FEYNMAN'S QUANTUM MECHANICS APPLIED TO SCATTERING PROBLEMS

by

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Abstract

This thesis consists of two independent parts, both of which are applications of the quantum mechanical methods developed recently by R. P. Feynman.

Part I is concerned with the non-relativistic theory, and applies Feynman's formalism to the simple problem of the scattering of a particle by a potential field. The method and results are compared with those of the familiar Born approximation. The two procedures are shown to be equivalent and to be valid under the same conditions. Feynman's formulae are used to calculate the first and second order terms of the scattered particle wave function, with an arbitrary scattering potential.

Part II uses the relativistic Feynman theory, and treats the scattering of positrons by electrons, and of two electrons. The calculation checks the work of H. J. Bhabha and C. Møller, who have obtained the same results by other methods. The differential cross-sections for the two scattering processes are calculated to first order, and an estimate is made of the feasibility of an experiment to determine whether the exchange effect described by Bhabha actually occurs in positron-electron scattering.

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F E Y N M A N ' S   Q U A N T U M   M E C H A N I C S
A P P L I E D   T O   S C A T T E R I N G   P R O B L E M S

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Part I

NON-RELATIVISTIC SCATTERING OF PARTICLES BY A POTENTIAL FIELD

Introduction.

Part I of this thesis deals with two methods of treating scattering problems in quantum mechanics which differ widely in approach. One, the well-known Born approximation, is a perturbation theory solution of the Schrödinger equation. The other, described in recent papers by Feynman,[1,2,3] deals directly with the solutions of the wave equation, rather than with the equation itself.

In view of the fact that Feynman's approach leads to a simplified formulation of quantum electrodynamical problems, it seems desirable to establish a link between this procedure and the familiar Born approximation. This thesis accordingly carried out explicitly Feynman's treatment of a simple scattering problem and demonstrates the equivalence of the two methods.

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1 R. P. Feynman, Rev. Mod. Phys. 20, 367, (1948).
2 R. P. Feynman, Phys. Rev. 76, 749, (1949), hereafter referred to as I.
3 R. P. Feynman, Phys. Rev. 76, 769, (1949), hereafter referred to as II.

The Born Approximation.

Suppose that a particle of mass m and initial momentum $\hbar\vec{k}$ is scattered by a potential field $V(\vec{r})$. To apply the Born approximation, we set $\psi(\vec{r},t) = u(\vec{r})e^{-i\omega t}$, with $\omega = \hbar k^2/2m$, and proceed to solve the time-independent Schrödinger equation

$$\{\nabla^2 + k^2 - (2m/\hbar^2)V(\vec{r})\}u(\vec{r}) = 0.$$

The method is well known, and we include here only those equations which will be compared directly with the Feynman solution. The wave function for the incident particle is $u_0 = e^{i\vec{k}\cdot\vec{r}}$. Green's solution of the first order equation is

$$u_1(\vec{r}_2) = (2m/\hbar^2)\int G(\vec{r}_2,\vec{r}_1)V(\vec{r}_1)u_0(\vec{r}_1)d\vec{r}_1, \tag{1}$$

where the Green's function $G(\vec{r}_2,\vec{r}_1) = -(4\pi|\vec{r}_2-\vec{r}_1|)^{-1}e^{ik|\vec{r}_2-\vec{r}_1|}$ satisfies the equation

$$\{\nabla_2^2 + k^2\}G(\vec{r}_2,\vec{r}_1) = \delta(\vec{r}_2-\vec{r}_1). \tag{2}$$

We write $d\vec{r} = dxdydz$; unless otherwise specified, the integration is to be taken over all space. The subscript 2 on ∇^2 means that it operates on the variables $\vec{r}_2$, and $\delta(\vec{r}_2-\vec{r}_1) = \delta(x_2-x_1)\delta(y_2-y_1)$ $\delta(z_2-z_1)$.

For solutions at large r_2 we make the approximations

$$|\vec{r}_2-\vec{r}_1| \approx r_2 - \vec{r}_1\cdot\vec{r}_2/r_2 \approx r_2 \tag{3}$$

in the exponent and denominator respectively of the Green's function. The first and second order solutions are then:

$$u_1(\vec{r}_2) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikr_2}}{r_2}\int V(\vec{r}_1)e^{i(\vec{k}-\vec{k}_2)\cdot\vec{r}_1}\,d\vec{r}_1 \tag{4}$$

$$u_2(\vec{r}_2) = \left(\frac{m}{2\pi\hbar^2}\right)^2\frac{e^{ikr_2}}{r_2}\int V(\vec{r}_3)e^{-i\vec{k}_2\cdot\vec{r}_3}\frac{e^{ikr_3}}{r_3}\int V(\vec{r}_1)e^{i(\vec{k}-\vec{k}_3)\cdot\vec{r}_1}\,d\vec{r}_1\,d\vec{r}_3 \tag{5}$$

where $\vec{k}_2 = k\vec{r}_2/r_2$, $\vec{k}_3 = k\vec{r}_3/r_3$.

Feynman's Formulation.

Feynman's method is based on the probability amplitude for a particle to move from one space-time point to another. This function, called the kernel, is a sum of contributions $e^{iS/\hbar}$ from each possible path, where S is the classical action along the path. Denoting the point $\vec{r}_1, t_1$ by 1 and $\vec{r}_2, t_2$ by 2, K(2,1) is the amplitude for a particle known to be at $\vec{r}_1$ at the time t_1 to arrive at $\vec{r}_2$ at the time t_2. The wave function $\psi(2)$, representing the amplitude for the particle to be found at $\vec{r}_2, t_2$, is then given by the expression

$$\psi(2) = \int K(2,1)\psi(1)d\vec{r}_1 . \tag{6}$$

Feynman shows that the kernel is that solution of

$$\{i\hbar\partial/\partial t_2 - H_2\}K(2,1) = i\hbar\delta(2,1) \tag{7}$$

which vanishes for $t_2 < t_1$, where H is the Hamiltonian operator for the particle (the subscript 2 indicating that it acts on the variables 2), and $\delta(2,1) = \delta(\vec{r}_2 - \vec{r}_1)\delta(t_2 - t_1)$. The kernel for a free particle is shown to be (for $t_2 > t_1$)

$$K_0(2,1) = \left[\frac{m}{2\pi i\hbar(t_2-t_1)}\right]^{\frac{3}{2}} \exp\left[\frac{im}{2\hbar}\frac{|\vec{r}_2-\vec{r}_1|^2}{(t_2-t_1)}\right]. \tag{8}$$

For a perturbation theory treatment of a particle in a potential $V(\vec{r})$, Feynman shows that the kernel may be expanded in increasing powers of V:

$$K(2,1) = K_0(2,1) + K_1(2,1) + K_2(2,1) + \dots \tag{9}$$

where

$$K_1(2,1) = -(i/\hbar)\int K_0(2,3)V(3)K_0(3,1)d\tau_3 \tag{10a}$$

$$K_2(2,1) = (-i/\hbar)^2\iint K_0(2,3)V(3)K_0(3,4)V(4)K_0(4,1)d\tau_3\, d\tau_4 \tag{10b}$$

etc., using $d\tau = d\vec{r}dt$. The time integration is limited to the interval $t_2 > t_3 > t_1$ in (10a), $t_2 > t_3 > t_4 > t_1$ in (10b), by the condition

$K(2,1) = 0$ for $t_2<t_1$.

The decisive advantage of Feynman's theory is the interpretation of the above formulae in terms of successive scattering. In (10a) for example we view the particle as moving freely from 1 to 3 (amplitude $K_o(3,1)$), being scattered by the potential at 3 (amplitude $-(i/\hbar)V(3)d\tau_3$), and moving as a free particle again to 2 ($K_o(2,3)$). This is summed over all points 3, thus giving the term $K_1(2,1)$. The successive terms of (9) are regarded as amplitudes for the particle to be scattered 0, 1, 2, etc. times by the potential V.

Corresponding to the expansion (9) of the kernel, the perturbed wave function may be written

$$\psi(2) = \psi_o(2) + \psi_1(2) + \psi_2(2) + \dots \qquad (11)$$

where $\psi_o(1)$ is the wave function of the incident free particle, and

$$\psi_o(2) = \int K_o(2,1)\psi_o(1)d\vec{r}_1 \qquad (12a)$$

$$\psi_1(2) = -(i/\hbar)\int K_o(2,1)V(1)\psi_o(1)d\tau_1 \qquad (12b)$$

etc.

<u>Comparison of the Methods.</u>

From the foregoing discussion it is possible to show a close correspondence between the two methods. The similarity of equations (12b) and (1) suggests identifying

$$G(\vec{r}_2,\vec{r}_1)e^{-i\omega t_2} \doteq -(i\hbar/2m)\int K_o(2,1)e^{-i\omega t_1}dt_1. \qquad (13)$$

Operating on (13) with $\{i\hbar\partial/\partial t_2 + (\hbar^2/2m)\nabla_2^2\}$ gives on the left, by (2),

$$(\hbar^2/2m)\{\nabla_2^2+k^2\}G(\vec{r}_2,\vec{r}_1)e^{-i\omega t_2} = (\hbar^2/2m)\delta(\vec{r}_2-\vec{r}_1)e^{-i\omega t_2},$$

and on the right, using (7) with $H=-(\hbar^2/2m)\nabla^2$,

$$-(i\hbar/2m)\int i\hbar\delta(2,1)e^{-i\omega t_1}dt_1 = (\hbar^2/2m)\delta(\vec{r}_2-\vec{r}_1)e^{-i\omega t_2}.$$

The equivalence of these two expressions does not prove equation (13), since an arbitrary solution of the Schrödinger equation can still be added to one side. It does show however that there is an exact parallel between the two methods in spite of the wide difference in approach. Without giving a full proof of equation (13), we will study the correspondence of the two methods from another point of view.

