C. 2

NONRELATIVISTIC QUARK MODEL CALCULATION OF THE $K^-P \to \Lambda\gamma$ AND $K^-P \to \Sigma^0\gamma$ BRANCHING RATIOS.

By

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Abstract

The radiative annihilation of K^-p atoms to $\Lambda\gamma$ and $\Sigma^0\gamma$ is investigated using a non-relativistic harmonic oscillator quark model. A nonrelativistic reduction of the first order Feynman diagrams is performed to yield a gauge invariant interaction, which is sandwiched between three quark wave functions. Pseudoscalar and pseudovector coupling schemes are used for the strong vertex and the effects of $SU(3)_{flavour}$ breaking is explored. We obtain results which are in agreement with experiment for the $\Sigma^0\gamma$ but are somewhat high for the $\Lambda\gamma$ calculation.

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Acknowledgement

Thanks to Harold Fearing. I appreciate his patience and generosity with his time.

Chapter 1

Baryon Wave Functions in the Nonrelativistic Quark Model.

1.1 Introduction

When a kaon is captured in the Coulomb field of a proton a K^-p atom is formed. Due to strong interaction effects, the K^-p system will eventually annihilate. The atom may decay through any one of the following reaction channels [1]:

$$\begin{aligned} K^-p &\rightarrow \Sigma^-\pi^+ \quad (0.47) \\ &\rightarrow \Sigma^0\pi^0 \quad (0.27) \\ &\rightarrow \Sigma^+\pi^- \quad (0.19) \\ &\rightarrow \Lambda\pi^0 \quad (0.07) \\ &\rightarrow \Sigma^0\gamma \quad (0.00144) \\ &\rightarrow \Lambda\gamma \quad (0.00086) \end{aligned}$$

The experimental branching ratio for each channel is given in parenthesis. These reactions are interesting in that they provide information on meson-nucleon interactions. In addition the kaon, since it contains a strange quark, has strangeness $S = -1$. This enables us to examine effects of the strange quark in strong interactions. Since these are some of the simplest reactions involving strange particles it would appear prudent to explore them further. We will look at the $K^-p \rightarrow \Lambda\gamma$ and $K^-p \rightarrow \Sigma^0\gamma$ branching ratios in this thesis.

The small branching ratios of the $\Lambda\gamma$ and $\Sigma^0\gamma$ reaction channels make them very difficult to determine experimentally. However, the new experiment by Whitehouse *et. al* [1] measures the $\Lambda\gamma$ and $\Sigma^0\gamma$ branching ratios to better than 1 in 10^4.

In chapter 2 we extract from a gauge invariant set of Feynman diagrams an interaction which can be used to act on three-quark wave functions. The form of the three-quark wave functions will be developed in chapter 1.

The impulse approximation is used to reduce the interaction to a sum of single quark transition operators. Both pseudoscalar and pseudovector coupling methods are employed at the strong vertices.

In chapter 3 we will detail our methods for evaluating the amplitude and compare our results with those of other calculations.

1.2 The Quark Model

There is overwhelming experimental evidence that baryons and mesons are made up of quarks. Baryons are bound states of three quarks; the mesons are comprised of a quark and an anti-quark.

Each quark comes in one of six different flavours, or type : u, d, s, c, t, b. Only the first three of which will be considered in this thesis. The proton is a baryon and consists of the quark combination uud; the neutron has ddu composition. The π^+ pion has composition $u\bar{d}$; the strange meson K^- consists of the combination $\bar{u}s$.

Quarks are spin $\frac{1}{2}$ particles. Three quarks will combine, by the conventional rules of addition of angular momentum, to a half-integral spin particle. Baryons, therefore, are fermions and obey Fermi statistics. Similarly mesons are integral-spin particles and obey Bose statistics.

A quark of a given flavour is in one of three possible colour states. The Δ^{++} has

quark composition *uuu* and therefore is symmetric in flavour. It is also symmetric in spin and in the ground state, symmetric in space. The colour degree of freedom allows one to construct a totally antisymmetric wave function by insisting that the colour part of the wave function is antisymmetric with respect to exchange of quark positions. Such a state will obey Fermi statistics.

1.3 Baryon Confinement

Quarks[2],[3] belong to the fundamental triplet representation $\begin{pmatrix} R \\ G \\ B \end{pmatrix}$ of the group $SU(3)_{colour}$. To form a baryon we combine three of these triplets. This gives a singlet state, two octets and a decuplet [4],

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}. \tag{1.1}$$

The confinement postulate states:

All hadrons and all physical states are colour singlets.

So the decuplet and octet states in (1.1) are not observed according to the confinement postulate. This rules out the possibility of observing states like diquarks or four-quark states. Since only colour singlet states exist in nature it follows that the confinement forces, between 3 quarks in a baryon, must depend on colour.

In the nonrelativistic quark model (NRQM) the quarks are confined in an oscillator potential whose slope is independent of flavour. The assumption [5] that the confinement potential is flavour independent (which is supported by studies of Q.C.D. on a lattice) means that the eigenstates of the zeroth order hamiltonian have flavour symmetry breaking only via explicit appearance of the quark masses in the kinetic energy part of H_0.

1.4 Symmetry and the Total Baryon Wave function

For a baryon (being a fermion) consisting of identical quarks, the total wave function must be antisymmetric with respect to the interchange of a pair of quarks. The wave function of a hadron has space, spin, flavour, and colour degrees of freedom. In a meson or baryon the colour degree of freedom separates out from the rest. In a baryon, the colour singlet wave function is a (3×3) determinant, antisymmetric under the exchange of a pair. This in turn means that the rest of the wave function, containing the space, spin, and flavour coordinates, be symmetric.

$$|qqq\rangle_A = |\text{colour}\rangle_A \times |\text{space,spin,flavour}\rangle_S$$

The colour wave function has the same form for all baryons [6] (R="red", G="Green", and B="Blue")

$$|\text{colour}\rangle_A = \frac{1}{\sqrt{6}}(RGB - RBG + BRG - BGR + GBR - GRB), \tag{1.2}$$

so we will suppress it henceforth.

In the $S = -1$ baryons[1] the strange quark mass differs from the non-strange quark mass. The three quarks are no longer all indistinguishable particles and so it is no longer necessary to construct baryon wavefunctions that are totally antisymmetric in space, spin, flavour, and colour. In this situation we are free to single out the strange quark as quark 3 and only the $1 \leftrightarrow 2$ symmetry of the states remains relevant. This is known as the *uds* basis and will be discussed further in §1.8.1.

1.5 $SU(3)$ Symmetry

The set of the eight traceless, hermitian, Gell-Mann 3×3 matrices generate the unimodular, unitary group in three dimensions, denoted $SU(3)$. For the group $SU(3)_{flavour}$

[1] Here S denotes the value of the strangeness quantum number for the state.

these operators act on the fundamental triplet, $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$. Here the u, d, and s quarks are considered three different quantum states of the same particle. This is analogous to isospin symmetry for the case of $SU(2)_{flavour}$. However in $SU(3)_{flavour}$ the symmetry is more approximate due to the significantly larger mass of the strange quark.

As a result of this symmetry breaking we treat the strange baryons in two ways:

- $m_u/m_s \approx 1$. Here all quarks in the baryon are indistinguishable and it is appropriate to use the fully symmetrized "$SU(6)$ basis".

- $m_u/m_s \approx 0.6$. Here the strange quark is distinguishable by virtue of its larger mass. This gives rise to the "*uds* basis" of Isgur and Karl where the symmetrization is carried out only between the two equal mass, light quarks.

The $SU(6)$ and *uds* bases are two physically distinct descriptions.

1.6 Zeroth Order Eigenstates

By taking the instantaneous limit of the Bethe-Salpeter (B.S.) equation one obtains [2] the three-particle Schrödinger equation with Breit-Fermi-type corrections. The Breit-Fermi Hamiltonian can be written (neglecting spin-orbit interactions)

$$H = \sum_{i=1}^{3} m_i + H_0 + U + H_{hyp}. \tag{1.3}$$

m_i is the mass of the i^{th} quark. H_0 contains the kinetic energy and a harmonic oscillator potential, which models confinement and 'asymptotic freedom'; U is some unknown potential which is included to incorporate the action of the Coulomb potential at short range and long range deviations from the harmonic oscillator potential; H_{hyp} is the

Q.C.D. analog of the hyperfine interaction. The colour hyperfine interaction between two quarks i and j in the same baryon, to order α_s is,

$$H_{hyp} = \frac{2}{3}\sum_{i<j}\frac{\alpha_s}{m_i m_j}\{\frac{8\pi}{3}\vec{S}_i\cdot\vec{S}_j\delta^3(\vec{r_{ij}}) + \frac{1}{r_{ij}^3}[\frac{3\vec{S}_i\cdot\vec{r_{ij}}\vec{S}_j\cdot\vec{r_{ij}}}{r_{ij}^2} - \vec{S}_i\cdot\vec{S}_j]\}. \tag{1.4}$$

Where $r_{ij} \equiv |\vec{r_i} - \vec{r_j}|$ and $\vec{r_i}$ is the position vector of the i^{th} quark in the baryon. $\vec{S_i} \equiv \frac{1}{2}\vec{\sigma}$ is the spin vector operator for the i^{th} quark in the baryon. α_s is the effective quark-gluon coupling constant. This piece will be discussed further in §1.11.

We now wish to construct eigenstates of the hamiltonian in the case when $SU(3)_{flavour}$ is a good symmetry and when it is broken.

1.6.1 The case of Unbroken $SU(3)_{flavour}$

In the NRQM the zeroth order basis states are generated by the hamiltonian,

$$H_0 = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + \frac{1}{2}K\sum_{i<j} \mid \vec{r_i} - \vec{r_j} \mid^2 . \tag{1.5}$$

$\vec{p_i}$, $\vec{r_i}$, m_i are the momentum, position and mass of the i^{th} quark.

In the $S = 0$ sector, or in $S = -1$ when $SU(3)_{flavour}$ is unbroken, the quark masses in the baryon are taken to be identical.

$$m_1 = m_2 = m_3 = m_u$$

We assume the eigenstates are of the form,

$$\Psi(r_1, r_2, r_3) = \psi(\vec{r_1}, \vec{r_2}, \vec{r_3})e^{-iEt} \tag{1.6}$$

That is we are assuming the baryon is in an eigenstate of total energy. In order to separate out the centre of mass motion it is convenient to transform to Jacobi coordinates,

$$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r_1} - \vec{r_2}),\ \vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r_1} + \vec{r_2} - 2\vec{r_3}),\ \vec{R} = \frac{1}{3}(\vec{r_1} + \vec{r_2} + \vec{r_3}). \tag{1.7}$$

$\vec{\rho}$ is antisymmetric with respect to interchange of the space coordinates of quarks 1 and 2 and is a representation of mixed permutation symmetry of type M_ρ, while $\vec{\lambda}$ is symmetric with respect to interchange of the space coordinates of quarks 1 and 2 and corresponds to symmetry M_λ. By symmetry we mean with respect to quark positions:

- ***S*** Fully symmetic. Exchanging any two quark positions gives the same state.
- ***A*** Fully antisymmetic. Exchanging any two quark positions gives the same state times -1.
- $\boldsymbol{M_\rho}$ The state transforms as $\vec{\rho}$. That is antisymmetric with respect to exchange of quarks 1 and 2 but has no definite symmetry with respect to exchange of other quark pairs. We will denote this symmetry with a superscript ρ on the state.
- $\boldsymbol{M_\lambda}$ The state transforms as $\vec{\lambda}$. That is symmetric with respect to exchange of quarks 1 and 2 but has no definite symmetry with respect to exchange of other quark pairs. We will denote this with a superscript λ on the state.

Define

$$\begin{aligned} M &\equiv 3m_u,\ m_\lambda \equiv \frac{3m_u m_s}{2m_u + m_s}, \\ \vec{P}_\rho &\equiv m_u \dot{\vec{\rho}},\ \vec{P}_\lambda \equiv m_\lambda \dot{\vec{\lambda}},\ \vec{P_{CM}} \equiv M\dot{\vec{R}} \\ \text{and } \dot{X} &\equiv \frac{\partial X}{\partial t}. \end{aligned} \tag{1.8}$$

In the case of unbroken $SU(3)_{flavour}$ $m_\lambda = m_u$. From equations (1.7) we get

$$\left.\begin{aligned} \vec{r_1} &= \vec{R} + \tfrac{\vec{\rho}}{\sqrt{2}} + \tfrac{\vec{\lambda}}{\sqrt{6}} \\ \vec{r_2} &= \vec{R} - \tfrac{\vec{\rho}}{\sqrt{2}} + \tfrac{\vec{\lambda}}{\sqrt{6}} \\ \vec{r_3} &= \vec{R} - \sqrt{\tfrac{2}{3}}\vec{\lambda} \end{aligned}\right\} \tag{1.9}$$

$$\begin{aligned}
\Rightarrow \vec{p_1} &= m_1\dot{\vec{r_1}} = m_1(\dot{\vec{R}} + \frac{\dot{\vec{\rho}}}{\sqrt{2}} + \frac{\dot{\vec{\lambda}}}{\sqrt{6}}) \\
\vec{p_2} &= m_2\dot{\vec{r_2}} = m_2(\dot{\vec{R}} - \frac{\dot{\vec{\rho}}}{\sqrt{2}} + \frac{\dot{\vec{\lambda}}}{\sqrt{6}}) \\
\vec{p_3} &= m_3\dot{\vec{r_3}} = m_3(\dot{\vec{R}} - \sqrt{\frac{2}{3}}\dot{\vec{\lambda}})
\end{aligned}$$

substituting these relations into equation (1.5), we find H_0 reduces to

$$H_0 = \frac{P_\rho^2}{2m_u} + \frac{P_\lambda^2}{2m_u} + \frac{3}{2}K(\rho^2 + \lambda^2) + \frac{P_{CM}^2}{2M}. \tag{1.10}$$

The last term of (1.10) corresponds to the centre of mass motion of the baryon and does not play any role in the intrinsic spectrum of the baryon[7]. The centre of mass motion is a plane wave[8]. The eigenstates of H_0 have the form,

$$\Psi(\vec{R}, \vec{\rho}, \vec{\lambda}) = e^{i\vec{P_{CM}}\cdot\vec{R}}\Phi_{NL}(\vec{\rho}, \vec{\lambda})e^{-iEt} \tag{1.11}$$

The elimination of the centre of mass coordinate $\vec{R}$ is crucial in the correct counting of the states. This is one reason why the nonrelativistic harmonic oscillator approach is so successful in baryon spectroscopy.

The hamiltonian has been reduced to that of two independent oscillators each with spring constant 3K. The oscillator energy spacings

$$\omega = \sqrt{\frac{3K}{m_u}} \tag{1.12}$$

for the ρ and λ oscillators are identical in the S=0 sector where $m_u \approx m_d$. The zeroth order energy of a state is then specified by the quantum number N

$$E_N = 3m_u + (N+3)\hbar\omega, \tag{1.13}$$

$$\text{where } N = N_\rho + N_\lambda = (2n_\rho + l_\rho) + (2n_\lambda + l_\lambda). \tag{1.14}$$

The principal quantum number of the ρ, λ oscillator, $n_{\rho,\lambda}$, takes on the values $0, 1, \cdots$ The orbital angular momentum quantum number of the ρ, λ oscillator, $l_{\rho,\lambda}$, takes on the values $0, 1, \cdots$

The wave function of an oscillator, for example an r (where $r = \rho$ or λ) oscillator, is [9],

$$\psi_{n_r l_r m_{l_r}}(\alpha r) = R_{n_r l_r}(\alpha r) Y_{l_r m_{l_r}}(\Omega_r), \tag{1.15}$$

$$\text{where } R_{n_r l_r}(\alpha r) = \mathcal{N}(\alpha r)^{l_r} L_{n_r}^{l_r+\frac{1}{2}}(\alpha r) e^{-\frac{1}{2}\alpha^2 r^2}, \tag{1.16}$$

$$\alpha \equiv (m\omega)^{\frac{1}{2}} \tag{1.17}$$

$$\text{and } \mathcal{N} = \sqrt{\frac{2\alpha^3\, n_r!}{\sqrt{\pi}(n_r + l_r + \frac{1}{2})(n_r + l_r - \frac{1}{2}) \ldots \frac{3}{2} \times \frac{1}{2}}}. \tag{1.18}$$

$Y_{l_r m_{l_r}}(\Omega_r)$ are the spherical harmonics. We have used the convention [10] that

$$Y^*_{l_r m_{l_r}}(\Omega_r) = (-1)^{m_{l_r}} Y_{l_r - m_{l_r}}(\Omega_r). \tag{1.19}$$

The $L_n^{l+\frac{1}{2}}(x)$ are the associated Laguerre polynomials. Defined in terms of Binomial coefficients these are

$$L_n^{l+\frac{1}{2}}(x) = \sum_{m=0}^{n} (-1)^m \begin{pmatrix} n + l + \frac{1}{2} \\ n - m \end{pmatrix} \frac{x^{2m}}{m!}.$$

The states, $\psi_{n_r l_r m_{l_r}}$, we require are listed in table 1.1.

The total spatial wave function consists of products of these λ and ρ oscillator states. The orbital angular momentum $\vec{L}$ of the baryon is obtained by coupling $\vec{l_\rho}$ and $\vec{l_\lambda}$ [11],

$$\vec{L} = \vec{l_\rho} + \vec{l_\lambda}, \tag{1.20}$$

$$\psi_{000}(\alpha\vec{r}) = \sqrt{\frac{4\alpha^3}{\sqrt{\pi}}}e^{-\frac{1}{2}\alpha^2r^2}Y_{00}(\Omega_r)$$
$$\psi_{100}(\alpha\vec{r}) = \sqrt{\frac{2}{3}}\sqrt{\frac{4\alpha^3}{\sqrt{\pi}}}(\frac{3}{2} - \alpha^2r^2)e^{-\frac{1}{2}\alpha^2r^2}Y_{00}(\Omega_r)$$
$$\psi_{01m}(\alpha\vec{r}) = \sqrt{\frac{2}{3}}\sqrt{\frac{4\alpha^3}{\sqrt{\pi}}}\alpha re^{-\frac{1}{2}\alpha^2r^2}Y_{1m}(\Omega_r)$$
$$\psi_{02m}(\alpha\vec{r}) = \sqrt{\frac{4}{15}}\sqrt{\frac{4\alpha^3}{\sqrt{\pi}}}\alpha^2r^2e^{-\frac{1}{2}\alpha^2r^2}Y_{2m}(\Omega_r)$$

Table 1.1: Normalized linear harmonic oscillator states $\psi_{n_r l_r m_{l_r}}(\alpha\vec{r})$ ($r = \rho$ or $r = \lambda$).

		l_ρ	l_λ	L
$N = 0$	$\psi_{00}(\vec{\rho})\psi_{00}(\vec{\lambda})$	0	0	0
$N = 1$	$\psi_{01}(\vec{\rho})\psi_{00}(\vec{\lambda})$	1	0	1
	$\psi_{00}(\vec{\rho})\psi_{01}(\vec{\lambda})$	0	1	1
$N = 2$	$\psi_{10}(\vec{\rho})\psi_{00}(\vec{\lambda})$	0	0	0
	$\psi_{00}(\vec{\rho})\psi_{10}(\vec{\lambda})$	0	0	0
	$[\psi_{01}(\vec{\rho})\psi_{01}(\vec{\lambda})]^{L=0}$	1	1	0
	$[\psi_{01}(\vec{\rho})\psi_{01}(\vec{\lambda})]^{L=1}$	1	1	1
	$[\psi_{01}(\vec{\rho})\psi_{01}(\vec{\lambda})]^{L=2}$	1	1	2
	$\psi_{02}(\vec{\rho})\psi_{00}(\vec{\lambda})$	2	0	2
	$\psi_{00}(\vec{\rho})\psi_{02}(\vec{\lambda})$	0	2	2

Table 1.2: Product wave functions $\psi_{n_r l_r}$ (we have omitted the m). $[\psi\ \psi]^L$ indicates coupling $\vec{l_\rho}$ and $\vec{l_\lambda}$ to total orbital angular momentum $\vec{L}$. N is obtained via equation (1.14) and the α dependence in the argument of ψ has been dropped.

To combine the harmonic oscillator wave functions to spatial states of definite permutation symmetry $\Phi^P_{NLM_L}$ we take linear combinations of the products

$$\psi_{n_\rho l_\rho m_{l_\rho}}(\vec{\rho})\psi_{n_\lambda l_\lambda m_{l_\lambda}}(\vec{\lambda}).$$

The well known prescriptions to multiply two mixed representations [12] are

$$\begin{aligned} S &= \lambda_1\lambda_2 + \rho_1\rho_2, \\ A &= \lambda_1\rho_2 - \rho_1\lambda_2, \\ M^\rho &= \rho_1\lambda_2 + \lambda_1\rho_2, \\ M^\lambda &= \lambda_1\lambda_2 - \rho_1\rho_2. \end{aligned}$$

The oscillator states must also be coupled to give states of good orbital angular momentum L. The total spatial wave function can be written as the linear combination,

$$\Phi^P_{NLM} = \mathcal{Z}\sum_{n_\rho,n_\lambda}\sum_{m_{l_\rho},m_{l_\lambda}}\langle l_\rho, m_{l_\rho}; l_\lambda, m_{l_\lambda}|L,M\rangle\psi_{n_\rho l_\rho m_{l_\rho}}\,\psi_{n_\lambda l_\lambda m_{l_\lambda}} \tag{1.21}$$

The first harmonic oscillator wave function in a product always denotes the $\vec{\rho}$ oscillator state, and the second the $\vec{\lambda}$ oscillator state. We will suppress the $\vec{\rho}$ or $\vec{\lambda}$ dependence from now on. $\langle l_\rho, m_{l_\rho}; l_\lambda, m_{l_\lambda}|L,M\rangle$ is a Clebsch-Gordan coefficient[13] obtained from the table in ref.[14] and $\mathcal{Z}$ is a normalization coefficient. The Condon-Shortley phase convention is used. Up to $N = 2$ the possible products are listed in table 1.2.

For the $N = 0$ case we can only have (see Table 1.1)

$$\Phi^S_{000} = \mathcal{Z}\langle 0,0;0,0|0,0\rangle\psi_{000}\psi_{000} = (\frac{\alpha}{\sqrt{\pi}})^3 e^{-\frac{1}{2}\alpha^2(\rho^2+\lambda^2)}.$$

The $e^{-\frac{1}{2}\alpha^2(\rho^2+\lambda^2)}$ is present in all the product wave functions. It is symmetric since

$$\rho^2 + \lambda^2 = \frac{1}{3}((\vec{r_1} - \vec{r_2})^2 + (\vec{r_1} - \vec{r_3})^2 + (\vec{r_2} - \vec{r_3})^2)$$

is invariant under transposition of quark positions. We write

$$\psi_{n_\rho l_\rho m_{l_\rho}}\,\psi_{n_\lambda l_\lambda m_{l_\lambda}} = \tilde{\psi}_{n_\rho l_\rho m_{l_\rho}}\tilde{\psi}_{n_\lambda l_\lambda m_{l_\lambda}}(\frac{\alpha}{\sqrt{\pi}})^3 e^{-\frac{1}{2}\alpha^2(\rho^2+\lambda^2)} \tag{1.22}$$

and the symmetry of the product depends only on the symmetry of $\tilde{\psi}_{n_\rho l_\rho m_{l_\rho}}\tilde{\psi}_{n_\lambda l_\lambda m_{l_\lambda}}$.

At $N=1$, $L=1$ we can only have

$$\begin{aligned}\Phi^{\rho}_{11M_L} &= \psi_{01M_L}\psi_{000}\\ \Phi^{\lambda}_{11M_L} &= \psi_{000}\psi_{01M_L}\end{aligned}$$

Since $\tilde{\psi}_{01M_L}\tilde{\psi}_{000} \sim \rho Y_{1M_L}(\Omega_\rho)$ which transforms as $\vec{\rho}$ under permutations. Similarly $\tilde{\psi}_{000}\tilde{\psi}_{01M_L} \sim \lambda Y_{1M_L}$ transforms as $\vec{\lambda}$ under permutations. The states $\Phi^{\rho}_{11M_L}$ and $\Phi^{\lambda}_{11M_L}$ are degenerate in the $S=0$ sector but cannot be combined to make a state of definite permutation symmetry.

At $N=2$ we have the states :

$L=0$

$$\begin{aligned}\Phi^{\rho}_{200} &= \sum_{m_{l_\rho},m_{l_\lambda}}\langle 1,m_{l_\rho};1,m_{l_\lambda}|0,0\rangle\psi_{01m_{l_\rho}}\psi_{01m_{l_\lambda}}\\ &= \frac{1}{\sqrt{3}}(\psi_{011}\psi_{01-1}+\psi_{01-1}\psi_{011}-\psi_{010}\psi_{010})\end{aligned}$$

$$\begin{aligned}\Phi^{\lambda}_{200} &= \mathcal{Z}\sum_{n_\rho,n_\lambda}\psi_{n_\rho 00}\,\psi_{n_\lambda 00}\\ &= \frac{1}{\sqrt{2}}(\psi_{100}\psi_{000}-\psi_{000}\psi_{100})\end{aligned}$$

$$\begin{aligned}\Phi^{S}_{200} &= \mathcal{Z}\sum_{n_\rho,n_\lambda}\psi_{n_\rho 00}\,\psi_{n_\lambda 00}\\ &= \frac{1}{\sqrt{2}}(\psi_{100}\psi_{000}+\psi_{000}\psi_{100})\end{aligned}$$

$L=1$

$$\Phi^{A}_{211} = \sum_{m_{l_\rho},m_{l_\lambda}}\langle 1,m_{l_\rho};1,m_{l_\lambda}|1,1\rangle\psi_{01m_{l_\rho}}\,\psi_{01m_{l_\lambda}}$$

$$= \frac{1}{\sqrt{2}}(\psi_{011}\psi_{010} - \psi_{010}\psi_{011}).$$

At $L = 2$ we can form

$$\begin{aligned}
\Phi^S_{222} &= \frac{1}{\sqrt{2}}(\psi_{022}\psi_{000} + \psi_{000}\psi_{022}) \\
\Phi^\lambda_{222} &= \frac{1}{\sqrt{2}}(\psi_{022}\psi_{000} - \psi_{000}\psi_{022}) \\
\Phi^\rho_{222} &= \psi_{011}\psi_{011}.
\end{aligned}$$

The relative signs of the ρ-type and λ-type wave functions for a given N and L are important. Otherwise, the overall phases are arbitrary. Our definitions differ by a minus sign for the states Φ^λ_{200}, Φ^S_{200} and Φ^ρ_{200} as compared to the phase convention of [15] and [16]. Note that there is an error in equations (A15) of reference [16]. The state $[\phi_{01}\phi_{01}]^{L=1}$ should have $\boldsymbol{A}$ symmetry and ψ^{22}_ρ should read $[\phi_{01}\phi_{01}]^{L=2}$.

