**COHERENT STATE PATH INTEGRAL
FOR THE HARMONIC OSCILLATOR
AND A SPIN PARTICLE IN A CONSTANT MAGNETIC FIELD**

By

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Abstract

The definition and formulas for the harmonic oscillator coherent states and spin coherent states are reviewed in detail. The path integral formalism is also reviewed with its relation and the partition function of a system is also reviewed. The harmonic oscillator coherent state path integral is evaluated exactly at the discrete level, and its relation with various regularizations is established. The use of harmonic oscillator coherent states and spin coherent states for the computation of the path integral for a particle of spin s put in a magnetic field is carried out in several ways, and a careful analysis of infinitesimal terms (in $1/N$ where N is the number of time slices) is done explicitly. The theory of the magnetic monopole and its relation with the spin system are explained, and the equivalence of these two systems is established up to infinitesimal order by the introduction of an exterior interaction to the monopole. This gives a new representation of a coherent state path integral in terms of a more familiar Feynman path integral. The coefficient of the topological term in the spin system appears explicitly without ambiguity, as being $2s$.

Table of Contents

Abstract	ii
List of Tables	v
List of Figures	vi
Acknowledgement	vii
1 Introduction	1
2 Review of Coherent States and Path Integrals	3
2.1 Harmonic Oscillator Coherent States	3
2.2 Spin Coherent States	6
2.3 Propagator, Path Integral and Partition Function	8
3 Coherent State Path Integral for the Harmonic Oscillator	11
3.1 Exact Result for the Discretisation	11
3.2 Continuum Limit	12
3.3 Semiclassical Approximation	16
3.4 Regularization	18
4 Coherent State Path Integral for Spin	26
4.1 Discretisation with Spin Coherent States	26
4.2 The Schwinger-Boson Model	27
4.3 Equivalence of the two Representations	28

4.4	Continuum Limit of the Spin Coherent States	30
4.5	Continuum Limit of the Schwinger-Boson Model	32
5	Path Integral for a Charged Particle in a Magnetic Monopole Field	39
5.1	Monopole Vector Potential	39
5.2	Monopole Angular Momentum	41
5.3	Path Integral for a Spin Particle in a Magnetic Field	43
5.4	Comparison with Coherent State Path Integral	45
6	Conclusion	47
	Bibliography	50
	Appendices	51
A	Identities for Determinants	51
B	Schwinger-Boson Model	54

List of Tables

3.1	Second order product regularization	24
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List of Figures

3.1	Discrete and continuous representation of the z variable	14
3.2	Contour of integration for the $\Gamma(x)$ function of the discrete determinant .	21

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Chapter 1

Introduction

The use of path integrals is a very active subject in physics. They have found many applications in quantum field theory, particle physics and condensed matter physics. The foundations have been studied for some time [7,8], and are by now well established.

Independently, in nuclear physics and quantum optics [9], some models have been studied using coherent states as a bridge between quantum theory and classical mechanics, obtaining a semi-classical representation of the theories. The theory of coherent states is very well known today, and does not present any mysteries by itself.

The introduction of coherent states as a tool for the evaluation of path integrals appears as an interesting alternative to the usual $|p\rangle$, $|q\rangle$ representation. Since these states exhibit classical behavior, we would expect the path integral to be easier to handle and, in particular, easier to approximate. However, their use has some difficulties [1]. A standard Lagrangian possesses a kinetic term which is quadratic in the velocity, like $m(\dot{\vec{r}})^2$. With coherent states, instead, we find a Lagrangian having a first order time derivative only, thus having a very different dynamical behavior and a different set of initial conditions. Actually, the Lagrange equations of motion for the coherent states are equivalent to the Hamilton equations for the coordinate and momentum variables, the latter variables being both included in the coherent state representation. So, a propagator computed with coherent states goes from an initial position and momentum to a final position and momentum, but such coordinates (in phase space) are usually not connected by a classical path! So, the difficulties with the coherent state path integral correspond to

the inclusion of these non-classical paths into the calculation, for a proper consideration of quantum mechanics.

In this thesis, I am going to look at these difficulties closely, and try to find some ways of properly evaluating these path integrals, by first using a discrete path integral, and then carefully examining the continuum limit.

In chapter 2, I will start by reviewing the theory of coherent states and path integrals. This will not cover new results, but is intended to introduce the notation.

In chapter 3, I will study the harmonic oscillator coherent state path integral in detail. I will explain different ways of regularizing the path integral, and compare these approximations with the discrete, but exact, path integral.

The same work will be done in chapter 4, for the coherent state path integral for a particle of spin s put in a constant magnetic field. For this path integral I use spin coherent states and the harmonic oscillator coherent states alternatively, where their equivalence will be made clear by using various methods. Specifically, for the harmonic oscillator coherent state path integral, the gauge symmetry will be studied and its connection with the topological term, appearing in this path integral, made clear. Furthermore, there has been some question about the coefficient of the topological term. This will be determined unequivocally in my calculation.

In chapter 5, I will review the theory of the magnetic monopole and indicate its use to represent a spin s particle. The path integral for this monopole system will be studied, and its relation with the system of chapter 4 will provide us with a new way of interpreting the regularization of the coherent state path integral.

Chapter 2

Review of Coherent States and Path Integrals

2.1 Harmonic Oscillator Coherent States

From the harmonic oscillator Hamiltonian, in M dimensions, $H = \sum_{k=1}^M (\frac{1}{2m} P_k^2 + \frac{m\omega^2}{2} Q_k^2) = \sum_{k=1}^M \omega (a_k^\dagger a_k + \hbar/2)$, with $a_k = \frac{1}{\sqrt{2}} (\sqrt{m\omega} Q_k + i \frac{P_k}{\sqrt{m\omega}})$ and $P_k = -i\hbar \frac{\partial}{\partial Q_k}$, we can single out a ground state $|0\rangle$:

$$a_k |0\rangle = 0, \quad \langle 0 | P_k | 0 \rangle = \langle 0 | Q_k | 0 \rangle = 0, \quad H |0\rangle = M\omega\hbar/2 |0\rangle$$

And then build up all the eigenstates of the Hamiltonian:

$$|n_1 \dots n_M\rangle = \frac{1}{\sqrt{n_1! \dots n_M!}} (a_1^\dagger / \sqrt{\hbar})^{n_1} \dots (a_M^\dagger / \sqrt{\hbar})^{n_M} |0\rangle$$

$$H |n_1 \dots n_M\rangle = \sum_{k=1}^M \omega \hbar (n_k + 1/2) |n_1 \dots n_M\rangle$$

But one of the drawbacks of these states is that they are not eigenstates of either the position or the momentum operator (Q and P). Furthermore, the commutation relation $[Q_k, P_l] = i\hbar \delta_{kl}$ prevents us from finding eigenstates for both of them. But it is possible to define a state, that we will call the coherent state $|p_i, q_j\rangle$, that will have a position and momentum, on average, given by some classical values (p_i, q_j) :

$$\langle p, q | P_i | p, q \rangle = p_i, \quad \langle p, q | Q_j | p, q \rangle = q_j \quad (2.1)$$

To find such a state, we can start with $|p, q\rangle = e^{-A} |\chi\rangle$ and the identity $e^A B e^{-A} = B + \frac{1}{1!}[A, B] + \frac{1}{2!}[A, [A, B]] + \dots$ that stops at the second term if $[A, B] = \text{c-number}$, and

then find that (2.1) imposes the conditions: $|\chi\rangle = |0\rangle$ and $A = \sum_{k=1}^M \frac{p_k Q_k - q_k P_k}{i\hbar}$, which gives:

$$|p, q\rangle = \exp\left\{\sum_{k=1}^M \frac{i}{\hbar}(p_k Q_k - q_k P_k)\right\} |0\rangle$$

In terms of $a_k, a_k^\dagger$ and the complex variable $z_k = \frac{1}{\sqrt{2}}(\sqrt{m\omega}q_k + \frac{ip_k}{\sqrt{m\omega}})$

$$|p, q\rangle \equiv |z\rangle = \exp\left\{\sum_{k=1}^M \frac{1}{\hbar}(z_k a_k^\dagger - z_k^* a_k)\right\} |0\rangle = \exp\left\{\frac{1}{\hbar}(a^\dagger z - z^\dagger a)\right\} |0\rangle \quad (2.2)$$

by working with columns $a = \begin{pmatrix} a_1 \\ \vdots \\ a_M \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ \vdots \\ z_M \end{pmatrix}$.¹

We can easily verify that $\langle z | (P_k - p_k)^2 | z \rangle = \langle z | (Q_k - q_k)^2 | z \rangle = \hbar/2$. So $|z\rangle$ is as close as possible to a classical state. If we use the identity $e^A e^B = e^{(A+B+[A,B]/2)}$ when $[A, B]$ = c-number, we can rewrite (2.2) as:

$$|z\rangle = e^{-\frac{1}{2\hbar} z^\dagger z} e^{\frac{a^\dagger z}{\hbar}} |0\rangle = e^{-\frac{1}{2\hbar} z^\dagger z} \sum_{n_1 \dots n_M=0}^{\infty} \frac{(z_1/\sqrt{\hbar})^{n_1}}{\sqrt{n_1!}} \dots \frac{(z_M/\sqrt{\hbar})^{n_M}}{\sqrt{n_M!}} |n_1 \dots n_M\rangle \quad (2.3)$$

From this we can show

$$a |z\rangle = z |z\rangle, \quad \langle z | a^\dagger = \langle z | z^* \text{ and } \langle z | z' \rangle = \exp\left\{\frac{1}{\hbar}(z^\dagger z' - z^\dagger z/2 - z'^\dagger z'/2)\right\} \quad (2.4)$$

and also

$$\begin{aligned} \int \frac{dz dz^\dagger}{(2\pi i \hbar)^M} |z\rangle \langle z| &= \int \prod_{k=1}^M \left(\frac{dz_k dz_k^*}{2\pi i \hbar} \right) |z\rangle \langle z| \\ &= \int \prod_{k=1}^M \left(\frac{dz_k dz_k^*}{2\pi i \hbar} \right) e^{-\frac{z^\dagger z}{\hbar}} \sum_{n_1, n_1', \dots, n_M=0}^{\infty} \prod_{k=1}^M \left(\frac{(z_k/\sqrt{\hbar})^{n_k} (z_k^*/\sqrt{\hbar})^{n_k'}}{\sqrt{n_k! n_k'!}} \right) |n_1 \dots n_M\rangle \langle n_1' \dots n_M'| \\ &= \sum_{n_1 \dots n_M=0}^{\infty} |n_1 \dots n_M\rangle \langle n_1 \dots n_M| \equiv I \end{aligned} \quad (2.5)$$

¹Sometimes it will be easier to work with one complex variable z , without any meaningful connection with its real (position) and imaginary (momentum) part.

So the $|z\rangle$ coherent states form an overcomplete set of states.

To summarize, the harmonic oscillator coherent states have the properties:

- Eigenstates of the a operator: $a|z\rangle = z|z\rangle$, so $\langle z|a|z\rangle = z$. (mean values of position and momentum given by p and q .)
- Are not orthonormal: $\langle z|z'\rangle \neq \delta(z - z')$
- Are overcomplete: $\int \frac{dz dz^*}{(2\pi i \hbar)^M} |z\rangle \langle z| = I$

Finally, let me prove the important following identity, that will be useful later:

For any complex $M \times M$ matrix σ :

$$\exp\{a^\dagger \sigma a / \hbar\} |z\rangle = \exp\left\{\frac{1}{2\hbar} z^\dagger (e^{\sigma^\dagger} e^\sigma - 1) z\right\} |e^\sigma z\rangle \quad (2.6)$$

Proof: first

$$\begin{aligned} e^{a^\dagger \sigma a / \hbar} (a^\dagger z / \hbar) &= e^{a^\dagger \sigma a / \hbar} (a^\dagger z / \hbar) e^{-a^\dagger \sigma a / \hbar} \times e^{a^\dagger \sigma a / \hbar} \\ &= a^\dagger / \hbar (1 + \sigma + \sigma^2 / 2! + \dots) z e^{a^\dagger \sigma a / \hbar} = (a^\dagger e^\sigma z / \hbar) e^{a^\dagger \sigma a / \hbar} \end{aligned}$$

So, using (2.3)

$$\begin{aligned} e^{a^\dagger \sigma a / \hbar} |z\rangle &= e^{-z^\dagger z / 2\hbar} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} e^{a^\dagger \sigma a / \hbar} (a^\dagger z / \hbar)^n |0\rangle \\ &= e^{-z^\dagger z / 2\hbar} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (a^\dagger e^\sigma z / \hbar)^n |0\rangle \\ &= e^{-z^\dagger z / 2\hbar} e^{(e^\sigma z)^\dagger (e^\sigma z) / 2\hbar} |e^\sigma z\rangle \end{aligned}$$

which gives (2.6).

2.2 Spin Coherent States

In the same spirit as the harmonic oscillator coherent states, all three components of the spin $\vec{J}$ (J_x, J_y, J_z) can not have definite eigenvalues for a given state of the system. But it is possible to define a state $|\vec{n}\rangle$ ($(\vec{n})^2 = 1$) such that:

$$\langle \vec{n} | \vec{J} | \vec{n} \rangle = s\hbar\vec{n} \text{ for a spin operator } \vec{J}, \text{ of spin } s.$$

To find such a state $|\vec{n}\rangle$, let us define a 'ground state' $|0\rangle = |s, s\rangle$, an eigenstate of projection $m = s$ of J_z : $J_z |0\rangle = s\hbar |0\rangle$. Then $\langle 0 | \vec{J} | 0 \rangle = s\hbar\vec{k}$, where $\vec{k} = (0, 0, 1)$, and we obtain a coherent state $|\vec{n}\rangle = |\theta, \phi\rangle$, using spherical coordinates, by performing the appropriate rotations:

$$|\theta, \phi\rangle = e^{-i\phi J_z/\hbar} e^{-i\theta J_y/\hbar} |0\rangle \quad (2.7)$$

For spin 1/2 (where $\vec{\sigma}$ are the Pauli matrices):

$$\begin{aligned} |\theta, \phi\rangle_{s=1/2} &= \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} \text{ for } \theta \neq \pi \\ &= \begin{pmatrix} e^{-i\phi} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \text{ for } \theta \neq 0 \end{aligned} \quad (2.8)$$

And we find

$$\begin{aligned} \langle \theta, \phi | \theta', \phi' \rangle_{1/2} &= \cos(\theta/2) \cos(\theta'/2) + \sin(\theta/2) \sin(\theta'/2) e^{-i(\phi-\phi')} \text{ for } \theta \neq \pi \\ &= \cos(\theta/2) \cos(\theta'/2) e^{i(\phi-\phi')} + \sin(\theta/2) \sin(\theta'/2) \text{ for } \theta \neq 0 \end{aligned}$$

with

$$\langle \theta, \phi | \vec{\sigma} | \theta, \phi \rangle_{1/2} = \vec{n} = \vec{i} \sin \theta \cos \phi + \vec{j} \sin \theta \sin \phi + \vec{k} \cos \theta$$

The two different representations are necessary because we need two coordinate patches to coordinatize sphere. The phase factor $e^{i\phi}$ between these two patches is purely topological. It will not change the physics in general.

