MEASUREMENT OF THE
$\pi^0$ ELECTROMAGNETIC TRANSITION FORM FACTOR

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Abstract

We present the result of a measurement of the \( \pi^0 \) electromagnetic transition form factor in the time-like region of momentum transfer. From a data sample of roughly 100,000 \( \pi^0 \rightarrow e^+e^-\gamma \) decays, observed in the SINDRUM I magnetic spectrometer at the Paul Scherrer Institute (Switzerland), we measure a value of the form factor slope \( a = 0.02 \pm 0.01 \) (stat) \( \pm 0.02 \) (sys). This measurement is consistent with both the results of the recent measurement by CELLO (DESY) in the space-like region, and with the vector meson dominance prediction of \( a \approx 0.03 \).
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Chapter 1

Introduction

Beginning with the Greek philosophers, humanity has speculated about the existence of basic building blocks of matter. The idea that matter might not be infinitely divisible, put forth by Democritus and hotly debated around 600 BC, was put aside until the recognition, during the 1800's, that all material substances are composed of "atoms" of various different "elements". In the early days, it was thought that the atoms were indivisible, and that, therefore, there existed about 100 different kinds which formed the basis of all matter. The discovery by J.J. Thompson in the 1890's [1] that different atoms could be forced to emit identical, very light, negatively charged "electrons", pointed to an underlying unifying structure. It was also obvious that the bulk of the mass of the atoms was therefore associated with positive charges. In 1911, Rutherford's famous scattering experiment [2] showed that the positive charge was confined to a very small region (radius $10^{-11}$ cm or less) of the atom, thus giving rise to the picture of the atom as a positive nucleus surrounded by negative electrons. This discovery was the starting point for the explosive development of atomic physics that culminated with the establishment of quantum mechanics in the late 1920's.

During the 1930's, the first accelerators were invented and built, leading to the discovery of a whole zoo of particles over the next 20 years. It was soon observed that one could group these particles in families exhibiting certain common properties. In 1964, Gell-Mann and Zweig [3,4] pointed out that the observed patterns could be understood if the particles were made up of smaller constituents. Three "quarks",

1
called "up", "down", and "strange", were enough to explain the observations. As experimentalists built larger and larger accelerators, probing the structure of matter at higher and higher energies, more quarks were added to the list.

As it now stands, the so-called Standard Model consists of three families of two quarks each; in order of increasing mass:

\[
\begin{pmatrix}
u \\ d \\ c \\ s \\ t \\ b
\end{pmatrix}
\]

With each family is associated a pair of "leptons", of which the electron is the most familiar:

\[
\begin{pmatrix}
e \\ \nu_e \\ \mu \\ \nu_\mu \\ \tau \\ \nu_\tau
\end{pmatrix}
\]

These quarks and leptons are, as far as we can tell, pointlike. Quarks group together to form "hadrons": three quarks make particles called "baryons", while quark-antiquark pairs can be bound to form "mesons". The light leptons stay single.

The three forces (on this tiny particle scale, gravity is negligible) which operate on the nuclear scale are understood in terms of the exchange of other particles

<table>
<thead>
<tr>
<th>force</th>
<th>particle exchanged</th>
</tr>
</thead>
<tbody>
<tr>
<td>electromagnetism</td>
<td>photon</td>
</tr>
<tr>
<td>weak</td>
<td>$W^\pm, Z^0$</td>
</tr>
<tr>
<td>strong</td>
<td>gluons</td>
</tr>
</tbody>
</table>

The rules which govern the exchange of these particles are embodied in the so-called "Standard Model" of particle physics.

The Standard Model comes in two parts. One explains the behaviour of the electromagnetic (which, in its marriage to quantum mechanics is called Quantum Electrodynamics, or QED) and weak forces and is hence called the Electroweak Model. The
other part predicts the effects of the strong force and is called, analogous to QED (and since gluons carry a quality dubbed "colour" by physicists), Quantum Chromodynamics (QCD). The Electroweak theory has proven fantastically successful in predicting, to very high accuracy, the results of all kinds of particle decays and collisions. QCD, which governs how quarks stick together in particles, is a much more difficult theory to calculate and hence is not so numerically precise in its predictions, but in spite of this physicists feel that they are on the right track.

The Standard Model, despite its successes, leaves many questions unanswered. It does not explain the observed family structure of the quarks and leptons. It does not explain their masses. It does not explain why we don't see quarks in groups of 4 or more. And it does not include gravity. While theoretical physicists labour to invent models which simultaneously answer these questions and unify all the forces into one mathematical construct, experimental physicists work hard to find holes in the existing Standard Model – discrepancies between the theoretical predictions and reality. Such holes may illuminate its shortcomings and provide clues towards the creation of a better theory. It is important, then, to attack the problem of particle structure. The postulated quark structure of the hadrons and mesons should have testable consequences.

1.1 Hadron Structure–Form Factors

In the classical method of studying particle structure, one bombards an object with electrons, à la Rutherford. For example, in order to probe the charge distribution of a hadron like a proton, one measures the angular distribution of the scattered electrons and compares it to the distribution obtained by scattering off of a point charge:

\[
\frac{d\sigma}{d\Omega} = \frac{d\sigma}{d\Omega_{\text{point}}} \times |F(q^2)|^2
\]  

(1.1)
Using quantum electrodynamics, the scattering from a point target, \( \frac{d\sigma}{dq_{\text{point}}} \), can be calculated to a high degree of accuracy. The function \( F \) is called the target's (in this case, the proton's) form factor, and describes its deviation from pointlikeness and hence gives an idea of the structure of the target. The form factor depends on the momentum transfer \( q^2 \). If \( q^2 \) is small then we find that \( F(q^2) \approx 1 \); in other words, the electron doesn't have enough energy to resolve the inner structure of the target and the target looks pointlike.

As a simple illustration, if we model the proton as a motionless (static and non-relativistic), spinless blob with a spherically symmetric charge distribution

\[
\rho(r) = e^{-mr}
\]

with \( m \) some constant (to be determined by the experiment), then the form factor is the fourier transform of the charge distribution:

\[
F(|q^2|) = \int r^2 \rho(r) e^{-iq \cdot r} d^3r
\]

\[
= \left( 1 - \frac{q^2}{m^2} \right)^{-2}
\]

where 1.4 is the so-called dipole function. For small \( q^2 \), equation 1.3 has the expansion

\[
F(q^2) = 1 - \frac{1}{6} |q^2| \langle r^2 \rangle + \text{order}(q^4)
\]

where

\[
\langle r^2 \rangle = \int r^2 \rho(r) dr
\]

Now, defining the form factor slope \( b \) by

\[
b \equiv \frac{dF}{dq^2} \big|_{q^2=0}
\]

we see that, in this simple case,

\[
b = -\frac{1}{6} \langle r^2 \rangle
\]
so that extraction of $b$, by measuring the form factor at different $q^2$ values, and fitting it with the dipole function and extrapolating back to $q^2 = 0$, allows the determination of the spread of the charge distribution of our simple "proton".

Real particles, however, are neither static nor (in general) spinless, so that the analysis is more complicated than that outlined here. The scattering electron may interact not only with the particle's charge, but also with its magnetic moment (due to its spin), and thus, in general, we may expect two form factors to come into play. Obviously, the relationship between the shape of the hypothesized charge distribution and the form factor slope is now not as simple as that given in equation 1.8 above; however, in a special reference frame (the "brick wall" or Breit frame) in which no energy is transferred to the target (and the magnitude of the projectile momentum is unchanged), a connection be derived between the magnetic moment distribution and the magnetic form factor, and between the electric charge distribution and the electric form factor. In scattering electrons from protons it is found that, in this frame, both form factors of the real proton are proportional to the dipole function, with

$$F(q^2) = \left(1 - \frac{q^2}{0.71 GeV^2}\right)^{-2}$$

(1.9)

Hence both distributions are approximately exponential. Determination of the form factor slopes allows determination of the "size" of the proton; it is found that both the magnetic and electric charge distributions have $\langle r^2 \rangle \approx (0.8 \times 10^{-13} \text{ cm})^2$.

In summary, then, we expect one form factor for spin 0 particles, and two form factors if the particle has spin $\frac{1}{2}$. The form factors can be related in some way to the particle's charge and magnetic moment distributions. In the case of the proton, it turns out that a reasonable assumption seems to be that these distributions are exponential.
1.2 Pion Form Factor–Particle Exchange

The charged pion is a spinless particle. We might expect, therefore, to be able to describe its electromagnetic structure by a single form factor. The assumption that works well for the proton; namely, that the charge distribution is exponential; does not work for the case of the \(\pi^\pm\). In fact, the experimental data fits a form factor of the form

\[
F(q^2) = \left(1 - \frac{q^2}{m^2}\right)^{-1}
\]

with \(m^2 \approx 0.56 \text{ GeV}^2\). The form factor above corresponds to a charge distribution of the form

\[
\rho(r) = \frac{1}{r} e^{-mr}
\]

the “Yukawa potential”, as opposed to the purely exponential case as for the proton.

One can also think of the form factor, instead of arising from some extended electromagnetic distribution, as arising from the exchange of particles. This picture is an extension of the quantum picture of photon exchange mediating the electromagnetic interaction between two charges. In this generalization, we allow the photon to assume some mass (which it may, as long as it is for a short enough time that the Heisenberg uncertainty principle is not violated). This shortens its range by an exponential factor \(e^{-mr}\), and we recover exactly the formulae above. We note that the mass \(m\) in equation 1.10 above is roughly \(m \approx 770 \text{ MeV}\), which corresponds to the mass of the \(\rho\) meson. Since the \(\rho\) meson has the same quantum numbers as the photon, it can be thought of as simply a heavy photon. The extended electromagnetic structure of the \(\pi^\pm\) can then be described by allowing the pion to emit a \(\rho\), which then scatters the incoming electron. In this picture, then, the pion is thought of as not as a single, rice-crispy-like object, but as some fuzzy cloud of short-lived \(\rho\) mesons. We depict this graphically in figure 1.1. Measurement of the form factor provides information about this fuzzy
cloud: how big it is, if any other mesons other than ρ's are present, etc.

\[ e^- e^- \rightarrow \pi \pi \gamma^* \rightarrow \pi \pi e^- e^- \]

\[ F(q^2) \]

Figure 1.1: Feynman diagram for electron-pion scattering. The blob represents the extended electromagnetic structure of the pion; its form factor. The $\gamma^*$ represents the heavy virtual photon which scatters the incoming electron. The process can be thought of as the sum of two processes shown; the blob contains ρ mesons.

In practice, it is not possible to use the reaction shown in figure 1.1 to study the form factor, since it is not possible to make a pion target. Instead, experimentalists make use of pion scattering from atomic electrons to produce the reaction shown in figure 1.2. This reaction is very closely related to the one shown in figure 1.1, differing only in the range of momentum transfer examined (in practise, pion scattering is limited to $q^2 < 0.2 \text{ GeV}^2$).

\[ \pi \rightarrow \pi e^- e^- \]

\[ F(q^*) \]

Figure 1.2: Feynman diagram for pion scattering. The blob represents the effect of the pion form factor.
Another reaction which may be used to study the charged pion form factor is the so-called "charge exchange" reaction and its relatives, shown in figure 1.3. The advantage of this reaction is that it can be used in many directions, as illustrated in figure 1.3: given that one can make $\pi^+$, $\pi^-$, and electron beams, as well as both proton and neutron targets, a myriad of experimental options are possible. Different ranges of $q^2$ can be probed, and the results of the different experiments can be combined to form a more complete picture of the pion form factor.

![Feynman diagrams for the charge exchange reaction.](image)

Figure 1.3: Feynman diagrams for the charge exchange reaction. a) through d) show the different reactions available to study the charged pion form factor. In reactions a) and b) the photon momentum $q^2$ is positive; in c) and d) it is negative.

As can be imagined, however, this reaction, since it involves neutrons and protons which are themselves extended objects even less pointlike than the pion, provides a more indirect and complicated way of extracting information on the pion form factor. In fact, the form factor one is studying is not the same one as may be measured in pion production or electron-pion scattering experiments. In the charge exchange case, the
Figure 1.4: In a) we show the Feynman diagram for the transition $A \rightarrow B\gamma^*$, followed by the decay of the virtual photon $\gamma^* \rightarrow e^+e^-$. In b) the same transition is depicted, where the particle $B$ is specifically a photon. Figure c) shows the production of a neutral meson through two virtual photons, one of which is nearly real (its energy is very close to its momentum). The range of momentum transfer of the virtual photon used to probe the meson structure is indicated.
form factor involved describes the transition of the incoming pion and the target particle A (proton or neutron) into a photon and the product particle B (neutron or proton). The form factor in question is thus known as a "transition" form factor (as opposed to the previous "static" form factor). It describes the electromagnetic properties of the cloud of virtual particles during the reaction $\pi + A \rightarrow B$.

1.3 Form Factors of Neutral Mesons

In the particle exchange picture, processes involving a single photon intermediate state such as those pictured in figures 1.1 and 1.2 are forbidden in the case of mesons with zero spin and charge like the $\pi^0$, both by charge conjugation invariance, and by conservation of angular momentum; the photon has a spin of 1, while the $\pi^0$ has spin 0. One cannot conserve angular momentum in allowing a $\pi^0$ to emit a single photon; hence these particles cannot couple to single photons and their static form factors are identically zero.

However, particle transitions or decays of the kind pictured in figure 1.4 (a and b) are allowed, since the parent meson and its decay product can have different spins and charge conjugation properties. One can therefore study the electromagnetic properties of neutral mesons by studying their transition form factors. Note, however, that it is not meaningful to relate the transition form factor to the "size" of a charge distribution, as is the case with the static form factor. The phenomena of particle decay and transition are quantum mechanical in nature, which the semi-classical picture of an electron scattering from a charge distribution has no way of describing. The transition form factor is an abstraction, albeit a useful one; given some detailed model of the internal structure of the meson, it is possible to predict the behaviour of the form factor, and hence to compare with experiment.
In the special case where the decay product is a photon (figure 1.4b), one is exploiting the fact that while the neutral meson cannot couple to a single photon, coupling to two photons is allowed. It is possible to construct another allowed situation, involving the production of the neutral meson by two photons, as shown in figure 1.4c. This type of production experiment requires an $e^+e^-$ collider. The short-lived meson is identified by its subsequent decay products. The two types of reactions, decay and production, complement each other since they probe the structure of the meson in different regions of momentum transfer.

1.4 Dalitz Decay of the $\pi^0$

The lowest-energy decay of the type pictured in figure 1.4b is the "Dalitz decay" of the $\pi^0$. This decay has historically been the method by which to study $\pi^0$ structure, since neutral pions are easily produced at medium-energy cyclotrons with $\pi^\pm$ beams.

In order to extract the form factor information, we will need the decay width appropriate for a pointlike $\pi^0$. This was first calculated by Dalitz [5] in 1951 and also by Kroll and Wada [6] in 1955. They isolated the pointlike part of the interaction by normalizing to the decay $\pi^0 \rightarrow \gamma\gamma$, obtaining, to lowest order in $\alpha$,

$$\frac{\Gamma(\pi^0 \rightarrow e^+e^-\gamma)}{\Gamma(\pi^0 \rightarrow \gamma\gamma)} = \frac{\alpha}{4\pi} \int_x^1 \int_{-\eta}^\eta \frac{(1-x)^3}{x} \left(1 + y^2 + \frac{r}{x}\right) |F(x)|^2 dx dy \quad (1.12)$$

$$= \frac{\alpha}{3\pi} \int_x^1 \frac{(1-x)^3}{x} \left(1 - \frac{r}{x}\right)^{1/2} \left(2 + \frac{r}{x}\right) |F(x)|^2 dx \quad (1.13)$$

$$= 0.0118 \quad (1.14)$$

where the final answer is obtained using $F(x) = 1$. Here $x$ is the invariant mass of the $e^+e^-$ pair, normalized to the $\pi^0$ mass and $y$ is the energy partition:

$$x = \frac{(p_- + p_+)^2}{m_{\pi^0}^2} \quad (1.15)$$
\[ = \frac{q^2}{m^2_{\pi^0}} \]
\[ \approx \frac{2|\vec{p}_+| |\vec{p}_-|(1 - \cos \varphi)}{m^2_{\pi^0}} \tag{1.16} \]

where \( \varphi \) is the opening angle of the \( e^+e^- \) pair, \( p \) and \( q \) are 4-momenta, \( \vec{p} \) is the 3-momentum. We also define the “energy partition” and the minimum \( x \) value as follows:

\[ y = \frac{E_- - E_+}{|\vec{p}_- + \vec{p}_+|} \tag{1.17} \]

and

\[ r = \frac{4m^2_e}{m^2_{\pi^0}}, \quad \eta = \sqrt{1 - \frac{r}{x}} \tag{1.18} \]

In figures 1.5 and 1.6 we show the resulting distributions in \( x \) and \( x \) versus \( y \). Equation 1.12 gives the rate of \( \pi^0 \to e^+e^-\gamma \) normalized to the decay \( \pi^0 \to \gamma\gamma \); the form factor \( F \) is normalized such that \( F(0) = 1 \). As in the case of the \( \pi^+ \) form factor, one posits a form factor \( F \) of the form

\[ F(q^2) = \left( 1 - \frac{q^2}{m^2} \right)^{-1} \]

or, writing \( a = \frac{m^2_{\pi^0}}{m^2} \) and in casting the above in terms of \( x \),

\[ F(x) = (1 - ax)^{-1} \tag{1.19} \]

For small \( x \) (low momentum transfer), we may expand

\[ F(x) \approx 1 + ax + \text{order}(x^2) \tag{1.20} \]

so that, by definition 1.7, the form factor slope is given by \( a \). We now turn to more detailed and complicated models of the \( \pi^0 \) structure to obtain theoretical predictions for \( a \).
Figure 1.5: The Kroll-Wada distribution. The surface plot shows the distribution in \( x \) and \( y \).
Figure 1.6: The projection of the Kroll-Wada distribution in \( x \) (note the logarithmic scale – the effect of \( a \) on a linear scale is invisible if the acceptance is uniform over all \( x \)). Here, the solid line shows the distribution with \( a = 0.0 \); the dashed line, the effect of \( a = 0.1 \).
Chapter 2

Historical Background

2.1 Theoretical Predictions

While the electromagnetic interactions between two pointlike particles can be described to very high accuracy by quantum electrodynamics, any calculation involving hadrons is only approximate. It seems that hadrons are complex objects, and their inner structure is not well-understood; models describing the inner structure of hadrons and how this manifests itself in their electromagnetic behaviour currently give only single-digit accuracy. These models take as their starting point the idea that the hadron consists of a cloud of "virtual" pointlike particles winking in and out of existence in accordance with the uncertainty principle, and that it is these virtual particles which interact with the electromagnetic field. In this way, the models can account for the deviation from point-like behaviour and give a prediction for the electromagnetic form factor of the hadron.

