rotator,

$$C_l = \frac{\hbar}{2r\sqrt{m}}\ \sqrt{l(l+1)} \tag{7.16}$$

The creation and annihilations operators for the rigid rotator can be formulated in a compact manner,

$$\eta^\dagger = \sum_l\ C_l \mid l+1 >< l \mid = \sum_l\ \frac{\hbar}{2\sqrt{I}}\ [\sqrt{l(l+1)}] \mid l+1 >< l \mid \tag{7.17}$$

and the annihilation operator will have the form

$$\eta = \sum_l\ C_{l-1} \mid l-1 >< l \mid = \sum_l\ \frac{\hbar}{2\sqrt{I}}\ [\sqrt{l(l-1)}]\ \mid l-1 >< l \mid \tag{7.18}$$

Chapter 8

Solving for the Lower Bound $\mid l >$ state of the Rigid Rotator

Define the lower bound state $\mid l >$ by the condition $\eta \ \mid l > = 0$. This means that

$$\eta \ \mid l >= \frac{\hbar}{2\sqrt{m}}(\frac{l}{r} - \frac{d}{dr}) \ \mid l >= 0 \tag{8.1}$$

Reformulate this defining equation we arrive at

$$(l - r \ \frac{d}{dr}) \ \mid l >= 0 \tag{8.2}$$

The solution to this differential equation is

$$\mid l >= C_l \ r^l \tag{8.3}$$

where C_l is some normalization constant.

Having found the lower bound state $\mid l >$ the next state is arrived at by application of the creation operator $\eta^\dagger$ to the lower bound state, namely

$$\mid 1 + 1 > = \frac{\eta^\dagger}{C_l} \ \mid l > \tag{8.4}$$

In terms of the lower bound state $\mid l >$ one can see by inspection that any other state of the rigid rotator is given by

$$| \, l+m >= \frac{(\eta^\dagger)^m}{\prod_{n=0}^{m-1} \; C_{l+n}} \;\; | \, l > \tag{8.5}$$

that is

$$| \, l+m >= (\frac{2\sqrt{I}}{\hbar})^m \;(\eta^\dagger)^m \frac{\sqrt{[(l+m)l]}}{\prod_{n=0}^{m} \;(l+n)} \;\; | \, 0 > \tag{8.6}$$

where $I = mr^2$ is the moment of inertia of the rigid rotator.

Chapter 9

Spherical Symmetric Potentials

The Algebraic Method is a straightforward technique to characterize spherically symmetric potentials. [4]

To begin, introduce the radial operators for $\hat{r}$

$$\hat{r} = (x^2 + y^2 + z^2)^{\frac{1}{2}} \tag{9.1}$$

and for $\hat{p}$

$$\hat{p} = \frac{x}{r}\, p_x + \frac{y}{r}\, p_y + \frac{z}{r}\, p_z \tag{9.2}$$

The operators satisfy the usual commutation relation $[\, \hat{r}, \hat{p}\,] = i\hbar$.

The momentum is given by

$$\hat{p} = -i\hbar\frac{\partial}{\partial r} - \frac{i\hbar}{r} \tag{9.3}$$

Consider a Hamiltonian with a spherically symmetric potential $V(r)$,

$$\hat{H} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(r) \tag{9.4}$$

We know from straightforward algebra that

$$r^2(p_x^2 + p_y^2 + p_z^2) = r^2p^2 + \hat{L}^2 \tag{9.5}$$

so then the Hamiltonian is given by

$$\hat{H} = \frac{1}{2m}(p^2 + \frac{\hat{L}^2}{r^2}) + V(r) \tag{9.6}$$

which, in terms of normalized eigenstates $\mid nl >$ of the principal quantum number n and the total angular momentum $\hat{L}$, becomes

$$\hat{H}_l = \frac{1}{2m}(p^2 + \frac{\hbar^2\ l(l+1)}{r^2}) + V(r) \tag{9.7}$$

Consider the operator $\hat{C}_l$ that takes eigenstate $\mid nl >$ onto $\mid n'l+1 >$ where n labels the energy. Then

$$\hat{C}_l^*\ \hat{C}_l = 2m\hat{H}_l + F_l \tag{9.8}$$

and

$$\hat{C}_l\ \hat{C}_l^* = 2m\hat{H}_l + G_l \tag{9.9}$$

where F_l and G_l are yet to be determined scalars.

We see then that

$$\hat{C}_l\ \hat{C}_l^*\ \hat{C}_l \mid nl >= (2mE_l^n + F_l)\ \ \hat{C}_l \mid nl > \tag{9.10}$$

with E_l^n the energy of the eigenstate $\mid nl >$ and that

$$\hat{C}_l\ \hat{C}_l\ \hat{C}_l^* \mid nl >= (2m\hat{H}_{l+1} + G_l)\ \ \hat{C}_l \mid nl > \tag{9.11}$$

Solving for $\hat{H}_{l+1}$ we get

$$\hat{H}_{l+1}\ \hat{C}_l \mid nl >= [\ E_l^n + \frac{(F_l - G_l)}{2m}\ ]\ \hat{C}_l \mid nl > \tag{9.12}$$

which means that $\hat{C}_l \mid nl >$ is an eigenstate of $\hat{H}_{l+1}$ with eigenvalue

$$E_l^{n\prime} = E_l^n + \frac{(F_l - G_l)}{2m} \tag{9.13}$$

If $F_l = G_l$ the energy is constant and $n\prime = n$. In this case $\mid nl >$ and $\hat{C}_l \ \mid nl >$ are states with the same energy and we have a degenerate system.

By inspection we see that

$$C_l \mid nl >= \lambda_l^n \mid n'l+1 > \tag{9.14}$$

Similarly, we can show that $C_l^* \mid n'l+1>$ is an eigenstate of H_l, that is

$$C_l^* \mid n'l+1 >= \mu_l^n \mid nl > \tag{9.15}$$

Note that λ_l^n and μ_l^n are complex conjugates of each other as can be seen from

$$\lambda_l^n = < n'l+1 \mid \; C_l \mid nl >=< nl \mid \; C_l^* \mid n'l+1 >^* = \mu_l^{n*} \tag{9.16}$$

From these equations we can derive the following simple relationship,

$$\mid \lambda_l^n \mid^2 = 2mE_l^n + F_l = 2mE_{l+1}^{n'} + G_l \tag{9.17}$$

which yields the recursion relationship

$$\mid \lambda_l^n \mid^2 - \mid \lambda_{l-1}^{n'} \mid^2 = F_l - G_{l-1} \tag{9.18}$$

If this series terminates at some stage with $C_l \mid nl >= 0$ then the energy is given by

$$E_l^n = \frac{-F_l}{2m} \tag{9.19}$$

It is necessary to determine whether the series terminates on a case by case basis .

Consider operators $\hat{C}_l$ linear in momentum p, namely

$$\hat{C}_l = p + f(r) \tag{9.20}$$

Since we know that

$$[\, f(r), p \,] = i\hbar \frac{df}{dr} \tag{9.21}$$

it is simple to see that

$$\hat{C}_l^* \hat{C}_l = p^2 - p(f + f^*) + i\hbar \frac{df^*}{dr} + f^* f \tag{9.22}$$

and similarly

$$\hat{C}_l\hat{C}_l^* = p^2 + p(f + f^*) + i\hbar\frac{df}{dr} + f^*f \tag{9.23}$$

Studying the Hamiltonian H_l given above we see that there are no terms linear in p so then we have the subsidiary condition

$$f = -f^* \tag{9.24}$$

In terms of our Hamiltonian we see then that these two equations for $C_l^*C_l$ and $C_lC_l^*$ yield

$$-i\hbar\frac{df}{dr} - f^2 = l(l+1)\frac{\hbar^2}{r^2} + 2mV + F_l \tag{9.25}$$

and

$$i\hbar\frac{df}{dr} - f^2 = (l+1)(l+2)\frac{\hbar^2}{r^2} + 2mV + G_l \tag{9.26}$$

The equations are easily solved to give

$$f(r) = i\hbar\frac{l+1}{r} + \frac{i}{2\hbar}(F_l - Gl)r + A \tag{9.27}$$

and

$$\begin{aligned} 2mV(r) = (F_l - G_l)^2\frac{r^2}{4\hbar^2} - \frac{iA}{\hbar}(F_l - G_l)r \\ +(l+1)(F_l - G_l) - \frac{1}{2}(F_l + G_l) - A^2 - 2i\hbar(l+1)\frac{A}{r} \end{aligned} \tag{9.28}$$

where A is an imaginary constant of integration.

The potential has to be independent of l, so the coefficients of r, r^2 and r^-1 must be independent of l. This restricts the possible values of $(F_l - G_l)$ and A in the following way: from the term in r^{-1} we that A is either 0 or proportional to $(l+1)$, and from the term in r^2 that $(F_l - Gl)$ is a constant. The coefficient of r, which contain the product of A and $(F_l - G_l)$ must now be zero showing that one or both of A and $(F_l - G_l)$ is zero. We shall examine these three cases.

Case 1: Free Field

Set $A = 0$ and $(F_l - G_l) = 0$ then

$$f(r) = i\hbar\frac{l+1}{r} \tag{9.29}$$

and

$$V(r) = -\frac{1}{4m}\,(F_l - G_l) = -\frac{F_l}{2m} \tag{9.30}$$

This means that the operator $\hat{C}_l$ is given by

$$\hat{C}_l = p + i\hbar(l+1)\frac{1}{r} \tag{9.31}$$

Since F_l is independent of r, and V of l they must be constant.

The Hamiltonian for the free field case is given by

$$\hat{H} = \frac{1}{2m}\,[\,p^2 + \frac{\hbar^2 l(l+1)}{r^2}\,] - \frac{F_l}{2m} \tag{9.32}$$

and the energy by

$$E_l^n =< nl \mid \; \hat{H}_l \; \mid nl >$$

$$= \frac{1}{2m}\,[< nl \mid \; p^2 \; \mid nl > + \; < nl \mid \; \frac{\hbar^2 l(l+1)}{r^2} \; \mid nl >] - \frac{F_l}{2m} \tag{9.33}$$

The first two terms are greater than or equal to zero and cannot be simultaneously zero. Thus apart from the state $l = 0$ and $p^2 \mid nl >= 0$, the energy is greater than $V_l = -\frac{F_l}{2m}$.

For the special case when $l = 0$ and $p^2 \mid nl >= 0$, we have only one state as $C_l \mid n0 >= 0$ with energy $E_l^n = -\frac{F_l}{2m}$.

We see that the series $\lambda_l^n, \lambda_{l+1}^n, \lambda_{l+2}^n, \cdots$ does not terminate and we have infinitely degenerate levels.

The energy of the free field case then is

$$E_l^n = \frac{1}{2m} [\mid \lambda_l^n \mid^2 - F_l \,] \tag{9.34}$$

but since we cannot find an expression for λ_l^n this means we have a continuous distribution of energy levels.

Case 2: Coulomb Potential Field

Set $F_l = G_l$ and $A \neq 0$ then the potential is given by

$$2mV(r) = -F_l - A^2 - 2i\hbar(l+1)\frac{A}{r} \tag{9.35}$$

Since we want the potential to be real and independent of l we must have

$$A = -\frac{iB}{(l+1)} \tag{9.36}$$

and

$$F_l = D + \frac{B^2}{(l+1)^2} \tag{9.37}$$

where B and D are themselves real. The potential then has the form

$$V(r) = -\frac{D}{2m} - \frac{B\hbar}{mr} \tag{9.38}$$

Notice that the first term is nothing more than a constant and so we are free to set it equal to zero, giving us

$$V(r) = -\frac{B\hbar}{mr} \tag{9.39}$$

and

$$F_l = G_l = \frac{B^2}{(l+1)^2} \tag{9.40}$$

and the operator $\hat{C}_l$ in this case is given by

$$\hat{C}_l = p + i\hbar(l+1)\frac{1}{r} - \frac{iB}{(l+1)} \tag{9.41}$$

Since $F_l = G_l$ the operator $\hat{C}_l$ leaves the energy unchanged and we have the recursion relationship

$$| \lambda_l^n |^2 - | \lambda_{l-1}^n |^2 = \frac{B^2}{(l+1)^2} - \frac{B^2}{(l)^2} = -\frac{B^2(2l+1)}{l^2(l+1)^2} \tag{9.42}$$

which has the solution

$$| \lambda_l^n | = \frac{B}{l+1} \; [\; 1 + K(l+1) \;]^{1/2} \tag{9.43}$$

where K is some real constant. This means that the energy is given by

$$E_l^n = \frac{KB^2}{2m} \tag{9.44}$$

The Hamiltonian for this system is

$$\hat{H}_l = \frac{1}{2m} \; [\; p^2 + \frac{\hbar l(l+1)}{r^2} \;] \; - \frac{B\hbar}{mr} \tag{9.45}$$

By inspection we see that if B is negative (a repulsive potential) the energy will be positive and if B is positive (an attractive potential) we can have either positive or negative energy.

If the energy is positive then K is positive, which means that λ_l^n can never be zero and we have infinite degeneracy.

If the energy is negative the square root in λ_l^n will vanish at $K(l_{max}+1) = -1$ which sets an upper bound to l. If we let $(l_{max}+1) = n$ then we have the familiar result

$$E_l^n = -\frac{B^2}{2m\; n^2} \tag{9.46}$$

with the operator $\hat{C}_l$ given by

$$\hat{C}_l \;\; | \, nl >= \frac{B}{n(l+1)} \; [\; (n+l+1)(n-l-1) \;]^{1/2} \;\; | \, nl+1 > \tag{9.47}$$

where $l = 0, 1, 2, \ldots, (n-1)$.

So we see then that for a repulsive potential we have an infinite number of degenerate, positive energy levels. For an attractive potential we have either a finite number of degenerate negative energy levels, or we have an infinite number of degenerate positive energy levels.

Case 3: The Isotropic Harmonic Oscillator

Let $A = 0$ but $F_l \neq G_l \neq 0$. Letting V(r) be independent of l we have

$$V(r) = \frac{B^2 r^2}{2m\hbar^2} \tag{9.48}$$

With

$$F_l = B\ (2l + 3) \tag{9.49}$$

and

$$G_l = B\ (2l + 1) \tag{9.50}$$

which on the surface appears to lead to an operator $\hat{C}_l$ of the form

$$\hat{C}_l = p + i\frac{\hbar(l+1)}{r} + \frac{iBr}{\hbar} \tag{9.51}$$

where B is a real constant. However, since V(r) is a quadratic potential in B we must be careful to take this into account. We have in fact two inequal operators $\hat{C}_l$ and $\hat{D}_l$ given by

$$\hat{C}_l = p + i\frac{\hbar(l+1)}{r} - \frac{iD^2 r}{\hbar} \tag{9.52}$$

and

$$\hat{D}_l = p + i\frac{\hbar(l+1)}{r} + \frac{iD^2 r}{\hbar} \tag{9.53}$$

with

$$\hat{C}_l \ \mid nl >= \lambda_l^n \mid n'l + 1 > \tag{9.54}$$

and

$$\hat{D}_l = \mid nl >= \mu_l^n \mid n^{"} l + 1 > \tag{9.55}$$

Since $F_l \neq G_l$, $n \neq\ n' \neq\ n^{"}$.

Consider the equations that result from the $\hat{C}_l$ operator, namely

$$E_{l+1}^{n'} = E_l^n - \frac{D^2}{m} \tag{9.56}$$

unless $\hat{C}_l \mid nl >= 0$, which may continue onto

$$E_{l+1}^{n''} = E_l^n - \frac{2D^2}{m} \tag{9.57}$$

unless $\hat{C}_{l+1} \mid n^{'}l+1 >= 0$, and so on. The energy is decreasing, but it can never be negative as all terms in the Hamiltonian have positive eigenvalues. This means that the series must terminate at some point with

$$E_l^n = -\frac{F_l}{2m} = (n + \frac{3}{2})D^2 \tag{9.58}$$

This means then that

$$E_{l+1}^{n'} = [\ (n-1) + \frac{3}{2})D^2 \tag{9.59}$$

which means that $n^{'} = n - 1$. It is easy to show that

$$\hat{C}_l \mid nl >= \lambda_l^n \mid n - 1l + 1 > \tag{9.60}$$

using the recursive relation (with $\lambda_n^n = 0$)

$$\mid \lambda_l^n \mid^2 - \mid \lambda_{l-1}^{n+1} \mid^2 = -4B^2 \tag{9.61}$$

To within a phase factor

$$\lambda_l^n = B[\ 2(n-1)\ ]^{1/2} \tag{9.62}$$

Studying the degeneracy of the system the allowed states are $\mid nn >, \mid nn-2 >, \mid nn-4 > \ldots$ so that l can only take the values $n, n-2, n-4, \ldots$ to 0 or to 1.

With regards to the operator $\hat{D}_l$ we see that

$$\hat{D}_l \mid nl >= \mu_l^n \mid n + 1l + 1 > \tag{9.63}$$

however we can show that this series does not terminate, which means we cannot perform useful calculations with the $\hat{D}_l$ operator.

Chapter 10

A Charged Particle in a Magnetic Field: Landau Levels

As an example of the Algebraic Method applied to a real discrete system, consider an electron in a magnetic field with a symmetric gauge given by

$$A = -\frac{B}{2}(-y, x, 0) \tag{10.1}$$

with the Hamiltonian given by

$$\hat{H} = \frac{1}{2m}(\hat{p}_x + \frac{eB}{2c})^2 + \frac{1}{2m}(\hat{p}_y - \frac{eB}{2c})^2 + \frac{1}{2m}\hat{p}_z^2 \tag{10.2}$$

The eigenvalue solution to this problem is known as the Landau Levels of charged particles in a magnetic field, and was solved in a complete and detailed fashion in 1930 by Dr. L. D. Landau. [5, 6]

We begin by defining for the conjugate momenta the following

$$\begin{aligned}
\pi_x &= \hat{p}_x + \frac{eB}{2c} \\
\pi_y &= \hat{p}_y - \frac{eB}{2c} \\
\pi_z &= \hat{p}_z = \hbar k_z
\end{aligned} \tag{10.3}$$

The equations of motion for this system are then given by

$$\frac{d}{dt}\pi_x = \frac{1}{i\hbar}[\pi_x, \hat{H}] = -\frac{eB}{mc}\pi_y = -\omega\pi_y$$
$$\frac{d}{dt}\pi_y = \frac{1}{i\hbar}[\pi_y, \hat{H}] = \frac{eB}{mc}\pi_x = \omega\pi_x$$
$$\frac{d}{dt}\pi_z = \frac{1}{i\hbar}[\pi_z, \hat{H}] = 0 \tag{10.4}$$

where

$$\omega = \frac{eB}{mc} \tag{10.5}$$

The solution to these equations of motion are

$$\pi_x = m\omega(y_0 - y)$$
$$\pi_y = m\omega(x - x_0)$$
$$\pi_x = \hat{p}_z = constant \tag{10.6}$$

The Hamiltonian can be rewritten as

$$\hat{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{e^2B^2}{8mc^2}(x^2 + y^2) + \frac{eB}{2mc}(x\hat{p}_y - y\hat{p}_x) = \hat{H}_T + \frac{\hat{p}_z^2}{2m} \tag{10.7}$$

Let us define two operators $\pi_\pm = \pi_x \pm i\pi_y$ as step up and step down operators such that

$$\hat{H}_T = \frac{1}{4m}(\pi_+\pi_- + \pi_-\pi_+) = \frac{1}{2m}\pi_+\pi_- + \frac{1}{2}\hbar\omega \tag{10.8}$$

where for state $\mid \phi_n >$

$$\mid \phi_n >= C_n\pi_+ \mid \phi_{n-1} > \tag{10.9}$$

with C_n being a normalization constant for the state. It is straightforward to show that

$$C_n = \frac{1}{(2nm\hbar\omega)^{\frac{1}{2}}} \tag{10.10}$$

Define the ground state of the system such that

$$\pi_- \mid \phi_0 >= 0$$
$$\pi_z \mid \phi_0 >= \hbar k \mid \phi_0 > \tag{10.11}$$

Then it is straightforward to show

$$\mid \phi_n >= \frac{1}{\sqrt{(2^n n! m\hbar\omega)}} \pi_+^n \mid \phi_0 > \tag{10.12}$$

To find the state $\mid \phi_0 >$ we apply the explicit form of π_- and π_z on state $\mid \phi_0 >$ so then

$$\pi_- \mid \phi_0 >= -i\hbar[\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} + \frac{1}{2\lambda^2}(x - iy)] \mid \phi_0 >$$
$$\pi_z = -i\hbar\frac{\partial}{\partial z} \mid \phi_0 > \tag{10.13}$$

so then the state $\mid \phi_0 >$ has the form

$$\mid \phi_0 >= f(x - iy) \exp\ [ikz]\ \exp\ [-\frac{(x^2 + y^2)}{4\lambda^2}] \tag{10.14}$$

where $f(x - iy)$ is a yet to be determined function which reflects the infinite degeneracy of the system and

$$\lambda = (\frac{\hbar}{m\omega})^{\frac{1}{2}} \tag{10.15}$$

If we impose an additional constraint on $\mid \phi_0 >$, namely $\mid \phi_0 >$ should be an eigenfunction of x_0 then

$$f(x - iy) = A\ \exp\ [-\frac{1}{4\lambda^2}\ (x - iy - 2a)^2] \tag{10.16}$$

where $x_0 \mid \phi_0 >= a \mid \phi_0 >$ and $x_0 \mid \phi_n >= a \mid \phi_n >$.

We therefore have a state $| \phi_0; a >$ given by

$$| \phi_0; a >= C_n(\exp[ikz]\ \exp[-\frac{1}{4\lambda^2}[(x^2+y^2)+(x-iy-2a)^2]]) \tag{10.17}$$

The normalization constant C_n is given by

$$C_n = (\frac{1}{\pi\lambda^2})^{\frac{1}{4}}\ \exp[\frac{a^2}{2\lambda^2}] \tag{10.18}$$

yielding

$$| \phi_0; a >= (\frac{1}{\pi\lambda^2})^{\frac{1}{4}}\ \exp[\frac{a^2}{2\lambda^2}]\ (\exp[ikz]\ \exp[-\frac{1}{4\lambda^2}[(x^2+y^2)+(x-iy-2a)^2\ ]\ ]) \tag{10.19}$$

To obtain the higher energy eigenfunctions $| \phi_n; a >$ we use the step up operator π_+ to arrive at

$$\begin{aligned} | \phi_n; a >= (\frac{i}{\sqrt{2}})^n(\frac{1}{\lambda n!\sqrt{\pi}})\ (\exp[ikz]\ \exp\ [-\frac{1}{4\lambda^2}[(x^2+y^2) \\ +(x-iy-2a)^2]\ ])H_N(\frac{x-a}{\lambda}) \end{aligned} \tag{10.20}$$

with the energy of level n given by

$$\begin{aligned} E(n:p_z) &= \hbar\omega(n+\frac{1}{2})+\frac{p_z^2}{2m} \\ &= \hbar(\frac{eB}{mc})(n+\frac{1}{2})+\frac{p_z^2}{2m} \end{aligned} \tag{10.21}$$

which describe the Landau Levels of an electron in a magnetic field.

Chapter 11

Second Quantization

The Algebraic Method can also be used to characterize Second Quantization of Boson and Fermion fields. [2, 3]

Define a number operator $\hat{N}$

$$\hat{N} = \eta^\dagger \eta \tag{11.1}$$

so that the number operator $\hat{N}$ is

$$\hat{N} = \frac{1}{2}(\alpha^2 + \beta^2) + \frac{i}{2}[\alpha, \beta] \tag{11.2}$$

We will denote the eigenvalue for the number operator $\hat{N}$ as ν_n. To focus on the character of bosons and fermions we shall study them separately.

Bosons : For bosons we have the commutator

$$[\eta, \eta^\dagger] = 1 \tag{11.3}$$

which means for bosons

$$\hat{A} = -i[\alpha, \beta] = 1 \tag{11.4}$$

and

$$\hat{S} = \alpha^2 + \beta^2 = 2\hat{N} + \hat{A} \tag{11.5}$$

so then $q_s = -q_a = \frac{1}{2}$ and $q_0 = 0$.

Applying the consistency relations we get

$$a_n = 1 \tag{11.6}$$

and

$$s_n = 2\nu_n + 1 \tag{11.7}$$

It follows then that $\nu_n \geq 0$ and $\nu_{n+1} - \nu_n = 1$. By iteration we see that

$$\nu_n = \nu_0 + n \tag{11.8}$$

with n = 0, 1, 2, ... It can immediately be seen that the lower bound is $\nu_0 = 0$, and that no upper bound exists.

The coefficient $\mid C_k \mid^2$ is given by

$$\mid C_n \mid^2 = \frac{1}{2}(s_n + a_n) = \nu_n + 1 = n + 1 \tag{11.9}$$

As we can see then for bosons the creation and annihilations operators can also be formulated in a compact manner. To within a phase factor, the creation operator will have the form

$$\eta^\dagger = \sum_n \sqrt{(n+1)} \mid n+1 >< n \mid \tag{11.10}$$

and the annihilation operator will have the form

$$\eta = \sum_n \sqrt{n} \ \mid n-1 >< n \mid \tag{11.11}$$

<u>Fermions :</u> For fermions we have the commutator

$$\eta\eta^\dagger - \eta^\dagger\eta = 1 \tag{11.12}$$

which for fermions means

$$\hat{A} = -i[\alpha, \beta] = \hat{S} - 2\hat{N} \tag{11.13}$$

and

$$\hat{S} = \alpha^2 + \beta^2 = 1 \tag{11.14}$$

Applying the consistency relations we get

$$a_n = 1 - 2\nu_n \tag{11.15}$$

and

$$s_n = 1 \tag{11.16}$$

Applying the consistency relations we get two conditions

$$1 + (1 - 2\nu_n) \geq 0 \tag{11.17}$$

and

$$1 - (1 - 2\nu_n) \geq 0 \tag{11.18}$$

which means that $0 \leq \nu_n \geq 1$. As well $2 - 2(\nu_{n+1} + \nu_n) = 0$ which means

$$\nu_{n+1} = 1 - \nu_n \tag{11.19}$$

This in turn means that $\nu = 0$ and $\nu_n = 1$. Since the spectrum is discrete, this shows that the only eigenvalues for the fermions number operator is $\nu_n = 0 \; or 1$. These two discrete eigenvalues alternate as the order number n grows as is seen in the expression $\nu_{n+1} = 1 - \nu_n$.

The coefficient $\mid C_k \mid^2$ is given by

$$\mid C_n \mid^2 = \frac{1}{2}(s_n + a_n) = 1 - \nu_n = \nu_{n+1} \tag{11.20}$$

Part II

Similarity Transformations

Chapter 12

Introduction

To describe a quantum system means choosing a Hilbert space of states on which the canonical variables are defined as operators. In turn, this means that definite representations of the canonical (or anti-) commutation relations has been chosen.

For the case of quantum systems with a denumerably finite number of degrees of freedom, all representations are unitary equivalent to each other. This fact may be used to our advantage when attempting to solve the Schrödinger equation for the quantum system.

The idea behind the Unitary Similarity Transformation method is to begin with a known eigenfunction and transform from a Hamiltonian with a simple form and structure to the more involved Hamiltonian and then use the transformation to find the transformed eigenfunction.

In the second part of this thesis I look at several examples of Unitary Similarity Transformations and how they can be used to simplify Hamiltonians describing quantum systems.

