

A SEMI-CLASSICAL ANALYSIS OF THE WILSON LOOP IN A
2+1 DIMENSIONAL YANG-MILLS THEORY
WITH A MONOPOLE GAS

By

Michael Perry Clark

B.Sc. Hons., Queen's University, Kingston, 1996

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES
DEPARTMENT OF PHYSICS & ASTRONOMY

We accept this thesis as conforming
to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

August 1997

© Michael Perry Clark, 1997

In presenting this thesis in partial fulfilment of the requirements for an advanced degree at the University of British Columbia, I agree that the Library shall make it freely available for reference and study. I further agree that permission for extensive copying of this thesis for scholarly purposes may be granted by the head of my department or by his or her representatives. It is understood that copying or publication of this thesis for financial gain shall not be allowed without my written permission.

Department of Physics & Astronomy
The University of British Columbia
2075 Wesbrook Place
Vancouver, Canada
V6T 1W5

Date:

August 19, 1997

Abstract

In this paper we consider a Wilson loop in a 2+1 dimensional pure Yang-Mills theory with an $SU(2)$ gauge group. The initial goal is to test a conjecture of A.M. Polyakov's which proposes that if one considers the field-strength, $F_{\mu\nu}^a$, and the gauge field, A_μ^a , as independent, random variables, then a sum over surfaces spanning the Wilson loop will re-introduce the Bianchi Identity. We do this by introducing an additional functional integral over a sigma model variable which unravels the path-ordering of the loop variables. Then, via a non-Abelian Stokes' theorem, we express the Wilson loop as a surface integral with separate functional integrals over both $F_{\mu\nu}^a$ and A_μ^a . At the semi-classical level, characterized by a large spin parameter, we find that the conjecture holds true – the Bianchi Identity arises as a natural constraint.

Secondly, we find that this reformulation of the Wilson loop naturally allows for an arbitrary distribution of monopoles. We treat both the cases of a single monopole and a monopole gas. In the latter case we demonstrate the confinement of quarks for states of half-odd-integer spin.

Table of Contents

Abstract	ii
Acknowledgements	v
1 Introduction — Some Background on the Problems at Hand	1
1.1 The Yang-Mills Field	1
1.2 The Wilson Loop and the Transformation of the Gauge Field	8
1.3 Quantizing the Yang-Mills Field	11
2 The Wilson Loop in a Yang-Mills Field Theory	13
2.1 The Problem of Confinement	13
2.2 Towards Solving the Wilson Loop Expression	17
2.2.1 A Coherent State Representation	18
2.2.2 Introducing a Sigma Model	21
3 Solving the Coherent State Path Integral in a Constant Gauge Field	26
4 Testing Polyakov's Conjecture – The Semiclassical Limit	31
4.1 Surface Independence	33
4.2 Variational Principles and the Method of Steepest Descent	34
4.3 The Derivation of the Classical Solutions	36
4.4 Giving Meaning to Monopoles	40
5 The Expectation Value of the Wilson Loop	42

5.1	Reformulating the Loop Expression – Semiclassical Limit	42
5.2	A Change to Comoving Coordinates	44
5.3	Deriving the Area Law	47
5.4	An Initial Approach to Fluctuations About the Classical Solutions	50
5.4.1	Faddeev-Popov Gauge Fixing	50
6	Monopoles and the Classical Solutions	53
6.1	Phase factor of the Wilson Loop for one Monopole	57
6.2	Charge Quantization	59
6.3	Many Monopoles and the Coloumb Gas	61
6.4	The Sine-Gordon Transformation and the Classical Solutions	63
6.5	A Final Look at Confinement	66
7	Conclusions	68
	Bibliography	71
	Appendix A – Properties of Coherent States	73
	Appendix B – Solving Functional Integrals	76
B.1	Functional Derivatives	76
B.2	Functional Integrals	77
B.3	Functional Determinants	78

Acknowledgements

First and foremost, I would like to thank my supervisor, Dr. Gordon Semenov. Without his countless hours of patient teaching and guidance this research thesis would never have become a reality. I am indebted to him for providing me with a large part of my skills and knowledge of field theory.

As well, over many conversations, Dr. Ariel Zhitnitsky provided invaluable advice and insight and for this I owe him my thanks.

I would also like to thank Todd Fugleburg, Sebastian Jaimungal and Lori Paniak for their helpful advice and witty conversations.

I thank Arnab Guha for his consistent moral support, and, last but certainly not least, I thank my parents for their support and guidance of every kind.

For his lasting personal influence and his love of science, this work is dedicated to the memory of my grandfather, Robert Ross Parker.

This work was funded in part by the Natural Sciences and Engineering Research Council of Canada.

Chapter 1

Introduction — Some Background on the Problems at Hand

In this project we investigate some of the properties of a Yang-Mills field (in three dimensions, for the most part). To this end, in the following pages, we present a brief outline and explanation of some field theoretical concepts, leading up to a discussion of the Yang-Mills non-Abelian gauge field theory. Also presented is a short introduction to some of the investigative tools at our disposal – most importantly, the Wilson loop.

1.1 The Yang-Mills Field

It is of great importance to have a firm understanding of the defining characteristics of the Yang-Mills field in order to understand both the motivation and the difficulties of an investigation into the behaviour of such a field. Begin by considering a simpler field theory – for example, the standard Dirac field, from which we shall shortly construct Quantum Electrodynamics:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad (1.1)$$

This Lagrangian, by inspection, is invariant under the following transformation:

$$\psi(x) \rightarrow e^{i\alpha}\psi(x) \quad (1.2)$$

which is known as a global phase invariance (or symmetry).

We need not stop here. We may generalize transformations of the type (1.2) to the much more powerful *local phase transformation* (*gauge transformation*):

$$\psi(x) \rightarrow e^{i\alpha(x)}\psi(x) \quad (1.3)$$

such that the phase rotation through angle $\alpha(x)$ may vary arbitrarily from point to point. However, due to the derivative operator in the kinetic term of (1.1) the symmetry is spoiled. We can remedy this by replacing ∂_μ with a generalized operator, D_μ , (called the *covariant derivative* for reasons which will become apparent later) which we will purposefully construct so as to maintain the symmetry under (1.3). Our requirement is that

$$D_\mu \psi \rightarrow e^{i\alpha(x)} D_\mu \psi \quad (1.4)$$

that is,

$$D_\mu \rightarrow e^{i\alpha(x)} D_\mu e^{-i\alpha(x)} \quad (1.5)$$

This is accomplished by specifying

$$D_\mu = \partial_\mu - ieA_\mu(x) \quad (1.6)$$

for some unknown function $A_\mu(x)$. In this case, the transformation properties of D_μ require that A_μ transforms as

$$A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \quad (1.7)$$

In a very natural way, our insistence that our Lagrangian have a local phase invariance has led to the introduction of $A_\mu(x)$, a *gauge field*.

We can now construct a kinetic term for this gauge field which has appeared as a natural consequence of invariance conditions. Of course, this new term for the Lagrangian must continue to preserve the symmetry.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (1.8)$$

satisfies this requirement. In fact, it is the *unique* gauge-invariant term with the minimum number of derivatives.

Putting all of these pieces together, we end up with¹:

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= \bar{\psi}(i\rlap{\not{D}} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu \\ &= \bar{\psi}(i\rlap{\not{D}} - m)\psi - \frac{1}{4}(F_{\mu\nu})^2\end{aligned}\tag{1.9}$$

It is this Lagrangian which describes *Quantum Electrodynamics*.

Applying the Euler-Lagrange equation to A_μ in (1.9) leads directly to the inhomogeneous Maxwell equations:

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi = ej^\nu\tag{1.10}$$

where j^ν denotes a current density (the conserved Dirac vector current).

We can also use the previous results to write down a gauge invariant form of the Schrödinger equation, the standard equation which describes the non-relativistic dynamics of a charged particle. From the classical Hamiltonian for a particle with charge in an electromagnetic field² we have

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 - eV\tag{1.11}$$

which allows us to write down the standard Schrödinger equation as

$$\left(\frac{1}{2m}(-i\vec{\nabla} + e\vec{A})^2 - eV\right)\psi = i\frac{\partial\psi}{\partial t}\tag{1.12}$$

which can be rewritten as

$$\frac{1}{2m}\vec{D}^2\psi = iD^0\psi\tag{1.13}$$

where D^μ is defined as before (with $A^0 = V$). This is a manifestly gauge invariant Schrödinger equation (this can be easily verified by plugging in the gauge transformed fields).

¹We make use of the standard notation, $\gamma^\mu\partial_\mu \equiv \rlap{\not{D}}$

²Note that $\hbar = c = 1$ throughout.

Thus, we have experienced the necessary appearance of a gauge field, A_μ , through the requirement for local gauge invariance. The importance of this cannot be overestimated. Despite the fact that in classical field theories only the field strengths (i.e. $F_{\mu\nu}$) are required, gauge fields are an integral part of quantum theory. In fact, as evidenced by the now famous Aharonov-Bohm effect, the existence of the gauge field is of *fundamental physical significance*.

Generalizing this discussion of local gauge symmetries naturally leads into the fascinating Yang-Mills theory. We generalize these previous concepts by defining a transformation analogous to (1.3), but under a *non-Abelian* symmetry group. This was first done in the 1950's, by C.N. Yang and R. Mills, who proposed that local phase invariance, as described above, can be generalized to local invariance under any continuous symmetry group. It is this powerful concept, encompassing the theoretical constructs of the Lie algebra and the Lie group³, that lies at the heart of the Yang-Mills theory. For clarity, as well as reasons of relevance to the rest of this paper, we will specialize to the case of an SU(2) (or O(3)) symmetry group. However, the following definitions easily generalize to other continuous symmetry groups.

To begin, consider a doublet of fields:

$$\psi = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix} \quad (1.14)$$

and then define an analogous, but more abstract, rotational transformation:

$$\psi \rightarrow \exp(i\alpha^a(x)\frac{\sigma^a}{2})\psi \quad (1.15)$$

³A *Lie group* is a continuously generated group, structured such that it contains elements arbitrarily close to the identity. Then, any general element may be reached by the continuous action of these infinitesimal elements. The Hermitean coefficients of the infinitesimal group parameters are called the generators of the group. The vector space spanned by the generators, with the operation of commutation, defines a *Lie algebra*.

(summation over repeated indices implied here and henceforth) where the σ^a are the familiar Pauli spin matrices, given by:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (1.16)$$

We then normalize our group generators, t^a , according to

$$\text{Tr } t^a t^b = \frac{1}{2} \delta^{ab} \quad (1.17)$$

which implies that our generators for the SU(2) Lie algebra will be $\frac{\sigma^1}{2}$, $\frac{\sigma^2}{2}$ and $\frac{\sigma^3}{2}$. Therefore any group element of SU(2) can be given by $e^{\frac{i}{2}(a\sigma^1+b\sigma^2+c\sigma^3)}$ for real parameters a , b , c . The finite number of generators is termed as *compact*, for the associated Lie group will be given by a finite-dimensional, compact manifold.

It is possible to construct a Lagrangian for a field theory with such a local gauge invariance as (1.15) – but the procedure will now have added difficulties due to the *three* orthogonal symmetry motions, which do not commute with one another. It is for this reason that we are said to have a *non-Abelian gauge field theory*.

Worth noting is the significance of Noether's theorem within these theories. In essence, Noether's theorem states that for each continuous symmetry of a Lagrangian, there will be a conserved quantity. For example, invariance under the transformation (1.2) is associated with the conservation of charge. It should therefore come as no surprise that invariance under (1.15) indicates conservation of charge density. At first, this would indicate that charged particles cannot move. However, as we know to be the case, charged particles *can* move (for example, electrons in quantum electrodynamics (QED)) – and this is due to the degeneracy of the QED vacuum (*non-invariance* under gauge transformations) [1].

As above, the normal derivative operator, $\partial_\mu \psi$, ceases to have any useful geometrical interpretation in theories with local gauge transformations and so instead we go on to

construct a much more meaningful *covariant derivative*, D_μ , suitable for a non-Abelian theory. The derivation of such an operator will not be included here, as it is similar in derivation to (1.6) and is shown in detail in countless field theory textbooks (see, for example, [2] or [3]) - instead we will simply quote the result for SU(2):

$$D_\mu = \partial_\mu + iA_\mu^i \frac{\sigma^i}{2} \quad (1.18)$$

It is most important to note that the existence of the covariant derivative necessarily implies the existence of three separate vector gauge fields, A_μ^i . In the language of differential geometry, these gauge fields which appear as the infinitesimal limits in comparisons of local symmetry transformations are called *connections*. We also define the non-Abelian field strength tensor in the following manner⁴:

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i - \epsilon^{ijk} A_\mu^j A_\nu^k \quad (1.19)$$

or, equivalently, for

$$A_\mu(x) = \sum_{a=1}^3 \frac{1}{2} \sigma^a A_\mu^a(x) \quad (1.20)$$

we have

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i[A_\mu(x), A_\nu(x)] \quad (1.21)$$

Unlike the Abelian case, this definition of the field strength is no longer a gauge-invariant quantity. However, it is possible to construct a gauge-invariant combination when designing a Lagrangian⁵:

$$\mathcal{L}_{\text{YM}} = \frac{1}{2e^2} \text{Tr} \left[(F_{\mu\nu}^i \frac{\sigma^i}{2})^2 \right] = \frac{1}{4e^2} (F_{\mu\nu}^i)^2 \quad (1.22)$$

⁴In the following equation, ϵ^{ijk} is the normal Levi-Cevita alternating symbol, a totally antisymmetric tensor. In general, for non-Abelian field theories, this is known as the *structure constant*. Structure constants f^{abc} are defined such that for t^a , the generators of some Lie group, $[t^a, t^b] = if^{abc}t^c$. In the particular case of SU(2) the structure constant happens to be ϵ^{ijk} .

⁵From now on, unless specified otherwise, we will be formulating our field theories in *Euclidean space*. The physical (local) world in which we live is described by the metric of *Minkowski space* but Euclidean theories offer several advantages over Minkowski formulations. Symbolically, this means that we consider matrix elements of $e^{-\beta H}$ instead of e^{-iHt} (where H is the Hamiltonian). This Euclidean formulation generally offers a more well-defined meaning to path integrals and correlators. It also makes...

where the constant factor e is to be identified as a coupling constant. This is the famous Yang-Mills Lagrangian. It contains both cubic and quartic terms in the gauge field and thus describes a nontrivial, interacting field theory – a wealth of physics. The action is then defined as

$$S = \int d^3x \mathcal{L}_{\text{YM}}(x) \quad (1.23)$$

At this point it is appropriate to remember that we are considering three (“2+1”) spacetime dimensions. As will be noted later, most of the results derived herein may be reinterpreted within a four dimensional theory – but with some added topological complications and a loss of pedagogical clarity that accompanies the inability to visualize higher dimensions. It is for these reasons that we restrict ourselves to three dimensions. Nevertheless, all of the material presented up to this point, and much to follow, applies equally well to four dimensions. However, as some of the later, and most important, calculations in this paper are specifically tailored to three dimensions, we specialize now to avoid confusion. Considering this, summations over indices (both Greek and Latin) will imply sums over 1,2,3 solely (we choose 1,2,3 – not 0,1,2 – so that the index choices will correspond to the chosen labelling of the Pauli sigma matrices in (1.16)).

The action, (1.23), can be shown to be stationary (that is, an extremal point wherein the variation vanishes – cf. Hamilton’s Principle [4]) on those configurations which obey the *Yang-Mills field equation*:

$$\partial_\mu F_{\mu\nu}^a(x) + \epsilon^{abc} A_\mu^b(x) F_{\mu\nu}^c(x) = 0 \quad (1.24)$$

more explicit the deep connection between quantum field theory and the statistical mechanics of critical systems (**Euclidean formulations also offer the technical ease of not differentiating between lowered and raised indices on tensors**). By analytically continuing the time parameter into the complex plane it is possible to rotate vectors (a *Wick rotation*) from Minkowski space into Euclidean space (and vice versa) thereby describing physics by a Euclidean metric, returning to Minkowski space whenever convenient.

Using antisymmetric properties, it is also possible to show:

$$\epsilon^{\mu\nu\lambda}(D_\mu F_{\nu\lambda})^a = 0 \quad (1.25)$$

This is a succinct version of the *Bianchi identity*. In an expanded form it reads:

$$\partial_\mu F_{\nu\lambda}^a(x) + \epsilon^{abc} A_\mu^b(x) F_{\nu\lambda}^c(x) + (\text{permutations}) = 0 \quad (1.26)$$

where the permutations are all symmetric permutations of the indices μ, ν, λ . This Bianchi identity is the analogue of the homogeneous Maxwell equations in electrodynamics. Note that both (1.24) and (1.26) are highly non-trivial, non-linear differential equations, even in this case without sources.

Most notably, the fact that non-Abelian (and thus Yang-Mills) fields contain non-Abelian degrees of freedom means that the gauge fields will necessarily interact with themselves. Thus even a pure gauge field theory (no sources or matter fields) will have non-trivial interactions and is not a free theory. It is for exactly this reason that the mediating bosons of the electromagnetic force – photons – are not self-interacting, whereas gluons, the mediating bosons of the non-Abelian theory of Quantum Chromodynamics (the “Strong Force”), do interact with one another and themselves.

Also, it is worth noting that since the field equation (1.24) is gauge invariant, no solution to this equation will ever specify a unique $A_\mu(x)$. Therefore, proper solutions to the field equation will be equivalence classes of $A_\mu(x)$ where two configurations belong to the same equivalence class if and only if they are related by a gauge transformation of the type (1.15).

1.2 The Wilson Loop and the Transformation of the Gauge Field

In order to understand the construction of the Wilson loop operator, one must understand how to make meaningful comparisons between different spacetime points in a theory with

local symmetries.

In order to deal effectively with the geometry of gauge invariance, one must construct a method for dealing with the different gauge transformations occurring at different points in spacetime. To do this we will construct a *comparator*, $U(y, x)$, which will convert the transformation law at point x to that at point y . That is,

$$U(y, x) \rightarrow V(y)U(y, x)V^\dagger(x) \quad (1.27)$$

for an arbitrary gauge transformation, $V(x)$. Again, we will now specialize to the non-Abelian (SU(2)) case as we construct our comparator, though the following arguments would apply equally well to an arbitrary gauge group.

