PARTICLE DETECTORS

IN THE THEORY OF QUANTUM FIELDS ON CURVED SPACETIMES

By

JOHN FRASER CANT

B.Sc., The University of St. Andrews, 1975
M.S., The University of California at Los Angeles, 1979

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Department of Physics
The University of British Columbia
1956 Main Mall
Vancouver, Canada
V6T 1Y3
Date 1 May 1988
Abstract

Particle Detectors
in the Theory of Quantum Fields on Curved Spacetimes

This work discusses aspects of a fundamental problem in the theory of quantum fields on curved spacetimes - that of giving physical meaning to the particle representations of the theory. In particular, the response of model particle detectors is analysed in detail.

Unruh (1976) first introduced the idea of a model particle detector in order to give an operational definition to particles. He found that even in flat spacetime, the excitation of a particle detector does not necessarily correspond to the presence of an energy carrier - an accelerating detector will excite in response to the zero-energy state of the Minkowski vacuum.

The central question I consider in this work is - where does the energy for the excitation of the accelerating detector come from? The accepted response has been that the accelerating force provides the energy. Evaluating the energy carried by the (conformally-invariant massless scalar) field after the interaction with the detector, however, I find that the detector excitation is compensated by an equal but opposite emission of negative energy.
This result suggests that there may be states of lesser energy than that of the Minkowski vacuum. To resolve this paradox, I argue that the emission of a detector following a more realistic trajectory than that of constant acceleration - one that starts and finishes in inertial motion - will in total be positive, although during periods of constant acceleration the detector will still emit negative energy. The Minkowski vacuum retains its status as the field state of lowest energy.

The second question I consider is the response of Unruh's detector in curved spacetime - is it possible to use such a detector to measure the energy carried by the field? In the particular case of a detector following a Killing trajectory, I find that there is a response to the energy of the field, but that there is also an inherent 'noise'. In a two dimensional model spacetime, I show that this 'noise' depends on the detector's acceleration and on the curvature of the spacetime, thereby encompassing previous results of Unruh (1976) and of Gibbons & Hawking (1977).
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I should like to acknowledge the help and support of my supervisor, Bill Unruh. I have learnt more about what constitutes good physics from Bill than from anyone. His insistence that physically meaningful results are obtained only through measurement has led me away from mathematical over-indulgence.

I should also like to thank Duncan Muirhead and Bruce Sharpe for help with mathematical details, and Rebecca Elson and Ralph Perkins for proof reading the text.
Einstein regarded it as a blemish on his theory that matter is represented in his equations,

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \]  

(0.0.1)

by the essentially non-geometric stress tensor. Lacking an alternative, theorists have of necessity adopted non-geometric models for the description of matter in general relativity. These models have been predominantly classical - based on hydrodynamics, classical Einstein-Maxwell theory, and relativistic kinetic theory.

Recently, however, matter being presumed to be governed by quantum mechanical laws, a great deal of work has been devoted to an analysis of how quantum fields propagate on curved spacetime backgrounds.

The theory that will be discussed here deals with quantum fields in the presence of a gravitational field characterized by a classical metric. This should be a reasonable approximation when one works at length scales greater than the Planck length (about \( 10^{-35} \) m.), and when it is acceptable to ignore the back-reaction on the metric to particle creation. This simplification leads to considerable insight without having to face the difficulties associated with quantizing the gravitational field itself.

### 50.1 Particle creation in cosmological models

Schrodinger (1940) was perhaps the first to realize that particles would be created by a gravitational field. He considered the
classical waves in an expanding universe and showed that a wave travelling in a given direction will produce a weaker wave travelling in the opposite direction, as if a reflection from the expanding space had occurred. In contrast to what happens in true reflection, however, the amplitude of the forward wave is increased by this process. He surmised (without second quantization) that this phenomenon corresponds to the creation of particles; one of the pair moves forward with the wave, while the other moves backward.

Later, on the basis of quantum field theory, the creation of particles in cosmological models was independently re-discovered by Parker (1966 - 1971) and by Sexl & Urbantke (1967, 1969).

Quantum processes are thought to be of particular significance in modelling the early stages of the universe. Firstly, there is the question of the existence of a cosmological singularity (of infinite curvature and density); whereas classical general relativity predicts that a singularity is almost inevitable, the application of quantum theory results in at least a weakening of the singularity, if not its disappearance (Fischetti, Hartle & Hu, 1979).

Secondly, there is the issue of the high degree of homogeneity and isotropy found in the universe today. The work of Zel'dovich & Starobinski (1971) and Parker & Hu (1978) analysing an initially anisotropic universe, shows that particles created within a few Planck times ($t_p = 10^{-44}$ sec.) of the initial singularity give rise to a very rapid return to isotropy within about $10^3 t_p$, independent of the initial
conditions.

Finally, particle creation processes occurring very shortly after the 'big bang' may provide an alternative explanation for the origin of the $3^0$ K background radiation (Parker, 1976).

**0.2 Quantum Processes and Thermodynamics**

Hawking's discovery (1974) that blackholes can evaporate by radiating thermally made the theory of quantum fields on curved spacetime backgrounds into one of the most exciting new areas of theoretical physics. This result completed the heuristic arguments of Bekenstein (1972-1974) linking gravitation, quantum field theory, and thermodynamics.

The 'no hair' theorems (Israel (1967), Muller zum Hagen et al (1973), Carter (1970), Hawking (1972), Robinson (1974, 1975)) imply that a blackhole formed by gravitational collapse will rapidly settle down to a quasi-stationary state characterized by only three parameters: mass $M$, angular momentum $J$, and charge $Q$. There are a large number of possible initial states of a collapsing body that could produce a blackhole of given $M$, $J$, $Q$.

Classically this number would be infinite, because a given blackhole could be formed from an indefinitely large number of particles of arbitrarily small mass. Quantum mechanics dictates, however, that for complete collapse, the energies of the constituent particles should have been such that their wavelengths were less than the size of the blackhole. Accordingly, the number of possible initial configurations
that could have formed the blackhole is finite, and it is possible to associate an entropy to the blackhole as a measure of how much information is lost to an external observer as the horizon forms.

To make the blackhole into a fully thermodynamic object, it remained to ascribe to it a temperature,

$$ T = \left[ \frac{\partial S}{\partial M} \right]_{\mathcal{Q}}^{-1} $$

where 'S' is the entropy of the blackhole.

Hawking found that quantum processes near the horizon do indeed result in the blackhole emitting thermal radiation at a temperature,

$$ T = \frac{1}{8\pi m} $$

(see Chapter 1 for a discussion of this process). Based on this temperature, the entropy of a blackhole is determined to be

$$ S = \frac{A}{4}, $$

where 'A' is the surface area of the event horizon.

SO.3 Particle detectors in curved spacetime - vacuum states

The extension of quantum field theory to curved spacetime backgrounds is hampered by two interlocking conceptual problems. The first is the problem of choosing a vacuum state for the theory, and consequently of deciding what a 'particle' is. The second is in defining the observable quantities of the theory, in particular, the energy-momentum tensor - when the theory is quantized this becomes mathematically undefined and a regularization scheme must be adopted.

In the unfamiliar setting of a quantum field propagating on a curved spacetime background, it is unclear how to make a correspondence between mathematical quantum states and concrete physical
situations. In general, neither of the conventional concepts of 'vacuum' nor 'particle' can be introduced in a physically meaningful way.

In flat spacetime, the vacuum state is determined by the demand that it be a state that is invariant under the action of the symmetry group of the manifold - the Poincare group. This vacuum is a no-particle state for all inertial observers. In the general case where there are no symmetries to appeal to, the choice of vacuum has to be made on other grounds. Typically, what some observers consider to be a no-particle state, other observers will find to be a many-particle state. The meaning to be ascribed to the usual mathematical formalisms becomes unclear.

To give operational meaning to the particles that are so difficult to define in curved spacetime, Unruh (1976) introduced the idea of a model particle detector. If a detector becomes excited, then it can be said to have detected a particle. To use a detector in a curved spacetime, it is clear that one should understand its workings in the more familiar setting of Minkowski spacetime. Unruh discovered that even in flat spacetime there can be unusual responses: he found that if the detector accelerates through the Minkowski vacuum, then the detector will respond as if it was in a thermal bath.

Much discussion of the meaning of this strange result has appeared in the literature. The main section of my work is concerned with the energetics of the situation. If a detector can excite in response to the zero-energy of the Mikowski vacuum, then where does the excitation energy come from? Either the detector must gain its energy from the
accelerating force, or it must emit negative energy.

By evaluating the energy-momentum tensor, I find that the excitation of the detector is compensated by an emission of negative energy. The accelerating force does not provide the excitation energy, but it may have to do extra work to compensate for an asymmetrical emission.

To understand this result, and to deal with the possibility of there being a state of energy lower than that of the Minkowski vacuum, I am lead to considering a more realistic model than that of a detector that accelerates for all time. Looking at a trajectory that starts and finishes in inertial motion, together with an interaction that is switched on and off, I argue that while during periods of constant acceleration, the detector does indeed emit negative energy, the emission during periods of changing acceleration is such as to ensure that the total emitted energy is positive. The Minkowski vacuum remains the lowest energy state.

**SO.4 Particle detectors in curved spacetime – measuring the energy-momentum tensor**

The formal definition of the energy-momentum tensor gains physical meaning only when an infinite quantity is subtracted. A number of regularization schemes have been adopted which appear to yield meaningful results in the cases of a few simple curved spacetime backgrounds.

In the second section of my work, I consider the question of whether a particle detector of the type introduced by Unruh (1976) can be used to
measure the energy carried by the field, as described by the energy-momentum tensor.

I am able to show that the response of the detector separates into two parts - first, a response to the energy of the field, and second, a 'noise' that I argue is a function of the acceleration of the detector and of the spacetime curvature in its vicinity.

This encompasses the results of Unruh (1976), who found that a detector *accelerating* through flat spacetime would excite in response to the Minkowski vacuum, and of Gibbons & Hawking (1977) who found that a detector moving *inertially* through de Sitter spacetime would excite in response to the de Sitter-invariant vacuum. In the first case, the response is a 'noise' due to the fact that the detector is accelerating; in the second, the response is a 'noise' due to the fact that the spacetime is curved.

**0.5 Structure of the thesis**

In Chapter One, I illustrate the construction of a theory of quantum fields on curved spacetime backgrounds by giving a number of examples that show the inherent peculiarities. The particle production and energy flows in the 'Blackhole Evaporation' process are discussed, and I describe the effects produced by accelerating mirrors in flat spacetime. I note that in this theory there can be states of negative energy density.

Chapter Two discusses Unruh's particle 'detector accelerating in flat spacetime and gives an analysis of the back-reaction of the interaction
on the field. I argue that at constant acceleration the detector emits negative energy, whilst overall the emission is positive due to the effects of changing acceleration as the detector speeds up and slows down.

Chapter Three discusses the possibility of using Unruh's particle detector as an energy measuring device in curved spacetime. I argue that the detector responds to the energy carried by the field, but that a 'noise' is also introduced dependent on the acceleration of the detector and the curvature of the spacetime.

The Appendices give the technical details of the calculations involved. A Glossary of unfamiliar technical terms is included.
Chapter One

Construction of a quantum field theory on curved spacetime backgrounds - examples

To illustrate the construction of a quantum field theory on a curved spacetime background, I give a number of examples.

The first discusses the 'Blackhole Evaporation' effect discovered by Hawking, and shows how vacuum conditions existing near to the horizon of a blackhole will be interpreted by distant observers not as a vacuum but as a thermal state - distant observers see the blackhole evaporating.

The second considers the problem of regularizing the energy-momentum tensor. The application of the renormalized expression to the Blackhole Evaporation effect (by Fulling, Davies & Unruh) confirms that the thermal radiation seen by distant observers does indeed carry the energy expected.

Finally, a third example emphasises the extreme care needed in this theory in distinguishing particles from energy-carriers. Fulling & Davies (1976) found that an accelerating mirror in flat spacetime will emit particles, but that these particles carry no energy due to coherence effects. An extension of this example also demonstrates that negative energy densities can arise in this theory.
Blackhole evaporation

In this example, I discuss the different possible definitions of vacua in the Schwarzschild blackhole spacetime, leading to a derivation of the 'Blackhole Evaporation' effect discovered by Hawking (1975). The particular treatment given here is essentially a paraphrasing of the work of Unruh (1976).

For simplicity I consider a model theory of scalar photons, in the belief that this is sufficient to display the essentials of the subject.

§1.1 Wave Equations

Consider a massless scalar field in a background metric specified by metric $g_{\mu\nu}$. Minimal coupling of the Klein-Gordon equation to the geometry gives

$$\square \phi = g_{\mu\nu} \nabla^\mu \nabla^\nu \phi = 0$$

(1.1.1)

where

$$\square \equiv g_{\mu\nu} \nabla^\mu \nabla^\nu = (-g)^{1/2} \nabla_\mu \left[ (-g)^{1/2} g^{\mu\nu} \nabla_\nu \right]$$

(1.1.2, 1.1.3)

and $\nabla^\mu$ is the covariant derivative. However, it is conventional to introduce a further coupling to the geometry through the scalar curvature, $R$, of the form,

$$(\square + \xi R) \phi = 0$$

(1.1.4)

where $\xi$ is an adjustable parameter. The choice $\xi = 0$ (two dimensions) or $\xi = 1/6$ (four dimensions) is made because the equation is then invariant under conformal transformations, $g_{\mu\nu} \rightarrow C(x) g_{\mu\nu}$, in the
massless limit.

The significance of conformal invariance can be seen by considering the line element,

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]  

(1.1.5)

The paths followed by massless particles are null lines with \( ds = 0 \). Clearly the null line structure is not changed by a conformal transformation, and consequently it seems desirable that the wave equation for massless particles should also display this property.

1.2 *The Schwarzschild blackhole spacetime*

The Schwarzschild blackhole metric

\[ ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta \ d\phi^2\right) \]  

(1.2.1)

is thought to describe the gravitational field outside of a spherically symmetric time-independent source of gravitational energy that manifests a Newtonian mass 'm' when viewed from a sufficient distance.

The \((t, r)\) coordinates run out at \( r = 2m \), as can be seen from the seeming singularity in the metric at this point. This is only a coordinate singularity, however, for it is possible to cover the manifold with Kruskal (null) coordinates, \((u, v)\), in whose terms the line element is
\[
    ds^2 = \frac{2m}{r} \exp\left(-\frac{r}{2m}\right) du dv - r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)
\]

(See Appendix B for a discussion of null coordinate systems).

where

\[ r = r(u, v) \]

through

\[ u, v = - (4m)^2 \left(\frac{r}{2m} - 1\right) \exp\left(\frac{r}{2m}\right) \]

In fact the \((u, v, \theta, \phi)\) coordinates cover the whole manifold, whereas the \((t, r, \theta, \phi)\) cover only a segment. By suppressing the uninteresting angular dependence, the situation can be displayed as in Figure 1.

---

Figure 1.2 The full blackhole spacetime

The \((u, v)\) coordinates cover the whole manifold with \(u = 0\) and \(v = 0\)
being the event horizons \((r - 2m)\). \(r = 0\) is a true singularity and
appears twice: once in the past and once in the future. The
Schwarzschild region is region (I), with mirror universe (II), and
blackhole interiors (III) and (IV).

Two dimensional models prove very useful in giving exact solutions to
situations that are not tractable in full generality. This approach of
suppressing the uninteresting angular dependence of the metric will be
used explicitly at certain points in my work.

S1.3 Quantization
The modified Klein-Gordon equation (1.1.2) may be derived from the
Lagrangian density,
\[
\mathcal{L} = \frac{1}{2} (-g)^{1/2} \left\{ g_{\mu\nu} \partial^\mu \Phi \partial_\nu \Phi - \Box R \Phi^2 \right\}
\]
whose complexified form admits a gauge symmetry,
\[
\Phi \rightarrow \exp(i\alpha) \cdot \Phi \quad \alpha = \text{constant}
\]
with corresponding conservation law
\[
S^k_{;k} = 0 \tag{1.3.3}
\]
where
\[
S_k \equiv -i \left\{ \Phi^*, \partial_k \Phi - \Phi^* \cdot \partial_k \Phi \right\}
\] (1.3.4)
The conservation of \(S\) allows a 'time'-independent
(hypersurface-independent) scalar product to be defined,
\[
(\Phi_1, \Phi_2) \equiv i \int \left( \Phi_1 \cdot \partial_\mu \Phi_2^* - \Phi_2 \cdot \partial_\mu \Phi_1^* \right) d\Sigma^\mu
\] (1.3.5)

This scalar product is not positive-definite for bosons, but becomes so
when restricted to an appropriate subspace of solutions to the wave
equations. Such solutions are termed 'positive frequency'.

The space of positive frequency solutions forms the Hilbert space of one-particle states, $H_1$. The $n$-particle spaces, $H_n$, are then formed as totally symmetric (bosons) $n$-fold tensor products of $H_1$ with itself. Second quantization then defines the positive frequency part of the field, $\Phi^+$, as an annihilation operator, mapping $H_n$ to $H_{n-1}$, and the negative frequency part, $\Phi^-$, as a creation operator mapping $H_n$ to $H_{n+1}$. In particular, a choice of positive frequency defines a 'vacuum state' through $\Phi^+ |0\rangle = 0$.

A major question in the theory of quantum fields on curved spacetimes is the problem of choosing a decomposition of solutions into positive and negative frequencies, or equivalently, of choosing a vacuum state.

§1.4 *Schwarzschild and Kruskal vacuum states*

In two dimensions, the massless wave equation can be written very simply in null Kruskal coordinates, $(u,v)$, as,

$$\partial_u \partial_v \Phi = 0$$

(1.4.1) (see Appendix B(b)).

This has solutions, $\{\exp(-iuu), \exp(-iuv)\}$, which can be used as basis functions for the field decomposition,

$$\Phi = \sum_{\omega} \left( \frac{\omega \cdot e^{-i\omega u}}{\sqrt{4\pi \omega}} + \frac{\omega^* \cdot e^{i\omega u}}{\sqrt{4\pi \omega}} \right) + (\sim)$$

(1.4.2)
where \( \left( \frac{2}{3} \right) \) denote right moving (u), and left moving (v) modes. In this way, a 'Kruskal vacuum state' can be defined by \( a_w |0\rangle_K = 0 \) for all \( w \).

Considering the \((t,r)\) coordinates in two dimensions,

\[
d s^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2
\]

let

\[
r^* \equiv r + 2m \cdot \ln \left( \frac{r}{2m} - 1 \right)
\]

then,

\[
d s^2 = \left(1 - \frac{2m}{r}\right) \left(dt^2 - dr^*^2\right) = \left(1 - \frac{2m}{r}\right) du^2 dv^2
\]

which defines a new set of null coordinates, \((u,v)\). In terms of these new coordinates, the wave equation remains invariant in form (see Appendix B),

\[
\frac{\partial^2 u}{\partial u^\lambda \partial u^\lambda} \bar{\Phi} = 0
\]

and one can define a 'Schwarzschild vacuum state', \( |0\rangle_S \), based on the mode solutions,

\[
\left\{ \frac{\exp(-i \frac{r}{4\pi} \frac{u}{\bar{\phi}})}{\sqrt{4\pi \bar{\phi}}} \right\} \cdot \bar{\exp}\left( -i \frac{r}{4\pi} \frac{v}{\bar{\phi}} \right)
\]

The different choices of positive frequency define two possible vacuum states, and one can ask how many Schwarzschild particles there are in the Kruskal vacuum state.
§1.5 Bogoliubov Transformations

If there exist two complete orthonormal sets of solutions to the wave equation on a region of spacetime, \( \{ f_i, f_i^* \} \) and \( \{ p_j, p_j^* \} \), then they are related by an expansion of one set in terms of the other,

\[
f_i = \sum_j (A_{ij} f_j + B_{ij} p_j^*)
\]

where 'A' and 'B' are scalar products and measure how much frequency mixing occurs.

The field can be expanded in terms of either set,

\[
\Phi = \sum_i (a_i f_i^* + a_i^* f_i)
\]

or equivalently,

\[
\Phi = \sum_j (b_j^* p_j + b_j p_j^*)
\]

A change of basis is known as a Bogoliubov transformation.

The 'vacuum' associated to the basis \( \{ f_i, f_i^* \} \) is defined by \( a_i|0_f>=0 \) for all 'i', while the 'vacuum' associated to the basis \( \{ p_j, p_j^* \} \) is defined by \( b_j|0_p>=0 \) for all 'j'. The number of 'b_j' particles in the 'a' vacuum is found to be,

\[
\mathcal{N}_j = \frac{<0|b_j^* b_j|0>_f}{f} = \sum_i |B_{ij}|^2
\]

Clearly the choice of basis has to be made carefully if the 'particles' associated to the representation are to have any more than a purely mathematical reality.

The relation between the Kruskal coordinates \( (u,v) \) and the
Schwarzschild coordinates $(u,v)$ is,
\begin{align*}
u &= -4m \cdot \exp \left( -\frac{\bar{u}}{4m} \right) \\
v &= 4m \cdot \exp \left( \frac{\bar{v}}{4m} \right)
\end{align*}
\tag{1.5.5}

and the Bogoliubov scalar products can be evaluated (see Appendix A(c)) as,
\begin{align*}
B_{\omega \lambda} &= \frac{(4m)^{-i4m\lambda - 1}}{2\pi} \sqrt{\lambda} \cdot \sqrt{i(4m\lambda)} \cdot \exp(-2\pi m \lambda) \cdot \omega^{-i4m\lambda} \\
A_{\omega \lambda} &= \exp(4\pi m \lambda) \cdot B_{\omega \lambda}
\end{align*}
\tag{1.5.6}

so that
\begin{equation}
\lambda_S^\alpha = \langle 0 | b_{\lambda}^+ b_{\lambda} | 0 \rangle_K = \sum_\omega |B_{\omega \lambda}|^2 = \left[ \exp \left( 8\pi m \lambda \right) - 1 \right]^{-1}
\tag{1.5.8}
\end{equation}

which is a Planck distribution at temperature $T = 1/(8\pi m)$.

The interpretation of this result is that if the $\Phi$ field is in the Kruskal vacuum state, then observers at infinity (for whom the $(t, r)$ coordinates are a natural coordinate system, and whose natural vacuum state and hence whose concept of particles is Schwarzschild) will view the Kruskal vacuum, at infinity, to be a thermal state at the Hawking temperature. They see the blackhole evaporating.

A number of points should be noted here:

\section{Discussion}

(i) This evaporation process clearly depends on the assertion that the $\Phi$ field is in the Kruskal vacuum state. That this should be the case has been argued by Unruh (1976), who examined the formation of a
blackhole from a collapsing dust cloud and thereby determined the state of the field after collapse (this process was first examined by Hawking (1975)).

(ii) As the (t,r) coordinates cover only one quarter of the manifold, the expansion of the $\Phi$ field in Schwarzschild modes is inadequate, and expansions appropriate to the other three quadrants must be included. As my questions about particle production have been restricted to region (I) only, I have left out these other expansions for simplicity. See Appendix A(a) for further discussion.

The Kruskal vacuum, being a thermal state for observers at infinity, implies that such observers see it as a mixed state. However, the Kruskal vacuum is a pure state of the field. In curved spacetimes, horizons may impose a degree of ignorance on observers. A pure state of the field on the whole spacetime may appear to be a mixed state to observers who remain on one side of the horizon - correlations that exist across the horizon will not be apparent to them.

(iii) A potentially confusing point is that a coordinate dependence has apparently appeared in the description of the physics. The particular coordinates used in the above example, Kruskal and Schwarzschild, are important because of their relation to the vacuum states under discussion. The Kruskal quantization is with respect to Kruskal modes; the Schwarzschild quantization is with respect to Schwarzschild modes.

The Schwarzschild coordinates have particular physical significance
to observers at infinity in that they are the natural (i.e. proper) measures of time and space for them - as \( r \to \infty \), \( ds^2 \to dt^2 - dr^2 \), which is the metric for Minkowski space.

Similarly, Kruskal coordinates are physically significant for observers near the past horizon. The right-moving (u) modes (those that escape to infinity as the thermal flux) are based on null coordinate, \( u^* \), which is an affine parameter on the past horizon - a 'proper' null coordinate for observers crossing the past horizon. (For more discussion on this point, see Unruh (1976)).

The energy-momentum tensor

S1.7 Introduction

The core of any theory of interacting fields is the set of currents that describes the interaction. Since the currents of general relativity are the components of the energy-momentum tensor, \( T^{\mu \nu} \), a fundamental task in developing the theory of quantum fields on curved spacetime backgrounds is to understand this tensor.

The energy-momentum tensor, like any current, is formally a bilinear product of operator-valued distributions (the field operators), and hence is meaningless. An infinite quantity has to be subtracted off through a renormalization scheme to obtain the physical current.

It has been known since the initial formulation of quantum electrodynamics that in the vacuum state, field strengths undergo random fluctuations analogous to the zero-point motion of harmonic oscillators. The subtraction process that sets the expectation value of
\[ \langle T^{\mu\nu} \rangle \] to zero in Minkowski spacetime amounts to disregarding the zero-point energy of the field oscillators.

Whereas the renormalization of \[ \langle 0 | T^{\mu\nu} | 0 \rangle \] to zero in flat spacetime seems natural in a system devoid of particles, it is not clear how it should be renormalized in curved spacetime, even assuming that one can somehow make a physically meaningful choice of a 'vacuum'. Introducing an external field, such as gravity, gives rise to a phenomenon known as 'vacuum polarization', where vacuum activity in the presence of the external field results in observable effects.

In flat spacetime, the first evidence of vacuum effects was provided by Casimir (1948), who found that there is a very weak attractive force between two parallel plane electrically-neutral conductors in a vacuum environment in flat spacetime.

B.S. deWitt's (1975) calculation of the energy-momentum tensor in this system yields (for the massless scalar field),

\[
\langle T^{\mu\nu} \rangle = \frac{3\lambda^4}{2\pi^2} \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{array} \right) + \frac{\pi^2}{1440 \alpha^2} \left( \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{array} \right)
\]

where 'a' is the separation of the plates in the z-direction, and '\lambda' is a high-frequency cut-off.
The cut-off dependent term is identical to the divergent expression obtained for \( \langle 0 | T_{\mu \nu} | 0 \rangle \) in a Minkowski spacetime empty of conductors, and hence may be identified as the part to be subtracted off, leaving the physical current. The finite remainder then constitutes the Casimir effect - a gas of negative energy density and negative pressure (tension) in the z-direction.

The Casimir energy is purely a vacuum energy - no real particles are involved, only virtual particles. Note that the energy density between the plates is negative. Quantum theory can therefore lead to violations of the hypotheses of the famous Hawking - Penrose theorems of classical gravity concerning the inevitability of singularities in spacetime.

\section*{1.8 Regularization and renormalization}

The formal expression for \( T_{\mu \nu} \) is,

\[
T_{\mu \nu} = 2 (-g)^{-1/2} \frac{\delta \mathcal{L}}{\delta g_{\mu \nu}} =
\]
where \( [ , ] \) is the anticommutator, and \( \ell \) is usually taken to be \( \frac{1}{6} \) to achieve conformal invariance in the massless limit (see Section 1.1).

This expression must first be 'regularized', meaning rewritten to be finite and well-defined. The two-point function, \( G(x,x') = \langle 0 | \Phi(x) \Phi(x') | 0 \rangle \), which is a well-behaved function of \( x,x' \) (so long as these points do not lie on each other's light cone) is used.

The regularized \( T_{\mu\nu} \) is defined as,

\[
[T_{\mu\nu}]_{\text{reg.}} \equiv \lim_{x \to x'} \{ \left( \frac{1}{2} - \ell \right) (g_{i\nu} \delta^{\sigma'} + g_{i\nu}^{'} \delta^{\sigma}) + 2(\ell - \frac{1}{2}) g_{\mu\nu} g_{i\sigma} - \ell (g_{\mu\nu} + g_{\mu\nu}^{'} ) + \frac{1}{4} g_{\mu\nu} (g_{i\sigma} - g_{i\sigma}^{'}). \}
\]

(1.8.1)

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\]

(1.8.2)

This expression is evaluated with the points \( x,x' \) separated along some curve, and then the points are brought back together again, renormalizing by throwing away suitable divergent terms. This technique is known as 'point-splitting'.

Christensen (1976) has shown that quartic, quadratic, linear and logarithmic divergences appear, together with terms that depend on the
details of how $x$ is separated from $x'$.

To use this information, the state must first be specified by constructing a Fock space based upon a set of modes $\{\phi_\omega^x, \phi_\omega^x\}$, and by giving the occupation numbers of the modes. A simple expansion of the field gives,

$$<\ell n_2\ell_3... | T_{\mu\nu}^{\text{reg}} | \ell n_2\ell_3... > = \sum_{\omega} T_{\mu\nu}^{\text{reg}} (\phi_\omega^x, \phi_\omega^x) + \sum_i 2n_i \cdot T_{\mu\nu}^{\text{reg}} (\phi_i^x, \phi_i^x)$$

(1.8.3)

where $T_{\mu\nu}^{\text{reg}} (\phi_i^x, \phi_i^x)$ is shorthand for (1.8.1) with the quadratic operators replaced by $\phi_i^x, \phi_i^x$.

Christensen's expression for the divergences is independent of the quantum state and may simply be subtracted from the result of the mode sums, (1.8.3), to give an expression that is independent of the distance of separation of the two points. Finally, the regularization can be relaxed, and the finite physical $<T_{\mu\nu}>$ recovered.

Section 1.9 The energy of blackhole evaporation

The blackhole evaporation process discussed in Section S1.5 furnishes an example of the use of the energy-momentum tensor and of the predictive power of two-dimensional model theories.

Davies, Fulling & Unruh (1976) have renormalized $<T_{\mu\nu}>$ in model two-dimensional spacetimes, and give the following prescription. A state with some of the properties expected of a 'vacuum state' can be
specified through a choice of positive frequency modes that go as 
\(\exp(-iwu), \exp(-iwv)\), where \((u,v)\) is a null coordinate system for the 
spacetime. If the metric is written as,

\[
ds^2 = C(u,v) \, du \, dv
\]

(1.9.1)
in this null coordinate system, where \('C(u,v)'\) is termed the 'conformal 
factor', then the energy-momentum tensor of this vacuum state is given 
by,

\[
\langle T^{\mu \nu} \rangle = \Theta^{\mu \nu} - \frac{3 u \nu R}{4 \pi}
\]

(1.9.2)
where \('\Theta'\) is,

\[
\Theta^{\mu \nu} = -(12\pi)^{-1} \, C^{1/2} \, \frac{\partial u^2}{2} \, (C^{-1/2})
\]

\[
\Theta^{\nu \mu} = -(12\pi)^{-1} \, C^{1/2} \, \frac{\partial v^2}{2} \, (C^{-1/2})
\]

\[
\Theta^{\mu \nu} = 0
\]

(1.9.3)

'\Theta' is not a geometric object, but contains all the global information
about the vacuum state chosen through the choice of positive frequency
modes based on the coordinate system \((u,v)\).

