HOLONOMY IN QUANTUM PHYSICS

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Abstract

Holonomy in nonrelativistic quantum mechanics is examined in the context of the adiabatic theorem. This theorem is proven for sufficiently regular unbounded hamiltonians. Then, simplifying to matrix hamiltonians, it is proven that the adiabatic theorem defines a connection on vector bundles constructed out of eigenspaces of the hamiltonian. Similar degeneracy regions, the natural base spaces for these bundles, are defined in terms of stratifications for the spaces of complex, hermitian matrices and real, symmetric matrices. The algebraic topology of similar degeneracy regions is studied in detail, and the results are used to classify and calculate all possible adiabatic phases for time-reversal invariant matrix hamiltonians in terms of the relevant topological data.

It is shown how vector bundles may be used to impose transversality on the helicity vector of a photon. This is used to give a calculation, which is consistent with transversality, of quantum adiabatic phase for photons in a coiled optical fibre. As an additional application, the importance of quantum adiabatic in the dynamical Jahn-Teller effect is briefly explained.

An introduction is given to some important aspects of algebraic topology, which are used herein. Moreover, a number of mathematical results for flag manifolds are obtained. These results are applied to quantum adiabatic holonomy.

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Introduction

Simply put, holonomy is the geometrical concept which refers to the property that upon transport around a loop, an object may not return to its original state. Specifically, holonomy pertains to the special form of transport, known as parallel transport. Quantum mechanics is not usually considered to be a geometrical theory, and therefore the recently discovered ubiquity of holonomy in quantum mechanics is surprising.

We shall study holonomy arising from the quantum adiabatic theorem—hence the name, quantum adiabatic holonomy. The results described in this thesis appear in the monograph [31], by R. R. Douglas and this author.

For quantum systems whose hamiltonians vary slowly with time, the adiabatic theorem provides an approximate solution to the time-dependent Schrödinger equation in terms of eigenfunctions of the hamiltonian. The evolution of a quantum system is completely determined by its hamiltonian, which is in general a hermitian operator on a separable Hilbert space: the state space of the system. Indeed, from a mathematical perspective, we can identify a quantum system with the hermitian operator that is its hamiltonian. A time-independent system is then represented as a point in the space of hermitian operators on the hilbert space. A time-dependent system is represented as path in this space of hermitian operators. The quantum adiabatic theorem defines a notion of parallel transport along this path.

Implicit in the adiabatic theorem is the condition that some distinguished eigenvalue is bounded away from the rest of the spectrum of the hamiltonian. This eigenvalue is usually distinguished by the initial state of the system. When viewed in terms of the space of hermitian operators, such a "non-crossing constraint" is geometrically very complicated. The nontrivial geometry of regions defined by such constraints is at the heart of quantum adiabatic holonomy. These regions are termed similar degeneracy regions.

In general, quantum adiabatic holonomy is difficult to compute because it depends on the details of the differential geometry of similar degeneracy regions. However, if the distinguished eigenvalue is nondegenerate, then the holonomy simplifies to a phase factor, the phase being called quantum adiabatic phase. Moreover, if the system is time-reversal invariant, then the adiabatic phase depends only on the most basic aspect of the topology of the similar degeneracy region. Specifically, a quantum system which is periodic in time is represented by a loop, and for time-reversal-invariant systems the adiabatic phase depends only on the homotopy class of this loop in the relevant similar degeneracy region.

Quantum adiabatic holonomy was first noticed in analyses of the dynamical Jahn-Teller effect of molecular physics [60]. (For a more complete list of references on the dynamical Jahn-Teller effect, see Subsection 3.d in Chapter III, [2], and [50].) Independently, it was argued that adiabatic phases arise in the wave functions of particles with spin coupled to a slowly rotating magnetic field [1]. An important contribution was made by M. V. Berry [8], who demonstrated that adiabatic phase is a general phenomenon,

which occurs in many quantum systems. In commenting on Berry's paper, B. Simon [81] argued that adiabatic phase is an example of holonomy.

Evidence of quantum adiabatic phase has been observed in a variety of experiments, many of which are reviewed in [2] and [50]. These include polarized photons in a coiled optical fibre [23], [88], nuclear quadrupole resonance spectra of rotating samples [89], neutron spin rotation [12], nuclear magnetic resonance [87], and half-odd integer quantization of pseudorotational levels in Na_3, which results from the Jahn-Teller effect [27]. In Subsection 3.c of Chapter II, we examine the example of photons in an optical fibre. The Jahn-Teller effect is discussed in Subsection 3.d of Chapter III.

Quantum adiabatic holonomy has figured prominently in recent theoretical work on the integer quantum hall effect [4], [5], [6], [81], the fractional quantum hall effect [3], [79], [78], vortices in 2-dimensional superfluids [43], superfluid ^{3}He [39], dipole-quadrupole interactions in spherical nuclei [58], optical pumping of atoms [76], and atoms in a slowly rotating electric field [64]. Also, it is now realized that holonomy is an important concept in gauge field theories. See [77] for a review of how chiral anomalies can be seen as holonomy in the hamiltonian formulation of chiral gauge theory.

Chapter I

Mathematical Preliminaries

This chapter provides a brief introduction to some of the mathematical concepts and results which are used in this thesis. Unfortunately, it is impossible to make this introduction complete, because of length considerations. To offset this, references to mathematical background are supplied throughout the thesis.

During this century, functional analysis and differential geometry have been extensively used in theoretical physics. However, until recently, algebraic topology has not been accorded the same attention. For this reason, many theoretical physicists are not as familiar with algebraic topology, and therefore the first section of this chapter is devoted to introducing some of the fundamental concepts of this field.

For Chapters II and III, we require a number of results on the algebraic topology of flag manifolds. These results are easy to prove by standard techniques; however, a suitable reference for them could not be found in the literature. Therefore, flag manifolds are the topic of discussion in the second section of this chapter.

§1 Algebraic Topology

The central theme of algebraic topology is to express various properties of topological spaces in terms of algebraic data. The algebraic data usually takes the form of collections of groups and homomorphisms between these groups. As might be expected, some information is lost when a topological space is analyzed algebraically. However, for some problems the essential features of a topological space may be encapsulated algebraically, and then it is possible to take advantage of the superior computational power of algebraic methods.

Three very important concepts in algebraic topology are homotopy, homology, and cohomology. In this section, we shall introduce homotopy theory, singular homology theory, singular cohomology theory, and de Rham cohomology theory. In addition, we shall state some important theorems which express the relationship between these theories.

Homotopy groups and singular homology groups are related by the Hurewicz homomorphisms, which are the subject of the Hurewicz theorem. This theorem will be used a number of times in this thesis. An elementary discussion of homotopy theory, singular homology, and the Hurewicz theorem is given in [38]. For a more advanced treatment of these subjects, we suggest [83] or [92].

The relationship between singular homology groups and singular cohomology groups is expressed through the universal coefficient theorem for cohomology. This theorem is a special case of a more general universal coefficient formula for functors of complexes. An excellent discussion of functors and universal coefficient formulae is given in [29, Chapt. VI].

Homotopy, singular homology, and singular cohomology theories are defined for an arbitrary topological space. For the special case of a differentiable manifold, differential forms and the exterior derivative operator may be used to define de Rham cohomology. A good introduction to de Rham cohomology is Chapter 1 in [16]. The de Rham theorem provides a natural isomorphism between the singular cohomology of a manifold and the de Rham cohomology.

This section does not cover all of the topics in algebraic topology which are prerequisite for a thorough reading of this thesis. Only the absolute versions of homotopy, singular homology, singular cohomology, and de Rham cohomology are presented, even though the relative versions of these theories are used in Chapter III. It is hoped that this elementary discussion will provide those physicists who are not versed in algebraic topology, with some feel for the subject. Relative homotopy theory is reviewed in [48], [83], and [92]. Relative singular homology and cohomology theory is presented in [29], [36], and [83].

We shall make extensive use of commutative diagrams, for which a good introduction is [61]. In particular, the five lemma will be used many times throughout the thesis, and the 3×3 lemma will be used in Chapter III.

Fibre bundles are now used extensively in theoretical physics, and this thesis is no exception. A comprehensive reference on fibre bundles is [49]. Two excellent treatments of vector bundles and characteristic classes are [20] and [66].

(a) Homotopy Theory

A pointed topological space is a nonempty topological space together with a distinguished element, which is called its base point. Let X and Y be pointed topological spaces with base points x_0 and y_0, respectively. A map f from X to Y is said to preserve base points if $f(x_0) = y_0$. We shall always assume that maps between topological spaces are continuous maps. Two maps $f\colon X \to Y$ and $g\colon X \to Y$ are homotopic (written $f \simeq g$) if there exists a map $F\colon X \times [0,1] \to Y$ such that

$$F(x,0) = f(x) \qquad \text{for all } x \in X,$$

$$F(x,1) = g(x) \qquad \text{for all } x \in X,$$

and

$$F(x_0,t) = y_0 \qquad \text{for all } t \in [0,1].$$

The relation $\simeq$ is an equivalence relation, partitioning the set of base-point-preserving maps from X to Y into disjoint equivalence classes, which are called homotopy classes. The set of all homotopy classes of base-point-preserving maps from X to Y is denoted by $[X;Y]$, and the homotopy class of a map f is written $[f]$.

If X and Y are path connected, then the following discussion is independent of which base points are chosen. However, if X and Y are not path connected, then the set of homotopy classes depends on which path components of X and Y contain the base points

x_0 and y_0, respectively. For a discussion of the role of base points in homotopy theory, see [83, Sect. 7.3].

As an important example, consider the set of homotopy classes of maps from the circle S^1 (with base point) to a topological space with base point x_0. The set $[S^1; X]$ has a natural group structure, which is defined as follows.

If S^1 is parameterized by $t \in [0,1]$, then a base-point-preserving map from S^1 to X may be written as $\sigma\colon [0,1] \longrightarrow X$, where $\sigma(0) = \sigma(1) = x_0$. The map σ defines a loop in X, which is based at x_0.

For two loops σ and τ, both based at x_0, the product loop $\sigma \cdot \tau$ is defined to be the loop obtained by first going around σ, and then going around τ. In other words,

$$\sigma \cdot \tau(t) = \begin{cases} \sigma(2t) & \text{if } 0 \le t \le 1/2 \\ \tau(2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

For two homotopy classes $[\sigma]$ and $[\tau]$ in $[S^1; X]$, we define the product to be

$$[\sigma] \cdot [\tau] = [\sigma \cdot \tau] \in [S^1; X] \,. \tag{1.1}$$

It is easy to check that this definition is independent of which loops are chosen to represent the homotopy classes, and hence we have a well-defined product on the elements of $[S^1; X]$. This set, along with the multiplication defined in (1.1), forms the fundamental group of X, which is denoted by $\pi_1(X)$.

Thus far, we have taken the limited perspective that homotopy theory associates with two spaces the set of homotopy classes of maps from one space to the other. This is only part of the story.

Take A to be a fixed topological space with a base point. We define a function from the collection of pointed topological spaces to the collection of sets by associating the set $[A; X]$ with the pointed topological space X. Furthermore, if X and Y are two pointed topological spaces, and $f\colon X \to Y$ is a base-point-preserving map, then we define an induced function

$$f_{\#}\colon [A; X] \longrightarrow [A; Y]$$

by $f_{\#}([g]) = [f \circ g]$ for all $[g] \in [A; X]$. It is left to the reader to check that $f_{\#}$ is well-defined.

By associating the set $[A; X]$ with every pointed topological space X, and the function $f_{\#}\colon [A; X] \to [A; Y]$ with every base-point-preserving function $f\colon X \to Y$, we have defined a covariant functor from the category of pointed topological spaces and base-point-preserving maps between them, to the category of sets and functions between them. Denoting this functor by π_A, we write $\pi_A(X) = [A; X]$, and $\pi_A(f) = f_{\#}$. This is the proper context in which to understand homotopy theory. Functors and categories are discussed in Chapter 1 of nearly every book on algebraic topology. For example, see [29], [61], and [83].

It is instructive to reconsider in the context of category theory our example of homotopy classes of maps from S^1 to a pointed topological space X. The covariant functor π_{S^1}, which is given the special notation π_1, assigns to each topological space the fundamental group $\pi_1(X)$. In addition, if f is a map from X to Y, the induced function $\pi_1(f) = f_{\#}$ is a group homomorphism from $\pi_1(X)$ to $\pi_1(Y)$. Therefore, we arrive at the remarkable

conclusion that π_1 is a covariant functor from the category of pointed topological spaces and maps, to the category of groups and homomorphisms.

It is by no means true that the functor π_A takes values in the category of groups and homomorphisms for all topological spaces, A. Indeed, the question, "For which topological spaces A does π_A take values in the category of groups and homomorphisms?" leads to the important topic of H-cogroups [47], [83, Sect. 1.6], [92, Sect. III.5].

To provide examples of H-cogroups, we introduce the concept of suspension. For a pointed topological space A with base point a_0, the suspension ΣA is defined to be the pointed topological space obtained from $A \times [0,1]$ by identifying $A \times \{0\} \cup A \times \{1\} \cup \{a_0\} \times [0,1]$ to a single point, which is taken to be the basepoint of ΣA. Important examples of suspensions are the n-dimensional spheres S^n, for $n \geq 1$. It is straightforward to show that S^n is homeomorphic to ΣS^{n-1}, for all $n \geq 1$. This computation is carried out explicitly in [83, Lemma 1.66].

For any pointed topological space A, its suspension ΣA is an H-cogroup. This is shown in most treatments of this subject, including [47, Chapt. 1], [83, Sect. 1.6], and [92, Sect. III.5]. Furthermore, the double suspension $\Sigma^2 A$ is an abelian H-cogroup, which implies that the functor $\pi_{\Sigma^2 A}$ takes values in the category of abelian groups. Therefore, we conclude that the functors π_{S^n} take values in the category of groups for $n \geq 1$, and in the subcategory of abelian groups for $n \geq 2$. Because of their importance, these functors are given the special notation π_n, and the groups $\pi_n(X)$ are called the homotopy groups of the topological space X.

It remains to consider the functor π_0, which is obtained from the set of homotopy classes of base-point-preserving maps from the sphere S^0 to an arbitrary pointed space X. Clearly, the set $\pi_0(X)$ is in one-to-one correspondence with the path components of X. There is no natural group structure on $\pi_0(X)$, and it should simply be regarded as a set with base point given by the homotopy class of the constant map.

Extensive use will be made of the homotopy functors π_n, for $n \geq 0$, and therefore we provide the following summary. The covariant functor π_n from the category of pointed topological spaces and maps takes values in the category of

$$\begin{cases} \text{sets with base point and base-point-preserving functions} & \text{if } n = 0 \\ \text{groups and homomorphisms} & \text{if } n = 1 \\ \text{abelian groups and homomorphisms} & \text{if } n \geq 2 \end{cases}$$

The basic philosophy remains the same for relative homotopy theory, which is defined on pairs of topological spaces with base points. However, there are important modifications, which the reader should investigate in references such as [48] or [92]. A pair of topological spaces (X, A) is defined to be a topological space X, along with a subspace $A \subset X$.[1] A base point for the pair (X, A) is a distinguished point in A. For the pointed pair (X, A), the relative homotopy group $\pi_n(X, A)$ is

$$\begin{cases} \text{a set with base point} & \text{if } n = 1 \\ \text{a group, not necessarily abelian} & \text{if } n = 2 \\ \text{an abelian group} & \text{if } n \geq 3 \end{cases}$$

[1] The term "subspace" will always mean topological subspace, rather than vector subspace.

Usually, $\pi_0(X, A)$ is not defined.

(b) Homology Theory

The most elegant formulation of homology theory is the axiomatic approach of Eilenberg and Steenrod [36]. Consider (X, A) and (Y, B) to be pairs of topological spaces, without base points. A map f from the pair (X, A) to the pair (Y, B) is a continuous map from X to Y which satisfies $f(A) \subset B$. The class of all pairs of topological spaces, along with maps between them, form the category of topological pairs. The pair $(X, \emptyset)$ is identified with the topological space X, and therefore the category of topological pairs contains the category of topological spaces as a subcategory. Furthermore, the category of pointed topological spaces is obtained as a subcategory of the category of topological pairs by identifying the pair $(X, \{x_0\})$ with the pointed topological space X, which has base point x_0.

A homology theory consists of a collection of functors $\{H_q \mid q \in \mathbf{Z}\}$, where each H_q is a covariant functor from the category of topological pairs (or a subcategory) to the category of abelian groups. In addition, a homology theory contains a collection of homomorphisms

$$\partial_q(X, A)\colon H_q(X, A) \longrightarrow H_{q-1}(A)\,,$$

which are called the boundary homomorphisms. When there is no possibility for confusion, all of the homomorphisms $\partial_q(X, A)$ will be denoted by ∂_*. Similarly, if $f\colon (X, A) \to (Y, B)$ is a map of topological pairs, then often the induced homomorphisms $H_q(f)$ will be collectively denoted by f_*.

A homology theory must satisfy the following five axioms.

Axiom 1. (Naturality of ∂_*) *For any map of pairs $f\colon (X,A) \to (Y,B)$, let $f|_A\colon A \to B$ denote the restriction of f to the subspace A. Then, the diagram of groups and homomorphisms,*

$$\begin{array}{ccc} H_q(X,A) & \xrightarrow{H_q(f)} & H_q(Y,B) \\ \Big\downarrow{\scriptstyle \partial_q(X,A)} & & \Big\downarrow{\scriptstyle \partial_q(Y,B)} \\ H_{q-1}(A) & \xrightarrow{H_{q-1}(f|_A)} & H_{q-1}(B) \end{array} \tag{1.2}$$

is commutative.

Commutativity of the diagram (1.2) simply means that the compositions $\partial_q(Y,B)\,H_q(f)$ and $H_{q-1}(f|_A)\,\partial_q(X,A)$ are identical homomorphisms from $H_q(X,A)$ to $H_{q-1}(B)$. In functorial language, Axiom 1 expresses the naturality of the boundary homomorphism.

Axiom 2. (Exactness Axiom) *For a topological pair (X,A), define the inclusion maps $i\colon A \hookrightarrow X$ and $j\colon X \hookrightarrow (X,A)$ by $i\colon a \mapsto a$ and $j\colon x \mapsto x$. These inclusions induce homomorphisms $H_q(i)$ and $H_q(j)$, and the sequence of groups and homomorphisms*

$$\begin{aligned} \cdots \xrightarrow{\partial_{q+1}} H_q(A) \xrightarrow{H_q(i)} H_q(X) \xrightarrow{H_q(j)} H_q(X,A) \xrightarrow{\partial_q} \\ H_{q-1}(A) \xrightarrow{H_{q-1}(i)} H_{q-1}(X) \xrightarrow{H_{q-1}(j)} H_{q-1}(X,A) \xrightarrow{\partial_{q-1}} \cdots \end{aligned} \tag{1.3}$$

is an exact sequence.

The sequence (1.3) is called the homology exact sequence of the pair (X,A).

Axiom 3. (Homotopy Axiom) *If $f\colon (X,A) \to (Y,B)$ and $g\colon (X,A) \to (Y,B)$ are homotopic maps, then the induced homomorphisms*

$$H_q(f)\colon H_q(X,A) \longrightarrow H_q(Y,B) \qquad \text{and} \qquad H_q(g)\colon H_q(X,A) \longrightarrow H_q(Y,B)$$

coincide for all $q \in \mathbf{Z}$.

Axiom 4. (Excision Axiom) *For any pair (X,A), let U be an open subset of X such that its closure is contained in the interior of A. Then, the inclusion*

$$k\colon (X-U, A-U) \hookrightarrow (X,A)$$

induces an isomorphism

$$H_q(k)\colon H_q(X-U, A-U) \cong H_q(X,A)\,, \qquad \text{for all } q \in \mathbf{Z}.$$

Axiom 5. (Dimension Axiom) *Let P be a one-point space, and recall that P is identified with the topological pair $(P,\emptyset)$. The homology groups of P are*

$$H_q(P) \cong \begin{cases} \mathbf{Z} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

It should now be demonstrated that homology theories exist. This will be established in the following subsection with the construction of singular homology theory. Also, it is important to understand the extent to which the homology theory is completely determined by the above five axioms. An extensive discussion of this question is given in [36].

(c) Singular Homology

The most common homology theory, and the one that is used in this thesis, is singular homology theory. For simplicity, we define singular homology theory on the subcategory of topological spaces. Almost any text on algebraic topology, such as [29], [36], or [83], will give a definition of singular homology on the full category of topological pairs, and provide a verification of the axioms. The reader should also enjoy [34], one of the original papers on singular homology theory.

Let $(x_0, x_1, \ldots, x_q)$ denote the usual cartesian coordinates for the Euclidean space $\mathbf{R}^{q+1}$. The standard q-simplex is defined to be the subspace

$$\Delta^q \stackrel{\text{def}}{=} \{(x_0, x_1, \ldots, x_q) \mid \sum_{i=0}^{q} x_i = 1 \quad \text{and} \quad x_i \geq 0\} .$$

Geometrically, Δ^q is the convex hull spanned by the vertices $e_0 = (1, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_q = (0, \ldots, 0, 1)$. For example, Δ^0 is a single point, Δ^1 is shown in Figure 1, and Δ^2 is shown in Figure 2.

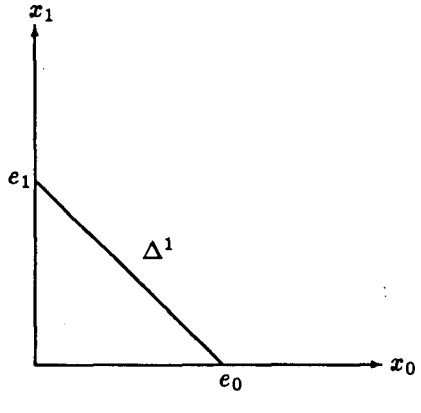

Figure 1. The standard 1-simplex.

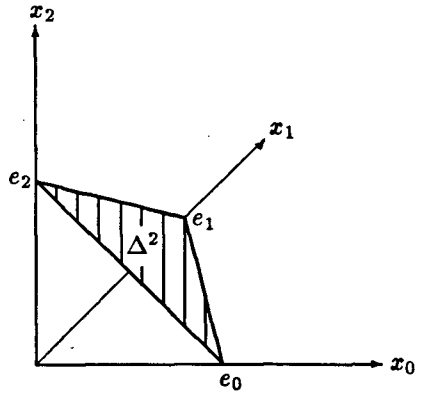

Figure 2. The standard 2-simplex.

A map $f\colon \Delta^p \to \Delta^q$ is said to be simplicial if it is the restriction of a linear function between Euclidean spaces, and maps vertices to vertices. Clearly, a simplicial map is completely determined by its action on vertices.

We define a collection of simplicial maps $f_q^i\colon \Delta^{q-1} \to \Delta^q$ for $i = 0, 1, \ldots, q$ by the vertex assignments

$$f_q^i(e_j) = \begin{cases} e_j & \text{if } j < i \\ e_{j+1} & \text{if } j \geq i \end{cases}$$

Hence, f_q^i maps Δ^{q-1} isometrically onto the $(q-1)$-dimensional face of Δ^q, which is opposite the vertex e_i.

The boundary of Δ^q, denoted by $\partial\Delta^q$, is constructed from the $(q-1)$-dimensional faces of Δ^q. An orientation of Δ^q is represented by an ordering of its vertices. The faces of Δ^q inherit an orientation from the orientation of Δ^q, and this orientation may possibly be different from the orientation inherited from Δ^{q-1} through the simplicial map f_q^i. Writing $\partial\Delta^q$ as a formal algebraic sum of the faces of Δ^q, and using a minus sign to indicate a change of orientation, we have that

$$\partial\Delta^q = \sum_{i=0}^{q} (-1)^i f_q^i(\Delta^{q-1}) \, .$$

For any topological space X, a singular q-simplex is defined to be a continuous map $\sigma\colon \Delta^q \to X$. The ith face of the singular q-simplex σ is the singular $(q-1)$-simplex

$$\sigma^{(i)} = \sigma \circ f_q^i\colon \Delta^{q-1} \longrightarrow X \, .$$

Following the formula for the boundary of Δ^q, the boundary of a singular q-simplex is defined to be the formal algebraic sum

$$\partial\sigma = \sum_{i=0}^{q} (-1)^i \, \sigma^{(i)} \, . \tag{1.4}$$

Our geometric intuition tells us that any reasonable definition of the boundary of a singular simplex should have the property that $\partial(\partial\sigma) = 0$ when interpreted as a formal algebraic sum. Indeed this is true, and the proof is straightforward algebra.

The first step in defining singular homology theory is to use the singular simplices to construct a chain complex. A chain complex C_* is a sequence of abelian groups $\{C_q \mid q \in \mathbf{Z}\}$, and homomorphisms $\partial_q \colon C_q \to C_{q-1}$, which have the property that the composition $\partial_q \partial_{q+1} \colon C_{q+1} \to C_{q-1}$ is the zero homomorphism for all $q \in \mathbf{Z}$. The elements of C_q are called q-chains, and the homomorphisms ∂_q are called boundary homomorphisms.

The singular chain complex for a topological space X is denoted by $S_*(X)$. The group of singular q-chains, $S_q(X)$ is defined to be the free abelian group generated by all of the singular q-simplices of X. For $q < 0$, there are no singular q-simplices, and hence $S_q(X)$ is defined to be the zero group. The elements of $S_q(X)$ are written as formal sums and differences of singular q-simplices. The boundary homomorphism $\partial_q \colon S_q(X) \to S_{q-1}(X)$ is defined by requiring that it map each generator of $S_q(X)$ to its boundary as defined in equation (1.4). It follows directly from the result that $\partial(\partial\sigma) = 0$ for any singular simplex σ, that the composition $\partial_q \partial_{q+1}$ is the zero homomorphism for all $q \in \mathbf{Z}$.

Consider a map f from a topological space X to a topological space Y. For each q, there is an induced homomorphism $S_q(f)\colon S_q(X) \to S_q(Y)$, which is defined by $S_q(f)\colon \sigma \mapsto \sigma \circ f$, for all singular q-simplices σ in $S_q(X)$. The collection of homomorphisms $\{S_q(f)\}$ is denoted by $S_*(f)$. It is easy to verify that $\partial_q S_q(f) = S_{q-1}(f)\partial_q$ for all $q \in \mathbf{Z}$. Any collection of homomorphisms defined on a chain complex and satisfying this property is called a chain map. Therefore, we have defined a covariant functor S_* from the category of topological spaces and maps to the category of chain complexes and chain maps.

Singular homology theory is defined in terms of the singular chain complex. The subgroup $Z_q(X) = \ker \partial_q$ is called the subgroup of q-cycles, and the subgroup $B_q = \operatorname{im} \partial_{q+1}$ is called the subgroup of q-boundaries. Because $\partial_q \partial_{q+1} = 0$, it follows that $B_q(X)$ is a subgroup of $Z_q(X)$. The singular homology group $H_q(X)$ is defined to be the quotient group $Z_q(X) \,/\, B_q(X)$, and the homology class of a cycle $z \in Z_q(X)$ is denoted by $\{z\}$.

Recall that if $f\colon X \to Y$ is a map between two topological spaces, then the induced homomorphism on the singular chain complexes, $S_*(f)\colon S_*(X) \to S_*(Y)$ is a chain map. This implies that for each q, there is a well-defined induced homomorphism from the quotient $Z_q(X) \,/\, B_q(X)$ to $Z_q(Y) \,/\, B_q(Y)$. This homomorphism is denoted by $H_q(f)\colon H_q(X) \to H_q(Y)$.

Having constructed a collection of functors from the category of topological spaces to the category of abelian groups, and a collection of boundary homomorphisms, we have

completed our definition of singular homology theory. It only remains to check the five axioms, for which a good reference is [36, Chapt. VII].

(d) Relation Between Homotopy and Homology

For any topological space X with base point x_0, the homotopy functors provide a sequence of groups $\pi_n(X)$, for $n = 1, 2, 3, \ldots$. In addition, if we ignore the base point x_0 and simply view X as a topological space, then the singular homology functors provide a sequence of abelian groups $H_n(X)$, where $n \in \mathbf{Z}$. The homotopy and singular homology groups are related by the Hurewicz homomorphisms

$$h_n \colon \pi_n(X) \longrightarrow H_n(X), \qquad \text{for } n \geq 1.$$

The Hurewicz homomorphisms are defined as follows. Let $S^n \subset \mathbf{R}^{n+1}$ denote the n-dimensional unit sphere, with base point. The sphere S^n is homeomorphic to $\partial\Delta^{n+1}$, the boundary of the standard $(n+1)$-simplex. Taking the orientation of S^n to be consistent with an outward pointing normal, we define $e \colon \partial\Delta^{n+1} \to S^n$ to be an orientation-preserving homeomorphism. The homeomorphism e defines a singular n-cycle, and its homology class is denoted by $\{e\}$. The homology class $\{e\}$ is a generator for $H_n(S^n)$, which is isomorphic to $\mathbf{Z}$, the abelian group of integers under addition.

Consider any base-point-preserving continuous map $f \colon S^n \to X$. It induces on homology a homomorphism $H_n(f) \colon H_n(S^n) \to H_n(X)$. The homotopy axiom implies that $H_n(f)$ depends only on the homotopy class $[f]$ in $\pi_n(X)$. Therefore, we define the Hurewicz homomorphism by $h_n \colon [f] \mapsto H_n(f)e$.

The Hurewicz homomorphisms are the subject of the Hurewicz theorem, one of the most important theorems in algebraic topology. A proof of the Hurewicz theorem for absolute homotopy and homology groups, and references to the original work of Hurewicz are provided in [34]. The Hurewicz theorem for topological pairs is covered in [48, Chapt. V], [83, Sect. 7.5], and [92, Sect. IV.7].

Before stating the absolute Hurewicz theorem, we introduce the concept of n-connectedness. A topological space X is said to be n-connected if and only if $\pi_k(X) = 0$ for all integers $0 \leq k \leq n$. Observe that 0-connectedness corresponds to path connectedness, and 1-connectedness corresponds to simply connectedness.

Theorem 1.5. (Absolute Hurewicz Theorem)

(i) $n \geq 2$: *If X is an $(n-1)$-connected topological space, then*

$$h_n \colon \pi_n(X) \longrightarrow H_n(X)$$

is an isomorphism, and

$$h_{n+1} \colon \pi_{n+1}(X) \longrightarrow H_{n+1}(X)$$

is an epimorphism.

(ii) $n = 1$: *If X is a path connected topological space, then*

$$h_1 \colon \pi_1(X) \longrightarrow H_1(X)$$

is an epimorphism, with kernel the commutator subgroup of $\pi_1(X)$.

The usual statement of the Hurewicz theorem does not contain the result in part (i) that h_{n+1} is an epimorphism. This is due to G. W. Whitehead [91]. Beware that if X is path connected and h_1 is an isomorphism, it does not follow that h_2 is an epimorphism. As a counterexample, consider the torus $T^2 = S^1 \times S^1$. The homotopy groups for the product of two topological spaces are given as the direct sum of the corresponding homotopy groups for the individual spaces,[2] and therefore $\pi_1(T^2) \cong \mathbf{Z} \oplus \mathbf{Z}$ and $\pi_2(T^2) = 0$. The Hurewicz theorem then implies that $h_1 \colon \pi_1(T^2) \to H_1(T^2)$ is an isomorphism. However, using a cellular decomposition,[3] it may be shown that $H_2(T^2) \cong \mathbf{Z}$. Hence, h_2 cannot possibly be an epimorphism.

(e) Cohomology Theory

In a manner which will be made precise, the concept of cohomology theory is dual to that of homology theory. Here, we shall use two cohomology theories: singular cohomology, which is defined in terms of homomorphisms on singular chains, and de Rham cohomology, which is defined in terms of differential forms.

Our definition of cohomology theory is based upon the axiomatic definition given by Eilenberg and Steenrod [36]. It is necessary to go further then simply give the dual version of the axioms for homology, by defining cohomology with an arbitrary coefficient group G.

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[2] This follows from a special case of the homotopy exact sequence for a fibration. The reader should become familiar with the homotopy exact sequence for a fibration, which is covered in most texts on homotopy theory, such as [48, Sect. V.6], [83, Sect. 7.2], or [92, Sect. IV.8].

[3] For a review of cellular decompositions, see [29, Chapt. V].

Let G be an abelian group. A cohomology theory with coefficients G is a collection of functors $\{H^q \mid q \in \mathbf{Z}\}$, where each H^q is a contravariant functor from the category of topological pairs (or some subcategory) to the category of abelian groups. For a topological pair (X, A), the qth cohomology group with coefficients G is written $H^q(X, A; G)$. If $f\colon (X, A) \to (Y, B)$ is a map of topological pairs, then the contravariant functors H^q induce homomorphisms

$$H^q(f; G)\colon H^q(Y, B; G) \longrightarrow H^q(X, A; G)\,.$$

Note that the direction of the arrow in $H^q(f; G)$ is reversed from that in f. This is what is meant by a contravariant functor. When there is no possibility for confusion, the homomorphisms $H^q(f; G)$ will be denoted collectively by f^*. In addition to the functors H^q, a cohomology theory has a collection of coboundary homomorphisms

$$\delta^q(X, A; G)\colon H^q(A; G) \longrightarrow H^{q+1}(X, A; G)\,.$$

Often, these homomorphisms will simply be denoted by δ^*.

A cohomology theory must satisfy the following five axioms.

Axiom 1. (Naturality of δ^*) *For any map of topological pairs $f\colon (X, A) \to (Y, B)$, let $f|_A\colon A \to B$ denote the restriction of f to the subspace A. Then, the diagram*

$$\begin{tikzcd}
H^q(B; G) \arrow[r, "H^q(f|_A;G)"] \arrow[d, "\delta^q(Y,B;G)"] & H^q(A; G) \arrow[d, "\delta^q(X,A;G)"] \\
H^{q+1}(Y, B; G) \arrow[r, "H^{q+1}(f;G)"] & H^{q+1}(X, A; G)
\end{tikzcd}$$

is commutative.

Axiom 2. (Exactness Axiom) *For a topological pair (X, A) define the inclusion maps $i\colon A \hookrightarrow X$ and $j\colon X \hookrightarrow (X, A)$ as in Axiom 2 for homology theory. Then, the sequence of groups and induced homomorphisms,*

$$\cdots \xrightarrow{i^*} H^{q-1}(A;G) \xrightarrow{\delta^*} H^q(X,A;G) \xrightarrow{j^*} H^q(X;G) \xrightarrow{i^*} H^q(A;G) \xrightarrow{\delta^*} H^{q+1}(X,A;G) \xrightarrow{j^*} \cdots \quad (1.7)$$

is an exact sequence.

The exact sequence (1.7) is called the cohomology exact sequence of the pair (X, A).

Axiom 3. (Homotopy Axiom) *If $f\colon (X,A) \to (Y,B)$ and $g\colon (X,A) \to (Y,B)$ are homotopic maps, then the induced homomorphisms $H^q(f;G)$ and $H^q(g;G)$ coincide for all $q \in \mathbf{Z}$.*

Axiom 4. (Excision Axiom) *For any pair (X, A), let U be any open subset of X such that its closure is contained in the interior of A. Then, the homomorphisms $H^q(k;G)$ induced by the inclusion*

$$k\colon (X - U, A - U) \hookrightarrow (X, A)\,,$$

are isomorphisms.

Axiom 5. (Dimension Axiom) *The cohomology groups of the one-point space, P are*

$$H^q(P;G) \cong \begin{cases} G & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

The most common cohomology theories are singular cohomology, Čech cohomology, and de Rham cohomology. Singular cohomology and de Rham cohomology will be defined in the remaining two subsections.

We remark that homology theory with an arbitrary coefficient group may be defined by generalizing in the obvious way the definition of homology theory in Subsection 1.b. In the context of homology theories with general coefficient groups, the definition in Subsection 1.b corresponds to homology theory with integer coefficients.

(f) Singular Cohomology

Just as singular homology theory is defined in terms of a chain complex, singular cohomology theory is defined in terms of a cochain complex. A cochain complex C^* is a sequence of abelian groups $\{C^q \mid q \in \mathbf{Z}\}$ and homomorphisms $\delta^q \colon C^q \to C^{q+1}$, which have the property that the composition $\delta^{q+1}\delta^q$ is the zero homomorphism for all $q \in \mathbf{Z}$.

A cochain complex may be obtained from a chain complex through the Hom functor. If C_* is a chain complex and G is an abelian group, then Hom is a functor of two arguments, contravariant in the first and covariant in the second, which maps the pair (C_*, G) to a cochain complex denoted by $\mathrm{Hom}(C_*, G)$.[4] For each $q \in \mathbf{Z}$, the group C^q of

[4] If $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are categories, and T is a functor from $\mathcal{B} \times \mathcal{C}$ to $\mathcal{D}$ which is contravariant in $\mathcal{B}$ and covariant in $\mathcal{C}$, then T is called a bifunctor. Hom is an example of a bifunctor.

q-cochains in the complex $\mathrm{Hom}(C_*, G)$ is defined to be the group of all homomorphisms from C_q to G. The coboundary homomorphism $\delta^q\colon C^q \to C^{q+1}$ is defined by $(\delta^q f)(c) = f(\partial_{q+1}c)$ for all $f \in \mathrm{Hom}(C_q, G)$ and $c \in C_{q+1}$. It is straightforward to verify that $\partial_{q+1}\partial_{q+2} = 0$ implies that $\delta^{q+1}\delta^q = 0$ for all $q \in \mathbf{Z}$.

If $S_*(X)$ is the singular chain complex of the topological space X, then the singular cochain complex with coefficient group G is defined as

$$S^*(X; G) \stackrel{\text{def}}{=} \mathrm{Hom}(S_*(X), G)\ .$$

Notice that it follows from the definition of the singular chain complex that $S^q(X; G) = 0$ for all $q < 0$.

The group of q-cochains contains the subgroup of q-cocycles, $Z^q(X; G) = \ker \delta^q$, and the subgroup of q-coboundaries, $B^q(X; G) = \mathrm{im}\, \delta^{q-1}$. Because $\delta^q\delta^{q-1} = 0$, it follows that $B^q(X; G) \subset Z^q(X; G)$, for each $q \in \mathbf{Z}$. The singular cohomology group with coefficient group G is defined to be the quotient group $H^q(X; G) = Z^q(X; G) \,/\, B^q(X; G)$.

If f is a continuous map from X to Y, the functors S_* and Hom provide an induced cochain homomorphism from $S^*(Y; G)$ to $S^*(X; G)$. This homomorphism of chain complexes induces a homomorphism $H^q(f; G)\colon H^q(Y; G) \to H^q(X; G)$ on the cohomology groups. Again, the notation $H^q(f; G)$ is usually abbreviated to f^*.

The relationship between singular homology and singular cohomology is given by the universal coefficient theorem for cohomology. Universal coefficient theorems are described in detail in [29], [61], and [83].

Before stating the universal coefficient theorem for cohomology, we must first introduce short exact sequences and the derived functor Ext. For simplicity, we shall not discuss derived functors in general, but rather give a somewhat parochial construction of the Ext functor. An excellent introduction to derived functors is [61, Chapt. XII]. An exhaustive treatment is given in [18].

A sequence of abelian groups and homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \tag{1.7}$$

is said to be exact at B if $\operatorname{im}\alpha = \ker\beta$. The exact sequence (1.7) is called a short exact sequence if moreover α is a monomorphism and β is an epimorphism. Sometimes this is indicated by writing the short exact sequence (1.7) as the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0\,,$$

which is exact at A, B, and C.

A homomorphism γ of short exact sequences is a triple of homomorphisms $(\gamma_1, \gamma_2, \gamma_3)$ such that

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow{\scriptstyle\gamma_1} & & \downarrow{\scriptstyle\gamma_2} & & \downarrow{\scriptstyle\gamma_3} \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

is a commutative diagram. If γ_1, γ_2, and γ_3 are isomorphisms, then the short exact sequences $A \longrightarrow B \longrightarrow C$ and $A' \longrightarrow B' \longrightarrow C'$ are said to be isomorphic. Note, that if γ_1 and γ_3 are isomorphisms, then it follows from the five lemma that γ_2 is also an isomorphism.

An important example of a short exact sequence is the direct sum short exact sequence

$$A \xrightarrow{\iota} A \oplus B \xrightarrow{\pi} B\,.$$

The monomorphism ι is defined by $\iota\colon a \mapsto (a, 0)$, and the epimorphism π is defined by $\pi\colon (a, b) \mapsto b$.