2.1.1 Vector Meson Dominance Model (VMD)

In the vector meson dominance model, the coupling of hadrons to the electromagnetic field's photons is supposed to proceed via intermediate vector meson states with the same quantum numbers as the photon. The model is highly phenomenological and as such relies on experimental input for the numerical values of the couplings of the various vector mesons to both the photon and hadron(s) being considered. The predictive power
of the VMD is thus limited by the accuracy of the experimental input.

![Diagram](image)

Figure 2.1: Diagram for $\pi^0 \rightarrow e^+e^-\gamma$ in the VMD. The blobs represent the 2- and 3-particle couplings of the vector mesons to the initial $\pi$ and final state photons.

To calculate the pion form factor in this model, the pion "cloud" is written as a sum over intermediate vector meson states. Using figure 2.1 one obtains [7] an expression for $F$ in terms of the matrix elements describing the transition from initial pion to final photon via the intermediate vector mesons $V$:

$$\Im F(q^2) \approx \sum_{V} \langle 0|j_{\alpha}|V\rangle \langle V|j_{\beta}|\pi^0\rangle$$

(2.1)

where $j$ is the hadronic electromagnetic current. Since we are concerned with low $q^2$, the standard approach is to concentrate on the lowest-mass intermediate states; the $\rho, \omega$ and $\phi$ mesons. The matrix elements in equation 2.1 above then reduce to the 2- and 3-particle couplings $f_{\rho\gamma}, f_{\omega\gamma}, f_{\phi\gamma}, f_{\rho\pi\gamma}, f_{\omega\pi\gamma}$ and $f_{\phi\pi\gamma}$, quantities whose magnitudes may be measured experimentally in other reactions. One thus obtains an expression for the form factor

$$F(q^2) = f_{\pi\pi\gamma} - q^2 \left( \frac{f_{\rho\gamma} f_{\rho\pi\gamma}}{m_{\rho}^2 - q^2} + \frac{f_{\omega\gamma} f_{\omega\pi\gamma}}{m_{\omega}^2 - q^2} + \frac{f_{\phi\gamma} f_{\phi\pi\gamma}}{m_{\phi}^2 - q^2} \right)$$

to be compared with the expansion for $F$.
The physical $\omega$ and $\phi$ mesons are mixtures of the $I = S = 0$ singlet (denoted here by $\omega_0$) and octet (here $\phi_0$) states:

$$|\omega\rangle = A|\omega_0\rangle - B|\phi_0\rangle$$

$$|\phi\rangle = B|\omega_0\rangle + A|\phi_0\rangle$$

with $A^2 + B^2 = 1$, so that one may identify the slope parameter $a$ as follows:

$$a = \frac{A f_{\rho \pi}}{m^2_{\pi}} \left[ \frac{1}{m^2_{\rho}} + \frac{B^2}{m^2_{\omega}} + \frac{A^2}{m^2_{\phi}} - \frac{AB f_{\omega_0 \pi} f_{\rho \pi}}{\sqrt{3} f_{\rho \pi}} \left( \frac{1}{m^2_{\omega}} - \frac{1}{m^2_{\phi}} \right) \right]$$

The magnitude of the factor $f_{\rho \pi} f_{\rho \pi}/f_{\pi \pi}$ has been measured to be of order 1, so that if one makes the further simplification that only the $\rho$ contributes, one may obtain an estimate for the slope parameter

$$a \approx \frac{m^2_{\pi}}{m^2_{\rho}} \approx +0.034$$

In fact, taking the full expression and using accepted and "reasonable" values for all parameters (an excellent discussion is given in [7]), the vector meson dominance model predicts that

$$|a| \leq 0.05 \quad (2.2)$$

The sign of $a$ depends on the overall sign of the couplings in the factor $f_{\rho \pi} f_{\rho \pi}/f_{\pi \pi}$. This cannot be experimentally obtained, but may be calculated by evaluating matrix elements of the form

$$f_{\rho \pi} \approx \langle \rho | j_\nu | \pi \rangle \approx \sum_n \langle \rho | j_\nu | n \rangle \langle n | J_\pi | 0 \rangle$$

i.e. they involve a sum over intermediate states. A simple approximation is to assume that only baryon-antibaryon states contribute to the sum, $|n\rangle = |N\rangle$. Since this leads
to roughly the correct experimentally measured magnitudes of the individual couplings $f_{\rho\pi\gamma}$ and $f_{\pi\gamma\gamma}$ theorists are fairly confident that the approximation is a valid one. Using the resulting couplings leads to the prediction that $a$ is positive. If, however, higher order states were to contribute significantly, then one could only calculate the couplings to within an order of magnitude and their signs would be completely unobtainable; hence it would not be possible to predict the sign of $a$. Even in this case, however, the prediction for the magnitude of $a$ stands as in equation 2.2.

2.1.2 The Quark Loop Model

In the quark loop model, the properties of hadrons are seen as the result of the point interactions between the constituent quarks (of which the model has six, each of which come in three different “colours”) and the photons of the electromagnetic field. The model led to the first theoretical understanding of the lifetime of the $\pi^0$, in $\pi^0 \rightarrow \gamma\gamma$, as set down by Adler [8] in 1969. In this framework, the $\pi^0$ is seen as a cloud of quarks, and in order to obtain the experimentally measured value of the lifetime, one must include the effect not only of the different quark species, but also of colour. The vindication of this seemingly unnecessary extra parameter was a major triumph of the quark model.

In order to examine the other decay modes of the $\pi^0$, one extends Adler’s analysis to the case where one or both of the photons are virtual, and hence massive. These virtual photons can then decay electromagnetically into, for example, an $e^+e^-$ pair. The decay of the $\pi^0$ into two virtual photons $\pi^0 \rightarrow \gamma^*\gamma^*$ is modelled according to the Feynman diagram of figure 2.2. The amplitude for the decay is [9]

$$A(s_0, s_1, s_2; M) = i e^2 (e_q^2) f_{\gamma\gamma} \epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\nu \epsilon_1^\rho \epsilon_2^\sigma F(s_1, s_2; M)$$

$$= 8ge^2 (e_q^2) M \epsilon_{\mu\nu\rho\sigma} k_1^\mu k_2^\nu \epsilon_1^\rho \epsilon_2^\sigma \times C$$
Figure 2.2: Feynman diagram for $\pi^0 \rightarrow e^+ e^- \gamma$ in the quark loop model. One sums over all possible quark species and colours.

with quark mass $M; k_1, k_2$ the momenta of the photons and $\epsilon_1, \epsilon_2$ their polarizations.

The average charge squared of the constituent quarks, including the colour factor, is given by $(e^2)$. The form factor $F$ is normalized such that $F(0,0;M) = 1$. For the purposes of estimation, one assumes that the only quarks that contribute are the $u, d$ pair, and one sets $M_u = M_d = M$.  

Equation 2.2 relates the form factor $F$ and coupling constant $f_{\gamma \gamma}$ to an integral $C$ involving the internal quark mass $M$ and the external masses $s_0 = (k_1 + k_2)^2, s_1 = k_1^2,$ and $s_2 = k_2^2$, where $C$ is given by

$$C = \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 - M^2 + i\epsilon][(q + k_1)^2 - M^2 + i\epsilon][(q + k_1 + k_2)^2 - M^2 + i\epsilon]}$$  \hspace{1cm} (2.4)$$

Standard techniques allow for the evaluation of $C$ and one can study special cases of the general expression:

1Generalizing equation 2.5 to real mesons, which are superpositions of six quark loops with different flavors, one obtains, instead of 2.5,

$$F(s,0;M) = \left\{ \sum_q \frac{4M_q g_{\pi q} Q^2_q}{s} \arcsin^2 \left( \sqrt{\frac{5}{2M_q}} \right) / \sum_q \frac{g_{\pi q} Q^2_q}{M_q} \right\}$$

with $g_{\pi q}$ the pion-quark coupling; a formula which does not lead to a significantly different result.
1. \( s_0 = m^2_{\pi^0}, s_1 = s_2 = 0 \). This corresponds to the decay of the \( \pi^0 \) into two real photons, and using expressions 2.4 and 2.4 one may obtain
\[
f_{\gamma\gamma} = -\frac{gM}{\pi^2m^2_{\pi^0}} \text{arcsin}^2 \frac{m_{\pi^0}}{2M}
\]
In the limit \( m_{\pi^0} \rightarrow 0 \) this gives the formula for the "triangle anomaly", which was set down by Adler [8] in the first successful calculation of the \( \pi^0 \) lifetime.

2. \( s_0 = m^2_{\pi^0}, s_1 = s, s_2 = 0 \). When \( 4m^2_e < s < m^2_{\pi^0} \), this corresponds to the Dalitz decay \( \pi^0 \rightarrow e^+e^-\gamma \). In this case one obtains an expression for the form factor
\[
F(s, 0; M) = \frac{m_{\pi^0}^2}{m^2_{\pi^0} - s} \left\{ 1 - \frac{\text{arcsin}^2(\sqrt{s}/2M)}{\text{arcsin}^2(m_{\pi^0}/2M)} \right\} \tag{2.5}
\]
In the limit that \( m_{\pi^0} \rightarrow 0 \) this becomes
\[
F(s, 0; M) = \frac{4M}{s} \text{arcsin}^2 \frac{\sqrt{s}}{2M}
\]
Expanding about \( s = 0 \), this gives
\[
F \approx 1 + \frac{1}{12M^2} s + \ldots
\]
whence one can immediately identify the slope parameter
\[
a = \frac{m_{\pi^0}^2}{12M^2}
\]
Note that the quark loop model predicts that \( a > 0 \). Further, if one chooses the quark masses \( M_u,d \approx 200 - 300 \text{ MeV} \), it is possible to match numerically the prediction of VMD (equation 2.2). The two models are thus equivalent in some sense. This matching is known as "\( Q^2 \) duality".

The results of the quark-loop calculations are only weakly dependent on the initial assumptions of \( m_{\pi^0} \rightarrow 0 \), and are applicable to other (heavier) meson form factors as well. Again, using the posited functional form of \( F \)
\[
F(q^2) = (1 - aq^2)^{-1}
\]
the VMD may be used to predict the form factors of the $\eta, \eta'$ and $\omega$; the theoretical results are shown in table 2.1, together with the existing experimental data. For heavier mesons, the form factor slope $a$ is conventionally expressed in dimensions of GeV$^{-2}$. We may convert to the more familiar dimensionless quantity by scaling the momentum transfer by the maximum allowed value $x = q^2 / q_{max}^2$, which, for the $\pi^0 \rightarrow e^+ e^- \gamma$ case, is $m_{*o}^2$. We see that the VMD/quark loop predictions are consistent with the experimental results.

Recently, a form factor has been measured in the decay $K_L \rightarrow e^+ e^- \gamma$ [10]; in this case the form factor is slightly more complex, involving contributions not only from $\rho$, $\omega$, and $\phi$ mesons, but also from $K_L \rightarrow K^* \gamma$ followed by $K^* \rightarrow \rho$, $\omega$, $\phi$ transitions. This leads to a quark-loop inspired prediction of a form factor of the form [11]

$$F(q^2) = \Gamma_{K_L \rightarrow \gamma\gamma} \left(1 - \frac{q^2}{m^2}\right)^{-1} + \alpha \cdot \Gamma_{K_L \rightarrow K^*\gamma}(q^2)$$

where the model predicts $m \approx m_{*o}$ as in the $\pi^0 \rightarrow e^+ e^- \gamma$ case, and $|\alpha| \approx 0.2 - 0.3$; the $\Gamma$ factors are experimentally measured decay rates, included for normalization. Reference [10] obtains values of $m = 610^{+55}_{-45}$ MeV (corresponding to $a = 2.7 \pm 0.4$ GeV$^{-2}$) and $\alpha = -0.28 \pm 0.13$.

2.2 Previous Measurements

We discuss the techniques used to measure the $\pi^0$ form factor in the region of timelike momentum transfer, using the Dalitz decay. A brief discussion of the single spacelike measurement can be found in the description of the previous experiments (experiment number 8).

There are two different approaches to measuring the form factor; each leads to a different experimental design.
Chapter 2. Historical Background

<table>
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<th>$a_{\text{expt}}$ (GeV$^{-2}$)</th>
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<td>$\omega$</td>
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<tr>
<td>$\pi^0$</td>
<td>1.7</td>
<td>see table 2.2</td>
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Table 2.1: Form factor slopes in VMD for several neutral mesons, together with the experimental results. The theoretical value for the $\pi^0$ form factor is included for comparison. See also figure 2.3.

1. One approach is to collect data over the entire range of invariant mass $x$ and to fit it, using equation 1.13 and the standard expansion of $F$ to extract a value for $a$. In order to see the small effect of $a$, one needs high statistics over all $x$ values, especially at the high end of the spectrum where the effect is the largest. However, because of the logarithmic scale of the invariant mass spectrum (figure 1.5), a huge preponderance of low $x$ (useless for the measurement of $a$) events will swamp the detector, unless some method of biasing the detector for larger invariant mass is implemented. Note that equation 1.13 assumes a normalization to the process $\pi^0 \rightarrow \gamma\gamma$. If this process is not simultaneously monitored and counted (effectively counting the number of neutral pions created), then one must introduce a normalization factor into the fit.

In order to see the full range of invariant mass (which means a full range of 3-momentum and opening angle $\varphi$, according to equation 1.15) a magnetic spectrometer is usually employed. This device comprises some sort of detector capable of position measurement surrounding, over as wide an angular range as possible,
the $\pi^0$ source. A magnetic field bends the electrons as they traverse the detector, thus allowing for momentum determination. In order to cut out the low-$x$ pairs, some sort of minimum opening angle requirement is usually made. If a normalization is to be measured, some way of counting either the $\pi^0$ production, or the incoming $\pi^-$ flux must be installed.

2. An alternative strategy is to perform an "integrated measurement" by choosing a specific value (or a small range of values) of the invariant mass and to measure the rate for $\pi^0 \rightarrow e^+e^-\gamma$ in this range, compared to the rate of $\pi^0 \rightarrow \gamma\gamma$. For such a measurement, one must have a way of counting the number of $\pi^0 \rightarrow \gamma\gamma$ events (or equivalently, the number of neutral pions produced). The advantage of this strategy is that fewer events are needed since one is concentrating on a specific range of $x$; however, background identification is usually difficult since the kinematical information available is limited to such a small range.

Two-arm experiments are suited for this type of measurement, the typical arrangement being two energy-sensitive devices (NaI crystals, for example) at some distance from and defining some angle about a target. There is no magnetic field; knowledge of the energy and the angle is enough to specify the invariant mass of the $e^+e^-$ pair (equation 1.15). A beam counter (counts the incoming $\pi^-$) or a NaI crystal (counts the decay $\pi^0 \rightarrow \gamma\gamma$) provides a normalization.

We now present a short description of the previous experiments performed to measure $\alpha$, together with a brief review of their salient features. Their final results are tabulated in table 2.2 and presented graphically in figure 2.3.

1. *N. Samios et al., 1961* [16]: an early magnetic spectrometer experiment done at the Nevis cyclotron at Columbia University (New York), using a 60-MeV $\pi^-$ beam
which was slowed down by polyethylene absorbers to stop in a liquid hydrogen bubble chamber, surrounded by a magnetic field of first 5.5, then 8.8 kG. The magnetic field was uniform to 4% and controlled to ±2%. Electron track momenta were measured from photographic plates. Using a sample of 3071 Dalitz decays, the authors extracted a value for the form factor slope by comparing the ratio of the number of events seen in $r < x < 0.1$ to the number seen in $0.1 < x < 1.0$ with the ratio obtained using the calculations of Kroll and Wada and Joseph. They obtained $a = -0.24 \pm 0.16$. Radiative corrections to the process $\pi^0 \rightarrow e^+e^-\gamma$ were not included in the analysis, nor was possible background from the decay $\pi^0 \rightarrow e^+e^-e^+e^-$. 

2. H. Kobrak et al., 1961 [17] : a similar experiment to the one above; a 68 MeV pion beam was brought to rest in a liquid hydrogen bubble chamber, surrounded by a 24.7 kG field, uniform to 0.5%. The energy resolution was approximately 2.4%, roughly a factor of three better than in the experiment of Samios et al.. A form factor measurement was done by fitting the data in $x$ with Joseph's theory, using a sample of 7676 Dalitz decays, resulting in $a = -0.15 \pm 0.10$. Background from $\pi^0 \rightarrow e^+e^-e^+e^-$ was not included, nor were radiative corrections.

3. S. Devons et al., 1969 [18] : a two arm experiment performed using a 150 MeV pion beam at the Nevis cyclotron. Each arm consisted of a pair of spark chambers mounted in front of a sodium iodide crystal, which functioned as total energy absorption spectrometers. The two arms, defining an opening angle of 120°, surrounded a liquid hydrogen target. A water Čerenkov detector rejected beam electrons, and a scintillation counter behind it counted the incoming $\pi^-$ beam. Facing the two sodium iodide arms was a lead plate gamma conversion spark chamber, to check for correlations between electron events and photon events.
After geometrical cuts, a sample of 2200 Dalitz decays was obtained (997 of which had converted photons associated with them). The energy and angle information was converted to invariant mass and a value of the form factor slope was extracted in two ways: firstly, by fitting the invariant mass spectrum with Joseph's theory a value of \( a = -0.10 \pm 0.09 \pm 0.13 \) (the first error is statistical, the second, systematic) was obtained; and secondly, by measuring the total rate of the Dalitz decay a value of \( a = 0.11 \pm 0.07 \pm 0.12 \) was measured. The authors combined these two results and obtained a final result \( a = 0.01 \pm 0.11 \).

