BIRTH OF UNIVERSES
WITH
NON-MINIMAL COUPLING

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Abstract

Part I : Models of cosmological inflation are plagued with a severe and seemingly unavoidable problem: in order to produce density perturbations of an amplitude consistent with large scale observations, the self-coupling λ of the *inflaton* field has to be tuned to an excessively small value. In all these models, however, the scalar field is taken to be minimally coupled to the scalar curvature (the curvature coupling ξ is set to zero). It is shown here that in the more general case of non-minimal coupling ($\xi \neq 0$), and within the framework of Linde's chaotic inflation, the constraint on the self-coupling can be relaxed by several orders of magnitude. This stems essentially from the fact that, contrary to common belief, the curvature coupling ξ can be almost arbitrarily large without upsetting the inflationary scenario. Non-minimal coupling may thus provide a relatively simple solution to the long-standing problem of excessive density perturbations in inflationary models.

Part II : The possibility of inflation during induced gravity spontaneous symmetry breaking with Zee's Lagrangian $\mathcal{L} = \xi\varphi^2 R/2 + \partial_\alpha\varphi\partial^\alpha\varphi/2 - \lambda(\varphi^2 - v^2)^2$ is confirmed. The 'ordinary' and 'chaotic' versions of this model are compared and found to differ substantially regarding the constraints on initial conditions and field parameters. The 'chaotic' version, which is very similar to the model discribed in Part I, proves to be more attractive in most respects. In particular, the belief that λ has to be excessively small is shown to be well founded *only* for 'ordinary' inflation.

Part III : The minisuperspace canonical quantization is applied to the $\lambda\varphi^4$ theory of Part I, where the usual assumption of minimal coupling between the curvature and the scalar field is dropped. Different sets of variables are used and special attention is paid

to the invariance of the quantization procedure and to the related issue of factor ordering in the Wheeler-DeWitt equation. It is shown that the generic features of Vilenkin's and Hartle and Hawking's wave functions are preserved in non-minimal coupling. The results also apply to the quartic induced gravity potential of Part II. It is noted that the minisuperspace wave function can be regarded, not as the wave function of the Universe, but rather as describing an ensemble of inflationary sub-universes. Predictions of classical initial conditions could then be made using an ordinary frequency interpretation of quantum probabilities.

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Acknowledgement

If any aspect of this work turns out to be a non-negligible contribution to science, the credit for it should go for a large part to W.G. Unruh. Here is why:

1) It is the dream of anybody involved in a scientific research to possess a magic potion that reveals which ideas are fertile and which are sterile or wrong. For over three years, Unruh administered to me this sometimes bitter but always fateful medicine. Let him be thanked for kindly putting his famous scientific intuition and mastery of theoretical physics at my disposal.

2) He has consistently striven, often at great cost, to let me meet some of the most distinguished personalities in the fields of Gravitation and Cosmology. This has had a far reaching effect on the work described in this thesis as well as on my general education.

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the names in the usual alphabetic order, without meaning to imply that they agree with the content of this thesis or are responsible for any mistakes it may contain:

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Chapter 0

To the layman

The hope in this chapter is to introduce the material covered in references [1,2,3], on which the present thesis is based, to the widest possible public. A large number of authors have contributed to the theories I briefly evoke here. Hence, it is probably more appropriate to postpone giving any references until we come back to these theories in more detail in the next chapters. The reader who is familiar with the terminology and formalism of physics may proceed directly to chapter 1.

1) General relativity and cosmology

To probe the birth of the Universe, its evolution up to the present day and its fate ('cosmology'), modern physicists assume Einstein's theory of 'general relativity' (or some modified version of it) to be correct everywhere and at all times. The essence of this theory is a revolutionary approach to the concepts of space and time: not only do space and time variations affect the matter, which has been known in the most ancient cosmogonies, but the converse is also true. One can write schematically

$$\textit{Spacetime} \rightleftharpoons \textit{matter content of the Universe.} \qquad (A)$$

This theory has had considerable success in explaining many astronomical observations (see below). Unfortunately, it has proved difficult so far to blend it with the physics of elementary particles. It is interesting, however, that both the successes of general relativity and its troubles are traceable to the feed-back mechanism that is apparent in

(A).

The statement *(A)* can be expressed mathematically as a set of ten equations to be solved simultaneously. This is very hard, if at all possible, to achieve in the general case of complex systems. A good example of an irregular system is the Universe at the human scale: we see the world around us to be non-homogeneous (different at different points of space) and non-isotropic (different in different directions of space). To make it easier to read the following, remember this:

> *The number of simultaneous equations that one has to solve in order to describe a given system increases with the degree of irregularity or complexity of that system.* *(B)*

Then, given this serious mathematical difficulty, one may wonder how physicists could have learned so much about the history of our observable Universe, a history that is known today with good confidence up to about the first *minute* of creation! The answer lies in a remarkable observational fact: at the largest scales (clusters of galaxies), the Universe seems to be extremely homogeneous and isotropic. This is a highly non-trivial astronomical observation although astronomers are by now quite used to it. In effect, the galaxy distribution could have been much less regular without affecting physical phenomena on the planetary scale (for example the evolution of life on Earth).

Recalling what we said in *(B)*, this means that the history of the observable Universe as a whole can be followed with reasonable accuracy by solving only a small number of simultaneous equations. This is how modern cosmology has been able to account for many observed facts such as the expansion of the Universe, its background temperature and the relative abundances of its chemical constituents.

However, that the Universe should be so regular in the first place remains a mystery to this day. In the following, I evoke some recent attempts to tackle this fundamental

problem.

2) Inflation

Imagine a balloon bearing some complicated design ('irregularities'). If one could inflate it until it reaches, say, the size of the Earth, then at least three observations could be made:

a) the surface of the balloon would start looking perfectly flat to us (spatial flatness);

b) it would no longer be possible to perceive that the rubber surface does bear a complicated design - instead, we would have the impression that the design is simply a series of infinite straight lines (local regularity);

c) previously microscopic defects in the rubber would be blown up to a macroscopic size and become apparent to the naked eye.

Inflationary cosmology is based on the idea that the Universe has experienced, sometime before what is usually called the Big Bang, a similar very large expansion. In close analogy with the above, some of the implications of this 'inflation' would be, respectively, that

a) the puzzling spatial flatness of the Universe would be explained;

b) the above mentioned problem of extreme regularity (homogeneity and isotropy) at the largest observable scales would be answered at least partially;

c) the existence today of intermediate scale inhomogeneities (such as galaxies) could be traced back to the inflationary growth of originally microscopic irregularities.

Cosmological inflation could thus account for many intriguing features of our observable Universe. However, one still has to show that it is likely to occur in the first place. Unfortunately, it turns out that this can be done only by assuming very special initial conditions for the Universe. In particular, one has to postulate the existence of small

homogeneous regions in the early Universe, which would have come about by some unknown physical process or 'by chance'. This is somewhat disappointing since one hope in introducing inflation was to *explain* the origin of homogeneity.

3) Quantum cosmology

It is clear from the above that there is a need to justify the initial conditions that one assumes in order to obtain inflation. One approach to this problem of finding a law for cosmological initial conditions is to assume that 'quantum physics' plays an important role in the birth and global evolution of the Universe. In 'quantum cosmology' one tries to determine the role of eventual quantum effects on the very spacetime structure.

Quantum mechanics is a branch of physics that can successfully predict many phenomena which, although counter-intuitive, are frequently observed in university laboratories. An example of such phenomena is the creation of negative and positive electrical charges in a box that originally contains no charges at all. Applied to cosmology, the formalism of quantum physics can lead to such concepts as the 'creation of the Universe from nothing'. However, quantum cosmology is still a very young and highly speculative branch of physics, and no fundamental problem has yet found a final answer in this framework.

One of the principle limitations of quantum models of the Universe developed so far is that many assumptions they make are similar to those needed for inflation. For example they postulate, without justification, a high level of regularity in the very early Universe. This is again intended to reduce the mathematical difficulty of the problem (see *(B)*). Thus, because these simplified models adopt rather than justify many assumptions that go into inflationary cosmology, they cannot themselves *prove* that the Universe was bound to inflate. In conclusion, the ultimate theory that would explain to us why *(A)* can be reduced for the early Universe to only a few simultaneous equations, why

initial conditions suitable for inflation should then hold and why the Universe should consequently look as it does today is still far out of reach.

4) Non-minimal coupling

As for the contribution of the present essay to these exciting developments, it is no false modesty to say that it is rather small. This is essentially because, here also, *(A)* is reduced without justification to only two simultaneous equations. I will now briefly indicate what improvements on previous works I hope to have achieved in this thesis.

At the starting point of any inflationary model there is the choice of a quantity, called the 'action', which after some appropriate manipulations leads to a set of equations describing the evolution of the model Universe - these equations are in fact just the mathematical expression of *(A)*. This action may contain 1) space-time terms, 2) matter terms and 3) hybrid terms which are combinations of space-time and matter components. Most models drop the third type of term from the action for the sake of simplifying the calculations. As a consequence, the interdependence between matter and spacetime is considerably weaker in these models than it could in principle be. This does not mean that the two are decoupled (see the second paragraph of this chapter). Rather, one usually says that there is 'minimal coupling' between them. 'Non-minimal coupling' is hence the more general case where hybrid terms (type 3 above) are taken into account.

Much fine tuning and many assumptions are necessary to make inflation work in the simple minimal coupling case. It is therefore understandable that the general feeling, before this research was completed, was that adding the extra complication of non-minimal coupling would make the house of cards of the inflationary scenario collapse. This presumption even received some apparent quantitative confirmation: calculations were conducted that seemingly proved inflation to be incompatible with non-minimal coupling, unless the latter were very weak. If true, this would be an additional set-back

for inflation. It would mean that minimal coupling should be added to the already long list of constraints imposed on physics for the sole purpose of producing inflation.

This thesis carries this somewhat unexpected but happy news: there exists at least one non-minimally coupled model of inflation that works as well as, and perhaps better than previous minimally coupled ones. This is essentially the content of Part I. In Part II, two versions of a specific non-minimally coupled model are analysed, and it is shown specifically why the previous pessimistic calculations were wrong. Finally, we develop in Part III a quantum cosmology (see above) with non-minimal coupling. Again, the conclusions are encouraging: all results previously proven for minimal coupling seem to be preserved in the more general case of non-minimal coupling.

I will stop short of making any (additional) specific claims since the man and the woman in the street, to whom this chapter is addressed, may not be in a position to critically appraise them. Rather, I unfortunately have to depart now from the kind casual reader and prepare to face the scrutiny of the specialists.

Part I

Classical Cosmology: Chaotic Inflation

Chapter 1

Introduction to chaotic inflation

Verily at first Chaos came to be...

Hesiod, VIIIth century BC.

It is a remarkable achievement of the standard Big Bang scenario to have given us an insight into the first few *seconds* of Creation. Many predictions of this model, including the primordial abundances of light chemical elements, the Hubble expansion law and the microwave background radiation, are in excellent agreement with observations. However, many fundamental cosmological problems remained unsolved in this theory. One example is the *horizon* or *causality* problem. This is the fact that regions of the Universe which in principle have never been in causal contact display extremely similar properties. The most spectacular case is the isotropy of the microwave background radiation: photons coming from parts of the sky that were presumably outside each other's horizon at the time of their emission, have nevertheless the same temperature within about one part in 10^5 .

A second example is the *flatness* problem: the observation that the spatial curvature of the Universe, after over 10 billion years of evolution, is still very close to the unstable zero value. The vanishing of the spatial curvature corresponds to the energy density ρ being equal to a certain critical value ρ_{cr} . The cases $\rho/\rho_{cr} < 1$, $= 1$ or > 1 correspond respectively to the Universe being spatially open, flat or closed. General Relativity predicts that even if the universe started off near flatness (e.g. $\rho/\rho_{cr} \sim 10^{-10}$ at the Planck time $t_{Pl} \sim 10^{-44}$ sec.), it should have today a definitely closed or open

spatial geometry ($\rho_{present}/\rho_{cr} >> 1$ or $<< 1$). Then, to be consistent with the observation that $\rho_{present}/\rho_{cr}$ is in fact of order unity, one has to assume that this ratio was extraordinarily close to one at early times ($\rho/\rho_{cr} \sim 10^{-59}$ at the Planck time).

A third problem is the origin of large scale structure in the Universe. To explain the hierarchy of structure observed today (galaxies, clusters of galaxies, voids etc.), a very special spectrum of initial density fluctuations has to be assumed without justification in the standard Big Bang model. First the fluctuations' spectrum must be scale-invariant. Second, and perhaps more troubling, their amplitude at early times must be much smaller than one would expect in any classical or quantum matter state.

There are several more so-called 'cosmological puzzles' that cannot be solved within the original Big Bang scenario. The examples cited above are those to be discussed in the following chapters, when we analyse a modified version of the Big Bang ('inflation') that helps solving some of these puzzles. However, the reader should keep in mind that inflation itself leaves several fundamental cosmological problems unanswered. An example is the extreme smallness of the present value of the cosmological constant.

In a refreshing blending of quantum field theory with the physics of the global space-time (see refs.[4-12]), Guth[13] suggested that a very early period of exponential expansion (inflation) could be at the source of several of the puzzling cosmological observations that remained unexplained within the standard Big Bang scenario. He suggested that such a very large expansion (quasi de Sitter phase) could be driven by the vacuum energy of some unified gauge theories scalar field (hereafter called φ). In this scenario, the 'inflaton' field is initially near an unstable maximum (false vacuum state) of its potential $V(\varphi)$. Inflation continues as long as the energy density is nearly constant. Then, the 'inflaton' undergoes a phase transition and settles at a constant spontaneous symmetry breaking (SSB) value (true vacuum state). In the process, the latent energy released by the strongly first order phase transition is converted into heat, thus initiating the hot Big

Bang.

Guth himself, among others, later realized that his original model (now referred to as *old inflation*) predicted a universe far too inhomogeneous. This was related to the sudden jump of the 'inflaton' field φ , at the end of the de Sitter phase, from the meta-stable maximum of the potential to the equilibrium value. Linde[14] and Albrecht and Steinhardt[15] then developed an improved model (the *new* inflation) where the spontaneous symmetry breaking (SSB) proceeds in a so-called *slow-rolling* fashion.

Motivated partly by difficulties with this scenario itself (see reviews [16-19]), and partly by a less anthropocentric perspective on the global structure of the universe, Linde[20-23] revived an ancient conception (e.g., ref.[24]) and proposed a picture where the cosmos appears as a large set of mini-universes, in each of which nature may have followed a different path of evolution.

The idea is to postulate the existence of regions filled with a scalar field taking a variety of values. Those domains where the field would be larger then a few times the Planck mass over significant volumes, were shown to inflate in much the same way (mathematically at least) as in the previous models, where the inflaton starts off near the zero value corresponding to the metastable state. This is the *chaotic* inflationary scenario.

One of the unexpected successes of inflation is the prediction of an almost scale-free spectrum of density perturbations, which may be what gravitational instability theory needs to reproduce the hierarchy of structure of the observable universe.

Related to this same issue, however, is a severe setback common to all these inflationary models: the magnitude of the resulting density perturbations is far too large to yield anything like our present universe, unless the parameters are finely tuned. For example, to account for galactic and cosmological microwave background observations, which set the fluctuations in $\Delta T/T$ to less than 10^{-4}, the self-coupling of the scalar field has to be

made about ten orders of magnitude smaller than typical field theory values.

A number of elaborate and sometimes contrived particle physics theories have been constructed in the hope of overcoming this difficulty (see refs.[16-19] and references therein). Examples are the 'geometric hierarchy model', 'primordial SSB' and supergravity. One feels, nevertheless, that inflation would become much less attractive if it were to lose its presumed genericity, and had to be based on evermore specific or exotic field theories. We propose here to remove one of the assumptions common to almost all previous models: that of minimal coupling between the inflaton field and the scalar curvature. We show that, for a given self-coupling, the magnitude of density perturbations could be thus improved by several orders of magnitude. Moreover, the situation regarding other constraints such as the large lower bound, in the chaotic picture, on the initial value of the inflaton, may also be improved.

A few months after the results of this research appeared in refs.[1,2], we received preprints by Linde[25] and Futamase and Maeda[26] evoking the possibility of non-minimally coupled inflation. There, no consideration was paid to the problem of density perturbations. No evaluation of the amount of inflation produced was performed, nor was the issue of related constraints on the non-minimal coupling studied. Instead, it was presumed that the scenario would develop along the lines of Accetta, Zoller, and Turner's treatment of induced gravity[27], where it is claimed that, for ordinary and chaotic inflation alike, the self-coupling is required to be at least as small as in Einstein's gravity models. However, as we shall see here and in Part II, we find that the couplings *do not* have to be small for the *chaotic* model to be successfully implemented. (The reader can find a clear illustration of our disagreement with ref.[27] in eqs.(107 to 110).)

We begin this first part by demonstrating the existence, for an arbitrary curvature coupling ξ, of a self-consistent inflationary solution in the chaotic picture (chapter 3). Next, we find that for all values of ξ the model yields sufficient expansion to solve

the usual cosmological puzzles (chapter 4). We then compare the amplitude of density perturbations and the self-coupling (λ) constraint in the minimal ($\xi = 0$) and non-minimal (especially $\xi > 1$) models, and show that the perturbations can be made considerably smaller and the λ constraint substantially weaker by chosing larger values for ξ (chapter 5). We also confirm numerically our analytical results.

Chapter 2

Classical evolution of matter and space-time geometry

Introducing a finite coupling between the scalar field φ and the scalar curvature R, we take the Lagrangian to be

$$\mathcal{L} = (\frac{1}{16\pi G} + \frac{1}{2}\xi\varphi^2)R + \frac{1}{2}\varphi_{;\alpha}\varphi^{;\alpha} - V(\varphi) \ . \tag{2.1}$$

$V(\varphi)$ is the potential energy of the scalar field φ . We choose the metric signature to be $(+ \ - \ - \ -)$. The sign of ξ is such that $\xi_c = -1/6$ is the conformal coupling value. 'Minimal coupling' corresponds to $\xi = 0$. However, as shown by Vilenkin and Ford in a study of curvature effects on phase transitions[28], it is the *conformal* coupling that represents the weakest physical coupling between matter and spacetime geometry. On the other hand, a negative value of ξ could lead to some interesting physics such as gravitational repulsion. This case was touched upon by Futamase and Maeda[26] and Linde[25] in recent preprints, and it needs further investigation. Here ξ is taken to be positive.

To facilitate the comparison with the existing literature we use the Ginzburg-Landau potential

$$V(\varphi) = \lambda\,(\varphi^2 - v^2)^2 \ . \tag{2.2}$$

λ is the dimensionless self-coupling of φ and v is the mass scale of the spontaneous

symmetry breaking. However, any potential with a similar quartic large φ behaviour would yield essentially the same results.

We assume a homogeneous distribution for the scalar field, and the line element is taken to be that of a Robertson-Walker universe

$$ds^2 = dt^2 - a^2(t)dx^2 \ , \tag{2.3}$$

where $a(t)$ is the scale factor and where the spatial curvature, which is unimportant in this context, is set to zero.

Variation of the action with respect to the gravitational degrees of freedom yields Einstein's equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -8\pi G T_{\alpha\beta} \ , \tag{2.4}$$

where $g_{\alpha\beta}$, $R_{\alpha\beta}$ and $T_{\alpha\beta}$ are the usual metric, Ricci, and energy-momentum tensors. The latter has the form (Smolin[29])

$$T_{\alpha\beta} = \varphi_{;\alpha}\varphi_{;\beta} - \frac{1}{2}g_{\alpha\beta}\varphi_{;\gamma}\varphi^{;\gamma} + g_{\alpha\beta}V(\varphi) + \xi\{(\varphi^2)_{;\alpha;\beta} + G_{\alpha\beta}\varphi^2 - g_{\alpha\beta}\Box(\varphi^2)\} \ . \tag{2.5}$$

Taking the time-time component of eq.(4) and eq.(5) gives the *energy* equation

$$3H^2[\frac{1}{8\pi G} + \xi\varphi^2] = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) - 6\xi H\varphi\dot{\varphi} \ . \tag{2.6}$$

Overdots denote time derivatives and $H \equiv \dot{a}/a$ is the Hubble expansion rate.