The n^{th} order perturbed solution is obtained by using for the kernel the first n+1 terms of the series (9). We now seek to modify the Hamiltonian of the particle by introducing a potential function $V_n(\vec{r})$ such that $K_o(2,1)+\ldots+K_n(2,1)$ is the exact kernel for a particle moving in the potential $V_n(\vec{r})$. Applying equation (7) gives

$$\{i\hbar\partial/\partial t_2 + (\hbar^2/2m)\nabla_2^2 - V_n(2)\}\{K_o(2,1)+\ldots+K_n(2,1)\} = i\hbar\delta(2,1) \quad (14)$$

which defines the function $V_n(\vec{r})$.

A general expression for the n^{th} order kernel is

$$K_n(2,1) = -(i/\hbar)\int K_o(2,3)V(3)K_{n-1}(3,1)d\tau_3.$$

Using the free particle form of equation (7) gives therefore

$$\{i\hbar\partial/\partial t_2 + (\hbar^2/2m)\nabla_2^2\}K_n(2,1) = V(2)K_{n-1}(2,1).$$

Thus we see that $V_n(2)$ is given by the equation

$$V_n(2)\{K_o(2,1)+\ldots+K_n(2,1)\} = V(2)\{K_o(2,1)+\ldots+K_{n-1}(2,1)\}.$$

Multiplying this equation by $\psi_o(1)$ and integrating over $d\vec{r}_1$, we obtain

$$V_n(2)\{\psi_o(2)+\ldots+\psi_n(2)\} = V(2)\{\psi_o(2)+\ldots+\psi_{n-1}(2)\}. \quad (15)$$

From the definition of V_n, the wave function $\psi_o+\ldots+\psi_n$ must satisfy the modified Schrödinger equation containing this potential:

$$\{i\hbar\partial/\partial t + (\hbar^2/2m)\nabla^2 - V_n\}\{\psi_o+\ldots+\psi_n\} = 0.$$

Using the expression (15) for V_n, this becomes:

$$\{i\hbar\partial/\partial t + (\hbar^2/2m)\nabla^2\}\{\psi_0+\ldots+\psi_n\} = V\{\psi_0+\ldots+\psi_{n-1}\},$$

and subtracting the corresponding equation with n lowered by one gives:

$$\{i\hbar\partial/\partial t + (\hbar^2/2m)\nabla^2\}\psi_n = V\,\psi_{n-1}. \qquad (16)$$

This is exactly the n^{th} order equation of the Born approximation. We have thus proved that the Born and Feynman solutions are identical, provided that the same incident particle wave function ψ_0 is chosen.

The condition for validity of either approximation is that the terms of the series expansion of the wave function (11) should decrease rapidly, i. e., that $|\psi_1| \ll |\psi_0| = 1$. Since the two methods give the same result for ψ_1, it is clear that they are valid under the same conditions.

Explicit solution by Feynman's method.

To illustrate the equivalence of the two methods, we will use Feynman's formulae to calculate explicity the first and second order terms of the wave function. We set $\psi_0(1) = e^{i(\vec{k}\cdot\vec{r}_1-\omega t_1)}$ for the wave function of the incident particle. The zero-order wave function (12a) should of course be the same. This is easily verified by direct integration using the expression (8) for the kernel.

The first order term, from (12b), is:

$$\psi_1(2) = -\frac{i}{\hbar}\int\!\!\int_{-\infty}^{t_2}\left[\frac{m}{2\pi i\hbar(t_2-t_1)}\right]^{\frac{3}{2}}\exp\left[\frac{im}{2\hbar}\,\frac{|\vec{r}_2-\vec{r}_1|^2}{(t_2-t_1)}\right]V(\vec{r}_1)e^{i(\vec{k}\cdot\vec{r}_1-\omega t_1)}d\vec{r}_1\,dt_1$$

$$-(i/\hbar)e^{-i\omega t_2}\int T(|\vec{r}_2-\vec{r}_1|)V(\vec{r}_1)\;e^{i\vec{k}\cdot\vec{r}_1}d\vec{r}_1$$

where

$$T(r) = \int_0^\infty (2\pi i\hbar t/m)^{-\frac{3}{2}}e^{imr^2/2\hbar t}e^{i\omega t}\,dt$$

(using $r = |\vec{r}_2 - \vec{r}_1|$, $t = t_2 - t_1$) is the integral on the right of equation (13), multiplied by $e^{i\omega t_2}$.

Now according to Feynman's interpretation of the perturbation formulae, $\psi_1(2)$ is due to particles scattered once at 1 and then proceeding as free particles to 2. Since a free particle has a definite velocity $\hbar k/m$, we should expect a contribution to $T(r)$ only from the neighbourhood of the point $t = (m/\hbar k)r$. We rearrange the exponent, and make the substitution $t = (m/\hbar k)r(1+\xi)$:

$$T(r) = \left(\frac{m}{2\pi i\hbar}\right)^{\frac{3}{2}} \int_0^\infty t^{-\frac{3}{2}} \exp\left[\tfrac{1}{2}ikr\frac{m}{\hbar k}\,\frac{r}{t} + \tfrac{1}{2}ikr\frac{\hbar k}{m}\,\frac{t}{r}\right]dt$$

$$= \left(\frac{m}{2\pi i\hbar}\right)^{\frac{3}{2}} \left(\frac{mr}{\hbar k}\right)^{-\frac{1}{2}} \int_{-1}^\infty (1+\xi)^{-\frac{3}{2}} \exp\left[\tfrac{1}{2}ikr\left(\frac{1}{1+\xi} + 1 + \xi\right)\right] d\xi$$

$$= \frac{m}{2\pi i\hbar}\left(\frac{kr}{2\pi i}\right)^{\frac{1}{2}} \frac{e^{ikr}}{r} \int_{-1}^\infty (1+\xi)^{-\frac{3}{2}}\, e^{\frac{1}{2}ikr\xi^2/(1+\xi)} d\xi$$

The coefficient of $i\xi$ in the exponent is $\frac{1}{2}kr\xi/(1+\xi)$. For fairly energetic particles at large distances r_2, $kr \gg 1$. Hence except for the region $\xi \ll 1$ the exponential is a very rapidly oscillating function of ξ. The function $(1+\xi)^{-\frac{3}{2}}$ varies slowly (except at $\xi \to -1$, and here the exponential oscillates with infinite frequency), and hence the integral vanishes except for very small values of ξ. Thus the physical reasoning of the previous paragraph is confirmed. For small ξ, we take $1+\xi \approx 1$, and then, since $kr \gg 1$, extend the lower limit to $-\infty$. The integral becomes *

$$T(r) = \frac{m}{2\pi i\hbar}\left(\frac{kr}{2\pi i}\right)^{\frac{1}{2}} \frac{e^{ikr}}{r} \int_{-\infty}^\infty e^{-(kr/2i)\xi^2} d\xi$$

$$= (m/2\pi i\hbar) e^{ikr}/r.$$

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* In spite of the approximations used, this result is exact. With $x = 1+\xi$, $p = -ia = -\frac{1}{2}ikr$, the integral is $\int_0^\infty x^{-\frac{3}{2}} e^{-px+ia/x} dx$. This can be evaluated by using a table of Laplace Transforms, such as W. Magnus and F. Oberhettinger, "Special Functions of Mathematical Physics" (Chelsea, New York, 1949), p. 127. The method used has the advantage of affording a physical interpretation.

Comparing this with the Green's function, we see that equation (13) is verified.

Applying the asymptotic approximations (3), the first order wave function becomes

$$\psi_1(2) = -\frac{m}{2\pi\hbar^2}\frac{e^{i(kr_2-\omega t_2)}}{r_2}\int V(\vec{r}_1)e^{i(\vec{k}-\vec{k}_2)\cdot\vec{r}_1}\,d\vec{r}_1 \tag{17}$$

which agrees exactly with the Born approximation result (4).

From (10b), the second order wave function is:

$$\begin{aligned}\psi_2(2) &= (-i/\hbar)^2\iint K_0(2,3)V(3)K_0(3,1)V(1)\psi_0(1)d\tau_1 d\tau_3\\ &= (-i/\hbar)^2 e^{-i\omega t_2}\iint T(|\vec{r}_2-\vec{r}_3|)V(\vec{r}_3)T(|\vec{r}_3-\vec{r}_1|)V(\vec{r}_1)e^{i\vec{k}\cdot\vec{r}_1}\,d\vec{r}_3\,d\vec{r}_1\\ &= \left(\frac{m}{2\pi\hbar^2}\right)^2\frac{e^{i(kr_2-\omega t_2)}}{r_2}\int V(\vec{r}_3)e^{-i\vec{k}_2\cdot\vec{r}_3}\frac{e^{ikr_3}}{r_3}\int V(\vec{r}_1)\\ &\qquad\qquad\times e^{i(\vec{k}-\vec{k}_3)\cdot\vec{r}_1}\,d\vec{r}_1\,d\vec{r}_3\end{aligned} \tag{18}$$

using again the approximations (3). This is identical with the Born result (5).