4. J. Burger, R. Garland et al., 1972 [19]: yet another experiment performed at the Nevis cyclotron, using a magnetic spectrometer consisting of 3 sets of acoustic spark chambers surrounding two sides and the bottom of a liquid hydrogen target to measure the electron momentum. The target and chambers were immersed in a uniform 3 kG magnetic field, calibrated to 0.1%. The authors quoted a momentum resolution of 2.7% at 70 MeV/c. A sodium iodide crystal monitored \( \pi^0 \rightarrow \gamma\gamma \) events, providing a normalization. The authors extracted a form factor \( a = 0.02 \pm 0.10 \) by a fit of 2437 events over the range \( 0.0 < x < 0.8 \). Radiative corrections, calculated according to Lautrup and Smith [20], were included in the analysis.

5. J. Fischer et al., 1978 [21], [23]: instead of using the reaction \( \pi^-p \rightarrow \pi^0n \), this experiment employed the decay \( K^+ \rightarrow \pi^+\pi^0 \) as the \( \pi^0 \) source. The CERN PS provided a 2.8 GeV kaon beam. The kaons decayed in flight in the apparatus, which consisted of a 4 m long decay region fitted with position sensitive proportional chambers in the middle and at both ends. The decay products were bent magnetically through a set of Čerenkov counters, scintillators, and spark chambers, enabling momentum determination. The magnetic field was calibrated to
center the $\pi^0$ mass to 0.3 MeV/c$^2$. A total of 31,458 $\pi^0 \to e^+e^-\gamma$ events were collected, and the full $x$ spectrum fit to obtain $a = 0.10 \pm 0.03$. The authors included the radiative corrections as calculated by Mikaelian and Smith [22]; their effect is to increase the value of $a$ by 0.05. No systematic error analysis was performed. The authors do not take into account any possible contamination due to $\pi^0 \to e^+e^-e^+e^-$.

6. P. Gumplinger et al., 1987 [24]: a two arm experiment performed at the TRIUMF cyclotron (Vancouver, Canada), with two large sodium iodide crystals preceded by sets of three wire chambers and scintillators defining an opening angle of 60°, then 130°, and finally 156° about a liquid hydrogen target. A 90 MeV/c $\pi^-$ beam was degraded to slow down in the target, and resulting $e^+e^-$ pairs were stopped in the NaI crystals. The wire chamber and scintillator information allowed for track traceback to the target. A smaller NaI crystal, faced with a collimator and charged-particle-veto counters sat off to the side, monitoring $\pi^0 \to \gamma\gamma$ events from the target for normalization. 10,402 events from the 60° data set, most of which consisted of $\pi^-p \to n e^+e^-$ events, were used to check the normalization of the simulation. The 130° sample, containing 11,736 events of which about 10,000 were $\pi^0 \to e^+e^-\gamma$ events, were used to find $a = -0.01_{-0.06}^{+0.08}$. The 156° data were not used due to unforeseen levels of $\pi^0 \to \gamma\gamma$ contamination and vertex reconstruction problems. The analysis included a thorough evaluation of backgrounds as well as radiative corrections (as formulated by Roberts and Smith [25]).

7. H. Fonvieille et al., 1989 [26]: a magnetic spectrometer consisting of two arms, each comprising a magnet surrounded on both sides by a series of drift chambers and scintillators, defined an angle of 110° out a liquid hydrogen target. The magnetic field permitted the momentum determination of $e^+e^-$ pairs over an
angular range of $50^\circ \leq \phi \leq 160^\circ$. The authors state a momentum resolution of 3.5% and a vertex resolution of roughly 9 mm. The magnetic field was calibrated to 0.1%. To eliminate background, an explicit cut of $x < 0.5$ was made. Two separate runs, comprising 18,346 and 18,353 $\pi^0 \rightarrow e^+e^-\gamma$, yielded $a = -0.021 \pm 0.036 \pm 0.056$ and $a = -0.205 \pm 0.032 \pm 0.050$, respectively, for a combined result of $a = -0.11 \pm 0.03 \pm 0.08$ (the first error is statistical, the second systematic).

Radiative corrections were included, using an approximate method formulated by the authors. Background calculations were also performed.

8. *CELLO Collaboration, 1991* [27]: this was a measurement performed at the DESY $e^+e^-$ collider in Hamburg, West Germany. Using the process pictured in figure 1.4c (production of a neutral pion by 2 photons, one of which is almost real) they were able to measure the form factor over a very large range of momentum transfer (from 0.5 GeV$^2$ to 2 GeV$^2$). The signature of a pion production event is that one of the beam electrons will be scattered into the endcap calorimeter close to the beam direction by emission of a heavy virtual photon (the other beam electron is hardly affected), together with two clean photons in the barrel calorimeter surrounding the entire detector from the subsequent decay $\pi^0 \rightarrow \gamma\gamma$. Using 137 events and fixing the pion lifetime, they obtain $a = 0.0326 \pm 0.0026$ (errors combined statistical and systematic).

While the CELLO result is of high accuracy and covers a wide range of momentum transfer, it is a measurement for *spacelike* momentum transfer only. It is of interest to determine the functional form of the $\pi^0$ form factor over the *entire range* of $q^2$; in the case of the charged pion, for example, much effort has gone into creating a consistent picture of the form factor for both negative and positive $q^2$ ([28] and the references therein).
Figure 2.3: Experimental results for the form factor slopes of heavier mesons in the region of timelike momentum transfer. The dotted line shows the range of VMD expectations. The experimental results for the $\pi^0$ form factor are also shown for comparison. The recent CELLO results (clustered around 1991) are all measured for spacelike momentum transfer. The vertical scale is the form factor divided by the square of the mass of the decaying meson, so that the results for different mesons may be compared.
Table 2.2: Summary of previous experiments to measure the form factor for the decay $\pi^0 \rightarrow e^+e^-\gamma$. See also figure 2.3.

In order to measure accurately an effect as small as the form factor slope in the timelike region of momentum transfer, one needs a detection system offering high momentum/energy resolution and very good energy calibration. A momentum uncertainty of 150 keV on 100 MeV electrons leads to an uncertainty of approximately 0.03 in $a$. In order to achieve this kind of calibration, if a magnetic field is to be used, one needs to know it to 0.1% at least. Also, since electrons lose approximately 300 keV per cm of liquid hydrogen traversed, it is desirable to know where in the target, to 0.5 cm or less, the event originated. Accurate calibration of the magnetic field and determination of the track length in hydrogen both require high statistics.

The earliest experiments to measure $a$ were done with very low numbers of events, resulting in high statistical errors as well as high systematic errors. These experiments were also done without taking into consideration the radiative corrections on the $\pi^0 \rightarrow e^+e^-\gamma$ process, corrections which are highly geometry-dependent and may be large. The first two experiments that did include the contributions due to these second-order corrections were those of Burger et al. and Fischer et al. However, the early theoretical work done on the radiative corrections was not directly applicable to
experiments, as the calculations did not take into account the fact that experiments have limited acceptance and do not detect all events with equal probability. Hence all the early results, up to and including the first high-statistics experiment performed by Fischer et al., are in doubt.

The more recent theoretical work of Roberts and Smith [25], connected with the Gumplinger experiment, and the approximate methods formulated by Fonvieille et al. do take into account experimental acceptance. Both experiments measure a negative slope for the form factor, contrary to theoretical expectations. Both experiments, however, have a rather poor vertex resolution, so that the systematic error on both experiments is large. The situation in the timelike region of momentum transfer remains unclear.
Chapter 3

Experimental Setup

3.1 Overview — General Principles

The form factor predictions and results in table 2.1 are scaled by the square of the mass of the decaying meson. In order to arrive at the more familiar unitless form factor discussed in chapter 1, we multiply the quoted $a$ by the mass squared of the decaying meson (in $GeV$), and we immediately see that the form factor influence on the invariant mass spectrum of the lepton pair increases with the mass of the decaying meson. One expects the lowest effect for the low-mass $\pi^0$. For the decay $\pi^0 \rightarrow e^+e^-\gamma$, the theoretical expectations outlined in the last chapter indicate that the form factor should lead to an increase in the number of events with increasing invariant mass $x$. However, one expects $a \approx 0.03$, so that it cannot change the partial decay rate by more than 6%, even at the highest allowed invariant mass $x = 1$ (see figure 1.5). For this reason alone, any measurement of the $\pi^0$ form factor requires large numbers of events. Further, since the decay products are an $e^+e^-$ pair, radiative corrections (higher-order Feynman diagrams involving extra internally or externally radiated photons) play an important role. These corrections also increase with invariant mass, going as $\ln\left(\frac{m_{\pi^0}\sqrt{s}}{m_e}\right)$, as will be discussed in chapter 7. In addition, there are many possible sources of background, including $e^+e^-$ pairs from $\pi^0 \rightarrow \gamma\gamma$ photon conversions in the detector material.

In designing an experiment to measure the $\pi^0$ form factor, it is essential to maximize the sensitivity to $a$ by ensuring that the detected events cover the full range of invariant
mass. One attempts to minimize the contributions from background processes by building a low-mass detection system; however, a full simulation is still required to assess all the background to the $e^+e^-\gamma$ sample. One further needs a clear understanding of the radiative corrections and of all systematic errors. The accuracy of the measurement hinges on the momentum/energy resolution of the detector. It is necessary to know to high precision the acceptance of the detector. This requires a thorough simulation of the experimental setup.

3.1.1 The $\pi^0$ Source

In order to measure the $\pi^0$ form factor, an intense $\pi^0$ source is necessary. Meson factories produce high-intensity $\pi^\pm$ and $K$ beams by illuminating solid targets with high energy proton beams, and magnetically separating the spray of resulting decay products into meson beams. These factories are thus ideal for high-statistics measurements. Two methods of obtaining the $\pi^0$ source have been used:

1. the reaction $\pi^-p \rightarrow \pi^0n$ at rest. This reaction produces roughly 6 neutral pions for every 10 $\pi^-$ incident on the target.

2. the reaction $K^+ \rightarrow \pi^+\pi^0$. The kaon is allowed to decay in flight; 21% of the time, it will result in a $\pi^0$.

Method 1 has the following advantages over method 2:

- approximately 3 times larger yield of $\pi^0$ per beam particle.

- much higher beam flux is attainable.

- much lower final state $\pi^0$ momentum, resulting in $e^+e^-$ pairs with larger opening angles, making track identification much easier and more accurate.
• lower reaction energy, resulting in far fewer photon conversions (background).

• no measurement of the incoming beam momentum is necessary to reconstruct the event.

Method 1 has these disadvantages:

• a target is required, leading to multiple scattering and potential photon conversion background.

• larger amount of background from $\pi^- p \rightarrow n e^+ e^-$ than from the corresponding process $K^+ \rightarrow \pi^+ e^+ e^-$. If a small target is used, multiple scattering and the number of photon conversions in the target will be small. Also, the potential $\pi^- p \rightarrow n e^+ e^-$ ("internal conversion") background has different kinematical limits and can therefore be discriminated against to a large degree by a smart trigger and by offline cuts. We cannot eliminate it completely, however, and therefore it must be simulated, as will be discussed in chapter 7. This is not a problem, since $\pi^- p \rightarrow n e^+ e^-$ has been calculated (at rest) and measured experimentally, so that it is well understood. Method 1, then, with its high $\pi^0$ flux and the resulting $e^+ e^-$ pairs with high opening angle resulting in accurate track reconstruction, is a good choice.

3.2 Experimental Setup

The form factor measurement is the last in a series of rare decay experiments performed with the SINDRUM I spectrometer at the Paul Scherrer Institut (PSI), Switzerland, throughout the eighties. The detector has been described in detail elsewhere [29]-[33]; we will provide only a brief description of the general design and function. SINDRUM I
is a magnetic spectrometer consisting of 5 concentric cylindrical wire chambers surrounded by a scintillator hodoscope, mounted inside a magnet solenoid. In figure 3.1 we show a detailed drawing of the entire device. The spectrometer was designed for detecting rare decays involving electrons in the final state, so that low detector mass was desirable to limit photon conversion background: a particle traversing all five chambers radially encounters only $5.4 \cdot 10^{-3}$ radiation lengths. In order to have as high an efficiency as possible for rare decay detection, SINDRUM I also has high solid angle coverage: 70% of $4\pi$.

3.2.1 Beam and Target

The $\pi E3$ beam at PSI provided approximately $1 \cdot 10^5$ 95 MeV/c $\pi^-$ per second, at a (low) primary proton current of roughly $1\mu A$ (maximum 250 $\mu A$). The $\pi^-$ flux that can be stopped at this energy in a depth 1.5 cm of liquid hydrogen, given that only 50% of the beam spot strikes the small target (radius 1.9 cm, length 12 cm), is roughly $1 \cdot 10^4$, leading to a $\pi^0$ rate of roughly 7000 per second ($1 \pi^0$ every 140 $\mu$s). This resulted in about 70 Dalitz pairs per second in the detector, which was as much as the data acquisition system could handle.

A system of four quadrupole magnets focussed the beam onto a small lead cone (the "moderator") mounted in front of a small liquid hydrogen target, as shown in figure 3.2. The purpose of the moderator was to slow the $\pi^-$ beam so that the particles would stop in the first 5 cm of the target. Since the beam diverges rapidly due to multiple scattering after passing through the lead, it was necessary to place the moderator as closely as possible to the target. For this reason, the moderator cone was integrated into the target design and did double duty as a vacuum window.
Figure 3.1: The SINDRUM I detector.
Figure 3.2: Detail of the target. Also shown are the lead moderator and the innermost wire chamber. The superinsulation around the vacuum cylinder is not shown.
The target itself consisted of a mylar cylinder (19 mm radius) of 0.12 mm wall thickness, with a spherical end. Surrounding the target was a 25 mm radius Makrolon vacuum cylinder (0.7 mm wall thickness), with the moderator functioning as the vacuum window. Several layers of superinsulation were wrapped around the vacuum cylinder to prevent ice buildup on its outer surface. Particles crossing the target and vacuum cylinder in the radial direction would traverse approximately $4 \times 10^{-3}$ radiation lengths of material. The entire target system could be wheeled in and out of the detector, enabling accurate repositioning between runs. The vacuum pressure and liquid hydrogen level were controlled and monitored by microcomputer throughout the experiment.

### 3.2.2 SINDRUM I Spectrometer

Surrounding the target were the 5 multiwire proportional chambers, which allowed for the reconstruction of the tracks of the particles passing through the detector. We summarize the attributes of the chambers in the table below. Each chamber consisted

<table>
<thead>
<tr>
<th>chamber</th>
<th>radius (cm)</th>
<th>length (cm)</th>
<th># wires</th>
<th>wire spacing (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.72</td>
<td>9.0</td>
<td>224</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>6.40</td>
<td>20.0</td>
<td>192</td>
<td>2</td>
</tr>
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<td>3</td>
<td>19.2</td>
<td>58.0</td>
<td>512</td>
<td>2</td>
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<tr>
<td>4</td>
<td>25.6</td>
<td>69.0</td>
<td>768</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>32.0</td>
<td>80.0</td>
<td>1024</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 3.1: Wire chamber specifications

of 2 concentric Rohacell cylinders faced with Kapton, on the facing sides of which were evaporated thin layers of aluminum which functioned as inner and outer cathode planes. The anode wires were strung between the 2 cylinders. The wire ends were fastened to fiberglass printed circuit rings (containing the necessary electronics) at either end of
Chapter 3. Experimental Setup

the chamber. Chambers 2, 3 and 5 had their inner and outer cathode planes etched into strips running at $+45^\circ$ and $-45^\circ$, respectively. The space between the Rohacell/Kapton cylinders was filled with chamber gas (argon-ethane-freon mix). A charged particle traversing the chamber would ionize the chamber gas, and the resulting free electrons would be accelerated towards the anode in the electric field between the cathode and anode strips. Very close to the anode, where the field was intense, secondary ionization would result in an avalanche of electric charges, causing a signal both on the anode wire (directly) and on the nearby cathodes (by induction). These signals were fed into a series of PCOS-III amplifiers, discriminators, and receivers, which processed them into a digital readout containing the hit wires' cluster midpoint and size. With the knowledge of the location of each wire, the wire numbers of the anode cluster then gave directly the $x-y$ position of the hit. The angled cathode strips of chambers 2, 3 and 5 enabled, in addition, the determination of the $z$ position of the hit.

The spatial resolution of the $\phi$ measurement was given by the 2 mm wire spacing ($\sigma \simeq 0.6$ mm), and the $z$-resolution was determined using cosmic rays to be $\sigma \simeq 0.3$ mm.

In addition to the cluster midpoint and size, the receivers generated FAST OR signals and LATCHED OR signals. For both signals, chamber wires were grouped into sectors which consisted of logically OR-ed neighbouring wires. For the FAST OR signal, only these OR signals were output. The LATCHED OR further required a coincidence with a gate signal. The 512 wires of chamber 3 were grouped into 16 sectors of 32 neighbouring wires which were FAST OR-ed, resulting in a 16-bit number pointing to an address in the Memory Lookup Unit (MLU) microprocessor. The contents of the $2^{16}$ possible addresses were 0 or 1, depending on whether the corresponding hit pattern was to be accepted or not. In this way, it was possible to test very quickly for two sectors with a minimum opening angle at chamber 3. Similarly, the LATCHED OR output
from chambers 2-5 were fed into the Track Preselector (TPS) microprocessor, which compared the signal with stored patterns (masks) of acceptable tracks. This enabled a fast track recognition in the $\phi$ plane. The masks were generated using simulated $e^+e^-$ pairs. A more detailed description of the operation of the MLU and the TPS can be found in reference [34].