Variation with respect to the matter results in the field equation

$$\ddot{\varphi} + 3H\dot{\varphi} - 6\xi(\dot{H} + 2H^2)\varphi + \frac{\partial V(\varphi)}{\partial \varphi} = 0 \ . \tag{2.7}$$

Finally, combining the time derivative of eq.(6) with eq.(7) yields the *momentum* equation

$$\dot{H}(\frac{1}{8\pi G} + \xi\varphi^2) = -\frac{1}{2}\dot{\varphi}^2 + \xi H\varphi\dot{\varphi} - \xi\dot{\varphi}^2 - \xi\varphi\ddot{\varphi} \ . \tag{2.8}$$

Let us now define

$$\psi \equiv 8\pi G\xi\varphi^2 = 8\pi\xi\frac{\varphi^2}{m_p^2} \ , \tag{2.9}$$

where $m_p = G^{-1/2}$ is the Planck mass, and change the independent variable from t to $\ln a$. For any function Q

$$\dot{Q} = Q'\frac{d\ln a}{dt} = HQ' \ , \tag{2.10}$$

$$\ddot{Q} = (H\dot{Q}') = H^2Q'' + HH'Q' \ , \tag{2.11}$$

where primes indicate derivatives with respect to $\ln a$.

The energy, momentum and field equations now become

$$1 + \frac{1}{\psi} = \frac{1}{24\xi}\frac{\psi'^2}{\psi^2} - \frac{\psi'}{\psi} + \frac{8\pi G}{3}\frac{V}{H^2\psi} \ , \tag{2.12}$$

$$2\frac{H'}{H}(1 + \frac{1}{\psi}) = -\frac{1}{4\xi}\frac{\psi'^2}{\psi^2} + \frac{\psi'}{\psi} - \frac{\psi''}{\psi} - \frac{H'}{H}\frac{\psi'}{\psi} \ , \tag{2.13}$$

$$12\xi(2 + \frac{H'}{H}) = \frac{\psi''}{\psi} + \frac{H'}{H}\frac{\psi'}{\psi} - \frac{1}{2}\frac{\psi'^2}{\psi^2} + 3\frac{\psi'}{\psi} + \frac{24\xi}{1 - \frac{v^2}{\varphi^2}}\frac{8\pi G}{3}\frac{V}{H^2\psi} \ . \tag{2.14}$$

We will concern ourselves with values of φ much larger then v. We can therefore use $v^2/\varphi^2 << 1$ to eliminate the potential term between eqs.(12) and (14). Combination with eq.(13) then serves to get rid of the ψ'' term. We thus obtain a simple formula for ψ, which clearly displays the singular nature of the conformal value of the coupling $\xi_c = -\frac{1}{6}$

$$\frac{1}{4\xi}(1+6\xi)\frac{\psi'^2}{\psi^2} - 4(1+6\xi)\frac{\psi'}{\psi} + 2(1+6\xi+\frac{1}{\psi})\frac{H'}{H} - \frac{24\xi}{\psi} = 0 \ , \tag{2.15}$$

for $\psi >> 8\pi\xi v^2/m_p^2$, (i.e. for $\varphi > v$).

It is worth noting that this formula gives us immediately the curious result that a conformally coupled scalar field behaves like a radiation dominated fluid in this cosmological context. For $\xi = -1/6$, eq.(13) yields

$$\frac{H'}{H} = -2 \ , \tag{2.16}$$

which implies that

$$H \propto \frac{1}{2t} \quad \text{and} \quad a(t) \propto t^{1/2} \ , \tag{2.17}$$

as advertised.

Chapter 3

Existence of a self-consistent inflationary solution

When $\varphi > m_p$ but $\psi < 1$, which may occur only if $\xi << 1$ (see eq.(9)), the model yields the same results, to first order, as in the minimally coupled case. In particular, the catastrophic λ constraint is unchanged. (One can use eq.(57) of chapter 5 to show that the constraint is $\lambda \sim 10^{-15}$ for a typical choice: $\delta\rho/\rho = 10^{-4}$, $C = 1$, $N_e = 70$ in eq.(57) which assumes $V(\varphi) = \lambda\varphi^4$.) However, a rather interesting situation develops in the domains where $\psi >> 1$. Note that this condition is easier to meet if ξ is not too small (see end of chapter 4 for an estimate of how small is small).

Let us first write the so-called slow-rolling approximation in terms of ψ. The usual inequalities

$$| \dot{H} |<< H^2 \ , \tag{3.18}$$

$$| \frac{\dot{\varphi}}{\varphi} |<< H \ , \tag{3.19}$$

$$\ddot{\varphi} << 3H\dot{\varphi} \ , \tag{3.20}$$

$$\frac{1}{2}\dot{\varphi}^2 << V \ , \tag{3.21}$$

now read, respectively

$$| \frac{H'}{H} |<< 1 \ , \tag{3.22}$$

$$| \frac{\psi'}{\psi} |<< 1 \ , \tag{3.23}$$

$$\psi'' << \psi' \ , \tag{3.24}$$

$$\frac{\psi'^2}{\psi^2} << \frac{24\xi}{\psi}\frac{8\pi G}{3}\frac{V}{H^2} \ . \tag{3.25}$$

Then, when $\psi >> 1$, one can easily see that eqs.(12) to (14) admit the solution

$$\psi' \approx -\frac{8\xi}{1+6\xi} \ . \tag{3.26}$$

A simple but important remark is that

$$0 < \frac{8\xi}{1+6\xi} < \frac{4}{3} \tag{3.27}$$

for all positive ξ's.

That this solution is self-consistent (i.e. obeys the assumptions 22-25) is easily verified. First, since eq.(13) gives here

$$\frac{H'}{H} \approx \frac{1}{2}\frac{\psi'}{\psi} \ , \tag{3.28}$$

we see, from (26),(27) and $\psi >> 1$, that (22) and (23) are immediately satisfied. On the other hand, the constancy of ψ' results in a $\psi'' = 0$; so (24) poses no problem. Finally, it is straightforward to compare the kinetic and potential terms, using the solution (26) together with $\psi >> 1$, and show that they are well within the limits implied by (25).

The above analytical results, as well as those of the following chapter, were confirmed numerically to a high accuracy (see figures 1 to 4).

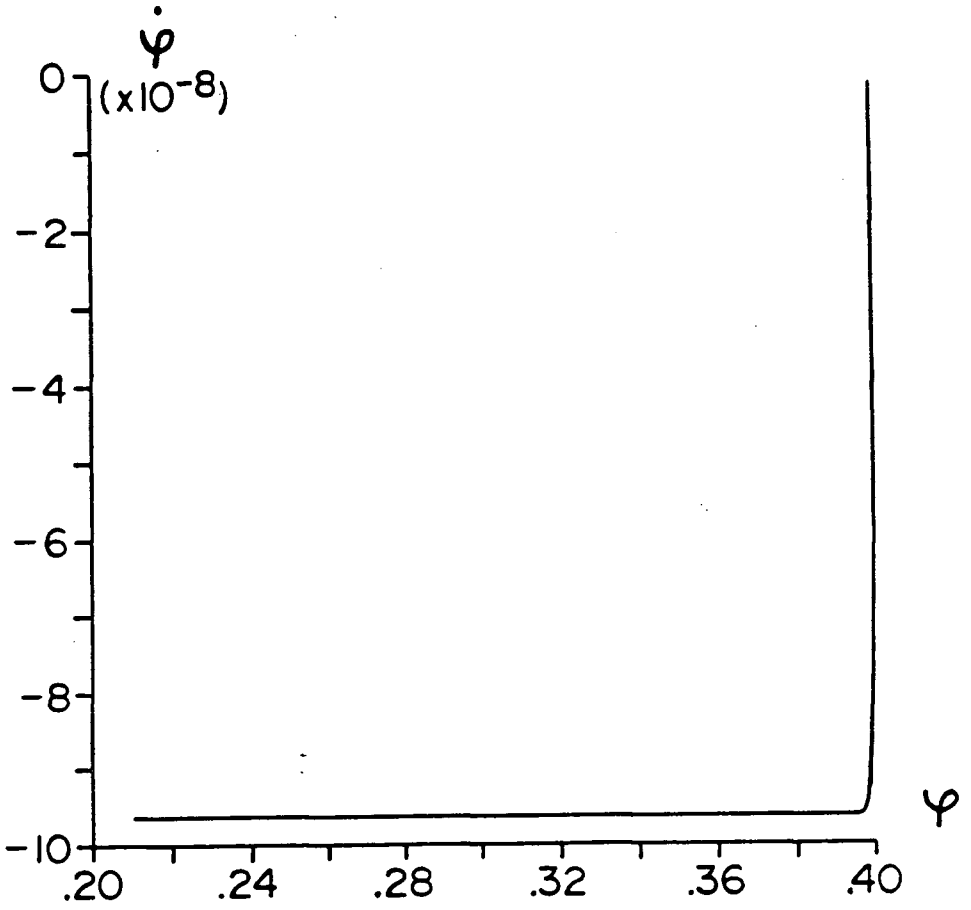

Figure 3.1: *Numerical solution of the exact equations of motion (6,7,8).* $8\pi G$ *is normalized to 1.* $\xi = 10^3$, $\lambda = 10^{-3}$, $v = 5 \times 10^{14} GeV$. $\dot{\varphi}$ *is started at zero. The constancy of the analytical solution (26) is confirmed. (One can see from fig.2 and eq.(48) that* $\psi = const. \times \dot{\varphi}$ *.)*

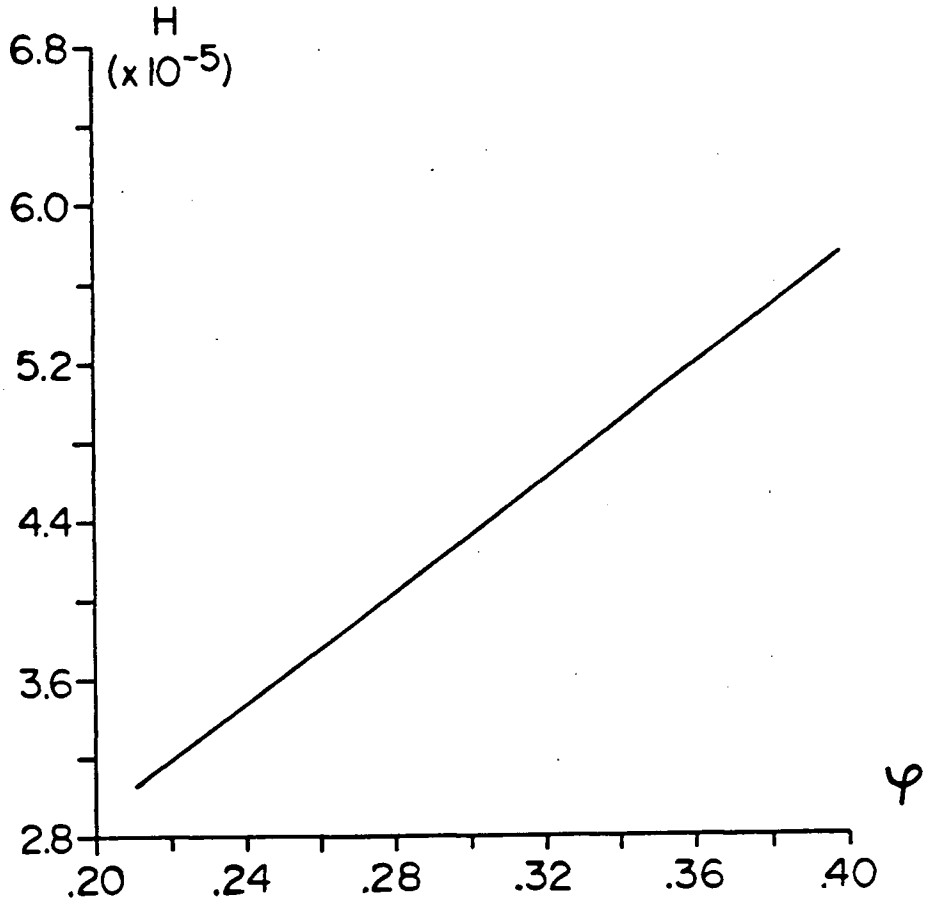

Figure 3.2: *The linearity in φ of the expansion rate H (see eq.(47)) is confirmed with a high accuracy. The parameters are those of fig.1 . Combining figures 1 and 2 shows that the slow-rolling conditions (22 to 25) are numerically well verified. Note that this monotonous variation of H does not induce any significant structure in the spectrum of density fluctuations (see ref.[30]).*

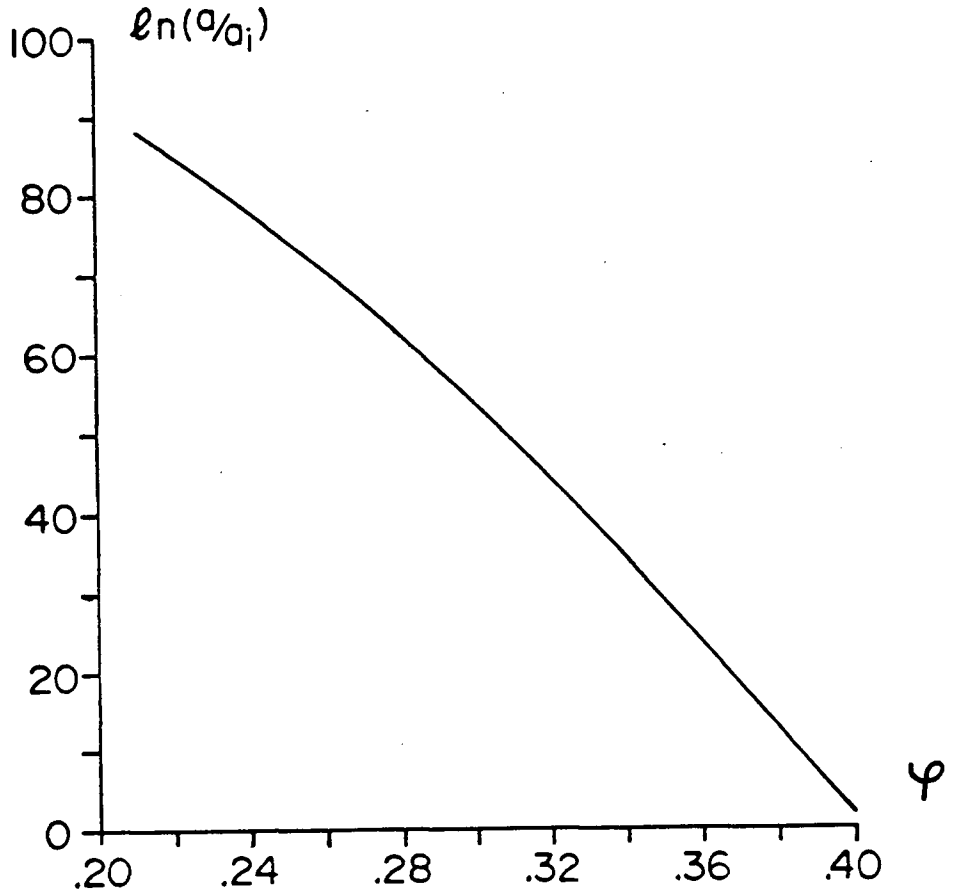

Figure 3.3: *The growth in the scale factor clearly exceeds* $N_e = 70$ *e-folds, even though the scalar field is initially just* $\varphi_{init} = 0.4$. *With our normalization, the value predicted by eq.(36) to be sufficient for obtaining* N_e *e-folds is* $\varphi_{init}^{min} \sim (N_e/\xi)^{1/2} \approx 0.26$ *(same parameters as figures 1 and 2). Eq.(36) is thus confirmed numerically. Note that* 0.26 *is only* 1% *of the normalized* $\xi = 0$ *value:* $\varphi_{init}^{min} \sim 25$. *We conclude that, in Linde's terminology,* $\xi > 1$ *allows many more mini-universes of the initial chaos to undergo enough inflation to resemble our observable Universe.*

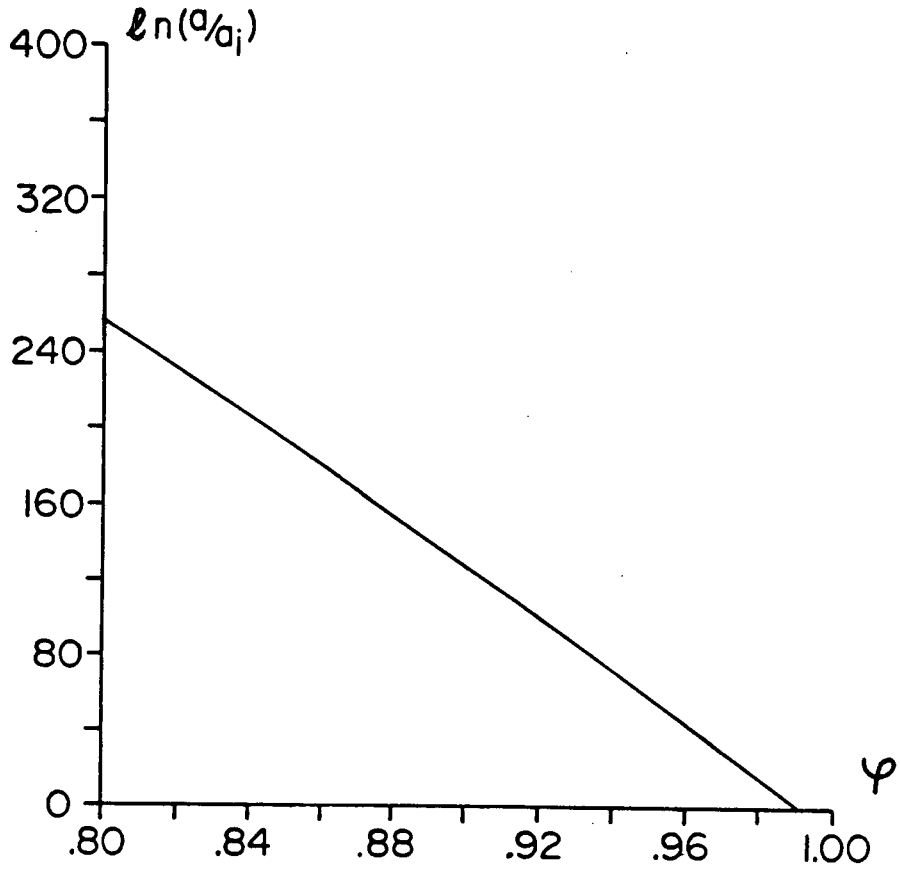

Figure 3.4: *See caption of figure 3. A growth in* a *of over* 250 *e-folds is obtained by taking* $\varphi_{init} = 1$ *, which is only* 4% *of* $\varphi_{init}^{min}|_{\xi=0} \sim 25$.

Chapter 4

Amount of inflation produced

As mentioned in chapter 1, the present spatial curvature of the Universe is exceedingly small ('flatness problem') and the causal horizon unexplainably large ('horizon problem') considering the age of the Universe. The inflationary model offers a possible solution to these puzzles[13], provided that the number of expansion e-folds is larger than $N_e \sim 60$ to 70 (the precise value of N_e depending on how the Universe eventually exits the quasi-de Sitter regime and reheats to high enough temperatures to allow the hot Big Bang scenario to proceed).