The two coordinate systems that were introduced to cover segments of
the blackhole spacetime are,

**Schwarzschild,**

\[
ds^2 = \left(1 - \frac{2m}{r}\right) \left(\frac{dt}{\sigma} - \frac{d\sigma}{dt}\right)^2
\]

\[
= \left(1 - \frac{2m}{r}\right) u^2 \, d\sigma^2
\]

(1.9.4)
and Kruskal,
\[ ds^2 = \frac{2m}{r} \cdot \exp\left(-\frac{m}{2}\right) \cdot du dv \]  
(1.9.5)

As discussed previously, the Schwarzschild coordinates cover only a quarter of the two dimensional manifold. To obtain the right-moving fluxes in the Schwarzschild and Kruskal vacua, the conformal factors

\[ c\equiv \left(1 - \frac{2m}{r}\right) \]

\[ C = \frac{2m}{\bar{r}} \cdot \exp\left(-\sqrt{2}m\bar{r}\right) \]  
(1.9.6)

respectively, are used in (1.9.2),

\[ S\langle 0 \mid T_{\tilde{u}\tilde{u}} \mid 0 \rangle_S = -(2\pi)^{-1} \cdot \bar{C}^{1/2} (\bar{C}^{-1/2}) \cdot \tilde{u}\tilde{u} \]

\[ K\langle 0 \mid T_{\tilde{u}\tilde{u}} \mid 0 \rangle_K = -(2\pi)^{-1} \cdot C^{1/2} (\bar{C}^{-1/2}) \cdot \tilde{u}\tilde{u} \]  
(1.9.7)

The evaluation gives,

\[ S\langle 0 \mid T_{\tilde{u}\tilde{u}} \mid 0 \rangle_S = (24\pi)^{-1} \cdot \left(\frac{3m^2}{2r^4} - \frac{m}{r^3}\right) \]

\[ K\langle 0 \mid T_{\tilde{u}\tilde{u}} \mid 0 \rangle_K = (24\pi)^{-1} \cdot \left(\frac{3m^2}{2r^4} - \frac{m}{r^3} + \frac{1}{32m^2}\right) \]  
(1.9.8)

(An analysis similar to this was given by Davies, Fulling & Unruh (1976), and Fulling (1977)),

where the tensorial properties of \( < T_{\tilde{u}\tilde{u}} > \) have been used to express \( K\langle 0 \mid T_{\tilde{u}\tilde{u}} \mid 0 \rangle_K \) in terms of the \((\tilde{u},\tilde{v})\) coordinate system.
The \( T_{00} \) component of \( \langle T_{\mu\nu} \rangle \) is the flux that ends up at future infinity. In the Schwarzschild vacuum, \( s \langle 0 | T_{00} | 0 \rangle_s, r \rightarrow \infty \rightarrow 0 \), as one would expect of a vacuum state for an observer in essentially flat spacetime. For the Kruskal vacuum, however,

\[
\langle 0 | T_{\text{rad}} | 0 \rangle_K, r \rightarrow 2m = 0
\]

\[
\langle 0 | T_{\text{rad}} | 0 \rangle_K, r \rightarrow \infty = \left( \frac{768\pi m^2}{\lambda} \right)^{-1}
\]

(1.9.9)

The interpretation of this is that the \( |0\rangle_K \) state corresponds to a vacuum state for observers crossing the past horizon of the blackhole: zero outward flux at \( r = 2m \). This demand, however, entails that there be an energy flux at infinity, which constitutes the Hawking radiation.

Although the energy tensor does not give information about the spectrum of the radiation, it has already been shown that the spectrum is thermal. To complete the connection between the energy tensor and the particle description given previously, the energy of a thermal spectrum at temperature \( T = 1/(8\pi m) \) is found to be,

\[
E = \left( \frac{2\pi}{\lambda} \right)^{-1} \int_{0}^{\infty} \frac{\omega}{(2\pi)^{1/2} (8\pi m \omega)^{-1/2}} d\omega = \left( \frac{768\pi m^2}{\lambda} \right)^{-1}
\]

(1.9.10)

(In four dimensions, an Albedo (reflectance) factor enters into this result).

The analysis shows that in the 'Blackhole Evaporation' process, the 'particles' carry precisely the energy that one would naively expect. In general, however, the relationship between particle fluxes and energy
fluxes is more subtle, as is demonstrated in the following example.

**S1.10 Quantum effects of an accelerating mirror**

Whereas in the case of blackhole evaporation the emitted particle flux carries energy, in general one has to be very careful not to confuse particles with energy-carriers. Fulling & Davies (1977) have discovered a system in flat spacetime where there is a particle flux that in fact carries no energy. The distinction between particles and energy carriers will be central to this thesis which is concerned with the energetics of the operation of particle detectors.

Fulling & Davies (1976) considered imposing a boundary condition of perfect reflection on a free field at a mirror: $\mathcal{D}(t, x(t)) = 0$, where $x(t)$ is the trajectory of the mirror. Here I will only consider the case of constant acceleration in a model two dimensional Minkowski spacetime.

Figure 1.10 Modes for the accelerating mirror
Trajectories of constant acceleration are hyperbolas, \(x^2 - t^2 = g^{-2}\), where 'g' is the proper acceleration of the trajectory (see Section S2.2). The easiest way to handle the boundary condition is to perform a conformal transformation of the coordinates, \((t, x)\), so that in the new coordinates, \((\tau, \varsigma)\), the mirror trajectory becomes a straight line, \(\tau = 0\), and the boundary condition becomes independent of 'time' \(T\).

Introducing the null Minkowski coordinates, \(u = t - x, v = t + x\), the wave equation for massless fields is very simply (see Appendix B(b)),

\[
\partial_u \partial_v \Phi = 0 \tag{1.10.1}
\]

Under a conformal transformation,

\[(u = t - x, v = t + x) \rightarrow (\hat{u} = \tau - \varsigma, \hat{v} = \tau + \varsigma),\]

the distance element becomes,

\[
ds^2 = du \cdot dv = \hat{C}(\hat{u}, \hat{v}). d\hat{u} d\hat{v} \tag{1.10.2}
\]

and the wave equation remains invariant in form,

\[
\partial_{\hat{u}} \partial_{\hat{v}} \Phi = 0 \Rightarrow \partial_{\hat{u}} \partial_{\hat{v}} \Phi = 0 \tag{1.10.3}
\]

(see Appendix B(b)).

Looking only at what happens to the right of the mirror for simplicity, the incoming left-moving modes will be taken to be the modes of the incoming vacuum,

\[
\left\{ \phi_{-\omega} \right\} = \left\{ \frac{e^{-i\omega \varsigma}}{\sqrt{4\pi \omega}} \right\} \tag{1.10.4}
\]

The right-moving modes, however, will have been affected by the presence of the boundary, and if it is possible to find a conformal
transformation of the form \( u \rightarrow u^A \); \( v \rightarrow v \) that straightens out the trajectory, the boundary condition becomes, \( \Phi (T, S = 0) = 0 \), and a complete set of positive frequency solutions to the wave equation satisfying the boundary condition and the initial vacuum condition will be,

\[
\left\{ \Psi_\omega \right\} = \left\{ (4\pi \omega)^{-1/2} \left( e^{-i\omega \sigma} - e^{-i\omega \hat{\sigma}} \right) \right\}
\]

(1.10.5)

Quantization then proceeds using these modes, thereby embodying the boundary conditions placed on the field.

In Appendix C, I find that the appropriate conformal transformation for the fluxes to the right of the mirror is given by,

\[
u \rightarrow \hat{u} (u) = -\frac{u^2}{u}
\]

(1.10.6)

and the Bogoliubov transformation between the modes (1.10.5) and the usual Minkowski modes has been evaluated by Davies & Fulling (1977) to give the number of particles in mode \( \gamma \) as,

\[
n_\gamma = \int_0^\infty |B_{0\gamma}|^2 \cdot d\lambda = \frac{g^2}{\pi} \int_0^\infty k_1^2(2g[3\lambda]^{1/2}) \cdot d\lambda
\]

(1.10.7)

with \( k_1 \) a MacDonald function (see Watson (1922)).

This expression diverges at the lower end of the range of integration. The accelerated trajectory is the integral curve of a Killing vector of the Minkowski spacetime, and due to this time-like symmetry, the particle production is constant over time, resulting in an infinite number of created particles over infinite time. To evaluate the rate of
particle production per unit time, the analysis would have to be repeated using wave packets (see for example, Hawking (1976)).

(Fulling, in a private communication, points out the mathematics leading to expression (1.10.7) are suspect, and that the argument on p.250 of the 1977 paper is necessary to show that the right-moving modes are not those of the vacuum).

The remarkable feature about the case of constant acceleration is that, in spite of the particle production, the hyperbolic trajectory produces no energy flux. To see this, the conformal factor from the line element in the new coordinates,

\[ ds^2 = g^{2/\alpha^2} \cdot d\alpha^2 d\nu = C d\alpha^2 d\nu \]  \hspace{1cm} (1.10.8)

is used in the expression for \( \langle T^{\alpha\beta} \rangle \),

\[ \langle T^{\alpha\beta} \rangle = - (12\pi)^{-1} \cdot C^{1/2} \left( \int C^{-1/2} \alpha^2 \right) \]  \hspace{1cm} (1.10.9)

to give \( \langle T^{\alpha\beta} \rangle = 0 \).

Clearly coherence effects are operating. This example illustrates that the connection between particles and energy in quantum field theory is much more subtle than in either non-relativistic quantum mechanics or in classical general relativity. One cannot, as some authors have attempted, treat energy questions merely by counting particles. This distinction will be of great importance in the context of particle detectors which will be discussed in the next two chapters.
1.1.1 Negative energy densities

Fulling & Davies (1976) also considered more general trajectories than constant acceleration and found that in general, the energy flux to the right of the mirror is given by,

\[ T_{uu} = -(L^2\pi)^{-1} \cdot \frac{(1-a^2)^{1/2}}{(1-v^2)^2} \cdot \frac{da}{dt} \]  
(1.11.1)

where 'v' is the velocity of the mirror, and 'a' the proper acceleration. Note that the mirror will radiate negative energy if its acceleration is increasing.

The emission of negative energy is purely a quantum phenomenon and opens up the possibility of unusual new physical processes not encountered in classical theory. Negative energies are not without precedent in quantum field theory - it is possible to construct many-particle states with negative or zero energy fluxes even in the absence of mirrors and the like (Epstein, Glaser & Jaffe (1965), Appendix A of Davies & Fulling (1977)).

If the emission is integrated over time between periods of inertial motion of the mirror, however, the emitted energy is always positive - the negative energy flux is restricted to finite intervals. Ford (1978) has shown that negative energy fluxes cannot be sustained for long enough to reduce the entropy of a hot body by more than would be expected on the basis of ordinary thermal fluctuations.
Chapter Two

Unruh's Particle Detector: the back-reaction on the field.

S2.1 Introduction:
To get away from purely formal prescriptions of positive frequency, and to give operational meaning to the 'particles' that are so difficult to define in curved spacetimes, Unruh (1976) introduced the idea of a model particle detector.

The model consists of a quantum system being carried along a time-like world line by an observer. This system is allowed to interact with a quantum field, $\Phi$, through an interaction of the form,

$$ L_{\text{INT}} = \xi . D . \Phi . \delta(\text{trajectory}) $$

(2.1.1)

where 'D' is a detector variable, and '$\xi$' gives the strength of the coupling. If the detector jumps to an excited state, then it can be said to have detected a quantum of energy - a particle.

Before considering the response of such a detector in curved spacetime, it is important to understand clearly its response in the familiar setting of flat spacetime. As might be expected, the detector gives a zero response to a field in the Minkowski vacuum state when the detector is moving inertially. Unruh showed, however, that when the detector accelerates (the $\Phi$ field still being in the Minkowski vacuum...
state), the response is no longer zero, but in fact the detector registers the presence of thermal radiation.

This result emphasizes the need for care in identifying particles with energy carriers. In the case of an accelerating detector, it is clear that it is the motion of the detector that is giving rise (directly or indirectly) to the excitation. One might demand that for an excitation to correspond to the presence of an energy-carrier, the detector should be inertial: in flat spacetime all geodesic detectors are equivalent. The generalization of this demand to curved spacetimes, however, results in the possibility that two equally valid geodesic detectors might disagree as to whether or not field quanta are present. The familiar concept of a 'particle' therefore becomes very difficult to articulate clearly in curved spacetime.

I return to the question of detectors in curved spacetime in Chapter 3, but here I consider the back-reaction of Unruh's detector on the field in flat spacetime. A process whereby a detector can become excited in response to the Minkowski vacuum, a state of zero energy, warrants further study - where does the excitation energy come from?

The detector is coupled to the field through the interaction (2.1.1) as well as being influenced by the force providing the acceleration. This force is the most obvious source of the excitation energy. The \( \Phi \) field is not left in the vacuum state by the interaction, however, and it is the energy of the final state of the field after interacting with the detector that I evaluate here.
Chapter Two

The calculation of the energy tensor is taken to second order in the coupling constant, $\mathcal{E}$, - the same order as Unruh's result. It turns out that in addition to an energy emission by the detector when it excites, there is also an energy emission to second order in $\mathcal{E}$ where the detector is subsequently found not to be excited. It is the overall emission to order $\mathcal{E}^2$ that I calculate here; that is, the final state of the detector is not observed.

I find that during periods of constant acceleration, the detector in fact emits negative energy. The accelerating force only does extra work if the energy emission is asymmetric.

This result suggests that there may be states of the field of less energy than the Minkowski vacuum. To resolve this situation I argue that the model has to be modified to ensure that the interaction ceases at early and late times. Changing to a more realistic model where the detector starts and finishes in inertial motion and where the interaction strength is switched on and off, I find that while at constant acceleration the emitted energy is negative, during periods of changing acceleration the emission is such as to ensure that the total emitted energy is positive.

*Structure of Chapter 2*

Section §2.2 gives a discussion of accelerated frames of reference. Section §2.3 describes a quantization scheme based on the proper time of an accelerated observer. Section §2.4 derives Unruh's result that an accelerating particle detector will respond to the Minkowski vacuum as if it was in a thermal bath. Sections §2.5 - §2.9 give a preliminary
discussion of the back-reaction of Unruh's detector on the $\Phi$ field. Section S2.10 gives the full four dimensional calculation of this back-reaction to show that there is an emission of negative energy.

S2.2 Accelerated frames of reference - the Rindler Wedge

For observers on trajectories of constant acceleration, there are regions of spacetime with which they cannot communicate - each trajectory has an associated event horizon. A coordination of the spacetime based on these trajectories splits the spacetime into regions known as the Rindler wedges, and shows a striking resemblance to the Schwarzschild coordination of the blackhole spacetime.

The proper acceleration, \('g'\), of a frame $\tilde{S}$, moving instantaneously at speed \('v'\) with respect to an inertial frame, $S$, is given by the Lorentz transformation, \(g = a/(1 - v^2)^{3/2}\), where \('a'\) is the acceleration as measured in frame $S$. Setting \(g = \text{constant}\) gives the differential equation,

\[
\frac{d^2x}{dt^2} = \text{constant}. \ (1 - (\frac{dx}{dt})^2)^{3/2}
\]

whose solution for time-like trajectories is \(x^2 - t^2 = g^{-2}\).

Observers following such hyperbolas of constant acceleration are confined to the regions, \(|t| < |x|\), and the lines \(t = \pm \sqrt{x}\) act as event horizons (See Figure 2.2 (b)). These disjoint regions are known as the Rindler wedges (R+), (R-). The future and past light cones of a point in (R-) never intersect region (R+) and hence observers confined to (R+) cannot communicate with events in (R-).
Rindler (1966) noted that there is a striking similarity in structure between the Schwarzschild blackhole spacetime, with its event horizons for an observer at infinity, and the Minkowski spacetime with its "event horizons" for accelerating observers (see Figure 2.2).

He showed that, just as region (I) of the blackhole spacetime may be coordinated by Schwarzschild coordinates, region (R+) of Minkowski spacetime can be covered by 'Rindler coordinates',

\[-\infty < t < \infty \ ; \quad \sigma > 0\]
\[t = \sigma \cdot \sinh (\tau \sigma)\]
\[x = \sigma \cdot \cosh (\tau \sigma)\]

\[(2.2.2)\]

Figure 2.2

(a) Kruskal / Schwarzschild
(b) Minkowski / Rindler
Expressed in the new coordinate system, the Minkowski line element,

\[ ds^2 = dt^2 - dx^2 - dr^2 - r^2 d\theta^2 \]  

(2.2.3)

becomes,

\[ ds^2 = (sg)^2 d\tau^2 - ds^2 - dr^2 - r^2 d\theta^2 \]  

(2.2.4)

\( \tau \) is then the proper time, and \( s \) the proper distance for an observer on a trajectory of constant \( s = s_0 \), having proper acceleration, \( g = \frac{s}{\tau} \).

The accelerated trajectories are integral curves of a Killing vector of the spacetime, that of Lorentz boosts in the \( x \)-direction. That \( \frac{\partial}{\partial \tau} \) is a Killing vector in \( \mathbb{R}^+ \) can be seen from the fact that the metric expressed in Rindler coordinates (2.2.4) is independent of \( \tau \).

S2.3 Fulling - Rindler quantization

Fulling (1973) has developed a quantum theory based on the 'Rindler wedge' segment of Minkowski spacetime and has shown that the resulting definitions of particles and vacuum state are different from those of the usual theory on the full Minkowski manifold.

The massless wave equation,

\[ \Box \Phi = 0 \]  

(2.3.1)

for a spacetime metric, \( g_{\mu\nu} \), is,
\[(\mathbf{g})^{-\frac{1}{2}} \cdot \partial_\mu \left[ (\mathbf{g})^{\frac{1}{2}} \cdot g^{\mu\nu} \cdot \partial_\nu \bar{\Phi} \right] = 0 \quad \text{(2.3.2)}\]

so that in the Rindler coordinates,

\[\left[ g^{\mu\nu} \right]^{-2} \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} - \frac{1}{S} \frac{\partial}{\partial \eta} \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} \bar{\Phi} = 0 \quad \text{(2.3.3)}\]

This has orthonormal solutions,

\[\left\{ \hat{\Phi}_x, \hat{\Phi}_x^* \right\} = \left\{ \left(2\pi^2\right)^{-\frac{1}{2}} \exp \left( \pm i\xi \tau \right), \right\}

\[\kappa \cdot \nu \cdot \exp \left( \pm i\xi \cdot \hat{r} \cdot \hat{\hat{r}} \right) \cdot \sinh \left( \frac{\pi \nu \eta}{g} \right) \right\} \quad \text{(2.3.4)}\]

with \(\mu = |\hat{\xi}|\), and \(K_{\kappa \nu} / g\) a modified Bessel function of the third kind of imaginary order (see Watson, 1922).

A field expansion in terms of these solutions of the form,

\[\bar{\Phi} = \sum_{\chi} \left( b_\chi \hat{\Phi}_\chi + b_\chi^* \hat{\Phi}_\chi^* \right) \quad \text{(2.3.5)}\]

(to simplify the notation, I suppress the \(\hat{r}\) indices) determines a separation into positive and negative frequency that is tied to the proper time of accelerated observers. The annihilation and creation operators

\(\left\{ b_\chi, b_\chi^* \right\}\) are used to construct a Fock basis that will act as a representation of the field algebra. The 'vacuum state', \(|0_{\mathcal{R}^+}\rangle\), is defined by the property that it is annihilated by all \(b_\chi\)'s; that is,
\[ b_{\lambda} | 0 >_{R^+} = 0, \text{ for all } \lambda. \]

Such a quantization scheme gives a theory for the Rindler wedge \((R^+)\), but does not determine the state of the field over the whole manifold. A similar structure must be established in the mirror Rindler wedge \((R^-)\), and these formalisms analytically continued into the regions \((F)\) and \((P)\). As the interaction to be considered here is restricted to a particular trajectory in region \((R^+)\), a Rindler quantization for any other region will not be needed, and I ignore this consideration from now on. (Unruh & Wald (1984) have studied effects of the interaction that propagate into region \((R^-)\); in their work a quantization valid in region \((R^-)\) was also required).

The relation between the basis for the Fulling-Rindler (FR) quantization and the usual Minkowski basis is found through Bogoliubov transformations (see Appendix A for further details).

Introduce the Minkowski basis, \([\phi_\omega, \phi_\omega^*]\),

\[ \phi_\omega = \frac{\exp[-i\omega E + i R \cdot x]}{[(2\pi)^3, 2\omega]^{1/2}} \]

in terms of which \(\overline{F}\) has the expansion,

\[ \overline{F} = \sum_\omega (\omega_\omega \phi_\omega + \omega^*_\omega \phi^*_\omega) \]

in \((R^+)\) the relation between the two sets of modes is given by,
\[ \Phi = \sum_{\lambda} (A_{\omega \lambda} \hat{\phi}_{\lambda} + B_{\omega \lambda} \hat{\phi}_{\lambda}^*) \]  

(2.3.8)

where 'A' and 'B' are scalar products between \( \langle \phi_{\omega}, \phi_{\omega}^* \rangle \) and \( \langle \hat{\phi}_{\lambda}, \hat{\phi}_{\lambda}^* \rangle \),

\[ A_{\omega \lambda} \equiv \langle \phi_{\omega}, \hat{\phi}_{\lambda} \rangle \quad ; \quad B_{\omega \lambda} \equiv \langle \phi_{\omega}^*, \hat{\phi}_{\lambda} \rangle \]  

(2.3.9)

and measure how much frequency mixing occurs. The number of FR particles in the Minkowski vacuum, \(|0\rangle_M\), is found to be,

\[ n_{\chi}^{FR} = \langle 0 | b_{\lambda}^\dagger b_{\lambda} | 0 \rangle_M = \sum_{\omega} |B_{\omega \lambda}|^2 \]  

(2.3.10)

and so if \( \sum |B_{\omega \lambda}|^2 \) is non-zero, as is the case here where,

\[ \sum |B_{\omega \lambda}|^2 = \left[ \sum_{\lambda} \alpha_{\lambda} \left( \frac{2\pi \lambda}{g} \right)^{-1} \right]^{-1} \]  

(2.3.11)

the particle content of the two theories is different. (This result is essentially the same as that found in Section §1.5).

(2.3.11) shows that the Minkowski vacuum appears as a thermal state at temperature \( T = g/2\pi \) in the FR theory. The inequivalence of the two quantization schemes is essentially the result of different choices of 'positive frequency': with respect to accelerated proper time, \( \tau \), in the FR quantization scheme, and with respect to inertial time, 't', in the usual Minkowski quantization.

In particular, the canonical momentum,
and the Hamiltonian,

\[ H = \int dx^3 \left( \pi \cdot \partial_\tau \Phi - \mathcal{L} \right) \]  

(2.3.13)

(where \( \mathcal{L} \) is the Lagrangian density) involve derivatives with respect to proper time, \( \tau \), rather than inertial time, 't'.

Fulling has drawn attention to various reasons why one might consider the FR particles to be no more than mathematical constructions, devoid of any physical significance. Rindler coordinates have a singularity at \( S = 0 \), and curves of constant \( S \) are not geodesics; the coordinate system does not cover the entire spacetime, but only the wedge between the null lines \( t = x, t = -x \); and at low energies, when the wavelength becomes comparable with the distance to the horizon, the global properties of the coordinate system will become relevant to the discussion of particles - at this point, the quantization scheme may be expected to break down.

S2.4 Unruh’s particle detector

The significance of Fulling's quantization of the Rindler wedge was discovered by Unruh (1976) when he showed that the thermal nature of the Minkowski vacuum in the Fulling-Rindler (FR) theory can be detected by an accelerating particle detector.

The model particle detector introduced by Unruh consists of a
Schrodinger particle confined to a small box that interacts with a field, \( \Phi \), propagating on the spacetime. If the particle starts in the ground state, \( E_0 \), the interaction with the field can subsequently cause it to jump to an excited state, \( E_j \). The response of the detector is taken to be the probability per unit proper time that the detector makes a transition between \( E_0 \) and \( E_j \).

The interaction Hamiltonian takes the form,

\[
H^{\text{int}} = \mathfrak{A} \int d^3 \mathbf{q} \left( \hat{q}^2 - \mathbf{z} \right) \Phi(x) \sqrt{-g} \ d^3 \mathbf{x}
\]  

(2.4.1)

where \( \hat{q} \) is the position operator for the particle in the box, and \( \mathbf{z} \) is the strength of the coupling. The interaction is clearly restricted to the interior of the box.

If the \( \Phi \) field is initially in the Minkowski vacuum state and the box moves inertially, then the detector will not be excited. If the detector accelerates, however, then there is a possibility of an excitation even if the \( \Phi \) field is again initially in the vacuum state. I now outline Unruh's calculation of this effect.

The equivalence principle requires that observers travelling with the accelerated detector should not be able to distinguish their acceleration from that produced by a gravitational field. Consequently the Schrodinger equation for the particle in the box is given in their proper coordinates by,
\[-i \frac{\partial \Phi}{\partial T} = \frac{1}{2m} \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right] + \gamma m \Phi\] \hspace{1cm} (2.4.2)

The box accelerates at \( g = \frac{g_0}{T} \), where \( g_0 \) is the position of the box in the \((T, g)\) coordinate system. To ease the notation, I will hereafter take the acceleration, \( 'g' \), to have the value, 1. The energies can later be suitably rescaled to accommodate a specific value for \( 'g' \).

The eigensolutions to this equation will be written as,

\[ \exp \left( -i E_j T \right), \Phi_{E_j}(\vec{r}) \] \hspace{1cm} (2.4.3)

with eigenvalue \( E_j \). The \( \Phi \) field is taken to be in the Minkowski vacuum state, \( |0\rangle_M \). The particle is confined to the interior of a small box, with the wave-function satisfying Dirichlet boundary conditions at the box walls.

The lowest order probability per unit proper time that the detector goes from state \( E_0 \) to state \( E_j \) is given by first order perturbation theory as,

\[ p_{E_j} = \lim_{T \to \infty} \frac{1}{T} \left| \int_0^T dT \cdot \exp \left[ -i (E_j - E_0) T \right] \right|^2 \left| \langle E_j | H^{\text{int}} | E_0 \rangle \right|^2 \] \hspace{1cm} (2.4.4)

By introducing two complete sets of detector eigenfunctions, \( |Q\rangle \), of the position operator, and \( |E\rangle \), of the free Hamiltonian, the interaction Hamiltonian can be written as,
\[
H^{\text{INT}} = \sum_{E_i, E_i'} \int d^3 x \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle
\]
(2.4.5)

which can be identified with,
\[
H^{\text{INT}} = \sum_{E_i, E_i'} \int d^3 x \left| \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle \left\langle E_i \left| \Phi(x) \right| E_i' \right\rangle \right|^2
\]
(2.4.6)

(2.4.4) then evaluates to,
\[
\rho_{E_j} = \lim_{T \to \infty} \frac{\hbar^2}{i} \sum_{\left| p \right\rangle} \int_0^T \left| \int d\zeta \exp \left( i [\tilde{E}_j - E_0] \zeta \right) \right|^2 \cdot \left. \left| \int_{\text{box}} d^3 x \hspace{1cm} \left\langle E_j \left| \Phi(x) \right| 0 \right\rangle M \hspace{1cm} \left\langle E_0 \left| \Phi(x) \right| 0 \right\rangle \right|^2
\]
(2.4.7)

where the sum is taken over all possible final states, \( \left| p \right\rangle \), of the \( \Phi \) field.

Expanding the \( \Phi \) field in Rindler coordinates, (Appendix A(b)), will allow the time integrals to be handled simply,
\[
\rho_{E_j} = \hbar \lim_{T \to \infty} \frac{\hbar^2}{i} \sum_{\left| p \right\rangle} \int_0^T \left| \int d\zeta \hspace{1cm} \left\langle E_j \left| \Phi(x) \right| 0 \right\rangle M \hspace{1cm} \left\langle E_0 \left| \Phi(x) \right| 0 \right\rangle \right|^2
\]
\[
\left. \left| \int_{\text{box}} d^3 x \hspace{1cm} \left\langle E_j \left| \Phi(x) \right| 0 \right\rangle M \hspace{1cm} \left\langle E_0 \left| \Phi(x) \right| 0 \right\rangle \right|^2 \right| \left| \int_{\text{box}} d^3 x \hspace{1cm} \left\langle E_j \left| \Phi(x) \right| 0 \right\rangle M \hspace{1cm} \left\langle E_0 \left| \Phi(x) \right| 0 \right\rangle \right|^2
\]
(2.4.8)
(By a similar analysis to what follows, upon integration over time the second time exponential becomes a delta function that cannot be satisfied, and hereafter I drop it from the calculation).

The Rindler annihilation operators do not annihilate the Minkowski vacuum; instead they must be re-expressed in terms of Minkowski creation / annihilation operators, using a Bogoliubov transformation. From Appendix A(b),

$$
\langle p | b_{R+}^\lambda | 0 \rangle_M = b_{p\lambda}^x = \frac{1}{\sqrt{\pi p}} \frac{\exp(i\lambda B_p)}{[\exp(2\pi \lambda) - 1]} \delta(k_x^2 + p^2)
$$

(2.4.9)

Distinguishing the two integration variables in the modulus squared by a prime, the integration over all possible final states of the $\Phi$ field may be performed.

Introduce a change of variables,

$$(p, p_x, p_y, p_z) \rightarrow (B_p, \mu, p_y, p_z, \xi)$$

$$b = \mu \coth B_p \quad j = p_x = \mu \sinh B_p$$

$$\mu = (b^2 + p_x^2 + \xi^2)$$

$$\sum_{|p\rangle} \rightarrow \int \mu \coth B_p \cdot dp_y \cdot dp_z \cdot dB_p$$

(2.4.10)

where $\xi$ is a small parameter that can eventually be set to zero, but for now prevents a coordinate singularity at $\mu = 0$. 