For arbitrary spin s :

We can use the spinor representation for a spin:

$$|\Psi\rangle = \begin{pmatrix} \psi(+s) \\ \vdots \\ \psi(-s) \end{pmatrix}, \quad \psi(\sigma) = \sqrt{\frac{(2s)!}{(s+\sigma)!(s-\sigma)!}} \overbrace{\psi^1 \dots \psi^1}^{s+\sigma} \overbrace{\psi^2 \dots \psi^2}^{s-\sigma}, \quad |\Psi\rangle_{s=1/2} = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

So ²

$$\psi(\sigma) = \sqrt{\frac{(2s)!}{(s+\sigma)!(s-\sigma)!}} e^{i(s-\sigma)\phi} \cos^{s+\sigma}(\theta/2) \sin^{s-\sigma}(\theta/2) \quad \text{for } \theta \neq \pi$$

$$= \sqrt{\frac{(2s)!}{(s+\sigma)!(s-\sigma)!}} e^{-i(s+\sigma)\phi} \cos^{s+\sigma}(\theta/2) \sin^{s-\sigma}(\theta/2) \quad \text{for } \theta \neq 0$$

$$|\theta, \phi\rangle_s = \begin{pmatrix} \vdots \\ \psi(\sigma) \\ \vdots \end{pmatrix} \quad (2.9)$$

The phase factor between these two patches is now $e^{2si\phi}$, where we can recognize $2s$ has a winding number. It can be checked that $\langle \theta, \phi | \vec{J} | \theta, \phi \rangle_s = s\hbar\vec{n}$ and also

$$\langle \theta, \phi | \theta', \phi' \rangle_s = \left[\langle \theta, \phi | \theta', \phi' \rangle_{1/2} \right]^{2s} \quad (2.10)$$

We also find $\langle \theta, \phi | (\vec{J})^2 | \theta, \phi \rangle_s - (\langle \theta, \phi | \vec{J} | \theta, \phi \rangle_s)^2 = \hbar^2[s(s+1) - s^2] = s\hbar^2$ the minimum value possible.

Finally, from (2.9), we can easily find the completeness relation:

$$\int_0^\pi \int_0^{2\pi} |\theta, \phi\rangle \langle \theta, \phi|_s \frac{d\phi \sin(\theta) d\theta}{\left(\frac{4\pi}{2s+1}\right)} \equiv I \quad (2.11)$$

²This might differ from other authors by a unitary transformation.

2.3 Propagator, Path Integral and Partition Function

In quantum mechanics, all the information for the evolution of a system can be stored in the propagator (transition amplitude) between an initial state $|q_i\rangle$ and a final state $|q_f\rangle$ at a time $t = t_f - t_i$ later, given by:

$$K_{i \rightarrow f}(t) = \langle q_i | e^{iHt/\hbar} | q_f \rangle$$

where H is the Hamiltonian of the system.

This allows Feynman, by using the completeness relation $\int \frac{dp dq}{2\pi\hbar} |p\rangle \langle p| q\rangle \langle q| = \int \frac{dp dq}{2\pi\hbar} e^{-ipq/\hbar} |p\rangle \langle q|$, to write a path integral (with $q_0 = q_i$ and $q_{N+1} = q_f$):

$$\begin{aligned} K_{i \rightarrow f}(t) &= \int dp_0 \prod_{k=1}^N \frac{dp_k dq_k}{2\pi\hbar} \langle q_i | e^{\frac{iHt}{\hbar(N+1)}} | p_0 \rangle e^{-ip_0 q_1/\hbar} \langle q_1 | e^{\frac{iHt}{\hbar(N+1)}} | p_1 \rangle e^{-ip_1 q_2/\hbar} \dots \\ &\quad \dots \langle q_N | e^{\frac{iHt}{\hbar(N+1)}} | p_N \rangle \langle p_N | q_f \rangle \\ &= \lim_{N \rightarrow \infty} \int dp_0 \prod_{k=1}^N \frac{dp_k dq_k}{2\pi\hbar} \exp \left\{ -\frac{i}{\hbar} \sum_{j=0}^N [p_j(q_{j+1} - q_j) - \left(\frac{t}{N+1}\right) H(p_j, q_j)] \right\} \\ &= \int dp_0 Dp Dq \exp \left\{ -\frac{i}{\hbar} \int_{t_i}^{t_f} (p\dot{q} - H(p, q)) dt \right\} \end{aligned} \quad (2.12)$$

where $H(p, q) = \frac{1}{2} \left[\frac{\langle p|H|q\rangle}{\langle p|q\rangle} + \frac{\langle q|H|p\rangle}{\langle q|p\rangle} \right]$ is the classical Hamiltonian, with boundary conditions $q(t_i) = q_i$, $q(t_f) = q_f$.

Furthermore if the Hamiltonian is of the form $H = \frac{p^2}{2m} + V(q)$ then we can perform the p integrations (being a Gaussian), giving:

$$\begin{aligned} K_{i \rightarrow f}(t) &= \lim_{N \rightarrow \infty} \sqrt{\frac{mi}{2\pi\hbar t}} \cdot \sqrt{N+1} \int \prod_{k=1}^N (dq_k \sqrt{\frac{mi(N+1)}{2\pi\hbar t}}) \\ &\quad \exp \left\{ -\frac{i}{\hbar} \left(\frac{t}{N+1}\right) \sum_{j=0}^N \left[\frac{m}{2} \left(\frac{q_{j+1} - q_j}{\frac{t}{N+1}}\right)^2 - V(q_j) \right] \right\} \\ &= M \int Dq \exp \left\{ -\frac{i}{\hbar} \int_{t_i}^{t_f} L(q, \dot{q}) dt \right\} \end{aligned} \quad (2.13)$$

where M is an infinite constant, and L is the classical Lagrangian.

This new representation has some difficulties. First of all, there is this infinite constant, $\sqrt{N+1}$, that can not even be absorbed in the measure of Dq . For example, if $V = 0$ we can perform the integrations:

$$\int \prod_{k=1}^N (dq_k \sqrt{\frac{mi(N+1)}{2\pi\hbar t}}) \exp\left\{-\frac{i}{\hbar} \left(\frac{t}{N+1}\right) \sum_{j=0}^N \frac{m}{2} \left(\frac{q_{j+1} - q_j}{\frac{t}{N+1}}\right)^2\right\} = \frac{e^{-i\frac{m(q_f - q_i)^2}{2\hbar t}}}{\sqrt{N+1}}$$

which gives:

$$K_{i \rightarrow f}(t) = \sqrt{\frac{mi}{2\pi\hbar t}} e^{-i\frac{m(q_f - q_i)^2}{2\hbar t}} \quad (2.14)$$

This clearly shows the fine tuned cancellation of $\sqrt{N+1}$ in this (simple) case. For a more complicated system, it could be expected to step through some divergences. Also, the $Dq (= \prod_k \sqrt{\frac{mi(N+1)}{2\pi\hbar t}} dq_k)$ measure, containing the N factor, indicates the difficulties that might appear by performing the $N \rightarrow \infty$ limit.

In statistical mechanics, at a temperature T , the information is stored, instead, in the partition function:

$$Z[\beta] = \text{tr}(e^{-\beta H}), \quad \beta = \frac{1}{kT}$$

Since the trace can be represented by $\int \langle q | () | q \rangle dq$, or more generally $\int e^{-ipq/\hbar} \langle q | () | p \rangle \frac{dp dq}{2\pi\hbar}$, so $\text{tr}(1)$ = number of states available, we find:

$$Z[\beta] = \int K_{i \rightarrow i}(i\beta\hbar) dq_i$$

which shows that the partition function is the integration over the initial state of the propagator that goes around a loop (initial $\equiv$ final) for a 'time' $t = i\beta\hbar$.

So in terms of the path integral formalism, we can find $Z[\beta]$ by inserting N resolution of unity ($\int | p \rangle \langle p | q \rangle \langle q | \frac{dp dq}{2\pi\hbar}$) in $e^{-\beta H}$. By using the Feynman path integral, we find:

$$\begin{aligned} Z[\beta] &= \lim_{N \rightarrow \infty} \int \prod_{k=1}^N \frac{dp_k dq_k}{2\pi\hbar} \exp\left\{\sum_{j=1}^N \left[-\frac{i}{\hbar} p_j (q_{j+1} - q_j) - \frac{\beta}{N} H(p_j, q_j)\right]\right\} \\ &= \int Dp Dq \exp\left\{-\int_0^\beta \left(\frac{i}{\hbar} p \dot{q} + H(p, q)\right) d\tau\right\} \end{aligned}$$

where now $\dot{q} = \lim_{N \rightarrow \infty} (\frac{q_{i+1} - q_i}{\beta/N})$ is a ‘temperature derivative’, and $q(0) = q(\beta)$, a loop in the q space for a ‘time’ β . Since the variable q , in the propagator (2.13), goes from an initial to a final position, without any condition on the velocity ($\dot{q}$) at these boundary points, the same indeterminacy will have to be applied for $Z[\beta]$. The variable q will leave the initial point $q(0)$, goes around a loop and comes back to $q(\beta) = q(0)$, but it does not mean that the curve will be smooth at the connecting point $q(0)$: $\dot{q}(0) \neq \dot{q}(\beta)$. Note that the phase space integration is more complete in this path integral, there is no extra dp_0 integration as in (2.12).

Applying this for our example, a free particle ($H = \frac{p^2}{2m}$), we find from (2.14):

$$Z[\beta] = \int dq_i \sqrt{\frac{m}{2\pi\hbar^2\beta}} e^{-\frac{m(q_i - q_i)^2}{2\hbar^2\beta}} = \sqrt{\frac{m}{2\pi\beta\hbar^2}} \cdot \int dq_i$$

or as we would do in statistical mechanics

$$= \int e^{-\beta H} \frac{dp dq}{2\pi\hbar} = \sqrt{\frac{m}{2\pi\beta\hbar^2}} \cdot \int dq$$

as expect.

Chapter 3

Coherent State Path Integral for the Harmonic Oscillator

3.1 Exact Result for the Discretisation

In this section, we will derive an exact expression for the partition function as a path integral, using resolution of unity and identities of the last chapter for coherent states, and write down the result in a suitable form that will be used to study the continuum limit of this new coherent state path integral.

We want to evaluate:

$$Z[\beta] = \text{tr}(e^{-\beta H}) \quad \text{where} \quad H = \omega(a^\dagger a + \hbar M/2)$$

$$= \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) \langle z_1 | e^{-\beta H/N} | z_2 \rangle \dots \langle z_N | e^{-\beta H/N} | z_1 \rangle$$

with (2.6) we find

$$\begin{aligned} Z[\beta] &= e^{-\beta \omega \hbar M/2} \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) e^{-\frac{z_1^\dagger z_1}{2\hbar} (1 - e^{-2\beta \omega \hbar/N})} \langle z_1 | e^{-\beta \omega \hbar/N} z_2 \rangle \dots \\ &= e^{-\beta \omega \hbar M/2} \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) \exp \left\{ \sum_{k=1}^N \left[-\frac{z_k^\dagger z_k}{2\hbar} (1 - e^{-2\beta \omega \hbar/N}) - \frac{z_k^\dagger z_k}{2\hbar} \right. \right. \\ &\quad \left. \left. - e^{-\beta \omega \hbar/N} \frac{z_{k+1}^\dagger z_{k+1}}{2\hbar} + e^{-\beta \omega \hbar/N} \frac{z_k^\dagger z_{k+1}}{\hbar} \right] \right\} \\ &= e^{-2\beta \omega \hbar M/2} \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) e^{-\frac{1}{\hbar} \sum_{k=1}^N [z_k^\dagger z_k - e^{-\beta \omega \hbar/N} z_k^\dagger z_{k+1}]} \end{aligned} \quad (3.15)$$

Also by changing $z_k \rightarrow z_k e^{\beta \omega \hbar/N}$, we find ¹

$$Z[\beta] = e^{\beta \omega \hbar M/2} \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) e^{-\frac{1}{\hbar} \sum_{k=1}^N [e^{\beta \omega \hbar/N} z_k^\dagger z_k - z_k^\dagger z_{k+1}]} \quad (3.16)$$

¹Note that the condition of periodicity implies the periodicity of the position and *momentum* as defined by (2.2) and (2.4), in contrast with the usual Feynman path integral.

This gives us an exact action, at the discrete level, for this path integral:

$$S = \sum_{k=1}^N [e^{\beta\omega\hbar/N} z_k^\dagger z_k - z_k^\dagger z_{k+1}] \quad (3.17)$$

These two formulas can be checked independently, using the determinants solved in Appendix A for a matrix of the form $A_{i,j} = \delta_{i,j} - e^{-\beta\omega\hbar/N} \delta_{i+1,j}$ or $A'_{i,j} = e^{\beta\omega\hbar/N} \delta_{i,j} - \delta_{i+1,j}$, because the path integral is a Gaussian:

$$\begin{aligned} Z[\beta] &= \frac{e^{-\beta\omega\hbar M/2}}{[\det(A)]^M} = \frac{e^{-\beta\omega\hbar M/2}}{(1 - e^{-\beta\omega\hbar})^M} = \frac{e^{\beta\omega\hbar M/2}}{[\det(A')]^M} = \frac{e^{\beta\omega\hbar M/2}}{(e^{\beta\omega\hbar} - 1)^M} \\ &= \frac{1}{[2 \sinh(\beta\omega\hbar/2)]^M} = \sum_{m_1, \dots, m_M=1}^{\infty} e^{-\beta\omega\hbar(m_1 + \dots + m_M + M/2)} = \text{tr}(e^{-\beta H}) \end{aligned} \quad (3.18)$$

3.2 Continuum Limit

When we are seeking a continuum limit of a path integral, we want to keep terms in the summation of the β/N order (excluding the extra β/N term for each ‘time’ derivative), so that we can approximate the summation by an integral $\int_0^\beta d\tau$.

For the path integrals (3.16) of the last section, this means the following:

$$\begin{aligned} Z[\beta] &\approx e^{\beta\omega\hbar M/2} \int Dz dz^\dagger e^{-\frac{1}{\hbar} \sum_{k=1}^N [z_k^\dagger (z_k - z_{k+1}) + \frac{\beta}{N} \omega\hbar z_k^\dagger z_k]} \\ &\approx e^{\beta\omega\hbar M/2} \int Dz dz^\dagger e^{-\frac{1}{\hbar} \int_0^\beta (-z^\dagger \dot{z} + \omega\hbar z^\dagger z) d\tau} \end{aligned} \quad (3.19)$$

But is $z_k^\dagger (z_k - z_{k+1})$ really becoming $-z^\dagger \dot{z} d\tau$? We can easily check that

$$\left(\int_0^\beta z^\dagger \dot{z} d\tau \right)^\dagger = \int_0^\beta \dot{z}^\dagger z d\tau = z^\dagger z \Big|_0^\beta - \int_0^\beta z^\dagger \dot{z} d\tau = - \left(\int_0^\beta z^\dagger \dot{z} d\tau \right)$$

so this is purely imaginary. But for the discrete counterpart

$$\sum_{k=1}^N z_k^\dagger (z_k - z_{k+1}) = \sum_{k=1}^N \left[\frac{1}{2} z_k^\dagger (z_k - z_{k+1}) - \frac{1}{2} (z_k - z_{k+1})^\dagger z_k + \frac{1}{2} |z_k - z_{k+1}|^2 \right] \quad (3.20)$$

clearly indicates the presence of a real term that is missing in the continuum limit! Even if this real contribution will appear only to the β/N order, it could be relevant for a convergence of the path integral.

For example $\prod_{k=1}^N (1 + \beta x/N) \rightarrow e^{\beta x}$ as $N \rightarrow \infty$, but doing the approximation $(1 + \beta x/N) \approx 1$ would give 1 instead. For the path integral each integration will contain a β/N term that will bring a finite modification at the end.