We require that our comparator satisfy the following equation, for a path joining x and y parameterized by τ ($\tau \in [0, 1]$):

$$\frac{\partial}{\partial \tau} U(\tau) = i \frac{dx_\mu(\tau)}{d\tau} A_\mu(x(\tau)) U(\tau) \quad (1.28)$$

with the boundary condition that $U(0) = 1$. This differential equation is solved by iterative methods using

$$\begin{aligned} U(\tau) &= 1 + \int_0^\tau d\tau' \frac{dU(\tau')}{d\tau'} = 1 + i \int_0^\tau A(\tau') U(\tau') d\tau' \\ &= 1 + i \int_0^\tau A(\tau') d\tau' + (i)^2 \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' A(\tau') A(\tau'') + \dots \end{aligned}$$

Where we have defined $A(\tau) = \frac{dx^\mu(\tau)}{d\tau} A_\mu(x(\tau))$. For an infinitesimal separation, this turns out to be

$$U(x + \epsilon n, x) = 1 + i \epsilon n^\mu A_\mu^i(x) \frac{\sigma^i}{2} + \mathcal{O}(\epsilon^2) \quad (1.29)$$

Actually, if we make use of the identity:

$$\begin{aligned} V(x + \epsilon n, x) V^\dagger(x) &= \left[(1 + \epsilon n^\mu \frac{\partial}{\partial x^\mu} + \mathcal{O}(\epsilon^2)) V(x) \right] V^\dagger(x) \\ &= 1 + \epsilon n^\mu V(x) \left(-\frac{\partial}{\partial x^\mu} V^\dagger(x) \right) + \mathcal{O}(\epsilon^2) \end{aligned}$$

then the substitution of (1.29) into (1.27) yields the transformation law for the non-Abelian gauge field components. This law turns out to be:

$$A_\mu^i(x) \frac{\sigma^i}{2} \rightarrow V(x) \left(A_\mu^i(x) \frac{\sigma^i}{2} - \frac{i}{e} \partial_\mu \right) V^\dagger(x) \quad (1.30)$$

Calculation of the derivative, $\partial_\mu V^\dagger(x)$ is generally non-trivial, due to the existence of non-Abelian components, and so the exponent will not necessarily commute with its derivative.

Integrating (1.29) along a path P of finite length, yields

$$\mathcal{P} \left\{ \exp(-i \int_P d\tau \frac{dx^\mu}{d\tau} A_\mu^i(x(\tau)) \frac{\sigma^i}{2}) \right\} \quad (1.31)$$

where we have introduced the symbol \mathcal{P} to denote *path-ordering*. Path-ordering becomes necessary because, in general, $A(\tau_1)A(\tau_2) \dots A(\tau_n)$ will not be a symmetric function of the τ_i because the A 's are matrix-valued and will not commute. Therefore, in order to have a physically meaningful and well-defined quantity, we path-order the exponential, according to the following definition. Firstly, parameterize the path such that

$$x_\mu = x_\mu(\tau) \text{ such that } \tau \in [0, 1]$$

then we define (1.31) as the power-series expansion of the exponential, with each term ordered such that higher values of our path parameter τ stand to the left. For example,

$$\mathcal{P} (A(\tau_1) \dots A(\tau_n)) = A(\tau_{i1}) A(\tau_{i2}) \dots A(\tau_{in})$$

such that

$$\tau_{i1} \geq \tau_{i2} \geq \dots \geq \tau_{in}$$

This is path-ordering.

Lastly, the quantity (1.31) is not yet gauge-invariant:

$$\mathcal{P} e^{-i \int_P d\tau \frac{dx^\mu}{d\tau} A_\mu^i(x(\tau)) \frac{\sigma^i}{2}} \rightarrow V \left(\mathcal{P} e^{-i \int_P d\tau \frac{dx^\mu}{d\tau} A_\mu^i(x(\tau)) \frac{\sigma^i}{2}} \right) V^\dagger \quad (1.32)$$

where the gauge matrices, V and V^\dagger , are evaluated on the final and initial points of the path, respectively, and are therefore different in general. However, we may define a gauge invariant quantity by choosing a *closed* path, and by taking the trace of the resulting quantity (due to the cyclic invariance of the trace operator). Therefore, choose P to be a closed loop Γ to get the following gauge-invariant quantity:

$$W[\Gamma] = \text{Tr } \mathcal{P} \left\{ \exp(-i \oint_{\Gamma} d\tau \frac{dx^\mu}{d\tau} A_\mu^i(x(\tau)) \frac{\sigma^i}{2}) \right\} \quad (1.33)$$

This is called the *Wilson loop*.

In a non-Abelian theory it is trivial to rewrite the Wilson loop, using Stokes' Theorem, as a surface integral, where surface Σ spans the closed Wilson line Γ ($\partial\Sigma = \Gamma$):

$$W[\Gamma]_{Abelian} = \exp(-i \frac{1}{2} \int_{\Sigma} d\sigma^{\mu\nu} F_{\mu\nu}) \quad (1.34)$$

The application of a *non-Abelian Stokes' Theorem* to $W[\Gamma]$ will be one of several results derived in the following chapter – a crucial step in this project.

1.3 Quantizing the Yang-Mills Field

In quantum Yang-Mills theory, unlike the classical version, $A_\mu(x)$ is considered as a *random variable*, the distribution of which is given by a *functional integral* (see below) over all classical field configurations. Each of these field configurations is weighted by a factor, in *Euclidean space*, of $e^{-S[A_\mu]}$ where $S[A_\mu]$ is the action of the Yang-Mills field, (1.23).

Therefore, the expectation value of any quantity $F[A_\mu]$ will be given by

$$\begin{aligned} \langle F[A] \rangle &= \frac{\int \prod_{x,a,\mu} [dA_\mu^a(x)] e^{-S[A]} F[A]}{\int \prod_{x,a,\mu} [dA_\mu^a(x)] e^{-S[A]}} \\ &= \frac{\int \mathcal{D}A e^{-S[A]} F[A]}{\int \mathcal{D}A e^{-S[A]}} \end{aligned} \quad (1.35)$$

A functional integral, of the type given above, $\int \mathcal{D}A$, is simply a method of denoting “a sum over all gauge field configurations A_μ^a ”. This is a very natural (though somewhat

ill-defined) extension of calculus into the space of functions. Since all possible gauge fields are thus included, the measure of the integral, $\mathcal{D}A$, is manifestly gauge invariant.

It is precisely because of this gauge invariance that one can show that correlators of gauge fields, $\langle A_\mu(x)A_\nu(y)\dots\rangle$, with separated spacetime points ($x \neq y$, etc.) will always equal zero. For example, consider a gauge transformation in the neighbourhood of x . Then

$$\langle A_\mu(x)\dots\rangle = \langle V(x)(A_\mu(x) + \frac{i}{e}\partial_\mu)V^\dagger(x)\dots\rangle$$

for *any* allowable gauge transformation $V(x)$. This can only be true if $\langle A_\mu(x)\dots\rangle = 0$. This is quite a powerful and far reaching statement – so long as we are dealing with correlations between *separated* points.

On the other hand, it is this gauge invariance which causes some serious difficulties. This is because the functional integral will redundantly integrate over a continuous infinity of equivalent field configurations (since, for example, in a U(1) gauge theory there are an infinite number of gauge fields $A_\mu(x) = \frac{1}{e}\partial_\mu\alpha(x)$, for an arbitrary scalar function $\alpha(x)$, all of which are physically equivalent to a gauge field of $A_\mu(x) = 0$). The solution to this problem lies in a procedure known as gauge-fixing - see, for example, the clear explanation of the Fadeev-Popov gauge-fixing method in [2], or the discussion of this in Section 5.4.1. A more complete explanation of the method used to calculate infinite-dimensional functional integrals is given in Appendix B.

Chapter 2

The Wilson Loop in a Yang-Mills Field Theory

2.1 The Problem of Confinement

The great success of the quark model, despite the failure of all attempts to detect a free quark, has led to a concerted search, now spanning decades, for a workable theory of quark confinement. This hypothesis of quark confinement proposes that hadrons are composed of *elementary particles called quarks*. The strong interactions (the theory of which is called *Quantum Chromodynamics*, or QCD) are then such that they permanently confine all quarks within bound states – hence the appearance of mesons and baryons. In QCD this statement of the confinement process is given in a more specific way: that the theory has a local gauge symmetry, called *colour*, as well as a global colour symmetry. Then quark confinement is given by the statement that all physical states must be colour singlets.

Though significant progress has been made, a complete theory of confinement in the continuous limit remains an ongoing challenge in field theoretical physics. Such a theory will explain the phenomenon that all colour-charged particles remain permanently confined within colour-neutral bound states due to some linearly increasing potential between colour sources. The search has now led to massive interest in both non-Abelian gauge fields, and string theory.

Much recent work has focused upon the conjecture that there exists an exact duality between confining gauge fields and strings, and results have followed from an analysis of

the loop space of these theories. That is, one considers the properties of the Wilson loop, a locally invariant, non-trivial function of the gauge field $A_\mu^a(\vec{x})$ which is given by:

$$W[\Gamma] = \text{Tr } \mathcal{P} \left\{ \exp \left(-i \int_0^1 d\tau \frac{dx^\mu}{d\tau} A_\mu^i(x(\tau)) \frac{\sigma^i}{2} \right) \right\} \quad (2.1)$$

where it is an important property that the Wilson loop depends upon the choice of the closed path Γ . Essentially, this quantity is related to the accumulation of non-Abelian phase by the wavefunction of a heavy quark when transported around a loop in spacetime. Equivalently, this can be viewed as the creation of a quark-antiquark pair, their spatial separation, and then their mutual annihilation after some length of time. Again, we parameterize the path of the quarks as:

$$x_\mu(\tau) \text{ such that } \tau \in [0, 1], \quad x_\mu(0) = x_\mu(1) \quad (2.2)$$

Using a scheme like this, in the 1970's, K. Wilson demonstrated that, in a lattice theory, an appropriate Lagrangian may be constructed out of elementary Wilson loops around the lattice sites [5, 6] – the difficulty has been in taking such a scheme from the lattice to the continuous limit.¹

Gauge interactions are classified by their asymptotic behaviour, as exhibited by large Wilson loops. For example, consider the creation of a quark-antiquark pair, their separation by a distance R (where R is large compared to any other mass scales in the theory) for a long time T , and then their subsequent annihilation. Then the free energy of this Wilson loop should be of the form $TV(R)$ for some function $V(R)$. We define this as a confining potential when the leading term in $V(R)$ is proportional to R . This yields the following *area law*:

$$\lim_{A \rightarrow \infty} (-\ln \langle W[\Gamma] \rangle) \sim \sigma_T A \quad (2.3)$$

¹Some effort has been made in reformulating Yang-Mills theory in terms of Wilson loops. The derived equations of motion are called the *Midgal-Makeenko equations*. However, little significant progress has yet been made in this direction.

where A is the area spanned by the Wilson loop and σ_T is a quantity known as the *string tension*. Such behaviour indicates permanent confinement of quarks into colour-singlet bound states, as any configurations of colour sources with large separations will be suppressed in the Euclidean functional integral by factors of the form $e^{-\sigma_T A}$ [2].

This type of behaviour can be contrasted with the non-confining phase, wherein one expects $V(R)$ to be a constant followed by other subleading terms. Then the constant is interpreted as the self-energy of the quarks, with the subleading terms indicating short-range interactions. This is known as *perimeter law* behaviour:

$$\lim_{A \rightarrow \infty} (-\ln \langle W[\Gamma] \rangle) \sim fP \quad (2.4)$$

where P is the perimeter of the Wilson loop and f is the free energy of a heavy quark.

The type of strong coupling behaviour as given by (2.3) can be exhibited by both Abelian and non-Abelian gauge theories – but in the case of non-Abelian theories we also have the property of *asymptotic freedom*, where a theory with weak coupling at small spatial scales flows into a strong coupling regime at large distance scales. And it is this sort of area law suppression of the Wilson loop (2.3) at large distances that has led to a search for an equivalent string theory. This is a very intuitive sort of reaction, for the area of the loop can be imagined to be comprised of lines of colour electric flux, swept out through time [7]. At a given point in time, this one-dimensional excitation is termed a *string* (hence the use of the term “string tension” in (2.3)).

If the Yang-Mills theory is truly equivalent to a string theory, then there should exist an effective string action, $S_{\text{eff}}[X_\mu]$ (where $X_\mu = X_\mu(\vec{\sigma})$ is an embedding function of the world sheet of the string, mapping from the disc (parameterized by $\vec{\sigma}$) to the surface D where $\Gamma = \partial D$). Then the expectation value of the Wilson loop, $\langle W[\Gamma] \rangle$, can be calculated as:

$$\begin{aligned}
\langle W_\Gamma \rangle &= \frac{\int [dA] \text{Tr} (\mathcal{P} e^{i \oint_\Gamma A}) e^{-S_{\text{YM}}[A]}}{\int [dA] e^{-S_{\text{YM}}[A]}} \\
&= \int [dX_\mu] e^{-S_{\text{eff}}[X_\mu]}
\end{aligned}$$

Unfortunately, the quantum properties of strings are extremely complicated and much work has yet to be done.

More recently, A.M. Polyakov investigated the connection between such fields and strings and determined that the connection between the two is intimately linked with a summation over all surfaces spanning a given Wilson loop in the field theory (where the loop integral has been converted to a surface integral via a non-Abelian Stokes theorem – this will be discussed in detail later in this paper) [8]. Even on an intuitive level, one may see how this could relate directly to a string functional integral, given the previously mentioned connection between the area of the Wilson loop and string-like lines of colour electric flux. More specifically, Polyakov raised the conjecture that the Bianchi Identity, which specifies how one constructs the field strength tensor $F_{\mu\nu}$ out of a gauge field A_μ , may be replaced by such a summation over surfaces.

We investigate a reformulation of the Bianchi Identity, in the non-Abelian Yang-Mills theory, through a semi-classical calculation of the expectation value of the Wilson loop, $\langle W[\Gamma] \rangle$. This is given by:

$$\langle W[\Gamma] \rangle = \frac{\int \mathcal{D}A_\mu(\vec{x}) e^{-S} \text{Tr} \mathcal{P} e^{i \oint_\Gamma A_\mu \cdot d\ell}}{\int \mathcal{D}A_\mu(\vec{x}) e^{-S}} \quad (2.5)$$

where S is the action of Yang-Mills field,

$$S = \int d^3x \frac{1}{2e^2} \text{Tr}(F_{\mu\nu})^2 \quad (2.6)$$

with $F_{\mu\nu}(x)$ as given by (1.21).

As a manifestly gauge invariant, and arguably fundamental quantity, the Wilson loop is a very natural object for which one would want to calculate the expectation value in the analysis of a field theory. However, the challenge remains to solve the expression for $\langle W[\Gamma] \rangle$. Though a semi-classical evaluation of this has been completed in the case of three dimensional compact $U(1)$ gauge theories [1, 9], this has yet to be done in the case of a Yang-Mills theory. Difficulties arise – namely the unravelling of the path ordering in this non-Abelian environment. In this paper we complete exactly such a calculation, using a saddle-point approximation in the semi-classical limit (large spin parameter j), through the use of a geometrical quantization scheme in a coherent state formulation. In the following pages these steps are outlined in detail.

2.2 Towards Solving the Wilson Loop Expression

In solving for the Wilson loop we will specialize to a $SU(2)$ spin algebra and reformulate the theory in a coherent state representation. Next, we unravel the path ordering of the loop integral via geometric quantization (with the unfortunate expense of introducing a new path integral), and thus arrive at a Cp^1 model. From there we use a Hopf map to move to a standard sigma model representation, where we will use a non-Abelian Stokes theorem to express all loop integrals as area integrals. In doing so, we rederive some of the earlier results of Diakonov and Petrov [10].² At this point the functional integral will be in a form where we may both test Polyakov's conjecture relating the Bianchi identity to a sum over surfaces, as well as solve for the equations of motion in the semi-classical limit.

²An earlier version of a non-Abelian Stokes' theorem was proposed by Simonov [11].

2.2.1 A Coherent State Representation

We begin by considering a standard $SU(2)$ spin algebra – take two copies of the annihilation and creation operators:

$$[a_\alpha^\dagger, a_\beta] = \delta_{\alpha\beta} \quad (2.7)$$

where $\alpha, \beta = 1, 2$. Then we define our coherent states $|z\rangle$ (eigenstates of the annihilation operator) as

$$|z\rangle = e^{a^\dagger z - \bar{z} a} |0\rangle \quad (2.8)$$

$$a \equiv \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad a^\dagger \equiv [a_1^\dagger \quad a_2^\dagger] \quad (2.9)$$

and, for a complete set of orthonormal basis vectors,

$$\frac{(a_1^\dagger)^m}{\sqrt{m!}} \frac{(a_2^\dagger)^n}{\sqrt{n!}} |0\rangle \quad \text{for } m, n = 0, 1, 2, \dots$$

we project onto the $2j + 1$ dimensional subspace

$$a_1^\dagger a_1 + a_2^\dagger a_2 = 2j \quad (2.10)$$

for some integer j (there will be $2j + 1$ states such that $m + n = 2j$) (later, our semi-classical approximation will consist in allowing $j \rightarrow \infty$). With these definitions we can define a spin operator, $S^a = a^\dagger (\frac{1}{2} \sigma^a) a$. This definition provides a representation of the spin algebra:

$$[S^a, S^b] = i\epsilon^{abc} S^c \quad (2.11)$$

For example,

$$\begin{aligned} [S^1, S^2] &= \frac{1}{4} ((a_1^\dagger a_2 + a_2^\dagger a_1)(ia_2^\dagger a_1 - ia_1^\dagger a_2) - (ia_2^\dagger a_1 - ia_1^\dagger a_2)(a_2^\dagger a_1 + a_1^\dagger a_2)) \\ &= \frac{i}{2} (a_1^\dagger a_1 - a_2^\dagger a_2) \\ &= i\epsilon^{123} S^3 \end{aligned}$$

and likewise for the other commutators. Also, the constraint (2.10) ensures that we maintain a definite spin j . That is, by simple algebra and the commutation relations given above,

$$\begin{aligned}\sum_{a=1}^3 S^a S^a &= \frac{1}{4}((a_1^\dagger a_1 + a_2^\dagger a_2)^2 + 2(a_1^\dagger a_1 + a_2^\dagger a_2)) \\ &= j(j+1)\end{aligned}\quad (2.12)$$

In fact, j is the parameter which determines the particular *representation* of $SU(2)$ being used. Next, we introduce gauge fields, using the notation:

$$A^a(t) = \sum_{\mu=1}^3 A_\mu^a(x(t)) \frac{dx_\mu(t)}{dt} \quad (2.13)$$

for the component of the gauge field tangential to the loop. For the time being, we can make the physical interpretation of a spin system, described by operator S^a , interacting with a time dependent, external “magnetic field”, $\vec{A}(t)$. Then define the Hamiltonian for such a system as

$$\begin{aligned}H &= S^a A^a(t) \\ &= \begin{bmatrix} a_1^\dagger & a_2^\dagger \end{bmatrix} \begin{bmatrix} \frac{1}{2} \sigma^a A^a(t) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\end{aligned}\quad (2.14)$$

In which case we can rewrite our loop integral as

$$\text{Tr } \mathcal{T} e^{i \int_0^\tau dt H(t)} \quad (2.15)$$

where time-ordering, \mathcal{T} has replaced the previous path-ordering, \mathcal{P} (this is exactly the same sort of operation as path-ordering, only it specifically orders the factors in each term of the expansion such that factors with a higher *time* parameter stand to the left).

Recalling the well-known properties of the overcomplete³ coherent state representation (see Appendix A), particularly that

$$e^{i\theta a^\dagger} |z\rangle = |e^{i\theta} z\rangle \quad (2.16)$$

³It is easy to show that $\int dz dz^* |z\rangle \langle z| = \pi$. See Appendix A for details.

we can express the trace of an operator in the coherent state representation, with an integral representation of the delta function $\delta(2j - a^\dagger a)$ that enforces (2.10) as

$$\text{Tr}(\Theta) = \int \frac{d\lambda d^2 z}{2\pi^2} e^{2ij\lambda} \langle z | \Theta | e^{-i\lambda} z \rangle \quad (2.17)$$

where λ is integrated over $[-\pi, \pi]$ and z extends over the complex plane. We now apply this formulation to our Wilson loop integral.