Then,

\[ \int \frac{\mu \cosh \Theta_\beta}{\beta} \cdot \exp \left( i \Theta_\beta (\lambda - \lambda') \right) \, d\Theta_\beta = 2\pi \delta (\lambda - \lambda') \]

(2.4.11)

leaving,

\[ p_{E_j} = \lim_{T \to \infty} \frac{2\xi^2}{\pi} \int_0^\infty \frac{d\lambda}{\left[ \exp \left( 2\pi \lambda \right) - 1 \right]} \left| \int_0^T \frac{i (E_j - E_0 - \lambda) T}{\pi} \right|^2 \]

\[ \int dp y dp z \left| \int_{box} dx^3 - q \left( \sinh \pi \lambda \right)^{1/2} \cdot K_{\pi \chi} (\mu \delta) \cdot \frac{\gamma^+ (x)}{\gamma^+ (x)} \cdot \gamma^- (x) \cdot \gamma^- (x) \cdot \exp (i \vec{p} \cdot \vec{x}) \right|^2 \]

\[ = 2\xi^2 \int_0^\infty d\Lambda \int dp y dp z \cdot \frac{\delta (\lambda - (E_j - E_0))}{\left[ \exp \left( 2\pi \Lambda \right) - 1 \right]} \cdot \sigma^2 (\lambda, p y, p z) \]

(2.4.12)

where I have identified \( \sigma^2 \) as the cross-section for the box to absorb the (normalized) mode,

\[ (2\pi^2)^{-1} \cdot (\sinh \pi \lambda)^{1/2} \cdot K_{\pi \chi} (\mu \delta) \cdot \exp (i \vec{p} \cdot \vec{x}) \]

(2.4.13)

(The details of how a cross-section is defined are discussed in Appendix D(a)). This is exactly what one would expect of a detector immersed in a thermal bath at temperature, \( T = g/2\pi \) (in the general case of \( g \neq 1 \), the energies are rescaled by \( g \)). The essential reason for this result is that the detector measures positive frequency with respect to its proper time, and not with respect to some universal time. For an accelerated observer, this definition of positive frequency is not equivalent to that of an inertial observer.
Apparently the similarity in structure between the Schwarzschild segment of the blackhole spacetime and the Rindler segment of Minkowski spacetime extends to the thermal effects originally found by Hawking (1975). Unlike the case of blackhole evaporation, however, it seems clear that there is no sense in which these FR particles can be thought of as carrying energy that contributes to the energy-momentum tensor. The Minkowski vacuum, from many considerations must be a state of zero energy.

Having presented the result that has motivated my work, I now consider the energetics of the situation.

§2.5 Back-reaction on the Field

The energetics of the back-reaction of the detection process on the field are considered. To simplify the discussion, I initially present a calculation based on a model two dimensional spacetime. I find that in two dimensions, the energy emission into the field is negative and of equal magnitude to the excitation energy of the detector.

Looking in detail at the calculation just presented, it can be seen that the result hinges on \[ M<pl|b_{R^+}|0\> M \] being non-zero, where \( b_{R^+} \) is a Fulling-Rindler annihilation operator. This operator does not annihilate the Minkowski vacuum, and the \( \Phi \) field is left in the state \( |p\>_M \) by the detection process. Inertial observers therefore see the detector jumping to its excited state by the emission of a \( \Phi \) quantum, not by absorption.
This result is independent of the means used to accelerate the detector, and depends on the acceleration itself. Notice that it is not the accelerating force that couples the ground state, $E_0$, to the excited state, $E_j$, but rather it is the $\vec{E}$ field that is producing the excitation of the detector. One expects that the one-particle state carries energy away from the detector and that the accelerating force provides the energy for both this process, and for the excitation. This was Unruh's (1976) original view (see also Birrell & Davies (1982), p.55). Where negative energy densities are possible, however, as discussed in Section §1.11, the energy carried away by the $\vec{E}$ field is worth considering carefully.

De Witt (1979) suggests that “if the detector were inert, possessing no degrees of freedom, the pattern of photon emissions would be that of a given accelerating source. When internal degrees of freedom are present, this pattern is altered: occasionally an emitted photon is softer than it would otherwise be. The detector has stolen some of its energy and passed to an excited state.” This explanation is misleading, however - the detector makes a transition away from the ground state only by the emission of a Minkowski particle. Consequently, this transition cannot be interpreted as being caused by self-absorption of Minkowski quanta previously emitted by Bremsstrahlung or other processes.

To add weight to the suggestion that things are not as simple as might be expected, consider the boundary conditions placed on the Schrödinger particle, $\vec{F} = 0$, at the box walls. Although the situation is rather
different in detail, Unruh & Wald (1982) (see also Fülling & Davies (1976)) found that imposing such a boundary condition on a massless scalar field, $\Delta \Phi = 0$, at an accelerating boundary (mirror) in a two dimensional model spacetime results in an energy emission by the mirror into the field dependent on the rate of change of acceleration of the mirror. A consequence is that at constant acceleration, no energy is emitted by the mirror (other than the purely classical Doppler shifting of existing radiation).

Although the box detector contains an interacting Schrodinger particle rather than a massless scalar field, if, by analogy, at constant acceleration no energy is given to the Schrodinger particle by the confining walls, then the accelerating force cannot be the agent supplying the energy for the excitation (the walls constitute the only means of contact between the accelerating force and the particle). If energy is conserved, the gain in energy of the box must be compensated by a negative energy emitted into the $\Phi$ field.

S2.6 Two dimensional calculation
I consider the energy of the state that the $\Phi$ field is left in after interacting with the detector. I firstly use a two dimensional model to display the essentials of the calculation.
I work in the Heisenberg picture, and use an in-out formalism to by-pass the difficulties of unperturbed and perturbed eigenstates not lying in the same Hilbert space. The in- and out-fields propagate on the whole of spacetime, and the perturbed field approaches the free in-field before the interaction, and approaches the free out-field after the interaction. As both the in- and out-fields satisfy identical commutation relations, they are related by a unitary transformation which Yang & Feldman (1950) identify as the S-matrix.

Consider a particle detector which is coupled to a massless scalar field in two dimensional Minkowski spacetime. The interaction Hamiltonian will be taken to be,

\[ \hat{H}^{\text{int}} = \sum_{E_0} \sum_{\varepsilon_0} e^{i(E-E')t} \langle E_0 \rangle \cdot \hat{\Phi}(x(t)) \cdot \hat{\Phi}^*(x(t')) \langle E_0 \rangle \]

(2.6.1)

where \( x(t) \) is the trajectory of the detector. (Here I include the time-dependence of the wave functions in the Hamiltonian).

The in-field is taken to be the Minkowski vacuum field of zero energy, and I calculate the lowest order contribution to the energy of the out-field. This energy is found from the two-point function, \( G(x,x') \), using the regularization scheme of Christensen (see Section S1.8). (Although this is flat spacetime, where the standard normal-ordering regularization could have been used on \( T_{\mu\alpha} \), I maintain a consistent approach by using the point-splitting technique throughout).

I calculate the lowest order correction to the two-point function,
\[ G(x,x'), \]
\[ G_{\text{out}}(x,x') = \langle \Phi^\text{in}(x) \Phi^\text{out}(x') | \Phi^\text{out}(x), \Phi^\text{in}(x') | E_0 \rangle | 0 \rangle_M \]

\[ = \sum_{n=0}^{\infty} S^{-1}(x) S S^{-1}(x') S | E_0 \rangle | 0 \rangle_M \] \hspace{1cm} (2.6.2)

with the S-matrix given by,
\[ S = \sum_{n=0}^{\infty} \frac{(i)^n}{n!} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \mathcal{P}(H^\text{in}(y_1) \ldots H^\text{in}(y_n)) \] \hspace{1cm} (2.6.3)

Expanding the S-matrix,
\[ S = 1 - \iota \varepsilon \int_{\text{box}} d_{y_1} \sum_{E} e^{-i(E-E')\tau_1} |E\rangle \langle E^*| \Phi(y_1) \Gamma_{E}(y_1) \Gamma_{E^*}(y_1) \]

\[ - \varepsilon^2 \int_{\text{box}} d_{y_1} \sum_{E_1, E_2} \langle E_1 | E_2 \rangle \Gamma_{E_1}(y_1) \Phi(y_1) \Gamma_{E_2}(y_1) \]

\[ \cdot \int_{\text{box}} d_{z_2} \sum_{E_1, E_2} e^{-i(E_1-E_2)^2\tau_2} |E_1\rangle \langle E_2| \Phi(y_1) \Phi(y_2) \Gamma_{E_1}(y_2) \Gamma_{E_2}(y_2) \]

\[ + O(\varepsilon^3) \] \hspace{1cm} (2.6.4)

so that (2.6.2) becomes,
\[ G_{\text{out}}(x,x') = \langle \Phi^\text{in}(x) \Phi^\text{in}(x') | 0 \rangle_M \]

\[ + \varepsilon^2 \sum_{E_1, E_2} \int_{\text{box}} d_{z_1} \int_{\text{box}} d_{z_2} \int_{\text{box}} d_{y_1} \int_{\text{box}} d_{y_2} e^{-i(E_1-E_2)^2\tau_2} |E_1\rangle \langle E_2| \Phi(y_1) \Phi(y_2) \Gamma_{E_1}(y_2) \Gamma_{E_2}(y_2) \]
Looking in detail at the terms involved, it can be seen that there is a contribution (to second order in $\varepsilon$, the order of Unruh's result) to the out-energy from a 1-particle state of the $\Phi$ field,

$$\sim \sum_{\omega, K} \frac{1}{M} \left\langle \Omega| \Phi(x) \Phi(x') | 1, K \right\rangle_M$$  \hspace{1cm} (2.6.5)

This corresponds to the situation discussed in Section 2.4 - the operation of Unruh's detector: the detector jumps to an excited state, and the $\Phi$ field is left in a one-particle state.

There are also, however, contributions to the same order in $\varepsilon$ of the form,

$$\sim \sum_{\omega, K} \frac{1}{M} \left\langle \Omega\Omega| \Phi(x) \Phi(x') | 1, K \right\rangle_M$$  \hspace{1cm} (2.6.7)

that is, cross terms between the vacuum and 2-particle field states. In
these cases, there has been an interaction, but the detector is finally found not to be excited.

Unruh & Wald (1984) (see Section §2.12) have considered the differing interpretations of the detection process offered by accelerating and inertial observers, and considered effects that propagate into the causally disconnected Rindler wedge, $R^-$. They emphasize that all terms to second order in $\Lambda$ have to be included to avoid nonsensical results.

Now using,

$$\Delta^{-}(x-y) \equiv \langle 0 | \Xi(x) \Xi(y) | 0 \rangle$$

$$\Delta^{+}(x-y) \equiv \langle 0 | [\Xi(x), \Xi(y)] | 0 \rangle$$

(2.6.8)

gives,

$$\quad G^{\text{out}}(x,x') - G^{\text{in}}(x,x') = -\Lambda^2 \sum_{E} \int_{y_1} dy_2 \sqrt{-g_2} \Delta^{-}(y_1-x) \Delta(y_2-x') + \int_{z_2} dz_2 \sqrt{-g_2} \Delta^{-}(x'-y_2) \Delta(x-y_1)$$

(2.6.9)

$G^{\text{in}}(x,x')$ gives rise to the incoming unperturbed energy of the Minkowski vacuum and contains all the divergences that will be subtracted off.
Expanding \( \Phi \) in Minkowski modes, \( \{ \phi_w(x), \phi_w^*(x) \} \),

\[
\Delta^-(y_1-x) = \sum_\omega \phi_\omega(y_1) \phi_\omega^*(x)
\]

\[
\Delta(y_2-x') = \sum_\omega \left[ \phi_\omega(y_2) \phi_\omega^*(x') - \phi_\omega(x') \phi_\omega^*(y_2) \right]
\]

(2.6.10)
dealing with (2.6.9) then involves evaluating such terms as

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{\Box} d\gamma \int_{\Box} d\gamma' \int_{\Box} d\gamma_1 \sum_\omega \phi_\omega(y_1) \phi_\omega^*(x) \sum_k \phi_k(y_2) \phi_k^*(x').
\]

(2.6.11)
The spatial integrals will be arranged to form partial cross-sections,

\[
\sigma(y_1, E_1 - E_0) = \sqrt{2\pi} \int_{\Box} d\gamma \sqrt{g_1} \cdot H_\gamma(s_1) \gamma_{E_1}^*(s_2) \gamma_{E_0}^*(s_1)
\]

(2.6.12)
where \( H_\gamma(y) \) is the spatial part of the Fulling-Rindler (FR) mode

\[
\hat{\phi}_\gamma = e^{i \gamma \tau} \cdot H_\gamma(s)
\]

This can be accomplished by expanding the Minkowski modes, \( \{ \phi_w(x), \phi_w^*(x) \} \), in terms of the FR modes, \( \{ \hat{\phi}_\gamma(x), \hat{\phi}_\gamma^*(x) \} \). In addition, such an expansion allows the time integrals to be performed simply.

(2.6.11) can be expanded as,
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{s_1} \int_{s_2} \int_{s_1} \int_{s_2} \sum_{\omega, k, \Sigma, \xi, e} \phi_\omega^*(x) \phi_k^*(x'). \]

where \( A \) and \( B \) are Bogoliubov coefficients for the transformation between Minkowski and Fulling-Rindler modes.

Identifying the partial cross-sections, and performing the time integrals gives,

\[ \sum_{\omega, k, \Sigma} \phi_{\omega}^*(x) \phi_k^*(x') \left\{ \frac{A_{\omega \Sigma} A_{k k}^*}{i(\sigma - \lambda)} \delta(\Sigma + \Sigma') \delta(\xi, \lambda) \sigma(\xi, \lambda^*) \right. \]

\[ + \frac{B_{\omega k}^*}{i(\sigma - \lambda)} \delta(\xi - \Sigma) \delta(\xi, \lambda) \sigma(\xi, \lambda^*) \]

\[ + \frac{B_{\omega k}^*}{-i(\sigma + \lambda)} \delta(\xi - \Sigma) \delta(\xi, \lambda) \sigma(\xi, \lambda^*) \]

\[ + \frac{A_{\omega k}^*}{-i(\sigma + \lambda)} \delta(\xi - \Sigma) \delta(\xi, \lambda) \sigma(\xi^*, \lambda^*) \right\} \]

(2.6.14)

Evaluating the remaining terms in (2.6.9) give three further contributions like the above. The crucial steps in simplifying the total expression are as follows.
(1) Summing over $k$,

$$
\sum_{k} \phi_{\omega}(x) \phi_{k}^{*}(x') \frac{B_{\omega \Sigma} A_{k} \delta}{i(\Sigma - \lambda)} \delta(\Sigma - \lambda) \cdot \sigma(\Sigma, \lambda) \cdot \sigma(\lambda, \lambda')
$$

$$
= \phi_{\omega}(x) \cdot \frac{\tilde{\phi}_{\Sigma}(x') B_{\omega \Sigma}}{i(\Sigma - \lambda)} \cdot \delta(\Sigma - \lambda) \cdot \sigma(\Sigma, \lambda) \cdot \sigma(\lambda, \lambda')
$$

(from (2.6.13))

(2.6.16)

using the Bogoliubov transformations between Minkowski modes and FR modes (see Appendix A(a)).

(2) Summing over $\gamma$,

$$
\sum_{\sigma} \phi_{\omega}(x) \tilde{\phi}_{\sigma}(x') \cdot \frac{\delta(\Sigma - \lambda)}{i(\Sigma - \lambda)} \cdot \delta(\Sigma - \lambda) \cdot \sigma(\Sigma, \lambda) \cdot \sigma(\lambda, \lambda')
$$

$$
+ \sum_{\sigma} \phi_{\omega}(x) \frac{\tilde{\phi}_{\sigma}(x') B_{\omega \Sigma}}{i(\Sigma + \lambda)} \cdot \delta(\Sigma + \lambda) \cdot \sigma(\Sigma, \lambda) \cdot \sigma(\lambda, \lambda')
$$

(from (2.6.13))

(2.6.17)

Now as

$$
\tilde{\phi}_{\sigma}(x') H_{\sigma}(\Sigma') = \frac{e^{-i \sigma \Sigma'}}{\sqrt{4\pi i} \sigma'} \cdot \frac{e^{-i \sigma (\Sigma'' - \Sigma')}}{\sqrt{4\pi i} \sigma}
$$
\[
\Phi_{\gamma}(s') H_{\tau}(s') = \frac{e^{-i \delta u'}}{\sqrt{4\pi \delta}} \cdot \frac{e^{-i \delta (u'' - z')}}{\sqrt{4\pi \delta}}
\]

(2.6.18)

where \( H_{\tau} \) has come from the cross-section and I am considering only right-moving \((u)\) modes (left-moving modes can be treated identically), the summations over positive \( \gamma \) can be turned into an integral over all \( \gamma \)

\[
\int d\gamma' \Phi_{\gamma'}(x) \cdot \frac{\Phi_{\gamma}(x)}{i \delta (\gamma - \gamma')} \cdot \delta(\gamma - \Sigma) \delta(\Sigma \lambda) \delta(\Sigma \lambda')
\]

(2.6.19)

(where the slash denotes the notational convenience of extending \( \Phi_{\gamma} \) to negative values of \( \gamma \) (see Appendix A(d)).

Looking at emissions at late times determines that an integration contour may be completed in the upper-half complex \( \gamma \)-plane, leading to the simplified result,

\[
\zeta^{\text{out}}(x,x') - \zeta^{\text{in}}(x,x') =
\]

\[-4\pi \varepsilon^2 \sum_{\omega, \lambda} \phi_{\omega}(x) \cdot B_{\omega \lambda} \cdot \Phi_{\lambda}(x) \cdot \delta^{2}(\lambda, \lambda')
\]

(2.6.20)

The right-moving \((u)\) fluxes are found by substituting (2.6.20) into the expression for \( \langle T_{uu}^{\omega} \rangle \). The Bogoliubov coefficient \( B_{\omega \lambda} \) between Minkowski and Rindler modes is evaluated in Appendix A(b). The \( \omega \)-integration may be performed,

\[
\int d\omega \cdot \Phi_{\omega}(u) \cdot B_{\omega \lambda} = \int d\omega \cdot \frac{e^{i\omega u}}{\sqrt{4\pi \omega}} \cdot \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \frac{1}{\sqrt{\omega}} \cdot \frac{\lambda}{\omega} \cdot e^{-\frac{i\lambda}{\sqrt{\omega}}}
\]
and similarly for \( \langle T^{\gamma \gamma} \rangle \).

Integrating over a hypersurface appropriate to catch emissions in time \( \Delta \tau \), the energy emitted by the detector in its frame is,

\[
\langle \Delta E_{\text{field}} \rangle = \int_{-\infty}^{\infty} d\lambda \left( T_{\mu \nu} + T_{\nu \mu} \right)
\]

\[
= -2\pi^2 \sum_{\lambda} \frac{\lambda \cdot \sigma^2(\lambda, \lambda^*)}{\left[ \exp(2\pi \lambda) - 1 \right]} \cdot \Delta \tau
\]

Now in time \( \Delta \tau \), the detector has gained an energy,

\[
\langle \Delta E_{\text{detector}} \rangle = \sum_{\lambda} \lambda \cdot P_\lambda
\]

which, from (2.4.12) is,

\[
\langle \Delta E_{\text{detector}} \rangle = 2\pi^2 \sum_{\lambda} \frac{\lambda \cdot \sigma^2(\lambda, \lambda^*)}{\left[ \exp(2\pi \lambda) - 1 \right]} \cdot \Delta \tau
\]

The comparison of (2.6.23) with (2.6.25), gives the central result of
this chapter - the emission energy is negative, and in two dimensions
is equal and opposite to the excitation energy of the detector.

Validity of the Model
More realistic models of the detection process are considered. Firstly I
consider modifying the interaction strength, $\xi$, to arrange that the
interaction is switched slowly on at early times, and slowly off at late
times. Secondly, I consider a more general trajectory than that of
constant acceleration - one that starts and finishes in inertial motion,
and argue that while negative energy is indeed emitted during periods
of constant acceleration, the emission during periods of changing
acceleration is sufficient to ensure that the total emitted energy is in
fact positive.

S2.7 Switching on and off the interaction
The calculation just presented indicates that in a model two
dimensional spacetime, particle detectors following trajectories of
constant acceleration will emit negative energy. Apart from the
novelty of negative energy itself, this would appear to suggest that the
$\Phi$ field could be finally left in a state of lesser energy than the
Minkowski vacuum, a state that is normally considered to be the state
of lowest energy.

This situation can be resolved by noting that the model as given
violates the assumption that the field approaches a free field at early
and late times - both the interaction and the acceleration continue for
all time.
In an attempt to construct a more realistic model, I first introduce an exponential switch on and switch off of the interaction,

$$\xi \rightarrow \xi \cdot \exp\left(-\alpha |\tau|\right)$$  \quad (2.7.1)

where \( \tau \) is the proper time along the trajectory, and \( \alpha \) is a small constant. The result of this analysis (see Appendix D(b)) is that, for small \( \alpha \), the fluxes are modified to,

$$\langle T_{\alpha \alpha} \rangle \rightarrow e^{-2\alpha |\alpha|} \cdot \langle T_{\alpha \alpha} \rangle$$

$$\langle T_{\gamma \gamma} \rangle \rightarrow e^{-2\alpha |\gamma|} \cdot \langle T_{\gamma \gamma} \rangle$$  \quad (2.7.2)

The fluxes die away exponentially at early and late times as measured in the accelerated coordinates. As measured in the inertial frame, however, the fluxes do not die away exponentially. In fact,

$$T_{\alpha \alpha} = \left(\frac{d\alpha}{du}\right)^2 \cdot T_{\alpha \alpha} \sim u^{-2-2\alpha} \quad \{ u \rightarrow \infty \}$$

$$T_{\gamma \gamma} = \left(\frac{d\gamma}{dv}\right)^2 \cdot T_{\gamma \gamma} \sim v^{-2+2\alpha} \quad \{ v \rightarrow \infty \}$$  \quad (2.7.3)

Thus, unless \( \alpha > 1 \) (or re-scaling the energies for a general acceleration, \( \alpha > 0 \)), the inertial observer sees infinite fluxes travelling along the null lines \( u = 0, \gamma = 0 \). The reason for this is that the relation between the proper times of the two observers is,

$$t = \sinh \mu \tau \quad (g = 1)$$  \quad (2.7.4)

which means that the inertial observer sees the interaction switching on/off at the rate,
that is, for small $\alpha$, the interaction strength remains essentially constant in inertial time, allowing infinite fluxes to build up along the lines $u = 0$, $v = 0$.

In addition, the problem of a negative energy out-state is not resolved.

S2.8 A more realistic trajectory

This modification to the model having proved to be inadequate (a more rapid switch on/off of $\xi$ leads to intractable mathematics), I secondly consider modifying the trajectory of the detector. An inertial detector does not become excited in response to the Minkowski vacuum; by considering a trajectory that starts and finishes in inertial motion, it should be possible to arrange that the interaction ceases at early and late times, as required.

The details of the calculation of the energy emission from a more general trajectory than that of constant acceleration are given in Appendix D(b). Essentially, a conformal mapping of the Minkowski null coordinates $(u, v)$ onto new coordinates $(\hat{u}, \hat{v})$ is made so that the trajectory becomes a straight line. New mode solutions $\{ \hat{\Phi}_x(\hat{u}), \hat{\Phi}_S(\hat{v}) \}$ are constructed based on this new coordinate system and the arguments leading to equation (2.6.20) remain true, 

$$\zeta^{out}(x, x') - \zeta^{in}(x, x') =$$
\[-4\pi z^2 \sum_{\omega, \lambda} \phi^*_{\omega}(x) \cdot B_{w}\lambda \cdot \Phi^*_{\lambda}(x') \cdot \delta^2(\lambda', \lambda)\]  

except that \( \Phi_{\lambda}(u), \Phi_{\lambda}(x) \) are no longer Rindler modes.

The total emitted energy to order \( \leq z \) can be found by integrating the flux obtained from (2.8.1) over a \( t = \text{constant} \) hypersurface at a late time when the detector has returned to inertial motion. This energy is given by,

\[ E = \int_{-\infty}^{\infty} dx \cdot T_{tt} = \int_{-\infty}^{\infty} dx \cdot (T_{uu} + T_{wv}) \]  

(2.8.2)

Now,

\[ \langle T_{uu} \rangle = \]

\[ \lim_{u \to \infty} \frac{2}{\omega} \cdot \frac{2}{\delta_{\omega}} \left\{ -8\pi z^2 \sum_{\omega, \lambda} \frac{e^{i\omega u}}{\omega, \lambda \sqrt{4\pi \omega}} \cdot \frac{e^{i\lambda u'}}{\sqrt{4\pi \lambda}} \cdot B_{w}\lambda \cdot \delta^2(\lambda', \lambda) \right\} \]  

(2.8.3)

The hypersurface integration can be performed to give,

\[ \int_{-\infty}^{\infty} T_{uu} \cdot dx = \int_{-\infty}^{\infty} T_{uu} \cdot du = 2\pi z^2 \sum_{\omega, \lambda} \omega \cdot B_{w}\lambda \cdot B_{w}\lambda \cdot \delta^2 \]  

(2.8.4)

(see Appendix A(c), and similarly for \( \langle T_{wv} \rangle \))

The emitted energy is therefore clearly positive if finite.

What goes wrong in the case of constant acceleration is that the summation over \( \omega \) is undefined (ie infinite).
In the next section, I argue that the energy emission from a detector that starts and finishes in inertial motion dies away at early and late times. I conclude that a detector following such a trajectory, together with switching the interaction on/off at early/late times, will ensure that the field approaches free fields. Under such conditions, the total energy emission from the detector is positive, and the Minkowski vacuum retains its status as the state of lowest energy.

S2.9 Details of the fluxes emitted by a general trajectory

I consider the detailed fluxes emitted by a detector following a trajectory of non-constant acceleration and find that a term depending on the rate of change of acceleration is instrumental in ensuring that the total energy emitted is positive. I find that for a detector starting and finishing in inertial motion, the energy fluxes die away at early and late times. I also find that during periods of constant acceleration, the emission is still negative.

To elucidate the meaning of (2.8.4), consider the energy of some state \(|s>|\) of the \( \Phi \) field as measured by inertial observers,

\[
E = \sum_{\omega} \langle s| \omega \cdot a_{\omega}^+ a_{\omega} |s> \\
= \sum_{\omega, \lambda, \Sigma} \langle s| (b_{\lambda}^+ A_{\omega, \lambda} - b_{\lambda} B_{\omega, \lambda}) (b_{\Sigma} A_{\omega, \Sigma} - b_{\Sigma}^+ B_{\omega, \Sigma}) |s> 
\]

(2.9.1)

where I have expanded the Minkowski \( \{a_{\omega}, a_{\omega}^+\} \) operators in terms of operators \( \{b_{\lambda}, b_{\lambda}^+\} \) appropriate to a quantization in the \( (\tilde{u}, \tilde{v}) \) coordinate system. If \(|s>|\) is the 'vacuum' of the latter
representation (i.e. $\langle b_\lambda | s \rangle = 0$, for all $b_\lambda$), then,

$$E = \sum_\omega \omega \cdot b_{\omega \lambda} \cdot \hat{B}_{\omega \lambda}$$

(2.9.2)

Consequently, the total emitted energy, (2.8.4), can be identified as the energy carried by the 'b' - vacuum, modified by the detector's cross-section - its ability to absorb modes of different energies.

Ignoring for the moment the presence of cross-section factors in (2.8.3), I determine the energy fluxes of the 'b'-vacuum by using the two dimensional result for $\langle T_{\lambda \lambda} \rangle$ (see Section § 1.9),

$$\langle T_{\lambda \lambda} \rangle = (-12\pi)^{-1} \cdot \hat{C}^{\nu_\lambda} \cdot \left[ \hat{C}^{\nu_\lambda} \right] \cdot a \hat{a}$$

$$\langle T_{\lambda \lambda} \rangle = (-12\pi)^{-1} \cdot \hat{C}^{\nu_\lambda} \cdot \left[ \hat{C}^{\nu_\lambda} \right] \cdot a \hat{a}$$

(2.9.3)

Here the relevant conformal factor is given by,

$$C = \left\{ \begin{array}{c} \left[ \frac{1 + \dot{x}(\hat{u})}{1 - \dot{x}(\hat{u})} \right] \left[ \frac{1 - \dot{x}(\hat{u})}{1 + \dot{x}(\hat{u})} \right] \end{array} \right\}^{\nu_\lambda}$$

(2.9.4)

where $\dot{x}(\hat{u})$, $\dot{x}(\hat{v})$ are the velocity of the detector to be considered as functions of the new null coordinates $(\hat{u}, \hat{v})$ (see Appendix D(b)), and the fluxes are found to be,

$$24\pi \langle T_{\lambda \lambda} \rangle = -\varepsilon^2 \left( \frac{a^2}{z} + \frac{dz}{dT} \right)$$

$$24\pi \langle T_{\lambda \lambda} \rangle = -\varepsilon^2 \left( \frac{a^2}{z} - \frac{dz}{dT} \right)$$

(2.9.5)

where 'a' is the proper acceleration of the detector, and $\frac{dz}{dT}$ is
proper rate of change of acceleration.

These fluxes can be integrated over a late-time hypersurface to obtain the total energy carried by the 'b'-vacuum,

$$E = \int_{-\infty}^{\infty} dx \left( T_{uu} + T_{v-v} \right)$$  \hspace{3cm} (2.9.6)

In the case of a trajectory that starts and finishes in inertial motion, this integral evaluates to,

$$E = \frac{\xi^2}{24\pi} \int_{-\infty}^{\infty} d\tau \cdot \alpha^2(\tau)$$  \hspace{3cm} (2.9.7)

Clearly the emitted energy is positive if finite.

In Appendix D(c), I find that, with certain provisos, (2.9.5) in fact represent the detailed fluxes emitted by the detector if again the presence of the cross-section factors are ignored (that is, if the cross-section is taken to be flat). Note that at constant acceleration, the emission is negative, and that a term depending on the rate of change of acceleration adds to the emission due to acceleration. This term is identical to that emission found by Fulling & Davies (1976) from a non-stationary mirror bounding a free field. It is this dependence on the rate of change of acceleration that makes the total emitted energy of a realistic trajectory positive.

The energy fluxes (2.9.5) die away as the detector returns to inertial motion. The relation between proper time and inertial time also approaches linearity. These facts indicate that a combination of
exponential switching of the interaction strength, together with starting from and returning to inertial motion will ensure that the field starts and finishes as essentially free.

The fact that the emitted energy depends on only the local values of the detector's acceleration and rate of change of acceleration and not on the past history of the detector is most likely a result of taking the cross section to be flat. The cross-section depends on the particular construction of the detector; taking the cross-section to be flat achieves a separation of the individual properties of a given detector from the basic kinematic effect it is responding to. Equivalently, one could sum over an ensemble of suitably normalized detectors, each responding to a single different frequency.

To check whether the energy fluxes (2.9.5) above bear any relation to the true emissions where the cross-section is taken into account, I now analyse a particularly simple trajectory.