Mounted on the outside chamber was a cylindrical hodoscope of 64 scintillator strips, each 88 cm long, 1 cm thick, and 3.3 cm wide. The ends of each strip were equipped with photomultiplier tubes, the output of which was fed into 64 “discriminator-meantimers”. These devices provided signals correlated to the time of the track’s traverse of the scintillator. The 64 hodoscope time signals were subsequently fed into electronics which allowed for the selection of events with a given number of hodoscope hits (at least 2 separate clusters) within a time period of 12 ns.

The chambers and hodoscope were mounted inside an iron solenoid magnet, which provided a uniform magnetic field parallel to the beam axis (and the anode wires) of 0.33 T. Charged particles thus described helical paths inside the detector. The magnet current was monitored and recorded by microcomputer throughout the experiment.

### 3.2.3 Trigger Logic

The scintillator hodoscope had a very fast response time, and was used to define the start of an event. Once two hodoscope hits within 12 ns were detected, the online computer (a PDP 11/44) began to check the rest of the (slower) electronics. The MLU had to show two hit sectors in chamber 3 with an opening angle of more than $67.5^\circ$. Then at least two hits in chamber 1 were required. The TPS selected events with at least one negative and one positive track. Upon passing all these requirements, the event was passed to the General Purpose Master (GPM), which used both the TPS and MLU information to apply a minimum $\phi$ opening angle cut of approximately $35^\circ$. 
Once the event had passed this trigger stage, the data was written into an event buffer on the PDP and passed to the online filter for further processing.

### 3.2.4 Online Filter

By using the full wire hit information (in contrast to the trigger's use of only the LATCHED and FAST OR's) from all 5 chambers and the hodoscope, the filter performed a more detailed track reconstruction in the \( r - \phi \) plane (as described in [35]) and calculated the distance of closest approach (DCA) of the track to the detector axis as well as its \( \phi \) emission angle at the DCA and transverse momentum \( p_t \). The emission angle was measured in the counterclockwise direction from the positive track to the negative one. The filter then applied more stringent cuts, requiring at least two tracks of opposite polarity with the following characteristics:

1. \( |DCA_{+,-}| \leq 25 \text{ mm} \)
2. \( |DCA_{+} + DCA_{-}| \leq 12 \text{ mm} \) (the DCA was negative if the axis was inside the curve of the track)
3. \( 35^\circ \leq \phi_{-} - \phi_{+} \leq 260^\circ \)
4. \( -4.0\text{ns} \leq t_{+} - t_{-} \leq 1.6\text{ns} \)

The first requirement rejected tracks which do not pass through the target, while the second ensured that a pair of correlated tracks existed. Small opening angle pairs were rejected, removing background and biasing the Dalitz sample towards higher invariant mass. A further hodoscope timing cut eliminated all but the very prompt tracks.
3.3 Data Acquisition

Data for the form factor measurement were taken at PSI during an dual-purpose experimental run lasting from the end of April to the end of October 1987, with a one-month cyclotron maintenance break. The long run period was necessary to measure the branching ratio of the rare decay $\pi^0 \rightarrow e^+e^-$, the results of which have been published [36]. During this run, the magnetic field was set to 0.33 T and the trigger conditions set as outlined above, and Dalitz data were taken on four separate occasions, each time for a day or less. The detector was taken apart between these runs for repairs to the wire chambers, resulting in four distinct data sets (labelled “geometries 2, 4, 5, and 6”), comprising a total of approximately $0.8 \times 10^6$ events.

These raw data were subsequently passed through an offline track recognition program to reconstruct the event kinematics precisely, which we will describe in the next chapter.
4.1 Overview

The offline analysis proceeded in two phases: first, the raw data was read from tape and a pattern recognition program (which incorporated detailed detector calibration information) translated the wire and hodoscope hit information into particle 3-momentum and event vertex location; second, the final analysis looped over all tracks in each event and selected the "best" pair, then applied final hard cuts (duplicating all cuts that went before, including those made by the online system) to these tracks and finally allowed for a detailed examination of the resulting kinematical distributions. We give a brief description of each of these elements; more details (especially concerning the calibration and pattern recognition software) can be found in [29] - [33].

4.2 Detector Calibration

In order to translate the wire hit information into \( x - y - z \) track coordinates precise knowledge of the wire locations is necessary. It was discovered, for example, that when the fiberglass anode wire prints at each end of the chambers were glued to form rings, the resulting space at the seam was not the same as the wire spacing. Also, the seams at either end of the chambers were rotated with respect to one another, inducing a slight twist in the anode wires. Corrections for the anode "gap" and "twist" had to be made for each of the five chambers in order to produce the correct \( x, y \) information.
The inner and outer cathode planes of chambers 2, 3 and 5 were also rotated slightly with respect to one another. In addition, minor variations occurred during the run period, when, on four separate occasions, the spectrometer was turned off and the inner chambers removed for repairs. Upon their reinsertion, the chambers were slightly rotated and offset with respect to one another.

The relative rotations and locations of the chambers were calibrated using cosmic ray data with no magnetic field (straight, throughgoing tracks). The following quantities were determined ($i = 1, 2, 3, 4, 5; j = 1, 2, 3, 4; k = 2, 3, 5; l = 2, 3$):

- $\phi^j_{\text{anodes}}$: the rotation of the $j^{th}$ chamber relative to chamber 5
- $\phi^k_{\text{cathodes}}$: the relative rotation between the inner and outer cathodes, for those chambers with $z$ information
- $\Delta x^j, \Delta y^j$: the $x$ and $y$ offsets of the midpoint of each chamber (relative to chamber 5)
- $\Delta z^l$: the offset in midpoint $z$ relative to chamber 5
- $\alpha^j_{\text{up,down}}$: the rotation of the anode prints at the up- and downstream end of each chamber
- $s^i_{\text{up,down}}$: the seam gap of the anode prints at the up- and downstream ends of each chamber

Typical $x, y$ offsets were on the order of 0.5 mm. The largest $z$ offsets were 1 mm, for chamber 3; those for chamber 2 were on the order of 0.5 mm. The largest rotation of 2.4 degrees (or 1.5 wires) was found for chamber 1; all other rotations were less than a single wire. The anode seam gaps were less than 0.5 mm (different from the wire
spacing), so that the twist produced in the anode over the length of the chamber was very slight.

The hodoscope time signals were also calibrated (for a more detailed description of the calibration procedure, see reference [33]); the digital output from the discriminators had to be translated into time signals in ns and corrected for $z$ position- and amplitude-dependence. After calibration, a time resolution $\sigma = 315$ ps per hodoscope scintillator strip was achieved.

4.3 Pattern Recognition and Track Fitting

With the calibration information in hand, accurate translation of the wire and hodoscope hits into $x, y, z$ and time co-ordinates was possible. The pattern recognition software fit helical tracks to this translated information. The pattern recognition and track fitting proceeded in three steps: first, wire hits were fit with circles in the $r - \phi$ plane; second, these results were combined with the $z$ hit information and data fit with straight lines in the the arc length $s$ versus $z$ plane. Finally, the vertex of every pair of $+$-- tracks was determined by extrapolating the fit tracks back into the target. A detailed description of the track fitting procedure may be found in [37,38]; a brief description follows.

4.3.1 $r - \phi$ Fit

The charged particles described circular tracks in the $\phi$ plane in the homogeneous magnetic field, the radius of which was proportional to the transverse momentum. The ionizing particle rarely caused more than 1 wire to give a signal. The pattern recognition algorithm first verified that the candidate track has hits in all 5 chambers as well as in the hodoscope. It then grouped hits in the first 3 chambers which fell
within a certain distance window into “triples”. The same was done for the outer 3 chambers. Subsequently, all triples sharing a common hit in chamber 3 were fit with circles; chamber 1 hits were weighted by a factor of 4 since the wire spacing was half that of the other chambers. The resulting fit was required to have an acceptable $\chi^2$. Those tracks for which the real chamber 1 hit differed from the circle fit projection by more than 0.5 mm were rejected, as were tracks with a fit DCA of more than 25 mm.

4.3.2 $z$ Fit

The ratio of the transverse to the longitudinal momentum was constant for a charged particle traversing a magnetic field uniform in $z$. Therefore, in the arc length $s$ versus $z$ plane, the particle track was a straight line. Once the $r - \phi$ fit was complete and the track fit with a circle, the arc length could be determined and the $s - z$ analysis performed. The determination of the $\theta$ angle of emission, combined with the results of the transverse momentum $P_t$ from the $r - \phi$ fit, allowed for the calculation of $P_z$.

At least two $z$ hits were required for a candidate track. Each anode signal induced signals on a few neighbouring cathodes, producing smeared out cathode “clusters”. All such clusters, on both the inner and outer cathode planes, were sought, and their centres determined. Dead and damaged cathode strips were interpolated over using the information from neighbouring strips. All inner and outer cathode clusters were combined into pairs, each pair defining an angle $\phi$ which was compared to all anode $\phi$ co-ordinates obtained from the wire hit information. Matching $\phi$ co-ordinates resulted in the assignment of $r - \phi - z$ coordinates to the hit. Once the track co-ordinates had been established, a straight line fit in $s - z$ was performed.
4.3.3 Vertex Fit

The final step in the event reconstruction was the determination of the event vertex. The point midway along the shortest line between the two tracks was chosen to be the event vertex.

Figure 4.1 shows the resulting distribution of event vertices for $e^+e^-$ pairs in $r - z$. The target is clearly visible. Also visible are the lead moderator in front of the target, the aluminum support ring at $(z \approx -110$ mm, $r \approx 20$ mm), the aluminum target mounting at $(z > -80$ mm), and chamber 1 at $(r \approx 35$ mm). These structures are visible due to the large numbers of photons from $\pi^0$ decays converting into $e^+e^-$ pairs.

The results of the track fit were written to a file, along with the raw data for each event.

4.4 Final Event Selection — Identification of Dalitz Events

The final analysis program reads the results of the reconstruction program from the file created and loops over all pairs of tracks with opposite sign, requiring that the event vertex of each combination is inside the target, and chooses the “best pair” on the basis of the $\chi^2$ of the vertex fit. This is not a stringent test, since at this stage of the analysis, most of the events have only a single pair of proper tracks. The final chosen event must meet the following requirements, which duplicate (and are more stringent than) the cuts applied by the online trigger and filter and the track fitting program:

- $45^\circ < \phi < 260^\circ$ : a cut which duplicates the action of the online trigger and filter at roughly $35^\circ < \phi < 260^\circ$.

- $P_t > 20$ MeV/c : SINDRUM's transverse momentum threshold is roughly 17 MeV/c, and this requirement ensures that any systematic error will be due to the
Figure 4.1: Distribution in $r - z$ of the distance of minimum approach for $e^+e^-$ pairs. The target, moderator, aluminum support ring, target support structure, and chamber 1 are clearly visible.
cut rather than to our understanding of the geometry of the detector.

- $-300 \text{ mm} < Z(5) < 300 \text{ mm}$: the particle track was required to lie well within the region of uniform magnetic field. By requiring the chamber 5 $z$ hit to be well away from the edges, we define a conical fiducial volume within which the magnetic field is uniform.

- $r < 19 \text{ mm}, -115 \text{ mm} < z < -80 \text{ mm}, 0 \text{ mm} < z + 104 + \sqrt{19^2 - r^2}$: cuts which ensure that the event happened well inside the target. The quantity $104 + \sqrt{19^2 - r^2}$ follows the curved upstream edge of the target.

After the reconstruction and final analysis cuts above, the four geometries 2, 4, 5 and 6 comprise 9294, 43968, 8899, and 43464 events, respectively. In figures 4.2 through 4.4, we show distributions of some of the kinematical variables of the resulting data. There is background remaining. Figure 4.2 shows the distribution in opening angle of the chosen $e^+e^-$ pairs. The large, sharp peak at $\varphi \approx 156^\circ$ is due to $\pi^0 \rightarrow \gamma\gamma$ events in which both the photons convert; the resulting leptons, one from each photon, when boosted into the laboratory frame, exhibit an opening angle of approximately $156^\circ$. In chapter 6 we discuss the cuts used to eliminate these events from the data sample. In figure 4.3 is shown the invariant mass of the $e^+e^-$ pair (normalized to the $\pi^0$ mass) versus the total energy of the pair plus the neutron,

$$E_{\text{tot}} = T_n + E_+ + E_-$$

$$= \sqrt{|\vec{P}_+ + \vec{P}_-|^2 + m_n^2} - m_n + E_+ + E_-$$

The events in the band at $E_{\text{tot}} \approx 130 \text{ MeV}$ are $\pi^- p \rightarrow n e^+ e^-$ events; for these 3-body events, the total energy of the event is constant and equal to the initial $\pi^- p$ energy. The width of this band is indicative of the resolution of the detector. The Dalitz events, in which the photon carries away energy, populate the slanted band.
Events from $\pi^0 \rightarrow e^+e^-\gamma$ cannot extend past $x = 1.0$; as $x$ increases, the $e^+e^-$ pair carries more and more of the energy until at $x = 1$, $E_\gamma = 0$ and the $\pi^0 \rightarrow e^+e^-\gamma$ and $\pi^-p \rightarrow ne^+e^-$ processes merge on the plot. The $ne^+e^-$ events may be separated from the $\pi^0 \rightarrow e^+e^-\gamma$ sample to a large degree by the requirement that $E_{tot} < 110$ MeV, but only at the expense of high invariant mass Dalitz events. This is not so desirable.

![Graph showing distribution in opening angle of $e^+e^-$ pairs.](image)

**Figure 4.2**: Distribution in opening angle of $e^+e^-$ pairs. The sharp peak at 156° is due to photon conversion events. The broad peak at 110° is due to the asymmetrical $\phi$ opening angle cut of the trigger.

**Figure 4.4** shows the distribution of transverse opening angle against the quantity 

$\text{"}E_t P_t\text{"}$,

$$E_t P_t = E_+ + E_- + |\vec{P}_+ + \vec{P}_-| + T_n$$
Figure 4.3: Distribution in $x$ and total energy of $e^+e^-$ pairs plus the neutron kinetic energy. The $n e^+e^-$ events populate the horizontal band at 130 MeV. The Dalitz data inhabit the slanted region up to $x = 1.0$. The curving branch at small $x$ and low total energy peeling away from the Dalitz region are events with an extra photon which radiates away more energy.
Figure 4.4: Distribution of $e^+e^-$ pairs in the quantity $E_tP_t$ and transverse opening angle. The Dalitz events are constrained to lie in the box between 107 and 163 MeV. The slanted bands are the $ne^+e^-$ events. The region below 107 MeV is inhabited by the radiative events.
The Dalitz events inhabit a boxlike region between $107 < E_t P_t < 163$, where the two extremes correspond to the cases where the neutron is emitted parallel and antiparallel to the $e^+e^-$ pair, respectively. If we apply the further restriction that $E_t P_t < 170$, we can eliminate $ne^+e^-$ events to a high degree, without the loss of the high invariant mass events. In order to assess the efficacy of these cuts, we must simulate both the $\pi^0 \rightarrow e^+e^-\gamma$ and the $\pi^-p \rightarrow ne^+e^-$ processes. The background also needs to be simulated. We turn now to a discussion of the simulation procedure.
Chapter 5

Data Simulation

In order to extract a parameter of the size of the form factor slope from the data, one must understand the data very well. In order to achieve this understanding, we use the CERN package GEANT for simulation of the detector response. A full description of this package and its implementation in our Monte Carlo simulation can be found in [39] and [40]; we give here a brief outline of the steps involved.

The main steps in simulating the data are as follows:

1. Choose the detector geometry, the magnetic field value, and the process to be modelled. Make any restrictions on the particle kinematics.

2. Assuming that the $\pi^- p$ atom is at rest, the energy available to the subsequent reactions is $E = m_{\pi^-} + m_p - B(\pi^- p) = 1077.83941$ MeV (the binding energy of the $\pi^- p$ system is approximately $0.4$ keV [41]), assign the 4-momenta of the resultant particles according to the appropriate matrix element and Lorentz transformations, within the kinematical restrictions imposed in step 1.

3. Decide on the location of the interaction.

4. Pass the particles through the detector, starting from the interaction point and ending when the particles leave the sensitive volume of the detector. Model the production of secondary particles such as photons (due to electron bremsstrahlung) and electrons (from photon pair production) and pass these through the detector as well.
5. As the particles lose energy in the various detector parts, sensitive and non-sensitive, model the signal produced in the sensitive ones.

6. Write the resultant response into a data file identical in format to that of the real data.

7. Run the trigger simulation programs. Read each event from the data file and either discard or accept it.

We now discuss these steps in more detail, highlighting the most important assumptions.

5.1 Deciding the Detector Geometry

The basic setup of the SINDRUM spectrometer, as discussed in the previous chapter and in reference [33] (and the references contained therein) was assumed to be fixed and conforming to the specifications. The slight chamber misalignments discussed in chapter 4 were taken into account. However, since the effect of the anode wire print gaps and twists resulted in less than a single wire difference in all but the most extreme cases, it was decided not to model them, but to include them only as corrections during the track reconstruction of the data.

The position of the target was determined on a run-by-run basis by examining the distribution of event vertices, and checking the location of the target edge. A typical distribution is pictured in figure 4.1. The target was found to be in the same position (to within 2 mm) for the 3 later geometries; for geometry 1 it was 4 mm further upstream.

The magnetic field was set to 3.313 kG throughout the experimental run, and the solenoid current monitored at 5 minute intervals during the run. This value was recorded in the data file of each accepted event. The measured magnetic field value
was estimated to be accurate to about 1% (using a field map produced in an earlier experiment using the SINDRUM detector [31]); however, as will be discussed in chapter 9, we desire an accuracy of less than 0.5%. In order to achieve this, the simulation was first run using the nominal value of 3.313 kG, whereupon the resultant events were used to calibrate the magnetic field for each of the four separate run periods. The simulation was then redone using the corrected field, in order to account for any changes in acceptance.

The magnetic field was uniform to better than 1% within the chosen fiducial volume inside the chambers. Since no accurate field map was made, we assume a uniform field for simulation purposes, and then apply a fiducial cut during the analysis.