A short exact sequence $A \longrightarrow B \longrightarrow C$ is said to be split if it is isomorphic, with the identity on A and C, to the short exact sequence $A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C$. Some properties of split short exact sequences are given in

Lemma 1.8. *A short exact sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is split iff any one of the following holds:*

(i) *There exists a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "\beta"] \arrow[d, "\gamma"] & C \arrow[d, equal] \\ A \arrow[r, "\iota"] & A \oplus C \arrow[r, "\pi"] & C \end{tikzcd}$$

(ii) *There exists a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, equal] & B \arrow[r, "\beta"] & C \arrow[d, equal] \\ A \arrow[r, "\iota"] & A \oplus C \arrow[r, "\pi"] \arrow[u, "\gamma'"'] & C \end{tikzcd}$$

(iii) *There exists a homomorphism $\beta'\colon C \to B$ such that $\beta\beta'$ is the identity on C. The homomorphism β' is called a right inverse for β.*

(iv) *There exists a homomorphism $\alpha'\colon B \to A$ such that $\alpha'\alpha$ is the identity on A. The homomorphism α' is called a left inverse for α.*

Proof. Parts (i) and (ii) follow from the five lemma. Parts (iii) and (iv) are proven in Proposition I.4.3 of [61]. □

This lemma allows us to prove the following useful result.

Proposition 1.9. *If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is a short exact sequence of abelian groups, and C is free, then this short exact sequence is split.*

Proof. Let $\{c_i\}$ be a basis for C. For each i, choose an element $b_i \in B$ such that $\beta b_i = c_i$. Recall that β is an epimorphism, and therefore $\beta^{-1}(c_i)$ is nonempty for all i. Define a homomorphism $\beta' \colon C \to B$ by requiring that $\beta' c_i = b_i$. Because C is freely generated, it follows that β' is well-defined. Obviously, β' is a right inverse for β, and therefore the proof follows from part (iii) of Lemma 1.8. □

If A and C are fixed abelian groups, then an extension of A by C is defined to be a short exact sequence $A \to B \to C$. Two extensions of A by C are said to be equivalent, if they are isomorphic as short exact sequences with the identity on A and C. The set of equivalence classes of extensions of A by C is denoted by $\mathrm{Ext}(C, A)$.

Let $A_1 \to B_1 \to C_1$ be an element of the equivalence class $E_1 \in \mathrm{Ext}(C_1, A_1)$, and $A_2 \to B_2 \to C_2$ be an element of the equivalence class $E_2 \in \mathrm{Ext}(C_2, A_2)$. The direct sum $E_1 \oplus E_2 \in \mathrm{Ext}(C_1 \oplus C_2, A_1 \oplus A_2)$ is defined to be the equivalence class of the short exact sequence

$$A_1 \oplus A_2 \to B_1 \oplus B_2 \to C_1 \oplus C_2 \,.$$

It is easy to check that $E_1 \oplus E_2$ is independent of the representative elements that are chosen for E_1 and E_2.

We now show that extensions may be used to define a bifunctor from pairs of abelian groups to the category of sets and set homomorphisms. Let $A \longrightarrow B \longrightarrow C$ be an extension of A by C, and denote its equivalence class by $E \in \mathrm{Ext}(C, A)$. For an abelian group C', consider a homomorphism $\phi\colon C' \to C$. Lemma III.1.2 in [61] states that there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B' & \longrightarrow & C' \\ \| & & \downarrow & & \downarrow{\scriptstyle \phi} \\ A & \longrightarrow & B & \longrightarrow & C \end{array} \tag{1.10}$$

for some abelian group B'. Furthermore, the equivalence class $E' \in \mathrm{Ext}(C', A)$ of the extension $A \longrightarrow B' \longrightarrow C'$, is uniquely determined by the class E and the homomorphism ϕ. This defines a morphism from $\mathrm{Ext}(C, A)$ to $\mathrm{Ext}(C', A)$.

Now, consider a homomorphism ψ from A to an abelian group A'. From Lemma III.1.4 in [61], we know that there exists a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow{\scriptstyle \psi} & & \downarrow & & \| \\ A' & \longrightarrow & B'' & \longrightarrow & C \end{array} \tag{1.11}$$

for some abelian group B''. Furthermore, the equivalence class $E'' \in \mathrm{Ext}(C, A')$ of the extension $A' \longrightarrow B'' \longrightarrow C$ is uniquely determined by the class E, and the homomorphism ψ. This defines a morphism from $\mathrm{Ext}(C, A)$ to $\mathrm{Ext}(C, A')$.

Suppose that we have two pairs of abelian groups (C, A) and (C', A'), and homomorphisms $\phi\colon C' \to C$ and $\psi\colon A \to A'$. If the morphism induced by (1.10) is applied first,

and then the morphism induced by (1.11), we obtain the composition of morphisms

$$\mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A) \longrightarrow \mathrm{Ext}(C', A') \,. \tag{1.12}$$

However, if the induced morphisms are computed in the opposite order, then we obtain the composition

$$\mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C, A') \longrightarrow \mathrm{Ext}(C', A') \,. \tag{1.13}$$

It turns out that the compositions (1.12) and (1.13) coincide [61, Lemma III.1.6], and they define a morphism which is denoted by

$$\mathrm{Ext}(\phi, \psi)\colon \mathrm{Ext}(C, A) \longrightarrow \mathrm{Ext}(C', A') \,.$$

Therefore, Ext is a bifunctor from pairs of abelian groups to the category of sets and morphisms.

The set $\mathrm{Ext}(C, A)$ has a natural group structure defined on it. In order to construct this group structure, we must first define the diagonal and codiagonal homomorphisms. For an abelian group G, the diagonal homomorphism $\Delta\colon G \to G \oplus G$ is $\Delta g = (g, g)$, and the codiagonal homomorphism $\Delta'\colon G \oplus G \to G$ is $\Delta'(g_1, g_2) = g_1 + g_2$. For $E_1, E_2 \in \mathrm{Ext}(C, A)$, the Bauer sum is defined by

$$E_1 + E_2 \stackrel{\text{def}}{=} \mathrm{Ext}(\Delta, \Delta')(E_1 \oplus E_2) \in \mathrm{Ext}(C, A) \,.$$

Under the Bauer sum, $\mathrm{Ext}(C, A)$ is an additive group [61, Thm. III.2.1]. The zero element of this group is the equivalence class of the split short exact sequence $A \xrightarrow{\iota} A \oplus C \xrightarrow{\pi} C$. Also, for group homomorphisms $\phi\colon C' \to C$ and $\psi\colon A \to A'$, the induced

map $\mathrm{Ext}(\phi, \psi)$ is a group homomorphism. Therefore, if $\mathcal{AG}$ is the category of abelian groups, then Ext is a bifunctor from $\mathcal{AG} \times \mathcal{AG}$ to $\mathcal{AG}$.

If C is a free abelian group, then it follows from Proposition 1.9 that $\mathrm{Ext}(C, A) = 0$ for all abelian groups A. Many of the Ext groups which arise in this thesis will fall under this example.

Now that the Ext functor has been defined, we are able to state the universal coefficient theorem for cohomology. This theorem is proven in many textbooks on algebraic topology, such as [29, p. 153], [61, Thm. III.4.1], and [83, Thm. 5.5.3].

Theorem 1.14. (Universal Coefficient Theorem for Cohomology) *If X is a topological space, and G is an abelian group, then there is a natural short exact sequence*

$$\mathrm{Ext}(H_{q-1}(X), G) \longrightarrow H^q(X; G) \longrightarrow \mathrm{Hom}(H_q(X), G) \ ,$$

which is split (although, not naturally) for all $q \in \mathbf{Z}$.

This theorem implies that the singular cohomology group $H^q(X; G)$ is isomorphic to $\mathrm{Hom}(H_q(X), G) \oplus \mathrm{Ext}(H_{q-1}(X), G)$.

Recall that it was remarked in Subsection 1.e, that singular homology may be defined with an arbitrary coefficient group. For singular homology with arbitrary coefficients, there is a universal coefficient theorem for homology. We will not review this theorem, as it is not required in this thesis. However, interested readers will find it discussed in most texts on algebraic topology, including [29], [61], and [83].

(g) De Rham Cohomology

De Rham cohomology is a cohomology theory defined on the category of smooth manifolds and smooth maps between them. Throughout this thesis, smooth shall always mean infinitely differentiable. De Rham cohomology satisfies the five axioms for cohomology in Subsection 1.e.

An elementary introduction to de Rham cohomology is given in Chapter 1 of [16]. Also suggested as references are de Rham's classic book [72], and [94]. References to many of the original papers on de Rham theory are given in [72].

Unlike the above references, we shall define de Rham cohomology with coefficients in the complex numbers $\mathbf{C}$, rather than the real numbers $\mathbf{R}$. For a smooth manifold $\mathcal{M}$, let $\Omega^p(\mathcal{M})$ denote the set of differential p-forms on $\mathcal{M}$ with complex coefficients. We shall allow p to range over $\mathbf{Z}$, and define $\Omega^0(\mathcal{M})$ to be the set of $\mathbf{C}$-valued smooth functions on $\mathcal{M}$, and $\Omega^q(\mathcal{M})$ to be zero for $q < 0$. Under the operation of addition of differential p-forms, $\Omega^p(\mathcal{M})$ is an abelian group. Indeed, it has the further structure of a vector space over $\mathbf{C}$.[5] The exterior differentiation operator

$$d\colon \Omega^p(\mathcal{M}) \longrightarrow \Omega^{p+1}(\mathcal{M})$$

is a group homomorphism for each $p \in \mathbf{Z}$. The collection of abelian groups $\{\Omega^p(\mathcal{M})\}$ and homomorphisms d form a cochain complex $\Omega^*(\mathcal{M})$, which is called the de Rham complex.

[5] Abelian groups and vector spaces are examples of $\mathcal{R}$-modules [61, Chapt. 1], where $\mathcal{R}$ is a commutative ring with unit. Homology and cohomology may generally by defined as collections of functors from the category of topological pairs to the category of $\mathcal{R}$-modules.

The de Rham cohomology of $\mathcal{M}$ is defined to be the cohomology of the cochain complex $\Omega^*(\mathcal{M})$. Specifically, $H^p_{DR}(\mathcal{M};\mathbf{C})$ is the quotient of the group $\ker d \subset \Omega^p(\mathcal{M})$ by the subgroup $\operatorname{im} d \subset \Omega^p(\mathcal{M})$. Since both $\ker d$ and $\operatorname{im} d$ are vector spaces over $\mathbf{C}$, it follows that $H^p_{DR}(\mathcal{M};\mathbf{C})$ also has the structure of a vector space over $\mathbf{C}$. To define H^p_{DR} as a contravariant functor, consider two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$, and a smooth map $f\colon \mathcal{M}_1 \to \mathcal{M}_2$. For $\omega \in \Omega^p(\mathcal{M}_2)$, the pullback[6] $f^*\omega$ is an element of $\Omega^p(\mathcal{M}_1)$. Furthermore, f^* commutes with d, which means that f^* is a cochain homomorphism. This implies that f^* induces a homomorphism

$$H^p_{DR}(f)\colon H^p_{DR}(\mathcal{M}_2;\mathbf{C}) \longrightarrow H^p_{DR}(\mathcal{M}_1;\mathbf{C})\,.$$

For a pair of manifolds $(\mathcal{M},\mathcal{N})$ with $\mathcal{N} \subset \mathcal{M}$, it is possible to define the relative de Rham cohomology $H^*_{DR}(\mathcal{M},\mathcal{N};\mathbf{C})$. For a definition of relative de Rham cohomology, see [16, pp. 78–79].

On the category of smooth manifolds, we have defined two cohomology theories with coefficients in $\mathbf{C}$: singular cohomology $H^*(\,\cdot\,;\mathbf{C})$ and de Rham cohomology $H^*_{DR}(\,\cdot\,;\mathbf{C})$. The relationship between these cohomology theories is the subject of the de Rham theorem [72, Chapt. IV], [94, Thm. IV.29A], which we shall now develop.

Recall that a singular p-simplex σ of a manifold $\mathcal{M}$ is a continuous map $\sigma\colon \Delta^p \to \mathcal{M}$. The singular chain complex constructed from all the singular simplices of $\mathcal{M}$ is $S_*(\mathcal{M})$. A smooth singular p-simplex σ is defined to be a map $\sigma\colon \Delta^p \to \mathcal{M}$, which can be extended to a smooth map defined on an open neighbourhood of Δ^p in $\mathbf{R}^{p+1}$. The

[6] Pullbacks of differential forms are defined in [16, p. 19].

chain complex constructed from all of the smooth singular simplices of $\mathcal{M}$ is denoted by $S_*^S(\mathcal{M})$. Obviously, $S_*^S(\mathcal{M})$ is a subcomplex of $S_*(\mathcal{M})$, and we denote the inclusion by $\epsilon\colon S_*^S(\mathcal{M}) \hookrightarrow S_*(\mathcal{M})$.

At this point, it is necessary to describe some general properties of chain complexes. Consider two chain complexes C and C' with boundary homomorphisms ∂ and ∂', respectively. Two chain maps $\tau, \eta\colon C \to C'$ are said to be chain homotopic if there exists a collection of homomorphisms $\{D_q \mid q \in \mathbf{Z}\}$ such that for all q,

$$\tau_q - \eta_q = \partial'_{q+1} D_q + D_{q-1}\partial_q \ .$$

Chain homotopy, which defines an equivalence relation on the set of all chain maps between C and C', is reviewed in [83, Sect. 4.2]. The category with chain complexes as its objects and homotopy classes of chain maps as its morphisms is called the homotopy category of chain complexes. If a chain map $\tau : C \to C'$ is an equivalence in the homotopy category of chain complexes, then it is called a chain equivalence, and C and C' are said to be chain equivalent. The usefulness of this concept lies in the fact that free chain complexes are chain equivalent if and only if their homologies are isomorphic [83, Thm. 4.6.10].

As a special case of Theorem II in [35], S. Eilenberg proved the following about the chain map ϵ.

Lemma 1.15. *The inclusion $\epsilon\colon S_*^S(\mathcal{M}) \hookrightarrow S_*(\mathcal{M})$ is a chain equivalence.*

We remark that for a chain complex constructed from singular cubes, the result corresponding to Lemma 1.15 is proven in [62, Appendix]. This proof is easily modified to give an alternate, and more modern proof of Lemma 1.15.

It follows from Lemma 1.15 that the induced cochain map

$$\mathrm{Hom}(\epsilon)\colon\ \mathrm{Hom}(S_*(\mathcal{M}),\mathbf{C}) \longrightarrow \mathrm{Hom}(S_*^S(\mathcal{M}),\mathbf{C})$$

is a cochain equivalence. Denoting the cohomology obtained from $S_S^*(\mathcal{M};\mathbf{C}) = \mathrm{Hom}(S_*^S(\mathcal{M}),\mathbf{C})$ by $H_S^*(\mathcal{M};\mathbf{C})$, this implies

Corollary 1.16. *The induced homomorphism $\epsilon^*\colon H^*(\mathcal{M};\mathbf{C}) \to H_S^*(\mathcal{M};\mathbf{C})$ is an isomorphism.* □

Given a p-form $\omega \in \Omega^p(\mathcal{M})$, we define a cochain $k \in S_S^p(\mathcal{M};\mathbf{C})$ by requiring that

$$k(\sigma) = \int_\sigma \omega = \int_{\Delta^p} \sigma^*\omega$$

for all smooth singular simplices $\sigma \in S_p^S(\mathcal{M})$. This defines a homomorphism $\psi\colon \Omega^p(\mathcal{M}) \to S_S^p(\mathcal{M})$. Stokes' theorem is just the statement that ψ is a cochain map. Therefore, ψ induces a homomorphism $\psi^*\colon H_{DR}^*(\mathcal{M};\mathbf{C}) \to H_S^*(\mathcal{M};\mathbf{C})$ on cohomology.

Theorem 1.17. (De Rham Theorem) *The de Rham homomorphism $\psi^*\colon H_{DR}^*(\mathcal{M};\mathbf{C}) \to H_S^*(\mathcal{M};\mathbf{C})$ is an isomorphism.*

Notice that Corollary 1.16 combined with the de Rham theorem imply that the composition

$$(\epsilon^*)^{-1}\psi^* \colon H^*_{DR}(\mathcal{M};\mathbf{C}) \longrightarrow H^*(\mathcal{M};\mathbf{C}) \tag{1.18}$$

is also an isomorphism. This result will be used in Chapter III.

The de Rham theorem also holds for relative cohomology groups. The relative version of the de Rham theorem is easily proven using the absolute de Rham theorem and the five lemma. For some hints on the proof, see [28, Chapt. 1], where the corresponding theorem is proven in the context of rational de Rham theory for arbitrary topological spaces.

§2 Flag Manifolds

In Chapters II and III, we shall need various results on the algebraic topology of flag manifolds. These results are not difficult to prove, and many of them represent a standard application of the concepts introduced in the preceding section. Indeed, with the exception of de Rham cohomology, this section should provide a good test of the reader's understanding of the previous section! Unfortunately, we have been unable to find a reference in the literature which provides a discussion of flag manifolds along the lines required here.

(a) Introduction to Flag Manifolds

To begin, we define complex flag manifolds. For positive integers $n_1, \ldots, n_p$, which satisfy $\sum_{i=1}^{p} n_i = n$, the set $F(n_1, \ldots, n_p)$ is defined to be the collection of all sequences of vector subspaces $V_1 \subset V_2 \subset \cdots \subset V_p = \mathbf{C}^n$ of the vector space $\mathbf{C}^n$ such that their complex dimensions satisfy $\dim_{\mathbf{C}}(V_j) = \sum_{i=1}^{j} n_i$. Given the usual inner product on $\mathbf{C}^n$, this set can be identified with the set of all ordered p-tuples $(E_1, E_2, \ldots, E_p)$ of mutually orthogonal subspaces of $\mathbf{C}^n$, which satisfy $\dim_{\mathbf{C}} E_i = n_i$. This identification is constructed by taking $E_1 = V_1$, and E_i to be the orthogonal complement of V_{i-1} in V_i, for $2 \leq i \leq p$.

In terms of the usual orthonormal basis for $\mathbf{C}^n$, there is a one-to-one correspondence between ordered, orthonormal complex n-frames $(e_1, e_2, \ldots, e_n)$ in $\mathbf{C}^n$, and elements in $\mathbf{U}(n)$, the group of $n \times n$ unitary matrices. Given an n-frame, we associate to it an element of $F(n_1, \ldots, n_p)$ by setting the subspace E_1 equal to the span of $(e_1, \ldots, e_{n_1})$, the subspace E_2 equal to the span of $(e_{n_1+1}, \ldots, e_{n_1+n_2})$, and so on. Let $N_i = \sum_{j=1}^{i} n_j$, and notice that two subframes, $(e_{N_i+1}, \ldots, e_{N_i+n_{i+1}})$ and $(e'_{N_i+1}, \ldots, e'_{N_i+n_{i+1}})$ give the same subspace $E_{n_{i+1}}$ if and only if they are related by an element of $\mathbf{U}(n_{i+1})$. Therefore as a set, $F(n_1, \ldots, n_p)$ corresponds to the quotient $\mathbf{U}(n)/\mathbf{U}(n_1) \times \mathbf{U}(n_2) \times \cdots \times \mathbf{U}(n_p)$. We define $\mathbf{U}(n_1) \times \cdots \times \mathbf{U}(n_p)$ to be the subgroup of all matrices which are block diagonal of the form

$$\begin{bmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & M_p \end{bmatrix} \tag{2.1}$$

where $M_i \in \mathbf{U}(n_i)$ for all $i = 1, \ldots, p$. The set $F(n_1, \ldots, n_p)$ can be given a complex analytic structure to make it into a connected, compact complex manifold, which is called a complex flag manifold [85, Sect. 1].

Now, we define real flag manifolds. For positive integers $n_1, \ldots n_p$, satisfying $\sum_{i=1}^{p} n_i = n$, the set $F'(n_1, \ldots, n_p)$ is defined to be the collection of all sequences of subspaces $V_1 \subset V_2 \subset \cdots \subset V_p = \mathbf{R}^n$ of the vector space $\mathbf{R}^n$ such that their real dimensions satisfy $\dim_{\mathbf{R}}(V_j) = \sum_{i=1}^{j} n_i$. Let $\mathbf{O}(n)$ denote the group of $n \times n$ orthogonal matrices. Then, after fixing the usual orthonormal basis for $\mathbf{R}^n$, we see by the same reasoning as for complex flag manifolds that as a set, $F'(n_1, \ldots n_p)$ can be identified with the quotient $\mathbf{O}(n)/\mathbf{O}(n_1) \times \cdots \times \mathbf{O}(n_p)$, where $\mathbf{O}(n_1) \times \cdots \times \mathbf{O}(n_p)$ is the subgroup consisting of all block diagonal matrices of the same form as (2.1), except that now $M_i \in \mathbf{O}(n_i)$, for all $i = 1, \ldots, p$. Because the quotient of a compact Lie group by a closed subgroup is a compact manifold, it follows that $F'(n_1, \ldots, n_p)$ is a connected, compact manifold, and it is called a real flag manifold.

In this thesis, we are primarily concerned with two special cases of flag manifolds: short flag manifolds and Grassmann manifolds. The complex and real, short flag manifolds, denoted by $F(p, q, r)$ and $F'(p, q, r)$, respectively, are flag manifolds which have at most three nonzero arguments. The complex and real Grassmann manifolds, $G(p, q)$ and $G'(p, q)$, respectively, are flag manifolds which have two nonzero arguments. We remark that even though a Grassmann manifold may be viewed as a special case of a short flag manifold, we shall assume, unless otherwise stated, that the arguments of F, F', G, and G' are strictly positive. The distinct symbols F and G, will be used for short flag manifolds and Grassmann manifolds, respectively. This is done because there

are important differences between short flag manifolds and Grassmann manifolds, which manifest themselves in their homotopy and homology groups.

Since $F(p,q,r)$ is diffeomorphic to $\mathbf{U}(p+q+r)/\mathbf{U}(p)\times\mathbf{U}(q)\times\mathbf{U}(r)$, and $F'(p,q,r)$ is diffeomorphic to $\mathbf{O}(p+q+r)/\mathbf{O}(p)\times\mathbf{O}(q)\times\mathbf{O}(r)$, it follows that these manifolds appear naturally as the base spaces in the following two fibre bundles:

$$\begin{array}{ccc} \mathbf{U}(p)\times\mathbf{U}(q)\times\mathbf{U}(r) & \xrightarrow{\ i\ } & \mathbf{U}(p+q+r) \\ & & \Big\downarrow{\scriptstyle p} \\ & & F(p,q,r) \end{array} \tag{2.2}$$

and

$$\begin{array}{ccc} \mathbf{O}(p)\times\mathbf{O}(q)\times\mathbf{O}(r) & \xrightarrow{\ i'\ } & \mathbf{O}(p+q+r) \\ & & \Big\downarrow{\scriptstyle p'} \\ & & F'(p,q,r) \end{array} \tag{2.3}$$

Similarly, the complex and real Grassmann manifolds are the base spaces of the fibre bundles

$$\begin{array}{ccc} \mathbf{U}(p)\times\mathbf{U}(q) & \xrightarrow{\ i\ } & \mathbf{U}(p+q) \\ & & \Big\downarrow{\scriptstyle p} \\ & & G(p,q) \end{array} \tag{2.4}$$

and

$$\begin{array}{ccc} \mathbf{O}(p)\times\mathbf{O}(q) & \xrightarrow{\ i'\ } & \mathbf{O}(p+q) \\ & & \Big\downarrow{\scriptstyle p'} \\ & & G'(p,q) \end{array} \tag{2.5}$$

respectively.

The above four fibre bundles are specific examples of fibre bundles in which the fibre, base space, and total space are all homogeneous spaces. More generally, if G is a compact Lie group with closed subgroups G_1 and G_2 such that $G_2 \subset G_1 \subset G$, then the natural projection $p\colon G/G_2 \to G/G_1$ is the projection of a fibre bundle, and the fibre inclusion

$i\colon G_1/G_2 \hookrightarrow G/G_2$ is the inclusion of the cosets. Further fibre bundles of this form, which are used in this thesis, are:

$$\begin{array}{ccc} G(p,q) & \xrightarrow{i_1} & F(p,q,r) \\ & & \downarrow{\scriptstyle p_1} \\ & & G(p+q,r) \end{array} \qquad \begin{array}{ccc} G'(p,q) & \xrightarrow{i_1'} & F'(p,q,r) \\ & & \downarrow{\scriptstyle p_1'} \\ & & G'(p+q,r) \end{array} \tag{2.6}$$

and

$$\begin{array}{ccc} G(q,r) & \xrightarrow{i_2} & F(p,q,r) \\ & & \downarrow{\scriptstyle p_2} \\ & & G(p,q+r) \end{array} \qquad \begin{array}{ccc} G'(q,r) & \xrightarrow{i_2'} & F'(p,q,r) \\ & & \downarrow{\scriptstyle p_2'} \\ & & G'(p,q+r) \end{array} \tag{2.7}$$

It is useful for us to consider two sorts of imbeddings of flag manifolds. One of them imbeds a real flag manifold into the complex flag manifold with the same arguments. The other imbeds a complex Grassmann manifold into a larger complex Grassmann manifold, and similarly for real Grassmann manifolds.

For any pair of flag manifolds (incuding Grassmann manifolds) $F'(n_1,\ldots,n_p)$ and $F(n_1,\ldots,n_p)$, the inclusion of $\mathbf{O}(k)$ in $\mathbf{U}(k)$ induces a smooth imbedding

$$j\colon F'(n_1,\ldots,n_p) \hookrightarrow F(n_1,\ldots,n_p)\,. \tag{2.8}$$

Notice that the real dimension of $F'(n_1,\ldots,n_p)$ is half of the real dimension of $F(n_1,\ldots,n_p)$. Indeed, $F(n_1,\ldots,n_p)$ is a complex algebraic variety defined over $\mathbf{R}$ [13, III.10.3], and j is simply the inclusion of its set of real points [13, V.15.3].

Now, for integers $1 \le a \le p$ and $1 \le b \le q$, we define the inclusion $G(a,b) \hookrightarrow G(p,q)$ as follows. Let $\theta\colon \mathbf{U}(a+b) \hookrightarrow \mathbf{U}(p+q)$ be the smooth imbedding which maps $W \in \mathbf{U}(a+b)$

to the $(p+q) \times (p+q)$ unitary matrix

$$\begin{bmatrix} I_{p-a} & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & I_{q-b} \end{bmatrix}$$

where I_j is the $j \times j$ identity matrix. Since the image under θ of the subgroup $\mathbf{U}(a) \times \mathbf{U}(b)$ is equal to the intersection of the image of $\mathbf{U}(a+b)$ with the subgroup $\mathbf{U}(p) \times \mathbf{U}(q)$, it follows that θ induces a smooth imbedding

$$\alpha\colon G(a,b) \hookrightarrow G(p,q)\,. \tag{2.9}$$

The restriction of α to the submanifold of real points defines the smooth imbedding

$$\alpha'\colon G'(a,b) \hookrightarrow G'(p,q)\,. \tag{2.10}$$

(b) Algebraic Topology of Flag Manifolds

We begin by computing the homotopy groups π_1 and π_2 for complex Grassmann manifolds. This is done by examining the homotopy exact sequence for the fibre bundle (2.4). For integers $1 \le a \le p$ and $1 \le b \le q$, the imbeddings θ and α induce the homomorphism from the homotopy exact sequence for $G(a,b)$ to the homotopy exact sequence for $G(p,q)$

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \pi_2(G(a,b)) \arrow[r, "\partial_\#"] \arrow[d, "\alpha_\#"] & \pi_1(\mathbf{U}(a) \times \mathbf{U}(b)) \arrow[r] \arrow[d, "\hat{\theta}_\#"] & \pi_1(\mathbf{U}(a+b)) \arrow[r] \arrow[d, "\theta_\#"] & \pi_1(G(a,b)) \arrow[r] \arrow[d, "\alpha_\#"] & 0 \arrow[d] \\
0 \arrow[r] & \pi_2(G(p,q)) \arrow[r, "\partial_\#"] & \pi_1(\mathbf{U}(p) \times \mathbf{U}(q)) \arrow[r] & \pi_1(\mathbf{U}(p+q)) \arrow[r] & \pi_1(G(p,q)) \arrow[r] & 0
\end{tikzcd} \tag{2.11}$$

where $\hat{\theta}$ denotes the restriction of θ to the subgroup $\mathbf{U}(a) \times \mathbf{U}(b)$. Notice that we have made use of the fact that π_2 of a Lie group is zero.

By the stability of the homotopy groups of $\mathbf{U}(n)$ [49, Thm. 7.4.1], $\hat{\theta}_\#$ and $\theta_\#$ are both isomorphisms. It then follows by diagram chasing,[7] that the homomorphisms $\alpha_\#\colon \pi_1(G(a,b)) \to \pi_1(G(p,q))$ and $\alpha_\#\colon \pi_2(G(a,b)) \to \pi_2(G(p,q))$ are also isomorphisms.

Specifically, by setting $a = b = 1$ and noticing that $G(1,1)$ is the complex projective space $\mathbf{C}P(1)$, which is diffeomorphic to S^2, we conclude that for all integers $p, q \geq 1$,

$$\pi_1(G(p,q)) = 0 \qquad \text{and} \qquad \pi_2(G(p,q)) \cong \mathbf{Z}\,. \tag{2.12}$$

Furthermore, because $\alpha_\#\colon \pi_2(S^2) \to \pi_2(G(p,q))$ is an isomorphism, the smooth imbedding $\alpha\colon S^2 \hookrightarrow G(p,q)$ represents a generator of $\pi_2(G(p,q))$.

We have shown that all of the complex Grassmann manifolds are simply connected. Therefore, the Hurewicz theorem and (2.12) imply that

$$H_1(G(p,q)) = 0 \qquad \text{and} \qquad H_2(G(p,q)) \cong \mathbf{Z}\,, \tag{2.13}$$

for all integers $p, q \geq 1$. The universal coefficient theorem for cohomology implies that for all integers $p, q \geq 1$, the first and second cohomology groups with integer coefficients are

$$H^1(G(p,q);\mathbf{Z}) = 0 \qquad \text{and} \qquad H^2(G(p,q);\mathbf{Z}) \cong \mathbf{Z}\,. \tag{2.14}$$

[7] "Diagram chasing" is a standard technique used in algebraic topology to prove results about commutative diagrams. For some examples of diagram chasing, see Section I.3 in [61].

For real Grassmann manifolds, it will only be necessary to compute the fundamental group, π_1. For integers $1 \leq a \leq p$ and $1 \leq b \leq q$, let $\theta' \colon \mathbf{O}(a+b) \hookrightarrow \mathbf{O}(p+q)$ denote the inclusion obtained by restricting θ to the subgroup $\mathbf{O}(a+b) \subset \mathbf{U}(a+b)$, and $\hat{\theta}'$ the restriction of θ' to the subgroup $\mathbf{O}(a) \times \mathbf{O}(b)$. The inclusions α', θ', and $\hat{\theta}'$ induce the following homomorphism between homotopy exact sequences obtained from the fibre bundle (2.5).

$$\begin{array}{ccccccccc}
\pi_1(\mathbf{O}(a) \times \mathbf{O}(b)) & \xrightarrow{i_\#} & \pi_1(\mathbf{O}(a+b)) & \xrightarrow{p'_\#} & \pi_1(G'(a,b)) & \xrightarrow{\partial_\#} & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \longrightarrow & \mathbf{Z}_2 \\
\downarrow{\hat{\theta}'}_\# & & \downarrow{\theta'}_\# & & \downarrow{\alpha'}_\# & & \cong\downarrow & & \cong\downarrow \\
\pi_1(\mathbf{O}(p) \times \mathbf{O}(q)) & \xrightarrow{i_\#} & \pi_1(\mathbf{O}(p+q)) & \xrightarrow{p'_\#} & \pi_1(G'(p,q)) & \xrightarrow{\partial_\#} & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \longrightarrow & \mathbf{Z}_2
\end{array} \tag{2.15}$$

In this commutative diagram, the last homomorphism in the top row is an epimorphism from $\pi_0(\mathbf{O}(a) \times \mathbf{O}(b)) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ onto $\pi_0(\mathbf{O}(a+b)) \cong \mathbf{Z}_2$. Similarly, the last homomorphism in the bottom row is an epimorphism from $\pi_0(\mathbf{O}(p) \times \mathbf{O}(q))$ onto $\pi_0(\mathbf{O}(p+q))$.

First, we shall assume that $a+b \geq 3$. Then, by the stability of the homotopy groups of $\mathbf{O}(n)$ [49, Thm. 7.4.1], the homomorphisms $i_\# \colon \pi_1(\mathbf{O}(a) \times \mathbf{O}(b)) \to \pi_1(\mathbf{O}(a+b))$ and $i_\# \colon \pi_1(\mathbf{O}(p) \times \mathbf{O}(q)) \to \pi_1(\mathbf{O}(p+q))$ are both epimorphisms. In both cases, this implies that $p'_\#$ is the zero homomorphism. Therefore by diagram chasing, $\alpha'_\#$ is an isomorphism. Furthermore, using (2.15), it is easy to compute that $\pi_1(G'(p,q))$ is $\mathbf{Z}_2$ if $p+q \geq 3$.

We now consider the remaining case that $a = b = 1$ and $p+q \geq 3$. The homomorphism $\theta'_\# \colon \pi_1(\mathbf{O}(2)) \to \pi_1(\mathbf{O}(p+q))$ is then an epimorphism [49, Thm. 7.4.1], and by diagram chasing, it follows that $\alpha'_\#$ is also an epimorphism. Also, $G'(1,1)$ is the real

projective space $\mathbf{R}P(1)$, which is diffeomorphic to S^1. Therefore, $\alpha'_{\#}\colon \pi_1(G'(1,1)) \to \pi_1(G'(p,q))$ is the epimorphism from $\mathbf{Z}$ to $\mathbf{Z}_2$.[8]

To conclude, the fundamental groups of real Grassmann manifolds are

$$\pi_1(G'(p,q)) \cong \begin{cases} \mathbf{Z} & \text{if } p = q = 1 \\ \mathbf{Z}_2 & \text{if } p + q \geq 3 \end{cases} \tag{2.16}$$

Since $\pi_1(G'(p,q))$ is abelian for all integers $p, q \geq 1$, the Hurewicz theorem implies that the first homology groups of real Grassmann manifolds are

$$H_1(G'(p,q)) \cong \begin{cases} \mathbf{Z} & \text{if } p = q = 1 \\ \mathbf{Z}_2 & \text{if } p + q \geq 3 \end{cases} \tag{2.17}$$

The universal coefficient theorem for cohomology implies that the first cohomology groups with integral coefficients are

$$H^1(G'(p,q);\mathbf{Z}) \cong \begin{cases} \mathbf{Z} & \text{if } p = q = 1 \\ 0 & \text{if } p + q \geq 3 \end{cases} \tag{2.18}$$

We now compute the first two homotopy groups of complex short flag manifolds. It follows immediately from the homotopy exact sequence for the fibre bundle $G(p,q) \hookrightarrow F(p,q,r) \to G(p+q,r)$ in (2.6), that $F(p,q,r)$ is simply connected. Indeed, by making an inductive argument using fibre bundles of this sort, we see that all of the complex flag manifolds are simply connected.

[8] There are only two homomorphisms from $\mathbf{Z}$ to $\mathbf{Z}_2$. One is the zero homomorphism, and the other is an epimorphism.

From the fibre bundle (2.2), we have the following short exact sequence for all integers $p, q, r \geq 1$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_2(F(p,q,r)) & \xrightarrow{\partial_\#} & \pi_1(\mathbf{U}(p) \times \mathbf{U}(q) \times \mathbf{U}(r)) & \longrightarrow & \pi_1(\mathbf{U}(p+q+r)) & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} & & \mathbf{Z} & & \end{array}$$

This short exact sequence is split because $\pi_1(\mathbf{U}(p+q+r))$ is free abelian, and we conclude that

$$\pi_2(F(p,q,r)) \cong \mathbf{Z} \oplus \mathbf{Z} \ , \tag{2.19}$$

for all integers $p, q, r \geq 1$.

The Hurewicz theorem implies that

$$H_1(F(p,q,r)) = 0 \qquad \text{and} \qquad H_2(F(p,q,r)) \cong \mathbf{Z} \oplus \mathbf{Z} \ , \tag{2.20}$$

for all integers $p, q, r \geq 1$. It follows from (2.20) and the universal coefficient theorem for cohomology that

$$H^1(F(p,q,r); \mathbf{Z}) = 0 \qquad \text{and} \qquad H^2(F(p,q,r); \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z} \ , \tag{2.21}$$

for all integers $p, q, r \geq 1$.

To compute the fundamental group of $F'(p,q,r)$, we shall first assume that $p + q + r \geq 4$. From the fibre bundle (2.3), we have the homotopy exact sequence

$$\pi_1(\mathbf{O}(p) \times \mathbf{O}(q) \times \mathbf{O}(r)) \xrightarrow{i'_\#} \pi_1(\mathbf{O}(p+q+r)) \xrightarrow{p'_\#} \pi_1(F'(p,q,r)) \xrightarrow{\partial_\#}$$
$$\pi_0(\mathbf{O}(p) \times \mathbf{O}(q) \times \mathbf{O}(r)) \xrightarrow{i'_\#} \pi_0(O(p+q+r))$$

Since at least one of the integers p, q, or r is greater than or equal to 2, it follows that the homomorphism $i'_{\#}\colon \pi_1(\mathbf{O}(p)\times\mathbf{O}(q)\times\mathbf{O}(r)) \to \pi_1(\mathbf{O}(p+q+r))$ is an epimorphism [49, Thm. 7.4.1]. Hence, $p'_{\#}$ is the zero homomorphism and we have the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(F'(p,q,r)) & \xrightarrow{\partial_{\#}} & \pi_0(\mathbf{O}(p)\times\mathbf{O}(q)\times\mathbf{O}(r)) & \xrightarrow{i'_{\#}} & \pi_0(\mathbf{O}(p+q+r)) & \longrightarrow & 0 \\ & & & & \| & & \| & & \\ & & & & \mathbf{Z}_2\oplus\mathbf{Z}_2\oplus\mathbf{Z}_2 & & \mathbf{Z}_2 & & \end{array}$$

This short exact sequence implies that $\pi_1(F'(p,q,r))$ is isomorphic to $\mathbf{Z}_2\oplus\mathbf{Z}_2$ for all integers $p,q,r\geq 1$, satisfying $p+q+r\geq 4$.

It now remains to compute the fundamental group of the closed 3-dimensional manifold $F'(1,1,1)$. By examining the homotopy exact sequence for the fibre bundle (2.3) with $p=q=r=1$, we see at once that $\pi_1(F'(1,1,1))$ is a group of order eight. In Proposition III.2.23, it will be shown using a Mayer-Vietoris exact sequence[9] that $H_1(F'(1,1,1))\cong\mathbf{Z}_2\oplus\mathbf{Z}_2$. Unfortunately, the proof of this result must be deferred until after similar degeneracy regions have been introduced in Section II.2. It follows from the Hurewicz theorem that $\pi_1(F'(1,1,1))$ must be a nonabelian group with commutator subgroup $\mathbf{Z}_2$.

Up to isomorphism, the only two nonabelian groups with eight elements are the quaternion group Q_8, and the dihedral group D_8 [44, p. 50]. Of these two groups, only Q_8 can be the fundamental group of a closed 3-dimensional manifold.[10] This is because for any closed 3-manifold $\mathcal{M}$, with finite fundamental group, the universal covering space

[9] The Mayer-Vietoris exact sequence for homology may be found in [29, Sect. III.8], [36, Sect. I.15], and [83, Sect. 4.6].

[10] The group Q_8 is isomorphic to the multiplicative group of the eight quaternions usually denoted by $\{\pm 1,\pm i,\pm j,\pm k\}$.

$\tilde{\mathcal{M}}$ has the homotopy type of S^3 [46, Thm. 3.6]. From this it follows that any element of order two in $\pi_1(\mathcal{M})$ must belong to the centre of $\pi_1(\mathcal{M})$ [65, Cor. 1]. However, D_8 contains an element of order two which is not in the centre of the group [44, p. 50].

We summarize our results for real, short flag manifolds as follows:

$$\pi_1(F'(p,q,r)) \cong \begin{cases} Q_8 & \text{if } p=q=r=1 \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } p+q+r \geq 4 \end{cases} \tag{2.22}$$

By the Hurewicz theorem, it follows that the first homology group is

$$H_1(F'(p,q,r)) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \ , \tag{2.23}$$

for all integers $p,q,r \geq 1$. The universal coefficient theorem for cohomology implies that

$$H^1(F'(p,q,r);\mathbf{Z}) = 0 \ , \tag{2.24}$$

for all integers $p,q,r \geq 1$.