Examples of the Similarity Transformation Method include a method to determine the ground state eigenfunction using a generating function, Electron-Spin Resonance, the Foldy and Wouthuysen Transformation and an approach first proposed by Wentzel and applied by Schwinger to describe the non-relativistic interaction of an electron with a field. Schwinger used this approach to solve for the Lamb shift of the electron in a central coulombic potential.

Chapter 13

The Similarity Transformation

Start with the Schrödinger equation ($\hbar = 1$)

$$i\frac{\partial\psi}{\partial t} = \hat{H}\ \psi \tag{13.1}$$

Consider the unitary transformation given by [7]

$$\psi^{'} = U\ \psi \tag{13.2}$$

Inverting this transformation we have

$$\psi = U^{\dagger}\ \psi^{'} \tag{13.3}$$

which in terms of the Schrödinger equation in Hamitonian form

$$i\frac{\partial U^{\dagger}\ \psi^{'}}{\partial t} = \hat{H}\ U^{\dagger}\ \psi^{'} \tag{13.4}$$

If the unitary transformation does not explicitly depend on time then

$$i\frac{\partial U^{\dagger}\ \psi^{'}}{\partial t} = iU^{\dagger}\ \frac{\partial\ \psi^{'}}{\partial t} \tag{13.5}$$

from which we find

$$i\frac{\partial\ \psi^{'}}{\partial t} = U\ \hat{H}\ U^{\dagger}\ \psi^{'} = \hat{H}^{\dagger}\ \psi^{'} \tag{13.6}$$

where $\hat{H}^\dagger = U\ \hat{H}\ U^\dagger$.

If the transformation depends explicitly on time we find

$$i\frac{\partial U^\dagger\ \psi^{'}}{\partial t} = i\frac{\partial U^\dagger}{\partial t}\ \psi^{'} + iU^\dagger\ \frac{\partial\ \psi^{'}}{\partial t} \tag{13.7}$$

so then

$$i\ U^\dagger\ \frac{\partial\ \psi^{'}}{\partial t} = \ \hat{H}\ U^\dagger\ \psi^{'} - i\frac{\partial U^\dagger}{\partial t}\ \psi^{'} \tag{13.8}$$

which in turn yields

$$\begin{aligned} i\ \frac{\partial\ \psi^{'}}{\partial t} &= \ U\ \hat{H}\ U^\dagger\ \psi^{'}\ - i\ U\ \frac{\partial U^\dagger}{\partial t}\ \psi^{'} \\ &= [\ \ U\ \hat{H}\ U^\dagger - i\ U\ \frac{\partial U^\dagger}{\partial t}\ ]\ \psi^{'} \end{aligned} \tag{13.9}$$

Chapter 14

The Fundamental Theorem of Algebra Applied to the QHO

The Fundamental Theorem of Algebra applied to second order differential equations states that an equation of the form

$$F = a\ \hat{D}^2\ + b\ \hat{D} + c \tag{14.1}$$

where $\hat{D} = \frac{d}{dx}$ is the differential operator, can always be expressed in terms of the product of two linear expressions in $\hat{D}$, namely

$$F = a\ \hat{D}^2\ + b\ \hat{D} + c\ =\ a_0\ (\hat{D} - r_1)(\hat{D} - r_2) \tag{14.2}$$

where r_1 and r_2 are the roots of the second order differential equation F.

Returning to the example of the Quantum Harmonic Oscillator, and taking note that

$$\hat{p} = -\hat{p}^\dagger = i\hbar \frac{d}{dx} \tag{14.3}$$

the Hamiltonian can be reformulated in the following fashion,

$$\hat{H} = a_0(\hat{p} - a)^\dagger\ (\hat{p} - a)\ = a_0(\hat{p}^\dagger - a^\dagger)\ (\hat{p} - a) \tag{14.4}$$

where $a_0 = \frac{1}{2m}$ and $a = im\omega x$.

This relationship is expected since

$$< n \mid \hat{H} \ \mid n >= E < n \mid n > \tag{14.5}$$

which sets the constraints

$$< n \mid \ (\hat{p} - a)^{\dagger} = (\hat{p} - a) \ \mid n > \tag{14.6}$$

When expanded out the Hamiltonian for the QHO becomes

$$\hat{H} = a_0(\hat{p}^{\dagger} \ \hat{p} - \hat{p}^{\dagger} \ a - a^{\dagger} \ \hat{p} + a^{\dagger} a) \tag{14.7}$$

however we know that $\hat{p}^{\dagger} \ \hat{p} = p^2$ and that $a^{\dagger} = -a$ so then this equation becomes

$$\hat{H} = a_0(p^2 + a^{\dagger} a + \hat{p} \ a + a \ \hat{p}) \tag{14.8}$$

Using the commutation relation for the a term

$$[a, p] = i\hbar \ \frac{\partial a}{\partial x} \tag{14.9}$$

we get

$$\hat{H} = a_0(p^2 + a^{\dagger} a + 2\hat{p} \ a + i\hbar \frac{\partial a}{\partial x}) = \ a_0(p^2 + a^{\dagger} a - i\hbar \frac{\partial a}{\partial x}) \tag{14.10}$$

We see then that

$$\hat{H} = a_0(\hat{p} - a)^{\dagger} \ (\hat{p} - a) \ = \ a_0(p^2 + a^{\dagger} a + \hbar m\omega) \tag{14.11}$$

So then

$$\hat{H} = \frac{1}{2m} \ p^2 + \frac{m\omega^2 x^2}{2} + \frac{\hbar\omega}{2} \tag{14.12}$$

where a rescaling of $\hat{H}$ can be done to remove the zero point energy

$$E_{zp} = \ \frac{\hbar\omega}{2} \tag{14.13}$$

Chapter 15

A Similarity Transformation Applied to the QHO

Consider now the term $(\hat{p} - a)$ to be an operator acting to the right on a state $\mid u >$ namely

$$(\hat{p} - a) \;\mid u >= (-i\hbar\frac{d}{dx} \;- a) \;\mid u >= -i\hbar(\frac{d}{dx} + \frac{a}{i\hbar}) \;\mid u > \tag{15.1}$$

Introduce a unitary transformation operator e^F such that

$$e^F \; e^{-F} = 1 \tag{15.2}$$

which we introduce between the operator and the state that it is operating on

$$-i\hbar(\frac{d}{dx} + \frac{a}{i\hbar}) \; e^F \; e^{-F} \;\mid u >= -i\hbar(\frac{d}{dx} + \frac{a}{i\hbar}) \; e^F \;\mid w > \tag{15.3}$$

where $\mid w >= \; e^{-F} \;\mid u >$.

Expanding out the expression we see

$$\begin{aligned} -i\hbar(\frac{d}{dx} + \frac{a}{i\hbar}) \; e^F \;\mid w > &= -i\hbar(e^F \; \frac{dF}{dx} + e^F \; D - \frac{ia}{\hbar} \; e^F) \;\mid w > \\ &= -i\hbar \; e^F \; (DF + D - \frac{ia}{\hbar}) \;\mid w > \end{aligned} \tag{15.4}$$

What we have is an expression of the form

$$(\hat{p}-a) \ \mid u>= \ e^{F} \ (\hat{p}-a \ + \ \hat{p} \ F) \mid w> \tag{15.5}$$

We are free to choose the function F to meet our needs.

Returning to the Quantum Harmonic Oscillator, let $\hat{p}\, F = a$ that is $DF = \frac{ia}{\hbar}$ where $a = im\omega x$, we get

$$\frac{dF}{dx} = \frac{ia}{\hbar} = \frac{-m\omega \ x}{\hbar} \tag{15.6}$$

The solution of this equation is

$$F = \int \frac{-m\omega \ x}{\hbar} \ dx = \frac{-m\omega \ x^2}{2\hbar} \tag{15.7}$$

We see then that

$$\mid w>= \ e^{-F} \ \mid u>= \exp \ [\ \frac{m\omega \ x^2}{2\hbar} \ ] \ \mid u> \tag{15.8}$$

So what we have found is

$$(\hat{p}-im\omega x) \ \mid u>= \ \exp \ [\ \frac{-m\omega \ x^2}{2\hbar} \ ] \ (\hat{p}) \ \exp \ [\ \frac{m\omega \ x^2}{2\hbar} \ ] \mid u> \tag{15.9}$$

It is now straightforward to find the state $\mid 0>$ by solving the equation

$$\hat{p} \ \ (\ [\ \exp \frac{m\omega \ x^2}{2\hbar} \ ] \ \mid 0>) = 0 \tag{15.10}$$

which has the solution

$$[\ \exp \frac{m\omega \ x^2}{2\hbar} \ ] \ \mid 0>= C \tag{15.11}$$

where C is a constant of integration, yielding (to within an arbitrary phase factor)

$$\mid 0>= \ C \ \exp \ [\ \frac{-m\omega x^2}{2\hbar} \ ] \tag{15.12}$$

In solving the equation for the state $\mid 0>$ we have used an operator to transform the lowest energy state, that is

$$\mid 0> \Rightarrow \ [\ \exp \frac{m\omega \ x^2}{2\hbar} \ ] \ \mid 0>= \mid w>_0 \tag{15.13}$$

Let us look again at the Hamiltonian $\hat{H}$

$$< u \mid \hat{H} \mid u >=< u \mid a_0(\hat{p}-a)^{\dagger} (\hat{p}-a) \mid u > \tag{15.14}$$

Applying the similarity transformation

$$(\hat{p}-a) \mid u >= e^F \hat{p} e^{-F} \mid u > \tag{15.15}$$

Consider now the adjoint, namely

$$(e^F \hat{p} e^{-F} \mid u >)^{\dagger} = (\mid u >)^{\dagger} (e^{-F})^{\dagger} (\hat{p})^{\dagger} (e^F)^{\dagger} =< u \mid e^F (\hat{p})^{\dagger} e^{-F} \tag{15.16}$$

This means the transformed Hamiltonian equation becomes

$$\begin{aligned} &< u \mid (\hat{p}-a)^{\dagger} (\hat{p}-a) \mid u > \\ &= < u \mid e^F (\hat{p})^{\dagger} e^{-F} e^F \hat{p} e^{-F} \mid u > \\ &=< w \mid (\hat{p})^{\dagger} \hat{p} \mid w > \end{aligned} \tag{15.17}$$

That is

$$< u \mid \hat{H} \mid u >= \frac{1}{2m} < w \mid (\hat{p})^{\dagger} \hat{p} \mid w > \tag{15.18}$$

Let us confirm this by straightforward calculation.

For the QHO we know that state $\mid n >$ in the number representation is given by

$$\mid n >= C_n H_n[(\sqrt{\frac{m\omega}{\hbar}})x] \exp \frac{-m\omega x^2}{2\hbar} \tag{15.19}$$

where H_n is the Hermite polynomial and C_n is the normalization constant for state $\mid n >$.

This means that the transformed state $\mid w >$ is given by

$$\begin{aligned} &\mid w >= e^{-F} \mid n > \\ &= \exp \frac{m\omega x^2}{2\hbar} [C_n H_n[(\sqrt{\frac{m\omega}{\hbar}})x] \exp \frac{-m\omega x^2}{2\hbar} \\ &= [C_n H_n[(\sqrt{\frac{m\omega}{\hbar}})x] \end{aligned} \tag{15.20}$$

Applying the momentum operator $\hat{p}$ to the state $\mid w >$ yields

$$\hat{p} \ \mid w >= -i \ 2n \ C_0 \ \sqrt{\frac{\hbar m\omega}{2^n n!}} \ H_{n-1}[(\sqrt{\frac{m\omega}{\hbar}})x\ ] \tag{15.21}$$

The Hamiltonian then is

$$\frac{1}{2m} \ < w \mid \ (\hat{p})^{\dagger -F} \ e^F \hat{p} \ \mid w >= \frac{1}{2m} \ \mid e^F \hat{p} \ \mid w > \mid^2 \tag{15.22}$$

In terms of the representation given above

$$\begin{aligned} & \frac{1}{2m} \ \mid e^F \hat{p} \ \mid w > \mid^2 \\ = \frac{1}{2m}[C_0 \sqrt{\frac{2n\hbar m\omega}{2^n \ n!}} \ ]^2 \int_{-\infty}^{\infty} & [H_{n-1}[(\sqrt{\frac{m\omega}{\hbar}})x\ ]^2 \ \exp \frac{-m\omega x^2}{\hbar} \ dx \\ & = \ n\hbar\omega \end{aligned} \tag{15.23}$$

as expected.

Chapter 16

Formal Expression for the Similarity Transformation

Having shown its application to the Quantum Harmonic Oscillator let us now express the similarity transformation operator in a more formal fashion.

Consider the Hamiltonian equation given by

$$< u \mid \ \hat{H}_0 \ \mid u >= \frac{1}{2m} \ < u \mid \ \hat{p}^{\dagger} \ \hat{p} \ \mid u > \tag{16.1}$$

Let us try to express a different Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ by transforming the Hamiltonian $\hat{H}_0$, using a unitary transformation e^F such that

$$< u \mid \ e^F \ e^{-F} \ \hat{H}_0 \ e^F \ e^{-F} \ \mid u > \Rightarrow < w \mid \ \hat{H}_0 + \hat{H}_1 \ \mid w > \tag{16.2}$$

where the state $\mid w >$ is given by

$$\mid w >= \ e^{-F} \ \mid u > \tag{16.3}$$

and where

$$e^{-F} \ \hat{H}_0 \ e^F = \hat{H}_0 + \hat{H}_1 \tag{16.4}$$

From the transformation we get [8]

$$e^{-F} \ \hat{H}_0 \ e^F = \ \hat{H}_0 + \ [\hat{H}_0, F] + \frac{1}{2!}[[\hat{H}_0, F], F] + \frac{1}{3!} \ [[[\hat{H}_0, F], F], F] + \ \dots \tag{16.5}$$

So then we must find F such that

$$e^{-F}\ \hat{H}_0\ e^{F} = \ \hat{H}_0 + \ [\hat{H}_0, F] + \frac{1}{2!}[[\hat{H}_0, F], F] + \ \ldots = \ \hat{H}_0 + \hat{H}_1 \tag{16.6}$$

or solving for $\hat{H}_1$

$$\hat{H}_1 = \ [\hat{H}_0, F] + \frac{1}{2!}[[\hat{H}_0, F], F] + \frac{1}{3!}\ [\ [\ [\hat{H}_0, F], F], F] + \ \ldots \tag{16.7}$$

The function F is made up of fundamental dynamical variables. It can be constructed from generalized coordinates and their conjugates or from creation and annihilation operators.

Let us return to the example of the Quantum Harmonic Oscillator with the Hamiltonian given by

$$\hat{H} = \ \frac{1}{2m}\hat{p}^2 + \frac{(m\omega)^2}{2\hbar}q^2 = \ \hat{H}_0 + \frac{(m\omega)^2}{2\hbar}q^2 \tag{16.8}$$

where $\hat{H}_0 = \ \frac{1}{2m}\hat{p}^2$. We shall construct the function F in terms of the position operator q.

Let us try the Ansatz $F = \alpha q^2$, so that

$$\frac{m\omega^2}{2}q^2 = \ [\hat{H}_0, \alpha\ q^2] + \frac{1}{2!}[[\hat{H}_0, \alpha\ q^2], \alpha\ q^2] + \ \ldots \tag{16.9}$$

Evaluating this expression term by term, [9]

$$[\hat{H}_0, \alpha\ q^2] = \ \frac{\alpha}{2m}\ [\hat{p}^2,\ q^2] = \ \frac{2i\hbar\alpha}{m}\ (\hat{p}q + q\hat{p}) \tag{16.10}$$

It is worth noting the symmetrization of this term. The next term is

$$\frac{1}{2!}[[\hat{H}_0, \alpha\ q^2], \alpha\ q^2] = \ \frac{i\hbar\alpha}{2m}\ [(\hat{p}q + q\hat{p}),\ \alpha\ q^2] \tag{16.11}$$

It can be seen that this term is equal to

$$\frac{1}{2!}[[\hat{H}_0, \alpha\ q^2], \alpha\ q^2] = \ \frac{2\hbar^2\alpha^2}{m}\ q^2 \tag{16.12}$$

All further terms in the expansion being zero since they commute with $F = \alpha\ q^2$.

This means then that the eigenvalue for the $| w >$ is given by

$$| w >= \exp(-\alpha q^2) \; | u > \tag{16.13}$$

where we see that

$$\frac{(m\omega)^2}{2m} q^2 = \frac{2\hbar^2 \alpha^2}{m} \, q^2 \tag{16.14}$$

so then

$$\alpha = \frac{(m\omega)}{2\hbar} \tag{16.15}$$

We have recovered the expression for the Quantum Harmonic Oscillator using a similarity transformation of the free field hamiltonian $\hat{H}_0$.

Chapter 17

The Generating Function

In terms of powers of the position operator q it can be shown that any term of the form $\alpha_n\ q^n$ will lead to an expression of the form

$$\hat{H}_1 = \frac{\hbar^2}{2m}\ [\ n(n-1)\alpha_n\ q^{n-2} + n^2\ \alpha_n^2\ q^{2n-2}\ ] \tag{17.1}$$

If you have a function f which is a polynomial in q then $f(q)$ provides the following expression

$$\hat{H}_1 = \frac{\hbar^2}{2m}\ [\ \frac{d^2 f}{dq^2} + (\frac{df}{dq})^2\ ] \tag{17.2}$$

Consider then the function F made up of a sum of powers in q, namely

$$F = \sum_{n=0}^{\infty}\ a_n\ q^n \tag{17.3}$$

We have then the following

$$\frac{d^2 f}{dq^2} = \sum_{n=2}^{\infty}\ n\ (n-1)\ a_n\ q^{n-2} \tag{17.4}$$

and

$$(\frac{df}{dq})^2 = [\ \sum_{n=1}^{\infty}\ na_n\ q^{n-1}\ ]\ [\ \sum_{m=1}\ ma_m\ q^{m-1}\ ] \tag{17.5}$$

so then we arrive at

$$\frac{\hbar^2}{2m}\left[\frac{d^2 f}{dq^2} + \left(\frac{df}{dq}\right)^2\right] =$$

$$\begin{aligned}
&\frac{\hbar^2}{2m}\ [\ (2a_2 + a_1^2)\ q^0 \\
&+ (6a_3 + 4a_1a_2)\ q \\
&+ (12a_4 + 6a_1a_3 + 4a_2^2)\ q^2 \\
&+ (20a_5 + 8a_1a_4 + 12a_2a_3)\ q^3 \\
&+ (30a_6 + 10a_1a_5 + 16a_2a_4 + 9a_3^2)\ q^4 \\
&+(42a_7 + 12a_1a_6 + 20a_2a_5 + 24a_3a_4)\ q^5 \\
&+(56a_8 + 14a_1a_7 + 24a_2a_6 + 30a_3a_5 + 16a_4^2)\ q^6 \\
&+(72a_9 + 16a_1a_8 + 28a_2a_7 + 36a_3a_6 + 40a_4a_5)\ q^7 \\
&+(90a_{10} + 18a_1a_9 + 32a_2a_8 + 42a_3a_7 + 48a_4a_6 + 25a_5^5)\ q^8 \\
&+ (110a_{11} + 20a_1a_{10} + 36a_2a_9 + 48a_3a_8 + 56a_4a_7 + 60a_5a_6)\ q^9 \\
&+ (132a_{12} + 22a_1a_{11} + 40a_2a_{10} + 54a_3a_9 + 64a_4a_8 + 70a_5a_7 + 36a_6^2)\ q^{10} \\
&+ \ldots\ ]
\end{aligned}$$

By inspection we see that the lead term in each power of q^n has a coefficient given by

$$(n+2)(n+1) \tag{17.6}$$

each mixed term $a_n a_m$ has a coefficient given by

$$2nm \tag{17.7}$$

and each repeated term $a_n a_n$ has the coefficient

$$n^2 \tag{17.8}$$

The mixed indices in each line start from the right with $a_1 a_m$ where $m+1 = n+2$ and continue up to $a_{n-2}a_{n-2}$ for even powers of n, or $a_{n-3}a_{n-2}$ for odd powers of n.

Chapter 18

The Hermite-Lindemmann Transcendental Theorem

Having derived an expression for a type of similarity transformation it is worth noticing the limitations set out by the Hermite-Lindemmann Transcendental Theorem which states that a function of the form

$$G(q) = \sum_{n} A_n \ \exp[F_n(q)] \tag{18.1}$$

cannot be equal to zero.

This means that the function $F(q)$ outlined above could not apply for state eigenfunctions that have a zero in the domain, that is, an eigenfunction that intercept or crosses the x-axis.

To accommodate eigenfunctions with a zero in the domain we need to consider functions of the form

$$D(q) = \sum_{n} \beta_n q^n \ \exp \sum_{m} \alpha_n \ q^n \tag{18.2}$$

or equivalently

$$D(q) = \exp\,[\ln \sum_{n} \beta_n q^n \ \ + \ \sum_{m} \alpha_n \ q^n] \tag{18.3}$$

It is a well known theorem that for a symmetric potential that the eigenfunction of lowest energy has no nodes. Only symmetric potentials will be considered in this thesis. [10]

Chapter 19

The Base State of the Hamiltonian

Before proceeding to study some example Hamiltonians it is worth noting the expression we get when we set each term of the function $F(q)$ in powers of q^n equal to zero, namely

$$(2a_2 + a_1^2) = (6a_3 + 4a_1 a_2) = (12a_4 + 6a_1 a_3 + 4a_2^2) = \ldots = 0 \qquad (19.1)$$

Expressed in terms of a_1 this yields the following exponential function

$$\begin{aligned} \Omega &= \exp\ [\ a_1 q - \frac{1}{2}a_1^2 q^2 + \frac{1}{3}a_1^3 q^3 - \frac{1}{4}a_1^4 q^4 + \ldots\ ] \\ &= \exp\ [\ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(a_1\ q)^n}{n}\ ] \end{aligned} \qquad (19.2)$$

The series in the exponential is the familiar Taylor series expansion of $\ln(1 + a_1 q)$ that is

$$\begin{aligned} \ln\ (1 + a_1 q) =&\ a_1 q - \frac{1}{2}(a_1\ q)^2 + \frac{1}{3}(a_1\ q)^3 - \frac{1}{4}(a_1\ q)^4 + \ldots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(a_1\ q)^n}{n} \end{aligned} \qquad (19.3)$$

provided $\mid a_1\ q \mid \leq 1$ and $a_1\ q \neq -1$.

We find then that

$$\Omega = \exp[\ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(a_1\ q)^n}{n}\ ] = \exp[\ \ln(1 + a_1 q)\ ] \qquad (19.4)$$

Let us now consider Ω to be an eigenstate of the Hamiltonian $\hat{H}$. We see that

$$\hat{H}_0 \ \mid \Omega > = \ \frac{1}{2m} \, \hat{p}^\dagger \ \hat{p} \ \mid \Omega > = 0 \tag{19.5}$$

Consider now the momentum operator acting on Ω, namely

$$\hat{p} \ \mid \Omega >= -i\hbar \frac{d}{dq} \ \mid \Omega >= -i\hbar \frac{a_1}{(1 + a_1 \ q)} \ \mid \Omega > \tag{19.6}$$

We can make the expression real by setting $a_1 = i$, in which case

$$\hat{p} \ \mid \Omega >= \frac{\hbar}{(1 + i \ q)} \ \mid \Omega > \tag{19.7}$$

(or we can set $a_1 = 0$ which leads to a trivial solution).

Let us set $a_1 = i$, to give a function

$$\mid \Omega >= \exp[\ \ln(1 + i \ q) \ ] \tag{19.8}$$

As one can see then for the momentum operator $\hat{p}$

$$< \Omega \mid \ \hat{p} \ \mid \Omega >= \ \hbar \tag{19.9}$$

If we need we can redefine the momentum operator $\hat{p}$ so that

$$\tilde{p} = \hat{p} - \hbar = -\hbar(1 + i \frac{d}{dq}) \tag{19.10}$$

in which case $< \Omega \mid \ \tilde{p} \ \mid \Omega >= 0$, while we still retain

$$[\ q, \hat{p}\ ] = [\ q, \tilde{p}\ ] = i\hbar \tag{19.11}$$

With the rescaling of the momentum operator $\hat{p} \Rightarrow \tilde{p}$ we have a rescaling of the Hamiltonian

$$\frac{1}{2m}\,\tilde{p}^{\dagger}\,\tilde{p} = \frac{1}{2m}\,(\hat{p}^{\dagger}\,\hat{p} + \hbar^2) \tag{19.12}$$

namely

$$\hat{H}_0 \Rightarrow \hat{H}_0 + \frac{\hbar^2}{2m} \tag{19.13}$$

For the original Hamiltonian $\hat{H}_0$ we have

$$< \Omega \mid \; \hat{H}_0 \; \mid \Omega >= 0 \tag{19.14}$$

and for the position operator $\hat{q}$ we have

$$< \Omega \mid \; \hat{q} \; \mid \Omega >= 0 \tag{19.15}$$

We can express $\mid \Omega >$ in the following equivalent fashion

$$\mid \Omega >= \rho \; \exp(i\theta) = \mid \Omega >_{norm} \tag{19.16}$$

where

$$\rho = \frac{1}{\sqrt{[1 + \mid q \mid^2]}} \tag{19.17}$$

and

$$\theta = \arctan(q) \tag{19.18}$$

In this fashion, we have defined a normalized *base state* $\mid \Omega >$ for the system.

Chapter 20

Example Hamiltonians

Let us now build up two example Hamiltonians using the generating function:

The Quantum Harmonic Oscillator

The expression

$$\frac{\hbar^2}{2m}[\ n(n-1)\ \alpha q^{n-2} + n^2\alpha_n^2 q^{2n-2}\ ] \tag{20.1}$$

provides a clue as to how best to build up the expression for any Hamiltonian $\hat{H}$.

Let us take as an example the Quantum Harmonic Oscillator where

$$\hat{H}_1 = \frac{1}{2}m\omega^2\ q^2 \tag{20.2}$$

If $n = 2$ we get two terms, one in q^0 and the second in q^2, so then $\alpha_2 \neq 0$ in the series for F.

We know for the QHO the zero point contribution is a term of order q^0, namely

$$\hat{H}_{ZP} = \frac{\hbar\omega}{2} \tag{20.3}$$

so by inspection we see that

$$\frac{\hbar^2}{2m}[\ 2a_2 + a_1^2] = \frac{\hbar\omega}{2} \tag{20.4}$$

or solving for a_2

$$a_2 = -\frac{1}{2}\ a_1^2 + \frac{m\omega}{2\hbar} \tag{20.5}$$

where we recognize the familiar term $\frac{m\omega}{2\hbar}$. The first term $-\frac{1}{2}\ a_1^2$ can be considered the base state contribution.

Continuing on and solving for a_3 we see

$$a_3 = \frac{1}{3}\ a_1^3 - \frac{1}{3}\ (\frac{m\omega}{\hbar})a_1 \tag{20.6}$$

and for a_4

$$a_4 = -\frac{1}{4}\ a_1^4 + \frac{1}{3}\ (\frac{m\omega}{\hbar})a_1^2 - \frac{1}{12}\ (\frac{m\omega}{\hbar})^2 \tag{20.7}$$

and for a_5

$$a_5 = \frac{1}{5}\ a_1^5 - \frac{1}{3}\ (\frac{m\omega}{\hbar})a_1^3 + \frac{2}{15}\ (\frac{m\omega}{\hbar})^2\ a_1 \tag{20.8}$$

and for a_5

$$a_5 = -\frac{1}{6}\ a_1^6 + \frac{1}{3}\ (\frac{m\omega}{\hbar})a_1^4 - \frac{17}{90}\ (\frac{m\omega}{\hbar})^2\ a_1^2 + \frac{1}{45}\ (\frac{m\omega}{\hbar})^3 \tag{20.9}$$

and so on to higher order.