The zeroth order eigenstates of H_0, in the $S = 0$ sector, are listed in table 1.3.

1.6.2 The $S = -1$ Baryons when $SU(3)_{flavour}$ is Broken

In this section we consider the case when $m_u/m_s \approx 0.6$. The analogue of equations (1.7) with $m_s \neq m_u$ are

$$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r_1} - \vec{r_2}),\ \vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r_1} + \vec{r_2} - 2\vec{r_3}),\ \vec{R} = \frac{m_u(\vec{r_1} + \vec{r_2}) + m_s\vec{r_3}}{M}, \quad (1.23)$$

$$\text{where } M = 2m_u + m_s. \quad (1.24)$$

Rearranging we get[2],

$$\left.\begin{aligned}
\vec{r_1} &= \vec{R} + \frac{\vec{\rho}}{\sqrt{2}} + \frac{\vec{\lambda}}{\sqrt{6}}\left(\frac{m_\lambda}{m_u}\right) \\
\vec{r_2} &= \vec{R} - \frac{\vec{\rho}}{\sqrt{2}} + \frac{\vec{\lambda}}{\sqrt{6}}\left(\frac{m_\lambda}{m_u}\right) \\
\vec{r_3} &= \vec{R} + \frac{\vec{\lambda}}{\sqrt{6}}\left(\frac{m_\lambda}{m_u} - 3\right)
\end{aligned}\right\} \quad (1.25)$$

[2]When $SU(3)_{flavour}$ is a good symmetry and $m_s = m_u = m_d = m_u$ we get $m_\lambda = m_u$ and (1.25) reduces to (1.9)

$$\tilde{\Phi}^{S}_{000} = \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) = 1$$

$$\tilde{\Phi}^{\lambda}_{11M_L} = \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{01M_L}(\vec{\lambda}) = 2\sqrt{\tfrac{2\pi}{3}}\alpha\lambda Y_{1M_L}(\Omega_\lambda)$$

$$\tilde{\Phi}^{\rho}_{11M_L} = \tilde{\psi}_{01M_L}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) = 2\sqrt{\tfrac{2\pi}{3}}\alpha\rho Y_{1M_L}(\Omega_\rho)$$

$$\begin{aligned}\tilde{\Phi}^{S}_{200} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{100}(\vec{\lambda}) + \tilde{\psi}_{100}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda})) \\ &= \tfrac{1}{\sqrt{3}}\alpha^2(\tfrac{3}{\alpha^2} - \rho^2 - \lambda^2)\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{\lambda}_{200} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{100}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) - \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{100}(\vec{\lambda})) \\ &= \tfrac{1}{\sqrt{3}}\alpha^2(\lambda^2 - \rho^2)\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{\rho}_{200} &= \tfrac{1}{\sqrt{3}}(\tilde{\psi}_{011}(\vec{\rho})\tilde{\psi}_{01-1}(\vec{\lambda}) + \tilde{\psi}_{01-1}(\vec{\rho})\tilde{\psi}_{011}(\vec{\lambda}) - \tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda})) \\ &= \tfrac{8\pi}{3\sqrt{3}}\alpha^2\rho\lambda(Y_{11}(\Omega_\rho)Y_{1-1}(\Omega_\lambda) + Y_{1-1}(\Omega_\rho)Y_{11}(\Omega_\lambda) - Y_{10}(\Omega_\rho)Y_{10}(\Omega_\lambda)) \\ &= -\tfrac{2}{\sqrt{3}}\alpha^2\vec{\rho}\cdot\vec{\lambda}\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{A}_{21\pm1} &= \pm\tfrac{1}{\sqrt{2}}(\tilde{\psi}_{01\pm1}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda}) - \tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{01\pm1}(\vec{\lambda})) \\ &= \pm\tfrac{4\pi}{3}\sqrt{2}\alpha^2\rho\lambda(Y_{1\pm1}(\Omega_\rho)Y_{10}(\Omega_\lambda) - Y_{10}(\Omega_\rho)Y_{1\pm1}(\Omega_\lambda))\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{A}_{210} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{011}(\vec{\rho})\tilde{\psi}_{01-1}(\vec{\lambda}) - \tilde{\psi}_{01-1}(\vec{\rho})\tilde{\psi}_{011}(\vec{\lambda})) \\ &= \tfrac{4\pi}{3}\sqrt{2}\alpha^2\rho\lambda(Y_{11}(\Omega_\rho)Y_{1-1}(\Omega_\lambda) - Y_{1-1}(\Omega_\rho)Y_{11}(\Omega_\lambda))\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{S}_{22M_L} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{02M_L}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) + \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{02M_L}(\vec{\lambda})) \\ &= 2\sqrt{\tfrac{2\pi}{15}}\alpha^2(\rho^2 Y_{2M_L}(\Omega_\rho) + \lambda^2 Y_{2M_L}(\Omega_\lambda))\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{\lambda}_{22M_L} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{02M_L}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) - \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{02M_L})(\vec{\lambda}) \\ &= 2\sqrt{\tfrac{2\pi}{15}}\alpha^2(\rho^2 Y_{2M_L}(\Omega_\rho) - \lambda^2 Y_{2M_L}(\Omega_\lambda))\end{aligned}$$

$$\tilde{\Phi}^{\rho}_{22\pm2} = \tilde{\psi}_{01\pm1}(\vec{\rho})\tilde{\psi}_{01\pm1}(\vec{\lambda}) = \tfrac{8\pi}{3}\alpha^2\rho\lambda Y_{1\pm1}(\Omega_\rho)Y_{1\pm1}(\Omega_\lambda)$$

$$\begin{aligned}\tilde{\Phi}^{\rho}_{22\pm1} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{01\pm1}(\vec{\lambda}) + \tilde{\psi}_{01\pm1}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda})) \\ &= \tfrac{4\pi}{3}\sqrt{2}\alpha^2\rho\lambda(Y_{10}(\Omega_\rho)Y_{1\pm1}(\Omega_\lambda) + Y_{1\pm1}(\Omega_\rho)Y_{10}(\Omega_\lambda))\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{\rho}_{220} &= \tfrac{1}{\sqrt{6}}(\tilde{\psi}_{01-1}(\vec{\rho})\tilde{\psi}_{011}(\vec{\lambda}) + \tilde{\psi}_{011}(\vec{\rho})\tilde{\psi}_{01-1}(\vec{\lambda}) + 2\tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda})) \\ &= \tfrac{4\pi}{3}\sqrt{\tfrac{2}{3}}\alpha^2\rho\lambda(Y_{1-1}(\Omega_\rho)Y_{11}(\Omega_\lambda) + Y_{11}(\Omega_\rho)Y_{1-1}(\Omega_\lambda) + 2Y_{10}(\Omega_\rho)Y_{10}(\Omega_\lambda))\end{aligned}$$

Table 1.3: Baryon space wave functions $\tilde{\Phi}^{\sigma}_{NLM_L} \equiv \Phi^{\sigma}_{NLM_L}/((\frac{\alpha}{\sqrt{\pi}})^3 e^{-\frac{1}{2}\alpha^2(\rho^2+\lambda^2)})$ when $SU(3)_{flavour}$ is unbroken.

$$\begin{aligned}
\Rightarrow \vec{p_1} &= m_1\dot{\vec{r_1}} = m_u(\dot{\vec{R}} + \frac{\dot{\vec{\rho}}}{\sqrt{2}} + \frac{\dot{\vec{\lambda}}}{\sqrt{6}}(\frac{m_\lambda}{m_u})) \\
\Rightarrow \vec{p_2} &= m_2\dot{\vec{r_2}} = m_u(\dot{\vec{R}} + \frac{\dot{\vec{\rho}}}{\sqrt{2}} + \frac{\dot{\vec{\lambda}}}{\sqrt{6}}(\frac{m_\lambda}{m_u})) \\
\Rightarrow \vec{p_3} &= m_3\dot{\vec{r_3}} = m_s(\dot{\vec{R}} + \frac{\dot{\vec{\lambda}}}{\sqrt{6}}(\frac{m_\lambda}{m_u} - 3)).
\end{aligned}$$

Now the harmonic oscillator hamiltonian, equation (1.5), becomes

$$H_0' = \frac{P_\rho^2}{2m_u} + \frac{P_\lambda^2}{2m_\lambda} + \frac{3}{2}K(\rho^2 + \lambda^2) + \frac{P_{CM}^2}{2M}, \tag{1.26}$$

where $\vec{P_{CM}} = M\dot{\vec{R}}$; and $\vec{P}_\rho = m_u\dot{\vec{\rho}}$ and $\vec{P}_\lambda = m_\lambda\dot{\vec{\lambda}}$.

This hamiltonian generates the same ρ oscillator states (table 1.1 with $r = \rho$) as for the $S = 0$ case. However, due to the higher mass of the strange quark,

$$\alpha \mapsto \alpha_\lambda = (3Km_\lambda)^{1/4}, \tag{1.27}$$

in the λ oscillator states. The same product states are formed but now the degeneracy between the λ and ρ normal modes has been broken,

$$\omega = \sqrt{\frac{3K}{m_u}},\ \omega_\lambda = \sqrt{\frac{3K}{m_\lambda}} \tag{1.28}$$

$$\text{and } E_N = 2m_u + m_s + (N_\rho + \frac{3}{2})\hbar\omega + (N_\lambda + \frac{3}{2})\hbar\omega_\lambda \tag{1.29}$$

Here $\omega_\lambda < \omega$ since $m_u < m_s$.

Because of this frequency splitting, the three states Φ^S_{200}, Φ^ρ_{200}, Φ^λ_{200}, which in the degenerate case ($S = 0$) have permutation symmetry S, M_ρ, M_λ respectively, break into $\lambda\lambda$, $\lambda\rho$, and $\rho\rho$ excitations. For example $\lambda\lambda$ corresponds to a double excitation in the λ oscillator (the ρ oscillator remains in the ground state); that is $N_\lambda = 2$. Although the complete permutation symmetry of the wave function is lost, we still have permutation symmetry between the u and d quarks. The zeroth order eigenstates of H_0 in the $S = -1$ sector for the case of broken $SU(3)_{flavour}$ are given in table 1.4.

$$\tilde{\Phi}^{S}_{000} = \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) = 1$$

$$\tilde{\Phi}^{\lambda}_{11M_L} = \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{01M_L}(\vec{\lambda}) = 2\sqrt{\tfrac{2\pi}{3}}\alpha_\lambda\lambda Y_{1M_L}(\Omega_\lambda)$$

$$\tilde{\Phi}^{\rho}_{11M_L} = \tilde{\psi}_{01M_L}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) = 2\sqrt{\tfrac{2\pi}{3}}\alpha\rho Y_{1M_L}(\Omega_\rho)$$

$$\tilde{\Phi}^{\lambda\lambda}_{200} = \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{100}(\vec{\lambda}) = \sqrt{\tfrac{2}{3}}\alpha_\lambda^2(\tfrac{3}{2\alpha_\lambda^2} - \lambda^2)$$

$$\tilde{\Phi}^{\rho\rho}_{200} = \tilde{\psi}_{100}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) = \sqrt{\tfrac{2}{3}}\alpha^2(\tfrac{3}{2\alpha^2} - \rho^2)$$

$$\begin{aligned}\tilde{\Phi}^{\rho\lambda}_{200} &= \tfrac{1}{\sqrt{3}}(\tilde{\psi}_{011}(\vec{\rho})\tilde{\psi}_{01-1}(\vec{\lambda}) + \tilde{\psi}_{01-1}(\vec{\rho})\tilde{\psi}_{011}(\vec{\lambda}) - \tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda})) \\ &= \tfrac{8\pi}{3\sqrt{3}}\alpha\alpha_\lambda\rho\lambda(Y_{11}(\Omega_\rho)Y_{1-1}(\Omega_\lambda) + Y_{1-1}(\Omega_\rho)Y_{11}(\Omega_\lambda) - Y_{10}(\Omega_\rho)Y_{10}(\Omega_\lambda))\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{\rho\lambda}_{21\pm1} &= \pm\tfrac{1}{\sqrt{2}}(\tilde{\psi}_{01\pm1}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda}) - \tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{01\pm1}(\vec{\lambda})) \\ &= \pm\tfrac{4\pi}{3}\sqrt{2}\alpha\alpha_\lambda\rho\lambda(Y_{1\pm1}(\Omega_\rho)Y_{10}(\Omega_\lambda) - Y_{10}(\Omega_\rho)Y_{1\pm1}(\Omega_\lambda))\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{\rho\lambda}_{210} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{011}(\vec{\rho})\tilde{\psi}_{01-1}(\vec{\lambda}) - \tilde{\psi}_{01-1}(\vec{\rho})\tilde{\psi}_{011}(\vec{\lambda})) \\ &= \tfrac{4\pi}{3}\sqrt{2}\alpha\alpha_\lambda\rho\lambda(Y_{11}(\Omega_\rho)Y_{1-1}(\Omega_\lambda) - Y_{1-1}(\Omega_\rho)Y_{11}(\Omega_\lambda))\end{aligned}$$

$$\tilde{\Phi}^{\lambda\lambda}_{22M_L} = \tilde{\psi}_{000}(\vec{\rho})\tilde{\psi}_{02M_L}(\vec{\lambda}) = \tfrac{4\pi}{\sqrt{15}}\alpha_\lambda^2\lambda^2 Y_{2M_L}(\Omega_\lambda)$$

$$\tilde{\Phi}^{\rho\rho}_{22M_L} = \tilde{\psi}_{02M_L}(\vec{\rho})\tilde{\psi}_{000}(\vec{\lambda}) = \tfrac{4\pi}{\sqrt{15}}\alpha^2\rho^2 Y_{2M_L}(\Omega_\rho)$$

$$\tilde{\Phi}^{\rho\lambda}_{22\pm2} = \tilde{\psi}_{01\pm1}(\vec{\rho})\tilde{\psi}_{01\pm1}(\vec{\lambda}) = \tfrac{8\pi}{3}\alpha\alpha_\lambda\rho\lambda Y_{1\pm1}(\Omega_\rho)Y_{1\pm1}(\Omega_\lambda)$$

$$\begin{aligned}\tilde{\Phi}^{\rho\lambda}_{22\pm1} &= \tfrac{1}{\sqrt{2}}(\tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{01\pm1}(\vec{\lambda}) + \tilde{\psi}_{01\pm1}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda})) \\ &= \tfrac{4\pi}{3}\sqrt{2}\alpha\alpha_\lambda\rho\lambda(Y_{10}(\Omega_\rho)Y_{1\pm1}(\Omega_\lambda) + Y_{1\pm1}(\Omega_\rho)Y_{10}(\Omega_\lambda))\end{aligned}$$

$$\begin{aligned}\tilde{\Phi}^{\rho\lambda}_{220} &= \tfrac{1}{\sqrt{6}}(\tilde{\psi}_{01-1}(\vec{\rho})\tilde{\psi}_{011}(\vec{\lambda}) + \tilde{\psi}_{011}(\vec{\rho})\tilde{\psi}_{01-1}(\vec{\lambda}) + 2\tilde{\psi}_{010}(\vec{\rho})\tilde{\psi}_{010}(\vec{\lambda})) \\ &= \tfrac{4\pi}{3}\sqrt{\tfrac{2}{3}}\alpha\alpha_\lambda\rho\lambda(Y_{1-1}(\Omega_\rho)Y_{11}(\Omega_\lambda) + Y_{11}(\Omega_\rho)Y_{1-1}(\Omega_\lambda) + 2Y_{10}(\Omega_\rho)Y_{10}(\Omega_\lambda))\end{aligned}$$

Table 1.4: Baryon space wave functions $\tilde{\Phi}^{\sigma_1\sigma_2}_{NLM_L} \equiv \Phi^{\sigma_1\sigma_2}_{NLM_L}/((\frac{\alpha\alpha_\lambda}{\pi})^{3/2}e^{-\frac{1}{2}(\alpha^2\rho^2+\alpha_\lambda^2\lambda^2)})$ in the $S=-1$ sector when $SU(3)_{flavour}$ is broken.

1.7 Deviations from Harmonic Confinement

The formula for the energy in the $S = 0$ (1.13) sector states that all multiplets with the same N are degenerate and have equal spacing $\hbar\omega$ between multiplets. This is not observed experimentally; these deviations from the harmonic oscillator spectrum are consistent [5] with the action of a short range attractive potential.

It can be shown [17] that any potential U, in first order perturbation theory, will split a harmonic spectrum into the same pattern. This pattern can be described by only three constants. For example

$$\begin{aligned} E_0 &= \text{hyperfine unperturbed level of the ground state.} \\ &= 3m_u + 3\omega + a = 1135\, MeV \end{aligned}$$

The ground state refers to the $N = 0$ zeroth-order eigenstate. E_0 is used as a fitting parameter to the baryon spectrum. Where a represents the energy shift due to the deviation from a harmonic potential.

By allowing the zeroth order energies of the seven (up to $N = 2$) supermultiplets[3]: (notation L_σ: L is the total orbital angular momentum of the state; σ the symmetry of the spatial wave function) $S_S, P_M, S_{S'}, S_M, P_A, D_S, D_M$ to be independent parameters, Isgur and Karl [17] found excellent agreement with the energy spacings of the supermultiplets predicted by first order perturbation theory. They then take the assumption, due to this confirmation of the first order result, that the harmonic oscillator wave functions remain an adequate approximation to the true zeroth order wave functions even though U is substantial.

The U perturbation affects only E_0. We will see however (§1.11) that H_{hyp} will mix these states.

[3]A supermultiplet contains states of various n_ρ, n_λ, l_ρ, l_λ that combine to N, L and symmetry type (M, A, S)

	J^P	B	Q/e	I	I_z	Y	S
u	$1/2^+$	$1/3$	$2/3$	$1/2$	$1/2$	$1/3$	0
d	$1/2^+$	$1/3$	$-1/3$	$1/2$	$-1/2$	$1/3$	0
s	$1/2^+$	$1/3$	$-1/3$	0	0	$-2/3$	-1

Table 1.5: Quantum numbers for the u, d and s quarks.

1.8 Flavour Wave Functions when $SU(3)_{flavour}$ is Unbroken

Consider the case of a baryon which consists of three quarks. Each of which may have one of the flavours u, d, or s. The $3^3 = 27$ states decompose into the irreducible representations [4] of $SU(3)_{flavour}$,

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1_A} \oplus \mathbf{8_{M_\rho}} \oplus \mathbf{8_{M_\lambda}} \oplus \mathbf{10_S}, \tag{1.30}$$

where the symmetry of the representation is indicated by a subscript. This is the same decomposition into multiplets as for the $SU(3)_{colour}$ case. Members of these representations must be combined to give the correct charge, strangeness and isospin of the resulting baryon.

From table 1.5 it can be seen that the proton must be some permutation of quarks uud. Since the u and d quarks are members of an isospin doublet, they can be coupled together to form states of isospin $I = \frac{1}{2}$ and $I = \frac{3}{2}$

$$|I, I_z\rangle^{I_{12}} = \sum_{I_{z1} I_{z2} I_{z3} I_{z12}} \langle I_1, I_{z1}; I_2, I_{z2} | I_{12}, I_{z12}\rangle \langle I_{12}, I_{z12}; I_3, I_{z3} | I, I_z\rangle |I_1, I_{z1}\rangle |I_2, I_{z2}\rangle |I_3, I_{z3}\rangle \tag{1.31}$$

For example using table 1.5 we get that

$$\begin{aligned} |I, I_z\rangle^1 = |\tfrac{1}{2}, \tfrac{1}{2}\rangle^1 &= \langle \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2} | 1, 1\rangle \langle 1, 1; \tfrac{1}{2}, -\tfrac{1}{2} | \tfrac{1}{2}, \tfrac{1}{2}\rangle |u\rangle |u\rangle |d\rangle \\ &+ \langle \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2} | 1, 0\rangle \langle 1, 0; \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2}, \tfrac{1}{2}\rangle |u\rangle |d\rangle |u\rangle \end{aligned}$$

$$+ \quad \langle \tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2} | 1, 0 \rangle \langle 1, 0; \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2}, \tfrac{1}{2} \rangle |d\rangle |u\rangle |u\rangle,$$

$$\text{therefore } |\tfrac{1}{2}, \tfrac{1}{2}\rangle^1 = -\frac{1}{\sqrt{6}}(udu + duu - 2uud) \equiv \phi_p^\lambda. \tag{1.32}$$

The λ superscript indicates that this state has M_λ symmetry. It is a member of the M_λ $SU(3)_{flavour}$ octet. This symmetry arises from choosing the symmetric intermediate isospin state when combining the isospin wave functions. Alternatively,

$$\begin{aligned} |\tfrac{1}{2}, \tfrac{1}{2}\rangle^0 &= \langle \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, -\tfrac{1}{2} | 0, 0 \rangle \langle 0, 0; \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2}, \tfrac{1}{2} \rangle |u\rangle |d\rangle |u\rangle \\ &+ \langle \tfrac{1}{2}, -\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2} | 0, 0 \rangle \langle 0, 0; \tfrac{1}{2}, \tfrac{1}{2} | \tfrac{1}{2}, \tfrac{1}{2} \rangle |d\rangle |u\rangle |u\rangle \\ &= \frac{1}{\sqrt{2}}(ud - du)u \equiv \phi_p^\rho. \end{aligned} \tag{1.33}$$

This state has M_ρ symmetry and is a member of the M_ρ $SU(3)_{flavour}$ octet.

The zeroth order eigenstate for the physical proton contains a mixture of the flavour states ϕ_p^ρ and ϕ_p^λ combined to give a totally symmetric space-spin-flavour wave function.

The antisymmetric $SU(3)_{flavour}$ singlet state can only be formed by a combination three quarks each with different flavour [4].

It is chosen to be,

$$\phi_\Lambda^A = \frac{1}{\sqrt{6}}(sdu - sud + usd - dsu + dus - uds). \tag{1.34}$$

All the states in the M_λ or the M_ρ octets can be generated by the application of the $SU(3)_{flavour}$ raising and lowering operators [2], $U_\pm$, $I_\pm$, $V_\pm$, on to one member of the $SU(3)_{flavour}$ multiplet. These operators act on states which are $SU(2)_{flavour}$ subgroups of $SU(3)_{flavour}$ (see Fig. 1.1). I_+ annihilates a d quark and creates a u quark. U_+ annihilates an s quark and creates a d quark. V_+ annihilates an s quark and creates a

u quark. That is

$$\begin{aligned}
&I_+|u\rangle = 0, \quad I_+|d\rangle = |u\rangle, \quad I_+|s\rangle = 0,\\
&I_-|u\rangle = |d\rangle, \quad I_-|d\rangle = 0, \quad I_-|s\rangle = 0,\\
&U_+|u\rangle = 0, \quad U_+|d\rangle = 0, \quad U_+|s\rangle = |d\rangle,\\
&U_-|u\rangle = 0, \quad U_-|d\rangle = |s\rangle, \quad U_-|s\rangle = 0,\\
&V_+|u\rangle = |s\rangle, \quad V_+|d\rangle = 0, \quad V_+|s\rangle = 0,\\
&V_-|u\rangle = 0, \quad V_-|d\rangle = 0, \quad V_-|s\rangle = |u\rangle.
\end{aligned}$$

When acting on a three particle wave function $|q_1q_2q_3\rangle$, such as a Baryon, $I_\pm, U_\pm, V_\pm$ act on each quark in turn,

$$J|q_1q_2q_3\rangle = J(|q_1\rangle)|q_2q_3\rangle + |q_1\rangle J(|q_2\rangle)|q_3\rangle + |q_1q_2\rangle J(|q_3\rangle). \tag{1.35}$$

Since the algebra of $SU(2)$ is the algebra of angular momentum, the standard relation for raising and lowering angular momentum operators applies to these operators, that is,

$$\tilde{J}_\pm|\tilde{J},\tilde{J}_z\rangle = \sqrt{\tilde{J}(\tilde{J}+1) - \tilde{J}_z(\tilde{J}_z \pm 1)}|\tilde{J},\tilde{J}_z \pm 1\rangle \tag{1.36}$$

Here $\tilde{J}$ and $\tilde{J}_z$ represent the total J and its z component for the state. The square root factor in (1.36) is $\sqrt{2}$ when acting on states within an $SU(2)_{flavour}$ triplet; 1 when acting on states within an $SU(2)_{flavour}$ doublet; and of course zero when stepping out of an $SU(3)_{flavour}$ multiplet. So U_- acting on the flavour wave function ϕ_p^λ gives the state $\phi_{\Sigma+}^\lambda$. The other members of the Σ isospin triplet can then be generated by applying I_- [18].

$$\begin{aligned}
U_-\phi_p^\lambda &= U_-(-\frac{1}{\sqrt{6}}(udu + duu - 2uud))\\
&= -\frac{1}{\sqrt{6}}(usu + suu - 2uus)\\
&= \phi_{\Sigma+}^\lambda,
\end{aligned} \tag{1.37}$$

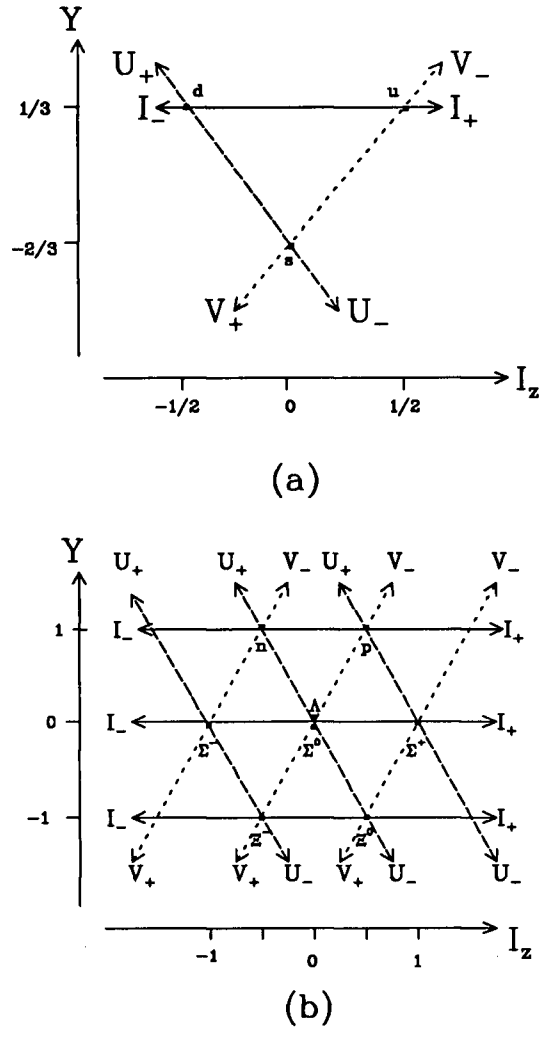


Figure 1.1: Illustration of the action of the $SU(3)_{flavour}$ raising and lowering operators in the $I_z - Y$ diagram ($Y \equiv B + S$) (a) of the quark triplet and (b) of the baryon octet.

$$\begin{aligned} I_-\phi^\lambda_{\Sigma^+} &= -\frac{1}{\sqrt{6}}(dsu + usd + sdu + sud - 2dus - 2uds) \\ &= \sqrt{2}\phi^\lambda_{\Sigma^0} \\ \Rightarrow \phi^\lambda_{\Sigma^0} &= -\frac{1}{\sqrt{12}}(sdu + sud + dsu + usd - 2(dus + uds)), \end{aligned} \tag{1.38}$$

$$\begin{aligned} I_-\phi^\lambda_{\Sigma^0} &= -\frac{1}{\sqrt{12}}(sdd + sdd + dsd + dsd - 2(dds + dds)) \\ &= \sqrt{2}\phi^\lambda_{\Sigma^-} \\ \Rightarrow \phi^\lambda_{\Sigma^-} &= -\frac{1}{\sqrt{6}}(sdd + dsd - 2dds). \end{aligned} \tag{1.39}$$

Λ is an isospin singlet state at the centre of the M_λ baryon octet. The centre member of the U-spin triplet which contains the neutron ($U = 1, U_z = 1$) and the Ξ^0 ($U = 1, U_z = -1$) is a linear combination of the isospin eigenstates Σ^0 and Λ,

$$\begin{aligned} |U = 1, U_z = 0\rangle &= \alpha|\phi^\lambda_{\Sigma^0}\rangle + \beta|\phi^\lambda_\Lambda\rangle \\ \Rightarrow U_-|U = 1, U_z = 1\rangle &= \sqrt{2}|U = 1, U_z = 0\rangle = \sqrt{2}(\alpha|\phi^\lambda_{\Sigma^0}\rangle + \beta|\phi^\lambda_\Lambda\rangle) \end{aligned} \tag{1.40}$$

$$\begin{aligned} U_-\phi^\lambda_n &= U_-(\frac{1}{\sqrt{6}}(dud + udd - 2ddu)) \\ &= \frac{1}{\sqrt{6}}(sud + dus + usd + uds - 2sdu - 2dsu) \end{aligned} \tag{1.41}$$

$$\Rightarrow I_+U_-\phi^\lambda_n = I_+\sqrt{2}(\alpha|\phi^\lambda_{\Sigma^0}\rangle + \beta|\phi^\lambda_\Lambda\rangle) = 2\alpha\phi^\lambda_{\Sigma^+} \tag{1.42}$$

because Λ is an isospin singlet.