We now prove a proposition about homomorphisms induced by the inclusion of the real points defined in (2.8). This proposition saves us the trouble of computing the second homotopy and homology groups of real Grassmann manifolds and real, short flag manifolds.

Proposition 2.25. *Let $j\colon G'(p,q) \hookrightarrow G(p,q)$ and $j\colon F'(p,q,r) \hookrightarrow F(p,q,r)$ be the inclusion of the real points defined in* (2.8). *Then for all integers $p,q,r \geq 1$, each of the*

following four induced homomorphisms is trivial (i.e. the zero homomorphism):

$$j_{\#}\colon \pi_2(G'(p,q)) \longrightarrow \pi_2(G(p,q)) \tag{2.26}$$

$$j_{\#}\colon \pi_2(F'(p,q,r)) \longrightarrow \pi_2(F(p,q,r)) \tag{2.27}$$

$$j_{*}\colon H_2(G'(p,q)) \longrightarrow H_2(G(p,q)) \tag{2.28}$$

$$j_{*}\colon H_2(F'(p,q,r)) \longrightarrow H_2(F(p,q,r)) \tag{2.29}$$

To prove Proposition 2.25, we require

Lemma 2.30. *For any n-fold covering, $\wp\colon \tilde{X} \to X$, the order of every element in the cokernel of the induced homomorphism $\wp_*\colon H_2(\tilde{X}) \to H_2(X)$, is a divisor of n.*

This lemma requires a knowledge of covering spaces. Two good reviews are [38, Sect. I.5] and [83, Chapt. 2].

Proof. Let ω be any singular 2-cycle in X. By barycentrically subdividing [83, Sect. 3.3] ω sufficiently often, we obtain a 2-cycle $\beta(\omega)$ with the property that the image of each singular 2-simplex σ_i in $\beta(\omega)$ is evenly covered [83, p. 62] by $\wp$. The 2-cycle $\beta(\omega)$ is equal to the formal sum $\sum_i \alpha_i\sigma_i$, where α_i are integers. Let $\{\tau_{ij} \mid j = 1,\ldots,n\}$ be the set of all singular simplices in $\tilde{X}$ such that $\wp \circ \tau_{ij} = \sigma_i$ for all j. Then, we define $\tilde{\omega} = \sum_{ij} \alpha_i\tau_{ij}$. Since each σ_i is evenly covered, it

follows that $\tilde{\omega}$ is a 2-cycle in $\tilde{X}$, and furthermore

$$\wp_*\{\tilde{\omega}\} = \{n\beta(\omega)\} = n\{\beta(\omega)\} = n\{\omega\} .$$

□

Proof of Proposition 2.25. We will prove that the homomorphisms (2.27) and (2.29) are trivial. A similar argument proves that (2.26) and (2.28) are trivial.

First, consider the homomorphism (2.27). We begin by noticing that the inclusion induced homomorphism

$$j_{\#}\colon \pi_1(\mathbf{O}(k)) \longrightarrow \pi_1(\mathbf{U}(k)) \tag{2.31}$$

is trivial for all integers $k \geq 1$. For $k = 1$, this is obvious because $\pi_1(\mathbf{O}(1)) = 0$. Also, $\pi_1(\mathbf{O}(k)) \cong \mathbf{Z}_2$ for all $k \geq 3$, and $\pi_1(\mathbf{U}(k)) \cong \mathbf{Z}$ for all $k \geq 1$, which implies that (2.31) is trivial for $k \geq 3$. For $k = 2$, we require the trivial principal bundles[11]

$$\begin{array}{ccc} \mathbf{SO}(2) & \longrightarrow & \mathbf{O}(2) \\ & & \downarrow{\scriptstyle \det} \\ & & \mathbf{O}(1) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{SU}(2) & \longrightarrow & \mathbf{U}(2) \\ & & \downarrow{\scriptstyle \det} \\ & & \mathbf{U}(1) \end{array}$$

which are associated with the determinant maps on $\mathbf{O}(2)$ and $\mathbf{U}(2)$. The homotopy exact sequences for these fibre bundles and the inclusion $j\colon \mathbf{O}(k) \hookrightarrow \mathbf{U}(k)$

[11] It is elementary to verify that both of these bundles are principal bundles. Obviously, the bundle over $\mathbf{O}(1)$ is trivial. Because the induced homomorphism $\det_{\#}\colon \pi_1(\mathbf{U}(2)) \to \pi_1(\mathbf{U}(1))$ is an epimorphism, it follows that the bundle over $\mathbf{U}(1)$ admits a section. Any principal bundle admitting a section is trivial.

provide the commutative diagram

$$\begin{tikzcd}
\pi_1(\mathbf{SO}(2)) \arrow[r, "\cong"] \arrow[d] & \pi_1(\mathbf{O}(2)) \arrow[r, "\det_\#"] \arrow[d, "j_\#"] & \pi_1(\mathbf{O}(1)) \arrow[r] \arrow[d, "0"] & \pi_0(\mathbf{SO}(2)) = 0 \arrow[d] \\
0 = \pi_1(\mathbf{SU}(2)) \arrow[r] & \pi_1(\mathbf{U}(2)) \arrow[r, "\det_\#", "\cong"'] & \pi_1(\mathbf{U}(1)) \arrow[r] & \pi_0(\mathbf{SU}(2)) = 0
\end{tikzcd}$$

which implies that $j_\#\colon \pi_1(\mathbf{O}(2)) \to \pi_1(\mathbf{U}(2))$ is the zero homomorphism.

The triviality of (2.27) follows from the triviality of (2.31), and the following commutative diagram whose rows are given by the homotopy exact sequences for the fibre bundles (2.2) and (2.3).

$$\begin{tikzcd}
0 = \pi_2(\mathbf{O}(p+q+r)) \arrow[r] \arrow[d] & \pi_2(F'(p,q,r)) \arrow[r, "\partial_\#"] \arrow[d, "j_\#"] & \pi_1(\mathbf{O}(p) \times \mathbf{O}(q) \times \mathbf{O}(r)) \arrow[d, "0"] \\
0 = \pi_2(\mathbf{U}(p+q+r)) \arrow[r] & \pi_2(F(p,q,r)) \arrow[r, "\partial_\#"] & \pi_1(\mathbf{U}(p) \times \mathbf{U}(q) \times \mathbf{U}(r))
\end{tikzcd}$$

We now show that the triviality of the homomorphism (2.27) implies the triviality of the homomorphism (2.29). Let $\wp\colon \tilde{F}'(p,q,r) \to F'(p,q,r)$ denote the universal covering space of $F'(p,q,r)$. Since $\pi_1(F'(p,q,r))$ is a finite group for all integers $p,q,r \geq 1$, it follows that $\wp$ is a finite covering. Therefore, from Lemma 2.30 we conclude that the cokernel of the induced homomorphism $\wp_*\colon H_2(\tilde{F}'(p,q,r)) \to H_2(F'(p,q,r))$ contains only elements of finite order.

Consider the commutative square

$$\begin{tikzcd}
\pi_2(\tilde{F}'(p,q,r)) \arrow[r, "(j\circ\wp)_\#"] \arrow[d, "\cong"] & \pi_2(F(p,q,r)) \arrow[d, "\cong"] \\
H_2(\tilde{F}'(p,q,r)) \arrow[r, "(j\circ\wp)_*"] & H_2(F(p,q,r))
\end{tikzcd}$$

where the Hurewicz homomorphisms (vertical maps) are isomorphisms because both spaces are simply connected. Triviality of (2.27), and commutativity of this square imply that $(j \circ \wp)_* \colon H_2(\tilde{F}'(p,q,r)) \to H_2(F(p,q,r))$ is the zero homomorphism. The homomorphism $(j \circ \wp)_*$ is equal to the composition of homomorphisms $j_* \, \wp_*$. Since the cokernel of $\wp_*$ contains only elements of finite order, and $H_2(F(p,q,r))$ is free abelian, it follows that j_* is the zero homomorphism for all integers $p, q, r \geq 1$. $\square$

We remark that the universal covering space of $F'(p,q,r)$ can be identified as a familiar space. If $p+q+r \geq 4$, then $\tilde{F}'(p,q,r)$ is the oriented real flag manifold which is diffeomorphic to $\mathbf{SO}(p+q+r)/\mathbf{SO}(p) \times \mathbf{SO}(q) \times \mathbf{SO}(r)$. If $p = q = r = 1$, then $\tilde{F}'(1,1,1)$ is diffeomorphic to S^3, which double covers $\mathbf{SO}(3) \approx \mathbf{R}P(3)$, the corresponding oriented, real flag manifold.

Chapter II

Quantum Adiabatic Holonomy

The quantum adiabatic theorem [14], [53] and its close relative the Born-Oppenheimer approximation [15], [24], [25], [82] are used extensively in nonrelativistic quantum mechanics to find approximate solutions to the Schrödinger equation. An interesting and important feature of the adiabatic limit of solutions to the time-dependent Schrödinger equation is adiabatic phase. In this chapter, we shall show that the appearance of adiabatic phase can be traced to the twisting of certain eigenspace line bundles.

Since the Born-Oppenheimer approximation is related to the adiabatic theorem, one should expect that nontriviality of the appropriate eigenspace line bundles will have important consequences for it. This is indeed the case, and the resulting phenomena have been observed in the Jahn-Teller effect, which arises because of the coupling between the vibrational modes of the nuclei and the electronic states of polyatomic molecules. References to this, and some of the other implications of adiabatic phase in physical systems are given in [50]. We shall be examining the examples of a particle with spin coupled to a magnetic field, and the molecular Jahn-Teller effect in more detail.

Quantum adiabatic phase is introduced through the adiabatic theorem in the first section of this chapter. The second and third sections are on regions of similar degeneracy and eigenspace line bundles, respectively. Both of these concepts play an important role

in our analysis of quantum adiabatic phase. Also in the third section, some of our results are applied to the theory for a photon propagating in a coiled optical fibre.

§1 Adiabatic Approximations in Quantum Mechanics

Adiabatic phases arise in quantum mechanics when we consider solutions to the adiabatic limit of the Schrödinger equation. The adiabatic theorem is reviewed and then used to define the adiabatic phase. In Subsection b, we review M. V. Berry's computation of the adiabatic phase for a general periodic 2-level quantum system. These results are required in Chapter III, and it is useful to have them described in our notation.

(a) The Adiabatic Theorem

The state vector $\psi(t)$ for a time-dependent quantum mechanical system with hamiltonian $H(t)$ evolves according to the Schrödinger equation

$$i\frac{d}{dt}\psi(t) = H(t)\psi(t) \ , \tag{1.1}$$

where we have set Planck's constant $\hbar$ equal to 1. The hamiltonian $H(t)$ shall be described as a continuous path of self-adjoint operators on a separable Hilbert space $\mathbf{H}$. Of course, $H(t)$ is not in general a bounded operator, which requires that we introduce a topology on the set of all self-adjoint operators on $\mathbf{H}$.

There is a standard procedure for defining a metric on the set of all closed operators on a Hilbert space [26], [56, Sect. IV.2], [69]. It exploits the fact that if T is a closed operator, then its graph $G(T)$ is a a closed linear subspace of $\mathbf{H} \times \mathbf{H}$. For two closed

operators S and T, let $P(S)$ and $P(T)$ denote the orthogonal projections onto $G(S)$ and $G(T)$, respectively. Then, denoting the operator norm for bounded operators on $\mathbf{H} \times \mathbf{H}$ by $\|\cdot\|$, the function $g(S,T) \stackrel{\text{def}}{=} \|P(S) - P(T)\|$ is called the gap between S and T [26]. It is easy to verify that $g(S,T)$ satisfies all of the properties of a metric, and the induced topology is called the gap topology on the space of closed operators on $\mathbf{H}$.[1]

The set of self-adjoint operators on a Hilbert space is a subset of the set of closed operators. We denote by $Herm(\mathbf{H})$ the topological space of all self-adjoint operators on a Hilbert space $\mathbf{H}$, with the gap topology. Beware that $Herm(\mathbf{H})$ is not a complete metric space. For an example of a Cauchy sequence which does not converge, see [56, Remark IV.2.10]. It is useful to know that the subset of bounded operators in $Herm(\mathbf{H})$ is an open subset, and that the the relative topology defined on this subset is equivalent to the norm topology [56, Remark IV.2.16].

Before stating the adiabatic theorem, it is necessary to introduce a notion of differentiability for continuous paths $T\colon [0,1] \to Herm(\mathbf{H})$. Because $Herm(\mathbf{H})$ is not a complete metric space, it is difficult to do this directly in terms of the gap metric. Instead, we define differentiability in terms of the resolvent of $T(t)$.

For a closed operator T, the resolvent set $\rho(T)$ is defined to be the set of complex numbers ζ for which $T - \zeta I$ is invertible, and $R(\zeta, T) \stackrel{\text{def}}{=} (T - \zeta I)^{-1}$ is a bounded operator. Here and throughout the remainder of this thesis, the identity operator is denoted by I. The operator $R(\zeta, T)$ is called the resolvent of T at ζ. Resolvents play an important role in the analysis of operators, and a good review is [56]. The resolvent set $\rho(T)$ is an

[1] This definition of $g(S,T)$ makes use of the Hilbert-space structure of $\mathbf{H}$. However, it is also possible to construct a metric for the set of closed operators on a Banach space. See [56] and [69].

open subset of the complex plane, and a fundamental property of $R(\zeta, T)$ is that it is a holomorphic function of ζ on each connected component of $\rho(T)$ [56, Thm. III.6.7]. The spectrum of T, denoted by $\sigma(T)$, is the complement in the complex plane of $\rho(T)$. If T is a hermitian operator, then the spectrum is contained in the real line, $\mathbf{R}$. We say that a path of self-adjoint operators $T\colon [0,1] \to Herm(\mathbf{H})$ is k-times differentiable in the norm resolvent sense, if the path of bounded operators $R(\zeta, T(s))$ is k-times norm differentiable for all complex numbers ζ with nonzero imaginary part. Norm differentiability is defined below.

The space of bounded operators on $\mathbf{H}$ is denoted by $\mathcal{B}(\mathbf{H})$. If B is a function from the real line $\mathbf{R}$ to $\mathcal{B}(\mathbf{H})$, then the norm derivative is defined by

$$\frac{d}{ds}B(s) = \lim_{\varepsilon \to 0} \frac{B(s+\varepsilon) - B(s)}{\varepsilon} ,$$

where the limit is taken in the norm topology on $\mathcal{B}(\mathbf{H})$. It is easy to verify that the norm derivative satisfies the following two properties. If $B(s)$ and $C(s)$ are two functions from $\mathbf{R}$ to $\mathcal{B}(\mathbf{H})$, which are at least once norm differentiable at s, then

$$\frac{d}{ds}\left(B(s)C(s)\right) = \frac{dB}{ds}(s)\, C(s) + B(s)\, \frac{dC}{ds}(s) . \tag{1.2}$$

Also, if $B(s)$ has a bounded inverse, then

$$\frac{d}{ds}B(s)^{-1} = -B(s)^{-1}\, \frac{dB}{ds}(s)\, B(s)^{-1} . \tag{1.3}$$

Using the identities (1.2) and (1.3), it is easy to prove

Lemma 1.4. *A path $B\colon [0,1] \to \mathcal{B}(\mathbf{H}) \cap Herm(\mathbf{H})$ is k-times differentiable in the norm resolvent sense iff it is k-times norm differentiable.* □

This lemma suggests that norm resolvent differentiability is a sensible generalization of norm differentiability to unbounded self-adjoint operators. If a path of self-adjoint operators is k-times differentiable in the norm resolvent sense, then we shall simply say that it is C^k. If it is C^∞, then it will be referred to as a smooth path.

The definition of norm resolvent differentiability for a path $T\colon \mathbf{R} \to Herm(\mathbf{H})$ requires that $R(\zeta, T(s))$ be norm differentiable for all ζ in the complex plane with the real line removed. The following lemma implies that it is necessary and sufficient for $R(\zeta_0, T(s))$ to be norm differentiable for some fixed ζ_0, with a nonzero imaginary part.

Lemma 1.5. *If $R(\zeta_0, T(s))$ is k-times norm differentiable for some $\zeta_0 \in \mathbf{C} - \mathbf{R}$, then $T(s)$ is C^k in the norm resolvent sense.*

Proof. If $\zeta, \zeta_0 \in \rho(T(s))$, and $\zeta \neq \zeta_0$, then from [56, Problem III.6.16] we have that $(\zeta - \zeta_0)^{-1}$ is contained in the resolvent set of $R(\zeta_0, T(s))$, and

$$R(\zeta, T(s)) = -(\zeta - \zeta_0)^{-1} - (\zeta - \zeta_0)^{-2}\, R(\,(\zeta - \zeta_0)^{-1}, R(\zeta_0, T(s))\,)\,.$$

This implies that if $R(\zeta_0, T(s))$ is norm differentiable for some $\zeta_0 \in \mathbf{C} - \mathbf{R}$, then $R(\zeta, T(s))$ is norm differentiable for all $\zeta \in \mathbf{C} - \mathbf{R}$. □

Having introduced the concept of differentiability in the norm resolvent sense, it is natural to say that a path $T(s)$ in $Herm(\mathbf{H})$ is continuous in the norm resolvent sense, or C^0, if $R(\zeta, T(s))$ is continuous in the norm topology for all $\zeta \in \mathbf{C} - \mathbf{R}$. For norm resolvent differentiability to be a sensible definition of differentiability on the topological space $Herm(\mathbf{H})$, we would like continuity in the norm resolvent sense to be equivalent to continuity in the gap metric g on $Herm(\mathbf{H})$. It follows from [56, Thm. IV.2.25] that these two definitions of continuity are equivalent.

Consider a path $T\colon [0,1] \to Herm(\mathbf{H})$, and assume that $T(s)$ has an isolated eigenvalue $\lambda(s)$ which remains bounded away from the remainder of the spectrum of $T(s)$ for all $s \in [0,1]$. Let $P(s)$ denote the orthogonal projection onto the eigenspace $E(s)$ of $\lambda(s)$.

Lemma 1.6. *If the path T is C^k, then the path of projection operators P is k-times norm differentiable.*

Proof. The projection $P(s)$ is given by the contour integral

$$P(s) = -\frac{1}{2\pi i} \oint_\Gamma R(\zeta, T(s))\, d\zeta \ ,$$

where Γ is a closed contour in $\rho(T(s))$, encircling $\lambda(s)$. Because Γ is compact, and $R(\zeta, T(s))$ is k-times norm differentiable in s, it follows that $P(s)$ is k-times norm differentiable. □

Differentiability of the projection $P(s)$ may be used to prove that the isolated eigenvalue $\lambda(s)$ is also a differentiable function of s.

Proposition 1.7. *If the path T is C^k, then the function $\lambda(s)$ is C^k.*

Proof. For an arbitrary $s_0 \in (0,1)$, let ϕ_0 be an eigenvector of $T(s_0)$ with eigenvalue $\lambda(s_0)$. There exists an $\varepsilon > 0$ such that $\|P(s)\phi_0\| > 0$ for all $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. Then, $\phi(s) = \|P(s)\phi_0\|^{-1} P(s)\phi_0$ is a k-times differentiable normalized eigenvector with eigenvalue $\lambda(s)$, for $s \in (s_0 - \varepsilon, s_0 + \varepsilon)$. The spectral theorem implies that for any fixed $\zeta \in \mathbf{C} - \mathbf{R}$, the resolvent satisfies

$$R(\zeta, T(s))\,\phi(s) = (\zeta - \lambda(s))^{-1}\phi(s)\,.$$

Therefore, $\lambda(s) = \zeta - (\phi(s), R(\zeta, T(s))\phi(s))^{-1}$, and $\lambda(s)$ is k-times differentiable at s_0. □

The goal of the adiabatic theorem is to approximate solutions to the Schrödinger equation for a slowly varying, time-dependent hamiltonian, by using eigenvectors of the hamiltonian. To this end, we follow Kato [53] and construct the adiabatic transformation. Given a C^k path $T\colon [0,1] \to Herm(\mathbf{H})$ with $k \geq 1$, and an isolated eigenvalue $\lambda(s)$ which is bounded away from the remainder of the spectrum of $T(s)$, there is an adiabatic transformation $A(s)$ associated with $\lambda(s)$. The transformation $A(s)$ will be defined so that it is a unitary operator which maps the eigenspace $E(0)$ isometrically onto the eigenspace $E(s)$.

Denote the projection onto the eigenspace $E(s)$ by $P(s)$ and the norm derivative $\frac{d}{ds}P(s)$ by $P'(s)$. Then, the adiabatic transform $A(s)$ is defined to be the unitary operator

generated by the anti-self-adjoint commutator $[P'(s), P(s)]$. This means that $A(s)$ is the unique solution to the differential equation

$$\frac{d}{ds}A(s) = [P'(s), P(s)]\, A(s)\,, \qquad A(0) = I\,. \tag{1.8}$$

Recall that I is the identity operator. Because $[P'(s), P(s)]$ is a bounded operator, the Dyson expansion [71, Thm. X.69] may be used to write down a norm-convergent series solution to (1.8). From the Dyson expansion, it is evident that if $T(s)$ is C^k, then $A(s)$ is k-times norm differentiable.

If $Q(s) = I - P(s)$, the adiabatic transformation satisfies the equivalent identities

$$P(s)A(s) = A(s)P(0) \qquad \text{and} \qquad Q(s)A(s) = A(s)Q(0)\,. \tag{1.9}$$

These identities, which are verified below, imply that $A(s)$ maps $E(0)$ isometrically onto $E(s)$, and $E^{\perp}(0)$ isometrically onto $E^{\perp}(s)$. In this computation, and throughout this thesis, the notation * is used to denote the adjoint of an operator. By differentiating $P(s) = P^2(s)$, it is easy to verify that $P'(s) = P'(s)P(s) + P(s)P'(s)$, which in turn implies that

$$P(s)P'(s)P(s) = 0\,. \tag{1.10}$$

Using (1.8) and these identities, compute the norm derivative

$$\begin{aligned}\frac{d}{ds}\left\{A^*(s)\,P(s)\,A(s)\right\} &= A^*(s)\,\left\{[P(s), P'(s)]\,P(s) + P'(s) + P(s)\,[P'(s), P(s)]\right\}\,A(s)\\ &= 0\end{aligned}$$

This implies that

$$A^*(s)\,P(s)\,A(s) = A^*(0)\,P(0)\,A(0) = P(0)\ ,$$

which establishes (1.9).

We return to considering solutions of the Schrödinger equation (1.1) as $H(t)$ evolves continuously from $H(0)$ to $H(\tau)$. Define the parameter $s \in [0,1]$ by $s = t/\tau$, and let $T(s) = H(s\tau)$. Then, the Schrödinger equation can be rewritten in the form

$$i\frac{d}{ds}\psi_\tau(s) = \tau T(s)\,\psi_\tau(s)\ . \tag{1.11}$$

The map $T\colon [0,1] \to Herm(\mathbf{H})$ defines a fixed path which is continuous in the gap topology. The time taken by the physical system to evolve along this path is τ, and therefore the limit in which the time-evolution of the system is slow corresponds to τ being large in (1.11).

There is an extensive literature on the problem of existence and uniqueness of solutions to the Schrödinger equation (1.1). Because the hamiltonian is not usually a bounded operator, it is not possible to use a Dyson expansion to construct solutions to (1.1), or (1.11). However, there are a number of theorems which provide sufficient conditions on $T(s)$ for (1.1), and hence (1.11), to have a unique solution for any specified initial data in the domain of $T(0)$ [54], [55], [71, Sect. X.12], [80, Sect. II.7], [98, Sect. XIV.4]. For our purposes, the most useful theorem is B. Simon's statement [80, Thm. II.21] of Yosida's theorem [98, Thm. XIV.4.2]. For simplicity, we state the theorem with slightly stronger hypotheses than did Simon.

Theorem 1.12. (Yosida) *Let $T\colon [0,1] \to Herm(\mathbf{H})$ be a C^1 path satisfying the following conditions:*

(i) The domain of $T(s)$ is independent of s. It is also assumed that this domain, denoted by $\mathcal{D}$, is dense in $\mathbf{H}$.

(ii) $T(s)$ is bounded from below uniformly in s. This means that there exists a fixed $\zeta_0 \in \mathbf{R}$ such that $T(s) \geq \zeta_0 + 1$ for all $s \in [0,1]$.

(iii) The operator $T(s)\,\frac{d}{ds}R(\zeta_0, T(s))$ is bounded uniformly in $s \in [0,1]$.

Given conditions (i), (ii) and (iii), the evolution equation

$$i\frac{d}{ds}\psi(s) = \tau\, T(s)\,\psi(s)\,, \qquad \psi(0) = \psi_0 \tag{1.13}$$

has a unique, strongly differentiable solution for all $\psi_o \in \mathcal{D}$. Moreover, $\psi(s) \in \mathcal{D}$ and $\|\psi(s)\| = \|\psi_o\|$ for all $s \in [0,1]$.

The solution to (1.13) may be used to define an s-dependent operator function $\mathcal{U}_\tau(s)\colon \mathcal{D} \to \mathcal{D}$ by $\psi(s) = \mathcal{U}_\tau(s)\,\psi_0$ for all $\psi_0 \in \mathcal{D}$. The subscript τ reminds us of the parametric dependence of (1.13) on τ. Because $\|\mathcal{U}_\tau(s)\,\psi_0\| = \|\psi_0\|$ for all $\psi_0 \in \mathcal{D}$, it follows that $\mathcal{U}_\tau(s)$ has a unique extension to a unitary operator on $\mathbf{H}$. Henceforth, the notation $\mathcal{U}_\tau(s)$ will be used for this extension, which is called the propagator for (1.13). Theorem 1.12 implies that $\mathcal{U}_\tau(s)$ is strongly continuous in s, and that $\mathcal{U}_\tau(s)\psi_0$ is strongly differentiable for all $\psi_0 \in \mathcal{D}$. We caution that the strong derivative of $\mathcal{U}_\tau(s)$ need not be defined outside of $\mathcal{D}$.

Recall, that the limit of large τ corresponds to the limit in which the time evolution of the system is slow. This limit is usually referred to as the adiabatic limit. The embodiment of the various adiabatic theorems is that under certain hypotheses, the appropriate adiabatic transformation may be used to approximate the propagator $\mathcal{U}_\tau$, in the adiabatic limit.

The first complete proof of an adiabatic theorem is due to Born and Fock [14]. In their theorem, it is assumed that the hamiltonian $H(t)$ depends smoothly on $t \in [0, \tau]$, and that for all t it is a bounded, self-adjoint operator with a spectrum consisting only of discrete eigenvalues which are nondegenerate, except possibly for isolated crossings. Write $T(s) = H(s\tau)$, and suppose that $T(s)$ has a nondegenerate eigenvalue $\lambda(s)$, which is bounded away from the rest of the spectrum for all $s \in [0, 1]$. Associated with $\lambda(s)$ is an eigenspace $E(s)$ and an adiabatic transformation $A(s)$. Born and Fock proved that in the adiabatic limit ($\tau \to \infty$), the restriction of the propagator $\mathcal{U}_\tau(s)$ to the subspace $E(0)$ is approximated by the same restriction of the unitary operator $\exp[-i\tau \int_0^s \lambda(r)\, dr]\, A(s)$. In the physics literature, the phase factor $\exp[-i\tau \int_0^s \lambda(r)\, dr]$ is usually termed the dynamical phase factor.

A substantially more general adiabatic theorem was proven by Kato [53]. It applies to bounded hamiltonians which have a distinguished eigenvalue $\lambda(s)$ of arbitrary multiplicity. This distinguished eigenvalue must be bounded away from the rest of the spectrum, which may be of an arbitrary nature. Kato's theorem has been generalized by Avron, Seiler, and Yaffe [7], who showed that suitably modified results hold when the eigenvalue $\lambda(s)$ is replaced by a finite band, containing both discrete and essential spectrum. It is

required that this band be bordered by gaps in the spectrum, and that the width of these gaps be bounded away from zero for all $s \in [0,1]$.

We provide an adiabatic theorem generalizing Kato's theorem to sufficiently regular hamiltonians, which are no longer required to be bounded. Let $T\colon [0,1] \to Herm(\mathbf{H})$ be a C^2 path satisfying conditions (i), (ii), and (iii) in Theorem 1.12. This guarantees that the evolution equation (1.13) has a unique solution for each specified initial value in the domain $\mathcal{D}$, and in addition provides enough regularity for the following adiabatic theorem. The notation $\sigma(T(s))$ is used below for the spectrum of $T(s)$.

Theorem 1.14. (Adiabatic Theorem) *Suppose that $T(s)$ has an eigenvalue $\lambda(s)$ of arbitrary multiplicity, which satisfies for all $s \in [0,1]$,*

$$\operatorname{dist}(\lambda(s)\,,\,\sigma(T(s)) - \{\lambda(s)\}) \geq \varepsilon\,,$$

for some constant $\varepsilon > 0$. Associated with the isolated eigenvalue $\lambda(s)$ is an eigenspace $E(s)$ and an adiabatic transformation $A(s)$. Note that $E(s) \subset \mathcal{D}$ for all $s \in [0,1]$. Then for all $\phi_0 \in E(0)$, there is the estimate

$$\left\| \mathcal{U}_\tau(1)\,\phi_0 - \exp[-i\tau \int_0^1 \lambda(x)\,dx]\,A(1)\,\phi_0 \right\| \leq \frac{K}{\tau}\,\|\phi_0\|\,,$$

where K is a constant independent of τ.

The following proof is a simple generalization of Kato's proof in [53].[2]

[2] We thank L. Rosen for suggesting a proof along these lines.

Proof. Recall, that it follows from (1.8) that $A(s)$ is twice norm differentiable, and that it follows from Theorem 1.12 that $\mathcal{U}_\tau(s)\psi_0$ is strongly differentiable for all $\psi_0 \in \mathcal{D}$. For any $\psi_0 \in \mathcal{D}$ and $\phi_0 \in E(0)$, this allows us to consider

$$\begin{aligned}
&\frac{d}{ds}\left(\mathcal{U}_\tau(s)\,\psi_0\ ,\ \exp[-i\tau\int_0^s \lambda(x)\,dx]\,A(s)\,\phi_0\right)\\
&\qquad = \left(-i\tau T(s)\,\mathcal{U}_\tau(s)\,\psi_0\ ,\ \exp[-i\tau\int_0^s \lambda(x)\,dx]\,A(s)\,\phi_0\right)\\
&\qquad\quad + \left(\mathcal{U}_\tau(s)\,\psi_0\ ,\ -i\tau\lambda(s)\exp[-i\tau\int_0^s \lambda(x)\,dx]\,A(s)\,\phi_0\right)\\
&\qquad\quad + \left(\mathcal{U}_\tau(s)\,\psi_0\ ,\ \exp[-i\tau\int_0^s \lambda(x)\,dx]\,A'(s)\,\phi_0\right)\\
&\qquad = \left(\mathcal{U}_\tau(s)\,\psi_0\ ,\ \exp[-i\tau\int_0^s \lambda(x)\,dx]\,A'(s)\,\phi_0\right),
\end{aligned} \tag{1.15}$$

We have used the fact that $A(s)\,\phi_0 \in E(s)$, and also introduced the notation $'$ for $\frac{d}{ds}$.

Let $P(s)$ be the projection onto the eigenspace $E(s)$. Then, (1.8) and (1.10) imply that (1.15) is equal to

$$\begin{aligned}
&\left(\mathcal{U}_\tau(s)\,\psi_0\ ,\ \exp[-i\tau\int_0^s \lambda(x)\,dx]\,P'(s)\,A(s)\,\phi_0\right)\\
&\qquad = \left(\exp[i\tau\int_0^s \lambda(x)\,dx]\,\mathcal{U}_\tau(s)\,\psi_0\ ,\ [I-P(s)]\,P'(s)\,A(s)\,\phi_0\right).
\end{aligned} \tag{1.16}$$

We now introduce the operator

$$S(s) \stackrel{\text{def}}{=} \lim_{\zeta\to\lambda(s)} R(\zeta, T(s))\,[I-P(s)]\,.$$

It is useful to observe that $S(s)$ is easily recognized in terms of familiar operators. To this end, note that $\lambda(s)$ is contained in the resolvent set of the restriction of $T(s)$ to the subspace $E^{\perp}(s)$, the orthogonal complement of $E(s)$. The operator $S(s)$ is the resolvent of $T(s)|_{E^{\perp}(s)}$, evaluated at $\lambda(s)$ [56, Sect. V.3.5]. From this it follows that $S(s)$ is a bounded operator with

$$\|S(s)\| = [\mathrm{dist}(\lambda(s), \sigma(T(s)) - \{\lambda(s)\})]^{-1} \ . \tag{1.17}$$

More results for $S(s)$ are proven in [56, Sect. III.6.5], where it is shown that

$$SP = PS = 0 \qquad \text{and} \qquad [T(s) - \lambda(s)]\, S(s) = I - P(s) \ . \tag{1.18}$$

Also, if Γ is any closed curve in $\rho(T(s))$, encircling only the isolated eigenvalue $\lambda(s)$, then

$$S(s) = \frac{1}{2\pi i} \oint_{\Gamma} R(\zeta, T(s)) \, \frac{d\zeta}{\zeta - \lambda} \ .$$

This formula implies that $S(s)$ is a C^2 function of s.

Substituting from (1.18) into (1.16) obtains

$$\begin{aligned} &\left(\exp[i\tau \int_0^s \lambda(x)\, dx]\, \mathcal{U}_\tau(s)\, \psi_0 \ , \ [T(s) - \lambda(s)]\, S(s)\, P'(s)\, A(s)\, \phi_0 \right) \\ &\qquad = \frac{-i}{\tau} \left(\frac{d}{ds} \{ \exp[i\tau \int_0^s \lambda(x)\, dx]\, \mathcal{U}_\tau(s) \}\, \psi_0 \ , \ S(s)\, P'(s)\, A(s)\, \phi_0 \right) . \end{aligned}$$

Therefore, the integral of $\frac{d}{ds}\left(\mathcal{U}_\tau(s)\, \psi_0 \, , \exp[-i\tau \int_0^s \lambda(x)\, dx]\, A(s)\, \phi_0\right)$ from 0 to 1

is equal to

$$\begin{aligned}
&\left(\mathcal{U}_\tau(1)\,\psi_0\,,\ \exp[-i\tau\int_0^1 \lambda(x)\,dx]\,A(1)\,\phi_0\right) - \left(\psi_0, \phi_0\right) \\
&\quad = \frac{-i}{\tau}\int_0^1 \left(\frac{d}{ds}\left\{\exp[i\tau\int_0^s \lambda(x)\,dx]\,\mathcal{U}_\tau(s)\right\}\psi_0\,,\ S(s)\,P'(s)\,A(s)\,\phi_0\right) ds \\
&\quad = \frac{-i}{\tau}\left[\left(\exp[i\tau\int_0^s \lambda(x)\,dx]\,\mathcal{U}_\tau(s)\,\psi_0\,,\ S(s)\,P'(s)\,A(s)\,\phi_0\right)\right]_0^1 \\
&\qquad + \frac{i}{\tau}\int_0^1 \left(\exp[i\tau\int_0^s \lambda(x)\,dx]\,\mathcal{U}_\tau(s)\,\psi_0\,,\ \frac{d}{ds}\left\{S(s)\,P'(s)\,A(s)\right\}\,\phi_0\right),
\end{aligned} \tag{1.19}$$

after integrating by parts.

It is easy to verify using (1.8), (1.10), and (1.18) that

$$\frac{d}{ds}\left\{S(s)\,P'(s)\,A(s)\right\}\,\phi_0 = \left\{S'(s)\,P'(s) + S(s)\,P''(s)\right\}\,A(s)\,\phi_0\,.$$

By substituting this identity into (1.19), and making use of (1.17), we obtain the bound

$$\left|\left(\mathcal{U}_\tau(1)\psi_0\,,\ \exp[-i\tau\int_0^1 \lambda(x)\,dx]\,A(1)\,\phi_0\right) - \left(\psi_0\,,\ \phi_0\right)\right| < \frac{K}{\tau}\,\|\psi_0\|\,\|\phi_0\|\,, \tag{1.20}$$

where

$$K = \frac{1}{\varepsilon}\left\{\|P'(1)\| + \|P'(0)\| + \sup_{s\in[0,1]}\|P''(s)\|\right\} + \sup_{s\in[0,1]}\|S'(s)\|\|P'(s)\|\,.$$

It is apparent from the form of K why we required that $T(s)$ be C^2, and that $\lambda(s)$ be bounded away from the rest of the spectrum. The bound (1.20) holds for

all $\psi_0 \in \mathcal{D}$. Therefore, it follows by a standard argument using the denseness of $\mathcal{D}$ that

$$\left\| \mathcal{U}_\tau^{-1}(1) \exp[-i\tau \int_0^1 \lambda(x)\, dx]\, A(1)\, \phi_0 - \phi_0 \right\| \leq \frac{K}{\tau} \|\phi_0\| \ .$$

□

The adiabatic theorem has particularly interesting implications when the hamiltonian $H(t)$ evolves periodically in time, with period τ. In this case, $T(0) = T(1)$, and T is a loop in $Herm(\mathbf{H})$. The unitary operator $\mathcal{U}_\tau(1)$ is called the monodromy operator for the periodic differential equation (1.11). The adiabatic theorem implies that for large τ, the monodromy operator $\mathcal{U}_\tau(1)$ is closely approximated by the dynamical phase factor times the monodromy operator $A(1)$ of the differential equation (1.8). Let m denote the dimension of $E(s)$ along the curve T, and now assume that m is finite. By hypothesis, m is a constant on T. If $E(0) = E(1)$, then it follows that the restriction of $A(1)$ to $E(0)$ is a unitary operator on $E(0)$. Hence, it may be represented as a matrix in $\mathbf{U}(m)$, the Lie group of $m \times m$ unitary matrices.

If $\lambda(s)$ is a nondegenerate eigenvalue for all $s \in [0,1]$, then $m = 1$ and $A(1)$ may be written in the form $e^{i\gamma}$ for some real number γ, which is called the quantum adiabatic phase.[3] Notice that γ is only defined modulo 2π, and may be assumed to live on the interval $[0, 2\pi)$. The first systematic analysis of quantum adiabatic phase was conducted by M. V. Berry in [8]. He examined the hamiltonian for a particle of arbitrary spin coupled to a periodically varying magnetic field. This hamiltonian has only nondegenerate energy

[3] In the literature, γ is also called Berry's phase.

levels, and he computed γ for each of these energy levels. For this class of hamiltonians, he made the important observation that γ is essentially geometrical in nature.

The differential equation (1.8) may be used to write down an integral formula for γ. Let $\phi(s) \in \mathbf{H}$ be a smooth family of normalized eigenvectors for the nondegenerate eigenvalue $\lambda(s)$. If T were a smooth open curve, then clearly it would always be possible to define $\phi(s)$ smoothly everywhere on T. However, since T is a loop, it is not obvious that it is always possible to define $\phi(s)$ such that it is single-valued and smooth everywhere on T. For now, we shall simply assume that such a $\phi(s)$ exists. This assumption will be justified in Section II.3.

In terms of this basis, $A(s)$ satisfies

$$A(s)\phi(0) = e^{i\theta(s)}\phi(s) \ , \tag{1.21}$$

where $\theta(s) \in \mathbf{R}$ is a phase defined on the loop T. The adiabatic phase for the eigenvalue $\lambda(s)$ is now $\gamma = \theta(1)$. Substituting (1.21) into the differential equation (1.8) gives the formula

$$\theta(s) = i \int_0^s \left(\phi(x), \frac{d}{dx}\phi(x) \right) dx \pmod{2\pi} \ .$$

Using the notation of differential forms, the adiabatic phase on the loop T is then

$$\gamma(T) = i \oint_T (\phi, d\phi) \pmod{2\pi} \ . \tag{1.22}$$

B. Simon observed that the 1-form $(\phi, d\phi)$ defines a connection on a hermitian line bundle, which when viewed as a subbundle of the trivial $\mathbf{H}$-bundle has the eigenspace E for its fibre [81]. The adiabatic phase is then understood as the holonomy of this

connection. An obvious question to ask is, "What should the base space of this line bundle be?" Roughly speaking, we would like the base space to be the largest region in $Herm(\mathbf{H})$ with the property that the distinguished eigenvalue λ remains nondegenerate. Such a region will be called a similar degeneracy region.

It was observed in [97] that for a degenerate eigenvalue, it is also interesting to look at the monodromy of equation (1.10). In this situation, $e^{i\gamma}$ generalizes to a matrix in $\mathbf{U}(m)$. This "nonabelian adiabatic phase" is discussed in [4], [58], [64], [75], [78], and [97]. While we define the appropriate similar degeneracy regions, we do not attempt to compute nonabelian adiabatic holonomy in this thesis.

