FINITE DENSITY EFFECTS IN GAUGE THEORIES

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Abstract

Various effects of finite fermionic densities in gauge theories are studied. The phase structure of $SU(N)$ gauge theories with fermions as a function of imaginary chemical potential is related to the confining properties of the theory. This phase structure is controlled by a remnant of the Z_N symmetry which is present in the absence of fermions. At high temperature the theory has a first-order phase transition as a function of imaginary chemical potential. This transition is expected to be absent in the low-temperature phase. It is shown that properties of the theory at nonzero fermion density can be deduced from its behaviour at finite imaginary chemical potential.

Anomalies in gauge theories are introduced using various two-dimensional models. In particular, the chiral Schwinger model is shown to be consistent despite being anomalous. The effects of finite densities in anomalous gauge theories is investigated. It is found that, contrary to some recent claims, an effective Hamiltonian (obtained by integrating out the fermions) cannot be obtained by the simple inclusion of a Chern-Simons term multiplying the fermionic chemical potential. The importance of dynamical effects is stressed and a mechanism for producing primordial magnetic field is suggested.

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Chapter 1

Introduction

The reductionist approach for describing the physical world has been very successful. A key factor to its success has been the discovery of the existence of various symmetries occuring in Nature. The reductionist approach can be traced back to the fifth century B.C. and the work of two Greek philosopher, Leucippus and Democritus, who introduced the notion of atoms. The atomic hypothesis remained mostly a philosophical concept until the eighteenth century when it was argued by Lavoisier that all substances were composed of chemical elements whose existence could be verified experimentally. The next crucial step was made in 1808 by Dalton with his atomic theory that incorporated the atomic weight as an essential characteristics of atoms. This allowed Mendeleev in 1869 to recognize a certain symmetry pattern in the occurence of those elements, symmetry which was to be explained by Bohr's atomic theory in 1922.

Bohr's atomic theory could be hailed as a triumph of reductionism as the number of fundamental building blocks was reduced to two: the electron and the proton. This simplicity wasn't to last very long as Chadwick discovered another fundamental particle, the neutron, in 1932. That same year Anderson observed the positron which had been predicted a few years before by Dirac. Physicists at that point had identified three basic interactions in nature: gravity, electro-magnetism and the strong interaction. Yukawa had postulated that the latter interaction, which acted as the basic glue for the nuclei, was mediated by a new particle called the pion. The muon was discovered shortly afterwards and was incorrectly identified as being the pion. This confusion wasn't to last too long as the real pion was soon observed. By the end of the 1950's, more than 200 of these "fundamental" building blocks had been identified mostly from the study of cosmic rays. On the theoretical side, quantum electro-dynamics had established itself as *the* description of electro-magnetism at the most fundamental level.

Just as Mendeleev helped uncover a deeper level of structure by exposing an existing symmetry pattern in the occurence of chemical elements, Gell-Mann and Zweig recognized an underlying symmetry of this particle zoo and postulated the existence of quarks. The existence of this underlying symmetry and other symmetries of the same type prompted a flurry of theoretical descriptions. By the early 1970's the dust settled and a model known as the standard model started its reign. In the standard model, the building blocks come in families. Each of these families contains 4 spin-$\frac{1}{2}$ fermions, two of these are the aforementioned quarks while the other two are called leptons. Four fundamental interactions describe the behaviour of these particles. We already mentioned gravity, electro-magnetism and the strong interaction. The fourth is called the weak interaction and is the interaction involved in some nuclear decays. These basic interactions have been described as resulting from some underlying symmetry in nature. The models describing these symmetries, ignoring gravity for the moment, are known as gauge theories and their interactions are mediated by the exchange of new types of particle called gauge bosons. The photon, massless mediator of the electro-magnetic interaction, is a familiar example of a gauge boson.

The underlying symmetries are described in the language of group theory. For the strong interaction, the group is $SU(3)_C$ where the label C identifies "colour" as the basic charge. The weak interaction and electromagnetism are grouped together in the so-called electro-weak interaction described by the group $SU(2)_L \times U(1)_Y$. The label L refers to the fact that only particles with left-handed helicity participate in the weak interaction while Y refers to a property called hypercharge which is partly related to electromagnetism. In order to preserve the symmetries, these gauge bosons have to be massless. However it is a fact of life that some of these, namely the W and the Z bosons, are massive. This is now understood to be the result of spontaneous symmetry breaking: the physical vacuum, one of the ground states of the theory, does not possess the full symmetry of the theory. A familiar example of spontaneous symmetry breaking occurs in ferromagnetism where, below a certain temperature the ground state presents a well defined magnetization which singles out a direction in space whereas the system usually possesses a larger spatial symmetry.

As a result of spontaneous symmetry breaking in the electro-weak interaction, we only see a remnant of the full symmetry namely the usual electro-magnetism described by the group $U(1)$. It is believed, however, that at high temperature the full symmetry of the electro-weak theory must be restored just as the spatial isotropy of a ferromagnet is restored once it is heated above its Curie temperature.

1.1 Quantum mechanical breakdown of symmetries: anomalies

The general features (particle content, symmetries, etc.) in field theories are generally expressed by means of a classical action. Quantum corrections to the classical action can be calculated leading to an effective action. In some cases, as a result of infinities arising in this type of calculation, the symmetries present in the classical action are not present in the effective action, once the infinities are properly regulated. This quantum mechanical breakdown of symmetries is called an anomaly. Examples of such broken symmetries are the conformal anomalies in gravity theory [15] and in condensed matter theory [107], the chiral gauge anomalies in particle physics [3, 14, 111], etc.

The anomaly phenomenon is also associated with the non-conservation of some current related to the broken symmetry. For the case of gauge theories this non-conservation

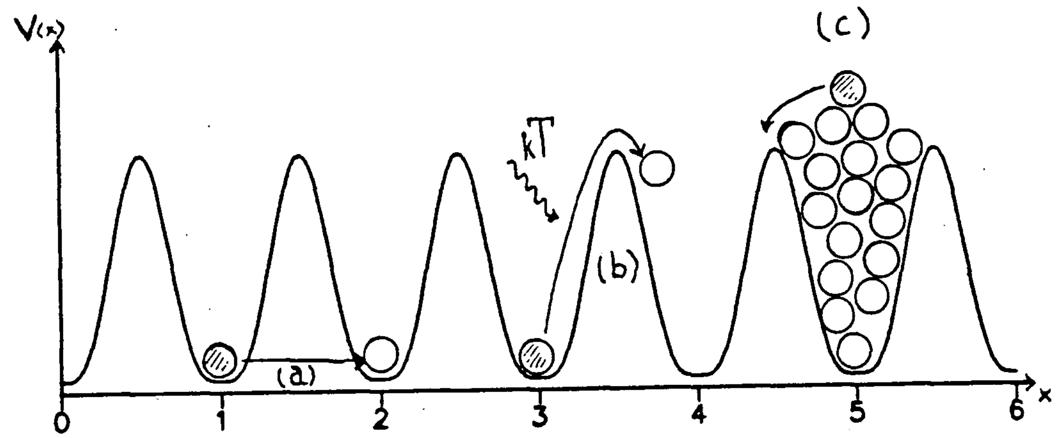


Figure 1.1: Three possible ways for a particle in a periodic potential well to hop from one minimum to another. (a) Tunnelling. (b) Thermal excitation. (c) Instability due to large density.

takes the general form

$$\partial_\mu J^\mu[\Psi] = \hbar f[A] \tag{1.1}$$

where $J[\Psi]$ is the non-conserved current associated to some matter field Ψ and A represents some gauge field. If we require the physical fields to have compact support, it is possible to integrate the above equation [20, 56] and to show that there is a change in the charge Q associated with the current $J[\Psi]$ given by

$$\Delta Q = N_{CS}[A_{\text{final}}] - N_{CS}[A_{\text{initial}}] \tag{1.2}$$

where the $N_{CS}[A]$ can be chosen, by an appropriate gauge transformation, to be integers. This leads to various interesting phenomena that can perhaps be illustrated by an analogy.

Consider a particle in a periodic potential (fig. 1.1) where each minimum is labelled by a different number. We will associate the label of the local well where the particle is located with the anomalous charge Q. Initially we consider the particle to be at rest at the minimum of the n^{th} well. Classically the particle is stationary and the corresponding charge Q is conserved. Quantum mechanically the particle can tunnel through to a different minimum, m, with the corresponding change in charge $\Delta Q = m - n$. However, because of its tunnelling nature, we expect this process to be exponentially suppressed.

Straight tunnelling is not the only process by which a particle can hop from one minimum to another. For example, if we imagine heating up this system, the particle could acquire enough thermal (kinetic) energy to go over the barrier between two minima without the exponential suppression associated with quantum mechanical tunnelling. We can also imagine having more than one particle, each of them with a finite size[1], so as to fill up a given well totally. Any extra particle that we attempt to add to this local well will not stay localised, the system being unstable.

The above processes are thought to have their counterpart in many quantum field theories. For example, in the electroweak theory, the baryon number is not conserved due to an anomaly. This non-conservation can occur as the result of some tunnelling events. In quantum field theory these tunnelling events are described in terms of solution to the classical equations in Euclidean space (instantons) [13]. However the exponential suppression of instanton-induced transitions makes these essentially unobservable (e.g. if the proton could decay only through instantons, its lifetime would be of the order of 10^{600} years!) [52, 53]

The thermal hopping mentioned before also has its counterpart in quantum field theory. It occurs via an unstable solution of the equation of motion in Euclidean space called a sphaleron [60]. Preliminary calculations [61, 8, 9] show this rate to be so large as to lead to an almost complete disappearance of any baryon asymmetry present in the early universe. At the present time, there is no obvious way to accomodate this result in any cosmological scenario while retaining consistency with astrophysical observations[2].

[1]Or, like fermions, filling up some distinct energy levels.

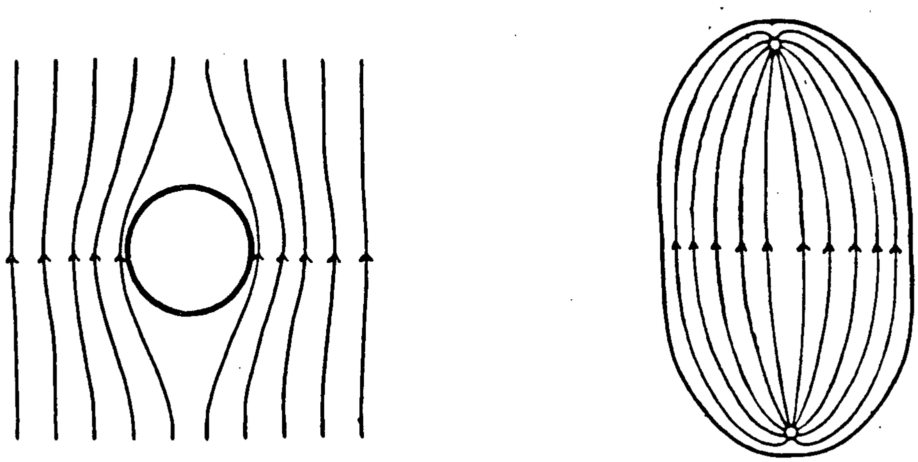
[2]See however [62].

Finally, the third mechanism for anomalous decay, namely decays induced by the presence of large densities, has been suggested to also have its counterpart in quantum field theory, more specifically in the electroweak theory [85, 86, 87, 67]. It has been argued that systems with large baryonic densities are inherently unstable due to the baryon current anomaly. We will look at this possibility in more detail in chapter 5.

1.2 The confining phase transition

The anomalies of the standard model are not its only interesting features. While the quark model has been very successful at explaining the profileration of hadrons, free isolated quarks yet remain to be observed. This has led to the notion of confinement: it is believed that all colour-charged objects form bound systems that are colour-neutral. This peculiar behaviour can be attributed to some properties of the vacuum in quantum field theories. In particular, in addition to carrying topological quantum numbers the vacuum has some (colour) electric polarizability and (colour) magnetic permeability. This can lead to screening (for QED) or anti-screening (for QCD) of charges. This is better illustrated by an analogy.

Consider a sphere of superconducting material placed in an external magnetic field. It is well known that, under reasonable conditions (where both the temperature and the external magnetic field are below a certain threshold), the field lines will be expelled from the interior of the sphere. This is known as the Meissner effect (see fig. 1.2). In QCD, the analogue of the superconducting material is the vacuum itself. If we place some colour-charged objects in the QCD vacuum, the vacuum tends to push the colour electric field away. Thus if for example we have two coloured objects of opposite charge such that one can act as a source for the field lines and the other one as a sink, the fields lines are confined to a limited region of space as opposed to extending over all space as



(a) (b)

Figure 1.2: Comparison between Meissner effect for a superconducting sphere (a) and confinement for a meson (b).

they do in electro-magnetism.

While the above analogy with superconductivity illustrates the confinement of quarks, it fails to illustrate the fact that quarks appear to behave as though they were free inside hadrons. Perhaps yet another analogy could help visualise this phenomenon.

Imagine the vacuum to be similar to water (liquid) while hadrons are like little steam bubbles. Inside the bubbles, water-vapour molecules behave essentially like the particle of an ideal gas and we have the analogue of asymptotic freedom. Yet outside the bubbles, these molecules cease to exhibit their ideal gas-like properties. This analogy can be taken even further.

If we imagine heating up this water-vapour system, we would create more and more steam bubbles leading to an eventual percolated state in which the vapour molecules could wander freely over all the system. A similar thing is believed to happen in QCD.

At large enough temperature or density it is believed that a phase transition takes place with the coloured objects being no longer confined. While there is no evidence for such a phase transition from experiments yet, its consequences for cosmology in general and nucleosynthesis [110] in particular could be very important. For instance, it has been suggested that if this phase transition is strongly first order, large local fluctuations in the baryon-to-photon ratio can arise leading to considerable changes on the limits of the allowable baryon density from nucleosynthesis[7, 4, 99]. Shock waves could also be formed in such a phase transition creating potentially important density fluctuations[16]. Even stranger events like the formation of "quarks nuggets" as a source of dark matter have been suggested to take place under some appropriate conditions[108].

In order to study properly these cosmological implications, a better understanding of the phase transition is needed. In particular one would like to have an order parameter characterizing the two phases and possibly allowing us to compute the order of the transition. For pure gauge theories, *i.e.* in the absence of fermions, such a quantity exists. It is called the Wilson Line [106, 76, 95] and has been studied extensively. Numerical results using Monte Carlo methods [96] show quite convincingly that the transition is first order for the $SU(3)$ gauge theory. However, a problem arises when one wants to introduce quarks into the picture. In this case, the Wilson Line can no longer be used as an order parameter (we will explain why later). In fact, it is not even sure that a phase transition in the thermodynamic sense does exist when quarks are present. There has been some indication from Monte Carlo simulations that there is indeed a phase transition when quarks are present and that it is a first order one but a consensus is far from being established on this point[97].

1.3 Overview of this thesis

This thesis is essentially divided in two parts. In the first part, we attempt to get some insight into the phase structure of QCD –and of other $SU(N)$ gauge theories. We do this in the second and third chapter. In this first part we totally ignore effects due to the non-trivial topology of gauge theories. To use our analogy of a particle in a periodic potential, this means that we concentrate on the physical phenomena that take place within a single well.

The content of the second chapter is a review[3]. Starting with an introduction to gauge theories we proceed to derive a partition function for these theories. We then introduce the Wilson Line and show that while it can be used as an order parameter for pure gauge theories, it fails to characterize deconfinement when dynamical fermions are included. A derivation of a one loop effective potential for the Wilson Line, based on the work of Nathan Weiss [101, 102, 103] follows.

The next chapter contains new material [80]. This work is essentially an extension to the effective potential calculation of the second chapter to include finite baryon density effects. We show that this leads naturally to the introduction of an imaginary chemical potential. The correspondingly modified effective potential is analyzed in order to deduce the phase structure of the theory.

The second part of this thesis starts in the fourth chapter and deals with anomalies in gauge theories. The fourth chapter contains an introduction to anomalies in some two-dimensional fermionic models with an abelian gauge symmetry. We show how the existence of these anomalies is linked to the existence of an infinite number of negative energy levels (the Dirac sea) in these system. We then review a remarkable property of certain two-dimensional theories, namely the equivalence between bosonic and fermionic

[3]However we try to present these known results using some original derivations whenever possible.

quantities. We conclude this chapter with some original work on anomalies in the chiral Schwinger model [64]. The fifth chapter is devoted to the study of the effects of anomalies in system with large fermionic density. Others have argued that such system are intrinsically unstable [85, 86, 87, 67]. By studying a "toy" model [81] we demonstrate that their argument is based on an incorrect premise. We then make some conjectures as to how our results translate into some more realistic theories. We refer the reader to Appendix A for a description of our notation.

Chapter 2

Confinement in gauge theories

The experimental cross-sections from deep inelastic lepton-nucleon scattering experiments are in agreement with a model in which hadrons are described as composite objects whose components are generically called partons. Although partons aren't produced as free particles in the final states of deep inelastic scatterings they behave as though they were only very weakly bound inside the target nucleons. These results appear to be explainable within the context of a gauge theory based on the symmetry group $SU(3)$, known as quantum chromodynamics (QCD).

It can be shown that gauge theories based on a non-abelian symmetry group[1] share a property known as asymptotic freedom. This simply means that the effective coupling constant in those theories depends on the energy scale in such a way as to be vanishingly small as the energy becomes infinite. The property of asymptotic freedom is believed to allow one to use perturbation theory with good reliability for calculating cross-sections at high energy. Despite the fact that all Yang-Mills theories exhibit asymptotic freedom, comparisons between theory and experiments yield only QCD as a reasonable candidate for describing the strong interaction.

In principle we should be able to use QCD to calculate all the hadron masses and their various decay rates. It turns out to be just about impossible to do so however as perturbation theory breaks down at low energies where the coupling constant becomes of order unity. Some attempts have been made using non-perturbative techniques but

[1] These are called Yang-Mills theories.

without too much success so far. In what follows, we look at QCD and other $SU(N)$ gauge theories and try to understand if a phase transition takes place as a function of temperature.

2.1 Introduction to gauge theories

The Lagrangian for a free massive fermi field is

$$L = \int d^3x \left[\overline{\Psi}(x)(i\gamma^\mu \partial_\mu - m)\Psi(x) \right]. \tag{2.1}$$

This Lagrangian possesses a global $U(1)$ invariance, *i.e.* it is invariant under the transformation

$$\begin{aligned} \Psi(x) &\longrightarrow \Psi^g(x) = e^{-i\theta}\Psi(x) \equiv \mathcal{U}\Psi(x) \\ \overline{\Psi}(x) &\longrightarrow \overline{\Psi}^g(x) = \overline{\Psi}(x)e^{i\theta} \equiv \overline{\Psi}(x)\mathcal{U}^{-1} \end{aligned} \tag{2.2}$$

where θ is a constant phase. This is very reminescent of ordinary quantum mechanics where the wave function is arbitrary up to an overall phase. We can consider more general symmetries in which $\mathcal{U}$ is taken to be an element of $SU(N)$[2], the group of $N \times N$ unitary matrices with unit determinant. We must also take $\Psi(x)$ to be an N-tuplet[3] of field operators

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_N(x) \end{pmatrix}. \tag{2.3}$$

We will call the internal degree of freedom of Ψ (the label specifying the position in the N-tuplet) *colour* and we will call *quarks* the fermions described by the components of $\Psi(x)$.

[2]We could easily generalize this to other Lie groups.

[3]We will only consider fermions in the fundamental representation.

It is also useful to consider theories in which the Lagrangian possesses a *local* symmetry *i.e.* we demand that the Lagrangian be invariant under the following transformation:

$$\begin{aligned} \Psi(x) &\longrightarrow \Psi^g(x) = \mathcal{U}(x)\Psi(x) \\ \overline{\Psi}(x) &\longrightarrow \overline{\Psi}^g(x) = \overline{\Psi}(x)\mathcal{U}^{-1}(x) \\ \mathcal{U}(x) &\in SU(N). \end{aligned} \tag{2.4}$$

In order to do this we need to modify our notion of derivative and introduce a so-called covariant derivative

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu - igA_\mu \tag{2.5}$$

where the potential A_μ transforms as

$$A_\mu \longrightarrow A_\mu^g = \mathcal{U}A_\mu\mathcal{U}^{-1} + \frac{i}{g}\mathcal{U}\partial_\mu\mathcal{U}^{-1}. \tag{2.6}$$

We take the Lagrangian to be

$$L = \int d^3x \left[\overline{\Psi}(x)(i\gamma^\mu D_\mu - m)\Psi(x)\right]. \tag{2.7}$$

which is invariant under the above transformation.

We now introduce the field strength $F_{\mu\nu}$

$$[D_\mu, D_\nu]\,\Psi \equiv (D_\mu D_\nu - D_\nu D_\mu)\Psi = -igF_{\mu\nu}\Psi. \tag{2.8}$$

It is then easy to verify that $F_{\mu\nu}$ can be written in terms of A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\,[A_\mu, A_\nu] \tag{2.9}$$

and that it transforms as

$$F_{\mu\nu} \longrightarrow F_{\mu\nu}^g = \mathcal{U}F_{\mu\nu}\mathcal{U}^{-1} \tag{2.10}$$

under a gauge transformation.

It will be convenient to write A_μ and $F_{\mu\nu}$ as linear combinations of the group generators λ^a:

$$\begin{aligned} A_\mu(x) &= A^a_\mu(x)\lambda^a \\ F_{\mu\nu}(x) &= F^a_{\mu\nu}(x)\lambda^a \end{aligned} \tag{2.11}$$

with the generators normalized as follows:

$$\mathrm{tr}(\lambda^a\lambda^b) = \frac{\delta^{ab}}{2}. \tag{2.12}$$

These generators satisfy the algebra

$$\left[\lambda^a, \lambda^b\right] = if^{abc}\lambda^c \tag{2.13}$$

where the f^{abc}'s are the totally antisymmetric structure constants of the group. The field strength can then be rewritten as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{2.14}$$

Using the field strength we can add one more Lorentz and gauge invariant term to the Lagrangian in order to make the gauge field dynamical. This yields the usual Yang-Mills Lagrangian

$$L = \int d^3x \left[\overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right]. \tag{2.15}$$

It is easy to derive the Euler-Lagrange equations of motion for this Lagrangian. Under variation with respect to A^a_μ, we have

$$\partial_\mu \frac{\delta L}{\delta \partial_\mu A^a_\nu} - \frac{\delta L}{\delta A^a_\nu} = \partial_\mu F^{a\nu\mu} - gf^{abc}A^b_\mu F^{c\mu\nu} - g\overline{\Psi}\gamma^\nu\lambda^a\Psi = 0. \tag{2.16}$$

The $\nu = 0$ component is the Yang-Mills analogue of Gauss' law. This is more easily seen if we define a color electric field, $E^a_i = F^a_{i0}$, and regard $g\overline{\Psi}\gamma^\mu\lambda^a\Psi$ as a current density. We thus rewrite this analogue of Gauss' law as

$$\vec{\nabla}\cdot\vec{E}^a + gf^{abc}\vec{A}^b\cdot\vec{E}^c = \rho^a \tag{2.17}$$

where $\rho^a(x)$, the zeroth component of the current density, is the charge density. If the symmetry group is $U(1)$, this reduces to the usual $\vec{\nabla}\cdot\vec{E}=\rho$ of electrodynamics.

We now proceed to derive a Hamiltonian for this system. The canonical momenta are:

$$\begin{aligned} \Pi_0^a &\equiv \frac{\delta L}{\delta \dot{A}_0^a} = 0 & \Pi_i^a &\equiv E_i^a \equiv \frac{\delta L}{\delta \dot{A}_i^a} = F_{0i}^a \\ \Pi_\Psi &\equiv \frac{\delta L}{\delta \dot{\Psi}} = i\Psi^\dagger & \Pi_{\Psi^\dagger} &\equiv \frac{\delta L}{\delta \dot{\Psi}^\dagger} = 0. \end{aligned} \tag{2.18}$$

We recognize $\Pi_0^a = 0$ as a constraint. We thus choose to derive the Hamiltonian using Dirac's method for constrained systems[32, 50]. A preliminary Hamiltonian is given by

$$H_0 = \int d^3x \left(E_i^a \dot{A}_i^a + \Pi_\psi \dot{\Psi}\right) - L. \tag{2.19}$$

Defining a colour magnetic field, $B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a$, we obtain after performing an integration by parts[4]

$$\begin{aligned} H_0 = \int d^3x \Big[&\overline{\Psi}(i\vec{\gamma}\cdot\vec{D}+m)\Psi + \frac{1}{2}(\vec{E}^a\cdot\vec{E}^a + \vec{B}^a\cdot\vec{B}^a) \\ &+A_0^a(\vec{\nabla}\cdot\vec{E}^a + gf^{abc}\vec{A}^b\cdot\vec{E}^c - g\Psi^\dagger\lambda^a\Psi)\Big]. \end{aligned}$$

We then postulate the equal time canonical commutation relations

$$\begin{aligned} \left[A_i^a(\vec{x},t), E_j^b(\vec{y},t)\right] &= -i\delta^{ab}\delta_{ij}\delta(\vec{x}-\vec{y}) \\ \left[A_0^a(\vec{x},t), \Pi_0^b(\vec{y},t)\right] &= i\delta^{ab}\delta(\vec{x}-\vec{y}) \\ \left[\Psi(\vec{x},t), \Psi^\dagger(\vec{y},t)\right]_+ &= \delta(\vec{x}-\vec{y}). \end{aligned} \tag{2.20}$$

At this point, the commutator for Π_0^a and A_0^a appears to be inconsistent with the constraints $\Pi_0^a = 0$.[5] We do not need to worry about this at this point. These constraints

[4]We will always assume suitable behavior of the fields at infinity allowing us to drop the surface terms.