It will be noted that the successive integrations involved in the higher order approximation do not lead to a more complicated time integral (as might be expected), but rather to simple products of the same function $T(r)$. This is due to the separation of the time dependence of equation (13) in the factor $e^{-i\omega t_2}$. It is clear, in fact, that this time integral of the kernel will always give the same function provided only that the wave function can be separated in this way. Thus this integration, the most difficult step in the Feynman calculation, need not be repeated in every problem.

Part II

THE SCATTERING OF POSITRONS BY ELECTRONS

Introduction.

In Part II Feynman's formalism for particles satisfying the Dirac wave equation will be applied to the typical problem of the first order scattering of two electrons. A particular case of this problem is the interaction between a single positive energy electron and an unoccupied negative energy state (i. e., a positron). This case is of special interest since it makes possible a direct verification of the hole theory of positrons. For if the positron is really a vacant negative energy electron state, then an exchange effect will occur in the interaction which will contribute an extra term to the scattering cross-section. If however the two particles are quite different, the exchange term will be absent.

This fact was pointed out by Bhabha,[4] who calculated the two cross-sections. This calculation will be repeated here using Feynman's simplified formalism. The cross-section for electron-electron scattering will also be obtained by a very slight change in the calculation and will be compared with the result given by Møller.[5]

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4 H. J. Bhabha, Proc. Roy. Soc. A 154, 195, (1936).
5 C. Møller, Ann. d. Phys. 14, 531, (1932).

Notation.

For relativistic problems the following notation is convenient: If p is a four-vector with components p_μ ($\mu = 1, 2, 3, 4$), then $\vec{p}$ denotes the three space components of p. We use the summation convention $p_\mu q_\mu = p_4 q_4 - \vec{p}\cdot\vec{q} \equiv p\cdot q$. Also, if p_μ is not a matrix, $\underline{p} = p_\mu \gamma_\mu$ is a matrix associated with the vector p, where γ_μ are the four Dirac matrices $\beta\vec{\alpha}, \beta$. The latter satisfy $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$, where $\delta_{44} = 1$, $\delta_{11} = \delta_{22} = \delta_{33} = -1$, and other $\delta_{\mu\nu}$ are zero. Then $\delta_{\mu\nu} p_\nu = p_\mu$, $\delta_{\mu\mu} = 4$. Note that $\underline{p}^2 = p\cdot p$ is a pure number, not a matrix.

We write $d\vec{p} = dp_1 dp_2 dp_3$ and $d^4p = d\vec{p}dp_4$. In particular, x_μ is the four-vector $\vec{r}$, t (we use natural units $\hbar = c = 1$, so that $x_4 = t$), $d\vec{r} = dxdydz$, and $d\tau = d\vec{r}dt$. Finally, $\nabla = \gamma_\mu \partial/\partial x_\mu = \beta\partial/\partial t + \beta\vec{\alpha}\cdot\vec{\nabla}$ where $\partial/\partial x_\mu$ means $\partial/\partial t$ for $\mu = 4$ and $-\partial/\partial x$, $-\partial/\partial y$, $-\partial/\partial z$ for $\mu = 1, 2, 3$.

Feynman's Formalism.

Feynman's relativistic formalism is essentially the same as that described in Part I, except that the wave function is now a solution of the Dirac equation having four components, and the kernel is a four by four matrix. The Dirac equation for a particle moving in the vector and scalar potentials (times e) A_μ is

$$(i\nabla - m)\psi = \underline{A}\,\psi \tag{1}$$

and the free particle kernel satisfies

$$(i\nabla_2 - m)K_+(2,1) = i\delta(2,1) \tag{2}$$

by analogy with equation (7) of Part I. The particular solution of this equation that must be chosen is (Feynman, I, p. 752)

$$K_+(2,1) = \sum_{E_n>0}\phi_n(2)\,\overline{\phi}_n(1)e^{-iE_n(t_2-t_1)} \quad \text{for } t_2>t_1$$
$$= -\sum_{E_n<0}\phi_n(2)\,\overline{\phi}_n(1)e^{-iE_n(t_2-t_1)} \quad \text{for } t_2<t_1 \qquad (3)$$

Here ϕ_n are eigenfunctions of the free particle Dirac equation, and E_n the corresponding energy values. It is convenient throughout to replace the usual Hermitian conjugate $\phi^\dagger$ by $\overline{\phi} = \phi^\dagger\beta$.

From (3) it is seen that electrons may be propagated either forwards in time with positive energies, or backwards in time with negative energies. An electron propagating towards the past is recognized as a positron. The above choice of the kernel is exactly equivalent to the hole theory of positrons.

The analogue of equation (6) of Part I is

$$\psi(2) = \int K_+(2,1)\underline{N}(1)\psi(1)d^3V, \qquad (4)$$

where the integration is over a 3-dimensional surface in space-time enclosing the point 2, and $N_\mu(1)$ is an inward unit normal to the surface at the point 1. In particular, if the surface consists of all space at a time $t_1<t_2$ and all space at a time $t_1'>t_2$, this becomes

$$\psi(2) = \int K_+(2,1)\beta\psi(1)d\vec{r}_1 - \int K_+(2,1')\beta\psi(1')d\vec{r}_1{}' \qquad (5)$$

Because of (3), only electron states in $\psi(1)$ and positron states in $\psi(1')$ contribute to the integrals.

If two particles are present, the amplitude that particle a goes from 1 to 3 while b goes from 2 to 4 (assuming no interaction) is the product (Feynman, I, p. 755)

$$K(3,4;1,2) = K_{+a}(3,1)K_{+b}(4,2) \qquad (6)$$

The subscripts a and b indicate that the matrices K_+ operate on the wave functions of particles a and b respectively. Matrices with subscript a and those with subscript b always commute.

Since the two particles are identical, another process is possible involving an exchange between them. The exclusion principle requires that the amplitudes for the two processes be subtracted, giving the net amplitude

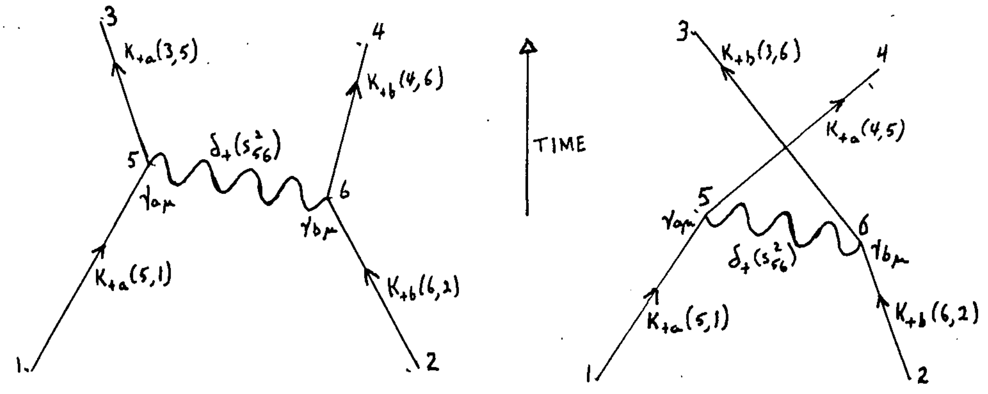
$$K(3,4;1,2) - K(4,3;1,2) \tag{7}$$

When the two particles interact, (6) no longer holds. The effect of the interaction of two electrons to first order in e^2 (regarded as the exchange of one virtual quantum) is given (Feynman, II, p. 772) by

$$K^{(1)}(3,4;1,2) = -ie^2 \iint K_{+a}(3,5)K_{+b}(4,6)\gamma_{a\mu}\gamma_{b\mu}K_{+a}(5,1)K_{+b}(6,2)\,\delta_+(s_{56}^2)\,d\tau_5\,d\tau_6 \tag{8}$$

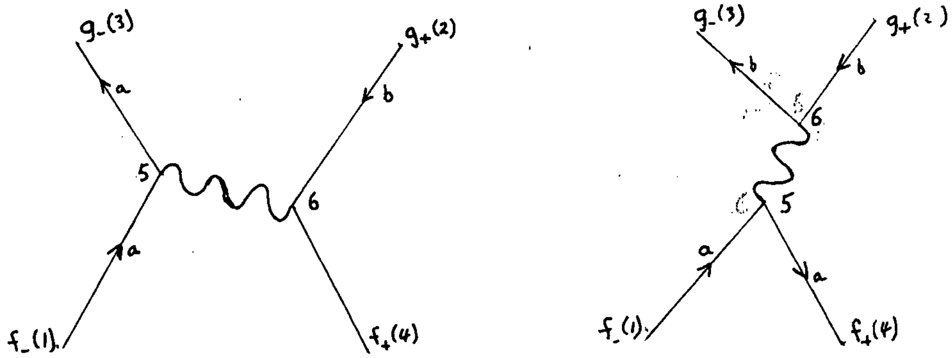
This formula can be interpreted as follows: Electron a travels as a free particle from 1 to 5 (amplitude $K_{+a}(5,1)$), emits a photon ($\gamma_{a\mu}$), and travels from 5 to 3 ($K_{+a}(3,5)$), while electron b goes from 2 to 6 ($K_{+b}(6,2)$), absorbs the photon ($\gamma_{b\mu}$), and goes on from 6 to 4 ($K_{+b}(4,6)$). Meanwhile the photon proceeds from 5 to 6, with amplitude $\delta_+(s_{56}^2)$. This is summed over all polarizations μ of the photon, and all points 5 and 6. If $t_5 > t_6$ we would say that b emits and a absorbs the photon, but this makes no difference in the formula.