To find the continuum limit with this extra real term, we just use (3.20) in the path integral (3.16) which gives the following limit:

$$Z[\beta] \approx e^{\beta \omega \hbar M/2} \int Dz Dz^\dagger e^{-\frac{1}{\hbar} \int_0^\beta (-z^\dagger \dot{z} + \omega \hbar z^\dagger z + \frac{\epsilon}{2} |\dot{z}|^2) dt} \quad (3.21)$$

where $\boxed{\epsilon = \beta/N}$. Because of the presence of the N in ϵ we do not really expect any dependence on ϵ after the evaluation of the path integral, but it should help to ‘smooth out’ the integrations.

Because we will be looking for corrections to the order ϵ in the path integral, like in (3.21), it will be important to keep track also of the boundary conditions at 0 and β . To approximate a summation by an integral, it will be useful to use the following relation, valid if $f(\tau)$ is smooth from 0 to β :

$$f_k = f((k - \frac{1}{2})\epsilon) \text{ for } k = 1 \dots N, \quad \sum_{k=1}^N f_k \epsilon = \int_0^\beta f(\tau) d\tau + O(\epsilon^3) \quad (3.22)$$

In our case, we will have to consider the variable z_k , that we would like to represent by a continuous curve $z(\tau)$. This representation appears on the figure 3.1. It is clear, then, that the value of $z_{N+1} = z((N + \frac{1}{2})\epsilon) \neq z_1 = z(\frac{1}{2}\epsilon)$, because $z(\tau)$ is not smooth at $\tau = 0$ and $\tau = \beta$, but z_{N+1} is instead analytically continued away.

If we write

$$z_{k+1} = z_k + \epsilon \dot{z}_k + \frac{\epsilon^2}{2} \ddot{z}_k + \dots$$

Figure 3.1: Discrete and continuous representation of the z variable

and use the identity (3.22), we can approximate the action (3.17) by:

$$S = \int_0^\beta [-z^\dagger \dot{z} - \frac{\epsilon}{2} z^\dagger \ddot{z} + \omega \hbar z^\dagger z + \frac{\epsilon}{2} \omega^2 \hbar^2 z^\dagger z] d\tau + z_N^\dagger (z_{N+1} - z_N)$$

The $z_N^\dagger (z_{N+1} - z_N)$ correction comes from the fact that the integral needs a smooth $z(\tau)$ function, that does not consider $z_{N+1} \equiv z_1$, this has to be corrected by 'hand':

$$z_{N+1} = z(\beta) + \frac{\epsilon}{2} \dot{z}(\beta) + O(\epsilon^2)$$

$$z_1 = z(0) + \frac{\epsilon}{2} \dot{z}(0) + O(\epsilon^2)$$

$$z_N = z(0) + O(\epsilon) = z(\beta) + O(\epsilon)$$

Thus to the ϵ order, we find

$$z_N^\dagger (z_{N+1} - z_N) = \frac{\epsilon}{2} [z^\dagger(\beta) \dot{z}(\beta) - z^\dagger(0) \dot{z}(0)] = \frac{\epsilon}{2} z^\dagger \dot{z} \Big|_0^\beta = \int_0^\beta \left[\frac{\epsilon}{2} (z^\dagger \ddot{z} + |\dot{z}|^2) \right] d\tau$$

This allows us to rewrite the action as

$$S = \int_0^\beta [-z^\dagger \dot{z} + \frac{\epsilon}{2} |\dot{z}|^2 + \omega \hbar z^\dagger z + \frac{\epsilon}{2} \omega^2 \hbar^2 z^\dagger z] d\tau$$

as we found in (3.21). These boundary contributions could be very important, as it will be seen in the next section. The $\frac{\epsilon}{2} \omega^2 \hbar^2 z^\dagger z$ term will be dropped since it is obviously negligible compared to $\omega \hbar z^\dagger z$.

Actually, if we consider the approximation $\int (-z^\dagger \dot{z} + \omega \hbar z^\dagger z + \alpha \epsilon |\dot{z}|^2) d\tau$ to the action, where α is a parameter that would connect the path integral (3.19) at $\alpha = 0$ to (3.21) at $\alpha = 1$ continuously, then the discrete path integral would contain:

$$\begin{aligned} & \sum_{k=1}^N \frac{1}{2} [z_k^\dagger (z_k - z_{k+1}) - (z_k - z_{k+1})^\dagger z_k + \alpha |z_k - z_{k+1}|^2] \\ &= \sum_{k=1}^N [\alpha z_k^\dagger z_k - \frac{\alpha+1}{2} z_k^\dagger z_{k+1} - \frac{\alpha-1}{2} z_{k+1}^\dagger z_k] \end{aligned}$$

instead of (3.20), where it is important to check that the boundary terms has been included in the calculation. The evaluation of the discrete determinant can be done exactly by using the identity (A.80) of appendix A, and simply gives

$$\text{determinant}^{1/M} = \left(\frac{\alpha+1}{2}\right)^N (e^{\beta\omega\hbar} - 1) + \left(\frac{\alpha-1}{2}\right)^N (e^{-\beta\omega\hbar} - 1)$$

This clearly indicates the effect of the ϵ term. When $\alpha = 1$, the determinant is exactly the expected one $(e^{\beta\omega\hbar} - 1)^M$. But when $0 < \alpha < 1$, the second term grows, and oscillates in sign as a function of N ! But $|\frac{\alpha+1}{2}|$ is bigger than $|\frac{\alpha-1}{2}|$, so the second term is still negligible. Furthermore we have to renormalize for the infinite constant $(\frac{\alpha+1}{2})^N$. But at the extreme case $\alpha = 0$, corresponding to (3.19), we have $(\frac{\alpha+1}{2}) = -(\frac{\alpha-1}{2}) = 1/2$ so each term has the same factor $(1/2)^N$, and the $(-1)^N$ of the second term produces $2(\cosh(\beta\omega\hbar) - 1) = [2\sinh(\beta\omega\hbar/2)]^2$ for N even and $2\sinh(\beta\omega\hbar)$ for N odd. Note that this unphysical oscillation will always be cancelled by taking the average. All of this

indicates the subtlety of the path integral (3.19), and the regularization introduced by the ϵ term.

I would also like to point out that if we start with the path integral (3.15), instead of (3.16), we would obtain the partition function (3.19) as a continuum limit, without including the ϵ corrections, but with $e^{-\beta\omega\hbar M/2}$ instead of $e^{\beta\omega\hbar M/2}$! This factor actually comes by doing the approximation $\omega\hbar z_k^\dagger z_{k+1} \approx \omega\hbar z_k^\dagger z_k$ for the continuum limit. So this example indicates how sensitive the discrete summation is to the discrete index ($k = 1, \dots, N$). A slight modification (like $k \rightarrow k+1$) might greatly affect the path integral.

3.3 Semiclassical Approximation

The action $S = \int_0^\beta (-z^\dagger \dot{z} + \omega\hbar z^\dagger z + \alpha \frac{\epsilon}{2} |\dot{z}|^2) d\tau$ in the path integral is evaluated along a continuous curve $z(\tau) \in C$ from $\tau = 0$ to $\tau = \beta$, with $z(0) = z(\beta)$, and a classical equation of motion is a curve $z(\tau)$ that minimizes the action S to S_c . The other trajectories $z(\tau)$ will give an action $S > S_c$, thus contributing less to the path integral. The semiclassical approximation is to consider only the classical solutions ($\int dz_c e^{-S_c}$) in the path integral.

More explicitly, let us write $z = z_c + \tilde{z}$ where z_c is the classical solution, with the boundary condition $z_c(0) = z_c(\beta) = z(0) = z(\beta) = z_0$. So $\tilde{z}$ will be a complex curve with $\tilde{z}(0) = \tilde{z}(\beta) = 0$. The integration will be separated as

$$\int Dz Dz^\dagger = \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) = \int \frac{dz_0 dz_0^\dagger}{(2\pi i \hbar)^M} \int \prod_{k=1}^{N-1} \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) = \int Dz_0 Dz_0^\dagger \int D\tilde{z} D\tilde{z}^\dagger$$

So the partition function is

$$Z[\beta] = e^{\beta\omega\hbar M/2} \int Dz_0 Dz_0^\dagger \int D\tilde{z} D\tilde{z}^\dagger e^{-\frac{1}{\hbar} S(z_c + \tilde{z})}$$

where

$$S(z_c + \tilde{z}) = S(z_c) + S(\tilde{z}) + \int_0^\beta [-(z_c^\dagger \dot{\tilde{z}} + \dot{\tilde{z}}^\dagger z_c) + \omega\hbar(z_c^\dagger \tilde{z} + \tilde{z}^\dagger z_c) + \alpha \frac{\epsilon}{2} (\dot{\tilde{z}}^\dagger \tilde{z} + \dot{\tilde{z}} \tilde{z}^\dagger)] d\tau$$

$$= S(z_c) + S(\tilde{z}) + \int_0^\beta \tilde{z}^\dagger (-\dot{z}_c + \omega \hbar z_c - \alpha \frac{\epsilon}{2} \tilde{z}_c) d\tau + \int_0^\beta (\dot{z}_c^\dagger + \omega \hbar z_c^\dagger - \alpha \frac{\epsilon}{2} \tilde{z}_c^\dagger) \tilde{z} d\tau$$

The second line is obtained after integration by parts and the use of the boundary conditions on $\tilde{z}$. The last two integrals are zero by the equation of motion of z_c and $z_c^\dagger$, so

$$Z[\beta] = e^{\beta \omega \hbar M/2} \int Dz_0 Dz_0^\dagger e^{-\frac{1}{\hbar} S_c(z_0)} \int D\tilde{z} D\tilde{z}^\dagger e^{-\frac{1}{\hbar} S(\tilde{z})}$$

The semiclassical approximation is to drop the $\tilde{z}$ path integral:

$$Z_{sc}[\beta] = e^{\beta \omega \hbar M/2} \int Dz_0 Dz_0^\dagger e^{-\frac{1}{\hbar} \int_0^\beta (-z_c^\dagger \dot{z}_c + \omega \hbar z_c^\dagger z_c + \alpha \frac{\epsilon}{2} |z_c|^2) d\tau}$$

where

$$-\dot{z}_c + \omega \hbar z_c - \alpha \frac{\epsilon}{2} \tilde{z}_c = 0 \quad \text{and} \quad \dot{z}_c^\dagger + \omega \hbar z_c^\dagger - \alpha \frac{\epsilon}{2} \tilde{z}_c^\dagger = 0$$

These two equations of motion correspond to the Euler-Lagrange equation (obviously?) for a Lagrangian: $L = -z^\dagger \dot{z} + \omega \hbar z^\dagger z + \alpha \frac{\epsilon}{2} |z|^2$. The fact that the equations of motion of z_c and $z_c^\dagger$ does not seem complex conjugate comes from the fact that $d\tau = -\frac{i}{\hbar} dt$, so τ transforms like a complex number, which explains the sign change in $\dot{z}_c$ in the equation of motion. But τ or β must still be considered as real, it is p and q of z that can get complexified in the search for extremal of the action. By integrating by parts $| \dot{z}_c |^2$, and using the z_c equation of motion, we can simplify S_c to

$$S_c = \lim_{\epsilon \rightarrow 0} \alpha \frac{\epsilon}{2} z_c^\dagger \dot{z}_c \Big|_0^\beta$$

For $\alpha = 0$, we trivially find $S_c = 0$. In fact the equation of motion becomes linear, and there exists no solution with $z_c(0) = z_c(\beta)$! So the semiclassical approximation is not applicable, or at the best it gives a constant ($\int Dz_0 Dz_0^\dagger e^0$). This indicates, again, the difficulties of the path integral (3.19).

For $\alpha = 1$, we find two independent solutions, $e^{\omega \hbar \tau}$ and $e^{-2\tau/\epsilon}$ for z_c , $e^{-\omega \hbar \tau}$ and $e^{2\tau/\epsilon}$ for $z_c^\dagger$. By considering ϵ very small and the boundary condition, we obtain:

$$z_c = z_0 [(1 - e^{-\beta \omega \hbar}) e^{-2\tau/\epsilon} + e^{\omega \hbar (\tau - \beta)}]$$

$$z_c^\dagger = z_0^\dagger [(1 - e^{-\beta\omega\hbar})e^{2(\tau-\beta)/\epsilon} + e^{-\omega\hbar\tau}]$$

So

$$S_c = 2 |z_0|^2 e^{-\beta\omega\hbar/2} \sinh(\beta\omega\hbar/2)$$

and

$$\begin{aligned} Z_{sc}[\beta] &= e^{\beta\omega\hbar M/2} \int \frac{dz_0 dz_0^\dagger}{(2\pi i\hbar)^M} e^{-\frac{|z_0|^2}{\hbar} 2e^{-\beta\omega\hbar/2} \sinh(\beta\omega\hbar/2)} = \left[\frac{e^{\beta\omega\hbar/2}}{2e^{-\beta\omega\hbar/2} \sinh(\beta\omega\hbar/2)} \right]^M \\ &= e^{\beta\omega\hbar M} \cdot Z[\beta] \end{aligned}$$

which is the right answer, except for a factor $e^{\beta\omega\hbar M}$. In fact it has been said that a semiclassical approximation for an harmonic oscillator should give the right result [1]. Actually the confusion arises by the approximation noted in the last section, a change of $\omega\hbar z_k^\dagger z_{k+1}$ to $\omega\hbar z_k^\dagger z_k$ in the path integral (3.15) will give an extra factor $e^{-\beta\omega\hbar M}$ to the partition function. These two approximations cancel each other to give an exact result! It is obvious from my calculation that the correct statement would be:

$$\int D\tilde{z} D\tilde{z}^\dagger e^{-\frac{1}{\hbar} S(\tilde{z})} = e^{-\beta\omega\hbar M}$$

where $\tilde{z}(0) = \tilde{z}(\beta) = 0$.

3.4 Regularization

So far, we were able to write down an exact expression for the discretisation of a path integral, and evaluate the determinant exactly afterwards. In more complicated cases, it will be necessary to evaluate a path integral by using some approximations, because the determinant will no longer be exactly solvable. In fact, these approximations will usually generate some divergences that we will have to regularize by using various techniques. Here, I will indicate a general way to regularize these divergences and the use of the contour integral regularization for the coherent state path integral of this chapter. I will also use the exact determinant found earlier to find the limitations of these regularizations.

The continuous approximation is derived in the following way; since the path integral uses periodic functions, $z(0) = z(\beta)$, we can use a set of eigenfunctions $\Psi_k(\tau) = e^{2\pi i k \tau / \beta}$, as in the appendix A for the discrete case. For a general path integral:

$$\int Dz Dz^\dagger e^{-\frac{1}{\hbar} \int_0^\beta z^\dagger (a + b \frac{\partial}{\partial t} + c \frac{\partial^2}{\partial t^2}) z d\tau} = \frac{1}{\det^M (a + b \frac{\partial}{\partial t} + c \frac{\partial^2}{\partial t^2})} \quad (3.23)$$

we obtain for the determinant:

$$\det = \prod_{k=-N/2}^{N/2} (\beta/N) [a + b \frac{2\pi i k}{\beta} + c (\frac{2\pi i k}{\beta})^2] \quad (3.24)$$

where β/N appears because an integration of N slices will give a factor $\Delta\tau = \beta/N$ in front of the Lagrangian. It is also necessary for keeping the right units. The product (3.24) is the expression that needs to be regularized.

But before analysing these products, let me start from the exact formula (A.80) for the product derived in appendix A for the path integral (3.16):

$$Z[\beta] = \frac{e^{\omega\beta\hbar M/2}}{[\det]^M}, \quad \text{with} \quad \det(x) = \prod_{k=1}^N \lambda_k = \prod_{k=1}^N (e^{x/N} - e^{2\pi i k/N}) = e^x - 1 \quad (3.25)$$

where $x = \beta\omega\hbar$.