(b) Matrix Hamiltonians

In many applications, a hamiltonian may be represented by a matrix, at least for the purpose of computing adiabatic phase. For example, a particle with spin s and mass m, interacting through its magnetic dipole moment with a time-dependent magnetic field $\vec{B}(t)$ has

$$H(t) = \frac{1}{2m}\nabla^2 - \mu\vec{B}(t)\cdot\vec{\mathbf{s}}$$

as its hamiltonian. The coupling constant μ depends on the charge, mass, and gyromagnetic ratio of the particle; $\vec{\mathbf{s}}$ is the vector spin matrix for spin s. The adiabatic phase is determined by the time-dependent $(2s+1)\times(2s+1)$ matrix $\mu\vec{B}(t)\cdot\vec{\mathbf{s}}$, because it commutes with the time-independent kinetic term $\frac{1}{2m}\nabla^2$. The hamiltonian $H(t)$ is closely related to the hamiltonian for a photon in a coiled optical fibre.

In the Born-Oppenheimer approximation [15], [24], [25], [82, Sect. 1.2] of molecular physics, the electronic part of the molecular hamiltonian is often approximated by a matrix. Adiabatic holonomy for eigenspaces of this matrix is important in Jahn-Teller theory, which describes the coupling between the vibrational modes of the nuclei and the quantum states of the electrons. The Jahn-Teller effect will be discussed further in Subsection III.3.d.

The above examples of a particle with spin interacting with a magnetic field, and the Jahn-Teller effect illustrate that there is some physical value to analyzing adiabatic holonomy for matrix hamiltonians. In addition, studying matrix hamiltonians allows us to examine various topological problems without becoming mired in analysis difficulties. For these reasons, much of the remainder of this thesis is devoted to matrix hamiltonians. Of course, our ultimate goal is to obtain results for operator-valued hamiltonians.

If a hamiltonian is represented as an $n \times n$ hermitian matrix, then the Hilbert space upon which it acts is $\mathbf{C}^n$, the vector space of complex n-vectors. A standard norm on the vector space of $n \times n$ matrices is the Hilbert-Schmidt norm, $\|B\| = [\operatorname{tr} B^* B]^{1/2}$, where * denotes conjugate transpose. We define $Herm(n, \mathbf{C})$ to be the normed space of $n \times n$ hermitian matrices with complex entries. It is clear that the adiabatic theorem (Theorem 1.14) holds for matrix hamiltonians with the operator norm replaced by the Hilbert-Schmidt norm. Contained in the real vector space $Herm(n, \mathbf{C})$ is $Herm(n, \mathbf{R})$, the vector subspace of $n \times n$ symmetric matrices with real entries. This subspace plays an important role for physical systems which exhibit time-reversal invariance, and therefore the topology of its similar degeneracy regions will be important in Chapter III.

The simpliest nontrivial example of a matrix hamiltonian is a 2×2 hamiltonian, which describes the dynamics of a 2-level quantum system. Adiabatic phases for 2-level systems may be computed in a straightforward manner, and this was done by Berry in [8]. His results are reviewed below.

Let I denote the 2×2 identity matrix, and σ^i, for $i = 1, 2, 3$, denote the Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then, any 2×2 hermitian matrix may be uniquely expressed in the form

$$H(\mathbf{x}) = \sum_{i=1}^{3} x_i \sigma^i + x_0 I \,, \tag{1.23}$$

where $\mathbf{x} = (x_0, x_1, x_2, x_3)$ is a coordinate vector in $\mathbf{R}^4$. From this we see that $Herm(2, \mathbf{C})$ is isomorphic to $\mathbf{R}^4$, and (1.23) is used to identify these two vector spaces.

The eigenvalues of $H(\mathbf{x})$ are $\lambda_\pm = \pm\|x\| + x_0$, where $x = (x_1, x_2, x_3)$, and $\|x\| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$. Let $\mathbf{R} \cdot I$ denote the 1-dimensional vector subspace of $Herm(n, \mathbf{C})$ which is spanned by I. Then, the adiabatic hypothesis is satisfied for both eigenvalues on all loops in $Herm(2, \mathbf{C}) - \mathbf{R} \cdot I$, the complement of $\mathbf{R} \cdot I$ in $Herm(2, \mathbf{C})$. Furthermore, if U is any contractible open set in $Herm(2, \mathbf{C}) - \mathbf{R} \cdot I$, then it is possible to construct on U a smooth family of normalized eigenvectors for both of the eigenvalues of $H(\mathbf{x})$. It should be emphasized, however, that it is not possible to define globally on $Herm(2, \mathbf{C}) - \mathbf{R} \cdot I$, a smooth family of normalized eigenvectors. We return to these facts in Section 3, where they will be discussed in more detail for general hermitian $n \times n$ matrices.

Denote by $\phi_+(\mathbf{x})$ and $\phi_-(\mathbf{x})$ the normalized eigenvectors on U for the upper and lower eigenvalues, respectively. Notice that both $\phi_+(\mathbf{x})$ and $\phi_-(\mathbf{x})$ may be chosen to be independent of x_0, which simply shifts the spectrum of $H(\mathbf{x})$. Therefore, the connection 1-form from the integrand in (1.22) takes the form

$$\mathcal{A}_+ = \sum_{j=1}^{3} \left(\phi_+(x)\,,\, \frac{\partial}{\partial x_j} \phi_+(x) \right) dx_j$$

for the upper eigenvalue on U. Similarly, the connection 1-form for the lower eigenvalue, $\mathcal{A}_-$ may also be chosen so that it does not contain a dx_0 term. This implies that for any loop T in $Herm(2, \mathbf{C}) - \mathbf{R} \cdot I$, the adiabatic phases for both of the eigenvalues are invariant under orthogonal projection of T onto $Herm_0(2, \mathbf{C})$, the subspace of 2×2 traceless, hermitian matrices. From the parameterization (1.23), it is clear that $Herm_0(2, \mathbf{C})$ is isomorphic to $\mathbf{R}^3$.

Suppose that we now wish to compute the adiabatic phases of the upper and lower eigenvalues for any loop T in $Herm(2, \mathbf{C}) - \mathbf{R} \cdot I$. By the above reasoning, it may be assumed without loss of generality that T is in $Herm_0(2, \mathbf{C}) - \{O\}$, where O is the origin in the coordinate system (x_1, x_2, x_3). Furthermore, it is assumed for now that T is a simple loop which is not knotted in $Herm_0(2, \mathbf{C}) - \{O\}$. A loop is said to be simple if it has no self-intersections. An extra subtlety occurs if T is knotted, and this will be dealt with in Subsection 3.b.

Since T is not knotted, there exists an imbedding D of the 2-dimensional unit disc, such that D spans T in the sense that $\partial D = T$. (We will occasionally abuse the notation by using D to denote both the map and the image of the map in $Herm_0(2, \mathbf{C}) - \{O\}$.)

Because D is contractible, it follows that we can define connection 1-forms $\mathcal{A}_+$ and $\mathcal{A}_-$ everywhere on D. In Section 3, we shall examine how one goes about defining these connection 1-forms in general. For the upper eigenvalue, we define the 2-form $\mathcal{K}_+ = d\mathcal{A}_+$. Then from Stokes' theorem and (1.22), the adiabatic phase for the upper eigenvalue is

$$\gamma_+(T) = i \iint_D \mathcal{K}_+ \pmod{2\pi} . \tag{1.24}$$

We remark that $\mathcal{K}_+$ may be interpreted as the curvature 2-form associated with the curvature of the line bundle mentioned at the end of the preceding subsection. This will be discussed in more detail in Section 3.

For 2×2 hamiltonians, the curvature 2-form may be computed directly [8], and the result is

$$\begin{aligned} \mathcal{K}_+ &= \frac{1}{4\|x\|^2} (\phi_+, (dH)\phi_-) \wedge (\phi_-, (dH)\phi_+) \\ &= \frac{i}{2\|x\|^3} [x_3 dx_1 \wedge dx_2 + x_2 dx_3 \wedge dx_1 + x_1 dx_2 \wedge dx_3] \end{aligned} \tag{1.25}$$

Also, it is easy to verify that for the lower eigenvalue, the curvature 2-form $\mathcal{K}_- = d\mathcal{A}_-$ is equal to $-\mathcal{K}_+$. Therefore, the adiabatic phase for the lower eigenvalue is $\gamma_-(T) = -\gamma_+(T)$.

The coordinate system (x_1, x_2, x_3) identifies D with a smooth disc in $\mathbf{R}^3 - \{(0,0,0)\}$. Let $\Omega(D)$ denote the solid angle subtended from the origin by this disc. Then, equations (1.24) and (1.25) imply that

$$\gamma_+(T) = -\frac{1}{2}\Omega(D) \pmod{2\pi} , \tag{1.26}$$

which is the result found by Berry [8].

Contained in $Herm(2, \mathbf{C})$ is $Herm(2, \mathbf{R})$, the vector subspace of 2×2 real, symmetric matrices. In terms of the coordinate system (x_0, x_1, x_2, x_3), this subspace is represented by $\{(x_0, x_1, x_2, x_3) \mid x_2 = 0\}$. Now, suppose that T is a smooth loop in $Herm_0(2, \mathbf{R}) - \{O\}$, where $Herm_0(2, \mathbf{R})$ is the vector subspace of 2×2 traceless, real, symmetric matrices. In terms of the (x_1, x_2, x_3)-coordinate system, $Herm_0(2, \mathbf{R})$ is the x_1x_3-plane, and T is a loop in the punctured x_1x_3-plane. If T is not simple, then it may be decomposed into simple loops, and (1.26) may be used to compute the adiabatic phase on each simple component. The total adiabatic phase is obtained by adding the adiabatic phases for all of the simple components (of course, taking orientation into account). From this it follows that $\gamma_+(T)$ is simply π times the number of times that T winds around the origin (modulo 2).

The number of times that T winds around the origin represents the homotopy class of T in $\pi_1(Herm_0(2, \mathbf{R}) - \{O\})$, which is isomorphic to $\mathbf{Z}$. This demonstrates that for any loop T in $Herm(2, \mathbf{R}) - \mathbf{R} \cdot I$, the adiabatic phases $\gamma_+(T)$ and $\gamma_-(T)$ depend only on the homotopy class of T in $\pi_1(Herm(2, \mathbf{R}) - \mathbf{R} \cdot I)$. As we shall see, this result generalizes to $n \times n$ hamiltonians.

Using the equation (1.25), observe that when restricted to $Herm_0(2, \mathbf{R}) - \{O\}$, the curvature 2-forms $\mathcal{K}_+$ and $\mathcal{K}_-$ are zero. More generally, $\mathcal{K}_+$ and $\mathcal{K}_-$ vanish when restricted to the direct sum of $\mathbf{R} \cdot I$ with any plane through the origin in the (x_1, x_2, x_3)-coordinate system. Such subspaces are examples of potentially real subspaces of $Herm(2, \mathbf{C})$, and these will be discussed in detail in Section III.1. We shall prove that

it is generally true that the adiabatic phase is a homotopy invariant for loops in any potentially real subspace of $n \times n$ hermitian matrices. Moreover, such loops represent all periodic, time-dependent hamiltonians which are time-reversal invariant, and have a nondegenerate eigenvalue.

§2 Regions of Similar Degeneracy

In this section, we shall give a precise definition of the similar degeneracy regions, and describe their topology and the topology of their intersections with $Herm(n, \mathbf{R})$.

The finite-dimensional spectral theorem implies that a hermitian matrix may be specified by giving its spectrum and a unitary matrix, although of course, the latter is not unique. This simple idea can be used to describe similar degeneracy regions in terms of the orbit space associated with the conjugate action of the unitary group on $Herm(n, \mathbf{C})$.[4] Likewise, the similar degeneracy regions in $Herm(n, \mathbf{R})$ will be described in terms of the orbit space associated with the conjugate action of the orthogonal group on $Herm(n, \mathbf{R})$.

(a) Orbit Spaces

First we shall discuss $Herm(n, \mathbf{C})$, and then state the corresponding results for $Herm(n, \mathbf{R})$. The real vector space $Herm(n, \mathbf{C})$ has real dimension n^2, and as before, we endow it with the Hilbert-Schmidt norm, $\|T\| = [\mathrm{tr}\, T^2]^{1/2}$. Let $Herm_0(n, \mathbf{C})$ denote

[4] This is essentially the well-known adjoint representation of $\mathbf{U}(n)$ on its Lie algebra.

the (n^2-1)-dimensional subspace of traceless matrices in $Herm(n,\mathbf{C})$. With respect to the inner product $(S,T) = \operatorname{tr}[S\,T]$, this space is the orthogonal complement to the line $\mathbf{R}\cdot I = \{\alpha I \mid \alpha \in \mathbf{R}\}$, which is the one-dimensional subspace spanned by the identity matrix, I.

The compact Lie group $\mathbf{U}(n)$ acts by conjugation as a smooth transformation group acting on the right of $Herm(n,\mathbf{C})$; that is, $U \in \mathbf{U}(n)$ takes $T \in Herm(n,\mathbf{C})$ to $T^U = U^{-1}\,T\,U$. The trace and the norm are well-defined, continuous functions on the orbit space $Herm(n,\mathbf{C})/\mathbf{U}(n)$, because $\operatorname{tr}(T^U) = \operatorname{tr} T$ and $\|T^U\| = \|T\|$. Orbit spaces are defined and discussed in [17, Sect. I.3]. Under the action of $\mathbf{U}(n)$, the vector subspace $\mathbf{R}\cdot I$ is the set of all fixed points.

Every matrix $T \in Herm(n,\mathbf{C}) - \mathbf{R}\cdot I$ can be expressed uniquely as $T = \rho B + \tau I$, with $B \in \mathcal{S}(n)$, the (n^2-2)-dimensional unit sphere in $Herm_0(n,\mathbf{C})$. The parameters $\tau,\rho \in \mathbf{R}$ are defined by $\tau = \operatorname{tr}(T)/n$ and $\rho = \|T-\tau I\| > 0$. Therefore, letting T correspond to (B,ρ,τ) defines the diffeomorphism

$$Herm(n,\mathbf{C}) - \mathbf{R}\cdot I \approx \mathcal{S}(n)\times\mathbf{R}^+\times\mathbf{R}\,, \tag{2.1}$$

where $\mathbf{R}^+$ is the set of strictly positive real numbers. Furthermore, since $T^U = \rho\cdot B^U + \tau\cdot I$ for any $U \in \mathbf{U}(n)$, the equivariant product decomposition (2.1) is inherited by the orbit space, inducing the homeomorphism

$$(Herm(n,\mathbf{C}) - \mathbf{R}\cdot I)/\mathbf{U}(n) \approx [\mathcal{S}(n)/\mathbf{U}(n)]\times\mathbf{R}^+\times\mathbf{R}\,.$$

From the spectral theorem, it follows that the $\mathbf{U}(n)$ orbits in $Herm(n, \mathbf{C})$ are in one-to-one correspondence with sets of n real numbers, which are in fact just the eigenvalues. This suggests that we may use the eigenvalues to parameterize the orbit space $(Herm(n, \mathbf{C}) - \mathbf{R} \cdot I)/\mathbf{U}(n)$. However, we find that it is far better to use a canonical barycentric coordinate system which allows explicit control over the behavior of orbits as eigenspaces change dimension in $Herm(n, \mathbf{C})$, and in particular near the set of fixed points $\mathbf{R} \cdot I$. Specifically for any $T \in Herm(n, \mathbf{C}) - \mathbf{R} \cdot I$, with eigenvalues denoted in non-decreasing order as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n \ , \tag{2.2}$$

we define the following parameters:

$$\begin{aligned} \tau &= \tau(T) = \mathrm{tr}(T)/n = \frac{1}{n} \sum_{j=1}^{n} \lambda_j \ , \\ \rho &= \rho(T) = \|T - \tau I\| = \left[\sum_{j=1}^{n} (\lambda_j^2 - \tau^2) \right]^{1/2} \ , \\ \beta &= \beta(T) = (b_0, b_1, \ldots, b_{n-3}, b_{n-2}) \ , \\ \text{where} \qquad b_k &= (\lambda_{k+2} - \lambda_{k+1})/(\lambda_n - \lambda_1) \ . \end{aligned} \tag{2.3}$$

Since $\sum_{k=0}^{n-2} b_k = 1$ and $b_k \geq 0$ for each $k = 0, 1, \ldots, (n-2)$, the barycentric coordinate vector β ranges over all the elements of Δ^{n-2}, the standard $(n-2)$-simplex in $\mathbf{R}^{n-1}$. It is easy to see that the coordinate system (β, ρ, τ) uniquely labels each orbit which is not a fixed point.

For each $T \in Herm(n, \mathbf{C}) - \mathbf{R} \cdot I$, we order the eigenspaces according to the ordering of eigenvalues in (2.2). Let n_j denote the dimension of the jth eigenspace. Then, $n_j \geq 1$

for each $j = 1, \ldots p$ and $\sum_{j=1}^{p} n_j = n$, where $p \geq 2$ is the number of eigenspaces or number of distinct eigenvalues of T. This associates with each orbit an *ordered* set of strictly positive integers $(n_1, n_2, \ldots, n_p)$, which we refer to as the degeneracy type of the orbit. If T is in an orbit of degeneracy type $(n_1, n_2, \ldots, n_p)$, then T is fixed under the action of a subgroup $U_T \subset \mathbf{U}(n)$, which is isomorphic to $\prod_{j=1}^{p} \mathbf{U}(n_j)$. This subgroup is called the isotropy subgroup [17, Sect. I.2] at T. Note that the integers $n_1, n_2, \ldots, n_p$ and p depend only on the barycentric coordinate $\beta = (b_0, \ldots, b_{n-2})$.

If S and T are two points in the same orbit, it follows that the isotropy subgroups U_S and U_T are conjugate subgroups in $\mathbf{U}(n)$. In fact, the collection of isotropy subgroups for all of the points in an orbit form a complete conjugacy class of $\mathbf{U}(n)$. This conjugacy class is called the isotropy type of the orbit [17, p. 42]. We point out that two orbits with degeneracy types $(n_1, \ldots, n_p)$ and $(m_1, \ldots, m_q)$ have the same isotropy type if and only if $p = q$ and the set $\{n_1, \ldots, n_p\}$ is equal to the set $\{m_1, \ldots, m_q\}$, without regard to order.

The orbit space $\mathcal{S}(n)/\mathbf{U}(n)$ has the simple and natural structure of Δ^{n-2}, with boundary faces in any subsimplex representing changes in the degeneracy type caused by eigenvalues becoming degenerate. Furthermore, the orbit space for all orbits whose isotropy type is the conjugacy class of the subgroup $\prod_{j=1}^{p} \mathbf{U}(n_j)$ in $\mathbf{U}(n)$ is the union of at most $p!$ mutually disjoint, $(p-2)$-dimensional open subsimplices[5] of Δ^{n-2}, where each

[5] The standard q-dimensional open simplex $\tilde{\Delta}^q \subset \mathbf{R}^{q+1}$ is defined by

$$\tilde{\Delta}^q = \{(x_0, x_1, \ldots, x_q) \mid \sum_{i=0}^{q} x_i = 1 \quad \text{and} \quad x_i > 0\} .$$

An open subsimplex of Δ^{n-2} is a simplicial map of $\tilde{\Delta}^q$ into Δ^{n-2}. We caution that the image of such

open subsimplex represents the orbit space for orbits of precisely one degeneracy type.

If β is a point in Δ^{n-2}, then β lies in a unique open subsimplex of Δ^{n-2}. This open subsimplex is denoted by $\chi(\beta)$, and its closure in Δ^{n-2} is the simplex determined by vertices associated to the positive barycentric coordinates of β. Now define $\Sigma(\beta)$ to be the union of all orbits in $\mathcal{S}(n)$ with barycentric coordinates in $\chi(\beta)$. Let $\wp\colon \Sigma \to \chi$ be a restriction of the projection onto the orbit space $\mathcal{S}(n)/\mathbf{U}(n)$. (When there is no ambiguity, we shall suppress the argument β from our notation.)

Lemma 2.4. *Σ is a smooth submanifold of the (n^2-2)-dimensional unit sphere, $\mathcal{S}(n)$, and the projection $\wp\colon \Sigma \to \chi$ is a trivial, smooth fibre bundle. Each fibre $\mathcal{F}(\beta) = \wp^{-1}(\beta)$, is a single orbit diffeomorphic to the complex flag manifold:*

$$\mathcal{F}(\beta) \approx F(n_1, n_2, \cdots, n_p) \approx \mathbf{U}(n) \Big/ \prod_{j=1}^{p} \mathbf{U}(n_j)\,.$$

Moreover, the inclusion $\mathcal{F}(\beta) \hookrightarrow \Sigma(\beta)$ is a homotopy equivalence.

Proof. This follows immediately from Corollary VI.2.5 in [17]. The fibre bundle is trivial because the base is contractible. For the same reason, the inclusion of any fibre $\mathcal{F}(\beta) \hookrightarrow \Sigma(\beta)$ is a homotopy equivalence. □

We remark that the real dimension of the fibre $\mathcal{F}(\beta)$ is $n^2 - \sum_{j=1}^{p} n_j^2$, and the dimension of Σ is $n^2 - 2 - \sum_{j=1}^{p}(n_j^2 - 1)$. Moreover, since $\chi(\beta)$ is diffeomorphic to

an open subsimplex is almost never an open set in Δ^{n-2}.

$(p-2)$-dimensional Euclidean space, it follows that

$$\Sigma(\beta) \approx F(n_1, n_2, \cdots, n_p) \times \mathbf{R}^{p-2} .$$

In defining the diffeomorphism in Lemma 2.4, we have taken the convention that the arguments of $F(n_1, n_2, \ldots, n_p)$ are ordered according to $(n_1, n_2, \ldots, n_p)$, the degeneracy type of the orbit. Of course, the flag manifolds obtained by permuting the arguments of F are all diffeomorphic to each other. Recall from Section I.2 that $F(n_1, n_2, \ldots, n_p)$ is naturally isomorphic to the set of ordered p-tuples $(E_1, E_2, \ldots, E_p)$ of mutually orthogonal vector subspaces in $\mathbf{C}^n$ with the property that $dim_{\mathbf{C}}(E_i) = n_i$. Thus, the diffeomorphism in Lemma 2.4 identifies E_i as the eigenspace of the associated eigenvalue.

By applying Lemma 2.4 to each open subsimplex of Δ^{n-2}, we obtain a stratification of $\mathcal{S}(n)$. For an example, consider Figure 3, which is a diagram of this stratification of $\mathcal{S}(4)$, the unit sphere in $Herm(4, \mathbf{C})$. It is interesting to note that since $\mathcal{S}(4)$ is diffeomorphic to the 14-dimensional sphere S^{14}, this stratification provides a decomposition of S^{14} into the seven subsets Σ_i for $i = 1, 2, \ldots, 7$. The straight arrows in this figure represent the projection maps $\wp$ for the fibre bundles in Lemma 3.4, and the hooked arrows represent the inclusion of the fibres.

We now consider the set of $n \times n$ real, symmetric matrices $Herm(n, \mathbf{R})$. It is a vector subspace of $Herm(n, \mathbf{C})$, with real dimension $(n^2 + n)/2$. The subspace $Herm(n, \mathbf{R})$ inherits its norm and inner product from $Herm(n, \mathbf{C})$. Let $Herm_0(n, \mathbf{R})$ denote the vector subspace of real, symmetric matrices with zero trace.

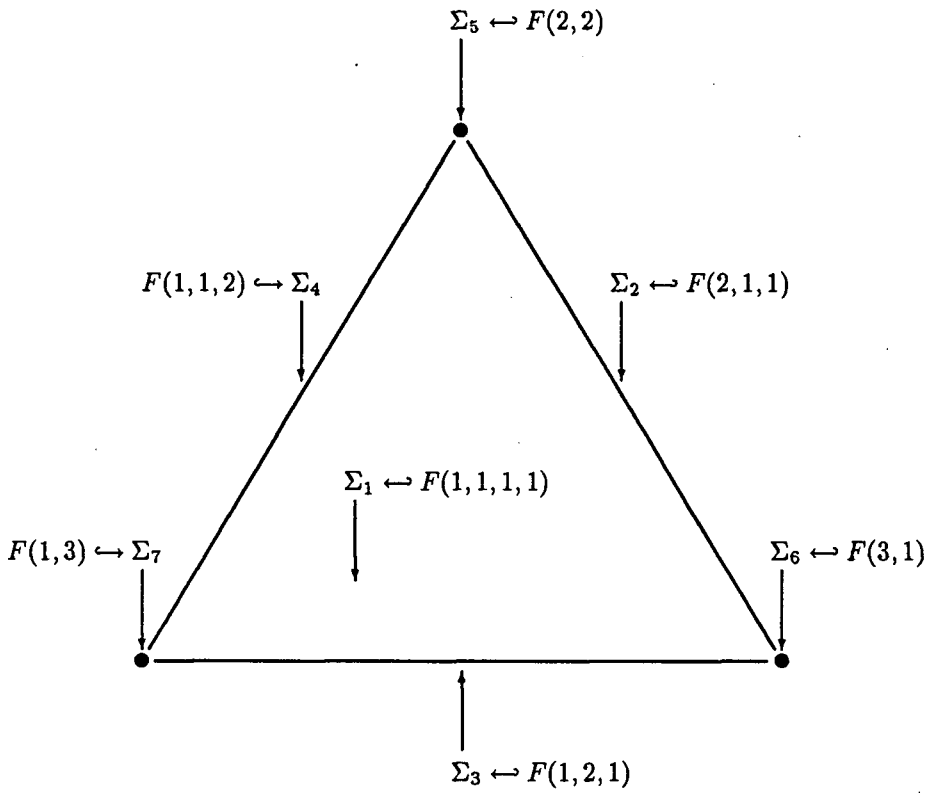

Figure 3. Stratification of $\mathcal{S}(4) \subset Herm(4, \mathbf{C})$.

The compact Lie group of $n \times n$ orthogonal matrices $\mathbf{O}(n)$ acts by conjugation as a smooth transformation group acting on the right of $Herm(n, \mathbf{R})$. Let $\mathcal{S}'(n)$ denote the $(n^2 + n - 4)/2$-dimensional unit sphere in $Herm_0(n, \mathbf{R})$. Furthermore, (2.1) restricts to the diffeomorphism

$$Herm(n, \mathbf{R}) - \mathbf{R} \cdot I \approx \mathcal{S}'(n) \times \mathbf{R}^+ \times \mathbf{R}\,. \tag{2.5}$$

As in our discussion of $Herm(n, \mathbf{C})$, we find that the orbit space of $Herm(n, \mathbf{R}) - \mathbf{R} \cdot I$

has the direct product decomposition

$$(Herm(n,\mathbf{R}) - \mathbf{R}\cdot I)/\mathbf{O}(n) \approx [\mathcal{S}'(n)/\mathbf{O}(n)] \times \mathbf{R}^+ \times \mathbf{R}\,,$$

which allows us to parameterize the orbit space of $Herm(n,\mathbf{R}) - \mathbf{R}\cdot I$ by the same coordinate system as is defined in (2.3). The orbit space $\mathcal{S}'(n)/\mathbf{O}(n)$ is again the $(n-2)$-dimensional standard simplex Δ^{n-2}.

Define $\Sigma'(\beta)$ to be the union of all orbits in $\mathcal{S}'(n)$ with barycentric coordinates in $\chi(\beta)$. Let $\wp'\colon \Sigma' \to \chi$ be the projection to the orbit space. The same reasoning as in Lemma 2.4 obtains the following result.

Lemma 2.6. *Σ' is a smooth submanifold of the $(n^2+n-4)/2$-dimensional unit sphere, $\mathcal{S}'(n)$, and the projection $\wp'\colon \Sigma'(\beta) \to \chi(\beta)$ is a trivial, smooth fibre bundle. Each fibre $\mathcal{F}'(\beta) = (\wp')^{-1}(\beta)$ is a single orbit diffeomorphic to the real flag manifold:*

$$\mathcal{F}'(\beta) = (\wp')^{-1}(\beta) \approx F'(n_1, n_2, \cdots, n_p) \approx \mathbf{O}(n)\Big/\prod_{j=1}^{p}\mathbf{O}(n_j)\,.$$

Moreover, the inclusion $\mathcal{F}'(\beta) \hookrightarrow \Sigma'(\beta)$ is a homotopy equivalence. □

Observe that the fibre $\mathcal{F}'(\beta)$ has real dimension $\frac{1}{2}\left(n^2 - \sum_{j=1}^{p} {n_j}^2\right)$, and that this is exactly half the real dimension of the fibre $\mathcal{F}(\beta)$ in Lemma 2.4. Indeed, as we pointed out in Section I.2, $F'(n_1,\ldots,n_p)$ is the set of real points in $F(n_1,\ldots,n_p)$, a complex variety defined over $\mathbf{R}$. Also, the dimension of Σ' is $\frac{1}{2}\left(n^2 - 4 - \sum_{j=1}^{p}({n_j}^2 - 2)\right)$, and the projection $\wp'$ is the restriction of $\wp$ to Σ'.

We caution the reader that our notation uses the index $\beta \in \Delta^{n-2}$ in two different ways. That is, if $\beta_j \in \Delta^{n-2}$ for $j = 1, 2$ are distinct barycentric coordinate vectors and $\chi(\beta_1) = \chi(\beta_2)$, then $\Sigma(\beta_1) = \Sigma(\beta_2)$, and $\Sigma'(\beta_1) = \Sigma'(\beta_2)$. Also, there are diffeomorphisms $\mathcal{F}(\beta_1) \approx \mathcal{F}(\beta_2)$ and $\mathcal{F}'(\beta_1) \approx \mathcal{F}'(\beta_2)$. However, $\mathcal{F}(\beta_1) \neq \mathcal{F}(\beta_2)$, and $\mathcal{F}'(\beta_1) \neq \mathcal{F}'(\beta_2)$, since they are distinct fibres, or orbits.

The results of this subsection provide a global generalization of the von Neumann-Wigner theorem [63], [68]. This theorem states that for an arbitrary hermitian matrix, it is only necessary to vary at most 3 parameters in order to cause two adjacent nondegenerate eigenvalues to cross. It also states that for real, symmetric matrices, it is only necessary to vary at most 2 parameters in order to cause two adjacent nondegenerate eigenvalues to cross. Using our stratifications for $Herm(n, \mathbf{C})$ and $Herm(n, \mathbf{R})$, it is easy to compute the relative codimension between regions with different degeneracy types.

(b) Definition of Similar Degeneracy Regions

Consider the adiabatic time evolution of an n-level quantum system, which initially has hamiltonian $B \in Herm(n, \mathbf{C})$. In the definition of adiabatic phase in Section 1, we distinguished some eigenspace of B. We are interested in the largest region in $Herm(n, \mathbf{C})$, containing B and all points in $Herm(n, \mathbf{C})$ which can be reached from B by a path along which the dimension of the distinguished eigenspace is constant. We termed such a region the similar degeneracy region (SD-region) of the distinguished eigenspace. Because such a region can be written as a union of orbits, it is most easily defined in terms of our parameterization of the orbit space.

Definition 2.7. *For $1 \leq k \leq k+d \leq n$, with $0 \leq d \leq n-2$, define $V(k, k+d) \subset \Delta^{n-2}$ to be the set of all $\beta = (b_0, b_1, \ldots, b_{n-2})$ which satisfy*

$$b_j = 0 \quad \text{if } k-1 \leq j \leq k+d-2\,,$$

$$b_{k-2} > 0 \quad \text{if } k \neq 1\,,$$

and

$$b_{k+d-1} > 0 \quad \text{if } k \neq n-d\,.$$

It is easy to see that $V(k, k+d)$ is the region in the orbit space for which the spectrum in (2.2) satisfies

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+d} < \lambda_{k+d+1}\,.$$

Of course, if either $k = 1$, or $k = n-d$, then one of the above inequalities is meaningless, and must be removed. The eigenspace associated with the eigenvalue $\lambda_k = \cdots = \lambda_{k+d}$ has constant dimension, $(d+1)$.

Notice that $V(k, k+d)$ is contained in the $(n-d-2)$-dimensional subsimplex $\sigma \subset \Delta^{n-2}$ defined by requiring that $b_j = 0$ for $k-1 \leq j \leq k+d-2$. Clearly, if $d = 0$, then $\sigma = \Delta^{n-2}$. We assume that the vertices of σ inherit their ordering from the ordering of the vertices $(w_0, ..., w_{n-2})$, of Δ^{n-2}, which in turn is consistent with the ordering of the eigenvalues in equation (2.2). If $k = 1$, then σ is the ordered simplex whose vertices are the last $n-d-1$ vertices of Δ^{n-2}, and $V(1, 1+d)$ is the open star[6] in σ of the first vertex v_0 of σ. Recall that v_0 is w_d, which is the $(d+1)$ vertex of Δ^{n-2}. Similarly, if

[6] Let x_i be the barycentric coordinate associated with the vertex v_i of Δ^q. The open star of v_i in Δ^q is defined to be the set $\{(x_0, x_1, \ldots, x_q) \in \Delta^q \mid x_i > 0\}$. Note that the open star of v_i is equal to the union all open subsimplices in Δ^q, which contain v_i in their boundary. Unlike open subsimplices, open stars of vertices are indeed open subsets in Δ^q.

$k = n - d$, then the vertices of σ are the first $n - d - 1$ vertices of Δ^{n-2} and $V(n-d, n)$ is the open star in σ of the last vertex v_{n-d-2} of σ. The vertex v_{n-d-2} is also w_{n-d-2}, the $(n-d-1)$ vertex of Δ^{n-2}. In the case $2 \leq k \leq n-d-1$, the region $V(k, k+d)$ is the intersection of the open stars for the vertices v_{k-2} and v_{k-1}, which are consecutive vertices in σ. In Δ^{n-2}, the vertices v_{k-2} and v_{k-1} are immediately before and after the d consecutive vertices of Δ^{n-2} which are not vertices of σ. Note that $V(k, k+d)$ is a contractible region in Δ^{n-2} because it is always either an open star or the intersection of two open stars in a subsimplex of Δ^{n-2}.

For example, consider the 2-simplex Δ^2 which represents the orbit spaces of the unit spheres $\mathcal{S}(4)$ and $\mathcal{S}'(4)$ in $Herm_0(4, \mathbf{C})$ and $Herm_0(4, \mathbf{R})$, respectively. The region in Δ^2 for which the first eigenvalue λ_1 is nondegenerate is $V(1,1)$: the open star of v_0 in Δ^2. This region is shaded in Figure 4.

Before writing down a careful definition of similar degeneracy regions, it is convenient to first define their projections onto the unit spheres $\mathcal{S}(n)$ and $\mathcal{S}'(n)$. We define the projected similar degeneracy regions, or PSD-regions, by

$$W(k, k+d) \stackrel{\text{def}}{=} \bigcup \{ \ \Sigma(\beta) \ \mid \ \beta \in V(k, k+d) \ \} \subset \mathcal{S}(n)$$

and

$$W'(k, k+d) \stackrel{\text{def}}{=} \bigcup \{ \ \Sigma'(\beta) \ \mid \ \beta \in V(k, k+d) \ \} \subset \mathcal{S}'(n) \ .$$

Similar degeneracy regions in $Herm(n, \mathbf{C})$ and $Herm(n, \mathbf{R})$ are defined in terms of PSD-regions by the diffeomorphisms (2.1) and (2.5), respectively. Let P denote the projection of $Herm(n, \mathbf{C}) - \mathbf{R} \cdot I$ onto $\mathcal{S}(n)$ that is defined by (2.1), and P' the restriction

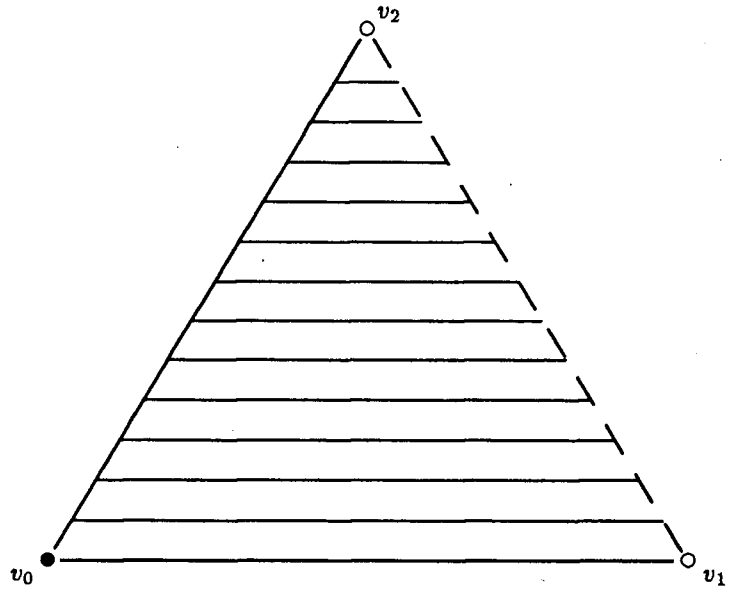

Figure 4. Open star of v_0 in Δ^2.

of P to $Herm(n, \mathbf{R}) - \mathbf{R} \cdot I$. Then, the SD-region $\mathcal{W}(k, k+d)$ in $Herm(n, \mathbf{C})$ is defined by

$$\mathcal{W}(k, k+d) \stackrel{\text{def}}{=} P^{-1}(W(k, k+d)) .$$

Similarly, the SD-region $\mathcal{W}'(k, k+d)$ in $Herm(n, \mathbf{R})$ is defined by

$$\mathcal{W}'(k, k+d) \stackrel{\text{def}}{=} (P')^{-1}(W'(k, k+d)) .$$

To keep our terminology concise, we shall refer to an SD-region in $Herm(n, \mathbf{R})$ as a real SD-region. Also, recall that the SD-regions $\mathcal{W}(k, k+d)$ and $\mathcal{W}'(k, k+d)$ are associated with an eigenvalue which has degeneracy $(d+1)$. Therefore, we will say that the SD-regions $\mathcal{W}(k, k+d)$ and $\mathcal{W}'(k, k+d)$ have degeneracy $(d+1)$. If $d = 0$, then we shall call $\mathcal{W}(k, k)$ and $W'(k, k)$ nondegenerate SD-regions.

(c) Deformation Retracts of Similar Degeneracy Regions

The existence of deformation retractions of the SD-regions $\mathcal{W}(k, k+d)$ and $\mathcal{W}'(k, k+d)$, and the PSD-regions $W(k, k+d)$ and $W'(k, k+d)$ will play an important role in our analysis of SD-regions. For a review of deformation retractions see [83, Sect. 1.4].

The first deformation retraction is an immediate result of the definitions of $\mathcal{W}(k, k+d)$ and $\mathcal{W}'(k, k+d)$.

Proposition 2.8. *The pair $(W(k, k+d), W'(k, k+d))$ is a strong deformation retract of the pair $(\mathcal{W}(k, k+d), \mathcal{W}'(k, k+d))$.* □

We remark that this implies that $\mathcal{W}(k, k+d) \simeq W(k, k+d)$ and $\mathcal{W}'(k, k+d) \simeq W'(k, k+d)$, where $\simeq$ denotes homotopy equivalence.

Our second deformation retraction is a retraction of $W(k, k+d)$ and $W'(k, k+d)$ onto either a single fibre, or the cartesian product of a single fibre with the open interval. There are two types of PSD-regions, depending on whether the associated eigenvalue is at one of the extremes of the spectrum (i.e. the smallest or largest eigenvalue), or otherwise. This will be reflected in the deformation retracts that we obtain. For ease of reference, we refer to $W(1, 1+d)$ and $W(n-d, n)$ as type I PSD-regions, while $W(k, k+d)$ with $2 \leq k \leq n-d-1$ will be called a type II PSD-region. The same terminology will also be applied to the real PSD-region $W'(k, k+d)$, and to the SD-regions, $\mathcal{W}(k, k+d)$ and $\mathcal{W}'(k, k+d)$.

Recall that the orbit space of a type I PSD-region is the open star associated with either the first vertex v_0, or the last vertex v_{n-d-2}, of an $(n-d-2)$-subsimplex $\sigma \subset \Delta^{n-2}$. In the first case, when $k = 1$ the orbit at v_0 is diffeomorphic to the complex Grassmann manifold $G(d+1, n-d-1)$. Taking liberties with the notation, we shall write $\mathcal{F}(v_0)$ for the orbit at the vertex v_0. In the second case, when $k = n-d$ the orbit $\mathcal{F}(v_{n-d-2})$ at v_{n-d-2} is diffeomorphic to the complex Grassmann manifold $G(n-d-1, d+1)$. Of course, $G(d+1, n-d-1)$ and $G(n-d-1, d+1)$ are diffeomorphic. We will show that $\mathcal{F}(v_0)$ and $\mathcal{F}(v_{n-d-2})$ are deformation retracts of their corresponding PSD-regions $W(1, d+1)$ and $W(n-d, n)$, respectively. Notice that in this way, the orbit at each vertex in Δ^{n-2} is a deformation retraction of exactly one type I PSD-region.