Consider the curve \( x(t) \) specified through,

\[
x(t) = B \left[ (t-x)^2 + A^2 \right]^{1/2} + \frac{1}{2} B^2 \left[ (t-x) - A \tan^{-1} \left( \frac{t-x}{A} \right) \right]
\]

(2.9.8)

with \( A, B \) constant. The velocity is given by,

\[
\dot{x} = \frac{\dot{p}^2 - 1}{\dot{p}^2 + 1} \quad ; \quad \dot{p} = 1 + \frac{B u}{(u^2 + A^2)^{1/2}}
\]

(2.9.9)

\((u = t - x)\) and this velocity remains below that of light if the condition \(|B| < 1\) is imposed.
From Appendix D(c), the appropriate conformal transformation 
\((u,v) \rightarrow \hat{(u,\hat{v})}\) is (I only solve explicitly for \(u \rightarrow u(\hat{u})\)),
\[
\frac{du}{du} = b(u) ; \quad \hat{u} = u + B (u^2 + A^2)^{1/2}
\] (2.9.10)

The condition \(|B| < 1\) also ensures that as \(u \rightarrow \pm \infty\), that 
\(u \rightarrow \text{constant}u\) (so that asymptotically, the detector's proper time 
coincides with Minkowski time) and ensures that there are no 
singular points of the transformation,
\[
\frac{du}{du} \neq 0 ; \quad \frac{d^2u}{du^2} \neq 0
\] (2.9.11)

Equation (2.8.3) may be written out in full as (Appendix A(c)),
\[
\left\langle T_{\alpha \alpha} \right\rangle = \lim_{\hat{u} \rightarrow u} \frac{\partial}{\partial \hat{u}} \frac{\partial}{\partial u} \int -8\pi \varepsilon^2 \sum_{\omega,\lambda} e^{i\omega u} \cdot \frac{e^{i\lambda \hat{u}}}{\sqrt{4\pi \omega}} \cdot \frac{e^{-i\lambda \hat{u}}}{\sqrt{4\pi \lambda}} \cdot
\]
\[-\left(\frac{1}{2\pi}\right) \sqrt{\frac{\omega}{\lambda}} \int_{-\infty}^{\infty} dx \cdot e^{-i\omega x} \cdot e^{-i\lambda (x + B (x^2 + A^2)^{1/2})} \cdot \delta_{\lambda}^2
\] (2.9.12)

Now an integral representation of \(K_1(x)\), a modified Bessel function 
of the third kind may be found as,
\[
K_1\left[A \left[(\omega + \lambda)^2 - \lambda^2 B^2\right]^{1/2}\right] =
\[[\omega + \lambda)^2 - \lambda^2 B^2]^{1/2} \cdot \int_{0}^{\infty} a \left[(\omega + \lambda)^2\right] e^{-i\lambda B (x^2 + A^2)^{1/2}} \cdot dx
\]
\[
\text{Re}(A) > 0 \quad \text{Re}(i\lambda B) > 0
\] (2.9.13)

and so, (2.9.12) reduces to,
\[ \langle T \sigma a \rangle = \frac{2}{\lambda^4} \frac{\partial}{\partial a} \frac{\partial}{\partial a} \sum_{n>0} (2i\lambda B A \lambda^2) \int_0^\infty dx \ (x^2 - \lambda^2 B^2)^{-1/2}. \]

\[ K_1 \left[ A (x^2 - \lambda^2 B^2)^{-1/2} \right] e^{-i\lambda u} e^{-i\lambda^2 u} e^{i\lambda^4 u} \delta^2(\lambda) \quad (2.9.14) \]

Unfortunately, I have not been able to integrate this expression in closed form. Figure 2.9.1 gives the results of a numerical integration of (2.9.14) for particular values of \( \lambda \).

The curves support the contention that the detailed energy fluxes emitted by the detector depend on the acceleration and the rate of change of acceleration, although it is clear that there is also a dependence on \( \lambda \) not found in the cases of constant acceleration, nor in the case where the cross-section is taken to be flat.

**Figure 2.9.1** Curves from (2.9.14) for particular values of \( \lambda \).
S2.10 Four dimensional calculation of the emission from a detector following a trajectory of constant acceleration.

To support the two-dimensional model given in the previous section, I now present a 4-dimensional calculation of the back-reaction of the detector on the $\Phi$ field. Due to the mathematical difficulties, I only consider a trajectory of constant acceleration. I find that the order $\varepsilon^2$ energy emission is again negative. The accelerating force does not provide the excitation energy, but may have to do extra work in reaction to an asymmetrical emission.

To evaluate the energy emitted, I integrate $< T_{\mu\nu} >$ over a region of a spacelike hypersurface at some late time $t = t_0$: a region appropriate to capture emissions from the detector during a time $\Delta \tau$ at $\tau = \tau_0$ (See Figure 2.10.1).
Again it is necessary to ensure that in the in-out formalism, the fields have become essentially free by the time measurement is made. As discussed in Section §2.9, a slow switch on or off of the interaction together with initially leaving and eventually returning to inertial motion should ensure that the fields are free at early and late times.

Causality ensures that the switch-off / decreasing acceleration will not affect the integrated <energy radiated> at time $t_0$ (Figure 2.10.1). If the switch-on and increasing acceleration is sufficiently slow and sufficiently far in the past, it seems reasonable to expect that the effects of the switch-on will be transient and again will not affect the energy to be integrated.

Having ensured that there is a suitable late time hypersurface, I can move it elsewhere in the spacetime to ease the mathematical difficulties. $\Phi_{\text{out}}(x)$ is the free field that the perturbed $\Phi$ field approaches at late times; however, $\Phi_{\text{out}}(x)$ is defined for all points of the spacetime, and integrating the out-energy across one hypersurface is equivalent to integrating it across any other (in flat spacetime, there is no gravitational particle creation).

For mathematical convenience, I choose to integrate the energy passing through an appropriate region of the null plane $t = x, x > 0$ (see Figure 2.10.1). This might appear to miss some of the energy being emitted in the x-direction, but in fact essentially no energy escapes to infinity while remaining in $(\mathbb{R}^+)$. 
Figure 2.10.2. The light cone at late time $t_0$

To see this, consider the light cone of emission from a point $(t_e, x_e)$ on the trajectory; this has the equation,

$$\left(t - t_e\right)^2 = \left(x - x_e\right)^2 + r^2$$

(2.10.1)

On the hypersurface $t = t_0$, this is the equation of a sphere, which is cut by the $x = t_0$ plane in a circle of radius,

$$r^2 = \left(t_0 - t_e\right)^2 - \left(t_0 - x_e\right)^2$$

(2.10.2)
The circle subtends an angle $\Theta$ at the centre of the sphere given by,

$$\sin^2 \Theta = \left[ \frac{(t_0 - t_e)^2 - (t_0 - x_e)^2}{(t_0 - t_e)^2} \right]$$

$$\Rightarrow \Theta^2 \sim \frac{2(x_e - t_e)}{t_0}, \text{ to large (2.10.3)}$$

and hence the portion of the sphere that has not crossed the $t = x$ hypersurface diminishes to zero at late times. In addition, it is straightforward to show that there is no delta-function emission in the forward direction.

As discussed in Chapter 1, the energy tensor of a quantized field is a sum of terms that formally involve products of two field operators acting at a single point. A regularization scheme has to be adopted to make sense of the formal expression, and again I use the point-splitting technique (Christensen (1976)).

The regularized tensor is defined in terms of the two-point function,

$$\mathcal{G}(x, x') = \langle 0 | \Phi(x) \Phi(x') | 0 \rangle$$

and $\langle T_{\mu\nu} \rangle$ is obtained from,

$$\langle T_{\mu\nu} \rangle_{\text{reg.}} = \lim_{\xi \to 0} \left\{ \left( \frac{1}{2} - \xi \right) (\mathcal{G}_{\mu\nu} + \mathcal{G}_{\nu\mu}) + \right.$$

$$+ 2\left( \xi - \frac{1}{4} \right) \cdot g_{\mu\nu} \cdot \mathcal{G}_{\sigma\sigma'} - \xi (\mathcal{G}_{\mu\nu} + \mathcal{G}_{\nu\mu}) \right\}$$
where $\xi$ is an adjustable parameter that is usually taken as $1/6$ in 4-dimensions (See Section 3.1.8). The Christensen result for the divergent terms in this regularized form of $\langle T_{\mu \nu} \rangle$ is subtracted from (2.10.5) before taking limits.

I now calculate the lowest order correction to the two-point function, $G(x,x')$:

\[
\xi^{\text{out}}(x,x') = \langle 0 | \langle E_0 | \Phi^{\text{out}}(x) \Phi^{\text{out}}(x') | E_0 \rangle | 0 \rangle \rangle
\]

\[
= \langle 0 | \langle E_0 | S^{-1} \Phi^{\text{in}}(x) S \Phi^{\text{in}}(x') S | E_0 \rangle | 0 \rangle \rangle
\]

The details of the calculation are identical to those of the last chapter, leading to equation (2.6.20):

\[
\xi^{\text{out}}(x,x') - \xi^{\text{in}}(x,x') =
\]

\[
-4\pi \xi^2 \sum_{\omega, \lambda} \phi_{\omega, \lambda}(x) \beta_{\omega, \lambda} \tilde{\phi}_{\lambda, \omega}(x') \sigma^{2}(x, x')
\]

Before using (2.10.7) in (2.10.5) to evaluate $\langle T_{\mu \nu}^{\text{out}} \rangle$ some of the integrals may be performed to yield, in $(R^+)$ (see Appendix E(a)),

\[
\xi^{\text{out}}(x,x') - \xi^{\text{in}}(x,x') = \frac{-\xi^2}{\pi^2} \sum_{\lambda} \frac{e^{-\lambda}}{\lambda} \sin \lambda \pi \lambda \cdot \lambda(\ell - \tau') \left| \int_{\Box} d^3 \tau \cdot d^3 \bar{\tau} \cdot d^3 \sigma \cdot \gamma^x \cdot \gamma^y \cdot \gamma^z \cdot \gamma^0 \right|^2.
\]
where ' \( \vec{r} \) ' and ' \( \vec{s} \) ' are coordinates for the integration over the box.

**S2.11 Calculation of the energy flux through \( t = x \)**

I determine the energy emission of the detector by integrating the energy fluxes passing through the hyperplane \( t = x \). As argued previously, no energy escapes to infinity while remaining in \( (R^+) \), and hence this integration picks up all the emitted energy.

The flux through \( t = x \) is given by,

\[
\int_{t}^{t_x} \int \mathcal{J}(\mathbf{r}) \, d\mathbf{r} \, = \left( \frac{x^2 - xt}{s^4} \right) T_{CC} + \left( \frac{t^2 - xt}{s^2} \right) T_{SS} + \left( \frac{(t-x)^2}{s^3} \right) T_{CS}
\]

Now, \( t = x \) is also the hyperplane, \( s = 0 \). A small- \( s \) expansion of (2.10.8) can be made, so that \( G(x,x') \) can be written as

\[
G = \sum_{\lambda} G_{\lambda} \cdot \mathbb{S}
\]

(Appendix E(b)), with

\[
\beta \equiv -\frac{e^2}{2\pi^2} \left[ \exp(2\pi\lambda) - 1 \right]^{-1} \int_{S} d\vec{s} \, d\vec{r} \, d\vec{\sigma} \, \mathcal{K}_{x'} \, \chi_{0} \, \gamma_{0} \left( 2\alpha \right) \mathbb{S}
\]

\[
e^{-i\lambda C} \cdot \frac{\partial}{\partial \mathbf{S}} = \frac{i\alpha}{\mathbf{S}} \cdot \chi + (2 + i \lambda) \mathbf{S} \cdot \gamma + O(s^2)
\]
\[
\frac{-i\lambda T}{2} \cdot \frac{\partial^2 \mathcal{G}}{\partial \kappa^2} = - \frac{(\lambda^2 + i\lambda)}{\kappa^2} X + (2 + 3i\lambda - \lambda^2) \gamma + O(\kappa)
\]
\[(2.11.3)\]

Where, to lowest order in the size of the detector, \((\mathcal{r} = 0, \mathcal{s} = 1)\),
\[
X = (r^2 + 1)^{-2} \quad ; \quad \gamma = - \frac{(2 - i\lambda)}{(r^2 + 1)^4} \cdot \kappa^2
\]
\[
X = X(\lambda, \kappa, R, \mathcal{s})
\]
\[
\mathcal{X}(\lambda, \mathcal{s}', R, \mathcal{s}') = X(\lambda, \mathcal{s}', R, \mathcal{s}')
\]
\[(2.11.4)\]

In (2.10.5) and (2.11.1) this gives,
\[
\mathcal{P} = \sum \beta \left\{ \frac{2\lambda^2}{\kappa^2} \cdot X - 2(\mathcal{s} - \frac{1}{4}) \mathcal{Q}, r \right\}_{\kappa = 0} +
- \frac{1}{4} \mathcal{S} \left( \mathcal{Q}_1 + \mathcal{Q}_2 r \right) \right\}_{\kappa = 0} +
+ \mathcal{S} \left( 1 - 3i\lambda \right) \mathcal{X} \right\}_{\kappa = 0} + O(\kappa^2)
\]
\[(2.11.5)\]

(Appendix E(c)).

which evaluates to,
\[
\sum \beta \left\{ \frac{2\lambda^2}{\kappa^2(1 + \kappa^2)^2} + \frac{2 \mathcal{S}}{(1 + \kappa^2)^3} + \frac{\kappa^2}{(1 + \kappa^2)^4} \cdot (2 - 16\mathcal{S} + 2\lambda^2) \right\}
\]
\[(2.11.6)\]
I evaluate the energy emitted by the detector at time $t_0$ in a time interval $\Delta \tau$. To determine the region of the $t = x$ plane to integrate the energy flux over, consider the equation of the light-cone emanating from the point $z = z_0$, $g = 1$ (again taking the acceleration $g = 1$),

$$ (t - t_0)^2 = (x - x_0)^2 + \eta^2 $$

(2.11.7)

This cone hits the $t = x$ plane at,

$$ \sqrt{S} = t + x = 2t = \frac{x_0^2 - t_0^2 + \eta^2}{x_0 - t_0} $$

(2.11.8)

and then using the equation for the trajectory ($g = 1$),

$$ \text{Figure 2.11 Emmission from } z_0 \text{ in } \Delta \tau $$
the equation of the intersection of the light-cone of emission from
the point \( \tau = \tau_0 \) on the trajectory and the null plane \( t = x \) is given by

\[
\mathbf{v} = (1+r^2) \cdot e \chi (\tau_0)
\]  

(2.11.10)
The energy passing through the appropriate region of the \( t = x \)
surface is now calculated from (2.11.6),

\[
\langle \Delta E \rangle = \pi \int_{0}^{\infty} d \tau^2 \int_{\mathcal{V}} \mathbf{\Pi} \omega x \cdot d \mathbf{v} \quad ; \quad \left. \frac{\Delta \mathbf{v}}{\mathbf{v}} \right|_{\mathcal{V}} = \Delta \mathcal{V}
\]

\[
= \sum_{\lambda} \pi \beta \int_{0}^{\infty} d \tau^2 \left\{ \frac{2 \lambda^2 e^{-\tau_0}}{(1+r^2)^3} + \left[ \frac{\lambda^2}{(1+r^2)^3} \cdot (2 \lambda^2 + 2 - 16 \pi) + \right. \\
+ \left. \frac{4 \pi}{(1+r^2)^3} \right] \cdot e^{\tau_0} \right\} \delta(\lambda - [E_j - E_0]). \Delta \mathcal{V}
\]

\[
\Rightarrow \langle \Delta E \rangle = \frac{-\lambda^2 \sum \lambda^2}{\pi} \cdot \frac{\lambda^2}{\lambda^2 \left[ \exp (2 \pi \lambda) - 1 \right]} \left[ \int_{0}^{\infty} d \tau^2 \int_{\mathcal{V}} \mathbf{\Pi} \omega x \cdot d \mathbf{v} \right] \left( \frac{1 - \tau_0}{\lambda^2} + 1 \right) \frac{1}{2} e^{\tau_0} \delta(\lambda - (E_j - E_0)). \Delta \mathcal{V}
\]

(2.11.11)

It remains to compare the energy gained by the detector in time
with the energy emitted into the field.
The energy gained by the detector (as measured in the detector's frame) to order $\varepsilon^2$ is given by,

$$\Delta E_{DET} = \sum_{E_j} (E_j - E_0) \cdot P_{E_j} \cdot \Delta \tau$$

(2.11.12)

where $P_{E_j}$ is the probability per unit proper time of excitation to state $E_j$. From equation (2.4.12),

$$\left( \frac{\Delta E_{DET}}{\Delta \tau} \right) = 2\varepsilon^2 \int_0^\infty d\lambda \int dp_y dp_z \cdot \frac{\delta(\lambda - (E_j - E_0))}{\exp(2\pi\lambda) - 1} \cdot \sigma^2(\lambda, p_y, p_z)$$

(2.11.13)

This expression can be approximated for a small box as (Appendix E(d)),

$$\left( \frac{\Delta E_{DET}}{\Delta \tau} \right) \sim \frac{\varepsilon^2}{\pi} \sum_{\lambda} \frac{\lambda^2}{\exp(2\pi\lambda) - 1} \left| \int_{box} dy^3 \int_{y_j}^{y_j+y} \lambda \cdot \gamma_{E_j} \gamma_{E_0} \right|^2 \delta(\lambda - (E_j - E_0))$$

(2.11.14)

To compare (2.11.11) with (2.11.14), note that in general the accelerated frame is Lorentz boosted from the $(t,x)$ frame that the energy emission (2.11.11) was evaluated in. Note also that the energy emission is not necessarily symmetric - that is, it may be emitted preferentially in the direction of acceleration, for example (I am indebted to Bill Unruh for this point).

To transform between the two frames it is necessary to take into account the Lorentz transformation of the energy-momentum 4-vector,

$$E' = \cosh(\tau_0) \cdot E - \sinh(\tau_0) \cdot P_x$$
where the primed frame is an inertial frame instantaneously co-moving with the detector at time \( T_0 \), and the unprimed is the \((t,x)\) frame. The velocity of the primed frame with respect to the unprimed is \( \sigma = \tanh(T_0) \).

In the \((t,x)\) frame, the \(x\)-component of the momentum of the emitted radiation can be found by integrating the momentum density, \( T_{x^2} - T_{x^x} \), over the same region of the \( t = x \) hypersurface as was used to determine the energy emission, to obtain,

\[
\Delta P_x = - \left( \frac{\Delta E_{\text{detector}}}{\Delta T} \right) \left[ \sinh(T_0) + \left( \frac{1-4\xi}{2\lambda^2} \right) \cdot T_0 \right] \cdot \Delta T
\]  

(2.11.16)

Transforming to the frame of the detector, I obtain,

\[
\Delta E'_{\text{field}} = - \left( \frac{\Delta E_{\text{detector}}}{\Delta T} \right) \left[ \left( \frac{1-4\xi}{\lambda^2} \right) + 1 \right] \cdot \Delta T
\]

\[
\Delta P_x'_{\text{field}} = - \left( \frac{\Delta E_{\text{detector}}}{\Delta T} \right) \left[ \left( \frac{1-4\xi}{\lambda^2} \right) \right] \cdot \Delta T
\]  

(2.11.17)

In general, the momentum transfer into the field is non-zero, and the accelerating force will feel this as an inertia (positive or negative) additional to that of the detector. Thus if \( \xi > 1/4 \), for example, \( \Delta P_x'_{\text{field}} \) is positive, the back-reaction on the detector is in the negative \(-x\) direction, and therefore the accelerating force needs to do more work than if the box were just a dead weight. This increased work appears as an energy emission additional to the negative emission.
compensating for the energy gain by the detector.

In the special case of the scalar field with parameter, $\xi = 1/4$, the energy emission into the field is exactly the negative of the excitation energy, the emission into the field is symmetric in the x-direction, and the accelerating force need do no extra work.

§2.12 Discussion

Summary

I have found that the energy emission of Unruh's particle detector following a trajectory of constant acceleration is negative. In general ($\xi \neq 1/4$), the emission into the field is not symmetric in the direction of acceleration, and consequently the force accelerating the detector needs to do more (or less) work, which energy appears as an additional emission into the field.

The emission of negative energy suggests that the overall energy of the out-state of the field will be lower than the zero energy of the Minkowski vacuum. To resolve this situation, I argued that the analysis presented is only valid if the interaction ceases at early and late times. I therefore discussed a more general and realistic trajectory in a model two dimensional spacetime, comprising a detector that starts and finishes in inertial motion with an interaction strength that is switched slowly on at early times and switched slowly off at late times.

Using the more general model, I found that the simplest form of a
switched interaction strength – a slow exponential growth and decay – changes insufficiently rapidly to ensure that the interaction ceases at early and late times. Nor does it cure the problem of the negative energy out-state. Considering the more general trajectory, however, I argued that starting and finishing in inertial motion would guarantee that the interaction ceases as required. Moreover, I found that the emission from a detector following the more general trajectory is positive overall, thereby preserving the Minkowski vacuum as the state of lowest energy.

Unruh & Wald (1984)

Unruh & Wald (1984) have looked at other aspects of the back-reaction process. The particle emitted when the detector excites has a substantial probability of being found in the causally disconnected Rindler wedge (R−) and carries positive energy there. This raises the possibility of being able to send a signal between causally disconnected regions. Taking into account the other processes of order $\xi^2$, however, Unruh & Wald find that the failure of the detector to be excited is correlated to a balancing negative energy propagating in (R−).

They also discuss the differing interpretations of the detector excitation given by inertial and co-moving accelerating observers. Inertial observers say that the field energy increased because the detector emitted a particle; accelerating observers say that the field energy decreased due to the absorption of a particle, but because a 'partial measurement' was performed during the absorption process, the net effect was an increase in the field energy.
The meaning of this 'partial measurement' is as follows - consider the result of measuring the energy of a state given by,

$$|0\rangle + \frac{1}{\sqrt{n}} |n\rangle, \quad n \text{ large}$$  \hspace{1cm} (2.12.5)

The initial expected energy is $\sim E$, where 'E' is the energy of a single quantum. If the detection of a quantum occurs, however, the expected field energy becomes $(n-1)E \gg E$, and thus the detection of a particle increases the field energy.

This is exactly the situation as seen by accelerating observers – the Minkowski vacuum appears to be a thermal state (not an eigen state of their Hamiltonian). The act of detection not only removes a particle from the field, but also performs a (partial) measurement of the state. Thus the act of detection increases the field energy from both the point of view of inertial observers, and from that of co-moving accelerated observers.

Note that this discussion only applies to the situation where the detector is finally found to be excited. The evaluation of the energy emission given in this chapter has assumed that the final state of the detector is not known, and so also picks up contributions where the detector would be finally found unexcited.
Chapter Three

Unruh's particle detector in curved spacetime - measuring the energy-momentum tensor

"The study of . . . particle detectors has exposed the nebulosity of the particle concept and suggests it should be abandoned completely. In its place, one would like to study as a probe to the field content, quantities like $\langle T_{\mu\nu} \rangle$, $\langle \Phi^2 \rangle$. The major problem then arises as to how these quantities are to be measured. We need a theory of model detectors for a variety of field-related physical quantities. So far as I know, there has been no attempt to tackle this interesting problem."


In this Chapter, I initiate a study of whether an Unruh-type detector can be used to measure the energy carried by a field in curved spacetime. I find that a detector moving on a Killing trajectory will respond to the true energy of the field, but in addition gives a response that is independent of the state of the field. I argue that the latter should be thought of as a noise inherent in the measurement process. In a two dimensional model, I show that this 'noise' is a function of the acceleration of the detector (encompassing Unruh's result for an accelerating detector in flat spacetime) and of the curvature of the spacetime (encompassing a result due to Gibbons & Hawking for the
response of a detector in de Sitter spacetime). A number of problems of interpretation are discussed.

S3.1 Introduction
Following the discovery of the thermal properties of field states in Schwarzschild spacetime, Gibbons & Hawking (1977) examined the application of quantum field theory to a cosmological model. They considered a particle detector moving inertially in de Sitter spacetime - an expanding spherically-symmetric vacuum solution ($<T_{\mu\nu}> = 0$) of Einstein's equations with positive cosmological constant, $\Lambda$ (see Section S3.2), and found that the detector will excite as if immersed in a thermal bath at temperature,

$$T = \chi (\Lambda / 12\pi^2)^{1/2}$$

(3.1.1)

where $\chi$ is a red-shift factor. They attributed the detector response to the presence of event horizons.

This finding is similar to Unruh's (1976) result of an accelerating detector's response to the Minkowski vacuum in that the de Sitter vacuum is devoid of energy fluxes, but it is perhaps stranger because the detector is moving inertially. Unlike the case of an accelerating detector in Minkowski spacetime, the motion of the detector cannot be held to be responsible for the excitations.

Hajicek (1977), in his theory of particle detection, distinguishes three possible sources of quantum particles - (i) the initial state of the field, (ii) production by the spacetime curvature and (iii) effects due to the detection process. Presumably, for example, the particles ending
up at infinity in the 'Blackhole Evaporation' process carry energy (as confirmed by an analysis of the energy-momentum tensor), whereas the particles detected by an accelerating detector in flat spacetime do not, and are only a product of the measuring process. The central question is whether or not one can distinguish between these sources of quanta, as measured by a particle detector.

Some authors (Candelas (1980), Grove & Ottewill (1981), Sciama, Candelas & Deutsch (1981)) consider Unruh's detector to be a "fluctuometer": an instrument that measures the power spectrum of the field, and hence of little value as a device that measures the field. Certainly, if acceleration or the presence of structures such as event horizons are inextricably linked to the response of the detector, it is difficult to see how any response due to the real energy content of the field might be distinguished.

On the other hand, Unruh's detector is, so far, the only measuring instrument introduced into the theory, and as such, it is surely worthwhile attempting a full understanding of its capabilities. In particular it seems implausible that the assertion of Gibbons & Hawking (1977) is true - that it is the "... loss of information about the quantum state ..." due to the presence of event horizons "... which is responsible for the thermal radiation that the observers see" in de Sitter space. The localized interaction of the detector with the field suggests, rather, that it is some local property of the field, or of the spacetime, that is more directly responsible.
To evaluate the response of Unruh's detector in the presence of a gravitational field, I evaluate the energy gained by an Unruh box detector in its interaction with a field propagating on curved spacetime.

In Section §3.2, I review the properties of de Sitter spacetime and the result of Gibbons & Hawking. Section §3.3 describes the model and shows that the detector responds to the true energy of the field but that there is an inherent noise. In Section §3.4 I argue that this 'noise' is a function of the motion of the detector and of the curvature of the spacetime. Section §3.5 discusses the interpretation of the results.

**S3.2 de Sitter Spacetime**

The gravitational field produced by a body that has collapsed to form a blackhole is so strong that light emitted from close to the blackhole is
drastically redshifted, and no energy reaches distant observers; there is a region bounded by an event horizon that is not visible to them. Event horizons of a different kind occur in cosmological models with a repulsive cosmological constant, $\Lambda$. The effect of $\Lambda$ is to cause the universe to expand so rapidly that for each observer there are regions from which light can never reach them. Unlike the blackhole spacetime where there is a unique event horizon, in an expanding universe each observer has their own "cosmological event horizon".

The metric of de Sitter spacetime is given through,

$$ds^2 = (1 - \Lambda r^2/3) \, dt^2 - \left(1 - \Lambda r^2/3\right)^{-1} dr^2 - r^2 d\Omega^2 \quad (3.2.1)$$

There is a coordinate singularity at $r = (3/\Lambda)^{1/2}$ but, just as in Schwarzschild spacetime, 'Kruskal type' coordinates $(u, v)$ can be introduced to cover the whole manifold. Suppressing the angular dependence, the distance element in 'Kruskal type' coordinates in two dimensions is,

$$ds^2 = \frac{3}{\Lambda} \left(u^2 - 1\right)^{-2} \, du \, dv \quad (3.2.2)$$

From (3.2.1), it can be seen that $\frac{\partial}{\partial t}$ is a timelike Killing vector, but, because of the symmetry properties of de Sitter spacetime, it is not unique. In fact, any timelike geodesic can be chosen as the origin of polar coordinates, and then the surfaces, $u = 0; v = 0$, will be the past and future cosmological event horizons for observers moving on this trajectory.

The value of $\langle T_{\mu\nu}\rangle$ for the vacuum state associated to a choice of
positive frequency based on the choice of coordinates \((u, v)\) (see Section S1.4) can be found by applying (1.9.3) with the conformal factor,

\[
C = \frac{3}{\lambda} (uv - 1)^{-2}
\]  

(3.2.3)

The result is that this 'Kruskal type' vacuum state is a state of zero energy fluxes: \(< T_{uu} > = < T_{vv} > = 0; < T_{uv} > \) is non-zero due to the presence of curvature, \(R = 2\lambda / 3\).

de Sitter spacetime is invariant under the action of the ten-parameter group, \(SO(4,1)\), and if one demands that the vacuum state be invariant under this group, (as is natural in analogy to Minkowski spacetime, where the vacuum is taken to be invariant under the full symmetry group of the manifold - the Poincare group), then the only possibility in 4-dimensions is

\[
< 0 | T_{\mu\nu} | 0 > \sim g_{\mu\nu}
\]  

(3.2.4)

(as confirmed explicitly by Dowker & Critchley (1976)). This is exactly the situation just described using the 'Kruskal type' vacuum state - the 'Kruskal type' vacuum is therefore also the de Sitter invariant vacuum.

If the field is in the de Sitter-invariant vacuum, then this being a 'Kruskal type' vacuum, in analogy with the 'Blackhole Evaporation' process, the state looks to a Killing observer like a thermal state at temperature,
\[ T = \chi \left( \frac{\Lambda}{12\pi^2} \right)^{1/2} \]  

(3.2.5)

with \( \chi \) a redshift factor (cf. Section §1.5).

This is the result that Gibbons & Hawking (1977) derived. It is not certain which state of the \( \Phi \) field Gibbons & Hawking were considering: they only specify that there should be no particles present on the surface at past null infinity, \( \mathcal{J}^- \); but, as Unruh has shown, particles are not unambiguously defined concepts in this theory. Remarks made in other papers (Gibbons (1979), Hajicek(1977)), however, indicate that they are indeed considering the de Sitter invariant state.

The unclear relation between the state of the field and the response of a particle detector has lead to much confusion. If the particles detected in de Sitter spacetime were energy carriers, an impossible situation would result - the analysis of Gibbons & Hawking shows that each inertial observer detects the same thermal radiation, whereas true thermal radiation has a preferred rest-frame.