[5]We don't explicitely make the distinction between weak and strong conditions for the constraints as it is not essential for our derivation.

simply tell us is that the physical space is a subspace of the full phase space and that the actual number of field degrees of freedom is less than what we would have expected at the outset.

Requiring the consistency of this set of constraints under time evolution gives us another set of constraints:

$$\begin{aligned} G^a(x) &\equiv -\dot{\Pi}_0^a(x) = -i\,[\Pi_0^a(x), H_0] = 0 \\ &= \vec{\nabla}\cdot\vec{E}^a + gf^{abc}\vec{A}^b\cdot\vec{E}^c - g\Psi^\dagger\lambda^a\Psi. \end{aligned} \tag{2.21}$$

We recognize once again the Yang-Mills analogue of Gauss' law which is satisfied at all times since $\dot{G}^a = 0$ as G^a commutes with the Hamiltonian. Note that this analogue of Gauss' law does not arise as an equation of motion for one of the dynamical fields in this Hamiltonian formulation.

This completes the set of constraints satisfied by our system. We are free to add all of the above constraints to our preliminary Hamiltonian and get

$$H = \int d^3x \left[\overline{\Psi}(i\vec{\gamma}\cdot\vec{D} + m)\Psi + \frac{1}{2}(\vec{E}^a\cdot\vec{E}^a + \vec{B}^a\cdot\vec{B}^a) + v_1^a\Pi_0^a + (v_2^a - A_0^a)G^a\right] \tag{2.22}$$

where v_1^a and v_2^a are some spacetime dependent Lagrange multipliers (or *gauge velocities*) undetermined at this point.

The Hamilton equations of motion partly fix the Lagrange multipliers as we have

$$\dot{A}_0^a(x) = -i[A_0^a(x), H] = v_1^a(x). \tag{2.23}$$

We do not have such an equation for $v_2^a(x)$. If we want to proceed with this canonical formulation and reduce the phase space to the physical subspace we have to choose some gauge conditions, $\gamma_\alpha(A_\mu, E_i, \Pi_0, \Psi, \Psi^\dagger) = 0$, and impose them as constraints that do not follow from the Lagrangian. We do however have some restriction on the form of these gauge conditions. If we denote a general constraint by $\chi_\alpha = 0$, we require that the matrix $M_{\alpha\beta} \equiv [\chi_\alpha, \chi_\beta]$ be well defined and non-singular in order to retain consistency. This will be shown (for a different model) in chapter 5. We will not need to deal with this technicality in this chapter as we won't pursue the canonical formulation much further.

We can look at what happens if we were to make a gauge choice such that $A_0^a(x) = 0$. We see that this gauge choice would reduce the number of degrees of freedom in a very natural way as the Hamiltonian would no longer depend on the conjugate pairs of variables A_0^a and Π_0^a. We could have proceeded differently and fixed this gauge before deriving a Hamiltonian as is most often done. However, in doing so, we would not have recovered Gauss' law directly. Rather, we would have to either invoke the knowledge we have from the Lagrangian formalism or, as Jackiw does in [57], notice that the resulting Hamiltonian still had some time independent gauge symmetry and recover Gauss' law as the generator of the corresponding transformation. In either case the usual procedure would then have been to impose Gauss' law as a constraint defining the physical states, not as part of the Hamiltonian. We won't have to do this as Gauss' law is already included in our Hamiltonian. This will become clearer when we derive a path integral representation for the partition function.

2.2 Gauge theories at finite temperature and the Wilson Line

In order to keep our discussion as simple as possible, we will ignore the dynamical contribution from the fermions and consider a pure gauge theory in a fixed background density of charges. By this we mean the theory described by the Hamiltonian

$$H = \int d^3x \left[\frac{1}{2}(\vec{E}^a \cdot \vec{E}^a + \vec{B}^a \cdot \vec{B}^a) + \dot{A}_0^a \Pi_0^a + v_2^a(\vec{\nabla} \cdot \vec{E}^a + g f^{abc} \vec{A}^b \cdot \vec{E}^c - \rho^a)\right] \quad (2.24)$$

where $\rho^a(x)$ is the fixed background charge density.

It is useful to study the partition function

$$Z = \mathrm{Tr}\exp(-\beta H). \quad (2.25)$$

The trace in this partition function is a restricted trace over the physical subspace. There are many different ways to do this restriction.

The first way would be to fix all the gauges at the Hamiltonian level thus having only physical degrees of freedom remaining. A second approach would be the introduction of a delta functional gauge fixing constraint in the functional integral – to be derived below – using the Faddeev-Popov approach[33]. Yet another method would be to partly fix the gauge at the Hamiltonian level, like setting $A_0^a = 0$, introduce Gauss' law by means of a projection operator in the functional integral and eventually fix the remaining gauges by various methods. We will proceed with yet another method where we try to retain the gauge freedom as long as we can without using the Faddeev-Popov procedure.

Writing the partition function in the $|A_\mu^a\rangle$ representation, we have

$$Z = \int \mathcal{D}A_\mu(\vec{x})\mathcal{D}v_2(\vec{x})\, \langle A_\mu|\, \delta(\cdot)\exp(-\beta H)\,|A_\mu\rangle. \tag{2.26}$$

In this expression we retained total gauge freedom and allowed all possible choices for the gauge velocities $v_2^a(\vec{x})$ by integrating over all such possible choices keeping in mind that the actual partition function has to be calculated only on some restricted physical subspace. This restriction (which also implies some specific choices for the gauge velocities) is represented by the functional delta function, $\delta(\cdot)$, unspecified at this point. This partition function factorizes nicely:

$$Z = Z_0 Z_1 \tag{2.27}$$

where

$$\begin{aligned} Z_0 &= \int \mathcal{D}A_0(\vec{x})\langle A_0|\exp(-\beta\int d^3x\, \dot{A}_0\Pi_0)|A_0\rangle \\ Z_1 &= \int \mathcal{D}\vec{A}(\vec{x})\mathcal{D}v_2(\vec{x})\,\langle\vec{A}|\,\delta(\cdot)\exp(-\beta[H_0+\int d^3x\, v_2^a G^a])\,|\vec{A}\rangle. \end{aligned}$$

This factorization appears to justify the usual gauge choice $A_0^a = 0$ as a –partial– reduction to the physical subspace is trivially realized in this fashion. Also, if we formally rewrite $v_2^a(\vec{x}) = i\theta(\vec{x})$ we see that

$$P_G = \int \mathcal{D}\theta(\vec{x}) \exp[i \int d^3x\, \theta^a G^a] \sim \delta(G^a) \tag{2.28}$$

is a projection operator that enforces Gauss' law. Note that we did not need to insert this operator "by hand" as it has to be done normally[46, 102].

The operator $\exp(-\beta H)$ may be interpreted as a propagation operator in imaginary time over an interval β. We use this fact to express the partition function as a Feynman path integral in Euclidean spacetime. We write

$$\exp(-\beta H) = \prod_{j=1}^{N} \exp(-\frac{\beta}{N} H) \tag{2.29}$$

and insert the identity operator

$$\mathcal{I} = \int \mathcal{D}A_{\mu,j} \mathcal{D}\Pi_{\mu,j} |A_{\mu,j}\rangle\langle A_{\mu,j}|\Pi_{\mu,j}\rangle\langle \Pi_{\mu,j}| \tag{2.30}$$

between each term of the product. We also allow for possibly different gauge choices at each Euclidean spacetime point by inserting

$$\int \mathcal{D}v_2 \delta(\cdot) \tag{2.31}$$

whenever we insert the above-mentioned identity operator. Using

$$\langle \Pi_\mu^a | A_\nu^b \rangle = \frac{\delta_{\mu\nu}\delta^{ab}}{2\pi} \exp\left[i \int d^3x\, \Pi_\mu^a A_\nu^b\right] \tag{2.32}$$

we get

$$\begin{aligned} Z = \int \prod_{j=1}^{N} \mathcal{D}A_{\mu,j}(\vec{x})\, \mathcal{D}\Pi_{\mu,j}(\vec{x})\, \mathcal{D}v_{2,j}(\vec{x}) \\ \times \exp\left[-\sum_{k=1}^{N} \int d^3x \left[\frac{\beta}{N} H(A_j, \Pi_j, v_j) - i\Pi_j(A_{j+1} - A_j)\right]\right] \delta(\cdot) \end{aligned}$$

up to some normalization constant. Since Z is a trace, we identify $A_{\mu,N+1}$ with $A_{\mu,1}$. Taking the limit as $N \to \infty$, we finally obtain

$$Z = \int_{P.B.C.} \mathcal{D}A_\mu(x)\, \mathcal{D}\Pi_\mu(x)\, \mathcal{D}v_2(x)\, \delta(\cdot) \exp\left[-\int_0^\beta d\tau \int d^3x \left(H - i\Pi_\mu \dot{A}_\mu\right)\right] \tag{2.33}$$

where the "dot" now represents Euclidean time derivative and the *P.B.C.* label is to remind us that we integrate only over A-fields that satisfy periodic boundary conditions in τ. Expanding out the Hamiltonian we can see that, upon restricting the integrand to the $\Pi_0 = 0$ hypersurface and making some explicit choice for A_0, the Π_0 and A_0 contribution is equal to 1:

$$\begin{aligned} Z = \int_{P.B.C.} \mathcal{D}A_0(x)\,\mathcal{D}\Pi_0(x) \int_{P.B.C.} \mathcal{D}\vec{A}(x)\,\mathcal{D}\vec{E}(x)\mathcal{D}v_2(x)\,\delta(\cdot) \\ \times \exp\left[-\int_0^\beta d\tau \int d^3x \left(\frac{1}{2}\left\{\vec{E}^a\cdot\vec{E}^a + \vec{B}^a\cdot\vec{B}^a\right\} + v_2^a G^a + i\dot{\vec{A}}^a\cdot\vec{E}^a - v_2^a\rho^a\right)\right] \quad . \end{aligned}$$

We now integrate over the momenta – thereby restricting the gauge velocities $v_2^a(x)$ to be functions only of the gauge fields $\vec{A}^a(x)$. These gaussian integrations give us

$$\begin{aligned} Z = \int_{P.B.C.} \mathcal{D}\vec{A}(x)\,\mathcal{D}v_2(x)\,\delta(\cdot)\exp\Bigg[-\int_0^\beta d\tau \int d^3x \Bigg(\frac{\vec{B}^a\cdot\vec{B}^a}{2} - v_2^a\rho^a \\ -\frac{1}{2}\left\{(i\dot{\vec{A}}^a - \vec{\nabla}v_2^a + gf^{abc}v_2^b\vec{A}^c)\cdot(i\dot{\vec{A}}^a - \vec{\nabla}v_2^a + gf^{ade}v_2^d\vec{A}^e)\right\}\Bigg)\Bigg] \quad . \end{aligned}$$

We now rename $v_2 \to iA_0$ and get the final form

$$Z = \int_{P.B.C.} \mathcal{D}A_\mu(x)\,\delta(\cdot)\exp(-S_E)\exp\left(i\int_0^\beta d\tau \int d^3x\, A_0^a\rho^a\right) \tag{2.34}$$

where S_E is the Euclidean action

$$S_E = \frac{1}{4}\int_0^\beta d\tau \int d^3x F_{\mu\nu}^a F_{\mu\nu}^a. \tag{2.35}$$

For the sake of simplicity we temporarily restrict our discussion to the group $U(1)$. We take $\rho(x)$ to be a collection of static quarks with unit charge

$$\rho(x) = \sum_n \delta(\vec{x} - \vec{x}_n) - \sum_m \delta(\vec{x} - \vec{y}_m) \tag{2.36}$$

and obtain

$$\begin{aligned} Z &= \int_{P.B.C.} \mathcal{D}A_\mu\,\delta(\cdot)\exp(-S_E)\prod_n \exp\left[i\int_0^\beta dt\, A_0(\vec{x}_n,t)\right]\prod_m \exp\left[-i\int_0^\beta dt\, A_0(\vec{x}_m,t)\right] \\ &= \int_{P.B.C.} \mathcal{D}A_\mu\,\delta(\cdot)\exp(-S_E)\prod_n L(\vec{x}_n)\prod_m L(\vec{y}_m)^\dagger \end{aligned} \tag{2.37}$$

where we have introduce the Wilson Line operator

$$L(\vec{x}) = \exp i \int_0^\beta d\tau A_0(\vec{x},\tau). \tag{2.38}$$

Using the definition for the free energy, $\beta F = -\log Z$, we get for the vacuum

$$\beta F_0 = -\log Z(\rho = 0) \tag{2.39}$$

and for a single quark

$$\beta(F_q - F_0) = -\log < L(\vec{x}) > . \tag{2.40}$$

Similarly, the free energy of a static $q\bar{q}$ pair is given by

$$\beta(F_{q\bar{q}}(\vec{r}) - F_0) = -\log < L(\vec{x})L^\dagger(\vec{x}+\vec{r}) > . \tag{2.41}$$

A confining phase is characterized by $< L >= 0$ (infinite free energy) whereas a non-zero value indicates deconfinement[76, 68, 69].[6]

The derivation of the Wilson Line operator is slightly complicated by the non-abelian nature in a general $SU(N)$ gauge theory ($N > 1$). A static quark field obeys the equation

$$\frac{d}{dt}\Psi \equiv \left[\frac{\partial}{\partial t} - iA_0\right]\Psi = 0 \tag{2.42}$$

whose solution is the time-ordered exponential

$$\Psi(\vec{x},t) = T\ \exp\left[ig\int_0^t d\tau A_0(\vec{x},\tau)\right]\Psi(\vec{x},0). \tag{2.43}$$

Using this result, one can easily show [70] that the Wilson Line operator – whose expectation value is equal to the exponential of the free energy of an isolated test quark – is given by

$$L(\vec{x}) = \frac{1}{N}\mathrm{tr}\,T\ \exp i\int_0^\beta d\tau A_0(\vec{x},\tau). \tag{2.44}$$

[6]Note however that one must not confuse $< L >= 0$ due to ultraviolet or infrared divergences with confinement [102].

The confining phase can be related to the existence of a global Z_N symmetry. By construction, the Yang-Mills action is invariant under any local gauge transformation. At finite temperature however there is a further restriction on the allowed form of the gauge transformations due to the periodicity requirement for the gauge fields in the path integral. Without loss of generality, we write $\mathcal{U}(\vec{x},\tau+\beta) = \mathcal{U}(\vec{x},\tau)z \equiv \mathcal{U}z$. The periodicity requirement $A^g_\mu(\vec{x},\tau+\beta) = A^g_\mu(\vec{x},\tau)$ then translates as

$$\mathcal{U}zA_\mu z^{-1}\mathcal{U}^{-1} + ig^{-1}\mathcal{U}z(\partial_\mu z^{-1})\mathcal{U}^{-1} = \mathcal{U}A_\mu\mathcal{U}^{-1} \tag{2.45}$$

This requires $\partial_\mu z = 0$ thus making z a global parameter. Furthermore z must be an element of the centre of the gauge group since we need $zA_\mu z^{-1} = A_\mu$. For $SU(N)$ this means

$$z = \exp\left(\frac{2\pi in}{N}\right) \in Z_N \tag{2.46}$$

whereas for $U(1)$, the centre is $U(1)$ itself. While the action is invariant under this global symmetry, the Wilson Line is not; it transforms as

$$L(\vec{x}) \to L^g(\vec{x}) = \frac{1}{N}\mathrm{tr}\,\mathcal{U}(\vec{x},0)P\exp\left[i\int_0^\beta d\tau\, A_0(\vec{x},\tau)\right]\mathcal{U}(\vec{x},\beta)^{-1}. \tag{2.47}$$

Thus under a Z_N transformation, $L \to z^{-1}L$. The existence of this symmetry implies that $< L >= 0$. Deconfinement must then be viewed as a spontaneous symmetry breaking phenomenon. We shall see later that this spontaneous symmetry breaking is indeed present in perturbation theory at high temperature.

In a theory where dynamical fermions are present, the situation isn't as clear. A dynamical antiquark will always neutralize a static – source – quark and yield a finite free energy[35, 11, 28, 51]. Thus $< L >$ is never zero when dynamical fermions are present. This can be linked to the fact that there is no global Z_N symmetry when dynamical fermions are present. We will come back to this point later.

The inclusion of dynamical fermions also introduces some minor changes in our derivation of a path integral representation for the partition function. The anticommuting properties of the fermi fields translate as anti-periodic boundary conditions for them instead of the periodic boundary condition for the gauge fields. We refer the reader to standard review in the literature [93, 70] for a complete derivation.

2.3 Effective potential at large temperature

As we mentioned before, it is believed that perturbation theory should yield a reasonably good description of QCD at large temperature because of its property of asymptotic freedom. It is possible to classify the terms in the perturbation expansion (Feynman diagrams) in terms of powers of the Planck constant $\hbar$ which is the same as the number of closed loops for a given class of diagrams[19]. We will only calculate the lowest order in this expansion for general $SU(N)$ gauge theories, thus obtaining a one-loop effective potential[55] for the Wilson Line[102]. However, instead of performing an explicit sum over all one-loop Feynman diagrams, we use the background field method [30, 1, 2] which yields the same result in a more compact way.

The idea of the background field method is to compute the effects of quantum fluctuations about a fixed classical background by means of the field expansion

$$A_\mu = \mathcal{A}_\mu + \hbar^{1/2} a_\mu \tag{2.48}$$

where $\mathcal{A}_\mu$ is the classical background field and a_μ represents the quantum fluctuations. Note that we have re-introduced $\hbar$ as a bookkeeping variable to keep track of the order of the expansion. The Euclidean action can be written as

$$S_E = S_{gauge} + S_{fermion} \tag{2.49}$$

where

$$S_{gauge} = \frac{\mathrm{tr}}{2} \int_0^\beta d\tau \int d^3x \, \Big\{ \mathcal{F}^2 + 4\hbar^{1/2} \left(\partial_\mu a_\nu - ig[a_\mu, \mathcal{A}_\nu] \right) \mathcal{F}_{\mu\nu} \\ + 2\hbar \left[2 \left(\partial_\mu a_\nu - ig[a_\mu, \mathcal{A}_\nu] \right)^2 - ig[a_\mu, a_\nu] \mathcal{F}_{\mu\nu} \right] + \mathcal{O}(\hbar^{3/2}) \Big\}$$

and

$$S_{fermion} = \int_0^\beta d\tau \int d^3x \, \left\{ \overline{\Psi} \left[\gamma_\mu (\partial_\mu - ig\mathcal{A}_\mu) + im \right] \Psi - ig\hbar^{1/2} \overline{\Psi} \gamma_\mu a_\mu \Psi \right\} \tag{2.50}$$

In the gauge part of the action $\mathcal{F} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$ is the classical field strength and the trace is over the internal $SU(N)$ symmetry group. The $\mathcal{O}(\hbar^{1/2})$ term can be integrated by parts to yield

$$\hbar^{1/2} \int_0^\beta d\tau \int d^3x \, a^a_\nu \left(\partial_\mu \mathcal{F}^a_{\nu\mu} + g f^{abc} \mathcal{A}^b_\nu \mathcal{F}^c_{\mu\nu} + ig\overline{\Psi}\gamma_\mu \Psi \right) \tag{2.51}$$

which vanishes by virtue of the classical equations of motion.

We wish to compute the partition function with an expansion around a classical background that minimizes the full action. The first step in determining this background is to fix a gauge. We might be tempted to go to the Weyl gauge by setting $A_0 = 0$. However we cannot do this at finite temperature since it would force A_i to violate the periodic boundary conditions in the functional integral. To see this, it is sufficient to consider gauge transformations for the group $U(1)$:

$$A_\mu \rightarrow A^g_\mu = A_\mu + \partial_\mu \theta. \tag{2.52}$$

We choose θ to be given by

$$\theta(\vec{x}, \tau) = \phi(\vec{x})\tau - \int_0^\tau dt \, A_0(\vec{x}, t) \tag{2.53}$$

which yields

$$\begin{aligned} A^g_0(\vec{x}, \tau) &= \phi(\vec{x}) \\ A^g_i(\vec{x}, \tau) &= A_i(\vec{x}, \tau) + \partial_i \left\{ \phi(\vec{x})\tau - \int_0^\tau dt \, A_0(\vec{x}, t) \right\}. \end{aligned} \tag{2.54}$$

The periodicity condition $A^g_\mu(\vec{x},0) = A^g_\mu(\vec{x},\beta)$ requires $\beta\phi(\vec{x}) = \int_0^\beta dt\, A_0(\vec{x},t)$ which is incompatible with setting $A^g_0 = 0$. However we can choose a gauge where A_0 is time independent. Furthermore for $SU(N)$ we can choose A_0 to be diagonal:

$$A_0(\vec{x}) = \begin{pmatrix} \Phi_1(\vec{x}) & & \\ & \ddots & \\ & & \Phi_N(\vec{x}) \end{pmatrix} \tag{2.55}$$

and subject to the condition $\exp[i\beta(\Phi_1 + \ldots + \Phi_N)] = 1$. In this gauge the classical action for the pure gauge term can be rewritten as

$$S_{gauge} = \mathrm{tr} \int_0^\beta d\tau \int d^3x \, \left[(\partial_0\mathcal{A}_i - \partial_i\mathcal{A}_0 + g[\mathcal{A}_0,\mathcal{A}_i])^2 + \mathcal{B}^2\right]. \tag{2.56}$$

Integrating the term $(\partial_0\mathcal{A}_i)\partial_i\mathcal{A}_0$ by parts, we see that it vanishes since $\mathcal{A}_0$ is time independent. Also, since $\mathcal{A}$ is diagonal, $(\partial_i\mathcal{A}_0)[\mathcal{A}_0,\mathcal{A}_i]$ is traceless which allows us to write

$$S_{gauge} = \mathrm{tr} \int_0^\beta d\tau \int d^3x \, \left[(\partial_i\mathcal{A}_0)^2 + (\partial_0\mathcal{A}_i + g[\mathcal{A}_0,\mathcal{A}_i])^2 + \mathcal{B}^2\right]. \tag{2.57}$$

It is clear that the above action will be minimized when $\vec{\nabla}\mathcal{A}_0 = 0$. If we make such a choice the classical action is minimized with $\mathcal{A}_i = 0$. Note that this result would not necessarily hold in other gauges. In this gauge, we write the field expansion for $A_0(\vec{x})$ as

$$A_0(\vec{x}) = \Phi(\vec{x}) = \begin{pmatrix} g^{-1}C_1 + \hbar^{1/2}\phi_1(\vec{x}) & & \\ & \ddots & \\ & & g^{-1}C_N + \hbar^{1/2}\phi_N(\vec{x}) \end{pmatrix} \tag{2.58}$$

where the C_j are constant. The partition function can then be written as

$$Z = \int [\mu(\beta\Phi(\vec{x}))]\, \mathcal{D}\phi \mathcal{D}a_i \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \exp\left[-\hbar^{-1}(S_{gauge} + S_{fermion})\right] \tag{2.59}$$

where we have written out explicitely the factors of $\hbar$ as well as the invariant measure for $SU(N)$, $[\mu(\beta\Phi(\vec{x}))]$[7]. This measure factor can be written as

$$[\mu(\beta\Phi(\vec{x}))] \;=\; \prod_{\vec{x}} \prod_{j<k}^{N} [1 - \cos(\beta\Phi_j - \beta\Phi_k)]$$

[7]see appendix B

$$
\begin{aligned}
&= \exp \sum_{j<k}^{N} \int d^3x \frac{d^3k}{(2\pi)^3} \{\ln\left[1-\cos(\beta C_j - \beta C_k)\right] \\
&\quad + \hbar^{1/2}\phi(\vec{x}) \cot\left(\frac{\beta C_j - \beta C_k)}{2}\right) + \mathcal{O}(\hbar)\Big\}
\end{aligned} \tag{2.60}
$$

using the results of appendix C. Keeping only terms up to $\mathcal{O}(\hbar^0)$ in the exponential the total action – including the contribution from the measure – is then quadratic in the fields to be integrated upon. The partition function can thus be brought in the form of a product of gaussian integrals

$$
Z = Z_{gauge}\, Z_{fermion} \tag{2.61}
$$

with

$$
\begin{aligned}
Z_{gauge} &= \int \mathcal{D}A exp\left[-A^T M A\right] \\
&= (\det M)^{-1/2}
\end{aligned} \tag{2.62}
$$

and

$$
\begin{aligned}
Z_{fermion} &= \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \exp\left[\overline{\Psi} N \Psi\right] \\
&= (\det N)\,.
\end{aligned} \tag{2.63}
$$

The details of the calculation are completely given in appendix C. Using the definition for the – Helmholtz – free energy, $F = -\beta^{-1}\ln Z$, we obtain for the effective action – the free energy density – the following result:

$$
V_{eff}(C) = V_{eff}^{boson}(C) + V_{eff}^{fermion}(C) \tag{2.64}
$$

where

$$
V_{eff}^{boson}(C) = \frac{\pi^2}{24\beta^4} \sum_{j,k=1}^{n} \left\{1 - \left(\left[\frac{\beta C_j - \beta C_k}{2\pi}\right]_{\text{mod } 2} - 1\right)^2\right\}^2 \tag{2.65}
$$

and

$$V_{eff}^{fermion}(C) = \frac{-\pi^2}{12\beta^4}\sum_{j=1}^{n}\left\{1-\left(\left[\frac{\beta C_j}{2\pi}+1\right]_{\mathrm{mod}\ 2}-1\right)^2\right\}^2 . \tag{2.66}$$

This is the result for massless fermions only. The fermionic contribution unfortunately cannot be expressed in closed form for the massive case.