The second type of PSD-region, $W(k, k+d)$ with $2 \leq k \leq n-d-1$, has a more complicated deformation retract. Here the orbit space is the intersection of two open stars, which are associated with the adjacent vertices v_{k-2} and v_{k-1} in the previously defined $(n-d-2)$-subsimplex $\sigma \subset \Delta^{n-2}$. Let χ^1 be the open 1-simplex connecting v_{k-2} and v_{k-1}. If $\Sigma^1 = \wp^{-1}(\chi^1)$, then by Lemma 2.4 the projection $\wp\colon \Sigma^1 \to \chi^1$ is a trivial, smooth fibre bundle and each fibre is diffeomorphic to $F(k-1, d+1, n-d-k)$. We will show in the proof of Theorem 2.9 that Σ^1 is a deformation retract of its corresponding PSD-region. In this manner, every 1-simplex of Δ^{n-2} corresponds to exactly one SD-region of type II.

All of the above deformation retracts restrict naturally to the corresponding PSD-regions in $\mathcal{S}'(n)$, and the results are summarized in

Theorem 2.9. *First for type I PSD-regions, the pair* $(\Sigma(v_0), \Sigma'(v_0))$ *is an equivariant, strong deformation retract of the pair* $(W(1,1+d), W'(1,1+d))$, *and the pair* $(\Sigma(v_{n-d-2}), \Sigma'(v_{n-d-2}))$ *is an equivariant, strong deformation retract of the pair* $(W(n-d,n), W'(n-d,n))$. *For type II PSD-regions, take any point* $\beta \in \chi^1$, *then the pair* $(\Sigma(\beta), \Sigma'(\beta))$ *is an equivariant, strong deformation retract of the pair* $(W(k,k+d), W'(k,k+d))$, *where* $2 \leq k \leq n-d-1$.

Proof. We first define our deformation retractions in the orbit space, and then equivariantly extend them to the corresponding PSD-regions in $\mathcal{S}(n)$ and $\mathcal{S}'(n)$. For $V(k,k+d)$, we consider the homotopy

$$\phi\colon V(k,k+d) \times [0,1] \to V(k,k+d)\ ,$$

where $\phi(\beta,t) = (\hat{b}_0, \ldots, \hat{b}_{n-2})$ and $\beta = (b_0, b_1, \ldots, b_{n-2}) \in V(k,k+d)$. The $\hat{b}_j$ are defined as follows:

(i) If $k=1$, we define $\hat{b}_j = tb_j$ for $0 \leq j \leq n-2$ and $j \neq d$, and $\hat{b}_d = 1-t+tb_d$.

(ii) If $k = n-d$, we define $\hat{b}_j = tb_j$ for $0 \leq j \leq n-2$ and $j \neq n-d-2$, and $\hat{b}_{n-d-2} = 1-t+tb_{n-d-2}$.

(iii) If $2 \leq k \leq n-d-1$, we define $\hat{b}_j = tb_j$ for $0 \leq j \leq n-2$, but $j \neq k-2$ and $j \neq k+d-1$, while

$$\hat{b}_{k-2} = \frac{b_{k-2}}{b_{k-2}+b_{k+d-1}}\,[1-t+t(b_{k-2}+b_{k+d-1})]$$

and

$$\hat{b}_{k+d-1} = \frac{b_{k+d-1}}{b_{k-2}+b_{k+d-1}}\,[1-t+t(b_{k-2}+b_{k+d-1})]\ .$$

It is easy to check that for (i) and (ii), $\phi(\beta, t)$ is a strong deformation retraction of $V(k, k+d)$ onto the required vertex, and that for (iii), $\phi(\beta, t)$ is a strong deformation retraction of $V(k, k+d)$ onto the interior of the required one-simplex.

We now lift our deformation retraction ϕ to a deformation retraction Φ of $W(k, k+d)$. Consider any $B \in W(k, k+d)$, and let D be the unique traceless, unit-norm diagonal matrix in the orbit of B with the property that the diagonal entries of D are ordered in nondecreasing order. Denote by β the coordinate in $V(k, k+d)$ associated with D, and let $\phi(\beta, t)$ be the path in $V(k, k+d)$ attached to β by the homotopy constructed above. Since there is a one-to-one correspondence between barycentric coordinates and traceless, unit-norm, diagonal matrices with nondecreasing entries along the diagonal, it follows that $\phi(\beta, t)$ defines a path of such diagonal matrices $\hat{D}(t)$.

Let U be any unitary matrix satisfying $B = U\,D\,U^{-1}$, and define the homotopy

$$\Phi\colon W(k, k+d) \times [0,1] \to W(k, k+d)$$

by $\Phi(B, t) = U\,\hat{D}(t)\,U^{-1}$. Note that U is not uniquely determined, but any such U will give the same result for $\Phi(B, t)$ because the isotropy subgroup for $\hat{D}(t)$ is the same for all $0 < t \leq 1$, and this subgroup is contained in the isotropy subgroup for $\hat{D}(0)$. Thus, Φ is well defined.

It is easy to check that in all three cases, the homotopy Φ is the strong deformation retraction required. The homotopy Φ is obviously equivariant by construction. In terms of the real PSD-regions $W'(k, k+d) \subset W(k, k+d)$, the

homotopy Φ is a relative homotopy since if $B \in W'(k, k+d)$, then $\Phi(B,t) \in W'(k, k+d)$ for all $t \in [0,1]$. $\square$

Theorem 2.9 establishes that the relative inclusion $(\Sigma, \Sigma') \hookrightarrow (W, W')$ is a homotopy equivalence. With Lemma 2.4, Lemma 2.6, and Proposition 2.8, this proves

Corollary 2.10. *The inclusion of the orbit pair into the corresponding SD-region pair*

$$i\colon (F(k-1, d+1, n-d-k), F'(k-1, d+1, n-d-k)) \hookrightarrow (\mathcal{W}(k, k+d), \mathcal{W}'(k, k+d))$$

is a homotopy equivalence. In the extreme case of $k = 1$ or $k = n-d$, it is understood that F and F' are replaced by the appropriate one of the two diffeomorphic Grassmann manifolds, $G(d+1, n-d-1) \approx G(n-d-1, d+1)$ and $G'(d+1, n-d-1) \approx G'(n-d-1, d+1)$, respectively. $\square$

We remark that Corollary 2.10 illustrates the prominent role played by short flag manifolds and Grassmann manifolds in adiabatic holonomy.

§3 Eigenspace Line Bundles

On each similar degeneracy region $\mathcal{W}(k, k+d)$ in $Herm(n, \mathbf{C})$, we construct a vector bundle, which when viewed as a subbundle of the trivial product bundle $\mathcal{W}(k, k+d) \times \mathbf{C}^n$, has fibre the $(d+1)$-dimensional eigenspace associated with the eigenvalue λ_k. We refer

to this bundle as the eigenspace vector bundle. If $d = 0$, then the eigenspace bundle is a complex line bundle, which we denote by ξ. We will restrict our attention to nondegenerate eigenvalues and their associated eigenspace line bundles, since in Chapter III, we are mostly concerned with adiabatic phase.

(a) Adiabatic Connection

The total space $\mathcal{E}(\xi)$ for the complex line bundle ξ over the nondegenerate SD-region $\mathcal{W}(k,k)$, is the subspace of $\mathcal{W}(k,k) \times \mathbf{C}^n$ given by $\{(B,\phi) \mid B\phi = \lambda_k\phi\}$. The projection map $\wp\colon \mathcal{E}(\xi) \to \mathcal{W}(k,k)$ is defined by $\wp\colon (B,\phi) \mapsto B$. For $B \in \mathcal{W}(k,k)$, the fibre $\mathcal{F}(B) = \wp^{-1}(B)$ is the eigenspace associated with the eigenvalue λ_k of B. In the remainder of this section, we shall simplify our notation by writing $\mathcal{W}$ for $\mathcal{W}(k,k)$.

To verify that ξ is a complex line bundle, it only remains to check local triviality. Recall from Lemma 1.6, that the projection operator $P(B)$ depends smoothly on B. Consider a point $B_0 \in \mathcal{W}$, and let ϕ_0 be a normalized eigenvector for the eigenvalue λ_k of B_0. Because $P(B)$ is a continuous function of B, there exists a neighbourhood U of B_0 such that $\|P(B)\phi_0\| > 0$ for all $B \in U$. Therefore, it is possible to set $\phi(B) = \|P(B)\phi_0\|^{-1}\, P(B)\phi_0$, and define a map h from $U \times \mathbf{C}$ to $\wp^{-1}(U)$ by

$$h\colon (B,z) \longmapsto (B, z\phi(B))\,.$$

It is easy to check that h is a diffeomorphism, and hence ξ satisfies local triviality, as required.

The usual inner product on the vector space $\mathbf{C}^n$ defines a natural hermitian structure on the line bundle ξ. We denote the vector space over $\mathbf{C}$ of smooth sections of ξ by $C^\infty(\xi)$. For sections $s_1, s_2 \in C^\infty(\xi)$, the inner product, which is written as (s_1, s_2), is an element of $C^\infty(\mathcal{W}, \mathbf{C})$, the space of all smooth functions from $\mathcal{W}$ to $\mathbf{C}$. We suggest references [20] and [66] as excellent reviews on hermitian line bundles.

Since the SD-region $\mathcal{W}$ is nondegenerate, it follows from Definition 2.7 that it is an open subset of the manifold $Herm(n, \mathbf{C})$. This implies that $\mathcal{W}$ is an open manifold, and τ^* is defined to be the cotangent bundle of this manifold.[7]

A connection on the complex line bundle ξ is a $\mathbf{C}$-linear map

$$\nabla \colon C^\infty(\xi) \longrightarrow C^\infty(\tau^* \otimes \xi)$$

which satisfies the Leibniz formula,

$$\nabla(fs) = df \otimes s + f\nabla s$$

for every $s \in C^\infty(\xi)$ and every $f \in C^\infty(\mathcal{W}, \mathbf{C})$. Geometrically, ∇ should be thought of as defining a notion of infinitesimal parallel displacement of the fibres of ξ. In this vein, a section s is called horizontal if $\nabla s = 0$.

The differential equation (1.8) will be used to define a connection ∇_A on ξ. This connection is first defined locally, and then extended to a global definition on ξ. On any m-manifold, there exists an open cover with the property that all finite, nonempty

[7] Note that degenerate SD-regions are not open subsets of $Herm(n, \mathbf{C})$. However, although it is significantly more difficult to prove, it is true that even degenerate SD-regions are submanifolds of $Herm(n, \mathbf{C})$ [32].

intersections of sets in the open cover are diffeomorphic to $\mathbf{R}^m$ [16, p. 42]. Such an open cover is called a good cover. Let $\{U_\alpha\}$ be a good cover of $\mathcal{W}$. For a particular open set U_α, the restriction $\xi|_{U_\alpha}$ defines a complex line bundle on U_α. However, because U_α is contractible, it follows that this line bundle is trivial and hence possesses a nonzero section, which when normalized with respect to the hermitian structure on $\xi|_{U_\alpha}$ is denoted by ϕ_α. Of course, ϕ_α is nothing more than a smooth family of normalized eigenvectors for the eigenvalue λ_k of the hermitian matrices in U_α.

The notation (s_1, ds_2) will be used to denote also the obvious product between a section s_1, and the vector of differential 1-forms $ds_2 \in C^\infty(\tau^* \otimes \xi)$. The complex 1-form $\mathcal{A}_\alpha = (\phi_\alpha, d\phi_\alpha)$ from the integrand in (1.22) defines a smooth section in $\tau^*|_{U_\alpha}$. We define a map

$$\nabla_\alpha \colon C^\infty(\xi|_{U_\alpha}) \longrightarrow C^\infty((\tau^* \otimes \xi)|_{U_\alpha})$$

by requiring that it satisfy the Leibniz formula, and map ϕ_α to $\mathcal{A}_\alpha \otimes \phi_\alpha$. Because any section $s \in C^\infty(\xi|_{U_\alpha})$ may be written in the form $s = f\phi_\alpha$ for some $f \in C^\infty(U_\alpha, \mathbf{C})$, this uniquely defines a connection on $\xi|_{U_\alpha}$.

Using the above procedure, we construct a connection on the restriction of ξ to every open set in the good cover $\{U_\alpha\}$. For all of these connections to piece together to give a connection on ξ, we must check that they agree on the overlaps of the open sets. Suppose that U_α and U_β are two open sets with a nonempty intersection, and consider normalized sections ϕ_α and ϕ_β of $\xi|_{U_\alpha}$ and $\xi|_{U_\beta}$, respectively. Then, in $\xi|_{U_\alpha \cap U_\beta}$, these sections are

related by $\phi_\beta = e^{i\theta}\phi_\alpha$, for some $\theta \in C^\infty(U_\alpha \cap U_\beta, \mathbf{R})$.[8] We compute that

$$\begin{aligned}\nabla_\alpha(e^{i\theta}\phi_\alpha) &= id\theta \otimes e^{i\theta}\phi_\alpha + (\phi_\alpha, d\phi_\alpha) \otimes e^{i\theta}\phi_\alpha \\ &= (\phi_\beta, d\phi_\beta) \otimes \phi_\beta \\ &= \nabla_\beta(\phi_\beta) \end{aligned} \tag{3.1}$$

Therefore, the collection of connections $\{\nabla_\alpha\}$ piece together to give a connection for ξ, and we denote this connection by ∇_A. Because of its relation to the adiabatic theorem, ∇_A is referred to as the adiabatic connection.

A connection ∇ on a hermitian line bundle is said to be compatible[9] with the hermitian structure, if for any two sections s_1 and s_2, the relation

$$d(s_1, s_2) = (s_1, \nabla s_2) + (\nabla s_1, s_2) \tag{3.2}$$

is satisfied. We remark that (3.2) is equivalent to requiring that the inner product (s_1, s_2) be constant, whenever s_1 and s_2 are horizontal sections [20, p. 44]. For the adiabatic connection, (3.2) corresponds to the requirement that $\mathcal{A}_\alpha + \overline{\mathcal{A}_\alpha} = 0$ on each open set U_α. It follows immediately from $\mathcal{A}_\alpha = (\phi_\alpha, d\phi_\alpha)$ that this requirement is satisfied, and therefore, ∇_A is a compatible connection on the hermitian line bundle ξ.

The connection 1-form $\mathcal{A}_\alpha$ represents the action of ∇_α in terms of ϕ_α, a normalized basis for sections of $\xi|_{U_\alpha}$. Consider a new normalized basis ϕ'_α, and let $\mathcal{A}'_\alpha$ be the connection 1-form relative to this basis. It is generally true that for a connection on a

[8] For readers who are familiar with the language of gauge theory, this is simply a local gauge transformation. Specifying a section ϕ_α, with respect to which the action of ∇_α is given, is called fixing a gauge.

[9] The term compatible is used in [66]. The same property is referred to as admissible in [20].

line bundle, the connection 1-form changes by an exact form under a change of basis. For ∇_α, observe that if we write $\phi'_\alpha = e^{i\theta}\phi_\alpha$, then $\mathcal{A}'_\alpha = \mathcal{A}_\alpha + i\,d\theta$. This implies that the 2-form $\mathcal{K}_\alpha = d\mathcal{A}_\alpha$ on U_α is independent of which basis is used to represent it. Also, if U_α and U_β have a nonempty intersection, then it follows that the 2-forms $\mathcal{K}_\alpha$ and $\mathcal{K}_\beta$ agree on $U_\alpha \cap U_\beta$. Therefore, all of the 2-forms $\mathcal{K}_\alpha$ piece together to give a smooth 2-form, which is globally defined on $\mathcal{W}$ and independent of the choice of local sections. This 2-form, which we denote by $\mathcal{K}$, is called the curvature 2-form for the connection ∇_A.[10]

Let $\mathcal{W}_I$ denote a type I SD-region for a nondegenerate eigenvalue. This means that in $Herm(n, \mathbf{C})$, the SD-region $\mathcal{W}_I$ is either $\mathcal{W}(1,1)$, or $\mathcal{W}(n,n)$. From Proposition 2.8 and Theorem 2.9, there is a strong deformation retract of $\mathcal{W}_I$ to a subspace which is a single orbit diffeomorphic to the complex projective space $\mathbf{C}P(n-1)$. Let $i\colon \mathbf{C}P(n-1) \hookrightarrow \mathcal{W}_I$ be the inclusion of this subspace. If ξ_I is the eigenspace line bundle over $\mathcal{W}_I$, then the pullback $i^*\xi_I$ defines a complex line bundle over $\mathbf{C}P(n-1)$.

Also defined over $\mathbf{C}P(n-1)$ is the canonical complex line bundle, γ_{n-1} [66, p. 159]. It is constructed as a subbundle of the trivial complex vector bundle $\mathbf{C}P(n-1) \times \mathbf{C}^n$ as follows. Recall from Section I.2 that $\mathbf{C}P(n-1)$ is the space of complex lines in $\mathbf{C}^n$. The total space of γ_{n-1} is defined as

$$\mathcal{E}(\gamma_{n-1}) = \{(v,x) \mid v \in \mathbf{C}P(n-1) \text{ and } x \in v \subset \mathbf{C}^n\}\ .$$

The projection map $\wp\colon \mathcal{E}(\gamma_{n-1}) \to \mathbf{C}P(n-1)$ is defined by $\wp\colon (v,x) \mapsto v$.

[10] The concept of curvature in vector bundles is much richer than indicated above. We suggest [20] and [66] for some informative reading.

The canonical line bundles γ_{n-1} are fundamental to the theory of complex line bundles. Indeed, they may be used to classify all isomorphism classes of line bundles over a given base space. More details on this construction are given in [20, Sect. 8], [49], and [66].

Given the importance of γ_{n-1}, the following result is very useful.

Proposition 3.3. *The complex line bundle $i^*\xi_I$ is isomorphic to γ_{n-1}.*

Proof. The proof of this theorem follows by inspection. Recall that Proposition 2.8 and Theorem 2.9 define a deformation retraction of $\mathcal{W}_I$ onto the subspace Σ, which consists of all matrices in $Herm(n, \mathbf{C})$ with two fixed eigenvalues λ_1 and λ_2, such that λ_1 is the nondegenerate eigenvalue associated with $\mathcal{W}_I$, and λ_2 is an $(n-1)$-fold degenerate eigenvalue. From Lemma 2.4, it follows that Σ is diffeomorphic to $\mathbf{C}P(n-1)$, where $\mathbf{C}P(n-1)$ is the subspace of all pairs (E_1, E_2) of orthogonal subspaces in $\mathbf{C}^n$, with the property that $\dim_{\mathbf{C}} E_1 = 1$ and $\dim_{\mathbf{C}} E_2 = n-1$. Under this diffeomorphism, E_1 is identified with the eigenspace of λ_1, and E_2 is identified with the eigenspace of λ_2. Therefore, $\mathcal{E}(i^*\xi_I)$ is diffeomorphic to $\mathcal{E}(\gamma_{n-1})$. This diffeomorphism restricts to a vector-space isomorphism on fibres, and hence $i^*\xi_I \cong \gamma_{n-1}$. □

Consider now any nondegenerate, type II SD-region in $Herm(n, \mathbf{C})$, and denote it by $\mathcal{W}_{II}$. Let $i\colon F(p,1,r) \hookrightarrow \mathcal{W}_{II}$ be the inclusion defined by Corollary 2.10. Of course, p and r satisfy $p+1+r=n$. The fibre bundles in (2.6) and (2.7) of Chapter I define

inclusions $i_1\colon \mathbf{C}P(p) \hookrightarrow F(p,1,r)$ and $i_2\colon \mathbf{C}P(r) \hookrightarrow F(p,1,r)$. Then, the eigenspace line bundle ξ_{II} over $\mathcal{W}_{II}$ satisfies

Proposition 3.4. *The complex line bundle $i_1^* i^* \xi_{II}$ is isomorphic to the canonical bundle γ_p over $\mathbf{C}P(p)$, and $i_2^* i^* \xi_{II}$ is isomorphic to the canonical bundle γ_r over $\mathbf{C}P(r)$.*

Proof. This proof is similar to the proof of Proposition 3.3. In Section I.2, we show that $F(p,1,r)$ is the space of all triples (E_1, E_2, E_3) of mutually orthogonal subspaces of $\mathbf{C}^n$ such that $\dim_{\mathbf{C}} E_1 = p$, $\dim_{\mathbf{C}} E_2 = 1$, and $\dim_{\mathbf{C}} E_3 = r$. The pullback $i^*\xi_{II}$ is isomorphic to the complex line bundle with total space $\{((E_1, E_2, E_3), v) \in F(p,1,r) \times \mathbf{C}^n \mid v \in E_2\}$. The proposition then follows immediately from the definitions of i_1 and i_2. □

(b) Holonomy and Stokes' Theorem

Recall from Section 1, that the adiabatic phase $\gamma(T)$ for a nondegenerate eigenvalue λ is defined by the $\mathbf{U}(1)$-valued monodromy matrix for the differential equation (1.8). B. Simon observed [81] that $\exp[-i\gamma(T)]$ may be given a more geometrical interpretation: specifically, that of the holonomy of the connection ∇_A on T. In the remainder of this section, we shall explain Simon's observation, and use Stokes' theorem to compute the adiabatic phase in terms of the curvature 2-form, $\mathcal{K}$.

The connection ∇_A defines a notion of parallel transport in the eigenspace line bundle ξ over $\mathcal{W}$, the SD-region associated with the nondegenerate eigenvalue λ. In general,

parallel transport defined by any connection satisfies the following properties [57, Proposition II.3.3]. If T is a C^1 curve in the base space, then parallel transport along the inverse curve T^{-1} is the inverse of parallel transport along T. Furthermore, if T and S are C^1 curves in the base space, then parallel transport along the composition $S \cdot T$ is the composition of parallel transport along S with parallel transport along T.

Suppose now that T is a loop in $\mathcal{W}$, with base point B. Then, the parallel transport generated by ∇_A of the fibre $\mathcal{F}(B)$ around T defines an isomorphism from $\mathcal{F}(B)$ onto itself. It follows from the previous paragraph, that there is a group structure on the set of all such isomorphisms. This group, which is denoted by $\Phi(B)$, is called the holonomy group of ∇_A with reference point B.

Each isomorphism from $\mathcal{F}(B)$ onto itself may be identified with an element of $\mathbf{GL}(1, \mathbf{C})$, the structure group of ξ. Therefore, $\Phi(B)$ is isomorphic with a subgroup of $\mathbf{GL}(1, \mathbf{C})$. We shall abuse the notation slightly, and write $\Phi(B)$ for this subgroup. Furthermore, because ∇_A is compatible with the hermitian structure on ξ, it follows that $\Phi(B)$ is actually a subgroup of the reduced structure group $\mathbf{U}(1)$. This is simply a geometrical restatement of the fact that solutions to the adiabatic differential equation (1.8) are unitary.

Let $\Gamma_B(T) \in \Phi(B)$ denote the holonomy of ∇_A on the loop T, with base point B. If we choose another base point C for T, it follows from the discussion in [57, Sect. II.4] that $\Gamma_B(T) \in \Phi(B)$ and $\Gamma_C(T) \in \Phi(C)$ are conjugate when viewed as elements of the reduced structure group of ξ. However, $\mathbf{U}(1)$ is abelian, and therefore $\Gamma_B(T) = \Gamma_C(T)$.

This implies that the holonomy of ∇_A defines a map Γ from the space of free loops in $\mathcal{W}$ to $\mathbf{U}(1)$.[11]

The loop T is 1-dimensional, and therefore it has no cohomology in dimension 2. This implies that the first Chern class of the restriction of ξ to T is zero. Line bundles are completely classified, up to isomorphism, by their first Chern class, and therefore we conclude that $\xi|_T$ is isomorphic to the trivial complex line bundle. It follows that there exists a smooth, normalized section $\phi \in C^\infty(\xi|_T)$, which is defined everywhere over T. This section gives rise to a smooth connection 1-form $\mathcal{A} = (\phi\,,\, d\phi)$. Because $\mathcal{A}$ is defined globally on T, it follows that the holonomy of ∇_A on T is given by the integral formula

$$\Gamma(T) = \exp[\oint_T \mathcal{A}] = \exp[-i\gamma(T)]\,. \tag{3.5}$$

This demonstrates the relationship between adiabatic phase and parallel displacement in the eigenspace line bundle.

It is now shown how Stokes' theorem may be used to give a formula for the adiabatic phase. Recall that a nondegenerate SD-region $\mathcal{W} \subset Herm(n, \mathbf{C})$ is an open n^2-dimensional manifold, without boundary. Because the situation when $n = 1$ is uninteresting, we shall assume that $n \geq 2$. Since the dimension of $\mathcal{W}$ is at least 4, it follows from the Whitney imbedding theorem [84, Sect. II.4], [93, Thm. 2] that any smooth loop T in $\mathcal{W}$ may be approximated arbitrarily closely by a simple loop.

[11] Holonomy in general vector bundles and principal bundles is more complicated than in line bundles, and it is typically not possible to represent it as a map from the space of free loops in the base space to the structure group. For a review of holonomy in principal bundles, see [57].

The holonomy $\Gamma(T)$ varies smoothly under a smooth homotopy of T.[12] Hence, we can approximate $\Gamma(T)$ arbitrarily closely by computing it on a simple loop which approximates T arbitrarily closely. It is therefore sufficient to compute the adiabatic phase only for simple, smooth loops in $\mathcal{W}$.

For the remainder of this section, we shall assume that T is any simple, smooth loop in $\mathcal{W}$. In the Section I.2, it was shown that all complex flag manifolds are simply connected. Along with Corollary 2.10, this implies that $\mathcal{W}$ is also simply connected, and hence T is null-homotopic. If $\dim \mathcal{W} \geq 5$ (i.e. $n \geq 3$), it follows from Theorem 7 in [93] that T may be extended to a smooth imbedding D of the 2-dimensional unit disc into $\mathcal{W}$, such that $T = \partial D$.

If $n = 2$, recall from Section 2 that both of the SD-regions $\mathcal{W}(1,1)$ and $\mathcal{W}(2,2)$ are equal to $Herm(2,\mathbf{C}) - \mathbf{R} \cdot I$, which is diffeomorphic to $\mathbf{R}^4$ with a line removed. It follows from the corollary to Theorem 1 in [40] that T bounds a smooth imbedding D of the 2-dimensional unit disc in $\mathbf{R}^4$. However, a disc and a line do not generically intersect in $\mathbf{R}^4$, and therefore by at most an arbitrarily small perturbation of D, the image of D can be made to lie entirely in $\mathcal{W}$.

At this point, it is appropriate to remark on a subtlety which was not discussed when 2×2 hamiltonians were considered in Subsection 1.b. It was argued that since the trace coordinate in $Herm(2,\mathbf{C})$ does not appear in the curvature 2-form $\mathcal{K}$, the trace may be ignored for the purposes of computing γ. For this reason, only loops in $Herm_0(2,\mathbf{C}) - \{O\}$ were considered. The space $Herm_0(2,\mathbf{C}) - \{O\}$ is diffeomorphic to

[12] This follows from the lemma on page 74 in [57].

$\mathbf{R}^3 - \{(0,0,0)\}$. Therefore, in general a simple loop T in $Herm_0(2,\mathbf{C}) - \{O\}$ may be knotted. If this is the case, then T does not bound a smooth disc in $Herm_0(2,\mathbf{C}) - \{O\}$, and it will not be possible to naively use Stokes' theorem in $Herm_0(2,\mathbf{C}) - \{O\}$ to compute γ. However, we have shown that T bounds a disc D in $Herm(2,\mathbf{C}) - \mathbf{R}\cdot I$, and therefore the trace degree of freedom may be used to unknot T. Note, that if T is knotted in $Herm_0(2,\mathbf{C})$, then D must necessarily extend outside of $Herm_0(2,\mathbf{C})$.

To conclude, we have shown that any smooth loop T in a nondegenerate SD-region $\mathcal{W}$ bounds a smoothly imbedded disc D in $\mathcal{W}$. Because D is contractible, it follows that the restriction of the eigenspace line bundle ξ to D is isomorphic to the trivial complex line bundle over D. This implies that there is a smooth, normalized section $\phi \in C^\infty(\xi|_D)$, which is defined everywhere over D.

Using the section ϕ, we obtain from ∇_A a smooth connection 1-form $\mathcal{A}$, defined everywhere on D. Note that the 1-form $\mathcal{A}$ depends on the disc that is used to span the loop T. In particular, suppose that we had chosen another disc D', with $\partial D' = T$. Let $\mathcal{A}'$ be the connection 1-form obtained from the construction of a normalized section of $\xi|_{D'}$. It need not be true that $\mathcal{A}$ and $\mathcal{A}'$ agree on T. Furthermore, although locally $\mathcal{A} - \mathcal{A}' = d\alpha$ for some 0-form α, it does not follow that $\oint_T \mathcal{A}$ is equal to $\oint_T \mathcal{A}'$. This is because it may not be possible to define α globally on T. Nevertheless, parallel transport generated by ∇_A is uniquely determined. Hence, the holonomy of ∇_A is independent of the connection 1-form. Therefore, we know that $\oint_T \mathcal{A} - \oint_T \mathcal{A}'$ is equal to an integer multiple of 2π. Keeping this in mind, we rewrite (3.5) as

$$\gamma(T) = i \oint_T \mathcal{A} \pmod{2\pi} .$$

Recall that the curvature 2-form $\mathcal{K}$ satisfies $\mathcal{K} = d\mathcal{A}$. Because $\mathcal{A}$ is defined smoothly everywhere on D, we may use Stokes' theorem to obtain the formula

$$\gamma(T) = i \iint_D \mathcal{K} \pmod{2\pi} . \tag{3.6}$$

As an aside, note that although thus far D has been taken to be a disc with boundary T, we could more generally consider D to be an arbitrary, compact, connected, orientable surface with boundary T. To derive (3.6), we required that $\xi|_D$ be a trivial line bundle. Since any connected 2-manifold with nonempty boundary has no cohomology in dimension 2,[13] it follows that the first Chern class of $\xi|_D$ is zero. Therefore, $\xi|_D$ is isomorphic to the trivial complex line bundle over D, and there exists a normalized section ϕ defined everywhere over D. Using ϕ, a connection 1-form, and a curvature 2-form may be constructed on D. Then, from Stoke's theorem, it follows that (3.6) holds if D is any compact, orientable surface with $\partial D = T$.

(c) Photons in an Optical Fibre

In [23], Chiao and Wu propose an experiment in which polarized light from a laser is injected into a helically wound optical fibre. They argued that because of the helical shape of the optical fibre, the wave function for a photon would acquire a quantum adiabatic phase. For a linearly polarized laser beam, this phase would cause a rotation of

[13] Indeed for an arbitrary coefficient group, the 2-dimensional cohomology of any connected 2-manifold with nonempty boundary is zero. This is proven by considering a triangulation of the manifold, and noting that it is possible to contract the triangles bordering the boundary, without changing the homotopy type of the manifold.

the plane of polarization. Subsequently, this rotation of polarization was experimentally verified by Tomita and Chiao [88].

The introduction of eigenspace line bundles in the previous two subsections provide the means for a careful computation of quantum adiabatic phase for photons in an optical fibre. As well, this example is an interesting application of line bundles to quantum physics.

Following Chiao and Wu, we consider an individual photon propagating in an optical fibre which has been wound in a helix. Furthermore, it is assumed that the length of the fibre, the radius of curvature of the helix, and the radius of torsion of the helix are all much larger than the diameter of the fibre. With these assumptions, it is reasonable to assume that the direction of propagation of the photon is well-defined, and specified by the tangent vector to the helix. We shall take the point of view that this is a problem in 1-dimensional quantum mechanics, where the direction of propagation is an external parameter which may be specified by the experimenter through the positioning of the optical fibre.

The two helicity states of a photon interact differently with a birefringent medium. We write the hamiltonian for the photon as

$$H = H_{\text{fr}} + H_{\text{int}} ,$$

where H_{int} describes the interaction of the helicity states with the medium of the optical fibre, and H_{fr} is the helicity-independent part of the hamiltonian. The helicity operator of a particle depends on the propagation direction, and therefore H_{int} varies with the

tangent vector to the optical fibre. This suggests that H_{int} may give rise to a quantum adiabatic phase. However, H_{fr} cannot contribute to such a phase, because it is a constant operator which commutes with H_{int}.

In order to construct H_{int}, note that the photon is a spin-1 particle. The helicity operator for a spin-1 particle is $\vec{k} \cdot \vec{\mathrm{s}}$, where $\vec{\mathrm{s}}$ is the vector spin matrix for spin-1, and $\vec{k}$ is a unit vector in $\mathbf{R}^3$, giving the propagation direction of the particle. A 3×3 matrix representation for $\vec{\mathrm{s}} = (s_1, s_2, s_3)$ is

$$s_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad s_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad \text{and} \qquad s_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{3.7}$$

In terms of this representation, $\vec{k} \cdot \vec{\mathrm{s}}$ is a 3×3 matrix parameterized by $\vec{k}$, which takes values in the unit sphere S^2. The eigenvalues of $\vec{k} \cdot \vec{\mathrm{s}}$ are $+1$, 0, and -1; the associated eigenspaces are denoted by $E_+(\vec{k})$, $E_0(\vec{k})$, and $E_-(\vec{k})$, respectively. Note that these eigenspaces depend on $\vec{k} \in S^2$.

The polarization vector of a photon must be perpendicular to the direction of propagation. This constraint, which is referred to as transversality, requires that the 0-helicity eigenvector must be projected out of the state space of the photon. Because the photon is massless, this constraint is relativistically invariant. Let $\mathcal{T}(\vec{k})$ denote the vector subspace of $\mathbf{C}^3$ defined by the span of $E_+(\vec{k})$ and $E_-(\vec{k})$. Then, the helicity operator for the photon is

$$\vec{k} \cdot \vec{\mathrm{s}}|_{\mathcal{T}(\vec{k})}, \tag{3.8}$$

and $H_{\text{int}} = \kappa\, \vec{k} \cdot \vec{\mathrm{s}}|_{\mathcal{T}(\vec{k})}$, where κ is related to the circular birefringence of the medium.

Observe that the transversality constraint applied in (3.8) depends on the parameter $\vec{k}$, which indicates that it is most naturally imposed through fibre bundle theory. Constructed over S^2 from the eigenspaces $E_+(\vec{k})$, $E_0(\vec{k})$, and $E_-(\vec{k})$ are the eigenspace line bundles ξ_+, ξ_0, and ξ_-, respectively. The coordinate representation of $\vec{k} \cdot \vec{\mathbf{s}}$, obtained by using (3.7) for $\vec{\mathbf{s}}$, defines an inclusion j of S^2 into the type II SD-region $\mathcal{W}(2,2)$ of $Herm(3,\mathbf{C})$. The line bundles ξ_+, ξ_0, and ξ_- are pullbacks over j of the appropriate eigenspace line bundles on $\mathcal{W}(2,2)$, which have been defined in Subsection 3.a.

For ξ_+, ξ_0, and ξ_-, there are defined adiabatic connections, which we denote by ∇_+, ∇_0, and ∇_-, respectively. Associated with ∇_+, ∇_0, and ∇_- are curvature 2-forms $\mathcal{K}_+$, $\mathcal{K}_0$, and $\mathcal{K}_-$, respectively. These 2-forms, which are globally defined on S^2, were calculated by Berry in [8], and the results are

$$\mathcal{K}_\pm = \pm i \, [k_3 dk_1 \wedge dk_2 + k_2 dk_3 \wedge dk_1 + k_1 dk_2 \wedge dk_3] \ , \tag{3.9}$$

and

$$\mathcal{K}_0 = 0 \ . \tag{3.10}$$

It follows from (3.9) that if T is a simple loop in S^2, and D is a disc spanning T such that $\partial D = T$, then the holonomy of ∇_+ on T is $\exp[-i\Omega(D)]$, where $\Omega(D)$ is the solid angle subtended from the origin by D. Similarly, the holonomy of ∇_- on T is $\exp[i\Omega(D)]$. Also, equation (3.10) and the fact that S^2 is simply connected imply that ∇_0 exhibits no holonomy for all loops in S^2.

Before continuing with the problem at hand, we require the following general result for hermitian vector bundles. Suppose that a hermitian n-plane bundle η has a compatible

connection ∇, which has no holonomy for all loops in the base manifold. In the fibre $\mathcal{F}_x$ over some point x in the base manifold, choose an orthonormal n-frame $(e_1, e_2, \ldots, e_n)$. Because ∇ has no holonomy, we can construct an orthonormal n-frame of sections for η by using ∇ to parallel transport $(e_1, e_2, \ldots, e_n)$ to all other fibres of η. This implies that η is a trivial vector bundle. It should be remarked that the converse to this result is not true. In other words, trivial vector bundles may have connections which exhibit holonomy.

In light of the above result, we reconsider the eigenspace line bundle ξ_0 over S^2. The connection ∇_0 has no holonomy, and therefore ξ_0 is a trivial vector bundle. The first Chern class $c_1(\xi_0)$ is an element of $H^2(S^2; \mathbf{Z}) \cong \mathbf{Z}$, and triviality of ξ_0 implies that $c_1(\xi_0) = 0$.

We now return to the transversality constraint in (3.8). Taking the vector space $\mathcal{T}(\vec{k})$ as the fibre over each $\vec{k} \in S^2$, we obtain a complex 2-plane bundle bundle over S^2. This vector bundle is equal to the Whitney sum $\xi_+ \oplus \xi_-$. It is desirable to express the operator in (3.8) as a 2×2 matrix which depends smoothly on $\vec{k} \in S^2$. This requires a smoothly $\vec{k}$-dependent orthonormal basis for $\mathcal{T}(\vec{k})$, which is of course nothing more than an orthonormal basis of sections for $\xi_+ \oplus \xi_-$. Therefore, H_{int} can be expressed as a smoothly $\vec{k}$-dependent matrix if $\xi_+ \oplus \xi_-$ is a trivial complex 2-plane bundle.

The Chern classes of $\xi_+ \oplus \xi_-$ are easily calculated using the product theorem for total Chern classes. Recall from the definition of eigenspace line bundles in Subsection 3.a, that $\xi_+ \oplus \xi_0 \oplus \xi_-$ is a trivial complex 3-plane bundle over S^2. Therefore, the product theorem and $c_1(\xi_0) = 0$ imply that both the first Chern class $c_1(\xi_+ \oplus \xi_-)$ and the second

Chern class $c_2(\xi_+ \oplus \xi_-)$ are zero. However, we caution that this does not necessarily imply that $\xi_+ \oplus \xi_-$ is trivial. Unlike with complex line bundles, the isomorphism class of a complex n-plane bundle is not completely determined by its total Chern class. We require some additional results.[14]

Lemma 3.11. *Every complex n-plane bundle η^n over S^2 is isomorphic to the Whitney sum $\eta^1 \oplus \varepsilon^{n-1}$, where ε^{n-1} is the trivial complex $(n-1)$-plane bundle on S^2, and η^1 is some complex line bundle, which is not necessarily trivial.*

Proof. For η^n, the fibre $\mathcal{F}_x$ over each point $x \in S^2$, is a complex n-plane. The collection of orthonormal $(n-1)$-frames in $\mathcal{F}_x$ form the Stieffel manifold $V_{n-1}(\mathcal{F}_x)$. All Stieffel manifolds are compact and connected. The manifolds $V_{n-1}(\mathcal{F}_x)$ are the fibres of the fibre bundle $V_{n-1}(\eta^n)$, which is one of the associated Stieffel bundles over S^2. More details on associated Stieffel bundles are given in [66, Sect. 12].

The notation for $V_{n-1}(\eta^n)$ is introduced through the diagram

$$\begin{array}{ccc} V_{n-1}(\mathbf{C}^n) & \xrightarrow{\ i\ } & \tilde{\mathcal{E}} \\ & & \Big\downarrow{\scriptstyle \wp} \\ & & S^2 \end{array}$$

The homotopy exact sequence for this bundle is

$$\cdots \pi_2(V_{n-1}(\mathbf{C}^n)) \xrightarrow{i_\#} \pi_2(\tilde{\mathcal{E}}) \xrightarrow{\wp_\#} \pi_2(S^2) \to \pi_1(V_{n-1}(\mathbf{C}^n)) \xrightarrow{i_\#} \pi_1(\tilde{\mathcal{E}}) \to \cdots \quad (3.12)$$

[14] We thank K. Y. Lam for bringing this lemma to our attention.