We discretize time, our domain of integration, in N slices ($N \rightarrow \infty$) such that

$$\mathcal{T} e^{i \int_0^\tau dt H(t)} = \prod_{k=1}^{N=\tau/\epsilon} e^{i\epsilon H(t_k)} \quad (2.18)$$

and therefore

$$\text{Tr} \mathcal{T} e^{i \int_0^\tau dt H(t)} = \int \left(\prod_{m=1}^N \frac{d\lambda_m d^2 z_m}{2\pi^2} \right) \left(\prod_{n=1}^N e^{2ij\lambda_n} \langle z_{n+1} | e^{i\epsilon H(t_n)} | e^{-i\lambda_n} z_n \rangle \right) \quad (2.19)$$

where $z_{N+1} = z_1$ since $\vec{x}(0) = \vec{x}(\tau)$. This discretized path integral (2.19) may be simplified by expanding $\langle z_{n+1} | e^{i\epsilon H(t_n)} | z_n \rangle$ to first order in ϵ , assuming $z_{n+1} \cong z_n + \epsilon \dot{z}_n$ (using the standard notation of a dot, $\dot{}$, to indicate a time derivative):

$$\langle z_{n+1} | e^{i\epsilon H(t_n)} | e^{-i\lambda_n} z_n \rangle \cong e^{-\frac{1}{2} \bar{z}_{n+1} z_{n+1} - \frac{1}{2} \bar{z}_n z_n + e^{-i\lambda_n} \bar{z}_{n+1} z_n + i\epsilon H(\bar{z}_{n+1}, e^{-i\lambda_n} z_n, t_n)} \quad (2.20)$$

$$\cong e^{\epsilon \{-\bar{z}_n \dot{z}_n - i\lambda \bar{z}_n z_n + iH(\bar{z}_n, z_n, t_n)\}} \quad (2.21)$$

Where we have also used the standard coherent state result (see Appendix A) that

$$\langle z' | z \rangle = e^{\bar{z}' z - \frac{1}{2}(\bar{z} z + \bar{z}' z')} \quad (2.22)$$

Also, rescale $\lambda_n : \lambda_n \rightarrow \epsilon \lambda_n$ so that the domain of λ is now $[-\pi/\epsilon, \pi/\epsilon]$. Then we are left with a continuous integral, with no more explicit time/path ordering:

$$\begin{aligned} \text{Tr} \mathcal{T} e^{i \int_0^\tau dt H(t)} &= \int \left(\prod_{m=1}^N \frac{d\lambda_m d^2 z_m}{2\pi^2} \right) e^{(\sum_{n=1}^N \epsilon \{2ij\lambda_n - \bar{z}_n \dot{z}_n - i\lambda_n \bar{z}_n z_n + iH(z_n, t_n)\})} \\ &= \int [d\lambda][d^2 z] e^{\int_0^\tau dt \{2ij\lambda - \bar{z} \dot{z} - i\lambda \bar{z} z + iH(z, t)\}} \end{aligned} \quad (2.23)$$

where, again, $z(\tau) = z(0)$, z is integrated over the complex plane, and now λ is integrated over the real line (due to the rescaling by ϵ).

Now do the integral in λ :

$$\begin{aligned} \int [d\lambda] [d^2 z] e^{\int_0^\tau dt \{2ij\lambda - \bar{z}\dot{z} - i\lambda\bar{z}z + iH\}} &= \int [d^2 z] \delta(\bar{z}z - 2j) e^{\int_0^\tau dt \{-\bar{z}\dot{z} + iH(z,t)\}} \\ &= \int [d^2 z] \delta(\bar{z}z - 1) e^{2ij \int_0^\tau dt \bar{z}(t)(i\partial_t + \frac{1}{2}\vec{\sigma} \cdot \vec{A}(t))z(t)} \end{aligned} \quad (2.24)$$

where, in the second line, we have rescaled z and have made use of the definition of our Hamiltonian (2.14). We also require the periodic boundary condition that $z(\tau) = z(0)$.

Note that so long as we require that $\chi(\tau) = \chi(0) + 2\pi N$, $N \in \mathbb{Z}$, and, of course, that $\bar{z}z = 1$, (2.24) has the simple gauge invariance:

$$z \rightarrow e^{i\chi(t)} z \quad \bar{z} \rightarrow \bar{z} e^{-i\chi(t)} \quad (2.25)$$

This is known as the Cp^1 model.

2.2.2 Introducing a Sigma Model

We now introduce the sigma model variable, $\vec{n} \in S^2$, and use it to replace our loop variable z . This is necessary because $z \in S^3$ clearly has too many degrees of freedom, as reflected by the gauge invariance (2.25). To do this we make use of the so-called Hopf map:

$$n^a = \bar{z} \sigma^a z \quad (2.26)$$

So that in spherical coordinates,

$$\vec{n} = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta) \quad (2.27)$$

$$z = e^{i\chi} \left(\cos\left(\frac{\theta}{2}\right) e^{-i\phi}, \sin\left(\frac{\theta}{2}\right) \right) \quad (2.28)$$

$$\bar{z} dz = i d\chi - i \frac{1}{2} (1 + \cos\theta) d\phi \quad (2.29)$$

$$\epsilon^{abc} n^a dn^b dn^c = 2 \sin\theta d\theta d\phi \quad (2.30)$$

Note, of course, that $\|\vec{n}\| = 1$, just as $\bar{z}z = 1$ before. This \vec{n} variable, which takes values over the surface (including the boundary) characterizes the instantaneous direction in colour space. Now in order to rewrite our path integral (2.24) in terms of \vec{n} , we define the traceless Hermitean matrix

$$n = \sum_a n^a \sigma^a. \quad (2.31)$$

where, immediately above, the n^a are the real scalar components of \vec{n} as given in (2.27). Then,

$$\text{Tr}(ndndn) = -4id(\cos\theta)d\phi \quad (2.32)$$

and therefore we have

$$\frac{1}{8} \int_D \text{Tr}(ndndn) = -\frac{i}{2} \oint_{\delta D} (1 + \cos\theta)d\phi \quad (2.33)$$

Let us try to reformulate our original path integral, expressed in terms of z and \bar{z} , in terms of n : From the above, we immediately notice that:

$$\begin{aligned} -2j \int d\tau \bar{z} \dot{z} &= -2j \oint_{\delta D} \bar{z} dz \\ &= -2j \oint_{\delta D} (id\chi - \frac{i}{2}(1 + \cos\theta)d\phi) \\ &= -\frac{j}{4} \int_D \text{Tr}(ndndn) + 2\pi i \cdot \text{integers} \end{aligned} \quad (2.34)$$

and also that since

$$\text{Tr} \frac{\sigma^a}{2} \frac{\sigma^b}{2} = \frac{1}{2} \delta^{ab} \quad (2.35)$$

we have

$$\text{Tr}(nA_\mu) = n^a A_\mu^a \quad (2.36)$$

Therefore, in terms of our new variable n (2.24) becomes

$$\int [dn] \delta(n^2 - 1) e^{-\frac{i}{4} \int_D \text{Tr}(ndndn) + ij \oint_{\delta D} d\ell \cdot \text{Tr}(nA)} \quad (2.37)$$

It is worth noting that the first term in the exponentiated action, above, is essentially a Witten form of the Wess-Zumino action. The Wess-Zumino action originates from an investigation into a non-linear σ -model, in an attempt to find a non-trivial infrared fixed point. Witten added a topological term, of the general type shown above, which suggested the existence of such a critical point in the theory. This is tangential to our discussion here, and is more fully developed in [12], however, in [10] and [13] it is pointed out that it is this Wess-Zumino type term which fixes the representation to which the spin of the probe quark in the Wilson loop belongs. As mentioned earlier, \vec{n} determines the instantaneous direction of the colour spin in colour space. But multiplying \vec{n} by j does not immediately guarantee a true quantum state in the j representation. The Wess-Zumino term is necessary to ensure this – and in our derivation it has arisen naturally.

Next, let $A \rightarrow iA$ so that our field is now *anti*-Hermitian. Also, rewrite our action, S , in the following manner:

$$S = \frac{j}{4} \int_D \text{Tr}(ndndn) + j \oint_{\partial D} \text{Tr}(nA) \quad (2.38)$$

$$= \frac{j}{4} \int_D \text{Tr}(ndndn + 4d(nA)) \quad (2.39)$$

Above, and subsequently, we find it useful to incorporate the terminology and notation of *differential forms*. We therefore define, from now on, that for $A_\mu = A_\mu^a \frac{\sigma^a}{2}$ and $F_{\mu\nu} = F_{\mu\nu}^a \frac{\sigma^a}{2}$ we take

$$A = A_\mu dx_\mu \quad (2.40)$$

$$F = F_{\mu\nu} dx_\mu \wedge dx_\nu \quad (2.41)$$

where the wedge product, \wedge , is defined to be an antisymmetrized tensor product. For example, $dx^1 \wedge dx^2 = dx^1 \otimes dx^2 - dx^2 \otimes dx^1$. This implies

$$F = 2dA + 2A^2 \quad (2.42)$$

$$Dn = dn + [A, n] \quad (2.43)$$

where we define $d = dx_\mu \partial_\mu$

Returning to (2.39), using these definitions and adding and subtracting terms of the form $[A, n]$, we can write our final formulation for the Wilson loop in this Yang-Mills gauge field, having applied, what is in effect, a non-Abelian Stokes' Theorem [10]:

$$\text{Tr}(\mathcal{P} e^{i \oint A \cdot d\ell}) = \int [dn] \delta(n^2 - 1) e^{-\frac{i}{4} \int_D \text{Tr}(ndndn) + ij \oint_{\partial D} \text{Tr}(nA) \cdot d\ell} \quad (2.44)$$

$$= \int [dn] \delta(n^2 - 1) e^{-\frac{i}{4} \int_D \text{Tr}(nDnDn + 2nF)} \quad (2.45)$$

In the above formulae we have ignored the denominator in the expression for the Wilson loop. This is satisfactory, and we will continue to do this, because it plays only a normalization role and is thus unimportant for our investigation (path integrals are always understood to have an arbitrary normalization in any case). From this expression it is possible to extract an expression for a hidden “Abelian-type” gauge-invariant field strength [10]:

$$G_{\mu\nu} = F_{\mu\nu}^a n^a - \epsilon^{abc} n^a (D_\mu n)^b (D_\nu n)^c \quad (2.46)$$

which coincides with the expression derived in the work by Polyakov and 't Hooft in the mid 1970's where \vec{n} played the role of the direction of the elementary Higgs field. The action is gauge invariant so long as n transforms under a gauge transformation $V(x)$ as

$$n \rightarrow V(x)nV^\dagger(x) \quad (2.47)$$

(and $F_{\mu\nu}(x) \rightarrow V(x)F_{\mu\nu}(x)V^\dagger(x)$, etc., as before).

Among other things, we have managed to derive an expression for the expectation value of the Wilson loop which allows us to consider all spin representations (as labelled by j) at once. This, in and of itself, is a very useful achievement. Also, and importantly, at large values of j the integral over n will be completely dominated by the saddle points

of the action (see the forthcoming discussion, in Chapter 4, on *Variational Principles and the Method of Steepest Descent*).

Chapter 3

Solving the Coherent State Path Integral in a Constant Gauge Field

Before going on to derive the Bianchi Identity via a sum over surfaces, we attempt to calculate the path integral (2.24), a coherent state formulation of the Wilson loop integral:

$$\int [d^2 z] \delta(z\bar{z} - 1) \exp(ij \int_0^\tau dt \bar{z} (i\partial_\tau + \frac{1}{2} \vec{\sigma} \cdot \vec{A}) z) \quad (3.1)$$

where we choose, for simplicity, our gauge field (which, in our previous notation, means $\vec{A}(t)$) to be a constant (of value A) along the 3^{rd} direction. This non-trivial calculation is useful in several respects. Firstly, it will justify our confidence in the coherent state formulation of the integral if the calculation produces a sensible answer. Secondly it provides a demonstration of the power of the path integral formalism, along with the somewhat technical difficulties associated with path integrals. Thirdly, it may stand as a model for calculations of this path integral in cases where the gauge field is not necessarily a constant.

We have:

$$\int dz d\bar{z} \delta(\bar{z}z - 1) e^{ij \int_0^\tau dt (\bar{z}_1 (i\partial_\tau + \frac{1}{2}A) z_1 + \bar{z}_2 (i\partial_\tau - \frac{1}{2}A) z_2)} \quad (3.2)$$

$$= \int dz d\bar{z} \int_{-\pi}^\pi \frac{d\lambda}{2\pi} e^{i \int_0^\tau dt (\frac{\lambda}{\tau} (\bar{z}z - 1) + j(\bar{z}_1 (i\partial_\tau + \frac{1}{2}A) z_1 + \bar{z}_2 (i\partial_\tau - \frac{1}{2}A) z_2))} \quad (3.3)$$

$$= \int_{-\pi/\tau}^{\pi/\tau} \tau \frac{d\lambda}{2\pi} (const)^2 \det^{-1}(i\partial_\tau + \frac{1}{2}A + \lambda) \det^{-1}(i\partial_\tau - \frac{1}{2}A + \lambda) e^{-2ij\lambda\tau} \quad (3.4)$$

$$= (const)^2 \int_{-\pi/\tau}^{\pi/\tau} \frac{d\lambda}{2\pi} \prod_n (\frac{2\pi n}{\tau} + \frac{1}{2}A + \lambda)^{-1} \prod_m (\frac{2\pi m}{\tau} - \frac{1}{2}A + \lambda)^{-1} e^{-2ij\lambda\tau} \quad (3.5)$$

where we have calculated our functional determinants using Bose statistics¹ and imposed periodic boundary conditions. Also note the rescaling of λ by $\frac{1}{\tau}$ in (3.4). It remains to

¹For a brief explanation of the calculation of functional determinants, see Appendix B.

calculate these non-trivial determinants which have, above, been expressed as an infinite product of *eigenvalues*. We will do this by what we will call the “derivative method” whereby we effectively renormalize by containing the infinite parts of the product within two constants of integration. The renormalization then consists in simply dropping the presumably infinite parts of these two constants (that is, they are removed via the implicit subtraction of counterterms). This is an entirely legitimate prescription for renormalization so long as the constants are independent of physical parameters (other than τ). The calculation is outlined in detail below:

$$\prod_n \left(\frac{2\pi n}{\tau} + \frac{1}{2}A + \lambda \right)^{-1} = \exp\left(-\sum_n \ln\left(\frac{2\pi n}{\tau} + \frac{1}{2}A + \lambda\right)\right) \quad (3.6)$$

$$= \exp\left(\frac{1}{4} \int dA \int dA \sum_n \frac{1}{\left(\frac{2\pi n}{\tau} + \frac{1}{2}A + \lambda\right)^2}\right) \quad (3.7)$$

$$= \exp\left(\frac{\tau^2}{16} \int dA \int dA \csc^2\left(\frac{\tau\pi}{2\pi}\left(\frac{1}{2}A + \lambda\right)\right)\right) \quad (3.8)$$

$$= \exp\left(-\frac{\tau}{4} \int dA \int dA \frac{\partial}{\partial A} \left(\cot\left(\frac{\tau}{2}\left(\frac{1}{2}A + \lambda\right)\right) + c_1\right)\right) \quad (3.9)$$

$$= \exp\left(-\frac{\tau}{4} \ln\left(\sin\left(\frac{\tau}{2}\left(\frac{1}{2}A + \lambda\right)\right)\right) \frac{4}{\tau} - \frac{\tau}{4}(c_1A + c_2)\right) \quad (3.10)$$

$$= \sin^{-1}\left(\frac{\tau}{2}\left(\frac{1}{2}A + \lambda\right)\right) C_1 C_2 \quad (3.11)$$

where we have used some basic results from trigonometric calculus, and, in the third line, an infinite series representation of the $\csc^2(x)$ function². It is worthwhile noting that the constants of integration, C_1 and C_2 are of the specific form:

$$C_1 = e^{-\frac{\tau}{4}c_1A} \quad C_2 = e^{-\frac{\tau}{4}c_2}$$

Now note that this derivative method has removed one of the symmetries of our original determinant. In (3.7) we have the symmetry $(\frac{1}{2}A + \lambda) \rightarrow (\frac{1}{2}A + \lambda) + \frac{2\pi}{\tau}m$ for $m \in \mathbb{Z}$. But, by inspection, in (3.11) this symmetry has been lost. Clearly, what we are missing must be hidden in the integration constants. Multiplying (3.11) by $e^{-i(\frac{1}{2}A+\lambda)\frac{\tau}{2}}$

²Gradshteyn and Ryzhik [14] p.44: $\pi^2 \csc^2 \pi x = \sum_{k=-\infty}^{\infty} \frac{1}{(x-k)^2}$

restores this symmetry (and this is clearly allowable, given the form of $C_1 C_2$ above). Aside from this (and a factor of $e^{-i(-\frac{1}{2}A+\lambda)\frac{\tau}{2}}$ in the case of the second determinant) we otherwise drop any remaining parts of the constants of integration, thereby accomplishing our goal of renormalization.

Thus, putting all of these pieces together, our path integral can now be written as:

$$\begin{aligned} & \int [d^2 z] \delta(z\bar{z} - 1) \exp(ij \int_0^\tau dt \bar{z} (i\partial_\tau + \frac{1}{2}\vec{\sigma} \cdot \vec{A}) z) \\ &= \tau (const)^2 \int_{-\pi/\tau}^{\pi/\tau} \frac{d\lambda}{2\pi} \sin^{-1}\left(\frac{\tau}{2}\left(\frac{1}{2}A + \lambda\right)\right) \sin^{-1}\left(\frac{\tau}{2}\left(-\frac{1}{2}A + \lambda\right)\right) e^{-2ij\lambda\tau} e^{-i\tau\lambda} \end{aligned} \quad (3.12)$$

$$= \tau (const)^2 \int_{-\pi/\tau}^{\pi/\tau} \frac{d\lambda}{2\pi} \frac{2}{(\cos(\frac{1}{2}\tau A) - \cos(\tau\lambda))} e^{2ij\lambda\tau} e^{-i\tau\lambda} \quad (3.13)$$

$$= \tau \frac{(const)^2}{\pi} \int_{-\pi/\tau}^{\pi/\tau} d\lambda \frac{e^{-2i(j+\frac{1}{2})\lambda\tau}}{\cos(\frac{1}{2}\tau A) - \cos(\tau\lambda)} \quad (3.14)$$

$$= \frac{(const)^2}{\pi} \int_{-\pi}^{\pi} d\lambda \frac{e^{-2i(j+\frac{1}{2})\lambda}}{\cos(\frac{1}{2}\tau A) - \cos(\lambda)} \quad (3.15)$$

Where, in the last step, we have rescaled λ by a factor of τ . Integral 3.15 is a non-trivial integral which we must evaluate using a contour integration in the complex plane. We define our complex variable z :

$$z = e^{i\lambda} \quad (3.16)$$

Which implies:

$$\cos(\lambda) = \frac{z^2 + 1}{2z} \quad (3.17)$$

$$d\lambda = \frac{-idz}{z} \quad (3.18)$$

$$e^{-2i(j+\frac{1}{2})\lambda} = z^{-2(j+\frac{1}{2})} \quad (3.19)$$

We use the unit circle, centred at the origin, as our contour (we label this contour C below). Then our integral becomes:

$$\frac{-2i(const)^2}{\pi} \oint_C dz \frac{z^{-2(j+\frac{1}{2})}}{2z \cos(\frac{1}{2}\tau A) - z^2 - 1} \quad (3.20)$$

We note that the denominator of the integrand vanishes at two points – both poles lying upon our proposed contour.

$$z_{poles} = \cos\left(\frac{1}{2}\tau A\right) \pm \frac{1}{2}\sqrt{4\cos^2\left(\frac{1}{2}\tau A\right) - 4} \quad (3.21)$$

$$= \cos\left(\frac{1}{2}\tau A\right) \pm i\sin\left(\frac{1}{2}\tau A\right) \quad (3.22)$$

The residues of the poles in the integrand, $f(z)$, are then calculated using the standard formula for a second order pole:

$$\text{residue} = \frac{d}{dz} ((z - z_{pole})f(z)) \big|_{z=z_{pole}}$$

The residues for the two poles turn out to be:

$$\left. \frac{z^{2(j+\frac{1}{2})}}{2z - 2\cos(\frac{1}{2}\tau A)} \right|_{z=z_{poles}} = \frac{e^{\mp iA(j+\frac{1}{2})\tau}}{\pm 2i\sin(\frac{1}{2}\tau A)} \quad (3.23)$$

Finally, in order to evaluate the integral, we use a Cauchy principal value prescription to pick up the two poles lying upon the unit circle contour. This is described in most introductory texts on complex variable theory (see, for example, [15]). The result is that we pick up one half of the sum of the two poles (that is, the contour integral will be equal to $(\frac{1}{2}2\pi i \sum (\text{residues on the contour}))$).