Some interesting facts from the detailed calculation are worth mentioning. First of all, we know that a massless spin-1 gauge boson has only two physical degrees of freedom. Yet, we computed the contribution from three degrees of freedom, $\vec{a}$. The longitudinal part of $\vec{a}$ can be easily be identified as being the unphysical one. This is verified by the explicit calculation which shows that the longitudinal part is cancelled by the contribution from the invariant measure term which acts as a ghost term. Secondly, at the classical level the action for the gauge background is identically zero for all values of C. This would seem to imply a larger symmetry than the Z_N symmetry we mentioned before. This larger symmetry is broken at the one-loop level leaving us with only the predicted Z_N symmetry for the effective potential. The consequences of this Z_N symmetry have been analyzed in [103].

We now examine the simplest case, $SU(2)$. We see that, using eq. 2.65 and eq. 2.66, V_{eff} depends only on one parameter – other than the temperature – namely $C \equiv C_1 = -C_2$. A plot of this effective potential as a function of C is provided on fig. 2.1. The gauge part is seen to have the Z_2 symmetry $C \to C + \frac{2\pi}{\beta}$ while the fermionic part doesn't. The Wilson Line for $SU(2)$ is given by

$$L = \cos\left(\frac{\beta C}{2}\right). \tag{2.67}$$

We see that at the minima of the effective potential for the gauge field $L = \pm 1$ and that the global Z_N symmetry we mentioned earlier is thus broken. The fact that the Wilson line is different from zero at the various minima of the effective potential for the pure gauge theory is also true for the more general $SU(N)$ case as well. We regard this as an

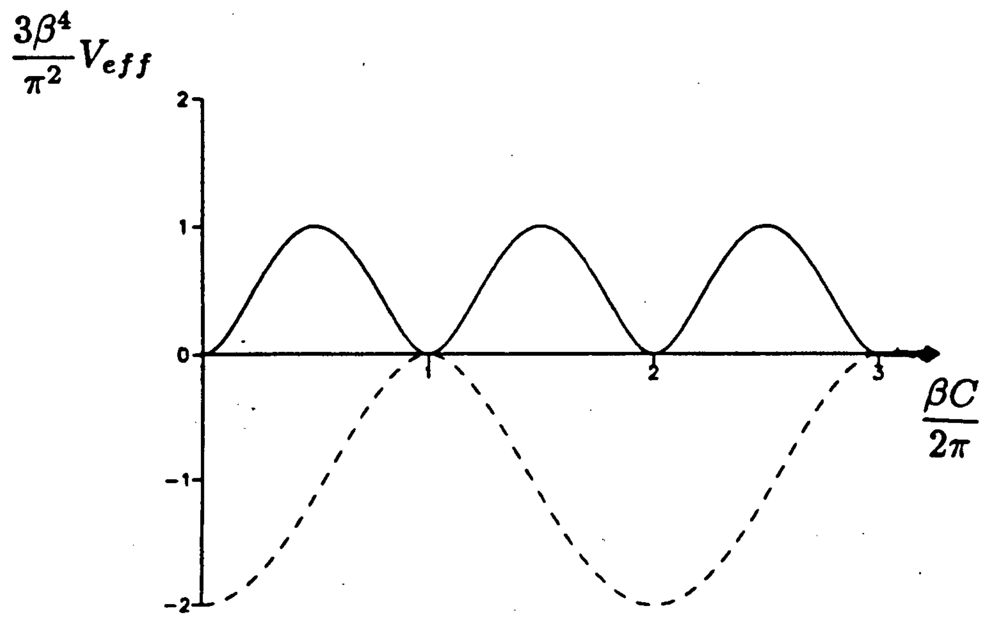


Figure 2.1: Sketch of the effective potential for $SU(2)$. The solid line is the bosonic contribution while the dashed line shows the contribution of a single massless fermion.

indication that deconfinement occurs at large temperature – where perturbation results are thought to be reliable – for the pure gauge theory. Note however that as soon as we introduce fermions, the local Z_N symmetry is broken as the effective potential no longer has a Z_N symmetry. There is no longer any global Z_N symmetry to be broken and the Wilson Line can't be used as an order parameter anymore.

Chapter 3

Imaginary chemical potential and phase structure of gauge theories

It would be very useful to be able to characterize unambiguously a (de-)confined phase when dynamical quarks are present, especially if we could do so as a function of the fermionic density. In order to study systems at finite density one usually performs a Legendre transformation to the Hamiltonian and looks at the Grand Canonical partition function

$$Z = \mathrm{Tr}\left(e^{-\beta(H+\mu N)}\right) \tag{3.1}$$

where μ is a fixed chemical potential. If the system is in thermodynamical equilibrium this implies that the average number of particles is conserved. We argued that the Wilson Line loses its meaning when dynamical fermions are present because a static source introduced into the system would always be neutralized by a dynamical quark. If we could somehow manage to constrain the dynamical quarks so that they couldn't neutralize the source, perhaps we could still use the Wilson Line – or an appropriate modification of it – as an order parameter. As it turns out, the Canonical formalism – fixed particle number – might just provide a natural framework for doing so. We will eventually come back to this point but first turn our attention towards a seemingly unphysical quantity.

3.1 Periodicity of the partition function as a function of θ

Consider the modified partition function

$$Z(\theta) = \mathrm{Tr}(e^{-\beta H + i\theta \hat{N}}) \tag{3.2}$$

where

$$\hat{N} = \int d^3x \, \Psi^\dagger \Psi \tag{3.3}$$

is the quark number operator. We see that θ essentially plays the rôle of an imaginary chemical potential. Since $\hat{N}$ has integer eigenvalues, $Z(\theta)$ has to be periodic with period 2π. However, if the spectrum contains only colour singlet states (*i.e.* if the eigenvalues of $\hat{N}$ are always multiples of N) then $Z(\theta)$ has periodicity $2\pi/N$ instead. We can see this in a physically interesting way. Consider the functional integral representation for $Z(\theta)$:

$$Z(\theta) = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\mathcal{D}A_\mu \exp\left(-\int_0^\beta d\tau \int d^3x \left[\overline{\Psi}(\gamma_\mu D_\mu + im)\Psi + \tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - i\frac{\theta}{\beta}\Psi^\dagger\Psi\right]\right) \tag{3.4}$$

with periodic boundary conditions for A_μ and anti-periodic boundary conditions for Ψ. This is the same partition function as one would obtain by introducing a fixed background $U(1)$ potential, θ, coupled to fermion number. This potential can not be gauged to zero at non-zero temperature due to the antiperiodic conditions on Ψ. We can visualize this situation by embedding the 4-dimensional space with (periodic) time interval $\tau \in (0, \beta)$ into a 5-dimensional space. The 4-dimensional space becomes a cylinder. We can now imagine putting a (5-dimensional) magnetic flux with magnitude θ through this cylinder. This corresponds to having a vector potential in the τ direction of magnitude θ/β. Now, suppose that the spectrum of the model contains isolated quarks. If we imagine transporting a quark "around" this cylinder, it will pick up a phase $\exp(i\theta)$. The physics is thus unchanged if $\theta \rightarrow \theta + 2\pi$ so that $Z(\theta)$ will have period 2π. On the other hand, if only colour singlets are present in the spectrum (colour singlets have fermion number zero modulo N) then moving a colour singlet object around the cylinder introduces a phase $\exp(i\theta N)$. Thus, the physics in this case is unchanged when $\theta \rightarrow \theta + 2\pi/N$ and $Z(\theta)$ will have periodicity $2\pi/N$ rather than just 2π. This would seem to indicate that the periodicity of $Z(\theta)$ is related to the confining properties of the system.

The situation is however somewhat more complicated. The point is that the Z_N symmetry of the action implies that $Z(\theta)$ *must* have periodicity $2\pi/N$ — this is reminiscent of the fact that for gauge theories without fermions, the Z_N symmetry implies that $< L >= 0$ always. To see this we first note that the θ dependence of the partition function in 3.4 can be eliminated from the action and transfered to the boundary conditions via the change of variables

$$\Psi(x,\tau) \longrightarrow e^{i\tau\theta/\beta}\Psi(x,\tau) \tag{3.5}$$

The partition function now becomes:

$$Z(\theta) = \int \mathcal{D}\Psi\mathcal{D}\overline{\Psi}\mathcal{D}A_\mu \exp\left(-\int d^4x \left[\overline{\Psi}(\gamma_\mu D_\mu + im)\Psi + \tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\right]\right) \tag{3.6}$$

with boundary conditions

$$\Psi(x,\beta) = -e^{i\theta}\Psi(x,0). \tag{3.7}$$

The periodicity of $Z(\theta)$ is proven by performing a gauge transformation

$$\begin{aligned} \Psi &\longrightarrow \mathcal{U}\Psi \\ A_\mu &\longrightarrow \mathcal{U}A_\mu\mathcal{U}^{-1} + \frac{i}{g}\mathcal{U}\partial_\mu\mathcal{U}^{-1} \end{aligned} \tag{3.8}$$

where $\mathcal{U}(x,\tau)$ are elements of $SU(N)$ with the property that

$$\mathcal{U}(x,\beta) = z\mathcal{U}(x,0) \quad ; \quad z = e^{i\theta+2\pi ik/N} \in Z_N. \tag{3.9}$$

Under this transformation both the action and the measure in eq. 3.6 are invariant whereas the boundary conditions are not:

$$\Psi(x,\beta) = -e^{i\theta+2\pi ik/N}\Psi(x,0). \tag{3.10}$$

Comparing eq. 3.7 with eq. 3.10 we conclude that $Z(\theta) = Z(\theta + 2\pi k/N)$; in other words, $Z(\theta)$ is periodic with period $2\pi/N$.

However we recall that in spite of the fact that the Z_N symmetry in pure gauge theories implies that $< L >$ is always zero, the symmetry is broken at high temperature leading to deconfinement. This suggests that the difference between a confining and a deconfining phase in gauge theories with fermions may manifest itself in the behaviour of the theory as a function of θ. In order to study this, we proceed to evaluate the free energy $F(\theta)$.

3.2 Phase structure as a function of θ

The free energy was calculated in the previous chapter in perturbation theory by means of a one-loop expansion. It is easy to extend this result to include an imaginary chemical potential. The modified action is given by

$$S_E + i\int d^3x\,\theta\Psi^\dagger\Psi = \int_0^\beta d\tau \int d^3x\left[\overline{\Psi}(\gamma_\mu\tilde{D}_\mu + im)\Psi + \tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}\right] \tag{3.11}$$

where

$$\tilde{D}_\mu = \partial_\mu - igA_\mu - i\delta_{\mu 0}\frac{\theta}{\beta}. \tag{3.12}$$

Thus in a gauge where A_0 is diagonal the effect of including θ is identical to shifting the matrix A_0 to $A_0 + \theta/(g\beta)$ in the fermionic part. The resulting one-loop potential is easily obtained:

$$V^\theta_{eff}(C) = V_{boson}(C) + V_{fermion}(C+\theta) \tag{3.13}$$

as given by 2.65 and 2.66. The formula for massive fermions is given by C.28. The extrema of V^θ_{eff} as a function of θ are easily found. For $\theta = 0$, V^θ_{eff} has a global minimum at $C_j = 0$ and local minima at $C_j = 2\pi k/\beta N$, the Z_N images of $C_j = 0$. Note that so far we have restricted our discussion to the case where only one fermionic species (or flavour) contributes to the effective potential. For N_f massless fermionic flavours, the fermionic part simply gets multiplied by N_f. When N_f gets sufficiently large, we no longer have

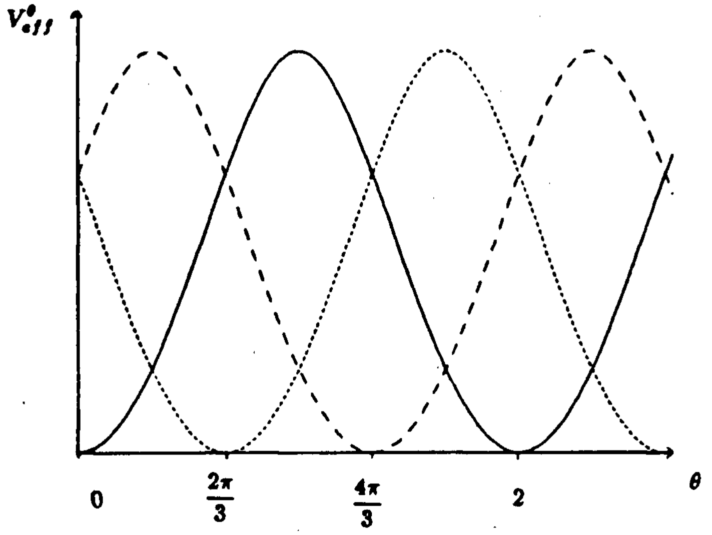


Figure 3.1: The effective potential as a function of θ at the three Z_3 images of $\phi = 0$ for the group $SU(3)$.

local minima at the Z_N images of $C_j = 0$. But this is quite irrelevant to our discussion since we are only interested in the global minimum of V^{θ}_{eff}. When θ is equal to $2\pi a/N$ this global minimum moves to $C_j = -2\pi[a]/(\beta N)$ where $[a]$ denotes the integer part of a.

We can now evaluate the free energy as a function of θ by first calculating the value of V^{θ}_{eff} at its various minima for the case $N_f = 1$. At $C_j = 0$ the effective potential is given by

$$V^{\theta}_{eff}(C=0) = -\frac{\pi^2 N}{12\beta^4}\left(1 - \frac{\theta^2}{\pi^2}\right)^2 \quad , \quad -\pi < \theta < \pi \tag{3.14}$$

with periodic extension to all value of θ. The value of V^{θ}_{eff} at the Z_N symmetric points of $C_j = 0$ is easily derived by performing a Z_N transformation:

$$V^{\theta}_{eff}(C = 2\pi k/\beta N) = V^{\theta - 2\pi k/N}_{eff}(C=0). \tag{3.15}$$

A plot of V^{θ}_{eff} as a function of θ for various values of k is provided on fig. 3.1 for the case of $SU(3)$. The multiphase structure is evident.

The free energy of the system as a function of θ is given by the curve with the smallest value of V^{θ}_{eff} at each value of θ.

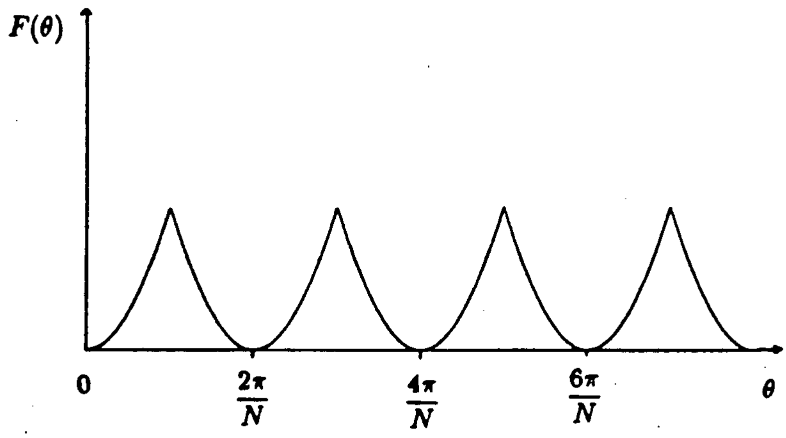


Figure 3.2: Free energy density as a function of θ for $SU(N)$.

It is shown on fig. 3.2. We can see that the free energy $F(\theta)$ is a continuous function of θ with cusps at $\theta = 2\pi(k + \frac{1}{2})/N$. This result implies that the system has a first order phase transition, for fixed temperature, as a function of θ. The Wilson Line is then discontinuous at these values of θ.

It would be interesting to study the θ-dependence of both the free energy and the Wilson Line in the low temperature limit. This limit corresponds to the strong coupling régime which doesn't allow for a treatment in perturbation theory. We hope that in the future, Monte-Carlo calculations will probe these features.

From the behaviour of the free energy we have the following expectation. At high temperature the free energy has the expected Z_N symmetry but shows some cusps. We would expect this to be a signal that at high temperature gauge theories have a first order phase transition as a function of θ. In the absence of Monte-Carlo simulations testing the low-temperature limit of these theories, we can look at the next best thing, namely analytical studies of the lattice gauge theory in the strong coupling limit. This has been done in [104] where it was found that both the Wilson Line and the effective potential were smooth functions of θ. Although we cannot draw firm conclusions for the continuum theory from this study it is nonetheless suggestive that in the low temperature

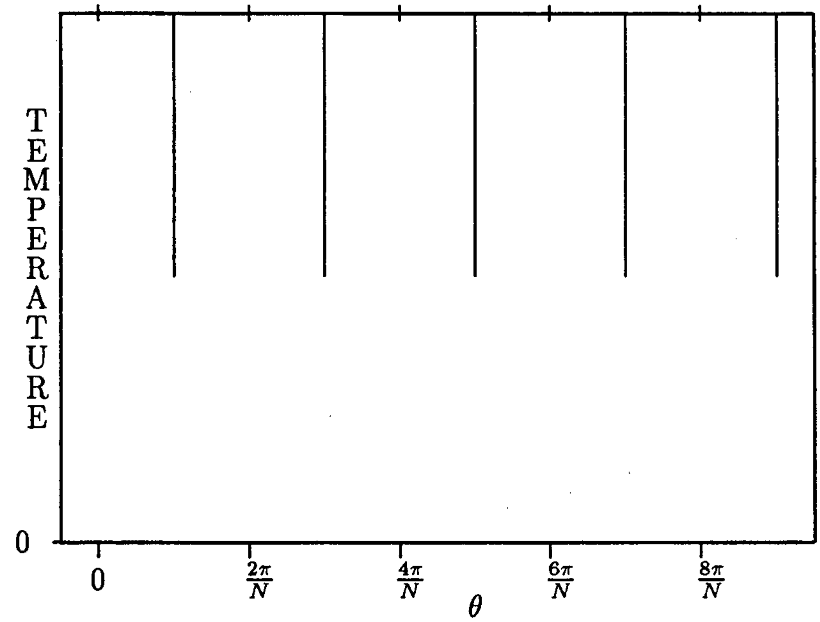


Figure 3.3: Sketch of the phase diagram for $SU(N)$ gauge theories with fermions as a function of temperature and imaginary chemical potential.

regime, both the free energy and the Wilson Line would be smooth functions of θ. We would thus expect no first-order transition to be present for this case. Fig. 3.3 contains a sketch of this aspect of the phase structure.

Even though we predict a different behaviour for the high temperature from the low temperature phases, it is clear from fig.3.3 that there is no evidence for a phase transition as a function of temperature at $\theta = 0$ which is the physical value of θ. It would be very interesting to test this hypothesis in a Monte Carlo simulation. It is difficult to simulate gauge theories at nonzero, real, chemical potential since the fermion determinant is complex. However when the chemical potential is imaginary, this determinant is real and a reasonable simulation may be possible.

3.3 Gauge theories at nonzero baryon density

We have just seen how we can get some insight into the "confining" properties of gauge theories by studying their behaviour as a function of an imaginary chemical potential. We now proceed to show how we can obtain results for the theory at finite baryon density from our knowledge of the behaviour of $Z(\theta)$. Consider the object

$$\begin{aligned} Z_B &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \, \exp(-i\theta B) Z(\theta) \\ &= \frac{1}{2\pi}\mathrm{Tr}\left[\mathrm{e}^{-\beta H}\int_0^{2\pi} d\theta \, \exp(i\theta[\hat{N}-B])\right] \\ &= \mathrm{Tr}\left[\mathrm{e}^{-\beta H}\delta(\hat{N}-B)\right]. \end{aligned} \tag{3.16}$$

Thus Z_B is the Canonical (as opposed to the Grand Canonical) partition function at fixed fermion number B. All thermodynamical properties should, in the infinite volume limit, be the same as those obtained at a fixed (real) chemical potential so chosen that the mean fermion number is B. Since the eigenvalues of $\hat{N}$ are integers, B must be an integer. Using the Z_N symmetry of $Z(\theta)$, we have

$$\begin{aligned} Z_B &= \frac{1}{2\pi}\int_a^b d\theta \, \exp(-i\theta B) Z(\theta + \tfrac{2\pi k}{N}) \\ &= \frac{1}{2\pi}\mathrm{e}^{2\pi i k B/N}\int_{a'}^{b'} d\phi \, \exp(-i\phi B) Z(\phi) \end{aligned} \tag{3.17}$$

where $b - a = b' - a' = 2\pi$. Due to the periodicity of the integrand, the exact values of the limits of integration are unimportant (as long as we integrate over one complete period) and we have

$$Z_B = \mathrm{e}^{2\pi i k B/N} Z_B. \tag{3.18}$$

We thus see that $Z_B = 0$ (infinite free energy) unless B is a multiple of N. The latter is the case if all quarks are bound into baryons. Note also that we have

$$< L >_{\mathrm{Canonical}} = \frac{\int d\theta \, \mathrm{e}^{-i\theta B} Z(\theta) L}{\int d\theta \, \mathrm{e}^{-i\theta B} Z(\theta)} \tag{3.19}$$

which yields under a Z_N transformation

$$< L >_{\text{Canonical}} = \mathrm{e}^{2\pi i k/N} < L >_{\text{Canonical}} . \tag{3.20}$$

The Wilson Line thus appears to recover its usefulness as an order parameter in the canonical formulation even in the presence of dynamical quarks since we recover the global Z_N symmetry present in pure gauge theories. We refer the reader to the work of Nathan Weiss [104] for more details on this.

For the high-temperature perturbative calculation which we did in the previous section, we can evaluate the free energy as a function of fermionic density for the massless fermion case. The free energy at fixed fermion number F_B is given by

$$\mathrm{e}^{-\beta F_B} = \int_0^{2\pi} d\theta \, \mathrm{e}^{i\theta B} \mathrm{e}^{-\beta F(\theta)}. \tag{3.21}$$

In the domain $-\pi/N < \theta < \pi/N$, $F(\theta)$ is given (according to eq. 3.14) by

$$F(\theta) = -\frac{V\pi^2 N N_f}{12\beta^4}\left(1 - \frac{\theta^2}{\pi^2}\right)^2 \tag{3.22}$$

where we have reintroduced N_f as the number of quark flavours. In performing the integral 3.21, $F(\theta)$ can thus be treated as an analytic function of θ on the whole complex plane. In the thermodynamic limit, $V \to \infty$, the integral can be done using the steepest descent method with only the first term of the expansion contributing. Writing $B = \rho V$, the stationary point of the phase in eq. 3.21 occurs when

$$i\rho = \frac{N N_f \theta}{3\beta^3}\left(1 - \frac{\theta^2}{\pi^2}\right). \tag{3.23}$$

This equation has the following three roots:

$$\begin{aligned} \theta_0 &= i\pi(a_+ - a_-) \\ \theta_\pm &= \tfrac{\pi}{2}\left[\sqrt{3}(a_+ + a_-) \pm i(a_+ - a_-)\right] \end{aligned} \tag{3.24}$$

where

$$a_\pm = \sqrt[3]{\sqrt{\frac{b^2}{4} + \frac{1}{27}} \pm \frac{b}{2}} \quad ; \quad b = \frac{3\beta^3 \rho}{\pi N N_f} \geq 0. \tag{3.25}$$

Note that $a_\pm, b$ are all real. The root of interest to us is the purely imaginary one, θ_0, which yields a real free energy. If we write $\mu = -i\theta_0/\beta$ we get

$$\rho = \frac{(2NN_f)}{6\beta^2}\mu\left(1 + \frac{\mu^2\beta^2}{\pi^2}\right) \tag{3.26}$$

and

$$\begin{aligned} F_B &\approx \frac{-i\theta_0 B}{\beta} + F(\theta_0) \\ &= V\frac{(2NN_f)\pi^2}{12\beta^4}\left[-\frac{1}{2} + \left(\frac{\mu\beta}{\pi}\right)^2 + \frac{3}{2}\left(\frac{\mu\beta}{\pi}\right)^4\right]. \end{aligned} \tag{3.27}$$

This is the same result –except not surprisingly for the constant term– which is obtained for an ideal gas of massless fermions (spin $\frac{1}{2}$) with degeneracy $2NN_f$ as is expected from this one-loop calculation [5]. In terms of the fermionic density, it is easy to show that for small fermionic density[1] , $F \propto \rho^2$ whereas for large density $F \propto \rho^{4/3}$.