Gibbons & Hawking appear to consider the 'particles' detected by the inertial observer in de Sitter spacetime to be energy carriers, contributing to \( T_{\mu \nu} \): "... different observers have different definitions of particles. It would appear that one cannot... construct a unique observer - independent renormalized energy - momentum tensor which can be put on the right hand side of the
It is clearly not true that the detected particles carry energy, however, for as has been shown, there are no energy carriers in the de Sitter-invariant vacuum. The situation is analogous to Unruh's result for an accelerating particle detector in flat spacetime: there are no energy carriers in the Minkowski vacuum, but yet the detector excites due to the effect of the zero-point fluctuations through the accelerated monopole interaction. The confusion stems from identifying 'particles', as excitations of a detector, with energy carriers which contribute to $T_{\mu\nu}$.

The question remains, however, as to the cause of the detector excitations in de Sitter spacetime.

S3.3 Response of a detector in curved spacetime

Introduction

I consider the response of a detector to a field propagating on a spacetime with a time-like Killing vector. The detector is taken to follow a Killing trajectory - an integral curve of the time-like symmetry. These restrictions allow for the repeated measurement of identical situations that is required to obtain an expectation value.

Design of the detector

To investigate the possibility of measuring the energy propagating in the field, a starting point in to consider the energy gained by the Unruh box detector discussed in Chapter Two.
Chapter Three

The particle wave function, \( \hat{\Psi} \), is represented as,

\[
\hat{\Psi} = \sum c_n(\tau) \exp(-iE_n\tau) \gamma_n(\vec{r})
\]

(3.3.1)

where \( \gamma_n \) are eigen functions of the free Hamiltonian, and \( c_n \) to order \( \xi \) are given by,

\[
c_n(\tau) = -i \int_0^\tau dz' e^{\frac{i}{\hbar} (E_n - E_0)z'} \langle E_n | H_{\text{int}}(\tau') | E_0 \rangle
\]

(3.3.2)

The order \( \xi^2 \) contribution to the energy of the state, \( \hat{\Psi} \), is then found to be,

\[
\langle E \rangle_{\xi^2} = \sum \left( E_n \cdot c_n^2 + 2 Re \left\{ c_n \cdot \frac{-iE_n}{\hbar} \right\} \langle \gamma_0 | H_{\text{int}} | \gamma_n \rangle \right)
\]

(3.3.3)

If measurement is made at late times when \( H_{\text{int}} \) is switched off, the second term disappears and the average energy gained per unit time is,

\[
\lim_{T \to \infty} \frac{\langle E \rangle}{T} = \lim_{T \to \infty} \frac{1}{T} \sum_n E_n \cdot c_n^2(\tau)
\]

(3.3.4)

For the original Unruh box detector with a coupling,

\[
H_{\text{int}} = \xi \int dx^3 \int_0^3 S^3 \left( \frac{A}{q^2 - \vec{x}^2} \right) \Phi(\vec{x})
\]

(3.3.5)
the rate of energy gain to order \( \xi^2 \) is found to be,

\[
\lim_{T \to \infty} \frac{\xi^2}{T} \sum_{E} E \int_{0}^{T} d\tau_1 \cdot \mathcal{X}_E \cdot \int_{0}^{T} d\tau_2 \cdot \mathcal{X}_E \cdot \Phi(\mathbf{x}, \mathbf{x}', E) \langle S | \Phi(\mathbf{x}) \Phi(\mathbf{x}') | S \rangle
\]

(3.3.6)

where \( |S\rangle \) is the state of the field, and the "Cross-section operation" involves spatial integrations over the particle wave functions, and the field (cf. (2.6.12)). The time integrations select annihilation / creation operators that are normal-ordered with respect to Killing time, \( \mathcal{T} \):

\[
\xi^2 \sum_{E} E \langle [X-S(\mathbf{x}, \mathbf{x}', E)] | S \rangle \ a_E^+ \Phi_E^{\dagger}(x) \ a_E \Phi_E(x, x') | S \rangle
\]

(3.3.7)

I now consider design changes to the detector and to its field coupling necessary to obtain a response to \( T_{\mu\nu} \).

\( T_{\mu\nu} \) contains both space and time derivatives of the fields; a time derivative brings down the premultiplying energy \( E \), but some response to the spatial derivatives has to be arranged. The cross-section muddles the response to the field with properties peculiar to the particular construction of the detector. A first requirement is for the cross-section to be proportional to energy, so as to allow a response to \( \langle S | \Phi_{\mathcal{T}}(x) \Phi_{\mathcal{E}}(x') | S \rangle \). This perhaps could be arranged by the use of an ensemble of detectors, each sensitive to a different frequency, and each suitably normalized.
Secondly, consider the response of the original Unruh detector to a state that is a coherent sum of left- and right-moving modes. If the detector is at a node of the standing wave, then it will not excite. This is as expected, for this type of detector responds to the particle number density, $\Phi \delta^2 \Phi$ which is zero at the node of a standing wave. The energy density is, however, non-zero.

What is required is to introduce some sort of spatial dependence on the field. Unruh (private communication) has suggested that the cross-section could be modified to allow no interference between left and right-moving modes.

In two dimensions, the average energy gain per unit time is,

$$\xi^2 \sum_{E} E \left[ -\mathbf{S}(\mathbf{x}, \mathbf{x'}, \mathbf{e}) \right] \frac{1}{4\pi E} \langle S | \hat{a}^+_E \exp(iE(t-g')) + \hat{a}^+_E \exp(-iE(t-g')) | S \rangle$$

and if no interference is allowed, this becomes,

$$\xi^2 \sum_{E} E \left[ -\mathbf{S}(\mathbf{x}, \mathbf{x'}, \mathbf{e}) \right] \frac{1}{4\pi E} \langle S | \hat{a}^+_E \exp(iE(t-g')) \hat{a}^+_E \exp(-iE(t-g')) + \hat{a}^+_E \exp(iE(t+g')) \hat{a}^+_E \exp(-iE(t+g')) | S \rangle$$

which approaches a response to the energy of the field normal-ordered with respect to the Killing vacuum (the significance of this normal ordering will be discussed shortly).
Such a non-interference restriction on the cross-section is insufficient in four-dimensions, however - the detector must respond differently to a wave packet that misses the detector from one that hits it, although the difference may be only in the relative phases of the modes comprising the packet (in two dimensions, a packet travelling in the detector's part of the spacetime must hit the detector). To pick up the various terms that comprise \( \mathcal{T}_{\mu \nu} \) in four dimensions, one must determine more complex restrictions on the cross-section, or use a variety of detectors with couplings to, say, spatial derivatives of the field.

Thirdly, the field state, \( |S\rangle \), may contain superpositions of quanta that the detector cannot respond to. If \( H_{\text{INT}} \) involves a linear coupling between the particle and the field, then \( c_n(T) \) is some function of \( \langle S| \hat{E}(x) \hat{E}(x') |S\rangle \) which can contain cross-terms such as,

\[
\langle \hat{\phi}| \hat{E}(x) \hat{E}(x') |\hat{\phi}\rangle
\]

with \( \phi \neq \beta \). Unless the coupling to the field is some complicated function of time, the integrations over all time result in the detector not giving a response.
A consequence is that to obtain a response to cross-terms, measurement of the wave function must needs be made at early times, taking into account the second term of (3.3.3).

In considering what properties the detector should have to be able to correctly resolve the field energy, it becomes apparent that at least some of the problems will be ameliorated if the detector itself is taken to have the properties of a field. That is, if the detector is a box containing the same field as that to be measured, and if it can be arranged to take a "snapshot" of the external field, then $T_{\mu \nu}$ for the field within the box will be that of the field to be measured.

*Normal-ordering with respect to the Killing vacuum.* Notice that the action of the time integrals is to select field operator products that are normal-ordered with respect to the Killing vacuum. The 'natural' particle picture for a detector moving on a Killing trajectory is based on 'Killing modes' of the field — that is, a separation between positive and negative frequencies based on the proper time of the detector. In analogy with the operation of the accelerating detector in flat spacetime discussed in Chapter Two, a detector in curved spacetime will excite in response to the presence of 'Killing particles' in the field.

In the following discussion, I show that the field-in-a-box detector responds not to the true energy of the external field, but to the energy normal-ordered with respect to the Killing vacuum.
That is, consider the formal energy - momentum tensor of the external field evaluated in the state, $|S\rangle$. This can be written as,

$$\langle S| T_{\mu\nu}| S\rangle = \langle S|_K T_{\mu\nu}^K | S\rangle + T_{\mu\nu}|_{\text{div}}$$

(3.3.12)

where $'K'$ denotes normal - ordering with respect to the Killing vacuum, and $'T_{\mu\nu}|_{\text{div}}'$ are divergent terms. The divergent terms are essentially sums of commutators of annihilation and creation operators, and are independent of the state; hence in the Killing vacuum,

$$\langle 0| T_{\mu\nu}| 0\rangle_K = \langle 0|_K T_{\mu\nu}^K | 0\rangle_K + T_{\mu\nu}|_{\text{div}}$$

$$= 0 + T_{\mu\nu}|_{\text{div}}$$

(3.3.13)

so that,

$$\langle S|_K T_{\mu\nu}^K | S\rangle = \langle S| T_{\mu\nu}| S\rangle - \langle 0| T_{\mu\nu}| 0\rangle_K$$

(3.3.14)

(The subtraction of (3.3.13) from (3.3.12) is clearly only valid when regularized operator products are used).

Upon renormalization of the right - hand terms, (3.3.14) shows that the energy of the state $|S\rangle$ normal ordered with respect to the Killing vacuum is equal to the difference between the true expectation value of $'T_{\mu\nu}'$ and that of $'T_{\mu\nu}|_{\text{div}}'$ evaluated in the Killing vacuum state.
The analysis presented here shows that it is the left-hand side of (3.3.14) that the field-in-a-box responds to. Consequently, any attempts to measure $\langle S| T_{\mu\nu} |S \rangle$ will involve $K \langle 0| T_{\mu\nu} |0 \rangle_K$, which I argue should be thought of as a 'noise'.

This result is not restricted to the particular model discussed here, but will be seen to apply to any similar model where the coupling between the fields is linear and does not involve complicated time dependencies.

**The model**

To analyse the response of a detector moving in a curved spacetime, I take the detector to be a field, $\Phi$, initially in its ground state, confined to a small box and interacting with the external field, $\Phi$, through,

$$\zeta^{\text{int}}(x) = \zeta \cdot \Phi(x) \cdot \Phi(x)$$

(3.3.15)

The analysis to be presented does not depend on which particular fields are used, thus the previous restriction to massless scalar fields may be removed.

Boundary conditions of perfect reflection are placed on the $\Phi$ field on the inside walls of the box, while the conditions placed on the outside walls are irrelevant to the analysis, as long as the interaction between $\Phi$ and $\Phi$ is confined to the interior of the box.

The detector field is taken to excite away from the ground state;
however, a more realistic detector would comprise a massive field exciting from the one-particle state of lowest energy. If the field particle was taken to be sufficiently massive that the probability of its annihilation is vanishingly small, then the one-particle state of lowest energy would act as the vacuum, as in the following analysis.

I restrict the spacetimes under consideration to those with a time-like Killing vector. The associated symmetry does not have to be global; it is sufficient that the Killing vector field has a complete integral curve, which the box moves along. An example, then, is the Killing vector $\frac{\partial}{\partial t}$ of the Schwarzschild segment of the eternal blackhole spacetime; a box following a trajectory of this symmetry hovers above the blackhole at constant distance ($r =$ constant). This restriction allows for the repeated measurement of identical situations that is required to obtain an expectation value. Indeed, it is hard to imagine how one could establish a connection between the formalism of quantum theory and the Born probability interpretation if there were no time-like symmetry.

Currents of the field typically involve products of the field operators at a point. In particular, the regularized energy-momentum tensor can be written as a function of the two-point function, $G(x,x')$ (see Section 1.8). To determine the effect of the interaction, I evaluate the two-point function of the detector at late times.

$G^{out}(x,x')$ is found from,
and Appendix F(a) gives the details of the calculations leading to the following.

\( G_{\text{out}}(x,x') \) evaluates to,

\[
G_{\text{out}}(x,x') = G_{\text{in}}(x,x') + \]

\[
-2 \int dy^4 \sqrt{g} \left\{ \int_{+1} dy_2 \sqrt{-g} \Delta^{-}(y_1,x) \Delta(y_2-x') + \right. \]

\[
+ \int_{+1} dy_2 \sqrt{-g} \Delta^{-}(y_2-x') \Delta(y_1-x'') \right\} \langle S | \Phi(y_1) \Phi(y_2) | S \rangle \]

\[
+ \xi^2 (x \leftrightarrow x') + O(\xi^4) \]  

where,

\[
\Delta^{-}(x-y) \equiv \langle 0 | \Phi(x) \Phi(y) | 0 \rangle \]

\[
\Delta(x-y) \equiv \langle 0 | \Phi(x) \Phi(y) | 0 \rangle \]  

(3.3.18)

If a particle representation is chosen based on a complete set of field modes, \( \Phi_\omega \), (which are assumed to be a complete set also for the expansion of 'Killing mode solutions'), the two-point function can be written as,

\[
\langle S | \Phi(x) \Phi(x') | S \rangle = \sum_{k} \phi_k(x) \phi_k^*(x') + \]

\[
+ \sum_{\alpha,\beta} N_{\alpha,\beta} \left( \phi_\alpha(x) \phi_\beta^*(x') + \phi_\alpha^*(x) \phi_\beta(x') \right) \]  

(3.3.19)

where the first term is the vacuum contribution, and the "N's" are numbers depending on the occupation number of modes, on which modes comprise a particular pure state, and on which pure states make up the
actual state of the field.

Evaluating the integrals in (3.3.17), the final result for the two-point function of the detection field is,

\[
\mathcal{G}^{\text{out}}(x,x') - \mathcal{G}^{\text{in}}(x,x') = \xi^2 \left\{ \sum_k D_k(x) D_k^*(x') + \sum_{\alpha\beta} N_{\alpha\beta} \left( D_\alpha(x) D_\beta^*(x') + D_\alpha^*(x) D_\beta(x') \right) + \sum_\lambda \gamma_\lambda(x) \gamma_\lambda^*(x') \cdot \sigma^2(\lambda,\lambda^*) \right\}
\]

(3.3.20)

where,

\[
D_k = \sum_\Sigma \left( A_{k\Sigma} \cdot \gamma_\lambda^*(x) \cdot \sigma(\Sigma,\Sigma^*) + B_{k\Sigma} \cdot \gamma_\lambda^*(x) \cdot \sigma(\Sigma^*,\Sigma) \right)
\]

(3.3.21)

In this expression, \( \{ \gamma_\lambda, \gamma_\lambda^* \} \) are detector field Killing modes, and 'A', 'B' are Bogoliubov coefficients between the chosen representation of the state of the external field and a representation associated to the Killing vector field. \( \sigma(\lambda,\lambda^*) \) is a cross-section term, being the response of the detector to absorbing an external field Killing mode of energy \( \lambda \),

\[
\sigma(\alpha,\beta) \equiv \sqrt{2\pi} \int_{b_{\alpha x}} d^3y \sqrt{g} \cdot h_\alpha(y) \cdot H_\beta^*(y)
\]

(3.3.22)

In this expression, 'h_\alpha' are detector spatial modes, and 'H_\beta' are \( \Phi \) field Killing spatial modes.
Now, expanding the two point function for the external field in terms of Killing modes for the region in which the detector moves, and normal ordering the creation and annihilation operators with respect to the Killing vacuum gives,

\[
\langle s | \Phi(x) \Phi(x') | s \rangle = \langle s | \sum_{\lambda, \gamma} \left( b_{\lambda} \Phi_{\lambda}(x) + b^{\dagger}_{\lambda} \Phi_{\lambda}^*(x) \right) \left( b_{\gamma} \Phi_{\gamma}(x') + b^{\dagger}_{\gamma} \Phi_{\gamma}^*(x') \right) | s \rangle
\]

\[
\Rightarrow \langle s | \sum_{k} \Phi_k(x) \Phi_k(x') | s \rangle = \langle s | \Phi(x) \Phi(x') | s \rangle + \langle s | \sum_{\lambda, \gamma} \left[ b_{\lambda} \cdot b_{\gamma} \right] \Phi_{\lambda}(x) \Phi_{\lambda}^*(x') | s \rangle
\]

\[
= \sum_k \rho_k(x) \rho_k^*(x') + \sum_{\alpha, \beta} N_{\alpha, \beta} \left[ \rho_{\alpha}(x) \rho_{\beta}^*(x') + \rho_{\beta}(x) \rho_{\alpha}^*(x') \right] - \sum_{\lambda} \Phi_{\lambda}(x) \Phi_{\lambda}^*(x')
\]

(3.3.23)

where,

\[
\rho_k(x) = \sum_{\Sigma} \left( A_{k\Sigma} \cdot \Phi_{\Sigma}(x) + B_{k\Sigma} \cdot \Phi_{\Sigma}^*(x) \right)
\]

(3.3.24)

Comparing (3.3.23) with the response of the detector, and '\(P_k\)' with '\(D_k\)', it can be seen that each Killing mode of the external field has been absorbed by the detector to the extent allowed by the detector's cross-section, and stimulated a corresponding detector field mode to appear in the box.
It is in this sense that the detector responds to the external field normal ordered with respect to the Killing vacuum.

The response of a detector to a current such as \( T_{\mu \nu} \), that is a function of the two-point function will therefore involve the energy of the Killing vacuum state, as shown in (3.3.14). This Killing energy is independent of the state of the external field. It is a term that will always be present, and consequently can be considered to be a noise inherent in the detection process.

**S3.4 Noise in the detection process**

Whatever the construction of the detector, it is straightforward to show that for a large class of detectors, the response will not include any contributions from operator products of the external field that are not normal-ordered with respect to the Killing vacuum.

To see this, consider such a term from (3.3.17) as,

\[
\sum_{\omega, k, \lambda} \int_{-\infty}^{\infty} dt_1 \left( \sum_{\omega_k} \right) \left( \sum_{\lambda} \right) e^{-i\omega t_1 - i\lambda t_1} e^{i\omega t_1 - i\lambda t_1} \sum_{\omega_k} \delta(\omega + k)
\]

(3.4.1)
which is zero.

Consequently, the best that one can hope for from a detector coupled to the external field in some manner that maintains the above time integrations is that it measure $T_{\mu \nu}$ of the external field normal ordered with respect to the Killing vacuum.

The response of a detector on a Killing trajectory thus involves a detection 'noise' that is essentially the negative of the energy of the vacuum state naturally associated to the Killing field ("essentially" because of the presence of cross-section terms). I now argue that this 'noise' is a function of the motion of the detector and of the curvature of the spacetime.

$\langle T_{\mu \nu} \rangle$ for a Killing vacuum state in a four dimensional spacetime has not yet been evaluated, so I consider only a two dimensional model, using a massless scalar field.

Appendix B(c) shows that if a portion of the spacetime admits a time-like Killing vector, $\partial / \partial t$ (with a complete integral curve), then this region may be coordinated by null coordinates, $(u, v)$, in whose terms the line element is

$$ds^2 = \mathcal{C}(u, v) \cdot d\mathcal{U} \; d\mathcal{V} = \mathcal{C}(u, v) (dt^2 - dx^2) \quad (3.4.2)$$

where $\mathcal{C}(u, v)$ has the property that $\mathcal{C}_{,\mathcal{U}} = - \mathcal{C}_{,\mathcal{V}}$.

From (1.9.3), $\langle T_{uu} \rangle$ is given by,
\[ 24\pi \langle T_{\hat{a}\hat{a}} \rangle = \mathcal{C}^{-1} \frac{\partial^2 \mathcal{C}}{\partial \mathcal{C}^2} - \frac{3}{2} \mathcal{C}^{-2} \left( \frac{\partial \mathcal{C}}{\partial \mathcal{C}} \right)^2 \]  

and using the fact that \( \mathcal{C}_{\hat{a}\hat{a}} = - \mathcal{C}_{\hat{a}\hat{a}} \), this can be written as,

\[ 24\pi \langle T_{\hat{a}\hat{a}} \rangle = -\mathcal{C}^{-1} \frac{\partial^2 \mathcal{C}}{\partial \mathcal{C}^2} + \frac{3}{2} \mathcal{C}^{-2} \left( \frac{\partial \mathcal{C}}{\partial \mathcal{C}} \right) \left( \frac{\partial \mathcal{C}}{\partial \mathcal{C}} \right) \]  

Now the scalar curvature is given by,

\[ R = 4 \left( \mathcal{C}^{-2} \mathcal{C}_{\hat{a}\hat{a}} - \mathcal{C}^{-3} \mathcal{C}_{\hat{a}\hat{b}} \mathcal{C}_{\hat{b}\hat{a}} \right) \]  

and the acceleration of an integral curve of the Killing vector field is

\[ a^2 = -\mathcal{C}^{-3} \mathcal{C}_{\hat{a}\hat{b}} \mathcal{C}_{\hat{b}\hat{a}} \]  

Hence,

\[ 48\pi \langle T_{\hat{a}\hat{a}} \rangle = -\mathcal{C} \left( a^2 + R/2 \right) \]  

Scaling the Killing time, 't', by a factor \( \mathcal{C}^{1/2}(x_0) \) to obtain the proper time of an observer following an integral curve, \( x = x_0 \), of the Killing field, I find that the energy density of the Killing vacuum state, and hence the 'noise', is a function of the acceleration of the Killing trajectories, and the curvature of the spacetime.

Although \( \langle T_{\hat{a}\hat{a}} \rangle \) for the vacuum state naturally associated to a Killing field in a 4-dimensional spacetime has not yet been evaluated, the expectation is that it will depend on the local curvature and the acceleration of the Killing trajectories, as
indicated by the two dimensional model. In four dimensions, a greater variety of motions is possible – the case of rotation, for example, has been analysed in flat spacetime (Letaw & Pfautsch (1980), Letaw (1981), Grove & Ottewill (1981)), and, as expected, the centripetal acceleration results in detector excitations in reaction to emitting synchrotron radiation.

This encompasses the results of Unruh and of Gibbons & Hawking – the effect of acceleration was found by Unruh (1976), while the effect of curvature was found by Gibbons & Hawking (1977), although they attributed their results in de Sitter spacetime to the presence of event horizons, rather than to the presence of curvature.

S3.5 Discussion
Problems involved in interpreting the results of this Chapter are discussed.

At first sight, the foregoing analysis suggests that if the expectation value of the energy gained by a detector following a Killing trajectory is determined by repeated measurement, and if the cross section of the box and the 'noise' due to the motion and the curvature of the spacetime can be taken into account, then this would constitute a measurement of the expectation value of the energy carried by the external field in the region of the detector. The measurement of the energy gained by the box might be accomplished by noting the increased force required to keep the box at constant acceleration. There are a number of reasons why this analysis is too
simplistic, however.

**Cross-section terms**
The presence of cross-section terms in the response of the detector muddles the kinematic effect that is being measured with properties peculiar to the particular detector in use.

To be able to take a "snapshot" of the external field, a sufficient (although excessively stringent) condition on the detector is that,

\[ \gamma_\Sigma (\xi) \cdot \sigma (\Sigma, \Sigma') = \widehat{\rho}_\Sigma (\xi), \quad \forall \Sigma \]  

(3.5.1)

If the detector were to cover the whole of the spacetime region that supports the Killing modes, \( \widehat{\rho}_\Sigma (\xi) \), and if \( \gamma_\Sigma (\xi) \) satisfied identical boundary conditions to \( \gamma_\Sigma (\xi) \), then the cross-sections would just be the normalization conditions for the spatial Killing modes, \( \sigma (\Sigma, \Sigma') \).

\[ \int d^3x \sqrt{-g} \cdot H_\Sigma (\xi) \cdot H_\Sigma (\xi) = \frac{1}{\Sigma} \]  

(3.5.2)

Condition (3.5.1) could then be arranged to be true if the interaction were modified to,

\[ \zeta^{INT} (\xi) = \xi \frac{\partial}{\partial \xi} (\xi) \]  

(3.5.3)

As the size of the detector is reduced, however, the boundary conditions on the detector field will differ from those of the external field and not all external field modes will be able to be represented within the detector. In addition, the cross-section will no longer be
uniform.

These problems can perhaps be circumvented by the use of an ensemble of detectors, each sensitive to a different frequency, and each suitably normalised.

The need for a readout system
Care is needed in interpreting the change in energy of the box's contents. $\langle T_{\text{out}} \rangle$ is the energy of the free field propagating within the box that the perturbed detector field approaches at late times, when the interaction has ceased. Whereas the external field carries energy away from the detector, the detector field, $\Phi$, remains in the interaction region for all time; in the model discussed, the detector field never in fact becomes a free field at late times.

The same problem arises with Unruh's Schrödinger particle detector. The analysis of the latter gives an expected rate of transition per unit time. The rate is, however, evaluated on the basis of the detector maintaining a constant acceleration, and continuing to interact for all time (or at least a very long time). The interpretation of the transition rate is that presumably the detector can be connected to some readout device that monitors the detector without disturbing its operation.

In the model under discussion here, a readout system that does not disturb the detector is also required. Such a readout system will, however, monitor the energy of the box field while the detector is still interacting. What connection there is between the energy of the
free out detector field and the energy carried by the interacting field at a particular time is also unclear in the absence of a model that includes a readout system.

Moreover, while the energy carried by the out-field is a function of space and time, it is of necessity measured at late times. As we have been looking only at the lowest order effect, however, the order $\xi^2$ contribution will propagate from the interaction region to late times without further change. After the many repeated measurements required to determine an expectation value, the result would be an average of $\langle T_{\mu\nu} \rangle$ over the trajectory of the detector.

**Modifying the interaction strength**

To ensure that the detector field approaches a free field at early and late times, alternatively, the interaction strength could be modified from the constant value used in the analysis presented. In that case, however, it is probable that the effects of switching on and off the interaction will colour the response of the detector at the time of measurement.

The question of whether a detector such as that discussed here can be sensibly used to measure the energy content of the field is clearly beset with problems that require further work.
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Appendix A

Bogoliubov Transformations

(a) Basic properties
(b) Coefficients for Minkowski/Rindler modes
(c) Coefficients for Kruskal/Schwarzschild modes
(d) Extending the range of integrations

Appendix A(a)

Basic properties

Consider two sets of mode solutions to the Klein-Gordon equation,
\{ F_\omega, F_\omega^* \} and \{ p_\lambda, p_\lambda^* \}. In the situations of interest in this work, 'F' and 'p' will be complete sets with supports intersecting regions 'S_F' and 'S_p' of a spacelike (Cauchy) surface, with \( S_p \subseteq S_F \). It is assumed that there is also a complete set \( \{ q_\gamma, q_\gamma^* \} \) with support intersecting the Cauchy surface on \( S_q = S_F - S_p \).

'P' can be expanded in terms of 'F' as,
\[
\begin{align*}
    p_\lambda &= A_{\omega \lambda} F_\omega - B_{\omega \lambda} F_\omega^* \\
    p_\lambda^* &= A_{\omega \lambda}^* F_\omega^* - B_{\omega \lambda}^* F_\omega
\end{align*}
\]  

(A.a.1)
The scalar product,
\[ \langle v_k, v' \rangle_{SF} = i \int_{t_0} dS_A \cdot \sqrt{g} \cdot g^{\mu \nu} \cdot v_k^\mu \delta \nu v'_k \]  
(A.a.2)
on some Cauchy surface, \( t_0 \leq S_F \), allows the identification,
\[ \langle F_\omega, \eta^\lambda \rangle_{SF} = A_{\omega \lambda} \quad ; \quad \langle F_\omega^*, \eta^\lambda \rangle_{SF} = B_{\omega \lambda} \]
\[ \langle F_\omega^*, \eta^\lambda \rangle_{SF} = -A_{\omega \lambda}^* \quad ; \quad \langle F_\omega, \eta^\lambda \rangle_{SF} = -B_{\omega \lambda}^* \]  
(A.a.3)
and, as the support of 'p' cuts the Cauchy surface only on \( S_p \), we also have,
\[ \langle \eta^\lambda, F_\omega \rangle_{S_p} = A_{\omega \lambda}^* \quad ; \quad \langle \eta^\lambda, F_\omega^* \rangle_{S_p} = B_{\omega \lambda}^* \]  
(A.a.4)
where \( \langle , \rangle_{S_p} \) is \( \langle , \rangle_{SF} \), with the Cauchy surface truncated to \( S_p \).
Now 'F' can be expanded in terms of 'p' and 'q' as,
\[ F_\omega = C_{\omega \lambda} \cdot \eta^\lambda + D_{\omega \lambda} \cdot \eta^\lambda + G_{\omega \lambda} \cdot q^\lambda + H_{\omega \lambda} \cdot q^\lambda \]  
(A.a.5)
But \( \langle \eta^\lambda, F_\omega \rangle_{S_p} = C_{\omega \lambda} = A_{\omega \lambda}^* \), and hence, projecting F onto \( S_p \) to give functions 'f' whose supports cut on \( S_p \),
\[ f_\omega = A_{\omega \lambda}^* \cdot \eta^\lambda + B_{\omega \lambda} \cdot \eta^\lambda \]
Appendix A

\[ \mathbf{f}_\omega = A_{\omega \lambda} \cdot \mathbf{p}_\lambda + B_{\omega \lambda} \cdot \mathbf{p}_\lambda \quad (A.a.6) \]

The projection of \( F \) onto \( f \) envisaged in this work is due to either the fact that the interaction region cuts the Cauchy surface on \( S_p \), or because measurements of interest are made in the causal future or past of \( S_p \).

As the support of \( p \) cuts on \( S_p \) only, equations \((A.a.1)\) are not affected by the projection of \( F \) onto \( f \), giving the consistency conditions, from \((A.a.1)\) and \((A.a.6)\),

\[ A_{\omega \lambda} A_{\omega \gamma} - B_{\omega \lambda} B_{\omega \gamma} = \delta(\lambda - \gamma) \]
\[ A_{\omega \lambda} B_{\omega \gamma} - B_{\omega \lambda} A_{\omega \gamma} = 0 \]

\((A.a.7)\)

(summation over repeated indices).