From (3.25), we can contemplate the fine tuning of the product of all the eigenvalues. On the complex plane these eigenvalues form a circle of radius 1, centered at $e^{x/N}$, equally spaced. Their modulus vary from $e^{x/N} - 1 \approx x/N \ll 1$ for $0 \approx k \ll N$, to $e^{x/N} + 1 \approx 2$ for $k \approx N/2$. So a huge number of cancellations must be involved, in this product, to give simply $e^x - 1$!

We will need also the $\Gamma(x)$ function defined by:

$$\Gamma(x) = \frac{d}{dx} \ln[\det(x)]$$

$$\begin{aligned}
&= \sum_{k=1}^N \frac{e^{x/N}}{N(e^{x/N} - e^{2\pi i k/N})} = \sum_{k=1}^N \frac{1}{N(1 - e^{\frac{2\pi i k - x}{N}})} \\
&= \frac{e^x}{e^x - 1}
\end{aligned} \tag{3.26}$$

that can be evaluated by contour integral with the function $\frac{1}{2} \cot(z/2)$, which has poles at $z = 2\pi n$ (n integer) with residue 1. This allows us to write:

$$\Gamma(x) = \sum_{k=1}^N \oint_{C(k)} \frac{\frac{1}{2} \cot(z/2)}{N(1 - e^{\frac{iz - x}{N}})} \frac{dz}{2\pi i}$$

where $C(k)$ is an oriented loop around $2\pi k$. The analysis of the integrand $f(z)$ shows that it contains others poles at $z = -ix + 2\pi nN$. Furthermore, $f(z)$ being a periodic function ($f(z + 2\pi N) = f(z)$), we can think of $f(z)$ as a function on a cylinder of circumference $2\pi N$ oriented along the imaginary axis. An integration along an imaginary line up and then down by another imaginary line shifted by $2\pi N$ will cancel each other. It can be checked that $f(z) \rightarrow 0$ as $\text{Im}(z) \rightarrow -\infty$ and $f(z) \rightarrow \frac{-i}{2N}$ as $\text{Im}(z) \rightarrow \infty$. So by using a contour integral, as shown in figure 3.2, we obtain

$$\begin{aligned}
\Gamma(x) &= - \int_0^{2\pi N} \frac{-i}{2N} \frac{d\text{Re}(z)}{2\pi i} - \oint_{-ix} \frac{\frac{1}{2} \cot(z/2)}{N(1 - e^{\frac{iz - x}{N}})} \frac{dz}{2\pi i} \\
&= \frac{1}{2} - \frac{i}{2} \cot(-ix/2) = \frac{e^x}{e^x - 1}
\end{aligned} \tag{3.27}$$

For a continuum limit approximation, we have to find some approximations to the product (3.25) that will contain all the physical properties of our model. But the fine tuning of this product shows that we can expect a lot of divergences. Thus, in these divergent cases, it is necessary to introduce a regularization that will throw away the divergent part, but keep the physics of the model, or in other words, the divergent term should not depend on any physical parameter.

For our problem, the eigenvalues λ_k become independent of x for large values of k ($\lambda_k \approx 2 + O(x/N)$) so the exact expression $e^{2\pi i k/N}$ for large k should not be necessary,

Figure 3.2: Contour of integration for the $\Gamma(x)$ function of the discrete determinant and then we should be able to use the approximation:

$$\exp\left\{\frac{2\pi i k}{N}\right\} \approx 1 + \left(\frac{2\pi i k}{N}\right) + \dots + \frac{1}{m!} \left(\frac{2\pi i k}{N}\right)^m$$

that I will call the m^{th} order approximation, and this should produce some meaningful results after regularization. In fact what is going on is the fact that the high frequency modes (large k) are not physically relevant in the path integral, and this enables us to keep only the continuous functions $z(\tau)$ in the path integral. The higher the approximation is, the better the 'discontinuous' (or fast oscillation) curves will be included properly in the path integral. If the discontinuous curves would contribute as much as the continuous ones (or even the classical solutions) then there would be no continuum limit at all!

If we use the 1st and 2nd order approximations to the product (3.25), and use an odd number of steps (time slices), $2N + 1$, so that we have a symmetric product, from

$k = -N$ to N , we obtain:

$$m = 1: \quad \det_1(x) = \prod_{k=-N}^N \left[\frac{x}{2N+1} - \frac{2\pi i k}{2N+1} \right] \quad (3.28)$$

$$m = 2: \quad \det_2(x) = \prod_{k=-N}^N \left[\frac{x}{2N+1} - \frac{2\pi i k}{2N+1} + \frac{2\pi^2 k^2}{(2N+1)^2} \right] \quad (3.29)$$

In fact, the approximations (3.28) and (3.29) correspond exactly to the path integral (3.19) and (3.21) respectively, with the approximation (3.23). So this approximation for the determinant has some justification now.

Let us first study (3.28). The idea is to evaluate the product (3.28) for a given N , and then divide the result by the same product, and same N , with $x = 0$. This divisor does not depend on x at all, so it will not change anything. And since the product becomes independent of x for large k , this divisor cancels exactly the divergent part that we want to throw away. More precisely:

$$\begin{aligned} \det_1(x) &= \prod_{k=-N}^N \left[\frac{x}{2N+1} - \frac{2\pi i k}{2N+1} \right] = \frac{x}{2N+1} \prod_{k=1}^N \left[\left(\frac{x}{2N+1} \right)^2 + \left(\frac{2\pi k}{2N+1} \right)^2 \right] \\ &= \frac{x}{2N+1} \prod_{k=1}^N \left[\frac{2\pi k}{2N+1} \right]^2 \prod_{k=1}^N \left[1 + \left(\frac{x}{2\pi k} \right)^2 \right] \end{aligned} \quad (3.30)$$

Since $\det_1(0) = 0$, we can not divide $\det_1(x)$ by $\det_1(0)$, but the relevant factor in (3.30) that needs to be divided is:

$$\frac{1}{2N+1} \prod_{k=1}^N \left(\frac{2\pi k}{2N+1} \right)^2 = \frac{(2\pi)^{2N}}{(2N+1)^{(2N+1)}} (N!)^2 \approx \left(\frac{\pi}{e} \right)^{2N+1}$$

and furthermore, see [13],

$$\lim_{N \rightarrow \infty} \prod_{k=1}^N \left[1 + \left(\frac{x}{2\pi k} \right)^2 \right] = \frac{\sinh(x/2)}{(x/2)}$$

So

$$\lim_{N \rightarrow \infty} \left(\frac{e}{\pi} \right)^{2N+1} \det_1(x) = \lim_{N \rightarrow \infty} \prod_{k=-N}^N \left[\frac{ex}{\pi(2N+1)} - \frac{2iek}{2N+1} \right] = 2 \sinh(x/2) \equiv DET_1(x) \quad (3.31)$$

Where $DET_1(x)$ is now the regularized determinant for the first order approximation, or for the path integral (3.19). We see that $\det(x) = e^{x/2} DET_1(x)$, so $DET_1(x)$ is almost right, except for the small factor $e^{x/2}$. Notice, again, the unusual coincidence of this factor appearing here and in front of the path integral (3.19). If we were to forget about this factor in (3.19), we would think that we have the exact partition function.

Since the divergence is a simple factor independent of x , the function (3.26), $\Gamma(x)$, for the approximation (3.28), should not contain any divergence at all!

$$\begin{aligned}
 \Gamma_1(x) &= \sum_{k=-N}^N \frac{1}{x - 2\pi i k} \\
 &= \sum_{k=-N}^N \oint_{C(k)} \frac{\frac{1}{2} \cot(z/2) dz}{x - iz} \frac{1}{2\pi i} \\
 &= - \oint_{z=-ix} \frac{\frac{1}{2} \cot(z/2) dz}{x - iz} \frac{1}{2\pi i} - \left(\sum_{k=-\infty}^{-N-1} + \sum_{k=N+1}^{\infty} \right) \frac{1}{x - 2\pi i k} \\
 &= \frac{1}{2} \frac{e^x + 1}{e^x - 1} - 2 \operatorname{Re} \left(\int_N^{\infty} \frac{du}{x - 2\pi i u} + O(1/N) \right)
 \end{aligned} \tag{3.32}$$

So

$$\lim_{N \rightarrow \infty} \Gamma_1(x) = \frac{1}{2} \frac{e^x + 1}{e^x - 1}$$

as given by $DET_1(x)$ in (3.31). Note also that the integrand of the contour integral is no longer periodic ($f(z + 2\pi N) \neq f(z)$), which indicates that we need the full Riemann sphere for the contour integral. The approximation, then, changes the topology of the domain of integration that is needed for the integration (this will be true for any order).

Now we can go further, and analyse the product (3.29). By using the same method that we used for (3.30), we can write:

$$\det_2(x) = -\frac{1}{2} \prod_{k=1}^N \left[\frac{2\pi^2 k}{(2N+1)^2} \right]^2 \prod_{k=1}^N \left[1 + \left(\frac{N}{2\pi k} \right)^2 \right] \prod_{k=-N}^N \left[1 + \frac{N(N+2x)}{(2\pi k - iN)^2} \right] \tag{3.33}$$

The product (3.33) has the same difficulty of (3.30) that $\det_2(x) \rightarrow 0$ for $x \rightarrow 0$. But even worse, the last two products do not converge to well known functions because terms

N	1000		5000			10000
x	0.1	0.5	1	2	5	10
$DET_2(x)$	0.1032846	0.5926451	1.434417	4.451911	59.72026	3615.019
$e^{-0.1807x} \det(x)$	0.1032875	0.5926791	1.434225	4.451257	59.72432	3615.389

Table 3.1: Second order product regularization

like $(\frac{N}{2\pi k})$ or $\frac{N(N+2x)}{2\pi k - iN}$ would not decrease to zero as k increase to N :

$$\prod_{k=1}^N [1 + (\frac{N}{2\pi k})^2] \neq \prod_{k=1}^{\infty} [1 + (\frac{N}{2\pi k})^2] = \frac{2 \sinh(N/2)}{N}$$

So the only way (by this method) to evaluate (3.29) is to apply the procedure directly without looking for simple solutions:

$$\begin{aligned} DET_2(x) &= \lim_{N \rightarrow \infty} x \prod_{k=-N, k \neq 0}^N \frac{[x - 2\pi i k + 2\pi^2 k^2 / (2N + 1)]}{[-2\pi i k + 2\pi^2 k^2 / (2N + 1)]} \\ &= \lim_{N \rightarrow \infty} x \prod_{k=1}^N \frac{[2\pi(2N + 1)k]^2 + [x(2N + 1) + 2\pi^2 k^2]^2}{[2\pi(2N + 1)k]^2 + [2\pi^2 k^2]^2} \end{aligned} \quad (3.34)$$

A numerical study of (3.34) indicates that $DET_2(x)$ is in between (3.31), $DET_1(x)$, and (3.25), the exact result. Since $DET_1(x) = e^{-x/2} \det(x)$, a simple exponential fit for $DET_2(x) = e^{-\gamma x} \det(x)$ shows a very accurate result for $\gamma = 0.1807 \pm 0.001$. My numerical study is summarised in the table 3.1, where $DET_2(x)$ is evaluated with the equation (3.34) for the indicated value of N .

A study of (3.33) with the use of $\Gamma(x)$ gives:

$$\begin{aligned} \Gamma_2(x) &= \sum_{k=-N}^N \frac{1}{x - 2\pi i k + 2\pi^2 k^2 / (2N + 1)} \\ &= - \left(\oint_{z \approx -ix} + \oint_{z \approx 4iN} \right) \frac{\frac{1}{2} \cot(z/2)}{x - iz + \frac{z^2}{2(2N+1)}} \frac{dz}{2\pi i} - \left(\sum_{k=-\infty}^{-N-1} + \sum_{k=N+1}^{\infty} \right) \frac{1}{x - 2\pi i k + \frac{2\pi^2 k^2}{2N+1}} \\ &= \frac{1}{2} \frac{e^x + 1}{e^x - 1} + \frac{1}{2} - 2 \operatorname{Re} \left(\int_N^{\infty} \frac{du}{x - 2\pi i u + \frac{2\pi^2 u^2}{2N+1}} + O(1/N) \right) \end{aligned}$$

then

$$\lim_{N \rightarrow \infty} \Gamma_2(x) = \frac{e^x}{e^x - 1} - \gamma$$

So $DET_2(x) = e^{-\gamma x} \det(x)$ as expected, and γ is given by:

$$\begin{aligned} \gamma &= \lim_{N \rightarrow \infty} 2Re \left(-\frac{1}{2\pi} \int_N^\infty \frac{du}{u(i - \frac{\pi u}{2N+1})} \right) \\ &= \lim_{N \rightarrow \infty} Re \left(-\frac{i}{\pi} \ln \left(\frac{-\pi}{2N+1} + \frac{i}{u} \right) \Big|_N^\infty \right) \\ &= \frac{1}{\pi} \arctan\left(\frac{2}{\pi}\right) \approx 0.18045 \end{aligned}$$

as found earlier!

If we were to check for higher order approximations, we would get better result; actually it is more likely that we would find a γ correction at each order, but with γ becoming smaller.

Notice that the product (3.29), with k running from $-\infty$ to ∞ instead, would give exactly the right answer (there would be no γ correction). It is very unlikely, however, that at any order of the approximation we would obtain an exact result (looking at (3.25)). This shows the importance of keeping k from $-N$ to N , for $2N+1$ steps, in the product.

There has been some study of these products by using a Riemann zeta function for regularizing some infinite products, such as $\prod_{k=-\infty}^\infty k$, but these formulas need the modification just noted, that k must run from $-\infty$ to ∞ right at the beginning [2,3]. This would generally lead to some errors, as already pointed out here (this has been found also by [2]). So I will not elaborate on the use of the Riemann Zeta function for the rest of my work. To find if it is applicable, we just look if the products contain the variable N , that could spoil its convergence. In these cases, a numerical study might gives some additional corrections to these path integrals.

Chapter 4

Coherent State Path Integral for Spin

4.1 Discretisation with Spin Coherent States

In the last chapter, we studied the properties of the harmonic oscillator coherent states using an harmonic oscillator Hamiltonian. We were able to obtain an exact path integral at the discrete level. In this chapter, we would like to study the spin coherent states. In hope of finding exact solution, it would be interesting to consider a simple Hamiltonian with a spin operator. More precisely, we will be interested in the partition function for a spin s particle in a constant magnetic field $\vec{B}$:

$$H = \mu \vec{B} \cdot \vec{J} \quad , \quad (\vec{J})^2 = \hbar^2 s(s+1)$$

$$Z[\beta] = \text{tr}(e^{-\beta H}) = \text{tr}(e^{-\beta(\mu \vec{B} \cdot \vec{J})})$$

With the help of the spin coherent states (2.9) and (2.11) we can write the following path integral:

$$Z[\beta] = \int \prod_{k=1}^N \frac{\sin(\theta_k) d\phi_k d\theta_k}{\left(\frac{4\pi}{2s+1}\right)} \langle \theta_k, \phi_k | e^{-\beta \mu \vec{B} \cdot \vec{J}/N} | \theta_{k+1}, \phi_{k+1} \rangle_s$$

By the properties of the spinorial representation, and the use of (2.10) we can transform it into:

$$Z[\beta] = \int \prod_{k=1}^N \frac{\sin(\theta_k) d\phi_k d\theta_k}{\left(\frac{4\pi}{2s+1}\right)} [\langle \theta_k, \phi_k | e^{-\frac{\beta \mu \hbar \vec{B} \cdot \vec{\sigma}}{2N}} | \theta_{k+1}, \phi_{k+1} \rangle_{1/2}]^{2s} \quad (4.35)$$

This is about as far as we can go at the discrete level. A continuum limit study of (4.35) will follow, but first let us try to analyse the same system by using the harmonic oscillator coherent states.