It is shown in [49, Thm. 7.5.1] that $\pi_1(V_{n-1}(\mathbf{C}^n)) = 0$ for all integers $n \geq 2$. Therefore, it follows from the exact sequence (3.12) that the induced homomorphism $\wp_{\#}\colon \pi_2(\tilde{\mathcal{E}}) \to \pi_2(S^2)$ is an epimorphism. This implies that there is no obstruction to constructing a section for the fibre bundle $V_{n-1}(\eta^n)$.

A section for $V_{n-1}(\eta^n)$ provides an orthonormal $(n-1)$-frame of sections for η^n. The span of this frame of sections gives the trivial vector bundle ε^{n-1} as a subbundle of η^n. The line bundle η^1 is taken to be the orthogonal complement of ε^{n-1} in η^n. Note that $\pi_1(V_n(\mathbf{C}^n)) \cong \mathbf{Z}$ [49, Prop. 7.11.3], which implies that η^1 need not be trivial. □

Consider the Chern classes $c_i(\eta^n)$ for any complex n-plane bundle η^n over S^2. For $i \geq 2$, the Chern classes $c_i(\eta^n)$ must vanish, because S^2 is 2-dimensional. Furthermore, it follows from Lemma 3.11 that the isomorphism class of η^n is completely determined by $c_1(\eta^n)$, which is equal to $c_1(\eta^1)$. Therefore, because $c_1(\xi_+ \oplus \xi_-) = 0$, it follows that the Whitney sum of eigenspace line bundles $\xi_+ \oplus \xi_-$ is a trivial complex 2-plane bundle.[15]

The triviality of $\xi_+ \oplus \xi_-$ guarantees the existence of a frame of sections, with respect to which H_{int} can be represented as a 2×2 matrix. Indeed, there are infinitely many such frames of sections. It will become apparent that it is not strictly necessary to write

[15] Although $\xi_+ \oplus \xi_-$ is trivial, the individual line bundles ξ_+ and ξ_- are not trivial. The first Chern classes of ξ_+ and ξ_- may be calculated by constructing normalized sections over the northern and southern hemispheres of S^2, and then computing the homotopy classes of the transition functions on the equator. The result is that $c_1(\xi_\pm) \in H^2(S^2; \mathbf{Z})$ are represented by the integers ± 2. Of course, which bundle is ascribed $+2$ and which bundle is ascribed -2 is simply a matter of convention.

down a specific matrix representation of H_{int}; however, for the sake of completeness, we do so. Observe that

$$\phi_1 = \frac{1}{\sqrt{2+2k_3^2}} \begin{bmatrix} 1+k_3^2 \\ \sqrt{2}k_3(k_1+ik_2) \\ (k_1+ik_2)^2 \end{bmatrix}, \qquad \phi_2 = \frac{1}{\sqrt{2+2k_3^2}} \begin{bmatrix} 0 \\ \sqrt{2}(k_1-ik_2) \\ -2k_3 \end{bmatrix}$$

is an orthonormal basis of sections for $\xi_+ \oplus \xi_-$. With respect to this basis,

$$H_{\text{int}}(\vec{k}) = \frac{\kappa}{1+k_3^2} \begin{bmatrix} 2k_3 & (k_1-ik_2)^2 \\ (k_1+ik_2)^2 & -2k_3 \end{bmatrix}.$$

It is easy to verify that the eigenvalues of this matrix are $\pm\kappa$, as expected.

Now that $H_{\text{int}}(\vec{k})$ has a matrix representation, quantum adiabatic phase for the photon is easily computed. Assume that the optical fibre is wound in a helix such that the unit tangent vector $\vec{k}$ describes a loop T in S^2. This loop is parameterized by s, the length along the fibre. With suitable normalization, it is arranged that $s \in [0,1]$.

We shall assume that T is a simple loop. If T were not simple, then it may be decomposed into simple components, and the following analysis carried out for each component. On a disc D spanning T in S^2, construct a family of normalized eigenvectors

$$\phi_+ = \begin{bmatrix} a(\vec{k}) \\ b(\vec{k}) \end{bmatrix}$$

for the eigenvalue $+\kappa$ of H_{int}. Associated with ϕ_+ is an adiabatic connection 1-form

$$\begin{aligned} \mathcal{A}_+ &= (\phi_+\,,\, d\phi_+) \\ &= (a\phi_1 + b\phi_2\,,\, d[a\phi_1 + b\phi_2])\ . \end{aligned}$$

Observe that the complex 3-vector $a\phi_1 + b\phi_2$ is an eigenvector for the 3×3 matrix $\vec{k}\cdot\vec{\mathbf{s}}$, with eigenvalue $+1$. Therefore, $\mathcal{A}_+$ is also an adiabatic connection 1-form for the eigenspace line bundle ξ_+. It follows from (3.9) that the quantum adiabatic phase acquired by $\phi_+(\vec{k})$ under parallel transport around T is $\gamma_+(T) = -\Omega(D) \pmod{2\pi}$. Similarly, if ϕ_- is an eigenvector associated with the lower eigenvalue of H_{int}, then ϕ_- acquires a quantum adiabatic phase of $-\gamma_+(T)$.

It is perhaps useful to recapitulate the above computation. By showing that the vector bundle $\xi_+ \oplus \xi_-$ is trivial, we demonstrated that the transversality constraint in (3.8) is "topologically trivial". The result of this is that for the positive and negative helicity states of a photon, the quantum adiabatic phase agrees with the phase for the corresponding helicity state of an ordinary spin-1 particle. Therefore, the phases computed here agree with the results of Chiao and Wu in [23], where transversality was not imposed on the hamiltonian.

It was shown in [23] that the adiabatic phases acquired by the helicity states of photons give rise to a rotation of the polarization plane for linearly polarized light. Let ψ_+ and ψ_- be positive and negative helicity eigenstates, respectively, for the total hamiltonian H, such that $H_{\text{fr}}\psi_+ = \epsilon\psi_+$ and $H_{\text{fr}}\psi_- = \epsilon\psi_-$. The wave function for linearly polarized light is a superposition of left-handed and right-handed polarization states. Therefore, the initial state of the photon is taken to be $\psi(0) = 2^{-\frac{1}{2}}\{\psi_+ + \psi_-\}$.

The adiabatic theorem implies that at the end of the optical fibre, the photon's final state vector $\psi(1)$ is approximately parallel to $\psi(0)$. The phase difference between the

adiabatic approximation to $\psi(1)$, and $\psi(0)$ is the sum of the adiabatic and dynamical phases. If τ is the optical length of the fibre, then

$$\psi(1) \simeq 2^{-\frac{1}{2}} \left\{ \exp[-i(\epsilon\tau + \kappa\tau - \gamma_+(T))]\, \psi_+ + \exp[-i(\epsilon\tau - \kappa\tau + \gamma_+(T))]\, \psi_- \right\} .$$

Squaring this gives

$$\|\psi(1)\|^2 \simeq \cos^2(\kappa\tau - \gamma_+(T)) . \tag{3.13}$$

Malus' law states that for linearly polarized light, its amplitude along a given direction varies as the square of the cosine of the angle that this direction makes with the plane of polarization.[16] Therefore, it follows from (3.13) that upon exiting the optical fibre, the photon's plane of polarization is rotated by $\theta = \kappa\tau - \gamma_+(T)$.

The contribution of the dynamical phase to θ is proportional to the circular birefringence κ. This is the usual optical activity which one would expect from a straight, birefringent optical fibre. However, the contribution from the adiabatic phase is independent of κ. Therefore, if the birefringence of the fibre is small enough, we expect that the dominant contribution to the rotation of polarization would be the adiabatic phase $\gamma_+(T)$. By varying the geometry of the helix, the dependence of the polarization rotation on the solid angle $\Omega(D)$ can be experimentally tested. This was the experiment conducted by Tomita and Chiao [88]. They found that within experimental error, the rotation angle of the polarization vector was indeed equal to $\gamma_+(T)$.

We remark that Ross [73] has performed an experiment similar to that of Tomita and Chiao, and his results agree with theirs. However, Ross and subsequently Haldane

[16] Malus' law is easily proven. See [51, Sect. 24.5].

[41] invoked a classical explanation for the rotation of polarization, which is based on a hypothesized law for parallel transport of polarization vectors. Berry [9] showed that in a certain approximation, this law may be derived from Maxwell's equations. This demonstrates that for large numbers of photons, the results obtained from quantum mechanics survive the classical limit, and may be interpreted within the context of classical electromagnetism [9], [22], [42].

To conclude, we note that Pancharatnam [70] has considered another geometrical procedure for rotating the polarization vector of light. The relationship between Pancharatnam's work and that presented here, is described in [10]. Some of the recent work on adiabatic phase in optics is reviewed in [21].

Chapter III

Quantum Adiabatic Phase and Time-Reversal Invariance

Isolated quantum systems may be described by a time-independent hamiltonian, and therefore they are invariant under time-reversal. Whether solutions of the Schrödinger equation for a time-dependent hamiltonian, or otherwise parameter-dependent hamiltonian, exhibit time-reversal invariance in the sense of Wigner [95], [96, Chapt. 26], depends on the nature of the time dependence, or parameter dependence. For example, the hamiltonian of a charged particle with spin coupled to time-dependent magnetic field (c.f. Subsection I.1.b) is not in general time-reversal invariant. However, in the Jahn-Teller effect the electronic part of the molecular hamiltonian is time-reversal invariant. The electronic hamiltonian $H(Q)$ depends on the configuration of the nuclei, which is specified by the vector parameter Q. The time-reversal invariance of the electronic hamiltonian is expressed through the existence of a time-reversal operator Θ which commutes with $H(Q)$ for all Q. Another time-reversal-invariant example is the Stark hamiltonian, which describes the coupling between atomic electron orbitals and a time-dependent electric field. This example has been examined by Mead [64] and Avron *et al.* [4].

In general, the adiabatic phase is difficult to compute because it depends on the details of the geometry of the similar degeneracy regions and the eigenspace line bundles over them. However, a remarkable simplification occurs for time-reversal invariant systems, where the adiabatic phase depends on the most basic aspect of the topology of the similar degeneracy regions. Specifically, recall that a periodic quantum system whose hamiltonian is represented by a matrix may be described by a loop in a space of hermitian matrices. For time-reversal invariant systems, the adiabatic phase depends on the homotopy class of this loop in the appropriate similar degeneracy region. These homotopy classes are given by the fundamental group of the similar degeneracy region.

In the first section of this chapter, we will characterize time-reversal invariant subspaces in similar degeneracy regions, and explain the simplifications that occur for adiabatic phase in these subspaces. In Section 2, we develop those results on the homotopy and homology of similar degeneracy regions, which are required to compute the adiabatic phase of an arbitrary time-reversal invariant system in terms of its homotopy class. This computation is completed in Section 3.

§1 Potentially Real Hamiltonians

Recall that the space of real, symmetric matrices $Herm(n, \mathbf{R})$ forms a vector subspace of $Herm(n, \mathbf{C})$. Consider any pair of nondegenerate SD-regions $(\mathcal{W}, \mathcal{W}')$, where as before, $\mathcal{W}'$ is the intersection of $\mathcal{W}$ with $Herm(n, \mathbf{R})$. In the first part of this section, it is shown that the restriction of the curvature 2-form $\mathcal{K}$ to $\mathcal{W}'$ is the zero form. Note that if T is a simple loop in $\mathcal{W}'$, then it does not follow from formula (II.3.6) and $\mathcal{K}|_{\mathcal{W}'} = 0$ that

$\gamma(T) = 0$. This is because $\mathcal{W}'$ is not simply connected, unlike $\mathcal{W}$. However, it does follow that $\gamma(T)$ depends only on the homotopy class of T in $\pi_1(\mathcal{W}')$.

The physical significance of $Herm(n, \mathbf{R})$ is that time-dependent quantum systems which are represented by curves in $Herm(n, \mathbf{R})$, are time-reversal invariant. In Subsection (b), Wigner's definition of time-reversal invariance is reviewed. More details may be found in [74, Sect. 29] and [96, Chapt. 26]. Also, we demonstrate the relationship between $Herm(n, \mathbf{R})$ and all other vector subspace in $Herm(n, \mathbf{C})$, which have the property that all curves in them describe time-reversal invariant systems. This relationship will allow us to reduce the problem of computing the adiabatic phase for all time-reversal invariant, periodic, quantum systems to a computation of the adiabatic phase for all loops in real SD-regions.

(a) The Adiabatic Curvature in Potentially Real Subspaces

For a pair of nondegenerate SD-regions $(\mathcal{W}, \mathcal{W}')$, let $j\colon \mathcal{W}' \hookrightarrow \mathcal{W}$ be the inclusion of $\mathcal{W}'$. Then, the eigenspace line bundle over $\mathcal{W}'$ is the pullback $\xi' = j^*\xi$. Of course, this means that ξ' is just the restriction of ξ to $\mathcal{W}'$. The adiabatic curvature 2-form for ξ' is denoted by $\mathcal{K}'$. It is simply the restriction of $\mathcal{K}$ to $\mathcal{W}$.

We now evaluate the curvature 2-form $\mathcal{K}'$. Let $\{U_\alpha\}$ be an open cover of $\mathcal{W}'$ such that $\xi'|_{U_\alpha}$ has a normalized section ϕ_α for each α. Obviously, such open covers exist (e.g., any good cover of $\mathcal{W}'$ has this property). Consider any U_α in this open cover. Because U_α is in $Herm(n, \mathbf{R})$, it follows that the normalized section, or eigenvector ϕ_α may be

chosen to be a real vector. Recall that the curvature 2-form is independent of the choice of local section that is used to compute it. Therefore on U_α,

$$\begin{aligned}\mathcal{K}' &= \frac{1}{2}d\left[(\phi_\alpha, d\phi_\alpha) + \overline{(d\phi_\alpha, \phi_\alpha)}\right] \\ &= \frac{1}{2}d\left[d(\phi_\alpha, \phi_\alpha)\right] \\ &= 0\end{aligned}$$

This implies that although the connection ∇_A on ξ has nontrivial curvature, its restriction to ξ' is flat.

Recall from Subsection II.1.b, that if T is a loop in $\mathcal{W}_2'$, the unique nondegenerate SD-region of $Herm(2, \mathbf{R})$, then $\gamma(T)$ depends only on the homotopy class of T in $\pi_1(\mathcal{W}_2')$. Suppose now that $n \geq 3$. A real, nondegenerate SD-region $\mathcal{W}' \subset Herm(n\mathbf{R})$ is an open, $(n^2+n)/2$-dimensional manifold without boundary. If T_1 and T_2 are two smooth loops in $\mathcal{W}'$, then by at most an arbitrarily small perturbation of T_1 and T_2, we can ensure that T_1 and T_2 are simple loops which do not intersect. If T_1 and T_2 are homotopic, it follows from the Whitney imbedding theorem [84, Sect. II.4], [93] that there is an imbedding Y of a 2-dimensional cylinder into $\mathcal{W}'$ such that T_1 is the boundary at one end of Y, and T_2 is the boundary at the other end. Because a 2-dimensional cylinder has no cohomology in dimension 2, it follows that the first Chern class of the restriction of ξ to Y is trivial. This implies that $\xi|_Y$ is isomorphic to the trivial complex line bundle over Y, and therefore a 1-form $\mathcal{A}$ for the connection ∇_A may be defined globally on Y. Applying Stokes' theorem on Y, and using flatness of ∇_A, we conclude that $\gamma(T_1) = \gamma(T_2)$. This proves that the adiabatic phase is a homotopy invariant in $\mathcal{W}'$. Another proof of this fact uses flatness of the connection ∇_A, combined with the holonomy theorem of Ambrose and Singer [57,

Sect. 8] to prove that the holonomy group is discrete. It then follows that the adiabatic phase is a homotopy invariant [57, p. 93].

As it has been remarked, $\mathcal{W}'$ is not simply connected. This fact will be proven in Section 2, where the fundamental groups for all of the SD-regions are computed. Our argument above suggests that once $\pi_1(\mathcal{W}')$ has been computed for each nondegenerate, real SD-region $\mathcal{W}'$, then it should be possible to compute $\gamma(T)$ in terms of the homotopy class of T in $\pi_1(\mathcal{W}')$. This is indeed the case, and these computations are carried out in Sections 2 and 3.

A natural question to ask is, "Are there any subspaces in $\mathcal{W}$, other than $\mathcal{W}'$, with the property that the adiabatic phase is a homotopy invariant for simple loops in that subspace?" Indeed, there are. Notice, that the conjugate action of a constant matrix $U \in \mathbf{U}(n)$ on $\mathcal{W}$ leaves the curvature 2-form $\mathcal{K}$ invariant. This motivates

Definition 1.1. *For $\mathcal{W}$, any SD-region in $Herm(n, \mathbf{C})$, let $\mathcal{W}' = \mathcal{W} \cap Herm(n, \mathbf{R})$. Then, a subspace $\mathcal{V}$ of $\mathcal{W}$ is said to be potentially real if there exists a constant matrix $U \in \mathbf{U}(n)$ such that $\mathcal{V} = U\,\mathcal{W}'\,U^*$.*

We remark that neither SD-regions, nor potentially real subspaces of SD-regions are vector spaces.

The restriction of $\mathcal{K}$ to any potentially real subspace in $\mathcal{W}$ is the zero form, which implies

Proposition 1.2. *If $\mathcal{V}$ is any potentially real subspace of a nondegenerate SD-region, and T is a simple loop in $\mathcal{V}$, then $\gamma(T)$ is determined by the homotopy class of T in $\pi_1(\mathcal{V})$.* □

(b) Time-Reversal Invariance

Before, continuing with our discussion of adiabatic phase, we shall give a brief introduction to time-reversal invariance in quantum mechanics, as it was first studied by E. P. Wigner in [95]. Consider a time-dependent hamiltonian represented by a path $T\colon [0,1] \to Herm(n, \mathbf{C})$, and let $\mathcal{U}(s)$ be the propogator for the Schrödinger equation

$$i\frac{d}{ds}\mathcal{U}(s) = T(s)\mathcal{U}(s)\ . \tag{1.3}$$

If it exists, the time-reversal operator Θ is defined to be an s-independent symmetry operator with the property that $\Theta\mathcal{U}\Theta^{-1}$ is the propogator for (1.3) with $T(s)$ replaced by $T(1-s)$. As a symmetry operator, Θ must preserve the transition probability between any two state vectors, which implies that Θ must be either a unitary or antiunitary operator. However, it is well-known that Θ cannot possibly be a unitary operator.[1] Therefore, the hamiltonian $T(s)$ is said to be time-reversal invariant if there exists an antiunitary operator Θ which meets the above requirements of a time-reversal operator.

Clearly, the hamiltonian $T(s)$ is time-reversal invariant if and only if there exists a time-reversal operator Θ such that $T(s)$ commutes with Θ for all s. Since any antiunitary

[1] For this result, and as well a general review of time-reversal invariance in quantum mechanics, see [74, Sect. 29] and [96, Chapt. 26]

operator may be written as the composition of a unitary operator and the complex conjugation operator,[2] it follows that a time-reversal operator exists if and only if there exists a constant unitary matrix U such that $U\,\overline{T(s)}\,U^* = T(s)$ for all $s \in [0,1]$. Of course, the corresponding time-reversal operator is then given by U composed with the complex conjugation operator.

Notice that taking U to be the identity provides a time-reversal operator for all paths in $Herm(n,\mathbf{R})$. Indeed, since the defining equation is linear over $\mathbf{R}$, it follows that time-reversal invariance is naturally associated, not with individual paths in $Herm(n,\mathbf{C})$, but rather with real vector subspace of $Herm(n,\mathbf{C})$.

Definition 1.4. *A subspace $\mathcal{S}$ of $Herm(n,\mathbf{C})$ is said to be time-reversal invariant if there exists a fixed $U \in \mathbf{U}(n)$ such that $U\,\overline{A}\,U^* = A$, for all $A \in \mathcal{S}$.*

The maximal time-reversal invariant subspaces in $Herm(n,\mathbf{C})$ are vector subspace obtained by fixing $U \in \mathbf{U}(n)$, and defining

$$J(U) \stackrel{\text{def}}{=} \{A \in Herm(n,\mathbf{C}) \mid U\,\overline{A}\,U^* = A\}\,.$$

For adiabatic phase, we are interested in the intersection of maximal time-reversal-invariant subspaces $J(U)$ with a nondegenerate SD-region $\mathcal{W}$. Furthermore, it is prudent to require that $J(U)$ be an irreducible subspace. By an irreducible set of matrices,

[2] A detailed discussion of antiunitary operators is given in [96].

we mean a set for which there does not exist a constant unitary matrix V such that conjugation by V puts every element of the set in the same block diagonal form.

To see why it is reasonable to require that $J(U)$ be irreducible, suppose that $J(U)$ were instead reducible. Then, $J(U) \cap \mathcal{W}$ would be a reducible subspace of $\mathcal{W}$, and any loop T in $J(U) \cap \mathcal{W}$ could be reduced into block diagonal form. This would imply that the Schrödinger equation for T could be decoupled into irreducible components. The adiabatic phase could then be obtained by examining the adiabatic limit of the appropriate component, and the problem would be reduced to calculating the adiabatic phase in $Herm(m, \mathbf{C})$, for some $m < n$.

Requiring that $J(U)$ be irreducible places the following restriction on U.

Lemma 1.5. *If $J(U)$ is irreducible, then U must be either symmetric or skew-symmetric.*

The proof of this lemma follows from a routine application of Schur's lemma. For a review of Schur's lemma, see [37].

Proof. For any $A \in J(U)$, note that $U\overline{A} = AU$. Taking the complex conjugate of this gives

$$\overline{U}A = \overline{A}\,\overline{U} = U^* A U \overline{U} .$$

Therefore,

$$U\overline{U}A = AU\overline{U} ,$$

for all $A \in J(U)$. Because $J(U)$ is irreducible, it follows from Schur's lemma that $U\,\overline{U} = zI$, where z is a complex number of modulus one. This implies that $U = zU^T$. Furthermore, the unitarity of U requires that $U\,U^* = I$, which in turn implies that $z^2 = 1$. Hence, $z = \pm 1$, and the lemma follows. □

Note that a complex SD-region $\mathcal{W}$ is invariant under the conjugate action of any unitary matrix. Therefore, any potentially real subspace of $\mathcal{W}$ is the intersection of $\mathcal{W}$ and a vector subspace conjugate to $Herm(n, \mathbf{R})$. This suggests that Definition 1.1 should be extended to say that a vector subspace $\mathcal{S}$ in $Herm(n, \mathbf{C})$ is potentially real if there exists a fixed $V \in \mathbf{U}(n)$ such that $\mathcal{S} = V\,Herm(n, \mathbf{R})\,V^*$. In this language, the potentially real subspaces of $\mathcal{W}$ are obtained from the intersection of potentially real vector subspace with $\mathcal{W}$.

Lemma 1.6. *An irreducible $J(U)$ is potentially real if and only if U is symmetric.*

Proof. First, assume that $J(U)$ is a potentially real vector subspace Therefore, $Herm(n, \mathbf{R}) = V^*\,J(U)\,V$ for some $V \in \mathbf{U}(n)$. It follows immediately from the definition of $J(U)$ that $V^*\,J(U)\,V = J(V^*\,U\,\overline{V})$. This implies that $J(V^*\,U\,\overline{V}) = Herm(n, \mathbf{R})$, and hence, $V^*\,U\,\overline{V}$ commutes with every real, symmetric matrix. From Schur's lemma, we conclude that $V^*\,U\,\overline{V}$ is equal to a complex number of modulus one times I. Therefore, U is symmetric.

Conversely, assume that U is a symmetric matrix. Any symmetric, unitary matrix may be written in the form $U = Q\,D\,Q^T$, where Q is an orthogonal matrix,

Q^T is the transpose of Q, and D is a diagonal unitary matrix[3] [96, p. 287]. Let D' be any one of the 2^n diagonal square roots of D. Then,

$$\begin{aligned} J(U) &= J(\,(QD')(QD')^T\,) \\ &= (QD')\,J(I)\,(QD')^* \\ &= (QD')\,Herm(n,\mathbf{R})\,(QD')^*\,, \end{aligned}$$

which completes the proof. □

We remark that Lemma 1.6 implies that if U is symmetric, then an irreducible $J(U)$ has a real structure defined on it. The remaining option that U be antisymmetric is also interesting, because then an irreducible $J(U)$ has a quaternionic structure defined on it. We shall not investigate this situation here; however, it is discussed in [4] and [33].

Obviously, all potentially real subspaces of $\mathcal{W}$ are time-reversal invariant. If $\mathcal{V} = U\mathcal{W}'U^*$, then a direct computation shows that $\mathcal{V} \subset J(UU^T)$. We now prove that if $\mathcal{W}$ is a nondegenerate SD-region, then the converse is also true.

Proposition 1.7. *If $\mathcal{W}$ is a nondegenerate SD-region, then all irreducible, time-reversal invariant subspaces of $\mathcal{W}$ are potentially real.*

Proof. By Lemma 1.5, all irreducible, time-reversal invariant subspaces of $\mathcal{W}$ are of the form $J(U) \cap \mathcal{W}$ for some $U \in \mathbf{U}(n)$, which is either symmetric or skew-symmetric. Consider any matrix $H \in J(U) \cap \mathcal{W}$. Since $\mathcal{W}$ is a nondegenerate

[3] The author thanks R. Westwick for drawing this fact to his attention.

SD-region, the matrix H has a distinguished nondegenerate eigenvalue, which we label μ. We denote the remainder of the eigenvalues of H, not necessarily in any particular order, by $\lambda_1, \ldots, \lambda_{n-1}$.

The spectral theorem implies that there exists a unitary matrix V such that $H = V\,D\,V^*$, where D is the diagonal matrix

$$\begin{bmatrix} \mu & 0 & \ldots & 0 \\ 0 & \lambda_1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_{n-1} \end{bmatrix}.$$

Because $H \in J(U)$, it follows that $U\,\overline{V}\,D\,V^T\,U^* = V\,D\,V^*$, which implies that

$$U = V\,B\,V^T \tag{1.8}$$

for some element B of the isotropy subgroup of $\mathbf{U}(n)$ at D.

Since U is either symmetric or skew-symmetric, it follows from (1.8) that B must also be either symmetric or skew-symmetric, respectively. However, B is of the form

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where $B_1 \in \mathbf{U}(1)$ and $B_2 \in \mathbf{U}(n-1)$. This implies that B is symmetric, and by Lemma 1.6, the subspace $J(U) \cap \mathcal{W}$ is potentially real. □

Note that in the proof of Proposition 1.7, it was essential that the SD-region $\mathcal{W}$ be nondegenerate. There exist degenerate SD-regions which have a nonempty intersection with some irreducible $J(U)$, where U is now taken to be skew-symmetric. For any

hamiltonian contained in such a $J(U)$, it follows from the quaternionic structure on $J(U)$ that the degeneracy of each of its eigenvalues is equal to an even integer [4]. Hamiltonians of this nature arise when considering Fermi systems with time-reversal invariance, and the aforementioned eveness of the degeneracies is called Kramers degeneracy [4], [64]. The appearance of quaternionic structure in Fermi systems with time-reversal invariance was first described by F. Dyson in [33]. The importance of this quaternionic structure to adiabatic holonomy in quadrupole hamiltonians is examined in [4].

The upshot of this section is that a computation of adiabatic phase for loops in each irreducible, time-reversal-invariant subspace of $\mathcal{W}$ may be reduced by a judicious choice of basis to a computation of adiabatic phase for loops in $\mathcal{W}'$. The remaining two sections of this chapter are devoted to computing adiabatic phase for any loop in $\mathcal{W}'$, in terms of its homotopy class in $\pi_1(\mathcal{W}')$.

§2 Homotopy and Homology of Similar Degeneracy Regions

Consider an arbitrary SD-region $\mathcal{W}$, and let $\mathcal{W}'$ be its intersection with $Herm(n, \mathbf{R})$. For the computations in this section, there is no advantage to assuming that the pair $(\mathcal{W}, \mathcal{W}')$ is nondegenerate. This section is devoted to studying the Hurewicz homomorphisms from the homotopy exact sequence to the homology exact sequence for $(\mathcal{W}, \mathcal{W}')$. The groups and homomorphisms of interest appear in the following commutative dia-

gram, which has exact rows.

$$\begin{array}{ccccccccc}
\pi_2(\mathcal{W}') & \longrightarrow & \pi_2(\mathcal{W}) & \longrightarrow & \pi_2(\mathcal{W},\mathcal{W}') & \xrightarrow{\partial_\#} & \pi_1(\mathcal{W}') & \longrightarrow & \pi_1(\mathcal{W}) \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} & & \downarrow \\
H_2(\mathcal{W}') & \longrightarrow & H_2(\mathcal{W}) & \longrightarrow & H_2(\mathcal{W},\mathcal{W}') & \xrightarrow{\partial_*} & H_1(\mathcal{W}') & \longrightarrow & H_1(\mathcal{W})
\end{array} \tag{2.1}$$

Once the groups and homomorphisms are identified, this diagram will provide detailed information about the topology of the pair of SD-regions $(\mathcal{W},\mathcal{W}')$. In particular, if $(\mathcal{W},\mathcal{W}')$ is a pair of nondegenerate SD-regions, then the diagram (2.1) will be used in Section 3 to compute the adiabatic phase $\gamma(T)$ for any loop T in $\mathcal{W}'$.

If $\mathcal{W}$ and $\mathcal{W}'$ are type II SD-regions, then the isomorphisms induced by the inclusions in Corollary 2.10 of Chapter II form an isomorphism of commutative diagrams between (2.1) and

$$\begin{array}{ccccccccc}
\pi_2(F') & \longrightarrow & \pi_2(F) & \longrightarrow & \pi_2(F,F') & \xrightarrow{\partial_\#} & \pi_1(F') & \longrightarrow & \pi_1(F) \\
\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle h'} & & \downarrow \\
H_2(F') & \longrightarrow & H_2(F) & \longrightarrow & H_2(F,F') & \xrightarrow{\partial_*} & H_1(F') & \longrightarrow & H_1(F)
\end{array} \tag{2.2}$$

To simplify our notation, we shall often not write the arguments of F, F', G, and G' explicitly. If $\mathcal{W}$ and $\mathcal{W}'$ are type I SD-regions, then we have an induced isomorphism between the commutative diagram (2.1) and the commutative diagram obtained by replacing F by G and F' by G' in (2.2).

In Section I.2, we showed that F' and G' are connected manifolds and that F and G are simply connected manifolds. From the Hurewicz theorem, it follows that h is an isomorphism, and that h' is an epimorphism. Using (2.2), a diagram chasing argument shows that h'' is also an epimorphism. This means that any relative homology class in

$H_2(F, F')$ can be represented by the continuous image of a 2-cell in F, with the image of its boundary circle in F'.

The same reasoning as above implies that $h''\colon \pi_2(G, G') \to H_2(G, G')$ is also an epimorphism. Therefore, any relative homology class in $H_2(G, G')$ can be represented as the continuous image of a 2-cell in G, with the image of its boundary circle in G'. We remind the reader, that by Corollary 2.10 of Chapter II, this same result holds for the pair of SD-regions $(\mathcal{W}, \mathcal{W}')$.

Using the formula (3.6) from Chapter II, a computation of the adiabatic phase for real hamiltonians, or more generally for hamiltonians with values in a potentially real subspace of hermitian matrices, requires us to integrate the adiabatic curvature 2-form over the relative 2-cycles described above. This motivates a detailed analysis of the commutative diagram (2.2), and the corresponding commutative diagram for Grassmann manifolds.

(a) Type I SD-Regions

If $(\mathcal{W}, \mathcal{W}')$ is a pair of type I SD-regions, then by Corollary 2.10 of Chapter II, the commutative diagram (2.1) is isomorphic to the following commutative diagram in which the vertical maps are Hurewicz homomorphisms and the rows are exact.

$$\begin{array}{ccccccccc}
\pi_2(G') & \xrightarrow{j_\#} & \pi_2(G) & \longrightarrow & \pi_2(G, G') & \xrightarrow{\partial_\#} & \pi_1(G') & \longrightarrow & \pi_1(G) = 0 \\
\downarrow & & \cong\downarrow h & & \downarrow h'' & & \downarrow h' & & \downarrow \\
H_2(G') & \xrightarrow{j_*} & H_2(G) & \longrightarrow & H_2(G, G') & \xrightarrow{\partial_*} & H_1(G') & \longrightarrow & H_1(G) = 0
\end{array} \tag{2.3}$$

Recall that because $G(p,q)$ is simply connected, it follows from the Hurewicz theorem that h is an isomorphism.

We now summarize some results for this commutative diagram which were proven in Section I.2. From (I.2.12) and (I.2.13), both $\pi_2(G)$ and $H_2(G)$ are isomorphic to $\mathbf{Z}$, for all integers $p, q \geq 1$. Furthermore, from (I.2.16) and (I.2.17), we have that h' is an isomorphism and

$$\pi_1(G'(p,q)) \cong H_1(G'(p,q)) \cong \begin{cases} \mathbf{Z}, & \text{if } p = q = 1 \\ \mathbf{Z}_2, & \text{if } p + q \geq 3 \end{cases}$$

Hence, we refer to the type I case as generic when $p + q \geq 3$. The special case that occurs when $p = q = 1$ is straightforward, and was discussed in Subsection II.1.b. We leave it to the reader to translate that discussion into the language of this section. For the remainder of this subsection, we assume that $p + q \geq 3$.

From (2.26) and (2.28) in Proposition 2.25 of Chapter I, the induced homomorphisms $j_\#$ and j_* in (2.3) are zero homomorphisms. Applying a generalization of the five lemma,[4] we conclude that the relative Hurewicz homomorphism h'' is an isomorphism.

Since $\pi_2(G, G') \cong H_2(G, G')$, it follows immediately that $\pi_2(G, G')$ is an abelian group, and furthermore must be isomorphic to either $\mathbf{Z} \oplus \mathbf{Z}_2$ or $\mathbf{Z}$. We shall now prove that it is the latter which is true. This will allow us to establish that with an appropriate choice of generators, the homomorphism $k_\#$ from $\pi_2(G)$ to $\pi_2(G, G')$ is multiplication

[4] As usually stated, the five lemma applies to commutative diagrams of abelian groups, and a priori $\pi_2(G, G')$ need not be abelian. However, with a little care it is easy to see that the standard diagram-chasing proof of the five lemma also applies to commutative diagrams which may possibly contain nonabelian groups. When quoting the five lemma in this chapter, it shall be this generalization which we have in mind.

by 2. As we shall see in the next section, this fact is crucial to the existence of a nontrivial adiabatic phase for real SD-regions.

We now show that in order to compute $H_2(G, G')$, and hence $\pi_2(G, G')$, it suffices to compute $H_2(\mathbf{C}P(2), \mathbf{R}P(2))$. For integers $1 \leq a \leq p$ and $1 \leq b \leq q$, define the pair inclusion

$$\alpha'' \colon (G(a,b), G'(a,b)) \hookrightarrow (G(p,q), G'(p,q)) \tag{2.4}$$

to be given by the inclusions α and α' defined in (I.2.9) and (I.2.10), respectively. The homomorphisms α_*, α'_*, and α''_*, which are induced on homology by the respective inclusions, give a homomorphism from the homology exact sequence for the pair $(G(a,b), G'(a,b)$ to the homology exact sequence for the pair $(G(p,q), G'(p,q))$. When $a = 1$ and $b = 2$, this defines the commutative diagram,

$$\begin{tikzcd}
0 \arrow[d, equal] & \mathbf{Z} \arrow[d, equal] & & & \mathbf{Z}_2 \arrow[d, equal] & 0 \arrow[d, equal] \\
H_2(\mathbf{R}P(2)) \arrow[r, "0"] \arrow[d] & H_2(\mathbf{C}P(2)) \arrow[r] \arrow[d, "\alpha_*"] & H_2(\mathbf{C}P(2), \mathbf{R}P(2)) \arrow[r, "\partial_*"] \arrow[d, "\alpha''_*"] & H_1(\mathbf{R}P(2)) \arrow[r] \arrow[d, "\alpha'_*"] & H_1(\mathbf{C}P(2)) \arrow[d] \\
H_2(G') \arrow[r, "0"] & H_2(G) \arrow[r] \arrow[d, equal] & H_2(G, G') \arrow[r, "\partial_*"] & H_1(G') \arrow[r] \arrow[d, equal] & H_1(G) \arrow[d, equal] \\
 & \mathbf{Z} & & \mathbf{Z}_2 & 0
\end{tikzcd}$$

Because the induced homomorphisms $\alpha_\#$ and $\alpha'_\#$ on the corresponding homotopy groups in (I.2.11) and (I.2.15) are isomorphisms, it follows from the Hurewicz theorem that α_* and α'_* are isomorphisms. Hence, by the five lemma, we conclude that α''_* is an isomorphism and $H_2(\mathbf{C}P(2), \mathbf{R}P(2)) \cong H_2(G, G')$.

To settle the issue of whether $H_2(\mathbf{C}P(2), \mathbf{R}P(2))$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_2$ or $\mathbf{Z}$, we compute the relative cohomology with coefficient group $\mathbf{Z}_2$ of the pair $(\mathbf{C}P(2), \mathbf{R}P(2))$, and then apply the universal coefficient theorem for cohomology.

Lemma 2.5. *Let $j\colon \mathbf{R}P(m) \hookrightarrow \mathbf{C}P(m)$ be the inclusion of the real points defined in* (I.2.8). *Then, the induced homomorphism*

$$j^*\colon H^2(\mathbf{C}P(m); \mathbf{Z}_2) \longrightarrow H^2(\mathbf{R}P(m); \mathbf{Z}_2)$$

is an isomorphism for all integers $m \geq 2$.

The result that $H_2(G, G') \cong \mathbf{Z}$ follows from

Corollary 2.6. *If $m \geq 2$, then $H_2(\mathbf{C}P(m), \mathbf{R}P(m)) \cong \mathbf{Z}$.*

Proof of Lemma 2.5. First, note that $H^2(\mathbf{C}P(m); \mathbf{Z}_2) \cong \mathbf{Z}_2$ for all $m \geq 1$, and that $H^2(\mathbf{R}P(m); \mathbf{Z}_2) \cong \mathbf{Z}_2$ for all $m \geq 2$. These results are well-known, and may be proven using either a CW-decomposition of $\mathbf{C}P(m)$ and $\mathbf{R}P(m)$ [29, p. 102], or by a Thom-Gysin exact sequence [83, p. 264]. To show that j^* is an isomorphism, it now suffices to prove that j^* is not the zero homomorphism.

Let γ_m be the canonical complex line bundle over $\mathbf{C}P(m)$, and $\hat{\gamma}_m$ the real, oriented 2-plane bundle obtained by restricting γ_m (i.e. pulling back over the inclusion j), and then ignoring the complex structure. The basic fact that $\mathbf{C}$ is isomorphic to $\mathbf{R} \oplus \mathbf{R}$, along with Lemma 3.2 in [66], implies that $\hat{\gamma}_m$ is isomorphic

to the Whitney sum $\gamma'_m \oplus \gamma'_m$, where γ'_m is the canonical real line bundle over $\mathbf{R}P(m)$.

By naturality of the Stieffel-Whitney classes, it follows that $j^*(w_2(\gamma_m)) = w_2(\gamma'_m \oplus \gamma'_m)$. The Whitney product theorem implies that $w_2(\gamma'_m \oplus \gamma'_m) = (w_1(\gamma'_m))^2$. For $m \geq 1$, the first Stieffel-Whitney class, $w_1(\gamma'_m)$ represents the unique nonzero element in $H^1(\mathbf{R}P(m); \mathbf{Z}_2)$. Furthermore, $H^*(\mathbf{R}P(m); \mathbf{Z}_2)$ is an algebra with unit over $\mathbf{Z}_2$, having $w_1(\gamma'_m)$ as its only generator, and $(w_1(\gamma'_m))^{m+1} = 0$ as its only relation [66, p. 42]. Therefore, if $m \geq 2$, we have that $w_2(\gamma'_m \oplus \gamma'_m) \neq 0$, and hence j^* is an isomorphism. □

Proof of Corollary 2.6. Recall that for $m \geq 2$, we have already demonstrated that $H_2(\mathbf{C}P(m), \mathbf{R}P(m))$ is isomorphic to either $\mathbf{Z} \oplus \mathbf{Z}_2$ or $\mathbf{Z}$. Since $H_1(\mathbf{C}P(m)) = 0$, it follows immediately from the reduced homology exact sequence for the pair $(\mathbf{C}P(m), \mathbf{R}P(m))$ that $H_1(\mathbf{C}P(m), \mathbf{R}P(m)) = 0$. Therefore, the universal coefficient theorem implies that $H^2(\mathbf{C}P(m), \mathbf{R}P(m); \mathbf{Z}_2)$ is isomorphic to either $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ or $\mathbf{Z}_2$.