Since

$$a(t) \propto \exp(\int H(t)dt) \,, \tag{4.29}$$

we demand that

$$\int H(t)dt > N_e \ . \tag{4.30}$$

The analysis is particularly simple within the approach of the previous chapter. Thus

$$\int H(t)dt = \int \frac{H}{\dot{\psi}} d\psi = \int \frac{d\psi}{\psi'} = \frac{1+6\xi}{8\xi}(\psi_i - \psi_f) \ , \tag{4.31}$$

where i and f refer to the onset and offset of the quasi-de Sitter phase. The condition (30) now reads

$$\frac{1+6\xi}{8\xi}(\psi_i - \psi_f) > N_e \ . \tag{4.32}$$

One still has to verify that $\psi >> 1$ remains true until the end of the e-folds considered. This can be easily done as follows. It is clear from chapter 3 that once exponential expansion starts, it will continue at least until ψ is of order unity (see below). Hence

$$\int_{\psi_i}^{\psi_f} H dt \geq \int_{\psi_i}^{\psi=1} H dt = \frac{1+6\xi}{8\xi}(\psi_i - 1) \ . \tag{4.33}$$

Since

$$\frac{1+6\xi}{8\xi} > \frac{3}{4} \ . \tag{4.34}$$

inflation will be large enough in all domains where

$$\psi_i - 1 > \frac{4}{3} N_e \ , \tag{4.35}$$

and where φ is sufficiently homogeneous. Thus the condition is essentially

$$\psi_i > N_e \ . \tag{4.36}$$

One can indeed show that, if $\xi > 1$, the condition $\psi \sim 1$ effectively signals the end of inflation. If we try to find a self-consistent inflationary solution with $\psi < 1$, the energy eq.(12) yields

$$\frac{1}{\psi} \approx \frac{8\pi G}{3} \frac{V}{H^2 \psi} \ , \tag{4.37}$$

that is

$$H^2 \approx \frac{8\pi G}{3} V \ . \tag{4.38}$$

From the field equation (14)

$$24\xi \approx 3\frac{\psi'}{\psi} + 8\xi \frac{8\pi G V}{H^2 \psi} \ , \tag{4.39}$$

so that the combination of the two equations gives

$$\frac{\psi'}{\psi} \approx 8\xi(1 - \frac{1}{\psi}) \ , \tag{4.40}$$

which means here that

$$| \frac{\psi'}{\psi} | > 1 \ , \tag{4.41}$$

in contradiction with the slow-rolling assumptions. So, for values of ξ not much smaller than one, the rollover does speed up at about $\psi \sim 1$, bringing inflation to an end.

This result can be easily understood in the following way. The condition $\psi >> 1$ that is necessary for inflation is just $\varphi > \varphi_c \equiv m_p/\sqrt{8\pi\xi}$ (see eq.(9)). For large ξ 's

$$\frac{m_p}{\sqrt{8\pi\xi}} < \frac{m_p}{3} \ . \tag{4.42}$$

$m_p/3$ is the minimal value of φ for inflation to occur in the $\xi = 0$ chaotic model (see e.g. Linde[16]). Hence, as φ goes under φ_c and the effect of non-minimal coupling in

eq.(1) becomes negligible, one is left with a scalar field that is both effectively minimally coupled and smaller than $m_p/3$. Inflation, therefore, is no longer possible.

One can now have a better feeling for what values of ξ should be considered as large or small. When $\varphi_c > m_p/3$, that is $\xi < 1/3$, inflation eventually proceeds along the lines of the minimally coupled model, as φ rolls down through the region

$$\frac{m_p}{3} < \varphi < \frac{m_p}{\sqrt{8\pi\xi}} \ . \tag{4.43}$$

For $\xi > 1/3$, on the other hand, the non-minimal coupling dominates the dynamics throughout inflation. An important consequence is that, according to whether $\xi < 1/3$ or $\xi > 1/3$, the subsequent reheating will be minimally or non-minimally coupled to the curvature. It is expected that successful reheating, which is usually difficult to achieve in inflationary models due to the very weak couplings of the inflaton, will be considerably easier in the present model, where the couplings are much less constrained. We plan to address this question in more detail elsewhere.

Chapter 5

Density perturbations

This issue was addressed by a number of authors (see e.g. [31-38] and references in reviews[16-19]) who arrive at equivalent results. Inflation is thought to make quantum fluctuations of the scalar field evolve into an almost scale-free spectrum of density perturbations, which then serve as seeds for galaxy formation. For minimal coupling (and for $\xi << 1$ in ref.[38]) the amplitude of the fractional density perturbations on some scale is found to be

$$\frac{\delta\rho}{\rho} = C\,\frac{H^2}{\dot{\varphi}}\,, \tag{5.44}$$

where C is a constant and $\delta\rho/\rho$ is the amplitude when the scale crosses back into the Hubble radius after inflation. The RHS is evaluated at the time when the scale crosses outside the Hubble radius during the de Sitter era.

We argue in this chapter that $\delta\rho/\rho$ is modified in two ways by $\xi > 1$, which turned out in chapters-3,4 to be the most interesting case:

1) $H^2/\dot{\varphi}$ is proportional to $\sqrt{\lambda/\xi}$ rather than to $\sqrt{\lambda}$,

2) the RHS of eq.(44) should be divided by a factor $\sqrt{1+6\xi}$.

We begin by proving the first statement above and deriving the resulting modifications in $\delta\rho/\rho$ and $\lambda_{constraint}$ (eqs.(59,60)). Then we show that the second statement is probably the right correction to eq.(44) when $\xi > 1$. Eqs.(83,84) thus obtained will be our final formulae for comparing the minimally coupled $\xi = 0$ and our $\xi > 1$ models.

Going back to eq.(44), it is clear that the precise value of C is important in determining how stringent the constraints from that equation are. In effect

$$\frac{H^2}{\dot{\varphi}} \propto \sqrt{\lambda/\xi} \ , \tag{5.45}$$

as we shall see, and thus (see eq.(44))

$$\lambda_{constraint} \propto C^{-2} \ . \tag{5.46}$$

A variety of values of C are adopted in the literature. For example, the results of ref.[27] were obtained using $C = 1$. Furthermore, C may vary substantially according to whether the scale of interest re-entered the Hubble radius when the Universe was radiation or matter dominated: in ref.[19], for example, C takes the values $1/\pi^{3/2}$ and $1/10\pi^{3/2}$ for these two eras respectively. In ref.[38], Luccin et al. obtain $C \approx 1/50$.

We emphasize that our results *do not assume any particular choice of* C - we are now going to show that whatever value of C is eventually adopted, the non-minimal coupling constraint on λ can be made weaker than the usual $\xi = 0$ constraint by several orders of magnitude.

First, we notice from chapter 3 that when inflation is under way H is simply proportional to φ

$$H \approx (\lambda/3\xi)^{1/2} \varphi \ . \tag{5.47}$$

Then, using (see eqs.(9,10))

$$\frac{2\dot{\varphi}}{H\varphi} = \frac{\psi'}{\psi} \ , \tag{5.48}$$

we can write

$$\frac{H^2}{\dot{\varphi}} = 2(\frac{\lambda}{3\xi})^{1/2}\frac{\psi(N_e)}{\psi'} \quad , \tag{5.49}$$

where $\psi(N_e)$ is the value of ψ at the beginning of the last N_e e-folds of expansion, which is the time where the scales of astronomical relevance cross outside the Hubble radius.

From the previous chapter we see that

$$\psi(N_e) = \frac{8\xi}{1+6\xi}N_e + 1 \sim N_e \quad , \tag{5.50}$$

since we are interested in $\xi > 1$. Hence, at first Hubble radius crossing we have

$$\frac{H^2}{\dot{\varphi}} \approx 2N_e\sqrt{\frac{\lambda}{3\xi}} \quad . \tag{5.51}$$

The results for minimal coupling can be easily recovered by setting $\xi = 0$ in eqs.(6,7,8). One obtains

$$\frac{H^2}{\dot{\varphi}} \approx 2(2\pi)^{3/2}\lambda^{1/2}\frac{\varphi^3}{m_p^3} \quad . \tag{5.52}$$

$\varphi(N_e)$, the value of φ at N_e e-folds before the end of inflation can be calculated from

$$\int_{\varphi(N_e)}^{\varphi_f} H\, dt = N_e \quad , \tag{5.53}$$

where φ_f is the value at the end of inflation which corresponds to (see e.g. ref.[19])

$$\frac{\partial^2 V}{\partial \varphi^2} \approx 9\, H^2 \ . \tag{5.54}$$

This implies that for minimal coupling one has

$$\varphi_f \approx \frac{m_p}{\sqrt{2\pi}} \ . \tag{5.55}$$

From eqs.(53,55) we find the $\xi = 0$ result

$$\varphi(N_e) \approx \sqrt{\frac{2N_e + 1}{2\pi}}\, m_p \approx \sqrt{\frac{N_e}{\pi}}\, m_p \ . \tag{5.56}$$

Hence the density perturbations cross back into the Hubble radius with amplitude

$$\frac{\delta\rho}{\rho}(\xi = 0) \ = \ 4\, C \sqrt{\frac{2}{3} N_e^3\, \lambda} \ . \tag{5.57}$$

Comparing this with the non-minimal coupling result (eq.(51))

$$\frac{\delta\rho}{\rho}(\xi > 1) \ = \ 2\, C\, N_e \sqrt{\frac{\lambda}{3\xi}} \ , \tag{5.58}$$

we derive the C independent formula

$$\frac{\delta\rho}{\rho}(\xi > 1) \ \approx \ \frac{1}{\sqrt{8\xi N_e}} \ \frac{\delta\rho}{\rho}(\xi = 0) \ . \tag{5.59}$$

The factor $8\, N_e$ alone exceeds 500, and ξ can be set to several powers of ten. We

conclude that *the amplitude of density perturbations from chaotic inflation, for a given self-coupling, may become much smaller when the assumption of minimal coupling is removed.* This effect will be made even stronger by eq.(83), which will replace eq.(59).

This very welcome feature (see introduction) also implies a modification, in the desirable direction, of the constraint on λ that eq.(44) usually generates. From eqs.(57,58)

$$\lambda^{(\xi>1)}_{constraint} \approx 8\, N_e\, \xi\ \lambda^{(\xi=0)}_{constraint}\ , \quad \forall C\ . \tag{5.60}$$

This equation is also independent of the conventional factor that multiplies λ in the potential ($1/8$ in our convention).

Note that our formulae are valid for all ξ 's such that

$$\frac{m_p^2}{8\pi\xi} > v^2\ . \tag{5.61}$$

For example, if v is the SU(5) mass scale $v \sim 5\times 10^{14}\ GeV$ then $(m_p^2/8\pi v^2) \sim 2\times 10^7$ and the λ constraint could be relaxed by up to ten orders of magnitude. The catastrophic $\lambda_{constraint} = 10^{-12}$ of ref.[16], e.g., keeping all conventions and parameters (except for ξ) unchanged, could thus be brought as close as desired to the ordinary GUT's range: $\lambda \sim 10^{-2}$. Once we include the correction '2)' to eq.(44) (see the beginning of this chapter), such a large value of ξ ($\sim 2\times 10^7$) will no longer be needed to achieve this improvement. A value of ξ of only 10^3 in eq.(84) (which will replace eq.(60)) yields over nine orders of magnitude in the relaxation of the λ constraint. Note finally that in a slightly simpler $V(\varphi) = \lambda\varphi^4$ theory ($v = 0$), our equations hold for ξ as large as desired. On the other hand, it is shown at the end of this chapter that the model can be easily made to work even if v is very large (i.e. (61) not satisfied).

Turning now to what we called modification '2)' at the beginning of the present chapter, we first address a concern that one may have about the large negative mass

$$m^2_{NMC} = -12\xi\, H^2 \quad , \tag{5.62}$$

apparently introduced by non-minimal coupling in the equation for the scalar field modes. It might be naively presumed to cause a fatal exponential growth in the fluctuations. It turns out, however, that this concern is unfounded. Consider the equation for small fluctuations (see e.g. ref.[30])

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \left(\frac{k^2}{a^2} + \frac{\partial^2 V}{\partial\varphi^2} - 12\xi H^2 \right) \delta\varphi - \xi\varphi\,\delta R + \frac{1}{2}\dot{\varphi}\dot{h} = 0 \, , \tag{5.63}$$

where δR is the perturbation in the curvature and h the metric perturbation in the synchronous gauge. In our model the $\partial^2 V/\partial\varphi^2$ term cancels the destablizing contributions of m^2_{NMC} . We will not examine these full coupled perturbation equations here, but note that the solutions for the long wavelength modes are closely related to the homogeneous mode solution, which was shown in chapter 3 to be stable (the inflaton is just a $k = 0$ mode).

A legitimate question, nevertheless, is whether eq.(44), which was derived assuming minimal coupling, still holds for the large non-minimal coupling cases we consider. The long wavelength density fluctuations arise from the quantum zero-point fluctuations in the field at very high frequencies due to the expansion of the Universe. It is not clear whether it is the fluctuations in the strongly coupled field φ rather than in some conformal field, for example, that should take the 'vacuum' form. It might be the case if ξ was zero at very early times and was 'switched on' due perhaps to renormalization

effects at later times (but before the inflationary era). However, if strong non-minimal coupling *was* present throughout the generation and evolution of the high frequency field fluctuations, then eqs.(44,59,60) will most likely have to be modified.

A convenient way to investigate this issue is to perform the conformal transformation

$$\tilde{g}_{\alpha\beta} = \Omega g_{\alpha\beta} \ , \ \tilde{g}^{\alpha\beta} = \Omega^{-1} g^{\alpha\beta} \ , \ \tilde{g} = \Omega^4 g \ , \tag{5.64}$$

where

$$\Omega \equiv 1 + 8\pi G \xi \varphi^2 \ . \tag{5.65}$$

The line element of the conformal space-time is

$$d\tilde{s}^2 = \Omega(dt^2 - a^2(t) dx^2). \tag{5.66}$$

This can be brought to the same form as eq.(3) by the following change of variables

$$t' = \int \Omega^{1/2} dt \quad \text{and} \quad a' = \Omega^{1/2} a \ . \tag{5.67}$$

Then

$$d\tilde{s}^2 = dt'^2 - a'^2(t') dx'^2 \ , \tag{5.68}$$

and the conformal metric in the primed coordinate system reads

$$\tilde{g}_{\alpha\beta}(x') = diag(1, -a'^2, -a'^2, -a'^2) \ . \tag{5.69}$$

The transformed scalar curvature is

$$\tilde{R} = \frac{1}{\Omega}\left\{ R + \frac{3}{2}\,\tilde{g}^{\alpha\beta}\,\frac{\Omega_{,\alpha}\Omega_{,\beta}}{\Omega} - 3\tilde{g}^{\alpha\beta}\Omega_{,\alpha,\beta} \right\} . \tag{5.70}$$

It is straightforward to show that the action in the conformal picture and in the primed coordinate system can then be written as

$$S = \int d^4x'\sqrt{-\tilde{g}}\,\left\{ \frac{\tilde{R}}{16\pi G} + \frac{3}{16\pi G}\,\frac{\Omega_{,\alpha}\Omega_{,\beta}}{2\Omega^2}\left(1 + \frac{1}{6\xi}\frac{\Omega}{\Omega - 1}\right)\tilde{g}^{\alpha\beta} - \frac{V(\Omega)}{\Omega^2} \right\} , \tag{5.71}$$

where we have used

$$\tilde{g}^{00}(x) = \tilde{g}^{00}(x')/\Omega , \tag{5.72}$$

$$\frac{d}{dt} = \Omega^{1/2}\frac{d}{dt'} , \tag{5.73}$$

$$d^4x = \Omega^{-1/2}d^4x' , \tag{5.74}$$

$$\tilde{g}(x) = \Omega^{1/2}g(x') . \tag{5.75}$$

Finally, to put the kinetic term in the canonical form, we define the field $\tilde{\varphi}$ by

$$(d\tilde{\varphi})^2 \equiv \frac{3}{16\pi G}\left\{ 1 + \frac{1}{6\xi}\frac{\Omega}{\Omega - 1} \right\}\left(\frac{d\Omega}{\Omega}\right)^2 . \tag{5.76}$$

In the limits of negligible ($\xi \approx 0$ or $\varphi < \varphi_c \equiv m_p/\sqrt{8\pi G}$) and dominant ($\xi >> 1$ or

$\varphi > \varphi_c$) non-minimal coupling we have respectively

$$\tilde{\varphi} \approx \varphi \quad \text{for} \quad \varphi < \varphi_c \ , \tag{5.77}$$

and

$$\tilde{\varphi} \approx \sqrt{\frac{3}{16\pi G}\left(1 + \frac{1}{6\xi}\right)} \ \ln \Omega \quad \text{for} \quad \varphi > \varphi_c \ . \tag{5.78}$$

We can now write the theory in terms of these new variables as a minimally coupled theory with the action

$$S = \int d^4x' \sqrt{-\tilde{g}} \left\{ \frac{\tilde{R}}{16\pi G} + \frac{1}{2}\, \tilde{\varphi}_{,\alpha} \tilde{\varphi}_{,\beta}\, \tilde{g}^{\alpha\beta} - \tilde{V}(\tilde{\varphi}) \right\} \ , \tag{5.79}$$

where

$$\tilde{V}(\tilde{\varphi}) \equiv \frac{V(\varphi)}{\Omega^2(\varphi)} \ , \tag{5.80}$$

and with the line element (68) (derivatives are taken with respect to the primed coordinates).

It should also be noted that any additional fields that would be present in a more complete theory would be affected by this conformal transformation, and could affect our conclusions. This issue will not be investigated here.

If we now adopt all the results of the minimally coupled theory, including the formula for the field fluctuations

$$\delta\tilde{\varphi} = \frac{\tilde{H}}{4\pi^{3/2}} \ , \tag{5.81}$$

where $\tilde{H}a' \equiv da'/dt'$, we obtain (using eqs.(67,78))

$$\frac{\delta\rho}{\rho} = C\,\frac{\tilde{H}^2}{d\tilde{\varphi}/dt'} = \frac{C}{(1+6\xi)^{1/2}}\,\frac{H^2}{d\varphi/dt} \ , \tag{5.82}$$

rather than eq.(44). Note that under the inflationary condition (19) we have $a'\tilde{H} \approx aH$ and that Hubble radius crossing corresponds to $aH = k$.

Thus, the improvement on the $\xi = 0$ models is likely to be even more substantial (for large ξ 's) than inferred from eqs.(59,60). The above conformal picture approach suggests that if the non-minimal coupling between the scalar field and the curvature is treated consistently starting from the quantum regime, the correct formulae are

$$\frac{\delta\rho}{\rho}(\xi > 1) \ \approx \ \frac{1}{\sqrt{8N_e\xi(1+6\xi)}}\ \frac{\delta\rho}{\rho}(\xi = 0) \ \approx \ \frac{1}{4\xi\sqrt{3N_e}}\ \frac{\delta\rho}{\rho}(\xi = 0) \ \ , \forall C \ , \tag{5.83}$$

and

$$\lambda^{(\xi>1)}_{constraint} \ \approx \ 8N_e\xi(1+6\xi)\ \lambda^{(\xi=0)}_{constraint} \ \approx \ 48N_e\xi^2\ \lambda^{(\xi=0)}_{constraint} \ , \ \ \forall C \ . \tag{5.84}$$

As an example, the choice $\xi = 10^3$ of figures 1 to 8, with the same choice of parameters and conventions that leads to $\lambda < 10^{-12}$ in ref.[16], would yield here the dramatically weaker constraint $\lambda < 10^{-3}$, rather than $\lambda < 10^{-7}$ which one would obtain from eq.(60).

Finally, we have also numerically verified that even if v is much larger than the SU(5) mass scale $v \approx 5 \times 10^{-15}\ GeV$, the results are not significantly altered. In fact, it is easy to show algebraically that all the attractive features of the model can be preserved

by moderately increasing the initial value of φ . When v is not negligible compared to φ_c , the solution (26) becomes

$$\dot{\varphi} = -\frac{4}{1+6\xi}\sqrt{\frac{\lambda\xi}{3}}\left(\frac{m_p^2}{8\pi\xi}+v^2\right) , \tag{5.85}$$

while the form of H (eq.(47)) is virtually unchanged. Because of this increase in $|\dot{\varphi}|$ the field has to be started at a larger φ_{init} to satisfy

$$\int H dt = \int_{\varphi_c}^{\varphi_{init}} \frac{H}{|\dot{\varphi}|} d\varphi > N_e . \tag{5.86}$$

However, the effect of this variation in $\dot{\varphi}$ cancels out in eq.(49) and hence the situation regarding density perturbations is unchanged. Figures 5,6,7,8 show that even in the extreme case where v^2 dominates over $m_p^2/8\pi\xi$ in eq.(85), the model is still well behaved and the $N_e \sim 70$ e-folds can be produced by merely increasing φ_{init} to about $m_p/5$. This is still 25 times smaller than the required initial value of φ when $\xi = 0$, namely $\varphi_{init}^{min} \approx 5\ m_p$. For example, in fig.8 we obtain about 150 e-folds by setting $\varphi_{init} \approx m_p/3$.