5.2 Generating the Particle Kinematics

Some minimal requirements were imposed on the initial lepton kinematics in order to cut down on the amount of computer time required to track the events. These requirements were set below the physical detector thresholds, so that the detector, trigger, and filter simulation and subsequent analysis would determine the event acceptance. The cuts imposed were the following:

- The initial electron momenta should lie within SINDRUM, and should be large enough to allow the lepton to hit the hodoscope. This limited the simulation to producing events with

\[ 27^\circ < \theta < 135^\circ \]

\[ 12 \text{ MeV}/c < P_t \]

Here \( \theta \) is the longitudinal angle of the emitted electron, and \( P_t \) is its momentum in the \( x - y \) plane.
• Since the trigger and filter applied a cut in transverse opening angle, we further restricted the simulation to generate events lying within the region:

\[ 15^\circ < \phi_t < 260^\circ \]

• The above restrictions on transverse momentum and opening angle result in an effective restriction on the minimum invariant mass of the \( e^+e^- \) pair of roughly 0.01. Since the simulation performs a numerical integration of the matrix element over this variable, we set the additional restriction \( x > 0.001 \) explicitly in order to reduce the computer time needed for the calculation.

We considered the following reactions:

1. \( \pi^0 \rightarrow e^+e^-\gamma \) according to the matrix element set out by Kroll and Wada [6], with the form factor slope set to zero.

2. \( \pi^0 \rightarrow e^+e^-\gamma\gamma^* \) and \( \pi^0 \rightarrow e^+e^-\gamma \gamma \), the first order radiative corrections to the above process, as calculated by Roberts and Smith [25].

3. \( \pi^-p \rightarrow ne^+e^- \) with first order radiative corrections \( \pi^-p \rightarrow ne^+e^-\gamma \) and \( \pi^-p \rightarrow ne^+e^-\gamma^* \), as formulated by Fonvieille et al. [26].

4. \( \pi^0 \rightarrow e^+e^-e^+e^- \) using the matrix element derived by Miyazaki [42].

5. \( \pi^0 \rightarrow e^+e^-\gamma \) where the photon was forced to undergo Compton scattering or pair production in a specified area of the detector.

6. \( \pi^0 \rightarrow \gamma\gamma \) where one or both of the photons were forced to undergo Compton scattering or pair production.

7. \( \pi^-p \rightarrow n\gamma \) where the 129 MeV photon is forced to undergo pair production.
In the case of the processes 4 through 7, the cuts imposed on the primary leptons outlined above were modified. For the process $\pi^0 \rightarrow e^+e^-e^+e^-$, we required only that at least one $e^+e^-$ combination fulfilled the restrictions listed. For the photon conversion background 5, 6 and 7 we waived all restrictions and generated the events in $4\pi$ with all possible momenta. A more complete discussion of the modelling of the background processes 3 to 6 can be found in chapter 7.

5.3 Stop Distribution

The events were generated in the liquid hydrogen target according to stop distribution found for the data. The event origin, as calculated by the reconstruction program, was plotted for each real event in an $r-z$ projection. This was digitized and used as the distribution function for the generation of the simulated events. The stop distribution was taken to be radially symmetric and identical for each of the 4 run periods, for each process.

During the final analysis, the data and simulation vertex distributions in the $r-z$ plane of both the processes $\pi^0 \rightarrow e^+e^-\gamma$ and $\pi^-p \rightarrow ne^+e^-$ for each of the 4 run periods, were plotted and digitized. By dividing the stop distribution of the data by that of the simulation, 8 sets of (2-dimensional) weights were generated. These weights were applied to each Monte Carlo stop distribution, so that the simulation would match the data as closely as possible. The importance of this matching will be discussed in chapter 9. A typical weight distribution can be seen in figure 5.1. Note that the weights are quite close to 1 for most regions of the target.
Figure 5.1: Two-dimensional weighting function for the simulation, designed to match the stop distribution to that of the data. The maximum height is roughly 7, the average is 0.8.
5.4 Modelling Detector Response

Once the detector geometry has been defined and entered into the GEANT package, the program will step each of the particles through the various detector parts, modeling its energy loss along the way. The production of secondary particles through pair production, bremsstrahlung, and Compton scattering is automatically performed. These secondary particles are also traced through the detector. Once a particle reaches a volume defined as "sensitive" by the user (for example, a chamber wire or hodoscope cell), control is passed to the user's subroutine, which then takes care of the specific detector response. We note here that the complex process of gas ionization, ion drift, and subsequent amplification and avalanche near a chamber wire has been approximated by recording the energy lost by the electron in the chamber and assigning it to the nearest wire. The response of the cathodes is approximated by a Gaussian response curve over 7 nearest neighbours on each side of the central strip. The cathode strips are simulated to be 100% efficient (no dead strips are simulated). A noise signal is added to the signal amplitude. The hodoscope signals are determined by assigning the energy lost in a particular scintillator by the particle to that scintillator.

The energies assigned to the anodes, cathodes, and hodoscope scintillators are translated into digital output identical in format to the output of the electronics and written to a file which can subsequently be read directly by the offline analysis programs.

5.5 Trigger Simulation

When the online trigger (TPS, MLU and GPM) and filter were being designed and the appropriate masks were being developed during the initial construction of the spectrometer, trigger simulation programs were written. Their purpose was to test the correctness and completeness of the masks (which were produced by simulating $e^+e^-$
pairs) and hence to verify the performance of the online $r-\phi$ pattern recognition. The programs were written by W. Bertl and H. Pruys of the SINDRUM I collaboration.

The programs take as input a raw data file with the format generated by the electronics. Different trigger conditions may be set during an initialization phase. We set the conditions appropriate to the Dalitz experiment outlined in chapter 3, and pass the simulated data through the trigger. The results are shown in figures 5.2a) through f), where we illustrate the action of the trigger both on the simulated Dalitz and $ne^+e^-$ data, as a function of momentum and transverse opening angle. We see that the trigger cuts harder on the $ne^+e^-$ data than on the Dalitz data: 78% of the $ne^+e^-$ events are lost in the trigger, compared to 55% of the Dalitz events. The higher the track momentum, the more likely it will be lost in the trigger. This is because the masks were produced by simulated $e^+e^-$ pairs from $\pi^0$ decay; such pairs will always have a smaller momentum range than the $e^+e^-$ pairs from $\pi^-p \rightarrow ne^+e^-$. This was an error. Had this cut been less stringent, a more thorough analysis of the $n^-p \rightarrow ne^+e^-$ data would have been attempted. Looking at the positron and electron transverse momentum distributions, we see further that the positive tracks are cut harder than the negative tracks; the masks, being simulated themselves, were not quite symmetric with respect to charge. Again this is exaggerated in the $ne^+e^-$ sample, because of the larger momentum range. The trigger simulation also applies a cut in transverse opening angle; however, from figure 5.2c) and f) we can see that the simulation's cut is not the same as the real trigger's.

The fact that the agreement between the data and simulation is poor at this point is not worrying, since much more stringent cuts will be made later on in the offline analysis. Any systematic errors due to the event selection will then be due to these cuts, and not due to the poorly-understood trigger. The reason for simulating the action of the trigger in the first place was to verify the reduction in the $ne^+e^-$ sample.
Figure 5.2: The figures illustrate the action of the simulated trigger. The histogram represents the simulated data before passing through the trigger, the points, after the trigger. In a) through c) we show the effect on the Dalitz simulation, while in d) through f) we show the $ne^+e^-$ simulation.
In addition to the online trigger, the data also passed through the online filter. This stage of the data acquisition software is not modelled, since, again, later cuts in the analysis will duplicate in a more stringent way the requirements imposed by the filter. During the data taking, every 4th event was written to tape by the online filter, regardless of its being accepted or not. In this manner, it is possible to test the efficiency of the later analysis cuts: we find that no events which were not passed by the filter found their way into the final data sample.
Radiative Corrections

6.1 Radiative Corrections for the Process $\pi^0 \rightarrow e^+ e^- \gamma$

Radiative corrections are processes such as those pictured in figures 6.1 and 6.2, processes involve more than two photons. It is less likely that these events will take place; radiative corrections, with Feynman diagrams involving extra vertices, are smaller by $\alpha \approx 1/137$. Note that the radiated photons pictured in figure 6.1 may be either external (hence in principle detectable) or internal. In both cases, they change the momentum of the electrons and hence modify the shape of the invariant mass spectrum. Since the slope parameter is itself a second-order effect, one would expect that the effect of these radiative corrections could be appreciable, and hence they must be included in any analysis that expects to extract a value for $\alpha$.

In the case of the decay $\pi^0 \rightarrow e^+ e^- \gamma$, the contributions to the matrix element corresponding to the diagrams shown have been evaluated exactly by Mikaelian and Smith and others [20,22,25] and the resulting corrections to the Kroll-Wada formula 1.11 tabulated [22], as shown in figures 6.3 and 6.4. We note that these analyses neglect the (form-factor dependent) corrections shown in figure 6.2. One analysis [43] claims that these diagrams make a large contribution to the form factor, while a later paper [44] comes to the conclusion that these graphs have an extra factor of $\frac{m_e^2}{m_\pi^2}$ associated with them, and can hence be safely ignored.

Since $\alpha$ is defined in terms of the invariant mass $x$, one could in principle apply the
Figure 6.1: First order radiative corrections to the process $\pi^0 \rightarrow e^+ e^- \gamma$. 
Figure 6.2: Two virtual photon loop graphs, corrections to $\pi^0 \rightarrow e^+e^-\gamma$. 
Figure 6.3: Two dimensional surface plot showing the percentage correction to the Kroll-Wada matrix element, as calculated in [18]. The surface is symmetric about $y = 0$. 

Figure 6.4: Corrections to the Kroll-Wada matrix element as a function of $x$, as calculated in Mikaelian and Smith [18].
results of Mikaelian and Smith's calculations in $x$ directly to the simulated, uncorrected spectrum as a multiplicative factor. The radiative corrections expressed as a percentage change over the "bare" spectrum are roughly linear in $x$ over the effective range of the data, so that the resulting value for $a$ increases by approximately 0.05. However, this naïve approach to the radiative corrections is not correct. As one does not actually simulate any radiative events using this method, the implicit assumption is that the acceptance and detection efficiencies for the radiative events are exactly the same as for the bare events. There is no \textit{a priori} reason for this to be so; the radiative events can be 4-body decays and as such are kinematically quite different from the bare 3-body Dalitz decays. Another serious drawback of this simplistic method is that, since one does not simulate any radiative events, it is not possible to check whether the bare events plus the corrections actually match the data in any kinematical region other than $x - y$ space. The applied multiplicative factor cannot create four-body decay events in the regions forbidden to the three-body bare events. In summary, it is not possible to gain any insight into the kinematical behaviour of the radiative corrections without generating some.

Before moving on to discussing the techniques used to generate radiative events, we first make a few observations concerning the diagrams shown. The radiative corrections pictured are those of second order in $a$ only. Higher order corrections of course exist, but each added photon line suppresses the amplitude of the event by a factor of $\sqrt{a}$ so that one usually considers only the leading order terms pictured. The diagrams divide themselves up naturally into two classes; "virtual" or "internal", and "bremsstrahlung" or "external" radiative corrections. Virtual radiative corrections are those in which the extra photon is reabsorbed onto one of the leptons and is therefore not experimentally detectable. The bremsstrahlung corrections lead to free photons which are in principle detectable. Both the virtual and bremsstrahlung corrections separate into a convergent
set of integrals and a divergent group. When one performs the integration numerically, the usual technique employed to cope with these divergent integrals is to introduce a photon cutoff energy $\Lambda$, below which the integral diverges, and above which it converges. Adding the divergent part of the bremsstrahlung contribution to the virtual corrections, one finds that the divergent parts of each cancel exactly (independent of the value of $\Lambda$), and the total contribution is finite.

In order to simulate radiative events, the results derived by Mikaelian and Smith are not sufficient as they are couched in terms of the two degrees of freedom $x$ and $y$. The virtual corrections and the bare process may be fully described by only two kinematical variables, but the 4-body bremsstrahlung corrections require one more parameter, $x_\gamma$, the invariant mass of the two photons, in order to specify the event kinematics completely. This variable has been integrated over to derive the surface illustrated in 6.3. We therefore use the code developed by L. Roberts [25] which performs the integrations numerically over phase space variables. The code separates the virtual and bremsstrahlung corrections and calculates the matrix element for each separately. The cutoff energy $\Lambda$ is set to a small value well below the experimental energy resolution. This corresponds to asserting that the bremsstrahlung events with photons of $E_\gamma < \Lambda$ cannot be experimentally distinguished from the virtual corrections, and hence may be safely used to cancel the divergent part of the virtual corrections. If $\Lambda$ is set too high, one in effect ignores bremsstrahlung events with low $E_\gamma$.

The integrations to be performed are quite complex; the virtual corrections involve numerical evaluation of a five-dimensional integral while the bremsstrahlung corrections are eight-dimensional. The code uses a generalized Monte Carlo integration technique to evaluate the integrals. The range of each integration variable is divided up into small cells (the so-called integration grid) and the contribution of each cell to the total integral is estimated on the basis of the fraction of randomly drawn points that
fall in it. These randomly drawn points are converted into the event kinematics; in this way, the integration package also functions as an event generator. As the accuracy of the desired integral is increased, a refining algorithm repartitions the integration space into smaller and smaller cells, focusing in on steep areas of the function. One must verify that this partitioning and refining algorithm does not introduce any bias into the result and skew the distribution of generated events.

In the case of the bremsstrahlung corrections, for example, the eight variables of integration are the energy, \( \cos \theta \), and \( \phi \) angle of the bremsstrahlung photon, the energies of the positron and electron, the opening angle between the electron and positron, and the \( \cos \theta \) and \( \phi \) angle of the electron. The program first models the 2-body decay of the initial \( \pi^0 \) into a virtual photon and a bremsstrahlung photon, and then allows the virtual photon to undergo a 3-body decay into \( e^+e^-\gamma \). Obviously, the distribution of events should be uniform in the \( \cos \theta \) and \( \phi \) angles of the bremsstrahlung photon; the initial 2-body decay of the \( \pi^0 \) does not favour any particular direction.

If these distributions are in any way skewed, the subsequent cuts modelling the detector acceptance will throw out or accept too many events and give an incorrect final event distribution. Note, however, that this skewing will not lead to an incorrect estimation of the integral! This is because the integrated function is flat in these variables; every point along the axis contributes the same amount to the final answer and favouring one point over another makes no difference. In the case of the bremsstrahlung corrections, it was found that the \( \cos \theta, \phi \) distributions were not flat. There was an error in the partitioning/refining algorithm of the integrating package which produced holes and spikes in the distribution of cells. Rather than rewrite the integrating package, it was decided simply to re-randomize these variables after the integration was finished but before the subsequent detector cuts, since, as pointed out above, the final answer would not be sensitive to this operation.
We run the code in an attempt to reproduce the results quoted by Mikaelian and Smith; see figure 6.5. The corroboration is not fantastic, but out to $x \approx 0.5$ (where most of the data is) the agreement was deemed acceptable. It must be noted that generating graphs such as those shown in 6.5 requires many hours of computer time. We find the total contribution to the matrix element for $\pi^0 \rightarrow e^+e^-\gamma$ to be

$$\Gamma_{tot} = \Gamma_0 + \Gamma_{rc}$$

$$= \Gamma_0 + \Gamma_v + \Gamma_b$$

$$= (6.2499 \pm 0.0026) \cdot 10^{-8}$$

$$+ (-0.9873 \pm 0.0004) \cdot 10^{-8} + (1.0341 \pm 0.0005) \cdot 10^{-8}$$  \hspace{1cm} (6.1)

The errors are statistical and are estimated by the Monte Carlo integrating package on the basis of the number of points drawn per integration region. The radiative corrections contribute a total $\Gamma_{rc} \approx 0.75\%$ to the total decay width, in agreement with the results of Mikaelian and Smith. We add the radiative events to the bare events generated according to the Kroll-Wada distribution in the proportions given by equation 6.1, and verify that these radiative events (the bremsstrahlung photons are tracked and allowed to interact with matter) now fill out the tails of the distributions as expected. In figure 6.6 we plot the energy of the lepton pair against their invariant mass, and figure 6.7 shows the total energy-momentum versus the opening angle of the lepton pair. The radiative events are clearly visible below the kinematical region allowed the bare events. In figure 6.8 we see that the tail fits the data very well.

It is instructive to plot the effect of the radiative corrections on the bare events, especially for the $x$ and $\phi$ distributions. In figure 6.9 we show the relative contribution to the invariant mass and opening angle spectrum from both the virtual and bremsstrahlung radiative corrections. We see that the corrections have a greater impact on $\phi$ than they do on $x$. This is presumably due to the fact that the extra photons being...
Figure 6.5: Verification of the published radiative corrections. The line are the corrections as calculated by Mikaelian and Smith [18] and the points are the result of the numerical integration using the program of Roberts and Smith [21].
Figure 6.6:Invariant mass of lepton pair versus the total energy, for the simulated events. We plot both the $ne^+e^-$ and the $e^+e^-\gamma$ events, both with radiative corrections.
Figure 6.7: $E_t P_t$ of lepton pair versus the transverse opening angle, for the simulated events. We plot both the $ne^+ e^-$ and the $e^+ e^- \gamma$ events, both with radiative corrections. The radiative tails are clearly visible.
Figure 6.8: The $E_tP_t$ distribution for the Dalitz events. The histogram is the simulation, the points, the data. Note that the radiative tail is very well fit.
emitted do not transfer much energy, but do transfer momentum, and will therefore kick the leptons a bit further apart.

Figure 6.9: The contribution to $x$ and $\phi$ from the virtual (solid line) and bremsstrahlung (dotted line) corrections to the total (triangles) simulation. The total contribution of the radiative corrections is shown by the shaded histogram.