Grouping the terms we have

$$\begin{aligned} F = a_1\ q - \frac{1}{2}\ a_1^2\ q^2 + \frac{1}{3}\ a_1^3\ q^3 - \frac{1}{4}\ a_1^4\ q^4 + \frac{1}{5}\ a_1^5\ q^5 - \frac{1}{6}\ a_1^6\ q^6 + \dots \\ + \frac{m\omega}{2\hbar}\ q^2 - \frac{1}{3}\ (\frac{m\omega}{\hbar})a_1\ q^3 + [\ \frac{1}{3}\ (\frac{m\omega}{\hbar})a_1^2 - \frac{1}{12}\ (\frac{m\omega}{\hbar})^2\ ]\ q^4 \\ +[\ -\frac{1}{3}\ (\frac{m\omega}{\hbar})a_1^3 + \frac{2}{15}\ (\frac{m\omega}{\hbar})^2\ a_1\ ]\ q^5 \\ +\ [\ \frac{1}{3}\ (\frac{m\omega}{\hbar})a_1^4 - \frac{17}{90}\ (\frac{m\omega}{\hbar})^2\ a_1^2 + \frac{1}{45}\ (\frac{m\omega}{\hbar})^3\ ]\ q^6 + \dots \end{aligned} \tag{20.10}$$

We recognize the first set of terms as being $\ln{(1 +\ a_1 q)}$.

Rearranging terms we get

$$F = \ln(1 + a_1 q) + \frac{m\omega}{2\hbar} q^2 [\, 1 - \frac{2}{3} \frac{a_1 q}{(1 + a_1 q)} + \frac{m\omega}{2\hbar} \vartheta(q^2) \,]$$

$$= \ln(1 + a_1 q) + \frac{m\omega}{2\hbar} q^2 [\, 1 - \frac{2}{3} \frac{i.q}{(1 + q^2)} - \frac{2}{3} \frac{q}{(1 + q^2)} + \frac{m\omega}{2\hbar} \vartheta(q^2) \quad (20.11)$$

We see then that in terms of the transformation

$$\exp(F) = \exp[\, \ln(1 + a_1 q) + \frac{m\omega}{2\hbar} q^2 [\, 1 - \frac{2}{3} \frac{a_1 q}{(1 + a_1 q)} + \frac{m\omega}{2\hbar} \vartheta(q^2) \,]] \quad (20.12)$$

or

$$\exp(F) = \exp[\ln(1 + a_1 q)] \exp[\frac{m\omega}{2\hbar} q^2 [\, 1 - \frac{2}{3} \frac{a_1 q}{(1 + a_1 q)} + \frac{m\omega}{2\hbar} \vartheta(q^2) \,]] \quad (20.13)$$

We can therefore express the lowest energy state of the Quantum Harmonic Oscillator as

$$| \, w >= \exp(-F) \, | \, \Omega >= \exp[\, - \ln(1 + a_1 q) \,] \exp[- \frac{m\omega}{2\hbar} q^2 [\, 1 - \frac{2}{3} \frac{a_1 q}{(1 + a_1 q)} + \frac{m\omega}{2\hbar} \vartheta(q^2) \,]] \quad | \, \Omega > \quad (20.14)$$

We are free to set the value of $a_1 = 0$ so then we have

$$| \, w >= \exp[- \frac{m\omega}{2\hbar} q^2 \,]$$

which, apart from a normalization term, is what we expect to find for the Quantum Harmonic Oscillator.

What has been done is starting with the base state $| \, \Omega >$ we have used a similarity transformation to transform the base state into the ground state of the Quantum Harmonic Oscillator.

Returning to the transformation of the Hamiltonian $\hat{H}_0$ we see

$$e^{-F}\ \hat{H}_0\ e^F = \frac{1}{2m}\ e^{-F}\hat{p}^\dagger\ e^F\ e^{-F}\hat{p}\ e^F \tag{20.15}$$

Expanding out each operator we have for the annihilation operator

$$e^{-F}\hat{p}\ e^F = \hat{p} + [\hat{p}, F] = \hat{p} - [F, \hat{p}] = \hat{p} - i\hbar\frac{d}{dq}F \tag{20.16}$$

and for the creation operator

$$e^{-F}\hat{p}^\dagger\ e^F = \hat{p}^\dagger + [\hat{p}^\dagger, F] = \hat{p}^\dagger - [F, \hat{p}^\dagger] = -\hat{p} + i\hbar\frac{d}{dq}F \tag{20.17}$$

which for the case of the Quantum Harmonic Oscillator yields ($a_1 = 0$)

$$e^{-F}\hat{p}\ e^F = \hat{p} - i\hbar\frac{d}{dq}F = \hat{p} - i\hbar\frac{m\omega}{\hbar}\ q \tag{20.18}$$

and

$$e^{-F}\hat{p}^\dagger\ e^F = -\hat{p} + i\hbar\frac{d}{dq}F = -\hat{p} + i\hbar\frac{m\omega}{\hbar}\ q \tag{20.19}$$

We can now go on to build up the other states of the Quantum Harmonic Oscillator in the usual fashion.

The $\hat{H}_1 = \lambda\ q$ Hamiltonian

Let us take as a second example the symmetric potential given by

$$\hat{H}_1 = \lambda\ \mid q\ \mid \tag{20.20}$$

We see then for this potential we start with a_1 and find for a_2

$$a_2 = -\frac{1}{2}a_1^2 \tag{20.21}$$

for a_3

$$a_3 = \frac{1}{3}a_1^3 + \frac{1}{3}(\frac{m\lambda}{\hbar^2}) \tag{20.22}$$

for a_4

$$a_4 = -\frac{1}{4}a_1^4 - \frac{1}{6}(\frac{m\lambda}{\hbar^2})a_1 \tag{20.23}$$

for a_5

$$a_5 = \frac{1}{5}a_1^5 + \frac{1}{6}(\frac{m\lambda}{\hbar^2})a_1^2 \tag{20.24}$$

for a_6

$$a_6 = -\frac{1}{6}a_1^6 - \frac{1}{6}(\frac{m\lambda}{\hbar^2})a_1^3 - \frac{1}{30}(\frac{m\lambda}{\hbar^2})^2 \tag{20.25}$$

and so on to higher order.

Grouping the terms we have

$$\begin{aligned} F = a_1\ q - \frac{1}{2}\ a_1^2\ q^2 + \frac{1}{3}\ a_1^3\ q^3 - \frac{1}{4}\ a_1^4\ q^4 + \frac{1}{5}\ a_1^5\ q^5 - \frac{1}{6}\ a_1^6\ q^6 + \ldots \\ +\frac{1}{3}(\frac{m\lambda}{\hbar^2})\ q^3 - \frac{1}{6}(\frac{m\lambda}{\hbar^2})a_1\ q^4 + \frac{1}{6}(\frac{m\lambda}{\hbar^2})a_1^2\ q^5 - \frac{1}{6}(\frac{m\lambda}{\hbar^2})a_1^3\ q^6 \\ -\frac{1}{30}(\frac{m\lambda}{\hbar^2})^2\ q^6 \end{aligned} \tag{20.26}$$

or

$$\begin{aligned} F = \ln(1 + a_1\ q) + \frac{1}{3}(\frac{m\lambda}{\hbar^2})\ q^3\ (1 - \frac{1}{2}a_1\ q + \frac{1}{2}a_1^2\ q^2 - \frac{1}{2}a_1^3\ q^3 + \ldots) \\ -\frac{1}{30}(\frac{m\lambda}{\hbar^2})^2\ q^6 \end{aligned} \tag{20.27}$$

By inspection we see that the next term is of the order

$$-\frac{4}{30}(\frac{m\lambda}{\hbar^2})^4\ q^7 \tag{20.28}$$

Repeating the same procedure as outlined above for the QHO we have for the state $\mid w >$

$$\mid w >= C_{norm}\ \exp[-\frac{1}{3}(\frac{m\lambda}{\hbar^2})\ q^3\ -\frac{1}{30}(\frac{m\lambda}{\hbar^2})^2\ q^6 - \vartheta(\frac{4}{30}(\frac{m\lambda}{\hbar^2})^4\ q^7) + \ldots] \tag{20.29}$$

where C_{norm} is a normalization constant.

Expanding out the momentum operators we see

$$e^{-F}\hat{p}\ e^F = \hat{p} - i\hbar\frac{d}{dq}F$$
$$\approx \hat{p} - i\hbar[\frac{m\lambda}{\hbar^2})\ q^2 + \frac{1}{5}(\frac{m\lambda}{\hbar^2})^2\ q^5 + \vartheta(\frac{28}{30}(\frac{m\lambda}{\hbar^2})^4\ q^6)] \tag{20.30}$$

and

$$e^{-F}\hat{p}^\dagger\ e^F = -\hat{p} + i\hbar\frac{d}{dq}F$$
$$\approx -\hat{p} + i\hbar[\frac{m\lambda}{\hbar^2})\ q^2 + \frac{1}{5}(\frac{m\lambda}{\hbar^2})^2\ q^5 + \vartheta(\frac{28}{30}(\frac{m\lambda}{\hbar^2})^4\ q^6)] \tag{20.31}$$

We can now go on to build up the other states of the system in the usual fashion.

The $\hat{H}_1 = \alpha\ q^2 + \beta\ q^4$ Hamiltonian

Let us take as a third example the potential given by $\hat{H}_1 = \alpha\ q^2 + \beta\ q^4$. We see then for this potential we start with a_1 and find for a_2

$$a_2 = -\frac{1}{2}a_1^2 \tag{20.32}$$

for a_3

$$a_3 = \frac{1}{3}a_1^3 \tag{20.33}$$

for a_4

$$a_4 = -\frac{1}{4}a_1^4 - \frac{1}{6}(\frac{m\alpha}{\hbar^2}) \tag{20.34}$$

for a_5

$$a_5 = \frac{1}{5}a_1^5 - \frac{1}{15}(\frac{m\alpha}{\hbar^2})a_1 \tag{20.35}$$

for a_6

$$a_6 = -\frac{1}{6}a_1^6 + \frac{1}{15}(\frac{m\alpha}{\hbar^2})\ a_1^2 + \frac{1}{15}(\frac{m\beta}{\hbar^2}) \tag{20.36}$$

for a_7

$$a_7 = \frac{1}{7}\ a_1^7 - \frac{1}{15}\ (\frac{m\alpha}{\hbar^2})\ a_1^3 - \frac{2}{105}(\frac{m\beta}{\hbar^2})\ \ a_1 \tag{20.37}$$

and so on to higher order.

Grouping the terms we have

$$\begin{aligned} F = a_1\ q - \frac{1}{2}\ a_1^2\ q^2 + \frac{1}{3}\ a_1^3\ q^3 - \frac{1}{4}\ a_1^4\ q^4 + \frac{1}{5}\ a_1^5\ q^5 - \frac{1}{6}\ a_1^6\ q^6 + \frac{1}{7}\ a_1^7\ q^7 + \dots \\ + \frac{1}{6}(\frac{m\alpha}{\hbar^2})\ q^4 - \frac{1}{15}(\frac{m\alpha}{\hbar^2})a_1\ q^5 + \frac{1}{15}(\frac{m\alpha}{\hbar^2})a_1^2\ q^6 - \frac{1}{15}(\frac{m\alpha}{\hbar^2})a_1^3\ q^7 \\ + \frac{1}{15}(\frac{m\beta}{\hbar^2})\ q^6 - \frac{2}{105}(\frac{m\beta}{\hbar^2})a_1\ q^7 + \dots \end{aligned} \tag{20.38}$$

or

$$F = ln(1 + a_1\ q) + \frac{1}{6}(\frac{m\alpha}{\hbar^2})\ q^4 - \frac{1}{15}(\frac{m\alpha}{\hbar^2})a_1\ q^5 + \frac{1}{15}(\frac{m\alpha}{\hbar^2})a_1^2\ q^6 - \frac{1}{15}(\frac{m\alpha}{\hbar^2})a_1^3\ q^7 + \frac{1}{15}(\frac{m\beta}{\hbar^2})\ q^6 - \frac{2}{105}(\frac{m\beta}{\hbar^2})a_1\ q^7 + \dots \tag{20.39}$$

We can simplify this equation to read

$$\begin{aligned} F = ln(1 + a_1\ q) &+ \frac{1}{6}(\frac{m\alpha}{\hbar^2})\ q^4(1 - \frac{2}{5}a_1\ q + \frac{2}{5}\ a_1^2\ q^2 + \dots) \\ &+ \frac{1}{15}(\frac{m\beta}{\hbar^2})\ q^6(1 - \frac{2}{7}a_1\ q + \frac{2}{7}\ a_1^2\ q^2 + \dots) \\ &+ \vartheta[(\frac{m\alpha}{\hbar^2})(\frac{m\beta}{\hbar^2})\ q^8] \end{aligned} \tag{20.40}$$

With $| w >= \exp(-F) \mid \Omega >$ we set $a_1 = 0$, to find

$$| w >= \exp[-\frac{1}{6}(\frac{m\alpha}{\hbar^2})\ q^4 - \frac{1}{15}(\frac{m\beta}{\hbar^2})\ q^6 - \vartheta[(\frac{m\alpha}{\hbar^2})(\frac{m\beta}{\hbar^2})\ q^8] \tag{20.41}$$

Let us find the momentum operators for this example.

$$\begin{aligned} e^{-F}\hat{p}\ e^F &= \hat{p} - i\hbar\frac{d}{dq}F \\ = \hat{p} - i\hbar\frac{d}{dq}[\frac{1}{6}(\frac{m\alpha}{\hbar^2})\ q^4 &+ \frac{1}{15}(\frac{m\beta}{\hbar^2})\ q^6 + \vartheta[(\frac{m\alpha}{\hbar^2})(\frac{m\beta}{\hbar^2})\ q^8] \\ = \hat{p} - i\hbar[\frac{2}{3}(\frac{m\alpha}{\hbar^2})\ q^3 &+ \frac{2}{5}(\frac{m\beta}{\hbar^2})\ q^5 + \vartheta[8(\frac{m\alpha}{\hbar^2})(\frac{m\beta}{\hbar^2})\ q^7] \end{aligned} \tag{20.42}$$

and

$$\begin{aligned} e^{-F}\hat{p}^\dagger\ e^F &= -\hat{p} + i\hbar\frac{d}{dq}F \\ = -\hat{p} + i\hbar\frac{d}{dq}[\frac{1}{6}(\frac{m\alpha}{\hbar^2})\ q^4 &+ \frac{1}{15}(\frac{m\beta}{\hbar^2})\ q^6 + \vartheta[(\frac{m\alpha}{\hbar^2})(\frac{m\beta}{\hbar^2})\ q^8] \\ = -\hat{p} + i\hbar[\frac{2}{3}(\frac{m\alpha}{\hbar^2})\ q^3 &+ \frac{2}{5}(\frac{m\beta}{\hbar^2})\ q^5 + \vartheta[8(\frac{m\alpha}{\hbar^2})(\frac{m\beta}{\hbar^2})\ q^7] \end{aligned} \tag{20.43}$$

We can now build up the other states of this system in the usual fashion.

Chapter 21

Describing Electron-Spin Resonance Using A Similarity Transformation

Consider a two level spin system of an electron in a magnetic field with a Hamiltonian given by [8]

$$\hat{H} = \frac{\gamma \hbar}{2} \; [\; B_0 \; \sigma_z + B_1 (\sigma_+ \; \exp(-i\omega t) + \sigma_- \; \exp(i\omega t)) \;] \tag{21.1}$$

The unperturbed energy eigenvalues are given by

$$E_{\pm} \; = \; \pm \frac{\hbar \omega_0}{2} \tag{21.2}$$

where $\omega_0 = \gamma B_0$, with B_0 being the unperturbed magnetic field and B_1 being the perturbing field.

To solve for the eigenstate of the Hamiltonian we shall use a similarity transformation given by

$$\mid \psi(t) >= \; \exp(-i\omega t \frac{\sigma_z}{2}) \; \mid \chi(t) > \tag{21.3}$$

The transformed Schrödinger equation becomes

$$\begin{aligned} i \frac{\partial \mid \chi(t) >}{\partial t} = \frac{1}{2} \; [\; (\omega_0 - \omega) \; \sigma_z + \gamma \; B_1 (\sigma_+ \; \exp(-i\omega t) \\ + \sigma_- \; \exp(i\omega t)) \;] \; \mid \chi(t) > \end{aligned} \tag{21.4}$$

where we have transformed the σ to

$$\sigma_\pm(t) = \exp(i\omega t\frac{\sigma_z}{2})\ \sigma_\pm\ \exp(-i\omega t\frac{\sigma_z}{2}) \tag{21.5}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. In the case of the $\sigma_\pm$ it is easy to show that

$$\sigma_\pm(t) = \ \sigma_\pm\ \exp(\pm i\omega t) \tag{21.6}$$

so then the transformed Hamiltonian becomes

$$\begin{aligned} i\frac{\partial \mid \chi(t) >}{\partial t} = \frac{1}{2}\ [\ (\omega_0 - \omega)\ \sigma_z + \gamma\ B_1(\sigma_+\ \exp(-i\omega t) \\ +\sigma_-\ \exp(i\omega t))\ ]\ \mid \chi(t) > \end{aligned} \tag{21.7}$$

It is worth noting that σ_z and $\sigma_\pm$ are in the Schrödinger picture.

The solution to the transformed Schrödinger equation is

$$\mid \chi(t) >= \exp[\ (i\Omega\frac{t}{2})(\cos(\theta\sigma_z) + \sin(\theta\sigma_z))\ ] \mid \psi(0) > \tag{21.8}$$

where

$$\begin{aligned} \Omega\cos(\theta) = (\omega_0 - \omega) \\ \Omega\sin\theta = \gamma B_1 \end{aligned} \tag{21.9}$$

from which we get the explicit expression

$$\begin{aligned} \mid \psi(t) >= \exp\ (i\omega\frac{\sigma_z t}{2})\ [\cos(\Omega\frac{t}{2}) - i\sin(\Omega\frac{t}{2})(\cos(\theta\sigma_z) \\ + \sin(\theta\sigma_z))\ ]\ \mid \psi(0) >= U(t,0)\ \mid \psi(0) > \end{aligned} \tag{21.10}$$

From this solution we can extract the Resonance behaviour of the two level spin system.

Chapter 22

Describing the Interaction of an Electron with a Field

We shall now use a Similarity Transformation to describe the interaction of an electron with a electromagnetic field. [11].

The Schrödinger equation for a non-relativistic particle interacting with a field is

$$\imath\hbar\partial_t\Psi(t) = [\frac{1}{2m_o}\,(p + \frac{e}{c}A)^2 + V(r) + H_{rad}]\Psi(t) \tag{22.1}$$

where $\partial_t\Psi(t) = \frac{\partial\Psi(t)}{\partial t}$. Upon making the Similarity Transformation

$$\Psi'(t) = \exp\,[\frac{iH_{rad}t}{\hbar}]\;\Psi(t) \tag{22.2}$$

$$A(t) = \exp\,[\frac{iH_{rad}t}{\hbar}]\;A\;\exp\,[-\frac{iH_{rad}t}{\hbar}]$$

the new state vector satisfies the equation

$$\imath\hbar\partial_t\Psi'(t) = H'\;\Psi'(t) \tag{22.3}$$

where the transformed Hamiltonian is given by

$$H' = \frac{p^2}{2m_o} + V(r) + \frac{e}{m_o c}\,(\vec{p}\cdot\vec{A(t)}) + \frac{e^2}{2m_o c^2}\,A^2(t) \tag{22.4}$$

Now perform a second Similarity Transformation to remove the "virtual effects" induced by the second term (dropping the primes):

$$\Psi(t) \to \exp\left[-iS(t)\right]\Psi(t) \tag{22.5}$$

in such a fashion as to define S(t) using the equality

$$\hbar\frac{\partial S}{\partial t} = \frac{e}{m_o c}\,(\vec{p}\cdot \vec{A(t)}) \tag{22.6}$$

Let the vector A(t) be derived from a vector potential Z(t), namely

$$\vec{A(t)} = \frac{\partial \vec{Z(t)}}{\partial t} \tag{22.7}$$

then, since the momentum operator is a time-independent operator,

$$S(t) = \frac{e}{m_o c}\,(\vec{p}\cdot \vec{Z(t)}) \tag{22.8}$$

Looking on the transformation operator [8, 28]

$$\exp iS(t)\,\frac{\partial \exp\left[-iS(t)\right]}{\partial t} = -i\frac{\partial S}{\partial t} + \frac{1}{2}[S, \frac{\partial S}{\partial t}] \tag{22.9}$$

$$\exp iS(t)\,\frac{\partial S}{\partial t}\exp\left[-iS(t)\right] = \frac{\partial S}{\partial t} + [S, \frac{\partial S}{\partial t}] \tag{22.10}$$

with the series terminating because $[A, \dot{A}]$ is a c-number.

The transformed Schrödinger equation is

$$\imath\hbar\partial_t\Psi(t) = [\,\frac{p^2}{2m_o} + \frac{\imath\hbar}{2}[S, \frac{\partial S}{\partial t}] + \exp\left[iS(t)\right]\,V(r)\,\exp\left[-iS(t)\right] + \frac{e^2}{2m_o c^2}\,\exp\left[iS(t)\right]\,A(t)^2\exp\left[-iS(t)\right]\,]\Psi(t) \tag{22.11}$$

Using the expressions for S(t) and its time dependence outlined above the transformed Schrödinger equation becomes

$$\imath\hbar\partial_t\Psi(t) = [\ \frac{p^2}{2m_o} + \frac{\imath\hbar}{2}(\frac{e}{\hbar m_0 c})^2[p\cdot Z(t), p\cdot A(t)] + \exp[iS(t)]\ V(r)\ \exp[-iS(t)]\ ]\Psi(t)$$
$$= [\ \frac{p^2}{2m_o} + \frac{\imath\hbar}{2}(\frac{e}{\hbar m_0 c})^2[p\cdot Z(t), p\cdot A(t)] + V(r+\frac{eZ}{mc})\ ]\ \Psi(t) \tag{22.12}$$

where the A^2 has been dropped.

In the dipole approximation, with

$$A(t) = \sum_{k\mu}\epsilon_{k\mu}c\sqrt{\frac{\hbar}{2\omega_k}}\ [\ a_{k\mu}\exp -i\omega_k t +\ a^*_{k\mu}\exp i\omega_k t\ ]$$
$$Z(t) = i\ \sum_{k\mu}\epsilon_{k\mu}c\sqrt{\frac{\hbar}{2\omega_k^3}}\ [\ a_{k\mu}\exp[-i\omega_k t] -\ a^*_{k\mu}\exp[i\omega_k t]\ ] \tag{22.13}$$

the commutator of A(t) and Z(t) (with a suitable upper limit to the integration) becomes

$$[A_l, Z_m] = \frac{-i}{3}\ \delta_{lm}\ \int_0^{\omega_1} d\omega\ \frac{\hbar}{\pi^2 c} \tag{22.14}$$

So then

$$[A_l, Z_m] = -\frac{i\hbar\omega_1}{3\pi^2 c}\ \delta_{lm} \tag{22.15}$$

In terms of $a = \frac{2\pi c}{\omega_1}$, the commutator is

$$\frac{i\hbar}{2}(\frac{e}{\hbar m_0 c})^2[p\cdot Z, p\cdot A\ ] = -\frac{\delta m}{m_0}\frac{\hat{p}^2}{2m_0} \tag{22.16}$$

with

$$\delta m = \frac{2e^2}{3\pi c^2 a} = \frac{8e^2}{3ac^2} \tag{22.17}$$

The kinetic energy of the electron will then be given by

$$\frac{\hat{p}^2}{2m_0}(1-\frac{\delta m}{m_0}) = \frac{\hat{p}^2}{2m} \tag{22.18}$$

where the electron mass has been renormalized with $m = m_0 + \delta m$ being the observed mass as opposed to m_0 the bare electron mass.

Note that the effects of the radiative corrections are included in the transformed expression for $V(r)$ namely

$$\exp iS(t)\ V(\hat{r})\ \exp -iS(t) = V(\hat{r} + \frac{e}{mc}\ \hat{Z}) \tag{22.19}$$

Upon expanding out the transformed expression for $V(r)$ the Schrödinger equation becomes

$$\imath\hbar\partial_t\Psi(t) = [\ \frac{p^2}{2m} + V(\hat{r}) + \frac{e}{mc}\ \hat{Z}\cdot\nabla V(\hat{r}) + \frac{1}{2}(\frac{e}{mc})^2\ (\hat{Z}\cdot\nabla)^2 V(\hat{r}) + \dots\ ]\ \Psi(t) \tag{22.20}$$

Note that there is no virtual interaction for a free electron because $\hat{Z}\cdot\nabla V(\hat{r})$ is proportional to the force acting on the electron and therefore to its acceleration. For a free electron all radiative corrections have been incorporated into the mass renormalization.

If no photons are present initially or finally the Schrödinger equation becomes

$$\imath\hbar\partial_t\Psi(t) = [\ \frac{p^2}{2m} + V(\hat{r}) + \frac{1}{2}(\frac{e}{mc})^2\ < (\hat{Z}\cdot\nabla)^2 >_{vac} V(\hat{r}) + \dots\ ]\ \Psi(t) \tag{22.21}$$

since in this case $< \hat{Z} >_{vac} = 0$.

The remaining leading vacuum expectation term in the Schrödinger equation can now be evaluated to yield

$$< (\frac{e}{mc}\hat{Z})^2 >_{vac} = (\frac{e}{mc})^2 \sum_k \frac{\hbar c^2}{2\omega_k^3} = \frac{2\alpha}{\pi}\ (\frac{\hbar}{mc})^2\ \ln\frac{\omega_1}{\omega_0} \tag{22.22}$$

so the Schrödinger equation becomes

$$\imath\hbar\partial_t \Psi(t) = [\ \frac{p^2}{2m} + V(\hat{r}) + \frac{2\alpha}{\pi}\ (\frac{\hbar}{mc})^2\ \ln\frac{\omega_1}{\omega_0}\nabla^2\ V(\hat{r})\ ]\ \Psi(t) \tag{22.23}$$

For the case

$$\begin{aligned} V(\hat{r}) &= -\frac{Ze^2}{r} \\ \nabla^2\ V(\hat{r}) &= 4\pi Ze^2\delta(r) \end{aligned} \tag{22.24}$$

the perturbing term results in the familiar Lamb shift which is given by

$$< H >= \frac{8}{3\pi}\ \alpha^3\ \frac{Z^4}{n^3}\ \ln[\frac{\hbar\omega_1}{\Delta E}] \tag{22.25}$$

Chapter 23

The Foldy-Wouthuysen Similarity Transformation

One of the more interesting uses of a Similarity Transformation was proposed by L. L. Foldy and S. A. Wouthuysen to decouple the Dirac equation into two two-component equations, where one of the two-component equations reduces to the Pauli description in the relativistic limit and the other describes the negative-energy states. [12]

Following the procedure outlined by Foldy and Wouthuysen we shall use a series of transformations to remove from the Dirac Hamiltonian all operators that couple the large components of the eigenfunction to the small components.

By convention the operators that do not couple large and small components are known as "even" operators and those that do are known as "odd" operators. For instance, 1, β, δ are "even" operators and α, γ, γ_5 are considered "odd" operators.