I_+ commutes with U_-. This can be seen from the matrix representation[19] of the raising and lowering operators which act on the $SU(3)_{flavour}$ triplet $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$,

$$I_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; U_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.43}$$

$$I_+U_-\phi_n^\lambda = U_-I_+\phi_n^\lambda = U_-\phi_p^\lambda = \phi_{\Sigma+}^\lambda \tag{1.44}$$

$$\text{Comparing (1.42) and (1.44)} \Rightarrow \alpha = \frac{1}{2}. \tag{1.45}$$

Now since $|\alpha|^2 + |\beta|^2 = 1$ we get $|\beta|^2 = \frac{3}{4}$

$$\Rightarrow \beta = \pm\frac{\sqrt{3}}{2} \tag{1.46}$$

The sign of β is not uniquely determined but we fix it to be positive for our flavour wave functions to agree with those of ref.[17]. From (1.40),

$$U_-\phi_n^\lambda = \frac{1}{\sqrt{2}}\phi_{\Sigma^0}^\lambda + \sqrt{\frac{3}{2}}\phi_\Lambda^\lambda. \tag{1.47}$$

This is the centre member of the U-spin triplet. From (1.38) and (1.41),

$$\Rightarrow \phi_\Lambda^\lambda = \frac{1}{2}(sud + usd - sdu - dsu). \tag{1.48}$$

Similarly for the M_ρ octet we can generate all the states ϕ_X^λ in the mixed antisymmetric octet. The baryon octet states are listed in table 1.6.

The decuplet (fully symmetric) states are simpler than the octet states since there are no mixed states. All the states can be generated by applying the raising and lowering $SU(3)_{flavour}$ operators to one of the members of the decuplet[6], for example uuu. These states are listed in table 1.7.

1.8.1 The *uds* Basis

As mentioned previously, it has been proposed [20],[21] that the *uds* is a more appropriate basis than the fully symmetrized $SU(6)$ basis when $SU(3)_{flavour}$ is broken.

In the *uds* basis the strange quark is treated as distinguishable and singled out as quark 3. We only symmetrize between the two equal mass (u and d quarks). In this

X	ϕ_X^λ	ϕ_X^ρ
p	$-\frac{1}{\sqrt{6}}(udu + duu - 2uud)$	$\frac{1}{\sqrt{2}}(udu - duu)$
n	$\frac{1}{\sqrt{6}}(dud + udd - 2ddu)$	$\frac{1}{\sqrt{2}}(udd - dud)$
Σ^+	$-\frac{1}{\sqrt{6}}(usu + suu - 2uus)$	$\frac{1}{\sqrt{2}}(usu - suu)$
Σ^0	$-\frac{1}{\sqrt{12}}(sdu + sud + dsu + usd$ $-2(dus + uds))$	$\frac{1}{2}(dsu + usd - sud - sdu)$
Σ^-	$-\frac{1}{\sqrt{6}}(sdd + dsd - 2dds)$	$\frac{1}{\sqrt{2}}(dsd - sdd)$
Λ	$\frac{1}{2}(sud + usd - sdu - dsu)$	$\frac{1}{\sqrt{12}}(sdu - sud + usd - dsu - 2(dus - uds))$
Ξ^-	$\frac{1}{\sqrt{6}}(dss + sds - 2ssd)$	$\frac{1}{\sqrt{2}}(dss - sds)$
Ξ^0	$\frac{1}{\sqrt{6}}(uss + sus - 2ssu)$	$\frac{1}{\sqrt{2}}(uss - sus)$

Table 1.6: Flavour wave functions in the baryon octet. The physical particle X is some mixture of the flavour states ϕ_X^λ and ϕ_X^ρ to be determined later.

$I = \frac{3}{2}$	ϕ_X^S
Δ^{++}	uuu
Δ^{+}	$\frac{1}{\sqrt{3}}(duu + udu + uud)$
Δ^{0}	$\frac{1}{\sqrt{3}}(ddu + dud + udd)$
Δ^{-}	ddd
$I = 1$	
Σ^{+}	$\frac{1}{\sqrt{3}}(uus + usu + suu)$
Σ^{0}	$\frac{1}{\sqrt{6}}(dus + uds + dsu + usd + sdu + sud)$
Σ^{-}	$\frac{1}{\sqrt{3}}(dds + dsd + sdd)$
$I = \frac{1}{2}$	
Ξ^{0}	$\frac{1}{\sqrt{3}}(ssu + sus + uss)$
Ξ^{-}	$\frac{1}{\sqrt{3}}(ssd + sds + dss)$
$I = 0$	
Ω^{-}	sss

Table 1.7: Fully symmetrized flavour wave functions in the baryon decuplet.

basis the only two flavour states are,

$$\phi_\Lambda \equiv \frac{1}{\sqrt{2}}(ud - du)s, \tag{1.49}$$

$$\text{and } \phi_\Sigma \equiv \frac{1}{\sqrt{2}}(ud + du)s. \tag{1.50}$$

ϕ_Λ has the u and d quarks coupling to isospin $I = 0$. ϕ_Λ therefore corresponds to the flavour wave function of a Λ particle. ϕ_Σ has $I = 1$ and so corresponds to a Σ^0.

1.9 Spin Wave Functions

Three spin $\frac{1}{2}$ particles (for example quarks) may give rise to a total spin[4] $S = \frac{3}{2}$ or $S = \frac{1}{2}$. As in the case of flavour, the spin wave function for a proton can be obtained by

[4]In this section S denotes the quantum number for the intrinsic spin magnitude.

combining spins of particle 1 and 2 to $S = 1$ and then coupling to the spin of particle 3 to yield $S = \frac{1}{2}$. This gives a spin wave function with M_λ type symmetry. Alternatively the spins of particles 1 and 2 can be combined to $S = 0$ and then coupled to the spin of particle 3 to yield $S = \frac{1}{2}$. This gives a spin wave function with M_ρ type symmetry.

For the spin up state

$$\begin{aligned}\chi^{\lambda}_{\frac{1}{2},\frac{1}{2}} \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle^1 &= \langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|1,1\rangle\langle 1,1;\tfrac{1}{2},-\tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle|\uparrow\rangle|\uparrow\rangle|\downarrow\rangle \\ &+ \langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}|1,0\rangle\langle 1,0;\tfrac{1}{2},\tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle \\ &+ \langle\tfrac{1}{2},-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|1,0\rangle\langle 1,0;\tfrac{1}{2},\tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle\end{aligned}$$

$$\text{therefore } \chi^{\lambda}_{\frac{1}{2},\frac{1}{2}} = -\frac{1}{\sqrt{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow). \tag{1.51}$$

The λ superscript indicates that this state has M_λ symmetry. This symmetry arises from choosing the symmetric intermediate spin state when combining the spin wave functions. Similarly,

$$\begin{aligned}\chi^{\rho}_{\frac{1}{2},\frac{1}{2}} &= \langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}|0,0\rangle\langle 0,0;\tfrac{1}{2},\tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle|\uparrow\rangle|\downarrow\rangle|\uparrow\rangle \\ &+ \langle\tfrac{1}{2},-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|0,0\rangle\langle 0,0;\tfrac{1}{2},\tfrac{1}{2}|\tfrac{1}{2},\tfrac{1}{2}\rangle|\downarrow\rangle|\uparrow\rangle|\uparrow\rangle \\ &= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\uparrow .\end{aligned} \tag{1.52}$$

This is analogous to the flavour wave function ϕ^{ρ}_{p}. Also,

$$\chi_{\frac{3}{2},\frac{3}{2}} = \langle\tfrac{1}{2},\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}|1,1\rangle\langle 1,1;\tfrac{1}{2},\tfrac{1}{2}|\tfrac{3}{2},\tfrac{3}{2}\rangle|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle = |\uparrow\uparrow\uparrow\rangle \text{ etc.} \tag{1.53}$$

States of different m_S follow from the Condon-Shortley convention. Note that for the case of coupling three spin $\frac{1}{2}$ particles together ($SU(2)_{spin}$), the restriction to two possible spin directions means that only mixed symmetric and fully symmetric irreducible representations of $SU(2)_{spin}$ can be formed. There is no fully antisymmetric state. That is

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4_S} \oplus \mathbf{2_{M_\lambda}} \oplus \mathbf{2_{M_\rho}}. \tag{1.54}$$

1.10 Combining Flavour, Spin and Spatial Wave Functions.

Each quark not only has 3 flavour types but two possible spin directions. We combine the $SU(2)_{spin}$ and the $SU(3)_{flavour}$ multiplet structure into $SU(6)_{SF}$ ($SF \equiv$*spin-flavour*) multiplets.

For baryons, Young diagram techniques[4] give,

$$\mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} = \mathbf{56_S} \oplus \mathbf{70_{M_\lambda}} \oplus \mathbf{70_{M_\rho}} \oplus \mathbf{20_A}. \tag{1.55}$$

In the $SU(6)$ basis we wish to construct states that are totally symmetric in space, spin and flavour so that when combined with the colour part we get a fully antisymmetric wave function. In the *uds* basis we construct states that are totally symmetric in space, spin and flavour only with respect to quarks one and two.

1.10.1 Permutation Group Addition Coefficients

To combine wave functions of different permutation symmetry it is convenient to introduce *Permutation group addition coefficients* [22]. These are completely analogous to the Clebsch-Gordan coefficients of the rotation group. For example to combine two wave functions of permutation symmetry P_1 and P_2 respectively, to permutation symmetry P then,

$$\psi^{P,\kappa} = \sum_{\kappa_1,\kappa_2} \begin{pmatrix} P_1 & P_2 & P \\ \kappa_1 & \kappa_2 & \kappa \end{pmatrix} \psi^{P_1,\kappa_1}\, \psi^{P_2,\kappa_2}, \tag{1.56}$$

$$\text{where } \kappa_i = \begin{cases} 1 & \text{if } P_i = A, S, M_\lambda \\ 2 & \text{if } P_i = M_\rho. \end{cases}$$

The non-zero coefficients with phase factors chosen such that they are all real are,

$$\begin{pmatrix} S & S & S \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} S & A & A \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A & S & A \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} A & A & S \\ 1 & 1 & 1 \end{pmatrix} = 1,$$

$$\begin{pmatrix} S & M & M \\ 1 & \kappa & \lambda \end{pmatrix} = \begin{pmatrix} M & S & M \\ \kappa & 1 & \lambda \end{pmatrix} = \sqrt{2} \begin{pmatrix} M & M & S \\ \kappa & \lambda & 1 \end{pmatrix} = \delta_{\kappa\lambda},$$

$$\begin{pmatrix} A & M & M \\ 1 & \kappa & \lambda \end{pmatrix} = \begin{pmatrix} M & A & M \\ \kappa & 1 & \lambda \end{pmatrix} = -\sqrt{2} \begin{pmatrix} M & M & A \\ \kappa & \lambda & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\kappa\lambda},$$

$$-\begin{pmatrix} M & M & M \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} M & M & M \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} M & M & M \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} M & M & M \\ 2 & 2 & 1 \end{pmatrix} = \tfrac{1}{\sqrt{2}}.$$

In addition the spin must also be coupled to the orbital angular momentum to yield total angular momentum J for the particle.

Using the notation[15]

$$|X_\mu^{2S+1}L_\sigma J^P\rangle,$$

where X is the particle n, p, Λ etc; μ the $SU(3)_{flavour}$ multiplicity (that is whether the flavour wave functions are in the octet, decuplet, or singlet irreducible representations of $SU(3)_{flavour}$ (see equation (1.30)); L, S, J is the orbital (S,P,D,F,... etc), spin, and total, angular momentum respectively; P is the parity of the spatial wave function; σ is the symmetry of the spatial wave function or the symmetry of the $SU(6)_{SF}$ multiplet.

$$\begin{aligned} |X_\mu^{2S+1}L_\sigma J^P\rangle \;=\; & \sum_{\kappa_1,\kappa_2,\kappa_3} \sum_{M_L,M_S} \begin{pmatrix} P_1 & P_2 & \sigma \\ \kappa_1 & \kappa_2 & \kappa_{12} \end{pmatrix} \begin{pmatrix} \sigma & \sigma & S \\ \kappa_{12} & \kappa_3 & 1 \end{pmatrix} \\ & \times \; \langle S, M_S; L, M_L | J, M_J \rangle \chi_{S,M_S}^{P_1,\kappa_1} \, \phi_X^{P_2,\kappa_2} \, \Phi_{NLM}^{\sigma,\kappa_3}. \end{aligned} \tag{1.57}$$

For our purposes we will need the $J^P = \frac{1}{2}^+$ states up to $N = 2$ (that is energy $2\hbar\omega$ above the ground state). $N = 1$ provide only $L = 1$ states and so have negative parity. For $L = 0$ all the Clebsch-Gordan coefficients give 1.

1.10.2 The $SU(6)$ Ground State

At the ground state ($N = 0$) where we have only Φ^S_{000} ($X = N, \Lambda, \Sigma^0$),

$$\begin{aligned}
|X_8{}^2S_S\tfrac{1}{2}^+\rangle &= \sum_{\kappa_1,\kappa_2} \begin{pmatrix} P_1 & P_2 & S \\ \kappa_1 & \kappa_2 & 1 \end{pmatrix} \begin{pmatrix} S & S & S \\ 1 & 1 & 1 \end{pmatrix} \langle \tfrac{1}{2},\tfrac{1}{2};0,0|\tfrac{1}{2},\tfrac{1}{2}\rangle \\
&\times \Phi^S_{000}\chi^{P_1,\kappa_1}_{\frac{1}{2},\frac{1}{2}}\phi^{P_2,\kappa_2}_X \\
&= \begin{pmatrix} M & M & S \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} S & S & S \\ 1 & 1 & 1 \end{pmatrix} \Phi^S_{000}\chi^{\rho}_{\frac{1}{2},\frac{1}{2}}\phi^{\rho}_X \\
&+ \begin{pmatrix} M & M & S \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} S & S & S \\ 1 & 1 & 1 \end{pmatrix} \Phi^S_{000}\chi^{\lambda}_{\frac{1}{2},\frac{1}{2}}\phi^{\lambda}_X \\
&= \frac{1}{\sqrt{2}}\Phi^S_{000}(\chi^{\rho}_{\frac{1}{2},\frac{1}{2}}\phi^{\rho}_X + \chi^{\lambda}_{\frac{1}{2},\frac{1}{2}}\phi^{\lambda}_X). \qquad (1.58)
\end{aligned}$$

1.10.3 The $SU(6)$ Excited States

Radial Excitations

At $N = 2$ we have,

$$\begin{aligned}
|X_8{}^2S_{S'}\tfrac{1}{2}^+\rangle &= \sum_{\kappa_1,\kappa_2} \begin{pmatrix} P_1 & P_2 & S \\ \kappa_1 & \kappa_2 & 1 \end{pmatrix} \begin{pmatrix} S & S & S \\ 1 & 1 & 1 \end{pmatrix} \langle \tfrac{1}{2},\tfrac{1}{2};0,0|\tfrac{1}{2},\tfrac{1}{2}\rangle \\
&\times \Phi^S_{200}\chi^{P_1,\kappa_1}_{\frac{1}{2},\frac{1}{2}}\phi^{P_2,\kappa_2}_X \\
&= \frac{1}{\sqrt{2}}\Phi^S_{200}(\chi^{\rho}_{\frac{1}{2},\frac{1}{2}}\phi^{\rho}_X + \chi^{\lambda}_{\frac{1}{2},\frac{1}{2}}\phi^{\lambda}_X) \qquad (1.59)
\end{aligned}$$

The prime on the S indicates we are referring to the excited ($N = 2$) symmetric spatial state, Φ^S_{200}, rather than the ground state symmetric spatial wave function, Φ^S_{000}.

$$|X_8{}^2S_M\tfrac{1}{2}^+\rangle = \sum_{\kappa_1,\kappa_2,\kappa_3} \begin{pmatrix} P_1 & P_2 & M \\ \kappa_1 & \kappa_2 & \kappa_{12} \end{pmatrix} \begin{pmatrix} M & M & S \\ \kappa_{12} & \kappa_3 & 1 \end{pmatrix}$$

$$\begin{aligned} &\times \ \langle \tfrac{1}{2},\tfrac{1}{2};0,0|\tfrac{1}{2},\tfrac{1}{2}\rangle \Phi_{200}^{M,\kappa_3} \chi_{\frac{1}{2},\frac{1}{2}}^{P_1,\kappa_1} \phi_N^{P_2,\kappa_2} \\ &= \ \frac{1}{2}\{\Phi_{200}^{\lambda}(\chi_{\frac{1}{2},\frac{1}{2}}^{\rho}\phi_X^{\rho} - \chi_{\frac{1}{2},\frac{1}{2}}^{\lambda}\phi_X^{\lambda}) + \Phi_{200}^{\rho}(\chi_{\frac{1}{2},\frac{1}{2}}^{\rho}\phi_X^{\lambda} + \chi_{\frac{1}{2},\frac{1}{2}}^{\lambda}\phi_X^{\rho})\}. \end{aligned} \tag{1.60}$$

For the $SU(3)_{flavour}$ decuplet (of which out of X only Σ^0 is a member)

$$\begin{aligned} |X_{10}^2 S_M \tfrac{1}{2}^+\rangle \ &= \ \sum_{\kappa_1,\kappa_3} \begin{pmatrix} P_1 & S & P_{12} \\ \kappa_1 & 1 & \kappa_{12} \end{pmatrix} \begin{pmatrix} P_{12} & M & S \\ \kappa_{12} & \kappa_3 & 1 \end{pmatrix} \\ &\times \ \langle \tfrac{1}{2},\tfrac{1}{2};0,0|\tfrac{1}{2},\tfrac{1}{2}\rangle \Phi_{200}^{M,\kappa_3} \chi_{\frac{1}{2},\frac{1}{2}}^{P_1,\kappa_1} \phi_X^{S} \\ &= \ \begin{pmatrix} M & S & M \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} M & M & S \\ 1 & 1 & 1 \end{pmatrix} \phi_X^S \chi_{\frac{1}{2},\frac{1}{2}}^{\lambda} \Phi_{200}^{\lambda} \\ &+ \ \begin{pmatrix} M & S & M \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} M & M & S \\ 2 & 2 & 1 \end{pmatrix} \phi_X^S \chi_{\frac{1}{2},\frac{1}{2}}^{\rho} \Phi_{200}^{\rho} \\ &= \ \frac{1}{\sqrt{2}} \phi_X^S (\chi_{\frac{1}{2},\frac{1}{2}}^{\lambda} \Phi_{200}^{\lambda} + \chi_{\frac{1}{2},\frac{1}{2}}^{\rho} \Phi_{200}^{\rho}). \end{aligned} \tag{1.61}$$

For the $SU(3)_{flavour}$ singlet irreducible representation,

$$\begin{aligned} |X_1^2 S_M \tfrac{1}{2}^+\rangle \ &= \ \sum_{\kappa_1,\kappa_3} \begin{pmatrix} P_1 & A & P_{12} \\ \kappa_1 & 1 & \kappa_{12} \end{pmatrix} \begin{pmatrix} P_{12} & M & S \\ \kappa_{12} & \kappa_3 & 1 \end{pmatrix} \\ &\times \ \langle \tfrac{1}{2},\tfrac{1}{2};0,0|\tfrac{1}{2},\tfrac{1}{2}\rangle \Phi_{200}^{M,\kappa_3} \chi_{\frac{1}{2},\frac{1}{2}}^{P_1,\kappa_1} \phi_X^{A} \\ &= \ \begin{pmatrix} M & A & M \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} M & M & S \\ 1 & 1 & 1 \end{pmatrix} \phi_X^A \chi_{\frac{1}{2},\frac{1}{2}}^{\lambda} \Phi_{200}^{\lambda} \\ &+ \ \begin{pmatrix} M & A & M \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} M & M & S \\ 2 & 2 & 1 \end{pmatrix} \phi_X^A \chi_{\frac{1}{2},\frac{1}{2}}^{\rho} \Phi_{200}^{\rho} \end{aligned}$$

$$= \frac{1}{\sqrt{2}}\phi_X^A(\chi^\rho_{\frac{1}{2},\frac{1}{2}}\Phi^\lambda_{200} - \chi^\lambda_{\frac{1}{2},\frac{1}{2}}\Phi^\rho_{200}). \tag{1.62}$$

Orbital Excitations

We combine $L = 2$, $S = \frac{3}{2}$ states to give $J^P = \frac{1}{2}^+$. In this case the Clebsch-Gordan coefficients are not all unity.

$$\begin{aligned}
|X_8{}^4D_M\tfrac{1}{2}^+\rangle &= \sum_{\kappa_1,\kappa_2,\kappa_3}\sum_{M_L,M_S}\begin{pmatrix} P_1 & P_2 & M \\ \kappa_1 & \kappa_2 & \kappa_{12}\end{pmatrix}\begin{pmatrix} M & M & S \\ \kappa_{12} & \kappa_3 & 1\end{pmatrix} \\
&\times \langle\tfrac{3}{2},M_S;2,M_L|\tfrac{1}{2},\tfrac{1}{2}\rangle\Phi^{M,\kappa_3}_{22M_L}\chi^{P_1,\kappa_1}_{\frac{3}{2},M_S}\phi_X^{P_2,\kappa_2} \\
&= \frac{1}{\sqrt{2}}\{-\frac{1}{\sqrt{10}}\chi^S_{\frac{3}{2},\frac{3}{2}}(\Phi^\lambda_{22-1}\phi^\lambda_X + \Phi^\rho_{22-1}\phi^\rho_X) \\
&+ \frac{1}{\sqrt{5}}\chi^S_{\frac{3}{2},\frac{1}{2}}(\Phi^\lambda_{220}\phi^\lambda_X + \Phi^\rho_{220}\phi^\rho_X) \\
&- \sqrt{\frac{3}{10}}\chi^S_{\frac{3}{2},-\frac{1}{2}}(\Phi^\lambda_{221}\phi^\lambda_X + \Phi^\rho_{221}\phi^\rho_X) \\
&+ \sqrt{\frac{2}{5}}\chi^S_{\frac{3}{2},-\frac{3}{2}}(\Phi^\lambda_{222}\phi^\lambda_X + \Phi^\rho_{222}\phi^\rho_X)\}.
\end{aligned} \tag{1.63}$$

For the $SU(3)_{flavour}$ decuplet,

$$\begin{aligned}
|X_{10}{}^4D_S\tfrac{1}{2}^+\rangle &= \sum_{M_L,M_S}\begin{pmatrix} S & S & S \\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix} S & S & S \\ 1 & 1 & 1\end{pmatrix} \\
&\times \langle\tfrac{3}{2},M_S;2,M_L|\tfrac{1}{2},\tfrac{1}{2}\rangle\Phi^S_{22M_L}\chi^S_{\frac{3}{2},M_S}\phi^S_X \\
&= \phi^S_X\{-\frac{1}{\sqrt{10}}\chi^S_{\frac{3}{2},\frac{3}{2}}\Phi^S_{22-1} + \frac{1}{\sqrt{5}}\chi^S_{\frac{3}{2},\frac{1}{2}}\Phi^S_{220} \\
&- \sqrt{\frac{3}{10}}\chi^S_{\frac{3}{2},-\frac{1}{2}}\Phi^S_{221} + \sqrt{\frac{2}{5}}\chi^S_{\frac{3}{2},-\frac{3}{2}}\Phi^S_{222}\}.
\end{aligned} \tag{1.64}$$

There exists no fully antisymmetric $L = 2$ spatial wave function to go with the $SU(3)_{flavour}$ singlet flavour state and the symmetric spin state.

We neglect the tensor part of the hyperfine interaction (§1.11), and so these orbital excitations will not contribute to the wave functions we use. We list them for completeness.

1.10.4 The *uds* Basis States

As mentioned previously when we take into account the different mass of the strange quark we require only symmetry with respect to the two light quarks which are in position 1 and 2. This corresponds to the total wave function possessing M_λ symmetry. For example, the ground state lambda has spatial wave function Φ^S_{000} and flavour wave function ϕ_Λ (in the *uds* basis). It therefore must combine with a χ^ρ spin wave function to give a total wave function with M_λ symmetry.

The strange states in the *uds* basis are given in Table 1.8.

1.11 Hyperfine Interactions

With the zeroth order eigenstates established we can now turn to calculating the effects of the hyperfine interaction.