6.2 Radiative Corrections for the Process $\pi^- p \rightarrow n e^+ e^-$

Radiative corrections to the internal conversion process $\pi^- p \rightarrow n e^+ e^-$ must also be considered, as they result in lepton pairs leaking down into the Dalitz decay region.
and contaminating it. This is especially important at higher values of $x$, since the two processes approach one another in energy as the lepton invariant mass increases. We expect higher contamination at higher $x$. Since this is the region where the sensitivity to $\alpha$ is highest, it is imperative that the $ne^+e^-$ radiative corrections be understood.

No exact theoretical calculations have been done for the $ne^+e^-$ radiative corrections, so approximate methods must be used to determine the shape of the radiative spectra. The approximate methods used for the $ne^+e^-$ process are quite general and equally applicable to the Dalitz decay. A more complete discussion of the method used is given in references [45,46]; we note here that the virtual corrections shown in figure 6.10 can be implemented by a multiplicative weighting factor (which is less than 1) carried along with each “bare” event:

$$\frac{d\sigma}{dx_{\text{virtual}}} = \frac{d\sigma}{dx_0} (1 + \delta(x)_{\text{virt}})$$

with

$$\delta(x)_{\text{virt}} = \frac{\alpha}{\pi} \times \left\{ \frac{\pi^2}{3} + \frac{13}{3} \log \left( \frac{m_e \sqrt{x}}{m_e} \right) - \frac{28}{9} \right\}$$

where one neglects the (small) dependence of the virtual corrections on $y$. The bremsstrahlung corrections pictured in figure 6.9 are modelled by allowing one of the leptons to emit a photon according to the probability distribution (expressed in the centre of mass of the emitted photon)

$$1 + \delta(x,y)_{\text{brems}} = \int_{0}^{E-m_e} P(q) dq$$

with

$$P(q) dq = \frac{A}{q} \left( \frac{q}{E} \right)^A \left( 1 - \frac{q}{E} + \frac{q^2}{2E^2} \right) dq$$

where

$$A = \frac{2\alpha}{\pi} \log \left( \frac{2E}{m_e} - \frac{1}{2} \right)$$

The bremsstrahlung photon is emitted with energy $q$ from a lepton of energy $E$. One imposes a threshold energy cut $q > \Delta$ ($\Delta \approx 0.1$ MeV) so that only those photons visible
Figure 6.10: Feynman diagrams for the radiative corrections to the process $\pi^- p \rightarrow ne^+e^-$. 
by the apparatus are modelled. Every event is forced to undergo bremsstrahlung. Its probability, as calculated by the equation above, is carried along as a weighting factor (again, always less than 1).

This approximate method is based on the assumption that the radiative corrections to $\pi^- p \rightarrow ne^+e^-$ correspond to those calculated in [46] for the process $e^+e^- \rightarrow \text{hadrons}$. Its accuracy is estimated by the authors to be roughly 10%-15%. In figure 6.11, we show the results of carrying the weighting factors along. Note that the shape of the (clearly visible) radiative tail is generally well simulated, although a slight underestimation of the tail seems to be occurring.
Figure 6.11: Total energy of the lepton pair for the $ne^+e^-$ events. The points show the data, the histogram is the simulation. The radiative tail is generally well fit, although an underestimation seems likely. The simulation is expected to be correct to roughly 15%.
Chapter 7

Background Considerations

Any process which produces $e^+e^-$ pairs may be considered a background to the Dalitz decays. Table 7.1 shows a list of all such processes. A major source of background are $\pi^0$ decays involving photons, where the $\gamma$ undergoes Compton scattering or pair production to produce other electrons. SINDRUM is a low-mass detector, so the probability of photon conversion in the wire chambers is very low. More probable are conversions in the target's aluminum support ring, in the lead degrader in front of the target, and in the liquid hydrogen and Mylar housing of the target. Another possible source of background are so-called "returning tracks"; electrons from beamline muon decays $\mu \rightarrow e\nu_\mu\bar{\nu}_e$.

<table>
<thead>
<tr>
<th>process</th>
<th>contribution to total sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^0 \rightarrow e^+e^-e^+e^-$</td>
<td>&lt; 10 events</td>
</tr>
<tr>
<td>$\pi^0 \rightarrow e^+e^-\gamma$ (compton)</td>
<td>&lt; 20 events</td>
</tr>
<tr>
<td>$\pi^0 \rightarrow e^+e^-\gamma$ (pair prod.)</td>
<td>&lt; 10 events</td>
</tr>
<tr>
<td>$\pi^0 \rightarrow \gamma\gamma$ (pair prod.)</td>
<td>&lt; 5 events</td>
</tr>
<tr>
<td>$\pi^0 \rightarrow \gamma\gamma$ (2x pair prod.)</td>
<td>&lt; 10 events</td>
</tr>
<tr>
<td>$\pi^0 \rightarrow \gamma\gamma$ (compton + pair prod.)</td>
<td>&lt; 15 events</td>
</tr>
<tr>
<td>$\pi^-p \rightarrow n\gamma$ (pair prod.)</td>
<td>&lt; 5 events</td>
</tr>
<tr>
<td>$\pi^-p \rightarrow ne^+e^-$</td>
<td>$\approx$ 2500 events</td>
</tr>
</tbody>
</table>

Table 7.1: Background processes considered and their calculated contribution to the total sample, which numbers roughly 100,000 events. Conversions are modelled in the liquid hydrogen target, in the target walls, and in chamber 1.

81
7.1 External Conversion Background

External conversion events, in which a photon from a pion decay $\pi^0 \rightarrow \gamma \gamma$ or $\pi^0 \rightarrow e^+e^-\gamma$ or from $\pi^-p \rightarrow n\gamma$ converts into an $e^+e^-$ pair or results in a Compton electron due to interaction with matter inside the detector, are a potential source of spurious $e^+e^-$ pairs. Such events occur in huge numbers in the detector’s massive parts, such as the aluminum target support ring and lead degrader clearly visible in figure 4.1. The constraints on $z$ and $r$ discussed in chapter 4 remove virtually all of the support ring and degrader conversions. A further requirement that the inner two chambers exhibit two and only two hits greatly reduces events with extra electrons (Dalitz events in which the photon converts, and $\pi^0 \rightarrow \gamma \gamma$ events in which both photons convert). This cut eliminates events with an extra electron track of momentum greater than about 2-6 MeV/c (depending on the event origin and emission angle). The minimum opening angle requirement practically eliminates the conversions from $\pi^-p \rightarrow n\gamma$, since the 129 MeV photon produces $e^+e^-$ pairs with opening angles smaller than 20°. The $\pi^0 \rightarrow \gamma \gamma$ conversions are also removed by this cut, although to a lesser extent since the photon is less energetic. We demonstrate this by simulating and then analyzing these processes to find the contributions given in the final column of table 7.1.

7.2 $\pi^0 \rightarrow e^+e^-e^+e^-$ Background

The largest contribution from the double Dalitz decay is expected from “crossed pairs”, where each lepton comes from a highly asymmetrical decay of one of the virtual photons. The remaining low energy electrons should be visible as hits in the inner chambers, so we expect that multiplicity cuts on the inner two chambers will result in clean events. In order to check this, double Dalitz events are generated according to the distribution functions given in [42] and passed through the analysis. After the requirement that
chambers 1 and 2 have only two hits each, we find that the contamination due to the double Dalitz decay is negligible.

7.3 Returning Tracks

In any pion beam, there will be muon and electron contamination emanating from the production target. The requirement that the $e^+e^-$ pair taken as a good event originates within the target eliminates the beam electrons. However, if a beam muon decays via $\mu \rightarrow \nu e$ in the hodoscope and the resulting electron passes through the target and continues back to the hodoscope on the opposite side, it mimics an $e^+e^-$ pair with an opening angle of 180 degrees. These “returning tracks” are a potential source of background. One eliminates most of them at the online trigger stage by requiring that the two hodoscope signals occur at the same time. The few remaining returning tracks, visible at $E_T P_t = m_\mu = 106$ and $\phi = 180^\circ$ in figure 4.4 (page 51), are removed by a requirement on the total $e^+e^-$ momentum; $P_{tot} > 10$ MeV/c.

7.4 Beam Buckets with more than 1 $\pi^0$

Roughly 1 in every $10^8$ beam buckets will contain two neutral pions. We therefore expect about 100 of these in our sample. This will result in only one Dalitz pair, so that background from this source is negligible.

7.5 $ne^+e^-$ Background

The theoretical ratio of the number of $ne^+e^-$ events to the number of $e^+e^-\gamma$ events is approximately 1. This ratio is reduced by the online trigger and filter, as discussed in chapter 5: the trigger cuts on maximum track curvature and therefore imposes a maximum momentum cut; since the $ne^+e^-$ events have higher $e^+e^-$ momenta, these
Chapter 7. Background Considerations

events are reduced in number. We fit for the resulting ratio of $n\pi^+\pi^-$ to $e^+e^-\gamma$

$$R = \frac{\#n\pi^+\pi^-\text{events}}{\#n\pi^+\pi^- + \#e^+e^-\gamma\text{events}}$$

including radiative corrections to both processes. The results of the fit are given in table 7.2 below, and a typical example is shown in figure 7.1.

<table>
<thead>
<tr>
<th>geometry</th>
<th>$N(n\pi^+\pi^-)$</th>
<th>$N(e^+e^-\gamma)$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>9230</td>
<td>11810</td>
<td>0.237 ± 0.007</td>
</tr>
<tr>
<td>4</td>
<td>52207</td>
<td>62087</td>
<td>0.235 ± 0.003</td>
</tr>
<tr>
<td>5</td>
<td>16495</td>
<td>18188</td>
<td>0.228 ± 0.007</td>
</tr>
<tr>
<td>6</td>
<td>16998</td>
<td>53178</td>
<td>0.218 ± 0.004</td>
</tr>
</tbody>
</table>

Table 7.2: The contribution of $\pi^-p \rightarrow n\pi^+\pi^-$ expressed as a fraction of the total sample. Other background contributions are negligible. The quantities $N$ provide a means for assessing the efficiency of the $E_tP_t$ separation cut; they are not products of the fit, and their magnitudes are of no special importance.

In order to remove as many of the $n\pi^+\pi^-$ events from the Dalitz sample, kinematical constraints must be imposed. From figure 4.4, we see that the Dalitz events inhabit a boxlike region from $107 < E_tP_t < 163$ MeV. By requiring $E_tP_t < 170$ MeV, we eliminate most of the $\pi^-p \rightarrow n\pi^+\pi^-$ events. There is still some contamination due to $n\pi^+\pi^-$ radiative events, in which one of the electrons loses energy by emission of a bremsstrahlung photon, thus populating the region below $E_{\text{total}} = 136$ MeV allowed to the "pure" $\pi^-p \rightarrow n\pi^+\pi^-$ process. These events must be simulated. The code used to simulate these events is approximate, accurate to roughly 20%. A brief discussion of the approximate treatment of the radiative corrections has been given in chapter 6.

We may then add the $n\pi^+\pi^-$ simulation to the $e^+e^-\gamma$ for any set of cuts (in particular, the cuts designed to eliminate the $n\pi^+\pi^-$ above) by

$$\text{total simulation} = \frac{(1 - R)}{N(e^+e^-\gamma)} \times (\# e^+e^-\gamma \text{ after cuts}) + \frac{(R)}{N(n\pi^+\pi^-)} \times (\# n\pi^+\pi^- \text{ after cuts})$$
Figure 7.1: The determination of the ratio of $ne^+e^-$ to $e^+e^-\gamma$. Radiative corrections to both reactions are included.

$$R = 0.2434 \pm 0.0027$$

$$\chi^2 = 63.67 \text{ for 49. dof}$$
where $R$ and $N$ are given in table 7.2. After all the cuts (those listed in chapter 4, the cut on $E_tP_t$, and the multiplicity cuts on the inner chambers) we find that approximately 2% of the total simulated data set is due to $ne^+e^-$. While this is a small contamination, it must be included, since the invariant mass distribution of the $ne^+e^-$ sample is radically different from that of the process $\pi^0 \rightarrow e^+e^-\gamma$. 
Chapter 8

Fitting Procedure

We divide the full data set into the four separate run periods (for each of which the chamber geometry was slightly different, as discussed in chapter 3) which we refer to as “geometries 2, 4, 5 and 6”. For each geometry we simulate double this number of events (including radiative corrections) as well as double the corresponding expected number of $n e^+ e^-$ events (including radiative corrections). Recall that other background contamination is negligible. The analysis as outlined in this chapter is performed for each separate geometry, and the results compared for consistency. Finally the results are combined and we quote an average value for $a$.

For completeness we list the cuts that are applied to both the simulation and the data in the final analysis:

1. $45^\circ < \phi_t < 260^\circ$, a cut which duplicates the action of the trigger and filter stages of the data acquisition hardware and eliminates photon conversion background.

2. $P_t > 20$ MeV, a duplication of SINDRUM’s transverse momentum threshold.

3. $-300$ mm $< Z(5) < 300$ mm : requiring the track to lie within the region of uniform magnetic field.

4. $r < 19$ mm, $-115$ mm $< z < -80$ mm, $0$ mm $< z + 104 + \sqrt{19^2 - r^2}$ : cuts which ensure that the event happened well inside the target.

5. $P_{tot} > 10$, which eliminates the rest of the $\mu$ decays.
6. Only 2 hits are allowed in each of the inner two chambers, a cut which eliminates any further background process involving extra electrons or photons, such as \( \pi^0 \rightarrow e^+e^-e^+e^- \), external conversion, and pair production.

7. \( E_{\text{tot}} + P_{\text{tot}} < 170 \) to separate the \( \pi^0 \rightarrow e^+e^-\gamma \) from the \( \pi^-p \rightarrow ne^+e^- \) process.

Application of these cuts results in 5837 events for geometry 2, 35873 events for geometry 4, 5957 events for geometry 5, and 26520 events for geometry 6, giving a total of 74187 \( \pi^0 \rightarrow e^+e^-\gamma \) events with a contamination due to \( ne^+e^- \) of roughly 2%.

In figures 8.1 and 8.2 we show the performance of the full simulation (including Dalitz and \( ne^+e^- \)) for various kinematical variables. The agreement with the data is excellent. We may now proceed to extract \( a \).

From chapter 1, the most obvious method of extracting \( a \) is to fit for the parameter in the invariant mass spectrum \( x \), since then we may extract \( a \) directly. Also, as discussed in chapter 6, the radiative corrections are expected to have a small effect. If we denote the distribution of the data by \( F \), and the Monte Carlo by \( f \), then we can write the fitting procedure schematically by

\[
F(x) = (1 + 2ax) \times f(x)
\]

We denote the \( a \) obtained by fitting in \( x \) by \( a_x \).

As a double check, it is useful to consider the analysis of the distribution in \( \phi \), where, as seen in in chapter 6, we expect the radiative corrections to have an appreciable effect. We can extract a slope parameter \( b \):

\[
F(\phi) = (1 + 2b\phi) \times f(\phi)
\]

Now \( b \) and \( a_x \) are not the same, although \( b \) is related to \( a_x \) in a measurable manner. We obtain the relationship between \( a_x \) and \( b \) by creating an artificial “data” sample; we set
Figure 8.1: The performance of the full simulation for various kinematical variables. All radiative corrections and backgrounds are included. The points are the data, the histogram, the simulation.
Figure 8.2: The performance of the full simulation for various kinematical variables. All radiative corrections and backgrounds are included. The points are the data, the histogram, the simulation.
a to some non-zero value in the Kroll-Wada matrix element, run the Monte Carlo, and generate a $\phi$ spectrum. We then fit for $b$. Repeating for a number of $a$ values allows us to map out the relationship between the two values. The relationship is linear over the range of $a$ we consider (from -0.1 to 0.1):

$$a = 1.65(1)b + 0.0000(3)$$

(8.1)

We denote the $a$ value obtained by fitting in $\phi$ and applying the relation above by $a_\phi$. The bracketed numbers in equation 8.1 indicate the statistical error on the last digit. Both the statistical and systematic errors on $b$ must be transformed to obtain the correct errors on $a^\phi$.

We employ two different fitting procedures for the parameters $a_x$ and $a_\phi$, which are now outlined. The two procedures stem from different philosophies of statistical analysis, and use different sets of mathematical tools to arrive at the final result. For each fitting method, we fit with and without the radiative corrections, in order to assess their contribution to the final result. If the analysis is properly done, we expect the final answers (16 of them, in $a_x$ and $a_\phi$ from each of 2 fitting methods and 4 separate data sets) to be consistent with each other.

8.1 Maximum Likelihood Technique – the Bayesian Approach

Bayesian analysis is performed by combining prior information about the parameters of the model (denoted by the vector $\theta$) with the information from the data sample into the “posterior distribution”. Model parameters are then estimated by maximizing the posterior with respect to the parameters $\theta$.

The prior information about the parameters, denoted $\pi(\theta)$, incorporates all previously known facts about the parameters; it may consist of previous results, subjective
bias, or any combination thereof. When no prior information is available and/or subjective bias is not appropriate, one uses a "noninformative" or "uniform" prior, which assigns equal probability to all values of $\theta$.

The information encapsulated in the data enters via the "likelihood function" $f(\theta|x)$, which expresses the probability of observing the data, given certain values of the model parameters. One can turn this around; this probability may be identified as being the likelihood of each of the parameters being the true value, given the data. It must be noted that this reversal-and-identification is based solely on intuition and not on any formal mathematical principles; however, it seems an eminently reasonable assumption to make and leads to very useful results.

Once a prior has been chosen and the likelihood function calculated, the posterior distribution is given by

$$\pi(\theta|x) = \text{normalizing constant} \times \pi(\theta)f(\theta|x)$$  \hspace{1cm} (8.2)

While the prior incorporates the beliefs about $\theta$ before the sample is observed, the posterior reflects the updated beliefs about the parameters after the experiment has been done. Bayesian parameter estimation defines the most likely parameter values as those which maximize the posterior distribution. Note that since derivatives will be taken, the normalizing constant drops out and is not important. Note also that, if one assumes a uniform prior, Bayesian maximum likelihood estimation consists of maximizing the likelihood function, in other words, Bayesian and Frequentist maximum likelihood parameter estimation are mathematically equivalent.