The expansion of field \( \Phi \) supported on \( S_F \) is given by,

\[ \Phi = a_\omega f_\omega + a_\omega^+ f_\omega^* \]
\[ \Phi = b_\lambda b_\lambda \mathbf{b}_\lambda + b_\lambda^+ b_\lambda^* + c_\lambda g_\lambda + c_\lambda^* g_\lambda^* \quad (A.a.8) \]

and the expansion coefficients (annihilation / creation operators upon second quantization) are related through,
Appendix A

\[ a_\omega = b_\lambda A_\omega \lambda - b_\lambda^+ B_\omega \lambda + c_\lambda \langle F_\omega, q_\lambda \rangle_{ SQ} \]
\[ + c_\lambda^+ \langle F_\omega, q_\lambda \rangle_{ SQ} \]

\[ a_\omega^+ = b_\lambda^+ A_\omega^+ - b_\lambda B_\omega \lambda + c_\lambda \langle F_\omega, q_\lambda \rangle_{ SQ} \]
\[ + c_\lambda^+ \langle F_\omega, q_\lambda^+ \rangle_{ SQ} \]  \hspace{1cm} (A.a.9)

\[ b_\lambda = a_\omega A_\omega \lambda + a_\omega^+ B_\omega \lambda \]

\[ b_\lambda^+ = a_\omega^+ A_\omega \lambda + a_\omega B_\omega \lambda \]  \hspace{1cm} (A.a.10)

Appendix A(b)

*Coefficients for Minkowski / Rindler modes:*

Normalized Minkowski modes are,

\[ \Phi_\omega = e^{-i\omega t} \frac{\hat{R} \cdot \hat{x}}{\sqrt{(2\pi)^3 \cdot 2\omega}} \]  \hspace{1cm} (A.b.1)

with \[ \omega = \left[ k_x^2 + k_y^2 + k_z^2 \right]^{1/2} \]

Normalized Rindler modes, solutions to,

\[ \left[ -\frac{1}{8} \frac{\partial^2}{\partial t^2} + \frac{1}{8} \frac{\partial^2}{\partial \theta^2} \cdot \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \Phi_\lambda = 0 \]  \hspace{1cm} (A.b.2)
are,

\[(2\pi^2)^{-1} (\sinh \pi \lambda)^{1/2} \exp(-i\lambda \tau). K_{i\lambda}(\mu \delta). \exp(i\delta^2 r^2)\]  

(A.b.3)

with

\[\mu = (y^2 + \tau^2)^{1/2}\]

and 'K_{i\lambda}' a Bessel function (see Watson (1922)).

Then, for example,

\[A_{\omega \lambda} \equiv \langle \phi_{\omega}, \tilde{\phi}_{\lambda} \rangle\]

\[= (\sinh \pi \lambda)^{1/2} \frac{2\pi^2 [(2\pi)^3 2\omega]^{1/2}}{\int_{\mathcal{C} = \text{const}} d\tau^2 \cdot d\delta \cdot \frac{e^{i(\delta^2 - \tau^2)}}{\sqrt{\tau^2}} \cdot \frac{e^{i\omega \tau - k_x \chi}}{\sqrt{2 \tau}} \cdot K_{i\lambda}(\mu \delta) \cdot \left[ \lambda - i\frac{2}{\sqrt{\delta}} (e^{i\omega \tau - k_x \chi}) \right]}\]

(A.b.4)

where 'C' = constant is a Cauchy surface for (R+). Changing from Minkowski to Rindler coordinates, and taking the Cauchy surface to be \(\tau = 0\), (A.b.4) equals,

\[\frac{(2\pi^2) (\sinh \pi \lambda)^{1/2}}{2\pi^2 [(2\pi)^3 2\omega]^{1/2}} \int_{\delta = 0} d\delta \cdot K_{i\lambda}(\mu \delta) \cdot e^{-ik_x \delta} \cdot (\lambda + \omega \delta) \cdot \delta(\delta^2 - \tau^2)\]

(A.b.5)

The integral over '\(\delta\)' has been evaluated by Candelas &
Appendix A

Raine (1975) to give,

\[
\tilde{A}_{\omega \lambda} = \frac{e^{-iA\beta} \delta(\vec{v}^2 - \vec{k}^2)}{\sqrt{2\pi} \omega (1 - e^{\pi(2\pi\lambda)})^{1/2}}
\]

where new momentum coordinates, \( A, B \) have been introduced through,

\[
\omega = \mu \cosh B \quad ; \quad k_x = \mu \sinh B
\]

Similarly,

\[
\beta_{\omega \lambda} = \langle \phi^{a^*}, \phi^a \rangle
\]

\[
= \frac{e^{-iA\beta} \delta(\vec{v}^2 + \vec{k}^2)}{\sqrt{2\pi} \omega (e^{\pi(2\pi\lambda)} - 1)^{1/2}}
\]

Appendix A(c)

Coefficients for Kruskal / Schwarzschild modes
(two dimensions)

It is not hard to show that there is no mixing between left-moving (u) and right-moving (v) modes. The scalar product on a spacelike hypersurface is given by (u modes),

\[
\tilde{A}_{\omega k} = \langle \frac{e^{-i\omega u}}{\sqrt{4\pi\omega}}, \frac{e^{-ik^a}}{\sqrt{4\pi k}} \rangle
\]

\[
= \frac{1}{4\pi \sqrt{\omega k}} \int_0^\infty \frac{dr}{2m} (1 - \frac{2m}{r})^{-1} e^{i\omega u} e^{-ik^a} (k + \omega \frac{du}{dk})
\]
By forming wave-packets, the boundary terms disappear, leaving

\[
\frac{1}{2\pi \sqrt{\omega}} \int_{0}^{\infty} du \cdot e^{i\omega u} \left( \frac{u}{4m} \right)^{\frac{-\omega k}{4m}} - 1
\]

\[
\Rightarrow
\]

\[
A_{wk} = \frac{(4m)^{-\frac{i4\pi m k}{4}}}{2\pi} \sqrt{k/\omega} \sum (i4m k) \cdot e^{2\pi m k} - i4mk
\]

Similarly,

\[
B_{wk} = e^{-4\pi mk} \cdot A_{wk}
\]

Appendix A(d)

Extending the range of integrations

(i) In two dimensions, consider the expressions,

\[
\int_{0}^{\infty} d\gamma \cdot \hat{\Phi}_\gamma^*(x) \cdot H_{\gamma}(s) = \int_{0}^{\infty} d\gamma \cdot \frac{e^{i\gamma \mu}}{\sqrt{4\pi \delta}} \cdot \frac{e^{-i\gamma (d-\gamma)}}{\sqrt{4\pi \delta}}
\]

(A.d.1)
Equations (A.d.1), (A.d.2) can be combined as,

$$
-\int_0^{d\gamma} d\gamma \cdot \Phi_2(\gamma) \cdot H_2(\gamma) = -\int_0^{d\gamma} \frac{e^{-i\delta_n}}{\sqrt{4\pi\delta}} \cdot \frac{e^{-i\delta(\gamma-\zeta)}}{\sqrt{4\pi\delta}}
$$

(Equation A.d.2)

which for convenience I write as,

$$
\int_0^{d\gamma} \frac{e^{i\delta_n} - e^{-i\delta(\gamma-\zeta)}}{4\pi\delta}
$$

(Equation A.d.3)

It is important to interpret the expression (A.d.4) as (A.d.3). Note that the $\gamma$ integration performed later gives contributions only at positive values of $\gamma$.

(ii) In two dimensions, consider the expressions,

$$
\int_0^{d\Sigma} A \omega^2 \cdot H_2(\gamma) = \int_0^{d\Sigma} \langle \lambda \omega^2, \Phi^2 \rangle \cdot \frac{e^{-i\delta(\gamma-\zeta)}}{\sqrt{4\pi\delta}}
$$

$$
= \int_0^{d\Sigma} \cdot \frac{1}{2\pi \sqrt{|\omega|}} \cdot \int_{d\Sigma} \frac{e^{-i\omega \cdot u}}{\sqrt{4\pi\delta}} \cdot \frac{e^{-i\delta(\gamma-\zeta)}}{\sqrt{4\pi\delta}}
$$

(Equation A.d.5)
and
\[
\int d\Sigma \cdot B \omega \Sigma \cdot H_\Sigma (\xi) = \int d\Sigma \cdot <\phi_{\omega}^\alpha, \Phi_{\Sigma^2}> \cdot \frac{\xi (\xi - 2)}{\sqrt{4\pi}}
\]
\[
= \int d\Sigma \cdot \frac{i}{2\pi} \frac{\xi}{\omega} \int d\xi' \cdot e^{-i \omega \xi'} \cdot e^{-i \xi \xi'} \cdot e^{-i \xi (\xi - 2)} \frac{\xi (\xi - 2)}{\sqrt{4\pi}}
\]
\[
(A.d.6)
\]

Equations (A.d.5) and (A.d.6) can be combined as,
\[
\int d\Sigma \cdot \frac{i}{2\pi} \frac{1}{\sqrt{\omega}} \int d\xi' \cdot e^{-i \omega \xi'} \cdot e^{-i \xi \xi'} \cdot e^{-i \xi (\xi - 2)} \frac{\xi (\xi - 2)}{\sqrt{4\pi}}
\]
\[
(A.d.7)
\]

which for convenience I write as,
\[
\int d\Sigma \cdot B \omega \Sigma \cdot H_\Sigma (\xi)
\]
\[
(A.d.8)
\]

It is important to interpret expression (A.d.8) as (A.d.7). Note that the \( \Sigma \) integration performed later gives a contribution only at positive values of \( \Sigma \).
Appendix B

Appendix B(a)

*The distance element in null coordinates in two dimensions*

In a two dimensional spacetime, it is always possible to find a (null) coordinate system, \((u, v)\), such that the distance element may be written as,

\[ ds^2 = C(u, v) \, du \, dv \]  \hspace{1cm} \text{(B.a.1)}

To verify this, suppose that a region of the manifold is coordinated by \((t, x)\), in terms of which the distance element may be written as,

\[ ds^2 = g_{tt} \, dt^2 + 2g_{tx} \, dt \, dx + g_{xx} \, dx^2 \]  \hspace{1cm} \text{(B.a.2)}

Null curves, \( t = t(x) \), satisfy,

\[ 0 = g_{tt} \left( \frac{dt}{dx} \right)^2 + 2g_{tx} \left( \frac{dt}{dx} \right) + g_{xx} \]  \hspace{1cm} \text{(B.a.3)}

which has two real solutions \((g_{tt} > 0; \, g_{xx} < 0)\),

\[ \frac{dt}{dx} = -g_{tx} \pm \sqrt{\frac{g_{xx} - g_{tt} \cdot g_{tx}}{g_{tt}}} \]  \hspace{1cm} \text{(B.a.4)}

Now introduce two functions \( u(x, t) \), \( v(x, t) \) with the properties that,
\[
\frac{\partial u}{\partial x} = -\left(\frac{dx}{dt}\right)_{(a)} \cdot \frac{\partial u}{\partial t}
\]
and
\[
\frac{\partial v}{\partial x} = -\left(\frac{dx}{dt}\right)_{(b)} \cdot \frac{\partial v}{\partial t}
\]
(B.a.5)

at any point, where (a) and (b) label the two solutions of (B.a.4).

Without explicitly showing that equations (B.a.5) are integrable, and that \((u, v)\) form a coordinate system (although it is clear that in a non-pathological spacetime region this will be the case), it is easy to show that in terms of these null coordinates, the distance element is now of the form, (B.a.1) with,

\[
C(u, v) = g_{tt}/\left(\frac{\partial u}{\partial t}\right)\left(\frac{\partial v}{\partial t}\right)
\]

(B.a.6)

Appendix B(b)

*The wave equation in null coordinates in two dimensions*

Write

\[
\text{\(ds^2 = C(u, v)du dv\).}
\]

(B.b.1)

(This is always possible - see Appendix B(a))

That is,

\[
g_{uv} = C/2 \; ; \; g_{tt} = g_{uu} = 0
\]

(B.b.2)

Then
Thus in two dimensions, the wave equation can be written in the simple form,

\[ \Box \Phi = 0 \Rightarrow \partial_u \partial_v \Phi = 0 \]  

Appendix B(c)

The distance element in a two dimensional spacetime with a time-like Killing vector

If a portion of the spacetime admits a time-like Killing vector, \( \partial_t \) (with a complete integral curve), then this region may be coordinated by null coordinates, \( (u, v) \), in whose terms the distance element is,

\[ ds^2 = \hat{C}(u,v) \, du \, dv = \hat{C}(u,v) \left( dt^2 - dx^2 \right) \]  

where \( \hat{C}(u,v) \) has the property that, \( \hat{C}, \hat{a} = - \hat{C}, \hat{v} \)
To verify this, note that if \( \frac{\partial}{\partial t} \) is a Killing vector, then there is a coordinate system, \((t, x)\), such that the distance element can be written as,

\[
ds^2 = A(t) \, dt^2 + B(t) \, dt \, dx + D(x) \, dx^2
\]

\[
= A(t) \left[ dt + \frac{B - \sqrt{B^2 - 4AD}}{2A} \, dx \right] \left[ dt + \frac{B + \sqrt{B^2 - 4AD}}{2A} \, dx \right]
\]

(B.c.3)

Now let

\[
\hat{t} = t + \int^x \left( \frac{B - \sqrt{B^2 - 4AD}}{2A} \right) \, dx'
\]

\[
\hat{x} = t + \int^x \left( \frac{B + \sqrt{B^2 - 4AD}}{2A} \right) \, dx'
\]

then

\[
\frac{\partial A}{\partial t} \mid_{x} = 0 \Rightarrow A_{\hat{t}} \hat{t} + A_{\hat{x}} \hat{x} = 0
\]

(B.c.6)
Appendix C

Conformal transformation to determine the emission from a mirror following a trajectory of constant acceleration (two dimensional mode, spacetime).

In a two dimensional spacetime with a massless field, the easiest way to handle the mirror boundary condition \( \Phi(x, x(t)) = 0 \), where \( x(t) \) is the trajectory of the mirror, is to perform a conformal transformation.

The transformation \((t, x) \rightarrow (\tau, \zeta)\) should be such that the mirror trajectory becomes a straight line \( \zeta = 0 \). In particular, let,

\[
(t-x) \equiv u = f(\tau-\zeta) \equiv f(\bar{u})
\]

\[
(t+x) \equiv \tau = h(\tau+\zeta) \equiv h(\bar{\zeta})
\]

(C.1)

where \( f \) and \( h \) are functions to be determined. The metric in the new coordinates is given through,

\[
ds^2 = dt^2 - dx^2 = f'(\tau-\zeta) \cdot h'(\tau+\zeta) (d\tau^2 - d\zeta^2)
\]

(C.2)

To determine \( f \) and \( h \), write the equation of the trajectory, \( x = x(t) \) or \( \zeta = 0 \) as,

\[
\frac{1}{2} [h(\tau) - f(\tau)] = x \left[ \frac{1}{2} (h(\tau) + f(\tau)) \right]
\]

(C.3)

If the incoming field is in the usual Minkowski vacuum state, then, to the right of the mirror, the left-moving modes go as \( \exp(i\omega \tau) \). To
incorporate this initial condition, \( h(\tau) \) is taken simply as \( h(\tau) = \tau \). That is, \( v = \frac{\Lambda}{\tau} \) to the right of the mirror, and only modes travelling to the right are affected by the mirror.

For the accelerated trajectory, \( x = (t^2 + \ell^2)^{1/2} \), equation (C.3) becomes,

\[
\frac{1}{2} \left( \tau - f(\tau) \right) = \left[ \frac{1}{4} \left( \tau + f(\tau) \right)^2 + g^2 \right]^{1/2}
\]  

(C.4)

with solution,

\[
f(\tau) = -\frac{g^2}{\tau}
\]  

(C.5)

so that the appropriate conformal transformation is given by,

\[
\sigma \mapsto \tilde{\sigma} = \sigma
\]

\[
\lambda \mapsto \tilde{\lambda} = -\frac{g^2}{\lambda}
\]

(C.6)
Appendix D

Appendix D(a)

Definition of a cross-section

To determine the correct definition of a cross-section for an Unruh detector to absorb a field mode of a particular energy, consider a detector moving inertially in flat spacetime. The probability of excitation per unit time in response to a state of \( n_\omega \) particles of energy \( \omega \) is given by,

\[
P_E = \lim_{T \to \infty} \sum_{\vec{p}} \frac{\xi^2}{T} \int_0^T dt \exp \left[ i \left( E - E_0 \right) T \right] \cdot \int \! d^3x \, \tilde{T}_E (\vec{x}) \langle \vec{p} | \tilde{\Phi} (\vec{x}) | n_\omega \rangle \tilde{T}_{E_0} (\vec{x})^\dagger
\]

Expand the \( \tilde{\Phi} \) field in Minkowski modes,

\[
\tilde{\Phi} (\vec{x}) = \sum_\xi (a_\xi e^{-i \vec{\xi} \cdot \vec{x}} \cdot \tilde{H}_\xi (\vec{x}) + a_\xi^\dagger e^{i \vec{\xi} \cdot \vec{x}} \cdot H_\xi^* (\vec{x}))
\]

Then the rate of excitation evaluates to,

\[
P_E = n_\omega \cdot \delta (E - E_0 - \omega) \frac{2 \pi}{\Delta} \int \! d^3x \tilde{T}_E^\dagger (\vec{x}) \tilde{H}_\omega (\vec{x}) \tilde{T}_{E_0} (\vec{x})
\]

As \( P_E \) is the probability per unit time that the detector makes a transition to a state of energy \( E \), then the cross-section, \( \sigma_E^2 \), which is the probability of absorption per unit proper time per unit particle
Appendix D

flux can be identified as,

\[ \sigma_E^2 = 2 \pi \int_{\text{box}} d^3x \, \tau_E^* (x) \, H_{E-E_0} (x) \, \tau_{E_0} (x) \]

(D.a.4)

In a situation where the detector is responding to field modes different from Minkowski modes (as in the case of an accelerating detector where the response is to Rindler mode functions), the cross-section is again taken to be given by (D.a.4), except that the spatial functions, \( H_\omega (x) \) are no longer Minkowskian.

Appendix D(b)

Emission of a particle detector undergoing non-stationary motion with a switched interaction strength: 2-dimensional model.

The in-out formalism assumes that the perturbed fields approach free fields at early and late times. Necessary conditions for this to be the case are that the interaction should have not yet started/ceased at early/late times. I attempt to ensure these conditions by modifying the interaction strength to accommodate an exponential fall-off with proper time:

\[ \xi \rightarrow \xi \, e^{-|C| \alpha} \]  

(D.b.1)

where \( C \) is the proper time along the trajectory, and \( \alpha \) is a small constant.

Unfortunately, I find that the modified interaction strength decays insufficiently rapidly (for small \( \alpha \)) to ensure switch off of the interaction as measured in the inertial frame. This is due to the
exponential relationship between inertial and accelerated times. For large $\omega$, the effects of the switch on/off are not transient, but pervade the detector emissions for all time. The interaction strength could, of course, be modified to decay more rapidly than the exponential fall-off considered here, but this would prove to be mathematically intractable.

Accordingly, I modify the problem further by considering a more general trajectory than constant acceleration — a trajectory that starts and finishes in inertial motion. An inertial detector does not excite in response to the Minkowski vacuum, and so one would expect that at the end points of such a trajectory, the interaction will be effectively switched off.

Consider a detector that starts at rest, accelerates, and finally decelerates, to end up moving at constant velocity. Unlike in the case of a detector following a Killing trajectory, if the detector undergoes non-stationary motion, the dimensions of the box cannot remain unambiguously fixed. Also, the temporal and spatial integrals involved in the perturbation calculation of the energy emitted cannot be separated. To be able to treat the problem, I have to idealize the detector by taking it to be essentially a single point. (This is the model used by de Witt (1979) and is perhaps not entirely unrealistic if the detector is thought of as being, say, an electron, with the field coupling to internal degrees of freedom such as spin. (Bell and Leinaas, 1983) have shown that accelerated electrons evince thermal effects just as does Unruh’s detector). In the full four-dimensional calculation of a detector at constant acceleration, I do not need to
impose this restriction as the walls of the detector follow Killing trajectories in that case.

Otherwise, the details of the model are as in Section §2.6, and I now consider how to evaluate equation (2.6.9).

To evaluate this expression, I make a change of coordinates 
\((u, v) \rightarrow (\hat{u}, \hat{v})\), with \(u = u(\hat{u})\); \(v = v(\hat{v})\) (so that the transformation is conformal), and so that, expressing \((\hat{u}, \hat{v})\) as \(\hat{u} = \tau - \mathcal{S}\); \(\hat{v} = \tau + \mathcal{S}\), the trajectory \(x = x(t)\) becomes the line, \(\mathcal{S} = 0\). Furthermore, I arrange that \('\hat{u}':\) and \('\hat{v}':\) increase as \('\tau':\) the proper time along the trajectory.

The reason for making this transformation is that now the interaction is essentially a delta-function centred on the line \(\mathcal{S} = 0\), and if the \((\phi_\omega, \phi^*_\omega)\) modes are expanded in terms of the \((\hat{u}, \hat{v})\) modes (ie. modes appropriate to a quantization in the \((\hat{u}, \hat{v})\) coordinates, \((\hat{\phi}_\omega, \hat{\phi}^*_\omega)\)), then the time integrals over proper time can be performed simply.

To determine the conformal transformation, write the trajectory as the curve, \(u = u(v)\). Then,

\[
\frac{du}{d\hat{u}} \bigg|_{\tau=\tau_j} = \frac{du}{dv} \bigg|_{\tau=\tau_j} \cdot \frac{dv}{d\hat{v}}
\]

(D.b.2)

The requirement that \(\mathcal{S} = 0\) implies that \(\hat{u} = \hat{v}\) on the
trajectory. Thus,
\[
\frac{du}{d\hat{u}} \big|_{\tau = \hat{a}} = \frac{du}{d\tau} \big|_{\tau = \hat{a}} \cdot \frac{d\tau}{d\hat{u}} \tag{D.b.3}
\]
Looking at the line element expressed in the new coordinates,
\[
ds^2 = du \cdot dv = \frac{du}{d\hat{u}} \cdot \frac{dv}{d\hat{v}} \cdot d\hat{u} d\hat{v} \tag{D.b.4}
\]
the requirement that \( \hat{u} \) and \( \hat{v} \) increase as the proper time along the trajectory gives the condition,
\[
\frac{du}{d\hat{u}} \big|_{\tau = \hat{a}} = \left[ \frac{dv}{d\hat{v}} \big|_{\tau = \hat{a}} \right]^{-1} \tag{D.b.5}
\]
From (D.b.5) and (D.b.3),
\[
\frac{du}{d\hat{u}} \big|_{\tau = \hat{a}} = \left[ \frac{du}{dv} \big|_{\tau = \hat{a}} \right]^{1/2} \tag{D.b.6}
\]
The appropriate conformal transformation is then given by,
\[
\frac{du}{d\hat{u}} = \left[ \frac{1 - \dot{x}(\hat{a})}{1 + \dot{x}(\hat{a})} \right]^{1/2} \quad ; \quad \frac{dv}{d\hat{v}} = \left[ \frac{1 + \dot{x}(\hat{a})}{1 - \dot{x}(\hat{a})} \right]^{1/2} \tag{D.b.7}
\]
where '\( \dot{x} \)' is the velocity of the detector (in the \((t,x)\) coordinate frame), to be regarded as a function of \( \hat{u} \) and \( \hat{v} \) respectively.

Rather than trying to keep track of all the terms resulting from an evaluation of (2.6.9), I evaluate a single term, and show how the remainder are absorbed to achieve the final result.
Now, (2.6.13) involves,
\[
\sum_{\omega, k} \phi_\omega (x) \phi_k^* (x) \int_{-\infty}^{\infty} d \tau_1 \xi (\tau_1) \int_{\tau_1}^{\infty} d \tau_2 \xi (\tau_2).
\]

\[
\sum_{\xi, \gamma} \left( A \omega_\xi \epsilon \xi (g_1) + B \omega_\xi \epsilon \xi (g_1) \right) \left( A_k \xi \epsilon \xi (s_2) + B_k \xi \epsilon \xi (s_2) \right) e^{i \lambda (\tau_1 - \tau_2)}
\]

(D.b.8)

where \( \lambda \) is written for \( (E' - E_0) \); \( \phi_k \) are Minkowski modes; \( A \) and \( B \) are Bogoliubov coefficients for the transformation between Minkowski modes, based on the \((u,v)\) coordinates, and the modes, \( \phi_{\xi} \), based on the \((u,v)\) coordinates; and \( H_\xi(s) \) is the spatial part of the \( \{ \phi_{\xi}, \phi_{\xi}^* \} \) modes.

Consider just the first term of (D.b.8),
\[
\sum_{\omega_1, k_1} \phi_\omega (x) \phi_k^* (x) \sum_{\xi_1, \gamma_1} \int_{-\infty}^{\infty} d \tau_1 \xi (\tau_1) e^{-i (\xi + \gamma) \tau_1}.
\]

\[
A \omega_\xi \cdot H_\xi (g_1) \int_{-\infty}^{\infty} d \tau_2 \xi (\tau_2) e^{-i (\xi - \gamma) \tau_2}.
\]

\[
A_k \xi \cdot H_\xi (s_2)
\]

(D.b.9)

To accommodate the discontinuous time-dependence of the interaction strength, the time integrals have to be split into three parts
\[
\int_{-\infty}^{\infty} d\tau_1 \, e^{i\tau_1} \int_{-\infty}^{\infty} d\tau_2 \, e^{i\tau_2} \Rightarrow \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} e^{i\tau_1} \int_{-\infty}^{\infty} e^{i\tau_2} + \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} e^{-i\tau_1} \int_{-\infty}^{\infty} e^{-i\tau_2} + \sqrt{\frac{\lambda}{2\pi}} \int_{-\infty}^{\infty} e^{i\tau_1} \int_{-\infty}^{\infty} e^{-i\tau_2}
\]

As an example, consider the first time integral,
\[
\int_{-\infty}^{\infty} d\tau_1 \cdot e^{-i(\Sigma+\lambda+2i\alpha)\tau_1} \int_{-\infty}^{\infty} d\tau_2 \cdot e^{-i(\Sigma-\lambda+2i\alpha)\tau_2} = \int_{0}^{\infty} d\chi \frac{e^{-i\chi}}{-i(\Sigma-\lambda+2i\alpha)} \left[ \frac{1}{(\Sigma+\lambda+2i\alpha)} - \frac{1}{(\Sigma+\lambda+2i\alpha)} \right] = \frac{1}{(\Sigma+\lambda+2i\alpha)(\Sigma+\lambda+2i\alpha)}
\]

Combining the three time integrals gives,
\[
\sum_{\omega, k_1, z_1, y} \frac{\phi(x) \phi^*(x') A_{\omega z_1} A_{k y} (2i\alpha)(2\lambda+\Sigma-\delta) \cdot \sigma(\Sigma, \lambda) \cdot \sigma(y, \lambda)}{2\pi (\Sigma+\delta+2i\alpha)(\Sigma+\lambda+2i\alpha)(\lambda-\delta+i\alpha)(\Sigma+\delta-2i\alpha)}
\]

where I have identified the spatial integrals as the partial cross-section of the box,
\[ \sigma(\xi, \lambda^* \gamma^*) = \sqrt{2\pi} \left| \int d^3 \mathbf{y} \cdot H_\lambda(\mathbf{y}) \, \mathcal{T}_\lambda(\mathbf{y}) \, \mathcal{T}_\lambda(\mathbf{y}) \right|^2 \]  

(D.b.13)

A similar term except with \( \phi_k^*(x) A_k^* \) replaced by \( -\phi_k^*(x') B_k^* \) is also obtained from (2.6.9), and consequently the integration over 'k' may be performed using the Bogoliubov transformation property,

\[ \phi_k^*(x) = \sum_k A_k^* \phi_k^*(x') - B_k^* \phi_k^*(x') \]  

(D.b.14)

(where \( \phi_k^*, \phi_k^* \) are the field modes of the \( (u,v) \) coordinate system (see Appendix Equation (A.a.3)), to give,

\[ \sum_{\omega, \Sigma, \gamma} \phi_{\omega}^*(x) \, \phi_{\gamma}^*(x') \cdot \frac{A_{\omega \Sigma} (2i\alpha)(2\lambda + \Sigma - \gamma)}{2\pi (\gamma + \Sigma + 2i\alpha)(\Sigma + \lambda + i\alpha)} \]  

(D.b.15)

Being careful as to the meaning of the expression, \( \phi_{\gamma}^* \) for negative values of '\( \gamma \)' (see Appendix A(d) for a discussion), it is possible to extend the range of the '\( \gamma \)' integration of (D.b.15) by adding a further term from (2.6.9),

\[ \sum_{\omega, \Sigma} \int_{-\infty}^{\infty} d\gamma \, \phi_{\omega}^*(x') \, A_{\omega \Sigma} \, \phi_{\gamma}^*(x') \, (2i\alpha)(2\lambda + \Sigma - \gamma) \cdot \frac{\sigma(\xi, \lambda^* \gamma^*) \sigma(\xi, \lambda^* \gamma^*)}{(\lambda - \gamma + i\alpha)(\Sigma + \gamma - 2i\alpha)} \]  

(D.b.16)
Finally, again being careful as to the meaning of the expression, 
\[ \phi_{\omega \Sigma} H_{\Sigma}^{(+)}(s) \]  for negative values of \( \Sigma \) (see Appendix A(d) for a discussion), it is possible to extend the range of the \( \Sigma \) - integration of (D.b.16) by adding a further term from (2.6.9) to give the final result, 

\[ \zeta^{\omega k}(x,x') = \zeta^{in}(x,x') = -2^2 \sum_{\omega,\lambda} \int_{-\infty}^{\infty} d\Sigma \int_{-\infty}^{\infty} d\sigma \frac{\phi^{\omega}(x) \phi_{\omega \Sigma} \phi^{\sigma}(x)(2i\alpha)(2\lambda-\Sigma-\delta)}{2\pi(\sigma-\Sigma+2i\alpha)(\lambda-\Sigma+i\alpha)} \]

\[ \frac{\sigma(\Sigma,\lambda) \sigma(\sigma,\lambda^*)}{(\lambda-\Sigma+i\alpha)(\sigma-\Sigma-2i\alpha)} \]  \hspace{1cm} (D.b.17)

To evaluate this expression, I integrate over one half of the complex \( \gamma \) - plane. Although the points \((x,x')\) have to be taken sufficiently far into the future that the field has become essentially free, this is not sufficient to determine that the contour should be completed in the upper-half complex \( \gamma \) - plane. This is because massless fields in two-dimensions propagate along lines of \( u = \) constant, \( v = \) constant.

Consider evaluating the left-moving fluxes, for example. To determine the energy emanating from an interaction at early times, because \( \hat{\phi}^{\omega}(\Sigma) \) goes as \( e^{\lambda (i\delta \alpha)} \) (with \( \lambda \) large and negative), the integration must be performed in the lower-half complex \( \gamma \) - plane (See Fig D.b).