Our final answer is:

$$\frac{-2i(const)^2}{\pi} \pi i \left(\frac{e^{-i\tau A(j+\frac{1}{2})} - e^{i\tau A(j+\frac{1}{2})}}{2i\sin(\frac{1}{2}\tau A)} \right) = -2(const)^2 \frac{\sin(\tau A(j + \frac{1}{2}))}{\sin(\frac{1}{2}\tau A)} \quad (3.24)$$

Despite the difficult path to achieve this result, it should not be a surprise. Up to a normalization constant (which will always be arbitrary for a path integral such as this),

it matches the simple spin sum one would expect³:

$$\begin{aligned}
\frac{1}{2j+1} \sum_{m=-j}^j e^{-imA\tau} &= \frac{1}{2j+1} \left(1 + 2 \left(\sum_{m=1}^j \cos(mA\tau) \right) \right) \\
&= \frac{1}{2j+1} \left(1 + 2 \left(-1 + \sum_{m=0}^j \cos(mA\tau) \right) \right) \\
&= \frac{1}{2j+1} \left(1 + 2 \left(-1 + \cos\left(\frac{1}{2}jA\tau\right) \frac{\sin\left(\frac{j+1}{2}A\tau\right)}{\sin\left(\frac{1}{2}A\tau\right)} \right) \right) \\
&= \frac{1}{2j+1} \left(-1 + \frac{(e^{\frac{1}{2}ijA\tau} + e^{-\frac{1}{2}ijA\tau}) \frac{1}{2i} (e^{i\frac{j+1}{2}A\tau} - e^{-i\frac{j+1}{2}A\tau})}{\sin\left(\frac{1}{2}A\tau\right)} \right) \\
&= \frac{1}{2j+1} \frac{\sin(\tau A(j + \frac{1}{2}))}{\sin(\frac{1}{2}\tau A)} \tag{3.25}
\end{aligned}$$

Thus our calculation, while not unexpected, has gone some distance in validating our coherent state formulation of the Wilson loop in an SU(2) gauge group Yang-Mills environment – in the case of a constant gauge field.

³To reach the third line we have used from Gradshteyn and Ryzhik, p.29 [14]:

$$\sum_{m=0}^{n-1} \cos(mX) = \cos\left(\frac{1}{2}(n-1)X\right) \sin\left(\frac{n}{2}X\right) \csc\left(\frac{1}{2}X\right)$$

Chapter 4

Testing Polyakov's Conjecture – The Semiclassical Limit

We have, thus far, succeeded in reformulating the path integral for the Wilson loop, using a sigma model variable and a geometrical quantization scheme, thereby unravelling the obstacle of path-ordering.

We begin by writing down the expression for the *expectation value* of the Wilson loop operator in Euclidean space, $\langle W[\Gamma] \rangle$, using results from (2.5) and (2.45):

$$\begin{aligned} \langle W[\Gamma] \rangle &= \int [dA] e^{-S[A]} \text{Tr}(\mathcal{P} e^{i \oint_{\Gamma} A}) \\ &= \int [dA][dn] \delta(n^2 - 1) e^{+ \int d^3x \frac{1}{2e^2} \text{Tr} F_{\mu\nu}^2 - \frac{i}{4} \int_D \text{Tr} (n Dn Dn + 2nF)} \end{aligned} \quad (4.1)$$

where the surface integrals are over a surface D such that D is formed by a smooth mapping from the disc to our three dimensional spacetime, and such that $\partial D = \Gamma$, the contour for the Wilson loop.¹ Note also that the F^2 term in the exponent now has a positive sign. This is an important, yet trivial step – the sign change follows simply from the redefinition of the gauge field, A , in order to have written down equation (2.38).²

We are now in a position to test Polyakov's conjecture [8] – namely, that our field strength tensor $F_{\mu\nu}$ and our gauge field A_μ may be uncoupled, both considered as independent random variables, and, so the conjecture goes, that subsequently the Bianchi identity (1.26) may be rederived through the introduction of a sum over surfaces and a

¹As discussed in Chapter 2, we continue to suppress the normalizing denominator in the expressions for the Wilson loop.

²The sign of the Yang-Mills action will play no role until Chapter 5, when we consider the variation in the field strength.

variational principle in the semiclassical limit. More explicitly, we replace (4.1) by:

$$\langle W[\Gamma] \rangle = \sum_D \int [dF][dA][dn] \delta(n^2 - 1) e^{+ \int d^3x \frac{1}{2e^2} \text{Tr} F_{\mu\nu}^2 - \frac{i}{4} \int_D \text{Tr} (n Dn Dn + 2nF)} \quad (4.2)$$

Here $F_{\mu\nu}^a(x)$ is no longer a function of the gauge field $A_\mu^a(x)$ but is independent of this; rather, it is a SU(2) Lie algebra valued 2-form:

$$\begin{aligned} F &= F_{\mu\nu} dx_\mu \wedge dx_\nu \\ &= F_{\mu\nu}^a \frac{\sigma^a}{2} dx_\mu \wedge dx_\nu \\ &\neq 2(dA + A^2) \end{aligned}$$

Summing over the surfaces D , in the semi-classical limit ($j \rightarrow \infty$), this will be shown to be an accurate reformulation of the Wilson loop, as the summation over surfaces within the functional integral shall re-enforce the Bianchi identity.

A question arises. In a normal application of Stokes' Theorem, the derived surface integral is independent of the particular surface chosen, so long as its boundary defines the original closed line integral. So, why do we now have a summation over surfaces? Why will some surfaces contribute more to the integral than others? Polyakov demonstrates this surface dependence explicitly, and then goes on to answer these questions [8]. Though dealing with Abelian gauge groups in this case, he goes on to say:

...What is surprising here is that we have found an explicit dependence on the surface Σ , while originally it was introduced as an unphysical object.

The origin of the paradox is the multivaluedness of the action ... we took into account only one branch at each spacetime point. The surface independence would be restored had we summed over all possible branches. The summation over branches can be replaced by the summation over surfaces. This is the heart of the connection between fields and strings in this problem.

We consider exactly this situation, except now in a Yang-Mills environment. Thus in our case, if $F = 2(dA + A^2)$ then the integrand would be independent of the different surfaces D (ie. the normal Stokes' Law). But when F is treated as a random variable it is exactly this sum over surfaces which re-introduces a constraint – the Bianchi Identity. In the limit that $j \rightarrow \infty$ the exponential in (4.2) becomes sharply peaked about the stationary points (that is, the condition that the first derivative(s) vanish, $\delta(\text{action})=0$). In this case, applying a variational principle to the action of the integral will obtain the solution (or the first, and most heavily weighted term in an expansion of the solution). It makes sense that in this semiclassical limit, only those surfaces which make extremal contributions to the exponentiated action in (4.2) need be considered as the domain of integration.

4.1 Surface Independence

Consider the action for our theory, given by:

$$S = - \int d^3x \frac{1}{2e^2} \text{Tr } F_{\mu\nu}^2(x) + \frac{j}{4} \int_D \text{Tr } (n Dn Dn + 2nF) \quad (4.3)$$

In considering the maximal value of our path integral for different surfaces D_i (over which we sum in (4.2)), we make use of a generalized triangle inequality for complex phases.

$$\left| e^{f(D_1)} + e^{f(D_2)} + \dots \right| \leq \left| e^{f(D_1)} \right| + \left| e^{f(D_2)} \right| + \dots \quad (4.4)$$

where the function $f(D_i)$ is free to take only complex values. This should be obvious, as complex phases may cause destructive interference. In the case when the equality holds in (4.4), it must be that

$$f(D_j) = f(D_k) \pm 2\pi iN \quad (4.5)$$

(for $N \in \mathbb{Z}$), for two surfaces in question.

In other words, the sum over surfaces will be maximized when all participating surfaces have an equal contribution – and this will occur when an integral over the difference of the two surfaces (which will be a closed surface, since the difference between any two open surfaces with a common boundary will be a closed surface equaling the union of the two surfaces) gives some multiple of 2π .

This implies a certain form of surface independence for the surface integral in our Wilson loop equation as we maximize our action. In requiring the stationary point (minimum value) of our action in accordance with the method of steepest-descent, we are insisting that the action be constant for perturbations of the surfaces, up to integer multiples of 2π .

4.2 Variational Principles and the Method of Steepest Descent

Hamilton's Principle (for a discussion of this, and the calculus of variations applied to a variety of systems, see [4]; the method of saddle point evaluation is covered in [3]) states that the evolution of a system is such that the action, $S = \int L dt$, has a stationary value for the correct (classical) path of the motion. This is equivalent to requiring that the variation of the action be zero. In anticipation of steps we are about to take in the evaluation of the Wilson loop path integral, we consider an action which is a functional of a field variable n , $S_j[n]$, then:

$$\begin{aligned} S_j[n + \delta n] - S_j[n] &= \delta S_j \\ &= \int dt \delta n(x) \left(\frac{\partial}{\partial n(x)} L_j[n] \right) \\ &= 0 \end{aligned}$$

This will be satisfied for an arbitrary, infinitesimal variation δn if and only if $\frac{\partial}{\partial n} L_j[n] = 0$. It is the solution(s) to this which will yield the classical equation(s) of motion for n . Of course, a similar strategy can be used with all other independent field variables.

Equivalently, the stationary points of the action may be found by replacing, for example, $S[n]$ by $S[n + \delta n]$, collecting all terms which are linear in δn and requiring that the coefficient of the term vanish. It is legitimate to infer from $X\delta n = 0$ that $X = 0$ due to the arbitrary profile of the variation, δn .

What is more, given the form of (4.2), as j gets very large the integral will be dominated by its saddle points. To be clear, consider an integral of the following type:

$$I = \int \mathcal{D}\phi e^{-jS[\phi]}$$

This integral can be well approximated by substituting into the action $S[\phi]$ those functions ϕ which minimize the action. And, for a parameter j as above, the larger j gets, the better the approximation becomes. This is the heart of the method of steepest-descent. In fact, if we expand the field ϕ about the saddle point ϕ^{sp} , such that $\phi = \phi^{sp} + \frac{1}{\sqrt{j}}\zeta$ then we have

$$jS[\phi] = jS[\phi^{sp}] + \frac{1}{2!} \frac{\delta S}{\delta\phi_1\delta\phi_2}(\phi^{sp})\zeta_1\zeta_2 + \dots$$

(The first functional derivative of the action being zero by the assumption of a saddle point.) Therefore,

$$I = e^{-jS[\phi^{sp}]} \int \mathcal{D}\phi e^{-\frac{1}{2!} \frac{\delta S}{\delta\phi_1\delta\phi_2}(\phi^{sp})\zeta_1\zeta_2 + \dots} \quad (4.6)$$

And thus the first term in the integral presents the classical solution evaluated directly at the saddle point, the second term is a Gaussian integral indicating fluctuations around the classical solution³ ... and then higher order terms after that.

³It is discussed in [16] (p.258) how variational principles can be used to define *entropy* in field theories, and that the quantum field begins to fluctuate (as \hbar becomes non-zero) due to this entropy... which is what truly makes ϕ a *random* variable.

4.3 The Derivation of the Classical Solutions

Let us slightly rewrite the integrand of the surface integral in (4.2) in preparation for the application of a variational principle.

$$\begin{aligned}
 \text{Tr}(nDnDn) &= \text{Tr}(ndndn + ndn[A, n] + n[A, n]dn + n[A, n][A, n]) \\
 &= \text{Tr}(4dnA + ndndn - 4nA^2) \\
 &= 4\text{Tr}(d(nA)) + \text{Tr}(ndndn - 4n(dA + A^2))
 \end{aligned} \tag{4.7}$$

where we have used the facts that $n^2 = 1$ and that both n and A_μ were defined with anticommutation properties (cf. the previous discussion of the spin algebra we have used to define our theory). It is trivial to work out by hand that though the commutator $\{n, A\} \neq 0$, $\text{Tr}\{n, A\} = 0$ now that A has been defined such that it is anti-Hermitian. Finally, in the last line, we have both added and subtracted the term $4dnA$. Therefore,

$$\begin{aligned}
 S_j &= \frac{j}{4} \int_D \text{Tr}(nDnDn + 2nF) \\
 &= \frac{j}{4} \int_D (4d(\text{Tr}(nA)) + \text{Tr}(ndndn + 2n(F - 2dA - 2A^2))) \\
 &= j \oint_{\delta D} \text{Tr}(nA) + \frac{j}{4} \int_D \text{Tr}(ndndn + 2n(F - 2dA - 2A^2))
 \end{aligned} \tag{4.8}$$

So that, now, we have

$$S = - \int d^3x \frac{1}{2e^2} \text{Tr} F_{\mu\nu}^2(x) + S_j \tag{4.9}$$

recalling that as j gets very large, we need only consider the variations in S_j in the search for the stationary points of the action so long as we are varying n and A .

We now apply a standard variational principle to S_j . This action is a function of our sigma variable n , the gauge field $A_\mu^a(x)$, and the field strength $F_{\mu\nu}^a(x)$ (which, for now, is considered to be independent of the gauge field $A_\mu^a(x)$). Each of these three quantities may be varied independently and we expect each to lead us to a classical equation of motion for these fields.

We begin with the variable n which parameterizes our surfaces. Vary n by δn within the boundary of the surface ($\delta n = 0$ on $\Gamma = \delta D$). In this case it is relatively straightforward to see that

$$\text{Tr } \delta n (F - 2dA - 2A^2) = 0 \quad (4.10)$$

since the first integral in S_j is only the surface boundary alone, where $\delta n = 0$, and because $\delta n d n d n = 0$ which should be intuitively obvious, since n lives on S^2 (only two degrees of freedom), and so by analogy with a triple scalar product of vectors with only two degrees of freedom, the product must vanish.

Equation (4.10) necessarily implies that

$$F = 2dA + 2A^2 + nf \quad (4.11)$$

or

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \epsilon^{abc} A_\mu^b A_\nu^c + n^a f_{\mu\nu} \quad (4.12)$$

where f is an unspecified colour-singlet 2-form (δn will necessarily be orthogonal to n since $n^2 = 1$, and thus $\text{Tr}(\delta n)n = 0$).

We now apply our variational principle to the gauge field, $A_\mu^a(x)$. We will do this by considering the action S_j , and everywhere replacing the gauge field A with $A + \delta A$ where δA is some small variation. Collecting all terms which are linear in δA and setting this equal to zero yields the stationary points of the action. The integrand, when all terms have been combined into the surface integral, is

$$(4d(\text{Tr}(nA)) + \text{Tr}(ndndn + 2n(F - 2dA - 2A^2)))$$

Clearly, we need only consider those terms which include a gauge field dependence: $\text{Tr}(4d(nA) - 4ndA - 4nA^2)$. By inspection, if we let $A \rightarrow A + \delta A$ and then collect only those terms which are linear in δA , the stationary points of the action, under a variation

in A are given by:

$$\begin{aligned}
 0 &= \text{Tr} (dn\delta A + nd(\delta A) - nd(\delta A) - n(\delta A)A - nA\delta A) \\
 &= \text{Tr} (dn\delta A - n(\delta A)A - nA\delta A) \\
 &= \text{Tr} ((dn + [A, n])\delta A) \\
 &= \text{Tr} (Dn \delta A)
 \end{aligned} \tag{4.13}$$

In deriving the second last line we have made use of the previously discussed fact that $\text{Tr}\{n, A\} = 0$. Due to the arbitrary profile of δA , this variational principle applied to A yields the following equation:

$$Dn = 0 \tag{4.14}$$

In light of this result, consider again (4.11):

$$\begin{aligned}
 DF &= D(2dA + 2A^2 + nf) \\
 &= D(nf) \\
 &= (Dn)f + n df \\
 &= n df
 \end{aligned}$$

This result will have great significance when considered in the light of the earlier discussion of surface independence. That discussion considering the maximization of the sum over surfaces indicated that as we vary our maximal surface we should do nothing more than pick up factors of $e^{2\pi i N}$. This in turn means that a volume integral within two surfaces, D and D_δ , (where the surface D_δ has slightly varied from D but such that $\partial D = \partial D_\delta = \Gamma$) should pick up at most *point* sources (monopoles) with an appropriate factor:

$$d(\text{Tr}(ndndn + 2n(F - 2dA - 2A^2))) = 0 \text{ modulo monopoles}$$

$$d(\text{Tr}(ndndn + 2n(F - 2dA - 2A^2))) = i\omega \sum_i 16\pi\delta(x - x_i) \quad (4.15)$$

$$d(\text{Tr}(\frac{1}{2}ndndn + f)) = i\omega \sum_i 8\pi\delta(x - x_i) \quad (4.16)$$

where we have defined a trivial 3-form, $\omega = dx^1 \wedge dx^2 \wedge dx^3$, and have made use of the general argument behind (4.5) as well as, in the last line, the result from (4.11).

Now the monopoles which appear in (4.16) may reside in the n field and/or the f field. Moreover, recognize that these singularities may be moved from f to n (and vice-versa) by a singular gauge transformation. Therefore, let us assume that they reside solely in the n field. Then, (4.16) indicates that:

$$df = 0 \quad (4.17)$$

and

$$\text{Tr } d(ndndn) = 16\pi i\omega \sum_i \delta(x - x_i) \quad (4.18)$$

Though it is not yet apparent, we have succeeded in rederiving the Bianchi identity as a constraint upon our functional integral. This will now be explained in detail.