We can also calculate the electric mass, $m_{\rm el}$, which is a measure of the screening of charges. We expect this screening to increase when more (anti-) fermions are present in the system. This means that $m_{\rm el}$ should increase with ρ (or with $|\mu|$). For fixed θ the electric mass is given by

$$m_{\rm el}^2 = \frac{g^2}{4}\left.\frac{\partial^2 V_{eff}}{\partial C_k^2}\right|_{C_j=0} \tag{3.28}$$

with k fixed. When performing this calculation one has to remember that $\sum C_j = 0$. This means that we can write $C_N = -C_1 - C_2 - \ldots - C_{N-1}$ and consider $C_1, \ldots, C_{N-1}$ as being independent. It is easy to show that we have

$$\left.\frac{\partial^2 V_{bos}}{\partial C_k^2}\right|_{C_j=0} = \frac{4N}{3\beta^2}$$

[1] More precisely for small b.

$$\left.\frac{\partial^2 V_{eff}}{\partial C_k^2}\right|_{C_j=0} = \frac{2N_f}{3\beta^2}\left(1-\frac{3\theta}{\pi^2}\right). \tag{3.29}$$

Since we are using the steepest descent method to evaluate the properties of this system at fixed fermion number, we simply insert θ_0 in the above equation so as to get

$$m_{\rm el}^2 = \frac{g^2}{3\beta^2}\left(N+\frac{N_f}{2}\right)+\frac{g^2 N_f \mu^2}{2\pi^2} \tag{3.30}$$

with $\mu = \pi(a_+ - a_-)/\beta$ as defined before. The ρ-independent part agrees with standard calculation [46] whereas we obtain the expected behaviour for the ρ-dependent part.

Even though we could have derived the same ρ-dependence of the free energy and of the electric mass using the Grand Canonical formulation, it is important to note that the imaginary chemical potential approach, motivated by the Canonical formulation, has provided us with some insight into the phase structure of the theory. It hasn't however given us any new information regarding possible phase transitions as a function of ρ as $T \to 0$. Perhaps Monte Carlo studies with an imaginary chemical potential would shed some light on this question.

Some interesting new results using the Canonical formulation have appeared recently in the literature [71]. Detailed calculations are provided for both the $SU(2)$ and the $SU(3)$ case using group-theoretical methods. A special emphasis is placed on finite volume effects which we need to understand very well if we want to extract any useful information regarding the phases of QCD in heavy ion collisions experiments. However special care must be taken when working at finite volume. For instance, the fact that symmetries are never broken in finite volume can hide some transitions that would be present in the thermodynamic limit. The effects of spatial boundary conditions (open, periodic, anti-periodic, etc.) and the discretness of the momentum spectrum modify the implementation of conservation laws. The authors of [71] ignore these issues and use the naïve approach of simply keeping V as a finite quantity. They then study the behaviour

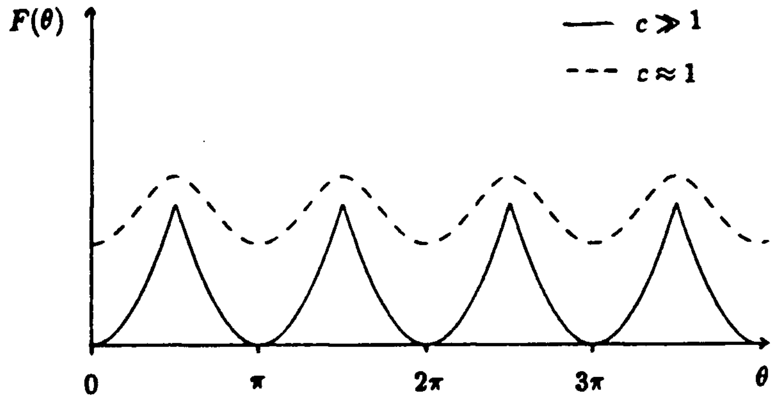


Figure 3.4: Sketch of the free energy as a fucntion of θ illustrating the finite volume effects for $SU(2)$. The parameter c is defined as $c = VT^3$ (adapted from [71]).

of the theories as a function of $c = VT^3$ as well as a function of θ. The cusping behaviour of the free energy is shown to be present only in the limit $c \to \infty$. This is illustrated on 3.4.

The Wilson Line is also shown to be a smooth function of θ as long as c is finite. It would be very interesting to see how these features depend on the boundary conditions and to see if they persist in higher-order loop calculations. It would also be interesting to see if the centre symmetry plays the same rôle in groups other than $SU(N)$. We hope that these questions will be addressed in the near future.

Chapter 4

Anomalies and the Schwinger model

The phenomenological success of the standard model and its apparent anomaly cancellation [39, 45] have led physicists to believe that the physical world is described by anomaly free gauge field theories. Anomaly cancellation which is associated with renormalizability, unitarity, Lorentz invariance and Einstein locality of the corresponding field theories, has been taken as a constraint in grand unified model building and has played a central rôle in the identification of certain superstring theories as candidates for ultimate unification [43, 44].

However the understanding of anomalous field theories has increased significantly over the last few years [12] and it has recently been suggested [34] that, properly treated, gauge theories with anomalies may form consistent quantum mechanical systems and in fact may have some useful properties. An important phenomenological application would be anomaly generated gauge symmetry breaking which may provide an alternative to the standard Higgs mechanism. For example, could the masses of the W and Z vector bosons be generated by an anomaly resulting from the absence of the top quark, rather than the standard picture of spontaneous symmetry breaking? We are unfortunately unable to answer this question at the present time. However, we can solve exactly some anomalous theories in 1+1-dimensions and show explicitely that theories with anomalies in the gauge current need not be inconsistent.

4.1 Anomalies and the Dirac sea

Consider the following 1+1-dimensional[1] Lagrangian

$$\mathcal{L} = \overline{\Psi} i\gamma^\mu \left[\partial_\mu - i(e_v + e_a\gamma^5)A_\mu\right] \Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.1}$$

defined on the spatial interval $[-\pi\ell, \pi\ell]$ where we take ℓ to be equal to 1 for simplicity. Ψ is a two-component spinor while the two-dimensional Dirac matrices are chosen as follows:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^5 \equiv \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.2}$$

If we choose the anti-symmetric tensor $\varepsilon^{\mu\nu}$ to satisfy $\varepsilon^{01} = -1$ it is easy to show that

$$\gamma_\nu\gamma^5 = \varepsilon_{\mu\nu}\gamma^\mu. \tag{4.3}$$

We can split this Lagrangian into separate contributions from left- and right-movers

$$\mathcal{L} = \overline{\Psi}_R i\gamma^\mu \left[\partial_\mu - i(e_v + e_a)A_\mu\right] \Psi_R + \overline{\Psi}_L i\gamma^\mu \left[\partial_\mu - i(e_v - e_a)A_\mu\right] \Psi_L - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.4}$$

where

$$\Psi_R = \begin{pmatrix} \psi(x,t) \\ 0 \end{pmatrix} \qquad \text{and} \qquad \Psi_L = \begin{pmatrix} 0 \\ \psi(x,t) \end{pmatrix}. \tag{4.5}$$

The reason behind our notation is the fact that, upon solving the equation of motions for the free spinors, we have

$$\Psi_R(x,t) = \Psi_R(x-t) \qquad\qquad \Psi_L(x,t) = \Psi_L(x+t). \tag{4.6}$$

We can then define a left (right) current $J_L = \overline{\Psi}_L\gamma^\mu\Psi_L$ ($J_R = \overline{\Psi}_R\gamma^\mu\Psi_R$). Since the Lagrangian splits up into two parts we would expect both the left and right current to be

[1] By 1+1-dimensional we mean one space and one time dimension.

separately conserved. It turns out however that upon quantization this is not the case. The theory is thus said to be anomalous.

The fact that this theory is anomalous can be demonstrated in many ways. For example, in the functional integral approach, the anomaly owes its existence to the lack of invariance of the fermionic measure under both axial and vector gauge transformations [37, 38, 31]. In the operator based approach of the second quantization procedure in quantum field theory, the origin of the anomaly can be linked to the existence of an infinite number of negative energy levels (the Dirac sea)[58]. We will use this second approach in our introduction to anomalies. The ideas described here are not new but we make an attempt at clarifying certain issues related to gauge invariance as they will be important later.

The Lagrangian is invariant (by construction) under the following gauge transformation:

$$\begin{aligned} A_\mu \longrightarrow A^g_\mu &= A_\mu + \partial_\mu \vartheta \\ \Psi \longrightarrow \Psi^g &= \exp\left[i(e_v + e_a \gamma^5)\vartheta\right] \Psi. \end{aligned} \tag{4.7}$$

Note that the existence of this symmetry prohibits the inclusion of a mass term for the fermi fields, $m\overline{\Psi}\Psi$, unless $e_a = 0$. The Euler-Lagrange equation for A_μ implies the existence of a conserved current:

$$0 = \partial_\mu J^\mu = e_R \partial_\mu J^\mu_R + e_L \partial_\mu J^\mu_L \tag{4.8}$$

with the corresponding conserved charge

$$Q = \int dx\, J^0 = e_R Q_R + e_L Q_L \tag{4.9}$$

where $e_R = e_v + e_a$ and $e_L = e_v - e_a$. This is a weaker requirement than having both Q_R and Q_L being independently conserved. Using the fact that we are in two dimensions, we write $A_\mu = \partial_\mu\theta + \varepsilon_{\mu\nu}\partial^\nu\xi$ and obtain

$$\mathcal{L} = \Psi_R^\dagger\left(i\partial_+ + e_R\left[\partial_+(\theta-\xi)\right]\right)\Psi_R + \Psi_L^\dagger\left(i\partial_- + e_L\left[\partial_-(\theta+\xi)\right]\right)\Psi_L + \tfrac{1}{2}(\Box\xi)(\Box\xi) \quad (4.10)$$

where $\partial_\pm = \partial_0 \pm \partial_1$. Under a gauge transformation, $\theta \to \theta^g = \theta + \vartheta$ while ξ remains unchanged, thus containing all the dynamics of the vector potential.

As an approximation, we ignore the term $\frac{1}{2}(\Box\xi)(\Box\xi)$ and take A_μ to be a time independent background field. By doing this, we loose the distinction between θ and ξ. Accordingly, we shall treat both θ and ξ on the same footing for the time being. This means that we choose not to fix a gauge at this point and keep θ arbitrary. The Hamiltonian for the system is easily obtained:

$$\mathcal{H} = \Psi_R^\dagger\left(-i\frac{\partial}{\partial x} - e_R\left[\frac{d}{dx}(\theta-\xi)\right]\right)\Psi_R + \Psi_L^\dagger\left(i\frac{\partial}{\partial x} + e_L\left[\frac{d}{dx}(\theta+\xi)\right]\right)\Psi_L. \quad (4.11)$$

In the first quantized theory Ψ is a wave-function satisfying a Schrödinger equation. Treating left and right movers separately, the eigenmodes are given by

$$\begin{aligned} H_R\Psi_{R,n} &= \left(-i\frac{\partial}{\partial x} - e_R\left[\frac{d}{dx}(\theta-\xi)\right]\right)\Psi_{R,n} = \omega_{R,n}\Psi_{R,n} \\ H_L\Psi_{L,m} &= \left(i\frac{\partial}{\partial x} + e_L\left[\frac{d}{dx}(\theta+\xi)\right]\right)\Psi_{L,m} = \omega_{L,m}\Psi_{L,m} \end{aligned} \quad (4.12)$$

which yields

$$\Psi_{R,n} = \exp(i\omega_{R,n}x + ie_R(\theta-\xi)) \qquad \Psi_{L,m} = \exp(-i\omega_{L,m}x + ie_L(\theta+\xi)). \quad (4.13)$$

Requiring periodic boundary conditions, *i.e.* $\Psi(\pi) = \Psi(-\pi)$, we find

$$\omega_{R,n} = n - e_R(\Theta - \Xi) \qquad \text{and} \qquad \omega_{L,m} = -m + e_L(\Theta + \Xi) \quad (4.14)$$

where $2\pi\Theta = \theta(\pi) - \theta(-\pi)$ and $2\pi\Xi = \xi(\pi) - \xi(-\pi)$. Note that in order to satisfy periodic boundary conditions, the gauge transformations have to satisfy the constraint

$$(|e_v| + |e_a|)[\theta(\pi) - \theta(-\pi)] = 2\pi p \quad (4.15)$$

with either $e_a = 0$ or $e_v/e_a = p/q$ (p, q integers).

Second quantization is obtained by expanding the wave function in terms of the eigenmodes

$$\Psi_c = \frac{1}{\sqrt{2\pi}} \sum_n a_{c,n} \psi_{c,n} \tag{4.16}$$

where the $a_{c,n}$ are the usual annihilation operators and the index c denotes the chirality (L or R). The $a_{c,n}$ satisfy the following anticommutation relations

$$\left\{a_{c,n}, a_{c',m}\right\} = 0 \qquad\qquad \left\{a^\dagger_{c,n}, a_{c',m}\right\} = \delta_{cc'}\delta_{nm}. \tag{4.17}$$

The Hamiltonian can then be written as

$$H = \sum_n \left(\omega_{R,n} a^\dagger_{R,n} a_{R,n} + \omega_{L,n} a^\dagger_{L,n} a_{L,n}\right). \tag{4.18}$$

We find it convenient (even though it is not necessary — [66, 63]) to redefine the creation and annihilation operators as follows

$$a_n \equiv a_{R,n} \qquad \text{and} \qquad b_n \equiv a^\dagger_{L,n} \tag{4.19}$$

and rewrite the Hamiltonian as $H = H_0 + H_I + k$ where

$$\begin{aligned} H_0 &= \sum_n \left[(n - e_R\Theta) a^\dagger_n a_n + (n + e_L\Theta) b^\dagger_n b_n\right] \\ H_I &= \Xi \sum_n \left[e_R a^\dagger_n a_n + e_L b^\dagger_n b_n\right] \\ k &= \sum_n \omega_{L,n}. \end{aligned} \tag{4.20}$$

As usual, we disregard the constant term, k, thus obtaining the "normal ordered" Hamiltonian, $: H : = H_0 + H_I$. We can similarly define a normal ordered charge operator

$$: Q : = e_R \sum_n a^\dagger_n a_n - e_L \sum_n b^\dagger_n b_n. \tag{4.21}$$

The ground state for this system is to be constructed by filling all the negative energy levels. With the above redefinition of the operators, a filled level above the Dirac sea is taken to be a particle while an empty level (or hole) in the Dirac sea represents an anti-particle.

Before defining the ground state, consider a state where all the energy levels of the right (left) movers are filled up to momentum $n = N$ ($m = M$). The charge of such a state is obtained by a simple counting argument as being

$$Q = e_r \sum_{n=-\infty}^{N} 1 - e_L \sum_{m=-\infty}^{M} 1 \tag{4.22}$$

and the energy is

$$E = \sum_{n=-\infty}^{N} \omega_{R,n} - \sum_{m=-\infty}^{M} \omega_{L,m}. \tag{4.23}$$

Both these quantities involve infinite sums which need to be regulated. We choose to regulate them by the heat kernel method which amounts to damping out the contribution of the negative energy levels as follows:

$$\begin{aligned} Q^\lambda &= e_R \sum_{n=-\infty}^{N} \exp(\lambda\omega_{R,n}) - e_L \sum_{m=-\infty}^{M} \exp(-\lambda\omega_{L,m}) \\ E^\lambda &= \sum_{n=-\infty}^{N} \omega_{R,n} \exp(\lambda\omega_{R,n}) - \sum_{m=-\infty}^{M} \omega_{L,m} \exp(-\lambda\omega_{L,m}). \end{aligned} \tag{4.24}$$

These expressions agree with the previous ones as λ goes to zero. These sums are easily evaluated and we define the regulated quantities in the following way:

$$\begin{aligned} Q^{\text{reg}} &= \lim_{\lambda\to 0} \left(Q^\lambda - \lambda^{-1}\right) = e_R Q_R^{\text{reg}} - e_L Q_L^{\text{reg}} \\ E^{\text{reg}} &= \lim_{\lambda\to 0} \left(E^\lambda + 2\lambda^{-2} + \tfrac{1}{12}\right) = \tfrac{1}{2}\left[(Q_R^{\text{reg}})^2 + (Q_L^{\text{reg}})^2\right] \end{aligned} \tag{4.25}$$

where

$$Q_R^{\text{reg}} = N + e_R(\Xi - \Theta) + \tfrac{1}{2} \qquad \text{and} \qquad Q_L^{\text{reg}} = M + e_L(\Xi + \Theta) + \tfrac{1}{2}. \tag{4.26}$$

The factor of $\frac{1}{2}$ that appears is due to the choice of periodic boundary conditions; if we had chosen anti-periodic boundary conditions, this factor would not have been present.

At this point, we reintroduce the distinction between dynamical and gauge degree of freedom. If $Q^{\rm reg}$ is to be a conserved quantity independently of the dynamical evolution of the system, we must have:

$$[(e_R)^2 - (e_L)^2]\Xi = 0. \tag{4.27}$$

Thus it would appear that both the vector Schwinger model[90] ($e_a = 0 \Longrightarrow e_R = e_L$) and the axial Schwinger model ($e_v = 0 \Longrightarrow e_R = -e_L$) can be consistent theories at the quantum level since the gauge symmetry present at the classical level survives the quantization procedure. On the other hand, the chiral Schwinger model ($e_a = \pm e_a \Longrightarrow e_R = 0$ or $e_L = 0$) would appear to be potentially inconsistent. We'll have more to say on this later in this chapter.

For definitiveness, let's look at the vector case and write $e_r = e_L = e$. The requirement that $Q^{\rm reg}$ be independent of the gauge choice apparently leads to $e\Theta = 0$. This is however more restrictive than needs to be. If we simply require $e\Theta = 0 \bmod p$ where p is an integer, we can still retain consistency. A choice of gauge is then viewed as a simple relabelling of the states. Allowable gauge choices thus belong to discrete sets called homotopy classes that are characterized by an integer related to the difference between the value of the gauge field at the two boundaries (winding number). If we go back to eq.4.15 we can see in fact that due to the periodicity condition, the allowed gauge transformations are consistent with this requirement.

While the vector charge $Q = e(Q_R - Q_L)$ is conserved, the axial charge $Q_5 = e(Q_R + Q_L)$ is anomalous as it depends on the backround dynamical field. For instance, the charges of the vacuum $|0,0\rangle$ in the $\Theta = 0$ gauge are:

$$Q = 0 \qquad \text{and} \qquad Q_5 = 2e\Xi + 1. \tag{4.28}$$

The reader may be worried that this background dependence of the axial charge is a consequence of using an improper regularization procedure. One could argue that the regulator should be independent of the dynamics *i.e.* that we should have chosen something like $\exp(\lambda n)$ instead of $\exp(\lambda[n + e\Xi])$. However such a regulator would have given a renormalized energy proportional to Ξ rather than Ξ^2. As a result, the energy would not have been bounded below as we could lower it with a spatially homogeneous constantly varying ($\dot{\xi} = \dot{\Xi} = cst$) dynamical field satisfying $\Box\xi = 0$. Furthermore, such a regulator would introduce an unphysical distinction between particle and anti-particle as the energy would no longer be proportional to the square of the charges.

4.2 Bose-Fermi equivalence in 1+1-dimensions.

While the study of fermions in a static background field allows for a simple understanding of anomalies, it does not tell us anything about the dynamical evolution of anomalous systems. The dynamics of such systems can be studied at a semi-classical level using a peculiarity of two dimensional field theories.

The Lagrangian for a free massless scalar (boson) field is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi. \tag{4.29}$$

This field satisfies the Klein-Gordon equation whose solution in two dimensions can be written as the sum of left- and right-movers as

$$\chi(x,t) = \chi_R(x-t) + \chi_L(x+t). \tag{4.30}$$

This is very similar to the solution for a free massless fermion in two dimensions. In fact, one can show [65, 49] that free Fermi fields can be written as

$$\begin{aligned} \Psi_R &= C_R \exp(i2\sqrt{\pi}\chi_R) \\ \Psi_L &= C_L \exp(-i2\sqrt{\pi}\chi_L) \end{aligned} \tag{4.31}$$

where the constants C_R and C_L include, amongst other things, a Klein factor giving the right anti-commutation properties to the Fermi fields. More generally, we have

$$\begin{aligned}
\overline{\Psi}\gamma^\mu\Psi &= \frac{1}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_\nu\chi && (4.32)\\
\overline{\Psi}\gamma^\mu\gamma^5\Psi &= -\frac{1}{\sqrt{\pi}}\partial^\mu\chi && (4.33)\\
\overline{\Psi}i\gamma^\mu\partial_\mu\Psi &= \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi && (4.34)\\
m^2\overline{\Psi}\Psi &= \mu m^2\cos\chi && (4.35)
\end{aligned}$$

where μ reflects some ambiguity in the regularization procedure.

This equivalence between Fermi and Bose fields can be extended to interacting theories as well. For instance, a bosonized Lagrangian equivalent to 4.1 [65, 49, 18] can be written as:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{\sqrt{\pi}}(e_v\varepsilon^{\mu\nu}\partial_\mu\chi + e_a\partial^\nu)A_\nu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2A_\mu A^\mu. \qquad (4.36)$$

The mass term for the photon has to be included in order to give the bosonized Lagrangian the correct symmetry and reflects the ambiguity in regularizing the fermionic current. One of the main advantages of working with the bosonized theory is that the effects of the anomaly are already present at the classical level. This should not be surprising as we have seen in the previous section that the origin of the anomaly can be understood as being a consequence of having to fill an infinite number of negative energy levels (the Dirac sea). Since we know that the Dirac sea is not a feature of bosonic theories, the anomaly must somehow already be included in the classical Lagrangian. In order to see this better, we briefly look at two special cases of the above Lagrangian.

4.2.1 The vector Schwinger model

One of the most studied two dimensional system is the vector Schwinger model[2], namely the theory described by the Lagrangian

$$\mathcal{L} = \overline{\Psi} i\gamma^{\mu}(\partial_{\mu} - ieA_{\mu}) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.37}$$

This Lagrangian is invariant under the following gauge transformation

$$\begin{aligned} A_{\mu} \rightarrow A_{\mu}^{g} &= A_{\mu} + \partial_{\mu}\vartheta \\ \Psi \rightarrow \Psi^{g} &= \exp(ie\vartheta)\Psi \end{aligned} \tag{4.38}$$

yielding the conserved vector current $J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi$. The axial current on the other hand, $J_{\mu}^{5} = \overline{\Psi}\gamma_{\mu}\gamma^{5}\Psi$, becomes anomalous after quantization and obeys

$$\partial^{\mu} J_{\mu}^{5} = -\frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}. \tag{4.39}$$

This can be shown using various methods. We will demonstrate here how it arises in the context of the bosonized theory. The Bose-equivalent Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}\partial_{\mu}\chi\partial^{\mu}\chi + \frac{e}{\sqrt{\pi}}A_{\mu}\varepsilon^{\mu\nu}\partial_{\nu}\chi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.40}$$

and is invariant (up to a total divergence term) under the gauge transformation

$$\begin{aligned} A_{\mu} \rightarrow A_{\mu}^{g} &= A_{\mu} + \partial_{\mu}\vartheta \\ \chi \rightarrow \chi^{g} &= \chi. \end{aligned} \tag{4.41}$$

The Euler-Lagrange equations for the system are

$$\Box\chi + \frac{e}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_{\nu}A_{\mu} = 0 \tag{4.42}$$

$$\partial_{\mu}F^{\nu\mu} - \frac{e}{\sqrt{\pi}}\varepsilon^{\mu\nu}\partial_{\mu}\chi = 0 \tag{4.43}$$

[2]For a review, see [66].

We can rewrite 4.43 as

$$\partial_\mu F^{\nu\mu} = eJ^\nu \tag{4.44}$$

from which it follows that J^μ is a conserved quantity. Using eq. 4.3 we can write $J_5^\mu = \varepsilon^{\mu\nu} J_\nu$ and further using eq. 4.42 we get

$$\partial_\mu J_5^\mu = \frac{e}{\pi}\varepsilon^{\mu\nu}\partial_\mu A_\nu = \frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu} \tag{4.45}$$

as advertised.