The FW Transformation Applied to A Free Dirac Particle

Consider the Dirac equation for a free particle given by the Dirac Hamiltonian

$$\hat{H} = \alpha \cdot \hat{p} + \beta m \tag{23.1}$$

Consider then a unitary similarity transformation $U = \exp(i\hat{S})$ with the operator $\hat{S}$ hermitian and not explicitly time-dependent, then

$$\psi' = \exp(iS)\ \psi \tag{23.2}$$

and ($\hbar = 1$ and $c = 1$)

$$i\frac{\partial\psi'}{\partial t} = \exp(iS)\ \hat{H}\ \psi = \exp(iS)\ \hat{H}\ \exp(-iS)\ \psi' = \hat{H}'\ \psi' \tag{23.3}$$

such that the Hamiltonian $\hat{H}'$ is to contain no odd operators by construction.

Try the Ansatz

$$\exp(iS) = \exp(\ \beta\ \alpha \cdot \hat{p}\ \theta(p)\ ) \tag{23.4}$$

so then the transformed Hamiltonian $\hat{H}'$ becomes

$$\begin{aligned} \hat{H}' &= \alpha \cdot \hat{p}\ (\ \cos\ [\ 2 \mid \hat{p} \mid \theta] - \frac{m}{\mid \hat{p} \mid} \sin\ [\ 2 \mid \hat{p} \mid \theta]) \\ &+\beta(m\ \cos\ [\ 2 \mid \hat{p} \mid \theta] + \ \mid \hat{p} \mid\ \sin\ [\ 2 \mid \hat{p} \mid \theta]) \end{aligned} \tag{23.5}$$

In order to eliminate the "odd" operator $(\alpha \cdot \hat{p})$ set

$$\tan[\ 2 \mid \hat{p} \mid \theta] = \frac{\mid \hat{p} \mid}{m} \tag{23.6}$$

which means the transformed Hamiltonian $\hat{H}'$ is

$$\hat{H}' = \beta\ \sqrt{[m^2 + p^2]} \tag{23.7}$$

which has eigenvalues which can easily be found using conventional methods.

Chapter 24

The FW Transformation Applied to an Electron in a Field

Consider now the Dirac Hamiltonian for a charged particle [12]

$$\hat{H} = \alpha \cdot (\hat{p} - e\ \hat{A}) + \beta m + e\ \phi \tag{24.1}$$

Note that

$$\beta\ \alpha \cdot (\hat{p} - e\ \hat{A}) = -\alpha \cdot (\hat{p} - e\ \hat{A})\ \beta \tag{24.2}$$

and $\beta\ e\ \phi = e\ \phi\ \beta$.

It is worth remembering that the Hamiltonian may be time dependent. In this case the operator $\hat{S}$ will also be time dependent and so then it is not possible to construct an operator $\hat{S}$ that will remove all operators from the transformed hamiltonian $\hat{H}^{'}$.

We are, however, able to expand out the hamiltonian in a power series in $\frac{1}{m}$ keeping terms to what ever order we wish in this non-relativistic expansion.

Introduce the time dependent transformation

$$\psi^{'} = \exp(iS)\ \psi \tag{24.3}$$

so then

$$i\frac{\partial}{\partial t}\ \exp[-i\hat{S}]\psi' = \hat{H}\ \psi = \hat{H}\ \exp(-iS)\ \psi'$$
$$= \exp(-iS)\ i\frac{\partial\psi'}{\partial t} + (i\ \frac{\partial}{\partial t}\ \exp[-i\hat{S}])\psi' \tag{24.4}$$

which yields

$$i\frac{\partial}{\partial t}\ \psi' = [\ \exp[i\hat{S}](\hat{H} - i\frac{\partial}{\partial t})\exp[-i\hat{S}]\ ]\ \psi' = \hat{H}'\ \psi' \tag{24.5}$$

Expanding out the expression we see

$$\begin{aligned} &[\ \exp[i\hat{S}](\hat{H} - i\frac{\partial}{\partial t})\exp[-i\hat{S}] = \\ &\hat{H} + i[\hat{S},\hat{H}] - \frac{1}{2!}[\hat{S},[\hat{S},\hat{H}]] - \frac{i}{3!}[\hat{S},[\hat{S},[\hat{S},\hat{H}]]] + \frac{1}{4!}[\hat{S}[\hat{S},[\hat{S},[\hat{S},\hat{H}]]]] + \ldots \\ &-\frac{\partial\hat{S}}{\partial t} - \frac{i}{2!}[\hat{S},\frac{\partial\hat{S}}{\partial t}] + \frac{1}{3!}[\hat{S},[\hat{S},[\hat{S},\frac{\partial\hat{S}}{\partial t}]] + \ldots \end{aligned} \tag{24.6}$$

We see that the first order term is given by

$$\beta m + e\ \phi + \vartheta + i[\hat{S},\beta]m \tag{24.7}$$

where we have let $\vartheta = \alpha\cdot(\hat{p} - e\ \hat{A})$.

To remove the odd term $\vartheta = \alpha\cdot(\hat{p} - e\ \hat{A})$ we choose $\hat{S} = -i\frac{\beta\vartheta}{2m}$. The remaining terms become

$$i[\hat{S},\hat{H}] = -\vartheta + \frac{\beta}{2m}[\vartheta, e\ \phi] + \frac{1}{m}\beta\vartheta^2 \tag{24.8}$$

and

$$\frac{i}{3!}[\hat{S},[\hat{S},[\hat{S},\hat{H}]]] = \frac{\beta\ \vartheta^2}{2m} - \frac{1}{8m^2}[\vartheta,[\vartheta, e\ \phi]] - \frac{1}{2m^2}\vartheta^3 \tag{24.9}$$

and

$$\frac{i}{3!}[\hat{S},[\hat{S},[\hat{S},\hat{H}]]] = \frac{\vartheta^3}{6m^2} - \frac{1}{6m^3}\beta\vartheta^4 \tag{24.10}$$

and

$$\frac{1}{4!}[\hat{S}[\hat{S},[\hat{S},[\hat{S},\hat{H}]]]] = \frac{\beta\vartheta^4}{24m^3} \tag{24.11}$$

while the next few terms in $\frac{\partial \hat{S}}{\partial t}$ are

$$-\frac{\partial \hat{S}}{\partial t} = \frac{i\beta}{2m}\frac{\partial \vartheta}{\partial t} \tag{24.12}$$

and

$$-\frac{i}{2!}[\hat{S},\frac{\partial \hat{S}}{\partial t}] = -\frac{i}{8m^2}[\vartheta,\frac{\partial \vartheta}{\partial t}] \tag{24.13}$$

By inspection it is possible to see that the odd terms now appear only to order $\frac{1}{m}$.

To reduce the odd terms further it is necessary to apply a second Foldy-Wouthuysen Transformation of the form

$$\hat{S}^{'} = -i\frac{\beta}{2m}[\frac{\beta}{2m}[\vartheta, e\ \phi] - \frac{\vartheta^3}{3m^2} + \frac{i\beta}{2m}\frac{\partial \vartheta}{\partial t}] \tag{24.14}$$

Under this second Transformation we see that

$$\begin{aligned} \hat{H}^{"} &= [\ \exp[i\hat{S}^{'}](\hat{H}^{'} - i\frac{\partial}{\partial t})\exp[-i\hat{S}^{'}] = \\ &\beta m + e\ \phi^{'} + \frac{\beta}{2m}[\vartheta^{'}, e\ \phi^{'}] + \frac{i\beta}{2m}\frac{\partial \vartheta^{'}}{\partial t} \\ &= \beta m + e\ \phi^{'} + \vartheta^{"} \end{aligned} \tag{24.15}$$

where $\vartheta^{"}$ is now of order $\frac{1}{m^2}$.

Applying a third transformation of the form

$$\hat{S}^{"} = \frac{-i\beta}{2m}\ \vartheta^{"} \tag{24.16}$$

results in the Hamiltonian

$$\tilde{H} = \beta(m + \frac{\vartheta^2}{2m} - \frac{\vartheta^4}{8m^3} + e\ \phi\ - \frac{1}{8m^2}[\vartheta,[\vartheta, e\ \phi]] - \frac{i}{8m^2}[\vartheta,\frac{\partial\ \vartheta}{\partial t}] \tag{24.17}$$

It is now possible to explicitly evaluate the terms,

$$\frac{\vartheta^2}{2m} = \frac{(p - eA)^2}{2m} - \frac{e}{2m}\delta \cdot B \tag{24.18}$$

and

$$\frac{1}{8m^2}([\vartheta, e\ \phi] + i\frac{\partial\ \vartheta}{\partial t}) = \frac{ie}{8m^2}\alpha \cdot E \tag{24.19}$$

so that

$$\begin{aligned} \frac{1}{8m^2}([\vartheta, [\vartheta, e\ \phi] + i\frac{\partial\ \vartheta}{\partial t}\ ]) &= [\vartheta, \frac{ie}{8m^2}\alpha \cdot E] \\ &= \frac{ie}{8m^2}[\alpha \cdot \hat{p}, \alpha \cdot E] = \\ \frac{e}{8m^2}\nabla \cdot E + \frac{ie}{8m^2}\ \delta \cdot\ \nabla \times E &+ \frac{e}{4m^2}\delta \cdot\ E\ \times p \end{aligned} \tag{24.20}$$

By inspection we have

$$\begin{aligned} \tilde{H} &= \beta(m + \frac{(p - eA)^2}{2m} - \frac{p^4}{8m^3}) + e\ \phi - \frac{e}{2m}\beta\delta \cdot B \\ &- \frac{ie}{8m^2}\ \delta \cdot\ \nabla \times E - \frac{e}{4m^2}\delta \cdot\ E\ \times p - \frac{e}{8m^2}\nabla \cdot E \end{aligned} \tag{24.21}$$

One is then free to interpret the individual terms of this transformed hamiltonian in the usual fashion.

Part III

The Bogoliubov Transformation

Chapter 25

Introduction

The Bogoliubov Transformation is used for Diagonalizing a Quadratic Bosonic Hamiltonian.

Several examples of Bogoliubov Transformations are presented, including phonons in a system of weakly interacting particles, as well as Electron Spin-Resonance.

The Bogoliubov Transformations will also be used in simple models of cosmological particle creation presented in a subsequent section of this thesis.

Chapter 26

The Bogoliubov Transformation

Let us consider a system of bosons with a quadratic Hamiltonian $\hat{H}$ with off diagonal elements in the creation operator $\eta^\dagger$ and annihilation operator η. The commutation relations between $\hat{H}$ and $\eta^\dagger$ will be of the form

$$[\hat{H}, \eta_k^\dagger] = \sum_i (\eta_i^\dagger \; A_{ik} + \eta_i \; B_{ik}) \tag{26.1}$$

The adjoint of this expression is

$$[\hat{H}, \eta_k] = -\sum_i \; (\eta_i \; A_{ik} + \eta_i^\dagger \; B_{ik}) \tag{26.2}$$

where for simplicity we shall consider only real A and B.

We would like to find a transformation that will diagonalize the Hamiltonian. [13] We can do this by introducing a new set of bosonic creation and annihilation operators $\alpha^\dagger$ and α so that

$$\alpha_\nu^\dagger = \sum_k \; (\eta_k^\dagger \; U_{k\nu} - \eta_k \; V_{k\nu}) \tag{26.3}$$

and

$$\alpha_\mu = \sum_k \; (\eta_k \; U_{k\nu} - \eta_k^\dagger \; V_{k\nu}) \tag{26.4}$$

which fulfill the following commutation relations

$$[\ \alpha_\nu^\dagger,\ \alpha_\mu^\dagger\ ] = 0 \tag{26.5}$$

$$[\ \alpha_\nu,\ \alpha_\mu\ ] = 0 \tag{26.6}$$

$$[\ \alpha_\nu,\ \alpha_\mu^\dagger\ ] = \delta_{\nu\mu} \tag{26.7}$$

and

$$[\hat{H},\ \alpha_\nu^\dagger\ ] = E_\nu\ \alpha_\nu^\dagger \tag{26.8}$$

What we see then is

$$[\hat{H},\ \alpha_\nu^\dagger\ ] = \sum_k \left(\ [\hat{H},\eta_k^\dagger]\ U_{k\nu} -\ [\hat{H},\eta_k]\ V_{k\nu}\right)\ =\ E_\nu \sum_k \left(\eta_k^\dagger\ U_{k\nu} - \eta_k\ V_{k\nu}\right) \tag{26.9}$$

Rewriting this relationship we see that

$$\sum_{k,i} [\ \eta_i^\dagger[(A_{ik} - \delta_{ik}E_\nu)\ U_{k\nu}\ + B_{ik}V_{k\nu}\ ] + \eta\ [\ B_{ik}U_{k\nu} + (A_{ik} + \delta_{ik}E_\nu)V_{k\nu}\ ]\ =\ 0 \tag{26.10}$$

This means that we obtain the following simultaneous equations

$$\sum_k [\ (A_{ik} - \delta_{ik}E_\nu)\ U_{k\nu}\ + B_{ik}V_{k\nu}\ ]\ =\ 0 \tag{26.11}$$

and

$$\sum_k [\ B_{ik}U_{k\nu} + (A_{ik} + \delta_{ik}E_\nu)V_{k\nu}\ ]\ =\ 0 \tag{26.12}$$

which must hold for all values of i.

We can express these equations in matrix form, so that

$$AU + BV = UE \tag{26.13}$$

and

$$BU + AV = -VE \tag{26.14}$$

where the matrix E is a diagonal matrix. It follows then that

$$(A + B)(U + V) = AU + BV + BU + AV = (U - V)E \tag{26.15}$$

Multiplying this expression on the left by (A-B) yields

$$\begin{aligned}(A - B)(A + B)(U + V) &= (A - B)(U - V)E \\ = (AU + BV + BU - BV)E &= \ (U + V)E^2\end{aligned} \tag{26.16}$$

Define a matrix W such that

$$W = (A - B)^{-1/2}\ (U + V) \tag{26.17}$$

so that

$$(U + V) = (A - B)^{1/2}\ W \tag{26.18}$$

We then see that

$$(A - B)^{1/2}\ (A + B)\ (A - B)^{1/2}\ W = W\ E^2 \tag{26.19}$$

or succinctly

$$MW = W\ E^2 \tag{26.20}$$

where

$$M = \ (A - B)^{1/2}\ (A + B)\ (A - B)^{1/2} \tag{26.21}$$

Since A and B are real matrices they must be symmetric in order that $\hat{H}$ shall be hermitian. This in turn means that M is real and symmetric and the problem of diagonalizing the matrix M is a standard one.

Having solved the expression $MW = WE^2$ for W we can generate $(U + V)$ and $(U - V)$ and in this way find transformation coefficients such that the expression $[\hat{H},\ \alpha_\nu^\dagger\ ] = \ E_\nu\ \alpha_\nu^\dagger$ can be satisfied.

Having satisfied this expression it is now necessary to satisfy the three commutation relations noted above. These three equations impose a rigid set of restrictions on the transformation coefficients U and V, namely

$$[\ \alpha_\nu^\dagger,\ \alpha_\mu^\dagger\ ] = \sum_k (U_{k\mu} V_{k\nu} - V_{k\mu} U_{k\nu}) = 0 \tag{26.22}$$

which in matrix notation is

$$U^T V - V^T U = 0 \tag{26.23}$$

and

$$[\ \alpha_\nu,\ \alpha_\mu^\dagger\ ] = \sum_k (U_{k\mu} U_{k\nu} - V_{k\mu} V_{k\nu}) = \delta_{\nu\mu} \tag{26.24}$$

which in matrix notation is

$$U^T U - V^T V = 0 \tag{26.25}$$

When diagonalizing the matrix M choose the normalization of the eigenvector in such a fashion that

$$W^T W = E^{-1} \tag{26.26}$$

This means that

$$W^T\ (A-B)^{1/2}\ (A+B)\ (A-B)^{1/2}\ W = \ E \tag{26.27}$$

Since A and B are symmetric matrices it follows that

$$[(A-B)^{1/2}]^T = (A-B)^{1/2} \tag{26.28}$$

so then

$$W^T\ (A-B)^{1/2} = (U+V)^T \tag{26.29}$$

This means that

$$(A+B)\ (A-B)^{1/2}\ W = \ (U+V)E \tag{26.30}$$

so we see that $(U+V)^T\ (U-V)=1$. The transpose of this expression can be written as

$$W^T\ (A-B)^{1/2}\ (A+B)\ = E(U-V)^T \tag{26.31}$$

This means that $(U-V)^T\ (U+V)=1$.

What this means is that provided we normalize the eigenvectors according to $W^TW = E^{-1}$ then the new creation and annihilation operators $\alpha^\dagger$ and α will automatically fulfill the boson commutation relations.

An Example Using the Bogoliubov Transformation

Consider phonons in a system of weakly interacting particles, described by the Hamiltonian [14].

$$\hat{H} = \sum_k \epsilon_k\ a_k^\dagger a_k + \frac{1}{2} \sum V(k_1 - k_1^\dagger)\ a_{k1}^\dagger\ a_{k2}^\dagger\ a_{k2^\dagger} a_{k1^\dagger}\ \Delta(k_1 + k_2 - k_{1^\dagger} - k_{2^\dagger}) \tag{26.32}$$

where the delta assures conservation of wavenumber. Let $N_0 = a_0^\dagger a_0$, and assuming $V_k = V_{-k}$ then the Hamiltonian can be written as

$$\hat{H} = \sum_k \epsilon_k\ a_k^\dagger a_k + \frac{1}{2}(N_0)^2\ V_0 + N_0 V_0\ \sum_k a_k^\dagger a_k$$

$$+N_0 \sum_k\ V_k\ a_k^\dagger\ a_k + \frac{1}{2} N_0 \sum_k V_k(\ a_k\ a_{-k} + a_k^\dagger\ a_{-k}^\dagger) + higherterms \tag{26.33}$$

where the summations do not include k =0.

Reading from left to right the terms in the Hamiltonian are the following:

(a) Kinetic energy:

$$\sum_k \epsilon_k\ a_k^\dagger a_k \tag{26.34}$$

(b) Interactions in the ground state:

$$a_0^\dagger\ a_0^\dagger\ a_0\ a_0 \tag{26.35}$$

(c) One particle not excited in the ground state:

$$a_0^\dagger \; a_k^\dagger \; a_k \; a_0 \tag{26.36}$$

(d) Exchange of one particle in the ground state:

$$a_k^\dagger \; a_0^\dagger \; a_k \; a_0 \tag{26.37}$$

and

$$a_0^\dagger \; a_k^\dagger \; a_o \; a_k \tag{26.38}$$

(e) Both initial and final particle in the ground state:

$$a_0^\dagger \; a_0^\dagger \; a_k \; a_{-k} \tag{26.39}$$

and

$$a_k^\dagger \; a_{-k}^\dagger \; a_o \; a_0 \tag{26.40}$$

Terms with three ground state operators are excluded by momentum conservation.

We can take the expectation value of $N_0 + \sum a_k^\dagger \; a_k$ to be the number of particles N in the system. Collecting terms we have a reduced Hamiltonian in bilinear form

$$\hat{H}_{red} = \frac{1}{2}(N_0)^2 \; V_0 + \sum_k (\epsilon_k + NV_k) a_k^\dagger \; a_k + \frac{1}{2} N \sum V_k (a_k \; a_{-k} + a_{-k}^\dagger a_k^\dagger) + \ldots \tag{26.41}$$

Express this reduced Hamiltonian in terms of new boson operators $\alpha^\dagger$ and α such that

$$[\hat{H}, \alpha_k^\dagger] = \lambda \; \alpha_k^\dagger \tag{26.42}$$

$$[\hat{H}, \alpha_k] = -\lambda \; \alpha_k \tag{26.43}$$

and

$$[\alpha_k, \alpha_k^\dagger] = \delta_{kk'} \tag{26.44}$$

The first two expressions are satisfied if the new Hamiltonian is written in diagonal form, namely

$$\hat{H}_{new} = \sum_k \lambda_k \ \alpha_k^\dagger \ \alpha_k \tag{26.45}$$

In terms of the reduced Hamiltonian $\hat{H}_{red}$ we have

$$\hat{H}_{red} - \frac{1}{2} N^2 \ V_0 = \sum_k \ \hat{H}_k \tag{26.46}$$

where

$$\hat{H}_k = \ \omega_0 \ (a_k^\dagger \ a_k + a_{-k}^\dagger \ a_{-k}) \ + \ \omega_1 (a_k \ a_{-k} + a_{-k}^\dagger a_k^\dagger) \tag{26.47}$$

with ω_0 and ω_1 given by

$$\omega_0 = \epsilon_k + NV_k \tag{26.48}$$

and

$$\omega_1 = NV_k \tag{26.49}$$

Make the transformations

$$\alpha_k = u_k a_k - v_k a_{-k}^\dagger \tag{26.50}$$

and

$$\alpha_k^\dagger = u_k a_k^\dagger - v_k a_{-k} \tag{26.51}$$

where u_k and v_k are real functions of k.

For these transformations we see that

$$[\ \alpha_k, \alpha_k^\dagger \ ] = u_k^2 - v_k^2 \tag{26.52}$$

which means we should choose u_k and v_k to make the commutation relation equal to 1. We can do this by letting

$$a_k = u_k \ \alpha_k + v_k \ \alpha_{-k}^\dagger \tag{26.53}$$

We see then that

$$[\hat{H}, \alpha_k^\dagger] = \ u_k(\omega_0\ a_k^\dagger + \omega_1\ a_{-k})$$
$$+v_k(\omega_0\ a_{-k} + \ \omega_1\ a_k^\dagger = \lambda(u_k\ a_k^\dagger - \ v_k\ \ a_{-k}) \tag{26.54}$$

This means then that

$$\omega_0\ u_k + \ \omega_1\ v_k = \lambda\ u_k \tag{26.55}$$

and

$$\omega_1\ u_k + \ \omega_0\ v_k = -\lambda\ v_k \tag{26.56}$$

These equations have a solution provided

$$\lambda^2 = \ \omega_0^2 - \ \omega_1^2 \ = \ (\epsilon_k + NV_k)^2 - (NV_k)^2 \tag{26.57}$$

The normalization of the commutation relation is assured if we choose

$$u_k = \cosh(\chi_k) \tag{26.58}$$

and

$$u_k = \sinh(\chi_k) \tag{26.59}$$

This means that the Hamiltonian $\hat{H}$ is diagonal if

$$\tanh(2\chi_k) = -\frac{NV_k}{\epsilon_k + NV_k} \tag{26.60}$$

The ground state in terms of the transformed annihilation operator α_k has the property

$$\alpha_k\ \mid \Phi_0 >= 0 \tag{26.61}$$

We see that

$$a_k^\dagger a_k = \; u_k^2 \; \alpha_k^\dagger \alpha_k + v_k^2 + v_k^2 \; \alpha_{-k}^\dagger \alpha_{-k} + u_k \; v_k(\; \alpha_k^\dagger \; \alpha_{-k}^\dagger + \alpha_{-k} \; \alpha_k) \qquad (26.62)$$

which means that the mixtures of excitation k in the ground state $\mid \Phi_0 >$ is given by

$$< \Phi_0 \mid \; a_k^\dagger a_k \mid \Phi_0 >= v_k^2 = \frac{1}{2}(\; \cosh(2\chi_k) - 1) \qquad (26.63)$$

where

$$\cosh(2\chi_k) = (\epsilon_k + NV_k)[(\epsilon_k + NV_k)^2 - N^2 V_k^2]^{-1/2} \qquad (26.64)$$

Chapter 27

Field Quantization and Spin-Resonance

Let us now quantize the radiation field and take into account the reaction of the spin with the field.

Consider spin 1/2 particles in a magnetic field with the main field in the z-direction. The energy of the field is given by [15]

$$\hat{H}_{field} = \hbar\omega a^\dagger a \tag{27.1}$$

The interaction energy is given by

$$\hat{H}_{int} = \hbar\omega_0 \frac{\sigma_z}{2} + \hbar\kappa(a^\dagger + a)\sigma_x \tag{27.2}$$

with κ a coupling constant. Let $\sigma_x = \sigma_+ + \sigma_-$ then

$$\begin{aligned} \hat{H} = \hat{H}_{field} + \hat{H}_{int} &= \hbar\omega a^\dagger a + \hbar\omega_0 \frac{\sigma_z}{2} \\ &+ \hbar\kappa(a^\dagger + a)(\sigma_+ + \sigma_-) \end{aligned} \tag{27.3}$$

This Hamiltonian can be approximated in a simpler form as

$$\begin{aligned} \hat{H} = \hat{H}_{field} + \hat{H}_{int} &= \hbar\omega a^\dagger a + \hbar\omega_0 \frac{\sigma_z}{2} \\ &+ \hbar\kappa(a^\dagger\, \sigma_- + a\, \sigma_+) \end{aligned} \tag{27.4}$$

This approximation is equivalent to decomposing the linearly polarized cavity RF field into two opposite circularly polarized waves and keeping only the one rotating in the same direction as the spin precession. This simplified Hamiltonian is still hermitian.

Let us split the Hamiltonian into two new operators

$$\hat{H} = \hbar(C_1 + C_2) \tag{27.5}$$

where

$$\begin{aligned} C_1 &= \omega(a^\dagger a + \frac{1}{2}\sigma_z) \\ C_2 &= \kappa(S_+ + S_-) - \frac{\Delta\omega}{2}\sigma_z \\ \Delta\omega &= \omega - \omega_0 \\ S_+ &= \sigma_+ a \\ S_- &= \sigma_- a^\dagger \end{aligned} \tag{27.6}$$

It can be shown as well that C_1 and C_2 commute with $\hat{H}$ and with each other.

When there is no coupling between the particle spin and the radiation field we have a complete set of basis states consisting of the states $\mid n >$ for the radiation field (with $a^\dagger a \mid n >= n \mid n >$, and the states $\mid \pm 1 >$ for the spins where $\sigma_z \mid \pm 1 >= \pm \mid \pm 1 >$. Using the usual procedure we can use this complete set of states to describe the state vectors of the coupled system.

We immediately find that for C_1

$$C_1 \ \mid n, \pm 1 >= \omega(a^\dagger a + \frac{\sigma_z}{2}) \ \mid n, \pm 1 >= \omega(n \pm \frac{1}{2}) \mid n, \pm 1 > \tag{27.7}$$

and so C_1 is diagonal in this representation.

It is easy to see that C_2 is not diagonal in this representation, but since C_1 and C_1 commute we can use a Bogoliubov Transformation, which is a special form of similarity transformation, to find a representation in which they are both diagonal by taking linear combinations of the eigenkets of C_1.

In this new representation, the transformed Hamiltonian is diagonal so that we can easily solve the Schrödinger equation for the system.

Consider the new state vectors given by

$$
\begin{aligned}
\mid \varphi(n,1) >&= \cos\theta_n \ \mid n+1,-1> + \sin\theta_n \ n,+1> \\
\mid \varphi(n,2) >&= -\sin\theta_n \mid n+1,-1> + \cos\theta_n \mid n,+1>
\end{aligned} \tag{27.8}
$$

where $\mid n+1,-1>$ corresponds to $n+1$ quanta in the field with spin-down and $\mid n,+1>$ corresponds to n quanta with spin-up.

The ground state is given by $\mid 0,-1>$ with no quanta and spin-down. This state is considered separately. The angle θ_n is a parameter where n may take on any value between 0 and ∞.