From (4.11) we have

$$F - 2dA - 2A^2 = nf \quad (4.19)$$

where, clearly, the choice of $f = 0$ will yield the Bianchi identity. But is this a justifiable choice? In fact, the presence of the term nf will have no overall bearing on the Bianchi identity, because of (4.17). Since $df = 0$, we take f as an exact differential 2-form and thus by Poincare's lemma, there exists some 1-form, a , such that f is the exterior derivative of a , and, as usual, $d^2a = 0$. Therefore, consider

$$\begin{aligned} DF &= D(2dA + 2A^2 + nf) \\ &= D(2dA + 2A^2) + Dn f + ndf \\ &= D(2dA + 2A^2) + 0 + nd^2a \\ &= D(2dA + 2A^2) \end{aligned} \quad (4.20)$$

where we have used the results that $d^2a = 0$ and that $Dn = 0$ from (4.14). Note how, before disconnecting F and A , we had $F = 2(dA + A^2)$ as in (2.42). Now, as before, $DF = D(2dA + 2A^2)$. So making use of (4.15) and (4.17), we have

$$DF = 0 \tag{4.21}$$

This leads directly to the *Bianchi Identity as originally stated* (1.25):

$$D_\mu F_{\nu\lambda} + D_\lambda F_{\mu\nu} + D_\nu F_{\lambda\mu} = 0 \tag{4.22}$$

As was our initial goal, having discarded the Bianchi identity earlier, considering F and A as independent field variables, we have succeeded in reproducing the identity via a sum over surfaces! The only complication is that along the way we have ended up introducing an arbitrary distribution of monopoles, which we have chosen to represent within the n field (4.18).

4.4 Giving Meaning to Monopoles

As the monopoles are points in 2+1 dimensional *spacetime*, they are truly events – fluctuations. The fact that our monopoles have turned out to be points – instantons – is a function of the fact that we have specialized to three dimensions. This same analysis would work in four dimensions, but then the monopoles would become lines, that is, the worldlines of point-like objects. Since magnetic flux is conserved, these worldlines must be either closed or infinite (the closed loops corresponding to the worldlines of monopole-antimonopole pairs). The physics of a dilute distribution of monopoles such as these is given on p.65-72 of [17] and later, in Chapter 6, we make an explicit connection to this work.

In [17], Polyakov explains the confining effect of a monopole condensate using a simple physical analogy: consider a superconductor, the ground state of which consists of

electrically charged fields (Cooper pairs). Now, it has been shown that the only way that a magnetic field may penetrate a superconductor is by forming thin filaments of quantized magnetic flux. The Ginzburg-Landau equations would then describe how, for two magnetic charges within the superconductor, their magnetic flux must be concentrated in such a filament connecting the two charges, with the interaction energy proportional to their separation. Now consider this analogy again but exchange the words “electric” and “magnetic.” The result will be the confinement of electric charges within a monopole condensate.

In fact, progress along these lines has been made in [10], from which this work has borrowed. Their work has been completed in relating the expectation value of the Wilson loop to path integrals over monopole trajectories. This is a step forward in providing a formal connection between the area law behaviour of the Wilson loop and monopole condensation. Until now, confinement due to monopole condensation has only been entirely understood in theories with spontaneous breaking of colour symmetry down to the $U(1)$ subgroup, allowing for Polyakov-'t Hooft monopoles. These monopole effects, and possible screening effects, will be discussed in Chapter 6.

Chapter 5

The Expectation Value of the Wilson Loop

We now attempt to calculate the expectation value of the Wilson loop in the semiclassical limit ($j \rightarrow \infty$). We thereby demonstrate the existence of the confining phase, as given by an area law behaviour, in the lowest-order, dominant term of the expectation value.

5.1 Reformulating the Loop Expression – Semiclassical Limit

Recall now the general form of the action of the Wilson loop expression, from (4.2):

$$\int \mathcal{D}F \mathcal{D}A \mathcal{D}n \delta(n^2 - 1) e^{\int d^3x \frac{1}{2e^2} \text{Tr} F_{\mu\nu}^2 - \frac{i}{4} \int_D \text{Tr}(n Dn Dn + 2nF)} \quad (5.1)$$

In the semiclassical limit the integrand will be sharply peaked near the minima of the action functional. Therefore the classical equation of motion for which we have solved ought to provide excellent approximations to our integral.

The integral simplifies to a great extent when we make use of our previous result from (4.14):

$$Dn = 0$$

So that we are left with

$$\int \mathcal{D}F \mathcal{D}n \delta(n^2 - 1) e^{\int d^3x \frac{1}{2e^2} F_{\mu\nu}^2 - \frac{i}{2} \int_D nF(x)} \quad (5.2)$$

The measure of the surface integral can be reformulated in the following way. Consider our surface to be parameterized by (σ^1, σ^2) , such that $x_\mu|_{\text{surface}} = X_\mu(\vec{\sigma}) = X_\mu(\sigma^1, \sigma^2)$

are the embedding functions of the surface within three-dimensional spacetime. Then

$$\begin{aligned} d^2\sigma_{\mu\nu} &= d\sigma^1 d\sigma^2 \left(\frac{\partial X_\mu}{\partial \sigma^2} \frac{\partial X_\nu}{\partial \sigma^1} - \frac{\partial X_\nu}{\partial \sigma^2} \frac{\partial X_\mu}{\partial \sigma^1} \right) \\ &= \epsilon_{\mu\nu} d^2(\text{Area}) \end{aligned} \quad (5.3)$$

and therefore define a quantity $E_{\mu\nu}^a(x)$ which can be identified as the current of a Nambu string:

$$E_{\mu\nu}^a(x) = \int d\sigma^1 d\sigma^2 \epsilon_{ij} \frac{\partial X_\mu(\vec{\sigma})}{\partial \sigma^i} \frac{\partial X_\nu(\vec{\sigma})}{\partial \sigma^j} \delta^3(x - X(\vec{\sigma})) n^a \quad (5.4)$$

allowing us to rewrite (5.2) as:

$$\int \mathcal{D}F \mathcal{D}n \, e^{\int d^3x \left(\frac{1}{2e^2} F_{\mu\nu}^2 - \frac{i}{2} E_{\mu\nu}^a(x) F_{\mu\nu}^a(x) \right)} \quad (5.5)$$

Finally, let us vary the field strength tensor, $F_{\mu\nu}^a(x)$, in this expression in order to derive the classical equation of motion.

$$\frac{\partial}{\partial F_{\mu\nu}^a} \mathcal{L}[F] = \frac{1}{e^2} F_{\mu\nu}^a(x) - \frac{j}{2} E_{\mu\nu}^a(x) = 0 \quad (5.6)$$

Yielding a final, surprisingly simple, expression:

$$F_{\mu\nu}^a(x) = \frac{je^2}{2} E_{\mu\nu}^a(x) \quad (5.7)$$

We can now reformulate the expression for the expectation value of the Wilson loop. Two comments should be made about this reformulation. First of all, we have undone all functional integrals, by having applied a first variational formula, and then substituted back into our action functional the classical equations of motion. Thus our new expression *only* represents the first, classical term in the Wilson loop. Secondly, we are systematically ignoring the denominator from (4.2). This is of no great consequence here, as it is included purely for purposes of normalization and thus this constant scaling factor is of no current interest.

Therefore,

$$\langle W[\Gamma] \rangle_{\text{Semi-Class}} = \exp \left(- \int d^3x \frac{j^2 e^2}{8} E_{\mu\nu}^2(x) \right) \quad (5.8)$$

is our somewhat simplified expression for the expectation value of the Wilson loop, given in terms of $E_{\mu\nu}^a(x)$, a string current.

5.2 A Change to Comoving Coordinates

Before we move on, it is advantageous to switch to a comoving coordinate system, wherein we borrow geometrical ideas from both [18] and [19]. That is, we are interested only in our given surface, and the surrounding space very close to that surface. Therefore we can make use of the *induced metric* of the surface. The method of calculating the components of an induced metric makes use of standard differential geometry – for a discussion of this, and other elements of differential geometry in a physical context, see both [20] and, for a more purely mathematical treatment, [21].

For an m -dimensional manifold M which is a submanifold of an n -dimensional Riemannian manifold N (metric g_N), we can define the natural metric for M , g_M . Let $f : M \rightarrow N$ be the embedding function which defines the structure of the submanifold M . Then:

$$g_{M\mu\nu}(x) = g_{N\alpha\beta}(f(x)) \frac{\partial f^\alpha}{\partial x^\mu} \frac{\partial f^\beta}{\partial x^\nu} \quad (5.9)$$

We use this to switch to our new comoving coordinates. The comoving coordinates are defined using two linearly independent tangent vectors, and the normal which they define, at each point on the surface. Therefore

$$x_\mu = X_\mu(\vec{\sigma}) + \rho N_\mu(\vec{\sigma}) \quad (5.10)$$

where X_μ are the embedding functions of our surface within the three-dimensional Euclidean space, and where ρ is a small parameter indicating distances normal to the surface

(small such that our coordinate systems do not overlap, which would occur for curved surfaces, for normal vectors sufficiently long to intersect) and N_μ is the normal vector given by:

$$N_\mu = B\epsilon_{ab}\epsilon_{\mu\nu\lambda}\partial_a X_\nu\partial_b X_\lambda \quad (5.11)$$

where B is a simple normalization factor, such that two properties hold true:

$$N^\mu N_\mu = 1 \quad (5.12)$$

$$N^\mu\partial_a X_\mu = 0 \quad (5.13)$$

The first requirement above is a statement of normalization while the second statement defines the orthogonality property of the normal vector. Both of these simple requirements will be crucial in the following calculations.

A change of coordinates will necessarily include a Jacobian factor in the transformation. In terms of the metric, this will be given by:

$$\int d^3x = \int \sqrt{\det G} d^2\sigma d\rho \quad (5.14)$$

where $G_{\mu\nu}$ is the metric of the space in terms of the new coordinates, σ^1 , σ^2 , ρ . It is relatively easy to see that this will be of the form

$$G_{\mu\nu} = \begin{bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.15)$$

That $G_{\mu\nu}$ is of this form follows immediately from (5.9). For example,

$$\begin{aligned} G_{33} &= \partial_\rho(X^\mu + \rho N^\mu)\partial_\rho(X_\mu + \rho N_\mu) \\ &= N^\mu N_\mu \\ &= 1 \end{aligned} \quad (5.16)$$

It remains to calculate h_{ab} . This too follows from (5.9) but requires a few tricks and manipulations. The calculation follows:

$$\begin{aligned}
 h_{ab} &= \partial_a x^\mu \partial_b x_\mu \\
 &= \partial_a (X^\mu + \rho N^\mu) \partial_b (X_\mu + \rho N_\mu) \\
 &= \partial_a X^\mu \partial_b X_\mu + 2\rho \partial_a X^\mu \partial_b N_\mu + \rho^2 \partial_a N^\mu \partial_b N_\mu
 \end{aligned} \tag{5.17}$$

In calculating the cross terms in the last line of the above expression, we commuted the indices a, b . This is allowable because, using (5.13):

$$\begin{aligned}
 N^\mu \partial_a X_\mu &= 0 \\
 \text{Therefore } \partial_b N^\mu \partial_a X_\mu + N^\mu \partial_b \partial_a X_\mu &= 0 \quad \text{or} \\
 \partial_b N^\mu \partial_a X_\mu &= -N^\mu \partial_b \partial_a X_\mu
 \end{aligned} \tag{5.18}$$

and since the right hand side of the last line is clearly symmetric under the interchange $a \leftrightarrow b$, so must the left hand side be symmetric under such an interchange of indices.

Now introduce a quantity known as the *extrinsic curvature*, K_{ab} . By definition,

$$\partial_a \partial_b X^\mu = \Gamma_{ab}^\nu \partial_\nu X^\mu + K_{ab} N^\mu \tag{5.19}$$

where Γ_{ab}^ν are the standard Christoffel symbols. Contracting both sides of the above equation with N_μ we find an expression for K_{ab} :

$$K_{ab} = N_\mu \partial_a \partial_b X^\mu \tag{5.20}$$

Therefore, using (5.18) and (5.20), we can immediately rewrite (5.17) as

$$h_{ab} = g_{ab} - 2\rho K_{ab} + \rho^2 \partial_a N^\mu \partial_b N_\mu \tag{5.21}$$

where g_{ab} is the *induced surface metric* easily calculable from (5.9). Lastly, notice that

$$K_{ac} K^c_b = N_\mu \partial_a \partial_c X^\mu \partial^c \partial_b X_\nu N_\mu$$

$$\begin{aligned}
&= \partial_a N_\mu \partial_b N_\nu \partial_c X^\mu \partial_c X^\nu \\
&= \partial_a N_\mu \partial_b N_\nu \delta^{\mu\nu} \\
&= \partial_a N_\mu \partial_b N^\mu
\end{aligned} \tag{5.22}$$

With this result (5.21) becomes

$$h_{ab} = g_{ab} - 2\rho K_{ab} + \rho^2 K_{ac} K^c_b \tag{5.23}$$

This gives us our full expression for $G_{\mu\nu}$. We need now only calculate the determinant. And clearly $\det G = \det h$. This tedious algebraic manipulation yields the following, very nice result:

$$\begin{aligned}
\det G &= \det h \\
&= (\det g) \left(1 - \rho \operatorname{Tr} K + \frac{1}{2} \rho^2 R\right)^2
\end{aligned} \tag{5.24}$$

where R , the *scalar curvature*, is defined as

$$R = (K^a_a)^2 - K^b_a K^a_b \tag{5.25}$$

Thus, our final result after all of these calculations for the coordinate transformation, is:

$$d^3x = d^2\sigma d\rho \sqrt{\det g} \left(1 - \rho \operatorname{Tr} K + \frac{1}{2} \rho^2 R\right) \tag{5.26}$$

Again, g_{ab} is the induced surface metric.

5.3 Deriving the Area Law

We will use this change of coordinates, to simplify our integral for the expectation value of the Wilson loop, and subsequently analyze the asymptotic behaviour of the expectation value in the limit of a large loop size.

Now since

$$\begin{aligned} E_{\mu\nu}^a E_{\mu\nu}^a \\ = \int d^2\sigma \int d^2\tilde{\sigma} \epsilon_{ij} \epsilon_{kl} \frac{\partial X_\mu(\vec{\sigma})}{\partial \sigma^i} \frac{\partial X_\nu(\vec{\sigma})}{\partial \sigma^j} \frac{\partial X_\mu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^k} \frac{\partial X_\nu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^l} \delta^3(x - X(\vec{\sigma})) \delta^3(x - X(\vec{\tilde{\sigma}})) n^a n^a \end{aligned} \quad (5.27)$$

We use this result, with the immediate simplification that $n^a n^a = 1$, to write our full integral as:

$$\begin{aligned} & - \int d^2\sigma' d\rho \sqrt{\det g} (1 - \rho \ln \kappa + \rho^2 \text{Tr } R) \frac{j^2 e^2}{8} E_{\mu\nu}^2 \\ = & - \int d^2\sigma' d\rho d^2\sigma d^2\tilde{\sigma} \sqrt{\det g} (1 - \rho \Omega + \rho^2 \Omega') \frac{j^2 e^2}{8} \epsilon_{ij} \epsilon_{kl} \frac{\partial X_\mu(\vec{\sigma})}{\partial \sigma^i} \frac{\partial X_\nu(\vec{\sigma})}{\partial \sigma^j} \frac{\partial X_\mu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^k} \frac{\partial X_\nu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^l} \\ & \delta^3(x - X(\vec{\sigma})) \delta^3(x - X(\vec{\tilde{\sigma}})) \\ = & - \int d^2\sigma' d\rho d^2\sigma d^2\tilde{\sigma} \sqrt{\det g} (1 - \rho \Omega + \rho^2 \Omega') \frac{j^2 e^2}{8} \epsilon_{ij} \epsilon_{kl} \frac{\partial X_\mu(\vec{\sigma})}{\partial \sigma^i} \frac{\partial X_\nu(\vec{\sigma})}{\partial \sigma^j} \frac{\partial X_\mu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^k} \frac{\partial X_\nu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^l} \\ & \delta^3(X(\vec{\sigma}') + \rho N_\mu(\vec{\sigma}') - X(\vec{\sigma})) \delta^3(X(\vec{\sigma}') + \rho N_\mu(\vec{\sigma}') - X(\vec{\tilde{\sigma}})) \end{aligned} \quad (5.28)$$

Clearly the delta functions will go some distance in simplifying the seven-dimensional integral. But before we accomplish this we must rewrite the delta functions in a much more friendly form. A variable change within a delta function must be accompanied by some sort of Jacobian to account for the transformation. That is,

$\delta(f(x)) = \left| \frac{df(x)}{dx} \right|^{-1} \delta(x - x_0)$. In this case, we have already spent some considerable time calculating the Jacobian. The delta functions transform as:

$$\delta^3(X(\vec{\sigma}') + \rho N(\vec{\sigma}') - X(\vec{\sigma})) = \frac{\delta^2(\vec{\sigma}' - \vec{\sigma}) \delta(\rho)}{\sqrt{\det g} (1 - \rho \text{Tr } K + \frac{1}{2} \rho^2 R)} \quad (5.29)$$

and likewise for the other delta function.

Rewrite (5.28) with the simplified delta functions and transformation factors. Also, evaluate the integral over ρ ; this will set ρ to zero while still leaving us with a divergent factor of $\delta(\rho = 0)$ due to the second delta function. We are left with:

$$\begin{aligned} & -\delta(0) \int d^2\sigma' d^2\sigma d^2\tilde{\sigma} \frac{1}{\sqrt{\det g}} \frac{j^2 e^2}{8} \delta^2(\vec{\sigma}' - \vec{\sigma}) \delta^2(\vec{\sigma}' - \vec{\tilde{\sigma}}) \epsilon_{ij} \epsilon_{kl} \frac{\partial X_\mu(\vec{\sigma})}{\partial \sigma^i} \frac{\partial X_\nu(\vec{\sigma})}{\partial \sigma^j} \frac{\partial X_\mu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^k} \frac{\partial X_\nu(\vec{\tilde{\sigma}})}{\partial \tilde{\sigma}^l} \\ = & -\delta(0) \int d^2\sigma' \frac{1}{\sqrt{\det g}} \frac{j^2 e^2}{8} \epsilon_{ij} \epsilon_{kl} \frac{\partial X_\mu(\vec{\sigma}')}{\partial \sigma'^i} \frac{\partial X_\nu(\vec{\sigma}')}{\partial \sigma'^j} \frac{\partial X_\mu(\vec{\sigma}')}{\partial \sigma'^k} \frac{\partial X_\nu(\vec{\sigma}')}{\partial \sigma'^l} \end{aligned} \quad (5.30)$$

A further simplification results when we consider the expanded version of the quantity $\det g$.

$$\begin{aligned}
 \det g &= \det(\partial_a X^\mu \partial_b X_\mu) \\
 &= \partial_1 X^\mu \partial_1 X_\mu \partial_2 X^\nu \partial_2 X_\nu - \partial_1 X^\mu \partial_2 X_\mu \partial_2 X^\nu \partial_1 X_\nu \\
 &= \frac{1}{2} \epsilon_{ij} \epsilon_{kl} \frac{\partial X_\mu(\vec{\sigma})}{\partial \sigma^i} \frac{\partial X_\nu(\vec{\sigma})}{\partial \sigma^j} \frac{\partial X_\mu(\vec{\sigma})}{\partial \sigma^k} \frac{\partial X_\nu(\vec{\sigma})}{\partial \sigma^l}
 \end{aligned}$$

Therefore the integral (5.30) becomes

$$-\delta(0) \int d^2 \sigma \frac{2 \det g}{\sqrt{\det g}} \frac{j^2 e^2}{8} = -\delta(0) \frac{j^2 e^2}{4} \int d^2 \sigma \sqrt{\det g} \quad (5.31)$$

But recall, along the lines of (5.14), that

$$\int d^2 \sigma \sqrt{\det g} = \text{Surface Area} \equiv A \quad (5.32)$$

And therefore we have the final result that

$$\langle W[\Gamma] \rangle \propto \exp \left(-\delta(0) \frac{j^2 e^2}{4} A \right) \quad (5.33)$$

where A is the area of the Wilson loop. The factor of $\delta(0)$ is infinite by definition. This can be regulated by a high momentum cutoff (which is equivalent to a cutoff at small distance scales), Λ . Therefore our answer becomes:

$$\langle W[\Gamma] \rangle \propto \exp \left(-\frac{j^2 e^2}{4} \Lambda A + \dots \text{higher order terms} \dots \right) \quad (5.34)$$

It is exactly this sort of area law behaviour which indicates a linear quark confining potential. Thus we have derived the lowest-order behaviour of the Wilson loop, finding that in this non-Abelian theory we appear to have confining behaviour. It should be noted, however, that at this point any possible spin (j) dependence appears to be concealed. In Chapter 6 we shall consider the explicit introduction of monopoles and in Section 6.5 confinement will be demonstrated to hold for half-odd-integer spin states.