4.2 The Schwinger-Boson Model

It is known in quantum mechanics that a two dimensional harmonic oscillator (see section 2.1) state $|n_1, n_2\rangle$ can be represented instead by two numbers m, s such that $n_1 = s + m, n_2 = s - m$, so m run from $-s$ to s by step of 1, and $2s$ is a positive integer. This looks very much like a spin representation, and in fact the representation of a spin s particle by using the two dimensional harmonic oscillator is called the Schwinger-Boson model, and it is explained in details in Appendix B, where we will be using harmonic oscillator coherent states in this section. By using the Schwinger-Boson representation we can evaluate the same partition function of last section:

$$Z[\beta] = \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^2} \right) \int_0^{2\pi} \frac{d\lambda_k}{2\pi} \langle z_k | e^{-\frac{\beta\mu}{2N} \vec{B} \cdot \vec{a}^\dagger \vec{\sigma} \vec{a} + i\lambda_k (a^\dagger a / \hbar - 2s)} | z_{k+1} \rangle$$

with the help of (2.6) we obtain

$$Z[\beta] = \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^2} \right) \int_0^{2\pi} \frac{d\lambda_k}{2\pi} \exp \left\{ -\frac{1}{\hbar} \sum_{k=1}^N [2si\hbar \lambda_k + z_k^\dagger z_k - e^{i\lambda_k} z_k^\dagger e^{-\frac{\beta\mu\hbar}{2N} \vec{B} \cdot \vec{\sigma}} z_{k+1}] \right\} \quad (4.36)$$

The λ integration in (4.36) is in fact a circle in the complex plane. And it will be useful, sometimes, to represent this circle with a given radius r , instead of $r = 1$ in (4.36). The path integral (4.36) is then modified to:

$$Z[\beta] = \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^2} \right) \int_0^{2\pi} \frac{d\lambda_k}{2\pi} \exp \left\{ -\frac{1}{\hbar} \sum_{k=1}^N [2si\hbar \ln(re^{i\lambda_k}) + z_k^\dagger z_k - r e^{i\lambda_k} z_k^\dagger e^{-\frac{\beta\mu\hbar}{2N} \vec{B} \cdot \vec{\sigma}} z_{k+1}] \right\} \quad (4.37)$$

The λ variable is in fact a gauge potential, to see this we can verify that (4.36) and (4.37) are invariant under the transformation:

$$z_k \rightarrow e^{i\alpha_k} z_k, \quad \lambda_k \rightarrow \lambda_k + \alpha_k - \alpha_{k+1} \quad (4.38)$$

It is important to realize that we were able to obtain, at the discrete level, an exact path integral with a discrete action:

$$S = \sum_{k=1}^N [2si\hbar\lambda_k + z_k^\dagger z_k - e^{i\lambda_k} z_k^\dagger e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} z_{k+1}] \quad (4.39)$$

where $\epsilon = \beta/N$ as usual.

4.3 Equivalence of the two Representations

The first question that one may ask is: are the two last representations really equivalent? Since (4.35) does not contain any gauge variable, λ , the first thing will be to integrate directly this variable in (4.36). To do so, let us call $w_k = e^{i\lambda_k}$, then

$$\begin{aligned} Z[\beta] &= \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i\hbar)^2} \right) \oint \frac{dw_k}{2\pi i} \frac{1}{w_k^{2s+1}} e^{\frac{1}{\hbar} \sum_{k=1}^N [w_k z_k^\dagger e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} z_{k+1} - z_k^\dagger z_k]} \\ &= \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i\hbar)^2} \right) \frac{(z_k^\dagger e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} z_{k+1}/\hbar)^{2s}}{(2s)!} e^{-\frac{1}{\hbar} \sum_{k=1}^N z_k^\dagger z_k} \end{aligned} \quad (4.40)$$

Now we can represent z_k by:

$$z_k = r_k e^{i\gamma_k} \tilde{z}_k \sqrt{\hbar} \quad (4.41)$$

Where $\tilde{z}$ is restricted to $\tilde{z}^\dagger \tilde{z} = 1$, thus $z^\dagger z = r^2 \hbar$. The γ is just a phase factor, in fact the Hopf phase, coming from the passage of C^2 to $CP^1 = S^2$. The Hopf phase will play an important role later on. Also we can check that $dz^\dagger dz = \hbar^2 i^2 r^3 \sin(\theta) dr d\gamma d\phi d\theta$. Putting everything into (4.40) gives:

$$Z[\beta] = \int_0^\pi \int_0^{2\pi} \prod_{k=1}^N \frac{\sin(\theta_k) d\phi_k d\theta_k}{8\pi^2} \int_0^\infty \int_0^{2\pi} \frac{d\gamma_k (r_k^2)^{2s+1} d(r_k^2)}{(2s)!} (\tilde{z}_k^\dagger e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} \tilde{z}_{k+1})^{2s} e^{-\sum_{k=1}^N r_k^2} \quad (4.42)$$

Integrating the phase γ_k and the radius r_k of z_k , that separates completely in the path integral, and reintroducing a state $|z\rangle$ later, gives

$$Z[\beta] = \int_0^\pi \int_0^{2\pi} \prod_{k=1}^N \frac{\sin(\theta_k) d\phi_k d\theta_k}{\left(\frac{4\pi}{2s+1}\right)} (\tilde{z}_k^\dagger e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} \tilde{z}_{k+1})^{2s}$$

$$= \int_0^\pi \int_0^{2\pi} \prod_{k=1}^N \frac{\sin(\theta_k) d\phi_k d\theta_k}{\left(\frac{4\pi}{2s+1}\right)} \langle \tilde{z}_k | e^{-\frac{\epsilon\mu}{2} \tilde{B} \cdot \tilde{a}^\dagger \tilde{\sigma}_a} | \tilde{z}_{k+1} \rangle^{2s} \quad (4.43)$$

where $|\tilde{z}_k\rangle$ is the same as $|z\rangle$ but restricted to the condition $z^\dagger z = 1$. The equivalence of (4.43) and (4.35) is then exactly demonstrated. Furthermore, this indicates that the $|\theta, \phi\rangle$ spin coherent state can be represented simply by a two dimensional harmonic oscillator coherent state, $|z\rangle$, with the restriction $z^\dagger z = 1$.

An other way of verifying (4.37) directly, and then (4.35) by the equivalence proved above, can be done by using the determinant formulas (A.77) in appendix A, with the path integral (4.37):

$$Z[\beta] = \int \prod_{k=1}^N \frac{d\lambda_k}{2\pi} \frac{1}{(re^{i\lambda_k})^{2s}} \frac{1}{\det[M_{ij,mn}]}$$

where $M_{ij,mn} = \delta_{i,j} \delta_{m,n} - re^{i\lambda_j} (e^{-\frac{\epsilon\mu}{2} \tilde{B} \cdot \tilde{\sigma}})_{mn} \delta_{i,j+1}$, with $i, j = 1$ to N cyclically, and $m, n = 1, 2$. With the help of (A.82) and (A.83), we can show that

$$\det[M] = (1 - r^N e^{i \sum_{k=1}^N \lambda_k} e^{-\frac{\epsilon\mu}{2} |\tilde{B}|}) (1 - r^N e^{i \sum_{k=1}^N \lambda_k} e^{\frac{\epsilon\mu}{2} |\tilde{B}|})$$

So by using $w_k = re^{i\lambda_k}$, we find

$$Z[\beta] = \oint \prod_{k=1}^N \frac{dw_k}{2\pi i} \frac{1}{w_k^{2s+1}} \frac{1}{(1 - e^{-\frac{\epsilon\mu}{2} |\tilde{B}|} \prod_{j=1}^N w_j) (1 - e^{\frac{\epsilon\mu}{2} |\tilde{B}|} \prod_{j=1}^N w_j)} \quad (4.44)$$

where the contour integral is a circle of radius r around the origin. Actually we can see that the determinant will create a pole in the integration when $r = e^{-\frac{\epsilon\mu |\tilde{B}|}{2}}$ (and also at $r = e^{\frac{\epsilon\mu |\tilde{B}|}{2}}$). This indicates that the Gaussian integral is no longer convergent over this value of r ! There is no surprise since the λ integration has been introduced to reduce the number of states from an harmonic oscillator (∞) to a spin s particle ($2s+1$), and we perform this λ integration after the z integration in (4.44) (which correspond to a trace). So it just happens that for $r < e^{-\frac{\epsilon\mu |\tilde{B}|}{2}}$, the integration converges, we can not hope for more. This explains why the path integral (4.36) would not give a correct answer if we integrate z first. Thus if we come back to our integral (4.44) and integrate w_1 , we

encounter two poles at:

$$w_1 = e^{\pm \frac{\beta\mu|\vec{B}|}{2}} (w_2 \dots w_N)^{-1}$$

and the integrand vanishes for $|w_1| \rightarrow \infty$ fast enough that we can deform the contour of integration and then use the residue theory. Which gives

$$\begin{aligned} Z[\beta] &= \oint \prod_{k=2}^N \frac{dw_k}{2\pi i w_k} \left[\frac{e^{-\beta\mu|\vec{B}|s}}{(1 - e^{\beta\mu|\vec{B}|})} + \frac{e^{\beta\mu|\vec{B}|s}}{(1 - e^{-\beta\mu|\vec{B}|})} \right] \\ &= \frac{\sinh(\frac{\beta\mu|\vec{B}|}{2}(2s+1))}{\sinh(\frac{\beta\mu|\vec{B}|}{2})} \\ &= \text{tr}(e^{-\beta\mu\vec{B} \cdot \vec{J}}) = \sum_{m=-s}^s e^{-\beta\mu|\vec{B}|m} \end{aligned}$$

as it should.

4.4 Continuum Limit of the Spin Coherent States

The path integral has been already calculated at the discrete level and appears at the equation (4.35). Now we have to work out a continuum limit approximation, which means that we have to find a continuum approximation to the expression:

$$\langle \theta_k, \phi_k | e^{-\frac{\epsilon\mu\hbar}{2} \vec{B} \cdot \vec{\sigma}} | \theta_{k+1}, \phi_{k+1} \rangle_{1/2} \quad (4.45)$$

In the continuum approximation, we can joint k to $k+1$, continuously, by using a Taylor series expansion of $| \theta_{k+1}, \phi_{k+1} \rangle$ around $| \theta_k, \phi_k \rangle$, using the first patch in (2.8), where, for simplification, I will remove the k index on all the variables:

$$\begin{aligned} | \theta_{k+1}, \phi_{k+1} \rangle &= | \theta, \phi \rangle + \frac{\epsilon}{2} \begin{pmatrix} -\sin(\theta/2)\dot{\theta} \\ (\cos(\theta/2)\dot{\theta} + 2i\sin(\theta/2)\dot{\phi})e^{i\phi} \end{pmatrix} \\ &\quad - \frac{\epsilon^2}{8} \begin{pmatrix} \cos(\theta/2)\dot{\theta}^2 + 2\sin(\theta/2)\ddot{\theta} \\ (\sin(\theta/2)\dot{\theta}^2 - 4i\cos(\theta/2)\dot{\theta}\dot{\phi} + 4\sin(\theta/2)\dot{\phi}^2 - 2\cos(\theta/2)\ddot{\theta} - 4i\sin(\theta/2)\ddot{\phi})e^{i\phi} \end{pmatrix} + \dots \end{aligned}$$

And also use the expansion:

$$e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} = 1 - \frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma} + \frac{\epsilon^2\mu^2\hbar^2}{8}|\vec{B}|^2 + \dots$$

So that (4.45) becomes, at the second order in ϵ :

$$\begin{aligned} & 1 - \frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{n} + \frac{\epsilon^2}{8}\mu^2\hbar^2|\vec{B}|^2 + \epsilon i \sin^2(\theta/2)\dot{\phi} - \frac{\epsilon^2}{8}(\dot{\theta}^2 - 2i \sin(\theta)\dot{\theta}\dot{\phi} + 4 \sin^2(\theta/2)\dot{\phi}^2 - 4i \sin^2(\theta/2)\ddot{\phi}) \\ & - \frac{\epsilon^2\mu\hbar}{4}[B_x(\cos(\theta)\cos(\phi)\dot{\theta} + i \sin(\phi)\dot{\theta} + i \sin(\theta)e^{i\phi}\dot{\phi}) \\ & + B_y(\cos(\theta)\sin(\phi)\dot{\theta} - i \cos(\phi)\dot{\theta} + \sin(\theta)e^{i\phi}\dot{\phi}) + B_z(-\sin(\theta)\dot{\theta} - 2i \sin^2(\theta/2)\dot{\phi})] \\ & = \exp \left\{ -\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{n} + \frac{\epsilon^2\mu^2\hbar^2}{8}(|\vec{B}|^2 - (\vec{B}\cdot\vec{n})^2) + \frac{i\epsilon}{2}(1 - \cos(\theta))\dot{\phi} - \frac{\epsilon^2}{8}(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2 \right. \\ & - 4i\frac{d}{d\tau}(\sin^2(\theta/2)\dot{\phi})) - \frac{\epsilon^2\mu\hbar}{4}[B_x(i \sin(\phi)\dot{\theta} + i \sin(\theta)\cos(\theta)\cos(\phi)\dot{\phi} + \frac{d}{d\tau}(\sin(\theta)\cos(\phi))) \\ & + B_y(-i \cos(\phi)\dot{\theta} + i \sin(\theta)\cos(\theta)\sin(\phi)\dot{\phi} + \frac{d}{d\tau}(\sin(\theta)\sin(\phi))) \\ & \left. + B_z(-i \sin^2(\theta)\dot{\phi} + \frac{d}{d\tau}(\cos(\theta))) \right\} \\ & = \exp \left\{ -\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{n} + \frac{\epsilon^2\mu^2\hbar^2}{8}(\vec{B}\times\vec{n})^2 - \frac{i\epsilon}{2}(\mp 1 + \cos(\theta))\dot{\phi} - \frac{\epsilon^2}{8}(\dot{n})^2 + \frac{i\epsilon^2\mu\hbar}{4}\vec{B}\cdot(\vec{n}\times\dot{\vec{n}}) \right\} \\ & \cdot \exp \left\{ \frac{d}{d\tau} \left(-\frac{\epsilon^2\mu\hbar}{4}\vec{B}\cdot\vec{n} + \frac{i\epsilon^2}{4}(\pm 1 - \cos(\theta))\dot{\phi} \right) \right\}_{\text{at the point } k} \end{aligned} \quad (4.46)$$

where the ± 1 has been introduced for taking into account the choice of the two patches in (2.8). (to get to the second patch, we multiply by $e^{i(\phi_k - \phi_{k+1})} = e^{-i\epsilon\dot{\phi} - \frac{i\epsilon^2}{2}\ddot{\phi} + \dots}$).