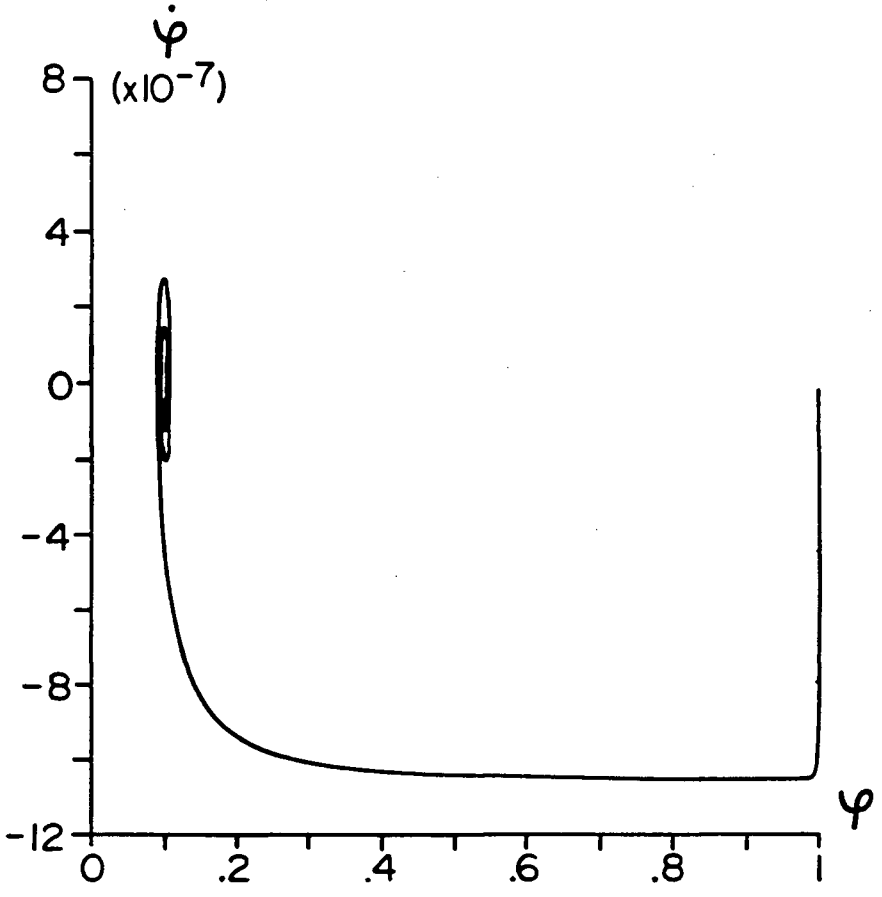

Figure 5.5: *The mass scale* v *is chosen to be large enough to dominate the solution (85):* $v^2 = 10\ m_p^2/8\pi\xi$. *The other parameters are those of fig.1 . During most of the inflationary phase,* φ *remains very close to its initial value. Thus, despite the apparent departure from fig.1, this figure does confirm that* $\dot{\varphi}$ *remains constant over most of the e-folds. (Here the coordinate* φ *is not a good measure of time.)*

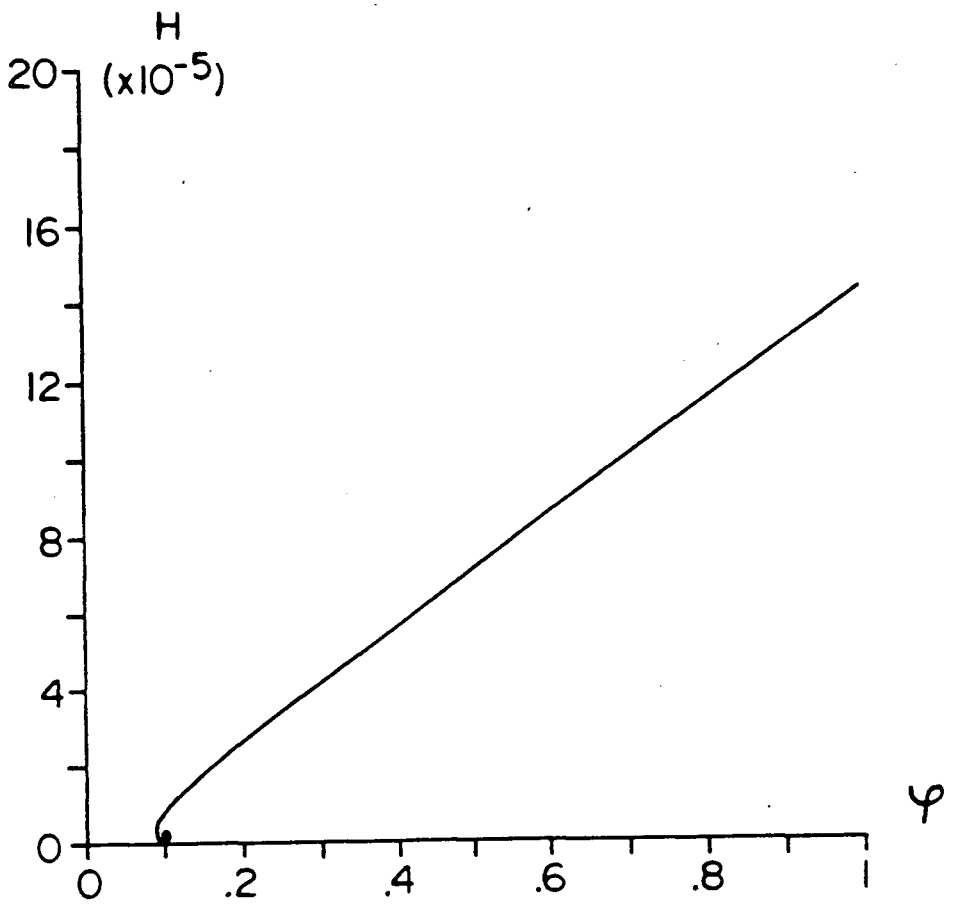

Figure 5.6: *The linearity of H is unaffected by the large increase in the SSB mass scale v (see parameters on figs.1,5). It is indeed easy to see from the inflationary equation $H^2 \approx V(\varphi)/3\xi\varphi^2$ and eq.(2) that a variation in v merely displaces vertically the straight line $H(\varphi)$.*

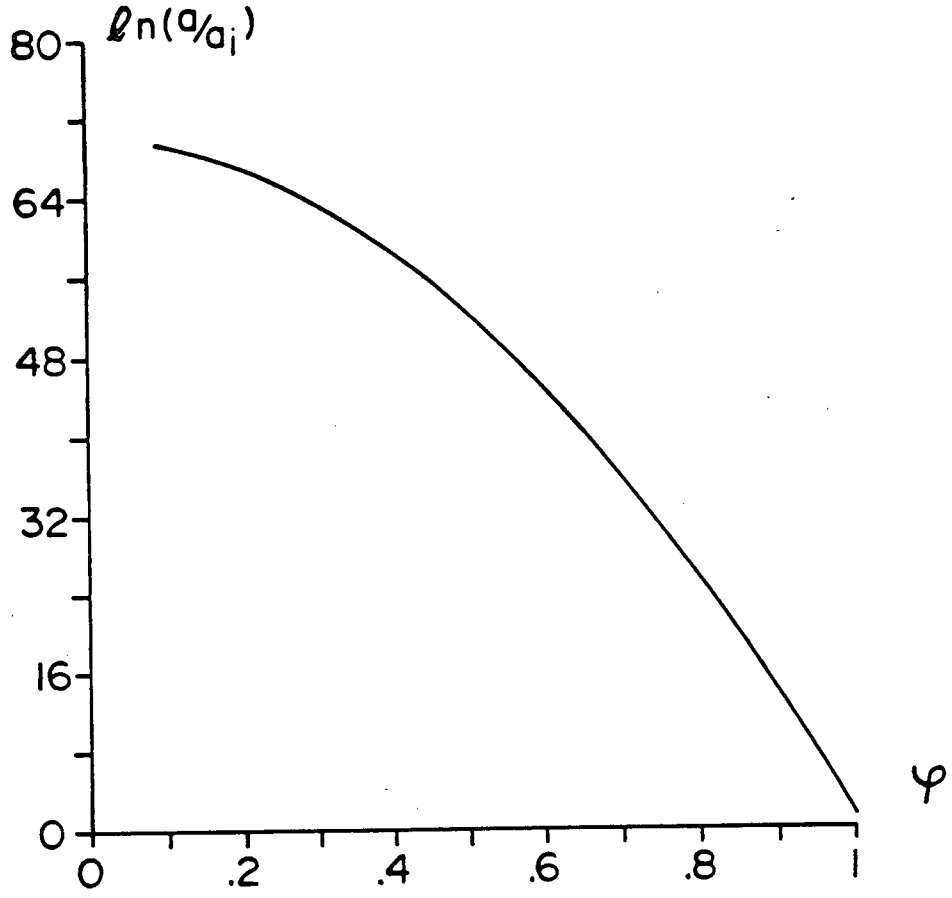

Figure 5.7: *With the parameters of figs.5,6 , the* $N \sim 70$ *e-folds can be easily obtained by setting* φ_{init} *to about* $m_p/5$ *, which is* ≈ 1 *with the normalization* $8\pi G = 1$.

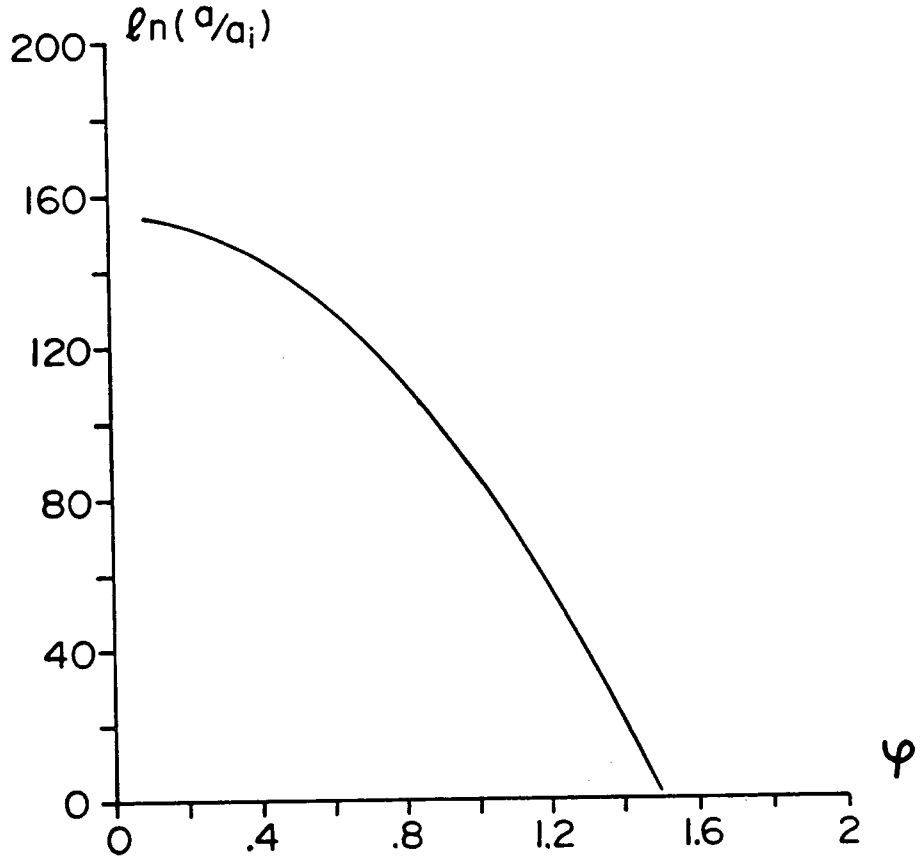

Figure 5.8: *A slight increase of the normalized* φ_{init} *from* 1 *(fig.7) to* 1.5 *yields a total growth in the scale factor of about* 150 *e-folds.*

Part II

Application to Induced Gravity: Ordinary versus Chaotic Inflation

Chapter 6

Introduction to induced gravity inflation

It has long been hoped that quantum field theory would shed some light on the origin of the constants of classical physics. In this spirit, a toy model has been proposed[39,29,40] where the gravitational constant results from a scalar field spontaneous symmetry breaking. In this model of 'induced gravity' (see also the earlier paper of Sakharov[41]), a $\xi R\varphi^2/2$ term (ξ is a dimensionless coupling, φ the scalar field and R the scalar curvature) replaces the Einstein-Hilbert component $R/16\pi G$ of the full Lagrangian. The field evolves towards the true minimum of the potential where it finally reaches the SSB equilibrium value $\varphi = v$ satisfying

$$\xi v^2 = 1/8\pi G \ . \tag{6.87}$$

On another front, the proposal of 'cosmological inflation'[4-13] has for a number of years now sustained a considerable interest in primordial phase transitions. Assuming a certain number of suitable initial conditions, these transitions are thought to drive the early universe into a period of quasi-exponential expansion which may be responsible for some the puzzling large-scale features of the observed universe such as spatial flatness, homogeneity and dilution of eventual topological defects predicted by Grand Unified Theories.

There is a growing feeling among many physicists that inflation is realizable within a wide variety of field theories (see reviews[16-19]). It is naturally tempting therefore

to investigate whether the induced gravity phase transition could also lead to a quasi-de Sitter era. A few authors have by now studied this possibility[42,27,38,2]. It was argued in ref.[27] that induced gravity inflation suffers from a problem common to most inflationary models: the inflaton couplings must be fine tuned to excessively small values. However, we have just described in Part I a 'chaotic'[20] inflationary model where these constraints can be considerably relaxed[1]. The non-minimally coupled scalar field model of Part I closely resembles induced gravity in a special case, that of *chaotic* inflation with a *large* coupling ξ . This led us to think that despite the pessimistic conclusions of ref.[27] the relaxation of the constraints in Part I may also apply to induced gravity[2]. It is worth noting nevertheless that the chaotic inflation versions of induced gravity and non-minimally coupled Einsteins' gravity models behave differently at the approach of reheating if the curvature coupling is only moderately large. Moreover, the ordinary inflation versions of the two models are completely different.

In this part of our thesis we compare ordinary[14,15] and chaotic[20] inflation in Zee's induced gravity theory (eq.(88)). We show that the stringent constraints of ref.[27] are indeed necessary for ordinary inflation but not for chaotic inflation. Thus, this latter turns out to be a much more appealing model. Our disagreement with ref.[27] is most clearly displayed in eqs.(107 to 110).

Finally, we have recently realized that prior to ref.[27] Spokoiny[42] had proposed a model of chaotic induced gravity inflation based on the Coleman-Weinberg potential, where the desirable smallness of the density perturbations' amplitude was obtained through field theory arguments rather than by constraining the couplings. Despite differences in the choice of the potential and in the general approach, many points of ref.[42] are independently confirmed on the chaotic inflation side of Part II.

Chapter 7

Self-consistent inflationary solutions

One can obtain the analog of Einstein's equations for this theory by varying the action of the induced gravity Lagrangian

$$L_{IG} = \xi\varphi^2 R/2 + \partial_\alpha\varphi\partial^\alpha\varphi/2 - \lambda(\varphi^2 - v^2)^2 \tag{7.88}$$

with respect to the gravitational degrees of freedom. Let us normalize the Newtonian constant G to $1/8\pi$ so that eq.(87) reads simply

$$v^2 = \frac{1}{\xi} \ . \tag{7.89}$$

The time-time Einstein equation reads (see also ref.[27])

$$H^2 = \frac{1}{6\xi}\frac{\dot{\varphi}^2}{\varphi^2} + \frac{V(\varphi)}{3\xi\varphi^2} - 2H\frac{\dot{\varphi}}{\varphi} \ . \tag{7.90}$$

Varying the action with respect to the matter yields (eq.(7))

$$\ddot{\varphi} + 3H\dot{\varphi} - 6\xi(\dot{H} + 2H^2)\varphi = -\frac{\partial V(\varphi)}{\partial\varphi} \ , \tag{7.91}$$

which can be combined with the 'energy' equation (90) to obtain the 'momentum' equation

$$\dot{H} = -\frac{1}{2}\left(\frac{1}{\xi}+2\right)\frac{\dot{\varphi}^2}{\varphi^2} + H\frac{\dot{\varphi}}{\varphi} - \frac{\ddot{\varphi}}{\varphi} \ . \tag{7.92}$$

To insure that the scale factor evolves much faster than the inflaton φ we adopt the 'slow-rolling' approximation

$$|\frac{\ddot{\varphi}}{\dot{\varphi}}| << H \ , \ |\frac{\dot{\varphi}}{\varphi}| << H \ , \ |\dot{H}| << H^2 \ , \ \dot{\varphi}^2 << V(\varphi) \ . \tag{7.93}$$

It is then straightforward to show that in this limit eqs.(90,91) have the approximate analytical solution

$$H \approx (\frac{\lambda}{3\xi})^{1/2}\varphi|1 - \frac{v^2}{\varphi^2}| \ , \tag{7.94}$$

$$|\dot{\varphi}| \approx \frac{4}{1+6\xi}\left(\frac{\lambda}{3\xi}\right)^{1/2} \ . \tag{7.95}$$

Only in the *small* ξ limit does eq.(95) reduce to eq.(7) of ref.[27] which reads

$$|\dot{\varphi}| = 4\left(\frac{\lambda\xi}{3}\right)^{1/2} v^2 \ . \tag{7.96}$$

$\dot{\varphi}$ is positive for the 'ordinary' scenario, where φ starts off near zero, and negative for the 'chaotic' model, where the initial value of φ is larger than v .

Upon checking self-consistency one discovers that the *ordinary* model cannot inflate at all unless $\xi << 1$. In contrast, the *chaotic* inflationary solution is self-consistent for all positive values of ξ . One can see how this comes about by writing the energy and field equations for the slow-rollover:

$$H^2 \approx \frac{V(\varphi)}{3\xi\varphi^2} \quad , \tag{7.97}$$

$$3H\dot{\varphi} - 12\xi H^2\varphi \approx -\frac{4V(\varphi)}{\varphi}\frac{1}{1-(v/\varphi)^2} \quad . \tag{7.98}$$

Denoting by φ_i the initial value of φ we distinguish the following two cases:

1) for $\varphi_i << v$ (*ordinary* inflation), the field equation (98) becomes

$$H\frac{\dot{\varphi}}{\varphi} \approx 4\xi(1+(\varphi/v)^2)H^2 \quad , \tag{7.99}$$

which combined with eq.(97) gives

$$\frac{\dot{\varphi}}{\varphi} \approx 4\xi H \quad , \tag{7.100}$$

which is consistent with (93) only if $\xi << 1/4$;

2) for $\varphi_i >> v$ (*chaotic* inflation)

$$\frac{1}{1-(v/\varphi)^2} \approx 1+(v/\varphi)^2 \quad , \tag{7.101}$$

and eqs.(97,98) yield

$$\frac{\dot{\varphi}}{\varphi} \approx -\frac{4}{\varphi^2} H \ . \tag{7.102}$$

This is consistent with (93) in all domains where $\varphi > 2$ *independently of* ξ .

Note also that to meet the $|\dot{H}| << H^2$ condition one must have $|\dot{\varphi}/\varphi| << \sqrt{\xi} H$ (see eq.(92)). Hence the initial conditions for *ordinary* inflation (which requires $\xi << 1$) need an even finer tuning than the usual $|\dot{\varphi}/\varphi| << H$.

As it will become clear in the next chapter, the difference in behaviour of the ordinary and chaotic versions of induced gravity inflation is related to the difference in φ dependence of the expansion rate (see eqs.(104,105) below).

Chapter 8

The causality and flatness Problems

In order for the model to solve cosmological problems such as the flatness and homogeneity puzzles, the scale factor $a(t)$ has to undergo a number N of e-folds that depends on details of the reheating process, but that is typically of the order of $N = 70$ (see chapter 4).

One can write the requirement for obtaining enough inflation as

$$\int_{\varphi_i}^{\varphi_f} H dt \approx \int_{\varphi_i}^{v} \frac{H}{\dot{\varphi}} d\varphi > N_e \ , \tag{8.103}$$

where the subscripts i and f indicate the onset and offset of inflation.