It has been demonstrated in similar theories (see, for example, [24]) that, to higher order, one expects terms of the form $e^{-(\sigma A + \rho P)}$ where P is the *perimeter* of the loop. However, for large loops, $e^{-(\sigma A + \rho P)} \approx e^{-\sigma A}$. It is for this reason that confining behaviour is classified by the *asymptotics* of the Wilson loop (note, however, that in our particular case, in the semi-classical approximation ($j \rightarrow \infty$), the first-order term will dominate regardless of loop size, due to the factor of j).

5.4 An Initial Approach to Fluctuations About the Classical Solutions

Clearly, the next obvious step is to calculate the next higher order quantum fluctuations about the classical solutions we have calculated for our fields. To do this, we must gauge fix the integral, as explained below, and then go on to calculate the leading term after the classical solution. Unfortunately, difficulties exist which preclude the success of this strategy, forcing us, in the next chapter, to adopt a new direction of investigation.

In order to illustrate the difficulties, we now *attempt* to calculate the next higher term in the solution to the functional integral over the gauge field. To do this we must *gauge-fix* the integral. We will use the Faddeev-Popov method. As will be seen, this will not prove to be adequate in continuing to analytically calculate the quantum fluctuations. The obstacle to the calculation will be discussed and further strategies suggested.

5.4.1 Faddeev-Popov Gauge Fixing

A problem arises because the Lagrangian in the functional integral is unchanged along an infinite number of directions in the space of field configurations due to invariance under gauge transformations. The solution is to apply a gauge-fixing condition which will restrict the integral to physically-inequivalent field configurations.

Consider some function of our gauge field, $G(A)$. Then our gauge fixing condition will

consist of insisting that $G(A) = 0$. Specifically, we use what is known as the *generalized Lorentz gauge condition*:

$$G(A) = \partial_\mu A_\mu^a(x) - \omega^a(x) = 0 \quad (5.35)$$

where $\omega^a(x)$ are some arbitrary, well-behaved, scalar functions.

Next, it is perfectly legal to insert the following unit identity into our functional integral:

$$1 = \int \mathcal{D}\alpha(x) \delta(G(A^\alpha)) \det \left(\frac{\delta G(A^\alpha)}{\delta \alpha} \right) \quad (5.36)$$

where $\alpha(x)$ is some local gauge transformation parameter. That is, our gauge field transforms as:

$$A_\mu^a \frac{\sigma^a}{2} \rightarrow (A^\alpha)_\mu^a \frac{\sigma^a}{2} = e^{i\alpha^a \frac{\sigma^a}{2}} \left(A_\mu^b \frac{\sigma^b}{2} + \frac{i}{e} \partial_\mu \right) e^{-i\alpha^c \frac{\sigma^c}{2}} \quad (5.37)$$

which, in infinitesimal form, is

$$A_\mu^a \rightarrow (A^\alpha)_\mu^a = A_\mu^a + \frac{1}{e} \partial_\mu \alpha^a + \epsilon^{abc} A_\mu^b \alpha^c \quad (5.38)$$

$$= A_\mu^a + \frac{1}{e} D_\mu \alpha^a \quad (5.39)$$

This is of great relevance, for, by taking a functional derivative, we find:

$$\frac{\delta G(A^\alpha)}{\delta \alpha} = \frac{1}{e} \partial_\mu D_\mu \quad (5.40)$$

We therefore need to calculate the quantity $\det(\frac{1}{e} \partial_\mu D_\mu)$. In non-Abelian gauge theories it is customary to represent this determinant as a Gaussian integral over *anti-commuting "ghost fields"*, c , and \bar{c} :

$$\det \left(\frac{1}{e} \partial_\mu D_\mu \right) = \int \mathcal{D}c \mathcal{D}\bar{c} e^{\int d^3x \bar{c} (\partial_\mu D_\mu) c} \quad (5.41)$$

Now, *any* scalar functions $\omega^a(x)$ are valid choices for our gauge condition. It is therefore legitimate to functionally integrate over all ω , with a Gaussian weighting function,

centred at $\omega = 0$. Our full integral, including this gauge-fixing, then becomes:

$$N(\xi) \int \mathcal{D}\omega e^{\int d^3x \frac{\omega^2}{2\xi}} \int \mathcal{D}\alpha \mathcal{D}A \mathcal{D}F \mathcal{D}n e^{\int d^3x (\frac{1}{2e^2} \text{Tr}(F_{\mu\nu})^2 + \bar{c} \partial_\mu D_\mu c)} e^{-\frac{i}{4} \int_D \text{Tr}(n D n D n + 2n F)} \delta(\partial_\mu A_\mu - \omega) \delta(n^2 - 1) \quad (5.42)$$

where $N(\xi)$ is an unimportant normalization function (for the recently introduced Gaussian integral), and ξ is any finite constant. Integrating over ω , making use of the delta function in the second line, we get:

$$N(\xi) \left(\int \mathcal{D}\alpha \right) \int \mathcal{D}F \mathcal{D}A \mathcal{D}n \delta(n^2 - 1) e^{\int d^3x (\frac{1}{2e^2} \text{Tr}(F_{\mu\nu})^2 - \frac{1}{2\xi} (\partial_\mu A_\mu^a)^2 + \bar{c} \partial_\mu D_\mu c) - \frac{i}{4} \int_D \text{Tr}(n D n D n + 2n F)} \quad (5.43)$$

Unfortunately, we have a problem. Despite the gauge fixing, this method will not prove satisfactory in calculating the first quantum fluctuations about the classical solutions. The problem can be easily understood by noting that the quantum fluctuations of the gauge field will be entirely undamped off of the surface of the Wilson loop. This means that the integral will be divergent at every point in spacetime which is not a part of the surface. One would expect a factor of the form

$$(\Lambda)^{V-tA} e^{-\sigma A} \quad (5.44)$$

where Λ is some ultraviolet cutoff for the gauge field integration, $(V - tA)$ is the volume of the space less the area of the surface of the Wilson loop (times some small thickness), and σ is a string tension as discussed and calculated in the previous chapter.

All of this suggests to us that somehow space-filling surfaces, which will enforce the Bianchi identity at all points in spacetime, might play some important role. Nevertheless, this difficulty is a major obstacle to further calculations. Some sort of regularization procedure is needed, and one possibility would be a lattice calculation – but this option will not be pursued further within this paper. Rather, we will forego the higher-level, quantum calculations in favour of a deeper analysis at the semi-classical level.

Chapter 6

Monopoles and the Classical Solutions

We are now able to explicitly take into account the effects of monopoles on the Wilson loop. We start by considering a single monopole, and then generalize this result to a monopole gas. The interactions provided by the monopoles will prove to be the key in unravelling some of the questions of confinement.

Given the achievements of the previous chapters, it is now possible to explicitly solve for expressions for the A , F and n fields, at the classical level, now including monopole contributions. The defining equations are:

$$\text{Tr } d(ndndn) = 16\pi i\omega \sum_i \delta(x - x_i) \quad (6.1)$$

$$Dn = 0 \quad (6.2)$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i[A_\mu(x), A_\nu(x)] \quad (6.3)$$

where these were initially derived in (4.18), (4.14) and (1.21) respectively (these equations were all found to hold, *at the classical level*, in Chapter 4).

We begin by deriving an expression for the n field. Initially we will do this for a single monopole at the origin...later treatments will include a monopole gas. It is useful to note that much of this calculation is analogous to earlier work completed by N.J. Snyderman [25], and, as such, these calculations can be shown to be essentially equivalent to the Georgi-Glashow model.

The matrix-valued field n must be of unit norm, with $\text{Tr } d(ndndn) = 16\pi i\omega\delta(r)$. By simple dimensional analysis there is a particularly obvious ansatz for n which will turn

out to be correct:

$$n = \frac{r^a \sigma^a}{r} \quad (6.4)$$

where \vec{r} is the vector from the monopole site to the coordinate of n . Then,

$$\begin{aligned} \text{Tr}(ndndn) &= \text{Tr}\left(\frac{r^a}{r} \sigma^a d\frac{r^b}{r} \sigma^b d\frac{r^c}{r} \sigma^c\right) \\ &= \text{Tr}\left(\frac{r^a}{r} d\frac{r^b}{r} d\frac{r^c}{r} (i\epsilon^{abd}(\delta^{dc} + i\epsilon^{dce} \sigma^e))\right) \\ &= i\epsilon^{abc} \frac{r^a}{r} d\frac{r^b}{r} d\frac{r^c}{r} \text{Tr}(\mathbf{1}) \\ &= 2i\epsilon^{abc} \frac{r^a}{r} d\frac{r^b}{r} d\frac{r^c}{r} \\ &= 2i\epsilon^{abc} \frac{r^a}{r} \frac{dr^b}{r} \frac{dr^c}{r} \end{aligned} \quad (6.5)$$

because, by a simple application of the chain rule,

$$d\frac{r^b}{r} = \frac{dr^b}{r} - \frac{r^b r^c dr^c}{(r)^3}$$

which, when used to expand $\epsilon^{abc} d\frac{r^b}{r} d\frac{r^c}{r}$, will cause terms other than those of the form $\frac{dr^b}{r}$ to vanish due to the antisymmetry of ϵ^{abc} . Also, we have used several times the simple result that $\sigma^a \sigma^b = \delta^{ab} + i\epsilon^{abc} \sigma^c$. To continue,

$$\begin{aligned} \text{Tr}(ndndn) &= 2i \frac{r^a}{(r)^3} (\epsilon^{abc} dr^b dr^c) \\ &= 4i \frac{r^a}{(r)^3} dS^a \end{aligned} \quad (6.6)$$

where $\epsilon^{abc} dr^b dr^c = dS^a$ is an area element defined by the surface of integration. Now consider the divergence of (6.6):

$$\begin{aligned} d\text{Tr}(ndndn) &= 2id \frac{r^a}{r^3} (\epsilon^{abc} dr^b dr^c) \\ &= 2i\epsilon^{abc} dr_e \partial_e \left(\frac{r^a}{(r)^3} \right) dr^b dr^c \\ &= 4i \partial_a \left(\frac{r^a}{(r)^3} \right) \omega \\ &= 16\pi i \omega \delta(r) \end{aligned} \quad (6.7)$$

where, as defined earlier, $\omega = dx^1 \wedge dx^2 \wedge dx^3$ and it is a fairly elementary result of calculus (Gauss' Law) that $\partial_a \left(\frac{r^a}{r^3} \right) = 4\pi\delta(r)$.

Therefore, we were correct in specifying $n = \frac{r^a \sigma^a}{r}$. Now use (6.2) to derive an expression for the gauge field. Clearly, we must have a non-zero gauge field if we are to maintain the condition $Dn = 0$ for a varying n field. We again borrow Synderman's ansatz (which will turn out to be the correct form for A_μ , but with a slightly altered multiplicative constant):

$$A_\mu \sim \frac{\epsilon^{\mu a \nu} \frac{\sigma^a}{2} r^\nu}{r^2} \quad (6.8)$$

The requirement that $Dn = 0$ implies that $dn = [n, A]$. In order to check that whether this is true, first calculate $[n, A]$ using (6.8):

$$\begin{aligned} [n, A] &= n^a \sigma^a A_\mu^b \frac{\sigma^b}{2} dr_\mu - A_\mu^b \frac{\sigma^b}{2} dr_\mu n^a \sigma^a \\ &= \left(\frac{r^a \sigma^a}{r} \frac{\epsilon^{\mu b \nu} \sigma^b r^\nu}{2r^2} - \frac{\epsilon^{\mu b \nu} \sigma^b r^\nu}{2r^2} \frac{r^a \sigma^a}{r} \right) dr_\mu \\ &= \frac{r^\lambda r^\nu}{r^3} \epsilon^{\mu b \nu} \epsilon^{\lambda b n} \sigma^n i dr_\mu \\ &= -i d \left(\frac{r^a \sigma^a}{r} \right) \end{aligned} \quad (6.9)$$

This last step is most easily demonstrated by direct calculation. For example, choose $n = 1$ above to get

$$\begin{aligned} \frac{r^\lambda r^\nu}{r^3} \epsilon^{\mu b \nu} \epsilon^{\lambda b 1} \sigma^1 i dr_\mu &= \left(-\frac{r^3 r^\nu}{(r)^3} \epsilon^{\mu 2 \nu} + \frac{r^2 r^\nu}{(r)^3} \epsilon^{\mu 3 \nu} \right) i dr_\mu \\ \text{for } \mu = 2, &\Rightarrow \frac{r^2 r^1}{(r)^3} i \end{aligned}$$

whereas $\partial_2 \frac{r^1}{r} = -\frac{r^1 r^2}{(r)^3}$. Therefore, since $dn = [n, A]$ and $dn = dr_\mu \partial_\mu \frac{r^a \sigma^a}{r}$, we have a final answer of:

$$A_\mu = \frac{i \epsilon^{\mu a \nu} \sigma^a r^\nu}{2(r)^2} \quad (6.10)$$

or, slightly more generally, for any non-matrix valued 1-form f^μ we can have

$A^\mu = \frac{i \epsilon^{\mu a \nu} \sigma^a r^\nu}{2(r)^2} + n^a \sigma^a f^\mu$. However, it has been our policy to assume that, through a

singular gauge transformation, we treat all monopoles within the n field, with none in the gauge field. Therefore, to keep (for now) $A_\mu(x)$ monopole-free, we choose $f^\mu = 0$.

Note that the coefficient for A_μ^a in (6.10) is imaginary. This is actually expected, for in Section 2.2.2, we defined the gauge field to be anti-Hermitean (we performed the transformation $A \rightarrow iA$). In the upcoming calculations, it will occasionally be advantageous to use a Hermitean formula for A_μ . To avoid confusion, for the remainder of this chapter, $A_\mu = \frac{i\epsilon^{\mu a \nu} \sigma^a r^\nu}{2(r)^2}$ as in (6.10). We will also define A_μ^{Herm} by the transformation $A \rightarrow A^{\text{Herm}} = -iA$. Therefore let

$$A_\mu^{\text{Herm}} = \frac{\epsilon^{\mu a \nu} \sigma^a r^\nu}{2(r)^2} \quad (6.11)$$

We now calculate the expression for the field strength tensor, $F_{\mu\nu}$. We will use A_{Herm}^μ for this as we can then use the standard formulation for the field strength tensor (6.3), ending up with, as we shall see, a fairly elegant result.¹

First, calculate:

$$\begin{aligned} i[A_{\text{Herm}}^\mu, A_{\text{Herm}}^\nu] &= i \frac{\epsilon^{\mu i \lambda} \sigma^i r^\lambda}{2(r)^2} \frac{\epsilon^{\nu b \omega} \sigma^b r^\omega}{2(r)^2} - i \frac{\epsilon^{\nu b \omega} \sigma^b r^\omega}{2(r)^2} \frac{\epsilon^{\mu i \lambda} \sigma^i r^\lambda}{2(r)^2} \\ &= \frac{-i}{4(r)^4} (\epsilon^{\nu b \omega} \epsilon^{\mu i \lambda} (\delta^{bi} + i\epsilon^{bij} \sigma^j) r^\omega r^\lambda - \epsilon^{\mu i \lambda} \epsilon^{\nu b \omega} (\delta^{ib} + i\epsilon^{ibj} \sigma^j) r^\omega r^\lambda) \\ &= \frac{1}{2(r)^4} \epsilon^{\nu b \omega} \epsilon^{\mu i \lambda} \epsilon^{bij} \sigma^j r^\omega r^\lambda \end{aligned} \quad (6.12)$$

And one may calculate that

$$\partial_\mu A_\nu^{\text{Herm}} - \partial_\nu A_\mu^{\text{Herm}} = \frac{\sigma^a \epsilon^{\nu a \mu}}{(r)^2} - \frac{\sigma^a \epsilon^{\nu a \lambda} r^\lambda r^\mu}{(r)^4} + \frac{\sigma^a \epsilon^{\mu a \lambda} r^\lambda r^\nu}{(r)^4} \quad (6.13)$$

Then,

$$F^{\mu\nu} = \partial^\mu A_{\text{Herm}}^\nu - \partial^\nu A_{\text{Herm}}^\mu + i[A_{\text{Herm}}^\mu, A_{\text{Herm}}^\nu]$$

¹The field strength, as a physically meaningful variable, makes much more sense as a Hermitean quantity. Note that $F_{\mu\nu}$ is Hermitean if the gauge field is Hermitean, but if the gauge field is not Hermitean then the field strength will have both Hermitean and non-Hermitean parts.

$$\begin{aligned}
&= \frac{\sigma^a \epsilon^{\nu a \mu}}{(r)^2} - \frac{\sigma^a \epsilon^{\nu a \lambda} r^\lambda r^\mu}{(r)^4} + \frac{\sigma^a \epsilon^{\mu a \lambda} r^\lambda r^\nu}{(r)^4} + \frac{1}{2(r)^4} \epsilon^{\nu b \omega} \epsilon^{\mu i \lambda} \epsilon^{b i j} \sigma^j r^\omega r^\lambda \\
&= \left(\frac{\epsilon^{\nu a \mu}}{(r)^2} - \frac{\epsilon^{\nu a \lambda} r^\lambda r^\mu}{(r)^4} + \frac{\epsilon^{\mu a \lambda} r^\lambda r^\nu}{(r)^4} + \frac{1}{2(r)^4} \epsilon^{\nu b \omega} \epsilon^{\mu i \lambda} \epsilon^{b i a} r^\omega r^\lambda \right) \sigma^a \\
&= + \epsilon^{\mu \nu \lambda} \frac{r^\lambda (r^a \frac{\sigma^a}{2})}{(r)^4}
\end{aligned} \tag{6.14}$$

where the last step can be most easily seen by direct calculation for each particular σ^a .