We see that the new states are diagonal and normalized, and are orthogonal to the ground state. We see as well that

$$
\begin{aligned}
C_1 \ \mid \varphi(n,1) >&= \ \omega(n+\frac{1}{2}) \ \mid \varphi(n,1)> \\
C_1 \ \mid \varphi(n,2) >&= \ \omega(n+\frac{1}{2}) \ \mid \varphi(n,2)> \\
C_1 \ \mid 0,-1 >&= \ -\frac{1}{2}\,\omega \mid 0,-1>
\end{aligned} \tag{27.9}
$$

We see that the two states $\mid \varphi(n,1)>$ and $\mid \varphi(n,2)>$ are degenerate eigenstates of C_1 irrespective of a choice in θ_n.

Applying C_2 to the new eigenstates we obtain

$$
\begin{aligned}
C_2 \ \mid \varphi(n,1) >= &(\kappa\surd(n+1) \ \sin\theta_n + \frac{\Delta\omega}{2}\cos\theta_n) \ \mid n+1,-1> \\
&+(\kappa\surd(n+1) \ \cos\theta_n - \frac{\Delta\omega}{2}\sin\theta_n) \ \mid n,+1> \\
C_2 \ \mid \varphi(n,2) >= &(\kappa\surd(n+1) \ \cos\theta_n - \frac{\Delta\omega}{2}\sin\theta_n) \ \mid n+1,-1> \\
&-(\kappa\surd(n+1) \ \sin\theta_n + \frac{\Delta\omega}{2}\cos\theta_n) \ \mid n,+1>
\end{aligned} \tag{27.10}
$$

where the composite operators S_+ and S_- were used, namely

$$
\begin{aligned}
S_+ \ \mid n+1,-1 >&= \surd(n+1) \ \mid n,+1> \\
S_+ \ \mid n,+1 >&= 0 \\
S_- \ \mid n+1,-1 >&= 0 \\
S_- \ \mid n,+1 >&= \surd(n+1) \ \mid n+1,-1>
\end{aligned} \tag{27.11}
$$

In order for $\mid \varphi(n,1)>$ and $\mid \varphi(n,2)>$ to be eigenvectors of C_2 we must be able to choose θ_n so that

$$\begin{aligned} C_2 \ \mid \varphi(n,1) > &= \lambda_n \ \mid \varphi(n,1) > \\ C_2 \ \mid \varphi(n,2) > &= \lambda_n^{'} \ \mid \varphi(n,2) > \end{aligned} \tag{27.12}$$

By inspection we see that if we let

$$\tan\theta_n = \frac{\kappa\sqrt{(n+1)}}{\frac{1}{2}\Delta\omega + \lambda_n} \tag{27.13}$$

and

$$\lambda_n = \sqrt{[(\frac{\Delta\omega}{2})^2 + \kappa^2(n+1)]} \tag{27.14}$$

then $\mid \varphi(n,1)>$ and $\mid \varphi(n,2)>$ are eigenkets of C_2 with eigenvalues $\pm\lambda_n$, and where $\lambda_n^{'} = -\lambda_n$.

For the ground state we have

$$C_2 \ \mid 0,-1 > = \frac{\Delta\omega}{2} \ \mid 0,-1 > \tag{27.15}$$

which we see is also an eigenket of C_2.

The eigenvalues of the transformed Hamiltonian then are

$$\begin{aligned} \hat{H} \ \mid \varphi(n,1) > &= \hbar[\omega(n+\frac{1}{2}) + \lambda_n] \ \mid \varphi(n,1) > \\ \hat{H} \ \mid \varphi(n,2) > &= \hbar[\omega(n+\frac{1}{2}) - \lambda_n] \ \mid \varphi(n,2) > \\ \hat{H} \ \mid 0,-1 > &= -\frac{\hbar\omega_0}{2} \ \mid 0,-1 > \end{aligned} \tag{27.16}$$

It is worth noticing that the eigenstates of the transformed Hamiltonian are mixtures of eigenstates of the unperturbed Hamiltonian $\hat{H}_0$.

It is possible to express the eigenstates $\mid n,\pm1>$ in terms of $\mid \varphi(n,1)>$ and $\mid \varphi(n,2)>$ where $(n = 0,1,2,\ldots,\infty)$, namely

$$\begin{aligned} \mid n+1,-1 > &= \cos\theta_n \ \mid \varphi(n,1) > - \sin\theta_n \ \mid \varphi(n,2) > \\ \mid n,+1 > &= \sin\theta_n \ \mid \varphi(n,1) > + \cos\theta_n \ \mid \varphi(n,2) > \end{aligned} \tag{27.17}$$

Consider now the transition probability between and initial state $| n, +1 >$ (a state with n quanta and spin-up) and a final state $| n+1, -1 >$ (a state with n+1 quanta and spin-down).

We see that

$$|< n+1, -1 | \exp(-\frac{i\hat{H}t}{\hbar}) \ | n, +1 >|^2 = \sin^2 2\theta_n \sin^2 \lambda_n t$$
$$= \frac{4\kappa^2(n+1)}{(\Delta\omega)^2 + 4\kappa^2(n+1)} \sin^2[\frac{t}{2} \sqrt{[(\Delta\omega)^2 + 4\kappa^2(n+1)]} \qquad (27.18)$$

It is worth noting that even if $n = 0$ there is a probability the spin will flip and emit a quanta of light which may later interact with the system.

The model outlined above therefore takes into account spontaneous emission of quanta.

Part IV

The Dressed Oscillator

Chapter 28

Introduction

It is possible to describe the coupling between an electromagnetic field and an oscillator in terms of a dressed oscillator. I shall consider a non-relativistic system of oscillators coupled linearly to a scalar field in ordinary Euclidean 3-space.

I start with an analysis of a non-relativistic system of oscillators confined in a reflecting sphere of radius R, and assume that the free space solution to the radiating oscillator is obtained by taking the radius of the cavity arbitrarily large in the R-dependent quantities. The limit of an arbitrarily large radius is taken as a description of the radiating oscillator in free space.

From a physical point of view we give a nonperturbative treatment to the oscillator radiation introducing some coordinates that allow to divide the coupled system into two parts, the "dressed" oscillator and the field, what makes unecessary to work directly with the concepts of "bare" oscillator, field and interaction to study the radiation process.

Chapter 29

An Exact Approach to Oscillator Radiation using A Contact Transformations

Consider a harmonic oscillator $q_0(t)$ with natural frequency ω_0 coupled linearly to a scalar field $\phi(x,t)$, the whole system confined to a cavity of radius R centred on the oscillator. [16]

The equations of motion are

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = 2\pi g c \int_0^R d^3r \ \phi(x,t)\delta(r) \tag{29.1}$$

$$\frac{\partial^2 \phi}{c^2 \partial t^2} - \nabla^2 \phi(r,t) = 2\pi g c \ q_0(t) \ \delta(r) \tag{29.2}$$

where g is a coupling constant.

Using Spherical Bessel functions in the domain $0 < r \leq R$ can be rewritten as a set of equations coupling the oscillator to the harmonic field modes, namely

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = \eta \sum_{i=0}^{\infty} \omega_i q_i(t) \tag{29.3}$$

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = \eta \ \omega_i q_0(t) \tag{29.4}$$

where $\eta = \sqrt{2g\Delta\omega}$ and $\Delta\omega = \pi c/R$ is the interval between two adjacent field modes, $\Delta\omega = \omega_{i+1} - \omega_i = \pi c/R$ in the spherical cavity.

Let us consider how a harmonic oscillator couples to N other oscillators. In the limit $N \to \infty$ we recover our original situation of the coupling oscillator field after an appropriate redefinition of diveregent quantities, in a manner similar to renormalization in field theory.

In terms of the cut-off N, the coupled equations are rewritten taking N as the upper limit instead of ∞ for any summation and the system of $N+1$ coupled oscillators q_0 and q_i is represented by the Hamiltonian,

$$H = \frac{1}{2}[p_0^2 + \omega_0^2 q_0^2 \ ] \ + \sum_{j=1}^{N} p_j^2 + \omega_j^2 q_j^2 - 2\eta\omega_j q_0 q_j \tag{29.5}$$

This Hamiltonian can be turned to principal axis by means of the canonical transformation

$$q_\mu = T_\mu^\nu Q_\nu$$

$$p_\mu = T_\mu^\nu P_\nu \tag{29.6}$$

performed by an orthonormal matrix $T = T_\mu^\nu$, $\mu = (0, k), k = 1, 2, \ldots N$. The subscript 0 and k refer to the oscillator and the harmonic modes of the field respectively.

Let r refer to the normal modes, $r = 0, \ldots N$, then the transformed Hamiltonian in the prinicpal axis is

$$H = \frac{1}{2}\sum_{r=0}^{N} (P_r^2 + \Omega_r^2 Q_r^2) \tag{29.7}$$

where the Ω_r are the normal frequencies corresponding to the possible collective oscillations modes of the coupled system.

Using the coordinate transformation $q_\mu = T_\mu^\nu Q_\nu$ in the equations of motions and making use of the normalization

$$\sum_{\mu=0}^{N} (T_\mu^\nu)^2 = 1 \tag{29.8}$$

we get the following conditions on the orthonormal matrix T

$$T_\mu^\nu = \frac{\eta\omega_j}{\omega_j^2 - \Omega_\nu^2} \, T_0^\nu \tag{29.9}$$

and

$$T_0^\nu = [1 + \sum_{j=1}^{N} \frac{\eta^2 \omega_j^2}{(\omega_j^2 - \Omega_\nu^2)^2}]^{-1/2} \tag{29.10}$$

and

$$\omega_0^2 - \Omega_\nu^2 = \eta^2 \sum_{j=1}^{N} \frac{\omega_j^2}{\omega_j^2 - \Omega_\nu^2} \tag{29.11}$$

There are $N+1$ solutions Ω_μ to these equations, corresponding to the $N+1$ normal collective oscillation modes.

Take $\Omega_\mu = \Omega$ so that we get

$$\omega_0^2 - N\eta^2 - \Omega^2 = \eta^2 \sum_{j=1}^{N} \frac{\Omega^2}{\omega_j^2 - \Omega^2} \tag{29.12}$$

It is easily seen that if $\omega_0^2 > N\eta^2$ then there are only positive solutions for Ω^2, which means that the system oscillates harmonically in all its modes.

There is also an oscillation mode whose amplitude varies exponentially and that does not allow stationary configurations. We shall disregard this case.

We shall assume in our model that $\omega_0^2 > N\eta^2$ and define a renormalized oscillator frequency $\overline{\omega}$ such that

$$\overline{\omega} = \sqrt{[\omega_0^2 - N\eta^2]} \tag{29.13}$$

In terms of the renormalized frequency then

$$\overline{\omega}^2 - \Omega_\nu^2 = \eta^2 \sum_{j=1}^{N} \frac{\omega_j^2}{\omega_j^2 - \Omega_\nu^2} \tag{29.14}$$

We get the transformation matrix elements for the oscillator-field system by taking the limit $N \to \infty$, namely

$$T_0^\nu = \frac{\Omega_\nu}{\sqrt{[\frac{R}{2\pi g c}(\Omega_\nu^2 - \overline{\omega}^2)^2 + \frac{1}{2}(3\Omega_\nu^2 - \overline{\omega}^2)^2 + \frac{\pi g R}{2c}\Omega_\nu^2}} \tag{29.15}$$

and

$$T_\mu^\nu = \frac{\eta\omega_j}{\omega_j^2 - \Omega_\nu^2}\, T_0^\nu \tag{29.16}$$

The Eigenfrequency Spectrum

Let us return to the coupling oscillator-field by taking the limit $N \to \infty$ in the relations outlined above. In this limit it becomes clear why we need the frequency renormalization, which serves as an analogue to mass renormalization in field theory. The infinite ω_0 is chosen in such a fashion as to make the renormalized frequency $\overline{\omega}$ finite.

Recall the solutions with respect to the variable Ω of the equation

$$\overline{\omega}^2 - \Omega^2 = \eta^2 \sum_{j=1}^{\infty} \frac{\Omega^2}{\omega_j^2 - \Omega^2} \tag{29.17}$$

give the collective modes. Let $\omega_j = j\frac{\pi c}{R}, j = 1, 2 \ldots, \infty$ and take the positive x such that $\Omega = x\frac{\pi c}{R}$ then using the Langevin identity

$$\sum_{j=1}^{\infty} \frac{x^2}{j^2 - \Omega^2} = \frac{1}{2}[1 - \pi x \cot(\pi x)] \tag{29.18}$$

the equation can be rewritten in the form

$$\cot(g\pi) = \frac{cx}{Rg} + \frac{1}{\pi x}[1 - \frac{R\overline{\omega}^2}{\pi g c}] \tag{29.19}$$

The secant curve corresponding to the right hand side cuts only once each branch of the cotangent on the left hand side of the equation.

Label the solutions $x_r = r + \epsilon_r$, where $0 < \epsilon_r < 1$, $r = 0, 1, 2, \ldots$ and the collective eigenfrequencies are

$$\Omega_r = (r + \epsilon_r)\frac{\pi c}{R} \tag{29.20}$$

where the ϵ_r satisfy the equation

$$\cot(\pi\epsilon_r) = \frac{\Omega_r^2 - \overline{\omega}^2}{\Omega_r \pi g} + \frac{c}{\Omega_r R} \tag{29.21}$$

The field $\phi(r, t)$ can be expressed in terms of the normal modes. Expanded out the field in terms of spherical Bessel Functions

$$\phi(r,t) = c \sum_{j=1}^{\infty} q_j(t)\phi_j(r) \tag{29.22}$$

where

$$\phi_j(r) = \frac{\sin(\mid r \mid \omega_j/c)}{r\sqrt{2\pi R}} \tag{29.23}$$

Using the principal axis transformation matrix together with the equations of motion we obtain an expression for the field in terms of an orthonormal basis associated to the collective normal modes, namely

$$\phi(r,t) = c \sum_{k=1}^{\infty} Q_k(t)\Phi_k(r) \tag{29.24}$$

where the normal collective Fourier mode is given by

$$\Phi_k(r) = \sum_j T_j^k \, \frac{\sin(\mid r \mid \omega_j/c)}{r\sqrt{2\pi R}} \tag{29.25}$$

which satisfy the following equation of motion

$$(-\frac{\Omega_k^2}{c^2} - \Delta)\phi_k(r) = 2\pi\sqrt{[\frac{g}{c}]}\ \delta(r)T_0^k \tag{29.26}$$

which has a solution of the form

$$\phi(r,t) = -\ \sqrt{[\frac{g}{c}]}\ \frac{T_0^k}{2\mid r \mid \sin\delta_k}\ \sin(\frac{\Omega_k}{c}\mid r \mid -\delta_k) \tag{29.27}$$

To determine the phase δ_k expand out the right hand side of this equation and comparing with the normal collective Fourier expression we see that the phase is set by the boundary condition, namely

$$\sin(\frac{\Omega_k}{2}R - \delta_k) = 0 \tag{29.28}$$

Recall that $\Omega_r = (r + \epsilon_r)\frac{\pi c}{R}$ it is easy to show that the phase $0 < \delta_k < \pi$ has the form

$$\delta_k = \pi\epsilon_k \tag{29.29}$$

Thus solving for the expansion of the field in terms of the normal collective modes yields,

$$\phi(r,t) = -\frac{\sqrt{gc}}{2} \sum_k \frac{Q_k \sin(\frac{\Omega_k}{c} \mid r \mid - \delta_k)}{\mid r \mid \sqrt{[\sin^2 \delta_k + (\frac{\eta R}{2c})^2 (1 - \frac{\sin \delta_k \cos \delta_k}{\Omega_k R/c})]}} \tag{29.30}$$

The infinite R limit

In the continuous formalism of free space the field normal modes Fourier components are given by

$$\phi_\Omega = H(\Omega) \int_0^\infty d\omega \frac{\omega}{(\omega^2 - \Omega^2)} \frac{\sin(\omega \mid r \mid /c)}{\mid r \mid} \tag{29.31}$$

where the leading term is given by

$$H(\Omega) = \frac{2g\Omega}{\sqrt{[(\Omega^2 - \varpi^2) + \pi g^2 \Omega^2]}} \tag{29.32}$$

Splitting $\frac{\omega}{\omega^2 - \Omega^2}$ into partial fractions we get

$$\begin{aligned} \phi_\Omega &= H(\Omega) \lim_{\epsilon \to 0} \int_{-\infty}^\infty d\omega \frac{1}{\omega - (\Omega \pm i\epsilon)} \frac{\sin(\omega \mid r \mid /c)}{\mid r \mid} \\ &= \int_{-\infty}^\infty d\omega \; \delta_\pm(\omega - \Omega) \frac{\sin(\omega \mid r \mid /c)}{\mid r \mid} \end{aligned} \tag{29.33}$$

where

$$\delta_\pm(\omega - \Omega) = \frac{1}{\pi} P[\frac{1}{\omega - \Omega}] \pm i \; \delta(\omega - \Omega) \tag{29.34}$$

where P is the Principal value.

The transformation matrix takes on the following form in the limit $r \to \infty$,

$$T_0^r = \lim_{\Delta\Omega \to 0} \frac{\sqrt{[2g]} \; \Omega \sqrt{[\Delta\Omega]}}{\sqrt{[(\Omega^2 - \varpi^2) + \pi^2 g^2 \Omega^2]}} \tag{29.35}$$

and

$$T_j^r = \frac{2g\omega_j \Delta\omega}{(\omega_j - \Omega_r)(\omega_j + \Omega_r)} \frac{\Omega_r}{\sqrt{[(\Omega^2 - \overline{\omega}^2) + \pi^2 g^2 \Omega^2]}} \tag{29.36}$$

where $\Delta\omega = \Delta\Omega = \frac{\pi c}{R}$.

For arbitrarily large R we note that $\Delta\omega \to 0$, so then the only non-vanishing matrix elements are those for which $\omega_j - \Omega_r \approx \Delta\omega$.

To arrive at an expression for the matrix elements in the limit $R \to \infty$ let us take R large enough so that $\Delta\omega \approx \Delta\Omega$ and consider the points of the spectrum of eigenfrequencies Ω inside and outside of a neighbourhood η of ω_j.

Note that when $R > \frac{2\pi c}{g}$ this means $\frac{\eta}{2} > \Delta\omega$, then we may consider R such that the neighbourhood $\frac{\eta}{2}$ of ω_j contains an integer number n of frequencies Ω_r, namely

$$n\Delta\omega = \frac{\eta}{2} = \sqrt{\frac{g\Delta\omega}{2}} \tag{29.37}$$

If R is arbitrarily large we see that $\frac{\eta}{2}$ is arbitrarily small, but n grows at the same rate, which means that the difference $\omega_i - \Omega_r$ outside the neighbourhood η of ω_i is arbitrarily larger than $\Delta\omega$, implying the corresponding matrix element T_i^r tends to zero. We also see that all frequencies Ω_r inside the neighbourhood of η of ω_i are arbitrarily close to ω_i, being an arbitrary large number N. Only the matrix elements T_i^r corresponding to these frequencies Ω_r inside the neighbourhood η of ω_i are different from zero.

For these we make a change of labels

$$\begin{aligned} r &= i - n \ (\omega_i - \frac{\eta}{2} < \Omega_r < \omega_i) \\ r &= i + n \ (\omega_i + \frac{\eta}{2} < \Omega_r < \omega_i) \end{aligned} \tag{29.38}$$

where $i = 1, 2, \ldots$. With the changed labels we get

$$T_i^i = \frac{1}{\epsilon_i} \frac{g\omega_i}{\sqrt{[(\Omega_r^2 - \overline{\omega}^2)^2 + \pi^2 g^2 \omega_i^2]}} \tag{29.39}$$

and

$$T_i^{i\pm n} = \frac{1}{n \pm \epsilon_i} \frac{\pm g\omega_i}{\sqrt{[(\Omega_r^2 - \overline{\omega}^2)^2 + \pi^2 g^2 \omega_i^2}]} \tag{29.40}$$

which satisfies the condition

$$\cot(\pi\epsilon_i) = \frac{\omega_i^2 - \overline{\omega}^2}{\omega_i \pi g} \tag{29.41}$$

Using the relationship

$$\pi^2 \csc^2(\pi\epsilon_i) = \frac{1}{\epsilon_i} + \sum_{n=1}^{\infty} [\frac{1}{(n+\epsilon_i)^2} + \frac{1}{(n-\epsilon_i)^2}] \tag{29.42}$$

the normalization condition for the relabelled matrix elements becomes

$$(T_i^i)^2 + \sum_{n=1}^{\infty} [(T_i^{i-n})^2 + (T_i^{i+n})^2] = 1 \tag{29.43}$$

and the orthogonality relation ($i \neq k$)

$$\sum_r T_i^r T_k^r = 0 \tag{29.44}$$

in the limit $R \to \infty$.

The Transformation matrix in the limit g = 0

For arbitrary R it is easily seen that $\lim_{g\to 0} T_0^r = 1$ if $\Omega_r = \overline{\omega}$ or 0 otherwise. We also see that the matrix element T_i^r for $i \neq r$ all vanish for g = 0. For small g for the diagonal terms T_i^i we have

$$T_i^i \approx \frac{1}{\epsilon_i} \frac{2g\Omega_i\omega_i}{(\Omega_i^2 - \overline{\omega}^2)\ (\omega_i + \Omega_i)} \tag{29.45}$$

or expanding ϵ_i for small g

$$T_i^i(g=0) = 1 \tag{29.46}$$

We see then that in the limit $R \to \infty$ the matrix element T_s^r remains an orthogonal matrix in the same sense as for finite R. With the choice of the procedure of taking the limit $R \to \infty$ using a discontinuous formalism from a confined solution, it can be seen that the matrix elements do not tend to the free space limit as it would in the case using a continuous formalism.

In the discontinuous formalism all non-vanishing matrix elements T_i^r are concentrated inside a neighbourhood η of ω_i and their spectrum is a quadratically summable and enumerable set. Also, the elements T_0^r are quadraticaaly integrable expressions.

Describing the Radiation Process Using the Dressing Formalism

Let us define coordinates q'_0 and q'_i as the coordinates of the dressed oscillator and the field. These dressed coordinates will provide a non-perturbative description of the oscillator-field system.

The general conditions that the dressed coordinates must satisfy are the following:

1) the coordinates q'_0 and q'_i should be linear functions of the coordinate modes Q_r,

2) the coordinates q'_0 and q'_i should allow the construction of orthogonal configurations corresponding to the separation of the system into two parts, the dressed oscillator and the field, and

3) the set should contain the ground state Γ_0.

The last condition restricts the transformation between coordinates q'_μ, $\mu = 0, 1, 2, \ldots$ and the collective ones Q_r to those that leave invariant the following quadratic form

$$\sum_r = \Omega_r\ Q_r^2 = \ \varpi(q'_0)^2 + \sum_i \ \omega(q'_i)^2 \tag{29.47}$$

Our configuration will behave in a first approximation as completely independent states, but they will evolve as time progresses, as if there are transitions between states, while the ground state Γ_0 is a fixed eignestate and does not evolve with time. The new coordinates q'_μ describes the dressed configuration of the oscillator and the field quanta.

The eigenstates of our system are represented by the normalized eigenfunctions

$$\phi_{n_0 n_1 n_2 \ldots}(Q,t) = \prod [N_{n_s}\ H_{n_s}(\sqrt{\frac{\Omega_s}{\hbar}}\ Q_s)]\Gamma_0 exp^{-i\sum_s n_s \Omega_s t} \tag{29.48}$$

where H_{n_s} is the n_s Hermite polynomial and N_{n_s} is a normalization coefficient given by

$$N_{n_s} = (2^{-n_s}\ n_s!)^{-\frac{1}{2}} \tag{29.49}$$

and Γ_0 is a normalization representation of the ground state

$$\Gamma_0 = exp[-\sum_s \frac{\Omega_s Q_s^2}{2\hbar} - \frac{1}{4} ln \frac{\Omega_s}{\pi\hbar}] \tag{29.50}$$

To describe the radiation process, havings as initial condition that the oscillator q_0 be excited, one considers the interaction term in the Hamiltonian written in terms of q_0, q_i as a perturbation, which induces transitions among the eigenstates of the free Hamiltonian. In this way it is possible to treat the problem approximately having as initial condition only that the "bare" or "undressed" oscillator be excited.

However, it is well known that this initial condition is physically not consistent due to the divergence of the "bare" or "undressed" oscillator frequency if there is an interaction with the field. One traditionally gets around this difficulty by a renormalization procedure, introducing by means of a perturbation, order by order corrections to the oscillator frequency.

It is possible to use an alternative procedure where we do not make explicit use of an interacting "bare" oscillator and field described by coordinates q_0 and q_i respectively. Instead we introduce "dressed" coordinates q_0' and q_i' for the "dressed" oscillator and field defined by

$$q'_\mu \sqrt{[\frac{\overline{\omega}_\mu}{\hbar}]} = \sum_r T_\mu^r \sqrt{[\frac{\Omega_r}{\hbar}]} \; Q_r \tag{29.51}$$

where $\overline{\omega}_\mu = \overline{\omega}, \omega_i$, and is valid for arbitrary R, while leaving invariant the quadratic form outlined above.

In terms of the bare coordinates, the dressed coordinates are expressed as

$$q'_\mu = \sum_\nu \alpha_{\mu\nu} \; q_\nu \tag{29.52}$$

where

$$\alpha_{\mu\nu} = \frac{1}{\sqrt{\overline{\omega}_\mu}} \sum_r T_\mu^r \, T_\nu^r \sqrt{\Omega_r} \tag{29.53}$$

As R becomes a large number we get for the coefficients $\alpha_{\mu\nu}$ the following: for α_{00}

$$\lim_{R\to\infty} \alpha_{00} = \frac{1}{\sqrt{\overline{\omega}}} \int_0^\infty \frac{2g\Omega^2\sqrt{\Omega}\; d\Omega}{(\Omega^2-\overline{\omega}^2)^2+\pi^2g^2\Omega^2} = A_{00}(\overline{\omega}, g) \tag{29.54}$$

for α_{i0}

$$\lim_{R\to\infty} \alpha_{i0} = \lim_{\Delta\omega\to\infty} \frac{1}{\sqrt{\omega_i}} \frac{(2g^2\omega_i^5\Delta\omega)^{\frac{1}{2}}}{(\omega_i^2-\overline{\omega}^2)^2+(\pi g)^2\omega_i^2} [\sum_{n=1}^{\infty} \frac{2\epsilon_i}{n^2-\epsilon_i^2} - \frac{1}{\epsilon_i}] \tag{29.55}$$

for α_{0i}

$$\lim_{R\to\infty} \alpha_{0i} = \lim_{\Delta\omega\to\infty} \frac{1}{\sqrt{\overline{\omega}}} \frac{(2g^2\omega_i^5\Delta\omega)^{\frac{1}{2}}}{(\omega_i^2-\overline{\omega}^2)^2+\pi^2 g^2\omega_i^2} [\sum_{n=1}^{\infty} \frac{2\epsilon_i}{n^2-\epsilon_i^2} - \frac{1}{\epsilon_i}] \tag{29.56}$$

and for α_{ik}

$$\lim_{R\to\infty} \alpha_{ik} = \delta_{ik} \tag{29.57}$$

Thus we can express the dressed coordinates q'_μ in terms of the bare ones q_μ in the limit $R \to \infty$

$$\begin{aligned} q'_0 &= A_{00}(\overline{\omega}, g)q_0 \\ q'_i &= q_i \end{aligned} \tag{29.58}$$

It is interesting to compare the finite radius and infinite radius cases.