Equation (8) can describe several processes, depending on the time relations of the points 1, 2, 3, and 4. Feynman represents these processes by simple space-time diagrams. Thus Fig. 1 illustrates the scattering of two electrons as described by (8), together with the interfering process whose amplitude is $K^{(1)}(4,3;1,2)$. The same kernels describe the interaction of an electron with a positron simply by reversing the time relation of points 2 and 4, as illustrated in Fig. 2. Positrons are distinguished by the direction of the arrows on their paths.



$K^{(1)}(3,4;1,2)$ $K^{(1)}(4,3;1,2)$

Fig. 1. Interaction of two electrons.



$K^{(1)}(3,4;1,2)$ $K^{(1)}(4,3;1,2)$

Fig. 2. Interaction of electron with positron.

General Expression for the Cross-section.

The principal aim of this thesis is to calculate the cross-section for the process illustrated in Fig. 2. We assume that initially (in time) there are present an electron in the state $f_-(1)$ and a positron in the state $f_+(4)$, and that finally these particles are found in the states $g_-(3)$ and $g_+(2)$ respectively. These states are shown in Fig. 2. The subscript + will be used always on quantities referring to positron states, and the subscript − for electron states.

It is necessary to compute the matrix element of the kernel given by (7) and (8) for the transition from the state $f_-(1)g_+(2)$ to the state $g_-(3)f_+(4)$. This matrix element is:

$$M_s = -ie^2\iint \overline{g}_{-a}(5)\overline{f}_{+b}(6)\gamma_{a\mu}\gamma_{b\mu}\delta_+(s_{56}^2)f_{-a}(5)g_{+b}(6)d\tau_5\, d\tau_6$$
$$+ie^2\iint \overline{f}_{+a}(5)\overline{g}_{-b}(6)\gamma_{a\mu}\gamma_{b\mu}\delta_+(s_{56}^2)f_{-a}(5)g_{+b}(6)d\tau_5\, d\tau_6, \qquad (9)$$

after carrying out the 3-surface integrals over $d\vec{r}_1$, $d\vec{r}_2$, $d\vec{r}_3$, $d\vec{r}_4$, such as:

$$\int K_{+a}(5,1)\beta_a f_{-a}(1)d\vec{r}_1 = f_{-a}(5)$$
$$-\int \overline{f}_{+b}(4)\beta_b K_{+b}(4,6)d\vec{r}_4 = \overline{f}_{+b}(6)$$

according to equation (5). The subscript s on M_s is used because this matrix element depends on the spins of the various states.

We take as the initial and final states of the two particles the Dirac free particle wave functions

$$f_\pm = L^{-\frac{3}{2}} u_\pm e^{-ip_\pm \cdot x}, \quad g_\pm = L^{-\frac{3}{2}} v_\pm e^{-iq_\pm \cdot x}, \qquad (10)$$

where $u_\pm$ and $v_\pm$ are constant spinors satisfying the Dirac equation $(\underline{p} - m)u = 0$, and $p_\pm$, $q_\pm$ are the momentum-energy four-vectors of the particles in each state. These solutions are normalized in a cubic box of volume L^3. The three space components of each momentum vector assume the discrete values $2\pi n/L$ (n an integer), while the fourth component varies continuously.

The Fourier transform of $\delta_+(s_{56}^2)$ is:

$$\delta_+(s_{56}^2) = -\pi^{-1}\int \underline{k}^{-2} e^{-ik\cdot(x_5-x_6)}\,(2\pi)^{-2}d^4k. \qquad (11)$$

Because of the box normalization, the integral over the three space components of k must be replaced by a sum, with $dk_\mu = 2\pi/L$:

$$\delta_+(s_{56}^2) = -2L^{-3}\sum_{k_1k_2k_3}\int \underline{k}^{-2} e^{-ik\cdot(x_5-x_6)}dk_4. \qquad (11a)$$

In order to obtain a transition probability per unit time, we assume that the interaction is "turned on" only for a finite time T. We now substitute the above expressions (10), (11a) into the matrix element M_5 and carry out the integrations over $d\tau_5$, $d\tau_6$ as follows: The space coordinates of each point are integrated over the range $-\frac{1}{2}L$ to $\frac{1}{2}L$. One time coordinate (say t_5) is integrated from 0 to T. The other (t_6) from $-\infty$ to $+\infty$. These integrals from the first term of M_5 are:

$$\iint L^{-\frac{3}{2}} e^{iq_-\cdot x_5} L^{-\frac{3}{2}} e^{ip_+\cdot x_6} L^{-3} e^{-ik\cdot(x_5-x_6)} L^{-\frac{3}{2}} e^{-ip_-\cdot x_5} L^{-\frac{3}{2}} e^{-iq_+\cdot x_6} d\tau_5 \, d\tau_6$$
$$= 2\pi L^{-3}\delta(q_{+4}-k_4-p_{+4})\,(e^{-iFT}-1)/(-iF)$$

provided $\vec{p}_-+\vec{k}-\vec{q}_- = \vec{q}_+-\vec{k}-\vec{p}_+ = 0$; otherwise the integral vanishes. We define $F = p_{-4}+k_4-q_{-4}$. The second term of M_5 is identical, except that f_+ and g_-, and therefore p_+ and q_-, are interchanged.

Thus in the first term $k = q_+-p_+$, and in the second term $k = q_+-q_-$. The other conditions are the same for both terms:

$$\vec{p}_- + \vec{q}_+ - \vec{p}_+ - \vec{q}_- = 0 \tag{12a}$$

$$p_{-4} + q_{+4} - p_{+4} - q_{-4} = F \tag{12b}$$

The first of these expresses the conservation of momentum, and the second defines the quantity F.

The final expression for M_5 in terms of the initial and final momenta is:

$$\begin{aligned} M_5 &= \frac{4\pi i e^2}{L^3}\frac{e^{-iFT}-1}{(-iF)}\left[\bar{v}_{-a}\bar{u}_{+b}\gamma_{a\mu}\gamma_{b\mu}(\underline{q}_+-\underline{p}_+)^{-2}u_{-a}v_{+b} - \bar{u}_{+a}\bar{v}_{-b}\gamma_{a\mu}\gamma_{b\mu}(\underline{q}_+-\underline{q}_-)^{-2}u_{-a}v_{+b}\right] \\ &= 4\pi i e^2 L^{-3}(e^{-iFT}-1)/(-iF)\Big[(\underline{q}_+-\underline{p}_+)^{-2}(\bar{v}_-\gamma_\mu u_-)(\bar{u}_+\gamma_\mu v_+) \\ &\qquad - (\underline{q}_+-\underline{q}_-)^{-2}(\bar{u}_+\gamma_\mu u_-)(\bar{v}_-\gamma_\mu v_+)\Big] \end{aligned} \tag{13}$$

The transition probability from the initial to the final state is $|M_5|^2 = \bar{M}_5 M_5$, and since $\bar{\gamma}_\nu = \gamma_\nu$ this is

$$\begin{aligned} |M_5|^2 = \frac{16\pi^2 e^4}{L^6}\frac{4\sin^2\frac{1}{2}FT}{F^2}\Big[&(\underline{q}_+-\underline{p}_+)^{-4}(\bar{u}_-\gamma_\nu v_-\bar{v}_-\gamma_\mu u_-)(\bar{v}_+\gamma_\nu u_+\bar{u}_+\gamma_\mu v_-) \\ &+ (\underline{q}_+-\underline{q}_-)^{-4}(\bar{u}_-\gamma_\nu u_+\bar{u}_+\gamma_\mu u_-)(\bar{v}_+\gamma_\nu v_-\bar{v}_-\gamma_\mu v_+) \\ &- (\underline{q}_+-\underline{p}_+)^{-2}(\underline{q}_+-\underline{q}_-)^{-2}\,2\mathrm{Re}(\bar{v}_-\gamma_\mu u_-\bar{u}_-\gamma_\nu u_+\bar{u}_+\gamma_\mu v_+\bar{v}_+\gamma_\nu v_-)\Big] \end{aligned} \tag{14}$$