Analogous to Q.E.D., in Q.C.D. one expects a hyperfine interaction of two quarks i and j in the same baryon, which to order α_s is,

$$H_{hyp} = \frac{2}{3}\sum_{i<j}\frac{\alpha_s}{m_i m_j}\{\frac{8\pi}{3}\vec{S}_i\cdot\vec{S}_j\delta^3(\vec{r_{ij}}) + \frac{1}{r^3_{ij}}[\frac{3\vec{S}_i\cdot\vec{r_{ij}}\vec{S}_j\cdot\vec{r_{ij}}}{r^2_{ij}} - \vec{S}_i\cdot\vec{S}_j]\}. \qquad (1.65)$$

Where $r_{ij} \equiv |\vec{r_i} - \vec{r_j}|$, $\vec{S}_i \equiv \frac{1}{2}\vec{\sigma}$ is the spin vector operator for the i^{th} quark in the baryon. α_s is the effective quark-gluon coupling constant, analogous to the electromagnetic coupling constant $\alpha = 1/137$, and is determined by calculating the $N - \Delta$ mass difference. α_s depends on how large an oscillator basis is chosen[7], we take up to $N = 2$ states.

The spin-orbit part of the potential has been neglected. Since it has been found experimentally that spin-orbit effects are reduced to a level of less than 10% of naive

$$|\Lambda^2 S\rangle = \Phi^S_{000}\phi_\Lambda\chi^\rho_{\frac{1}{2},\frac{1}{2}} \text{ the } \Lambda \text{ ground state}$$

$$|\Lambda^2 S_{\rho\lambda}\rangle = \Phi^{\rho\lambda}_{200}\phi_\Lambda\chi^\lambda_{\frac{1}{2},\frac{1}{2}}$$

$$|\Lambda^2 S_{\rho\rho}\rangle = \Phi^{\rho\rho}_{200}\phi_\Lambda\chi^\rho_{\frac{1}{2},\frac{1}{2}}$$

$$|\Lambda^2 S_{\lambda\lambda}\rangle = \Phi^{\lambda\lambda}_{200}\phi_\Lambda\chi^\rho_{\frac{1}{2},\frac{1}{2}}$$

$$|\Lambda^4 D_{\rho\lambda}\rangle = \textstyle\sum_m \langle 2, m; \frac{3}{2}, (\frac{1}{2} - m)|\frac{1}{2}, \frac{1}{2}\rangle \Phi^{\rho\lambda}_{200}\phi_\Lambda\chi^S_{\frac{3}{2},(\frac{1}{2}-m)}$$

$$|\Sigma^2 S\rangle = \Phi^S_{000}\phi_\Sigma\chi^\lambda_{\frac{1}{2},\frac{1}{2}} \text{ the } \Sigma \text{ ground state}$$

$$|\Sigma^2 S_{\rho\lambda}\rangle = \Phi^{\rho\lambda}_{200}\phi_\Sigma\chi^\rho_{\frac{1}{2},\frac{1}{2}}$$

$$|\Sigma^2 S_{\rho\rho}\rangle = \Phi^{\rho\rho}_{200}\phi_\Sigma\chi^\lambda_{\frac{1}{2},\frac{1}{2}}$$

$$|\Sigma^2 S_{\lambda\lambda}\rangle = \Phi^{\lambda\lambda}_{200}\phi_\Sigma\chi^\lambda_{\frac{1}{2},\frac{1}{2}}$$

$$|\Sigma^4 D_{\rho\rho}\rangle = \textstyle\sum_m \langle 2, m; \frac{3}{2}, (\frac{1}{2} - m)|\frac{1}{2}, \frac{1}{2}\rangle \Phi^{\rho\rho}_{200}\phi_\Sigma\chi^S_{\frac{3}{2},(\frac{1}{2}-m)}$$

$$|\Sigma^4 D_{\lambda\lambda}\rangle = \textstyle\sum_m \langle 2, m; \frac{3}{2}, (\frac{1}{2} - m)|\frac{1}{2}, \frac{1}{2}\rangle \Phi^{\lambda\lambda}_{200}\phi_\Sigma\chi^S_{\frac{3}{2},(\frac{1}{2}-m)}$$

Table 1.8: The uds basis states we will need, combined so that they have M_λ symmetry. Notation $|X^{2S+1}L_{\sigma_1\sigma_2}\rangle$.

expectations of one-gluon-exchange [17]. Isgur and Karl speculate that Thomas precession may cancel the spin-orbit effect.

The Fermi contact term (the first term in equation (1.65) is operative only when the pair (i & j) have zero orbital angular momentum.

The second term is the tensor term. There is an effective 'colour-magnetic' field due to the spin of the quark. It is directly analogous to the tensor term present in the nucleon-nucleon potential. The tensor force couples S and D states of the same J^P. The tensor operator gives zero when acting upon singlet states. We do not include effects of this tensor operator as its effect is negligible in the ground state baryons [23]. As a result there will be no D state admixtures in our wave functions.

In the $S = 0$ or $S = -1$ sectors (1.65) can be written

$$H_{hyp} = \sum_{i<j}(1-(1-x)(\delta_{is}+\delta_{js}))\tilde{H}^{ij}_{hyp} \tag{1.66}$$

$$\text{where } \tilde{H}^{ij}_{hyp} \equiv \frac{2}{3}\frac{\alpha_s}{m_u^2}\{\frac{8\pi}{3}\vec{S}_i\cdot\vec{S}_j\delta^3(\vec{r_{ij}})+\frac{1}{r_{ij}^3}[\frac{3\vec{S}_i\cdot\vec{r_{ij}}\vec{S}_j\cdot\vec{r_{ij}}}{r_{ij}^2}-\vec{S}_i\cdot\vec{S}_j]\} \tag{1.67}$$

$$\text{and } x \equiv m_u/m_s. \tag{1.68}$$

H_{hyp} will not connect $L = 1$ and $L = 0$ states so matrix elements between the ground state ($N = 0$) and $N = 1$ states will give zero.

1.11.1 Hyperfine Mixing in the $SU(6)$ basis

In the $SU(6)$ basis, H_{hyp} will lead to mixing within a given $SU(6)$ multiplet and also *between* different $SU(6)$ multiplets with the same isospin and J^P[21].

In the $SU(6)$ basis $x = 1$, so the hyperfine matrix elements between the states ϕ_m and ϕ_n, take the form:

$$H^{SU(6)}_{mn} = \langle\phi^{SU(6)}_m(1,2,3)|(\tilde{H}^{12}_{hyp}+\tilde{H}^{13}_{hyp}+\tilde{H}^{23}_{hyp})|\phi^{SU(6)}_n(1,2,3)\rangle.$$

$(1,2,3)$ denotes quark positions. In particular,

$$\begin{aligned}\langle\phi_m^{SU(6)}(1,2,3)|\tilde{H}_{hyp}^{12}|\phi_n^{SU(6)}(1,2,3)\rangle &= \langle\phi_m^{SU(6)}(1,3,2)|\tilde{H}_{hyp}^{13}|\phi_n^{SU(6)}(1,3,2)\rangle \\ &= (-1)^2\langle\phi_m^{SU(6)}(1,2,3)|\tilde{H}_{hyp}^{13}|\phi_n^{SU(6)}(1,2,3)\rangle\end{aligned}$$

Where we have relabeled the quark positions and then used the complete permutational anti-symmetry of the $SU(6)$ baryon wave functions. That is the matrix elements $\tilde{H}_{hyp}^{13} = \tilde{H}_{hyp}^{12}$. Similarly we get $\tilde{H}_{hyp}^{23} = \tilde{H}_{hyp}^{12}$. Therefore

$$\langle\phi_m^{SU(6)}(1,2,3)|\sum_{i<j}\tilde{H}_{hyp}^{ij}|\phi_0^{SU(6)}(1,2,3)\rangle = 3\langle\phi_m^{SU(6)}(1,2,3)|\tilde{H}_{hyp}^{12}|\phi_0^{SU(6)}(1,2,3)\rangle. \tag{1.69}$$

$SU(6)$ Basis Compositions.

H_{hyp} has non-diagonal matrix elements between different supermultiplets leading to wave function distortions and second order mass shifts. Using

$$\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$$

and the relation

$$\delta^3(a\vec{x}) = \frac{1}{|a|^3}\delta(\vec{x}) \tag{1.70}$$

in (1.67) with $i = 1$, $j = 2$, we get,

$$\tilde{H}_{12} = \vec{S}_1 \cdot \vec{S}_2\,\delta^3(\vec{\rho})\left(\frac{2\pi\sqrt{\pi}}{3\alpha^3}\right)\delta \tag{1.71}$$

$$\text{with } \delta \equiv \frac{4\alpha_s\alpha^3}{3\sqrt{2\pi}m_u^2} \tag{1.72}$$

We also need the relations:

$$\langle\chi_m^\rho|\vec{S}_1 \cdot \vec{S}_2|\chi_{m'}^\rho\rangle = -\frac{3}{4}\delta_{m,m'}, \tag{1.73}$$

$$\langle \chi^{\lambda}_{m} | \vec{S}_1 \cdot \vec{S}_2 | \chi^{\lambda}_{m'} \rangle = \frac{1}{4}\delta_{m,m'}, \tag{1.74}$$

$$\langle \Phi^{S}_{000} | \delta^3(\vec{\rho}) | \Phi^{S}_{000} \rangle = \psi_{000}(0)\psi_{000}(0) = \frac{\alpha^3}{\pi\sqrt{\pi}}, \tag{1.75}$$

$$\langle \Phi^{S}_{200} | \delta^3(\vec{\rho}) | \Phi^{S}_{000} \rangle = \frac{\sqrt{3}}{2}\frac{\alpha^3}{\pi\sqrt{\pi}}, \tag{1.76}$$

$$\langle \Phi^{\rho}_{200} | \delta^3(\vec{\rho}) | \Phi^{S}_{000} \rangle = 0, \tag{1.77}$$

$$\langle \Phi^{\lambda}_{200} | \delta^3(\vec{\rho}) | \Phi^{S}_{000} \rangle = \frac{\sqrt{3}}{2}\frac{\alpha^3}{\pi\sqrt{\pi}}. \tag{1.78}$$

The relevant hyperfine matrix elements are

$$\langle X_8^2 S_S | H_{hyp} | X_8^2 S_S \rangle = -\frac{1}{2}\delta, \tag{1.79}$$

$$\langle X_8^2 S_{S'} | H_{hyp} | X_8^2 S_S \rangle = -\frac{\sqrt{3}}{4}\delta, \tag{1.80}$$

$$\langle X_8^2 S_M | H_{hyp} | X_8^2 S_S \rangle = -\frac{\sqrt{6}}{4}\delta, \tag{1.81}$$

$$\langle \Lambda_1^2 S_M | H_{hyp} | \Lambda_8^2 S_S \rangle = 0, \tag{1.82}$$

$$\langle \Sigma_{10}^2 S_M | H_{hyp} | \Sigma_{10}^2 S_S \rangle = 0, \tag{1.83}$$

where $X = N, \Lambda, \Sigma^0$. Using the masses and compositions of the excited states from ref.[17] given in table 1.9, along with the zeroth order ground state nucleon wave function, as a basis and taking $\delta = 260\,MeV$ [23] we get, for the nulceon hyperfine mixing matrix,

$$\begin{pmatrix} 1005 & 138.5 & 132.8 \\ 138.5 & 1405 & 0 \\ 132.8 & 0 & 1705 \end{pmatrix} \begin{pmatrix} N_0 \\ N(1405) \\ N(1705) \end{pmatrix}. \tag{1.84}$$

N_0 represents the zeroth order ground state nucleon wave function, $|N_8^2 S_S\rangle$.

We have used the results

$$1005 = 1135 + \langle N_8^2 S_S | H_{hyp} | N_8^2 S_S \rangle$$

X	$X_8^2S_{S'}$	$X_8^2S_{S_M}$	$X_1^2S_{S_M}$	$X_{10}^2S_{S_M}$
$N(1405)$	-0.99	-0.17	—	—
$N(1705)$	$+0.15$	-0.94	—	—
$\Lambda(1555)$	$+0.99$	$+0.02$	$+0.10$	—
$\Lambda(1740)$	$+0.09$	$+0.30$	-0.95	—
$\Lambda(1860)$	-0.04	$+0.91$	$+0.29$	—
$\Sigma(1640)$	-0.97	-0.23	—	$+0.08$
$\Sigma(1910)$	-0.20	$+0.91$	—	$+0.17$
$\Sigma(1995)$	-0.08	$+0.27$	—	-0.23

Table 1.9: Excited baryon compositions, in the $SU(6)$ basis (from ref. [17]), adjusted to our phase convention.

$$\begin{aligned} 138.5 &= -0.99\langle X_8^2S_{S'}|H_{hyp}|X_8^2S_S\rangle - 0.17\langle X_8^2S_M|H_{hyp}|X_8^2S_S\rangle \\ 132.8 &= +0.15\langle X_8^2S_{S'}|H_{hyp}|X_8^2S_S\rangle - 0.94\langle X_8^2S_M|H|X_8^2S_S\rangle \end{aligned}$$

The matrix in (1.84) has

$$N(941) = +0.95N_8^2S_S + 0.25N_8^2S_{S'} + 0.20N_8^2S_M \tag{1.85}$$

as its lowest eigenvalue (in MeV) and corresponding eigenvector. Similarly for the Λ and Σ^0 in the $SU(6)$ basis we obtain

$$\Lambda(957) = +0.97\Lambda_8^2S_S + 0.18\Lambda_8^2S_{S'} + 0.16\Lambda_8^2S_M - 0.01\Lambda_1^2S_M, \tag{1.86}$$

$$\Sigma(957) = +0.97\Sigma_8^2S_S + 0.17\Sigma_8^2S_{S'} + 0.17\Sigma_8^2S_M - 0.00\Sigma_{10}^2S_M. \tag{1.87}$$

The low value for the mass is a result of neglecting the mass difference between the two light quarks and the strange quark.

1.11.2 Hyperfine Mixing in the *uds* Basis

When $SU(3)_{flavour}$ is broken we must use the *uds* basis and we can no longer use (1.69). Since the strange quark is always in the third position in this basis, we have using (1.66)

$$H_{mn}^{uds} = \langle\phi_m^{uds}(1,2,3)|(\tilde{H}_{hyp}^{12} + x(\tilde{H}_{hyp}^{13} + \tilde{H}_{hyp}^{23}))|\phi_n^{uds}(1,2,3)\rangle$$

$$= \langle\phi_m^{uds}(1,2,3)|(\tilde{H}_{hyp}^{12} + x(P_{23}^{\dagger}\tilde{H}_{hyp}^{12}P_{23} + P_{13}^{\dagger}\tilde{H}_{hyp}^{12}P_{13}))|\phi_n^{uds}(1,2,3)\rangle \quad (1.88)$$

$\{|\phi_j^{uds}(1,2,3)\rangle\}$ are the excited uds wave functions up to $N = 2$, see table 1.8; P_{ij} are permutation operators, transposing quark positions i and j. We have

$$\begin{aligned}\langle\phi_m^{uds}(1,2,3)|\tilde{H}_{hyp}^{23}|\phi_n^{uds}(1,2,3)\rangle &= \langle\phi_m^{uds}(2,1,3)|\tilde{H}_{hyp}^{13}|\phi_n^{uds}(2,1,3)\rangle \\ &= (-1)^2\langle\phi_m^{uds}(1,2,3)|\tilde{H}_{hyp}^{13}|\phi_n^{uds}(1,2,3)\rangle\end{aligned}$$

After relabeling and transposing the quark positions 1 and 2 and noting that the uds wave functions are antisymmetric with respect to exchange of quark positions 1 and 2 only. Therefore

$$H_{mn}^{uds} = \langle\phi_m^{uds}(1,2,3)|(\tilde{H}_{hyp}^{12} + 2xP_{13}^{\dagger}\tilde{H}_{hyp}^{12}P_{13})|\phi_0^{uds}(1,2,3)\rangle \quad (1.89)$$

uds Basis Compositions

The effects of $\alpha \neq \alpha_\lambda$ are small in the $S = -1$ sector [24] and are neglected in the matrix elements of H_{hyp}. We will need to know how a function F (say) behaves under the permutation operator P_{13}. It can be easily verified that

$$P_{13}F^{\rho} = \frac{1}{2}\rho - \frac{\sqrt{3}}{2}\lambda, \quad (1.90)$$

$$P_{13}F^{\rho\rho} = \frac{1}{4}\rho\rho - \frac{3}{4}\lambda\lambda - \frac{\sqrt{3}}{2}\rho\lambda, \quad (1.91)$$

$$P_{13}F^{\lambda} = -\frac{1}{2}\lambda - \frac{\sqrt{3}}{2}\rho, \quad (1.92)$$

$$P_{13}F^{\lambda\lambda} = \frac{1}{4}\lambda\lambda + \frac{3}{4}\rho\rho + \frac{\sqrt{3}}{2}\rho\lambda. \quad (1.93)$$

Along with the relations

$$\langle\Phi_{200}^{\lambda\lambda}|\delta^3(\vec{\rho})|\Phi_{000}^{S}\rangle = 0, \quad (1.94)$$

$$\langle\Phi_{200}^{\rho\rho}|\delta^3(\vec{\rho})|\Phi_{000}^{S}\rangle = \sqrt{\frac{3}{2}}\frac{\alpha^3}{\pi\sqrt{\pi}}, \quad (1.95)$$

$$\langle\Phi_{200}^{\rho\lambda}|\delta^3(\vec{\rho})|\Phi_{000}^{S}\rangle = 0, \quad (1.96)$$

we get for the *uds* hyperfine matrix elements

$$\langle \Lambda^2 S|H|\Lambda^2 S\rangle = -\frac{1}{2}\delta, \tag{1.97}$$

$$\langle \Lambda^2 S_{\rho\rho}|H|\Lambda^2 S\rangle = -\frac{\sqrt{6}}{4}\delta, \tag{1.98}$$

$$\langle \Lambda^2 S_{\lambda\lambda}|H|\Lambda^2 S\rangle = 0, \tag{1.99}$$

$$\langle \Lambda^2 S_{\rho\lambda}|H|\Lambda^2 S\rangle = -x\frac{\sqrt{3}}{4}\delta, \tag{1.100}$$

$$\langle \Sigma^2 S|H|\Sigma^2 S\rangle = \frac{1-4x}{6}\delta, \tag{1.101}$$

$$\langle \Sigma^2 S_{\rho\rho}|H|\Sigma^2 S\rangle = \frac{\sqrt{6}}{12}(1-x)\delta, \tag{1.102}$$

$$\langle \Sigma^2 S_{\lambda\lambda}|H|\Sigma^2 S\rangle = -x\frac{\sqrt{6}}{4}\delta, \tag{1.103}$$

$$\langle \Sigma^2 S_{\rho\lambda}|H|\Sigma^2 S\rangle = -x\frac{\sqrt{3}}{4}\delta. \tag{1.104}$$

Due to the higher mass of the strange quark the zeroth-order energy ($1135\,MeV$) now becomes[5]

$$1135 + m_s - m_u - (1-x)\frac{p_3^2}{2m_u} = 1295\,MeV$$

Using the $N = 2$ wave functions for the Λ and Σ^0 in the *uds* basis (table 1.10) we can construct the mixing matrix. When diagonalized this gives[6],

$$\Lambda(1113) \approx 0.95\Lambda^2 S + 0.07\Lambda^2 S_{\lambda\lambda} + 0.28\Lambda^2 S_{\rho\rho} + 0.08\Lambda^2 S_{\rho\lambda}, \tag{1.105}$$

$$\Sigma^0(1209) \approx +0.98\Sigma^2 S + 0.17\Sigma^2 S_{\lambda\lambda} + 0.02\Sigma^2 S_{\rho\rho} + 0.11\Sigma^2 S_{\rho\lambda}. \tag{1.106}$$

for the ground state eigenvectors in the *uds* basis.

We now take these compositions (1.85)–(1.106) and apply them to our problem of calculating the $K^- p \to \Lambda\gamma$ and the $K^- p \to \Sigma^0\gamma$ branching ratios.

---

[5] A quark in a harmonic oscillator potential has ground state energy $\frac{p_3^2}{2m} = \frac{1}{2}\hbar\omega = \frac{1}{2}\frac{\alpha^2}{m}$, ($\hbar = 1$) which gives $p_3 = \alpha$.

[6] Reference uses $\alpha = 0.32$, $m_u = 0.33$, $m_s = 0.55$ (GeV) but they point out that if $m_s - m_u$ is increased to $0.28 GeV$ with a corresponding increase in α to 0.41 there is little change in the compositions.

X	$X^2S_{\lambda\lambda}$	$X^2S_{\rho\rho}$	$X^2S_{\rho\lambda}$
$\Lambda(1555)$	-0.75	-0.66	-0.09
$\Lambda(1740)$	$+0.56$	-0.69	$+0.46$
$\Lambda(1860)$	$+0.34$	-0.28	-0.85
$\Sigma(1640)$	-0.84	-0.53	-0.11
$\Sigma(1910)$	$+0.23$	-0.51	$+0.56$
$\Sigma(1995)$	$+0.19$	-0.31	$+0.02$

Table 1.10: Excited baryon compositions, in the *uds* basis (from ref. [17]), adjusted to our phase convention.

Chapter 2

Theory

2.1 Method

We wish to calculate the branching ratios for the radiative capture reactions $K^- p \rightarrow Y\gamma$ within the N.Q.M., where Y represents Λ or Σ^0. Within the context of this model we picture one of the u quarks of the proton being transformed into an s quark and so creating a Λ or Σ^0. We take the approximation (as in ref.[15]) that the kaon is a point-like particle and ignore its internal quark structure. Also we take the standard hypothesis that photo-emission occurs via the de-excitation of a single quark.

Since we are using the nonrelativistic wave functions given in chapter 1, it is appropriate to develop a nonrelativistic operator from the

$$K^- + u \rightarrow s + \gamma$$

interaction on a single quark. This operator will then be 'sandwiched' between the nonrelativistic wave functions corresponding to the proton and the Y. We will procure this operator from a nonrelativistic reduction of the interaction obtained from the lowest order Feynman diagrams contributing to the process (see Fig. 2.2).

In Dirac notation the amplitude for the process is,

$$\mathcal{S}_{Yp} = \langle \Psi_Y | \sum_{i=1}^{3} V^{(i)} | \Psi_p \rangle \tag{2.107}$$

where the sum is over the three quarks in the baryons and $V^{(i)}$ is the single quark transition operator which acts on the space, spin, and flavour of the i^{th} quark.

2.2 Symmetry considerations

The Y and proton wave functions are given by three body wave functions, the form of which was described in the previous chapter. In coordinate representation they are dependent on the coordinates of each of the three quarks, suppressing the spin-flavour dependence of the baryon wave functions we get,

$$\begin{aligned} \mathcal{S}_{Yp} &= \int \langle \Psi_Y | \vec{r}_1, \vec{r}_2, \vec{r}_3, t \rangle \langle \vec{r}_1, \vec{r}_2, \vec{r}_3, t | (V^{(1)} + V^{(2)} + V^{(3)}) | \vec{r}_1', \vec{r}_2', \vec{r}_3', t' \rangle \\ &\times \langle \vec{r}_1', \vec{r}_2', \vec{r}_3', t' | \Psi_p \rangle \, d^3\vec{r}_1 \, d^3\vec{r}_2 \, d^3\vec{r}_3 \, dt \, d^3\vec{r}_1' \, d^3\vec{r}_2' \, d^3\vec{r}_3' \, dt' \end{aligned} \tag{2.108}$$

where $\vec{r}_i$ is the position vector of the i^{th} quark in the Y; $\vec{r}_i'$ is the position vector of the i^{th} quark in the proton; and the interaction in coordinate representation, $\langle \vec{r}_1, \vec{r}_2, \vec{r}_3, t | (V^{(1)} + V^{(2)} + V^{(3)}) | \vec{r}_1', \vec{r}_2', \vec{r}_3', t' \rangle$, is a sum of single quark transition operators

$$\begin{aligned} \langle \vec{r}_1, \vec{r}_2, \vec{r}_3, t | (V^{(1)} + V^{(2)} + V^{(3)}) | \vec{r}_1', \vec{r}_2', \vec{r}_3', t' \rangle &= V^{(1)}(\vec{r}_1, \vec{r}_1', t, t') \delta^3(\vec{r}_2 - \vec{r}_2') \delta^3(\vec{r}_3 - \vec{r}_3') \\ &+ V^{(2)}(\vec{r}_2, \vec{r}_2', t, t') \delta^3(\vec{r}_1 - \vec{r}_1') \delta^3(\vec{r}_3 - \vec{r}_3') \\ &+ V^{(3)}(\vec{r}_3, \vec{r}_3', t, t') \delta^3(\vec{r}_1 - \vec{r}_1') \delta^3(\vec{r}_2 - \vec{r}_2'). \end{aligned}$$

$SU(6)$ Basis.