1) For $\varphi_i << v$ the expansion rate takes the form

$$H \approx (\frac{\lambda}{3\xi})^{1/2}(\frac{1}{\xi\varphi} - \varphi) \approx (\frac{\lambda}{3\xi})^{1/2}\frac{1}{\xi\varphi} \ . \tag{8.104}$$

Hence, using (95,104,103) one obtains the constraint

$$\frac{1+6\xi}{4\xi}(\ln\frac{v}{\varphi_i} - \frac{1}{2}) > N_e \ . \tag{8.105}$$

Except for the $(1+6\xi)$ term, this is the same result as eq.(11b) of ref.[27]. The condition (105) will be satisfied if $\xi << 1$, which we showed in chapter 7 to be required

by self-consistency in the first place.

2) For $\varphi_i >> v$ eq.(94) yields

$$H \approx (\frac{\lambda}{3\xi})^{1/2}\varphi(1 - \frac{1}{\xi\varphi^2}) \approx (\frac{\lambda}{3\xi})^{1/2}\varphi \ . \tag{8.106}$$

The $(1 + 6\xi)$ factor now *has* to be included in the calculations since ξ can be large. The condition (103) then writes

$$\frac{1+6\xi}{8}(\varphi_i^2 - v^2(1 - 2\ln\frac{v}{\varphi}) \approx \frac{1+6\xi}{8}\varphi^2 > N_e \ , \tag{8.107}$$

or equivalently

$$\frac{1+6\xi}{8\xi}\frac{\varphi^2}{v^2} > N_e \ . \tag{8.108}$$

Since $\{1 + 6\xi\}/8\xi > 3/4$ for all positive values of ξ , (108) means that the chaotic model produces enough inflation *automatically* and *independently of* ξ . Note also that (107) is easier to satisfy when $v^2 = 1/\xi$ is small, that is when ξ is large.

The authors of ref.[27] derive the constraint (eq.(13b) therein)

$$\frac{1}{8\xi}\{\frac{\varphi_i^2}{v^2} - 1 - 2\ln(\varphi_i/v)\} > N_e \ , \tag{8.109}$$

and then claim that $\xi << 1$ insures enough inflation. However, using eq.(89) one can rewrite (109) as

$$\varphi_i^2 > 8N_e + \frac{1}{\xi}\left(1 + 2\ln\frac{\varphi}{v}\right) \ . \tag{8.110}$$

Not only does this condition not imply that ξ should be small, but it is more easily satisfied for larger values of ξ . The $(1 + 6\xi)$ term should then be included in (110), which would yield just the constraint (107).

Chapter 9

Density perturbations' constraints

As mentioned in chapter 5, the existence of a de-Sitter stage in the early Universe was shown by several authors to produce an almost scale-free spectrum of density perturbations. Unfortunately, all inflationary models with simple field theories seem to yield unacceptibaly large fluctuation amplitudes, unless the inflaton couplings are tuned to extremely small values. We shall now compare, in this context, the ordinary and chaotic versions of induced gravity inflation.

1) *Ordinary inflation*

We have already seen that this model can work only if $\xi << 1$. More precisely, one can infer from (105) that ξ should be smaller than $1/4N_e$. It is easy to show (see ref.[27]) that the scalar field at N_e e-folds before the end of inflation is then given by

$$\varphi_{|N_e} \approx 1 - \sqrt{4N_e\xi} \quad \text{for } \xi << \frac{1}{4N_e} \ . \tag{9.111}$$

Using eqs.(94,95,111) we find

$$\frac{H^2}{\dot{\varphi}}|_{N_e} \approx 4N_e\sqrt{\frac{\lambda}{3\xi}} \ . \tag{9.112}$$

The self-coupling constraint resulting from eq.(44) (chapter 5) thus reads

$$\lambda_{constraint} \approx \frac{3/16}{C^2N_e^2}\,\xi\left(\frac{\delta\rho}{\rho}\right)^2 \quad \text{for } \varphi_i < v \text{ and } \xi << \frac{1}{4N_e} \ . \tag{9.113}$$

If we take for example

$$C = 1 \ , \quad N_e = 70 \quad \text{and} \quad \frac{\delta\rho}{\rho} = 10^{-4} \tag{9.114}$$

then (113) implies that

$$\lambda_{constraint} \approx 4 \times 10^{-13}\, \xi \ , \quad \text{with} \quad \xi << 4 \times 10^{-3} \ . \tag{9.115}$$

This represents an excessively stringent constraint on λ .

2) *Chaotic inflation*

In the limit of very small values of ξ , this model is similar (regarding perturbations) to its ordinary inflation counterpart we have just investigated (ref.[27]). In particular, the unattractive fact that the smallness of ξ imposes a severe constraint on λ is unchanged. However, the important difference here is that we are free to choose ξ to have any positive value. If we thus take $\xi >> 1$, the model becomes similar to the *strong* non-minimal coupling version of the Einstein's gravity model described in Part I. We argued there that for $\xi > 1$ eq.(44) should be replaced by (eq.(82))

$$\frac{\delta\rho}{\rho} = \frac{C}{(1+6\xi)^{1/2}} \frac{H^2}{\dot{\varphi}} \ . \tag{9.116}$$

From (107) we have

$$\varphi^2|_{N_e} \approx \frac{8N_e}{1+6\xi} \ . \tag{9.117}$$

Then, combining eqs.(95,106,116,117) we find

$$\lambda_{constraint} \approx \frac{9/2}{C^2 N_e^2} \xi^2 \left(\frac{\delta\rho}{\rho}\right)^2 \quad \text{for } \varphi_i > v \text{ and } \xi >> 1 \ . \tag{9.118}$$

We are now free to relax the self-coupling constraint by making a suitable choice of ξ . For example, if we again adopt the values (114) then eq.(118) yields

$$\lambda_{constraint} \approx \xi^2 \times 10^{-11} \ . \tag{9.119}$$

Hence, if desired $\lambda_{constraint}$ can be made to fall in the familiar GUT's range ($\sim 10^{-2}$) by choosing $\xi = 10^4$.

Part III

Quantum Cosmology: Minisuperspace Quantization

Chapter 10

Introduction to quantum cosmology

The working hypothesis in quantum cosmology is that the global structure of the Universe is quantum in nature. For a satisfactory investigation of this idea we first need to bridge the gap between quantum physics and gravity, the dominant interaction at large scales. Unfortunately, progress in the quantum gravity program has been very slow since a formal framework was set for it in 1967-68. DeWitt[43] and Wheeler[44] put forward the notion of wave function of the Universe $\Psi(g^{(3)}, \varphi_l)$ defined on 'superspace' - the set of all three-geometries and matter configurations. The quantization of the General Relativity Hamiltonian constraint yields an equation (the Wheeler-DeWitt equation) for this time-independent wave function. The ultimate goal is to explain, in terms of quantum probabilities, why the gravitational field is as we observe it today.

There are three phases to this project: we must 1) propose a law for boundary conditions, 2) solve the Wheeler-DeWitt equation with a large number of degrees of freedom, 3) find a clear probabilistic interpretation of the solution. As far as the general (infinite dimensional) superspace is concerned, there has been little progress beyond this formal stage on any of these three fronts.

We know that standard cosmology (classical matter and geometry), especially if an inflationary phase (quantum matter, classical geometry)[4-13] is included, can explain many large-scale features of the observable Universe[13] by using a very small number of degrees of freedom. For example, during the exponential expansion that presumably preceded the hot Big Bang, the matter and the geometry can accurately be represented

by just a scalar field φ and the scale factor a of a Robertson-Walker metric. As a fringe benefit, small inhomogeneous quantum fluctuations of φ can be made to account for galaxy formation (see e.g. [31-38] and refs. in [16-19]).

The impressive successes of classical cosmology are nevertheless obtained at a high price: the extreme fine tuning of parameters (e.g. scalar field couplings) and initial conditions (see reviews[16,19]).

Thus, as an attempt to derive these initial conditions from first principles, it is natural to consider building quantum gravitational models with an artificially reduced number of dimensions (say a and φ). This is the 'minisuperspace' approach[45]. It has many attractive features, especially if fluctuations are included. For example, it is consistent with quantum field theory in curved spacetimes. It also provides an arena where different conceptions of the birth of the Universe (i.e. different proposals of boundary conditions for Ψ) can be tested.

However, minisuperspace models (see Halliwell's bibliography[46]) rely on a strong assumption: the tremendous reduction of superspace dimensions is expected to follow from the yet unknown complete theory. There is nothing to indicate that this will be the case. It is quite possible, for example, that the Universe was highly inhomogeneous and anisotropic when it first became classical. In this case, the minisuperspace wave function $\Psi(a,\varphi)$ would be irrelevant to the real quantum history of the Universe.

This chaotic kind of Universe is precisely the starting point of Linde's inflationary scenario[20-23]. In chapter 14, we suggest that $\Psi(a,\varphi)$ may play a role even in the chaotic Universe. This could be realized, for example, by combining Linde's inflationary Universe with Vilenkin's 'creation from nothing'. We do not propose here any explicit mechanism for such a synthesis. We merely remark that minisuperspace wave functions could be much easier to interpret if they are no longer attributed the role of wave functions of the *whole* Universe.

Most previous models assume that a and φ are decoupled at the Lagrangian level ('minimal coupling'). We showed in Part I that this assumption can be dropped without harm from the $\lambda\varphi^4$ inflationary model. New attractive features may even be introduced in this way if the coupling is strong. In the remainder of this thesis we study the quantum cosmology of the non-minimally coupled model. One natural question to ask is whether the generic minimal coupling results are preserved when the minisuperspace variables are strongly coupled - the answer is yes.

In chapter 11, we set the classical theory both in the physical picture $\{a, \varphi\}$ and in a conformal picture $\{\tilde{a}, \tilde{\varphi}\}$ where the equations are 'diagonal'. That it is instructive to carry the calculations in the two pictures as far as possible becomes clear in chapter 12. There, we canonically quantize the system and investigate the invariance of the quantization procedure under the conformal transformation. The tunneling solution[47-50] for the non-minimally coupled $\lambda\varphi^4$ theory is derived in chapter 13 following Vilenkin's approach[47,51,52]. Chapter 14 is the most speculative part of this work. There we adopt the view that $\Psi(a, \varphi)$ is not properly speaking the wave function of the Universe and this allows us to give it a more straightforward probabilistic interpretation. By using Hartle and Hawking's wave function[53-55], we also illustrate how drastically the predictions depend on the choice of boundary conditions. Next, we consider the transition to the classical inflationary phase in chapter 15 and allude to the quantum cosmology of the chaotic induced gravity model in chapter 16. Finally, we comment in chapter 17 on whether this approach really has the power to predict inflation, as it is sometimes claimed.

Chapter 11

The classical Hamiltonian

Let us consider a homogeneous and isotropic universe with the Robertson-Walker line element

$$ds^2 = N^2(t)dt^2 - a^2(t)dx^2 \ , \tag{11.120}$$

with the matter content represented by a homogeneous scalar field φ . We write the bare action as

$$S_b = \int d^4x\sqrt{-g}\left\{ \frac{R}{2} + \frac{1}{2}\xi R\varphi^2 + \frac{1}{2}\varphi_{;\alpha}\varphi^{;\alpha} - V(\varphi) \right\} \ . \tag{11.121}$$

Our units are such that

$$\hbar = c = 8\pi G = 1 \ . \tag{11.122}$$

Greek letters run from 0 to 3 and Latin ones from 1 to 3 .

Most minisuperspace models investigated in the literature[46] use either minimally coupled ($\xi = 0$) theories with *slowly varying* potentials or free fields with a *conformal* coupling (that is $\xi_c = -1/6$ in the sign conventions of eq.(121)).

For $\xi < 0$, gravitation effectively switches signs at large values of φ . It is still unclear whether the model would then be viable[25,26]. Motivated by the results of Part I, we focus here on the case of large positive values of ξ with potentials dominated by a quartic term

$$V(\varphi) = \lambda\varphi^4 \ . \tag{11.123}$$

As it is clear from eq.(121), the non-minimal coupling between φ and R can be seen as a renormalization of the gravitational constant. It is then natural to think of using

the conformal transformation (64,65) to absorb the φ dependent prefactor of R in the Lagrangian (see also ref.[26]), solve the problem in the conformal picture and finally transform the results back into the physical picture. The advantage of this procedure is of course to 'diagonalize' the field equations. However, the correspondence between the two pictures is not trivial in the quantum theory. We will present the classical equations in the two pictures and later touch on the issue of invariance of the quantization procedure.

We will be considering closed Robertson-Walker cosmologies, so the scalar curvature writes

$$R = \frac{6}{a^2}(1 + \dot{a}^2 + a\ddot{a}) \ , \tag{11.124}$$

dots indicating time derivatives. Homogeneity and isotropy also imply that

$$\varphi_{,i} = 0 \ , \ (i = 1, 2, 3) \ \ and \ \int d^4x\{ \ \} = 2\pi^2 \int dt\{ \ \} \ . \tag{11.125}$$

Going back to the action (121), we first eliminate the term in $\ddot{a}$ by partial integration. This will generate a surface term which is a function of $\dot{a}$, namely

$$S.T. = \int d^3x \ \{3(1 + \xi\varphi^2)a^2\dot{a}\} \ . \tag{11.126}$$

When applying the variational principle to obtain the field equations, the variations of the metric are set to zero on the boundary, but not the variations of its derivatives. Hence, one should add a surface term to the bare action in order to cancel $S.T.$ above[56] - the general form of the full action should be

$$S = S_b \ + \int d^3x \ h^{1/2} \ (1 + \xi\varphi^2) \ K \ . \tag{11.127}$$

h is the determinant of h_{ij} , the metric on the three-surface. K is the trace of the corresponding extrinsic curvature:

$$K = \frac{1}{2} h^{ab} \frac{\partial h_{ab}}{\partial t} \ , \tag{11.128}$$

which in our case is just

$$K = 3\,\frac{\dot{a}}{a} \ . \tag{11.129}$$

After integration by parts the Lagrangian $L \equiv dS/dt$ writes

$$L = 2\pi^2 N a^3 \left\{ 3\Omega \left(\frac{1}{a^2} - \frac{H^2}{N^2} \right) - \frac{6\xi\varphi\dot{\varphi}H}{N^2} + \frac{1}{2}\frac{\dot{\varphi}^2}{N^2} - V(\varphi) \right\} \ , \tag{11.130}$$

where $H \equiv \dot{a}/a$ is the expansion rate and

$$\Omega \equiv 1 + \xi\varphi^2 \ . \tag{11.131}$$

The canonical momenta can be derived in the usual way:

$$P_a \equiv \frac{\partial L}{\partial \dot{a}} = -\frac{12\pi^2 a^2}{N}\,(\Omega H + \xi\varphi\dot{\varphi}) \ , \tag{11.132}$$

$$P_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = \frac{2\pi^2 a^3}{N}\,(\dot{\varphi} - 6\xi\varphi H) \ . \tag{11.133}$$

By inverting eqs.(132,133) one obtains, as expected, a system that is coupled through ξ :

$$H = -\frac{N}{2\pi^2 a^3 \widehat{\Omega}} \left(\frac{a}{6} P_a + \xi\varphi P_\varphi \right) \ , \tag{11.134}$$

$$\dot{\varphi} = \frac{N}{2\pi^2 a^3 \widehat{\Omega}}\,(-\xi\varphi a P_a + \Omega P_\varphi) \ , \tag{11.135}$$

where

$$\widehat{\Omega} \equiv 1 + (1 + 6\xi)\xi\varphi^2 \ . \tag{11.136}$$

The Lagrangian is related to the Hamiltonian $\mathcal{H}$ on minisuperspace through

$$L = aHP_a + \dot{\varphi}P_\varphi - N\mathcal{H} \ . \tag{11.137}$$

Thus the lapse function N acts as a a Lagrange multiplier. Finally, by combining eqs.(130,134-137), we obtain the super-Hamiltonian in the physical picture:

$$\mathcal{H} = -\left(24\pi^2 a^3 \widehat{\Omega}\right)^{-1} \left\{ a^2 P_a^2 - 6\Omega P_\varphi^2 + 6a\Omega_{,\varphi} P_a P_\varphi \right\} - 6\pi^2 a \left\{ \Omega - \frac{a^2 V(\varphi)}{3} \right\} \ , \tag{11.138}$$

where $\Omega_{,\varphi} \equiv d\Omega/d\varphi$. Variation with respect to N then yields the classical Hamiltonian constraint

$$\mathcal{H} = 0 \ . \tag{11.139}$$

At the classical level, as mentioned above, the non-minimally coupled solutions can be inferred from those of minimal coupling via a conformal transformation. For the quantum theory, however, the extrapolation of results between the conformal and the physical pictures is not trivial: the invariance of the quantization procedure has to be addressed. This will lead us in the next chapter to deal with the familiar "problem" of factor ordering in the Wheeler-DeWitt equation. Although the final semi-classical probability densities presented in this thesis are not sensitive to operator ordering, these considerations may be of interest if the non-minimally coupled full quantum theory is analysed in some other context.

So let us perform the conformal transformation (eq.(64))

$$\tilde{g}_{\alpha\beta} = \Omega g_{\alpha\beta} \ , \ \tilde{g}^{\alpha\beta} = \Omega^{-1} g^{\alpha\beta} \ , \ \tilde{g} = \Omega^4 g \ . \tag{11.140}$$

The line element of the conformal space-time can be written as

$$d\tilde{s}^2 = \widetilde{N}^2 dt^2 - \tilde{a}^2 dx^2 \ , \tag{11.141}$$

where

$$\widetilde{N} \equiv \Omega^{1/2} N \quad , \quad \tilde{a} \equiv \Omega^{1/2} a \ . \tag{11.142}$$

Since this transformation is extensively used in the following, it might be useful here to remind the reader of the main related formulae (see chapter 5).

The scalar curvature transforms as follows:

$$\tilde{R} = \frac{1}{\Omega} \left\{ R + \frac{3}{2} \tilde{g}^{\alpha\beta} \frac{\Omega_{,\alpha}\Omega_{,\beta}}{\Omega} - 3\tilde{g}^{\alpha\beta}\Omega_{,\alpha,\beta} \right\} \ . \tag{11.143}$$

From eqs.(121,140,143) we obtain the bare conformal action

$$\tilde{S}_b = \int d^4x\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{2}+\frac{3}{2}\frac{\Omega_{,\alpha}\Omega_{,\beta}}{2\Omega^2}\left(1+\frac{1}{6\xi}\frac{\Omega}{\Omega-1}\right)\tilde{g}^{\alpha\beta}-\frac{V(\Omega)}{\Omega^2}\right\} . \tag{11.144}$$

One can put the kinetic part of this action in the canonical form by redefining the scalar field as follows

$$(d\tilde{\varphi})^2 \equiv \frac{3}{2}\left\{1+\frac{1}{6\xi}\frac{\Omega}{\Omega-1}\right\}\left(\frac{d\Omega}{\Omega}\right)^2 . \tag{11.145}$$

When non-minimal coupling is negligible, that is when φ is less than the critical value (see eq.(131))

$$\varphi_{cr} \equiv \frac{1}{\sqrt{\xi}} , \tag{11.146}$$

we have

$$\tilde{\varphi} \approx \varphi \qquad (\varphi < \varphi_{cr}) . \tag{11.147}$$

Then the minimal coupling results apply approximately and the conformal and physical pictures can be identified ($\Omega \to 1$). In the remainder of this work we focus on the domain ($\varphi > \varphi_{cr}$) where non-minimal coupling effects are important. In this case eq.(145) yields

$$\tilde{\varphi} \approx \sqrt{\frac{3}{2}\left(1+\frac{1}{6\xi}\right)}\,\ln\Omega \qquad (\varphi > \varphi_{cr}) . \tag{11.148}$$

Finally, the transform of eq.(126) gives just the Gibbons-Hawking boundary term[56] for the Hilbert-Einstein action in conformal space. We have thus reduced the classical non-minimally coupled theory to the usual minimally coupled form

$$\tilde{S} = \int d^4x\,\sqrt{-\tilde{g}}\left\{\frac{\tilde{R}}{2}+\frac{1}{2}\,\tilde{\varphi}_{,\alpha}\tilde{\varphi}_{,\beta}\,\tilde{g}^{\alpha\beta}-\tilde{V}(\tilde{\varphi})\right\} + \int d^3x\,\tilde{h}^{1/2}\tilde{K} , \tag{11.149}$$

where

$$\tilde{V}(\tilde{\varphi}) \equiv \frac{V(\varphi)}{\Omega^2} . \tag{11.150}$$

The conformal super-Hamiltonian may now be simply read off eq.(138) using $\xi = 0$. One finds

$$\widetilde{\mathcal{H}} = -\left(24\pi^2\tilde{a}^3\right)^{-1} \left\{ \tilde{a}^2 P_{\tilde{a}}^2 - 6P_{\tilde{\varphi}}^2 \right\} - 6\pi^2\tilde{a} \left\{ 1 - \frac{1}{3}\,\tilde{a}^2\widetilde{V}(\tilde{\varphi}) \right\} \quad , \tag{11.151}$$

a well known minimal coupling result (e.g. refs.[51,52]).