This transition probability can be summed over final state spins and averaged over initial state spins by using the projection operators $(2p_4)^{-1}(\underline{p}+m)$.* The result is

$$|M|^2 = \tfrac{1}{4}\sum_s |M_s|^2 = \pi^2 L^{-6} e^4 (p_{-4}p_{+4}q_{-4}q_{+4})^{-1} (\sin^2 \tfrac{1}{2}FT/F^2)S \tag{15}$$

where

$$\begin{aligned} S = {} & (\underline{q}_+-\underline{p}_+)^{-4}\,\mathrm{Sp}\{(\underline{p}_-+m)\gamma_\nu(\underline{q}_-+m)\gamma_\mu\}\,\mathrm{Sp}\{(\underline{q}_++m)\gamma_\nu(\underline{p}_++m)\gamma_\mu\} \\ & + (\underline{q}_+-\underline{q}_-)^{-4}\,\mathrm{Sp}\{(\underline{p}_-+m)\gamma_\nu(\underline{p}_++m)\gamma_\mu\}\,\mathrm{Sp}\{(\underline{q}_++m)\gamma_\nu(\underline{q}_-+m)\gamma_\mu\} \\ & - (\underline{q}_+-\underline{p}_+)^{-2}(\underline{q}_+-\underline{q}_-)^{-2}\,2\mathrm{Re}\,\mathrm{Sp}\{(\underline{q}_-+m)\gamma_\mu(\underline{p}_-+m)\gamma_\nu(\underline{p}_++m)\gamma_\mu(\underline{q}_++m)\gamma_\nu\} \end{aligned} \tag{16}$$

The first term only of S would be obtained if the two interacting particles were not identical. The second and last terms are due to the exchange effect.

The spurs occurring in $|M|^2$ are evaluated with the help of the following relations for the spurs of the matrices γ_μ and their products:

$$\begin{aligned} & \mathrm{Sp}(\gamma_\mu) = 0; \quad \mathrm{Sp}(\gamma_\lambda\gamma_\mu\gamma_\nu) = 0; \quad \text{etc., for odd products;} \\ & \mathrm{Sp}(\gamma_\mu\gamma_\nu) = 4\delta_{\mu\nu}; \quad \mathrm{Sp}(\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\nu) = 4(\delta_{\lambda\mu}\delta_{\rho\nu} - \delta_{\lambda\rho}\delta_{\mu\nu} + \delta_{\lambda\nu}\delta_{\mu\rho}); \\ & \mathrm{Sp}(\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\tau) = \delta_{\lambda\mu}\mathrm{Sp}(\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\tau) - \delta_{\lambda\rho}\mathrm{Sp}(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\tau) \\ & \qquad + \delta_{\lambda\nu}\mathrm{Sp}(\gamma_\mu\gamma_\rho\gamma_\sigma\gamma_\tau) - \delta_{\lambda\sigma}\mathrm{Sp}(\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\tau) + \delta_{\lambda\tau}\mathrm{Sp}(\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma); \\ & \text{etc., for even products.} \end{aligned} \tag{17}$$

The result (see Appendix A) is

$$\begin{aligned} S = 32\Big[& (\underline{q}_+-\underline{p}_+)^{-4}\{(p_-\cdot q_+)(p_+\cdot q_-) + (p_-\cdot p_+)(q_-\cdot q_+) - m^2(p_-\cdot q_-) - m^2(p_+\cdot q_+) + 2m^4\} \\ & + (\underline{q}_+-\underline{q}_-)^{-4}\{(p_-\cdot q_+)(p_+\cdot q_-) + (p_-\cdot q_-)(p_+\cdot q_+) - m^2(p_-\cdot p_+) - m^2(q_-\cdot q_+) + 2m^4\} \\ & - (\underline{q}_+-\underline{p}_-)^{-2}(\underline{q}_+-\underline{q}_-)^{-2}\{-2(p_-\cdot q_+)(p_+\cdot q_-) \\ & \quad + m^2[(p_-\cdot q_-) + (p_+\cdot q_-) + (q_-\cdot q_+) + (p_-\cdot p_+) + (p_+\cdot q_-) + (p_+\cdot q_+)] - 2m^4\}\Big] \end{aligned} \tag{18}$$

Now $|M|^2$ is the probability for a transition in the time T

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* This projection operator includes the correction factor p_4/m required by Feynman's normalization (I, p. 757-8).

from a state characterized by the momenta p_-, p_+ to a state q_-, q_+. To obtain a differential scattering cross-section we must find the transition probability per unit time and per unit incident flux, summed over all final states in which the particles move within a specified solid angle. We may write

$$dQ = (JT)^{-1} \int |M|^2 \frac{dn}{dq_{-4}} dq_{-4} \tag{19}$$

The incident flux is $J = v_r/L^3$ where v_r is the relative velocity of the colliding particles. Because of the momentum conservation condition, we need to sum over the final states of only one particle, say the electron. We multiply by the energy density of states (dn/dq_{-4}) and integrate over the energy dq_{-4}. The familiar formula for the number of momentum states in a box of side L, within a solid angle $d\Omega$, is

$$\frac{dn}{dq_{-4}} = \left(\frac{L}{2\pi}\right)^3 (q_{-4}^2 - m^2) \frac{d\sqrt{q_{-4}^2 - m^2}}{dq_{-4}} d\Omega \tag{20}$$

$$= (L/2\pi)^3 q_{-4} \sqrt{q_{-4}^2 - m^2}\, d\Omega$$

Recalling that $F = p_{-4} + q_{+4} - p_{+4} - q_{-4}$, (12b) we put $dq_{-4} = dF$.

The cross-section becomes

$$dQ = \frac{\pi^2 e^4}{L^6 JT} \left(\frac{L}{2\pi}\right)^3 d\Omega \int \frac{q_{-4}\sqrt{q_{-4}^2 - m^2}}{p_{+4}p_{-4}q_{+4}q_{-4}} \frac{\sin^2 \frac{1}{2}FT}{F^2} \quad SdF \tag{21}$$

As usual we make use of the fact that the only important contribution to the integral is at $F = 0$, and that all functions except $\sin^2\frac{1}{2}FT/F^2$ are slowly varying and may be taken outside the integral. The integral becomes $\int_{-\infty}^{\infty} (\sin^2\frac{1}{2}FT/F^2)dF = \frac{1}{2}\pi T$. We therefore have

$$dQ = \frac{e^4}{16L^3 J} \frac{\sqrt{q_{-4}^2 - m^2}}{p_{+4}p_{-4}q_{+4}} \quad Sd\Omega \tag{22}$$

and the condition $F = 0$ expresses the conservation of energy:

$$p_{-4} + q_{+4} - p_{+4} - q_{-4} = 0. \tag{23}$$

Cross-section for Positron-electron Scattering.

We will now find an expression for dQ in the center of mass coordinate system of the two particles. Let the two particles move in opposite directions along the z-axis with equal velocities v and energies $-p_{+4} = p_{-4} = E = \gamma m$. (It is here that we first make explicit use of the fact that p_+ describes a negative energy state.) Let the electron be scattered through an angle θ. Then $p_{\pm 1} = p_{\pm 2} = 0$, $p_{+3} = p_{-3} = -\sqrt{E^2-m^2}$.* We apply the conservation laws (12a), (23) $p_- + q_+ = p_+ + q_-$. Then

$$q_{+i} = p_{+i} - p_{-i} + q_{-i} = q_{-i} \text{ for } i = 1,2,3$$

$$q_{+4} = p_{+4} - p_{-4} + q_{-4} = q_{-4} - 2E$$

Since $q_+ q_+ = q_- q_- = m^2$, we have

$$q_{+4}^2 = m^2 + \sum_i q_{+i}^2 = m^2 + \sum_i q_{-i}^2 = q_{-4}^2,$$

or $q_{+4} = \pm q_{-4}$, and therefore $q_{-4} = -q_{+4} = E$. Finally we have $\sum_i p_{-i} q_{-i} = (E^2-m^2)\cos\theta$. We then get the following expressions for the scalar products of the various four-vector momenta:

$$\begin{aligned}
p_- q_- = p_+ q_+ &= E^2-(E^2-m^2)\cos\theta = m^2[\gamma^2-(\gamma^2-1)\cos\theta] \\
p_- q_+ = p_+ q_- &= -E^2-(E^2-m^2)\cos\theta = -m^2[\gamma^2+(\gamma^2-1)\cos\theta] \qquad (24)\\
p_- p_+ = q_- q_+ &= -E^2-(E^2-m^2) \qquad = -m^2(2\gamma^2-1)
\end{aligned}$$

Introducing these values into the expression (18) for S gives

$$\begin{aligned}
S = 4\Big[&\{(\gamma^2-1)^2 \sin^4\tfrac{1}{2}\theta\}^{-1}\{1+4(\gamma^2-1)\cos^4\tfrac{1}{2}\theta + 2(\gamma^2-1)^2\ (1+\cos^4\tfrac{1}{2}\theta)\} \\
&+ \gamma^{-4}\{3+4(\gamma^2-1)+(\gamma^2-1)^2(1+\cos^2\theta)\} \qquad (25)\\
&- \{\gamma^2(\gamma^2-1)\sin^2\tfrac{1}{2}\theta\}^{-1}\{3+8(\gamma^2-1)\cos^2\tfrac{1}{2}\theta+4(\gamma^2-1)^2\cos^4\tfrac{1}{2}\theta\}\Big]
\end{aligned}$$