It is interesting to solve these classical equations of motion and to compare the results with our knowledge of the quantum system. Introducing the electric field, $E = F^{01}$, the equations of motion can be rewritten as

$$\begin{aligned} \Box\chi + gE &= 0 \\ \partial_0(E - g\chi) &= 0 \\ \partial_1(E - g\chi) &= 0 \end{aligned} \tag{4.46}$$

where $g = e/\sqrt{\pi}$. We work on the finite spatial interval, $[-\pi\ell, \pi\ell]$, and require that no net current of either type flows in or out of the system *i.e.*

$$\begin{aligned} \int_{-\pi\ell}^{\pi\ell} dx\,\partial_1 J^1 &= J^1(\pi\ell) - J^1(-\pi\ell) = 0 \\ \int_{-\pi\ell}^{\pi\ell} dx\,\partial_1 J_5^1 &= J_5^1(\pi\ell) - J_5^1(-\pi\ell) = 0. \end{aligned} \tag{4.47}$$

We also introduce the conserved and anomalous charges

$$\begin{aligned} Q \equiv \int_{-\pi\ell}^{\pi\ell} dx\,J^0 &= \frac{1}{\sqrt{\pi}}[\chi(-\pi\ell) - \chi(\pi\ell)] \quad ; \quad \dot{Q} = 0 \\ Q_5(t) \equiv \int_{-\pi\ell}^{\pi\ell} dx\,J_5^0 &= -\frac{1}{\sqrt{\pi}}\int_{-\pi\ell}^{\pi\ell} dx\,\partial_0\chi \quad ; \quad \dot{Q}_5(t=0) = 0. \end{aligned} \tag{4.48}$$

The solution is:

$$\begin{aligned} \chi = {} & \frac{1}{\sqrt{2\pi\ell}}\sum_{n\neq 0}\frac{1}{\sqrt{\omega_n}}[a_n \exp(-i\omega_n t + ik_n x) + a_n^* \exp(+i\omega_n t - ik_n x)] \\ & - \frac{E_0}{g} - \frac{Q_5(0)\sin gt}{2\sqrt{\pi}\ell g} - \frac{\sqrt{\pi}Q\sinh gx}{2\sinh g\pi\ell} \end{aligned} \tag{4.49}$$

where $k_n = n/\ell$,$\omega_n^2 = g^2 + k_n^2$ and $E = g\chi + E_0$. The energy is given by:

$$\begin{aligned} H &= \tfrac{1}{2}\int_{-\pi\ell}^{\pi\ell} dx\,[(\dot{\chi})^2 + (\chi')^2 + E^2] \\ &= \sum_{n\neq 0}\omega_n a_n^* a_n + \frac{Q_5^2(0)}{4\ell} + \frac{\pi g Q^2}{4}\coth g\pi\ell. \end{aligned} \tag{4.50}$$

Note that if we require periodic boundary conditions, we are forced to set $Q = 0$ and the contribution from the zeroth mode of the energy agrees with what we had found previously in eq.4.23. We note also that the zero mode of the current density is

$$J^0 = \frac{-1}{\sqrt{\pi}}\partial_1\chi = \tfrac{1}{2}Qg\frac{\cosh gx}{\sinh g\pi\ell} \tag{4.51}$$

which shows that, in the limit where $\ell \to \infty$, the charge density is concentrated near the boundary; the system behaves like a metal with the charges being expelled to the boundaries being localized within the screening length g^{-1}. This is how confinement of charges manifests itself in this bosonized version of the vector Schwinger model, with no physical charge density being observed in the infinite volume limit. The dynamical evolution of the anomalous charge is given by:

$$Q_5 = Q_5(0)\cos(gt). \tag{4.52}$$

The oscillatory behaviour of the anomalous charge is due to the back-reaction of the Fermi fields onto the gauge fields (and vice versa). If the gauge field is fixed as a background external configuration, it is found that there is creation of a net anomalous charge. [72, 100, 6]. This back-reaction will be very important in the next chapter when we will study the effect of finite fermionic density in anomalous systems.

4.2.2 The axial Schwinger model

The axial Schwinger model is defined by the Lagrangian

$$\mathcal{L} = \overline{\Psi} i\gamma^\mu(\partial_\mu - ie\gamma^5 A_\mu)\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{4.53}$$

This Lagrangian is invariant under an axial gauge transformation

$$\begin{aligned} A_\mu \to A_\mu^g &= A_\mu + \partial_\mu \vartheta \\ \Psi \to \Psi^g &= \exp(ie\gamma^5\vartheta)\Psi. \end{aligned} \tag{4.54}$$

As we mentioned earlier, the existence of this symmetry prohibits the inclusion of a mass term for the fermions. In terms of conserved and anomalous currents, vector and axial-vector current see their rôle being interchanged from the vector Schwinger model. The Bose-equivalent Lagrangian for this system is

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{e}{\sqrt{\pi}}(\partial^\nu \chi) A_\nu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{e^2}{2\pi}A_\mu A^\mu \tag{4.55}$$

which is invariant under the axial gauge transformation

$$\begin{aligned} A_\mu \to A_\mu^g &= A_\mu + \partial_\mu \vartheta \\ \chi \to \chi^g &= \chi + \frac{e}{\sqrt{\pi}}\vartheta. \end{aligned} \tag{4.56}$$

As we mentioned earlier (see eq. 4.35), a mass term for the Fermi fields translates in the Bose language into a term proportional to $\cos(\chi)$ which is not invariant (as expected) under the axial gauge transformation. The Euler-Lagrange equations of motion for this system are easily derived:

$$\Box \chi - \frac{e}{\sqrt{\pi}}\partial^\mu A_\mu = 0 \tag{4.57}$$

$$\partial_\mu F^{\nu\mu} + \frac{e}{\sqrt{\pi}}(\partial^\nu \chi - \frac{e}{\sqrt{\pi}}A^\nu) \equiv \partial_\mu F^{\nu\mu} - eJ_5^\nu = 0. \tag{4.58}$$

We thus see that the axial current, J_5^μ, is conserved. The vector current on the other hand obeys the anomaly equation:

$$\begin{aligned} J_\mu = \varepsilon_{\mu\nu}J_5^\nu &= \frac{\varepsilon_{\mu\nu}e}{\sqrt{\pi}}(\partial^\nu \chi - \frac{e}{\sqrt{\pi}}A^\nu) \\ \Longrightarrow \qquad \partial_\mu J^\mu &= -\frac{e^2}{2\pi}\varepsilon^{\mu\nu}\partial_\mu A_\nu = -\frac{e^2}{\pi}\varepsilon^{\mu\nu}F_{\mu\nu}. \end{aligned} \tag{4.59}$$

It is crucial to note here that it is the requirement of axial gauge invariance which gives a topologically non-trivial contribution to the vector current making it non-conserved. The vector current naïvely obtained from the Bose-Fermi equivalence for free fields (eq. 4.32) is still conserved but is not gauge invariant and therefore is not the physical current. We will come back to this point in the next chapter. Once again, it is interesting to solve the classical equations of motion and to clearly expose the equivalence of the vector and axial models. Writing $g = e/\sqrt{\pi}$, the equations of motion can be written as

$$\begin{aligned} \left(\Box + g^2\right) J_5^\nu &= 0 \\ \partial_\nu J_5^\nu &= 0. \end{aligned} \tag{4.60}$$

As we did in the vector case, we require no net flow of either current in or out of the system and obtain

$$\begin{aligned} J_5^0 &= \frac{1}{\sqrt{2\pi\ell}} \sum_{n\neq 0} \alpha_n [a_n \exp(-i\omega_n t + ik_n x) + a_n^* \exp(+i\omega_n t - ik_n x)] + \frac{Q_5 \cosh gx}{2\sinh g\pi\ell} \\ J_5^1 &= -\frac{1}{\sqrt{2\pi\ell}} \sum_{n\neq 0} \frac{\omega_n \alpha_n}{\sqrt{k_n}} [a_n \exp(-i\omega_n t + ik_n x) + a_n^* \exp(+i\omega_n t - ik_n x)] - \frac{Q(0)\cos gt}{2\sqrt{\pi}\ell g} \end{aligned}$$

where

$$k_n = \frac{n}{\ell} \quad ; \quad \omega_n^2 = k_n^2 + g^2 \quad ; \quad \alpha_n^2 = \frac{g^2 k_n^2}{2\pi\omega_n(\omega_n^2 + k_n^2)}. \tag{4.61}$$

The energy is given by

$$\begin{aligned} H &= \frac{\pi}{2} \int_{-\pi\ell}^{\pi\ell} dx \, \left[(J_5^0)^2 + (J_5^1)^2 + g^{-2}(\partial_0 J_5^1 + \partial_1 J_5^0)^2\right] \\ &= \sum_{n\neq 0} \omega_n a_n^* a_n + \frac{Q^2(0)}{4\ell} + \frac{\pi g Q_5^2}{4} \coth g\pi\ell. \end{aligned} \tag{4.62}$$

Comparing with the vector case, we see that the two models are equivalent up to an interchange between axial and vector charges.

4.3 Introduction to the chiral Schwinger model

The chiral Schwinger model describes quantum electrodynamics with a chiral[3] fermion in two spacetime dimensions. The Lagrangian is

$$\mathcal{L}_0 = \int dx \left\{ \Psi^\dagger i\sigma^\mu(\partial_\mu + i\sqrt{\pi}eA_\mu)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right\} \tag{4.63}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\sigma^0 = 1$, $\sigma^1 = \varepsilon = \pm 1$ with the + (–) sign corresponding to a coupling to right- (left-) movers. It is important to note that we consider the minimal chiral Schwinger model *i.e.* Ψ is a single component fermion.

As we mentioned in section 4.1, the chiral Schwinger model appears to be potentially inconsistent since the charge associated with gauge invariance is not conserved[4]. This model has been examined by canonical methods [47] and more recently was investigated by Jackiw and Rajaraman [59] using the Feynman path integral representation of the theory. The latter authors showed that for a certain choice of counterterms the theory is Lorentz invariant and unitary. It has further been argued that the gauge variance of the fermionic part of the path integral measure which, upon using the Fadeev-Popov proceedure to fix the gauge, yields a Wess-Zumino term in the gauge field action [10].

However use of the naïve path integral representation requires canonical justification. At the Hamiltonian level the theory involves second class gauge constraints which lead to nontrivial modification of the equations of motion. It has been solved [78, 40, 41, 48, 17] using some bosonization techniques which, in our opinion, are not entirely satisfactory in this case. As we have just seen, a Bose equivalent Lagrangian includes both left- and right-movers, thus introducing an extra $\frac{1}{2}$ configuration degree of freedom. It is not *a*

[3]In two dimensions the term chiral means an eigenstate of the projection operator $(1-\gamma^5)/2$ by analogy with the four-dimensional case.

[4]In section 4.1, we looked at theories where both left and right movers were present. However, it is trivial to reduce the system to include only a single component and show the same potential problems to be present.

priori obvious that the latter modification is not responsible for the apparent consistency of the model.

At the classical level the Lagrangian contains a vector gauge field and a chiral fermion, representing $2\frac{1}{2}$ configuration space degrees of freedom. There are two first class constraints each of which remove one degree of freedom resulting in a physical $\frac{1}{2}$ degree of freedom - a charged Weyl fermion with a relativistic Coulomb self-interaction. The classical theory can be formulated as a consistent, causal, Lorentz covariant initial value problem.

At the quantum level the Schwinger term in the gauge constraint algebra changes the two first class constraints into two second class constraints which together remove 1 degree of freedom. Therefore the quantum theory has $1\frac{1}{2}$ physical degrees of freedom. Lorentz invariance is not manifest in this reduction and it is not obvious that it is maintained by the quantization. Furthermore, the presence of the extra degrees of freedom would seem to indicate that the quantized theory loses the geometrical interpretation of a gauge theory. A characteristic of a gauge theory is that there is a redundancy in the number of dynamical variables - this is what gives rise to the gauge invariance and results in the need to impose constraints. In the quantum theory the gauge constraint is second class and therefore does not eliminate as many degrees of freedom as it did in the classical theory.

Constrained systems are normally handled at the classical level using Dirac's [32, 50] method of eliminating extraneous degrees of freedom. When second class constraints are present one quantizes by subsequently identifying the classical Dirac brackets with commutation relations. In the present case Schwinger terms arise which alter the constraint algebra at the quantum level. This requires that we modify Dirac's method so that it can be applied directly to the quantum theory.

As is desirable we remain with fermionic variables. We implement infrared regularization of the theory by taking the space as a circle, S^1, and add counterterms to the Lagrangian. We solve the model with the counterterms and find that the Hamiltonian is hermitian and positive with a Lorentz invariant spectrum only for a certain range of the parameters. The spectrum contains a single massless chiral scalar and a single massive boson. These resemble the particle content discovered in reference [78, 40, 41] with the exception of the absence of one of the chiral Bose degrees of freedom.

There have been several other attempts to quantize the chiral Schwinger model which have failed to obtain a Lorentz invariant solution [48, 74, 75]. Some of these can be regarded as special cases of the model which we consider here with particular values of the counterterms [48, 75]. Our present work shows how to restore Lorentz invariance in these models by adding counterterms to the action.

A further gauge invariant solution which recognized the origin of the anomaly as an induced quantum curvature and added Lorentz noninvariant nonlocal counterterms obtained a noncovariant spectrum with a hermitian Hamiltonian [74]. The solution has a $\frac{1}{2}$ degree of freedom which, with the mass generation inherent in the model, is incompatible with Lorentz invariance. (A chiral particle has a Lorentz invariant spectrum only when it is massless.) It is apparent that the quantum curvature cannot be cancelled by adding local counterterms to the gauge theory action.

4.4 Solution of the Chiral Schwinger Model

The fermion bilinear operators which occur in the Lagrangian and later in the Hamiltonian must be defined with some regularization. We shall choose a particular regularization and parameterize the difference between the one we choose and others by adding local counterterms to the Lagrangian

$$\mathcal{L}_{ct} = \int dx \left\{ \frac{a}{2} A_\mu A^\mu + \frac{b}{2} (A_\varepsilon)^2 \right\} \tag{4.64}$$

where $A_\varepsilon = A_0 + \varepsilon A_1$. We note that the first term is Lorentz invariant but has the conventionally undesirable feature of breaking gauge invariance. Here, we expect that the quantization of the theory breaks gauge invariance so we cannot exclude dynamical generation of such a term. We shall show that the model is consistent for a large range of the this parameter. Addition of this counterterm was also considered in references [59] and [78] with a similar conclusion. Although their treatment (using bosonization techniques) did not require additional counterterms we require the second gauge variant, Lorentz variant counterterm. The model will be found to have a Lorentz invariant spectrum only for a particular nonzero value of b.

The full Lagrangian is:

$$\mathcal{L} = \int dx \left\{ \Psi^\dagger (i\partial_\varepsilon - \sqrt{\pi} e A_\varepsilon)\Psi + \frac{1}{2}(\partial_0 A_1 - \partial_1 A_0)^2 + \frac{a}{2}(A_0^2 - A_1^2) + \frac{b}{2}(A_\varepsilon)^2 \right\} \tag{4.65}$$

The canonical momenta are:

$$\begin{aligned} \Pi_0(x) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 A_0(x))} &= 0 \qquad\qquad \Pi_\Psi(x) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 \Psi(x))} = i\Psi^\dagger(x) \\ \Pi_1(x) \equiv \frac{\delta \mathcal{L}}{\delta(\partial_0 A_1(x))} &= \partial_0 A_1(x) - \partial_1 A_0(x) \end{aligned} \tag{4.66}$$

We have one primary constraint, $\Pi_0 = 0$. The Hamiltonian is

$$\begin{aligned} H_0 &= \int dx \left\{ \Pi_1 \partial_0 A_1 + i\Psi^\dagger \partial_0 \Psi \right\} - L \\ &= \int dx \left\{ \frac{\Pi_1^2}{2} + \Pi_1 \partial_1 A_0 - \varepsilon i \Psi^\dagger \partial_1 \Psi + \sqrt{\pi} e \Psi^\dagger \Psi A_\varepsilon + \frac{a}{2}(A_1^2 - A_0^2) - \frac{b}{2} A_\varepsilon^2 \right\} \end{aligned} \tag{4.67}$$

We work on the compact one dimensional space S^1 with length $2\pi\ell$ and do a plane wave expansion of our variables.

$$\Psi(x) = \frac{1}{\sqrt{2\pi\ell}} \sum_{k=-\infty}^{\infty} a_k \exp(i\varepsilon_f k x \ell^{-1})$$

$$
\begin{aligned}
A_j(x) &= \frac{1}{\sqrt{2\pi\ell}} \sum_{k=-\infty}^{\infty} A_j(k) \exp(i\varepsilon_f k x \ell^{-1}) \\
\Pi_j(x) &= \frac{1}{\sqrt{2\pi\ell}} \sum_{k=-\infty}^{\infty} \Pi_j(k) \exp(i\varepsilon_f k x \ell^{-1})
\end{aligned}
\tag{4.68}
$$

where $\varepsilon_f = \pm 1$ (the reason for keeping this sign arbitrary will become clear at the end). We also define

$$
\rho^\dagger(n) = \int_0^{2\pi\ell} \Psi^\dagger \Psi \exp(i\varepsilon_f n x \ell^{-1})\, dx = \sum_{k=-\infty}^{\infty} a_{k+n}^\dagger a_k \tag{4.69}
$$

This allows us to write

$$
\begin{aligned}
H_0 = \sum_n \Big\{ & \frac{1}{2}\Pi_1^\dagger(n)\Pi_1(n) + \frac{i\varepsilon_f n}{\ell}\Pi_1^\dagger(n)A_0(n) + \frac{\varepsilon\varepsilon_f n}{\ell}a_n^\dagger a_n - \frac{b}{2}A_\varepsilon^\dagger(n)A_\varepsilon(n) \\
& + \frac{e}{\sqrt{2\ell}}\rho^\dagger(n)A_\varepsilon(n) + \frac{a}{2}\left(A_1^\dagger(n)A_1(n) - A_0^\dagger(n)A_0(n)\right)\Big\}
\end{aligned}
\tag{4.70}
$$

We also take our field variables to be quantum operators obeying the canonical commutations relations:

$$
\begin{aligned}
\left[A_j(x), \Pi_k(y)\right] = \delta_{jk}\delta(x-y) \quad &\longleftrightarrow \quad \left[A_j(n), \Pi_k^\dagger(m)\right] = i\delta_{jk}\delta_{nm} \\
\left[\Psi(x), \Psi^\dagger(y)\right]_+ = i\delta(x-y) \quad &\longleftrightarrow \quad \left[a_n^\dagger, a_m\right]_+ = \delta_{nm}.
\end{aligned}
\tag{4.71}
$$

Defining the Fock vacuum as the state annihilated by

$$
\begin{aligned}
a_{\varepsilon_0 n} \mid 0\rangle = \; & 0 \qquad\qquad n \geq 0 \\
a_{\varepsilon_0 n}^\dagger \mid 0\rangle = \; & 0 \qquad\qquad n < 0
\end{aligned}
\tag{4.72}
$$

with $\varepsilon_0 = \pm 1$, we can easily derive the following (see for instance [66]) anomalous commutator for the charge density

$$
[\rho(p), \rho^\dagger(q)] = \varepsilon_0 p \delta_{p-q} \tag{4.73}
$$

and the Sommerfeld-Sugawara formula [94]

$$
\sum_n n a_n^\dagger a_n = \frac{1}{2}\varepsilon_0 \sum_n \rho^\dagger(n)\rho(n) \tag{4.74}
$$

The right hand side of eq. 4.74 is generated by quantum effects, *i.e.* the charge density has vanishing classical Poisson brackets with itself. Furthermore, the Sommerfeld-Sugawara formula holds only because of the operator structure of the charge densities.

We further define (for $n \neq 0$)

$$\sigma(n) = \frac{\rho(n)}{\sqrt{|n|}} \quad \longrightarrow \quad [\sigma(p), \sigma^\dagger(q)] = \varepsilon_0 \delta_{p-q} \tag{4.75}$$

so that we can write the Hamiltonian as

$$H_0 = h_0(0) + \sum_{n>0} h_0(n) \tag{4.76}$$

where

$$h_0(0) = \frac{1}{2}\Pi_1(0)^2 + \frac{\varepsilon\varepsilon_0\varepsilon_f}{2\ell}\rho(0)^2 + \frac{e}{\sqrt{2\ell}}\rho(0)A_\varepsilon(0) + \frac{a}{2}\left(A_1(0)^2 - A_0(0)^2\right) - \frac{b}{2}A_\varepsilon(0)^2 \tag{4.77}$$

and

$$\begin{aligned} h_0(n) = \Pi_1^\dagger(n)\Pi_1(n) \quad &+ \quad \frac{\varepsilon\varepsilon_0\varepsilon_f n}{\ell}\sigma^\dagger(n)\sigma(n) + \frac{e\sqrt{n}}{\sqrt{2\ell}}\left(\sigma^\dagger(n)A_\varepsilon(n) + A_\varepsilon^\dagger(n)\sigma(n)\right) \\ &+ \quad \frac{i\varepsilon_f n}{\ell}\left(\Pi_1^\dagger(n)A_0(n) - A_0^\dagger(n)\Pi_1(n)\right) - bA_\varepsilon^\dagger(n)A_\varepsilon(n) \\ &\qquad + a\left(A_1^\dagger(n)A_1(n) - A_0^\dagger(n)A_0(n)\right) \end{aligned} \tag{4.78}$$

We do not worry at this point about the possible appearance of constant terms arising from commuting operators; this is taken care of by field independent normal ordering.

We now proceed to implement Dirac's procedure directly at the quantum level. We first look at $h_0(n)$ for $n \neq 0$ and set $\ell = 1$. Consistency of our constraint $\Pi_0 = 0$ under time evolution requires $[\Pi_0, H_0] = 0$. We thus define (dropping the irrelevant n label)

$$G \equiv -i[h_0, \Pi_0] = -i\varepsilon_f n\Pi_1 + \frac{e\sqrt{n}}{\sqrt{2}}\sigma - aA_0 - bA_\varepsilon = 0 \tag{4.79}$$

so that

$$[G, \Pi_0^\dagger] = a + b$$

and

$$[G, G^{\dagger}] = \frac{1}{2}\varepsilon_0 e^2 n + 2nb\varepsilon\varepsilon_f \tag{4.80}$$

Note that $G = 0$ is the analogue of Gauss' law for our system. If $a + b \neq 0$ then $\Pi_0 = 0$ and $G = 0$ form a complete set of second class constraints. Let us assume this to be the case. The complete Hamiltonian can then be written as

$$\begin{aligned} h_0 = \Pi_1^{\dagger}\Pi_1 &+ \varepsilon\varepsilon_0\varepsilon_f n\sigma^{\dagger}\sigma + \frac{e\sqrt{n}}{\sqrt{2}}\left(\sigma^{\dagger}A_{\varepsilon} + A_{\varepsilon}^{\dagger}\sigma\right) \\ &+ i\varepsilon_f n\left(\Pi_1^{\dagger}A_0 - A_0^{\dagger}\Pi_1\right) - bA_{\varepsilon}^{\dagger}A_{\varepsilon} \\ &+ a\left(A_1^{\dagger}A_1 - A_0^{\dagger}A_0\right) + v_0^{\dagger}\Pi_0 + \Pi_0^{\dagger}v_0 + v_1^{\dagger}G + G^{\dagger}v_1 \end{aligned} \tag{4.81}$$

where v_0 and v_1 are some arbitrary Lagrange multipliers. It is straightforward to reduce h to the physical subspace by means of a canonical transformation. Such a transformation preserving the commutator structure is implemented as

$$\tilde{\mathcal{O}} = e^{iS}\mathcal{O}e^{-iS} = \mathcal{O} + i[S, \mathcal{O}] - \tfrac{1}{2!}\left[S, [S, \mathcal{O}]\right] + \ldots \tag{4.82}$$

with $S = S^{\dagger}$. An obvious choice is to look for an S that leaves Π_0 unchanged while transforming A_0 into a linear combination of the two constraints, Π_0 and G. We find

$$S = (a+b)^{-1}\left[b\varepsilon A_1 + i\varepsilon_f n\Pi_1 - e\sqrt{\frac{n}{2}}\varepsilon\sigma\right]\Pi_0^{\dagger} + \text{h.c.} \tag{4.83}$$

and the resulting transformation is

$$\begin{aligned} \tilde{A}_0 &= A_0 + \frac{b\varepsilon}{a+b}A_1 + \frac{i\varepsilon_f n}{a+b}\Pi_1 - \frac{e\varepsilon}{a+b}\sqrt{\frac{n}{2}}\sigma - \frac{i}{2(a+b)}\left(\frac{2bn\varepsilon_f}{a+b} - \varepsilon\varepsilon_0 e\sqrt{\frac{n}{2}}\right)\Pi_0 \\ \tilde{\Pi}_0 &= \Pi_0 \\ \tilde{A}_1 &= A_1 - \frac{i\varepsilon_f n}{a+b}\Pi_0 \\ \tilde{\Pi}_1 &= \Pi_1 - \frac{b\varepsilon}{a+b}\Pi_0 \\ \tilde{\sigma} &= \sigma + \frac{ie\varepsilon\varepsilon_0}{a+b}\sqrt{\frac{n}{2}}\Pi_0 \end{aligned} \tag{4.84}$$

yielding

$$h = h_{physical}(\tilde{\Pi}_1, \tilde{A}_1, \tilde{\sigma}) + v_2^\dagger \tilde{\Pi}_0 + \tilde{\Pi}_0 v_2 + v_3^\dagger G + G^\dagger v_3. \tag{4.85}$$

Since the constraints have effectively decoupled we can now set them to zero and identify the Hamiltonian with $h_{physical}$ without needing to modify the commutator structure for consistency. The Hamiltonian is thus written as (dropping the tildes)

$$\begin{aligned} h &= \Pi_1^\dagger \Pi_1 \left(1 + \frac{n^2}{a+b}\right) + \sigma^\dagger \sigma \left(\varepsilon\varepsilon_0\varepsilon_f + \frac{e^2}{2(a+b)}\right) n + i\left(A_1^\dagger \Pi_1 b - \Pi_1^\dagger A_1\right) \frac{nb\varepsilon\varepsilon_f}{a+b} \\ &+ A_1^\dagger A_1 \frac{a^2}{a+b} + \left(\sigma^\dagger A_1 + A_1^\dagger \sigma\right) \frac{ea\varepsilon\sqrt{n}}{\sqrt{2}(a+b)} + i\left(\Pi_1^\dagger \sigma - \sigma^\dagger \Pi_1\right) \frac{en^{3/2}\varepsilon_f}{\sqrt{2}(a+b)} \end{aligned} \tag{4.86}$$

Performing the following canonical transformation

$$\begin{aligned} A_1 &= A - \beta\Sigma - \tfrac{i\beta^2\varepsilon_0}{2}\Pi \\ \sigma &= \Sigma + i\varepsilon_0\beta\Pi \qquad \text{with} \qquad \beta = \frac{e\sqrt{n}\varepsilon}{\sqrt{2}a} \\ \Pi_1 &= \Pi \end{aligned} \tag{4.87}$$

we get

$$\begin{aligned} h &= \Pi^\dagger \Pi \left(1 + \frac{n^2(4a - \varepsilon\varepsilon_0\varepsilon_f e^2)^2}{16a^2(a+b)}\right) + i(A^\dagger \Pi - \Pi^\dagger A) \frac{\varepsilon_0 n(e^2 + 4b\varepsilon\varepsilon_0\varepsilon_f)}{4(a+b)} \\ &+ \Sigma^\dagger \Sigma n\varepsilon\varepsilon_0\varepsilon_f + A^\dagger A \frac{a^2}{a+b}. \end{aligned} \tag{4.88}$$

We can define some new operators, $f_{1,2} = \dfrac{\gamma A}{\sqrt{2}} \pm \dfrac{i\Pi}{\gamma\sqrt{2}}$, that obey $\left[f_1, f_2^\dagger\right] = 0$ and $\left[f_1, f_1^\dagger\right] = -\left[f_2, f_2^\dagger\right] = 1$. If we choose γ to be given by

$$\gamma = \sqrt{\frac{16a^4(a+b)}{16a^2(a+b) + (4a - \varepsilon\varepsilon_0\varepsilon_f e^2)^2}}$$

we get

$$h = E_1 f_1^\dagger f_1 + E_2 f_2^\dagger f_2 + n\varepsilon\varepsilon_0\varepsilon_f \Sigma^\dagger \Sigma \tag{4.89}$$

where

$$E_{2,1} = \pm\frac{\varepsilon_0 n(e^2 + 4\varepsilon\varepsilon_0\varepsilon_f b)}{4(a+b)} + \sqrt{\frac{(4a - \varepsilon\varepsilon_0\varepsilon_f e^2)^2 n^2}{16(a+b)^2} + \frac{a^2}{(a+b)}}$$

So we see that, in general, the spectrum is not Lorentz invariant. This lack of Lorentz invariance is deeply connected with the fact that this theory has an anomaly which is manifest in the non-vanishing commutator $[G, G^\dagger]$. It is intriguing that with the choice of b such that $4b + \varepsilon\varepsilon_0\varepsilon_f e^2 = 0$ the gauge generators commute $[G, G^\dagger] = 0$ and we recover a Lorentz invariant spectrum

$$h = Ef_1^\dagger f_1 + Ef_2^\dagger f_2 + n\Sigma^\dagger\Sigma \tag{4.90}$$

with

$$E = \sqrt{\frac{n^2}{\ell^2} + m^2} \qquad \text{where} \qquad m^2 = \frac{a^2}{a - \frac{e^2}{4}}.$$

Note that we had to choose $\varepsilon\varepsilon_0\varepsilon_f = +1$ in order to obtain a positive definite Hamiltonian. We see that a is then restricted to values greater than $\frac{e^2}{4}$. The analysis of the $n = 0$ sector is trivial. Proceding as above, one easily obtains

$$h(0) = m\left(f^\dagger(0)f(0) + \frac{1}{2}\right) + \frac{1}{2\ell}\rho(0)^2 \tag{4.91}$$

which is consistent with the $n \neq 0$ sector. The extension of our result from S^1 to the real line is quite trivial as we simply replace n/ℓ by p the momentum.