Credible regions, or confidence intervals, on the parameters describe the uncertainty of the result of the estimation. A 100(1 - $\alpha$)% credible region for $\theta$ is defined by the limits $\theta_i$, $\theta_f$, where

$$1 - \alpha \leq \int_{\theta_i}^{\theta_f} \pi(\theta|x) \, d\theta$$  \hspace{1cm} (8.3)
Chapter 8. Fitting Procedure

Let us now move on to specifics. Using maximum likelihood considerations, it is fairly straightforward to derive a simple expression for $a$ in terms of the first- and second-order moments of the spectrum to be fit. We present here an extended version of the argument found in reference [19]. We begin by calculating the likelihood function.

The likelihood $L_k$ of observing $n$ independent events in a category $k$ is Poisson distributed:

$$L_k = e^{-p_k} \frac{p_k^{n_k}}{n_k!}$$

The total likelihood of observing a data sample with $M$ categories is then

$$L = \prod_{k=1}^{M} e^{-p_k} \frac{p_k^{n_k}}{n_k!}$$

Hence

$$\ln L = - \sum_k p_k + \sum_k n_k \ln p_k - \sum_k \ln(n_k!)$$

Defining the category $k$ to be the interval $x_k < x < x_k + \delta$ in the experimental spectrum $f(x)$, we may write the probability $p_k$ of finding an event in the category $k$ as

$$p_k = \int_{x_k}^{x_k+\delta} f(x) \, dx \approx f(x_k) \delta$$

for $\delta$ small enough. Substituting, we find

$$\ln L = - \sum_k \left( \int_{x_k}^{x_k+\delta} f(x) \, dx \right) + \sum_k n_k \ln \{f(x_k) \delta\} - \sum_k \ln(n_k!)$$

Now, in terms of the parameter $a$, $f(x) = f_0(x)(1 + 2ax)$. Thus

$$\ln L = - \int_0^{x_{\text{max}}} f_0(x)(1 + 2ax) \, dx + \sum_k n_k \ln \{f_0(x_k)(1 + 2ax_k) \delta\} - \sum_k \ln(n_k!)$$

We now have the likelihood function for the data in terms of the parameter $a$. We forget the results of all previous experiments and choose a uniform prior. Our posterior is
then equal to the likelihood function, and the most likely value for $a$ may be found by maximizing with respect to $a$:

$$0 = \frac{\partial \ln L}{\partial a}$$

$$= -2 \int_0^{x_{\text{max}}} x f_0(x) \, dx + 2 \sum_k \frac{n_k x_k}{1 + 2ax_k}$$

Hence

$$\int_0^{x_{\text{max}}} x f_0(x) \, dx = \sum_k \frac{n_k x_k}{1 + 2ax_k}$$

Since $a$ is small, we approximate

$$\frac{1}{1 + 2ax} \approx 1 - 2ax$$

giving

$$\int_0^{x_{\text{max}}} x f_0(x) \, dx \approx \sum_k n_k x_k (1 - 2ax_k)$$

$$= \sum_k n_k x_k - 2a \sum_k n_k x_k^2$$

Observing that $\int f_0(x) \, dx = N_0$, the number of Monte Carlo events in the spectrum, we may rewrite this in terms of moments:

$$N_0 \bar{x}_0 = N \bar{x} - 2aN x^2$$

$$\left( \frac{N_0}{N} \right) \bar{x}_0 = \bar{x} - 2a \bar{x}^2 \quad (8.5)$$

Now, since this experiment was performed without any absolute normalization, we do not know $\frac{N_0}{N}$; that is, we have no way of calibrating $N_0$, the number of Monte Carlo events generated, to $N$, the final number of Dalitz decays seen. All is not lost, however: the number of events seen experimentally must be

$$N \equiv \int f_0(x)(1 + 2ax) \, dx$$

$$= \int f_0(x) \, dx + 2a \int x f_0(x) \, dx$$

$$= N_0 + 2a N_0 \bar{x}_0$$

$$= N_0(1 + 2a \bar{x}_0) \quad (8.6)$$
which quickly gives a simple expression for $\frac{N}{N_0}$. Substituting into equation 8.5 gives
\[
\frac{1}{1 + 2a\bar{x}_0} \bar{x}_0 = \bar{x} - 2a\bar{x}^2
\]
\[
\Rightarrow \bar{x}_0 = (1 + 2a\bar{x}_0)(\bar{x} - 2a\bar{x}^2)
\]
\[
\approx \bar{x} - 2a\bar{x}^2 + 2a\bar{x}_0\bar{x}^2
\]
\[
\Rightarrow \bar{x}_0 - \bar{x} = 2a(x^2 - \bar{x}_0\bar{x})
\]
and hence
\[
a = \frac{1}{2} \left( \frac{\bar{x}_0 - \bar{x}}{x^2 - \bar{x}_0\bar{x}} \right) \quad (8.7)
\]
\[
\frac{N}{N_0} = 1 + \bar{x}_0 \left( \frac{\bar{x}_0 - \bar{x}}{x^2 - \bar{x}_0\bar{x}} \right) \quad (8.8)
\]
to first order.

We may now establish the classical "1-σ" ($\alpha = 0.32$) credible region for $a$. In order to do this, we need the likelihood function, which is the exponential of equation 8.4. In order to simplify the calculations, we observe that equation 8.4 may be written as
\[
\ln L = -\int_0^{x_{\text{max}}} f_0(x)(1 + 2ax) \, dx + \sum_k n_k \ln \{f_0(x_k)\} + \sum_k n_k \ln \{(1 + 2ax_k)\}
\]
\[
+ \sum_k n_k \ln \{\delta\} - \sum_k \ln(n_k!)
\]
\[
= -\sum_k f_0(x_k)(1 + 2ax_k) + \sum_k n_k \ln \{(1 + 2ax_k)\} + \text{constant} \quad (8.9)
\]
where we have approximated the integral by a discrete sum, and have separated the terms that depend on $a$ from those that do not. When we exponentiate both sides, the constant term will simply adjust the amplitude of the resulting curve. We shall ignore it, choosing instead to define the amplitude by the requirement that
\[
\int L \, da = 1 \quad (8.10)
\]
The likelihood function, as given by equation 8.9, is simple to calculate numerically. The likelihood function is sharply peaked, and may be approximated by a normal
distribution in the parameter $a$. This is convenient, since, for a normal distribution, equation 8.3 leads us to

$$0.683 = \int_{a_0-\sigma}^{a_0+\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{(a-a_0)}{2\sigma^2} \right\} da$$

In other words, the standard deviation of the normal distribution defines the 68.3% credible region of the parameter. We may therefore easily find the standard deviation by taking the second derivative of the likelihood function:

$$0 = \frac{d^2}{da^2} \left[ \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{(a-a_0)}{2\sigma^2} \right\} \right]_\sigma \approx \frac{d^2L}{da^2}_\sigma$$

Note that we must be careful about the normalization: the quantity $\frac{N}{N_0}$ is implicitly included as a parameter in the above equations. This quantity is not of especial interest; in the above, we have set the normalizations to their most likely value (according to equation 8.8) and have then calculated the subsequent error on $a$. We note further that the two parameters are correlated, and the correlation is given by equation 8.6.

One may simply substitute $x$ by $\phi$ in the above derivation to extract the maximum likelihood parameter estimation for the $\phi$ case. The two parameters $a_x$ and $a_\phi$ are not the same, as pointed out at the beginning of this chapter, but we may convert from one to the other using equation 8.1. We apply the analysis to both the corrected and uncorrected Monte Carlo spectra. Contamination from $\pi^- p \to n e^+ e^-$ is included. Figure 8.3 and table 8.1 summarize the resulting $a_\phi$ and $a_x$ for the various geometries. We do not show the results for the normalizations since they are of no real interest; all geometries and fitting procedures give consistent results. The four separate geometries give consistent results. Fitting in $x$ and $\phi$ also leads to consistent results. Note that the radiative corrections have a much larger effect in $\phi$ than in $x$, and that we do not obtain consistent results until we include them!
Figure 8.3: Graphical summary of the results of the maximum likelihood fitting for $a_x$ (white boxes) and $a_\phi$ (dark boxes) for each of the four geometries. On the left are the results without radiative corrections, on the right the results including the radiative corrections.

<table>
<thead>
<tr>
<th>geometry</th>
<th>$a_x$</th>
<th>$a_\phi$</th>
<th>$a_x$</th>
<th>$a_\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.140 ± 0.026</td>
<td>0.166 ± 0.018</td>
<td>0.023 ± 0.026</td>
<td>-0.003 ± 0.018</td>
</tr>
<tr>
<td>4</td>
<td>0.031 ± 0.010</td>
<td>0.121 ± 0.007</td>
<td>0.008 ± 0.010</td>
<td>0.002 ± 0.007</td>
</tr>
<tr>
<td>5</td>
<td>0.031 ± 0.025</td>
<td>0.141 ± 0.018</td>
<td>0.076 ± 0.025</td>
<td>0.098 ± 0.018</td>
</tr>
<tr>
<td>6</td>
<td>0.027 ± 0.012</td>
<td>0.123 ± 0.008</td>
<td>0.008 ± 0.012</td>
<td>0.020 ± 0.008</td>
</tr>
</tbody>
</table>

Table 8.1: Results for $a_x$ and $a_\phi$ from maximum likelihood fitting, for each of the various geometries, with and without radiative corrections. Errors are statistical only.
8.2 $\chi^2$ Minimization — the Frequentist Approach

The starting point for $\chi^2$ minimization is once again the likelihood function. If we assume that each measured data point $n_k = f(x_k)$ has an associated error $\sigma_k = \sqrt{n_k}$, and we further assume that this measurement error is independently random and is distributed around $n_k$ in a Gaussian manner, then we may write the likelihood of observing $n_k$ events in category $k$ in terms of the parameter $a$ as

$$L_k = \exp \left\{ -\frac{1}{2} \frac{f(x_k) - \frac{N}{N_0} f_0(x_k)(1 + 2ax_k)}{\sigma_k} \right\}^2$$

where $\delta$ is the width of the category. The total likelihood over $M$ categories is then

$$L = \prod_{k=1}^{M} \exp \left\{ -\frac{1}{2} \frac{f(x_k) - \frac{N}{N_0} f_0(x_k)(1 + 2ax_k)}{\sigma_k} \right\}^2$$

As before, we choose the most likely parameters $a$ and $\frac{N}{N_0}$ as those which maximize this likelihood function. Maximizing the likelihood function is equivalent to minimizing the negative of the log likelihood function:

$$-\ln L = \sum_{k=1}^{M} \left\{ -\frac{1}{2} \frac{f(x_k) - \frac{N}{N_0} f_0(x_k)(1 + 2ax_k)}{\sigma_k} \right\}^2 - M \ln \delta$$

We may drop the constant factor of $1/2$ and $M \ln \delta$; they will not affect the minimization. We then minimize the so-called "$\chi^2$" function

$$\chi^2 = \sum_{k=1}^{M} \left\{ \frac{f(x_k) - \frac{N}{N_0} f_0(x_k)(1 + 2ax_k)}{\sigma_k} \right\}^2$$

---

1We assume that the number of entries $n_k$ in each category is governed by Poisson statistics. The standard deviation $\sigma^2$ of a Poisson distribution is equal to the mean $\mu$; in the limiting case of large data samples, the number of entries observed $n_k$ approaches $\mu$ and hence $\sigma = \sqrt{n_k}$. Note that for $n_k$ less than about 20 the Poisson distribution becomes noticeably skewed, and this approximation is no longer valid. One must then calculate $\mu$, given that $n_k$ is the most likely value of the Poisson distribution. Since we have only 2 or 3 bins (at the extreme ends of the distributions) out of a total of 100 that have $n_k \leq 20$, and so the approximation was deemed adequate.
Chapter 8. Fitting Procedure

We use a standard iterative algorithm which chooses different values of the parameters $a$ and $\frac{N}{N_0}$, calculates the chi-squared, and uses this information to steer itself towards the minimum $\chi^2$ and the associated parameter values.

In order to assess the errors on the parameters thus found, the standard practise is to perturb the values of the parameters slightly and observe the change in $\Delta \chi^2$ defines some 2-dimensional confidence region in the parameter space. However, we are not interested in the confidence region of the 2 parameters jointly, but in the confidence region of each of the parameters by themselves. To evaluate this, we hold each of the parameters fixed, in turn, and find the amount by which we must vary the other to induce a change in $\chi^2$ of 1. The magic number 1 appears due to the fact that a $\chi^2$ distribution with one degree of freedom (the single parameter) is the square of a normally distributed quantity: $\Delta \chi^2 < 1$ happens 68.3% of the time (the 1-$\sigma$ level), $\Delta \chi^2 < 4$ happens 95.4% of the time (the 2-$\sigma$ level), etc. All of this $\chi^2$ statistical analysis is standard fare, and we make use of the PLOTDATA [48] analysis package from TRIUMF, which which allows basic interactive fitting and plotting. Many advanced statistical analysis packages exist, capable of handling scores of parameters and complicated fits (most notably MINUIT); however, since ours is a fairly simple problem involving only two parameters and no pathological functions, the basic package is quite sufficient.

As in the case of maximum likelihood fitting, the data is divided up into the 4 geometries. Each data set is fit in $x$ and $\phi$ (converting via equation 8.1), with and without radiative corrections. The fit results are summarized in table 8.2 and in figure 8.4. We note that the results in $x$ and $\phi$ are consistent, and that, once again, the radiative corrections are needed for this consistency.
Figure 8.4: Graphical summary of the results of the $\chi^2$ fitting for $a_x$ (white boxes) and $a_\phi$ (dark boxes) for each of the four geometries. On the left are the results without radiative corrections, on the left the results including the radiative corrections.

Table 8.2: Results from $\chi^2$ minimization, for each of the various geometries, with and without radiative corrections. Errors are statistical only. The $\chi^2$ per degree of freedom varies from 1.2 to 1.7 for the various fits.
8.3 Summary of Fitting Results

The two tables 8.1 and 8.2 are summarized in table 8.3 and figure 8.5 below. We obtain consistent results between all the data sets and the fits in $x$ and in $\phi$. The statistical error of the $\phi$ fit in $\chi^2$ is larger; this is not surprising since we expect a somewhat washed-out effect in $\phi$ due to the fact that $x$ and $\phi$ are not uniquely correlated.

<table>
<thead>
<tr>
<th>geometry</th>
<th>$\chi^2$ minimization</th>
<th>maximum likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$a_x = -0.067 \pm 0.034$</td>
<td>$a_x = 0.026 \pm 0.036$</td>
</tr>
<tr>
<td>4</td>
<td>$a_x = -0.005 \pm 0.018$</td>
<td>$a_x = 0.007 \pm 0.017$</td>
</tr>
<tr>
<td>5</td>
<td>$a_x = 0.003 \pm 0.036$</td>
<td>$a_x = 0.076 \pm 0.029$</td>
</tr>
<tr>
<td>6</td>
<td>$a_x = -0.012 \pm 0.019$</td>
<td>$a_x = 0.009 \pm 0.019$</td>
</tr>
<tr>
<td>all</td>
<td>$a_x = -0.014 \pm 0.012$</td>
<td>$a_x = 0.019 \pm 0.011$</td>
</tr>
</tbody>
</table>

Table 8.3: Summary of the fit results for each geometry, fitting method, and spectrum. Radiative corrections and $\pi^- p \rightarrow n e^+ e^-$ background are included. Individual errors are statistical only. The summary value given is the error-weighted average, and its error takes into account the statistical error only.

We also see that the two fitting procedures also give consistent central values. The error obtained using maximum likelihood is however much smaller than the corresponding statistical error from the $\chi^2$ fit, an effect especially pronounced in the $\phi$ fits. Since we have not been able to find any flaw in either the maximum likelihood or $\chi^2$ analysis, we have adopted a more robust method by which to estimate the statistical errors. We generate several hundred "new" data spectra by allowing the data spectrum in $x$ and $\phi$ to vary on a bin-to-bin basis, within its 1-$\sigma$ error bars. We fit this "new" data, by both methods, using the simulation, and tabulate the resulting values for $a$ and the normalization. The distributions in $a$ and the normalization are Gaussian, and we take their central value and their standard deviations as the true $a$ and statistical error of
Figure 8.5: Summary of results for all four geometries, in $x$ and $\phi$. The four points to the left of the double line represent the results for geometries 2, 4, 5, and 6, respectively; the point to the right indicates the error-weighted average. Its small error bar shows the statistical error, and the larger one shows the combined statistical error and standard error of the fluctuations. The results in $x$ and $\phi$ are averaged to obtain the final results.
the measurement. Our results are now independent of fitting method and spectrum, as shown in table 8.3. The central values are essentially unchanged. We note that this error estimation produces results midway between those obtained from the maximum likelihood and $\chi^2$ analyses.

Combining the results of all the geometries, taking an error-weighted average over the $x$ and $\phi$ spectra, we obtain the 4 results noted in the last line of table 8.3. We combine the maximum likelihood result with the $\chi^2$ result by simple averaging, and we take the total statistical error to be the $\chi^2$ error, since the statistical errors obtained by all methods are similar. The fluctuation error is taken to be the average of the fluctuation in the $\chi^2$ and maximum likelihood methods. We obtain:

$$a_x = 0.003 \pm 0.011 \pm 0.028$$  \hspace{1cm} (8.12)

$$a_\phi = 0.027 \pm 0.013 \pm 0.054$$  \hspace{1cm} (8.13)

We turn to an evaluation of the systematic errors in the next chapter.
Chapter 9

Evaluation of Systematic Errors

Systematic errors are due to the incorrect or incomplete knowledge of the behaviour of the detector. Since both the transverse momentum $P_t$ and the transverse opening angle $\phi$, enter directly into the calculation of the invariant mass spectrum $x$ and the opening angle distribution $\phi$, anything which affects these quantities will have a direct impact on the measurement of the form factor slope. The systematic errors may be either time dependent (arising from the variation in the detector setup from run to run) or time independent (due to problems with the analysis or the simulation). The scatter in the results obtained in the last chapter (shown in figure 8.3 and indicated in the results 8.12,8.13) gives a minimum value for the combined run-dependent systematic errors. We double-check the magnitude of these errors by varying the quantities

1. stop distribution
2. target location
3. chamber locations
4. magnetic field
5. analysis cuts

and noting the effect on the result. The results of our systematic error analysis are tabulated in table 9.1.
Table 9.1: Systematic errors from various sources. The table is divided into time-dependent errors (top) and time-independent errors (bottom).