The poles in the \( \gamma \) -plane are at,
\[ \gamma = \Sigma - 2i\alpha \]
$\gamma = \lambda + i\alpha$

$\gamma = \Sigma + 2i\alpha$

(D.b.18)

(and at $\gamma = 0$ from the fact that $\frac{d}{d\gamma} \tilde{G}^\dagger \sim 1/\gamma$. This pole can be ignored, because on taking derivatives of $G(x,x')$ to obtain $\langle T_{\mu\nu} \rangle$, there is no contribution).

The emission at early times is found by integrating over the lower-half complex $'\gamma'$-plane ($'u' \sim -\infty$, sufficiently large and negative that $\tilde{G}^\dagger (u)$ dominates the $'\gamma'$-dependence of (D.b.17)). The residue from the pole at $\gamma = \Sigma - 2i\alpha$ gives, from (D.b.17),

$$\frac{\bar{e}^2}{2} \sum \int_{-\infty}^{\infty} d\Sigma \cdot \frac{\phi^\dagger (\lambda) \cdot \delta_{\nu \Sigma} \cdot \tilde{G}^\dagger (\Sigma - 2i\alpha) (\lambda')}{(\lambda - \Sigma + 3i\alpha)} \cdot \sigma (\Sigma^\dagger, \lambda) \sigma (\Sigma - 2i\alpha, \lambda')$$

(D.b.19)
Now integrating over the lower-half complex \( \Sigma \)-plane leaves,
\[
\xi^{out}(x,x') - \xi^{in}(x,x') = \]
\[
-\frac{2\pi}{\alpha} \sum_{\omega,\lambda} \phi_{\omega,\lambda}(x) \beta \omega (\lambda+3i\alpha) \hat{\Phi}_{\lambda+3i\alpha}(x').
\]
\[
\sigma(\lambda+3i\alpha, \lambda) \sigma(\lambda+\alpha, \lambda)
\]
(D.b.20)

(Note that as \( \alpha \to 0 \), this reduces to (2.6.20), although the modes based on the \((u,v)\) coordinates are different in this case.

The emission from late times is found by integrating over the upper-half \( \gamma \)-plane \( \tilde{u} \sim +\infty \), sufficiently large and positive that \( \hat{\Phi}_{\gamma}^{\ast}(\tilde{u}) \) dominates the \( \gamma \)-dependence of (D.b.17). The residue from the pole at \( \gamma = \Sigma + 2i\alpha \) gives, from (D.b.17),
\[
-\frac{2\pi}{\alpha} \sum_{\omega,\lambda} \int_{-\infty}^{\infty} d\Sigma \frac{\phi_{\omega,\lambda}(x)}{\beta \omega \Sigma} \frac{\hat{\Phi}_{\Sigma+2i\alpha}^{\ast}(x')}{(\lambda - \Sigma + i\alpha)}.
\]
\[
\sigma(\Sigma, \lambda) \sigma(\Sigma + 2i\alpha, \lambda^\ast)
\]
(D.b.21)

Now, integrating over the upper half complex \( \Sigma \)-plane leaves,
\[
\xi^{out}(x,x') - \xi^{in}(x,x') = \]
\[
-\frac{2\pi}{\alpha} \sum_{\omega,\lambda} \phi_{\omega,\lambda}(x) \beta \omega (\lambda+i\alpha) \hat{\Phi}_{\lambda+i\alpha}^{\ast}(x').
\]
\[
\sigma(\Sigma + i\alpha, \lambda) \sigma(\lambda + 3i\alpha, \lambda^\ast)
\]
(D.b.22)
Appendix D

Again, as \( \varepsilon \to 0 \), this reduces to (2.6.20).

Finally, the residue at \( \zeta = \lambda + \bar{\alpha} \) gives,

\[
\frac{-c^2}{2} \sum_{\omega_{1},\omega} \int_{-\infty}^{\infty} \frac{(2i\omega) \Phi^*(\lambda') \beta \omega \Sigma \Phi_{(\lambda+\bar{\alpha})}(\lambda')}{(\lambda - \Sigma + 3i\alpha)(\lambda - \Sigma + i\alpha)}
\]

This last cannot yet be integrated over \( \Sigma' \), however.

Case (a). Uniform Acceleration with switch on/off effects

I consider first the case of uniform acceleration with switch on/off at early/late times.

The equation of a trajectory of constant acceleration, \( 'g' \), is given by

\[
x^2 - t^2 = g^2
\]

leading to a conformal transformation (from (D.b.7)),

\[
u = -\frac{1}{g} e^{-g\bar{\alpha}}
\]

\[
v = \frac{1}{g} e^{g\bar{\alpha}}
\]

(D.b.25)

(which is of course just the transformation between Minkowski and
Rindler coordinates).

The \( \mathcal{W} \) integration in (D.b.20), (D.b.22), and (D.b.23) can be performed (\( g = 1 \)),

\[
\int_0^\infty d\omega \, \mathcal{W}(\omega) \mathcal{W} = \int_0^\infty d\omega \, \frac{1}{2\pi} \sqrt{\frac{\lambda}{\omega}} \cdot \mathcal{W}(\omega) \cdot e^{-i\lambda/2}
\]

\[
= \frac{e^{-i\lambda/2}}{2\sqrt{4\pi \lambda}} \cdot \left( \frac{e^{-\pi \lambda}}{\sinh \pi \lambda} \right)
\]

(c.f. from Appendix A(c))

(D.b.26)

The contribution of (D.b.23) to the emission at late times can then be found by integrating \( \Sigma \) over the lower-half \( \Sigma \) complex plane. The contour misses the poles at \( \Sigma = \lambda + 3i\alpha \), \( \Sigma = \lambda + i\alpha \), and so (D.b.23) gives no contribution. The left-moving (order \( \leq 2 \)) energy flux from (D.b.20) is found to be,

\[
\langle T_a \dot{a} \rangle_{\text{early}} = -\frac{\pi \ell^2}{4} \sum \frac{1}{\lambda} \Re \left\{ \frac{(\lambda + 3i\alpha)(\lambda + i\alpha)}{\exp[2\pi(\lambda + 3i\alpha)] - 1} \right\} e^{2\alpha / i} \cdot \sigma(\lambda + 3i\alpha, \lambda) \cdot \sigma(\lambda + i\alpha, \lambda^*)
\]

(D.b.27)

and that from (D.b.22) is,

\[
\langle T_a \dot{a} \rangle_{\text{late}} = -\frac{\pi \ell^2}{4} \sum \frac{1}{\lambda} \Re \left\{ \frac{(\lambda + 3i\alpha)(\lambda + i\alpha)}{\exp[2\pi(\lambda + 3i\alpha)] - 1} \right\} e^{-2\alpha / i} \cdot \sigma(\lambda + i\alpha, \lambda) \cdot \sigma(\lambda + 3i\alpha, \lambda^*)
\]
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(D.b.28)  

(and similarly for \( \langle T_{\mu \nu} \rangle \)).

Note that \( \langle T_{\mu \nu} \rangle \) must be measured in the causal future of the interaction.

For small \( \lambda \), the left-moving fluxes are given by

\[
\langle T_{\mu \nu} \rangle \sim \frac{\pi \varepsilon^2}{4} \sum_{\lambda} \frac{\lambda \cdot e^{-2\lambda \hat{u}}}{\left[ \exp(2\pi \lambda) - 1 \right]} \cdot \delta^2(\lambda, \lambda^*)
\]

(D.b.29)

This is the central result of Chapter Two – the energy emission of the detector is negative; in two dimensions, the emission is equal and opposite to the excitation energy of the detector.

The fluxes die away exponentially at early / late times as measured in the accelerated coordinates,

\[
\begin{align*}
T_{\mu \nu} & \sim e^{2\lambda \hat{u}} \quad ; \quad \hat{u} \to +\infty \\
T_{\mu \nu} & \sim e^{-2\lambda \hat{u}} \quad ; \quad \hat{u} \to -\infty
\end{align*}
\]

(D.b.30)

As measured in the inertial frame, however, the fluxes do not die away exponentially, in fact,

\[
T_{\mu \nu} = \left( \frac{d\hat{u}}{du} \right)^2, \quad T_{\mu \nu} \sim u^{-2-2\lambda} \quad \{ \hat{u} \to \infty \}
\]
Thus, unless $\alpha > 1$, (or re-scaling the energies for a general acceleration, $\alpha > g$), the inertial observer sees infinite fluxes travelling along the null lines $u = 0$, $v = 0$. The reason for this is that the relation between the proper times of the two observers is,

$$t = \sinh \tau \quad ; \quad (g = g = 1)$$

(D.b.32)

which means that the inertial observer sees the interaction switching on/off at the rate,

$$e^{\alpha \tau} \sim t^{-\alpha} \quad ; \quad \tau \to -\infty$$

$$e^{-\alpha \tau} \sim t^{-\alpha} \quad ; \quad \tau \to +\infty$$

(D.b.33)

that is, for small $\alpha$, the interaction strength remains essentially constant in inertial time, allowing infinite fluxes to build up along the lines $u = 0$, $v = 0$.

In addition, it is clear that if $\alpha$ takes on a value that is significantly larger than zero, then the effects of the switch on/off will not be transient - the values of $\lambda$ are displaced by an amount.
dependent on the speed of switching. Equations (D.b.27) and (D.b.28) are, however, only valid at early and late times; moreover, the modified interaction strength, (D.b.1), has a discontinuous derivative at \( \zeta = 0 \), which could be responsible for the non-transient effect. The question of whether or not the effects of switching are transient (as one would expect), requires further work.

**Case (b). Non-uniform acceleration**

Secondly, I consider the case of a trajectory that starts and finishes in inertial motion, but with a switch on/off parameter, \( \alpha \), sufficiently small that equations (D.b.20) and (D.b.22) are essentially identical,

\[
\zeta^{\text{out}}(x,x') - \zeta^{\text{in}}(x,x') =
\]

\[
-4\pi \sum_{\omega, \lambda} \phi_{\omega}(x) \cdot \beta_{\omega, \lambda} \cdot \Phi_{\lambda}(x') \cdot \sigma(\lambda, \lambda) \cdot \sigma(\lambda', \lambda')
\]  

(D.b.34)

and equation (D.b.23) disappears.

This is equation (2.8.1) discussed further in Chapter Two, §2.8.

**Appendix D(c)**

Integration of (2.8.1) in the case of a flat cross-section.

The presence of the cross-section terms in (2.8.1) make the
Integrations impossible to perform for a general trajectory. However, there is a sense in which the cross-section of a particular detector is more a property of the particular construction of the detector than of the "basic response" it exhibits. Consequently it is perhaps meaningful to integrate (2.8.1)) by taking the cross-section to be flat,

\[ \sigma^2 = 1 \quad \forall \lambda \quad \text{(D.c.1)} \]

and thereby gain some understanding of this "basic response".

Writing out \( b_{\omega} \) in full (see Appendix A(c)),

\[
\zeta^{\text{out}}(\tau, x') - \zeta^{\text{in}}(\tau, x') = -8\pi^2 \sum_{\lambda} \int_0^\infty dw \frac{e^{i\omega w}}{\sqrt{4\pi w}} \left[ \frac{1}{2\pi i} \int_{\infty}^{0} dw' \frac{e^{i\omega w'} - e^{-i\lambda w'}}{w' - \lambda} \right] \frac{e^{i\lambda w'}}{\sqrt{4\pi \lambda}}
\quad \text{(D.c.2)}
\]

I first interchange the orders of integration and validate the \( \omega \) integral by giving \( \omega \) a small negative imaginary part. Performing the \( \omega \) integral to give a gamma function,

\[
\zeta^{\text{out}}(\tau, x') - \zeta^{\text{in}}(\tau, x') = \epsilon \xi^2 \int_0^\infty d\lambda \int_{\infty}^{0} dw' \frac{e^{i\lambda (\omega' - w')}}{(\omega' - \omega - i\xi)}
\quad \text{(D.c.3)}
\]
\( \langle T_{\alpha\bar{\alpha}} \rangle \) is obtained from \( G(x, x') \) through,

\[
\langle T_{\alpha\bar{\alpha}} \rangle = \lim_{\alpha \to \bar{\alpha}} \frac{2}{\partial \alpha} \frac{2}{\partial \bar{\alpha}} \cdot G(x, x')
\]

(D.c.4)

and as the energy of the incoming vacuum is zero,

\[
\langle T_{\alpha\bar{\alpha}} \rangle^{\text{out}} = -\varepsilon^2 \lim_{\alpha \to \bar{\alpha}} \int_0^\infty d\lambda \int_0^\infty du \frac{e^{i\lambda (\alpha - \bar{\alpha})}}{(u - u + i\varepsilon)^2} \cdot \frac{du}{d\alpha}
\]

(D.c.5)

Performing the \( \lambda \) integration leaves,

\[
\langle T_{\alpha\bar{\alpha}} \rangle^{\text{out}} = i \varepsilon^2 \int_{-\infty}^\infty du \frac{\gamma(\alpha) \cdot \frac{du}{d\alpha}}{(u - u + i\varepsilon)^2 (\alpha - \bar{\alpha} + i\varepsilon)}
\]

(D.c.6)

Now, (2.6.9) includes the complex conjugate as a factor of 2, hence, if the mapping \( u \to \bar{\alpha} \) is analytic near \( \text{Im}(u) = 0 \) and the pole at \( \bar{\alpha} = u \) is isolated, then the flux may be found from,

\[
\langle T_{\alpha\bar{\alpha}} \rangle^{\text{out}} = i \varepsilon^2 \int_{C} du \frac{\frac{du}{d\alpha}}{(u - u)(\bar{\alpha} - \bar{\alpha})}
\]

(D.c.7)

I follow the evaluation with a proof that the in the case of the particular trajectory of equation (2.9.8), the pole is isolated.

where the contour is,
That \( u = u(\hat{u}) \) is conformal implies that \( u \) is also a monotonic function of \( \hat{u} \) and hence the poles at \( \tilde{u} = u \) and \( \hat{\tilde{u}} = \hat{u} \) coincide.

The contribution from the triple pole is,

\[
-\pi \xi^2 \frac{du}{d\hat{u}} \frac{d^2 u}{d\hat{u}^2} \left[ \frac{du}{d\hat{u}} \left( \frac{\hat{u} - \hat{\tilde{u}}}{\hat{u} - u} \right)^2 \right] \bigg|_{\hat{u} = \hat{\tilde{u}}}
\]

which evaluates to

\[
= -\pi \xi^2 \left\{ \frac{3}{2} \frac{d^2 u}{d\hat{u}^2} \left( \frac{du}{d\hat{u}} \right)^2 - \frac{d^3 u}{d\hat{u}^3} \left( \frac{du}{d\hat{u}} \right) \right\}
\]

\[\text{(D.c.8)}\]

\[
= -\pi \xi^2 \left( \frac{da}{d\xi} + \frac{a^2}{2} \right)
\]

\[\text{(D.c.9)}\]
Lemma:

The pole at \( \hat{u} = \hat{u} \) is isolated for the trajectory of equation (2.9.8) if \( B < 1 \).

Write \( z = u - u \) and consider the function

\[
\hat{u} - \hat{u} = u + B \left( u^2 + A^2 \right)^{1/2} - u - B \left( u^2 + A^2 \right)^{1/2}
\]

(D.c.10)

To find the zeros of 'z' in a region of \( z = 0 \), I use the

Theorem of Rouche:
If \( f(z) \) and \( g(z) \) are analytic inside and on a simple closed curve, \( C \), and if \( \left| g(z) \right| < \left| f(z) \right| \) on \( C \), then \( f(z) \) and \( f(z) + g(z) \) have the same number of zeros inside \( C \).

Let \( f(z) = z \), and

\[
g(z) = -B \left[ \left( z + u \right)^2 + A^2 \right]^{1/2} + B \left( u^2 + A^2 \right)^{1/2}
\]

(D.c.11)
For small $z$,

$$g(z) \sim \left\{ \frac{-B u}{(u^2 + A^2)^{1/2}} \right\} \cdot z$$

which has maximum values of $+/ - Bz$, and so $|g(z)| < |Bz| < |z|$ if $B < 1$, which is just the condition that the trajectory should have a velocity less than that of light.
Appendix E

Appendix E(a)

Integrations to achieve equation (2.10.8) for $G(x,x')$

This section gives the integrations involved in equation (2.10.7),

$$
\eta^{\text{out}}(x,x') - \eta^{\text{in}}(x,x') =
$$

$$
-4\pi\hbar^2 \sum_{\omega,\lambda} \phi_\omega^+(x) \cdot \omega \lambda \cdot \phi^*_\lambda(x') \cdot \sigma^2 (\lambda,\lambda) + (\text{c.c.}) + (x \leftrightarrow x')
$$

(Ea.1)

In future, I include the complex conjugate and the interchanged $x \leftrightarrow x'$ form of (Ea.1) as a factor of 4, remembering the significance of this.

Looking at (Ea.1) in full (complex conjugate) - the evaluation of the Bogoliubov coefficients is given in Appendix A(b), and I introduce new variables through,

$$
\begin{align*}
\omega &= \mu \cosh \theta \quad ; \quad k_x = \mu \sinh \theta \\
ky &= \mu \cos \theta \quad ; \quad k_z = \mu \sin \theta
\end{align*}
$$

(Ea.2)

$$
\begin{align*}
\eta^{\text{out}}(x,x') - \eta^{\text{in}}(x,x') = \\
-4\pi\hbar^2 \int dk^3 \sum_{\lambda} \frac{e^{i\lambda \theta}}{2\pi \omega [\exp(2\pi \lambda) - 1]^{1/2}} \cdot e^{-i\omega t} \cdot ik_x \cdot ik_z \\
\end{align*}
$$
\[ \int d\gamma^2 K_{\text{in}}(\mu g) \frac{(\sinh \lambda)^{y_2}}{2\pi^2} \cdot e^{i\frac{\pi}{2} - i\lambda} \cdot e^{-i\frac{\lambda z}{2}} \]

\[ \sigma(\lambda, \mu) \cdot \sigma(\lambda, \mu, g) \]

(E.a.3)

where

\[ \sigma(\lambda, \mu, g) \equiv \sqrt{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\bar{g} \cdot d\bar{\omega} \cdot \bar{\sigma} \]

\[ \left( \frac{\sinh \lambda}{2\pi^2} \right)^{\frac{1}{2}} K_{\text{in}}(\mu g) e^{i\bar{\gamma} \cdot \bar{r}} \cdot \bar{r} \cdot \bar{\omega} \bar{g} \]

(E.a.4)

(A tilde denotes a box variable).

To simplify this, I perform, \text{k-integrations} over \( \kappa g \)

\[ \int_0^{2\pi} e^{i\kappa \cdot (\bar{r} + \bar{R})} \cdot d\theta = \int_0^{2\pi} e^{i\mu R \cos \theta} \cdot d\theta \]

\[ = 2\pi \int_0^R (\mu R) \]

\[ \bar{R} \equiv \bar{r} + \bar{\omega} \]

(E.a.5)

over \( \kappa g \)

\[ \int_{-\infty}^{\infty} \exp \left[ -i\mu \bar{R} \cdot \bar{g} \sin \alpha \bar{C} + i \mu \sin \alpha \bar{C} \cdot \cos \bar{R} \right] e^{i\bar{\gamma} \cdot \bar{R}} \]

(E.a.6)
where I have changed to Rindler coordinates.

(E.a.6) equals,
\[ \int_{-\infty}^{\infty} d\theta \exp \left[ i\mu g \sinh(\pi \theta/2) \right] e^{i\lambda(2-\theta)} e^{i\lambda T} \]
\[ = 2\pi e^{-\pi \lambda/2} K_i(\mu g) \]

(Watson (1922), p. 182)
So that (E.a.3) becomes,
\[ -\frac{\pi^2}{\pi^4} \sum_{\lambda} e^{-\pi \lambda} \sinh \pi \lambda \cdot e^{i\lambda(2-\theta)} \left\{ \int_{\text{box}} dy^3 \right\} \]
\[ \int_{0}^{\infty} d\mu J_0(\mu R) K_i(\mu g) K_i(\mu g') \int_{0}^{\infty} d\mu' J_0(\mu' R') K_i(\mu' g) K_i(\mu' g') \]

which is the required result, (2.10.8).

Appendix E(b)

Small 'approximation to \( G(x, x') \)

EMOT (1953) II, p. 67 (30) give the following evaluation of the integral of (E.a.8):
\[ \int_{0}^{\infty} d\mu J_0(\mu R) K_i(\mu g) K_i(\mu g) = \]
where
\[
\frac{\pi^{1/2}}{2^{3/2} (\bar{g}^2) (u^2-1)^{1/4}} \cdot P_{i\lambda}^{-1/2} (u)
\]

(E.b.1)

Abrahamowitz & Stegun (8.6.9).

Expanding this in powers of \( \bar{g} \), (E.b.1) can be written as,

\[
P_{i\lambda}^{-1/2} (u) = \left( \frac{\alpha}{\pi} \right)^{1/2} \frac{1}{2i\lambda} \cdot \left( \frac{u^2-1}{2i\lambda} \right)^{-1/4}
\]

\[
\left\{ \left[ u + (u^2-1)^{1/2} \right]^{i\lambda} - \left[ u + (u^2-1)^{1/2} \right]^{-i\lambda} \right\}
\]

(E.b.2)

Consequently, \( G(x,x') \), for small \( \bar{g}, \bar{g}' \) looks as,

\[
\frac{\pi}{2i \cdot \sinh(\pi \lambda)} \left[ (X + \bar{g}^2 Y) - (X^* + \bar{g}^* Y^*) \right]
\]

(E.b.3)

where,
\[
X = \frac{(g \bar{g})^{i\lambda}}{(R^2 + \bar{g}^2)^{1+i\lambda}} \quad \text{and} \quad \gamma = \frac{(g \bar{g})^{i\lambda}}{(R^2 + \bar{g}^2)^{2+i\lambda}} - \frac{R^2 (2+i\lambda) (g \bar{g})^{i\lambda}}{(R^2 + \bar{g}^2)^{3+i\lambda}}
\]

(E.b.4)
\[ G^\text{out}(x,x') - G^\text{in}(x,x') \approx e^{i\lambda(2-\epsilon')} \left[ (X+g^2\bar{Y}) - (X^*+g^2\bar{Y}^*) \right] \]

(E.b.5)

where

\[
\bar{x} \equiv X(-\lambda,g',r',\bar{s}'), \\
\bar{y} \equiv Y(-\lambda,g',r',\bar{s}^*)
\]

(E.b.6)

Because of the peculiar coordinate system, however, care must be exercised in interpreting this expression. There are contributions from left-moving modes of the form,

\[ e^{i\lambda Z} \cdot g ; e^{-i\lambda Z'} ; e^{i\lambda} \]

- these are the modes that cross the future horizon, \( t = x, x > 0 \) due to emissions from the detector. There are also contributions from right-moving modes of the form,

\[ e^{i\lambda Z} ; e^{-i\lambda} ; e^{i\lambda Z'} \cdot g \]

On the future horizon, these modes would be moving on the horizon; however, there is no source for such modes (except from the detector at very late times). This is a manifestation of the peculiar coordinate system: \( g = 0 \) is also the past horizon, \( t = -x, x > 0 \), and the right-moving modes are in fact crossing the past horizon.

It would seem that there is no source for such modes either, except that the formalism deals with the free field that the perturbed field approaches at late times. This free field propagates on the whole of
the spacetime (including in the causal shadow of the detector) and so the right-moving modes in question are in fact the right-moving modes emitted by the detector - propagated backwards in time to the past horizon, with the detector itself removed.

Such modes will eventually cross the future horizon (as left moving modes - a further peculiarity of the Rindler coordinates). Including the right-moving modes in the calculation would therefore amount to double-counting, and hence I discard them from the calculations.

\[ G(x, x') \text{ is then taken to be,} \]

\[ G^{\text{out}}(x, x') - G^{\text{in}}(x, x') \sim e^{i\lambda(\tau-\zeta)} \]

\[ (X + g^2 Y)(\bar{X}^* + g'^2 \bar{Y}^*) \]
Appendix E (c)

Evaluation of the energy flux

The energy flux through the line $t = x$ is,

$$\mathcal{T} = \left( \frac{x^2 - x t}{s^2} \right) \mathcal{T}_{12} + \left( \frac{t^2 - x t}{s^2} \right) \mathcal{T}_{33} + \left( \frac{1 - x^2}{s^3} \right) \mathcal{T}_{13} \mathcal{T}_{23} \mathcal{T}_{33}$$

(E.c.1)

and $\langle \mathcal{T}_{12} \rangle$ is obtained from,

$$\lim_{x \to \infty} \left\{ \left( \frac{1}{2} - \frac{e}{3} \right) \left( g_{i
u} + i \mu \lambda \Omega \right) + 2 \left( \frac{e}{2} - \frac{1}{4} \right) g_{i\lambda} g_{i\mu} \mathcal{G} \left( \sigma \partial_{i} \sigma' + \sigma' \partial_{i} \sigma \right) \right\}$$

(E.c.2)

(Christensen (1976); see Section 8.1.8), where the semicolon denotes covariant differentiation, and $\frac{1}{3}$ is an adjustable parameter, taken to be $1/6$ to achieve conformal invariance.

Taking,

$$q \sim e^{\frac{i}{3} \lambda \left( t - t' \right)} \left( X + s^2 Y \right) \left( \bar{X} + s^2 \bar{Y} \right)$$

$$\frac{\partial q}{\partial s} \sim e^{\frac{i}{3} \lambda \left( t - t' \right)} \left( \frac{i}{3} \lambda \cdot X + \left( \frac{i}{3} \lambda + 2 \right) s Y \right) \left( \bar{X} + s^2 \bar{Y} \right)$$
\[ \frac{\partial^2 G}{\partial s^2} \sim e^{i \lambda (t-x)} \left[ -\frac{(\lambda^2 + i \lambda)}{s^2} X + (2 + 3i \lambda - \lambda^2) Y \right] (\chi + s^2 \gamma) \]

(E.c.3)

then flux (E.c.1) is:
\[
\sum \beta \left[ \left( \frac{t^2 - x^2}{s^4} + \lambda^2 \frac{(x^2 - x^2)}{s^4} + \lambda^2 \frac{(x^2 - x^2)^2}{s^4} \right) XX + \right]
\]
\[
\left\{ \left( \frac{(x^2 - x^2)}{s^2} \right) \left( \lambda^2 - 3i \lambda - i \lambda + \xi \right) + \left( \frac{(x^2 - x^2)}{s^2} \right) \left( \lambda^2 + i \lambda + i \lambda + \xi \right) \right\} \chi \gamma + \]
\[
\left\{ \left( \frac{(x^2 - x^2)}{s^2} \right) \left( \lambda^2 + 3i \lambda - i \lambda + \xi \right) + \left( \frac{(x^2 - x^2)}{s^2} \right) \left( \lambda^2 - i \lambda + i \lambda + \xi \right) \right\} \chi \gamma + \]
\[
\left. - 2 \left( \frac{x^2 - x^2}{s^4} \right) G_{rr'} + \frac{1}{r^2} \left( \frac{x^2 - x^2}{s^4} \right) \left( r + G_{r00} \right) - \frac{1}{r} \left( \frac{x^2 - x^2}{s^4} \right) \left( r + G_{r00} \right) + \right. \]
\[
\left. + \frac{1}{r^2} \left( \frac{x^2 - x^2}{s^4} \right) \left( r + G_{r00} \right) \right] \]

(E.c.4)

\( r, \theta \) derivatives

Remembering that \( \vec{R} = \vec{r} - \vec{r} \), take \( \vec{R} = \vec{r} \) as a zeroth approximation, for a small box. The 'r' derivatives are,
\[
-2 \left( \frac{x^2 - x^2}{s^4} \right) G_{rr'} - \frac{1}{4} \left( \frac{x^2 - x^2}{s^4} \right) \left( G_{rr} + G_{rr'} + \frac{1}{r} G_{r0} + \frac{1}{r} G_{r0'} \right) \]

(E.c.5)

as, in this approximation, \( G(x,x') \) is independent of '\( \theta \)'.

---

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Again, in the zeroth approximation, taking \( \tilde{S}, \tilde{S}' = 1 \):

\[
G \sim (r^2 + 1)^{-\lambda - 1} \cdot (r^2 + 1)^{\lambda - 1}
\]

and (E.c.5) gives,

\[
\frac{2 \xi}{(r^2 + 1)^3} + \frac{r^2 (2 - 12 \xi + 2 \lambda^2 - 6 \xi \lambda^2)}{(r^2 + 1)^4}
\]

The energy flux from (E.c.4) in this lowest order approximation at \( S = 0 \) is then,

\[
F_{\text{flux}} \sim \left( \frac{2 \lambda^2}{r^2} x \bar{x} + (\xi - 3 \lambda \xi) x \bar{y} + \frac{(\xi + 3 \lambda \xi)}{(1 + r^2)^2} \right) x \bar{y}
\]

\[
+ \left[ \frac{2 \lambda^2}{r^2} x \bar{x} + \frac{2 \xi}{(r^2 + 1)^3} + \frac{r^2}{(r^2 + 1)^4} (2 - 12 \xi + 2 \lambda^2 - 6 \xi \lambda^2) \right]
\]

\[
\left[ \frac{2 \lambda^2}{r^2} + \frac{2 \xi}{(r^2 + 1)^3} + \frac{r^2}{(r^2 + 1)^4} \cdot (2 + 2 \lambda^2 - 16 \xi) \right]
\]

which is equation (2.11.6).
Appendix E(d)

Small box approximation to detector excitation.

The result of Unruh's calculation is that the rate of excitation to a state $E_j$ is (2.4.12),

$$P_j = \frac{\varepsilon^2}{2\pi^3} \int_0^\infty d\lambda \, dy \, dz \, \frac{\delta(\lambda - E_j + E_0)}{\left[\exp(2\pi \lambda) - 1\right]}.$$

The energy absorbed per unit proper time is then,

$$E = \frac{\varepsilon^2}{2\pi^3} \sum_k \frac{\lambda}{\left[\exp(2\pi \lambda) - 1\right]} \int dy \, dz.$$

The energy absorbed per unit proper time is then,

$$E = \frac{\varepsilon^2}{2\pi^3} \sum_k \frac{\lambda}{\left[\exp(2\pi \lambda) - 1\right]} \int dy \, dz.$$

I now obtain an approximation to this expression, taking the detector to be small.

Change coordinates to,

$$\rho_y = \mu \cos \Theta \quad ; \quad \rho_z = \mu \sin \Theta.$$
Appendix E

\[
\frac{\Pi}{\tilde{R}} = \frac{\Pi}{\tilde{R}} - \frac{\Pi}{\tilde{R}}
\]  
(E.d.3)

Perform the \( \int \rho \theta \) integration to give \( \int \zeta_R \rho \) (See Appendix E(a)).