In the case of finite R, the coordinates q'_0 and q'_i are all dressed, in that they are all collective, both the oscillator and field modes cannot be separated.

In the infinite radius limit $R \to \infty$, the coordinates q'_0 describes a dressed oscillator, while the dressed harmonic modes of the field, described by the coordinates q'_i are identical to the bare field modes. In the limit $R \to \infty$ the field retains its bare field modes while the oscillator is accompanied by a cloud of field quanta.

Therefore we identify the coordinates q'_0 as the coordinates describing the oscillator dressed by the field. The systems is divided into dressed oscillator and field. There is no interaction between them, the interaction being absorbed in the dressing cloud of the oscillator. We can use the dressed coordinates to describe the Radiation Process.

The Dressed Oscillator and Radiation

Let us define for a fixed instant the complete orthonormal set of functions

$$\phi_{\kappa_0\kappa_1\ldots}(Q,t) = \prod_{\mu} [N_{\kappa_\mu}\ H_{\kappa_\mu}(\sqrt{\frac{\overline{\omega}_\mu}{\hbar}}\ q'_\mu)]\ \Gamma_0 \tag{29.59}$$

where $q'_\mu = q'_0, q'_i$, and $\overline{\omega}_\mu = \overline{\omega}$. This function can be written as linear combinations of the eigenfunctions of the coupled system (let t = 0), namely

$$\psi_{\kappa_0\kappa_1\ldots}(q') = \sum_{n_0 n_1\ldots} T^{n_0 n_1\ldots}_{\kappa_0\kappa_1\ldots}(0)\ \phi_{n_0 n_1\ldots}(Q,0) \tag{29.60}$$

where the coefficients are given by

$$T^{n_0 n_1\ldots}_{\kappa_0\kappa_1\ldots}(0) = \int dQ\ \psi_{\kappa_0\kappa_1\ldots}\phi_{n_0 n_1\ldots} \tag{29.61}$$

with the integral extending over the entire Q-space.

Let us consider the case where there is only one dressed oscillator q'_μ in an excited state

$$\psi_{0\ldots 0N(\mu)0\ldots}(q') = N_N H_N(\sqrt{[\frac{\overline{\omega}_\mu}{\hbar}]}q'_\mu)\Gamma_0 \tag{29.62}$$

Using the relationship

$$\frac{1}{m!}[\sum_r (T^r_\mu)^2]^{\frac{m}{2}}\ H_n[\frac{\sum_r T^r_\mu\sqrt{[\frac{\Omega_r}{\hbar}]}Q_r}{\sqrt{\sum_r (T^r_\mu)^2}}]$$
$$= \sum_{m_0+m_1+\ldots=N} \frac{(T^0_\mu)^{m_0}\ (T^1_\mu)^{m_1}\ldots}{m_0!m_1!\ldots}\ H_{m_0}(\sqrt{[\frac{\Omega_0}{\hbar}]}Q_0)\ H_{m_1}(\sqrt{[\frac{\Omega_1}{\hbar}]}Q_1)\ldots \tag{29.63}$$

from which we get

$$T^{n_0 n_1\ldots}_{0\ldots 0N(\mu)0\ldots} = (\frac{m!}{n_0!n_1!\ldots})^{\frac{1}{2}}\ (T^0_\mu)^{n_0}\ (T^1_\mu)^{n_1}\cdots \tag{29.64}$$

where the subscripts $\mu = 0, i$ refer respectively to the dressed oscillator and the harmonic modes of the field, with the quantum numbers subject to the constraint $n_0 + n_1 + \cdots = N$.

Let us now look at the simple case of $N = 1$ and look at the behaviour of the "dressed" oscillator q_0' in the N-th excited state.

In the case of $N = 1$, let Γ_1^μ be the configuration of the "dressed" oscillator q_μ' in the first excited state, so that the time evolution of this state is given by

$$\Gamma_1^\mu = \sum_\nu f^{\mu\nu}(t)\ \Gamma_1^\mu(0)$$
$$f^{\mu\nu}(t) = \sum_s\ T_\mu^s\ T_\nu^s\ \exp(-i\Omega_s t) \tag{29.65}$$

We see then that as time progresses the excited "dressed" oscillator shares its energy amongst itself and all other "dressed" oscillators. Starting in its first excited state at time $t = 0$, the "dressed" oscillator's decay rate may be evaluated from its time evolution operator, namely

$$\Gamma_0^\mu = \sum_\nu f^{0\nu}(t)\ \Gamma_1^\mu(0) \tag{29.66}$$

where $f^{00}(t)$ is the probability that the "dressed" oscillator shall be excited and $f^{0\nu}(t)$ is the probability that the "dressed" oscillator shall have radiated away a field quanta of frequency ω_ν. Under this formalism we have a radiation process that is a simple exact time evolution of the system.

Evaluating $f^{00}(t)$ we have

$$f^{00}(t) = \int_0^\infty \frac{2g\Omega^2 \exp(-i\Omega t)}{(\Omega^2 - \omega^2)^2 + \pi^2 g^2 \Omega^2}\ d\Omega \tag{29.67}$$

which for large time t and arbitrary coupling constant g, yields

$$\begin{aligned} \mid f^{00}(t) \mid^2 = &\ \exp(-\pi g t)\ (1 + (\frac{\pi g}{2\varpi})^2) \\ + \exp(-\pi g t)\ &\frac{8g}{\varpi^4 t^3}\ (\sin(\varpi t) + \frac{\pi g}{2\varpi}\ \cos(\varpi t)) \\ &+ (\frac{4g}{\varpi^4 t^3})^2 \end{aligned} \tag{29.68}$$

for the oscillator decay probability where $\varpi = \surd(\omega^2 - (\frac{\pi g}{2})^2)$.

In the weak coupling regime where $g \ll \varpi$ we find

$$\mid f^{00}(t) \mid^2 \approx\ \exp(-\pi g t) \tag{29.69}$$

as expected.

Chapter 30

Transitions Due to a Forcing Function

Let us now consider an oscillator which at time $t = -\infty$ is in the ground state. We shall now try to find the probability that at time $t = +\infty$ the oscillator will be in the nth excited state if it has been subjected to a force f(t) such that $| f(t) | \to 0$ as $t \to \pm\infty$. [17]

Consider a Hamiltonian given by

$$\hat{H} = \frac{1}{2\mu}\hat{P}^2 + \frac{1}{2}\mu\omega^2 \, \hat{x}^2 - f(t) \, \hat{x} \tag{30.1}$$

which in terms of the creation and annihilation operators can be expressed as

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}) - f(t)\sqrt{\frac{\hbar}{2\mu\omega}}(\hat{a}^\dagger \; + \hat{a}) \tag{30.2}$$

We shall try to find a solution to the Schrödinger equation in the form of a compound function namely

$$\Psi(t) = c(t) \exp[\alpha(t) \; \hat{a}^\dagger] \; \exp[\; \hat{a}] \exp[\gamma(t) \; \hat{a}^\dagger \hat{a}] \; \Psi(-\infty) \tag{30.3}$$

In differentiating the operator acting on $\Psi(-\infty)$ with respect to time we must remember that the creation and annihilation operators do not commute, namely

$$[\; \hat{a}, \; \hat{a}^\dagger \;] = 1 \tag{30.4}$$

The derivative of $\Psi(t)$ with respect to time can be put into the form

$$\frac{d\Psi}{dt} = \hat{G}\ \exp[\alpha(t)\ \hat{a}^{\dagger}]\ \exp[\ \hat{a}]\exp[\gamma(t)\ \hat{a}^{\dagger}\hat{a}]\ \Psi(-\infty) \tag{30.5}$$

where the operator $\hat{G}$ is given by

$$\begin{aligned} \hat{G} = c\ \frac{d\gamma}{dt}\ \hat{a}^{\dagger}\hat{a}\ + c(\ \frac{d\alpha}{dt} - \alpha\ \frac{d\gamma}{dt})\ \hat{a}^{\dagger} \\ + c(\ \frac{d\beta}{dt} - \beta\ \frac{d\gamma}{dt})\ \hat{a} + \frac{d\ c}{dt} - c\ \frac{d\gamma}{dt}\ \alpha\beta - c\ \frac{d\beta}{dt}\alpha \end{aligned} \tag{30.6}$$

Equating the terms in the expression $i\hbar\hat{G} = \hat{H}$ we find

$$\begin{aligned} \frac{d\gamma}{dt} &= i\omega \\ \frac{d\alpha}{dt} + i\omega\alpha &= \frac{i}{\sqrt{(2\hbar\mu\omega)}}\ f(t) \\ \frac{d\beta}{dt} - i\omega\beta &= \frac{i}{\sqrt{(2\hbar\mu\omega)}}\ f(t) \\ \frac{\frac{dc}{dt}}{c} &= \frac{-i\omega}{2} + \alpha(\ \frac{d\beta}{dt} - i\omega\beta) \end{aligned} \tag{30.7}$$

Given the initial conditions

$$\begin{aligned} \alpha(-\infty) &= 0 \\ \beta(-\infty) &= 0 \\ \mid c(-\infty) \mid &= 0 \end{aligned} \tag{30.8}$$

we find the following

$$\begin{aligned} \alpha(t) &= \frac{i\exp(-i\omega t)}{\sqrt{(2\hbar\mu\omega)}} \int_{-\infty}^{t} f(t^{'})\ \exp(i\omega t^{'})\ dt^{'} \\ \beta(t) &= \frac{i\exp(i\omega t)}{\sqrt{(2\hbar\mu\omega)}} \int_{-\infty}^{t} f(t^{'})\ \exp(-i\omega t^{'})\ dt^{'} \\ \gamma(t) &= -i\omega t \end{aligned} \tag{30.9}$$

and for $c(t)$

$$c(t) = \exp(-\frac{i\omega t}{2})\ \exp[-\frac{1}{(2\hbar\mu\omega)} \int_{-\infty}^{t} f(t')\ \exp(-i\omega t')\ dt'$$
$$\int_{-\infty}^{t'} f(t'')\ \exp(i\omega t'')\ dt''\ ]$$
$$(30.10)$$

The probability of transition from the state $\Psi(-\infty)$ to the nth excited state for $t = +\infty$ is

$$W_n = \lim_{t\to\infty}\ |\int \Psi^*\Psi\ dx\ |^2$$

$$= \lim_{t\to\infty}\ |\, c \int \Psi^*\ \exp[\alpha(t)\ \hat{a}^\dagger]\ \exp[\ \hat{a}]\exp[\gamma(t)\ \hat{a}^\dagger\hat{a}]\ \Psi(-\infty)\ dx\ \ |^2 \quad (30.11)$$

If the initial state $\Psi(-\infty)$ was a ground state Ψ_0 then,

$$\exp[\ \hat{a}]\exp[\gamma(t)\ \hat{a}^\dagger\hat{a}]\ \Psi_0 = \Psi_0 \quad (30.12)$$

since the annihilation operator applied on the ground state is equal to zero.

Using the normalized wave function

$$\Psi_n = \frac{1}{\sqrt(n!)}\ (\hat{a}^\dagger)^n\ \Psi_0 \quad (30.13)$$

then

$$\exp[\alpha(t)\ \hat{a}^\dagger]\ \Psi_0 = \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt(m!)}\ \Psi_m \quad (30.14)$$

So then we find for the probability of the oscillator being in excited state n having started in ground state 0 is

$$W_{n0} = \lim_{t\to\infty} \frac{1}{n!}\ |\ c(t)\ |^2\ |\ \alpha(t)\ |^{2n} \quad (30.15)$$

where

$$|\ c(t)\ |^2 = \exp[-\frac{1}{\sqrt(2\hbar\mu\omega)}\ |\int_{-\infty}^{+\infty} f(t)\ \exp(i\omega t)\ dt\ \ |^2] = \exp(-\nu) \quad (30.16)$$

and

$$| \alpha(t) |^2 = \frac{1}{\sqrt{(2\hbar\mu\omega)}} \; | \int_{-\infty}^{+\infty} f(t) \; \exp(i\omega t) \; dt \; |^2 = \nu \tag{30.17}$$

We therefore obtain for W_{n0}

$$W_{n0} = \frac{\nu^n}{n!} \; \exp(-\nu) \tag{30.18}$$

which is the familiar Poisson distribution.

Two Examples of Forcing Functions

Let us look at two examples of forcing functions f(t) and solve for ν.

For a Gaussian type forcing function we have

$$f(t) = f_0 \; \exp[-\frac{t^2}{\tau^2}] \tag{30.19}$$

which yields a ν of

$$\nu = \frac{\pi\tau^2 f_0^2}{(2\hbar\mu\omega)} \; \exp(-\frac{\omega^2\tau^2}{2}) \tag{30.20}$$

while for a simpler type of forcing function

$$f(t) = f_0 \; \frac{1}{[1 + \frac{t^2}{\tau^2}]} \tag{30.21}$$

we get

$$\nu = \frac{\pi\tau^2 f_0^2}{(2\hbar\mu\omega)} \; \exp(-2\omega\tau) \tag{30.22}$$

As can be seen, the time evolution of a forcing function $f(t)$ will result in a finite probability of excitations of higher energy states.

Where we may have started in the distant past with only the ground state we have in the distant future higher energy states which are now populated. [30]

Part V

Cosmological Particle Creation

Chapter 31

Introduction

In this Part of the thesis I look at uniform acceleration in Rindler space, particle creation in two simple models of a Friedmann-Robertson-Walker Universe, as well as a hypothesis that gravity is an induced quantum effect.

Chapter 32

Particle Creation in a Two-Dimensional FRW Universe

Consider the two-dimensional Friedmann-Robertson-Walker universe with line element

$$ds^2 = dt^2 - a^2(t)dx^2 \tag{32.1}$$

where the space is expanding uniformly as described by the scalar function $a^2(t)$. Using the conformal time parameter η defined as $\eta = dt/a(t)$ we see that the line element is given by

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2) = C(\eta)(d\eta^2 - dx^2) \tag{32.2}$$

where $C(\eta)$ is a conformal scale factor.

As a simple model suppose that the conformal scale factor is given by [18]

$$C(\eta) = A + B \tanh \rho\eta \tag{32.3}$$

where A, B and ρ are constants. This model represents an asymptotically static universe that undergoes a period of smooth expansion. We see that in the far past and far future we have

$$C(\eta) \to A \pm B \tag{32.4}$$

Since $C(\eta)$ is not a function of x, spatial translation invariance is a spacetime symmetry. This means that we can separate the variables in the scalar function.

Consider the scalar field equation given by

$$[\ \Box_x + m^2 + \xi R(x)\ ]\phi = 0 \tag{32.5}$$

where

$$\begin{aligned} \Box_x\ \phi &= g^{\mu\nu}\ \nabla_\mu\ \nabla_\nu \\ &= \frac{1}{\sqrt{(-g)}}\ \partial_\mu[\sqrt{(-g)}\ g^{\mu\nu}\ \partial_\nu] \end{aligned} \tag{32.6}$$

with g being the determinant of the metric, and R is the Ricci scalar.

Use as an ansatz the function $u_k(\eta, x)$ for ϕ

$$u_k(\eta, x) = \frac{1}{2\pi}\exp(ikx)\chi_k(\eta) \tag{32.7}$$

In terms of $\chi_k(\eta)$ we have

$$\frac{d^2}{d\eta^2}\ \chi_k(\eta) + (k^2 + C(\eta)m^2)\chi_k = 0 \tag{32.8}$$

The solution of this ordinary differential equation can be given in terms of hypergeometric function. [5]

In the remote past we have the asymptotic function which behaves like a positive frequency solution ($\eta \to -\infty$)

$$u_k^{in}(\eta, x) \to \frac{1}{\sqrt{(4\pi\omega_{in})}}\ \exp(ikx - i\omega_{in}\eta) \tag{32.9}$$

where

$$\begin{aligned} \omega_{in} &= [k^2 + m^2(A-B)]^{\frac{1}{2}} \\ \omega_{out} &= [k^2 + m^2(A+B)]^{\frac{1}{2}} \\ \omega_\pm &= \frac{1}{2}(\omega_{out} \pm \omega_{in}) \end{aligned} \tag{32.10}$$

Those modes that behave like positive frequency Minkowski modes in the out region ($\eta \to \infty$) are given by the asymptotic form

$$u_k^{out}(\eta, x) \to \frac{1}{\sqrt{(4\pi\omega_{out})}} \ \exp(ikx - i\omega_{out}\eta) \tag{32.11}$$

As we can see u_k^{in} and u_k^{out} are not equal, however it is possible to express u_k^{in} in terms of u_k^{out} using a Bogolubov transformation and the linear transformation properties of hypergeometric functions, namely

$$u_k^{in}(\eta, x) = \alpha_k \ u_k^{out}(\eta, x) + \beta_k \ u_{-k}^{out*}(\eta, x) \tag{32.12}$$

where

$$\begin{aligned} \alpha_k &= (\frac{\omega_{out}}{\omega_{in}})^{\frac{1}{2}} \frac{\Gamma(1-(i\frac{\omega_{in}}{\rho}))\Gamma(-i\frac{\omega_{out}}{\rho})}{\Gamma(-i\frac{\omega_+}{\rho}))\Gamma(1-(i\frac{\omega_+}{\rho}))} \\ \beta_k &= (\frac{\omega_{out}}{\omega_{in}})^{\frac{1}{2}} \frac{\Gamma(1-(i\frac{\omega_{in}}{\rho}))\Gamma(i\frac{\omega_{out}}{\rho})}{\Gamma(i\frac{\omega_-}{\rho})\Gamma(1+(i\frac{\omega_-}{\rho}))} \end{aligned} \tag{32.13}$$

We can express the amplitudes in terms of hyperbolic sinh functions, namely

$$\begin{aligned} \mid \alpha_k \mid^2 &= \frac{\sinh^2(\frac{\pi\omega_+}{\rho})}{\sinh(\frac{\pi\omega_{in}}{\rho})\sinh(\frac{\pi\omega_{out}}{\rho})} \\ \mid \beta_k \mid^2 &= \frac{\sinh^2(\frac{\pi\omega_-}{\rho})}{\sinh(\frac{\pi\omega_{in}}{\rho})\sinh(\frac{\pi\omega_{out}}{\rho})} \end{aligned} \tag{32.14}$$

In the way of interpretation, in the remote past, where the spacetime is flat, all inertial particle detectors will register no particles, so that unaccelerated observers there would identify the quantum state with the real vacuum.

In contrast, in the remote future region ($\eta \to +\infty$) unaccelerated particle detectors there will register the presence of quanta where the expected number is given by $\mid \beta_k \mid^2$.

We can interpret this as the quantum creation of particles into the mode k which has come about as a result of the isotropic expansion of the universe. This is a very enlightening, albeit naive model of particle creation due to the expansion of the universe.

Chapter 33

Particle Creation in a Four-Dimensional Model Universe

We have seen in the previous section that for a scalar field satisfying the equation

$$\frac{d^2}{d\eta^2}\,\chi_k + \omega_k^2(\eta)\chi_k = 0 \tag{33.1}$$

where

$$\omega_k^2 = (k^2 + C(\eta)m^2) \tag{33.2}$$

we have particle creation due to the evolution in the conformal scale factor $C(\eta)$.

The scalar field equation is similar in form to that of a quantum harmonic oscillator with a time-dependent frequency. Such equations can be solved by the WKB method in the adiabatic limit, or by any of the methods outlined in this thesis.

Consider an asymptotically non-static four-dimensional model with the conformal scale factor $C(\eta)$ given by [18]

$$C(\eta) = a^2 + b^2\,\eta^2 \tag{33.3}$$

with $-\infty < \eta < \infty$, and where a and b are constants.

In the asymptotic reqions $\eta \to \pm\infty$, the model is equivalent to the radiation-dominated Friedmann model $a(t) \to \sqrt{C(t)} \propto \sqrt{t}$.

In the adiabatic approximation we see that the zeroth order approximation is valid when

$$\omega_k^2(\eta) = [k^2 + m^2\ (a^2 + b^2\ \eta^2\ )\ ] \tag{33.4}$$

for large values of η or large mb, or for large values of λ where λ is given by

$$\lambda = \frac{k^2 + m^2\ a^2}{mb} \tag{33.5}$$

The zeroth order solution in the large λ limit is

$$\chi_k^{(0)} \to \frac{1}{(2mb\lambda)^{\frac{1}{4}}}\ \exp[-i\eta\sqrt{(mb\lambda)}] \tag{33.6}$$

where η is fixed.

While the exact solution can be expressed in terms of the parabolic cylinder function D let us look at the asymptotic distant past and distant future approximations:

In the large $\mid \eta \mid$ limit the zeroth order solution approaches

$$\chi_k^{(0)} \to \frac{1}{(2mb \mid \eta \mid)^{\frac{1}{2}}}\ \exp[\mp \frac{im\eta^2}{2}] \tag{33.7}$$

which is equal to the exact solution $\eta_k^{(in)}$ as $\eta \to -\infty$ and $\eta_k^{(out)}$ as $\eta \to \infty$.

Solving for the Bogoliubov coefficient yields

$$\mid \beta_k \mid^2 = \exp[-\pi\ (\frac{k^2}{mb} + \frac{ma^2}{b}\ )\ ] \tag{33.8}$$

Spectrum of a Non-Relativistic Gas

A closer look at $\mid \beta_k \mid^2$ tells us that this spectrum is the same as that for a non-relativistic gas of particles with momentum

$$\frac{k}{\sqrt{C(\eta)}} \tag{33.9}$$

at a chemical potential of

$$-\frac{ma^2}{2C(\eta)} \tag{33.10}$$

and temperature

$$\frac{b}{2\pi C k_b} \tag{33.11}$$

where k_b is the Boltzman constant.

In the next section we shall look more closely at this spectrum temperature.

Chapter 34

Uniform Acceleration in Rindler Space

In the case of a scalar field the Lorentz Invariant spectral function is

$$\pi^2 f_0(\omega) = \frac{1}{2}\hbar c^2/\omega \tag{34.1}$$

Consider a scalar field of the form

$$\Phi(r,t) = \int d^3k f_0(\omega) cos(ik * r - i\omega t + i\Theta_k) \tag{34.2}$$

The time average value of the amplitude of the field is given by the correlation function

$$< \Phi(0,t) * \ \Phi(0,t) >= 1/2 \int d^3k f_0(\omega) \tag{34.3}$$

which is Lorentz Invariant.

Now consider an accelerating frame of reference moving along the x-axis with uniform acceleration a (Rindler Space) where

$$x(\tau) = c^2/a \ \cosh(a\tau/c) \tag{34.4}$$

and

$$v(\tau) = c \ \tanh(a\tau/c) \tag{34.5}$$

where $\gamma = \surd(1 - v^2/c^2) = \cosh(a\tau/c)$.

It is also straightforward to show that

$$\omega' = \omega \ \cosh(a\tau/c) - ck_x \sinh(a\tau/c) \tag{34.6}$$

and that

$$k_{x'} = k_x \ \cosh(a\tau/c) - \omega/c \ \sinh(a\tau/c) \tag{34.7}$$

The transformed correlation function is

$$< \Phi(0,t)^* \ \Phi(0,t) >_{accelerated} = -(\hbar/\pi c)(a/2c)^2 \cosh^2(a\tau/c) \tag{34.8}$$

Compare this to the scalar correlation function of the system at rest in a Zero Point Thermal Field:

$$< \Phi(0,t)^* \ \Phi(0,t) >_{ZPThermal} = -(\hbar/\pi c)(\pi KT/\hbar)^2 \cosh^2(\pi KT\tau/\hbar) \tag{34.9}$$

If we compare the two correlation functions we find that they are identical in functional form provided the acceleration and temperature are related by the expression

$$T = \hbar a/(2\pi Kc) \tag{34.10}$$

This relation is known as the Unruh-Davies Temperature relation. [27]

Now consider the Lorentz Invariant spectral function for an electromagnetic field,

$$\pi^2 H_0^2(\omega) = 1/2\hbar\omega \tag{34.11}$$

The transformed electromagnetic correlation function is of the form (i,j = 1,2,3):

$$\begin{aligned} &< E_i(0,t)E_j(0,t) > = < B_i(0,t)B_j(0,t) > \\ &\qquad = \delta_{ij} 4\hbar/(\pi c^3)(a/2c)^4 \ {}^4(a\tau/c) \end{aligned} \tag{34.12}$$

where the cross terms are of the form

$$< E_i(0,t)B_j(0,t) > = 0 \tag{34.13}$$

(csch is the hyperbolic cosecant).

Compare this to correlation function of the system at rest in a Zero Point Thermal field,

$$< E_i(0,t)E_j(0,t) >=< B_i(0,t)B_j(0,t) >$$
$$= \delta_{ij} 4\hbar/(\pi c^3)(\pi KT/\hbar)^4(\csc^4(\pi KT/\hbar) + 2/3\csc^2(\pi KT/\hbar)) \tag{34.14}$$

where again the cross terms are of the form

$$< E_i(0,t)B_j(0,t) >= 0 \tag{34.15}$$

Notice the additional term. The question is how to interpret the functional form and the additional term.

In the case of an electromagnetic field, the spectrum seen by the detector accelerating through a Zero Point electromagnetic field is

$$\pi^2 H^2_{accel}(\omega, a) = 1/2\hbar\omega(1 + (a/c\omega)^2)\coth(\pi c\omega/a) \tag{34.16}$$

If we express the acceleration in terms of the relation $T = \hbar a/(2\pi k_b c)$ (the Unruh-Davies Temperature) then the spectrum as seen by the accelerated system is

$$\pi^2 H^2_{accel}(\omega, a) = 1/2\hbar\omega(1 + (2\pi KT/\hbar\omega)^2)\coth(\hbar\omega/2KT) \tag{34.17}$$

rather than the unaccelerated Zero Point Thermal spectrum

$$\pi^2 H^2_{at_rest}(\omega, 0) = 1/2\hbar\omega\coth(\hbar\omega/2KT) \tag{34.18}$$

Note that acceleration adds a new term to the Zero Point Thermal spectrum and that the two spectrums agree at the higher frequencies $\hbar\omega \gg KT$.

In an accelerating frame there is an event horizon in the sense that in certain directions events occuring beyond a certain distance from the observer can never be reported to the observer by light signals due to dilation.

The observer is running away with ever increasing speed from these space-time events and modulated light signals carrying information can never catch up with the observer. These modes are frozen out and the spectral distribution of eigenvalues change.

A careful study of the situation shows that it is the long wavelength electromagnetic waves that are cut-off by the event horizon. As a result the accelerated spectrum H_{accel} does not go over to the energy equipartition at low frequency found with the unaccelerated Zero Point Thermal spectrum H_{atrest}.

Given that acceleration and temperature are related by the expression

$$T = \frac{\hbar a}{(2\pi k_b \ c)} \tag{34.19}$$

at the event horizon of a Schwarzchild black hole we have an acceleration of [21]

$$a = \frac{1}{4GM} \tag{34.20}$$

which means that an observer will find that a black hole will produce particles at a temperature of

$$T = \frac{\hbar}{(8\pi GMk_b \ c)} \tag{34.21}$$

This effect is known as Hawking Radiation.

In terms of the gravitational field, the vacuum around a black hole becomes unstable at the Schwarzchild radius and particles are produced with a thermal spectrum.

Chapter 35

Is Gravity an Induced Quantum Effect

In a 1968 paper Andrei Sakharov roposed [26] that gravity is not a separately existing fundamental force but rather an induced effect associated with fluctuations of the vacuum state. Sakharov's Proposal is discussed in greater detail in an appendix to this thesis and in a paper by the author. [20]

Einstein's Principle of Equivalence requires a modification to the Zero Point Field due to gravitational mass. Following the reasoning set out by Sakharov and Puthoff, in this section we shall show that gravity is not a separately existing fundamental force but rather an induced effect associated with Zero Point Fluctuations of the vacuum.