Chapter 12

Non-minimally coupled Wheeler-DeWitt equation

Let $\Psi(a,\varphi)$ be the wave function on our two-dimensional minisuperspace[45]. One can quantize the system by making the redefinitions

$$P_a \equiv -i\partial_a \equiv -i\frac{\partial}{\partial a} \quad , \quad P_\varphi \equiv -i\partial_\varphi \equiv -i\frac{\partial}{\partial \varphi} \quad . \tag{12.152}$$

Then the classical Hamiltonian constraint eq.(139) becomes the Wheeler-DeWitt (WD) equation[43,44]

$$\mathcal{H}\Psi(a,\varphi) = 0 \quad . \tag{12.153}$$

Using eq.(138,153) we can now write an equation for the wave function of the Universe in the physical picture, i.e. with the variables $\{N, a, \varphi\}$:

$$\left[\partial_a^2 - \frac{6}{a^2}\Omega(\varphi)\partial_\varphi^2 + \frac{6\Omega_{,\varphi}}{a}\partial_a\partial_\varphi - 144\pi^4 a^2 \widehat{\Omega}(\varphi)\left\{ \Omega(\varphi) - \frac{a^2 V(\varphi)}{3} \right\} \right] \Psi(a,\varphi) = 0 \; . \tag{12.154}$$

However, given the ambiguity in the ordering of operators, first derivative terms could also in principle be added to eq.(154). One way to fix the ordering is to require that the theory be invariant with respect to redefinitions of the minisuperspace variables a, φ and of the lapse function N . This can be achieved by setting (e.g. ref.[45])

$$\mathcal{H} = -\nabla^2 \; + \; \eta \, \mathcal{R} \; - \; \mathcal{U}(a,\varphi) \; , \tag{12.155}$$

where ∇^2 and $\mathcal{R}$ are the Laplacian and scalar curvature defined by the metric on minisuperspace (see eq.(157) below). $\mathcal{U}(a,\varphi)$ is the potential term in eq.(138) and η is a dimensionless constant.

Invariance with respect to a and φ reparameterizations is secured by using the covariant operator ∇^2. It is the N invariance that necessitates the introduction of the $\eta\,\mathcal{R}$ term (see ref.[57]). It has been argued[45,57] that η should take the 'conformal' value

$$\eta = \frac{n-2}{4(n-1)} \ , \tag{12.156}$$

where n is the dimension of minisuperspace. Thus, the two-dimensional model we are considering in this thesis falls into the simplest case: $\eta = 0$. Note however that the question is left unresolved for the simplest and most thoroughly studied model: Robertson-Walker space-time with a cosmological constant, for which $n = 1$.

To apply this prescription to the physical picture we first read the super-metric and the superpotential from eq.(138):

$$\mathcal{G}^{aa} = -\frac{1}{a\widehat{\Omega}} \ , \ \mathcal{G}^{a\varphi} = \mathcal{G}^{\varphi a} = -3\frac{\Omega_{,\varphi}}{a^2\widehat{\Omega}} \ , \ \mathcal{G}^{\varphi\varphi} = 6\frac{\Omega}{a^3\widehat{\Omega}} \ , \tag{12.157}$$

and

$$\mathcal{U}(a,\varphi) \equiv 144\pi^4 a\left(\Omega - \frac{a^2V(\varphi)}{3}\right) \ . \tag{12.158}$$

The Laplacian is

$$\nabla^2 \equiv \frac{1}{\sqrt{-\mathcal{G}}}\,\frac{\partial}{\partial\gamma^l}\left(\mathcal{G}^{lm}\sqrt{-\mathcal{G}}\,\frac{\partial}{\partial\gamma^m}\right) \ , \tag{12.159}$$

where $l,m = a,\varphi$ and $\gamma^a = a$, $\gamma^\varphi = \varphi$. From eqs.(155-159) we finally derive the 'covariant' WD equation

$$\left[\partial_a^2 + \frac{1}{a}\partial_a + \frac{6\Omega_{,\varphi}}{a}\,\partial_a\partial_\varphi - \frac{3\Omega_{,\varphi}}{a^2}\,\partial_\varphi \right.$$
$$\left. -\frac{6\Omega}{a^2}\,\partial_\varphi^2 - 144\pi^4 a^2\Omega\widehat{\Omega}\left\{1 - \frac{a^2V(\varphi)}{3\Omega}\right\}\right]\Psi(a,\varphi) = 0 \ . \tag{12.160}$$

A worthwhile remark is that the conformal transformation of the metric: $\{a,N\} \to \{\tilde{a},\widetilde{N}\}$ (see eqs.(140-142)) is not necessary to obtain a diagonal quantum theory - the

redefinition of N is not required. This can be seen by writing the classical Lagrangian in the picture $\{\tilde{a}, \tilde{\varphi}, N\}$ and then quantizing. One finds

$$\nabla^2_{\{\tilde{a},\tilde{\varphi},N\}} = \Omega^{1/2} \left(-\frac{1}{\tilde{a}}\partial^2_{\tilde{a}} - \frac{1}{\tilde{a}^2}\partial_{\tilde{a}} + \frac{6}{\tilde{a}^3}\partial^2_{\tilde{\varphi}} \right) , \tag{12.161}$$

and

$$\mathcal{U}_{\{\tilde{a},\tilde{\varphi},N\}} = 144\pi^4\Omega^{1/2}\tilde{a} \left(1 - \frac{\tilde{a}^2\widetilde{V}(\tilde{\varphi})}{3} \right) . \tag{12.162}$$

While it is obvious from eqs.(142,150,158,162) that

$$\mathcal{U}_{\{\tilde{a},\tilde{\varphi},N\}} = \mathcal{U}_{\{a,\varphi,N\}} , \tag{12.163}$$

it is somewhat tedious but straightforward to verify that

$$\nabla^2_{\{\tilde{a},\tilde{\varphi},N\}} = \nabla^2_{\{a,\varphi,N\}} \tag{12.164}$$

as it should.

Because our mini-superspace is two-dimensional ($\eta = 0$), the above formulae are simply related to those of the conformal picture $\{\tilde{a}, \tilde{\varphi}, \widetilde{N}\}$. Namely, one can show (from eqs.(151,157-159) that

$$\nabla^2_{\{\tilde{a},\tilde{\varphi},\widetilde{N}\}} = \Omega^{-1/2}\, \nabla^2_{\{a,\varphi,N\}} , \tag{12.165}$$

$$\mathcal{U}_{\{\tilde{a},\tilde{\varphi},\widetilde{N}\}} = \Omega^{-1/2}\, \mathcal{U}_{\{a,\varphi,N\}} . \tag{12.166}$$

Eqs.(163-166) imply that the $\{\tilde{a}, \tilde{\varphi}, \widetilde{N}\}$ and $\{\tilde{a}, \tilde{\varphi}, N\}$ pictures yield the same diagonal WD equation, namely

$$\left[\partial^2_{\tilde{a}} + \frac{1}{\tilde{a}}\,\partial_{\tilde{a}} - \frac{6}{\tilde{a}^2}\,\partial^2_{\tilde{\varphi}} - 144\pi^4\,\tilde{a}^2 \left\{ 1 - \frac{\tilde{a}^2\widetilde{V}(\tilde{\varphi})}{3} \right\} \right] \Psi(\tilde{a}, \tilde{\varphi}) = 0 . \tag{12.167}$$

To summarize, this prescription for the ordering of operators in the WD equation insures that, if two theories are transforms of each other at the classical level, they remain so after quantization. In practice, this eliminates the ambiguity concerning the

first order derivatives in the WD equation. The issue is of particular importance when, as it is the case here, more than one set of variables is used to study a minisuperspace model.

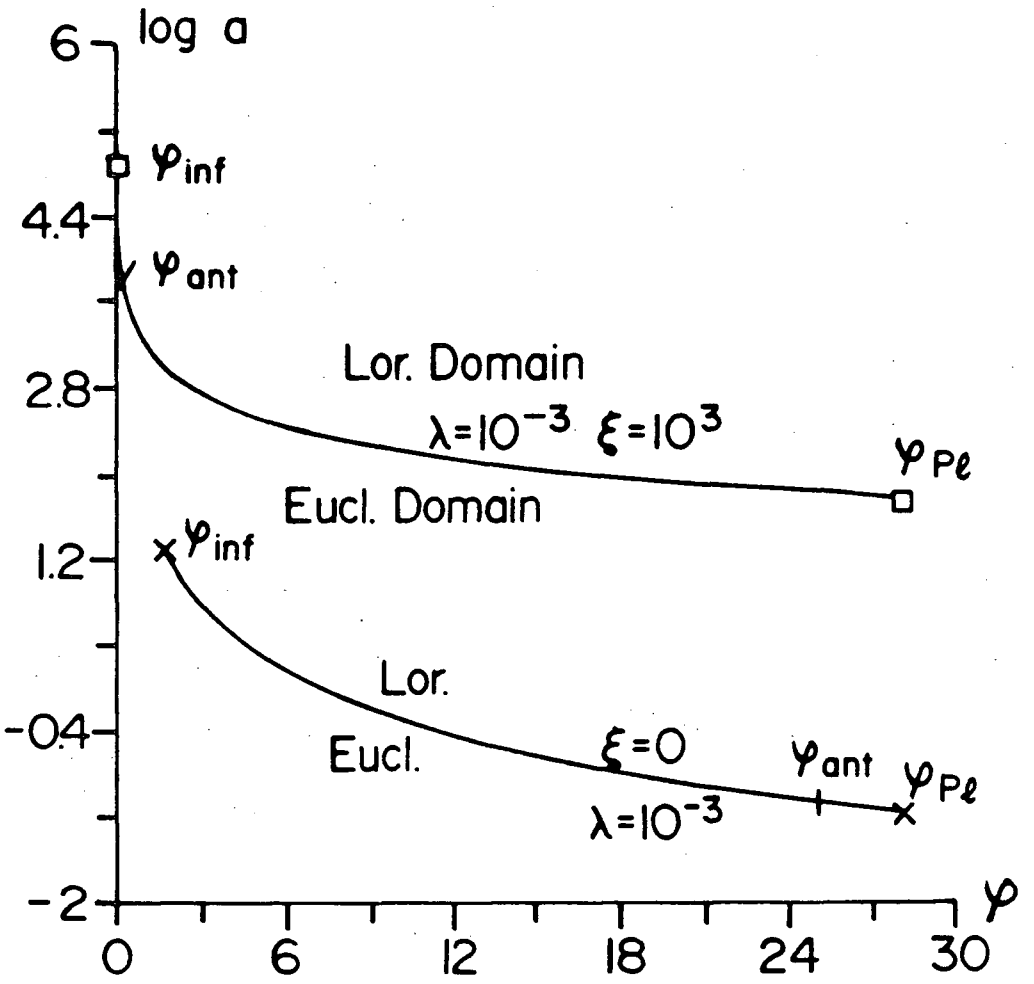

Figure 12.9: *The line of zero WD potential divides minisuperspace into Euclidian (positive potential, exponential modes) and Lorentzian (negative potential, oscillatory modes) domains. In most minisuperspace models (including the present) the solutions are valid only for* $\varphi_{inf} < \varphi < \varphi_{Pl}$. *But this covers most of the sub-Planckian minisuperspace* ($0 < \varphi < \varphi_{Pl}$) *if* $\xi >> 1$.

Chapter 13

'Creation from nothing'

A key feature of strong non-minimal coupling is that the potential term in the conformal WD equation (167) is $\tilde{\varphi}$ independent to a high accuracy. This follows from

$$\tilde{V}(\tilde{\varphi}) = \frac{V(\varphi)}{\Omega^2} \approx \frac{\lambda}{\xi^2}\left(1 - 2\,\frac{\varphi_{cr}^2}{\varphi^2}\right) \approx \frac{\lambda}{\xi^2} \quad (\varphi > \varphi_{cr}) \ . \tag{13.168}$$

Most minisuperspace models assume the non-gravitational energy to be either a cosmological constant or a slowly varying matter potential. This condition is naturally met in the present model when viewed in an $\tilde{a}$ picture. However, there is an interesting departure from the previous models: the solutions in the physical picture will extend over a domain of minisuperspace where the matter potential is *not* slowly varying (see the end of this chapter).

One should also note that our model has some similarities with the quantum cosmology of higher derivative gravity[58-62]. This is essentially because the R^2 term in the latter represents two scalar degrees of freedom (in our case the two scalars are R and φ).

We shall first look for solutions using the zeroth order WD potential (see eqs.(167,168))

$$\tilde{\mathcal{V}}_0(\tilde{a}) = -144\pi^4\tilde{a}^2\left(1 - \frac{\lambda}{3\xi^2}\,\tilde{a}^2\right) \ , \tag{13.169}$$

and then seek first order approximate solutions. To zeroth order, the WD equation (167) separates as follows (see also ref.[62])

$$\left[\,\partial_{\tilde{a}}^2 + \frac{1}{\tilde{a}}\,\partial_{\tilde{a}} + \tilde{\mathcal{V}}_0(\tilde{a}) - \frac{\sigma}{\tilde{a}^2}\,\right]\,\Psi(\tilde{a}) = 0 \ , \tag{13.170}$$

$$\left[\partial_{\tilde{\varphi}}^2 - \frac{\sigma}{6} \right] \Psi(\tilde{\varphi}) = 0 \ , \tag{13.171}$$

with

$$\Psi(\tilde{a})\Psi(\tilde{\varphi}) = \Psi(\tilde{a}, \tilde{\varphi}) \ . \tag{13.172}$$

σ is an arbitrary separation constant. The solutions will depend strongly on the choice of σ which 1) redefines the potential for $\Psi(\tilde{a})$; 2) determines whether $\Psi(\tilde{\varphi})$ is exponential, linear or oscillatory (respectively: $\sigma > 0$, $= 0$ or < 0). Hence, the choice of a particular value for the separation constant is an important part of the boundary conditions.

There has been a number of proposals for a law of initial conditions of the Universe[63-65,47,53,66-68,21,22]. Some of the most thoroughly analyzed in the recent literature are Vilenkin's[47,66] and Hartle and Hawking's[53] proposals. In the rest of this chapter I will give Vilenkin's solution for the present model. The HH wave function will be considered in the next chapter.

The minisuperspace is divided into Euclidian regions (positive WD potential, exponential modes) and Lorentzian regions (negative WD potential, oscillatory modes). The solutions are matched across the lines of zero potential where the WKB approximation breaks down. Note that these lines determine sharply the scope of classical trajectories only when the matter kinetic terms are negligible.

Vilenkin's program is to implement explicitly the notion of creation of the Universe from 'nothing'[69,71]. His wave function describes a quantum tunneling from a zero size to a finite size state through a potential barrier on minisuperspace. In the Lorentzian domain (i.e. outside the barrier), such a wave function includes only modes that 'move away' from the Euclidian regions (i.e. from the barrier). It thus describes only expanding universes. An additional boundary condition is that the wave function must be independent of the matter field in the zero size state.

The only value of the separation constant in eqs.(170-172) that is compatible with the above boundary conditions is $\sigma = 0$. This is for two reasons:

1) the effective potential in eq.(170) would otherwise diverge at the origin $\tilde{a} = 0$, thus hampering the quantum tunneling scenario;

2) unless $\sigma = 0$ in eq.(171), the wave function could not be made independent of $\tilde{\varphi}$ at $\tilde{a} = 0$ as required above.

Having fixed the boundary conditions we shall now write Vilenkin's wave function (Ψ_V) explicitly. One can already have an idea of the profile of its density ($|\Psi_V|^2$) by considering the shape of the potential barrier in the physical picture. This three-dimensional barrier (axes a , φ and $\mathcal{V}(a,\varphi)$) is delimited by the curve $a = a(\varphi)$ where the superpotential is zero (see fig.9). To zeroth order in $(\varphi_{cr}/\varphi)^2$ that curve is

$$a(\varphi) = \sqrt{\frac{3\xi}{\lambda}}\,\frac{1}{\varphi} \quad . \tag{13.173}$$

Thus the barrier vanishes asymptotically at large values of φ . The Universe is therefore 'more likely' (chapter 14) to nucleate with $\varphi >> \varphi_{cr}$. This is an initial configuration that is very favourable to non-minimally coupled chaotic inflation[1] (see Part I).

With $\sigma = 0$, the first order WKB solution of eq.(170) in the Lorentzian domain gives

$$\Psi_V(\tilde{a}) \propto e^{i\pi/4}\,\tilde{a}^{-1/2}\,\left(-\tilde{\mathcal{V}}_0\right)^{-1/4}\,\exp\left(-i\int^{\tilde{a}} d\tilde{a}\,\sqrt{-\tilde{\mathcal{V}}_0}\right) \quad , \quad \tilde{\mathcal{V}}_0 < 0 \ . \tag{13.174}$$

We have kept only the outgoing mode, in accordance with the tunneling boundary conditions (see above).

This solution, when continued into the Euclidian domain, connects to the decaying mode

$$\Psi_V(\tilde{a}) \propto \tilde{a}^{-1/2}\tilde{\mathcal{V}}_0^{-1/4}\,\exp\left(-\int^{\tilde{a}} d\tilde{a}\,\sqrt{\tilde{\mathcal{V}}_0}\right) \quad , \quad \tilde{\mathcal{V}}_0 > 0 \ . \tag{13.175}$$

Strictly speaking, the Lorentzian outgoing mode connects to a combination of decaying and growing Euclidian modes. However, the growing component turns out to be negligible for small values of $\tilde{a}$, so we disregard it here for simplicity.

The integration constants in eqs.(174,175) can be fixed by solving eq.(170) in the small $\tilde{a}$ limit. There we have

$$\tilde{\mathcal{V}}_0(\tilde{a}) \approx -144\pi^4\,\tilde{a}^2 \quad , \quad \tilde{a}^2 << \frac{3\xi^2}{\lambda} \ . \tag{13.176}$$

Hence the two linearly independent solutions are the modified Bessel functions

$$K_0(6\pi^2\tilde{a}^2) \quad , \quad I_0(6\pi^2\tilde{a}^2) \ . \tag{13.177}$$

Using the series expansions of these functions one finds

$$\Psi_V(\tilde{a}) \propto -\ln(6\pi^2\tilde{a}^2) \quad \text{for} \quad \tilde{a}^2 << \frac{1}{6\pi^2} \ , \tag{13.178}$$

and

$$\Psi_V(\tilde{a}) \propto \tilde{a}^{-1}\,\exp\left\{-6\pi^2\tilde{a}^2\right\} \quad \text{for} \quad \frac{1}{6\pi^2} << \tilde{a}^2 << 1 \ . \tag{13.179}$$

(For our purpose, both proportionality constants are positive).

On the other hand, the requirement that the wave function should be $\tilde{\varphi}$ -independent at $\tilde{a} = 0$ implies that $\Psi(\tilde{\varphi})$ is not only linear ($\sigma = 0$ in eq.(171)), but *constant* over minisuperspace.