The incident flux is $J = 2v/L^3 = 2\sqrt{E^2-m^2}/L^3 E$. The cross-section

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* Because of Feynman's treatment of positrons, the momentum as well as the energy of a positron has the opposite sign to that of an electron following the same path.

dQ therefore becomes

$$dQ = (e^4/8E^2)d\Omega(\tfrac{1}{4}S). \tag{26}$$

To express the cross-section in terms of a laboratory coordinate system in which the electron is initially at rest, we apply a Lorentz transformation. Fig. 3 shows the various momenta in the two coordinate systems. We denote quantities in the laboratory system by primes. The positron and electron are

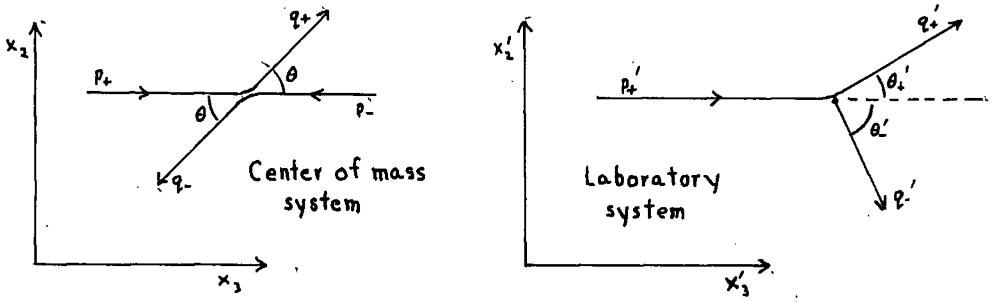

Fig. 3. The transformation from center of mass to laboratory coordinates.

scattered through angles θ_+' and θ_-' respectively with the direction of the incident positron. If we let $-p_{+4}' = 2mT+m$, $q_{-4}' = 2mV+m$, then since $2m = 1.02$ Mev., T and V are very nearly the kinetic energies in Mev. of the incident positron and the scattered electron respectively.

The relative velocity of the laboratory system with respect to the center of mass system is $-v = -\sqrt{\gamma^2-1}/\gamma$. Hence $(1-v^2)^{-\frac{1}{2}} = \gamma$. The Lorentz transformation equations are therefore:

$$\begin{aligned}
-m(2T+1) &= p_{+4}' = (1-v^2)^{-\frac{1}{2}}(p_{+4}+vp_{+3}) = -m(2\gamma^2-1)\\
m(2V+1) &= q_{-4}' = (1-v^2)^{-\frac{1}{2}}(q_{-4}+vq_{-3}) = m[\gamma^2-(\gamma^2-1)\cos\theta]\\
-m\sqrt{(2V+1)^2-1}\,\sin\theta_-' &= q_{-2}' = q_{-2} = -m\sqrt{\gamma^2-1}\,\sin\theta\\
-m\sqrt{(2T-2V+1)^2-1}\,\sin\theta_+' &= q_{+2}' = q_{+2} = -m\sqrt{\gamma^2-1}\,\sin\theta
\end{aligned} \tag{27}$$

These equations give:

$$\begin{aligned} T &= \gamma^2 - 1 \\ V &= (\gamma^2-1)\sin \tfrac{1}{2}\theta \\ T-V &= (\gamma^2-1)\cos \tfrac{1}{2}\theta \end{aligned} \tag{28}$$

$$\begin{aligned} \cot\theta_-' &= \gamma \tan\tfrac{1}{2}\theta \\ \cot\theta_+' &= \gamma \cot\tfrac{1}{2}\theta \end{aligned} \tag{29}$$

The relation between the two scattering angles is therefore

$$\cot\theta_-' \cot\theta_+' = T + 1 \tag{30}$$

Since the cross-section is an area perpendicular to the relative velocity of the two coordinate systems, $dQ' = dQ$.

Substituting the above values into the expression for the cross-section, and arranging the result in powers of the energy V transferred to the electron, we get

$$\begin{aligned} dQ = \tfrac{1}{8} r_0^2 d\Omega (T+1)^{-3} \Big[&(T+1)^2 (2T+1)^2 V^{-2} - (T+1)(8T^2+16T+7)V^{-1} \\ &+ (12T^2+24T+13) - 4(2T+1)V + 4V^2 \Big] \end{aligned} \tag{31}$$

where $r_0 = e^2/mc^2$ is the classical radius of the electron. If the exchange effect is neglected by taking only the first term of S in equation (25), the cross-section becomes

$$dQ_0 = \tfrac{1}{8} r_0^2 d\Omega (T+1)^{-1} \left[(2T+1)^2 V^{-2} - 4(T+1)V^{-1} + 2 \right] \tag{32}$$

The cross-section for electron-electron scattering is derived in Appendix B.

Comparison with Other Results.

The preceding values of the cross-sections are in agreement, up to a constant factor, with those of Bhabha[4] and Møller.[5]* (The results given here are in each case just twice those of the

* The notation of Møller and Bhabha differs from that used here. Their γ^*, θ^* are our γ, θ. Their γ equals 2T+1. Bhabha's ϵ is V/T.

other authors.) Within this factor, Bhabha's equation (15), p. 202, is identical with (26) and (25) above, and Møller's equation (74), p. 568, is equivalent to the electron-electron cross-section (B2) given in Appendix B.

Certain values given by Mott and Massey[6] appear to be incorrect. Their expressions (15) for scattering neglecting exchange and (16) for positron-electron scattering both contain a number of errors.

Feasibility of an Experiment.

To facilitate the plotting of numerical results, we introduce as a variable $\epsilon = V/T$, the fraction of the kinetic energy of the incident positron that is transferred to the electron. Each of the preceding cross-sections may be written

$$dQ = \tfrac{1}{8} r_0^2 \, d\Omega \, \frac{(2T+1)^2}{T^2(T+1)} \, \frac{1}{\epsilon^2} \, \phi\,(T,\epsilon) \tag{33}$$

where the function $\phi(T,\epsilon)$ has the values

$$\phi_0 = 1 - \left(1 - \frac{1}{(2T+1)^2}\right)\epsilon + \tfrac{1}{2}\left(\frac{2T}{2T+1}\right)^2 \epsilon^2 \tag{34}$$

for scattering with no exchange effect, and

$$\begin{aligned}\phi_+ = 1 - \left(1 - \frac{1}{(2T+1)^2}\right)\left(2 - \frac{1}{(2T+2)^2}\right)\epsilon + \left(\frac{2T}{2T+1}\right)^2\left(3 + \frac{1}{(2T+2)^2}\right)\epsilon^2 \\ - 2\left(\frac{2T}{2T+1}\right)\left(\frac{T}{T+1}\right)^2\epsilon^3 + \left(\frac{2T}{2T+1}\right)^2\left(\frac{T}{T+1}\right)^2\epsilon^4\end{aligned} \tag{35}$$

for positron-electron scattering with exchange.

If no attempt were made to distinguish positrons from electrons in a scattering experiment, the measurements would correspond

6 N. F. Mott and H. S. W. Massey, "The Theory of Atomic Collisions," (Second Edition, Oxford, 1949) pp. 371-2.

to one of the functions

$$\overline{\phi}_o = \phi_o(\epsilon) + \left(\frac{\epsilon}{1-\epsilon}\right)^2 \phi_o(1-\epsilon) \tag{36}$$

$$\overline{\phi}_+ = \phi_+(\epsilon) + \left(\frac{\epsilon}{1-\epsilon}\right)^2 \phi_+(1-\epsilon) \tag{37}$$

depending on whether or not the exchange effect is present. For electron-electron scattering, $\phi(T,\epsilon)$ in (33) becomes

$$\phi_- = \overline{\phi}_o + \left(\frac{2T-1}{2T+1}\right)\left(\frac{\epsilon}{1-\epsilon}\right) \tag{38}$$

In equations (36) to (38), we may take ϵT to be the kinetic energy (in Mev) of the least energetic of the two scattered particles, whether it is a positron or an electron.

Since $\phi(T,\epsilon) \approx 1$ for most values of T and ϵ, the quantity $\frac{1}{\phi}\frac{dQ}{d\Omega}$ gives the order of magnitude of the scattering cross-section per unit solid angle. This quantity is plotted (on a logarithmic scale) against ϵ for several values of T in Fig. 4.

In Fig. 5, ϕ_o and ϕ_+ are plotted as functions of ϵ, and in Fig. 6 ϕ_-, $\overline{\phi}_o$, and $\overline{\phi}_+$ are plotted for $\epsilon \leq 0.5$, all for several values of T.

The relation between ϵ and the scattering angle is

$$\begin{aligned}\epsilon &= \sin^2\tfrac{1}{2}\theta \\ &= \left[1 + (T+1)\tan^2\alpha\right]^{-1}\end{aligned}$$

where α is the scattering angle of the particle with energy ϵT ($\alpha = \theta_-'$ or θ_+' according as $\epsilon T = V$ or $T-V$). The angle for the other particle is of course the same function of $1-\epsilon$. An angle scale for different values of T is shown as well as the ϵ scale in Fig. 5.