When we use the $SU(6)$, or fully antisymmetrized, description of the baryon wave functions we can use symmetry considerations to simplify the interaction. Using the condensed notation,

$$\begin{aligned} \Psi(r_i) &\equiv \Psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, t), \\ d\tau &\equiv d^3\vec{r}_1 \, d^3\vec{r}_2 \, d^3\vec{r}_3 \, dt, \\ \text{and } d\tau' &\equiv d^3\vec{r}_1' \, d^3\vec{r}_2' \, d^3\vec{r}_3' \, dt', \end{aligned}$$

we have

$$\mathcal{S}_{Yp} = \int \bar{\Psi}_Y(r_i) \{ V^{(1)}(\vec{r}_1, \vec{r}_1', t, t') \delta^3(\vec{r}_2 - \vec{r}_2') \delta^3(\vec{r}_3 - \vec{r}_3')$$

$$
\begin{aligned}
&+ \; V^{(2)}(\vec{r}_2,\vec{r}_2',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_3-\vec{r}_3') \\
&+ \; V^{(3)}(\vec{r}_3,\vec{r}_3',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_2-\vec{r}_2')\} \\
&\times \; \Psi_p(r_i')\,d\tau\,d\tau'.
\end{aligned}
$$

In particular

$$
\begin{aligned}
&\int \; \bar{\Psi}_Y(\vec{r}_1,\vec{r}_2,\vec{r}_3,t)V^{(1)}(\vec{r}_1,\vec{r}_1',t,t')\delta^3(\vec{r}_2-\vec{r}_2')\delta^3(\vec{r}_3-\vec{r}_3')\Psi_p(\vec{r}_1',\vec{r}_2',\vec{r}_3',t')\,d\tau\,d\tau' \\
&= \int \; \bar{\Psi}_Y(\vec{r}_2,\vec{r}_1,\vec{r}_3,t)V^{(2)}(\vec{r}_2,\vec{r}_2',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_3-\vec{r}_3')\Psi_p(\vec{r}_2',\vec{r}_1',\vec{r}_3',t')\,d\tau\,d\tau',
\end{aligned}
$$

where we have just relabeled quarks 1 and 2. Making a transposition of any two quark positions in an antisymmetric wave function will result in a change of sign of the wave function. Transposing $\vec{r}_1 \leftrightarrow \vec{r}_2$ and $\vec{r}_1' \leftrightarrow \vec{r}_2'$ in the last expression we get

$$
\begin{aligned}
(-1)^2 \quad &\int \bar{\Psi}_Y(\vec{r}_1,\vec{r}_2,\vec{r}_3,t)V^{(1)}(\vec{r}_2,\vec{r}_2',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_3-\vec{r}_3')\Psi_p(\vec{r}_1',\vec{r}_2',\vec{r}_3',t')\,d\tau\,d\tau' \\
= \quad &\int \bar{\Psi}_Y(\vec{r}_1,\vec{r}_2,\vec{r}_3,t)V^{(1)}(\vec{r}_1,\vec{r}_1',t,t')\delta^3(\vec{r}_2-\vec{r}_2')\delta^3(\vec{r}_3-\vec{r}_3')\Psi_p(\vec{r}_1',\vec{r}_2',\vec{r}_3',t')\,d\tau\,d\tau'.
\end{aligned}
$$

By applying the same procedure, of changing the labels and transposing, to quarks 1 and 3 and quarks 2 and 3 we obtain the result

$$
\begin{aligned}
\mathcal{S}_{Y_p} \;&=\; \int \bar{\Psi}_Y(r_i)\{V^{(1)}(\vec{r}_1,\vec{r}_1',t,t')\delta^3(\vec{r}_2-\vec{r}_2')\delta^3(\vec{r}_3-\vec{r}_3') && (2.109)\\
&+\; V^{(2)}(\vec{r}_2,\vec{r}_2',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_3-\vec{r}_3') && (2.110)\\
&+\; V^{(3)}(\vec{r}_3,\vec{r}_3',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_2-\vec{r}_2')\} && (2.111)\\
&\times\; \Psi_p(r_i')\,d\tau\,d\tau'. \\
&=\; 3\int \bar{\Psi}_Y(r_i)V^{(3)}(\vec{r}_3,\vec{r}_3',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_2-\vec{r}_2')\Psi_p(r_i')\,d\tau\,d\tau'. && (2.112)
\end{aligned}
$$

uds **Basis.**

When we use the *uds* basis wave functions the strange quark is always in the third position in the strange baryon. Since the model involves the transformation of one of

the u quarks in the nucleon, to an s quark it follows that only the term

$$\int \bar{\Psi}_Y(r_i)V^{(3)}(\vec{r}_3,\vec{r}_3',t,t')\delta^3(\vec{r}_1-\vec{r}_1')\delta^3(\vec{r}_2-\vec{r}_2')\Psi_p(r_i')\,d\tau\,d\tau'$$

can contribute to the amplitude.

Since only the interaction on the third quark is relevant, we drop the superscript on all references to V henceforth.

2.3 The Impulse Approximation

By singling out quark 3 in this manner, quarks 1 and 2 are being treated as spectator quarks. This amounts to using the impulse approximation of nuclear physics. It states that the interaction occurs over a small enough time period that the momentum-energy absorbed by the third quark has insufficient time to redistribute to quarks 1 and 2. On physical grounds this statement can be justified [25] by noting that the energy uncertainty associated with the absorption of a kaon is of the order of the kaon mass. Therefore the strong interaction takes place over a time interval of the order of $1/m_K \approx 2\,GeV^{-1}$. This interval is much shorter than the characteristic periodicity time associated with the binding of the quark which is of the order of $1/(m_p - 3m_u) \approx 20\,GeV^{-1}$ (taking $m_u = 0.33\,GeV$). Therefore the interaction may be thought of as an impulse during which the binding forces are unable to play an important role. The target quark is thus regarded as free and the binding forces serve only to determine what momentum components are present in its wave function.

2.4 Fourier and Jacobi Transformations

We now Fourier transform the interaction $V(r_3,r_3')$ in (2.112) to momentum space

$$V(p,p') = \int \langle p|r_3\rangle\langle r_3|V|r_3'\rangle\langle r_3'|p'\rangle\, d^4r_3'\, d^4r_3. \tag{2.113}$$

Where the closure relations

$$\int |r_3'\rangle\langle r_3'|\, d^4r_3' = 1 \text{ and } \int |r_3\rangle\langle r_3|\, d^4r_3 = 1 \tag{2.114}$$

have been inserted. We get

$$V(p,p') = \int e^{ip\cdot r_3} V(r_3', r_3) e^{-ip'\cdot r_3'} \frac{d^4r_3'\, d^4r_3}{(2\pi)^4} \tag{2.115}$$

and the inverse relationship is

$$V(\vec{r}_3, \vec{r}_3', t, t') = \int e^{-ip\cdot r_3} V(p,p') e^{ip'\cdot r_3'} \frac{d^4p'\, d^4p}{(2\pi)^4}. \tag{2.116}$$

Now using (2.116) in equation (2.112) yields,

$$\begin{aligned} \mathcal{S}_{Yp} &= \frac{b}{(2\pi)^4} \int \bar{\Psi}_Y(r_i)\delta^3(\vec{r}_1 - \vec{r}_1')\delta^3(\vec{r}_2 - \vec{r}_2') e^{-ir_3\cdot p} V(p,p') e^{ir_3'\cdot p'} \\ &\times \Psi_p(r_i')\, d\tau\, d\tau'\, d^4p\, d^4p'. \end{aligned}$$

Where $b = 3$ when using $SU(6)$ wave functions and $b = 1$ when using uds wave functions. Separating out the time dependence in the bound state baryon wave functions (equation (1.6)),

$$\begin{aligned} \bar{\Psi}_Y(r_i) &\equiv \bar{\psi}_Y(\vec{r}_i) e^{iEt} \\ \text{and } \Psi_p(r_i') &\equiv \psi_p(\vec{r}_i') e^{-iE't'}, \end{aligned}$$

we obtain

$$\begin{aligned} \mathcal{S}_{Yp} &= \frac{b}{(2\pi)^4} \int \bar{\psi}_Y(\vec{r}_i) e^{i(E_1+E_2+E_3)t} \delta^3(\vec{r}_1 - \vec{r}_1')\delta^3(\vec{r}_2 - \vec{r}_2') \\ &\times e^{-itp^0} e^{i\vec{r}_3\cdot\vec{p}} V(p,p') e^{it'(p^0)'} e^{-i\vec{r}_3'\cdot\vec{p}'} \\ &\times \psi_p(\vec{r}_i') e^{-i(E_1'+E_2'+E_3')t'}\, dt\, dt'\, d^3\vec{r}_i\, d^3\vec{r}_i'\, d^3\vec{p}\, dp^0\, d^3\vec{p}'\, d(p^0)'. \end{aligned} \tag{2.117}$$

Since we have a single quark interaction we have $E_1 = E_1'$ and $E_2 = E_2'$. Performing the time integrals we get,

$$\mathcal{S}_{Yp} = \frac{b}{(2\pi)^2} \int \delta(E_3 - p^0)\delta(E_3' - (p^0)')\bar{\psi}_Y(\vec{r}_i)\delta^3(\vec{r}_1 - \vec{r}_1')$$

$$\times \quad \delta^3(\vec{r}_2 - \vec{r}_2')e^{i\vec{r}_3\cdot\vec{p}}V(p,p')e^{-i\vec{r}_3'\cdot\vec{p}'}\psi_p(\vec{r}_i')\, d^3\vec{r}_i\, d^3\vec{r}_i'\, d^3\vec{p}\, d^3\vec{p}'\, dp^0\, d(p^0)'. \tag{2.118}$$

E_1 and E_2 have been dropped from the time dependence as p_0 is the zeroth component of the 4-vector, p, which must correspond to the energy of the interacting quark (E_3) only.

2.4.1 When $SU(3)_{flavour}$ is Broken

We first develop a form for the amplitude when $m_s \neq m_u$; appropriate for use with uds wave functions for the Y. We then generalize the amplitude to the case of unbroken $SU(3)_{flavour}$ symmetry by letting $m_s \to m_u$.

We switch to the Jacobi coordinates introduced in §1.7,

$$\begin{aligned}
\vec{\rho} &= \frac{1}{\sqrt{2}}(\vec{r_1} - \vec{r_2}),\ \vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r_1} + \vec{r_2} - 2\vec{r_3}),\ \vec{R} = \frac{m_u(\vec{r_1} + \vec{r_2}) + m_s\vec{r_3}}{2m_u + m_s}, \\
\vec{\rho}' &= \frac{1}{\sqrt{2}}(\vec{r_1}' - \vec{r_2}'),\ \vec{\lambda}' = \frac{1}{\sqrt{6}}(\vec{r_1}' + \vec{r_2}' - 2\vec{r_3}'),\ \vec{R}' = \frac{1}{3}(\vec{r_1}' + \vec{r_2}' + \vec{r_3}').
\end{aligned}$$

$$\begin{aligned}
\Rightarrow (\vec{r_1}, \vec{r_2}, \vec{r_3}, \vec{r_1}', \vec{r_2}', \vec{r_3}') &\mapsto (\vec{\rho}, \vec{\lambda}, \vec{R}, \vec{\rho}', \vec{\lambda}', \vec{R}'), \\
\psi_Y(\vec{r_1}, \vec{r_2}, \vec{r_3}) &\mapsto \Psi_Y(\vec{\rho}, \vec{\lambda}, \vec{R}) \\
\text{and } \psi_p(\vec{r_1}', \vec{r_2}', \vec{r_3}') &\mapsto \Psi_p(\vec{\rho}', \vec{\lambda}', \vec{R}'),
\end{aligned}$$

and using the relations derived in §1.9,

$$\begin{aligned}
\vec{r_1}' &= \vec{R}' + \frac{\vec{\rho}'}{\sqrt{2}} + \frac{\vec{\lambda}'}{\sqrt{6}}, \\
\vec{r_2}' &= \vec{R}' - \frac{\vec{\rho}'}{\sqrt{2}} + \frac{\vec{\lambda}'}{\sqrt{6}}, \\
\vec{r_3}' &= \vec{R}' - \sqrt{\frac{2}{3}}\vec{\lambda}', \\
\text{and } \vec{r_1} &= \vec{R} + \frac{\vec{\rho}}{\sqrt{2}} + \frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u}),
\end{aligned}$$

$$\vec{r_2} = \vec{R} - \frac{\vec{\rho}}{\sqrt{2}} + \frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u}),$$
$$\vec{r_3} = \vec{R} + \frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u} - 3),$$
$$\text{with } m_\lambda \equiv \frac{3m_u m_s}{2m_u + m_s},$$

we get,

$$\begin{aligned}
\mathcal{S}_{Yp} &= \frac{bJ\,J'}{(2\pi)^2}\int \delta(E_3 - p^0)\delta(E_3' - (p^0)') \\
&\times \bar{\Psi}_Y(\vec{\rho},\vec{\lambda},\vec{R})\delta^3(\vec{R} - \vec{R}' + \frac{1}{\sqrt{2}}(\vec{\rho} - \vec{\rho}') + \frac{1}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda} - \vec{\lambda}')) \\
&\times \delta^3(\vec{R} - \vec{R}' - \frac{1}{\sqrt{2}}(\vec{\rho} - \vec{\rho}') + \frac{1}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda} - \vec{\lambda}'))e^{i(\vec{R} + \frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u} - 3))\cdot\vec{p}}V(p,p') \\
&\times e^{-i(\vec{R}' - \sqrt{\frac{2}{3}}\vec{\lambda}')\cdot\vec{p}'}\Psi_p(\vec{\rho}',\vec{\lambda}',\vec{R}') \\
&\times d^3\vec{\rho}\,d^3\vec{\lambda}\,d^3\vec{R}\,d^3\vec{\rho}'\,d^3\vec{\lambda}'\,d^3\vec{R}'\,d^3\vec{p}\,d^3\vec{p}'\,dp^0\,d(p^0)' \\
&= \frac{8bJ\,J'}{(2\pi)^2}\int \bar{\Psi}_Y(\vec{\rho},\vec{\lambda},\vec{R})\delta^3(2\sqrt{2}(\vec{R} - \vec{R}') + \frac{2}{\sqrt{3}}((\frac{m_\lambda}{m_u})\vec{\lambda} - \vec{\lambda}')) \\
&\times e^{i(\vec{R} + \frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u} - 3))\cdot\vec{p}}V(\vec{p}, p^0 = E_3, \vec{p}', (p^0)' = E_3') \\
&\times e^{-i(\vec{R}' - \sqrt{\frac{2}{3}}\vec{\lambda}')\cdot\vec{p}'}\Psi_p(\vec{\rho}' = \vec{\rho} + \sqrt{2}(\vec{R} - \vec{R}') + \frac{1}{\sqrt{3}}((\frac{m_\lambda}{m_u})\vec{\lambda} - \vec{\lambda}'),\vec{\lambda}',\vec{R}') \\
&\times d^3\vec{\rho}\,d^3\vec{\lambda}\,d^3\vec{R}\,d^3\vec{\lambda}'\,d^3\vec{R}'\,d^3\vec{p}\,d^3\vec{p}'. \qquad (2.119)
\end{aligned}$$

Where we have integrated with respect to p^0, $(p^0)'$ and $\vec{\rho}'$ and the Dirac delta function property (1.70)) has been invoked. J and J' are the Jacobians

$$J \equiv \begin{vmatrix} \frac{\partial r_1}{\partial R} & \frac{\partial r_1}{\partial \rho} & \frac{\partial r_1}{\partial \lambda} \\ \frac{\partial r_2}{\partial R} & \frac{\partial r_2}{\partial \rho} & \frac{\partial r_2}{\partial \lambda} \\ \frac{\partial r_3}{\partial R} & \frac{\partial r_3}{\partial \rho} & \frac{\partial r_3}{\partial \lambda} \end{vmatrix}^3 = \begin{vmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{m_\lambda}{\sqrt{6}m_u} \\ 1 & -\frac{1}{\sqrt{2}} & \frac{m_\lambda}{\sqrt{6}m_u} \\ 1 & 0 & \frac{1}{\sqrt{6}}(\frac{m_\lambda}{m_u} - 3) \end{vmatrix}^3 = \sqrt{3}^3. \qquad (2.120)$$

$$J' \equiv \begin{vmatrix} \frac{\partial r_1'}{\partial R'} & \frac{\partial r_1'}{\partial \rho'} & \frac{\partial r_1'}{\partial \lambda'} \\ \frac{\partial r_2'}{\partial R'} & \frac{\partial r_2'}{\partial \rho'} & \frac{\partial r_2'}{\partial \lambda'} \\ \frac{\partial r_3'}{\partial R'} & \frac{\partial r_3'}{\partial \rho'} & \frac{\partial r_3'}{\partial \lambda'} \end{vmatrix}^3 = \begin{vmatrix} 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 1 & 0 & -\sqrt{\frac{2}{3}} \end{vmatrix}^3 = \sqrt{3}^3. \qquad (2.121)$$

Evaluating the $\vec{R}'$ integral we get

$$\begin{aligned} \mathcal{S}_{Yp} &= \frac{27\times 8b}{(2\sqrt{2})^3(2\pi)^2}\int \bar{\Psi}_Y(\vec{\rho},\vec{\lambda},\vec{R})e^{i(\vec{R}+\frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u}-3))\cdot\vec{p}}V(\vec{p},p^0=E_3,\vec{p}',(p^0)'=E_3') \\ &\times\ e^{-i(\vec{R}+\frac{1}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}')-\sqrt{\frac{2}{3}}\vec{\lambda}')\cdot\vec{p}'}\Psi_p(\vec{\rho}'=\vec{\rho},\vec{\lambda}',\vec{R}'=\vec{R}+\frac{1}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}')) \\ &\times\ d^3\vec{\rho}\,d^3\vec{\lambda}\,d^3\vec{R}\,d^3\vec{\lambda}'\,d^3\vec{p}\,d^3\vec{p}' \end{aligned} \tag{2.122}$$

$$\begin{aligned} \mathcal{S}_{Yp} &= \frac{27b}{2\sqrt{2}(2\pi)^2}\int \bar{\Psi}_Y(\vec{\rho},\vec{\lambda},\vec{R})e^{i(\vec{R}+\frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u}-3))\cdot\vec{p}}V(\vec{p},p^0=E_3,\vec{p}',(p^0)'=E_3') \\ &\times\ e^{-i(\vec{R}+\frac{1}{\sqrt{6}}(\frac{m_\lambda}{m_u})\vec{\lambda}-\sqrt{\frac{3}{2}}\vec{\lambda}')\cdot\vec{p}'}\Psi_p(\vec{\rho}'=\vec{\rho},\vec{\lambda}',\vec{R}'=\vec{R}+\frac{1}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}')) \\ &\times\ d^3\vec{\rho}\,d^3\vec{\lambda}\,d^3\vec{R}\,d^3\vec{\lambda}'\,d^3\vec{p}\,d^3\vec{p}'. \end{aligned} \tag{2.123}$$

The Jacobian transformation allows us to separate out the centre of mass motion, since

$$\bar{\Psi}_Y(\vec{\rho},\vec{\lambda},\vec{R}) = N_Y\bar{\Phi}_Y(\vec{\rho},\vec{\lambda})e^{-i\vec{P}_Y\cdot\vec{R}} \tag{2.124}$$

$$\text{and } \Psi_p(\vec{\rho}',\vec{\lambda}',\vec{R}') = N_p\Phi_p(\vec{\rho}',\vec{\lambda}')e^{i\vec{P}_p\cdot\vec{R}'} \tag{2.125}$$

$$= N_p\Phi_p(\vec{\rho},\vec{\lambda}')e^{i\vec{P}_p\cdot(\vec{R}+\frac{1}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}'))} \tag{2.126}$$

Where $\vec{P}_Y$ is the centre of mass momentum for the Y final state and $\vec{P}_p$ is the centre of mass momentum for the proton. N_Y is a normalization coefficient for the Y and N_p is a normalization coefficient for the proton.

The $\Phi_X(\vec{\rho},\vec{\lambda})$ contain normalized flavour and spin wave functions. The harmonic oscillator functions are normalized, over all space, to unity. The normalization coefficient N_X, therefore, contains a $\frac{1}{\sqrt{\mathcal{V}}}$ from the box normalization of the centre of mass plane wave. It will also include normalization factors associated with the Dirac spinors (§2.10.2).

$$\begin{aligned} \mathcal{S}_{Yp} &= \frac{27bN_YN_p}{2\sqrt{2}(2\pi)^2}\int \bar{\Phi}_Y(\vec{\rho},\vec{\lambda})e^{-i\vec{P}_Y\cdot\vec{R}}e^{i(\vec{R}+\frac{\vec{\lambda}}{\sqrt{6}}(\frac{m_\lambda}{m_u}-3))\cdot\vec{p}}V(\vec{p},p^0=E_3,\vec{p}',(p^0)'=E_3') \\ &\times\ e^{-i(\vec{R}+\frac{1}{\sqrt{6}}(\frac{m_\lambda}{m_u})\vec{\lambda}-\sqrt{\frac{3}{2}}\vec{\lambda}')\cdot\vec{p}'}\Phi_p(\vec{\rho},\vec{\lambda}')e^{i\vec{P}_p\cdot(\vec{R}+\frac{1}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}'))} \end{aligned}$$

$$
\begin{aligned}
&\times \quad d^3\vec{\rho}\, d^3\vec{\lambda}\, d^3\vec{R}\, d^3\vec{\lambda}'\, d^3\vec{p}\, d^3\vec{p}' \\
&= \quad \frac{27bN_Y N_p(2\pi)}{2\sqrt{2}} \int \delta^3(\vec{P}_Y - \vec{p} + \vec{p}' - \vec{P}_p)\bar{\Phi}_Y(\vec{\rho},\vec{\lambda})e^{i\vec{\lambda}\cdot\vec{p}\frac{1}{\sqrt{6}}(\frac{m_\lambda}{m_u}-3)} \\
&\times \quad V(\vec{p}, p^0 = E_3, \vec{p}', (p^0)' = E_3')e^{\frac{-i}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-3\vec{\lambda}')\cdot\vec{p}'}\Phi_p(\vec{\rho},\vec{\lambda}') \\
&\times \quad e^{\frac{i}{\sqrt{6}}\vec{P}_p\cdot((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}')}\, d^3\vec{\rho}\, d^3\vec{\lambda}\, d^3\vec{\lambda}'\, d^3\vec{p}\, d^3\vec{p}' \\
&= \quad \frac{27bN_Y N_p(2\pi)}{2\sqrt{2}} \int \bar{\Phi}_Y(\vec{\rho},\vec{\lambda})e^{i\vec{\lambda}\cdot\vec{p}\frac{1}{\sqrt{6}}(\frac{m_\lambda}{m_u}-3)}V(\vec{p}, p^0 = E_3, \vec{p}' = \vec{P}_p + \vec{p} - \vec{P}_Y, (p^0)' = E_3') \\
&\times \quad e^{-\frac{i}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-3\vec{\lambda}')\cdot(\vec{P}_p+\vec{p}-\vec{P}_Y)}\Phi_p(\vec{\rho},\vec{\lambda}')e^{\frac{i}{\sqrt{6}}\vec{P}_p\cdot((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}')}\, d^3\vec{\rho}\, d^3\vec{\lambda}\, d^3\vec{\lambda}'\, d^3\vec{p}. \qquad (2.127)
\end{aligned}
$$

The delta function

$$\delta^3(\vec{P}_Y - \vec{p} + \vec{p}' - \vec{P}_p) = \delta^3((\vec{P}_p - \vec{p}') - (\vec{P}_Y - \vec{p}))$$

tells us that the momenta of quarks 1 and 2 remains constant throughout the process and allows us to perform the integration over $\vec{p}'$.

2.4.2 When $SU(3)_{flavour}$ is Unbroken

If $SU(6)$ basis wave functions are used for the Y then (2.127) becomes (with $m_s \to m$, $\Rightarrow m_\lambda \to m$)

$$
\begin{aligned}
\mathcal{S}_{Yp}^{SU(6)} &= \frac{27bN_Y N_p(2\pi)}{2\sqrt{2}} \int \bar{\Phi}_Y(\vec{\rho},\vec{\lambda})e^{-i\sqrt{\frac{2}{3}}\vec{\lambda}\cdot\vec{p}} \\
&\times \quad V(\vec{p}, p^0 = E_3, \vec{p}' = \vec{P}_p + \vec{p} - \vec{P}_Y, (p^0)' = E_3') \\
&\times \quad e^{\frac{-i}{\sqrt{6}}(\vec{\lambda}-3\vec{\lambda}')\cdot(\vec{P}_p+\vec{p}-\vec{P}_Y)}\Phi_p(\vec{\rho},\vec{\lambda}')e^{\frac{i}{\sqrt{6}}\vec{P}_p\cdot(\vec{\lambda}-\vec{\lambda}')} \\
&\times \quad d^3\vec{\rho}\, d^3\vec{\lambda}\, d^3\vec{\lambda}'\, d^3\vec{p}. \qquad (2.128)
\end{aligned}
$$

We will return to the last two expressions after deriving the explicit form of $V(p,p')$.

2.5 The Form of the Interactions.

We will now derive the form of the vertex functions and propagators for the interacting quarks and the kaon.

2.5.1 Equations satisfied by an interacting field.

Spin $\frac{1}{2}$ particles:

Spin $\frac{1}{2}$ particles such as quarks obey the Dirac equation. The Dirac equation for a free particle is

$$(\gamma_\mu p^\mu - m)\psi = 0, \tag{2.129}$$

where m is the rest mass and p the 4-momentum of the Dirac particle. When the particle is in the presence of a potential V, the Dirac equation using Bjorken and Drell[26] notation becomes,

$$(\gamma_\mu p^\mu - m)\psi = V\psi. \tag{2.130}$$

Integral-spin particles:

Spin zero particles such as the kaon obey the Klein-Gordon (KG) equation. The KG equation for a free particle,

$$(\Box + m_K^2)\psi_K = 0, \tag{2.131}$$

where m_K is the kaon rest mass and $\Box \equiv -p^\mu p_\mu = \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}$. In the presence of a potential V the KG equation becomes,

$$(\Box + m_K^2)\psi_K = -V\psi_K \tag{2.132}$$

2.5.2 Form of the vertex functions

The electromagnetic interaction for a spin $\frac{1}{2}$ particle:

The minimal substitution[1], $p^\mu \rightarrow p^\mu - eA^\mu$, introduces the coupling of a Dirac particle, charge e, to the electromagnetic field A^μ. Using the Feynman slash notation [26], $\not{A} \equiv \gamma_\mu A^\mu = \gamma^0 A^0 - \vec{\gamma} \cdot \vec{A}$, we get from (2.129) and (2.130)

$$(\not{p} - m)\psi = e\not{A}\psi \tag{2.133}$$

$$\Rightarrow V_{EM}^{Dirac} = e\not{A}. \tag{2.134}$$

This is the form of the electromagnetic interaction of a Dirac particle such as the spin $\frac{1}{2}$ quark.

The electromagnetic interaction for a spinless particle:

Again we use the substitution $p^\mu \rightarrow p^\mu - e_K A^\mu$, this time into (2.131) (e_K is the charge on the kaon), giving,

$$\begin{aligned} & [\ (i\frac{\partial}{\partial x_\mu} - e_K A^\mu(x))^2 - m_K^2]\psi_K(x) = 0 \\ \Rightarrow\ & [\ \Box + m_K^2 + e_K(A^\mu(x) i\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial x_\mu}A_\mu(x)) - e_K^2 A^\mu(x)A_\mu(x)]\psi_K(x) = 0. \end{aligned}$$

So to first order (neglecting terms $O(e^2) \propto (\frac{1}{137})$) and using (2.132)

$$V_{EM}^{KG}(x) = ie_K(A^\mu(x)\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x_\mu}A_\mu(x)). \tag{2.135}$$

The form of the strong interaction:

For the strong ($K^- p$) vertex, consider the meson field in analogy with the electromagnetic potential[26].

[1] This substitution is necessary to preserve local phase (gauge) invariance of the QED Lagrangian [6].

The conventional choice for the electromagnetic transition current that generates A^μ in equations (2.134) and (2.135) is,

$$J^\mu = e\bar{\psi}^f \gamma^\mu \psi^i. \tag{2.136}$$

Where ψ^i is the wave function of the particle before interacting with the electromagnetic field. ψ^f is the wave function of the particle after interacting with the electromagnetic field.

(2.136) relates to the electromagnetic 4-vector potential, A^μ, through Maxwell's equations. In the Lorentz gauge Maxwell's equations are

$$\Box A^\mu = J^\mu \tag{2.137}$$

$$= e\bar{\psi}^f \gamma^\mu \psi^i. \tag{2.138}$$

For example in the case of photon exchange in electron scattering from a Dirac proton, ψ^f and ψ^i would represent the final and initial proton wave functions, and e the charge on the proton. (2.138) would then define the Møller potential of the Dirac proton.