The above analysis is singular for the particular combination of counterterms where $a + b = 0$ which we must analyze separately. This critical value corresponds to the critical value $a = 1$ of references [59, 78, 40, 41] where the mass diverges. We consider only the critical value $4b = -e^2$ as, once again, only for this value do we get a Lorentz invariant Hamiltonian. Since

$$[\Pi_0, G^\dagger] = [G, G^\dagger] = 0 \tag{4.92}$$

we then get a third constraint obtained by commutating Gauss' Law constraint with the Hamiltonian –remember that in order to obtain a consistent theory, using Dirac's formulation, we must be able to add terms proportional to the constraints to the Hamiltonian such that all the constraints commute (up to terms proportional to themselves).– The third constraint turns out to be simply $\Pi_1 = 0$. Together with Gauss'Law constraint, they form a complete set of second class constraints. The other constraint, $\Pi_0 = 0$, is a first class constraint which is associated to a gauge freedom for our system. We can fix the corresponding gauge by setting $A_0 = 0$ and use the other two constraints to eliminate both A_1 and Π_1 from the system so that we get quite simply

$$H = \frac{1}{2\ell}\rho(0)^2 + \sum_{n>0} \frac{n}{\ell}\sigma^\dagger(n)\sigma(n) \tag{4.93}$$

Looking back at the case $a + b \neq 0$ and taking the limit as a goes to $-b$, we see that the mass goes to infinity and that the corresponding boson effectively decouples. This is consistent with the result obtained above if we had put $a + b = 0$ at the outset.

4.5 Discussion

In summary we have obtained a Lorentz invariant unitary solution of the chiral Schwinger model with minimal degrees of freedom by adding gauge and Lorentz variant counterterms to the bare Lagrangian. This can be interpreted as a particular choice of regularization which is implemented at the Hamiltonian level and is implicit in the gauge field independent Sommerfeld-Sugawara formula 4.74. It requires that the renormalization of the fermionic charge and the fermionic Hamiltonian operators are independent of the gauge fields. The difference between this and other regularizations is then parameterized by the local counterterms.

The spectrum we obtain agrees with that found previously by Jackiw and Rajaraman, and Girotti et al. [59, 78, 40, 41] except we find a massless chiral scalar instead of a massless scalar. The extra degree of freedom found previously is introduced by bosonization. Incidently this results in a different constraint structure as we have 2 second class or 2 second class plus 1 first class where Girotti et al. find 2 second class or 4 second class constraints. However as one first class constraint eliminates as many degrees of freedom (one from the constraint and one from the gauge fixing condition) as two second class constraints it is not surprising that the resulting theories show equivalence. A point that is not clear to us is the connection between the extra degree of freedom and the absence of a Lorentz variant counterterm.

Chapter 5

Anomalous theories at finite density

The anomalous non-conservation of the baryon number in the standard model has been known to exist for a certain time. This non-conservation was first studied in the context of instanton transitions[52, 53]. Since instantons correspond to tunnelling, it was found that the corresponding anomalous decay rate was exponentially suppressed by the spontaneous symmetry breaking in the standard model so as to be totally unobservable. However, it was shown that in some other cases (at high temperature [62], in the decays of some heavy particles [84], in the presence of magnetic monopoles [82, 23], etc.) anomalous baryon-violating processes took place without any exponential suppression. More recently it has been argued by Rubakov and his various co-workers [85, 86, 87, 67] that such processes were also unsuppressed at high densities. As a result it was found that the normal state of cold fermionic matter is unstable at sufficiently high fermion density. The basis of their argument stems from the fact that, because of the axial anomaly of the baryon current, both the fermion number and the vacuum energy of the baryon levels in the Dirac sea receive a contribution from the Chern-Simons number [22, 24]

$$N_{CS} = \frac{g^2}{32\pi^2} \int d^3x \, \epsilon^{ijk} \left(F^a_{ij} A^a_k - \tfrac{2}{3} \epsilon^{abc} A^a_i A^b_j A^c_k \right) . \tag{5.1}$$

We have seen already the corresponding phenomenon in two-dimensional theories in section 4.1. In fact we could repeat the analysis of that section in the $A_0 = 0$ and we'd find that the gauge-field dependence of the fermionic charge would be in the form of the

Chern-Simons number

$$N_{CS} = \frac{e}{2\pi} \int dx\, A_1. \tag{5.2}$$

Since the energy of the (anomalous) fermions receive a contribution proportional to the Chern-Simons number, it can be argued that an effective Hamiltonian for the gauge fields should include a term proportional to μN_{CS} where μ is the chemical potential for the fermions. Such a term was indeed found to occur in the context of perturbation theory at finite temperature and density in ref.[79]. This was criticized in [73] where it was pointed out that the inclusion of such a term led to a gauge-variant statistical mechanics. More recently, Rutherford [88, 89] showed that when non-perturbative effects were taken into account, the Chern-Simons term was always accompanied by an integer valued index whose effect was to subtract the integer part of N_{CS}. The effective Hamiltonian resulting from this computation is fully gauge invariant but has a non-trivial oscillatory behaviour as a function of the value of the background gauge field. In the light of this result, we find that the validity of any description of phenomena occuring as a result of including gauge-variants term in an effective Hamiltonian has to be questioned.

5.1 Introduction to a toy model

In order to get a better understanding of stability of anomalous theories at high density we study a simple 1+1-dimensional toy model having all of the basic characteristics of the physically interesting four-dimensional theories. This model is the axial Schwinger model to which we couple a complex scalar (Higgs) field. The Lagrangian is

$$\mathcal{L} = \overline{\Psi} i\gamma^\mu(\partial_\mu - ie\gamma^5 A_\mu)\Psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^* - \lambda(|\phi|^2 - c^2)^2 \tag{5.3}$$

where $D_\mu = \partial_\mu - ieA_\mu$. This model will be regulated such that the gauged current, $\overline{\Psi}\gamma^\mu\gamma^5\Psi$, is conserved while the vector current is beset by the anomaly

$$\partial_\mu \overline{\Psi}\gamma^\mu\Psi = \frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}. \tag{5.4}$$

This model was introduced by Rubakov in [87]. Under the assumption that $c \gg 1$ and that the system is neutral with respect to the gauged charge, *i.e.* $< \overline{\Psi}\gamma^0\gamma^5\Psi >= 0$, Rubakov makes the following claims:

- at small n_f (n_f is the anomalous charge density, $\overline{\Psi}\gamma^0\Psi$) the instanton-like transitions are negligible;
- the free energy density contains a non-trivial contribution proportional to the Chern-Simons density;
- the field strength vanishes in the case of neutral matter;
- at large enough n_f, the system is unstable and decay in a state having a vanishing fermionic density.

We will carefully examine the last three claims as well as the general issue of gauge invariance in this system. We will look at the first claim later, when we make connection to the physics of four-dimensional systems. Our discussion is based on the Bose-equivalent Lagrangian[18]

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \chi - \tfrac{e}{\sqrt{\pi}}A_\mu)^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^* - \lambda(|\phi|^2 - c^2)^2. \tag{5.5}$$

In his study of this model, Rubakov ignores all tunneling events and shows that the system is unstable at large enough fermionic density. This approximation amounts to studying the classical system described by the above bosonized Lagrangian which is what we will do. We will look at the validity of this approximation later.

5.2 Hamiltonian solution

The first step in our study is to derive a Hamiltonian. In this derivation we shall handle the gauge constraints using Dirac's method for constrained systems [32, 50]. It is convenient to express the Higgs field in terms of its real and imaginary part as follows:

$$\phi = \tfrac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \tag{5.6}$$

where both ϕ_1 and ϕ_2 are real. In terms of these fields, the Lagrangian is rewritten as

$$\begin{aligned} L &= \int dx\, \Big\{\frac{1}{2}(\partial_\mu \chi - \frac{e}{\sqrt{\pi}}A_\mu)^2 + \frac{1}{2}[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2] + eA_\mu(\phi_2\partial^\mu\phi_1 - \phi_1\partial^\mu\phi_2) \\ &\qquad - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2A_\mu A^\mu(\phi_1^2+\phi_2^2) + \frac{1}{4}\lambda(\phi_1^2+\phi_2^2-2c^2)^2\Big\}. \end{aligned} \tag{5.7}$$

The canonical momenta are

$$\begin{aligned} \Pi_0 &\equiv \frac{\delta L}{\delta \dot{A}_0} \approx 0 & E &\equiv \frac{\delta L}{\delta \dot{A}_1} = \dot{A}_1 - A_0' \\ \Pi &\equiv \frac{\delta L}{\delta \dot{\chi}} = \dot{\chi} - gA_0 & P_1 &\equiv \frac{\delta L}{\delta \dot{\phi}_1} = \dot{\phi}_1 + eA_0\phi_2 \\ P_2 &\equiv \frac{\delta L}{\delta \dot{\phi}_2} = \dot{\phi}_2 - eA_0\phi_1. \end{aligned} \tag{5.8}$$

We write $g = e/\sqrt{\pi}$ and obtain the preliminary Hamiltonian

$$\begin{aligned} H_0 &= \int dx\, \left\{\Pi\dot{\chi} + E\dot{A}_1 + P_1\dot{\phi}_1 + P_2\dot{\phi}_2\right\} - L \\ &= \int dx \Big\{\tfrac{1}{2}(\Pi^2 + E^2 + P_1^2 + P_2^2) + A_0\left[g\Pi - E' + e(P_2\phi_1 - P_1\phi_2)\right] \\ &\qquad + \tfrac{1}{2}\left[(\phi_1')^2 + (\phi_2')^2 + e^2A_1^2(\phi_1^2+\phi_2^2)\right] + eA_1(\phi_2\phi_1' - \phi_1\phi_2') \\ &\qquad\qquad + \tfrac{1}{2}(\chi' - gA_1)^2 + \tfrac{1}{4}\lambda(\phi_1^2+\phi_2^2-2c^2)^2\Big\} \end{aligned} \tag{5.9}$$

where we have integrated the $A_0'E$ term by parts and discarded the surface term. We postulate the fundamental equal time Poisson brackets between canonically conjugate variables[1]

$$\{Q(x,t), P(y,t)\} = \delta(x-y). \tag{5.10}$$

[1] we will drop the explicit time dependence in what follows.

We note that the constraint $\Pi_0(x) \approx 0$ is inconsistent at this stage with the Poisson bracket

$$\{A_0(x), \Pi_0(y)\} = \delta(x - y). \tag{5.11}$$

This will be taken care of shortly. The time evolution of physical quantities is governed by Hamilton's equations of motion

$$\dot{f}(x) = \{f(x), H\}. \tag{5.12}$$

In particular, consistency of the $\Pi_0 \approx 0$ constraint under time evolution yields Gauss's law for this system

$$G = -\dot{\Pi}_0 = g\Pi - E' + e(P_2\phi_1 - P_1\phi_2) \approx 0. \tag{5.13}$$

It is easy to verify that $\dot{G} \approx \{G, \Pi_0\} \approx 0$. In Dirac terminology we thus have two first class constraints. We are free to add these constraints to our preliminary Hamiltonian and get

$$H = H_0 + \int dx \, (v_0 \Pi_0 + v_1 G) \tag{5.14}$$

where $v_1(x)$ and $v_2(x)$ are arbitrary Lagrange multipliers. The presence of these constraints tells us that there is a redundancy in the number of dynamical variables. The physical evolution of these variables take place on an hypersurface embedded in the larger hypervolume of phase space. The reduction of the phase space to the physical subspace entails a choice of the Lagrange multipliers as well as some gauge fixing.

5.2.1 Unitary gauge

Our first gauge choice is the unitary gauge, $\phi_2 = 0$. This is the same gauge that was used in [87]. A subsequent constraint is obtained by looking at the defining equation for P_2, $\dot{\phi}_2 = P_2 + eA_0\phi_1 \approx 0$. We thus have the following four constraints

$$C_1 = \Pi_0 \approx 0$$

$$\begin{aligned} C_2 &= g\Pi - E' + eP_2\phi_1 \approx 0 \\ C_3 &= \phi_2 \approx 0 \\ C_4 &= P_2 + eA_0\phi_1 \approx 0. \end{aligned} \tag{5.15}$$

They yield the matrix $C_{ij}(x,y) = \{C_i(x), C_j(y)\}$

$$C(x,y) = e\delta(x-y)\begin{pmatrix} 0 & 0 & 0 & -\phi_1(x) \\ 0 & 0 & -\phi_1(x) & 0 \\ 0 & \phi_1(x) & 0 & 1 \\ \phi_1(x) & 0 & -1 & 0 \end{pmatrix}. \tag{5.16}$$

Provided that $\phi_1 \neq 0$ this matrix is invertible in the sense

$$\int dz\, C_{ij}(x,z)C^{-1}_{jk}(z,y) = \delta_{ik}\delta(x-y). \tag{5.17}$$

The inverse is given by

$$C^{-1}(x,y) = \frac{\delta(x-y)}{e\phi_1(x)^2}\begin{pmatrix} 0 & -1 & 0 & \phi_1(x) \\ 1 & 0 & \phi_1(x) & 0 \\ 0 & -\phi_1(x) & 0 & 0 \\ -\phi_1(x) & 0 & 0 & 0 \end{pmatrix}. \tag{5.18}$$

Using this inverse, we can define a Dirac bracket, $\{,\}_D$, as follows

$$\{A(x), B(y)\}_D = \{A(x), B(y)\} - \int dz\, dw\, \{A(x), C_i(z)\}C^{-1}_{ij}(z,w)\{C_j(w), B(y)\}. \tag{5.19}$$

The Dirac bracket is designed in such a way that $\{A(x), C_i(y)\}_D$ yields a vanishing result for all constraints. The consistency of the constraint is thus guaranteed. We can thus use the constraints to eliminate two of the redundant variables

$$P_2 = \frac{E' - g\Pi}{e\phi_1} \quad \text{and} \quad A_0 = \frac{g\Pi - E'}{e^2\phi_1^2}. \tag{5.20}$$

This yields the following Hamiltonian

$$\begin{aligned} H &= \int dx \Big\{ \frac{1}{2} \left[\Pi^2 + E^2 + P_1^2 + \left((e\phi_1)^{-1}(E' - g\Pi) \right)^2 \right] + \tfrac{1}{2}(\chi' - gA_1)^2 \\ &\quad + \tfrac{1}{2} \left[(\phi_1')^2 + eA_1^2\phi_1^2 \right] + \tfrac{1}{4}\lambda(\phi_1^2 - 2c^2)^2 \Big\}. \end{aligned} \tag{5.21}$$

As it turns out, the Dirac brackets of the remaining variables with themselves are the same as the original Poisson brackets. Note however that the Hamiltonian becomes singular when ϕ_1 vanishes. This is not surprising since the gauge we've chosen is ill-defined for such a case.

We are now in a position to make connection with Rubakov's argument. The first step is to ignore all dynamical contributions. This is done by setting all momenta to zero which yields the energy density

$$\mathcal{E} = \tfrac{1}{2}(\chi' - gA_1)^2 + \tfrac{1}{2} \left[(\phi_1')^2 + eA_1^2\phi_1^2 \right] + \tfrac{1}{4}\lambda \left(\phi_1^2 - 2c^2 \right). \tag{5.22}$$

Using the Bose-Fermi equivalence, we identify $\sqrt{\pi}\chi' = \overline{\Psi}\gamma^0\Psi = n_f$ as the anomalous fermion number density in the absence of gauge fields. The "real" fermion number density is given by $n_r = n_f - \frac{e}{\pi}A_1$ where we recognize $\frac{e}{\pi}A_1$ as being twice the Chern-Simons density.

The energy density is minimized by homogeneous Higgs fields yielding

$$\begin{aligned} \mathcal{E} &= \frac{\pi}{2}\left(n_f - \frac{e}{\pi}A_1 \right)^2 + \frac{e^2}{2}A_1^2\phi_1^2 + \frac{\lambda}{4}\left(\phi_1^2 - 2c^2 \right) \\ &\equiv \frac{\pi}{2}\left(n_f - \frac{e}{\pi}A_1 \right)^2 + \mathcal{E}_B. \end{aligned} \tag{5.23}$$

The reader should recognize eq. (3.12) of ref. [87].

The free energy density for this system is given by $f = \mathcal{E} - \mu n$ where the chemical potential is given by

$$\mu = \frac{\partial \mathcal{E}}{\partial n}. \tag{5.24}$$

However, there appears to be some ambiguity as to which n we should choose. Our preference would naturally be with the gauge invariant n_r yielding

$$\mu_r = \pi n_r \tag{5.25}$$

and

$$f = \frac{-\mu_r^2}{2\pi} + \mathcal{E}_B. \tag{5.26}$$

We recognize the first term as the free energy density for free massless fermions in 1+1-dimensions. The fact that our system is anomalous doesn't seem to imply anything special for the free energy. On the other hand, if we choose n_f, we find

$$\mu_f = \pi \left(n_f - \frac{e}{\pi} A_1 \right) \tag{5.27}$$

and

$$f = \frac{-\mu_f^2}{2\pi} + \mathcal{E}_B - e\mu_f A_1. \tag{5.28}$$

The free energy now contains a non-trivial contribution proportional to the Chern-Simons density for this system. Rubakov argues that, in spite of its gauge dependence, n_f makes physical sense. Indeed, we know that of the 5 field degrees of freedom originally present in out Lagragian, only 3 are physical as a consequence of gauge invariance. Since we have fixed all the gauges, we should be left with only 3 physical degrees of freedom. This is what we have with χ, A_1 and ϕ_1. Thus χ (or n_f) makes physical sense for this system. However, *it does not mean that we have to identify* n_f *with the fermionic density of our system.* It is true that when $A_1 = 0$, n_f and n_r coincide but in such cases, there is a contribution to the energy density from the field strength even in the case of neutral matter. This is a consequence of the anomaly and has been shown to occur in section (4.1.1). We will come back to this point shortly.

The behaviour of the energy density (5.23) as a function of n_f has been analyzed in [87]. For small n_f, the minimum of $\mathcal{E}$ is at $\phi_1 \approx \sqrt{2}c$, $A_1 \approx n_f/(2\phi^2)$ such that

$n_f \approx n_r \neq 0$. On the other hand, for large enough n_f, the minimum is at $\phi_1 = 0$, $eA_1 = \pi n_f$ such that $n_r = 0$. Rubakov argues that this indicates that the system is stable for small fermionic densities but not for large enough ones. Note that in the large density case, the expression that we derived for the energy density is incorrect as our gauge is ill-defined. We also find puzzling the fact that the energy is not always minimized in the absence of fermions. In order to clarify this whole situation,we find it more convenient to study the system in a different gauge.

5.2.2 Topological gauge

A more convenient gauge choice is provided by what we call the topological gauge. As we shall see this gauge choice is a very natural one and is always well defined. This gauge choice essentially corresponds to setting $\chi = 0$. This means that the anomalous fermion number is purely topological in nature, $N = \int dx\, A_1$. In the previously used notation, this gauge choice amounts to setting $n_f = 0$. The reader might object to this choice since the instability of the fermionic system is described using Rubakov's argument as a consequence of having a large n_f. Even though we stressed before that we should not identify n_f with the fermionic density, we wish to demonstrate quite clearly that our argument is not based on any gauge artifact. To this end, we first perform the following canonical transformation:

$$
\begin{aligned}
\tilde{A}_0 &= A_0 & \tilde{\Pi}_0 &= \Pi_0 \\
\tilde{A} &= A_1 - g^{-1}\chi' & \tilde{E} &= E \\
\tilde{\chi} &= \chi & \tilde{\Pi} &= \Pi - g^{-1}E' + eg^{-1}(P_2\phi_1 - P_1\phi_2) \\
\tilde{\phi}_1 &= \cos(eg^{-1}\chi)\phi_1 + \sin(eg^{-1}\chi)\phi_2 & \tilde{P}_1 &= \cos(eg^{-1}\chi)P_1 + \sin(eg^{-1}\chi)P_2 \\
\tilde{\phi}_2 &= -\sin(eg^{-1}\chi)\phi_1 + \cos(eg^{-1}\chi)\phi_2 & \tilde{P}_2 &= -\sin(eg^{-1}\chi)P_1 + \cos(eg^{-1}\chi)P_2
\end{aligned}
\tag{5.29}
$$

Such a transformation preserves the structure of the Poisson brackets. Note that $\mathcal{X}$ has not been altered by this transformation. This means if n_f $(\mathcal{X}')$ is to have any meaning on its own, this meaning should have been preserved by this transformation. The Hamiltonian is now written as (dropping the tildes)

$$
\begin{aligned}
H_0 \;=\; \int dx \Big\{ & \tfrac{1}{2}\left(E^2 + P_1^2 + P_2^2 + g^{-2}\left[E' + e(P_1\phi_2 - P_2\phi_1)\right]\right) + v_0\Pi_0 + v_2\Pi \\
& + \tfrac{1}{2}\left[(\phi_1')^2 + (\phi_2')^2 + e^2A^2(\phi_1^2 + \phi_2^2)\right] + eA(\phi_2\phi_1' - \phi_1\phi_2') \\
& + \tfrac{1}{2}g^2A^2 - \tfrac{1}{4}\lambda(\phi_1^2 + \phi_2^2 - 2c^2)^2 \Big\}
\end{aligned} \tag{5.30}
$$

where v_2 is a new Lagrange multiplier, linear combination of the old one, v_1, and of some of the dynamical fields. Note that $\mathcal{X}$ (and hence n_F) no longer explicitely appears as a dynamical field. The anomalous fermion number density is now given by $n = A$ which is gauge invariant. We can set the two Lagrange multiplier to zero and reduce the Hamiltonian to the physical subspace in a natural fashion. Note that this is then exactly the same Hamiltonian that we would have obtained by making the gauge choice $\mathcal{X} = 0$ before doing any canonical transformation. The fourth constraint needed to define the Dirac brackets would have been obtained from the defining equation for Π:

$$\dot{\mathcal{X}} = \Pi + gA_0 \approx 0. \tag{5.31}$$

As in the previous case, we look at the energy density ignoring the dynamical terms (momenta). We find that it is minimized not by spatially homogeneous Higgs fields but by the following:

$$
\begin{aligned}
\phi_1 &= \varphi \cos \int^x eA(y)\, dy \\
\phi_2 &= \varphi \sin \int^x eA(y)\, dy
\end{aligned} \tag{5.32}
$$

where $\varphi' = 0$. The energy density is then given by

$$\mathcal{E} = \frac{e^2}{2\pi}A^2 + \frac{\lambda}{4}(\varphi^2 - 2c^2)^2. \tag{5.33}$$

This expression is always minimized in the absence of anomalous charges ($A = 0$), in agreement with our physical intuition. Furthermore, we do not see how one can justify the inclusion of of a term proportional to the Chern-Simons density in the free energy using the above expression.