Consider the equation of motion for an oscillating charged particle given by

$$\frac{d^2q}{dt^2} + v_0^2 q - \frac{\Gamma d^3 q}{dt^3} = \Gamma^* E \tag{35.1}$$

where $q = q(t)$ is the oscillator coordinate, v_0 is the natural frequancy of the oscillator, Γ is the damping coefficient

$$\Gamma = \frac{e^2}{(6\pi\epsilon_0 m_e c^3)} \tag{35.2}$$

Now consider the kinetic energy W_{kin} of the particle motion due to fluctuations induced by the Zero Point electromganetic field,

$$W_{kin} = \frac{1}{2m_0}\,\frac{d^2q}{dt^2} = \frac{1}{(12\pi\Gamma\epsilon_0 c^3)}\,(\frac{dp}{dt})^2 \tag{35.3}$$

where $p = eq$ is the dipole moment of the oscillator.

Written in this form it is worth noting that the energy equation refers to the global properties of the oscillator (p, v_0 and the damping constant Γ) and does not involve individual properties such as mass or charge.

Using the Zero Point Electromagnetic fields E^{ZP} and B^{ZP} and solving for the time average value for $< (d/dtp - x)^2 >$ yields

$$< (d/dtp_x)^2 > \simeq 6\epsilon_0 c^3 \hbar (\Gamma\omega_c)^2 \tag{35.4}$$

where ω_c is some characteristic frequency. In two-dimensions the particle motion due to fluctuations induced by the Zero Point electromagnetic field is

$$< (d/dtp)^2 >_{two-dimen} = 2 < (d/dtp)^2 >_{one-dimen} \tag{35.5}$$

The time average value for the internal energy of the oscillator, expressed in terms of its global properties is given by

$$< Energy > = \hbar\Gamma\omega_c^2/\pi \tag{35.6}$$

The energy calculated in this fashion is a transverse self-energy of the particle motion due to fluctuations induced by the Zero Point electromagnetic field. Using the expression Einstein expression $E = m_G c^2$ gives

$$m_G = \hbar\Gamma\omega_c^2/(\pi c^2) \tag{35.7}$$

In Puthoff's interpretation of Sakharov's Proposal, the oscillator's mass is of dynamical origin, originating in the motion response of the charged particle to the motion induced by the Zero Point electromagnetic field. It is the internal motion of the charged oscillator that contributes to the effective mass of the oscillator through the mass-energy equivalence outlined in the Einstein expression $E = mc^2$.

The lowest order interaction between a charged particle and a Zero Point Field that produces a far field effect is the dipole interaction. Of the dipole-field terms, the $1/r^4$ term predominates at large distances.

In expanding out the dipole field distribution there is a term proportional to $1/r^2$ which is the radiation field associated with the Zero Point Fluctuation driven dipole. This radiation just replaces that being absorbed from the background on a detailed-balanced basis.

The energy density Δw_d in the two-dimensional far field dipole-field interaction is

$$\Delta w_d = (3\hbar c\Gamma^2 cos^2\Theta)/(2\pi^2 r^4) \int_0^{\omega_c} d\omega\ \omega \tag{35.8}$$

where ω_c is a characteristic frequency used as a cut-off frequency to avoid divergence.

Averaged over the net contribution of randomly oriented individual Zero Point particle motion, and integrated over the solid angle, we have an overall spectral density of

$$\Delta\rho_{d'} = \omega(\hbar c\Gamma^2)/(2\pi^2 r^4) \tag{35.9}$$

Using the relationship for mass m_g and Γ we have

$$\Delta\rho_{d'} = \omega(c_g^{52})/(2\pi^2\omega_c^4 r^4) \tag{35.10}$$

Recall the expression for the accelerated Zero Point Thermal spectrum for the electromagnetic field and set $T = 0$

$$\pi^2 H^2_{accel}(\omega, a) = 1/2\hbar\omega(1 + (a/c\omega)^2) \tag{35.11}$$

Multiply this expression by the density of normal modes (ω^2/π^2c^3) and equate the contribution from the acceleration term $1/2\hbar\omega(a/c\omega)^2$ to the expression $\Delta\rho_{d'}$ yielding

$$\hbar a^2/(\pi^2c^5) = (c^5 m_G^2)/(\hbar\omega_c^4 r^4) \tag{35.12}$$

Now let $a = Gm_G/r^2$ and solve for ω_c

$$\omega_c = \sqrt{}(\pi c^5/\hbar G) \tag{35.13}$$

On the basis of heuristic and dimensional considerations Sakharov proposed that a vacuum fluctuation model for gravitation would have a characteristic cut-off frequency ω_c of this form. Solving for the gravitational constant G we have

$$G = \pi c^5/\hbar\omega_c^2 \tag{35.14}$$

Studying the Zero Point fluctuation induced dipole field at the position of particle A due to the fluctuating motion of a second similar particle, particle B, leads to an expression for the potential energy of the interaction of the form:

$$U = -9\hbar c^3\Gamma^3/(4\pi)\ Re(\int_0^{u_c} du\ (exp^{-2uR})/R) \tag{35.15}$$

where $u = -i\ \omega/c$ and $u_c = -i\ \omega_c/c$.

For two-dimensional Zero Point dipole motion the attracting potential is given by

$$U = -1/2\ \delta(1-\cos(2R))/R^3 = -\delta/R((\sin R)/R)^2 \tag{35.16}$$

where the parameter δ is given by

$$\delta = \hbar\Gamma^2\omega_c^3/\pi \tag{35.17}$$

and the scale parameter R is $R = r\omega_c/c$.

With the gravitational potential thus defined, the gravitational force is given by the classical expression

$$F_g = -\partial U/\partial r \tag{35.18}$$

The gravitational potential has the desired $1/r$ dependence modified by a form factor $((\sin R)/R)^2$ which has a characteristic length on the order of the fundamental length $\Lambda_{EG} \simeq 2.82 \times 10^{-34} cm$.

If we extract the leading terms from both U and F_g we arrive with the following:

$$U = -\hbar c\Gamma^2\omega_c^2/(\pi r) \tag{35.19}$$

$$F_G = -\hbar c\Gamma^2\omega_c^2/(\pi r^2) \tag{35.20}$$

Using Sakharov's characteristic cut-off frequency

$$\omega_c = \sqrt{(\frac{\pi c^5}{\hbar G})} \tag{35.21}$$

and the expression for the gravitational mass derived above

$$m_G = \frac{\hbar \Gamma \omega_c^2}{(\pi c^2)} \tag{35.22}$$

we arrive at the familiar expression for the gravitational force,

$$F_G = \frac{-G(m_A m_B)}{r^2} \tag{35.23}$$

which is Newton's Law of Gravitational Attraction between two bodies of similar mass.

For dissimmilar masses we modify the force equation F_G to read

$$F_G = \frac{-\hbar c \Gamma_1 \Gamma_2 \omega_c^2}{(\pi r^2)} \tag{35.24}$$

and solve in a similar fashion to arrive at

$$F_G = \frac{-G(m_1 m_2)}{r^2} \tag{35.25}$$

which is Newton's Law of Gravitational Attraction between two bodies of dissimilar masses.

A more detailed discussion of the consequence of this proposal is outlined in the paper *Zero Point Fluctuations and the Suspended Charge Paradox* [20] which is reproduced in abridged form as an appendix to this thesis.

Part VI

Summary and Conclusions

Chapter 36

Summary and Conclusions

Exact solutions for the Schödinger equation are known for a rather restricted set of interaction Hamiltonian or potentials, so the standard problem we are faced with is to find a good approximation in place of an exact solution.

Many of our descriptions of quantum systems have been influenced by the Quantum Harmonic Oscillator (QHO). For quite a wide range of quantum systems, it is valid to look for an initial approximation in the form of an oscillator basis, that is, a stable quantum system in a well chosen representation can be described by some set of harmonic oscillators with a spectrum of frequencies.

Many systems may be treated as a set of oscillators with a frequency defined by a mass parameter. The interaction does not change the oscillator nature of the underlying quantum field, but only redefine their masses and other physical characteristics.

The first four sections of this thesis build up the mathematical tools, namely the Algebraic Method, the Bogoliubov transformation and the "dressed" oscillator approach, for Part Five in which I look at Particle Creation in two Simple Models of a Friedmann-Robertson-Walker Universe as well as a hypothesis that Gravity is an Induced Quantum Effect.

Part VII

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Part VIII

Appendices

Appendix A

Some Observable Effects of Zero Point Fluctuation

The quantum mechanical Zero-Point variation of the field in empty space gives rise to fluctuating electric and magnetic fields [28] whose mean square value at a point in space is given by

$$< E^2 >_{Av}=< B^2 >_{Av}= 2\hbar c/\pi \int_0^\infty dk\ k^2 \tag{A.1}$$

where k refers to the wave number of the quanta of electromagnetic energy. The contribution from the frequency to the mean square fluctuation in the range c dk is explicitly outlined.

The mean square value of the electric and magnetic field at a point in space is derived by ascribing to each normal mode of the radiation field an energy which is just the Zero-Point energy of an ocsillator with the frequency of the normal mode. The total energy can be written either as a sum or as an integration of the electromagnetic energy density.

To describe the non-relativistic motion of a free charged particle, such as an electron, in a fluctuating Zero-Point field, let q be its position vector, then

$$md^2/dt^2q = eE \tag{A.2}$$

where e is the charge of the electron and E is the fluctuating electric field whose mean square value is given above. We have disregarded any damping or back reaction in the motion of the fluctuating electron.

Since this equation of motion is linear we can regard it as a classical equation for the quantum mechanical expectation value of the position of the electron given by q. For a given harmonic component of E (i.e. $E(\omega)e^{i\omega t}$) the solution of the equation of motion is straightforward.

Fluctuation in the Position of the Particle

Performing this integration, find the value for $< q^2 >_{Av}$ for the given harmonic component and sum over the frequencies using the mean square value given above. You arrive at an expression for the mean square fluctuation in the position of a free electron given by

$$< (\Delta q)^2 >_{Av} = (2/\pi)\ \alpha(\hbar/mc)^2 \int_{k_0}^{\kappa} dk/k \tag{A.3}$$

where $\alpha = (e^2/\hbar c)$. In this integral for the mean square fluctuation in the position of a free electron we have assumed both a lower cut-off k_0 and an upper cut-off $\kappa = mc/\hbar$.

The Lamb Shift

The magnitude for this mean square fluctuation in the position of a free electron will be very small for most values to the lower cut-off k_0, however an observable effect known as the Lamb shift arises when an electron moves in a potential with a large curvature such as the ground state of a hydrogen or helium atom.

Consider the motion of an electron in a statis field specified by a potential V(q). The position of the electron consists of two contributions, one from the orbital motion plus a second smaller contribution which fluctuates randomly in time.

Let q be the orbital part and Δq be the random part, then the instantaneous potential energy $V(q + \Delta q)$ is given by

$$V(q + \Delta q) = [1 + \Delta q \cdot \nabla + 1/2(\Delta q \cdot \nabla)^2 + \cdots]\ V(q) \tag{A.4}$$

The effective potential energy that the electron sees is the average of $V(q + \Delta q)$ over all values of Δq. Since Δq has an isotropic spatial distribution we get

$$< V(q + \Delta q) >_{Av} = [1 + 1/6 < (\Delta q)^2 >_{Av} \nabla^2 + \cdots]\ V(q) \tag{A.5}$$

As can be seen, the existence of the fluctuations in the position of the electron effectively modifies the potential in which it moves by adding a term proportional to the Laplacian of the potential energy.

Consider the perturbation to the ground state potential energy of a hydrogen atom caused by the fluctuations in the position of an orbiting electron. Given the static potential of the nucleus

$$V(q) = -(1/4\pi\epsilon_0)\ (e^2/r) \tag{A.6}$$

Retaining the first order term in the effective potential energy, the correction to the potential energy becomes

$$\Delta V(q) = (e^2/3\pi\epsilon_0)\ \alpha\ (\hbar/mc)^2 \ln\ [mc^2/\hbar c k_0]\delta(q) \tag{A.7}$$

The correction to the energy W of the stationary state is given by

$$\Delta W = (e^2/3\pi\epsilon_0)\ \alpha\ (\hbar/mc)^2 \ln\ [mc^2/\hbar c k_0]\ \mid \Psi(0) \mid^2 \tag{A.8}$$

By letting the quantity $\hbar c k_0$ equal to the average excitation energy of 17.8 Rydbergs for hydrogen this expression becomes identical with the expression derived by Bethe for the level shift in hydrogen.

Reviewing the equation it becomes obvious that the fluctuation in the position of the electron must act to weaken the effect of the potential energy. This results in an observable shift in the energy levels which were first measured some fifty years ago at Cornell University.

Low Energy Compton Scattering

The fluctuations in the position of the electron results in a spreading out of the electronic charge and the current that participates in electromagnetic interactions. Consider the modification to the transition probability in non-relativistic Compton scattering due to the fluctuations in the position of the electron.

Under the non-relativistic Compton scattering a free electron executes a steady forced oscillation under the action of an incident light wave and emits a scattered light wave of the same frequency. The effect of the position fluctuation is twofold. The electron behaves like a distributed charge with a mean square radius $< (\Delta q)^2 >_{Av}$. It therefore interacts less strongly with the incident light and radiates a weaker scattered wave than what would be expected from a point particle.

To estimate the magnitude in this reduction in the interaction between the electron and the incident light we consider the change in the phase factor for the wave introduced by the averaging over the position fluctuation of the electron. This yields

$$< \exp[ik \cdot (q + \Delta q)] >_{Av} = \exp(ik \cdot q) \ < [1 + \Delta q \cdot \nabla + 1/2(\Delta q \cdot \nabla)^2 + \cdots] >_{Av} \tag{A.9}$$

which can be reformulated to read

$$< \ expik \cdot (q + \Delta q) >_{Av} \simeq \exp(ik \cdot q) \ [1 - 1/6 \ k^2 < (\Delta q)^2 >_{Av}] \tag{A.10}$$

The correction involves the product of the mean square position fluctuation with the Laplacian of the space function describing the interaction. We see that the amplitude of the oscillation and the amplitude of the scattered wave will suffer a fractional reduction equal to

$$-1/6 \ k^2 < (\Delta q)^2 >_{Av} \tag{A.11}$$

The resultant reduction will be twice as large and the reduction in the scattered intensity or cross section will be twice as large again , so that the fractional change in the Compton scattering cross section will be

$$\Delta\sigma/\sigma = -2/3k^2 < (\Delta q)^2 >_{Av} = 4/(3\pi\epsilon_0)\alpha(\hbar k/mc)^2 \ln \ [mc/\hbar k_0] \tag{A.12}$$

We observe that the angular distribution will remain unchanged since the scattering retains its dipole character. There remains the problem of determining the lower cut-off k_0.

Frequencies of fluctuation higher than the frequency of the incident light wave will effectively spread out the scattering charge, while frequencies below this limit will only displace the scattering charge in a random fashion, therefore set $k_0 = k$ to get

$$\Delta\sigma/\sigma = 4/(3\pi\epsilon_0)\alpha(\hbar k/mc)^2 \ln \ [mc/\hbar k] \tag{A.13}$$

Note that the correction goes to zero strongly at low frequencies.

The Interaction Between a Spin and a Magnetic Field

Fluctuations of the radiation field change the effective potential energy of interactions between an external magnetic field and the spin and orbital magnetic moment of an electron.

Consider a simple system which consists of an angular momentum with an associated magnetic moment. Let $1/2\hbar\sigma$ be the angular momentum operator. Then the equation of motion for the angular momentum operator is given by

$$d/dt\ \sigma = (e/mc)B \times \sigma \tag{A.14}$$

where B is the instantaneous magnetic field intensity arising from the fluctuations in the Zero-Point field. The correction $\Delta\sigma$ to the angular momentum operator is given by

$$\Delta\sigma = (e/mc)B \times \sigma \tag{A.15}$$

The mean square fluctuation in the unit spin operator $< (\Delta\sigma) >_{Av}$ is then given by

$$< (\Delta\sigma)^2 >_{Av} = \sigma^2 \alpha 2/\pi(\hbar/mc)^2 \int_0^{\kappa} dkk \tag{A.16}$$

The upper limit to the integral has been made finite rather than infinite. We define the mean square angle of fluctuation $< (\Delta\Theta)^2 >_{Av}$ by the expression

$$< (\Delta\Theta)^2 >_{Av} = < (\Delta\sigma)^2 >_{Av} /\sigma^2 = \alpha/\pi(\hbar/mc)^2\kappa^2 \tag{A.17}$$

The energy W of a particle with spin in a magnetic field is given by $W = (e\hbar/2mc) \mid \sigma \mid cos\Theta$ where Θ is the angle between the spin direction and the direction of the external magnetic field.

It is immediately apparent that the fluctuations in the magnetic field affects the average value of $\cos\Theta$. This means that $cos\Theta$ is to be replaced by an average value $< \cos\Theta >_{Av}$ which is taken as an average over the fluctuations in the spin vector.

It can be shown that

$$< \cos\Theta >_{Av} = < \cos\Theta \cos\Delta\Theta + \sin\Theta \cos\phi >_{Av} \tag{A.18}$$

This average is easy to assess given that the average of $cos\phi$ must vanish because of the isotropy of the fluctuation. Given that $\Delta\Theta$ is small, $\cos\Delta\Theta$ can be respresented by the first two terms in its series expansion so that

$$< \cos\Theta >_{Av} = < \cos\Theta[1 - 1/2 < (\Delta\Theta)^2 >_{Av}] > = < \cos\Theta[1 - \alpha/2\pi(\hbar\kappa/mc)^2 > \tag{A.19}$$

We see that the correction to the orientation energy of the orbital spin consists in a reduction in the magnitude of the energy proportional to the energy itself. It is convenient to consider the effect of the interaction of the orbital spin with the radiation field as consisting of an alteration of the magnetic moment of the electron μ. If we set $\kappa = 1/a = \alpha(mc/\hbar)$ then

$$\Delta\mu/\mu = -(1/2\pi)\ \alpha^3 \tag{A.20}$$

which is a very small effect. For the intrinsic spin of the electron the alteration of the magnetic moment of the electron μ is given by

$$\Delta\mu/\mu = -(1/2\pi)\ \alpha \tag{A.21}$$

Appendix B

Zero Point Fluctuations and the Suspended Charge Paradox

From electrodynamics we know that an accelerating charge with colinear velocity and acceleration radiates with the following angular dependence: [20]

$$\frac{dP}{d\Omega} = \frac{e^2 a^2 \sin^2\theta}{(4\pi c^3(1-\beta\cos\theta)^5)} \tag{B.1}$$

where a is the acceleration, $\beta = \mathrm{v/c}$ and θ the angle measured in the direction of motion.

In the limit of small velocity and integrated over the solid angle the total power radiated by a nonrelativistic accelerated charge is given by the familiar Larmor formula:

$$P = \frac{2e^2a^2}{(3c^3)} \tag{B.2}$$

Einstein's Principle of Equivalence tells us that an accelerating and a gravitational frames of reference should be equivalent on a local scale.

The "Paradox of the Suspended Charge" arises when we set a = g, the acceleration due to gravity, then we would expect the suspended charge to radiate away its power at a rate proportional to g^2, namely

$$P \simeq e^2 g^2 / c^3 \tag{B.3}$$

something we of course do not see.

If as it appears that an electron has no finite size then the "Paradox of the Suspended Charge" carries down to a very small length scale.

What is Zero Point Fluctuation?

The random radiation involved in Zero Point Fluctuations is not connected with temperature radiation, but exists in the vacuum at the absolute zero of Thermal Temperature. These fluctuations are a result of the Heisenberg Uncertainty relation $\Delta E \Delta t \geq \hbar$, although the underlying physical cause is yet not fully understood.

This random radiation is considered as real as thermal radiation. Some of the observable effects of this Fluctuation is outlined in an appendix to this thesis.

A special aspect of Zero Point Fluctuation is that its spectrum is Lorentz Invariant. This means that for a given field type every inertial observer, irrespective of their velocity, finds the same spectrum for the Zero Point Field. [22]

For a scalar field the Lorentz Invariant spectral function is given by: [19]

$$\pi^2 f_0(\omega) = \frac{1}{2}\hbar c^2/\omega \tag{B.4}$$

For an electromagnetic field the Lorentz Invariant spectral function is given by the fàmilar equation:

$$\pi^2 h_0(\omega) = \frac{1}{2}\hbar\omega \tag{B.5}$$

Stability of Ground State

One of the great questions of quantum and classical physics is why atoms do not radiate away all their energy and collapse down to zero.

It appears that Zero Point Fluctuation prevents the collapse. To describe this effect requires both a classical and a quantum description of the process of absorption and radiation from an atom. [23, 24]

Define the relationship between the quantum expectation value and the classical probability by the following:

$$< \Psi \mid f(r) \mid \Psi > = \int d^m r \; P_\Psi(r) f(r) \tag{B.6}$$

Let $f(r) = \exp(-is * r)$ so then

$$< \Psi \mid \exp(-is * r) \mid \Psi >= \int d^m r \; P_\Psi(r) \; \exp(-is * r) = g(s) \tag{B.7}$$

where g(s) is a generating function. So then

$$1/(2\pi)^m \int d^m r \; g(s) \; \exp(-is * r) = P_\Psi(r) \tag{B.8}$$

where m is the dimensionality of the space. You recognize this relationship to be a fourier transformation from the r-space to the s-space.

The nonrelativistic equation of motion for an oscillating charged particle is given by

$$\frac{d^2 q}{dt^2} + v_0^2 q - \Gamma \frac{d^3 q}{dt^3} = \Gamma^* E \tag{B.9}$$

where q = q(t) is the oscillator coordinate, v_0 is the natural frequancy of the oscillator, Γ is the damping coefficient

$$\Gamma = \frac{e^2}{(6\pi\epsilon_0 m_e c^3)} \tag{B.10}$$

and Γ^* is the driving coefficient

$$\Gamma^* = \frac{e}{m_e} \tag{B.11}$$

Fourier transform the equation of motion and solve for $q(\nu)$

$$q(\nu) = H(\nu) E(\nu) \tag{B.12}$$

where the dispersion relation $H(\nu)$ is given by

$$H(\nu) = \frac{\Gamma^*}{(v_0^2 - v^2 + i\Gamma v^3)} \tag{B.13}$$

Describe the zero point electric field E^{ZP} by a traveling wave

$$E^{ZP}(r,t) = Re \sum \epsilon_k \sigma_k w_k \; \exp(ik * r - i\omega t) \tag{B.14}$$

with $w_k = u_k + iv_k$, ϵ_k a unit vector in the direction of propagation and σ_k is the polarization.

We shall introduce the intensity of the E^{ZP} field by letting $u_k = (\sqrt{I_k})\cos(\Theta_k)$ and $V_k = (\sqrt{I_k})\sin(\Theta_k)$, where Θ_k is a random phase subject to a constant distribution in $[0, 2\pi]$, and I_k is the intensity per mode.

Set r=0 (to suppress the k*r term in $E^{ZP}(r,t)$) and taking the Fourier transform of $E^{ZP}(r,t)$ yields

$$E^{ZP}(\nu) = \pi \sum \epsilon_k \sigma_k (\delta(\omega + \nu)(u_k + iv_k) + (\delta(\omega - \nu)(u_k - iv_k)) \tag{B.15}$$

So then solving for $q(t)$

$$q(t) = 1/(2\pi)^m \int d\nu \; exp(i\nu t) \; H(\nu)E(\nu) = \sum \epsilon_k \sigma_k Re(\omega_k \zeta^*(\omega)) \tag{B.16}$$

where $\zeta(\omega) = \exp(i\omega t)H(\omega)$.

The generating function $g(s) =< \Psi \mid$
$exp(-is * r) \mid \Psi >$ for this distribution is given

$$\begin{aligned} g(s) = \Pi_k (1/(2\pi)^m) \int du_k \int dv_k \; \exp((-u_k/2 - v_k/2) \\ -(is * \epsilon_k \sigma_k (u_k \; Re(\zeta(\omega)) + v_k Re(\zeta(\omega)))) \end{aligned} \tag{B.17}$$

or

$$g(s) = \Pi_k \; \exp(- \mid s * \; \epsilon_k \sigma_k \; H(\omega) \mid^2 /2) \tag{B.18}$$

It is worth noting that this result is exact and that there has been no requirement for $\hbar$ (Planck's constant) up to this point.

In an unbounded space ($L \gg \lambda$), the mode product Π_k in the generating function g(s) can be converted to an integral within the exponential, so that

$$g(s) = \exp[-\frac{s^2 \hbar}{12\pi^2 \epsilon_0 c^3}] \int d\omega \omega^3 \mid H(\omega) \mid^2) \tag{B.19}$$

where the dispersion equation is given by

$$\mid H(\omega) \mid^2 = \mid \frac{\Gamma^*}{(\omega_0^2 - \omega^2 + i\Gamma\omega^3)} \mid^2 \tag{B.20}$$

Notice that Planck's constant $\hbar$ has been introduced for the first time in the generating function and is being used as a scaling factor.

Reformulating the dispersion equation

$$\mid H(\omega) \mid^2 = \frac{(\Gamma^*)^2}{((\omega_0^2 - \omega^2)^2 + (\Gamma\omega^3)^2)} \tag{B.21}$$

Due to the smallness of the damping term Γ in the denominator the integrand is sharply peaked at $\omega = \omega_0$.

Using the resonance approximation

$$\int_0^\infty d\omega\ \omega^3 \mid H(\omega) \mid^2 \simeq \int_{-\infty}^\infty d\omega\ \omega_0^3 \mid H(\omega_0) \mid^2 \simeq \frac{\pi\Gamma^{*2}}{(2\Gamma\omega_0)} \tag{B.22}$$

Then

$$\int_0^\infty d\omega\ \omega^3 \mid H(\omega) \mid^2 \simeq \frac{3\pi^2\epsilon_0 c^3}{(m_e\omega_0)} \tag{B.23}$$

The generating function in the resonance approximation is

$$g(s) \simeq \exp(-\frac{s^2\hbar}{4m_e\omega_0}) = \exp(-s^2\sigma_q^2) \tag{B.24}$$

and the probability distribution becomes

$$P(q) = \frac{1}{(2\pi\sigma_q^2)^{1/2}} \exp(-\frac{q^2}{2\sigma_q^2}) \tag{B.25}$$

where $\sigma_q^2 = \frac{\hbar}{2m_e\omega_0}$.

Note that this distribution agrees in form with that predicted by quantum mechanics for the nonrelativistic harmonic oscillator in the ground state.

If we consider the ground state of the Bohr atom as modeled by a pair of orthogonal one-dimensional harmonic oscillators then the two-dimensional distribution becomes:

$$P(q_x, q_y) = \frac{1}{(2\pi\sigma_q^2)} \exp[\,-\frac{q_x^2 + q_y^2}{2\sigma_q^2}\,] \tag{B.26}$$

Compare this with the quantum probability for a two-dimensional quantum oscillator in the ground state

$$P_0(x,y) = (\frac{m_e\omega_0}{\pi\hbar})\ \exp[\,-\frac{m\omega_0(x^2+y^2)}{\hbar}\,] \tag{B.27}$$

and note that they have similar functional form.