We can check the magnetic component of $F_{\mu\nu}$ by taking the projection of $F_{\mu\nu}$ in the electromagnetic direction – that is, the abelian (U(1)) direction. This is accomplished by projecting along the direction of the n field, $F \cdot n$. The answer is very reassuring:

$$B^\lambda = \frac{1}{2} \epsilon^{\mu \nu \lambda} F_{\mu \nu} \cdot n \tag{6.15}$$

$$\begin{aligned}
&= \frac{1}{2} \epsilon^{\mu \nu \lambda} \epsilon^{\mu \nu \xi} \frac{r^\xi r^a}{(r)^4} \frac{\sigma^a}{2} \frac{r^b \sigma^b}{r} \\
&= \frac{r^\lambda r^a}{(r)^4} \frac{r^b}{2r} (\delta^{ab} + i \epsilon^{abc} \sigma^c) \\
&= \frac{r^\lambda}{2(r)^3} \mathbf{1}
\end{aligned} \tag{6.16}$$

which is, up to a constant, exactly what one would expect for the field of a magnetic monopole (remember that B^λ is matrix-valued (hence the factor of $\mathbf{1}$ as a reminder) and so $\text{Tr } B^\lambda = \frac{r^\lambda}{(r)^3}$).

6.1 Phase factor of the Wilson Loop for one Monopole

Using expressions (6.4) and (6.10) we may calculate the phase factor that occurs in the expression for the Wilson loop: $-\frac{i}{4} \int_D \text{Tr}(n D n D n + 2 n F)$ (see (5.1)). It is absolutely necessary to use the anti-Hermitian expression for the gauge field in this calculation, as expression (5.1) was derived using an explicitly anti-Hermitian gauge field.

Using (2.39) and (2.45), we complete this calculation using:

$$\begin{aligned}
\text{Tr}(nDnDn + 2nF) &= \text{Tr}(ndndn + 4d(nA)) \\
&= \text{Tr}(ndndn + 4dnA + 4ndA)
\end{aligned} \tag{6.17}$$

Relatively mechanical calculations yield the following results:

$$\text{Tr}(ndndn) = 2i\epsilon^{abc} \frac{r^a}{(r)^3} dr^b dr^c \tag{6.18}$$

(this was derived in (6.6)). Likewise,

$$\begin{aligned}
\text{Tr}(4dnA) &= 4 \left(\frac{i\epsilon^{abc} r^c dr^d dr^a}{(r)^3} \delta^{db} + \frac{i\epsilon^{abc} r^c r^d r^e dr^e dr^a}{(r)^5} \delta^{db} \right) \\
&= \frac{4i\epsilon^{abc} r^c dr^b dr^a}{(r)^3}
\end{aligned} \tag{6.19}$$

where the second term in the first line must vanish by an anti-symmetric argument ($r^c r^d \delta^{bd}$ is symmetric in b and c while ϵ^{abc} is not). Lastly,

$$dA = \left(\frac{i\epsilon^{\mu a \lambda} \sigma^a}{2(r)^2} - \frac{i\epsilon^{\mu a \nu} \sigma^a r^\nu r^\lambda}{(r)^4} \right) dr^\lambda dr^\mu$$

and so

$$\begin{aligned}
\text{Tr}(4ndA) &= \left(\frac{4i\epsilon^{\mu a \lambda} r^b}{(r)^3} \delta^{ab} - \frac{8i\epsilon^{\mu a \nu} r^\nu r^\lambda r^b}{(r)^5} \delta^{ab} \right) dr^\lambda dr^\mu \\
&= \frac{4i\epsilon^{\mu a \lambda} r^a}{(r)^3} dr^\lambda dr^\mu
\end{aligned} \tag{6.20}$$

where, once again, the second term in the first line vanishes by the obvious symmetry argument.

Putting all of this together,

$$\begin{aligned}
\text{Tr}(ndndn + 4dnA + 4ndA) &= (2 - 4 + 4) \frac{i\epsilon^{abc} r^a dr^b dr^c}{(r)^3} \\
&= 4i \frac{r^a}{(r)^3} dS^a
\end{aligned} \tag{6.21}$$

and therefore

$$\begin{aligned}
 \exp\left(-\frac{j}{4}\int_D \text{Tr}(nDnDn + 2nF)\right) &= \exp\left(-\frac{j}{4}\int_D \text{Tr}(ndndn + 4d(nA))\right) \\
 &= \exp\left(-ij\int_D \frac{r^a}{(r)^3}dS^a\right) \\
 &= \exp(-ij\Omega)
 \end{aligned} \tag{6.22}$$

where Ω is the solid angle subtended by the Wilson loop from the monopole site (in our case, the origin). Note that (6.22) remains single-valued for changes of 4π in Ω . This is important, as this was a requirement upon which we insisted in defining our monopoles. The solid angle, Ω , is only defined modulo 4π , but the Wilson loop formula is indifferent to this, as well it should be ($e^{-ij(\Omega+4\pi)} = e^{-ij\Omega}e^{-4\pi ij} = e^{-ij\Omega}$).

6.2 Charge Quantization

It is an empirical fact that charges appear to be quantized. This quantization of electric and magnetic charges appears naturally in compact theories – for example, in the Abelian case, compact QED. In the case of non-Abelian field theories, such as this one, the compactness directly follows from the structure of the Lie algebra (cf. the brief discussion of Lie algebras in Section 1.1). In fact, as discussed in [17, 29], as the group structure of $SU(2)$ is a sphere, an Abelian subgroup will be some circle on this sphere, and thus necessarily compact. According to Gauss' theorem, the quantization of charge will necessarily imply the quantization of electric and magnetic flux.

We now, by considering the magnetic flux of the single monopole at the origin, explicitly demonstrate this magnetic charge quantization. For a similar derivation, and a longer discussion of the topological significance, see [27].

For a quantum of magnetic charge, g , we have by Gauss' theorem:

$$g = \int_{S_\infty^2} \vec{B} \cdot d\vec{S} \tag{6.23}$$

where we are integrating over a 2-sphere at the boundary of our three-dimensional space. We have, to this point, been suppressing the electrical coupling constant, e , in our calculations. Including this now we have:

$$\begin{aligned}
 g &= \int_{S_\infty^2} \vec{B} \cdot d\vec{S} \\
 &= \frac{1}{e} \int_{S_\infty^2} \frac{r^\lambda}{(r)^3} dS^\lambda \\
 &= \frac{4\pi m}{e} \quad m \in \mathcal{Z}
 \end{aligned} \tag{6.24}$$

and therefore we have a standard Dirac-type quantization condition:

$$eg = 4\pi m \quad m \in \mathcal{Z} \tag{6.25}$$

where m is some integer which acts as a winding number. The idea behind this is that as the coordinate \vec{x} covers the sphere S_∞^2 exactly once as the integral is evaluated, n maps this coordinate to a point on the unit sphere (since $n^2 = 1$). The number of times that n covers the unit sphere must be an integer, or otherwise the n field would not be single-valued. Now, in *our* particular case, with the derived expressions for the n , A and F fields, it is clear that $m = 1$. But the argument holds for more general gauge field configurations.

In fact, as discussed in [27], the integer m is called *Kronecker's index* and it is a topologically important quantity characterizing the mapping by n to the unit sphere.² This is to be expected, since, from the beginning, the monopoles did not arise out of dynamical concerns, but rather from the topological characteristics of the n fields.

²There are at least three different, but equivalent, ways of characterizing the topology of the mapping $n(r) : S_\infty^2 \rightarrow S_{R=1}^2$. These are the *Kronecker index*, the *Brouwer degree*, and the *Poincaré-Hopf index*. It is not relevant to give a full explanation of these here – see [27] for a clear explanation and discussion.

6.3 Many Monopoles and the Coloumb Gas

We will now begin to consider the treatment of an arbitrary distribution of monopoles. To do so, it is far easier to treat the monopoles within the gauge field, instead of dealing primarily with the n field. As has been mentioned before, the mathematical location of the monopoles may be moved between the gauge and n fields by a gauge transformation.

We will perform a rotational gauge transformation which will transform the radial n field ($\vec{n} \propto \hat{r}$) into a constant field pointing in the positive direction along the third (\hat{z}) axis. As one would guess, this transformation is singular along the negative half of the third axis. In performing this transformation we draw upon the results of both Boulware *et al* [28] and Polyakov [29].

The gauge transformation, $V(\vec{r})$, is given by a standard rotational operator:

$$V(\vec{r}) = e^{-i\frac{\sigma^3}{2}\phi} e^{i\frac{\sigma^2}{2}\theta} e^{i\frac{\sigma^3}{2}\phi} \quad (6.26)$$

where

$$\hat{r} = \cos \theta \hat{z} + \sin \theta (\cos \phi \hat{x} + \sin \phi \hat{y})$$

Then,

$$n \rightarrow V n V^\dagger = \delta^{a3} \sigma^a \quad (6.27)$$

$$\begin{aligned} A_\mu &\rightarrow V \left(A_\mu^i(x) \frac{\sigma^i}{2} - i \partial_\mu \right) V^\dagger \\ &= i \delta^{a3} \epsilon^{\mu 3 \nu} \left(\frac{r^\mu}{r(r - r^3)} \right) \frac{\sigma^a}{2} \end{aligned} \quad (6.28)$$

The advantage of this is that in the gauge field the monopoles appear as normal point sources of a Coulomb-type interaction and thus an arbitrary distribution of monopoles may be built up from a simple superposition of isolated point sources [17, 29].

Given this gauge field, the field strength becomes effectively Abelian

($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$). Indeed, only one index is required to label the non-zero components

of the field strength tensor. Therefore, for a single monopole the field strength is given by:

$$F_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda^3 = \frac{1}{2} \frac{x_\mu}{(x)^3} - 2\pi \delta_{\mu 3} \theta(x_3) \delta(x_1) \delta(x_2) \quad (6.29)$$

where the second term is recognized as a standard Dirac string.³ Clearly, a distribution of an arbitrary number of well-separated monopoles and anti-monopoles will be a superposition of these single-monopole fields. Therefore, for (anti-)monopoles located at points $\{r_a\}$ we have:

$$F_\mu = \sum_a \frac{1}{2} q_a \frac{(x - x_a)_\mu}{|x - x_a|^3} - 2\pi \delta_{\mu 3} \sum_a q_a \theta(x_3 - x_{3a}) \delta(x_1 - x_{1a}) \delta(x_2 - x_{2a}) \quad (6.30)$$

where $q_a = \pm 1$ for monopoles/anti-monopoles.

The action for such a monopole distribution will have two distinct components – one representing the self-energy of the monopoles, the other representing their Coulomb interactions. As is generally recognized, the Dirac string singularities do not contribute to the energy or dynamics of the field. The self-energy term for the monopoles will provide some formally divergent factor with which we shall not be seriously concerned in the following calculations. However, we include it now for the sake of completeness. We will have:

$$S = S_{\text{se}} + S_{\text{int}} \quad (6.31)$$

where

$$S_{\text{se}} = \Delta(m) \sum_a q_a^2 \quad (6.32)$$

is the self-energy term and $\Delta(m)$ is some function which depends upon the mass, m , of the monopoles. Then, for the interaction term, removing the singular Dirac strings, we

³A Dirac string is a non-observable, singular line, which can be visualized as a tightly wound solenoid starting at the monopole and stretching off to infinity. Inclusion of such an object accounts for the magnetic flux of the monopole and thus allows one to treat the monopole within conventional electromagnetic theory.

have:

$$\begin{aligned}
S_{\text{int}} &= \frac{1}{2e^2} \int (F_\mu - F_\mu^{\text{sing.}})^2 d^3x \\
&= \sum_{a \neq b} \frac{q_a q_b}{8e^2} \int d^3x \frac{(x - x_a)}{|x - x_a|^3} \frac{(x - x_b)}{|x - x_b|^3} \\
&= \frac{\pi}{2e^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|}
\end{aligned} \tag{6.33}$$

where the region of integration in the first two lines excludes the monopole sites and some small region surrounding them.

The next step consists in assembling a grand canonical ensemble, by summing over all possible monopoles and anti-monopoles, using Boltzmann statistics. In doing so we end up transforming our theory into a Sine-Gordon model.

6.4 The Sine-Gordon Transformation and the Classical Solutions

In the following calculations, as we transform to a Sine-Gordon model and then solve for the classical solutions, we follow quite closely the work of Polyakov [29].

From (6.22), (6.32) and (6.33) we have

$$\langle W \rangle = \frac{1}{Z} \sum_{N, q_i} (e^{-\Delta \sum q^2})^N \frac{1}{N!} \int \prod_{j=1}^N dx_j \exp \left(-\frac{\pi}{2e^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|} - i j q_a \Omega \right) \tag{6.34}$$

where the factor of $\frac{1}{N!}$ was introduced to account for a sum over all possible N *indistinguishable* monopoles, and where

$$Z = \sum_{N, q_i} (e^{-\Delta \sum q^2})^N \frac{1}{N!} \int \prod_{j=1}^N dx_j \exp \left(-\frac{\pi i}{2e^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|} \right)$$

is the partition function.

Now, recalling the Gaussian integration formula (cf. Appendix B):

$$\int \prod_i dx_i \exp \left(-\frac{1}{2} \sum_{i,j} A_{ij} x_i x_j + i \sum_i b_i x_i \right) = (\det A)^{-\frac{1}{2}} \exp \left(-\sum_{i,j} A_{ij}^{-1} b_i b_j \right)$$

and noticing that

$$\nabla^2 \frac{1}{|x_a - x_b|} = -4\pi\delta(x_a - x_b)$$

we are led to transform (6.34) to the following integral over a field $\chi(x)$:

$$\langle W \rangle = \frac{1}{Z} \int \mathcal{D}\chi(x) \exp\left(-\frac{e^2}{4\pi^2} \int d^3x (\nabla\chi)^2\right) \sum_{N, q_a} \frac{(e^{-\Delta \sum q^2})^N}{N!} \prod_{j=1}^N dx_j e^{-i \sum_a q_a \chi} e^{-\sum_a i j \Omega q_a} \quad (6.35)$$

Now consider the factor:

$$\frac{1}{N!} \sum_{N, q_a} e^{-i \sum_a q_a (\chi + j\Omega)} = \frac{1}{N!} \sum_N \left(e^{i(\chi + j\Omega)} + e^{-i(\chi + j\Omega)} \right)^N \quad (6.36)$$

$$= e^{2 \cos(\chi + j\Omega)} \quad (6.37)$$

We have, therefore,

$$\langle W \rangle = \frac{1}{Z} \int \mathcal{D}\chi(x) e^{-\frac{e^2}{4\pi^2} \int ((\nabla\chi)^2 - M^2 \cos(\chi + j\Omega)) d^3x} \quad (6.38)$$

where

$$M^2 = \frac{8\pi^2}{e^2} (e^{-\Delta(m)} \sum q^2) \quad (6.39)$$

We have succeeded in deriving a version of the Sine-Gordon action:

$$S_{\text{SG}} = \int (\nabla\chi \cdot \nabla\chi - M^2 \cos(\chi + j\Omega)) d^3x$$

Again, we may apply the saddle-point approximation to this action, provided that the constant $\frac{e^2}{4\pi^2}$ is sufficiently large. Trivially, the classical equation of motion is given by:

$$\nabla^2 \chi + M^2 \sin(\chi + j\Omega) = 0 \quad (6.40)$$

Given this, we now proceed to solve for the field $\chi(x)$ which will solve this equation.

We consider solutions in the region of the large Wilson loop, away from the boundary. That is, within a distance of several multiples of $\frac{1}{M}$, and certainly distances much smaller

than the size of the loop. Within this region the solid angle function, $\Omega(x)$ only takes on the values $\pm 2\pi$. Take the \hat{z} axis as running through the centre of the loop, orthogonal to the loop surface. Then, far from the boundaries, $\chi(x)$ (classical) has essentially one degree of freedom (z). Denote $\chi' = \frac{\partial \chi}{\partial z}$. Then multiply both sides of (6.40) by χ' and integrate. We get:

$$\frac{1}{2}(\chi')^2 - M^2 \cos(\chi + j\Omega) + E = 0 \quad (6.41)$$

for some constant of integration, E . This constant is fixed by the boundary conditions which require that at spatial (or temporal) infinity, $\Omega = 0$ (the solid angle of the loop approaches zero as we move away an infinite distance) and, at infinity, the potential $\cos(\chi + j\Omega)$ should be minimized, and lastly, all derivatives of $\chi(x \rightarrow \infty)$ should vanish. This necessarily implies that $E = -M^2$.

Now use the standard trigonometric identity

$$\cos a - \cos b = -2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{a-b}{2}\right)$$

which implies $2 \sin^2(\frac{a}{2}) = 1 - \cos a$. Therefore we can rewrite (6.41) as

$$(\chi')^2 = 4M^2 \sin^2\left(\frac{\chi + j\Omega}{2}\right) \quad (6.42)$$

which implies

$$\int \frac{\pm d\chi}{2M \sin\left(\frac{\chi + j\Omega}{2}\right)} = \int dz \quad (6.43)$$

This can be integrated using the formula $\int \frac{dz}{\sin z} = \ln \tan\left(\frac{z}{2}\right)$. The solution is

$$\chi = \pm 4 \arctan(e^{\mp Mz}) - j\Omega(z) \quad (6.44)$$

It remains to choose the appropriate signs for the solution such that the χ field is continuous across $z = 0$. It is clear that, to allow for well-behaved solutions at infinity and a continuous solution at $z = 0$, we must choose our solutions as:

$$\chi(z) = \begin{cases} +4 \arctan(e^{-Mz}) - j\Omega(z) & \text{if } z > 0 \\ -4 \arctan(e^{+Mz}) - j\Omega(z) & \text{if } z < 0 \end{cases} \quad (6.45)$$

remembering that the solid angle, $\Omega(z)$, will tend to zero at large distances from the loop, and, close to the loop, will take on a value of 2π on one side of the loop, discontinuously switching to -2π on the other side of the loop.

6.5 A Final Look at Confinement

We will now demonstrate that (6.45) provides for area law behaviour of large Wilson loops, this being the working definition of confinement. Moreover, the existence of spin dependence will be indicated.

Consider the change in the field $\chi(z)$ from $z = +\epsilon$ to $z = -\epsilon$ for some small positive constant ϵ . For z small and positive,

$$\chi|_{z=0+} = 4\left(\frac{\pi}{4} + \pi n\right) - 2\pi j$$

and for z small and negative,

$$\chi|_{z=0-} = -4\left(\frac{\pi}{4} + \pi m\right) + 2\pi j$$

where m and n are some arbitrary integers, indicative of the periodicity of the tangent function. For χ to be continuous at $z = 0$ it must be the case that

$$\pi + 4\pi n - 2\pi j = -\pi + \pi m + 2\pi j$$

or

$$2\pi + 4\pi(n - m) - 4\pi j = 0 \tag{6.46}$$

This equation can only be satisfied when j is a half-odd-integer. Clearly, this solution cannot be applied to a system with integer-valued j . While this does not constitute a rigorous proof of non-confinement for states of integer spin, it does clearly indicate that the theory differentiates between integer and half-odd-integer spins and goes some

distance in justifying the expected and intuitive result that integer spin states should remain unconfined.