As it has been explained in the last chapter, we have to correct for the boundary conditions, when we are passing from N to $N + 1 \equiv 1$. Which means that we have to add the following term to the action of the path integral:

$$\begin{aligned} & \frac{\epsilon}{2} \left[\langle \theta, \phi | \frac{d}{d\tau} | \theta, \phi \rangle \Big|_0 - \langle \theta, \phi | \frac{d}{d\tau} | \theta, \phi \rangle \Big|_\beta \right] \\ & = -\frac{i\epsilon}{4} (\pm 1 - \cos(\theta)\dot{\phi}) \Big|_0^\beta = -\int_0^\beta \left[\frac{d}{d\tau} \left(\frac{i\epsilon}{4} (\pm 1 - \cos(\theta)) \right) \right] d\tau \end{aligned}$$

Multiplying (4.46), for $k = 1$ to N , and using the integral approximation gives a continuum limit approximation, where it could be noted that the boundary term, above, cancels the last term in the second bracket in (4.46), while the first term vanishes by the boundary condition, $\vec{n}(0) = \vec{n}(\beta)$. We finally obtains the continuum limit approximation of the spin coherent state path integral:

$$Z[\beta] = \int D\vec{n} \delta((\vec{n})^2 - 1) \exp \left\{ - \int_0^\beta [is(\mp 1 + \cos(\theta))\dot{\phi} + \mu\hbar s \vec{B} \cdot \vec{n} + \frac{\epsilon s}{4}(\dot{\vec{n}})^2 - \frac{\epsilon\mu^2\hbar^2 s}{4}(\vec{B} \times \vec{n})^2 - \frac{i\epsilon\mu\hbar s}{2}\vec{B} \cdot (\vec{n} \times \dot{\vec{n}})] d\tau \right\} \quad (4.47)$$

As in (3.21), the ϵ terms can be thought of as a regulator for the path integral. And in fact, for the two remaining terms, one is $\vec{B} \cdot \vec{n}$ which can be either positive or negative, and the other one is purely imaginary. Then without these ϵ terms there really would not be any convergent terms for the path integral at all, which would make the continuum approximation meaningless.

The term $\int_0^\beta (1 + \cos(\theta))\dot{\phi} d\tau$ in (4.47) actually represent the area on the sphere enclosed by the vector $\vec{n}$ in its closed loop motion, in the South pole side. While $\int_0^\beta (-1 + \cos(\theta))\dot{\phi} d\tau$ represent minus the area seen from the North pole. These two terms always differ by a multiple of 4π , leaving the path integral single valued (since $2s$ is an integer). It is very interesting to notice that this term is purely topological. Its relation with the Hopf phase will be made clear at the next section.

4.5 Continuum Limit of the Schwinger-Boson Model

In a first approach, I will start with the action (4.39) of the path integral (4.36). Then, try to find a covariant way to rewrite this action, in the same spirit of section 3.2, that will be suitable for a continuum limit approximation. Here, of course, we might expect some difficulties coming from the gauge variable λ , that has to be integrated out to really

get a spin s particle.

The first thing to do is to find a covariant derivative of z , that will transform in the same way z transform upon the gauge transformation (4.38). If λ were zero, this derivative should become a normal derivative. Like in the last section, we just have to connect z_k to z_{k+1} continuously, but by using a covariant derivative to take into account the gauge term $e^{i\lambda_k}$:

$$e^{i\lambda_k} z_{k+1} = e^{\epsilon D} z_k = z_k + \epsilon D z_k + \frac{\epsilon^2}{2} D^2 z_k + \dots \quad (4.48)$$

So now the gauge transformation is:

$$z_k \rightarrow e^{i\alpha_k} z_k, \quad \lambda_k \rightarrow \lambda_k + \alpha_k - \alpha_{k+1}, \quad D z_k \rightarrow e^{i\alpha_k} D z_k \quad (4.49)$$

A second important point is to consider a non-trivial Hopf phase in z . Usually, we should have $z(0) = z(\beta)$, but let us consider instead the boundary condition $z(0) = e^{i\gamma} z(\beta)$, so that the norm of z and also its representation on the sphere (θ, ϕ) still agrees at 0 and β . This phase, γ , does not change anything physically, and since a phase in z is locally unphysical, the γ phase might, and will, represent only a topological phase. To achieve this transformation, let us consider the same gauge transformation (4.49) but considering this time α_1 and α_{N+1} as being completely independent. Furthermore, in the case of a continuous gauge transformation we really have to consider $\alpha(0)$ as different from $\alpha(\beta)$, in general. This phase is absorbed by the λ_k gauge transformation (4.49). The only problem, is that usually

$$\sum_{k=1}^N \lambda_k \rightarrow \sum_{k=1}^N \lambda_k$$

Since now $\alpha_1 \neq \alpha_{N+1}$, this is no longer true, and we have to correct this problem by implementing the local gauge transformation by a global gauge transformation

$$\sum_{k=1}^N \lambda_k \rightarrow \sum_{k=1}^N \lambda_k + \sum_{k=1}^N (\alpha_{k+1} - \alpha_k) \quad (4.50)$$

in the path integral (4.36).

Let us write the action (4.39) with the help of the covariant derivative (4.48), to second order in ϵ :

$$\begin{aligned}
S &= \sum_{k=1}^N [2si\hbar\lambda_k + z_k^\dagger z_k - z_k^\dagger e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} e^{\epsilon D} z_k] + z_N^\dagger e^{-\frac{\epsilon\mu\hbar}{2}\vec{B}\cdot\vec{\sigma}} (e^{i\lambda_{N+1}} z_{N+1} - e^{i\lambda_1} z_1) \\
&= \sum_{k=1}^N [2si\hbar\lambda_k + \frac{\epsilon\mu\hbar}{2}\vec{B}\cdot z_k^\dagger \vec{\sigma} z_k - \epsilon z_k^\dagger D z_k - \frac{\epsilon^2\mu^2\hbar^2}{8}|\vec{B}|^2 z_k^\dagger z_k \\
&\quad - \frac{\epsilon^2}{2} z_k^\dagger D^2 z_k + \frac{\epsilon^2\mu\hbar}{2}\vec{B}\cdot z_k^\dagger \vec{\sigma} D z_k] + \frac{\epsilon}{2} z^\dagger D z|_0^\beta \\
&= \int_0^\beta [\frac{2si\hbar\lambda}{\epsilon} + \frac{\mu\hbar}{2}\vec{B}\cdot z^\dagger \vec{\sigma} z - z^\dagger D z \\
&\quad - \frac{\epsilon\mu^2\hbar^2}{8}|\vec{B}|^2 z^\dagger z + \frac{\epsilon}{2} D z^\dagger D z + \frac{\epsilon\mu\hbar}{2}\vec{B}\cdot z^\dagger \vec{\sigma} D z] d\tau
\end{aligned} \tag{4.51}$$

By expanding $e^{\epsilon D}$ in power of D , this is like expanding in power of λ , because D depends on λ by its definition (4.48). So, at this stage, we can reintroduce λ in (4.51) by its dependence in D , and then integrate it out. We can rewrite (4.48) as

$$e^{i\lambda_k} e^{\epsilon \frac{\partial}{\partial \tau}}(z_k) = e^{\epsilon D}(z_k) \tag{4.52}$$

By expanding on both side we find for D , in first order in ϵ :

$$D = \frac{\partial}{\partial \tau} + i\frac{\lambda}{\epsilon} \tag{4.53}$$

This is an approximation for λ , thus we can not expect an exact result for the remaining calculation of the path integral, but this should give a good approximation. Putting (4.53) back into (4.51) gives:

$$\begin{aligned}
S &= \int_0^\beta [\frac{2si\hbar\lambda}{\epsilon} + \frac{\mu\hbar}{2}\vec{B}\cdot z^\dagger \vec{\sigma} z - z^\dagger \dot{z} - \frac{i\lambda}{\epsilon} z^\dagger z - \frac{\epsilon\mu^2\hbar^2}{8}|\vec{B}|^2 z^\dagger z \\
&\quad + \frac{\epsilon}{2}|\dot{z}|^2 - \frac{i\lambda}{2}(\dot{z}^\dagger z - z^\dagger \dot{z}) + \frac{1}{2\epsilon}\lambda^2 z^\dagger z + \frac{\epsilon\mu\hbar}{2}\vec{B}\cdot z^\dagger \vec{\sigma} \dot{z} + \frac{i\mu\hbar}{2}\lambda \vec{B}\cdot z^\dagger \vec{\sigma} z] d\tau
\end{aligned}$$

Then, at this point, we can use the Hopf phase to set

$$z^\dagger \dot{z} - \dot{z}^\dagger z = 0 \quad (4.54)$$

Note that (4.54) correspond only to one constraint because its complex conjugate gives the same constraint (unlike a constraint like $z^\dagger \dot{z} = 0$). Then we have to add the new phase (4.50), $\int_0^\beta \dot{\alpha} d\tau$, to the action. The value of $\dot{\alpha}$ will be evaluated later. This leave us with the action

$$S = \int_0^\beta L d\tau$$

where

$$\begin{aligned} L &= \frac{i\lambda}{\epsilon} (2s\hbar - z^\dagger z + \frac{\epsilon\mu\hbar}{2} \vec{B} \cdot z^\dagger \vec{\sigma} z) + \frac{\lambda^2}{2\epsilon} z^\dagger z \\ &+ 2si\hbar\dot{\alpha} + \frac{\mu\hbar}{2} \vec{B} \cdot z^\dagger \vec{\sigma} z - \frac{\epsilon\mu^2\hbar^2}{8} |\vec{B}|^2 z^\dagger z + \frac{\epsilon}{2} |\dot{z}|^2 + \frac{\epsilon\mu\hbar}{2} \vec{B} \cdot z^\dagger \vec{\sigma} \dot{z} \\ &= L_\lambda + L_z + L_s \end{aligned}$$

where we will consider three parts for the Lagrangian, namely

$$\begin{aligned} L_\lambda &= \frac{z^\dagger z}{2\epsilon} \left[\lambda + \frac{i}{z^\dagger z} (2s\hbar - z^\dagger z + \frac{\epsilon\mu\hbar}{2} \vec{B} \cdot z^\dagger \vec{\sigma} z) \right]^2 \\ L_z &= \frac{(z^\dagger z - 2s\hbar)^2}{2\epsilon z^\dagger z} \\ L_s &= 2si\hbar\dot{\alpha} + s\mu\vec{B} \cdot \frac{z^\dagger \vec{\sigma} z}{z^\dagger z} - \frac{\epsilon\mu^2\hbar^2}{8z^\dagger z} (|\vec{B}|^2 (z^\dagger z)^2 - (\vec{B} \cdot z^\dagger \vec{\sigma} z)^2) + \frac{\epsilon}{2} |\dot{z}|^2 + \frac{\epsilon\mu\hbar}{2} \vec{B} \cdot z^\dagger \vec{\sigma} \dot{z} \end{aligned}$$

The L_z contribution to the path integral is of the form

$$e^{-\frac{1}{\hbar} L_z \Delta\tau} = e^{-\frac{(z^\dagger z - 2s\hbar)^2}{2\hbar z^\dagger z} \left(\frac{\Delta\tau}{\epsilon}\right)} \quad (4.55)$$

which indicates that we have $z^\dagger z = 2s\hbar + O(\hbar\sqrt{\epsilon/\Delta\tau})$. Then, if we look at a scale $\Delta\tau \gg \epsilon$, we find that $z^\dagger z$ is very well peaked at $2s\hbar$. However, if we set $\Delta\tau = \epsilon$, as we use to do for the discretisation, we find $z^\dagger z = \hbar(2s + O(1))$, so only the classical limit, $s \rightarrow \infty$, is well peaked. For low values of s , instead, the norm of z is less well defined.

In fact, this is not such a surprise, since if we take the $z^\dagger z = r^2 = R$ dependence of the path integral (4.42), after λ has been integrated out,

$$\frac{R^{2s}}{(2s)!} e^{-R} = f(R)$$

and find an approximation around its maximum:

$$f'(R) = 0 \Rightarrow R_0 = 2s, \quad f(R_0) = \frac{(2s)^{2s}}{(2s)!} e^{-2s} \approx \frac{1}{\sqrt{4\pi s}}, \quad f''(R_0) = -\frac{f(R_0)}{R_0}$$

so

$$\begin{aligned} f(R) &\approx f(R_0) + \frac{1}{2}(R - R_0)^2 \left(-\frac{f(R_0)}{R_0}\right) \\ &\approx f(R_0) e^{-\frac{(R-R_0)^2}{2R_0}} \approx \frac{1}{\sqrt{4\pi s}} e^{-\frac{(R-R_0)^2}{2R}} \end{aligned} \quad (4.56)$$

which is just equation (4.55) for $R = z^\dagger z / \hbar$, up to a constant $\frac{1}{\sqrt{4\pi s}}$. In the path integral (4.42) we integrated R directly since it was completely decoupled to the rest of the path integral. Here we notice that the term in ϵ^0 in L_s does not depend on $z^\dagger z$ at all, thanks to the λ integration that brought corrections in this respect. The remaining terms are of the order ϵ , so they do not really influence the path integral, they only regularize it. Furthermore, a more elaborate expansion up to order ϵ^2 would also correct these terms, as it did for the first order, since we know from (4.42) that the $z^\dagger z = r^2$ variable completely decouples in the path integral. So it would make sense to simply set $z^\dagger z = 2s\hbar$ for the remaining of the calculation.

Then for L_λ , we find its contribution to the path integral, after considering the new constraint above, to be a simple Gaussian:

$$\approx \int_{-\pi}^{\pi} \prod_{k=1}^N \frac{d\lambda_k}{2\pi} e^{-\sum_{k=1}^N s\lambda_k^2} \approx \int_{-\infty}^{\infty} \prod_{k=1}^N \frac{d\lambda_k}{2\pi} e^{-\sum_{k=1}^N s\lambda_k^2} = \prod_{k=1}^N \frac{1}{\sqrt{4\pi s}}$$

which is just the missing factor in (4.55) that appears in (4.56)!

Then we finally obtain the path integral, after rescaling z to $z^\dagger z = \hbar$,

$$Z[\beta] = \int Dz^\dagger Dz \delta(z^\dagger z / \hbar - 1) \exp \left\{ -\frac{1}{\hbar} \int_0^\beta [2si\hbar \dot{\alpha} + s\mu\hbar \vec{B} \cdot z^\dagger \vec{\sigma} z] \right.$$

$$-\frac{\epsilon s \mu^2 \hbar^2}{4} (|\vec{B}|^2 (z^\dagger z)^2 - (\vec{B} \cdot z^\dagger \vec{\sigma} z)^2) + \epsilon s |\dot{z}|^2 + \epsilon s \mu \hbar \vec{B} \cdot z^\dagger \vec{\sigma} \dot{z} d\tau \Big\} \quad (4.57)$$

and where $z(\beta) = z(0)e^{i \int_0^\beta \dot{\alpha} d\tau}$, which is determined by the constraint (4.54).

Before continuing with (4.57), let me rederive it in an other way, which is less physical and does not represent the significance of λ , but gives what we want directly.