The 'Klein-Gordon' analogy to (2.138) is ($A^\mu \leftrightarrow \psi_K$),

$$(\Box + m_K^2)\psi_K = \bar{\psi}_Y \Gamma \psi_p \tag{2.139}$$

(compare to equation (10.11) of ref.[26]). Also,

$$\Gamma \equiv g_{Kus} i\gamma_5 \text{ for pseudoscalar (PS) coupling,} \tag{2.140}$$

$$\Gamma \equiv \tilde{g}_{Kus} \not{q} i\gamma_5 \text{ for pseudovector (PV) coupling,} \tag{2.141}$$

$$\text{where } q \equiv p - p' \text{ and } \gamma_5 \equiv i\gamma_0\gamma_1\gamma_2\gamma_3$$

$p'(p)$ represents the up(strange) quark momentum, immediately before(after) the strong interaction. $g_{Kus}/(\tilde{g}_{Kus})$ represents the strong coupling $Ku \rightarrow s$ constant in the pseudoscalar/(pseudovector) coupling scheme for the process $K^- p \rightarrow Y\gamma$. It is the analogue

of e in (2.138). g_{Kus} and $\tilde{g}_{Kus}$ are related by noting that the interactions taken over free quark states must be equal in the PS and PV coupling schemes. That is

$$\begin{aligned}\bar{u}(p)g_{Kus}\gamma_5 u(p') &\equiv \bar{u}(p)\tilde{g}_{Kus}\not{q}\gamma_5 u(p') \\ &= \bar{u}(p)\tilde{g}_{Kus}(\not{p}-\not{p}')\gamma_5 u(p').\end{aligned}$$

Since the $u(p)$ and $u(p')$ are solutions of the free Dirac equation

$$(\not{p}' - m_u)u(p') = 0 \tag{2.142}$$

and for the conjugate relation,

$$\bar{u}(p)(\not{p} - m_s) = 0, \tag{2.143}$$

and noting the anticommutation relation

$$\gamma_\mu\gamma_5 + \gamma_5\gamma_\mu = 0, \tag{2.144}$$

we get that

$$\tilde{g}_{Kus} = \frac{g_{Kus}}{m_s + m_u}. \tag{2.145}$$

Now the equation for a proton absorbing a kaon (analogous to equation (2.133)), can be written,

$$(\not{p} - m_p)\psi_p = (\psi_K\Gamma)\psi_p. \tag{2.146}$$

In the strong (K^-p) interaction a u quark in the proton is transformed into an s quark. This corresponds to a V_+ operator (see §1.8) acting on the third quark of the proton. It can be thought of as the combination of creation and annihilation operators

$$a_s^\dagger a_u,$$

acting on the third quark of the proton. Where $a_s^\dagger$ creates an s quark and a_u annihilates a u quark.

Therefore the form of the strong interaction can be written, from (2.146),

$$V_{ST} = \Gamma V_+ \psi_K. \tag{2.147}$$

2.5.3 The Feynman Propagators.

The Feynman propagator for a spin $\frac{1}{2}$ particle.

The Feynman propagator $S_X(y-x)$ (or Green's function) for a Dirac particle X, satisfies the Green's function equation

$$(i \not\nabla_y - m_X) S_X(y-x) = \delta^4(y-x), \tag{2.148}$$

where ∇_y is the 4-vector gradient

$$\not\nabla_y \equiv \gamma_0 \frac{\partial}{\partial t} - \vec{\gamma}.\vec{\nabla}_y$$

This operator acts on functions of y only. $S_X(y-x)$ represents the wavefunction $\psi(y)$ at space-time point y produced by a unit source ($\psi(x) \equiv \delta^4(y-x)$) at space-time point x; the evolution of $\psi(x)$ is governed by the free Dirac equation (2.129).

By Fourier transforming to momentum space we get[26],

$$S_X(y-x) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq.(y-x)}}{\not q - m_X}$$

Now,

$$\begin{aligned} \frac{1}{\not q - m} &= \frac{\not q + m}{(\not q - m)(\not q + m)} \\ &= \frac{\not q + m}{q^2 - m^2}, \end{aligned}$$

and by the anticommutation relation

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2.149}$$

it is easily found that

$$\not{q}^2 = q.q = q^2. \tag{2.150}$$

$$\Rightarrow S_X(y-x) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq.(y-x)}}{q^2 - m_X^2 + i\epsilon}(\not{q} + m_X). \tag{2.151}$$

The small positive imaginary part added to the denominator in equation (2.151) assures that (2.151) meets the desired boundary condition of propagating only the positive frequencies forward in time (particles) and the negative frequencies backward (antiparticles).

The Feynman Propagator for a Klein-Gordon particle.

The Feynman propagator for a Klein-Gordon particle K is the solution of the equation,

$$(\Box_y + m_K^2)\Delta_K(y-x) = -\delta^4(y-x). \tag{2.152}$$

$\Delta_K(y-x)$ represents the wavefunction $\psi(y)$ at space-time point y produced by a unit source ($\psi(x) \equiv \delta^4(y-x)$) at space-time point x; the evolution of $\psi(x)$ is governed by the free Klein-Gordon equation (2.131). By once again, Fourier transforming to momentum space we find[26],

$$\Delta_K(y-x) = \int \frac{d^4q}{(2\pi)^4} \frac{e^{-iq.(y-x)}}{q^2 - m_K^2 + i\epsilon}. \tag{2.153}$$

2.5.4 The kaon wave function

The kaon and proton form a hydrogenic atom immediately prior to capture. Leon and Bethe [27] suggest that the kaon capture by the proton is most likely from an S state.

In fact, the fraction of K mesons reacting from P states is less than 1%. However, the capture can occur from one of many different states with principal quantum number n. The bound state wave function for the kaon can be written

$$\psi_K(z) = N_K \tilde{\psi}_K(\vec{z}) e^{-iE_K t}, \tag{2.154}$$

where N_K is a normalization constant. The probability density for a Klein-Gordon particle is given by

$$\rho = i(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}) \tag{2.155}$$

which for ψ given by equation (2.154) is

$$\rho = 2E_K |N_K|^2 |\tilde{\psi}_K(\vec{z})|^2.$$

Normalizing to a box of volume $\mathcal{V}$ we get

$$N_K = \frac{1}{\sqrt{2E_K}} \tag{2.156}$$

$$\text{with } \int_{\mathcal{V}} |\tilde{\psi}_K(\vec{z})|^2 \, d^3\vec{z} = 1. \tag{2.157}$$

In terms of its Fourier transform

$$\begin{aligned} \psi_K(z) &= N_K e^{-iE_K t} \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{p}_K \cdot \vec{z}} \phi_K(\vec{p}_K) \, d^3\vec{p}_K \\ &= \frac{N_K}{(2\pi)^{3/2}} \int e^{-ip_K \cdot z} \phi_K(\vec{p}_K) \, d^3\vec{p}_K. \end{aligned} \tag{2.158}$$

Where $\phi_K(\vec{p}_K)$ is the bound state momentum space wave function.

2.5.5 The photon wave function

$A^\mu(w)$ has the form, with box normalization [26],

$$A^\mu(w) = \frac{\epsilon^\mu e^{ik.w}}{\sqrt{2k^0 \mathcal{V}}} = N_\gamma \epsilon^\mu e^{ik.w}, \tag{2.159}$$

$$\text{with } N_\gamma \equiv \frac{1}{\sqrt{2k^0 \mathcal{V}}}. \tag{2.160}$$

(2.160) is the photon "wave function" and describes the emission of a photon with momentum $\vec{k}$, energy k^0, and polarization 4-vector ϵ^μ.

2.6 Derivation of the explicit form of V

We now derive the explicit form of the interactions corresponding to the Feynman diagrams in Fig. 2.2, first in coordinate, and then in momentum representation.

From the Feynman graphs (Fig. 2.2) it can be seen that the structure of the interaction V_i, associated with the i^{th} graph is,

$$V_1(r_3', r_3) = V_{EM}(r_3) S_s(r_3 - r_3') V_{ST}(r_3'), \tag{2.161}$$

$$V_2(r_3', r_3) = V_{ST}(r_3) S_u(r_3 - r_3') V_{EM}(r_3'), \tag{2.162}$$

$$V_3(r_3', r_3) = V_{ST}(r_3) \delta^4(r_3 - r_3'), \tag{2.163}$$

$$V_4(r_3', r_3) = V_{ST}(r_3) \delta^4(r_3 - r_3') V_{EM}(r_3'). \tag{2.164}$$

V_{EM} is the form of the interaction at the electromagnetic vertex.

V_{ST} is the form of the interaction at the strong vertex.

$S_s(r_3 - r_3')$ is the Feynman propagator for a Dirac particle — the strange quark.

$S_u(r_3 - r_3')$ is the Feynman propagator for a Dirac particle — an up quark.

Note that all the interactions act only on the quark in the third position, as was shown in §2.2.

2.6.1 Coordinate Representation

Using equations (2.134), (2.151), and (2.147), and since the interactions act only on the third quark of the Y and proton, we get[2],

$$V_1(r_3', r_3) = V_{EM}^{Dirac}(r_3) S_s(r_3 - r_3') V_{ST}(r_3')$$

[2]Suppressing the $+i\epsilon$ term in the denominator.

$$
\begin{aligned}
&= N_K N_\gamma e_s \not{\epsilon} e^{ik.r_3} \int \frac{d^4q}{(2\pi)^4} \frac{d^3\vec{p}_K}{(2\pi)^{3/2}} \frac{e^{-iq.(r_3-r_3')}}{q^2-m_s^2} (\not{q}+m_s)\Gamma V_+ e^{-ip_K.r_3'} \phi_K(\vec{p}_K) \\
&= \frac{N_K N_\gamma e_s}{(2\pi)^{11/2}} \int d^4q\, d^3\vec{p}_K \, \frac{e^{ik.r_3-iq.(r_3-r_3')-ip_K.r_3'}}{q^2-m_s^2} \not{\epsilon}(\not{q}+m_s)\Gamma V_+ \phi_K(\vec{p}_K),
\end{aligned}
\tag{2.165}
$$

where m_s, e_s is the mass and charge of the strange quark.

$$
\begin{aligned}
V_2(r_3', r_3) &= V_{ST}(r_3) S_u(r_3 - r_3') V_{EM}^{Dirac}(r_3') \\
&= N_K N_\gamma \Gamma V_+ \int \frac{d^4q}{(2\pi)^4} \frac{d^3\vec{p}_K}{(2\pi)^{3/2}} \frac{e^{-ip_K.r_3} e^{-iq.(r_3-r_3')}}{q^2-m_u^2} (\not{q}+m_u) e_u \not{\epsilon} e^{ik.r_3'} \phi_K(\vec{p}_K) \\
&= \frac{N_K N_\gamma e_u}{(2\pi)^{11/2}} \int d^4q\, d^3\vec{p}_K \, \frac{e^{-ip_K.r_3-iq.(r_3-r_3')+ik.r_3'}}{q^2-m_u^2} \Gamma V_+ (\not{q}+m_u) \not{\epsilon} \phi_K(\vec{p}_K),
\end{aligned}
\tag{2.166}
$$

where m_u, e_u is the mass and charge of the up quark. Using equation (2.147),

$$
V_3(r_3', r_3) = V_{ST}(r_3) \delta^4(r_3 - r_3') \tag{2.167}
$$

The kaon wave function, $\psi_K(r_3)$, to be inserted into V_{ST} of equation (2.167) for graph (3) is a solution of the Klein-Gordon equation in the presence of a potential V_{EM}^{KG}, equation (2.132).

$$
(\Box_{r_3} + m_K^2)\psi_K(r_3) = -V_{EM}^{KG}(r_3)\psi_K(r_3). \tag{2.168}
$$

From equation (2.152) we have,

$$
(\Box_{r_3} + m_K^2) \int \Delta_K(r_3 - w) V_{EM}^{KG}(w) \psi_K(w)\, d^4w = -V_{EM}^{KG}(r_3)\psi_K(r_3) \tag{2.169}
$$

comparing equations (2.168) and (2.169), we get that the kaon wave function at r_3 after scattering off the potential V_{EM}^{KG} at w is,

$$
\psi_K(r_3) = \int \Delta_K(r_3 - w) V(w) \psi_K(w)\, d^4w. \tag{2.170}
$$

Where the form of the interaction between the kaon and the electromagnetic field at w is known from equation (2.135).

$$V(w) = ie_K(A^\mu(w)\frac{\partial}{\partial w^\mu} + \frac{\partial}{\partial w_\mu}A_\mu(w))$$

Using this along with (2.170) and (2.153) we get,

$$\begin{aligned}\psi_K(r_3) &= \frac{1}{(2\pi)^4}\int d^4q\, d^4w\, \frac{e^{-iq.(r_3-w)}}{q^2-m_K^2} ie_K(A^\mu(w)\frac{\partial}{\partial w^\mu} + \frac{\partial}{\partial w_\mu}A_\mu(w))\psi_K(w) \\ &= \frac{1}{(2\pi)^4}\int d^4q\, d^4w\, \frac{e^{-iq.(r_3-w)}}{q^2-m_K^2} ie_K(A^\mu(w)\frac{\vec{\partial}}{\partial w^\mu} - \frac{\overleftarrow{\partial}}{\partial w_\mu}A_\mu(w))\psi_K(w).\end{aligned} \tag{2.171}$$

Where $\frac{\overleftarrow{\partial}}{\partial w^\mu}$ indicates the derivative operator acts to the left and $\frac{\vec{\partial}}{\partial w^\mu}$ indicates the derivative operator acts to the right. The minus sign arises from integrating

$$\int d^4q\, d^4w\, \frac{e^{-iq.(r_3-w)}}{q^2-m_K^2} ie_K \frac{\partial}{\partial w_\mu}(A_\mu(w)\psi_K(w))$$

by parts. The boundary term is omitted as the potential A^μ is taken to vanish as $t, |\vec{w}| \to \pm\infty$.

Substituting for $\psi_K(w)$ and $A^\mu(w)$ into (2.171) and evaluating the derivatives we get,

$$\begin{aligned}\psi_K(r_3) &= \frac{N_K N_\gamma e_K}{(2\pi)^{11/2}}\int d^4q\, d^3\vec{p}_K\, d^4w\, \frac{e^{-iq.(r_3-w)}}{q^2-m_K^2} e^{-ip_K.w}(p_K+q)_\mu \epsilon^\mu e^{ik.w}\phi_K(\vec{p}_K) \\ &= \frac{N_K N_\gamma e_K}{(2\pi)^{11/2}}\int d^4q\, d^3\vec{p}_K\, \frac{e^{-iq.r_3}}{q^2-m_K^2+i\epsilon}(p_K+q)_\mu \epsilon^\mu (2\pi)^4\delta^4(q-p_K+k)\phi_K(\vec{p}_K) \\ &= \frac{N_K N_\gamma e_K}{(2\pi)^{3/2}}\int d^3\vec{p}_K\, \frac{e^{-ir_3.(p_K-k)}}{(p_K-k)^2-m_K^2+i\epsilon}(2p_K-k)_\mu \epsilon^\mu \phi_K(\vec{p}_K),\end{aligned} \tag{2.172}$$

after performing the integrals over w and q. Therefore,

$$\begin{aligned}V_3(r_3, r_3') &= N_K N_\gamma \int \frac{d^3\vec{p}_K}{(2\pi)^{3/2}}\Gamma V_+ e^{-ip_K.r_3}\phi_K(\vec{p}_K) e_K e^{ik.r_3}\frac{(2p_K-k).\epsilon}{(p_K-k)^2-m_K^2}\,\delta^4(r_3-r_3') \\ &= N_K N_\gamma e_K \int \frac{d^3\vec{p}_K}{(2\pi)^{3/2}}\frac{e^{-i(p_K-k).r_3}}{(p_K-k)^2-m_K^2}\Gamma V_+\phi_K(\vec{p}_K)(2p_K-k).\epsilon\,\delta^4(r_3-r_3')\end{aligned} \tag{2.173}$$

Where m_K, e_K is the mass and charge of the kaon.

The contact term comes from the minimal substitution

$$p_\mu \mapsto p_\mu - eA_\mu$$

in the kaon-nucleon pseudovector Lagrangian [28]. That is,

$$\begin{aligned} \Gamma &= i\tilde{g}_{Kus}\, \not{p}_K\gamma_5 = i\tilde{g}_{Kus}(\not{p} - \not{p}')\gamma_5 \\ &\mapsto i\tilde{g}_{Kus}(\not{p} - e_s\, \not{A} - \not{p}' + e_u\, \not{A})\gamma_5, \end{aligned} \tag{2.174}$$

The term

$$i\tilde{g}_{Kus}(-e_s\, \not{A} + e_u\, \not{A})\gamma_5$$

in (2.174) corresponds to the radiative process, graph (4) of Fig.2.2. Since $e_s - e_u = e_K$ we get the result

$$V_4(r_3', r_3) = -N_K N_\gamma e_K\, \not{e} e^{ik.r_3} \int \frac{d^3\vec{p}_K}{(2\pi)^{3/2}} i\tilde{g}_{Kus}\gamma_5 V_+ e^{-ip_K.r_3'} \phi_K(\vec{p}_K)\delta^4(r_3 - r_3') \tag{2.175}$$

2.6.2 Momentum Representation

In the next section we wish to examine the effect of a gauge transformation on our interactions $V^{(i)}$ $(i = 1, 2, 3)$. To do this it is convenient to Fourier transform the interactions V_i to momentum space. By inserting the closure relations (2.114) in V_1 (2.165) we get,

$$\begin{aligned} V_1(p, p') &= \frac{e_s N_K N_\gamma}{(2\pi)^{19/2}} \int d^4q\, d^4r_3'\, d^4r_3\, d^3\vec{p}_K\, \frac{e^{ik.r_3 - iq.(r_3 - r_3') - ip_K.r_3' - ip'.r_3' + ip.r_3}}{q^2 - m_s^2} \\ &\times \not{\epsilon}(\not{q} + m_s)\Gamma V_+ \phi_K(\vec{p}_K) \\ &= \frac{e_s N_K N_\gamma}{(2\pi)^{19/2}} \int d^4q\, d^4r_3'\, d^3\vec{p}_K\, \frac{e^{i(q - p_K - p').r_3'}}{q^2 - m_s^2}\, \not{\epsilon}(\not{q} + m_s)\Gamma V_+ \phi_K(\vec{p}_K) \end{aligned}$$

$$
\begin{aligned}
&\times\ (2\pi)^4\,\delta^4(k-q+p) \\
&= \frac{e_s N_K N_\gamma}{(2\pi)^{11/2}} \int d^4r_3'\, d^3\vec{p}_K\, \frac{e^{-i(k+p-p_K-p').r_3'}}{(k+p)^2-m_s^2}\, \not{(}\not{k}+\not{p}+m_s)\Gamma V_+ \phi_K(\vec{p}_K) \\
&= \frac{e_s N_K N_\gamma}{(2\pi)^{3/2}} \int d^3\vec{p}_K\, \frac{\not{(}\not{k}+\not{p}+m_s)\Gamma V_+}{(k+p)^2-m_s^2}\, \delta^4(p_K+p'-(k+p))\phi_K(\vec{p}_K).
\end{aligned}
\tag{2.176}
$$

For V_2 we have, using (2.166),

$$
\begin{aligned}
V_2(p,p') &= \frac{e_u N_K N_\gamma}{(2\pi)^{19/2}} \int d^4q\, d^4r_3'\, d^4r_3\, d^3\vec{p}_K\, \frac{e^{ik.r_3'-iq.(r_3-r_3')-ip_K.r_3-ip'.r_3'+ip.r_3}}{q^2-m_u^2} \\
&\times\ \Gamma V_+(\not{q}+m_u)\ \not{\epsilon}\phi_K(\vec{p}_K) \\
&= \frac{e_u N_K N_\gamma}{(2\pi)^{19/2}} \int d^4q\, d^4r_3'\, d^3\vec{p}_K\, \frac{e^{-i(q+k-p').r_3'}}{q^2-m_u^2}\Gamma V_+(\not{q}+m_u)\ \not{\epsilon}\phi_K(\vec{p}_K) \\
&\times\ (2\pi)^4\,\delta^4(q-(p-p_K)) \\
&= \frac{e_u N_K N_\gamma}{(2\pi)^{11/2}} \int d^4r_3' d^3\vec{p}_K\, \frac{e^{i(p-p_K+k-p').r_3'}}{(p-p_K)^2-m_u^2}\Gamma V_+(\not{p}-\not{p}_K+m_u)\ \not{\epsilon}\phi_K(\vec{p}_K) \\
&= \frac{e_u N_K N_\gamma}{(2\pi)^{3/2}} \int d^3\vec{p}_K\, \frac{\Gamma V_+(\not{p}-\not{p}_K+m_u)\ \not{\epsilon}}{(p-p_K)^2-m_u^2}\, \delta^4(p_K+p'-(k+p))\phi_K(\vec{p}_K).
\end{aligned}
\tag{2.177}
$$

With (2.173),

$$
\begin{aligned}
V_3(p,p') &= \frac{e_K N_K N_\gamma}{(2\pi)^{11/2}} \int d^4r_3'\, d^4r_3\, d^3\vec{p}_K\, \frac{e^{-i(p_K-k+p').r_3'} e^{ip.r_3}}{(p_K-k)^2-m_K^2} \\
&\times\ \Gamma V_+(2p_K-k).\epsilon\,\delta^4(r_3-r_3')\phi_K(\vec{p}_K) \\
&= \frac{e_K N_K N_\gamma}{(2\pi)^{11/2}} \int d^4r_3'\, d^3\vec{p}_K\, \frac{e^{-i(p_K-k+p'-p).r_3'}}{(p_K-k)^2-m_K^2}\Gamma V_+(2p_K-k).\epsilon\phi_K(\vec{p}_K) \\
&= \frac{e_K N_K N_\gamma}{(2\pi)^{3/2}} \int d^3\vec{p}_K\, \frac{\Gamma V_+(2p_K-k).\epsilon}{(p_K-k)^2-m_K^2}\, \delta^4(p_K+p'-(k+p))\phi_K(\vec{p}_K).
\end{aligned}
\tag{2.178}
$$

And for V_4 we get from (2.175),

$$\begin{aligned} V_4(p,p') &= \frac{-e_K N_K N_\gamma}{(2\pi)^{11/2}} \int d^3\vec{p}_K\, d^4r_3\, d^4r_3'\ \not{e}^{ik.r_3} i\tilde{g}_{Kus}\gamma_5 V_+ e^{-ip'.r_3'+ip.r_3-ip_K.r_3'} \\ &\times\ \phi_K(\vec{p}_K)\delta^4(r_3-r_3') \\ &= \frac{-e_K N_K N_\gamma}{(2\pi)^{11/2}} \int d^3\vec{p}_K\, d^4r_3'\ \not{e}^{ik.r_3'} i\tilde{g}_{Kus}\gamma_5 V_+ e^{-ip'.r_3'+ip.r_3'-ip_K.r_3'}\phi_K(\vec{p}_K) \\ &= -\frac{e_K N_K N_\gamma}{(2\pi)^{3/2}} \int d^3\vec{p}_K\ \not{e} i\tilde{g}_{Kus}\gamma_5 V_+ \phi_K(\vec{p}_K)\delta^4(p_K+p'-(k+p)). \end{aligned} \tag{2.179}$$

Notice the appearance of the energy-momentum conserving delta function for these processes. From this it can be seen that p corresponds to the 4-momentum of the s quark in the outgoing baryon (Y); and p' corresponds to the 4-momentum of the u quark in the incoming baryon (p).

2.7 Gauge Invariance.

2.7.1 Choice of Gauge

In the Lorentz gauge specified by

$$\partial_\mu A^\mu = 0 \tag{2.180}$$

we get the constraint, using (2.160),

$$k.\epsilon = 0. \tag{2.181}$$

This is a covariant condition which must be true in any frame. Within this gauge we can exploit the residual gauge freedom [29],

$$A^\mu \mapsto A^\mu - \partial^\mu \chi, \tag{2.182}$$

provided χ satisfies

$$\Box\chi = 0, \tag{2.183}$$

which keeps us in the Lorentz gauge. For a real photon satisfying the free field equation [29],

$$\Box A^{\mu} = 0, \tag{2.184}$$

with plane wave solutions of the form (2.160), the photon 4-momentum k must satisfy, the massless condition,

$$k_{\mu}k^{\mu} = 0 \Rightarrow k^{0} = |\vec{k}|. \tag{2.185}$$

The transformation (2.182) corresponds, by equations (2.181) and (2.185), to

$$\epsilon^{\mu} \mapsto \epsilon^{\mu} + \beta k^{\mu}. \tag{2.186}$$

We choose β so that ϵ^{0} vanishes. This fixes A^{μ}. It is known as the Coulomb gauge and makes manifest the transverse nature of the electromagnetic field.

2.7.2 Proof of Gauge Invariance

To check invariance under a gauge transformation (within the Lorentz gauge (2.180)), we note that the amplitude $\mathcal{S}^{i}_{Yp}$, corresponding to the i^{th} graph in Fig. 2.2, is linear in the polarization vector ϵ. That is

$$\begin{aligned} \mathcal{S}^{i}_{Yp} &= \epsilon_{\mu} T_{i}^{\mu} \\ &\mapsto (\epsilon_{\mu} + \beta k_{\mu}) T_{i}^{\mu} \end{aligned}$$

under a gauge transformation. Then if the set of Feynman diagrams in Fig. 2.2 is gauge invariant we must have

$$\sum_{i=1}^{N} k_{\mu} T_{i}^{\mu} = 0, \tag{2.187}$$

where N is the number of graphs.

We ignore binding between the quarks for the purpose of determining the behaviour of the interactions under a gauge transformation[3]. Replacing ϵ with k in (2.176)

$$\begin{aligned} k_\mu T_1^\mu &= \bar{u}(p)V_1^{\epsilon\mapsto k}(p,p')u(p') \\ &= \frac{e_s N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K \frac{\not{k}(\not{k}+\not{p}+m_s)\Gamma V_+\phi_K(\vec{p}_K)}{(k+p)^2-m_s^2}\,\delta^4(p_K+p'-(k+p))u(p'), \end{aligned}$$

where $u(p)$ and $u(p')$ are Dirac 4-spinors which are solutions of the free Dirac equation for the third quark. Using (2.150) and (2.185) we get,

$$\begin{aligned} \bar{u}(p)V_1^{\epsilon\mapsto k}(p,p')u(p') &= \frac{e_s N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K \frac{(m_s\not{k}+\not{k}\not{p})\Gamma V_+\phi_K(\vec{p}_K)}{p^2+2k.p-m_s^2} \\ &\times\ \delta^4(p_K+p'-(k+p))u(p'). \end{aligned}$$

Using the anticommutation relation for the Dirac gamma matrices (2.149) we find that

$$\not{k}\not{p} = 2k.p - \not{p}\not{k}. \tag{2.188}$$

Employing (2.188) and the Einstein mass-energy relation

$$p^2 = m_s^2, \tag{2.189}$$

$$\text{and } (p')^2 = m_u^2, \tag{2.190}$$

we get,

$$\begin{aligned} \bar{u}(p)V_1^{\epsilon\mapsto k}(p,p')u(p') &= \frac{e_s N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K \frac{(2k.p-(\not{p}-m_s)\not{k})\Gamma V_+\phi_K(\vec{p}_K)}{2k.p} \\ &\times\ \delta^4(p_K+p'-(k+p))u(p'). \end{aligned}$$

Using (2.143) we get

$$\bar{u}(p)V_1^{\epsilon\mapsto k}(p,p')u(p') = \frac{e_s N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K\,\Gamma V_+\phi_K(\vec{p}_K)u(p')\,\delta^4(p_K+p'-(k+p)). \tag{2.191}$$

[3]However, in general we will lose gauge invariance when we fold the interaction over three quark wave functions.