Using the Bose-equivalent approach, we can do more than simply study the behaviour of a static energy density. We can in fact study the full dynamics of the system, at least classically. The dynamical evolution is governed by Hamilton's equations of motion:

$$\begin{aligned}
\dot{A} &= E - B' \\
\dot{E} &= -\frac{e^2}{\pi}A[1 + \pi(\phi_1^2 + \phi_2^2)] + e(\phi_2'\phi_1 - \phi_1'\phi_2) \\
\dot{\phi}_1 &= P_1 + \phi_2 B \\
\dot{\phi}_2 &= P_2 - \phi_1 B \\
\dot{P}_1 &= eP_2B + \phi_1'' - e^2A^2\phi_1 + eA'\phi_2 + 2eA\phi_2' - \lambda\phi_1(\phi_1^2 + \phi_2^2 - 2c^2) \\
\dot{P}_2 &= -eP_1B + \phi_2'' - e^2A^2\phi_2 - eA'\phi_1 - 2eA\phi_1' - \lambda\phi_2(\phi_1^2 + \phi_2^2 - 2c^2)
\end{aligned} \tag{5.34}$$

where

$$B = \frac{\pi}{e^2}\left[E' + e(P_1\phi_2 - P_2\phi_1)\right]'.$$

These equations, involving only first-order time derivatives, are cast in a perfect form to be solved numerically. We have done so for the very special case where the system is spatially homogeneous. In this case, without further loss of generality, we can set $\phi_2 = P_2 = 0$ and obtain a very simple system of ordinary differential equations:

$$\begin{aligned}
\dot{\mathcal{A}} &= \mathcal{F} \\
\dot{\mathcal{F}} &= -g^2\mathcal{A}(1 + \Phi^2)
\end{aligned}$$

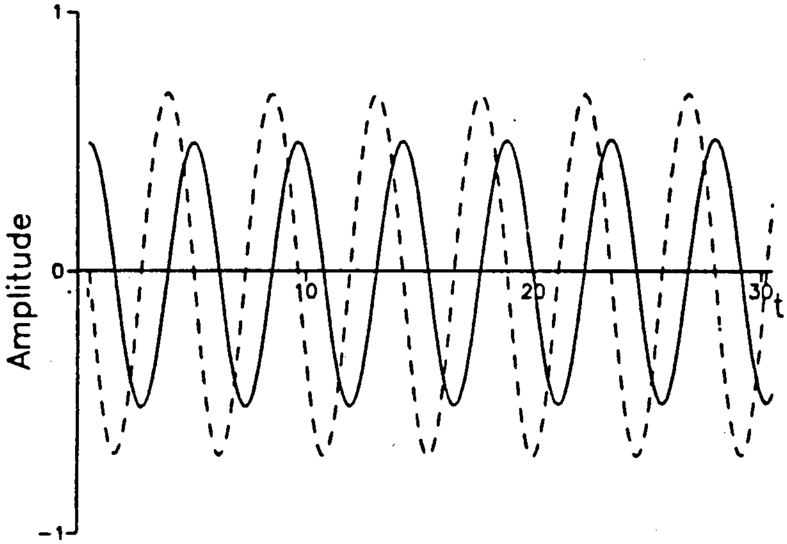


Figure 5.1: Oscillations of the fermion number (solid line) and the electric field (dashed line) as a function of time. The initial conditions were $\mathcal{F}$=P=0, $\mathcal{A} = 0.5$, $\Phi = 1$ with $m = \Lambda = g = 1$. Although we cannot see it on this scale, there are some small variations in amplitude. See appendix D for more details.

$$\begin{aligned}\dot{\Phi} &= P \\ \dot{P} &= -g^2\mathcal{A}\Phi - \Lambda\Phi\left(\Phi^2 - m^2\right)\end{aligned} \tag{5.35}$$

where

$$\sqrt[3]{\pi}\mathcal{A} = A, \quad \sqrt[3]{\pi}\mathcal{F} = E, \quad \sqrt{\pi}\Phi = \phi_1, \quad \sqrt{\pi}P = P_1$$
$$\Lambda = \sqrt[3]{\pi}\lambda, \quad \pi^2 m^2 = 2c^2 \qquad \text{and} \qquad \pi g^2 = e^2.$$

The behaviour of the rescaled fermion number $\mathcal{A}$ and electric field $\mathcal{F}$ is shown on figure 5.1 for a particular set of initial conditions. The important thing to notice is that there is a back reaction of the (Bose-equivalent) fermi field onto the gauge fields causing oscillations in the latter. An electric field is thus induced in contradiction with one of Rubakov's claims. Although the effects of the non-linearity of the equations of motion are too small to be seen on figure 5.1, they can yield very peculiar result as is shown in Appendix D.

5.3 Implications for four dimensional theories

Physics in two dimensions is very different from physics in four dimensions. For instance the Bose-Fermi equivalence that we described does not exists in four-dimensional theories. Also, the Higgs phenomena does not take place in two-dimensions because of instanton effects[21, 77]. These instantons effects in fact dominate the physics in two dimensions giving rise to confinement. Because of this fact, our semi-classical results are likely to be fairly different from what would happen in a fully quantized theory. Another peculiarity of two-dimensions is the allowed existence of a non-vanishing background electric field. Such an electric field in four dimension could not survive because of the dielectric breakdown of the vacuum[27]. As a consequence of all these differences, one has to be very careful in extrapolating results from two-dimensional theories to our four-dimensional world.

There are however some features of two-dimensional theories that we can confidently assess to be present as well in higher dimensions. For instance, the issue of gauge invariance should be independent of the dimensionality. We thus believe that the inclusion of a term proportional to the Chern-Simons number as a mean to obtain an effective Hamiltonian is incorrect. We would further question the validity of any prediction obtained from the study of such a Hamiltonian as we believe that non-perturbative effects are of the same magnitude as the perturbatives ones (so as to cancel their gauge variance) and can not be ignored. We also question the validity of studying a static effective Hamiltonian as is done in [85, 86, 87, 67] in order to understand the effects of anomalies. As we have seen, back reaction of the Fermi fields onto the gauge fields produce dynamical fields which have non-negligible effects in two dimensions. This back-reaction is shown to be present as well in the fully quantized Schwinger model[109, 18]. In fact, we believe that this back-reaction and the resulting oscillations in the anomalous fermion number could yield some very interesting results for cosmology.

The type of results we have in mind are best illustrated by the following scenario. We take for starting point a time at which the temperature was below the Grand Unification scale and where a baryon asymmetry had developped. Assuming that baryon-violating processes are unsuppressed at high temperature (as has been suggested in [62, 8, 9]) due to the electro-weak anomaly, we suggest that non-vanishing oscillating electric and magnetic field would be present (the analogue of the two-dimensional electric field is of the form $\vec{E} \cdot \vec{B}$ in four-dimensions). The presence of such oscillating fields and the corresponding oscillations in the baryon number might in fact be necessary in order that a net baryon number exists if the baryon-number violating interactions occur rapidly enough to yield thermodynamical equilibrium (a model implementing baryogenesis based on a similar type of oscillations was introduced in [25, 26]). As the universe cools down, the anomalous processes become negligible and we are left with a net baryon number as well as with both an electric and a magnetic field background. The dielectric breakdown of the vacuum would cancel the electric field leaving only a magnetic field. The existence of a primordial magnetic field not only isn't ruled out by observations but in fact appears to be required in order to explain the bisymmetric spiral structure of the magnetic field of many galaxies [92]. We do not know of any other mechanism that could produce such primordial magnetic fields.

The above scenario is very tentative and leaves many questions unanswered.[2] However we feel that it provides an interesting motivation for studying dynamical effects related to anomalous phenomena. Unfortunately, we do not know at this time how to properly address these questions in any realistic four-dimensional model.

---

[2]For instance, one might expect the formation of domains having different baryon number in apparent contradiction with current observations.

Chapter 6

Conclusion

In our study of the effects of finite densities in gauge theories we managed to answer some of the questions that we mentioned in the introduction. We have shown how one can characterize a confining phase in QCD and other related $SU(N)$ gauge theories even in the presence of fermions. In particular, we have shown that the phase structure as a function of an imaginary chemical potential for the fermions is related to the confining properties of these theories. Our results indicate that at high temperature these theories should have a first-order phase transition as a function of imaginary chemical potential whereas we wouldn't expect such a transition to be present in the low-temperature phase. We have also shown how we can deduce the properties of these theories at nonzero fermion density from their behaviour at finite imaginary chemical potential.

We have also gained a substantial understanding of anomalous gauge theories and have clarified (we hope) many issues related to gauge invariance. In the process we even managed to show that theories that lose their gauge invariance through quantization can nonetheless be consistent. We have also suggested a new mechanism for producing primordial magnetic fields in the early universe.

However, we have left many questions unanswered. For example, does the phase structure that we've uncovered for $SU(N)$ gauge theories at the one-loop level survive at higher orders? Do we have the same type of global symmetry and does it play the same rôle in other gauge theories, like those based on the exceptional groups E_6 and E_8? Can we realistically take into account finite volume effects and potentially verify them in heavy ion collisions?

From our study of anomalous theories, further questions arise. Issues related to possible mass generation for gauge bosons in four-dimensional theories through anomalies need clarification. The dynamical effects related to the electroweak anomaly in the early universe need also to be addressed. These and other related questions fully deserve a thorough study. We hope that this will be done in the near future.

Appendix A

Notation

Throughout this thesis, we use an implicit summation convention over repeated indices. Greek indices run from 1 to 4. Staggered repeated indices are used for Minkowski space whereas non-staggered ones denote Euclidean space. Our choice for the Minkowski metric is taken to be $g_{\mu\nu}$ = diag(1,-1,-1,-1) in four spacetime dimensions and $g_{\mu\nu}$ = diag(1,-1) in two spacetime dimensions. Any Euclidean metric is taken to be of positive signature. Roman indices $a, b, c, \ldots$ are used as group indices although we sometimes use i, j, k as spatial indices; this should usually be clear from the context. The metric used for summations over group indices is of positive signature.

As mentioned in the introduction, we choose to work in a system of units where $\hbar = c = k_B = 1$. The inverse of the temperature, T, is denoted by $\beta = T^{-1}$. We use interchangeably the terms Lagrangian and Lagrangian density as well as Hamiltonian and Hamiltonian density. An integration over space-time is often understood even when it is not explicitely written; this should be always clear from the context. For instance, the derivation of the Euler-Lagrange equations of motion is to be obtained from the functional derivation of the action $S = \int dt\, L$ even though we generally write L instead of S.

Appendix B

Invariant measure for $SU(N)$

When performing an integration over all possible group elements it is important to insure that the result does not depend on the parametrization of the group elements. In order to do so, we need to introduce an invariant measure –also called Haar measure–. The reader is referred to [105, 98] for more details on this procedure in general. We provide here a derivation for the invariant measure for $SU(N)$ as we found it lacking from the standard literature.

By invariance we mean that given a set of variables $\mathbf{a} = \{a_1, a_2, \ldots, a_r\}$ used to parametrize any group element $\mathcal{U}$ we must have

$$I = \int D\mathbf{a}\, f[\mathcal{U}(\mathbf{a})] = \int D\mathbf{a}\, f[\mathcal{U}(\mathbf{c})] \tag{B.1}$$

where $\mathbf{c}$ denotes the parameters of the transformed group elements $\mathcal{U}(\mathbf{c}) = \mathcal{U}(\mathbf{a})\mathcal{U}(\mathbf{b})$ for any fixed $\mathbf{b}$. Writing the measure as

$$D\mathbf{a} = \mu(\mathbf{a})d\mathbf{a} = \mu(\mathbf{a})da_1 da_2 \ldots da_r \tag{B.2}$$

we obtain quite trivially

$$I = \int d\mathbf{a}\, \mu(\mathbf{a}) f[\mathcal{U}(\mathbf{a})] = \int d\mathbf{c}\, \mu(\mathbf{c}) f[\mathcal{U}(\mathbf{c})] = \int d\mathbf{a} \left|\frac{\partial \mathbf{c}}{\partial \mathbf{a}}\right| \mu(\mathbf{c}) f[\mathcal{U}(\mathbf{c})] \tag{B.3}$$

where

$$\left|\frac{\partial \mathbf{c}}{\partial \mathbf{a}}\right| = \begin{vmatrix} \dfrac{\partial c_1}{\partial a_1} & \cdots & \dfrac{\partial c_1}{\partial a_r} \\ \vdots & & \vdots \\ \dfrac{\partial c_r}{\partial a_1} & \cdots & \dfrac{\partial c_r}{\partial a_r} \end{vmatrix} \tag{B.4}$$

is the Jacobian of the transformation. In order for eq. [...] to be valid $\mu(\mathbf{c})$ must satisfy the condition

$$\mu(\mathbf{c})\left|\frac{\partial \mathbf{c}}{\partial \mathbf{a}}\right| = \mu(\mathbf{a}) \tag{B.5}$$

for all $\mathbf{a}$. In particular, we may choose

$$\mu(\mathbf{c}) = \left|\frac{\partial \mathbf{c}}{\partial \mathbf{a}}\right|^{-1}_{\mathbf{a}=0}. \tag{B.6}$$

We now proceed to derive the invariant measure for both $U(N)$ and $SU(N)$. We first note that any matrix $\mathcal{U} \in U(N)$ can be brought to diagonal form by a unitary transformation $\mathcal{V}$: $D = \mathcal{V}\mathcal{U}\mathcal{V}^{-1}$ where $\mathcal{D}$ can be written as $D_{jk} = \delta_{jk}\exp(i\theta_j)$. So each element of $U(N)$ can be uniquely characterized by the set of N real parameters $\{\theta_j\}$. These are the parameters we integrate over. However, since a $U(N)$ transformation generally involves N^2 parameters, the Jacobian has to depend on N^2 parameters; the remaining $N^2 - N$ parameters are those needed to define the unitary transformation diagonalizing $\mathcal{U}$. For $SU(N)$ the only modification is that we have one less parameter as we have the constraint $\sum \theta_j = 0 \mod 2\pi$ which comes from requiring the determinant to be unity.

In order to keep the notation simple, we first consider the group $U(3)$. The first step is to choose a basis by considering a group element $\mathcal{U}(\mathbf{b})$ with matrix elements $\mathcal{U}_{jk} = \delta_{jk}\exp(ib_j)$ where the b_j are 3 $-N$ for $U(N)-$ real numbers. Since we are interested in the limit $\mathbf{a} \to 0$ we need only consider a matrix $\mathcal{U}(\mathbf{a})$ that is infinitesimally close to the identity:

$$\mathcal{U}(\mathbf{a}) = \begin{pmatrix} 1 - ia_{11} & a_{12} & a_{13} \\ -a_{12}^* & 1 - ia_{22} & a_{23} \\ -a_{13}^* & -a_{23}^* & 1 - ia_{33} \end{pmatrix} \tag{B.7}$$

where $| a_{ij} | \ll 1$ and the a_{jj} are real. To first order we can rewrite this matrix as a product of $U(2)$ transformations:

$$
\begin{aligned}
\mathcal{U}(\mathrm{a}) &= \mathcal{U}(\mathrm{d})\mathcal{U}(\mathrm{e})\mathcal{U}(\mathrm{f}) \\
&= \begin{pmatrix} 1-id_{11} & a_{12} & 0 \\ -a_{12}^* & 1-id_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1-ie_{11} & 0 & a_{13} \\ 0 & 1 & 0 \\ -a_{13}^* & 1 & 1-ie_{33} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-if_{22} & a_{23} \\ 0 & -a_{23}^* & 1-if_{33} \end{pmatrix}.
\end{aligned}
$$

It is very easy to see how to generalize this result: any element of $U(N)$ infinitesimally close to the identity can be written to first order as a product of $\frac{N(N-1)}{2}$ $U(2)$ transformations. The corresponding Jacobian for such a transformation can then be written by using the chain rule as a product of Jacobians for $U(2)$ transformations. We can thus now concentrate on $U(2)$ transformations. Working only to first order in a_{jk} we write

$$
\begin{aligned}
\mathcal{U}(\mathbf{c}) &= \mathcal{U}(\mathrm{a})\mathcal{U}(\mathrm{b}) \\
\begin{pmatrix} \mathrm{e}^{i(b_1+a_{11})} & ia_{12}\mathrm{e}^{ib_2} \\ ia_{12}^*\mathrm{e}^{ib_2} & \mathrm{e}^{i(b_2+a_{22})} \end{pmatrix} &= \begin{pmatrix} 1+ia_{11} & ia_{12} \\ ia_{12}^* & 1+ia_{22} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{ib_1} & 0 \\ 0 & \mathrm{e}^{b_2} \end{pmatrix}.
\end{aligned}
\tag{B.8}
$$

$\mathcal{U}(\mathbf{c})$ is easily diagonalized and we have

$$
\begin{aligned}
\mathcal{V}(\mathbf{c}) &= \begin{pmatrix} 1 & \left(\frac{ia_{12}}{\mathrm{e}^{i(b_1-b_2)}-1}\right) \\ \left(\frac{ia_{12}^*}{\mathrm{e}^{i(b_1-b_2)}-1}\right) & 1 \end{pmatrix} \\
D(\mathbf{c}) &= \begin{pmatrix} \mathrm{e}^{i(b_1+a_{11})} & 0 \\ 0 & \mathrm{e}^{i(b_2+a_{22})} \end{pmatrix}.
\end{aligned}
\tag{B.9}
$$

Our choice of parameters is $\mathbf{c} = \{b_1 + a_{11}, b_2 + a_{22}, \mathcal{V}_{12}, \mathcal{V}_{21}\}$ and $\mathbf{a} = \{a_{11}, a_{22}, a_{12}, a_{12}^*\}$. The Jacobian is easily calculated and the measure is then given by:

$$
\mu(\mathbf{b}) = \left|\mathrm{e}^{ib_1} - \mathrm{e}^{ib_2}\right|^2. \tag{B.10}
$$

The measure for $SU(N)$ is thus given by

$$
\mu(\mathbf{b}) = \prod_{j<k}^{N} \left|\mathrm{e}^{ib_j} - \mathrm{e}^{ib_k}\right|^2 \tag{B.11}
$$

subject to the condition

$$\sum_{j=1}^{N} b_j = 0 \mod 2\pi.$$

Appendix C

Computation of the free energy

We wish to compute the free energy $-\beta F = \ln Z$ with Z as defined by equation 2.63. In order to do so, it is very convenient to separate out the longitudinal part of $\vec{a}$ from the two transverse compontents. The longitudinal part of $\vec{a}$ is given by $a_{||} = \vec{a} \cdot \mathbf{e}_{||}$ where

$$\mathbf{e}_{||} = \frac{(\partial_1 \hat{\imath} + \partial_2 \hat{\jmath} + \partial_3 \hat{\mathbf{k}})}{(\vec{\nabla}^2)^{1/2}}. \tag{C.1}$$

We get similar expressions for the transverse components. It is then easy to factor out the partition function

$$Z = Z_\phi Z_\perp^2 Z_{||} Z_{fermion} Z_{measure} \tag{C.2}$$

with the various subpartition functions given by

$$\begin{aligned}
Z_\phi &= \int \mathcal{D}\phi \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x\, \phi \nabla^2 \phi\right\} \\
Z_\perp &= \int \mathcal{D}a_\perp \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x\, \left[(\partial_0 a_\perp + g[\Phi, a_\perp])^2 + a_\perp \nabla^2 a_\perp\right]\right\} \\
Z_{||} &= \int \mathcal{D}a_{||} \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x\, \left[(\partial_0 a_{||} + g[\Phi, a_{||}])^2\right]\right\} \\
Z_{fermion} &= \int \mathcal{D}\overline{\Psi}\mathcal{D}\Psi \exp\left\{-\frac{1}{2}\int_0^\beta d\tau \int d^3x\, \overline{\Psi}(i\gamma_\mu \partial_\mu + \gamma_0 \Phi - m)\Psi\right\} \\
Z_{measure} &= \exp\left\{\sum_{j>k}^{N} \int d^3x \frac{d^3k}{(2\pi)^3} \ln\left[1 - \cos\left(\frac{\beta C_j - \beta C_k}{2}\right)\right]\right\}.
\end{aligned} \tag{C.3}$$

In the expression for $Z_\perp$ and $Z_{||}$, a has $N^2 - 1$ independent components over which we have to integrate. Using the fact that Φ is diagonal and after doing an integration by parts it is possible to diagonalize $Z_\perp$ (or $Z_\|$) further and obtain

$$Z_\perp = Z_{00}^{-1} \prod_{j,k=1}^{N} Z_{jk} \tag{C.4}$$

where

$$Z_{jk} = \int \mathcal{D}\tilde{a} \exp\left\{ -\frac{1}{2} \int_0^\beta \int d^3x \, \left[\tilde{a} \left([\partial_0 + C_j - C_k]^2 + \nabla^2 \right) \tilde{a} \right] \right\}. \tag{C.5}$$

The above expression was obtained by considering each component of $a_\perp$ ($a_\|$) as independent. however, in doing so, we are overcounting by one and so we subtract the (normalized) trace which is denoted by Z_{00}. We can rewrite this result as

$$Z_\perp = \int \mathcal{D}\tilde{a} \exp\left\{ -\frac{1}{2} \int_0^\beta \int d^3x \left[\tilde{a} \left(\sum_{j,k=1}^{N} [\partial_0 + C_j - C_k]^2 + \nabla^2 \right) \tilde{a} - \tilde{a}\partial^2 \tilde{a} \right] \right\} \tag{C.6}$$

with a similar expression for $Z_\|$. The full partition function is now expressed in a form allowing an easy evaluation using the basic formulae

$$\begin{aligned} Z_{boson} &= \int da \exp\left\{ -\frac{1}{2} a^T M a \right\} \\ &= (\det M)^{-1/2} \end{aligned} \tag{C.7}$$

and

$$\begin{aligned} Z_{fermion} &= \int d\overline{\Psi} d\Psi \exp\left\{ \overline{\Psi} N \Psi \right\} \\ &= \det N. \end{aligned} \tag{C.8}$$

It is now quite easy to evaluate the various determinants and obtain an integral representation for the free energy. To see how to do this, consider the trace of the operator ∂^2 in the $|x\rangle$ representation:

$$\mathrm{Tr}\partial^2 = \int_0^\beta \int d^3x \, \langle x|\partial^2|x\rangle. \tag{C.9}$$

Inserting the identity $\mathcal{I} = \sum_n |\omega_n\rangle\langle\omega_n| \int d^3k \, |\vec{k}\rangle\langle\vec{k}|$, we get

$$\mathrm{Tr}\partial^2 = \int_0^\beta \int d^3x \sum_n \int d^3k \langle x|\vec{k}\rangle |\omega_n\rangle \partial^2 \langle\omega_n|\langle\vec{k}|x\rangle. \tag{C.10}$$

We have two different cases to consider depending on the periodicity nature of the function considered. If we are dealing with periodic functions, we have

$$\langle\omega_n|\langle\vec{k}|x\rangle = \frac{\exp(i\omega_n^+\tau)}{\beta}\frac{\exp(i\vec{k}\cdot\vec{x})}{(2\pi)^{3/2}} \tag{C.11}$$

with $\omega_n^+ = 2\pi n\beta^{-1}$ whereas if we are dealing with anti-periodic functions we obtain a similar expression with $\omega_n^- = (2n+1)\pi\beta^{-1}$. The resulting trace becomes

$$\mathrm{Tr}\partial_\pm^2 = -V\beta\sum_n\int\frac{d^3k}{(2\pi)^3}\left[(\omega_n^\pm)^2 + k^2\right] \tag{C.12}$$

where $V = \int d^3x$ is the spatial volume. Using the well known identity $\det\mathcal{O} = \exp[\mathrm{Tr}\ln\mathcal{O}]$, we get the following contributions to the free energy:

$$\begin{aligned}
F_\phi &= \frac{NV}{2\beta}\int\frac{d^3k}{(2\pi)^3}\ln k^2 \\
F_\perp &= \frac{V}{\beta}\int\frac{d^3k}{(2\pi)^3}\left\{\sum_{j,k=1}^{N}\ln\left([\omega_n^+ + C_j - C_k]^2 + k^2\right) - \ln[(\omega_n^+)^2 + k^2]\right\} \\
F_\parallel &= \frac{V}{2\beta}\int\frac{d^3k}{(2\pi)^3}\left\{\sum_{j,k=1}^{N}\ln[\omega_n^+ + C_j - C_k]^2 - \ln(\omega_n^+)^2\right\} \\
F_{measure} &= -\sum_{j<k}^{N}\frac{V}{2\beta}\int\frac{d^3k}{(2\pi)^3}\ln\left[1-\cos\left(\frac{\beta C_j - \beta C_k}{2}\right)\right] \\
F_{fermion} &= \frac{2V}{\beta}\int\frac{d^3k}{(2\pi)^3}\left\{\sum_{j=1}^{N}\ln\left([\omega_n^- - C_j]^2 + k^2 + m^2\right)\right\}.
\end{aligned} \tag{C.13}$$

$$\tag{C.14}$$

The result for $F_{fermion}$ comes from the fact that in four dimensions we have

$$\det(i\gamma_\mu D_\mu - m) = det(-D_\mu D_\mu + m^2)^2. \tag{C.15}$$

The value of F at $C = 0$ is simply the free energy for an ideal Bose (Fermi) gas. We will ignore this contribution and study the C-dependence by looking at $F(C) - F(0)$. To this end we first compute the following definite integral:

$$I(C) = \sum_{n=-\infty}^{\infty}\int\frac{d^3k}{(2\pi)^3}\ln\left[\left(\frac{2\pi n}{\beta} + C\right)^2 + k^2\right]. \tag{C.16}$$

The C-dependent part of this integral will be obtained by considering $I(C)-I(0)$

$$I(C)-I(0)=\int\frac{d^3k}{(2\pi)^3}\left[S(C)-S(0)\right] \tag{C.17}$$

where

$$S(C)=\sum_{n=-\infty}^{\infty}\ln\left[\left(\frac{2\pi n}{\beta}+C\right)^2+k^2\right]. \tag{C.18}$$

Taking a derivative with respect to k

$$\frac{dS}{dk}=\sum_{n=-\infty}^{\infty}\frac{2k}{\left(\frac{2\pi n}{\beta}+C\right)^2+k^2} \tag{C.19}$$

we can evaluate this new sum by standard contour integral techniques. This is done by introducing

$$f(z)=\frac{2k\cos z}{\left[\left(\frac{2\pi n}{\beta}+C\right)^2+k^2\right]\sin z} \tag{C.20}$$

which has single poles at $z=n\pi,-\frac{\beta}{2}(C\pm ik)$. Now, consider $J=\oint_\Gamma f(z)\,dz$ where Γ is illustrated on fig. C.1. It is easy to show that in the limit as $R\to\infty$, J vanishes and that the infinite sum (eq. C.19) is equal to minus the sum of the residues at the two poles away from the real axis

$$\frac{dS}{dk}=-\mathrm{Res}\left[f\left(z=-\frac{\beta}{2}(C+ik)\right)\right]-\mathrm{Res}\left[f\left(z=-\frac{\beta}{2}(C-ik)\right)\right]. \tag{C.21}$$

This is easily evaluated and yields

$$\frac{dS}{dk}=\beta k\frac{\sinh\beta k}{\cosh\beta k-\cos\beta C} \tag{C.22}$$

from which we get

$$S(C)=\ln(\cosh\beta k-\cos\beta C) \tag{C.23}$$

up to some irrelevant additive C-independent constant[1]. We thus have

$$I(C) \;=\; \int\frac{d^3k}{(2\pi)^3}\ln(\cosh\beta k-\cos\beta C)$$

[1]This is most easily seen by noting that $I(C)-I(0)$ has to converge to some finite value. A C-dependent contribution to $S(C)$ would clearly spoil the convergence.