We have derived a quantum mechanical ground state probability distribution for a simple atom using classical field theory. The underlying mathematics of the Gaussian function and the fact that a Gaussian function carries over to a Gaussian function under a Fourier transform is worth remembering.

Power Absorbed from the Zero Point Field

Now look at the time-average power that is being absorbed from a Zero Point electric field E^{ZP} by a harmonic oscillator:

$$< P_{abs} >= e < E^{ZP} * v > \tag{B.28}$$

where E^{ZP} and B^{ZP} (the Zero Point magnetic field) are given by

$$\begin{aligned} E^{ZP}(r,t) &= Re \sum d^3k\epsilon_k\sqrt{I_k}\ \exp(ik*r - i\omega t + i\Theta_k) \\ B^{ZP}(r,t) &= Re \sum d^3k(k\times\epsilon_k)\sqrt{I_k}\ \exp(ik*r - i\omega t + i\Theta_k) \end{aligned} \tag{B.29}$$

Let us now use a classical intensity function I_k

$$I_k = \frac{h_0^2(\omega)}{4\pi\epsilon_0} = \frac{\hbar\omega}{8\pi^3\epsilon_0} \tag{B.30}$$

where you recall $h_0^2(\omega)$ is the Lorentz Invariant spectral function for the vector Electromagnetic field.

The time average power absorbed by the oscillator is then

$$< P_{abs} >= (\frac{\Gamma\hbar}{\pi}) \int_0^\infty d\omega \frac{\Gamma\omega^7}{((\omega_0^2-\omega^2)^2 + (\Gamma\omega^3)^2)} \tag{B.31}$$

This integral is strongly peaked at ω_0 so then the resonance approximation can be used and the integral becomes

$$< P_{abs} >\simeq 1/2 \int_{-\infty}^\infty d\omega \frac{\Gamma\omega_0^3(\omega_0^2/2)}{((\omega_0^2-\omega^2)^2 + (\Gamma\omega_0^2)^2)} \tag{B.32}$$

This integral describes a Lorentzian line shape and is equal to $\pi\omega_0^3$, so then

$$< P_{abs} > \simeq \frac{\Gamma\hbar\omega_0^3}{2} = \frac{e^2\hbar\omega_0^3}{12\pi\epsilon_0 m_e c^3} \tag{B.33}$$

This is an expression for the absorption of electromagnetic power from the Zero Point field by a one dimensional charged harmonic oscillator.

Model the ground state motion of the Bohr atom with radius r_0 by a pair of one dimensional harmonic oscillators describing circular motion. The time averaged electromagnetic power absorbed for a two dimensional oscillator is then simply

$$< P_{abs} >_{two-dimen} = 2 * < P_{abs} >_{one-dimen} \tag{B.34}$$

The time averaged electromagnetic power radiated by an electron in circular motion with acceleration a is given by the well known expression

$$< P_{rad} > = \frac{e^2(r_0\omega_0^2)^2}{6\pi\epsilon_0 m_e c^3} \tag{B.35}$$

Now consider the ratio of the time-average radiated power to the time-average absorbed power,

$$< P_{rad} > / < P_{abs} > = m_e r_0^2 \omega_0 / \hbar \tag{B.36}$$

We know that the ground state of an atom constitutes a stable state. We now also see that it is a state that is in a dynamic balance between the electromagnetic energy that is being radiated into the Zero Point Field and the electromagnetic energy that is being absorbed from the Zero Point Field, such that

$$\frac{< P_{rad} >}{< P_{abs} >} = \frac{m_e r_0^2 \omega_0}{\hbar} = 1 \tag{B.37}$$

We recognize this as the angular momentum quantization condition first introduced by Niels Bohr in 1913

$$m_e r_0^2 \omega_0 = n\ \hbar \tag{B.38}$$

with $n = 1$.

As outlined, atoms do not collapse down to zero size due to a detailed-balance between the electromagnetic energy that is being radiated into the Zero Point Field and the electromagnetic energy that is being absorbed back from to the Zero Point Field.

Zero Point/Thermal Spectral Function

Up until this point thermal temperature has not been included in the spectral function. Let us now add Planck's thermal temperature spectrum to the Zero Point electromagnetic spectrum:

$$\pi^2 H_0^2(\omega, T) = 1/2\ \hbar\omega + \hbar\omega(exp(\hbar\omega/k_b T) - 1) \tag{B.39}$$

where T is the thermal temperature and K is Boltzman's constant. We can reformulate this spectral function to read

$$\pi^2 H_0^2(\omega, T) = 1/2\ \hbar\omega \coth(1/2\hbar\omega/_b T) \tag{B.40}$$

which is a combined Zero Point/Thermal Spectral Function.

Unruh-Davies Temperature

In the case of a scalar field the Lorentz Invariant spectral function is as outlined above, namely [27]

$$\pi^2 f_0(\omega) = \frac{\hbar c^2}{2\omega} \tag{B.41}$$

Consider a scalar field of the form

$$\Phi(r, t) = \int d^3k f_0(\omega)\ \cos(ik * r - i\omega t + i\Theta_k) \tag{B.42}$$

The time average value of the amplitude of the field is given by the correlation function (a correlation function is to classical field theory what an expectation function is to quantum field theory)

$$< \Phi(0, t) *\ \Phi(0, t) >= \frac{1}{2} \int d^3k f_0(\omega) \tag{B.43}$$

which we know to be Lorentz Invariant.

Now consider an accelerating frame of reference moving along the x-axis with uniform acceleration a. It can be shown that

$$x(\tau) = \frac{c^2}{a}\ \cosh(a\tau/c) \tag{B.44}$$

and that

$$v(\tau) = c \ \tanh(a\tau/c) \tag{B.45}$$

where $\gamma = \sqrt{(1 - v^2/c^2)} = cosh(a\tau/c)$.

It is also straightforward to show that

$$\omega\prime = \omega \ \cosh(a\tau/c) - ck_x \sinh(a\tau/c) \tag{B.46}$$

and that

$$k_x\prime = k_x \ \cosh(a\tau/c) - \omega/c \ \sinh(a\tau/c) \tag{B.47}$$

The transformed correlation function is

$$< \Phi(0,t)^* \ \Phi(0,t) >_{accelerated} = -\frac{\hbar}{\pi c} \ (\frac{a}{2c})^2 \ \cosh^2(a\tau/c) \tag{B.48}$$

Compare this to the scalar correlation function of the system at rest in a Zero Point Thermal Field:

$$< \Phi(0,t)^* \ \Phi(0,t) >_{ZPThermal} = -\frac{\hbar}{\pi c} \ (\frac{\pi k_b T}{\hbar})^2 \ \cosh^2(\pi k_b T\tau/\hbar) \tag{B.49}$$

If we compare the two correlation functions we find that they are identical in functional form provided the acceleration and temperature are related by the expression

$$T = \frac{\hbar a}{2\pi k_b c} \tag{B.50}$$

This relation is known as the Unruh-Davies Temperature relation.

Now consider the Lorentz Invariant spectral function for an electromagnetic field,

$$\pi^2 H_0^2(\omega) = 1/2\hbar\omega \tag{B.51}$$

The transformed electromagnetic correlation function is of the form (i,j = 1,2,3):

$$< E_i(0,t)E_j(0,t) > = < B_i(0,t)B_j(0,t) > = \delta_{ij} 4\hbar/(\pi c^3)(a/2c)^4 \ csch^4(a\tau/c) \tag{B.52}$$

where the cross terms are of the form

$$< E_i(0,t)B_j(0,t) >= 0 \tag{B.53}$$

(csch is the hyperbolic cosecant).

Compare this to the correlation function of the system at rest in a Zero Point Thermal field,

$$< E_i(0,t)E_j(0,t) >=< B_i(0,t)B_j(0,t) >$$
$$= \delta_{ij}4\hbar/(\pi c^3)(\pi KT/\hbar)^4(\csc^4(\pi KT/\hbar) + 2/3\csc^2(\pi KT/\hbar)) \tag{B.54}$$

where again the cross terms are of the form

$$< E_i(0,t)B_j(0,t) >= 0 \tag{B.55}$$

Notice the additional term. The question is how to interpret the functional form and the additional term. A clue is to be found in the result for a scalar field and the Unruh-Davies Temperature relation.

In the case of an electromagnetic field, the spectrum seen by the detector such as Casimir plates accelerating through a Zero Point electromagnetic field is

$$\pi^2 H^2_{accel}(\omega, a) = 1/2\hbar\omega(1 + (a/c\omega)^2)\coth(\pi c\omega/a) \tag{B.56}$$

If we express the acceleration in terms of the Unruh-Davies Temperature relation $T = \hbar a/(2\pi Kc)$ then the Zero Point Thermal spectrum as seen by the accelerated system is

$$\pi^2 H^2_{accel}(\omega, a) = 1/2\hbar\omega(1 + (2\pi KT/\hbar\omega)^2)\coth(\hbar\omega/2KT) \tag{B.57}$$

rather than the unaccelerated Zero Point Thermal spectrum

$$\pi^2 H^2_{at_rest}(\omega, 0) = 1/2\hbar\omega\coth(\hbar\omega/2KT) \tag{B.58}$$

Note that acceleration adds a new term to the Zero Point Thermal spectrum and that the two spectrums agree at the higher frequencies $\hbar\omega \gg KT$.

In an accelerating frame there is an event horizon in the sense that in certain directions events occuring beyond a certain distance from the observer can never be reported to the observer by light signals due to dilation. The observer is running away with ever increasing speed from these space-time events and modulated light signals carrying information can never catch up with the observer. These modes are frozen out and the spectral distribution of eigenvalues change.

A careful study of the situation shows that it is the long wavelength electromagnetic waves that are cut-off by the event horizon. As a result the accelerated spectrum H_{accel} does not go over to the energy equipartition at low frequency found with the unaccelerated Zero Point Thermal spectrum H_{atrest}.

Sakharov's Proposal

Andrei Sakharov's Proposal is that gravity is not a separately existing fundamental force but rather an induced effect associated with fluctuations of the vacuum state. Sakharov's Proposal was discussed in detail in a paper written by H.E. Puthoff of the Institute for Advanced Studies in Austin, Texas. [26, 25]

Consider again the equation of motion for an oscillating charged particle given by

$$d^2/dt^2q + v_0^2q - \Gamma d^3/dt^3q = \Gamma^* E \tag{B.59}$$

where $q = q(t)$ is the oscilator coordinate, v_0 is the natural frequancy of the oscillator, Γ is the damping coeeficient

$$\Gamma = e^2/(6\pi\epsilon_0 m_e c^3) \tag{B.60}$$

Now consider the kinetic energy W_{kin} of the particle motion due to fluctuations induced by the Zero Point electromganetic field,

$$W_{kin} = 1/2m_0 d^2/dt^2q = 1/2(d/dtq)^2/m_0 = (d/dtp)^2/(12\pi\Gamma\epsilon_0 c^3) \tag{B.61}$$

where $p = eq$ is the dipole moment of the oscillator. Written in this form it is worth noting that the energy equation refers to the global properties of the oscillator (p, v_0 and the damping constant Γ) and does not involve individual properties such as mass or charge.

Using the Zero Point Electromagnetic fields E^{ZP} and B^{ZP} outlined above in section 4. and solving for the time average value for $< (d/dtp - x)^2 >$ yields

$$< (d/dtp_x)^2 > \simeq 6\epsilon_0 c^3 \hbar (\Gamma\omega_c)^2 \tag{B.62}$$

where ω_c is some characteristic frequency. In two-dimensions the particle motion due to fluctuations induced by the Zero Point electromagnetic field is

$$< (d/dtp)^2 >_{two-dimen} = 2 < (d/dtp)^2 >_{one-dimen} \tag{B.63}$$

The time average value for the internal energy of the oscillator, expressed in terms of its global properties is given by

$$< Energy > = \hbar\Gamma\omega_c^2/\pi \tag{B.64}$$

The energy calculated in this fashion is a transverse self-energy of the particle motion due to fluctuations induced by the Zero Point electromagnetic field. Using the expression Einstein expression $E = m_G c^2$ gives

$$m_G = \hbar\Gamma\omega_c^2/(\pi c^2) \tag{B.65}$$

In Puthoff's interpretation of Sakharov's Proposal, the oscillator's mass is of dynamical origin, originating in the motion response of the charged particle to the motion induced by the Zero Point electromagnetic field. It is the internal motion of the charged oscillator that contributes to the effective mass of the oscillator through the mass-energy equivalence outlined in the Einstein expression $E = mc^2$.

The lowest order interaction between a charged particle and a Zero Point Field that produces a far field effect is the dipole interaction. Of the dipole-field terms, the $1/r^4$ term predominates at large distances. In expanding out the dipole field distribution there is a term proportional to $1/r^2$ which is the radiation field associated with the Zero Point Fluctuation driven dipole. This radiation just replaces that being absorbed from the background on a detailed-balanced basis.

The energy density Δw_d in the two-dimensional far field dipole-field interaction is

$$\Delta w_d = (3\hbar c\Gamma^2 cos^2\Theta)/(2\pi^2 r^4) \int_0^{\omega_c} d\omega\ \omega \tag{B.66}$$

where ω_c is a characteristic frequency used as a cut-off frequency to avoid divergence.

Averaged over the net contribution of randomly oriented individual Zero Point particle motion, and integrated over the solid angle, we have an overall spectral density of

$$\Delta\rho_{d'} = \omega(\hbar c\Gamma^2)/(2\pi^2 r^4) \tag{B.67}$$

Using the relationship for mass m_g and Γ we have

$$\Delta\rho_{d'} = \omega(c_g^{5\,2})/(2\pi^2\omega_c^4 r^4) \tag{B.68}$$

Recall the expression for the accelerated Zero Point Thermal spectrum for the electromagnetic field and set $T = 0$

$$\pi^2 H_{accel}^2(\omega, a) = 1/2\hbar\omega(1 + (a/c\omega)^2) \tag{B.69}$$

Multiply this expression by the density of normal modes (ω^2/π^2c^3) and equate the contribution from the acceleration term $1/2\hbar\omega(a/c\omega)^2$ to the expression $\Delta\rho_{d'}$ yielding

$$\hbar a^2/(\pi^2c^5) = (c^5 m_G^2)/(\hbar\omega_c^4 r^4) \tag{B.70}$$

Now let $a = Gm_G/r^2$ and solve for ω_c

$$\omega_c = \surd(\pi c^5/\hbar G) \tag{B.71}$$

On the basis of heuristic and dimensional considerations Sakharov proposed that a vacuum fluctuation model for gravitation would have a characteristic cut-off frequency ω_c of this form. Solving for the gravitational constant G we have

$$G = \pi c^5/\hbar\omega_c^2 \tag{B.72}$$

The Suspended Charge Revisted

Let us now look again at the contribution from the acceleration term in the expression $\Delta\rho_{d'}$ given by

$$\Delta\rho_{d'} = (1/2\ \hbar\omega)a^2 2\pi^2 c^5 \tag{B.73}$$

Consider the following expression relating time average differential power to the incident flux,

$$< DP/D\Omega >=< d\sigma/d\Omega >< incidentflux > \tag{B.74}$$

where for a massless relativistic particle

$$< incidentflux >= c < Energy > \tag{B.75}$$

This means that

$$(e^2(\sin^2\theta)/(4\pi c^3(1-\beta\cos\theta)^5)a^2 =< d\sigma/d\Omega > c \int d\omega \Delta\rho_{d'} \tag{B.76}$$

Integrate over ω to get

$$< Energy >= \int_0^{\omega} d\omega\Delta\rho_{d'} = \hbar\omega^2 a^2/(4\pi^2 c^5) \tag{B.77}$$

Solving for the differential cross section $< d\sigma/d\Omega >$ yields

$$< d\sigma/d\Omega >= (e^2/\hbar c)(\hbar\pi c^2/\omega^2)\sin^2\theta/(1-\beta\cos\theta)^5 \tag{B.78}$$

Notice that the dependence on the square of the acceleration drops out.

Replace $(e^2/\hbar c)$ by the fine structure constant α and ω^2 by the Sakharov characteristic cut-off frequency ω_c^2 and you get the following:

$$< d\sigma/d\Omega >= (\alpha G\hbar/c^3)(\sin^2\theta/(1-\beta\cos\theta)^5) \tag{B.79}$$

The angle θ points in the direction of the gravitational gradient. This is an interesting expression in that it connects the fine structure constant α with the gravitational constant G.

We shall call this expression the Electro-Gravitational differential cross section. [20]

The Electro-Gravitational Cross Section

For $\beta \ll 1$ and integrated over the solid angle, the Electro-Gravitational differential cross section has the value

$$\sigma_{EG}(\beta \ll 1) \simeq 4\pi\alpha G\hbar/3c^2 \simeq 7.92 \times 10^{-68} cm^2 \tag{B.80}$$

This Electro-Gravitational differential cross section is many orders of magnitude smaller than cross sections for typical electromagnetic interactions.

This exceedingly small cross section is due to gravity not being a separately existing fundamental force but rather an induced effect associated with fluctuations in the Zero Point electromagnetic field.

A fundamental length is given by the square root of the Electro-Gravitational differential cross section, namely

$$\Lambda_{EG} = \sqrt{\sigma_{EG}} \simeq 2.82 \times 10^{-34} cm \tag{B.81}$$

which is on the order of the Planck length $\Lambda_{Planck} \simeq 10^{-33} cm$.

It is also worth noting that the Electro-Gravitational differential cross section indicates that if a particle does not participate in the electromagnetic interaction then it does not have gravitational mass.

If this hypothesis is indeed correct then under the Eötvos Principle of the Equivalence of Inertial to Gravitational Mass, a particle that does not participate in the electromagnetic interaction would not have inertial mass.

A particle such as a neutrino is essentially massless under this proposal, what little mass it may have being a result of the Electro-weak interaction.

Gravitational Force

Studying the Zero Point fluctuation induced dipole field at the position of particle A due to the fluctuating motion of a second similar particle, particle B, leads to an expression for the potential energy of the interaction of the form: [25]

$$U = -\frac{9\hbar c^3 \Gamma^3}{4\pi} \; Re\left(\int_0^{u_c} du \; \frac{exp^{-2uR}}{R} \right) \tag{B.82}$$

where $u = -i\ \omega/c$ and $u_c = -i\ \omega_c/c$.

For two-dimensional Zero Point dipole motion the attracting potential is given by

$$U = -1/2\ \Delta\ \frac{1-\cos(2R)}{R^3} = -\Delta\ \frac{1}{R}\ (\frac{\sin R}{R})^2 \tag{B.83}$$

where the parameter Δ is given by

$$\Delta = \frac{\hbar \Gamma^2 \omega_c^3}{\pi} \tag{B.84}$$

and the scale parameter R is given by

$$R = \frac{r\omega_c}{c} \tag{B.85}$$

where r is the distance between the two dipoles.

With the gravitational potential thus defined, the gravitational force is given by the classical expression

$$F_g = -\frac{\partial U}{\partial r} \tag{B.86}$$

The gravitational potential has the desired $1/r$ dependence modified by a form factor $((\sin R)/R)^2$ which has a characteristic length on the order of the fundamental length $\Lambda_{EG} \simeq 2.82 \times 10^{-34} cm$.

If we extract the leading terms from both U and F_g we arrive with the following:

$$U = -\frac{\hbar c \Gamma^2 \omega_c^2}{\pi r} \tag{B.87}$$

$$F_G = -\frac{\hbar c \Gamma^2 \omega_c^2}{\pi r^2} \tag{B.88}$$

Using Sakharov's characteristic cut-off frequency

$$\omega_c = \sqrt{(\frac{\pi c^5}{\hbar G})} \tag{B.89}$$

and the expression for the gravitational mass derived above

$$m_G = \frac{\hbar \Gamma \omega_c^2}{\pi c^2} \tag{B.90}$$

we arrive at the familiar expression for the gravitational force,

$$F_G = \frac{-G m_A m_B}{r^2} \tag{B.91}$$

which is Newton's Law of Gravitational Attraction between two bodies of similar mass.

For dissimilar masses we modify the force equation F_G to read

$$F_G = \frac{-\hbar c \Gamma_1 \Gamma_2 \omega_c^2}{\pi r^2} \tag{B.92}$$

and solve in a similar fashion to arrive at

$$F_G = \frac{-G m_1 m_2}{r^2} \tag{B.93}$$

which is Newton's Law of Gravitational Attraction between two bodies of dissimilar masses.

Appendix C

Why the Cut-off?

What could be the cause of the cut-off in ω considered in Sakharov's proposal?

For a particle of diameter $2R > 0$ spectral components of the electromagnetic Zero-Point random radiation whose wavelength λ is smaller than the size 2R of the particle cannot be effective in producing translational motion of the particle. [29]

Only spectral components with wavelengths $\lambda > 2R$ can be responsible for translational motion of the particle as a whole. Spectral components smaller than the size of the particle can only be effective in producing internal deformation or rearrangements.

Therefore, despite the spectral divergence of the Zero-Point spectrum, and its associated infinite energy density, a natural cut-off should appear that is related to the size of the particle.

Convergence of the Form Factor

A convergence form factor can be obtained by finding an upper bound to the energy available from the electromagnetic Zero-Point field for a charged particle of non-vanishing size.

Model the particle as a homogenously charged sphere. Since the spectrum of the electromagnetic Zero-Point field is Lorentz Invariant we are not concerned about velocity effects and we can work in the frame of reference where the particle is instantaneously at rest.

Let the particle have a small non-zero volume v, with $v \ll V$, where V is the electromagnetic cavity volume and let the time duration of the interaction be a short non-zero time interval $\tau > 0$.

To study the translational effect of the field on the particle consider the jth field component

$$(E_j)_{v,\tau} = \frac{1}{v\tau}\int_v d^3\bar{x}\int_\tau dtE_j(\bar{x},t) \tag{C.1}$$

The expectation value of E_j^2 is given by averaging over the volume of the particle and the duration of measurement so that

$$<0 \mid (E_j)^2_{v,\tau} \mid 0> = \frac{1}{(v\tau)^2}\int_{v,\tau}\int_{v,\tau} d^3\bar{x}\ d^3\bar{x}'dtdt' <0 \mid E_j(\bar{x},t)E_j(\bar{x}',t') \mid 0> \tag{C.2}$$

The matrix element in the integrand is

$$<0 \mid E_j(\bar{x},t)E_j(\bar{x}',t') \mid 0> = \frac{2\pi}{V}\sum_{s,\lambda}\hbar\omega_s exp(i[\bar{k}_s\cdot(\bar{x}-\bar{x}')-\omega_s(t-t')])(\hat{e}_{s\lambda}\cdot\hat{e}_j) \tag{C.3}$$

It follows then that

$$<0 \mid (E_j)^2_{v,\tau} \mid 0>$$

$$= \frac{2\pi}{V}\ \frac{1}{(v\tau)^2}\int_{v,\tau}\int_{v,\tau} d^3\bar{x}\ d^3\bar{x}'dtdt'\ 2\sum_{s,\lambda}\hbar\omega_s exp(i[\bar{k}_s\cdot(\bar{x}-\bar{x}')-\omega_s(t-t'])) \tag{C.4}$$

Carrying out the time integration and replacing the summation over s by an integration and expressing the energy density $u =< E^2 > /4\pi$ and recalling that the total energy density is $u = V^{-1}\sum 1/2\hbar\omega_s$ we arrive at an expression for the average electromagnetic Zero-Point energy density over the volume of the particle, namely

$$<u> = \frac{\hbar c}{(2\pi)^3v^2}\int_v\int_v d^3\bar{x}'$$

$$\int_k d\bar{k}exp(i\bar{k}\cdot(\bar{x}-\bar{x}'))\frac{\sin^2(1/2ck\tau)}{(1/2ck\tau)^2} \tag{C.5}$$

We integrate first over $\bar{k}$, letting the k_3 axis be parallel to the vector $\bar{x}-\bar{x}'$, to yield

$$<u> = \frac{\hbar c}{(2\pi)^3v^2}\int_v d^3x\ \int_v \bar{x}'\int_k dkd\Theta_k d\phi_k k^3\sin\Theta_k\ \frac{\sin^2(1/2ck\tau)}{(1/2ck\tau)^2}\ exp(ik \mid \bar{x}-\bar{x}' \mid \cos\Theta_k) \tag{C.6}$$

where the integration is taken over an infinite sphere in k-space.

Since

$$\int_{-1}^{1} exp(ik \mid \bar{x} - \bar{x}' \mid \mu) d\mu = 2\frac{\sin(k \mid \bar{x} - \bar{x}' \mid)}{k \mid \bar{x} - \bar{x}' \mid} \tag{C.7}$$

we obtain

$$< u >= \frac{2\hbar c}{(2\pi v)^2} \int_v d^3x \int_v d\bar{x}' \int_{k=0}^{\infty} dk k^3 \, \frac{\sin^2(1/2ck\tau)}{(1/2ck\tau)^2} \, \frac{\sin(k \mid \bar{x} - \bar{x}' \mid)}{k \mid \bar{x} - \bar{x}' \mid} \tag{C.8}$$

This equation gives a divergent expression if $v = 0$ or $\tau = 0$.

This follows that when the particle volume is very small and the time of the interaction is very short, there is no averaging of the high-frequency components and the full field acts on the particle.

In contrast, when v and τ are finite, non-zero quantities, the high-frequency components give no contribution to the energy available from the field.

In the case of a spherical particle it can be shown expanding out the volume integral in terms of spherical Bessel, spherical Hankel and spherical Harmonics that the energy density becomes

$$< u >= \frac{9\hbar c}{(2\pi)^2 R^4} \int_0^{\infty} \frac{\sin^2(1/2ck\tau)}{(1/2ck\tau)^2} \, [\frac{\sin(kR)}{(kR)} - \cos(kR)]^2 \, \frac{dk}{k} \tag{C.9}$$

A signal takes a maximum time $2R/c$ in traversing the particle.

It can be assumed that the maximum time of detection approximately corresponds to this amount, then

$$< u >= \frac{9\hbar c}{(2\pi)^2 R^4} \int_0^{\infty} \frac{\sin^2(\alpha)}{(\alpha)^2} \, [\frac{\sin(alpha)}{(\alpha)} - \cos(\alpha)]^2 \, \frac{d\alpha}{\alpha} \tag{C.10}$$

where $\alpha = kR$.

For the case $R > 0$ the energy available from the field, $U = v < u >$ remains bounded, namely

$$U = v \int_0^\infty \gamma(\omega)\rho(\omega)d\omega \tag{C.11}$$

where ρ is the Zero-Point spectrum and

$$\gamma(\omega) = \gamma[\alpha] = \frac{9}{\alpha^4}(\frac{\sin\alpha}{\alpha})^2\ (\frac{\sin\alpha}{\alpha} - \cos\alpha)^2 \tag{C.12}$$

where $\alpha = \omega R/c$.

A cut-off occurs around the critical wavelength $\lambda_c \equiv 2R$, that is at a critical frequency

$$\omega_c = \frac{\pi c}{R} \tag{C.13}$$

Thus wavelengths smaller than the size of the particle produce internal deformation or rearrangements and do not directly contribute to the translational motion of the particle.

Estimate of the Size of Fundamental Particles

If we equate the critical frequency to the Sakharov frequency we can solve for R namely

$$R = \sqrt{[\frac{\pi\hbar G}{c^3}]} \tag{C.14}$$

giving an estimate for the size of fundamental particles like electrons and quarks on the order of

$$R \approx 2.85 \times 10^{-33} cm \tag{C.15}$$

where we have assumed that constants like $\hbar$ and G are not running constants.

It is interesting that with the Sakharov frequency we can use the Gravitational constant in an estimate of electrons and quark size.