In the section 4.3 we derived the equivalence of the spin coherent state and harmonic oscillator coherent state path integral, by integrating the λ variable, the Hopf phase and the norm r of z in the later path integral. The result appears at the equation (4.43) where the z variable is restricted to $z^\dagger z = \hbar$. Then we can reintroduce the full z variable by insertion (by brute force!) of a delta function, $\delta(z^\dagger z/\hbar - 1)$, in the path integral and rewrite the effective Lagrangian with an unrestricted z variable:

$$Z[\beta] = \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^2} \right) (2s+1) \delta(z_k^\dagger z_k/\hbar - 1) (z_k^\dagger e^{-\frac{\epsilon \mu \hbar}{2} \vec{B} \cdot \vec{\sigma}} z_{k+1}/\hbar)^{2s}$$

Then we follow the same procedure, we set $z_{k+1} = e^{\epsilon \frac{\vec{\sigma}}{\hbar} z_k}$ and find, up to second order in ϵ :

$$z_k^\dagger e^{-\frac{\epsilon \mu \hbar}{2} \vec{B} \cdot \vec{\sigma}} z_{k+1} = \hbar - \frac{\epsilon \mu \hbar}{2} \vec{B} \cdot \vec{\sigma} z_k + \epsilon z_k^\dagger \dot{z}_k + \frac{\epsilon^2 \mu^2 \hbar^2}{8} |\vec{B}|^2 z_k^\dagger z_k - \frac{\epsilon^2 \mu \hbar}{2} \vec{B} \cdot z_k^\dagger \vec{\sigma} \dot{z}_k + \frac{\epsilon^2}{2} z_k^\dagger \ddot{z}_k$$

where I assumed $z^\dagger z = \hbar$ for the first term. By taking the logarithm of this expression, to put everything in the exponential, and including the boundary conditions, we obtain the following path integral:

$$\begin{aligned} Z[\beta] = \int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^2} \right) (2s+1) \delta(z_k^\dagger z_k/\hbar - 1) \exp \Big\{ & -\frac{1}{\hbar} \sum_{k=1}^N \epsilon [s \mu \hbar \vec{B} \cdot z_k^\dagger \vec{\sigma} z_k - 2s z_k^\dagger \dot{z}_k \\ & - \frac{\epsilon s \mu^2 \hbar^2}{4} (|\vec{B}|^2 \hbar - (\vec{B} \cdot z_k^\dagger \vec{\sigma} z_k)^2) - \epsilon s z_k^\dagger \ddot{z}_k + \epsilon s \mu \hbar \vec{B} \cdot z_k^\dagger \vec{\sigma} \dot{z}_k \\ & - 2s \epsilon z_k^\dagger \dot{z}_k (-\mu \hbar \vec{B} \cdot z_k^\dagger \vec{\sigma} z_k + z_k^\dagger \dot{z}_k)] d\tau \Big\} \cdot e^{-\frac{1}{\hbar} \int_0^\beta [\epsilon s (z^\dagger \dot{z} + |\dot{z}|^2)] d\tau} \end{aligned}$$

And if we use the Hopf phase to set $(z^\dagger \dot{z} - \dot{z}^\dagger z) = 0$, which implies $z^\dagger \dot{z} = 0$ with $z^\dagger z = \hbar$, we find that the above path integral becomes completely identical with equation (4.57)!

Now, let me complete the derivation of the equation above. For the z variable we can use the same representation as the spin coherent states (2.8), by including also the Hopf phase:

$$\begin{aligned} z &= \sqrt{\hbar} e^{i\alpha} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix} \quad \text{for } \theta \neq \pi \\ &= \sqrt{\hbar} e^{i\alpha} \begin{pmatrix} \cos(\theta/2) e^{-i\phi} \\ \sin(\theta/2) \end{pmatrix} \quad \text{for } \theta \neq 0 \end{aligned}$$

Then

$$(z^\dagger \dot{z} - \dot{z}^\dagger z) = \hbar i \left[\dot{\alpha} + \frac{1}{2} (\mp 1 - \cos(\theta)) \dot{\phi} \right] = 0$$

which gives the Hopf phase contribution to the path integral

$$\dot{\alpha} = \frac{1}{2} (\pm 1 + \cos(\theta)) \dot{\phi} \quad (4.58)$$

Furthermore, in this gauge, we can checked that

$$\vec{n} = z^\dagger \vec{\sigma} z / \hbar, \quad (\vec{n})^2 = 1, \quad (z^\dagger \vec{\sigma} \dot{z} - \dot{z}^\dagger \vec{\sigma} z) / \hbar = -i \vec{n} \times \dot{\vec{n}}, \quad \dot{z}^\dagger \dot{z} = (\dot{\vec{n}})^2 / 4 \quad (4.59)$$

We can use the fact that $\vec{n}(0) = \vec{n}(\beta)$ and $z^\dagger \dot{z} = 0$ to integrate by parts some terms in (4.57). After that we can use (4.59) to write the path integral as:

$$\begin{aligned} Z[\beta] &= \int D\vec{n} \delta((\vec{n})^2 - 1) \exp \left\{ - \int_0^\beta [i s (\mp 1 + \cos(\theta)) \dot{\phi} + \mu \hbar s \vec{B} \cdot \vec{n} \right. \\ &\quad \left. + \frac{\epsilon s}{4} (\dot{\vec{n}})^2 - \frac{\epsilon \mu^2 \hbar^2 s}{4} (\vec{B} \times \vec{n})^2 - \frac{i \epsilon \mu \hbar s}{2} \vec{B} \cdot (\vec{n} \times \dot{\vec{n}})] d\tau \right\} \end{aligned} \quad (4.60)$$

which is exactly the same as the spin coherent state path integral continuum limit, that appears at the equation (4.47).

Chapter 5

Path Integral for a Charged Particle in a Magnetic Monopole Field

5.1 Monopole Vector Potential

In this chapter, I will demonstrate that the physics of a charged particle in the field of a magnetic monopole is related to the spin system that we are studying.

A magnetic monopole field is, like a point charge electric field, given by the equation:

$$\vec{B}_m = \frac{g\vec{r}}{r^3} \quad (5.61)$$

where g is the magnetic charge. Since we will be interested in the gauge field (like λ of last section) we want to represent this field by a vector potential, $\vec{A}_m$, such that

$$\text{rot}(\vec{A}_m) = \vec{B}_m \quad (5.62)$$

Mathematically, we find that the 2-form $\mathbf{F}$ ($F_{ij} = \epsilon_{ijk}B_k$) is closed, $d\mathbf{F} = 0$, and if (5.62) is defined throughout all the space, then $\mathbf{F}$ is exact, $\mathbf{F} = d\mathbf{A}$. So the second cohomology group of the space will indicate if there is some $\vec{B}$ field that has no solution for $\vec{A}$ (closed 2-form that are not exact). The space here is R^3 , but from (5.61) we notice that there is a singularity at $r = 0$. So we have to remove this point to get a well defined vector potential. Which leaves us with $R^3 - \{0\} = S^2 \times [0, \infty]$. It is known that:

$$H^2(S^2 \times [0, \infty]) = H^2(S^2) = R$$

which means that there is some field $\vec{B}$ that will have no solution for (5.62), valid everywhere. The class to which $\vec{B}_m$ belongs (in R), in the cohomology group, is given by the

integration

$$\int_S \mathbf{F} = \int_S \vec{B}_m \cdot d\vec{S} = 4\pi g$$

This is actually the value (up to a constant) of the magnetic charge! Then, for non-zero magnetic charge, there is no solution for the vector potential valid everywhere on the space ($s^2 \times [0, \infty]$). This is known as the Dirac string (since for any solution there is a divergence along a string starting at the origin and going to infinity!).

This is not the end of the story, since all our study has been done classically. In quantum mechanics, the vector potential enters the theory as a gauge field:

$$\vec{D}\psi = (\vec{\nabla} - \frac{ie}{\hbar} \vec{A})\Psi, \quad \Psi \rightarrow e^{\frac{ie}{\hbar}\chi}\Psi, \quad \vec{A} \rightarrow \vec{A} + \vec{\nabla}\chi \Rightarrow \vec{D}\Psi \rightarrow e^{\frac{ie}{\hbar}\chi}\vec{D}\Psi \quad (5.63)$$

So the gauge field $\vec{A}$ can be changed by some gauge transformation (5.63) by a quantity $\vec{\nabla}\chi$ for an arbitrary field χ (or $\mathbf{A} \rightarrow \mathbf{A} + d\chi$). This does not seem to change the equation (5.62), since $\text{rot}(\vec{\nabla}\chi) = 0$ (or $d(d\chi) = 0$). In quantum mechanics, however, the χ appears in the wave function in the exponential $e^{\frac{ie}{\hbar}\chi}$. So χ does not have to be a function, but simply a section of the vector bundle (the wave function on the space). In other words, χ is defined only modulo $\frac{2\pi\hbar}{e}$.

Let us solve (5.62) in two patches, using polar coordinates:

$$\vec{A}_m \cdot d\vec{x} = g(1 - \cos(\theta))d\phi \quad \text{for } \theta \neq \pi$$

$$\vec{A}'_m \cdot d\vec{x} = -g(1 + \cos(\theta))d\phi \quad \text{for } \theta \neq 0$$

These two vector potentials differ only by:

$$(\vec{A}_m - \vec{A}'_m) \cdot d\vec{x} = 2gd\phi = d(2g\phi)$$

or

$$\vec{A}_m = \vec{A}'_m + \vec{\nabla}(2g\phi)$$

So, as far as quantum mechanics is concerned, the $\vec{A}_m$ and $\vec{A}'_m$ gauge fields are the same, as long as $2g\phi$ is single valued everywhere modulo $\frac{2\pi\hbar}{e}$, which means:

$$\frac{eg}{\hbar} = \pm s \quad \text{for } s = 0, 1/2, 1, 3/2, \dots \quad (5.64)$$

must be fulfilled, this is the Dirac quantisation condition [4].

For simplicity I will consider $\frac{eg}{\hbar} = s$ (by choosing the appropriate sign of e). Now if we consider a charge, e , moving in this magnetic field, the Hamiltonian is simply given by:

$$H_0 = \frac{m}{2}(\vec{v})^2 = \frac{1}{2m}(\vec{p} - e\vec{A}_m)^2 = -\frac{\hbar^2}{2m}(\vec{D})^2 \quad (5.65)$$

What would be interesting now is to express H_0 in terms of the angular momentum, $\vec{L}$, of the field.

5.2 Monopole Angular Momentum

We should be able to find $\vec{L}$ in term of $\vec{D}$ and $\vec{r}$. We have the commutation relations:

$$[r_i, r_j] = 0 \quad , \quad [D_i, r_j] = \delta_{ij} \quad , \quad [D_i, D_j] = -is\epsilon_{ijk}\frac{r_k}{r^3} \quad (5.66)$$

We expect the angular momentum to have a commutation relation with $\vec{r}$ and $\vec{D}$ such that they transform as a vector

$$[L_i, D_j] = i\epsilon_{ijk}D_k \quad , \quad [L_i, r_j] = i\epsilon_{ijk}r_k \quad (5.67)$$

In fact, the commutation relations (5.67), determined completely the commutation relations of the components of $\vec{L}$ between themselves and the Hamiltonian, by using the Jacobi identity and the irreducibility of the $\vec{r}$ and $\vec{D}$ variables:

$$[L_i, L_j] = i\epsilon_{ijk}L_k \quad , \quad [L_i, H_0] = 0$$

The first choice for $\vec{L}$ would be simply

$$\vec{L} \stackrel{?}{=} -i\hbar\vec{r} \times \vec{D} = \vec{r} \times (\vec{p} - e\vec{A}_m)$$

But it turns out to be wrong, mainly because of the presence of the magnetic potential vector $\vec{A}_m$, as it could be noticed already in (5.66). The correct answer is in fact given by

$$\vec{L} = -i\hbar\vec{r} \times \vec{D} - s\hbar\frac{\vec{r}}{r} \quad (5.68)$$

This expression for the angular momentum is actually true also classically (obviously?) since

$$\frac{d}{dt}(m\vec{r} \times \dot{\vec{r}}) = m\vec{r} \times \ddot{\vec{r}} = \vec{r} \times (e\dot{\vec{r}} \times B) = \frac{s\hbar}{r^3}\vec{r} \times (\dot{\vec{r}} \times \vec{r}) = \frac{d}{dt}(s\hbar\frac{\vec{r}}{r})$$

In quantum mechanics we just have to verify the commutation relations (5.67) to convince ourselves.

A special study has to be done to see which values of l can really occur. We know that $2l$ must be an integer since $\vec{L}$ follows the spin algebra. But a more careful construction [5] of the representation of $\vec{L}$ defined by (5.68) actually shows that

$$l = s, s + 1, s + 2, \dots$$

(we must have a state $|l, s\rangle$ to construct the representation!)

Now, we know that $(\vec{L})^2 = \hbar^2 l(l + 1)$ where l is one of the values mentioned above. By using the equation (5.68) instead, we find

$$(\vec{L})^2 = -\hbar^2(\vec{r} \times \vec{D})^2 + \hbar^2 s^2$$

Furthermore, we have

$$\begin{aligned} \vec{D} \cdot \vec{D} &= D_i \delta_{ij} D_j = D_i \frac{r_i r_j - \epsilon_{inm} r_n \epsilon_{mkj} r_k}{r^2} D_j \\ &= (\vec{D} \cdot \vec{r}) \left(\frac{1}{r^2} \right) (\vec{r} \cdot \vec{D}) - (\vec{D} \times \vec{r}) \left(\frac{1}{r^2} \right) (\vec{r} \cdot \vec{D}) \end{aligned}$$

$$= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} (\vec{r} \times \vec{D})^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{s^2 - l(l+1)}{r^2}$$

So, this enable us to write the Hamiltonian (5.65) as:

$$\begin{aligned} H_0 &= \frac{1}{2m} (-i\hbar \vec{D})^2 = -\frac{\hbar^2}{2m} (\vec{D})^2 \\ &= -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{\hbar^2(l(l+1) - s^2)}{2mr^2} \end{aligned} \quad (5.69)$$

5.3 Path Integral for a Spin Particle in a Magnetic Field

What would be the Hamiltonian of our magnetic monopole system, if we put it in a constant magnetic field? We already solved the system for a magnetic potential $\vec{A}_m$, so a constant magnetic field $\vec{B}$, corresponding to a vector potential $\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$, will simply shift $\vec{A}_m \rightarrow \vec{A}_m + \frac{1}{2} \vec{B} \times \vec{r}$ in the Hamiltonian (5.65):

$$\begin{aligned} H_s &= \frac{1}{2m} (\vec{p} - e\vec{A}_m - \frac{e}{2} \vec{B} \times \vec{r})^2 \\ &= H_0 - \frac{e}{2m} (\vec{p} - e\vec{A}_m) \cdot (\vec{B} \times \vec{r}) + \frac{e^2}{8m} (\vec{B} \times \vec{r})^2 \\ &= H_0 - \frac{e}{2m} \vec{B} \cdot \vec{L} - \frac{es\hbar}{2m} \frac{\vec{r} \cdot \vec{B}}{r} + \frac{e^2}{8m} (\vec{B} \times \vec{r})^2 \end{aligned} \quad (5.70)$$

Here appears finally the reward of all our work in this chapter, the $\vec{B} \cdot \vec{L}$ term that we are studying, but there are still two problems. One is the last two terms in (5.70), they have nothing to do with our model of last chapter. The only simple way of correcting this is to add an interaction potential to the Hamiltonian, that will cancel them. The second problem is more important, because we need our particle to have a definite spin, s , and the Hamiltonian (5.70) does not guarantee this constraint (other values of l might appear). The contribution of l in the Hamiltonian is given by the term $\frac{\hbar^2}{2mr^2} (l(l+1) - s^2)$ in H_0 , in (5.69). The smaller l is, the smaller the energy contribution of this term. So if $\frac{\hbar^2}{mr^2} \gg \beta$, which means mr^2 is small enough, the values of $l > s$ (where $l = s$ is the

ground state) will not contribute much to the partition function, and we would be able to consider the system in a state $l = s$ with $E = E_s$, more precisely we must have $\beta E_s \gg 1$.