Now, consider the integration over \( \lambda \).

\[
\int d\lambda \int \zeta_R \left( \rho \theta \right) \tilde{K}_\lambda \left( \rho \theta \right) \tilde{K}_\lambda \left( \rho \theta \right)
\]

\[
= \pi^{1/2} \Gamma(1+i\lambda) \Gamma(1-i\lambda) \frac{\rho^{-1/2}}{2^{3/2}(\tilde{g} \tilde{g}') (\tilde{u}^2 - 1)^{1/4}} \rho^{-1/2} \left( \tilde{u} \right)
\]  
(E.d.4)

(EMOT (1953) I, p.67 (30))

with \( 2 \tilde{g} \tilde{g}' = \tilde{u} \equiv \tilde{R}^2 + \tilde{g}^2 + \tilde{g}'^2 \)

Here 'P' is an Associated Legendre Polynomial that can be expressed as an Hypergeometric Function,

\[
\rho^{-1/2} \left( \tilde{u} \right) \equiv \Gamma^{-1} \left( \frac{3}{2} \right) \left( \frac{\tilde{u} - 1}{\tilde{u} + 1} \right)^{1/4} \cdot \sum_{2} \frac{1}{1} \left( \tilde{u} + \frac{1}{2}, -i \lambda + \frac{1}{2}; \frac{3}{2}; \frac{1}{2} - \frac{1}{2} \right)
\]  
(E.d.5)

(EMOT (1953) I, p.147)

that, finally, can be rewritten simply as,
\[
\frac{\Gamma}{2\lambda} \left( i\lambda + \frac{i}{2} - i\lambda + \frac{i}{2} - \frac{3}{2} \right) \left( \sin x \right)^2 = \frac{\sin (2i\lambda x)}{2i\lambda \sin x}
\]

(E.d.6)

(EMOT (1953) I, p.101 (12))

so that (E.d.4) is

\[
\frac{\pi \lambda}{\sqrt{2 \sin \pi \lambda \left( \frac{x}{3} \right) \left( \frac{x}{3} + 1 \right)^{1/2}}} \cdot \frac{\sin \left[ 2i\lambda \cdot \sin^{-1} \left( \frac{1}{2} - \frac{1}{2} \lambda x \right)^{1/2} \right]}{2i\lambda \left[ \frac{1}{2} - \frac{1}{2} \lambda x \right]^{1/2}}
\]

(E.d.7)

Now, for a small box, \( \tilde{\omega} \sim 1 \), and performing an expansion of (E.d.7) in terms of the variable \( \tilde{y} = \left( \frac{1}{2} - \frac{1}{2} \lambda x \right)^{1/2} \), the energy absorbed per unit proper time is,

\[
E = \frac{e^2}{\pi} \sum \frac{\lambda^2}{\exp(2\pi \lambda) - 1} \int_{\text{box}} d^3x \sqrt{-g} \int_{\text{box}} d^3x' \sqrt{-g} \cdot \left( 1 + \frac{2}{3} \tilde{y}^2 (1+\lambda^2) + O(\tilde{y}^4) \right) \frac{\lambda^4}{(\tilde{x} \tilde{x}')}
\]

(E.d.8)
Appendix F

Appendix F(a)

Evaluation of the two point function of Section 83.3

I evaluate
\[ G^{\text{out}}(x,x') = \langle 0 | \prod_{\text{box}}^{\text{out}}(x) \prod_{\text{box}}^{\text{out}}(x') | 0 \rangle_{\text{box}} \]

\[ = \langle 0 | S^{-1} \prod_{\text{box}}^{\text{in}}(x) SS^{-1} \prod_{\text{box}}^{\text{in}}(x') S | 0 \rangle_{\text{box}} \]  \hspace{1cm} (F.a.1)

where the S matrix is given by

\[ S = \sum_{n=0}^{\infty} \frac{1}{n!} (-i)^n \int \prod_{\text{box}} dy_i \sqrt{g} \cdots \prod_{\text{box}} dy_l \sqrt{g} P \left( H_{\text{in}}^1 (y_1) \cdots H_{\text{in}}^n (y_n) \right) \]

\[ = 1 - i \xi \int_{\text{box}} dy_1 \sqrt{g} \Phi(y_1) \bar{\Phi}(y_1) + \]

\[ - \xi^2 \int_{\text{box}} dy_1 \sqrt{g} \Phi(y_1) \bar{\Phi}(y_1) \int_{\text{box}} dy_2 \sqrt{g} \Phi(y_2) \bar{\Phi}(y_2) + O(\xi^3) \]  \hspace{1cm} (F.a.2)

Defining,

\[ \langle 0 | \Phi(x) \bar{\Phi}(y) | 0 \rangle_{\text{box}} \equiv \Delta^{-}(x-y) \]
\[
\langle 0 | \mathcal{F}(x) \mathcal{F}(y) \rangle_{\text{box}} \equiv \Delta(x-y)
\]

\[(F.a.3)\]

gives,

\[
\begin{align*}
\zeta_{\text{out}}(x,x') - \zeta_{\text{in}}(x,x') &= -\varepsilon^2 \int dy_1 \sqrt{-g} \\
\frac{2}{\sqrt{t_1}} \int dy_2 \sqrt{-g} \Delta^-(y_1-x) \Delta(y_2-x') + \int dy_2 \sqrt{-g} \Delta^-(y_2-y_1) \Delta(x-y_1)
\end{align*}
\]

\[(F.a.4)\]

where '\(|S\rangle\)' is the state of the \(\mathcal{F}\) field.

From (F.a.4), consider a typical integral,

\[
\begin{align*}
\frac{2}{\sqrt{t_1}} \int dy_2 \sqrt{-g} \Delta^-(y_1-x) \Delta^-(y_2-x') \langle S | \mathcal{F}(y_1) \mathcal{F}(y_2) | S \rangle
\end{align*}
\]

\[(F.a.5)\]

The two-point function involving the \(\mathcal{F}\) field in (F.a.5) can be evaluated simply by expanding \(\mathcal{F}\) in terms of mode solutions appropriate to the particular state representation chosen,

\[
\{ \phi_K, \phi_K^* \},
\]

\[
\langle S | \mathcal{F}(y_1) \mathcal{F}(y_2) | S \rangle = \sum_K \phi_K(x) \phi_K^*(x') +
\]
\[ + \sum_{\alpha \beta} \Lambda_{\alpha \beta} \left( \phi_\alpha(x) \phi^*_\beta(x') + \phi^*_\alpha(x) \phi_\beta(x') \right) \]  

(F.a.6)

(See Section 53.3 for discussion).

From (F.a.5), consider,

\[ \int \frac{dy_1 \sqrt{g}}{e_{\gamma_1}} \int \frac{dy_2 \sqrt{g}}{e_{\gamma_2}} \sum_{\lambda, \sigma} \gamma^\lambda_{\lambda}(y_1) \gamma^\sigma_{\sigma}(y_2) \gamma^\lambda_{\lambda}(x_1) \gamma^\sigma_{\sigma}(x_2) \phi_\lambda(y_1) \phi^*_\sigma(y_2) \]  

(F.a.7)

where \( \{ \gamma^\lambda_{\lambda}, \gamma^\sigma_{\sigma} \} \) are box field modes associated to the Killing vector field an integral curve of which the box is following.

Re-expanding the \( \overrightarrow{\Phi} \) field in terms of Killing modes, \( \{ \overrightarrow{\phi}_\lambda, \overrightarrow{\phi}^*_\lambda \} \), by using the Bogoliubov coefficients 'A' and 'B', (Appendix A), and choosing coordinates so that \( \tau \) is the proper time along the Killing trajectory, (F.a.7) becomes,

\[ \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 \sum_{\lambda, \sigma, \lambda_1, \lambda_2} \left\{ e^{-i \lambda \tau_1} \gamma^\lambda_{\lambda}(x_1) \gamma^\sigma_{\sigma}(x_2) e^{-i \sigma \tau_2} \right\} \]

\[ \left[ A_{\lambda, \sigma} e^{-i \tau_1} \cdot \sigma(\lambda, \sigma) + B_{\lambda, \sigma} e^{-i \tau_1} \cdot \sigma(\lambda, \sigma) \right] \]

\[ \left[ A_{\lambda, \sigma} e^{-i \tau_2} \cdot \sigma(\lambda, \sigma) + B_{\lambda, \sigma} e^{-i \tau_2} \cdot \sigma(\lambda, \sigma) \right] \]

(F.a.8)

where
\[ \sigma(\alpha, \beta^*) = \int_{\text{box}} \sqrt{-g} \, dy^3 \, h_\alpha(y) \, H_\beta(y). \]

'\(h_\alpha(y)\)' are box spatial modes; and '\(H_\beta(y)\)' are \(\Phi\) field Killing spatial modes.

Evaluating the \(\int \mathcal{Z}\) integrals gives,
\[
\sum_{\lambda, \delta, \Sigma} \gamma_{\lambda}^* \gamma_{\delta} \left\{ A_{\lambda, \delta} \delta_{(\Sigma - \lambda - \delta - \Sigma)} + i(\Sigma - \lambda - \delta) \cdot \sigma(\lambda, \delta) \sigma(\delta, \Sigma) \right\} \]
\[ + B_{\lambda, \delta} \delta_{(\Sigma + \delta - \lambda - \delta)} \cdot \sigma(\lambda, \delta) \sigma(\delta, \Sigma) \]
\[ + B_{\lambda, \delta} \delta_{(\Sigma - \lambda - \delta)} \cdot \sigma(\lambda, \delta) \sigma(\delta, \Sigma) \]
\[ + B_{\lambda, \delta} \delta_{(\Sigma + \delta - \lambda - \delta)} \cdot \sigma(\lambda, \delta) \sigma(\delta, \Sigma) \]

where the frequencies have been given a small complex part, ')0', to ensure coverage of the time integrals.

Ideally, '\(\lambda\)' and '\(\delta\)' are discrete due to the boundary conditions on the \(\Phi\) field at the walls of the box. More realistically, however, '\(\lambda\)' and '\(\delta\)' will be continuous variables with a sharply peaked weighting function giving an almost discrete spectrum. Similarly, the delta distributions will in reality be sharply peaked functions. Henceforth, I represent these discrete spectra through a weighting function lumped in with the cross-sections, so that I can consider '\(\lambda\)' and '\(\delta\)' to be continuous variables.
The other terms of (F.a.4) may be evaluated similarly.

Collecting terms in $B_{\lambda \gamma} B_{\beta \Sigma}^{\ast}$, for example, results in

\[-\xi^2 B_{\lambda \gamma} B_{\beta \Sigma}^{\ast} \sum_{\lambda, \gamma} \left\{ \frac{\gamma_{\lambda}^{\ast} \gamma_{\gamma}^{\ast} \delta(\gamma - \lambda - \Sigma - \sigma)}{i(\Sigma + i \xi)} \cdot \sigma(\lambda, \gamma^{\ast}) \sigma(\gamma^{\ast}, \Sigma) + \right.\]

\[-\gamma_{\lambda}^{\ast} \gamma_{\gamma}^{\ast} \delta(\gamma + \lambda - \Sigma) \cdot \sigma(\lambda, \gamma^{\ast}) \sigma(\gamma^{\ast}, \Sigma) + \]

\[+ \frac{\gamma_{\lambda}^{\ast} \gamma_{\gamma}^{\ast} \delta(\gamma + \lambda + \Sigma - \sigma)}{i(\lambda - \Sigma - i \xi)} \cdot \sigma(\lambda, \gamma^{\ast}) \sigma(\gamma^{\ast}, \Sigma) + \]

\[- \gamma_{\lambda}^{\ast} \gamma_{\gamma}^{\ast} \delta(\gamma - \Sigma + \lambda - \sigma) \cdot \sigma(\lambda, \gamma^{\ast}) \sigma(\gamma^{\ast}, \Sigma) \right\} \]

(F.a.10)

Appendix F(b) shows that each term is odd in $\lambda$ and $\gamma$, and so (F.a.10) can be simplified to,

\[-\xi^2 B_{\lambda \gamma} B_{\beta \Sigma}^{\ast} \sum_{\lambda} \int_{-\infty}^{\infty} d\gamma \gamma_{\lambda}^{\ast} \gamma_{\gamma}^{\ast} \delta(\gamma + \lambda - \Sigma - \sigma) \cdot \sigma(\lambda, \gamma^{\ast}) \sigma(\gamma^{\ast}, \Sigma) + \]
Similarly, the terms in $A_x A_y A_{x'}$ simplify to,

\[- \xi^2 A_x A_y A_{x'} \sum_{\lambda} \left\{ \int_{-\infty}^{+\infty} d\sigma \frac{\gamma_{\lambda}^* \bar{\gamma}_{\lambda} \delta(\Sigma - \sigma - \Sigma')}{i(\sigma - \Sigma - i\epsilon)} \sigma(\lambda, \Sigma') \sigma(\Sigma', \Sigma) \right\} \]

\[- \int_{-\infty}^{+\infty} d\sigma \frac{\gamma_{\lambda} \bar{\gamma}_{\lambda} \delta(\Sigma - \sigma + \Sigma + \lambda)}{i(\sigma - \Sigma + i\epsilon)} \sigma(\lambda, \Sigma') \sigma(\Sigma', \Sigma) \right\} \]

\[\text{(F.a.12)}\]

I now consider two cases:

(i) Vacuum contribution to (F.a.6) - 'k' is summed over

In which case the properties of Bogoliubov transformations, and interchanging 'q', 'Σ' allow (F.a.12) to be written as,

\[- \xi^2 B_{k\sigma} B_{k\epsilon} \sum_{\lambda} \left\{ \int_{-\infty}^{+\infty} d\sigma \frac{\gamma_{\lambda}^* \bar{\gamma}_{\lambda} \delta(\Sigma - \lambda - \Sigma - \Sigma')}{i(\sigma - \Sigma - i\epsilon)} \sigma(\lambda, \Sigma') \sigma(\Sigma', \Sigma) + \right\} \]

\[\text{(F.a.11)}\]
Combining (F.a.11) with the first two terms of (F.a.13), and using the fact that each is odd in \( \lambda \) (Appendix F(b)) gives:

\[
-\pi^2 B_{k\bar{\gamma}} B_{k\bar{\Sigma}} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma_{\lambda} \gamma_{\bar{\lambda}} \delta(\lambda - \bar{\lambda} - \bar{\Sigma})}{i(\lambda - \bar{\Sigma} + i0)} \cdot \delta(\lambda, \bar{\gamma} \bar{\lambda}) \sigma(\bar{\lambda}, \bar{\Sigma}) \right. \\
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\gamma_{\lambda} \gamma_{\bar{\lambda}} \delta(\lambda + \bar{\lambda} - \bar{\Sigma})}{i(\lambda - \bar{\Sigma} + i0)} \cdot \delta(\lambda, \bar{\gamma} \bar{\lambda}) \sigma(\bar{\lambda}, \bar{\Sigma}) \right\}
\]

(F.a.14)

Integration over the appropriate half complex \( \gamma \) – plane leaves,

\[
2\pi^2 B_{k\bar{\gamma}} B_{k\bar{\Sigma}} \gamma_{\bar{\gamma}} \gamma_{\bar{\Sigma}} \cdot \delta(\lambda, \bar{\gamma} \bar{\lambda}) \sigma(\bar{\lambda}, \bar{\Sigma})
\]

(F.a.15)

The second two terms of (F.a.13) give,
\[
\sum_{\lambda} \tau_{\lambda}^{\ast} \tau_{\lambda} \delta(s-\Sigma) B_{\lambda g} B_{\lambda \Sigma} \left( \frac{1}{i[(\lambda+\Sigma)+i0]} - \frac{1}{i[(\lambda+\Sigma)-i0]} \right) \sigma^2
\]

\[
= 2\pi \sum_{\lambda} \tau_{\lambda}^{\ast} \tau_{\lambda} \delta(s-\Sigma) B_{\lambda g} B_{\lambda \Sigma} \sigma^2 \delta(\lambda+\Sigma)
\]

which is zero.

(ii) The non-vacuum contribution

In this case, the third term from (F.a.4) has to be included,

\[
-\xi^2 B_{\alpha g} B_{\beta \Sigma} \sum_{\lambda} \left\{ \int_{-\infty}^{\infty} \frac{\delta(s+\tau-\lambda-\Sigma)}{i(s+\tau+i0)} \tau_{\tau}^{\ast} \tau_{\tau} \sigma(\delta^{\ast}, \Sigma) \sigma(\lambda, \delta^2) \right\}
\]

\[
-\int_{-\infty}^{\infty} d\delta^2 \frac{\delta(\lambda-\delta-\Sigma+\delta^2)}{i(s+\Sigma-i0)} \tau_{\delta}^{\ast} \tau_{\delta} \sigma(\delta, \delta^2) \sigma(\lambda, \delta^2)
\]

(F.a.16)

and combining with (F.a.11) gives,
Appendix F

\[-\pi^2 \beta \lambda g \beta \gamma \Sigma \left\{ \int_0^\infty \int_0^\infty \frac{\gamma_\lambda^* \gamma_\lambda \delta(g+\xi-\lambda-\Sigma)}{i(\xi-\Sigma+i0)} \sigma(\xi,\Sigma) \sigma(\lambda,\zeta^*) \right\}

\[-\int_0^\infty \int_0^\infty \frac{\gamma_\lambda^* \gamma_\lambda \delta(g-\xi+\lambda-\Sigma) \sigma(\xi,\Sigma) \sigma(\lambda,\zeta^*)}{i(\xi-\lambda-i0)} \right\}\]  

(F.a.18)

Integration over the appropriate half of the complex plane leaves,

\[2\pi^2 \beta \lambda g \beta \gamma \Sigma \gamma_\lambda^* \gamma_\lambda \sigma(\xi,\Sigma^*) \sigma(\Sigma^*,\Sigma)\]

and similarly for the terms involving \(A_{\beta \gamma} A_{\beta \gamma}^\Sigma\)

Completing the evaluation of all the terms from (F.a.4) gives in total,

\[\pi^2 \sum_{\xi,\lambda,\Sigma} \left\{ \beta_{K\lambda} A_{K\Sigma} \gamma_\lambda^* (\xi) \gamma_\Sigma (\lambda^*) \delta(\xi,\Sigma^*) \sigma(\xi,\Sigma^*) \right\} +

+ \beta_{K\lambda} A_{K\Sigma} \gamma_\lambda (\xi) \gamma_\Sigma (\lambda^*) \delta(\xi,\Sigma^*) \sigma(\xi,\Sigma^*) \right\} +

+ 2 \beta_{K\lambda} A_{K\Sigma} \gamma_\lambda^* (\xi) \gamma_\Sigma (\lambda^*) \sigma(\xi,\Sigma^*) \sigma(\xi,\Sigma^*) \right\} +

+ \pi^2 \sum_{\alpha,\beta} N_{\alpha \beta} \left[ \Delta \alpha (\lambda^*) \Delta \beta (\xi^*) + \Delta \alpha (\xi^*) \Delta \beta (\lambda^*) \right]\]

where

\[D_{K\lambda}(\xi) = \sum_{\Sigma} \left( \beta_{K\Sigma} \gamma_\Sigma (\xi) \sigma(\xi,\Sigma^*) + \beta_{K\Sigma} \gamma_\Sigma^* (\xi) \sigma(\xi,\Sigma^*) \right)\]
Finally, using the properties of the Bogoliubov transformations,

$$\pi \Sigma \sum_{k_i \lambda \Sigma} 2 B_{k\lambda} B_{k\lambda} \gamma_i^{\ast} \gamma_i \sigma^2$$

can be written as,

$$\pi \Sigma \sum_{k_i \lambda \Sigma} \left\{ A_{k\lambda} A_{k\lambda} \gamma_i^{\ast} \gamma_i \sigma(\lambda, \lambda) \sigma(\Sigma, \Sigma^\ast) + B_{k\lambda} B_{k\lambda} \gamma_i^{\ast} \gamma_i \sigma(\lambda, \lambda) \sigma(\Sigma^\ast, \Sigma) + -\delta(\Sigma - \lambda) \gamma_i^{\ast} \gamma_i \sigma(\lambda, \lambda) \sigma(\Sigma^\ast, \Sigma) \right\}$$

leading to equation (3.3.20).

Appendix F(b)

Equation (F.a.10) is odd in the energies

I show that, for a sufficiently small box containing a massive scalar field, equation (F.a.10) is odd in the energies, \( \lambda \) and \( \Sigma \). Restricting the analysis to a small box, and, shortly, to a massive scalar field, is purely to give the simplest treatment; this result may also hold in less restrictive circumstances.

For a sufficiently small box, the physical situation as seen by
co-moving observers will be indistinguishable from constant acceleration in flat spacetime.

The relevant line element in the observers' local coordinates is,

\[ ds^2 = (g^2) dt^2 - ds^2 - dz^2 - dy^2 \]  \hspace{1cm} (F.b.1)

(cf. equation (2.2.4)), and the wave equation is:

\[ \nabla^2 \Phi + (m^2 + \xi R) \Phi = 0 \]  \hspace{1cm} (F.b.2)

where the scalar curvature, \( R \) can be taken to be constant over the volume of the box.

I look for solutions to the wave equation of the form,

\[ \Phi = \exp \left( \pm i \frac{\delta}{h} \right) \sin \left( k_2 \cdot \vec{z} \right) \sin \left( k_y \cdot \vec{y} \right) \]  \hspace{1cm} (F.b.3)

leading to,

\[ \left[ g^2 \gamma^2 - \left( \frac{dL}{dy} \right)^2 + i \left( \frac{d^2L}{dy^2} \right) - C^2 e^{2y} \right] \sin [h(s)] = 0 \]  \hspace{1cm} (F.b.4)

where \( y = h \gamma \); \( C^2 = m^2 + \xi R + k_2^2 + k_y^2 \)

Approximating \( h(\gamma) \) as a slowly varying function of \( \gamma \), a WKB
approximation gives,

\[ \Theta(s) \sim \frac{1}{\sqrt{K(s)}} \cdot \sin \left[ \int k(s') ds' \right] \]

where,

\[ k(s') = \frac{1}{s} \left( g^2 y^2 - c^2 \right)^{1/2} \]

Now,

\[ \int k(s') ds' = a \sqrt{1 - \frac{c^2 s^2}{g^2 y^2}} - \ln \left| \frac{1 + \sqrt{1 - \frac{c^2 s^2}{g^2 y^2}}}{cs/a y} \right| \quad + \text{const.} \]

(F.b.6)

The boundary condition at the first box wall, \( s = 1 \), fixes the constant, while the boundary condition at the second wall, at \( s = 1 + \ell \), fixes the possible values of \( \delta \).

Finally,

\[ \Theta(s) \sim \left[ g \delta \left( 1 - \frac{c^2 s^2}{g^2 y^2} \right) \right]^{-1/2} \cdot \sin \left[ a \gamma \sqrt{1 - \frac{c^2 s^2}{g^2 y^2}} + \right. \\
\left. - \sqrt{1 - \frac{c^2}{g^2 y^2}} - \ln \left| \frac{1}{g} \left( 1 + \sqrt{1 - \frac{c^2 s^2}{g^2 y^2}} \right) \right| \right] \]

which is of the form,

\[ \Theta \sim \left[ g \delta f \left( \frac{g^2 y^2}{s^2} \right) \right]^{-1/2} \cdot \sin \left[ g \delta f' \left( \frac{g^2 y^2}{s^2} \right) \right] \cdot \sin k_x \cdot \sin k_y y \]
Normalization requires that,

\[ 1 = 2 \gamma \int_{g=1}^{g=1+\varepsilon} d\gamma^3 \sin k_x \sin k_y . \]

\[ \frac{\sin^2 \left[ g \gamma \gamma' \left( \frac{g^2}{\gamma^2} \right) \right]}{g^2 \gamma^2 \gamma'^{2}} f \left( \frac{g^2}{\gamma^2} \right) \]

which is of the form,

\[ 1 = 2 \gamma f'' \left( \gamma^2 \right) \]

and hence \( \gamma \) is odd in \( \gamma \), as required.

Appendix F(c)

_Emission of a particle detector undergoing non-stationary motion in a curved spacetime (two dimensions)._  

Here I give an analysis of the emission of a detector following an arbitrary trajectory in a curved spacetime.

The argument proceeds exactly as in Appendix D(b), the only difference being that the required conformal mapping is different.

Consider, for simplicity, a situation where the field is in the vacuum state associated to modes that go as \( \exp(i \omega u), \exp(i \omega v) \)
(the extension to a many particle state is straightforward). Write the metric as,

\[ ds^2 = c(u, v) \, du \, dv \]  

where \((u, v)\) are the special coordinates related to the state of the \(\Phi\) field, and let the detector be on a trajectory \(v = v(u)\).

To satisfy the conditions for the conformal transformation, \((u, v) \rightarrow (\hat{u}, \hat{v})\) of Appendix D(b), the appropriate transformation is

\[ \frac{du}{d\hat{u}} = \left[ C \left. \frac{dv}{du} \right|_{\tau_0} \right]^{-1/2} (\hat{u}) \]

(the analysis for \(\nu\)-modes is similar), where the trajectory is now parameterized by \(\hat{u}\) so that \(C\) is implicitly a function of \(\hat{u}\) through

\[ C(\hat{u}) = C(u(\hat{u}), v(\hat{u})|_{\tau_0}) \]

with

\[ v(\hat{u}) = v(u(\hat{u})|_{\tau_0}) \]

where \(v(u)\) is the equation of the trajectory.

Taking the cross-section to be flat, the right-moving emitted energy flux, \(\langle T^\nu_\mu \rangle\), can be found using,
as the appropriate conformal factor, and \( \langle T_{\tilde{u}\tilde{u}} \rangle \) is found from (1.9.2) to be given by,

\[
4 \pi \int_0^\infty \langle T_{\tilde{u}\tilde{u}} \rangle \\tilde{u}^2 = -C \left( \frac{du}{du} \right)^3 \left\{ -\frac{5}{4} (C,u)^2 \left( \frac{du}{du} \right)^2 +
\right.
\]

\[
\left. -\frac{5}{2} C,u C,\sigma \left( \frac{du}{du} \right)^3 \frac{1}{2} C,u \cdot C \cdot \frac{d^2r}{du^2} - \frac{5}{4} (C,u)^2 \left( \frac{du}{du} \right)^4 +
\right.
\]

\[
\left. + \frac{1}{2} C,\sigma \cdot C \cdot \frac{d^2r}{du^2} \left( \frac{du}{du} \right)^2 - \frac{5}{4} C \cdot \frac{d^2r}{du^2} + C \cdot C,u \cdot \frac{d^2r}{du^2} \right\} + C \cdot C,u \cdot \frac{d^2r}{du^2} \right\} +
\]

\[
+ 2C \left( \frac{du}{du} \right)^3 C,u + C,\tilde{u} \cdot C \left( \frac{du}{du} \right)^4 + C^2 \left( \frac{du}{du} \right)^4 \right\} +
\]

\[
\left. + 2C \left( \frac{du}{du} \right)^3 C,u + C,\tilde{u} \cdot C \left( \frac{du}{du} \right)^4 + C^2 \left( \frac{du}{du} \right)^4 \right\} +
\]

Apart from the restriction to two dimensions, no conditions have so far been placed on the spacetime metric. To identify the terms in equation (F.c.5), I now consider only those spacetimes that have a time-like Killing vector in the region that the detector moves through. This region can be coordinated by (new) coordinates, \((\tilde{u}, \tilde{v})\), so that the distance element is,

\[
ds^2 = A(\tilde{u}, \tilde{v}) \ d\tilde{u} \ d\tilde{v}
\]

and \( A(\tilde{u}, \tilde{v}) \) has the property that \( A,\tilde{u} + A,\tilde{v} = 0 \) (see Appendix B(c))
Expressing (F.c.6) in terms of the coordinates \((\tilde{u}, \tilde{v})\), and using the metric form (F.c.6), a lengthy but straightforward calculation allows the order \(\xi^2\) term in the flux emission to identified as,

\[
\langle T_{uu} \rangle_{\xi^2} = \frac{-1}{(4\cdot8\pi)} \left( g^2 + \frac{R}{2} + 2 \frac{dg}{d\xi} \right) - \langle T_{uu}(\Phi) \rangle
\]  

\[(F.c.8)\]

where \(g\) is the acceleration of the trajectory; \(\frac{dg}{d\xi}\) is the proper rate of change of acceleration, \(R\) is the scalar curvature, and \(\langle T_{uu}(\Phi) \rangle\) is the retarded energy flux of the vacuum state of the \(\Phi\) field.

At constant acceleration, therefore, the energy emission is just the negative of the energy absorbed by the detector (as measured in the proper frame of the detector), and consequently, the box walls emit no quantum energy at this order.
Glossary

Event Horizon:
The boundary between those events that given observers can influence
and be influenced by, and the rest.

Fulling-Rindler:
The quantization scheme developed by Fulling (1973) based on the
Rindler (1966) coordinates for accelerating observers (see Section
§2.3).

Globally hyperbolic:
A technical requirement on the spacetime to ensure the existence of a
Cauchy surface (see Hawking & Ellis (1973) for a full discussion).

Killing vector:
The generator of a group of isometries of the manifold - a Killing
vector, \( K^\mu \), generates a diffeomorphism that leaves the metric
unchanged. This can be summarized by the statement,

\[ \mathcal{L}_K g = 0 \]

where \( \mathcal{L}_K g \) is the Lie derivative of the metric with respect to \( K^\mu \).

Notation:
Spacetime is a four-dimensional pseudo-Riemannian space \((M,g)\) with
a metric of normal form \( \text{diag}(1,-1,-1,-1) \).
Units are such that $\hbar = c = G = k = 1$.

Null coordinates:
Coordinates for a region of spacetime with the property that lines of constant coordinate are null. In two-dimensional Minkowski spacetime, null coordinates $u = t - r$ (retarded), and $v = t + r$ (advanced) may be introduced so that the line element looks as,

$$ds^2 = dt^2 - dr^2 = du dv$$

Scalar curvature:
A contraction of the Riemann curvature tensor,

Riemann: $R(\xi, \eta) \equiv \partial_\xi (\partial_\eta Z) - \partial_\eta (\partial_\xi Z) - \partial_{[\xi, \eta]} Z$

Ricci: $R^{\alpha \beta} \equiv R_{\mu \alpha \beta}^\beta$

Scalar curvature: $R \equiv R^{\alpha \beta} = g^{\alpha \nu} R_{\mu \alpha \beta}^\beta$

Surface gravity:
A measure of the gravitational field strength at an event horizon. If $V^\alpha$ is the generator of the horizon, then the surface gravity, $K_c$, is defined by,

$$V_{\mu} V^\nu \equiv K_c V^\alpha$$