Replacing ϵ with k in (2.177)

$$\begin{aligned}\bar{u}(p)V_2^{\epsilon\mapsto k}(p,p')u(p') &= \frac{e_u N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K \frac{\Gamma V_+\phi_K(\vec{p}_K)(\not{p}-\not{p}_K+m_u)\not{k}}{(p-p_K)^2-m_u^2} \\ &\times \delta^4(p_K+p'-(k+p))u(p').\end{aligned}$$

Noting that $p-p_K=p'-k$ and using (2.150), (2.185), (2.190), and (2.188) we get,

$$\begin{aligned}\bar{u}(p)V_2^{\epsilon\mapsto k}(p,p')u(p') &= \frac{e_u N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K \frac{\Gamma V_+\phi_K(\vec{p}_K)(2p'.k-\not{k}(\not{p}'-m_u))}{-2p'.k} \\ &\times \delta^4(p_K+p'-(k+p))u(p').\end{aligned}$$

From the relation (2.142),

$$\bar{u}(p)V_2^{\epsilon\mapsto k}(p,p')u(p') = -\frac{e_u N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K\,\Gamma V_+\phi_K(\vec{p}_K)u(p')\,\delta^4(p_K+p'-(k+p)). \tag{2.192}$$

Replacing ϵ with k in (2.178)

$$\begin{aligned}\bar{u}(p)V_3^{\epsilon\mapsto k}(p,p')u(p') &= \frac{e_K N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K \frac{\Gamma V_+\phi_K(\vec{p}_K)(2p_K-k).k}{(p_K-k)^2-m_K^2} \\ &\times \delta^4(p_K+p'-(k+p))u(p').\end{aligned}$$

As before we get,

$$\bar{u}(p)V_3^{\epsilon\mapsto k}(p,p')u(p') = -\frac{e_K N_K N_\gamma}{(2\pi)^{3/2}}\bar{u}(p)\int d^3\vec{p}_K\,\Gamma V_+\phi_K(\vec{p}_K)u(p')\,\delta^4(p_K+p'-(k+p)). \tag{2.193}$$

With PS Coupling

Here we use the substitution (2.140)

$$\Gamma \mapsto g_{Kus}i\gamma_5.$$

From our condition for gauge invariance (2.187), and noting that $e_u = \frac{2}{3}e, e_s = -\frac{1}{3}e, e_K = -e$, where $-e$ is the charge on the electron, we have,

$$\begin{aligned}\sum_{i=1}^{3} k_\mu T_i^\mu &= \bar{u}(p)(V_1^{\epsilon\mapsto k}(p,p') + V_2^{\epsilon\mapsto k}(p,p') + V_3^{\epsilon\mapsto k}(p,p'))u(p') \\ &= \frac{N_K N_\gamma}{(2\pi)^{3/2}} i g_{Kus}\bar{u}(p)(e_s - e_u - e_K)\int d^3\vec{p}_K\, \gamma_5 V_+ \phi_K(\vec{p}_K)u(p') \\ &\times\ \delta^4(p_K + p' - (k+p)) = 0.\end{aligned}$$

Thus our interaction is gauge invariant at the level of free quarks, in PS coupling, using only the first three diagrams of Fig. 2.2.

With PV Coupling

Here we use the substitution (2.141)

$$\Gamma \mapsto \tilde{g}_{Kus}\, \not{q} i\gamma_5.$$

For graphs (1) and (2) of Fig. 2.2 $q = p_K$. However for graph (3), $q = p_K - k$, since the kaon radiates prior to capture. Now we have

$$\begin{aligned}\sum_{i=1}^{3} k_\mu T_i^\mu &= \bar{u}(p)(V_1^{\epsilon\mapsto k}(p,p') + V_2^{\epsilon\mapsto k}(p,p') + V_3^{\epsilon\mapsto k}(p,p'))u(p') \\ &= \frac{N_K N_\gamma}{(2\pi)^{3/2}} i\tilde{g}_{Kus}\bar{u}(p)\int d^3\vec{p}_K\,(e_s\, \not{p}_K - e_u\, \not{p}_K - e_K(\not{p}_K - \not{k}))\gamma_5 V_+ \phi_K(\vec{p}_K)u(p') \\ &\times\ \delta^4(p_K + p' - (k+p)) \\ &= \frac{N_K N_\gamma}{(2\pi)^{3/2}} i\tilde{g}_{Kus}\bar{u}(p)\int d^3\vec{p}_K\, e_K\, \not{k}\gamma_5 V_+ \phi_K(\vec{p}_K)u(p')\,\delta^4(p_K + p' - (k+p)) \\ &\neq\ 0.\end{aligned}$$

Therefore we need the contact graph (Fig. 2.2 (4)) in order to obtain a gauge invariant interaction in the PV coupling scheme.

$$V_4^{\epsilon\mapsto k}(p,p') = -\frac{N_K N_\gamma}{(2\pi)^{3/2}} i\tilde{g}_{Kus}\bar{u}(p)\int d^3\vec{p}_K\, e_K\, \not{k}\gamma_5 V_+ \phi_K(\vec{p}_K)u(p')\,\delta^4(p_K + p' - (k+p)). \tag{2.194}$$

2.8 Kinematics

In this section we derive the magnitude of the photon momentum in the kaonic atom rest frame.

The photon momentum is kinematically constrained to take on the value, $|\vec{k}| = k^0 \approx 281MeV$ when $Y = \Lambda$, and $k^0 \approx 219MeV$ when $Y = \Sigma^0$. This can be seen from conservation of energy-momentum in the K^-p atom rest frame,

$$P_p^\mu + p_K^\mu = k^\mu + P_Y^\mu = 0. \tag{2.195}$$

The zeroth component gives,

$$m_K + m_p = k^0 + \sqrt{|\vec{P_Y}|^2 + m_Y^2}, \tag{2.196}$$

and the 3-vector components give,

$$\vec{P_Y} + \vec{k} = 0 \Rightarrow |\vec{P_Y}| = |\vec{k}|. \tag{2.197}$$

From (2.196) we get,

$$\begin{aligned} (m_K + m_p)^2 + (k^0)^2 - 2k^0(m_p + m_K) &= |\vec{P_Y}|^2 + m_Y^2, \\ \text{with } k^0 = |\vec{P_Y}| \Rightarrow k^0 &= \frac{(m_p + m_K)^2 - m_Y^2}{2(m_p + m_K)}. \end{aligned} \tag{2.198}$$

Inserting the values [30],

$$\begin{aligned} m_p &= 938.27231MeV, \\ m_K &= 493.67MeV, \\ m_\Lambda &= 1115.63MeV, \\ m_{\Sigma^0} &= 1192.55MeV, \end{aligned}$$

we get, for the reaction $K^-p \to \Lambda\gamma$, $|\vec{k}| = k^0 = 281MeV$ and for the reaction $K^-p \to \Sigma^0\gamma$, $k^0 = 219MeV$.

2.9 Observations

It is generally assumed [31],[32] that the kaon momentum is approximately zero prior to the formation of the kaonic atom. This can be seen heuristically by modeling the kaonic atom with the Bohr model[33]. The binding energy of the kaon will then be given by,

$$E_B = -\frac{\mu_K}{\mu_e}\frac{13.6}{n^2}eV = \frac{-8.63 \times 10^3}{n^2}eV, \tag{2.199}$$

where μ_K is the proton-kaon reduced mass; μ_e the proton-electron reduced mass; and n is the principal quantum number of the kaonic orbit. The total energy of the atom is the sum of the kinetic and potential energies,

$$E_B = K + U.$$

Taking a circular orbit

$$\begin{aligned}
\frac{mv^2}{r} &= \frac{e^2}{4\pi\epsilon_0 r^2} \\
\text{along with } U &= \frac{-e^2}{4\pi\epsilon_0 r} \text{ and } K = \frac{1}{2}mv^2 \\
\text{gives } E_B &= \frac{e^2}{8\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} \\
&= -K = \frac{-|\vec{p_K}|^2}{2\mu_K} \\
\\
\Rightarrow |\vec{p_K}| &= \sqrt{-2\mu_K E_B} \\
&= \sqrt{\frac{5.591 \times 10^{12}}{n^2}}eV.
\end{aligned}$$

As mentioned in §2.5.4 almost all captures take place from a relative S state. In addition, half the captures are said to take place for principal quantum number $n > 10$ and less than 4% survive to $n = 4$. We take then, as a reasonable estimate, $n = 10$.

This gives, for the kaon momentum,

$$|\vec{p_K}| \approx 0.236 MeV.$$

It can be seen that $|\vec{p_K}|$ is far smaller than the other momenta ($|\vec{P_Y}| = |\vec{k}| \approx 281 MeV$) and masses ($m_u = 330 MeV$, $m_s = 550 MeV$) in the problem; we therefore neglect $|\vec{p_K}|$.

Since $|\vec{p_K}| \approx 0$ we take the normalized momentum space wave function to be

$$\phi_K(\vec{p}_K) = \sqrt{\frac{(2\pi)^3}{\mathcal{V}}}\delta^3(\vec{p}_K). \tag{2.200}$$

Inserting this into (2.158) we get

$$\psi_K(\vec{z}, t) = N_K e^{-iE_K t}\tilde{\psi}_K(\vec{0}). \tag{2.201}$$

$$\text{with } \tilde{\psi}_K(\vec{0}) = \frac{1}{\sqrt{\mathcal{V}}} \tag{2.202}$$

so that it retains the correct dimensions. That is the radial wave function of the kaon is approximately constant and so must go as $1/\sqrt{\mathcal{V}}$ to satisfy equation (2.157).

From our choice of the Coulomb gauge we had (§2.7.1)

$$k.\epsilon = 0 \text{ and } \epsilon_0 = 0.$$

Inserting these conditions into $V_3(p, p')$ (2.178) and letting $\vec{p}_K \to 0$, we see that the contribution to the amplitude from graph (3) of Fig. 2.2 is zero. However, it was necessary to include this diagram to preserve gauge invariance (§2.7). For $V_1(p, p')$, $V_2(p, p')$ and $V_4(p, p')$ we get

$$V_1(p, p') = e_s N_K N_\gamma \frac{\not{\epsilon}(\not{k} + \not{p} + m_s)\Gamma V_+}{(k+p)^2 - m_s^2}\delta^4(p_K + p' - (k+p))\psi_K(0), \tag{2.203}$$

$$V_2(p, p') = e_u N_K N_\gamma \frac{\Gamma V_+(\not{p} - \not{p}_K + m_u)\not{\epsilon}}{(p - p_K)^2 - m_u^2}\delta^4(p_K + p' - (k+p))\psi_K(0), \tag{2.204}$$

$$V_4(p, p') = -e_K N_K N_\gamma \not{\epsilon} i\tilde{g}_{Kus}\gamma_5\,\delta^4(p_K + p' - (k+p))V_+\psi_K(0). \tag{2.205}$$

2.10 The Nonrelativistic Reduction.

In this section we approximate V by an expansion in p/m, consistent with our use of nonrelativistic wave functions.

2.10.1 Validity of the Nonrelativistic Reduction.

The NQM assumes the quarks are approximately at rest in the hadron rest frame. That is the hadron has negligible internal momentum. However most models tend to predict that the momentum of a quark in a hadron, can be sizeable. For example a particle of mass M, localized within a volume of radius R, has momentum p, by the uncertainty relation,

$$\langle p^2 \rangle^{\frac{1}{2}} \sim \frac{1}{R}. \tag{2.206}$$

The identity[9], $\Delta p \equiv \langle (p - \langle p \rangle)^2 \rangle^{\frac{1}{2}}$ with $\langle p \rangle = 0$ in the hadron rest frame, has been used. R is the radius of the ground state wave function. The analysis of various hadronic processes indicates [34] that,

$$R^2 \sim 6\text{–}12 GeV^{-2} \Rightarrow \frac{1}{R} \sim 300 MeV. \tag{2.207}$$

In our calculations we use the constituent quark mass[4] [24]

$$\begin{aligned} m_u &= 330\, MeV \\ m_s &= 550\, MeV \end{aligned}$$

So we get,

$$\frac{\langle p^2 \rangle^{\frac{1}{2}}}{m_q} \leq 1. \tag{2.208}$$

[4]Note that $3m_u > m_p$ due to the binding energy of the baryon.

This suggests that taking $(p/m)^2 \ll 1$ may be reasonable, although not $(p/m) \ll 1$.

In addition Capstick and Isgur [11] point out that the hadronic wave functions (of chapter 1) derived from the harmonic confinement potential depend only on the quark coordinates. In QCD however, the hadronic wave function must also depend on the state of the glue.

Despite these shortcomings in the NQM, the agreement with experiment (predicting masses, magnetic moments, and decay amplitudes) has been excellent. It is claimed [11], that this can be attributed to choices of effective parameters such as quark constituent masses and α_s (which can absorb the effect of relativistic modifications of spin dependent interactions).

Looking at the derivation of the effective Hamiltonian one can see that the typical approximations like

$$\sqrt{m^2 + |\vec{p}|^2} \rightarrow m + \frac{|\vec{p}|^2}{2m}$$

are not that bad for $(p/m) = 1$ (6% difference). Furthermore, some of the approximations will probably be compensated by a renormalization of parameters [35].

Koniuk[36] points out that relativistic models, once their parameters have been chosen, give essentially the same results as nonrelativistic models. In addition, it is argued [5], that neglected terms in the Hamiltonian equation (1.3)

$$H = \sum_i m_i + H_0 + H_{hyp},$$

such as relativistic corrections and other one-gluon exchange effects, seem to be relatively unimportant at the level of 10–20%.

From its considerable success in the past it appears the model can give good qualitative agreement with the observed properties of low-lying baryons. With the caveat, however, that the model should not be take too seriously quantitatively. It is still a

source of some debate as to why the NQM works as well as it does. We believe that it is a acceptable model of confinement; suitable for our purposes.

2.10.2 The Nonrelativistic Reduction Prescription.

In chapter 1 we treated spin nonrelativistically, that is we introduced it in a purely 'ad hoc' fashion. Had we used the Dirac equation for our wave functions, spin would have arisen naturally. The Dirac equation for a quark of mass m_q, in the presence of a potential, V_{int} is,

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}\cdot\vec{p} + \beta m_q + V_{int})\psi. \tag{2.209}$$

V_{int} represents the internal confinement with relativistic corrections experienced by the quarks in the baryon. In the representation

$$\psi = \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix}, \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.210}$$

where $\vec{\sigma}$ are the Pauli spin matrices and ϕ and χ are two component spinors, (2.209) becomes

$$\begin{aligned} i\frac{\partial}{\partial t}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} &= \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} m_q \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} + V_{int}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} \\ &= \vec{\sigma}\cdot\vec{p}\begin{pmatrix} \tilde{\chi} \\ \tilde{\phi} \end{pmatrix} + m_q \begin{pmatrix} \tilde{\phi} \\ -\tilde{\chi} \end{pmatrix} + V_{int}\begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix}. \end{aligned} \tag{2.211}$$

We separate out the time dependence in ψ

$$\psi = \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \end{pmatrix} = e^{-iE_q t}\begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{2.212}$$

where ϕ and χ are constant with respect to time and E_q is the total energy of the state ψ. Inserting (2.212) into (2.211) and cancelling the common exponential factor we get,

$$E_q \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \vec{\sigma} \cdot \vec{p} \begin{pmatrix} \chi \\ \phi \end{pmatrix} + m_q \begin{pmatrix} \phi \\ -\chi \end{pmatrix} + V_{int} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{2.213}$$

The second equation of (2.213) gives,

$$\chi = \frac{\vec{\sigma} \cdot \vec{p}}{E_q + m_q - V_{int}} \phi \tag{2.214}$$

χ are the "small" components of the Dirac 4-spinors. They are reduced by v/c compared to the "large" components ϕ. Because of relation (2.208) we conclude that the small components of the quark Dirac spinors are of the same order as the large.

To treat this relativistic situation we follow Yaouanc *et. al.*'s [34] prescription of replacing the Pauli spinors χ_i in the spin wave functions

$$\chi_{i+} = \uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_{i-} = \downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by Dirac spinors

$$u = \sqrt{\frac{E_i + m_i}{2m_i}} \begin{pmatrix} \chi_i \\ \frac{\vec{\sigma}_i \cdot \vec{p}_i}{E_i + m_i} \chi_i \end{pmatrix}. \tag{2.215}$$

The subscript i refers to the i^{th} quark in the baryon. We have choosen normalization so that

$$\bar{u}u = 1, \tag{2.216}$$

as in Bjorken and Drell [26]. This yields an over all normalization factor, from normalizing the total baryon wave function,

$$|N|^2 \int \Psi^\dagger \Psi \, d\mathcal{V} \;=\; 1$$

$$\Rightarrow N_p = \frac{1}{\sqrt{\mathcal{V}}}\sqrt{\frac{m_u}{E_1'}}\sqrt{\frac{m_u}{E_2'}}\sqrt{\frac{m_u}{E_3'}},$$
$$N_Y = \frac{1}{\sqrt{\mathcal{V}}}\sqrt{\frac{m_u}{E_1}}\sqrt{\frac{m_u}{E_2}}\sqrt{\frac{m_s}{E_3}}.$$

Using (2.215) amounts to adopting the spinor structure of free quarks. Since we are using the impulse approximation (§2.3) this seems reasonable.

The interaction involves only the third quark. Therefore the spectator quark spinors, as a result of (2.216), give unity when folded together. We are left with

$$\int \bar{\Psi}_Y(p)V(p,p')\Psi_p(p')\,d^4p\,d^4p' = \sqrt{\frac{(E_3+m_s)(E_3'+m_u)}{4m_um_s}}\int \phi_Y^\dagger(p) \times \left\{(1\ \frac{-\vec{\sigma}\cdot\vec{p}}{E_3+m_s})V(p,p')\begin{pmatrix}1\\ \frac{\vec{\sigma}\cdot\vec{p}'}{E_3'+m_u}\end{pmatrix}\right\}\phi_p(p')\,d^4p\,d^4p'. \tag{2.217}$$

This is expanded and a nonrelativistic reduction carried out. We

- Expand the interaction and discard terms of order $(p/m)^2$ or higher,
- treat the denominator of the propagators and phase space factors relativistically.

Therefore the expression of the full matrix element is valid up to order $(p/m)^2$ (p is any momentum and m any mass).

Applying this prescription to the { } in (2.217), we get for $V = V_1$ (equation (2.203)),

$$(1\ \frac{-\vec{\sigma}\cdot\vec{p}}{E_3+m_s})\ \not{\epsilon}(\not{k}+\not{p}+m_s)\gamma_5\begin{pmatrix}1\\ \frac{\vec{\sigma}\cdot\vec{p}'}{E_3'+m_u}\end{pmatrix}$$

we get (with $\epsilon^0 = 0$)

$$(1\ \frac{-\vec{\sigma}\cdot\vec{p}}{E_3+m_s})\begin{pmatrix}0 & -\vec{\sigma}\cdot\vec{\epsilon}\\ \vec{\sigma}\cdot\vec{\epsilon} & 0\end{pmatrix}\begin{pmatrix}m_s+k^0+p^0 & -\vec{\sigma}.(\vec{k}+\vec{p})\\ \vec{\sigma}.(\vec{k}+\vec{p}) & m_s-k^0-p^0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\ \frac{\vec{\sigma}\cdot\vec{p}'}{E_3'+m_u}\end{pmatrix}$$

$$= -\frac{\vec{\sigma}\cdot\vec{p}}{E_3+m_s}\vec{\sigma}\cdot\vec{\epsilon}(m_s+k^0+p^0)\frac{\vec{\sigma}\cdot\vec{p'}}{E_3'+m_u}+\frac{\vec{\sigma}\cdot\vec{p}}{E_3+m_s}\vec{\sigma}\cdot\vec{\epsilon}\,\vec{\sigma}.(\vec{k}+\vec{p})$$
$$- \vec{\sigma}\cdot\vec{\epsilon}\,\vec{\sigma}.(\vec{k}+\vec{p})\frac{\vec{\sigma}\cdot\vec{p'}}{E_3'+m_u}-\vec{\sigma}\cdot\vec{\epsilon}(m_s-k^0-p^0). \tag{2.218}$$

Now we neglect terms $O(p/m)^2$ or higher, and get

$$V_1^{PS}(p,p') = \frac{-N_K N_\gamma e_s i g_{Kus} V_+ \psi_K(0)\vec{\sigma}\cdot\vec{\epsilon}(m_s-k^0-p^0)}{(k+p)^2-m_s^2}\delta^4(p_K+p'-(k+p)) \tag{2.219}$$

in PS coupling. With PV coupling we get

$$(1 \quad \frac{-\vec{\sigma}\cdot\vec{p}}{E_3+m_s})\begin{pmatrix} 0 & -\vec{\sigma}\cdot\vec{\epsilon} \\ \vec{\sigma}\cdot\vec{\epsilon} & 0 \end{pmatrix}\begin{pmatrix} m_s+k^0+p^0 & -\vec{\sigma}.(\vec{k}+\vec{p}) \\ \vec{\sigma}.(\vec{k}+\vec{p}) & m_s-k^0-p^0 \end{pmatrix}$$
$$\times \begin{pmatrix} E_K & 0 \\ 0 & -E_K \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p'}}{E_3'+m_u} \end{pmatrix}$$

$$\Rightarrow V_1^{PV}(p,p') = \frac{N_K N_\gamma e_s i\tilde{g}_{Kus} V_+\psi_K(0)\vec{\sigma}\cdot\vec{\epsilon}\,m_K(m_s-k^0-p^0)}{(k+p)^2-m_s^2}\delta^4(p_K+p'-(k+p)) \tag{2.220}$$

$$= -\frac{\tilde{g}_{Kus}}{g_{Kus}}m_K V_1^{PS}(p,p')$$

since $E_K = m_K$ when $\vec{p}_K = 0$.

Similarly we get from equation (2.204),

$$V_2^{PS}(p,p') = \frac{N_K N_\gamma e_u i g_{Kus} V_+\psi_K(0)\vec{\sigma}\cdot\vec{\epsilon}(m_K-p^0+m_u)}{(p-p_K)^2-m_u^2}\delta^4(p_K+p'-(k+p)) \tag{2.221}$$

for PS coupling and

$$V_2^{PV}(p,p') = \frac{N_K N_\gamma e_u i\tilde{g}_{Kus} V_+\psi_K(0)\vec{\sigma}\cdot\vec{\epsilon}\,m_K(m_K-p^0+m_u)}{(p-p_K)^2-m_u^2}\delta^4(p_K+p'-(k+p)) \tag{2.222}$$

$$= \frac{\tilde{g}_{Kus}}{g_{Kus}}m_K V_2^{PS}(p,p')$$

for PV coupling. Recall from equation (2.145) that

$$\frac{\tilde{g}_{Kus}}{g_{Kus}} = \frac{1}{m_s + m_u}.$$

For the contact graph the interaction (2.205) becomes

$$V_4^{PV}(p,p') = N_K N_\gamma e_K i\tilde{g}_{Kus} V_+ \psi_K(0)\vec{\sigma}\cdot\vec{\epsilon}\,\delta^4(p_K + p' - (k+p)) \tag{2.223}$$

in the nonrelativistic approximation[5].