To accomplish this task we could consider $m \rightarrow 0$, but we do not know if r would not become very large. In fact, a solution of the Hamiltonian H_0 is well known [6] and indicates an unbounded system! So we really need something that will keep the particle close to the monopole. To do so, we can impose an additional interaction. Several choices are possible and though many people can study the different potentials, ultimately they should produce the same result. The idea actually would be to put a particle on a sphere, $r = r_0$. Physically, this means that we have to put a steep potential (harmonic for example) around $r = r_0$, such that the particle will stay in the first (ground state) energy level of that potential. This will just mean a shift of $E_h = \frac{1}{2}\hbar\omega_h$ ($\beta E_h \gg 1$) for the remaining part of the Hamiltonian, and a particle constraint to move on a sphere. All together, we will be interested in an interaction potential of the form:

$$V = \frac{es\hbar}{2m} \frac{\vec{r} \cdot \vec{B}}{r} - \frac{e^2}{8m} (\vec{B} \times \vec{r})^2 + E_s \frac{(r - r_0)^2}{2\sigma^2}$$

$$E_s = \frac{\hbar^2 s}{2mr_0^2} \quad , \quad E_h = \frac{1}{2}\hbar\omega_h = \frac{1}{2}\hbar\sqrt{\frac{E_s}{m\sigma^2}} = \frac{\hbar^2}{2mr_0\sigma}\sqrt{s/2} = E_s\left(\frac{r_0/\sqrt{2s}}{\sigma}\right) \quad (5.71)$$

The variable σ controls the steepness of the potential, and also indicates the value of the uncertainty of the radius $r = r_0$ constraint. We therefore must have $\sigma \ll r_0$, which gives $E_h \gg E_s$, usually. However, if $s \rightarrow \infty$, the classical limit, we can choose $\sigma = r_0/\sqrt{2s}$ and obtain $E_h = E_s$. In other words, the harmonic potential, that we added here, might even be present, in (5.70), at the classical limit and responsible for the E_s ground state energy!

So we obtain the following Hamiltonian

$$H = H_s + V = (E_h + E_s) - \frac{e}{2m} \vec{B} \cdot \vec{L} \quad , \quad (\vec{L})^2 = \hbar^2 s(s+1) \quad (5.72)$$

Now, let us write what is the Lagrangian corresponding to (5.72). We already know what to do, since it is simply a particle moving in a magnetic field and a potential, which has a Lagrangian given by

$$\begin{aligned}
 L &= \frac{m}{2}(\dot{\vec{r}})^2 + e(\vec{A}_m + \vec{A}) \cdot \dot{\vec{r}} - V \\
 &= \frac{m}{2}(\dot{\vec{r}})^2 + \frac{e}{2}(\vec{B} \times \vec{r}) \cdot \dot{\vec{r}} - \frac{es\hbar}{2m} \frac{\vec{r} \cdot \vec{B}}{r} + \frac{e^2}{8m}(\vec{B} \times \vec{r})^2 + \hbar s(\pm 1 - \cos(\theta))\dot{\phi} \\
 &= \frac{m}{2}(\dot{\vec{r}} + \frac{e}{2m}\vec{B} \times \vec{r})^2 - \frac{es\hbar}{2m} \frac{\vec{r} \cdot \vec{B}}{r} + \hbar s(\pm 1 - \cos(\theta))\dot{\phi}
 \end{aligned} \tag{5.73}$$

where I dropped the harmonic potential term, I will introduce it in the path integral as a $\delta(r - r_0)$ function. Since we are looking for the partition function, we have to introduce the Euclidean Lagrangian, by changing $\frac{d}{dt} = -\frac{i}{\hbar} \frac{d}{d\tau}$ in the path integral (5.73), which gives

$$\begin{aligned}
 Z[\beta] &= \text{tr}(e^{-\beta H}) = \text{tr}(e^{-\beta(-\frac{e}{2m}\vec{B} \cdot \vec{L})}) \cdot e^{-\beta(E_h + E_s)} \\
 &= \int D\vec{r} \delta(r - r_0) e^{-\int_0^\beta L_E d\tau}
 \end{aligned}$$

where

$$L_E = \frac{m}{2\hbar^2}(\dot{\vec{r}})^2 + \frac{ie}{2\hbar}(\vec{B} \times \vec{r}) \cdot \dot{\vec{r}} + \frac{es\hbar}{2m} \frac{\vec{r} \cdot \vec{B}}{r} - \frac{e^2}{8m}(\vec{B} \times \vec{r})^2 + is(\pm 1 - \cos(\theta))\dot{\phi} \tag{5.74}$$

and $\vec{r}(0) = \vec{r}(\beta)$ as usual.

5.4 Comparison with Coherent State Path Integral

The comparison of the path integrals (5.74) and (4.60) or (4.47) indicates how similar are these path integrals. In fact, they are identical if we make the correspondence:

$$\mu = -\frac{e}{2m} \quad , \quad \epsilon = \frac{2mr_0^2}{\hbar^2 s} \quad , \quad \vec{r} = -r_0 \vec{n} \tag{5.75}$$

The topological terms differ by a minus sign, which is normal since the $\vec{r}$ differ by a minus sign (and a constant) with $\vec{n}$, indicated in (5.75). If $eg/\hbar = -s$, instead of our convention, then μ and $\vec{r}$ will change sign in (5.75).

The value of μ is not a surprise, since the Hamiltonian (5.72) contain this factor in front of the $\vec{B} \cdot \vec{L}$ term. The very interesting revelation of this calculation is the ϵ correspondence, which can be rewrite as:

$$\epsilon = \frac{1}{E_s} \quad \text{or} \quad \beta E_s = N \quad (5.76)$$

It was already known that $\beta E_s \gg 1$ and (5.76) indicates this statement in term of the N variable. Furthermore, if we put (5.76) back into the harmonic potential in (5.71) and use the value of $\sigma = r_0/\sqrt{2s}$, valid at the classical limit, we find exactly the equation (4.55) or (4.56) for the radial part of the path integral! Again the classical limit is very well defined, but the low spin limit is more tricky, and needs some 'artificial' constraints to be well defined.

In the partition function (5.74), the presence of $e^{-\beta E_s} = e^{-N}$ indicates that the path integral measure must be renormalised by e , to exactly correspond to the coherent state path integral. On an other hand, the $e^{-\beta E_h}$ come from the harmonic potential that we *added* to the Hamiltonian, so there is no surprise if we find this extra term.

Since the kinetic energy comes mainly from the spin of the particle, thus E_s , the kinetic term $\frac{es}{4}(\dot{\vec{n}})^2$ in the coherent state path integral, will contribute to the order one, which can not really be neglected in the calculations.

Chapter 6

Conclusion

The purpose of this work has been the study of path integral evaluated with coherent states. This has been done by looking at two solvable problems: the harmonic oscillator and the particle with spin in a constant magnetic field systems.

The coherent states represent the system so well, that the path integral vanishes for classical trajectories. The real problem has been, then, to include the quantum trajectories in the path integral calculations, to get the quantum corrections, if not a completely quantum result. It is clear that the classical limit of these path integrals are very well defined, but it is useless to use path integrals to find only classical solutions.

These quantum corrections have been taken into account, in my work, by keeping the ϵ terms in the path integral and then taking the limit $\epsilon \rightarrow 0$ at the end of the calculations. These terms create a bridge between these classical and purely quantum paths, by making these trajectories smooth enough so that we can use a continuum limit approximation.

The use of a lattice regularization gives us a way to obtain an exact discrete action. This method has been used before with success [10,11], but its discrete level has never been studied very deeply. It seems that a careful analysis of the discretisation gives some useful corrections, and allows an interesting comparison between various continuous approximations and the exact solution. This explains why the ϵ terms regularized the path integral appropriately, the resulting path integral is closer to its discrete version.

For the harmonic oscillator coherent state path integral, it has been noticed that the

simple definition of the classical Hamiltonian $H(z)$ in term of the quantum one H as $H(z_k) = \langle z_k | H | z_k \rangle$, or $H(z_k) = \frac{\langle z_k | H | z_{k+1} \rangle}{\langle z_k | z_{k+1} \rangle}$ affect the ground state in the path integral. It has been demonstrated that the $\epsilon |\dot{z}|^2$ term helps to regularize the path integral. This is particularly apparent by going back to a discrete level, at the section 3.2, or a semiclassical approximation, at the section 3.3.

The application of the same procedure to a spin s particle in a constant magnetic field by the use of spin coherent states or harmonic oscillator coherent states produced the same continuum limit, up to order ϵ terms, after the integration of the appropriate variables in the latter path integral. We were able to extract the meaning of the λ_k variables as a gauge field, and construct a covariant derivative for the harmonic oscillator coherent states. The topological action, $2s(\pm 1 - \cos(\theta))\dot{\phi}$, has been shown to be related to the Hopf phase of this variable, z . This gave us a mapping of this spin path integral into a CP^1 model. The coefficient of this topological action appeared clearly as being $2s$.

Furthermore, a study of magnetic monopole has been reproduced in details. It has been explained how it is possible to obtain a spin s particle representation, using a monopole field and a specific interaction: $V = \frac{es\hbar}{2m} \frac{\vec{r} \cdot \vec{B}}{r} - \frac{e^2}{8m} (\vec{B} \times \vec{r})^2 + E_s \frac{(r-r_0)^2}{2\sigma^2}$. This identification has been studied before, but never up to the ϵ order [14,15]. We showed a complete correspondence between this monopole path integral, in position space representation, and the same spin system path integral using coherent states, indicating even more the relevance of the ϵ terms as a regulator. The $\epsilon \rightarrow 0$ limit is imposed by a $r_0 \rightarrow 0$ limit, or $m \rightarrow 0$ with $\frac{e}{m}$ fixed, on the radius of the sphere on which the particle moves around the monopole. In this limit, the $\frac{1}{2}m(\dot{\vec{r}})^2$ will not necessarily go to zero, since the particle might simply spin faster, by conservation of angular momentum.

The radial part of the motion of the particle, for low value of the spin, is not very well peaked at a given radius, however, it completely decoupled in the path integral. Which

gives us a unambiguous path integral for the tangential motion.

In the future, it would be interesting to apply this path integral method to some statistical models, like the Heisenberg model or the spin chain, that are currently under study, since they could be relevant to high temperature superconductors. The comparison of some of these studies, like [12], and the ones using coherent state path integrals might give some new insights into these models.

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Appendix A

Identities for Determinants

Gaussian integrals have the very useful property that:

$$\int \prod_{k=1}^N \left(\frac{dz_k dz_k^\dagger}{(2\pi i \hbar)^M} \right) \exp \left\{ -\frac{1}{\hbar} \sum_{k,l=1}^N \sum_{i,j=1}^M (z_k^\dagger)_i (M_{kl})_{ij} (z_l)_j \right\} = \frac{1}{\det[M_{kl,ij}]} \quad (\text{A.77})$$

Thus, the evaluation of various determinants might give an alternative verification of some Gaussian path integrals.

In most cases, the M_{kl} matrix is non-zero for $|k-l| = 0$ or 1 only, so let us study the determinant of

$$\tilde{M}_{(N)} = \begin{pmatrix} A & -B & 0 & \dots & 0 \\ -C & A & -B & \dots & 0 \\ 0 & -C & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A \end{pmatrix}_{(N)}, \quad M_{(N)} = \begin{pmatrix} A & -B & 0 & \dots & -C \\ -C & A & -B & \dots & 0 \\ 0 & -C & A & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B & 0 & 0 & \dots & A \end{pmatrix}_{(N)} \quad (\text{A.78})$$

or

$$= A\delta_{i,j} - B\delta_{i,j-1} - C\delta_{i,j+1} \quad i, j = 0 \text{ to } N$$

where i, j assume cyclic boundary conditions, $N+1 \equiv 1$, $N-1 \equiv -1$, for the $M_{(N)}$ matrix.

Let us call

$$\tilde{D}_N = \det[\tilde{M}_{(N)}] \quad , \quad D_N = \det[M_{(N)}]$$

The evaluation of these determinants can be done by using the well known recursive formulas. Expanding along the first line for $\tilde{D}_N$ gives

$$\tilde{D}_N = A\tilde{D}_{N-1} - BC\tilde{D}_{N-2}$$

which can be solved by inserting a solution of the form λ^N , that produces the following constraint:

$$\lambda^2 = A\lambda - BC \Rightarrow \lambda = \lambda_{\pm} = \frac{1}{2}(A \pm \sqrt{A^2 - 4BC})$$

Since $\tilde{D}_1 = A$, $\tilde{D}_2 = A^2 - BC$, we find the general solution for this determinant identity

$$\det[\tilde{M}_{(N)}] = \tilde{D}_N = \frac{1}{\lambda_+ - \lambda_-}(\lambda_+^{N+1} - \lambda_-^{N+1}) \quad (\text{A.79})$$

For D_N , we procede in the same way

$$D_N = A\tilde{D}_{N-1} - 2BC\tilde{D}_{N-2} - (B^N + C^N)$$

with $A\lambda_{\pm} - 2BC = (\lambda_+ - \lambda_-)(\pm\lambda_{\pm})$ we find for the cyclic determinant D_N , the identity:

$$\det[M_{(N)}] = D_N = (\lambda_+)^N + (\lambda_-)^N - B^N - C^N \quad (\text{A.80})$$

We can, furthermore, find the eigenvectors and eigenvalues of the $M_{(N)}$ matrix, because of its cyclic boundary condition. These are simply

$$(\Psi_{(N)}^k)_j = e^{\frac{2\pi i}{N}kj}, \quad M_{(N)}\Psi_{(N)}^k = \lambda_k\Psi_{(N)}^k = (A - Be^{\frac{2\pi i}{N}k} - Ce^{-\frac{2\pi i}{N}k})\Psi_{(N)}^k$$

This gives another identity for the determinant:

$$D_N = \prod_{k=1}^N \lambda_k = \prod_{k=1}^N (A - Be^{\frac{2\pi i}{N}k} - Ce^{-\frac{2\pi i}{N}k}) \quad (\text{A.81})$$

If the A , B and C are not simple complex numbers, but submatrices, we can still solve the determinant under one condition: that we can diagonalize all three submatrices,

A , B , C , at the same time. In other words, $(SAS^{-1})_{ij} = \delta_{ij}A_i$, $(SBS^{-1})_{ij} = \delta_{ij}B_i$ and $(SCS^{-1})_{ij} = \delta_{ij}C_i$ with the same S matrix. Then the determinant is simply

$$\det[M_{(N)}] = \prod_{i=1}^M \det^{(i)}[M_{(N)}] \quad (\text{A.82})$$

where $\det^{(i)}[M_{(N)}]$ is the determinant (A.80) with the use of the A_i , B_i , C_i variables.

If $C = 0$, the previous determinant (A.79) and (A.80) can be simplified significantly:

$$\det[\tilde{M}_{(N)}] = A^N, \quad \det[M_{(N)}] = A^N - B^N$$

Actually, if all the A and B 's are different, we can still easily prove, like (A.79) or (A.80), that

$$\det \begin{bmatrix} A_1 & -B_1 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots \\ -B_N & 0 & \cdots & A_N \end{bmatrix}_{(N)} = \left(\prod_{k=1}^N A_k \right) - \left(\prod_{k=1}^N B_k \right) \quad (\text{A.83})$$

In cases of submatrices, we can still use (A.82) with (A.83), but again, as long as we can diagonalize all the A_k , B_k ($k = 1$ to N) at the same time.

Appendix B

Schwinger-Boson Model

Let us consider a set of two dimensional creation and destruction operators:

$$[a_i, a_j^\dagger] = \hbar \delta_{ij} \quad , \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \quad , \quad \text{for } i, j = 1, 2$$

Then we can build up a spin vector, $\vec{J}$, by

$$\vec{J} = a^\dagger \frac{\vec{\sigma}}{2} a \tag{B.84}$$

We can verify the identity

$$(\vec{J})^2 = \frac{a^\dagger a}{2} \left(\frac{a^\dagger a}{2} + 1 \right) \tag{B.85}$$

This indicates that we can use (B.84) to represent a spin s angular momentum operator on a set of states, $|\Psi\rangle$, build up by $a_i^\dagger$, if we have the constraint

$$a^\dagger a |\Psi\rangle = 2s |\Psi\rangle$$

In other words, we have to work in a subspace represented by:

$$|\Psi\rangle \rightarrow \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{i\lambda(a^\dagger a - 2s)} |\Psi\rangle \tag{B.86}$$