Now, consider the following cohomology exact sequence for the pair $(\mathbf{C}P(m), \mathbf{R}P(m))$ with coefficient group $\mathbf{Z}_2$ (coefficients are suppressed in the notation of this diagram),

$$\begin{array}{ccccccc} H^1(\mathbf{R}P(m)) & \xrightarrow{\delta^*} & H^2(\mathbf{C}P(m), \mathbf{R}P(m)) & \xrightarrow{k^*} & H^2(\mathbf{C}P(m)) & \xrightarrow{j^*} & H^2(\mathbf{R}P(m)) \\ \| & & & & \| & & \| \\ \mathbf{Z}_2 & & & & \mathbf{Z}_2 & & \mathbf{Z}_2 \end{array}$$

From Lemma 2.5, we know that j^* is an isomorphism. It follows that k^* is the zero homomorphism, and δ^* is an epimorphism from $\mathbf{Z}_2$ to $H^2(\mathbf{C}P(m), \mathbf{R}P(m); \mathbf{Z}_2)$. This eliminates the possibility that $H^2(\mathbf{C}P(m), \mathbf{R}P(m); \mathbf{Z}_2)$ is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. $\square$

We make use of a commutative diagram to summarize our results for generic, type I SD-regions. Denoting a pair of generic, type I SD-regions by $(\mathcal{W}_g, \mathcal{W}'_g)$, we have established

$$\begin{tikzcd}
 & \mathbf{Z} \arrow[d, equal] & \mathbf{Z} \arrow[d, equal] & \mathbf{Z}_2 \arrow[d, equal] & 0 \arrow[d, equal] \\
\pi_2(\mathcal{W}'_g) \arrow[r, "0"] \arrow[d] & \pi_2(\mathcal{W}_g) \arrow[r, "\times 2"] \arrow[d, "h", "\cong"'] & \pi_2(\mathcal{W}_g, \mathcal{W}'_g) \arrow[r, "\partial_\#"] \arrow[d, "h''", "\cong"'] & \pi_1(\mathcal{W}'_g) \arrow[r] \arrow[d, "h'", "\cong"'] & \pi_1(\mathcal{W}_g) \arrow[d] \\
H_2(\mathcal{W}'_g) \arrow[r, "0"] & H_2(\mathcal{W}_g) \arrow[r, "\times 2"] \arrow[d, equal] & H_2(\mathcal{W}_g, \mathcal{W}'_g) \arrow[r, "\partial_*"] \arrow[d, equal] & H_1(\mathcal{W}'_g) \arrow[r] \arrow[d, equal] & H_1(\mathcal{W}_g) \arrow[d, equal] \\
 & \mathbf{Z} & \mathbf{Z} & \mathbf{Z}_2 & 0
\end{tikzcd} \tag{2.7}$$

The rows in this commutative diagram are exact.

To conclude, we compare (2.7) with the corresponding commutative diagram for the nongeneric case of 2×2 hamiltonians. The SD-regions $\mathcal{W}(1,1)$ and $\mathcal{W}(2,2)$ correspond to the same region in $Herm(2, \mathbf{C})$, which we denote by $\mathcal{W}_2$. The intersection of $\mathcal{W}_2$ with $Herm(2, \mathbf{R})$ is denoted by $\mathcal{W}'_2$. Recall that $\pi_1(\mathcal{W}'_2) \cong H_1(\mathcal{W}'_2) \cong \mathbf{Z}$, which is not like the generic situation. However, as we showed in Subsection II.1.b, the computations for

2×2 hamiltonians can be done directly, and we obtain the following results.

$$\begin{tikzcd}
0 \arrow[d, equal] & \mathbf{Z} \arrow[d, equal] & \mathbf{Z} \oplus \mathbf{Z} \arrow[d, equal] & \mathbf{Z} \arrow[d, equal] & 0 \arrow[d, equal] \\
\pi_2(\mathcal{W}_2') \arrow[r] \arrow[d] & \pi_2(\mathcal{W}_2) \arrow[r] \arrow[d, "h", "\cong"'] & \pi_2(\mathcal{W}_2, \mathcal{W}_2') \arrow[r, "\partial_\#"] \arrow[d, "h''", "\cong"'] & \pi_1(\mathcal{W}_2') \arrow[r] \arrow[d, "h'", "\cong"'] & \pi_1(\mathcal{W}_2) \arrow[d] \\
H_2(\mathcal{W}_2') \arrow[r] \arrow[d, equal] & H_2(\mathcal{W}_2) \arrow[r] \arrow[d, equal] & H_2(\mathcal{W}_2, \mathcal{W}_2') \arrow[r, "\partial_*"] \arrow[d, equal] & H_1(\mathcal{W}_2') \arrow[r] \arrow[d, equal] & H_1(\mathcal{W}_2) \arrow[d, equal] \\
0 & \mathbf{Z} & \mathbf{Z} \oplus \mathbf{Z} & \mathbf{Z} & 0
\end{tikzcd} \tag{2.8}$$

(b) Type II SD-Regions

Now we turn our attention to the case when $\mathcal{W}$ and $\mathcal{W}'$ are type II SD-regions. Recall that by Corollary II.2.10, the commutative diagram (2.1) is now isomorphic to the commutative diagram (2.2). For convenience, we write this diagram again.

$$\begin{tikzcd}
\pi_2(F') \arrow[r, "j_\#"] \arrow[d] & \pi_2(F) \arrow[r] \arrow[d, "h", "\cong"'] & \pi_2(F, F') \arrow[r, "\partial_\#"] \arrow[d, "h''"] & \pi_1(F') \arrow[r] \arrow[d, "h'"] & \pi_1(F) = 0 \arrow[d] \\
H_2(F') \arrow[r, "j_*"] & H_2(F) \arrow[r] & H_2(F, F') \arrow[r, "\partial_*"] & H_1(F') \arrow[r] & H_1(F) = 0
\end{tikzcd} \tag{2.9}$$

Recall that since $F(p, q, r)$ is simply connected, the Hurewicz theorem implies that h is an isomorphism for all integers $p, q, r \geq 1$.

Using various results from Section I.2, we learn the following about the commutative diagram (2.9). From (2.27) and (2.29) in Proposition I.2.25, both $j_\#$ and j_* are the zero homomorphism. Furthermore, from (I.2.19), we have that $\pi_2(F(p, q, r)) \cong \mathbf{Z} \oplus \mathbf{Z}$ for all integers $p, q, r \geq 1$. From (I.2.22) and (I.2.23), it follows that for $p + q + r \geq 4$, the Hurewicz homomorphism h' is an isomorphism, and $\pi_1(F'(p, q, r)) \cong H_1(F'(p, q, r)) \cong$

$\mathbf{Z}_2 \oplus \mathbf{Z}_2$. By the generalized five lemma, this implies that the Hurewicz homomorphism h'' is also an isomorphism for $p+q+r \geq 4$.

For $p = q = r = 1$, we have from (I.2.22) that $\pi_1(F'(1,1,1)) \cong Q_8$, where Q_8 is the quaternion group. Recall that Q_8 is a nonabelian group consisting of 8 elements. In this case, $h' \colon \pi_1(F'(1,1,1)) \to H_1(F'(1,1,1))$ is an epimorphism with kernel the commutator subgroup of Q_8. Note that the commutator subgroup of Q_8 is $\mathbf{Z}_2$. It follows by diagram chasing that $\pi_2(F(1,1,1), F'(1,1,1))$ is also nonabelian, and

$$h'' \colon \pi_2(F(1,1,1), F'(1,1,1)) \longrightarrow H_2(F(1,1,1), F'(1,1,1))$$

is an epimorphism. Without much difficulty, it can be shown directly that the kernel of h'' is the commutator subgroup of $\pi_2(F(1,1,1), F'(1,1,1))$. Thus, we are led to consider the type II case as generic when $p+q+r \geq 4$. As in our discussion of the type I case, we examine the generic situation first and then return to the special case $p = q = r = 1$.

Since $p+q+r \geq 4$ for the generic type II case, it follows that at least one of the integers p, q, or r is greater than or equal to 2. We shall assume that $q \geq 2$. However, if we had assumed that $p \geq 2$, or that $r \geq 2$, then there is a discussion which is almost identical to that which we formulate below. Indeed, we remind the reader that permuting the arguments of $F(p,q,r)$ and $F'(p,q,r)$ produce flag manifolds which are diffeomorphic to $F(p,q,r)$ and $F'(p,q,r)$, respectively.

As the reader might suspect, the type II case may in some sense be thought of as the direct sum of two type I cases. To make this precise, we shall make use of two fibre

bundles, both of which have a short flag manifold fibred over a Grassmann manifold, with fibre another Grassmann manifold.

First, using the homotopy exact sequence for the fibre bundles in (I.2.6), we obtain the following diagram. To simplify notation, we denote $F(p,q,r)$ and $F'(p,q,r)$ by F and F', respectively.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi_2(G(p,q)) \arrow[r, "\times 2"] \arrow[d, "i_{1\#}"] & \pi_2(G(p,q),G'(p,q)) \arrow[r, "\partial_\#"] \arrow[d, "i''_{1\#}"] & \pi_1(G'(p,q)) \arrow[r] \arrow[d, "i'_{1\#}"] & 0 \\
0 \arrow[r] & \pi_2(F) \arrow[r] \arrow[d, "p_{1\#}"] & \pi_2(F,F') \arrow[r] \arrow[d, "p''_{1\#}"] & \pi_1(F') \arrow[r] \arrow[d, "p'_{1\#}"] & 0 \\
0 \arrow[r] & \pi_2(G(p+q,r)) \arrow[r, "\times 2"] \arrow[d] & \pi_2(G(p+q,r),G'(p+q,r)) \arrow[r, "\partial_\#"] \arrow[d] & \pi_1(G'(p+q,r)) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.10}$$

The homomorphisms in this diagram are defined as follows. Each of the three rows are obtained from the homotopy exact sequences for the corresponding pair of spaces. The left and right columns are obtained from the homotopy exact sequences for the fibre bundles in (I.2.6). The homomorphism $i''_{1\#}$ is defined to be induced by the inclusion of the pair $i''_1\colon (G(p,q),G'(p,q)) \hookrightarrow (F,F'))$, and $p''_{1\#}$ is defined to be the homomorphism induced by the projection of the pair $p''_{1\#}\colon (F,F') \to (G(p+q,r),G'(p+q,r))$. With these definitions, (2.10) is obviously a commutative diagram.

Exactness of the rows follows from Proposition 2.25 of Chapter I, and the fact that complex flag manifolds are simply connected. From the homotopy exact sequences for

the fibre bundles in (I.2.6), it follows that $p_{1\#}$ and $p'_{1\#}$ are epimorphisms. Therefore, to show exactness of the left and right columns, it suffices to show that $i_{1\#}$ and $i'_{1\#}$ are monomorphisms.

From (I.2.12), we have that $\pi_2(G(p,q)) \cong \pi_2(G(p+q,r)) \cong \mathbf{Z}$, and from (I.2.19) that $\pi_2(F) \cong \mathbf{Z} \oplus \mathbf{Z}$, for all integers $p, q, r \geq 1$. This implies that $i_{1\#}$ is monic, and hence the left column is a short exact sequence.

For all integers p, q, r such that $p + q \geq 3$, and $r \geq 1$, we have from (I.2.16) and (I.2.22) that $\pi_1(G'(p,q)) \cong \pi_1(G'(p+q,r)) \cong \mathbf{Z}_2$, and that $\pi_1(F') \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. This implies that $i'_{1\#}$ is monic, and hence the right column is also a short exact sequence. We remark that here it is important that either p or q is greater than or equal to 2.

In the previous subsection, it was computed that $\pi_2(G(p,q), G'(p,q)) \cong \pi_2(G(p+q,r), G'(p+q,r)) \cong \mathbf{Z}$. Therefore, commutativity of the diagram (2.10) implies that the composition $p''_{1\#}\, i''_{1\#}$ is the zero homomorphism. From this it follows by a corollary of the 3×3 lemma [61, Exercise I.5.2] that the middle column is a short exact sequence. Indeed, since $\pi_2(G(p+q,r), G'(p+q,r))$ is free abelian, this short exact sequence must be split, and therefore $\pi_2(F(p,q,r), F'(p,q,r))$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ for all integers $p, q, r \geq 1$, satisfying $p + q + r \geq 4$.

At this point, we have computed all of the groups in the diagram (2.9). However, it will be useful to explore the relationship between the type I case and the type II case more fully. By examining how the assumption that $q \geq 2$ was used above, we realize that there is another commutative diagram of the form of (2.10) which we may construct.

The following diagram is constructed by making use of the fibre bundles in (I.2.7).

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \pi_2(G(q,r)) \arrow[r, "\times 2"] \arrow[d, "i_{2\#}"] & \pi_2(G(q,r), G'(q,r)) \arrow[r, "\partial_\#"] \arrow[d, "i''_{2\#}"] & \pi_1(G'(q,r)) \arrow[r] \arrow[d, "i'_{2\#}"] & 0 \\
0 \arrow[r] & \pi_2(F) \arrow[r] \arrow[d, "p_{2\#}"] & \pi_2(F, F') \arrow[r] \arrow[d, "p''_{2\#}"] & \pi_1(F') \arrow[r] \arrow[d, "p'_{2\#}"] & 0 \\
0 \arrow[r] & \pi_2(G(p,q+r)) \arrow[r, "\times 2"] \arrow[d] & \pi_2(G(p,q+r), G'(p,q+r)) \arrow[r, "\partial_\#"] \arrow[d] & \pi_1(G'(p,q+r)) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.11}$$

The homomorphisms in this diagram are defined in an analogous fashion to those in (2.10). Notice that the diagrams (2.10) and (2.11) share the same middle row. This suggests that we should think of these two diagrams as being in perpendicular planes, meeting in their middle row, and thus comprising a 3-dimensional commutative diagram.

Consider the left cross-section of this three-dimensional diagram. The fibre inclusion $i_2\colon G(q,r) \hookrightarrow F(p,q,r)$ is defined in terms of the appropriate fibre bundle in (I.2.7). By direct inspection, it is easy to see that the composition $p_1 \circ i_2\colon G(q,r) \to G(p+q,r)$ is the same map as the inclusion $\alpha\colon G(q,r) \hookrightarrow G(p+q,r)$ defined in (I.2.9).

From the commutative diagram (I.2.11), it follows that for all integers $p, q, r \geq 1$, the induced composition

$$\pi_2(G(q,r)) \xrightarrow{i_{2\#}} \pi_2(F(p,q,r)) \xrightarrow{p_{1\#}} \pi_2(G(p+q,r)) \tag{2.12}$$

is an isomorphism. Similarly, the composition

$$\pi_2(G(p,q)) \xrightarrow{i_{1\#}} \pi_2(F(p,q,r)) \xrightarrow{p_{2\#}} \pi_2(G(p,q+r)) \tag{2.13}$$

is also an isomorphism. This implies that the left cross-section of the 3-dimensional commutative diagram formed by (2.10) and (2.11) provides a direct sum decomposition[5] of $\pi_2(F(p,q,r))$ This decomposition is summarized by the commutative diagram

$$\begin{array}{ccccc} \pi_2(G(p,q)) & \xrightarrow{i_{1\#}} & \pi_2(F) & \xleftarrow{i_{2\#}} & \pi_2(G(q,r)) \\ \cong\Big\downarrow\alpha_\# & & \Big\| & & \cong\Big\downarrow\alpha_\# \\ \pi_2(G(p,q+r)) & \xleftarrow{p_{2\#}} & \pi_2(F) & \xrightarrow{p_{1\#}} & \pi_2(G(p+q,r)) \end{array} \tag{2.14}$$

which is a direct sum diagram for all integers $p, q, r \geq 1$.

We apply a similar argument to the right cross-section of the 3-dimensional diagram formed by (2.10) and (2.11), and conclude that the induced compositions

$$\pi_1(G'(q,r)) \xrightarrow{i'_{2\#}} \pi_1(F'(p,q,r)) \xrightarrow{p'_{1\#}} \pi_1(G'(p+q,r)) \tag{2.15}$$

and

$$\pi_1(G'(p,q)) \xrightarrow{i'_{1\#}} \pi_1(F'(p,q,r)) \xrightarrow{p'_{2\#}} \pi_1(G'(p,q+r)) \tag{2.16}$$

are isomorphisms as long as p, q, and r are integers satisfying $p, r \geq 1$, and $q \geq 2$.

[5] Direct sum decompositions are reviewed in [61, Sect. I.4]

Therefore, we have the direct sum diagram

$$\begin{array}{ccccc} \pi_1(G'(p,q)) & \xrightarrow{i_{1\#}} & \pi_1(F')) & \xleftarrow{i_{2\#}} & \pi_1(G'(q,r)) \\ \cong\Big\downarrow \alpha'_{\#} & & \Big\| & & \cong\Big\downarrow \alpha'_{\#} \\ \pi_1(G'(p,q+r)) & \xleftarrow{p'_{2\#}} & \pi_1(F') & \xrightarrow{p'_{1\#}} & \pi_1(G'(p+q,r)) \end{array} \tag{2.17}$$

for all integers $p, r \geq 1$, and $q \geq 2$. The isomorphisms denoted by $\alpha'_{\#}$ are induced by the inclusion α', which is defined in (I.2.10). We remark that if $p = q = r = 1$, then (2.17) is not a direct sum diagram.

We now show that the middle cross-section in the 3-dimensional diagram formed by (2.10) and (2.11) is also a direct sum decomposition for all integers $p, r \geq 1$ and $q \geq 2$. Because the compositions (2.12), (2.13), (2.15), and (2.16) are isomorphisms, the five lemma implies that the induced compositions

$$\pi_2(G(q,r), G'(q,r)) \xrightarrow{i''_{2\#}} \pi_2(F(p,q,r), F'(p,q,r)) \xrightarrow{p''_{1\#}} \pi_2(G(p+q,r), G'(p+q,r))$$

and

$$\pi_2(G(p,q), G'(p,q)) \xrightarrow{i''_{1\#}} \pi_2(F(p,q,r), F'(p,q,r)) \xrightarrow{p''_{2\#}} \pi_2(G(p,q+r), G'(p,q+r))$$

are also isomorphisms. Indeed, they are isomorphisms induced on homotopy by the relative inclusion α'' defined in (2.4). Therefore, for all integers $p, r \geq 1$ and $q \geq 2$, the commutative diagram

$$\begin{array}{ccccc} \pi_2(G(p,q), G'(p,q)) & \xrightarrow{i''_{1\#}} & \pi_2(F, F') & \xleftarrow{i''_{2\#}} & \pi_2(G(q,r), G'(q,r)) \\ \cong\Big\downarrow \alpha''_{\#} & & \Big\| & & \cong\Big\downarrow \alpha''_{\#} \\ \pi_2(G(p,q+r), G'(p,q+r)) & \xleftarrow{p''_{2\#}} & \pi_2(F, F') & \xrightarrow{p''_{1\#}} & \pi_2(G(p+q,r), G'(p+q,r)) \end{array} \tag{2.18}$$

is a direct sum diagram.

The direct sum diagrams (2.14), (2.17), and (2.18) imply that for all integers $p, r \geq 1$ and $q \geq 2$, the commutative diagrams (2.10) and (2.11) provide direct sum splittings of each other.

Proposition 2.19. *Let F and F' denote $F(p,q,r)$ and $F'(p,q,r)$, respectively. Then, for all integers $p, q, r \geq 1$, the commutative diagram*

$$\begin{tikzcd} H_2(G(p,q)) \arrow[r, "i_{1*}"] \arrow[d, "\alpha_*", "\cong"'] & H_2(F) \arrow[d, equal] & H_2(G(q,r)) \arrow[l, "i_{2*}"'] \arrow[d, "\alpha_*", "\cong"'] \\ H_2(G(p,q+r)) & H_2(F) \arrow[l, "p_{2*}"'] \arrow[r, "p_{1*}"] & H_2(G(p+q,r)) \end{tikzcd} \tag{2.20}$$

is a direct sum diagram.

For all integers $p, r \geq 1$ and $q \geq 2$, the following two commutative diagrams are direct sum diagrams:

$$\begin{tikzcd} H_1(G'(p,q)) \arrow[r, "i'_{1*}"] \arrow[d, "\alpha'_*", "\cong"'] & H_1(F')) \arrow[d, equal] & H_1(G'(q,r)) \arrow[l, "i'_{2*}"'] \arrow[d, "\alpha'_*", "\cong"'] \\ H_1(G'(p,q+r)) & H_1(F') \arrow[l, "p'_{2*}"'] \arrow[r, "p'_{1*}"] & H_1(G'(p+q,r)) \end{tikzcd} \tag{2.21}$$

and

$$\begin{tikzcd} H_2(G(p,q),G'(p,q)) \arrow[r, "i''_{1*}"] \arrow[d, "\alpha''_*", "\cong"'] & H_2(F,F') \arrow[d, equal] & H_2(G(q,r),G'(q,r)) \arrow[l, "i''_{2*}"'] \arrow[d, "\alpha''_*", "\cong"'] \\ H_2(G(p,q+r),G'(p,q+r)) & H_2(F,F') \arrow[l, "p''_{2*}"'] \arrow[r, "p''_{1*}"] & H_2(G(p+q,r),G'(p+q,r)) \end{tikzcd} \tag{2.22}$$

Proof. The Hurewicz theorem implies that (2.20) is isomorphic to the direct sum diagram (2.14). Similarly, (2.21) is isomorphic to the direct sum diagram (2.17).

It follows from the commutative diagrams (2.3) and (2.9) that (2.22) is isomorphic to the direct sum diagram (2.18). □

If $p = q = r = 1$, the middle and right columns of the 3×3 commutative diagrams (2.10) and (2.11) are no longer short exact sequences. Indeed, because the homotopy groups $\pi_1(F'(1,1,1))$ and $\pi_2(F(1,1,1), F'(1,1,1))$ are nonabelian, they cannot possibly be a direct sum of abelian groups. However for homology, we have the following direct sum decompositions.

Proposition 2.23. *The diagrams*

$$H_1(G'(1,2)) \xleftarrow{p'_{2*}} H_1(F'(1,1,1)) \xrightarrow{p'_{1*}} H_1(G'(2,1)) \tag{2.24}$$

and

$$H_2(G(1,2), G'(1,2)) \xleftarrow{p''_{2*}} H_2(F(1,1,1), F'(1,1,1)) \xrightarrow{p''_{1*}} H_2(G(2,1), G'(2,1)) \tag{2.25}$$

are both projective direct sum diagrams.

Proof. We shall prove the proposition for diagram (2.25). A similar argument obtains the result for (2.24).

Notice that the pairs of PSD-regions $(W(3,3), W'(3,3))$ and $(W(1,1), W'(1,1))$ are homeomorphic to the pairs of open-ended mapping cylinders for the relative projections

$$p_1 \colon (F(1,1,1), F'(1,1,1)) \longrightarrow (G(2,1), G'(2,1))$$

and

$$p_2\colon (F(1,1,1), F'(1,1,1)) \longrightarrow (G(1,2), G'(1,2)) ,$$

respectively. Furthermore, the union $(W(1,1), W'(1,1)) \cup (W(3,3), W'(3,3))$ is homeomorphic to (S^7, S^4). It follows from Lemma 2.4 in Chapter II, that $W(1,1) \cap W(3,3)$ is homeomorphic to the product space $F(1,1,1) \times (0,1)$, where $(0,1)$ denotes the open, unit interval in $\mathbf{R}$. Also, Lemma II.2.6 implies that $W'(1,1) \cap W'(3,3)$ is homeomorphic to the product space $F'(1,1,1) \times (0,1)$. Choosing a pair of fibres, we define the relative inclusions

$$i^1\colon (F(1,1,1), F'(1,1,1)) \hookrightarrow (W(1,1), W'(1,1))$$

and

$$i^2\colon (F(1,1,1), F'(1,1,1)) \hookrightarrow (W(3,3), W'(3,3)) .$$

Because $(W(1,1), W'(1,1)) \cap (W(3,3), W'(3,3))$ is homotopy equivalent to $(F(1,1,1), F'(1,1,1))$, the relative Mayer-Vietoris exact sequence for homology[6] gives

$$\begin{aligned} H_3(S^7, S^4) \longrightarrow H_2(F(1,1,1), F'(1,1,1)) \xrightarrow{\varphi} \\ H_2(W(1,1), W'(1,1)) \oplus H_2(W(3,3), W'(3,3)) \longrightarrow H_2(S^7, S^4) \end{aligned} \tag{2.26}$$

The homomorphism φ is defined by $\varphi\colon z \mapsto (i^1_* z, i^2_* z)$ where i^1_* and i^2_* are the homomorphisms induced by i^1 and i^2, respectively, on the 2-dimensional relative homology.

[6] The relative Mayer-Vietoris exact sequence for homology is reviewed in [83, Sect. 4.6].

From the homology exact sequence for the pair (S^7, S^4), it follows immediately that $H_3(S^7, S^4) = H_2(S^7, S^4) = 0$. Hence, φ is an isomorphism, and from the exact sequence (2.26) we obtain the projective direct sum diagram

$$H_2(W(1,1), W'(1,1)) \xleftarrow{i_*^1} H_2(F(1,1,1), F'(1,1,1)) \xrightarrow{i_*^2} H_2(W(3,3), W'(3,3)) \tag{2.27}$$

Recall from Lemma II.2.4, Lemma II.2.6, and Theorem II.2.9 that the pair $(G(1,2), G'(1,2))$ is a deformation retract of the pair $(W(1,1), W'(1,1))$, and that $(G(2,1), G'(2,1))$ is a deformation retract of $(W(3,3), W'(3,3))$. This implies that the direct sum diagram (2.27) is isomorphic to the diagram (2.25). $\square$

We conclude this section by using two commutative diagrams to summarize the situation for type II SD-regions. For $n \geq 4$, the complex, type II SD-regions in $Herm(n, \mathbf{C})$, and the real, type II SD-regions in $Herm(n, \mathbf{R})$ are all generic. We have shown that a pair of generic, type II SD-regions $(\mathcal{W}_g, \mathcal{W}'_g)$ satisfy the following commutative diagram, which has exact rows.

$$\begin{tikzcd}
 & \mathbf{Z} \oplus \mathbf{Z} \arrow[d, equal] & \mathbf{Z} \oplus \mathbf{Z} \arrow[d, equal] & \mathbf{Z}_2 \oplus \mathbf{Z}_2 \arrow[d, equal] & 0 \arrow[d, equal] \\
\pi_2(\mathcal{W}'_g) \arrow[r, "0"] \arrow[d] & \pi_2(\mathcal{W}_g) \arrow[r, "\times 2"] \arrow[d, "h", "\cong"'] & \pi_2(\mathcal{W}_g, \mathcal{W}'_g) \arrow[r, "\partial_\#"] \arrow[d, "h''", "\cong"'] & \pi_1(\mathcal{W}'_g) \arrow[r] \arrow[d, "h'", "\cong"'] & \pi_1(\mathcal{W}_g) \arrow[d] \\
H_2(\mathcal{W}'_g) \arrow[r, "0"] & H_2(\mathcal{W}_g) \arrow[r, "\times 2"] \arrow[d, equal] & H_2(\mathcal{W}_g, \mathcal{W}'_g) \arrow[r, "\partial_*"] \arrow[d, equal] & H_1(\mathcal{W}'_g) \arrow[r] \arrow[d, equal] & H_1(\mathcal{W}_g) \arrow[d, equal] \\
 & \mathbf{Z} \oplus \mathbf{Z} & \mathbf{Z} \oplus \mathbf{Z} & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & 0
\end{tikzcd} \tag{2.28}$$

There is only one type II SD-region in $Herm(3, \mathbf{C})$, and it is the nongeneric, complex, SD-region which is homotopy equivalent to $F(1,1,1)$. Also, the unique type II SD-region in $Herm(3, \mathbf{R})$ is the nongeneric, real, SD-region which is homotopy equivalent to $F'(1,1,1)$. We denote this pair of SD-regions by $(\mathcal{W}_3, \mathcal{W}_3')$. The homotopy and homology in dimensions one and two for the pair $(\mathcal{W}_3, \mathcal{W}_3')$ are described by the commutative diagram with exact rows,

$$\begin{tikzcd}
 & \mathbf{Z} \oplus \mathbf{Z} \arrow[d, equal] & [\text{nonabelian}] \arrow[d, equal] & Q_8 \arrow[d, equal] & 0 \arrow[d, equal] \\
\pi_2(\mathcal{W}_3') \arrow[r, "0"] \arrow[d] & \pi_2(\mathcal{W}_3) \arrow[r] \arrow[d, "h", "\cong"'] & \pi_2(\mathcal{W}_3, \mathcal{W}_3') \arrow[r, "\partial_\#"] \arrow[d, "h''", "\text{epic}"'] & \pi_1(\mathcal{W}_3') \arrow[r] \arrow[d, "h'", "\text{epic}"'] & \pi_1(\mathcal{W}_3) \arrow[d] \\
H_2(\mathcal{W}_3') \arrow[r, "0"] & H_2(\mathcal{W}_3) \arrow[r, "\times 2"] \arrow[d, equal] & H_2(\mathcal{W}_3, \mathcal{W}_3') \arrow[r, "\partial_*"] \arrow[d, equal] & H_1(\mathcal{W}_3') \arrow[r] \arrow[d, equal] & H_1(\mathcal{W}_3) \arrow[d, equal] \\
 & \mathbf{Z} \oplus \mathbf{Z} & \mathbf{Z} \oplus \mathbf{Z} & \mathbf{Z}_2 \oplus \mathbf{Z}_2 & 0
\end{tikzcd} \tag{2.29}$$

§3 Computation of Adiabatic Phase

In Section 1, it was shown that a physical system described by an irreducible, time-dependent matrix hamiltonian $H(t)$, with values in a nondegenerate SD-region $\mathcal{W}$ of $Herm(n, \mathbf{C})$, exhibits time-reversal invariance if and only if the path of hermitian matrices $H(t)$ is in a potentially real subspace of $\mathcal{W}$, for all t. Furthermore, we showed that to study paths in any potentially real subspaces of $\mathcal{W}$, it suffices to consider paths in the nondegenerate, real SD-region $\mathcal{W}' = \mathcal{W} \cap Herm(n, \mathbf{R})$. The purpose of this section is to compute the adiabatic phase on all loops in each of the nondegenerate, real SD-regions of $Herm(n, \mathbf{C})$.

Consider any pair of nondegenerate SD-regions $(\mathcal{W}, \mathcal{W}')$, and let T be a simple, smooth loop in $\mathcal{W}'$. It was proven in Subsection II.3.b that there exists a smooth imbedding D of the 2-dimensional unit disc into $\mathcal{W}$ such that $\partial D = T$. Furthermore, viewing T as a smooth, oriented 1-cycle, the orientation on T induces an orientation on D. From equation (II.3.6), the adiabatic phase $\gamma(T)$ is given by the integral of the adiabatic curvature 2-form $\mathcal{K}$ over D.

The disc D represents a relative homology class in $H_2(\mathcal{W}, \mathcal{W}')$. Our approach to computing $\gamma(T)$ is to relate $\mathcal{K}$ to a relative cohomology class in $H^2(\mathcal{W}, \mathcal{W}'; \mathbf{Z})$, and then evaluate the cap product between this cohomology class and a choice of generator or generators for $H_2(\mathcal{W}, \mathcal{W}')$. Thus, we obtain $\gamma(T)$ expressed in terms of the homology class of D in $H_2(\mathcal{W}, \mathcal{W}')$. Our knowledge of the homomorphisms in the commutative diagram (2.1) is used to relate $\gamma(T)$ to the homotopy class of T in $\pi_1(\mathcal{W}')$.

For a complex vector bundle, the relationship between curvature and cohomology is expressed by the Chern-Weil description of characteristic classes. A brief review, which is well suited to our needs, is given in [66, Appendix C]. For complex line bundles, Chern-Weil theory relies on the fact that if η is a complex line bundle with connection ∇, defined over a base manifold $\mathcal{M}$, then its curvature 2-form $\mathcal{K}$ is a globally-defined, closed 2-form on $\mathcal{M}$. This implies that $\mathcal{K}$ represents a cohomology class $\{\mathcal{K}\} \in H^2(\mathcal{M}; \mathbf{C})$. Furthermore, there is the remarkable fact that the cohomology class $\{\mathcal{K}\}$ is independent of the connection ∇, and depends only on the isomorphism class of the bundle η [66, p. 298].

The computation of the adiabatic phase for simple, smooth loops is easily extended to all smooth loops, not necessarily simple. Indeed, because the adiabatic theorem (Theorem II.1.14) holds for twice continuously differentiable loops, we can also relax the requirement that the loops be smooth. However, this generalization is obvious, and we will not bother with it further. If T is an arbitrary, smooth loop in $\mathcal{W}'$, and $n \geq 2$, it follows from the Whitney imbedding theorem [84, Sect. II.4], [93, Thm. 2] that T may be approximated arbitrarily closely by a simple, smooth loop T', which also lies entirely in $\mathcal{W}'$. Of course, T and T' are homotopic in $\mathcal{W}'$. Furthermore, we can choose T' such that $\gamma(T')$ approximates $\gamma(T)$, arbitrarily closely. This allows us to obtain $\gamma(T)$ in terms of the homotopy class of T in $\pi_1(\mathcal{W}')$. As in the previous section, we divide our analysis into first considering type I SD-regions, and then type II SD-regions.

(a) Type I SD-Regions

The two nondegenerate, type I SD-regions in $Herm(n, \mathbf{R})$ are $\mathcal{W}'(1,1)$ and $\mathcal{W}'(n,n)$. We shall denote both of these regions by $\mathcal{W}'$, and their corresponding complex SD-region by $\mathcal{W}$. By Proposition II.2.8 and Theorem II.2.9, there is a relative, strong deformation retraction of the pair $(\mathcal{W}, \mathcal{W}')$ to the pair $(\mathbf{C}P(n-1), \mathbf{R}P(n-1))$. We denote the relative retraction mapping by

$$r\colon (\mathcal{W}, \mathcal{W}') \longrightarrow (\mathbf{C}P(n-1), \mathbf{R}P(n-1)) \,,$$

and the corresponding relative inclusion by

$$i\colon (\mathbf{C}P(n-1), \mathbf{R}P(n-1)) \hookrightarrow (\mathcal{W}, \mathcal{W}') \,.$$

In Subsection II.3.a, we defined the eigenspace line bundle ξ over $\mathcal{W}$. By representing the adiabatic connection in a basis of local sections, we obtained the adiabatic curvature 2-form $\mathcal{K}$ on $\mathcal{W}$. The curvature form $\mathcal{K}$ is closed, and hence represents a de Rham cohomology class $\{\mathcal{K}\} \in H^2_{DR}(\mathcal{W}; \mathbf{C})$. Furthermore, it follows from Chern-Weil theory that the cohomology class $\{\mathcal{K}\}$ is independent of the connection ∇_A. Perhaps surprisingly, this means that the cohomology class of any curvature 2-form on ξ is identical to the cohomology class of the adiabatic curvature 2-form.

The de Rham isomorphism theorem (Theorem I.1.17) implies that the de Rham cohomology group $H^2_{DR}(\mathcal{W}; \mathbf{C})$ is naturally isomorphic to the singular cohomology group $H^2(\mathcal{W}; \mathbf{C})$. Let $\phi\colon H^2(\mathcal{W}; \mathbf{C}) \to H^2_{DR}(\mathcal{W}; \mathbf{C})$ be the inverse of the isomorphism defined in (I.1.18), and denote by $\zeta\colon \mathbf{Z} \hookrightarrow \mathbf{C}$ the inclusion of the integers into the complex plane. Because $H^2(\mathcal{W}; \mathbf{Z}) \cong H^2(\mathbf{C}P(n-1); \mathbf{Z})$, it follows from (I.2.14) that $H^2(\mathcal{W}; \mathbf{Z})$ is isomorphic to $\mathbf{Z}$, and is therefore a free, finitely generated abelian group. This implies that the induced homomorphism $\zeta_*\colon H^2(\mathcal{W}; \mathbf{Z}) \to H^2(\mathcal{W}; \mathbf{C})$ is a monomorphism, which allows us to identify the integral, singular cohomology classes with their images under $\phi\,\zeta_*$. Under this identification, the cohomology classes in $H^2(\mathcal{W}; \mathbf{Z})$ are represented by closed 2-forms on $\mathcal{W}$. Of course, not all closed 2-forms will represent an integral cohomology class.

Consider the induced line bundle $i^*\xi$, obtained by restricting ξ to $\mathbf{C}P(n-1)$. By Proposition II.3.3, the line bundle $i^*\xi$ is isomorphic to the canonical line bundle γ_{n-1} over $\mathbf{C}P(n-1)$, and furthermore, $i^*\nabla_A$ is a connection on γ_{n-1}. Thus, Chern-Weil theory implies that the cohomology class $\{\frac{1}{2\pi i} i^*\mathcal{K}\} \in H^2_{DR}(\mathbf{C}P(n-1); \mathbf{C})$ is equal to the image of the first Chern class $c_1(\gamma_{n-1}) \in H^2(\mathbf{C}P(n-1); \mathbf{Z})$ under the the monomorphism $\phi\,\zeta_*$

[66, p. 306]. Moreover, the cohomology group $H^2(\mathbf{C}P(n-1);\mathbf{Z}) \cong \mathbf{Z}$ is freely generated by $c_1(\gamma_{n-1})$ [66, Thm. 14.4]. By identifying integral cohomology classes with their images under $\phi\,\zeta_*$, we conclude that $\{\frac{1}{2\pi i}i^*\mathcal{K}\}$ generates $H^2(\mathbf{C}P(n-1);\mathbf{Z})$.

We have shown that $\frac{1}{2\pi i}$ times the restriction of the adiabatic curvature 2-form to the canonical line bundle over $\mathbf{C}P(n-1)$ gives a cohomology class which generates $H^2(\mathbf{C}P(n-1);\mathbf{Z})$. However, because $\mathbf{C}P(n-1)$ is a strong deformation retract of $\mathcal{W}$, it follows that the induced homomorphism $r^*\colon H^2(\mathbf{C}P(n-1);\mathbf{Z}) \to H^2(\mathcal{W};\mathbf{Z})$ is an isomorphism, and the homomorphism r^*i^* is equal to the identity on $H^2(\mathcal{W};\mathbf{Z})$. Thus, the cohomology class $r^*\{\frac{1}{2\pi i}i^*\mathcal{K}\} = \{\frac{1}{2\pi i}\mathcal{K}\}$ generates $H^2(\mathcal{W};\mathbf{Z})$.

First, we consider a pair of generic, nondegenerate, type I SD-regions $(\mathcal{W}_g, \mathcal{W}_g')$, and examine the relative cohomology group $H^2(\mathcal{W}_g, \mathcal{W}_g';\mathbf{Z})$. Recall that type I SD-regions are generic when $n \geq 3$.

Lemma 3.1. *If $(\mathcal{W}_g, \mathcal{W}_g')$ is a pair of generic, nondegenerate, type I SD-regions, then the closed 2-form $\frac{1}{\pi i}\mathcal{K}$ represents an integral, relative cohomology class $\{\frac{1}{\pi i}\mathcal{K}\}$, which generates $H^2(\mathcal{W}_g, \mathcal{W}_g';\mathbf{Z}) \cong \mathbf{Z}$.*

Proof. The bottom row of the commutative diagram (2.7) gives the homology exact sequence

$$\begin{array}{ccccccc} H_2(\mathcal{W}_g) & \xrightarrow{k_*} & H_2(\mathcal{W}_g, \mathcal{W}_g') & \xrightarrow{\partial_*} & H_1(\mathcal{W}_g') & \longrightarrow & H_1(\mathcal{W}_g) \\ \| & & \| & & \| & & \| \\ \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z}_2 & & 0 \end{array} \tag{3.2}$$

With an appropriate choice of generators, the homomorphism k_* is multiplication by 2. Applying the contravariant functor $\mathrm{Hom}(\,\cdot\,,\mathbf{Z})$ to this exact sequence, we obtain the homomorphism

$$\mathrm{Hom}(k_*,\mathbf{Z})\colon\ \mathrm{Hom}(H_2(\mathcal{W}_g,\mathcal{W}_g')\,,\,\mathbf{Z}) \longrightarrow \mathrm{Hom}(H_2(\mathcal{W}_g)\,,\,\mathbf{Z})\,, \tag{3.3}$$

where $\mathrm{Hom}(H_2(\mathcal{W}_g,\mathcal{W}_g')\,,\,\mathbf{Z})$ and $\mathrm{Hom}(H_2(\mathcal{W}_g)\,,\,\mathbf{Z})$ are both isomorphic to $\mathbf{Z}$. It follows immediately from the definition of the functor Hom, that the homomorphism (3.3) is also multiplication by 2.