A possible experiment would consist of directing a well-collimated and reasonably monoenergetic beam of positrons onto a scattering foil, and then recording the scattered positrons and

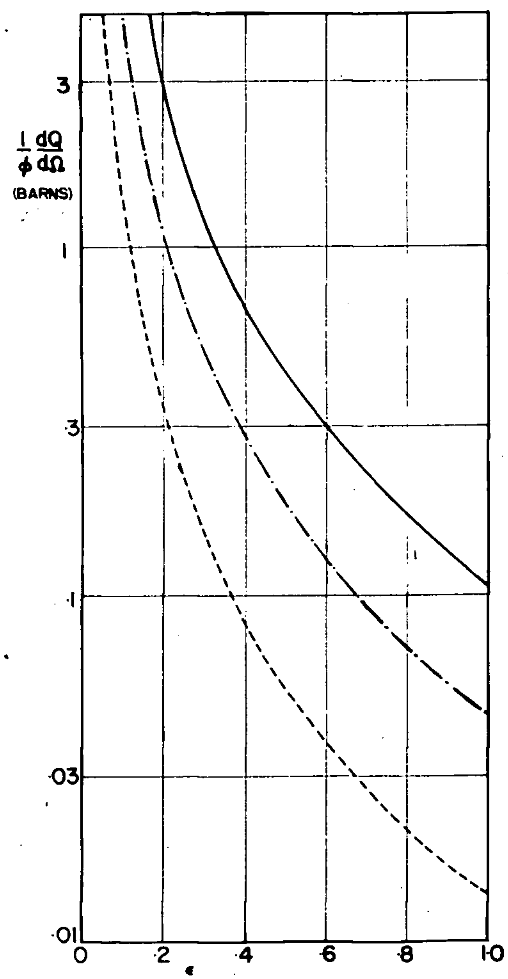

Fig. 4. Magnitude of the Cross-section per Unit Solid Angle.

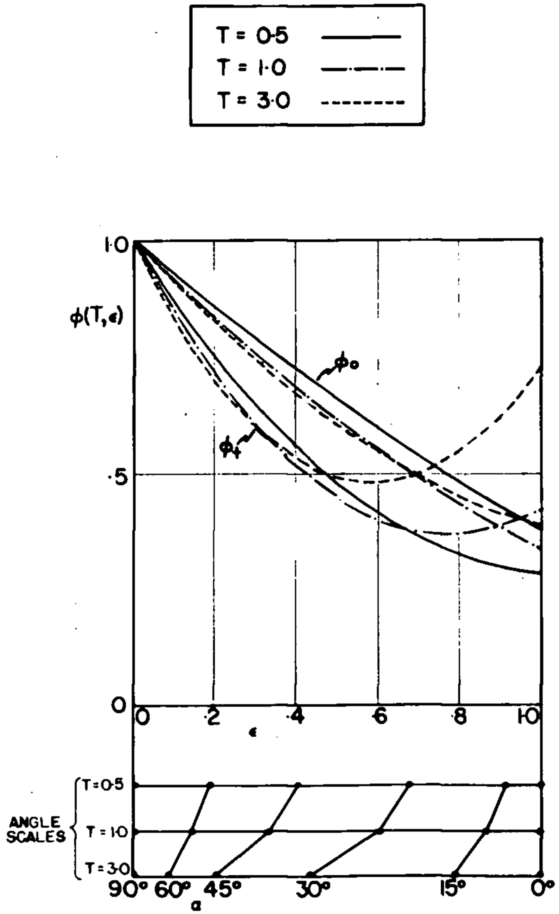

Fig. 5. ϕ_+ and ϕ_0 as Functions of ϵ and α.

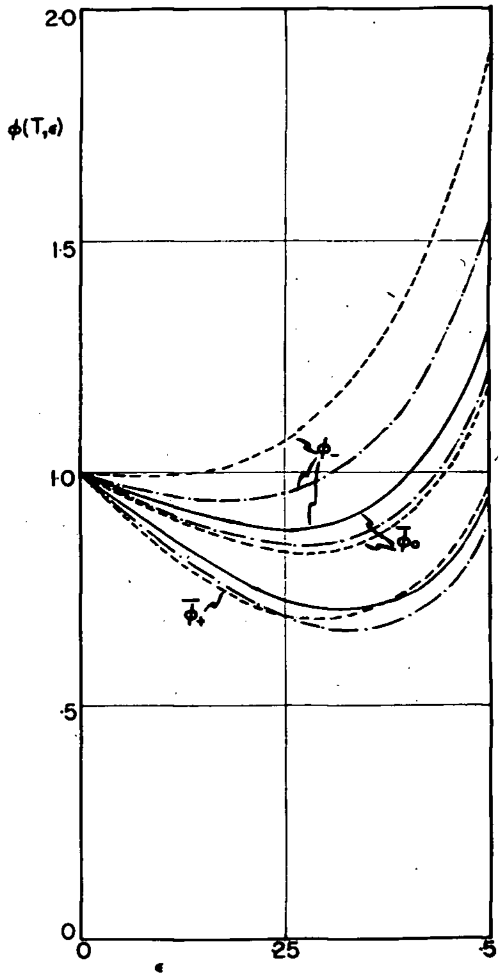

Fig. 6. ϕ_-, $\bar{\phi}_0$ and $\bar{\phi}_+$ as Functions of ϵ.

electrons at various angles by means of counters in coincidence. The atomic electrons in the foil will act as free electrons provided that the energies of both particles after the collision are much larger than the binding energies involved. This sets a lower limit of say 50 to 100 kev for V and T-V. The foil should be of a low Z material (e.g., carbon) to keep the binding energies small, and also to reduce scattering by nuclei. The latter has the same order of magnitude as scattering by electrons, but is proportional to Z^2 instead of Z (see Mott and Massey,[6] p. 81).

It is clear from Figs. 4, 5, and 6 that a successful experiment, while not impossible, would require careful technique. The cross-section is very small, and the difference effect to be sought for is at most points not large. The aim should be to compare the relative shapes of the experimental and theoretical curves for ϕ as a function of ϵ, rather than to make an absolute measurement of the cross-section.

This means that an experiment using the curves in Fig. 5, (i.e., distinguishing positrons from electrons) would be most likely to succeed at energies of 3 Mev and greater, for which ϕ_+ increases as $\epsilon \to 1$. The two curves for $T = \frac{1}{2}$, for instance, although they differ by about 40%, are nearly parallel in the range $0.2 < \epsilon < 0.8$. Unfortunately, the positrons from the most common emitters have energies under 1 Mev.

Reference to Fig. 6 suggests that low energy positrons might be used by comparing the scattering of positrons with that of electrons (without distinguishing the two particles in the positron case). At $T = \frac{1}{2}$ the curves for ϕ_- and $\overline{\phi_0}$ coincide exactly, while $\overline{\phi}_+$ differs appreciably. This method, in contrast to that using Fig. 5, would be less advantageous at higher energies.

Appendix A: Evaluation of Spurs.

It is required to evaluate the spurs in the expression (16) for S_ξ using the relations (17) for the spurs of products of the matrices γ_μ.

One of the spurs in the first term of S becomes

$$\begin{aligned} Sp\{(p_{-\lambda}\gamma_\lambda+m)\gamma_\nu(q_{-\rho}\gamma_\rho+m)\gamma_\mu\} &= 4p_{-\lambda}q_{-\rho}(\delta_{\lambda\nu}\delta_{\rho\mu}-\delta_{\lambda\rho}\delta_{\nu\mu}+\delta_{\lambda\mu}\delta_{\rho\nu})+4m^2\delta_{\mu\nu} \\ &= 4(p_{-\nu}q_{-\mu}+p_{-\mu}q_{-\nu})+4(m^2-p_-\cdot q_-)\delta_{\mu\nu} \end{aligned}$$

The other factor simply has q_+, p_+ replacing p_-, q_-. The first term of S is therefore

$$\begin{aligned} &16(\underline{q}_+-\underline{p}_+)^{-4}\{p_{-\nu}q_{-\mu}+p_{-\mu}q_{-\nu}+(m^2-p_-\cdot q_-)\delta_{\mu\nu}\}\{q_{+\nu}p_{+\mu}+q_{+\mu}p_{+\nu}+(m^2-p_+\cdot q_+)\delta_{\mu\nu}\} \\ &=32(\underline{q}_+-\underline{p}_+)^{-4}\{(p_-\cdot q_+)(p_+\cdot q_-)+(p_-\cdot p_+)(q_-\cdot q_+)-m^2(p_-\cdot q_-)-m^2(p_+\cdot q_+)+2m^4\} \end{aligned} \tag{A1}$$

The second term of S differs only in that p_+ and q_- are interchanged. It is therefore

$$32(\underline{q}_+-\underline{q}_-)^{-4}\{(p_-\cdot q_+)(p_+\cdot q_-)+(p_-\cdot q_-)(p_+\cdot q_+)-m^2(p_-\cdot p_+)-m^2(q_-\cdot q_+)+2m^4\} \tag{A2}$$