### 9.1 Time Dependent Systematic Errors

#### 9.1.1 Stop Distribution

The deeper inside the target an electron is generated, the more material (and distance) it must traverse to reach the hodoscope and trigger the electronics. Those events which originate at small target radii, therefore, tend to have higher momenta than those which occur at the outer edges of the target. Since the form factor slope is directly dependent on the momentum distribution of the electrons, it is vital that the stop distribution be correctly modelled. Since we know the final stop distribution, we can adjust the simulation to match the data as closely as possible using the scheme of weights outlined in chapter 5. By adjusting these weights to the point where the stop distributions become statistically different, we conclude that the error in $a$ due to improper knowledge of the experimental stop distribution is as tabulated in 9.1; we see no significant systematic error.
9.1.2 Target Location

The experimental stop distribution is measured with respect to the centre of SINDRUM. The position of the target must also be fixed relative to this point. In other words, specifying the stop distribution does not fix the target location and hence the stop distribution may be moved relative to the ends of the target. If, for instance, the target is shifted slightly downstream, those events generated at the upstream end will traverse less hydrogen, and have a correspondingly higher momentum than those events coming from the back of the target. Minimum-ionizing electrons lose approximately 300 keV for every centimetre of liquid hydrogen traversed; a 0.5 cm shift in the target, therefore, can induce a 150 keV shift in the mean track energy. If we arrange the cuts on the target vertex and the longitudinal emission angle in such a way as to remove most of the events coming from the very tip of the target, we may minimize this systematic error. By moving the target by 2 mm in the simulation and analyzing the result, we obtain the systematic error quoted in the table. It is small.

9.1.3 Chamber Geometries

As discussed in chapters 4 and 5, SINDRUM's wire chambers were calibrated for the run-dependent $x, y, z$ offsets and rotations relative to chamber 5, as well as for the run-independent anode print gaps and twists. The offsets and chamber rotations represent a much larger effect than the anode print gaps and twists. These calibrations were determined using cosmic rays. The rotations are accurate to roughly $1 \cdot 10^{-4}$ mrad, and the $x, y, z$ offsets to 0.5 mm. The largest rotation and $x, y$ offset was found for chamber 1; 40 mrad 0.9 mm, respectively. The largest $z$ offset (1.8 mm) was measured for chamber 3. In comparison, the largest twist was measured for chamber 3 (0.19 mm), and the anode print gaps were determined to vary from $< 0.1$ mm (chamber 1).
to 0.5 mm (chamber 4). We redo the track fit with these geometry calibrations set to their limiting cases for the run periods for which the calibrations are a maximum (the "worst case"). Extracting the variation of $a$ with respect to this change in each of the calibrations leads to the results tabulated in 9.1. The uncertainty in the relative chamber positioning is the largest source of systematic error.

9.1.4 Magnetic Field

The value of the magnetic field enters directly into the calculation of the electron momentum during the track fitting. We can calibrate the magnetic field using either the $ne^+e^-$ data or the $\pi^0 \rightarrow e^+e^-\gamma$ data. We note that since the magnet was turned off and on each time that the spectrometer was taken apart, we might expect 4 different calibration values.

Calibration on the $ne^+e^-$ peak

We choose an energy variable in which the $ne^+e^-$ data is as sharply peaked as possible. The appropriate kinematical variable, from the considerations outlined in chapter 3, is the total energy of the final state

$$E_{tot} = E_+ + E_- + T_n$$

a quantity which was also useful in checking the accuracy of the $ne^+e^-$ radiative corrections. We perform a one-parameter fit using $\chi^2$-minimization to extract a momentum calibration. The result of one of the fits is shown in figure 9.1. The results for the 4 different run periods are tabulated in table 9.2. The error quoted is statistical only.
Figure 9.1: The result of the calibration on the $n\bar{e}^+e^-$ peak for geometry 4. The net change to the magnetic field is about 0.5%. On the left, before the calibration, on the right, after.

Figure 9.2: The result of the calibration on the Dalitz box for geometry 4. The net change to the magnetic field is about 0.5%. On the left, before the calibration, on the right, after.
Calibration on the $\pi^0 \rightarrow e^+e^-\gamma$ peak

We may also calibrate using the $\pi^0 \rightarrow e^+e^-\gamma$ data, a procedure which may be preferable since we will calibrate the energy range in which we are interested, thus obviating the need to worry about a possible energy dependence of the calibration.

We perform a one-parameter $\chi^2$-minimization on the quantity $E_tP_t$, discussed in chapter 3. The $\pi^0 \rightarrow e^+e^-\gamma$ process is restricted to a box with very sharp edges, and it is the location of these edges which determines the fit. The fit is, however, sensitive to changes in the $n^e_+e^-$ contamination and to addition of the radiative corrections, so that a comparison with the $n^e_+e^-$ calibration results is nice to have. The results are shown in figure 9.2 and tabulated in table 9.2. We obtain consistent results; this

<table>
<thead>
<tr>
<th>geometry</th>
<th>$e^+e^-\gamma$ calibration</th>
<th>$n^e_+e^-$ calibration</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.0020 ± 0.0010</td>
<td>1.0012 ± 0.0010</td>
</tr>
<tr>
<td>4</td>
<td>1.0024 ± 0.0005</td>
<td>1.0028 ± 0.0005</td>
</tr>
<tr>
<td>5</td>
<td>1.0032 ± 0.0010</td>
<td>1.0012 ± 0.0010</td>
</tr>
<tr>
<td>6</td>
<td>1.0036 ± 0.0007</td>
<td>1.0068 ± 0.0007</td>
</tr>
</tbody>
</table>

Table 9.2: Results of the magnetic field calibrations for the various runs. The magnet was turned off and on between the various runs.

verifies our initial guess that the calibration is independent of energy, and hence can be attributed to a simple scale factor on the magnetic field. Notice that the changes to the field are small; on the order of 0.5%. This is entirely consistent with the accuracy of the magnet current monitor installed during the experimental run.

Armed with these magnetic field scaling factors, we return to the Monte Carlo simulation and reset the field for each run period. This is a necessary step since the detector acceptance is a function of magnetic field. In order to obtain the resulting error in $\alpha$, we run through the analysis once with the magnetic field set to its limit.
We obtain the result shown in table 9.1.

A further source of error is the nonuniformity of the magnetic field. An accurate field map was not made for SINDRUM I, although one existed for the previous incarnation of the detector [31]. From this map, we estimate that within the central region of the chambers (within a cone defined by excluding the outer 10 cm of chamber 5) the magnetic field is uniform to better than 1%. We may verify this by repeating the magnetic field calibration for different fiducial volumes of the detector; we see a variation consistent with the above assertion. However, the statistics are not sufficient for this to be an accurate method of mapping the field. We apply a fiducial cut and calibrate on the interior region.

9.1.5 Analysis Cuts

In order to assess the sensitivity of $a$ to the values of the final cuts, we need to know the detector resolution of these quantities. These are simple to obtain; we simply record the exact value of the kinematical variable in question before modelling the detector response, and then plot the difference between the exact "remembered" value and the reconstructed value. This is of course the resolution of the simulation, not of the actual detector, but provides a useful estimate. SINDRUM's momentum (and energy) resolution is approximately 5% at 100 MeV. The angular resolution is approximately 3.5°.

The quantity $E_tP_t$ controls the amount of $ne^+e^-$ contamination in the final sample; a 5% resolution, together with the estimation that the radiative tail of the $ne^+e^-$ process is understood to no better than 15% leads us to the systematic error due to $ne^+e^-$ contamination stated in table 9.1.

Because the trigger is not understood well enough, we apply a final cut in transverse opening angle $\phi_t$, more stringent than the one the trigger makes. Thus the final result
is dominated by the error associated with this cut rather than by the error due to the poor knowledge of the action of the trigger. By varying the opening angle cut by the stated resolution and finding the resulting change in the form factor slope, we arrive at the conclusion that the systematic error due to this cut is negligible.

The lower transverse momentum threshold is a function of the hodoscope radius and the vertex position. In order to minimize systematic error due to uncertainty in these quantities we apply an explicit requirement of $P_t > 20$ MeV/c. Varying this quantity by 10% results in no appreciable systematic error.

9.2 Time Independent Systematic Errors

Run-independent systematic errors will increase the overall error on the final result quoted in 8.12,8.13 and shown in figure 8.5. We elaborate briefly on their evaluation.

1. *Chamber construction*: The chambers exhibit twists and wire spacing irregularities, which are constant from run to run. These are small in comparison to the aforementioned offsets and rotations, and hence they are not modelled but included in the track reconstruction of the data only. Eliminating these corrections gives negligible systematic effect.

2. *Hardware trigger*: The hardware trigger removes a large number of events. It applies an opening angle cut in the transverse plane, as well as a non-linear and charge-asymmetric momentum cut. The $\pi^-p \rightarrow ne^+e^-$ data, which have a larger electron momentum, are reduced in number by a factor of roughly 3, and therefore provide a good test of our trigger simulation program. We run this simulation on the $ne^+e^-$ Monte Carlo; if subsequent results match the real data well, we may be confident that the trigger is properly understood. From figures 9.3 and 9.4, we see that the action of the trigger is understood, at least for the $ne^+e^-$ of geometry...
4. Similar results are achieved for geometries 2 and 5. We are confident that the trigger is understood, and that the systematic error induced by it is small.

3. **Contamination from ne\(^+\)e\(^-\)**: The radiative ne\(^+\)e\(^-\) events contaminate the Dalitz sample. These events must be well understood. From figure 9.4h) we see that the long radiative tail is accurate to within roughly 15%. By increasing the number of ne\(^+\)e\(^-\) tail events by 15% and reanalyzing the data, we obtain the systematic error shown in the table.

4. **Radiative corrections to π\(^0\) → e\(^+\)e\(^-\)γ**: The results of the numerical integration of the matrix elements of the π\(^0\) → e\(^+\)e\(^-\)γ radiative corrections have been checked against published semi-analytical values [22], and agree to within 5%. Furthermore, the radiative tail in \(E_tP_t\) is very well modelled. The systematic error induced by the uncertainty in the π\(^0\) → e\(^+\)e\(^-\)γ radiative corrections must be small.

9.3 **Other Systematic Errors**

Other sources of systematic errors have been investigated and found to be negligible. These include

- **Possible error due to the multiplicity cuts on the inner chambers.** The possible source of error depends on how the simulation decides on wire hits. The energy loss as calculated by GEANT is deposited directly into the cathode and anode strips of the detector. This discrete deposited energy is converted into a Gaussian response function and then transformed into a cluster of cathode signals, which are converted into wire hits by the analysis package during the track reconstruction. We have checked that the real data and simulation produce similar
Figure 9.3: The performance of the $e^+e^-$ simulation (with radiative corrections) for some areas of phase space, for geometry 4. The points are the data; the histogram, the simulation. Since the agreement between the data and the Monte Carlo is good, we are confident that we understand the action of the trigger.
Figure 9.4: The performance of the $n\epsilon^+\epsilon^-$ simulation (with radiative corrections) for some areas of phase space, for geometry 4. The points are the data; the histogram, the simulation. Since the agreement between the data and the Monte Carlo is good, we are confident that we understand the action of the trigger.
cluster sizes, and that hence the analysis thresholds for deciding true wire hits are correct. We are confident that we do not miss real hits.

- **Possible errors due to misalignment of the hodoscope.** Since the outermost chamber (number 5) was mounted onto the inside of the hodoscope, it is highly unlikely that the hodoscope was moved or rotated in any way relative to it (calibrations were done relative to chamber 5). Furthermore, since the hodoscope was used only as a trigger condition, not as part of the track reconstruction, fine-tuning its location is unnecessary.

- **Errors due to binning.** When fitting for \( a \) in \( x \) and \( \phi \) we bin the data into 100 and 90 bins, respectively. Since both the resolution in \( x \) and in \( \phi \) is much smaller than the size of one bin, we expect no systematic effect from "edge" events being misassigned to neighbouring bins. We have checked that increasing the bin size by a factor of two does not affect the result of the fit.

- **Possible dependence on detector \( \phi \) quadrant.** We divide the data up into four samples according to the quadrant of emission of the electron to study the effect on \( a \). We see no systematic effect.

### 9.4 Summary

Table 9.1 summarizes the contributions to the total systematic error from various sources. Note that the standard deviation of the scatter (indicated in the results 8.12, 8.13) is in agreement with the magnitude of the time-dependent errors. We see that the error on \( a_x \) is dominated by the uncertainty in the value of the magnetic field; this is not surprising since \( x \) is a linear function of the momentum. On the other hand, since the opening angle is only indirectly dependent on the magnetic field (the acceptance
in $\phi_t$ is weakly dependent on the magnetic field since the trigger also cuts on track curvature), $a_\phi$ is relatively free of this systematic error. However $a_\phi$ is more sensitive to the calibrations of the chamber rotations and offsets than is $a_x$. Our two results are thus

$$a_x = 0.003 \pm 0.011 \pm 0.022$$

$$a_\phi = 0.027 \pm 0.013 \pm 0.047$$

Taking the error weighted average of the two central values and quoting the smaller errors, we obtain our final result

$$a = 0.02 \pm 0.01 \pm 0.02$$
Chapter 10

Summary and Conclusions

The basic difficulty with an experiment attempting to measure the $\pi^0$ form factor via the Dalitz decay is the limited range of momentum transfer available to probe the pion structure. The virtual photon used to study the pion structure is limited to energies below the pion mass. It is difficult to resolve the quark structure of the meson at such low energies, and the results of previous experiments (table 2.2) are, accordingly, inconclusive. We point out that the result we obtain is also consistent with a structureless $\pi^0$, although it represents a substantial improvement in the understanding of systematic errors. Could the error bars be reduced in a future experiment?

Data for the form factor measurement were taken over only three days. It would be a very simple matter to reduce the statistical errors by a factor of 3 by running for a month; however, the reduction of the systematic errors is not so simple. In our experiment, the availability of a large number of events, spread over multiple runs, combined with our fits for $a$ in two spectra using two different methods, provides many double checks and is essential to our understanding of the systematic errors, the largest of which were the magnetic field, the chamber calibrations, and the uncertainty in the $\pi^- p \rightarrow ne^+e^-$ radiative tail.

An accurate field map might have been helpful in reducing the error associated with the field's non-uniformity; however, since the dependence of $a$ on the value of the field is linear, and since we use the same data set for the calibration and for the extraction of $a$, the use of the average value of the field should not induce a large systematic error.
It is the change in acceptance due to the nonuniformity of the field which causes the systematic error. A field map would have reduced the need for fiducial chamber cuts and would therefore have increased the number of events.

The uncertainty in the relative positions of the chambers proves to be the largest source of error in this measurement. The measurement of $a$ using the $\phi$ distribution is especially sensitive to the relative chamber rotations. The reconstructed $z$ momentum is quite sensitive to the $z$ positions of the chambers, and has a considerable influence on the determination of $a_z$. The chamber geometries were calibrated using cosmic rays, and it is difficult to see how these calibrations could have been improved.

The uncertainty in the $\pi^- p \rightarrow ne^+ e^-$ radiative tail is a systematic error that could be reduced in future experiments. Because the trigger cut so heavily on the $ne^+ e^-$ data, a detailed understanding of this process is difficult to achieve. The acceptance of the detector for the $ne^+ e^-$ data is very sensitive to the magnetic field and to the chamber geometries. Combined with a slight difference in trigger acceptance between electrons and positrons of high momentum, this leads to the observed charge asymmetry in the $\pi^- p \rightarrow ne^+ e^-$ data (see figures 9.3 and 9.4). While this asymmetry is reasonably well understood for geometries 2, and 4 and 5, for geometry 6 it is much more pronounced and is not well simulated. It is not surprising, then, that the radiative tail of this process is understood to no better than 20%. A more careful choice of trigger conditions, plus additional running with a reversed magnetic field would have aided considerably in an understanding of this important background process.

It would be difficult, but not impossible, to improve the systematic and statistical errors by 30 or 50%. Whether this is necessary, however, is not clear, since, by time invariance, we expect the form factor for pion decay $\pi^0 \rightarrow \gamma \gamma^*$ to be no different from the form factor for pion production $\gamma \gamma^* \rightarrow \pi^0$. By using pion production, one not only probes the pion at large (negative) momentum transfer, but also avoids all of the
systematic errors outlined above (introducing, of course, others). This has recently been done by the CELLO collaboration, who find a value for $a$ of $0.0326 \pm 0.0026$ [27], in excellent agreement with theoretical expectations.

In short, then, our result of $a = 0.02 \pm 0.01 \pm 0.02$ serves to clear up past discrepancies, and brings the measurement of the $\pi^0$ form factor in the timelike region of momentum transfer into line with both the theoretical expectations and the recent (spacelike) CELLO result. It seems that the structure of the neutral pion can be understood in the framework of the Standard Model.
Bibliography


[29] Ch. Grab: A Search for the Decay $\mu^+ \rightarrow e^+ e^+ e^-$, Doctoral Thesis, Universität Zürich, 1985
[30] N. Kraus: The Rare Decay $\mu^+ \rightarrow e^+ \bar{\nu}_\mu \nu_e e^+ e^-$, Doctoral Thesis, Universität Zürich, 1985
Bibliography


[33] C. Niebuhr: A Search for the Rare Decay \( \pi^0 \rightarrow e^+e^- \), Doctoral Thesis, ETH Zürich, 1989


[40] R. Brun, F. Bruyant, M. Maire, A. C. McPherson, and P. Zanarini, GEANT 3 package, CERN Data Handling Division, DD/EE/84-1