Finally, by matching eqs.(174,175,178) we obtain for the Euclidian domain ($\tilde{a}^2 < 3\xi^2/\lambda$)

$$\Psi_V \propto \tilde{a}^{-1}\left(1-\frac{\lambda}{3\xi^2}\,\tilde{a}^2\right)^{-1/4}\exp\left\{-12\pi^2\,\frac{\xi^2}{\lambda}\left[1-\left(1-\frac{\lambda}{3\xi^2}\,\tilde{a}^2\right)^{3/2}\right]\right\} \tag{13.180}$$

and for the Lorentzian domain ($\tilde{a}^2 > 3\xi^2/\lambda$)

$$\Psi_V \propto e^{i\pi/4}\,\tilde{a}^{-1}\left(-1+\frac{\lambda}{3\xi^2}\,\tilde{a}^2\right)^{-1/4}\exp\left\{-12\pi^2\,\frac{\xi^2}{\lambda}\left[1+i\left(-1+\frac{\lambda}{3\xi^2}\,\tilde{a}^2\right)^{3/2}\right]\right\} \ . \tag{13.181}$$

We now argue that replacing λ/ξ^2 by $\tilde{V}(\tilde{\varphi})$ in eqs.(180,181) yields a good approximate solution to the full WD equation (167). This is because 1) for small $\tilde{a}$, eq.(167) separates even if the exact potential $\tilde{V}(\tilde{\varphi})$ is used. Again, the boundary conditions imply that $\Psi_V(\tilde{\varphi})$ is just a constant. Thus $\Psi_V(\tilde{a},\tilde{\varphi})$ is in fact $\tilde{\varphi}$ -independent in this limit: it is identical to the zeroth order wave function of eq.(178); 2) for larger values of $\tilde{a}$, the $\tilde{\varphi}$ derivative term in the WD equation (167) can be neglected because of the very slow variation of $\tilde{V}(\tilde{\varphi})$ in the considered region[51,52].

There is an interesting twist here: our solution extends over a domain of minisuperspace where the *physical* potential $V(\varphi)$ is *not* slowly varying. In effect, in minimally coupled models one can neglect the matter derivatives in the WD equation only if[51,52]

$$\partial_\varphi V(\varphi) << \left(\frac{4\pi G}{3}\right)^{1/2} V(\varphi) \ . \tag{13.182}$$

This corresponds to $\varphi > 4\sqrt{6}$ in our conventions (see eqs.(122,123)). On the other hand, the non-minimal coupling solutions we obtained via the conformal transformation are valid for $\varphi > \xi^{-1/2}$. Thus, at the same level of confidence as in minimal coupling, our non-minimal coupling solutions cover an additional domain, namely

$$\frac{1}{\sqrt{\xi}} < \varphi < 4\sqrt{6} \ . \tag{13.183}$$

This window may be several orders of magnitude wide for the preferred range of ξ , which is $\xi >> 1$ (see Part I).

Chapter 14

Predictions

A major difficulty in quantum cosmology is to interpret the wave function of the Universe. The problem is two-fold: 1) the wave function is time-independent, in contrast with the case of ordinary quantum mechanics; 2) we have only one copy of the Universe, and hence it is unclear how to interpret probabilities calculated from this wave function .

DeWitt[43] suggested solving the first difficulty by introducing a conserved probability current on minisuperspace, with probability distributions defined on hypersurfaces of dimension $n-1$ (n is the minisuperspace dimension). This procedure leads to a problem of negative probabilities analogous to that of the Klein-Gordon equation. More recently, Vilenkin proposed a version of this approach which could be free of negative probabilities[51,72]: the variables are divided into quantum and semi-classical, and one of the latter plays the role of time. For the tunneling wave function, the fixed- $\tilde{a}$ probability density $\rho_V(\tilde{\varphi}|\tilde{a})$ can be defined through the minisuperspace probability current $j^{\tilde{a}}$ by

$$\rho_V(\tilde{\varphi}|\tilde{a}) = j^{\tilde{a}} = \frac{i}{2}\left(\bar{\Psi}\,\partial_{\tilde{a}}\Psi - \Psi\,\partial_{\tilde{a}}\bar{\Psi}\right) \quad . \tag{14.184}$$

Here, bars indicate complex conjugation and Ψ is the Lorentzian wave function

$$\Psi_V \propto e^{i\pi/4}\,\tilde{a}^{-1}\left(-1+\frac{\tilde{a}^2\tilde{V}(\tilde{\varphi})}{3}\right)^{-1/4}\exp\left\{-\frac{12\pi^2}{\tilde{V}(\tilde{\varphi})}\left[1+i\left(-1+\frac{\tilde{a}^2\tilde{V}(\tilde{\varphi})}{3}\right)^{3/2}\right]\right\} \quad . \tag{14.185}$$

Since we are interested in providing initial conditions for classical cosmology, we

consider distributions over the line

$$\tilde{a} = \sqrt{\frac{3}{\tilde{V}(\tilde{\varphi})}} \tag{14.186}$$

where the Lorentzian trajectories originate. Combining eqs.(168,184-186) and transforming the results back into the physical picture one obtains, to first order in φ ,

$$P_V(\varphi_1, \varphi_2) \propto \int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{\varphi} \exp\left\{-12\pi^2 \frac{\xi^2}{\lambda} \left(1 + 2\frac{\varphi_{cr}^2}{\varphi^2}\right)\right\} . \tag{14.187}$$

This is the 'probability for the Universe to nucleate' (whatever that means, see below) with a value of the scalar field comprised between φ_1 and φ_2 , in the non-minimally coupled $\lambda\varphi^4$ model.

As we said earlier, it is not clear how to interpret such a probability, since we have here the analogue of a single-trial quantum experiment. One possibility is to invoke the so-called 'many worlds' or Everett interpretation of quantum mechanics[73]. Hartle[74] suggested that the wave function of the Universe may have a predictive power if it is sharply peaked about certain correlations. In the following, we try looking at $\Psi(a, \varphi)$ from a different perspective. Essentially, we remark that it may be easier to find an interpretation for $\Psi(a, \varphi)$ than for $\Psi(g^{(3)}, \varphi_l)$ (see chapter 10).

The formula (187) is hard to interpret only if $\Psi(a, \varphi)$ is considered to be the wave function of the *whole* Universe. We now suggest to abandon that concept. Instead, one could seek in general to associate minisuperspace wave functions with homogeneous subregions of the Universe. The set of these sub-universes would have to constitute, at least approximately, a statistical ensemble. Then eq.(187) may be given an ordinary frequency interpretation.

To give an example of a framework in which this could be achieved, let us attempt a *formal* synthesis of Linde's chaotic inflation and Vilenkin's 'creation from nothing'[75].

Consider Linde's chaotic inflationary Universe in its simplest form[20]. When it emerges from the Planck era, it is filled with a scalar field φ (which is assumed to

dominate the dynamics) that takes a wide range of values in different regions of space. Linde showed that inflation will occur whenever φ is 1) homogeneous over a large enough domain and 2) greater than a given minimal value (φ_{inf} in the following). Furthermore, in order to produce enough e-folds to solve the cosmological puzzles[13], it suffices that φ be greater than an other given value (that we call 'anthropic' for short): $\varphi_{ant} > \varphi_{inf}$. This scenario works with many matter potentials, including (123).

For the non-minimally coupled $\lambda\varphi^4$ version of Linde's model[1] (see chapters 3 and 4), one has: $\varphi_{inf} = \varphi_{cr} = 1/\sqrt{\xi}$ and $\varphi_{ant} = \sqrt{N_e/\xi}$, where N_e is the desired number of e-folds. To be self-consistent in our semi-classical treatment we must also restrict φ to be less than φ_{Pl} , the value for which the field potential reaches the Planck scale. In our conventions $\varphi_{Pl} = (8\pi)^{1/2}\lambda^{-1/4}$.

The picture then is that of a globally very inhomogeneous Universe that contains many (smooth) inflating sub-regions. Let us write symbolically the set of these inflationary sub-universes as

$$\mathcal{S} = \{\varphi_{inf} < \varphi_{init} < \varphi_{Pl}\} \quad , \tag{14.188}$$

where φ_{init} is the value of the field at the onset of inflation. If we now assume that these sub-universes were born by quantum tunneling, then their histories may be described by $\Psi_V(a,\varphi)$. This function can thus be given an ordinary probabilistic interpretation: it determines internal frequencies in $\mathcal{S}$.

The above interpretation of $\Psi_V(a,\varphi)$ implicitly assumes that $\mathcal{S}$ constitutes a statistical ensemble. In other words, we have neglected eventual correlations between the sub-universes[76]. This can only be justified within some model of the *global* structure of the Universe. Such a model should have many more degrees of freedom than our two-dimensional minisuperspace.

It should perhaps be added that our sub-universes are *not* the inflationary domains

generated by the random walk fluctuations of φ (Linde's 'mini-universes'[21-23]). What happens to the sub-universes after quantum birth *from 'nothing'*, including eventual self-reproduction, is irrelevant in our context.

We close this digression by a final remark: in the above illustrative example we considered that $\Psi_V(a,\varphi)$ describes 'locally' (i.e. at the scale of an inflationary sub-universe) certain parts of the global spacetime. However, the idea of tunneling from a zero-size state ('nothing') is probably more compatible with the creation of separate disconnected three-geometries. Thus, one way to approach (or to avoid) the problem of interpretation is to consider that eq.(187) gives frequencies in a set of isolated closed sub-universes. Just to show that this high degree of speculation *has* been reached before, we mention that the concept of isolated closed universes was considered in the context of 'false vacuum bubbles' dynamics[77-80] ('child universes').

We now go back to the somewhat less speculative (by cosmological standards) internal physics of the sub-universes. Clearly, we cannot hope to *predict* inflation in the approach we have used (see chapters 15 and 17). It seems rather that the only question we can ask here is: if a large number of uncorrelated inflationary sub-universes are born by quantum tunneling, how many will be 'anthropic'?

For a quantitative answer one can use the normalization

$$P(\varphi_{inf},\varphi_{Pl}) = 1 \tag{14.189}$$

in eq.(187) and write

$$P_V(\varphi_{ant},\varphi_{Pl}) = \frac{1}{C_V}\int_{\varphi_{ant}}^{\varphi_{Pl}} \frac{d\varphi}{\varphi}\ \exp\left\{-24\pi^2\,\frac{\xi}{\lambda\varphi^2}\right\}\ , \tag{14.190}$$

where

$$C_V = \int_{\varphi_{inf}}^{\varphi_{Pl}} \frac{d\varphi}{\varphi}\ \exp\left\{-24\pi^2\,\frac{\xi}{\lambda\varphi^2}\right\}\ . \tag{14.191}$$

Obviously, these equations would not apply if $\varphi_{ant} > \varphi_{Pl}$, that is if $(\lambda/\xi^2) > (8\pi/N_e)^2$. However, this situation is ruled out by the analysis of non-minimally coupled inflation (chapter 5) - it produces density perturbations of too large an amplitude. Using the values $\xi = 10^3$ and $\lambda = 10^{-3}$, which are considered typical in Part I, one finds

$$P_V(\varphi_{ant}, \varphi_{Pl}) \approx 1 \tag{14.192}$$

to an accuracy of about one part in 10^{10^7} ! This depends qualitatively very little on the particular choice of parameters (as long as $(\lambda/\xi^2) < (8\pi/N_e)^2$).

The result (192) relies heavily on the choice of boundary conditions. We illustrate this in the following by considering the Hartle-Hawking minisuperspace wave function $\Psi_{HH}(a, \varphi)$.

Hartle and Hawking's proposal prescribes that the value of the wave function of the Universe for the spatial and matter configurations $g^{(3)}$ and φ is given by the path integral[53-55]

$$\Psi(g^{(3)}, \varphi) = \int \mathcal{D}g^{(4)}\mathcal{D}\varphi \ \exp\{S_E(g^{(4)}, \varphi)\} \ . \tag{14.193}$$

The integration is over the Euclidian actions of all compact four-geometries and regular matter configurations that are bounded only by the argument of Ψ .

The gravitational part of S_E causes the integral (193) to diverge[81], hence no straightforward implementation of this program seems possible. As a remedy, it was proposed to integrate over complex metrics, and it was believed that the proposal would then yield a unique wave function. However, it has recently emerged[82] that depending on the choice of integration contours, the Hartle-Hawking proposal can yield either Ψ_V or what is usually called the Hartle-Hawking wave function (Ψ_{HH}). This follows from a detailed analysis of the lapse function integral.

The complex integration method was applied extensively to Robertson-Walker universes with a cosmological constant[46,82]. This is approximately the case of our model

as seen from the conformal picture. From eqs.(149,168) and ref.[53] we find, to zeroth order in the $\tilde{\varphi}$ dependence, the Lorentzian solution

$$\Psi_{HH} \propto \tilde{a}^{-1} \left(\frac{\lambda}{3\xi^2} \tilde{a}^2 - 1 \right)^{-1/4} \exp\left\{ 12\pi^2 \frac{\xi^2}{\lambda} \right\} \cos\left\{ 12\pi^2 \frac{\xi^2}{\lambda} \left(\frac{\lambda}{3\xi^2} \tilde{a}^2 - 1 \right)^{3/2} - \frac{\pi}{4} \right\}$$

$$\text{for} \quad \tilde{a}^2 > \frac{3\xi^2}{\lambda} \ . \tag{14.194}$$

Unlike the tunneling case, the solution here is real - it is a superposition of contracting and expanding modes of equal weights.

As in chapter 13, the first order solution is approximately equal to $\Psi_{HH}(\tilde{a}, \tilde{\varphi})$ above with λ/ξ^2 multiplied by $(1 - 2\varphi_{cr}^2/\varphi^2)$. The minisuperspace current (eq.(184)) vanishes in this case (Ψ_{HH} real). Nevertheless, one can still derive the analogue of eqs.(190,191) from eq.(184) by keeping only the expanding modes[52]. One thus obtains

$$P_{HH}(\varphi_{ant}, \varphi_{Pl}) = \frac{1}{C_{HH}} \int_{\varphi_{ant}}^{\varphi_{Pl}} \frac{d\varphi}{\varphi} \exp\left\{ 24\pi^2 \frac{\xi}{\lambda\varphi^2} \right\} \ , \tag{14.195}$$

where

$$C_{HH} = \int_{\varphi_{inf}}^{\varphi_{Pl}} \frac{d\varphi}{\varphi} \exp\left\{ 24\pi^2 \frac{\xi}{\lambda\varphi^2} \right\} \ . \tag{14.196}$$

So, if some physically motivated choice of boundary conditions (or of integration contours) eventually singles out Ψ_{HH} as the right solution to the WD equation, how much would eq.(192) change? From eqs.(189,195,196) we find this time a vanishingly small frequency of anthropic sub-universes. Again, the result is not significantly sensitive to the choice of parameters. The same values of λ and ξ as in eq.(192) yield

$$P_{HH}(\varphi_{ant}, \varphi_{Pl}) \sim 10^{-10^9} \ ! \tag{14.197}$$

(The term '!' is usually missing in the quantum cosmology literature.)

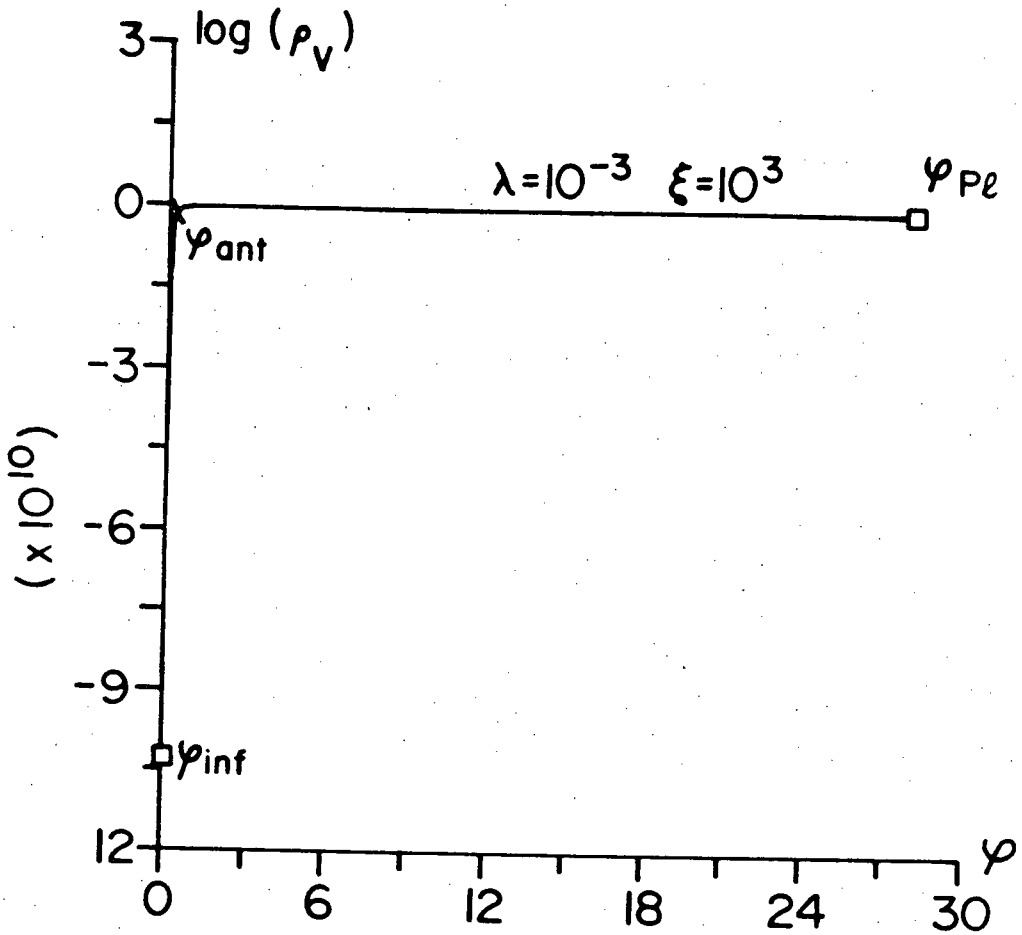

Figure 14.10: *A typical profile of non-minimal coupling tunneling probability density. The predicted frequency of anthropic sub-universes is* $P_V(\varphi_{ant}, \varphi_{Pl}) \approx 1$.

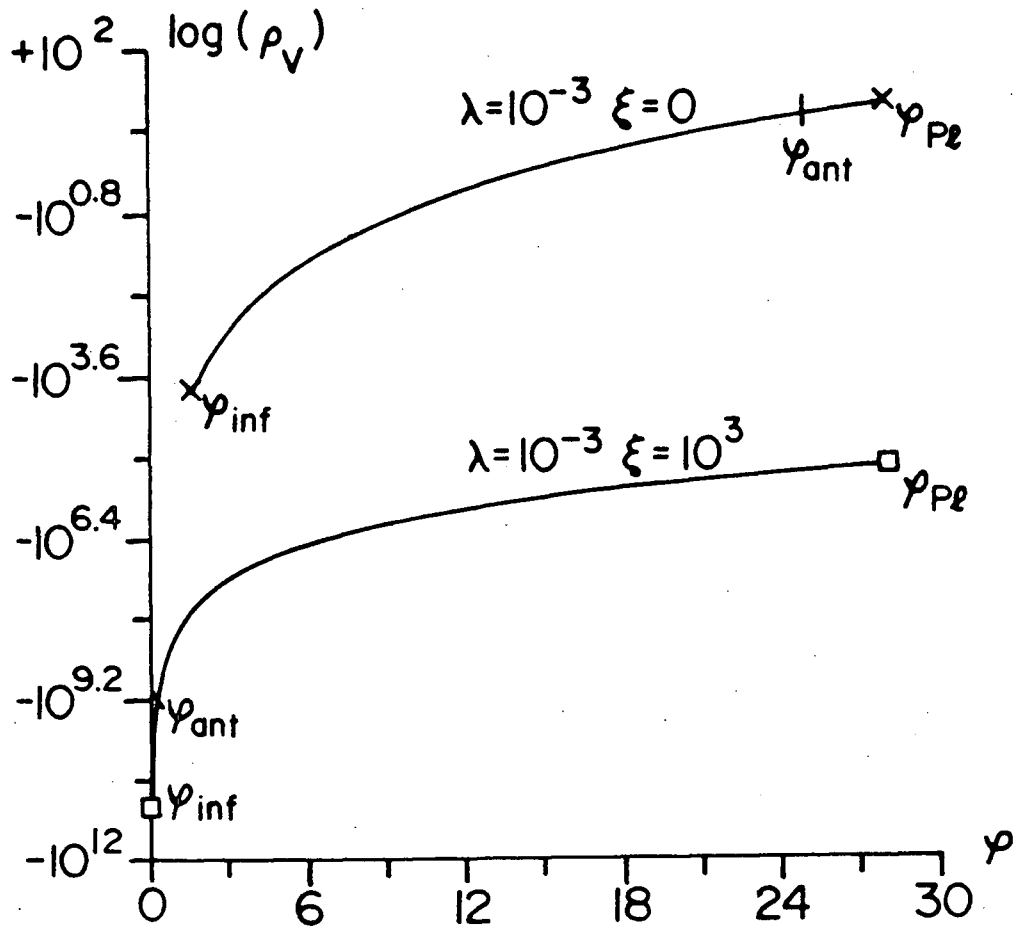

Figure 14.11: *The tunneling probability density is plotted in double-log scale for minimal and non-minimal coupling. With* $\lambda = 10^{-3}$ *the non-minimal coupling frequency of anthropic sub-universes is* $P_V(\varphi_{ant}, \varphi_{Pl})|_{(\xi=10^3)} \approx 1$ *, whereas for minimal coupling* $P_V(\varphi_{ant}, \varphi_{Pl})|_{(\xi=0)} \approx 0$ *. The situation is very different in the Hartle-Hawking case (fig.13). As mentioned in fig.9, the non-minimal coupling solution extends over to practically all the sub-Planckian minisuperspace* ($0 < \varphi < \varphi_{Pl}$).