Lastly, substitution of (6.45) into the Wilson loop expression, (6.38), yields:

$$\langle W \rangle = e^{-\gamma A} \quad (6.47)$$

where $\gamma = \left(\frac{e^2}{4\pi^2}\right) M \int_{-\infty}^{\infty} (\chi''(x-y) + M^2 \cos(\chi(y) + j\Omega)) dy$ and where A is the surface area of the loop. This result follows from the fact that $\chi(z)$ is a function of z only and so in (6.38) the integration over x and y will yield the surface area of the loop (further outside of the loop in these directions we assume the trivial solution which will integrate to zero).

Therefore, we have clearly demonstrated the confinement of quarks in this monopole gas in cases of the spin j being a half-odd-integer. For integer j we have demonstrated the inadequacy of the solution, which is in agreement with the well-established expectation that integer-spin quarks should be unconfined due to screening by $j = 1$ gluons.

Chapter 7

Conclusions

In considering the expectation value of the Wilson loop, $\langle W[\Gamma] \rangle$, in a 2+1 dimensional SU(2) Yang-Mills theory we have derived several interesting results, all at the purely semi-classical level. We managed to reformulate the Wilson loop as a surface integral with an explicit spin parameter, j . Then, in order to test Polyakov's conjecture, we considered $F_{\mu\nu}^a$ and A_μ^a as independent, random variables. For large values of the spin-parameter the Wilson loop expression may be evaluated using a saddle-point approximation. By this method we calculated the classical equations of motion to discover that, indeed, to this level, the Bianchi Identity is indeed re-introduced as a natural constraint:

$$D_\mu F_{\nu\lambda} + D_\lambda F_{\mu\nu} + D_\nu F_{\lambda\mu} = 0$$

Put another way, at the level of the saddle-point approximation $F_{\mu\nu}$ and A_μ are related by the standard relationship:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$$

We also demonstrated, in Chapter 5 an area law for the expectation value of the Wilson loop:

$$\lim_{A \rightarrow \infty} (-\ln \langle W[\Gamma] \rangle) \sim \sigma_T A$$

where A is the area spanned by the Wilson loop and σ_T is a quantity known as the *string tension*, thereby indicating a confining potential. However, any possible spin-dependence was concealed at this preliminary stage.

Secondly, and very importantly, we noted that this reformulation of the Wilson loop allowed for an arbitrary distribution of monopoles – singular points in the three dimensional spacetime. Using the classical equations of motion derived in Chapter 4, we derived explicit expressions for the fields in the case of a single monopole at the origin. These expressions yielded a magnetic field for the monopole of

$$\text{Tr } B^\mu = \frac{r^\mu}{(r)^3}$$

and a phase factor associated with the Wilson loop expression:

$$\exp\left(-\frac{j}{4} \int_D \text{Tr}(nDnDn + 2nF)\right) = \exp(-ij\Omega)$$

where Ω is the solid angle subtended by the Wilson loop from the monopole site. We also derived the very standard charge quantization condition:

$$eg = 4\pi m \quad m \in \mathcal{Z}$$

Next, a monopole gas was introduced by considering a simple superposition of monopole fields in the gauge field. This was shown to yield a Sine-Gordon action:

$$\langle W \rangle = \frac{1}{Z} \int \mathcal{D}\chi(x) e^{-\frac{e^2}{4\pi^2} \int ((\nabla\chi)^2 - M^2 \cos(\chi + j\Omega)) d^3x}$$

the classical solutions of which indicated confinement of states with half-odd-integer spin:

$$\chi(z) = \begin{cases} +4 \arctan(e^{-Mz}) - j\Omega(z) & \text{if } z > 0 \\ -4 \arctan(e^{+Mz}) - j\Omega(z) & \text{if } z < 0 \end{cases}$$

indicating, for $j = m + \frac{1}{2}$, $m = 0, 1, 2 \dots$ that:

$$\langle W \rangle = e^{-\gamma A}$$

where γ is a particular string tension and where A is the surface area of the Wilson loop. In the case of integer spin, it was demonstrated that these solutions are inadequate, with the expectation that such states are not confined at all.

While answering some interesting questions, this work also raises a number of questions, suggesting directions for future investigations. The most obvious, though difficult, next step is to consider the quantum fluctuations about the classical solutions derived in this paper. It will be interesting indeed to determine whether the higher-order terms in the solutions radically alter the behaviour illustrated by the classical solutions. As well, it remains to give a more rigorous proof of the deconfinement of quarks with integral spin within the framework of this particular theory.

At the beginning of this paper it was claimed that the Yang-Mills field theory concealed a wealth of interesting physics. We have considered the field strength and gauge fields to be independent, random variables and subsequently derived the Bianchi Identity in a sum over surfaces of the Wilson loop. We have demonstrated the natural appearance of monopoles. And we have linked this to earlier work using a Sine-Gordon treatment of a monopole gas. Indeed, the Yang-Mills field and the simple Wilson loop have yielded much fascinating physics — fascinating physics which is far from exhausted.

Bibliography

- [1] A.M. Polyakov, Nucl. Phys. **B120**, 429 (1977).
- [2] M.E. Peskin & D.V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, New York, 1995).
- [3] P. Ramond, *Field Theory: A Modern Primer* (Addison-Wesley Publishing Company, New York, 1989).
- [4] H. Goldstein, *Classical Mechanics*, 2nd Ed. (Addison-Wesley Publishing Co., Don Mills, 1980).
- [5] K.Wilson, Phys. Rev. **D10**, 2445 (1974).
- [6] M. Creutz, *Quarks, Gluons and Lattices* (Cambridge University Press, Cambridge, 1983).
- [7] A.A. Migdal, Phys. Repts. **102**, 199 (1983).
- [8] A.M. Polyakov "Confining Strings" July 1996 preprint hep-th/9607049.
- [9] A.M. Polyakov, Phys. Lett. **59B**, 82 (1975).
- [10] D.I. Diakonov & V.Yu. Petrov, Phys. Lett. B **224**, 131 (1989).
D.I. Diakonov & V.Yu. Petrov, June 1996 preprint hep-th/9606104.
- [11] Yu.A. Simonov, Sov. J. Nucl. Phys. **50**, 134 (1989).
- [12] C. Itzykson & J.M. Drouffe, *Statistical Field Theory, Vol.2* (Cambridge University Press, Cambridge, 1989).
- [13] D.I. Diakonov & V.Yu. Petrov, Phys. Lett. B **242**, 425 (1990).
- [14] I.S. Gradshteyn & I.M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press Inc., Toronto, 1980).
- [15] G. Arfken, *Mathematical Methods for Physicists* (Academic Press Inc., Toronto, 1985).
- [16] G. Roepstorff, *Path Integral Approach to Quantum Physics – An Introduction* (Springer-Verlag, New York, 1991).

- [17] A.M. Polyakov, *Gauge Fields and Strings* (Harwood Academic Publishers, New York, 1987).
- [18] H. Arodź, Phys. Rev. **D49**, 4154 (1994). (pre-print hep-th/9309089).
- [19] S. Jaimungal, private communication.
- [20] M. Nakahara, *Geometry, Topology and Physics* (Adam Hilger, New York, 1990).
- [21] E. Kreyszig, *Introduction to Differential Geometry and Riemannian Geometry* (University of Toronto Press, Toronto, 1968).
- [22] L.C. Lew Yan Voon, *An Investigation of Coherent State Path Integrals as Applied to a Harmonic Oscillator and a Single Spin*, (University of British Columbia thesis, Vancouver, 1987).
- [23] A. Perelomov, *Generalized Coherent States and their Applications* (Springer-Verlag, New York, 1986).
- [24] S.R. Wadia, *Gauge Theories of Quark Confinement in 2+1 Dim: A Comparative Study of $SU(2)$ Yang-Mills and Yang-Mills-Higgs Systems*, from *Particles and Fields 2*, Banff 1981 Conference Proceedings, A.Z. Capri & A.N. Kamal, ed., (Plenum Press, New York, 1983). 673.
- [25] N. J. Snyderman, Nuc. Phys. **B218**, 381 (1983).
- [26] S. Coleman, *The Magnetic Monopole Fifty Years Later* (in *The Unity of the Fundamental Forces*, ed. A. Zichichi) (Plenum Press, New York, 1983).
- [27] J. Arafune, P.G.O. Freund & C.J. Goebel, J. Math. Phys. **16**, 433 (1975).
- [28] D.G. Boulware, L.S. Brown, R.N. Cahn, S.D. Ellis, C. Lee, Phys. Rev. **D14**, 2708 (1976).
- [29] A.M. Polyakov, Nuc. Phys. **B120**, 429 (1977).

Appendix A

Properties of Coherent States

A brief introduction to coherent states, and some of their more relevant properties is presented here. The set of coherent states is a well developed area of research, and of wide application. It is not our aim here to delve deeply into this topic, but merely to present information directly relevant to the calculations in this paper. For further information on coherent state representations see [22] or, for a more thorough treatment, [23].

Given an annihilation and creation operator, a and a^\dagger , respectively, of Bose statistics in a Fock space,

$$[a, a^\dagger] = 1 \quad a|0\rangle = 0 \quad (\text{A.1})$$

with basis vectors:

$$\left\{ \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \quad n = 0, 1, 2, \dots \right\} \quad (\text{A.2})$$

Then the coherent states, $|z\rangle$, are defined as the eigenstates of the annihilation operator of this Bose algebra.

$$a|z\rangle = z|z\rangle \quad (\text{A.3})$$

and can be generated by the following unitary transformation:

$$|z\rangle = e^{za^\dagger - z^*a} |0\rangle \quad z \in \mathcal{C} \quad (\text{A.4})$$

$$= e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} (a^\dagger)^n |0\rangle \quad (\text{A.5})$$

The full set of coherent states is overcomplete:

$$\int d^2z |z\rangle \langle z| = \pi \quad (\text{A.6})$$

as we will now demonstrate in detail. To continue, we make use of the Baker-Cambell-Hausdorff formula. This states that

$$\begin{aligned} \text{if } [A, [A, B]] = 0 \text{ and } [B, [A, B]] = 0 \\ \text{then } e^{A+B} = e^A e^B e^{-\frac{1}{2}[A, B]} \end{aligned} \quad (\text{A.7})$$

Therefore, using (A.1),

$$\begin{aligned} e^{a^\dagger z - az^*} |0\rangle &= e^{a^\dagger z} e^{-az^*} e^{-\frac{1}{2}zz^*} |0\rangle \\ &= e^{-\frac{1}{2}zz^*} e^{a^\dagger z} |0\rangle \end{aligned} \quad (\text{A.8})$$

and likewise for $\langle 0|e^{-za^\dagger + z^*a} = \langle 0|e^{z^*a} e^{-\frac{1}{2}zz^*}$. Therefore,

$$\int d^2z |z\rangle\langle z| = \int e^{a^\dagger z} |0\rangle\langle 0| e^{az^*} e^{-zz^*} dz dz^* \quad (\text{A.9})$$

Now expanding the exponentials on the right-hand side of the above equation, and keeping in mind that $(a^\dagger)^n |0\rangle = \sqrt{n!} |n\rangle$, we complete the calculation:

$$\begin{aligned} \int d^2a |z\rangle\langle z| &= \int d^2z \sum_n \left(\frac{1}{n!}\right)^2 e^{-zz^*} (za^\dagger)^n |0\rangle\langle 0| (az^*)^n \\ &= \sum_n \left(\frac{1}{n!}\right) |n\rangle\langle n| \int d^2z e^{-zz^*} (zz^*)^n \\ &= \sum_n \left(\frac{1}{n!}\right) (2\pi) |n\rangle\langle n| \int_0^\infty dr e^{-r^2} r^{2n+1} \\ &= \sum_n \left(\frac{1}{n!}\right) \pi |n\rangle\langle n| \int_0^\infty dx e^{-x} x^n \\ &= \sum_n \left(\frac{1}{n!}\right) |n\rangle\langle n| \pi \Gamma(n+1) \\ &= \sum_n |n\rangle\langle n| \pi \end{aligned} \quad (\text{A.10})$$

$$= \pi \quad (\text{A.11})$$

Therefore $\int d^2z |z\rangle\langle z| = \pi$.

There are a variety of other useful coherent state relations, all derivable using similar methods. A short list of useful relations follows:

Using (A.8), we have,

$$\begin{aligned} \langle z'|z\rangle &= e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|z'|^2} \langle 0|e^{az'^*} e^{a^\dagger z} |0\rangle \\ &= e^{-\frac{1}{2}(|z'|^2 + |z|^2)} e^{z'^* z} \end{aligned} \quad (\text{A.12})$$

Also,

$$e^{i\omega a^\dagger at}|z\rangle = |e^{i\omega t}z\rangle \quad (\text{A.13})$$

$$\text{The identity, } \mathcal{I} = \int \frac{d^2z}{\pi} |z\rangle\langle z| \quad (\text{from (A.11)}) \quad (\text{A.14})$$

$$\text{Tr}(\Theta) = \int \frac{d^2z}{\pi} \langle z|\Theta|z\rangle \quad (\text{A.15})$$

Appendix B

Solving Functional Integrals

Following is a brief discussion of the defining characteristics and methods for solving functional integrals – indispensable tools in field theoretical research. As such, functional derivatives, functional integrals, and functional determinants will be treated in order.

A functional is simply a function which *maps functions to numbers*. For example, we have the standard action functional, $S[X(t)]$. This functional takes as its *argument* functions $X(t)$ (it is conventional to write the argument of a functional in square brackets). Likewise, a functional integral will integrate, not over numbers, but over a *space of functions*. To continue with the example, a propagation amplitude would be given by:

$$U(x_a, x_b, T) = \int \mathcal{D}X(t) e^{iS[X(t)]}$$

where we are integrating over all possible paths of propagation, $X(t)$, each path weighted by a complex phase $e^{iS[X(t)]}$ (it is also conventional to write the measure of a functional integral using the script \mathcal{D} – in this paper we use both this notation, and the equivalent one, using square brackets, $[dX(t)]$).

B.1 Functional Derivatives

It is also possible to take *functional derivatives* of functionals. The functional derivative, denoted $\frac{\delta}{\delta X(t)}$, obeys the following axioms:

$$\frac{\delta}{\delta X(t)} X(T) = \delta(t - T) \tag{B.1}$$

or

$$\frac{\delta}{\delta X(t)} \int dT X(T) Y(T) = Y(t) \tag{B.2}$$

The normal rules for derivatives (ie. the product rule) also apply to functional derivatives.

When a functional depends upon a derivative, before taking a functional derivative one integrates by parts. Therefore:

$$\frac{\delta}{\delta X(t)} \int dT \partial_\mu X(T) V^\mu(T) = -\partial_\mu V^\mu(t) \quad (\text{B.3})$$

The functional derivative will be a very important tool in solving functional integrals perturbatively, by a saddle point approximation.

B.2 Functional Integrals

How does one solve a functional integral? A standard method of solving this somewhat ill-defined quantity uses perturbation theory. Section 4.2 provides a brief discussion of the saddle point method. In this method, functional derivatives are used to calculate a first term in the perturbative expansion, with higher order terms following. The second term is Gaussian. The full arguments of Section 4.2 will not be repeated here, but we quote the result (4.6):

$$\int \mathcal{D}\phi e^{-jS[\phi]} = e^{-jS[\phi^{sp}]} \int \mathcal{D}\phi e^{-\frac{1}{2!} \frac{\delta S}{\delta \phi_1 \delta \phi_2}(\phi^{sp}) \zeta_1 \zeta_2 + \dots \text{higher order terms} \dots} \quad (\text{B.4})$$

where $\phi = \phi^{sp} + \frac{1}{\sqrt{j}}\zeta$, j is some large parameter (thereby justifying the saddlepoint approximation) and where ϕ^{sp} is the saddle point as given by the vanishing of the first functional determinant ($\frac{\delta}{\delta \phi} S[\phi]|_{\phi^{sp}} = 0$).

Then given (B.4), the next step is to figure out how to calculate a Gaussian functional integral. This is done by considering the solution to a normal Gaussian integral, where we know that $\int dx e^{-ax^2} = \sqrt{\frac{\pi}{a}}$, and then extrapolating to the infinite dimensional case of a functional integral. Consider then

$$\left(\prod_k \int d\beta_k \right) \exp(-\beta_i A_{ij} \beta_j) \quad (\text{B.5})$$

where A is a symmetric matrix with eigenvalues a_i . Then write $\beta_i = O_{ij}x_j$ where O is the orthogonal matrix which diagonalizes A . Then:

$$\begin{aligned} \left(\prod_k \int d\beta_k \right) \exp(-\beta_i A_{ij} \beta_j) &= \left(\prod_k \int dx_k \right) \exp(-\sum_i a_i x_i^2) \\ &= \prod_i \left(\int dx \exp(-a_i x_i^2) \right) \\ &= \prod_i \sqrt{\frac{\pi}{a_i}} \end{aligned} \tag{B.6}$$

$$= (\text{const})(\det A)^{-\frac{1}{2}} \tag{B.7}$$

B.3 Functional Determinants

At first bizarre looking, functional determinants can readily be reformulated into a more intuitive format. Realize first that for a matrix B , with eigenvalues b_i , by diagonalizing the matrix the determinant may be expressed as the product of the eigenvalues. Therefore,

$$\det B = \prod_i b_i \tag{B.8}$$

$$\begin{aligned} &= \exp \left(\sum_i \ln b_i \right) \\ &= \exp[\text{Tr}(\ln B)] \end{aligned} \tag{B.9}$$

where the logarithm of a matrix is defined by its power series. These expressions, (B.8) and (B.9), provide methods for evaluating a functional determinant. For example, in Chapter 3, specifically (3.4), the determinant $\det^{-1}(i\partial_\tau + \frac{1}{2}A + \lambda)$ is replaced with the *product of eigenvalues*, $\prod_n (\frac{2\pi n}{\tau} + \frac{1}{2}A + \lambda)$. Then, of course, the challenge of evaluating this infinite product remains...but the task of rewriting the determinant in some mathematically well-defined way has been accomplished.

As another quick example, consider $\det(\partial^2 + \Lambda^2)$ (the Klein-Gordon operator) for

some constant Λ (a mass term). Then

$$\begin{aligned}
 \det(\partial^2 + \Lambda^2) &= \exp[\text{Tr}(\ln(\partial^2 + \Lambda^2))] \\
 &= \exp\left[\sum_k \ln(-k^2 + m^2)\right] \\
 &= \exp\left[(VT) \int \frac{d^d k}{(2\pi)^d} \ln(-k^2 + \Lambda^2)\right] \quad (\text{B.10})
 \end{aligned}$$

which illustrates the case of a continuous spectrum of eigenvalues. Here the factor (VT) represents the d -dimensional volume of the functional integral. Though there is no need to complete the calculation here, (B.10) may be subsequently evaluated using a dimensional regularization procedure, after a Wick rotation.

Also, very importantly, it is possible to evaluate a functional integral by a Feynman diagram expansion. A concrete example of this is given in [2], p.304. We will not demonstrate this in detail here, but the determinant, in some important cases, may be expressed as the *exponentiated sum of all one-loop diagrams*. In this case, the single loop is in the field over which the Gaussian integration occurs, and all numbers of external legs are provided by any other fields in the determinant which were not integrated.