By extending the exact sequence (3.2) further to the right, we observe that $H_1(\mathcal{W}_g,\mathcal{W}_g') = 0$. Thus, the universal coefficient theorem for cohomology implies that

$$H^2(\mathcal{W}_g;\mathbf{Z}) \cong \mathrm{Hom}(H_2(\mathcal{W}_g)\,,\,\mathbf{Z})$$

and

$$H^2(\mathcal{W}_g,\mathcal{W}_g';\mathbf{Z}) \cong \mathrm{Hom}(H_2(\mathcal{W}_g,\mathcal{W}_g')\,,\,\mathbf{Z})\,.$$

Since $\{\frac{1}{2\pi i}\mathcal{K}\}$ generates $H^2(\mathcal{W}_g;\mathbf{Z})$, we conclude from the homomorphism (3.3) that $\{\frac{1}{\pi i}\mathcal{K}\}$ generates $H^2(\mathcal{W}_g,\mathcal{W}_g';\mathbf{Z})$. □

There are two possible homology classes which may be chosen as a generator of $H_2(\mathcal{W}_g,\mathcal{W}_g')$. We claim that one of these two generators, ν is naturally dual to the generator $\{\frac{1}{\pi i}\mathcal{K}\}$ for $H^2(\mathcal{W}_g,\mathcal{W}_g';\mathbf{Z})$, in the sense that $\left\langle\{\frac{1}{\pi i}\mathcal{K}\}\,,\,\nu\right\rangle = 1$. The product $\langle\,\cdot\,,\,\cdot\,\rangle$ between cohomology and homology, is the usual scalar product defined as evaluation of the cochain on the chain.

It is crucial for the existence of a nontrivial adiabatic phase that one of the two possible generators of $H_2(\mathcal{W}_g, \mathcal{W}'_g)$ be dual to $\{\frac{1}{\pi i}\mathcal{K}\}$. This claim may be proven in at least two ways. We present in detail a geometric argument that uses the imbedding $\alpha\colon \mathbf{C}P(1) \hookrightarrow \mathbf{C}P(n-1)$, which is defined in (I.2.9). Poincaré duality for $\mathbf{C}P(1)$ allows us to explicitly construct a dual generator for $H_2(\mathcal{W}_g, \mathcal{W}'_g)$. Alternatively, a more algebraic argument using the universal coefficient theorem proves the same result. First, the geometric argument will be used to prove Theorem 3.5 and Corollary 3.6. Then, since the more algebraic argument clarifies some aspects of the proof, it will be briefly described in the remarks following Corollary 3.6.

Consider the imbedding $\alpha\colon \mathbf{C}P(1) \hookrightarrow \mathbf{C}P(n-1)$, and recall that $\mathbf{C}P(1)$ is diffeomorphic to the sphere S^2. The 2-form $\alpha^* i^* \mathcal{K}$ is the restriction of $\mathcal{K}$ to $\mathbf{C}P(1)$. From the definition of $\mathcal{K}$ in Subsection II.3.a, it follows that $\mathcal{K}$ is invariant under the conjugate action of a constant $U \in \mathbf{U}(n)$ on $\mathcal{W}_g$. Hence, the 2-form $\alpha^* i^* \mathcal{K}$ is invariant under the usual action of $\mathbf{U}(2)$ on the homogeneous space $\mathbf{C}P(1)$. This implies that $\alpha^* i^* \mathcal{K}$ is a nowhere vanishing 2-form on $\mathbf{C}P(1)$. For, if $\alpha^* i^* \mathcal{K}$ vanished anywhere on $\mathbf{C}P(1)$, then it would follow that it must be identically zero on $\mathbf{C}P(1)$. This would mean that $\{\alpha^* i^* \mathcal{K}\}$ would be the trivial cohomology class in $H^2_{DR}(\mathcal{W}_g; \mathbf{C})$, which we have shown not to be the case.

In general, a choice of orientation for an orientable, n-dimensional manifold is uniquely determined by a nowhere vanishing n-form on the manifold.[7] Therefore, the 2-form $\alpha^* i^* \mathcal{K}$ defines an orientation on $\mathbf{C}P(1) \approx S^2$. The cohomology class $u = \{(\frac{1}{2\pi i})\alpha^* i^* \mathcal{K}\} \in$

[7] This well-known fact is explained in [16, p. 29].

$H^2(S^2;\mathbf{Z})$ is called the fundamental cohomology class of this oriented S^2. Given a fundamental cohomology class, there is a unique choice of generator $\mu \in H_2(S^2) \cong \mathbf{Z}$, which is dual in the sense that $\langle u, \mu \rangle = 1$. The homology class μ is called the fundamental homology class. More details on duality, and fundamental cohomology and homology classes are given in [38] and [66].

The homomorphism $i_*\alpha_*\colon H_2(S^2) \to H_2(\mathcal{W}_g)$ is an isomorphism, and hence $i_*\alpha_*\mu$ defines a preferred choice of generator for $H_2(\mathcal{W}_g)$. The homology exact sequence for the pair $(\mathcal{W}_g, \mathcal{W}_g')$ defines a homomorphism $k_*\colon H_2(\mathcal{W}_g) \to H_2(\mathcal{W}_g, \mathcal{W}_g')$, which we have shown to be multiplication by 2. There are two possible homology classes which may be chosen as a generator for $H_2(\mathcal{W}_g, \mathcal{W}_g')$. However, a choice of generator for $H_2(\mathcal{W}_g)$ defines a preferred choice of generator, $\nu \in H_2(\mathcal{W}_g, \mathcal{W}_g')$ by the condition that $k_*i_*\alpha_*\mu = 2\nu$.

Having fixed a choice of generator for $H_2(\mathcal{W}_g, \mathcal{W}_g')$, we have established a one-to-one correspondence between relative homology classes in $H_2(\mathcal{W}_g, \mathcal{W}_g')$, and integers. For a relative 2-cycle D, we denote by $\{D\}$ both the homology class and the integer obtained by this correspondence.

The cap product[8] defines a homomorphism from $H^2(\mathcal{W}_g, \mathcal{W}_g'; \mathbf{Z}) \otimes H_2(\mathcal{W}_g, \mathcal{W}_g')$ to $H_0(\mathcal{W}_g)$. For $c \in H^2(\mathcal{W}_g, \mathcal{W}_g'; \mathbf{Z})$ and $\sigma \in H_2(\mathcal{W}_g, \mathcal{W}_g')$, this homomorphism is denoted by $c \otimes \sigma \mapsto c \frown \sigma$. If ε is the augmentation on $H_0(\mathcal{W}_g)$, then the cap product satisfies

$$\varepsilon(c \frown \sigma) = \langle c, \sigma \rangle \,. \tag{3.4}$$

Of course, since $\mathcal{W}_g$ is path connected, the augmentation ε is an isomorphism.

[8] The cap product is reviewed in [29, Sect. VII.12] and [83, p. 254].

Let D be a smooth disc which represents the generator $\nu \in H_2(\mathcal{W}_g, \mathcal{W}'_g)$. Recall that because the Hurewicz homomorphism h'' in the commutative diagram (2.1) is an epimorphism, any relative homology class in $H_2(\mathcal{W}_g, \mathcal{W}'_g)$ may be represented by the smooth image of a disc in $\mathcal{W}_g$, with its boundary in $\mathcal{W}'_g$. Also, the Whitney imbedding theorem implies that we can assume that the disc is imbedded in $\mathcal{W}_g$. The de Rham isomorphism theorem for relative cohomology implies that

$$\int_D \frac{1}{\pi i}\mathcal{K} = \left\langle \{\frac{1}{\pi i}\mathcal{K}\}, \nu \right\rangle ,$$

where $\{\frac{1}{\pi i}\mathcal{K}\}$ denotes the cohomology class in $H^2(\mathcal{W}_g, \mathcal{W}'_g; \mathbf{Z})$ represented by $\frac{1}{\pi i}\mathcal{K}$. By (3.4), we have that

$$\int_D \frac{1}{\pi i}\mathcal{K} = \varepsilon(\{\frac{1}{\pi i}\mathcal{K}\} \frown \nu) .$$

Naturality of the cap product implies that

$$\varepsilon(\{\frac{1}{\pi i}\mathcal{K}\} \frown \nu) = \varepsilon(\{\frac{1}{2\pi i}\mathcal{K}\} \frown i_*\alpha_*\mu) ,$$

where $\{\frac{1}{2\pi i}\mathcal{K}\} \in H^2(\mathcal{W}_g; \mathbf{Z})$ and $i_*\alpha_*\mu \in H_2(\mathcal{W}_g)$. Also by naturality, we obtain

$$\begin{aligned} \varepsilon(\{\frac{1}{2\pi i}\mathcal{K}\} \frown i_*\alpha_*\mu) &= \varepsilon(\alpha^* i^* \{\frac{1}{2\pi i}\mathcal{K}\} \frown \mu) \\ &= \varepsilon(u \frown \mu) = \langle u, \mu \rangle = 1 \end{aligned}$$

Let D now be a smooth imbedding of the unit disc which represents an arbitrary relative homology class in $H_2(\mathcal{W}_g, \mathcal{W}'_g)$. Then, for the pair of generic, nondegenerate, type I SD-regions $(\mathcal{W}_g, \mathcal{W}'_g)$, we have proven

Theorem 3.5. *Take ν as the generator of $H_2(\mathcal{W}_g, \mathcal{W}'_g)$, and let $\{D\}$ represent the integer corresponding to the relative homology class $\{D\} \in H_2(\mathcal{W}_g, \mathcal{W}'_g)$. Then, the integral of the adiabatic curvature 2-form $\mathcal{K}$ over D is given by*

$$\int_D \mathcal{K} = \{D\}\, \pi i \, .$$

□

We are now able to compute the adiabatic phase for any simple loop T in the SD-region $\mathcal{W}'_g$. Any such loop is a 1-cycle in $\mathcal{W}'_g$, and hence represents a homology class $\{T\} \in H_1(\mathcal{W}'_g)$. Recall from the commutative diagram (2.7) that $H_1(\mathcal{W}'_g)$ is equal to $\mathbf{Z}_2$. Therefore, each homology class $\{T\}$ is uniquely represented by either 0 or 1, depending on whether it is trivial, or not, respectively. We use the notation $\{T\}$ to also denote this integer (mod 2) in $\mathbf{Z}_2$.

For the smooth, simple loop T in $\mathcal{W}'_g$, there exists a smooth imbedding D of the unit disc into $\mathcal{W}_g$ with the property that $T = \partial D$. The disc D represents a relative homology class $\{D\} \in H_2(\mathcal{W}_g, \mathcal{W}'_g)$. The homology classes $\{T\}$ and $\{D\}$ are related by the boundary homomorphism ∂_* in the short exact sequence

$$0 \longrightarrow H_2(\mathcal{W}_g) \xrightarrow{\times 2} H_2(\mathcal{W}_g, \mathcal{W}'_g) \xrightarrow{\partial_*} H_1(\mathcal{W}_g) \longrightarrow 0$$

which is obtained from the bottom row of the commutative diagram (2.7). This short exact sequence, along with equation (II.3.6), and Theorem 3.5, provides $\gamma(T)$ for all simple, smooth loops in $\mathcal{W}'_g$. By using the Whitney imbedding theorem to extend this result to all smooth loops in $\mathcal{W}'_g$, we prove

Corollary 3.6. *Let T be a smooth loop in a nondegenerate, type I SD-region in $Herm(n, \mathbf{R})$ for $n \geq 3$. Then, the adiabatic phase is $\gamma(T) = \{T\}\,\pi$.* □

We remark that it is not necessary to specify (mod 2π) in the above formula for $\gamma(T)$, because $\{T\}$ is an element of $\mathbf{Z}_2$. Also, the result in Corollary 3.6 is independent of the choice of generator for $H_2(\mathcal{W}_g, \mathcal{W}_g')$. This is not surprising, because the physics should not depend on the generators that are chosen for homology groups.

The aforementioned algebraic proof for Corollary 3.6 is now sketched. Using the universal coefficient theorem, it illustrates that the central idea behind Corollary 3.6 is the existence of a basis for $H^2(\mathcal{W}_g, \mathcal{W}_g')$, which is dual to $\{\frac{1}{\pi i}\mathcal{K}\}$. Define the homomorphism f from $H_2(\mathcal{W}_g, \mathcal{W}_g') \cong \mathbf{Z}$ to $\mathbf{Z}$ by

$$f\colon \sigma \longmapsto \left\langle \{\frac{1}{\pi i}\mathcal{K}\}, \sigma \right\rangle .$$

Then, it is straightforward to verify that

$$\gamma(T) = \deg(f)\,\{T\}\,\pi \pmod{2\pi} ,$$

where $\deg(f)$ is the degree of f. However, it follows from the universal coefficient theorem, that if (X, Y) is a pair of topological spaces, $H_n(X, Y)$ is free abelian, and $H_{n-1}(X, Y)$ is torsion-free, then there is an isomorphism between $H_n(X, Y)$ and $H^n(X, Y; \mathbf{Z})$, which allows us to identify dual bases for these two free abelian groups. This implies that $\deg(f) = 1$, and Corollary 3.6 follows.

Now, consider the case when $n = 2$. The single type I SD-region in $Herm(2, \mathbf{C})$ is denoted by $\mathcal{W}_2$, and $\mathcal{W}_2'$ denotes its intersection with $Herm(2, \mathbf{R})$. From the commutative diagram (2.8), we have the homology short exact sequence

$$\begin{array}{ccccccccc} H_2(\mathcal{W}_2') & \longrightarrow & H_2(\mathcal{W}_2) & \longrightarrow & H_2(\mathcal{W}_2, \mathcal{W}_2') & \longrightarrow & H_1(\mathcal{W}_2') & \longrightarrow & H_1(\mathcal{W}_2) \\ \| & & \| & & \| & & \| & & \| \\ 0 & & \mathbf{Z} & & \mathbf{Z} \oplus \mathbf{Z} & & \mathbf{Z} & & 0 \end{array}$$

Because this short exact sequence is split, an analogue of Theorem 3.5 for nongeneric, type I SD-regions would be significantly different. However, in Subsection II.1.b, the adiabatic phase for any smooth loop $T \subset \mathcal{W}_2$ was computed by directly evaluating the integral $\int_D \mathcal{K}$. The results of this computation are summarized in

Proposition 3.7. *Let T be any smooth loop in the SD-region $\mathcal{W}_2' \subset Herm(2, \mathbf{R})$. By choosing one of the two possible choices of generator for $H_1(\mathcal{W}_2') \cong \mathbf{Z}$, we assign an integer to each homology class in $H_1(\mathcal{W}_2')$. The integer associated with the homology class $\{T\} \in H_1(\mathcal{W}_2')$ is denoted by $\{T\}$. Then, the adiabatic phase is*

$$\gamma(T) = \{T\}\, \pi \pmod{2\pi}\,.$$

□

We remark that in Proposition 3.7, the choice of generator for $H_1(\mathcal{W}_2')$ is irrelevant, because $\gamma(T)$ is only defined modulo 2π.

(b) Type II SD-Regions

Let $\mathcal{W}$ denote a nondegenerate, type II SD-region in $Herm(n, \mathbf{C})$, and $\mathcal{W}'$ its intersection with $Herm(n, \mathbf{R})$. In this subsection, the pair of type II SD-regions $(\mathcal{W}, \mathcal{W}')$ may be either generic or nongeneric. This is because the commutative diagrams (2.28) and (2.29) differ only in homotopy, and it is homology which is used for computing the adiabatic phase. From Proposition 2.8 and Theorem 2.9 in Chapter II, we have a relative retraction mapping

$$r\colon (\mathcal{W}, \mathcal{W}') \longrightarrow (F(p,1,r), F'(p,1,r))$$

and the corresponding relative inclusion

$$i\colon (F(p,1,r), F'(p,1,r)) \hookrightarrow (\mathcal{W}, \mathcal{W}')\,, \qquad \text{where } p, r \geq 1.$$

Let i_1 and i_2 denote the fibre inclusions in the fibre bundles

$$\begin{array}{ccc} \mathbf{C}P(p) & \xrightarrow{i_1} & F(p,1,r) \\ & & \downarrow{\scriptstyle p_1} \\ & & G(p+1,r) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{C}P(r) & \xrightarrow{i_2} & F(p,1,r) \\ & & \downarrow{\scriptstyle p_2} \\ & & G(p,r+1) \end{array}$$

which are defined in (I.2.6) and (I.2.7), respectively. From the direct sum splitting in Proposition 2.19 and the universal coefficient theorem for cohomology, we obtain the direct sum diagram

$$\begin{array}{ccccc} H^2(\mathbf{C}P(p);\mathbf{Z}) & \xleftarrow{i_1^*} & H^2(F(p,1,r);\mathbf{Z}) & \xrightarrow{i_2^*} & H^2(\mathbf{C}P(r);\mathbf{Z}) \\ \uparrow{\scriptstyle \alpha_1^*} & & \| & & \uparrow{\scriptstyle \alpha_2^*} \\ H^2(G(p,r+1);\mathbf{Z}) & \xrightarrow{p_2^*} & H^2(F(p,1,r);\mathbf{Z}) & \xleftarrow{p_1^*} & H^2(G(p+1,r);\mathbf{Z}) \end{array} \tag{3.8}$$

where i_1^*, i_2^*, p_1^*, and p_2^* are the homomorphisms induced on cohomology by the maps i_1, i_2, p_1, and p_2, respectively. The vertical homomorphisms α_1^* and α_2^* are the isomorphisms induced on cohomology by the two appropriate inclusions defined by (2.9) in Chapter I. We define homomorphisms φ_1 and φ_2 by $\varphi_1 = p_2^*(\alpha_1^*)^{-1}$ and $\varphi_2 = p_1^*(\alpha_2^*)^{-1}$. Because (3.8) is a direct sum diagram, it follows that the composition $i_1^*\varphi_1$ is equal to the identity on $H^2(\mathbf{C}P(p);\mathbf{Z})$, the composition $i_2^*\varphi_2$ is equal to the identity on $H^2(\mathbf{C}P(r);\mathbf{Z})$, and $\varphi_1 i_1^* + \varphi_2 i_2^*$ is equal to the identity on $H^2(F(p,1,r);\mathbf{Z})$.

The adiabatic curvature 2-form on the eigenspace line bundle ξ over $\mathcal{W}$ is denoted by $\mathcal{K}$. The induced line bundles $i_1^* i^*\xi$ and $i_2^* i^*\xi$ are the restrictions of the eigenspace line bundle to the subspaces $\mathbf{C}P(p)$ and $\mathbf{C}P(r)$, respectively. It follows by Proposition II.3.4 that $i_1^* i^*\xi$ is isomorphic to the canonical line bundle γ_p over $\mathbf{C}P(p)$, and $i_2^* i^*\xi$ is isomorphic to the canonical line bundle γ_r over $\mathbf{C}P(r)$. The adiabatic curvature 2-form $\mathcal{K}$ restricts to give the curvature 2-forms $\mathcal{K}_1 = i_1^* i^*\mathcal{K}$ and $\mathcal{K}_2 = i_2^* i^*\mathcal{K}$ for γ_p and γ_r, respectively. By the same reasoning as in the generic type I case, the cohomology class $\{\frac{1}{2\pi i}\mathcal{K}_1\}$ is equal to the first Chern class $c_1(\gamma_p)$, and therefore generates $H^2(\mathbf{C}P(p);\mathbf{Z})$. Similarly, the cohomology class $\{\frac{1}{2\pi i}\mathcal{K}_2\}$ generates $H^2(\mathbf{C}P(r);\mathbf{Z})$.

From the direct sum splitting (3.8), we have that the two cohomology classes $\varphi_1\{\frac{1}{2\pi i}\mathcal{K}_1\}$ and $\varphi_2\{\frac{1}{2\pi i}\mathcal{K}_2\}$ generate $H^2(F(p,1,r);\mathbf{Z})$. Furthermore, using the fact that the homomorphism $\varphi_1 i_1^* + \varphi_2 i_2^*$ is equal to the identity on $H^2(F(p,1,r);\mathbf{Z})$, it follows that the de Rham cohomology class $i^*\{\frac{1}{2\pi i}\mathcal{K}\}$ is integral, and is given by

$$i^*\{\frac{1}{2\pi i}\mathcal{K}\} = \varphi_1\{\frac{1}{2\pi i}\mathcal{K}_1\} + \varphi_2\{\frac{1}{2\pi i}\mathcal{K}_2\} \,. \tag{3.9}$$

The homomorphism r^*, induced by the retraction mapping of $(\mathcal{W}, \mathcal{W}')$ to $(F(p,1,r), F'(p,1,r))$, is an isomorphism, and hence the cohomology classes $r^*\varphi_1\{\frac{1}{2\pi i}\mathcal{K}_1\}$ and $r^*\varphi_2\{\frac{1}{2\pi i}\mathcal{K}_2\}$ generate $H^2(\mathcal{W}; \mathbf{Z})$. Also because $(r^*)^{-1} = i^*$, it follows from (3.9) that $\{\frac{1}{2\pi i}\mathcal{K}\}$ is an integral cohomology class, and

$$\{\frac{1}{2\pi i}\mathcal{K}\} = r^*\varphi_1\{\frac{1}{2\pi i}\mathcal{K}_1\} + r^*\varphi_2\{\frac{1}{2\pi i}\mathcal{K}_2\} \,. \tag{3.10}$$

.

Our computation of $\gamma(T)$ for a nondegenerate, real, type II SD-region $\mathcal{W}$ closely parallels the computation for type I SD-regions in the previous subsection. Generators for the relative cohomology group $H^2(\mathcal{W}, \mathcal{W}'; \mathbf{Z})$ are determined in

Lemma 3.11. *Let $(\mathcal{W}, \mathcal{W}')$ be a pair of nondegenerate type II SD-regions. Then, the cohomology classes $r^*\varphi_1\{\frac{1}{\pi i}\mathcal{K}_1\}$ and $r^*\varphi_2\{\frac{1}{\pi i}\mathcal{K}_2\}$ are integral, relative cohomology classes, and they generate $H^2((\mathcal{W}, \mathcal{W}'); \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$.*

Proof. The proof of this lemma is almost identical to the proof of Lemma 3.1. However, it uses the bottom rows of the commutative diagrams (2.28) and (2.29).

□

As in the type I case, generators for $H_2(\mathcal{W}, \mathcal{W}')$, which are dual to $r^*\varphi_1\{\frac{1}{\pi i}\mathcal{K}_1\}$ and $r^*\varphi_2\{\frac{1}{\pi i}\mathcal{K}_2\}$, may be constructed either geometrically using Poincaré duality, or algebraically using the universal coefficient theorem. The arguments are similar to those used in the type I case, and to avoid being pedantic, only the geometric construction will be described.

In order to construct generators for $H_2(\mathcal{W}, \mathcal{W}')$, consider the imbeddings $\alpha_3 \colon S^2 \hookrightarrow \mathbf{C}P(p)$ and $\alpha_4 \colon S^2 \hookrightarrow \mathbf{C}P(r)$, which are two examples of the smooth imbedding α, defined in (I.2.9). Following the generic type I case, we observe that the cohomology classes $\alpha_3^*\{\frac{1}{2\pi i}\mathcal{K}_1\} \in H^2(S^2; \mathbf{Z})$ and $\alpha_4^*\{\frac{1}{2\pi i}\mathcal{K}_2\} \in H^2(S^2; \mathbf{Z})$ are both fundamental cohomology classes for S^2. Fundamental homology classes $\mu_1 \in H_2(S^2)$ and $\mu_2 \in H_2(S^2)$ are defined by the conditions that $\langle \alpha_3^*\{\frac{1}{2\pi i}\mathcal{K}_1\}, \mu_1 \rangle = 1$, and $\langle \alpha_4^*\{\frac{1}{2\pi i}\mathcal{K}_2\}, \mu_2 \rangle = 1$. The homology classes μ_1 and μ_2 define preferred generators for $H_2(\mathcal{W})$ by the homomorphisms in the diagram

$$\begin{tikzcd}
 & H_2(\mathcal{W}) & \\
H_2(\mathbf{C}P(p)) \arrow[r, "i_{1*}"] & H_2(F(p,1,r)) & H_2(\mathbf{C}P(r)) \arrow[l, "i_{2*}"'] \\
H_2(S^2) \arrow[u, "\alpha_{3*}", "\cong"'] & & H_2(S^2) \arrow[u, "\alpha_{4*}", "\cong"'] \\
 & & \\
 & H_2(\mathcal{W}) \arrow[uu, "i_*", "\cong"'] &
\end{tikzcd}$$

This defines generators $i_* i_{1*} \alpha_{3*} \mu_1$ and $i_* i_{2*} \alpha_{4*} \mu_2$ for $H_2(\mathcal{W})$. We remark that the non-homotopic imbeddings $i_1 \circ \alpha_3 \colon S^2 \hookrightarrow F(p,1,r)$ and $i_2 \circ \alpha_4 \colon S^2 \hookrightarrow F(p,1,r)$ represent generators for $\pi_2(F(p,1,r))$, which is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

From the homology exact sequence for the pair $(\mathcal{W}, \mathcal{W}')$, we have the homomorphism $k_* \colon H_2(\mathcal{W}) \to H_2(\mathcal{W}, \mathcal{W}')$, which is multiplication by 2. This defines generators ν_1 and ν_2 for $H_2(\mathcal{W}, \mathcal{W}')$ by the conditions that $k_* i_* i_{1*} \alpha_{3*} \mu_1 = 2\nu_1$ and $k_* i_* i_{2*} \alpha_{4*} \mu_2 = 2\nu_2$. Using (3.10), a computation which is similar to the proof of Theorem 3.5, proves

Theorem 3.12. *Consider a pair of nondegerate type II SD-regions $(\mathcal{W}, \mathcal{W}')$, and let D be any smooth, relative 2-cycle. Taking ν_1 and ν_2 as generators for $H_2(\mathcal{W}, \mathcal{W}')$, the*

homology class $\{D\} \in H_2(\mathcal{W}, \mathcal{W}')$ *is represented by a pair of integers* $(\{D\}_1, \{D\}_2) \in \mathbf{Z} \oplus \mathbf{Z}$. *Then, the integral of the adiabatic curvature 2-form* $\mathcal{K}$ *over* D *is*

$$\int_D \mathcal{K} = (\{D\}_1 + \{D\}_2)\,\pi i\,.$$

□

The homology group $H_1(\mathcal{W}')$ is related to $H_2(\mathcal{W}, \mathcal{W}')$ by the boundary homomorphism ∂_* in the short exact sequence

$$0 \longrightarrow H_2(\mathcal{W}) \xrightarrow{\times 2} H_2(\mathcal{W}, \mathcal{W}') \xrightarrow{\partial_*} H_1(\mathcal{W}') \longrightarrow 0\,.$$

This defines the generators $\partial_*\nu_1$ and $\partial_*\nu_2$ for $H_1(\mathcal{W}') \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Then, for any smooth loop T in $\mathcal{W}'$, the homology class $\{T\} \in H_1(\mathcal{W}')$ is represented by $(\{T\}_1, \{T\}_2)$, a pair of integers (mod 2) in $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. An immediate result of formula (II.3.6) and Theorem 3.12 is

Corollary 3.13. *For* T *a smooth loop in a nondegenerate, type II SD-region in* $Herm(n, \mathbf{R})$, *where* $n \geq 3$, *the adiabatic phase is given by*

$$\gamma(T) = (\{T\}_1 + \{T\}_2)\,\pi \pmod{2\pi}\,.$$

□

(c) Adiabatic Phase in Terms of Homotopy Classes

To recapitulate, the hamiltonian of a periodic quantum system with a finite number of energy levels is described by a loop $T\colon S^1 \to Herm(n, \mathbf{C})$. If $T(s)$ has an eigenvalue $\lambda(s)$, which is nondegenerate for all $s \in S^1$, then associated with this eigenvalue is an adiabatic phase $\gamma(T)$. Furthermore, if this system is also time-reversal invariant, then there exists a choice of basis for $\mathbf{C}^n$ such that with respect to this basis, T is a loop in $Herm(n, \mathbf{R})$.

Corresponding to the eigenvalue $\lambda(s)$ is a nondegenerate, real SD-region $\mathcal{W}'$. We have computed $\gamma(T)$ in terms of the homology class $\{T\} \in H_1(\mathcal{W}')$. However, from a physical standpoint, it may be more useful to obtain $\gamma(T)$ in terms of the homotopy class $[T] \in \pi_1(\mathcal{W}')$. We remind the reader that two loops T_0 and T_1 in $\mathcal{W}'$ are in the same homotopy class if and only if there is a homotopy of loops $\Phi\colon [0,1] \times [0,1] \to Herm(n, \mathbf{R})$ such that $\Phi(0,t) = T_0(t)$, $\Phi(1,t) = T_1(t)$, $\Phi(s,0) = \Phi(s,1)$ for all $s \in [0,1]$, and the eigenvalue $\lambda(s,t)$ of $\Phi(s,t)$ remains nondegenerate for all $(s,t) \in [0,1] \times [0,1]$. Therefore, a homotopy of the loop T correspond to a continuous perturbation of the hamiltonian, which preserves the periodicity, the time-reversal invariance, and the nondegeneracy of λ. For a specific time-reversal invariant quantum system, it may be possible to use such a homotopy to simplify the hamiltonian to the extent that its homotopy class in $\pi_1(\mathcal{W}')$ can be identified.

We now summarize our results for the adiabatic phase $\gamma(T)$ in terms of the homotopy class of T in $\pi_1(\mathcal{W}')$. These result are obtained by using the the Hurewicz homomorphisms in the commutative diagrams (2.7), (2.8), (2.28), and (2.29) to relate

the homotopy classes in $\pi_1(\mathcal{W}')$ to the homology classes in $H_1(\mathcal{W}')$, and thus obtain $\gamma(T)$ in terms of $[T] \in \pi_1(\mathcal{W}')$.

If $\mathcal{W}_2'$ is the nondegenerate SD-region in $Herm(2, \mathbf{R})$, it follows from (2.8) that $\pi_1(\mathcal{W}_2') \cong \mathbf{Z}$. Furthermore, the Hurewicz homomorphism $h' \colon \pi_1(\mathcal{W}_2') \to H_1(\mathcal{W}_2')$ is an isomorphism. By choosing one of the two possible generators for $\pi_1(\mathcal{W}_2')$, we associate an integer, denoted by $[T]$, with each homotopy class $[T] \in \pi_1(\mathcal{W}_2')$. It follows from Proposition 3.7 that the adiabatic phase for a loop $T \subset \mathcal{W}_2'$ is

$$\gamma(T) = [T]\,\pi \pmod{2\pi}\,.$$

Of course, because $\gamma(T)$ is only defined modulo 2π, this result is independent of the choice of generator for $\pi_1(\mathcal{W}_2')$.

We now consider $\mathcal{W}_g'$ to be a nondegenerate, type I SD-region in $Herm(n, \mathbf{R})$ for $n \geq 3$. From the commutative diagram (2.7), we know that $\pi_1(\mathcal{W}_g') \cong \mathbf{Z}_2$, and the Hurewicz homomorphism $h' \colon \pi_1(\mathcal{W}_g') \to H_1(\mathcal{W}_g')$ is an isomorphism. Therefore, it follows from Corollary 3.6 that

$$\gamma(T) = \begin{cases} 0 & \text{if } T \text{ belongs to the trivial class in } \pi_1(\mathcal{W}_g'). \\ \pi & \text{if } T \text{ belongs to the unique nontrivial class in } \pi_1(\mathcal{W}_g'). \end{cases}$$

Now, consider $\mathcal{W}_3'$, the nondegenerate, type II SD-region in $Herm(3, \mathbf{R})$. Notice that all three eigenvalues $\lambda_1 < \lambda_2 < \lambda_3$ are nondegenerate in $\mathcal{W}_3'$. For a loop T in $\mathcal{W}_3'$, we denote by $\gamma_i(T)$ the adiabatic phase associated the eigenvalue λ_i. From the commutative

diagram (2.29), recall that $\pi_1(\mathcal{W}_3')$ is isomorphic to the nonabelian, 8-element quaternion group, Q_8. This implies that there are eight homotopy classes of loops in $\mathcal{W}_3'$. Furthermore, the Hurewicz homomorphism $h'\colon \pi_1(\mathcal{W}_3') \to H_1(\mathcal{W}_3')$ is an epimorphism with kernel the two-element commutator subgroup of Q_8.

The three adiabatic phases for each of the eight homotopy classes in $\pi_1(\mathcal{W}_3')$ are as follows. If T is in one of the two homotopy classes which comprise the commutator subgroup, then $\gamma_i(T) = 0 \pmod{2\pi}$ for $i = 1,2,3$. In Subsection III.3.b, we defined generators $\partial_*\nu_1$ and $\partial_*\nu_2$ for $H_1(\mathcal{W}_3')$. If T is an element of one of the two homotopy classes in the preimage of $\partial_*\nu_1$ under the Hurewicz homomorphism h', then $\gamma_1(T) = \gamma_2(T) = \pi \pmod{2\pi}$, and $\gamma_3(T) = 0 \pmod{2\pi}$. If T is an element of one of the two homotopy classes in $(h')^{-1}\partial_*\nu_2$, then $\gamma_1(T) = 0 \pmod{2\pi}$, and $\gamma_2(T) = \gamma_3(T) = \pi \pmod{2\pi}$. There are only two remaining homotopy classes in $\pi_1(\mathcal{W}_3')$, and these lie in the preimage of $\partial_*\nu_1 + \partial_*\nu_2$. If T is in one of these two homotopy classes, then $\gamma_1(T) = \gamma_3(T) = \pi \pmod{2\pi}$, and $\gamma_2(T) = 0 \pmod{2\pi}$.

Finally, consider a loop T in a nondegenerate, type II SD-region $\mathcal{W}_g'$ in $Herm(n,\mathbf{R})$ for $n \geq 4$. The adiabatic phase associated with the eigenvalue λ, which corresponds to $\mathcal{W}_g'$, is denoted by $\gamma(T)$. From (2.28), the Hurewicz homomorphism $h'\colon \pi_1(\mathcal{W}_g') \to H_1(\mathcal{W}_g')$ is an isomorphism, and therefore $(h')^{-1}\partial_*\nu_1$ and $(h')^{-1}\partial_*\nu_2$ are generators of $\pi_1(\mathcal{W}_g') \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. Given this choice of generators, a homotopy class $[T] \in \pi_1(\mathcal{W}_g')$ is represented by $([T]_1, [T]_2)$, a pair of integers (mod 2) in $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. The adiabatic phase for λ is then

$$\gamma(T) = ([T]_1 + [T]_2)\,\pi \pmod{2\pi}\,.$$

(d) Jahn-Teller Effect

Quantum adiabatic holonomy is important in the Jahn-Teller effect of molecular physics. Furthermore, it is a time-reversal-invariant manifestation of adiabatic holonomy, assuming that no external magnetic fields are applied to the molecular system. Therefore, the results of this section may be applied to the Jahn-Teller effect.[9] This work is currently in progress, and only a brief outline of the methods will be presented here. Further details will appear in [31].

The motion of electrons within a molecule is described by a hamiltonian which depends on the positions of the nuclei. It follows that the electronic states must be coupled with the rotational and vibrational states of the nuclei. This coupling is the dynamical Jahn-Teller effect, and it has important consequences for the vibrational and rotational spectra of the molecule. Since the original investigation of the dynamical Jahn-Teller effect by Longuet-Higgins, Öpik, Pryce, and Sack in [60], there has developed an extensive literature on the subject. Some good reviews are [11], [52], [59], [86], and [90].

We consider a molecule with N nuclei, that are all assumed to have mass M, for the sake of simplicity. The positions of the nuclei are described by the coordinate vector $Q \in \mathbf{R}^{3N}$. The hamiltonian is

$$H_{\text{mol}} = -\frac{1}{2M}\nabla_Q^2 + H(Q) \ ,$$

where the operator $H(Q) = -\frac{1}{2m}\nabla_q^2 + V(Q,q)$ depends parametrically on the nuclear coordinates Q. The electronic mass and coordinate vectors are m and q, respectively.

[9] We thank the referees of [31] for encouraging us to consider this problem.

The total potential energy $V(Q,q)$ contains the coupling between the electrons and the nuclei.

Let $\phi(Q,q)$ be a Q-dependent eigenfunction satisfying

$$H(Q)\,\phi(Q,q) = \lambda(Q)\,\phi(Q,q)\,, \qquad \text{for all } Q.$$

It is assumed that $\lambda(Q)$ is a nondegenerate eigenvalue, that is bounded away from the rest of the spectrum of $H(Q)$ for all Q.[10] At least locally, $\phi(Q)$ defines an adiabatic connection 1-form $\mathcal{A} = (\phi(Q)\,,\, d\phi(Q))$. If $\chi(Q)$ is a wave function satisfying

$$\left[-\frac{1}{2M}(\nabla_Q + \mathcal{A})^2 + \lambda(Q)\right]\chi(Q) = \epsilon\,\chi(Q)\,, \tag{3.14}$$

then the Born-Oppenheimer approximation implies that $\psi(Q,q) = \chi(Q)\,\phi(Q,q)$ approximates an eigenfunction of H_{mol}, with eigenvalue ϵ.

In most applications, $H(Q)$ is time-reversal invariant. Therefore, locally $\mathcal{A}$ may be chosen to be 0. For this reason, the usual statement of the Born-Oppenheimer approximation does not include the 1-form $\mathcal{A}$ in (3.14). The importance of the adiabatic connection 1-form in the Born-Oppenheimer approximation was demonstrated in [67].

Recall from Section II.3, that the 1-form $\mathcal{A}$ is globally defined on Q-space only if the the eigenspace line bundle associated with $\phi(Q)$ is trivial. This is not usually the case. Therefore, (3.14) is not really a single differential equation, but rather a collection of differential equations, defined on coordinate neighbourhoods in Q-space.

[10] We have assumed that $\epsilon(Q)$ is nondegenerate for simplicity. Degenerate eigenvalues may be handled by replacing the eigenvector $\phi(Q)$ with an orthonormal frame for the associated eigenspace.

Throughout this thesis, we have seen the importance of taking particular care when making expansions in terms of parameter-dependent eigenvectors. In general, the eigenfunction $\phi(Q,q)$ will have nontrivial holonomy when parallel transported around loops in Q-space. If $H(Q)$ is time-reversal invariant and $\lambda(Q)$ is nondegenerate, then this holonomy will be restricted to changes in sign. However, the approximate molecular eigenfunction $\psi(Q,q) = \chi(Q)\,\phi(Q,q)$ must be a single-valued function of (Q,q). This is achieved by choosing appropriate boundary conditions for the differential equation (3.14) defined on the various coordinate neighbourhoods of Q-space. For example, if $\phi(Q,q)$ were to change sign when parallel transported around a loop T, then it would be necessary to look for solutions to (3.14) which were odd functions on T. The spectrum of a differential operator depends on its boundary conditions, and therefore we see that quantum adiabatic phase will influence the molecular spectrum.

Unfortunately, $H(Q)$ is a hermitian operator on an infinite-dimensional Hilbert space, and not a matrix. Hence, it seems that the Jahn-Teller effect requires an analysis of adiabatic holonomy for hermitian operators. However, the spectrum of a molecular hamiltonian often consists of an infinite number of almost degenerate multiplets, separated by relatively large gaps. The usual procedure in Jahn-Teller analysis is to neglect mixing between eigenfunctions of different multiplets. In doing this, Q is fixed to some value Q_0, usually an equilibrium position for the nuclei. Then a Q-independent orthonormal basis is chosen for the span of those eigenfunctions of $H(Q_0)$ which are associated with the multiplet of interest. In terms of this basis, $H(Q)$ may then be represented as a Q-dependent matrix. Adiabatic holonomy is computed using this matrix. Therefore, many of the results in this chapter have direct application to Jahn-Teller theory.

Implications of adiabatic holonomy have been worked out for rotating diatoms [67], and for the $E \otimes \varepsilon$ Jahn-Teller effect, in which an electronic doublet E is coupled to a pair of vibrational modes ε [45], [60]. For both of these problems, $H(Q)$ may be represented as a 2×2 matrix. Hence, it is straightforward to compute adiabatic phases in terms of explicit eigenvectors, as was done in Subsection II.1.b. For more complicated systems, such an approach becomes difficult, if not impossible.[11]

Experimental consequences of the $E \otimes \varepsilon$ Jahn-Teller effect have been observed in the spectra of a number of molecular systems [2], [45]. The most compelling evidence is the half-odd integer quantization of pseudorotational levels in Na_3 [27].

[11] Some explicit families of eigenvectors have been computed at great effort for Jahn-Teller systems involving electronic triplets. For example, see [52]. These families of eigenvectors have been used to compute adiabatic connection 1-forms in [19].

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