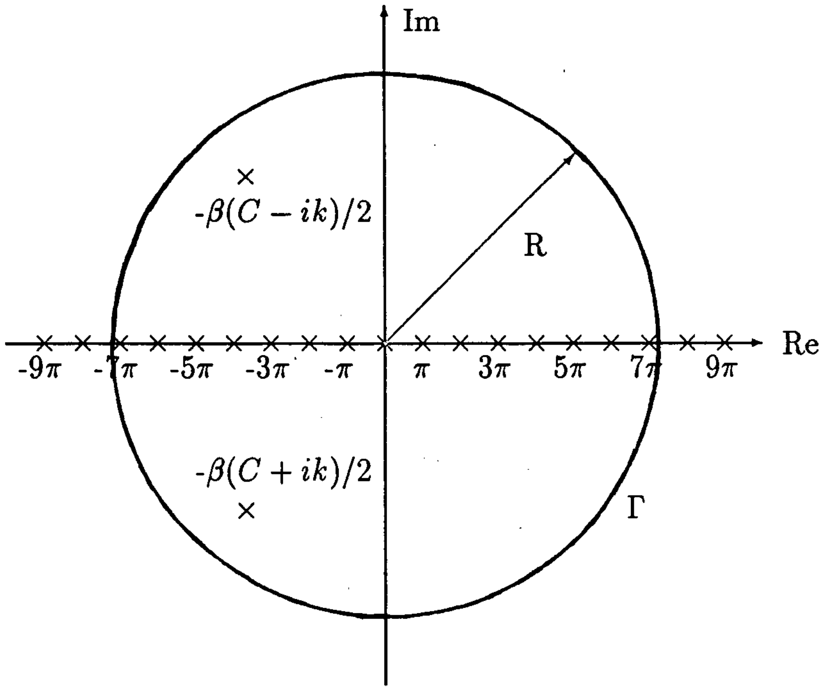


Figure C.1: Contour of integration for evaluating the infinite series [eq. C.19] .

$$= \int \frac{d^3k}{(2\pi)^3}\left[\beta k - \ln 2 + \ln(1 - \mathrm{e}^{-\beta(k+iC)}) + \ln(1 - \mathrm{e}^{-\beta(k-iC)})\right]. \qquad \text{(C.24)}$$

Dropping the first two terms since they are C-independent, we Taylor expand the logarithms, interchange the order of summation and integration and get

$$I(C) = \frac{1}{2\pi^2\beta^3}\sum_{n=1}^{\infty}\frac{\cos n\beta C}{n^4} \quad + \; \mathrm{C-independent\ \ terms.} \qquad \text{(C.25)}$$

We recognize this as some Fourier expansion which can be rewritten as[2]

$$I(C) = \frac{\pi^2}{90\beta^3} - \frac{\pi^2}{3\beta^3}\left[\frac{\beta C}{2\pi}\right]^2_{\mathrm{mod1}}\left(\left[\frac{\beta C}{2\pi}\right]^2_{\mathrm{mod1}} - 1\right)^2. \qquad \text{(C.26)}$$

Going back to the expression for the free energy, it is quite straightforward now to show that the longitudinal part is cancelled by the measure term and obtain the bosonic

[2]See formula 1.4436 in Gradshteyn and Ryzhik [42]

contribution as

$$F_{boson} = \frac{V\pi^2}{24\beta^4}\sum_{j,k=1}^{N}\left\{1-\left(\left[\frac{\beta C_j - \beta C_k}{2\pi}\right]_{\mathrm{mod}\ 2} - 1\right)^2\right\}^2 . \tag{C.27}$$

The fermionic part is handled in a similar fashion yielding

$$F_{fermion}(C) = -\frac{2V}{\beta}\sum_{j=1}^{N}\int\frac{d^3k}{(2\pi)^3}\left[\ln(1-\mathrm{e}^{-\beta(\Omega+iC)}) + \ln(1-\mathrm{e}^{-\beta(\Omega-iC)})\right] \tag{C.28}$$

with $\Omega = (k^2+m^2)^{1/2}$. This integral can be evaluated in the case $m = 0$ and we obtain

$$F_{fermion}(C) = \frac{V\pi^2}{12\beta^4}\sum_{j=1}^{N}\left\{1-\left(\left[\frac{\beta C_j}{2\pi}+1\right]_{\mathrm{mod}\ 2} - 1\right)^2\right\}^2 . \tag{C.29}$$

Appendix D

Examples of non-linear oscillations

As we mentioned in chapter 5, the back-reaction of the Fermi fields onto the gauge fields creates some time-dependent oscillatory variations of the anomalous charge. We have studied the homogeneous case with various initial conditions in order to see if the system would reach some steady state where the oscillations would either cease or settle around a non-vanishing value for the charge. We haven't seen any clear indication that this would happen. However, some interesting behaviour arises because of the non-linearity of the equations, espcially when the total energy in the system is slightly higher than the potential barrier between the two minima of the rescaled Higgs. We recall that the equations of motion for the rescaled fields are

$$\begin{aligned}
\dot{\mathcal{A}} &= \mathcal{F} \\
\dot{\mathcal{F}} &= -g^2\mathcal{A}(1+\Phi^2) \\
\dot{\Phi} &= P \\
\dot{P} &= -g^2\mathcal{A}\Phi - \Lambda\Phi\left(\Phi^2 - m^2\right).
\end{aligned} \tag{D.1}$$

We have chosen for initial conditions $\mathcal{E} = P = 0$ and $\Phi = m$ with $\Lambda = g = 1$. We have looked at two different values of m, with various initial values for the rescaled anomalous charge. We show here a subset of the cases we looked at. As a convention, the solid line shows the time variations of the rescaled anomalous charge, the long dotted line shows the location of the two minima (in the absence of gauge fields) for Φ ($\pm m$), and the short dotted line shows the behaviour of Φ.

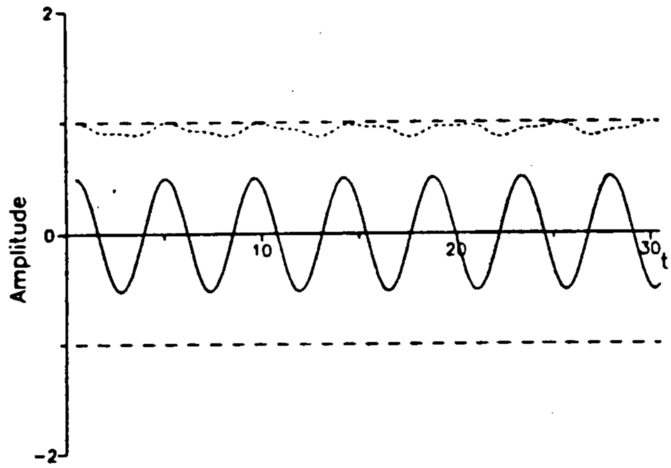


Figure D.1: $\mathcal{A}$= 0.5, m=1. The non-linearity of the equations has very little effect on the oscillations of the anomalous charge.

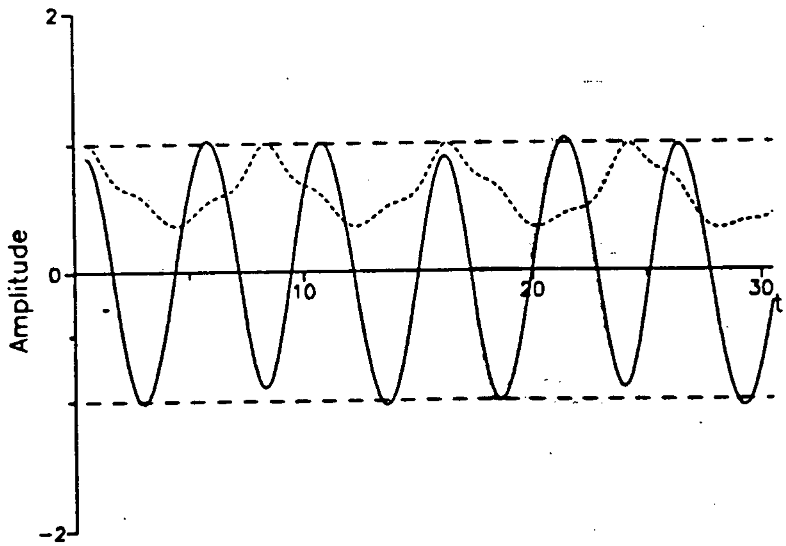


Figure D.2: $\mathcal{A}$=0.9, m=1. Some small variations in amplitude appear.

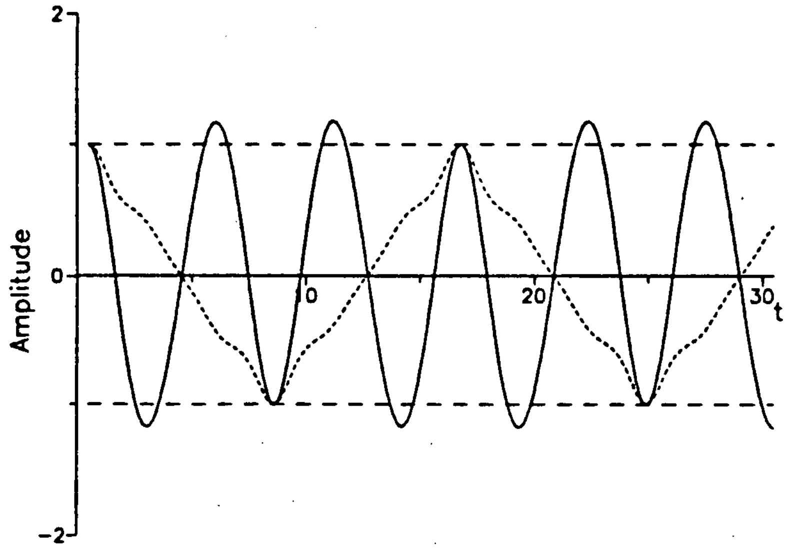


Figure D.3: $\mathcal{A}$=1, m=1.

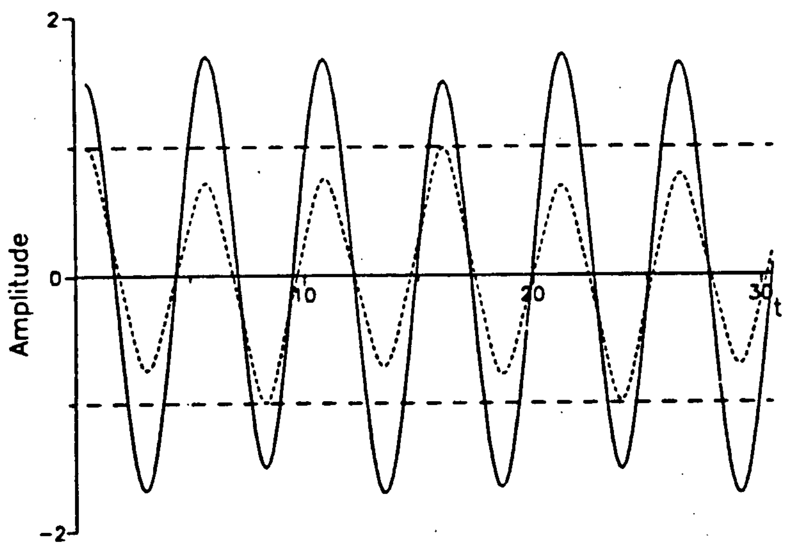


Figure D.4: $\mathcal{A}$=1.5, m=1. The two fields appear to be synchronized.

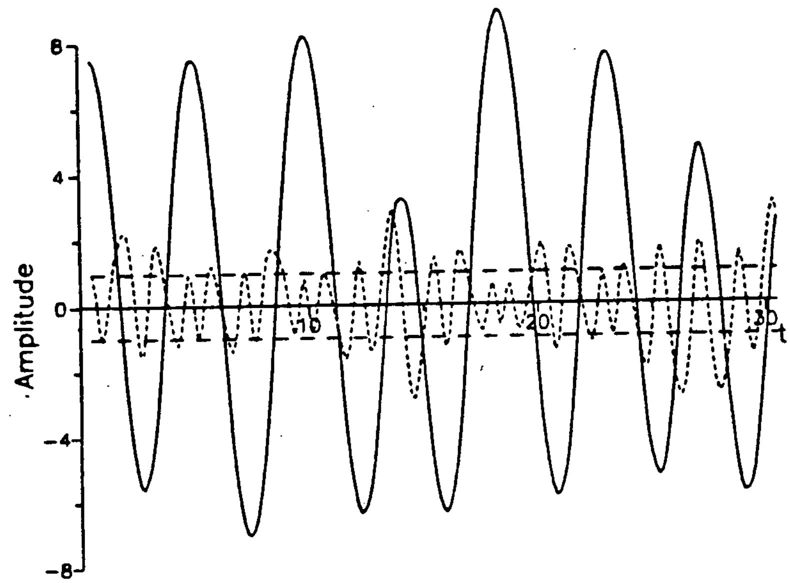


Figure D.5: $\mathcal{A}$=7.5, m=1. The apparent synchronization showed in the previous figure is absent. Note that the period of oscillation has shortened.

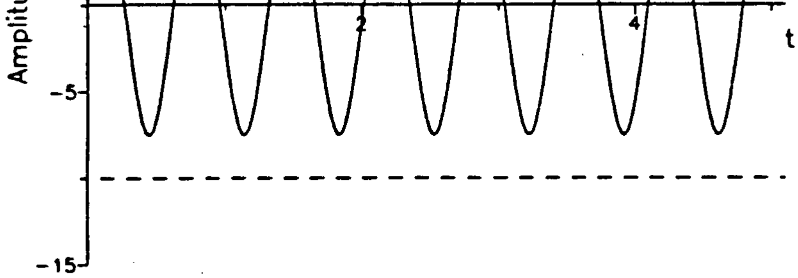


Figure D.6: $\mathcal{A}$=7, m=10.

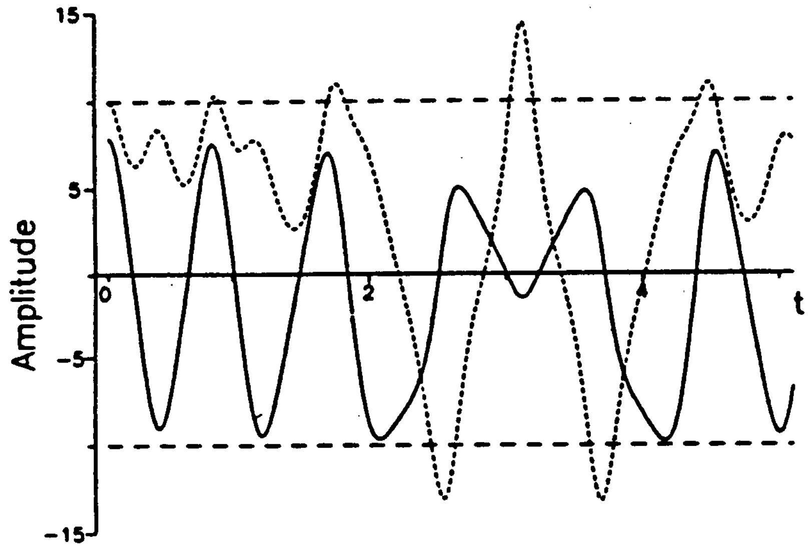


Figure D.7: $\mathcal{A}$=7.9, m=10. The non-linearity effects are becoming important.

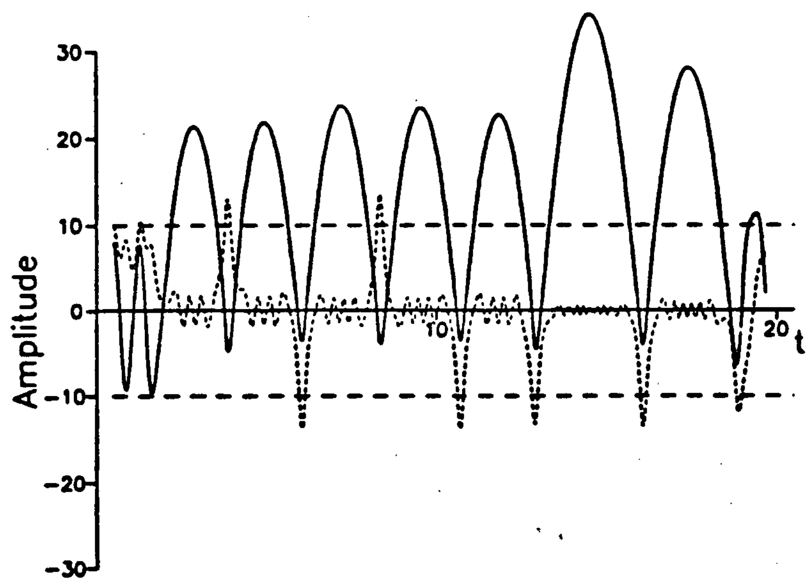


Figure D.8: $\mathcal{A}$=8.0, m=10. The charge appears to oscillate around a positive value. However, this behaviour doesn't persists.

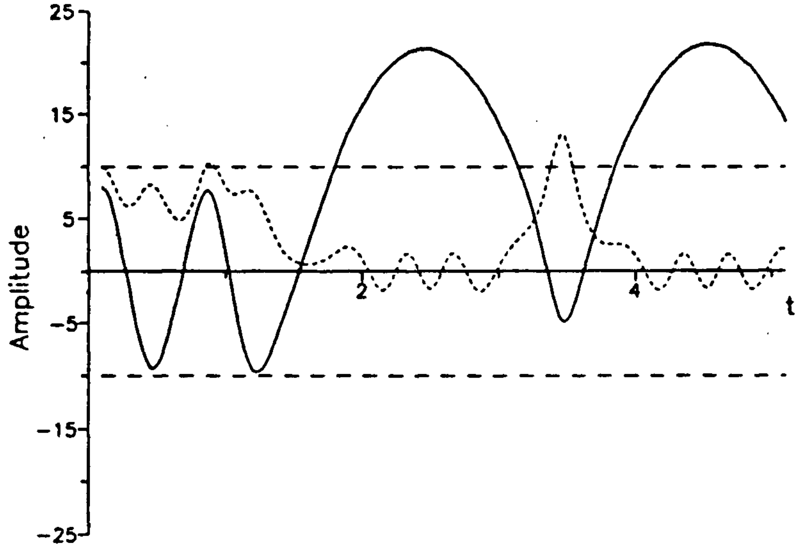


Figure D.9: $\mathcal{A}$=8.0, m=10. Compare this with the next figure. It shows that a small change in the initial conditions can produce large variations after only a short time. The horizontal scale is the same for both figures.

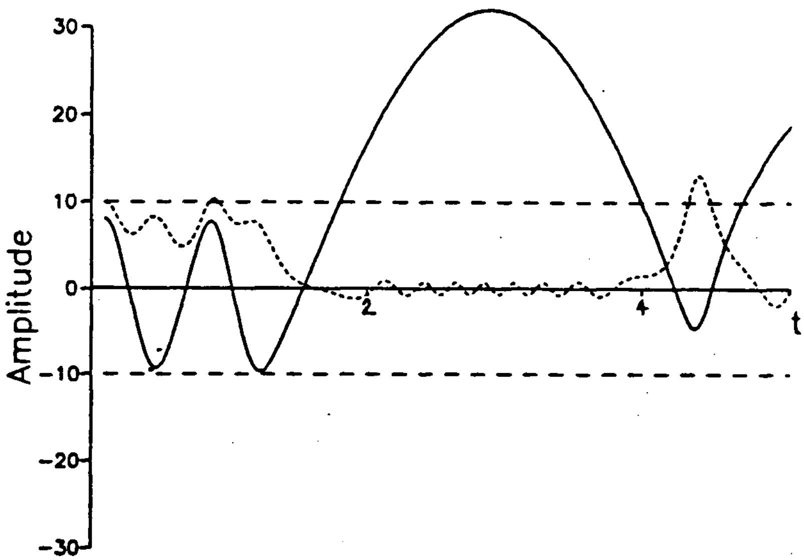


Figure D.10: $\mathcal{A}$=8.01, m=10.

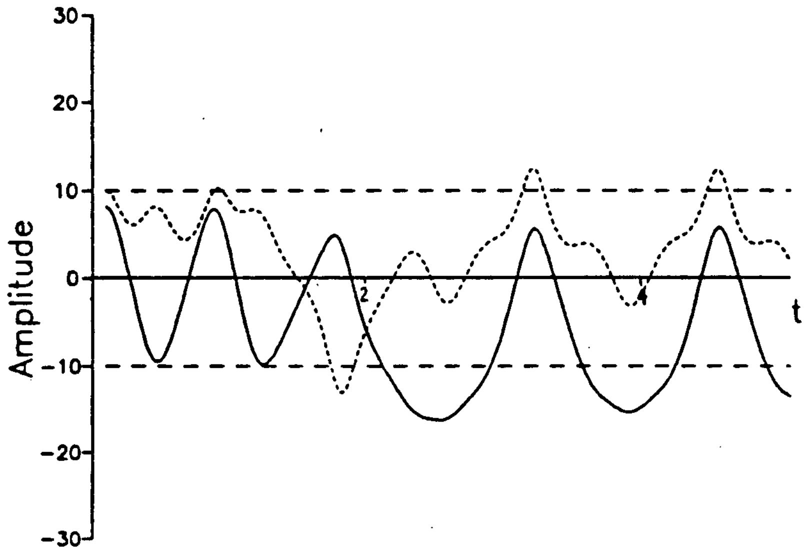


Figure D.11: $\mathcal{A}$=8.1, m=10. The initial behaviour is very similar to the previous two cases but any similarity is lost after only two full oscillations.

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