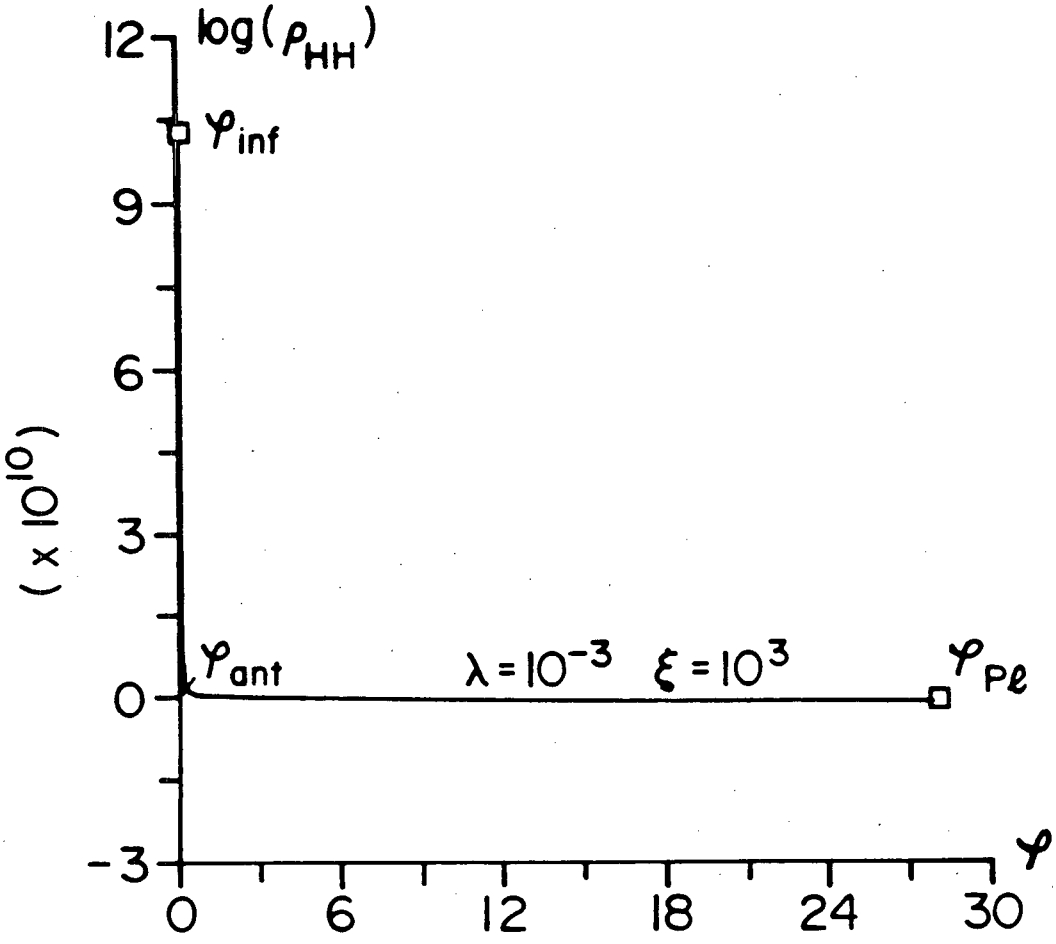

Figure 14.12: *A typical non-minimal coupling Hartle-Hawking probability density. The predicted frequency of anthropic sub-universes is* $P_{HH}(\varphi_{ant}, \varphi_{Pl}) \approx 0$.

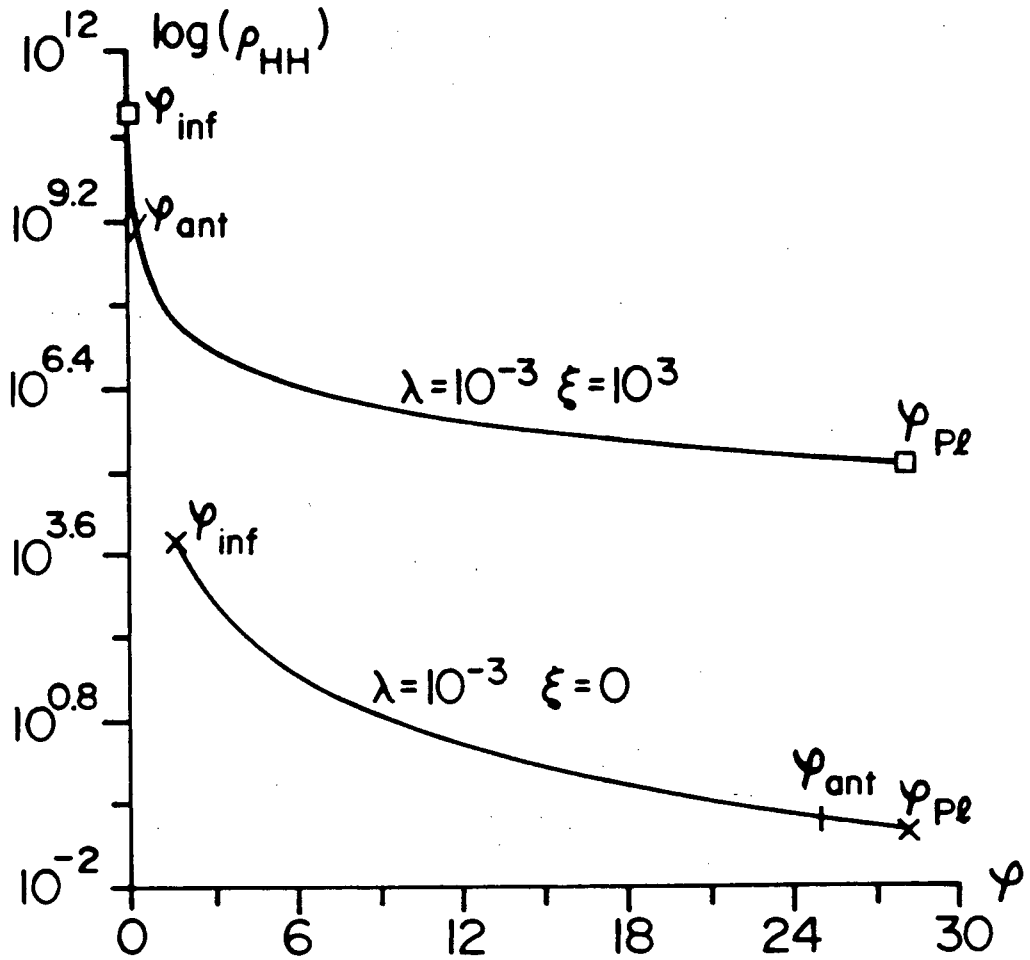

Figure 14.13: *This is the Hartle-Hawking analog of fig.11 . This time, both minimal and non-minimal coupling yield* $P_{HH}(\varphi_{ant}, \varphi_{Pl}) \approx 0$. *As noted in figs.9,11 , the 'physical' potential* $V(\varphi)$ *is* not *slowly varying in the sector* $\varphi_{inf}|_{(\xi>>1)} < \varphi < \varphi_{inf}|_{(\xi=0)}$.

Chapter 15

The subsequent non-minimally coupled inflation

Once a given sub-universe (chapter 14) has reached its initial classical size, it starts evolving according to the equations of motion that follow from the action (127) (see chapter 2). The initial conditions for these equations can be found from the correlations of the semi-classical Lorentzian wave function - this is indeed the main aim of quantum cosmology.

It is easy to see that the matter field starts off with a vanishing kinetic energy. In effect, the wave function is peaked around classical trajectories with momentum

$$P_{\tilde{\varphi}} \propto \frac{\partial \tilde{S}_V}{\partial \tilde{\varphi}} \quad , \tag{15.198}$$

where $\tilde{S}_V$ is the logarithm of the oscillatory term in eq.(185). Note that $\tilde{S}_V$ depends on $\tilde{\varphi}$ only through $\tilde{V}(\tilde{\varphi})$, which is constant to a high accuracy in the domain of interest ($\varphi > \varphi_{cr}$, see eq.(168)). Thus, at least in models comparable to ours, the inflationary initial condition $\dot{\tilde{\varphi}} \approx 0$ is not a surprising outcome. Rather, it is directly traceable to the fact that the minisuperspace is restricted 'by hand' to the domain of constant potential.

Given the initial condition $\dot{\varphi} \approx 0$ (which follows from $\dot{\tilde{\varphi}} \approx 0$ and eqs.(131,148)) and the restriction $\varphi > \varphi_{cr} = \xi^{-1/2}$ that we imposed throughout, we can infer from Part I that the subsequent classical evolution will be inflationary.

We find it suitable, after this investigation of the quantum era, to summarize here for the reader the main features of the subsequent classical inflation (Part I). But first, we

should note that the three-geometry was assumed there to be flat, whereas here we consider closed universes. However, because of the rapid expansion, spatial curvature terms become quickly negligible. Hence, the Lorentzian trajectories on our minisuperspace are accurately described by Part I.

Let us first consider small positive values of ξ , say $0 < \xi < 1/3$. Then, when non-minimal coupling becomes negligible ($\xi\varphi^2 < 1$), φ is still large enough for minimally coupled inflation to proceed ($\varphi > m_{Pl}/3 \approx \sqrt{3}$ in our conventions). Thus, inflation in this case goes through both a non-minimally coupled and a minimally coupled phase. The most interesting range of ξ turns out to be $\xi >> 1$. Then inflation is non-minimally coupled throughout. Here are two attractive features of this case:

1) the constraint on the self-coupling (dictated by the amplitude of density perturbations) is relaxed considerably: $\lambda_{constraint} \mid_{\xi>1} \approx 48 N_e \xi^2 \, \lambda_{constraint} \mid_{\xi=0}$. If $\xi = 10^3$, for example, then the excessive constraint: $\lambda \sim 10^{-12}$ in the minimally coupled model of ref.[16] is relaxed to $\lambda \sim 10^{-3}$;

2) the minimal value for obtaining N_e e-folds (the 'anthropic' value) is $\varphi_{ant} \equiv \sqrt{N_e/\xi}$. (Recall that the condition for obtaining any inflation at all is $\varphi < 1/\sqrt{\xi} = \varphi_{cr}$.)

In the terminology of chapter 14 one may conclude that, compared to minimal coupling, non-minimal coupling allows many more of the homogeneous regions $\{0 < \varphi_{initial} < \varphi_{Pl}\}$ to become anthropic sub-universes.

Chapter 16

Application to induced gravity

For the comfort of the reader, we begin by summarizing the main features of induced gravity classical (inflationary) cosmology (see Part II).

In this model, the gravitational constant G is induced dynamically in the early Universe via a scalar field φ [29,39-41]. Including the appropriate boundary term (analogous to eq.(127)), the action for this mechanism can be written as

$$S_{IG} = \int d^4x\sqrt{-g}\left\{ \frac{1}{2}\xi R\varphi^2 + \frac{1}{2}\varphi_{;\alpha}\varphi^{;\alpha} - \lambda\left(\varphi^2 - \frac{1}{\xi}\right)^2 \right\}$$
$$+ \int d^3x\, h^{1/2}\xi\varphi^2 K \quad . \tag{16.199}$$

The field φ rolls down until it reaches the potential's minimum. It eventually thermalizes there and acquires the constant value $\xi^{-1/2}$. The now constant quantity $\xi\varphi^2$ can be replaced in eq.(199) by $(8\pi G)^{-1}$ $(= 1$ in our conventions): the Newtonian constant has been generated.

Both 'new' inflation[14,15] ($\xi\varphi^2_{initial} < 1$) and chaotic inflation[20] ($\xi\varphi^2_{initial} > 1$) are possible in this model[42,27,38,2]. However, the 'new' version is at a net disadvantage[2]: it only works with very small values of ξ and λ . This case will not be considered here.

The chaotic version of induced gravity, on the other hand, is similar to chaotic non-minimally coupled inflation (see Part I) in the regime $\xi\varphi^2_{initial} >> 1$, with essentially two differences: 1) in non-minimal coupling the potential minimum mass scale is arbitrary (it is set to zero in eq.(123)) and more importantly it is independent of ξ , while in induced

gravity it *has* to be set to $\xi^{-1/2}$; 2) the equality

$$\Omega(\varphi) = \xi\varphi^2 \ , \tag{16.200}$$

which was only approximate in our non-minimal coupling calculations, holds exactly in induced gravity. This means in particular that the correspondence between the conformal and the physical pictures is simpler (e.g. eq.(148) holds exactly).

One may now redefine the variables a and φ in the same way as in non-minimally coupled ordinary gravity (eq.(142)), except that Ω is given by eq.(200). Then obviously all the results of Part III apply equally well to the chaotic induced gravity model. Perhaps the only non-trivial step here is that the conformal potential turns out to be the same as in non-minimal coupling *to first order* (see eq.(168)):

$$\tilde{V}(\tilde{\varphi})_{IG} = \frac{\lambda\left(\varphi^2 - 1/\xi\right)^2}{\xi^2\varphi^4} \approx \frac{\lambda}{\xi^2}\left(1 - 2\frac{2}{\xi\varphi^2}\right) \ . \tag{16.201}$$

Chapter 17

Limitations of the quantized model

The $\lambda\varphi^4$ potential has the convenient feature of making the conformal potential $\tilde{V}(\tilde{\varphi})$ extremely flat. Our conclusions should also hold for any matter potential dominated at large values of φ by a quartic term. This model seems to provide a natural example where the matter derivatives in the WD equation can be neglected[51,52]. Moreover, it is interesting that one thus obtains solutions over a domain where the *physical* potential $V(\varphi)$ is *not* slowly varying. However, we think that only a genuine two-dimensional integration would definitely settle the issue of the wave function's φ dependence. (Note that such an integration has been investigated in a different context by Halliwell[83,84].)

It is often claimed that one of the successes of minisuperspace cosmology is that it "predicts" inflation. We feel however that this cannot be said of any of the analytical models we are aware of, including ours. The reason is simple: the minisuperspace is restricted to the domain where the matter potential (in our case the conformal potential $\tilde{V}(\tilde{\varphi})$) is flat. But this is precisely the domain where the classical behaviour is *already* known to be inflationary. Moreover, this restriction is also what is responsible for the inflationary initial condition $\dot{\varphi} = 0$ (see chapter 15). Whether the density profile resembles that of Vilenkin's or Hartle and Hawking's wave function is irrelevant here- as long as the distribution over the *non-inflationary* regions remains unknown, we cannot truly evaluate the likelihood of inflation.

We might also add that in our opinion, even if one could show (e.g. numerically; see also refs.[83,84]) that inflationary regions of minisuperspace have a larger measure

than non-inflationary ones, it would still be too optimistic to claim that inflation is predicted from first principles. In effect, one should keep in mind that many of the restrictions imposed on these models are biased in favour of inflation: 1) infinitely many degrees of freedom are disposed of, hence homogeneity (or sometimes quasi-homogeneity) is built into the model; 2) the wave function is made *by hand* (as part of the boundary conditions) to be independent of φ over some domain of minisuperspace (for us, at $a = 0$). This is at least partially responsible for the weak φ dependence of the solution in the Lorentzian domain, which in turn translates into a negligible classical kinetic term (chapter 15) - a desirable initial condition for primordial inflation. We conclude that this approach is hardly convincing as a means of comparing inflationary initial conditions to non-inflationary ones - the claim that inflation is "predicted" can be true only in some weak sense.

Chapter 18

General conclusions

We have shown in Part I that by dropping the usual assumption of minimal coupling between the inflaton and the gravitational field, the familiar severe constraints on the shape of the inflaton's potential due to density fluctuations can be considerably relaxed for chaotic inflation. Despite previous belief on this matter, a strong coupling produces a stable consistent model for inflation with many appealing features. The naive concern that the large negative mass introduced by non-minimal coupling may destabilize the matter field modes (including the homogeneous mode, the inflaton) turns out to be unfounded - the model is in fact most attractive for the largest values of the coupling ξ . In effect, considerations regarding the amplitude of density perturbations constrain the *ratio* λ/ξ^2 rather than λ . Thus, by making a suitable choice of ξ , one can allow the self-coupling λ to be as large as desired. It is found that for large ξ the amplitude of density perturbations is much smaller than in $\xi = 0$ models: $(\delta\rho/\rho)\,|_{\xi>1} \approx (48N_e\xi^2)^{-1/2}\,(\delta\rho/\rho)\,|_{\xi=0}$, where $N_e \sim 70$. For example, this represents a drop of over four orders of magnitude for $\xi = 10^3$. This same value results in a dramatic nine orders of magnitude weakening of the constraint on λ according to our formula: $\lambda_{constraint}\,|_{\xi>1} \approx 48N_e\xi^2\,\lambda_{constraint}\,|_{\xi=0}$. Thus, non-minimal coupling seems to provide a simple solution to the long-standing problem of density perturbations over-production in the inflationary scenario.

In Part II we showed that the chaotic version of induced-gravity inflation (with a quartic potential) is more attractive than its ordinary counterpart, in that it requires

much weaker constraints on initial conditions and inflaton couplings. In particular, 1) for the *ordinary* model the very existence of an inflationary solution, as well as the requirement of producing enough inflation, forces ξ to be small. This in turn leads to a very stringent constraint on λ if the density perturbations are to fall within the limits inferred from large scale observations. 2) For the *chaotic* model it is legitimate to choose ξ to be as large as desired. Hence the constraint on λ can be made several orders of magnitude weaker than in the ordinary model (eqs.(113,118)). Thus we disagree with the previous belief[27] that these two versions of induced-gravity inflation possess basically the same unattractive features.

Our first task in Part III was to write a non-minimally coupled Wheeler-DeWitt equation that is invariant under redefinitions of minisuperspace variables. We then used this freedom to solve this equation in a configuration space where it is diagonal.

We have emphasized that both this and previous minisuperspace models have a rather limited predictive power. Still, we have tried to make the best of the wave function $\Psi(a, \varphi)$ by giving it a meaning that makes it easier to interpret probabilistically. To illustrate our idea, we speculated that instead of being the 'wave function of the Universe', this minisuperspace wave function could describe the quantum creation of inflationary sub-universes in Linde's chaotic Universe. In the process, however, it was assumed without proof that such sub-universes would be uncorrelated.

We then tried to determine what questions can effectively be answered in this approach and we found that, although the final non-minimal coupling distributions differ from minimal coupling ones, they yield qualitatively the same predictions:

1) According to Hartle and Hawking's wave function, virtually none of the sub-universes should be anthropic (i.e. be like the sub-universe containing our observable Universe). Nevertheless, given the strong limitations of the minisuperspace approach (chapters 15,17), it would be premature to rule out the H-H solution on this ground.

2) The prediction from Vilenkin's wave function is also in a sense extreme: virtually all sub-universes should be anthropic. However, this result is a positive point for Vilenkin's proposal only because this latter has many intuitively appealing features. But, as a rule, it would be unwise to focus on designing ad hoc boundary conditions that make our observable Universe the only possible outcome of quantum cosmology. Imagine a solar physicist with no access to observations from any other star but the sun. He could well design, modulo some fine tuning of parameters and initial conditions, a model of stellar evolution that *inevitably* produces sun-like stars. A temptation of this kind naturally arises in fields of knowledge with little experimental control. Once he would have successfully completed his program, our solar physicist would have the wrong theory and a wrong (and rather boring) picture of the Milky Way.

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