The spur in the last term of S can be expanded as follows:

$$\begin{aligned} &q_{-\lambda}p_{-\rho}p_{+\sigma}q_{+\tau}Sp(\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\tau\gamma_\nu)+m^4Sp(\gamma_\mu\gamma_\nu\gamma_\mu\gamma_\nu) \\ &\quad+m^2q_{-\lambda}p_{-\rho}Sp(\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\mu\gamma_\nu)+m^2q_{-\lambda}p_{+\sigma}Sp(\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\nu) \\ &\quad+m^2q_{-\lambda}q_{+\tau}Sp(\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\mu\gamma_\tau\gamma_\nu)+m^2p_{-\rho}p_{+\sigma}Sp(\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\nu) \\ &\quad+m^2p_{-\rho}q_{+\tau}Sp(\gamma_\mu\gamma_\rho\gamma_\nu\gamma_\mu\gamma_\tau\gamma_\nu)+m^2p_{+\sigma}q_{+\tau}Sp(\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\mu\gamma_\tau\gamma_\nu) \end{aligned} \tag{A3}$$

To evaluate these terms we will make use of the fact that the spur of a product is unchanged by a cyclic permutation of the matrices in the product. We first set down the results of a summation over certain indices in some of the equations (17):

$$\mathrm{Sp}(\gamma_\lambda \gamma_\rho \gamma_\mu \gamma_\mu) = 4\mathrm{Sp}(\gamma_\lambda \gamma_\rho) = 16\,\delta_{\lambda\rho}$$
$$\mathrm{Sp}(\gamma_\lambda \gamma_\mu \gamma_\rho \gamma_\mu) = 4(\delta_{\lambda\mu}\delta_{\rho\mu} - \delta_{\lambda\rho}\delta_{\mu\mu} + \delta_{\lambda\mu}\delta_{\mu\rho}) = -8\,\delta_{\lambda\rho} \tag{A4}$$
$$\mathrm{Sp}(\gamma_\mu \gamma_\nu \gamma_\mu \gamma_\nu) = -8\,\delta_{\mu\mu} = -32$$

$$\mathrm{Sp}(\gamma_\lambda \gamma_\rho \gamma_\mu \gamma_\sigma \gamma_\tau \gamma_\mu) = 16\delta_{\lambda\rho}\delta_{\sigma\tau} - 4(\delta_{\rho\sigma}\delta_{\tau\lambda} - \delta_{\rho\tau}\delta_{\sigma\lambda} + \delta_{\rho\lambda}\delta_{\sigma\tau}) - 8\,\delta_{\lambda\sigma}\delta_{\rho\tau} + 8\,\delta_{\rho\sigma}\delta_{\lambda\tau} + 4(\delta_{\rho\lambda}\delta_{\sigma\tau} - \delta_{\rho\sigma}\delta_{\lambda\tau} + \delta_{\rho\tau}\delta_{\lambda\sigma}) \tag{A5a}$$
$$= 16\,\delta_{\lambda\rho}\delta_{\sigma\tau}$$

$$\mathrm{Sp}(\gamma_\lambda \gamma_\mu \gamma_\rho \gamma_\sigma \gamma_\tau \gamma_\mu) = 4(\delta_{\rho\sigma}\delta_{\tau\lambda} - \delta_{\rho\tau}\delta_{\sigma\lambda} + \delta_{\rho\lambda}\delta_{\sigma\tau}) - 16\,\delta_{\lambda\rho}\delta_{\sigma\tau} + 16\,\delta_{\lambda\sigma}\delta_{\rho\tau} - 16\,\delta_{\lambda\tau}\delta_{\rho\sigma} + 4(\delta_{\lambda\rho}\delta_{\sigma\tau} - \delta_{\lambda\sigma}\delta_{\rho\tau} + \delta_{\lambda\tau}\delta_{\rho\sigma}) \tag{A5b}$$
$$= -8(\delta_{\lambda\rho}\delta_{\sigma\tau} - \delta_{\lambda\sigma}\delta_{\rho\tau} + \delta_{\lambda\tau}\delta_{\rho\sigma})$$

The six coefficients of m^2 in the expression (A3) can be brought by cyclic permutations into one of the following two forms, obtained from equation (A5a):

$$\begin{aligned} \mathrm{Sp}(\gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\sigma \gamma_\mu \gamma_\nu) &= 16\,\delta_{\lambda\mu}\delta_{\sigma\mu} = 16\,\delta_{\lambda\sigma} \\ \mathrm{Sp}(\gamma_\lambda \gamma_\mu \gamma_\rho \gamma_\nu \gamma_\mu \gamma_\nu) &= 16\,\delta_{\nu\lambda}\delta_{\rho\nu} = 16\,\delta_{\lambda\rho} \end{aligned} \tag{A6}$$

From the preceding expressions the first term of (A3) can be calculated:

$$\begin{aligned} \mathrm{Sp}(\gamma_\lambda \gamma_\mu \gamma_\rho \gamma_\nu \gamma_\sigma \gamma_\mu \gamma_\tau \gamma_\nu) = {} & -8(\delta_{\rho\sigma}\delta_{\lambda\tau} - \delta_{\rho\lambda}\delta_{\sigma\tau} + \delta_{\rho\tau}\delta_{\sigma\lambda}) - 16\,\delta_{\lambda\rho}\delta_{\sigma\tau} + 16\,\delta_{\rho\sigma}\delta_{\tau\lambda} \\ & -16\,\delta_{\lambda\sigma}\delta_{\rho\tau} + 16\,\delta_{\lambda\rho}\delta_{\sigma\tau} - 16\,\delta_{\lambda\tau}\delta_{\rho\sigma} \\ & -8(\delta_{\rho\lambda}\delta_{\sigma\tau} - \delta_{\rho\sigma}\delta_{\lambda\tau} + \delta_{\rho\tau}\delta_{\lambda\sigma}) \\ = {} & -32\,\delta_{\lambda\sigma}\delta_{\rho\tau} \end{aligned} \tag{A7}$$

The complete expression for the spur in the third term of S is finally obtained by substituting in (A3). It is:

$$-32(p_-\cdot q_+)(p_+\cdot q_-) + 16m^2\left[p_-\cdot q_- + p_+\cdot q_- + q_-\cdot q_+ + p_-\cdot p_+ + p_-\cdot q_+ + p_+\cdot q_+\right] - 32m^4 \tag{A8}$$

Since this is of course real, the third term of S is

$$-32(\underline{q}_+ - \underline{p}_+)^{-2}(\underline{q}_+ - \underline{q}_-)^{-2}\Big\{-2(p_-\cdot q_+)(p_+\cdot q_-) + m^2\left[p_-\cdot q_- + p_+\cdot q_- + q_-\cdot q_+ + p_-\cdot p_+ + p_-\cdot q_+ + p_+\cdot q_+\right] - 2m^4\Big\} \tag{A9}$$

Combination of the expressions (A1), (A2), and (A9) obtained for the three terms of S gives equation (18).

Appendix B. Electron-electron Scattering.

The preceding calculation can be easily adapted to give the cross-section for electron-electron scattering. The derivation of all equations up to (23) is exactly the same, except that p_-, q_+ are now the initial momenta and q_-, p_+ the final momenta, and of course all energies are positive.

We then put $p_{-4} = q_{+4} = E$, $p_{-3} = -q_{+3} = \sqrt{E^2-m^2}$, $p_{-1,2} = q_{+1,2} = 0$, and $\vec{p}_- \cdot \vec{q}_- = (E^2-m^2)\cos\theta$. The scalar products of the momenta are then:

$$
\begin{aligned}
p_- q_- = p_+ q_+ &= m^2\left[\gamma^2-(\gamma^2-1)\cos\theta\right] = m^2(2V_1+1) \\
p_- q_+ = p_+ q_- &= m^2(2\gamma^2-1) \qquad\qquad\qquad = m^2(2T+1) \\
p_- p_+ = q_- q_+ &= m^2\left[\gamma^2+(\gamma^2-1)\cos\theta\right] = m^2(2V_2+1)
\end{aligned}
\tag{B1}
$$

The Lorentz transformation to a laboratory coordinate system is carried out as before. V_1 and V_2, which replace V and $T-V$ in equation (28), are the kinetic energies in Mev of the two electrons after the collision.

We note that the scalar products $p_- q_- = p_+ q_+$ are exactly the same as in the equation (24), and that the other two pairs are interchanged and reversed in sign. Since the latter two occur symmetrically and always squared in the first term of S, the first term of the electron-electron cross-section is just dQ_0 with V_1 replacing V. The second term is identical, except that V_2 replaces V_1.

The electron-electron scattering cross-section is therefore:

$$
\begin{aligned}
dQ_- = \tfrac{1}{8} r_0^2\, d\Omega (T+1)^{-1}\Big[&\left\{(2T+1)^2 V_1^{-2} - 4(T+1)V_1^{-1} + 2\right\} \\
&+ \left\{(2T+1)^2 V_2^{-2} - 4(T+1)V_2^{-1} + 2\right\} - (4T^2-1)(V_1 V_2)^{-1}\Big] \\
= \tfrac{1}{8} r_0^2\, d\Omega (T+1)^{-1}\Big[&T^2(2T+1)^2(V_1 V_2)^{-2} - (8T^2+12T+3)(V_1 V_2)^{-1} + 4\Big]
\end{aligned}
\tag{B2}
$$

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