2.11 The Problem.

Now that we know the form of the interactions and their Fourier transforms we can substitute them into (2.127). We write

$$V(p,p') \equiv \tilde{V}(p,p')\, N_K N_\gamma \psi_K(0)\delta^4(p_K + p' - (k+p)). \tag{2.224}$$

V is the total interaction:

$$\begin{aligned} V^{PS} &= V_1^{PS} + V_2^{PS} \text{ for PS coupling,} \\ V^{PV} &= V_1^{PV} + V_2^{PV} + V_4^{PV} \text{ for PV coupling.} \end{aligned}$$

We get,

$$\begin{aligned} \mathcal{S}_{Yp} &= ZC(2\pi)^4\,\delta^4(p_K + P_p - (k + P_Y)) \\ &\times \int \Phi_Y^\dagger(\vec{\rho},\vec{\lambda}) e^{i\vec{\lambda}\cdot\vec{p}\frac{1}{\sqrt{6}}(\frac{m_\lambda}{m_u}-3)} \tilde{V}(\vec{p}, p^0 = E_3, \vec{p}' = \vec{P}_p + \vec{p} - \vec{P}_Y, (p^0)' = E_3') \\ &\times e^{\frac{-i}{\sqrt{6}}((\frac{m_\lambda}{m_u})\vec{\lambda}-3\vec{\lambda}')\cdot(\vec{P}_p+\vec{p}-\vec{P}_Y)} \Phi_p(\vec{\rho},\vec{\lambda}') e^{\frac{i}{\sqrt{6}}\vec{P}_p\cdot((\frac{m_\lambda}{m_u})\vec{\lambda}-\vec{\lambda}')}\, d^3\vec{\rho}\, d^3\vec{\lambda}\, d^3\vec{\lambda}'\, d^3\vec{p}. \end{aligned} \tag{2.225}$$

That is

$$\mathcal{S}_{Yp} = Z(2\pi)^4\delta^4(p_K + P_p - (k + P_Y))\mathcal{M} \tag{2.226}$$

[5]If we use the on-shell condition (2.142) before we take the nonrelativistic reduction, we find that the sum $V_1^{PV} + V_2^{PV} + V_4^{PV}$ reduces to $V_1^{PS} + V_2^{PS}$. Since we intend to use bound state quark wave functions we discard this approach and the PV amplitude will differ from PS amplitude.

where

$$\mathcal{M} \equiv C \int I_Y(\vec{\rho},\vec{p})\, \tilde{V}\, I_p(\vec{\rho},\vec{p})\, d^3\vec{\rho}\, d^3\vec{p}, \tag{2.227}$$

$$Z \equiv N_K N_\gamma N_p N_Y \sqrt{\frac{(E_3+m_s)(E_3'+m_u)}{4m_u m_s}}\psi_K(0), \tag{2.228}$$

$$I_Y(\vec{\rho},\vec{p}) \equiv \int \Phi_Y^\dagger(\vec{\rho},\vec{\lambda}) e^{i\frac{\vec{\lambda}}{\sqrt{6}}\cdot(\frac{m_\lambda}{m_u}\vec{P}_Y - 3\vec{p})}\, d^3\vec{\lambda}, \tag{2.229}$$

$$I_p(\vec{\rho},\vec{p}) \equiv \int \Phi_p(\vec{\rho},\vec{\lambda}') e^{\frac{i\vec{\lambda}'}{\sqrt{6}}\cdot(2\vec{P}_p + 3(\vec{p}-\vec{P}_Y))}\, d^3\vec{\lambda}', \tag{2.230}$$

$$\text{and } C \equiv \frac{27b}{(2\pi)^3 2\sqrt{2}}. \tag{2.231}$$

The normalization constants are given by

$$\begin{aligned} N_K &= \frac{1}{\sqrt{2m_K}}, \\ N_\gamma &= \frac{1}{\sqrt{2k^0\mathcal{V}}}, \\ N_p &= \frac{1}{\sqrt{\mathcal{V}}}\sqrt{\frac{m_u}{E_1'}}\sqrt{\frac{m_u}{E_2'}}\sqrt{\frac{m_u}{E_3'}}, \\ N_Y &= \frac{1}{\sqrt{\mathcal{V}}}\sqrt{\frac{m_u}{E_1}}\sqrt{\frac{m_u}{E_2}}\sqrt{\frac{m_s}{E_3}}. \end{aligned}$$

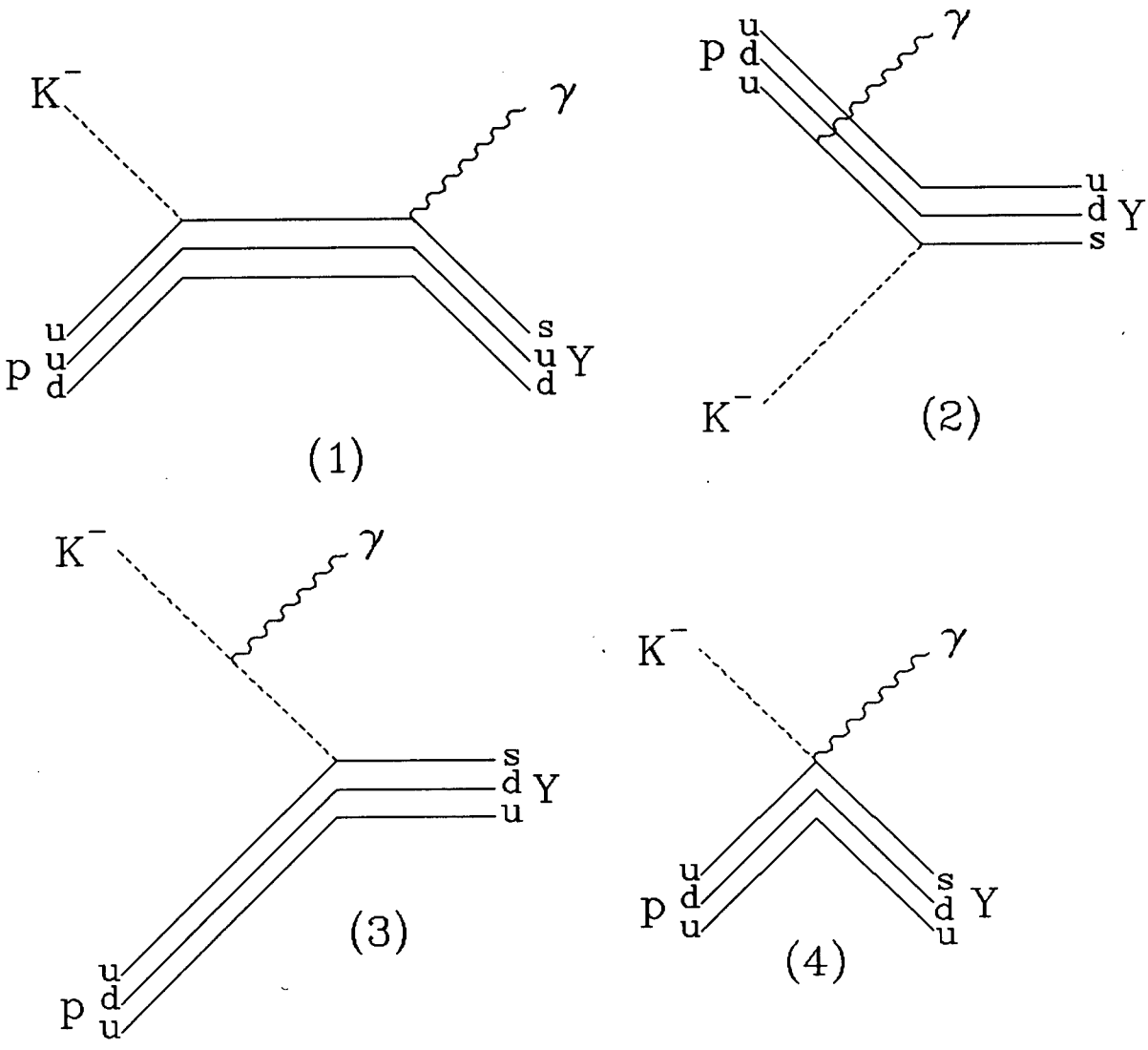


Figure 2.2: Feynman diagrams contributing to the process $K^- p \to Y\gamma$.

Chapter 3

Calculations

In this chapter we wish to evaluate the integral (2.227) in order to get an expression for the invariant amplitude $\mathcal{M}$. This calculation can be divided into evaluating the flavour, space, and spin contributions.

3.1 Flavour Space

The only flavour dependent piece in $\tilde{V}$ is V_+ (see §1.8). V_+ acts in the flavour space of the third quark; transforming a u quark to an s quark. From table 1.6, table 1.7, and equation (1.34) we get the flavour matrix elements listed in table 3.11.

3.2 Momentum Space

With the flavour matrix elements determined we have, for the momentum space part of the amplitude,

$$\mathcal{M}_{MS} = C \int \left[\int \Phi_Y^\dagger e^{i\vec{\lambda}\cdot\vec{q}_Y} \, d^3\vec{\lambda} \right] \tilde{V} \left[\int \Phi_p e^{i\vec{\lambda}'\cdot\vec{q}_p} \, d^3\vec{\lambda}' \right] d^3\vec{p}\, d^3\vec{p}' \tag{3.232}$$

$$\text{where } \vec{q}_Y \equiv \frac{1}{\sqrt{6}}\left(\left(\frac{m_\lambda}{m_u}\right)\vec{P}_Y - 3\vec{p}\right) \tag{3.233}$$

$$\vec{q}_p \equiv \frac{3}{\sqrt{6}}(\vec{p} - \vec{P}_Y). \tag{3.234}$$

We have already taken care of the flavour dependent part of the potential V_+. Also note that the spin dependent term $\vec{\sigma}\cdot\vec{\epsilon}$ in the potential, acts only in the spin space of

$$
\begin{aligned}
&\langle\phi_\Lambda^\rho|V_+|\phi_p^\rho\rangle = \sqrt{\tfrac{2}{3}}, \quad && \langle\phi_\Lambda^\lambda|V_+|\phi_p^\rho\rangle = 0,\\
&\langle\phi_\Lambda|V_+|\phi_p^\rho\rangle = 1, && \langle\phi_\Lambda^\rho|V_+|\phi_p^\lambda\rangle = 0,\\
&\langle\phi_\Lambda^\lambda|V_+|\phi_p^\lambda\rangle = 0, && \langle\phi_\Lambda|V_+|\phi_p^\lambda\rangle = 0,\\
&\langle\phi_\Lambda^A|V_+|\phi_p^\lambda\rangle = 0, && \langle\phi_\Lambda^A|V_+|\phi_p^\rho\rangle = -\tfrac{1}{\sqrt{3}},\\
&\langle\phi_\Sigma^\rho|V_+|\phi_p^\rho\rangle = 0, && \langle\phi_\Sigma^\lambda|V_+|\phi_p^\rho\rangle = 0,\\
&\langle\phi_\Sigma|V_+|\phi_p^\rho\rangle = 0, && \langle\phi_\Sigma^\rho|V_+|\phi_p^\lambda\rangle = 0,\\
&\langle\phi_\Sigma^\lambda|V_+|\phi_p^\lambda\rangle = -\tfrac{\sqrt{2}}{3}, && \langle\phi_\Sigma|V_+|\phi_p^\lambda\rangle = -\tfrac{1}{\sqrt{3}},\\
&\langle\phi_\Sigma^S|V_+|\phi_p^\lambda\rangle = -\tfrac{1}{3}, && \langle\phi_\Sigma^S|V_+|\phi_p^\rho\rangle = 0.
\end{aligned}
$$

Table 3.11: Flavour matrix elements

the third quark. Therefore we separate out the spin piece and denote it

$$\langle SP\rangle \equiv \langle\chi_f^J|\vec{\sigma}\cdot\vec{\epsilon}|\chi_i^{J'}\rangle.$$

$\chi^{J=0}$ denotes χ^ρ and $\chi^{J=1}$ corresponds to χ^λ. This term will be evaluated in §3.3.

Since $\tilde{V}$ is a function of $\vec{p}$ only, we can use the orthonormality of the $\vec{\rho}$ oscillator wave functions to write the general momentum space matrix element (3.232) in terms of the $\vec{\lambda}$ oscillator wave functions. This will involve combinations of the terms

$$a_Y \equiv \int \psi_{000}(\vec{\lambda})e^{i\vec{q}_Y\cdot\vec{\lambda}}\,d^3\vec{\lambda}, \tag{3.235}$$

$$a_p \equiv \int \psi_{000}(\vec{\lambda}')e^{i\vec{q}_p\cdot\vec{\lambda}'}\,d^3\vec{\lambda}', \tag{3.236}$$

$$b_Y \equiv \int \psi_{100}(\vec{\lambda})e^{i\vec{q}_Y\cdot\vec{\lambda}}\,d^3\vec{\lambda}, \tag{3.237}$$

$$b_p \equiv \int \psi_{100}(\vec{\lambda}')e^{i\vec{q}_p\cdot\vec{\lambda}'}\,d^3\vec{\lambda}', \tag{3.238}$$

$$c_Y \equiv \int \psi_{010}(\vec{\lambda})e^{i\vec{q}_Y\cdot\vec{\lambda}}\,d^3\vec{\lambda}, \tag{3.239}$$

$$c_p \equiv \int \psi_{010}(\vec{\lambda}')e^{i\vec{q}_p\cdot\vec{\lambda}'}\,d^3\vec{\lambda}'. \tag{3.240}$$

Integrals involving the harmonic oscillator wave function ψ_{01m} with $m \neq 0$ give zero

when integrated over all space. Now defining

$$a \equiv \int a_Y \tilde{V} a_p \, d^3\vec{p}, \tag{3.241}$$

$$b \equiv \int a_Y \tilde{V} b_p \, d^3\vec{p}, \tag{3.242}$$

$$c \equiv \int b_Y \tilde{V} a_p \, d^3\vec{p}, \tag{3.243}$$

$$d \equiv \int b_Y \tilde{V} b_p \, d^3\vec{p}, \tag{3.244}$$

$$g \equiv \int c_Y \tilde{V} c_p \, d^3\vec{p}, \tag{3.245}$$

$$\text{and } Q \equiv e^{i\vec{q}_Y \cdot \vec{\lambda}} \tilde{V} e^{i\vec{q}_p \cdot \vec{\lambda}'}. \tag{3.246}$$

From table 1.3 and table 1.4, we obtain table 3.12.

See Appendix B for details on the evaluation of a_Y, a_p, b_Y, b_p, c_Y, c_p. The results are

$$a = \left(\frac{4\pi}{\alpha\alpha_\lambda}\right)^{\frac{3}{2}} I_1, \tag{3.247}$$

$$b = \frac{4}{\sqrt{3}}\left(\frac{2\pi}{\alpha\alpha_\lambda}\right)^{\frac{3}{2}} \left(\frac{I_2}{\alpha^2} - \frac{3}{2} I_1\right), \tag{3.248}$$

$$c = \frac{4}{\sqrt{3}}\left(\frac{2\pi}{\alpha\alpha_\lambda}\right)^{\frac{3}{2}} \left(\frac{I_3}{\alpha_\lambda^2} - \frac{3}{2} I_1\right), \tag{3.249}$$

$$d = \frac{16}{3\alpha_\lambda^2\alpha^2}\left(\frac{\pi}{\alpha\alpha_\lambda}\right)^{\frac{3}{2}} \left(I_4 - \frac{3}{2}(\alpha^2 I_3 + \alpha_\lambda^2 I_2) + \frac{9}{4}\alpha^2\alpha_\lambda^2 I_1\right), \tag{3.250}$$

$$g = \frac{-16}{\alpha_\lambda \alpha}\left(\frac{\pi}{\alpha\alpha_\lambda}\right)^{\frac{3}{2}} I_5. \tag{3.251}$$

Where

$$I_J \equiv \sum_k I_{Jk} \text{ (k is an index which sums over the diagrams),}$$

$$\text{and } I_{1k} \equiv \int e^{-\frac{|\vec{q}_Y|^2}{2\alpha_\lambda^2}} \tilde{V}_k e^{-\frac{|\vec{q}_p|^2}{2\alpha^2}} \, d^3\vec{p}, \tag{3.252}$$

$$I_{2k} \equiv \int |\vec{q}_p|^2 e^{-\frac{|\vec{q}_Y|^2}{2\alpha_\lambda^2}} \tilde{V}_k e^{-\frac{|\vec{q}_p|^2}{2\alpha^2}} \, d^3\vec{p}, \tag{3.253}$$

$$I_{3k} \equiv \int |\vec{q}_Y|^2 e^{-\frac{|\vec{q}_Y|^2}{2\alpha_\lambda^2}} \tilde{V}_k e^{-\frac{|\vec{q}_p|^2}{2\alpha^2}} \, d^3\vec{p}, \tag{3.254}$$

$$
\begin{aligned}
&\langle \Phi^S_{000}|Q|\Phi^S_{000}\rangle = a, && \langle \Phi^S_{000}|Q|\Phi^S_{200}\rangle = \tfrac{1}{\sqrt{2}}b,\\
&\langle \Phi^S_{000}|Q|\Phi^\lambda_{200}\rangle = -\tfrac{1}{\sqrt{2}}b, && \langle \Phi^S_{000}|Q|\Phi^\rho_{200}\rangle = 0,\\
&\langle \Phi^S_{200}|Q|\Phi^S_{000}\rangle = \tfrac{1}{\sqrt{2}}c, && \langle \Phi^S_{200}|Q|\Phi^S_{200}\rangle = \tfrac{1}{2}(a+d),\\
&\langle \Phi^S_{200}|Q|\Phi^\lambda_{200}\rangle = \tfrac{1}{2}(a-d), && \langle \Phi^S_{200}|Q|\Phi^\rho_{200}\rangle = 0,\\
&\langle \Phi^\lambda_{200}|Q|\Phi^S_{000}\rangle = -\tfrac{1}{\sqrt{2}}c, && \langle \Phi^\lambda_{200}|Q|\Phi^S_{200}\rangle = \tfrac{1}{2}(a-d),\\
&\langle \Phi^\lambda_{200}|Q|\Phi^\lambda_{200}\rangle = \tfrac{1}{2}(a+d), && \langle \Phi^\lambda_{200}|Q|\Phi^\rho_{200}\rangle = 0,\\
&\langle \Phi^\rho_{200}|Q|\Phi^S_{000}\rangle = 0, && \langle \Phi^\rho_{200}|Q|\Phi^S_{200}\rangle = 0,\\
&\langle \Phi^\rho_{200}|Q|\Phi^\lambda_{200}\rangle = 0, && \langle \Phi^\rho_{200}|Q|\Phi^\rho_{200}\rangle = g/3,\\
&\langle \Phi^{\rho\rho}_{200}|Q|\Phi^S_{000}\rangle = 0, && \langle \Phi^{\rho\rho}_{200}|Q|\Phi^S_{200}\rangle = \tfrac{1}{\sqrt{2}}a,\\
&\langle \Phi^{\rho\rho}_{200}|Q|\Phi^\lambda_{200}\rangle = \tfrac{1}{\sqrt{2}}a, && \langle \Phi^{\rho\rho}_{200}|Q|\Phi^\rho_{200}\rangle = 0,\\
&\langle \Phi^{\lambda\lambda}_{200}|Q|\Phi^S_{000}\rangle = c, && \langle \Phi^{\lambda\lambda}_{200}|Q|\Phi^S_{200}\rangle = \tfrac{1}{\sqrt{2}}d,\\
&\langle \Phi^{\lambda\lambda}_{200}|Q|\Phi^\lambda_{200}\rangle = -\tfrac{1}{\sqrt{2}}d, && \langle \Phi^{\lambda\lambda}_{200}|Q|\Phi^\rho_{200}\rangle = 0,\\
&\langle \Phi^{\rho\lambda}_{200}|Q|\Phi^S_{000}\rangle = 0, && \langle \Phi^{\rho\lambda}_{200}|Q|\Phi^S_{200}\rangle = 0,\\
&\langle \Phi^{\rho\lambda}_{200}|Q|\Phi^\lambda_{200}\rangle = 0, && \langle \Phi^{\rho\lambda}_{200}|Q|\Phi^\rho_{200}\rangle = g/3.
\end{aligned}
$$

Table 3.12: Matrix elements of the harmonic oscillator wave functions in terms of a, b, c, d, g. The Dirac bracket denotes integration over $\vec{\rho}$, $\vec{\lambda}$, $\vec{\lambda}'$ and $\vec{p}$.

$$I_{4k} \equiv \int |\vec{q}_p|^2 |\vec{q}_Y|^2 e^{-\frac{|\vec{q}_Y|^2}{2\alpha_\lambda^2}} \tilde{V}_k e^{-\frac{|\vec{q}_p|^2}{2\alpha^2}} d^3\vec{p}, \tag{3.255}$$

$$I_{5k} \equiv \int |\vec{q}_p||\vec{q}_Y| e^{-\frac{|\vec{q}_Y|^2}{2\alpha_\lambda^2}} \tilde{V}_k e^{-\frac{|\vec{q}_p|^2}{2\alpha^2}} d^3\vec{p}. \tag{3.256}$$

Since neither $\tilde{V}, \vec{q}_Y$ or $\vec{q}_p$ have any ϕ dependence the integral over ϕ simply yields 2π. The piece which is common to all the momentum integrals

$$e^{-\frac{|\vec{q}_Y|^2}{2\alpha_\lambda^2}} e^{-\frac{|\vec{q}_p|^2}{2\alpha^2}}$$

can be written

$$e^{-\frac{|\vec{q}_Y|^2}{2\alpha_\lambda^2}} = e^{-\frac{1}{12\alpha_\lambda^2}((\frac{m_\lambda}{m_u})^2|\vec{P}_Y|^2 - 6(\frac{m_\lambda}{m_u})\vec{p}.\vec{P}_Y + 9|\vec{p}|^2)} = A_y e^{-a_1 p^2 + \vec{b}_1.\vec{p}}$$

where

$$A_y \equiv e^{-\frac{1}{12\alpha_\lambda^2}(\frac{m_\lambda}{m_u})^2|\vec{P}_Y|^2}, \; a_1 \equiv \frac{3}{4\alpha_\lambda^2}, \; \vec{b}_1 \equiv \frac{1}{2\alpha_\lambda^2}(\frac{m_\lambda}{m_u})\vec{P}_Y.$$

Similarly

$$e^{-\frac{|\vec{q}_p|^2}{2\alpha^2}} = e^{-\frac{3}{4\alpha^2}(|\vec{P}_Y|^2 + |\vec{p}|^2 - 2\vec{p}.\vec{P}_Y)} = A_p e^{-a_1' p^2 + \vec{b}_1'.\vec{p}}$$

$$\text{where } A_p \equiv e^{-\frac{3}{4\alpha^2}|\vec{P}_Y|^2}, \; a_1' \equiv \frac{3}{4\alpha^2}, \; \vec{b}_1' \equiv \frac{3}{2\alpha^2}\vec{P}_Y.$$

Finally we get

$$h \equiv a_1 + a_1', \tag{3.257}$$

$$\vec{s}_2 \equiv \vec{b}_1 + \vec{b}_1' = (\frac{m_\lambda}{m_u} + \frac{3\alpha_\lambda^2}{\alpha^2})\vec{P}_Y. \tag{3.258}$$

3.2.1 Including the Potentials

With PS coupling we had, from the previous chapter,

$$\begin{aligned} \tilde{V}^{PS} &= \tilde{V}_1^{PS} + \tilde{V}_2^{PS} \\ &= ig_{Kus} V_+ \vec{\sigma} \cdot \vec{\epsilon} \left\{ \frac{-e_s(m_s - k_0 - E_3)}{(k+p)^2 - m_s^2 + i\epsilon} + \frac{e_u(m_K + m_u - E_3)}{(p - p_K)^2 - m_u^2 + i\epsilon} \right\}. \end{aligned}$$

With PV coupling

$$\begin{aligned}\tilde{V}^{PV} &= \tilde{V}_1^{PV} + \tilde{V}_2^{PV} + \tilde{V}_4^{PV} \\ &= i\tilde{g}_{Kus}V_+\vec{\sigma}\cdot\vec{\epsilon}m_K\left\{\frac{e_s(m_s-k_0-E_3)}{(k+p)^2-m_s^2+i\epsilon}+\frac{e_u(m_K+m_u-E_3)}{(p-p_K)^2-m_u^2+i\epsilon}+\frac{e_K}{m_K}\right\}.\end{aligned}$$

Diagram 1: radiating s quark

The denominator of $\tilde{V}_1$ can be written

$$\begin{aligned}(k+p)^2-m_s^2+i\epsilon &= (k^0+E_3)^2-|\vec{k}+\vec{p}|^2-m_s^2+i\epsilon \\ &= -(|\vec{u}|^2-(k^0+E_3)^2+m_s^2-i\epsilon),\end{aligned}$$

where we have employed the substitution

$$\vec{u}\equiv\vec{p}+\vec{k}$$

Now we get after integration of equations (3.252)–(3.256) with respect to the θ and ϕ coordinates of vector $\vec{u}$. (See Appendix B for details)

$$I_{J1} \equiv ig_{cc}C_2\lim_{\epsilon\to 0}\int_0^\infty\frac{G_J(u)}{u^2-\lambda_1^2-i\epsilon}du \;\; J=1,\cdots,4 \tag{3.259}$$

$$\text{with } \lambda_1^2 \equiv (k^0+E_3)^2-m_s^2$$

$$\text{and } g_{cc} \to g_{Kus} \text{ in PS coupling}$$

$$g_{cc} \to m_K\tilde{g}_{Kus} \text{ in PV coupling.}$$

$G_J(u)$ has the form[1]

$$G_J(u) \equiv 2\pi A_pA_y e^{-hu^2}\sum_{i=1}^{5}\eta_{iJ}(u^ie^{s_1u}+(-u)^ie^{-s_1u}), \tag{3.260}$$

$$\text{where } C_2 \equiv e_s(m_s-k^0-E_3)e^{(c_1-h)k^2}, \tag{3.261}$$

$$\vec{s_1} \equiv (2h-c_1)\vec{k}, \tag{3.262}$$

$$c_1 \equiv \left(\frac{m_\lambda}{m_u}+\frac{3\alpha_\lambda^2}{\alpha^2}\right)\frac{1}{2\alpha_\lambda^2}. \tag{3.263}$$

[1]For $J=5$ the θ and u integrals must be done numerically.

The η_{iJ} are constants (but depend on J) defined through equations (B.306–B.309) and u^i denotes u raised to the power of i.

The integrand has a pole at $u = \lambda_1$. To deal with this we separate the integrand into two parts: one containing the pole, and a part which is bounded over the whole integration range. That is

$$\lim_{\epsilon\to 0}\int_0^\infty \frac{G_J(u)}{u^2-\lambda_1^2-i\epsilon}\,du = \int_0^\infty \frac{G_J(u)-G_J(\lambda_1)}{u^2-\lambda_1^2}\,du + \lim_{\epsilon\to 0}\int_0^\infty \frac{G_J(\lambda_1)}{u^2-\lambda_1^2-i\epsilon}\,du. \quad (3.264)$$

The first integral in (3.264) can be evaluated numerically (Appendix A), the integrand is plotted in §3.2.2.

The second integral in (3.264) can be evaluated by contour integration techniques (see Appendix B). The result is

$$G_J(\lambda_1)\int_0^\infty \frac{1}{u^2-\lambda_1^2-i\epsilon}\,du = \frac{G_J(\lambda_1)\pi i}{2\lambda_1}. \quad (3.265)$$

Therefore

$$I_{J1} = ig_{cc}C_2\left[\int_0^\infty \frac{G_J(u)-G_J(\lambda_1)}{u^2-\lambda_1^2}\,du + \frac{G_J(\lambda_1)\pi i}{2\lambda_1}\right]. \quad (3.266)$$

Diagram 2: radiating u quark

The denominator of $\tilde{V}_2$ can be written

$$(p-p_K)^2 - m_u^2 + i\epsilon = -(|\vec{p}|^2 + (m_u^2 - (E_3 - m_K)^2) - i\epsilon).$$

Now defining

$$\lambda_2^2 \equiv m_u^2 - (E_3 - m_K)^2 \quad (3.267)$$

$$\text{and } C_3 \equiv -e_u(m_K + m_u - E_3). \quad (3.268)$$

$\lambda_2^2 > 0$ so the integrand has no singularity and therefore can be integrated directly. We get after performing the angular integrals

$$I_{J2} = ig_{cc}C_3 \lim_{\epsilon\to 0}\int_0^\infty \frac{F_J(p)}{p^2+\lambda_2^2-i\epsilon}dp,\ J = 1,\cdots,4 \quad (3.269)$$

The integration over θ in I_{52} must be carried out numerically. $F_J(p)$ has the form

$$F_J(p) \equiv 2\pi A_p A_y e^{-hp^2} \sum_{i=1}^{5} \eta'_{iJ}(p^i e^{s_2 p} + (-p)^i e^{-s_2 p}), \tag{3.270}$$

$$\tag{3.271}$$

and the η'_{iJ} are constants and $s_2 \equiv |\vec{s_2}| > 0$.

The integrand of the integral in equation (3.269) is plotted in §3.2.2.

Diagram 4: the contact term

$\tilde{V}_4$ contains no propagator, so we get simply

$$I_{J4} = \frac{ig_{cc}e_K}{m_K} \int_0^\infty F_J(p)\, dp. \tag{3.272}$$

We make the replacement, since E_3 is the energy of the third quark in the Y baryon,

$$E_3 \to m_s.$$

3.2.2 Plots of the Integrands

We perform Gaussian numerical integration on the following integrands. As can be seen from the plots, they are well behaved and converge rapidly to zero above momenta of $1\,GeV$. The plots labelled (a) show the integrands of I_{1k} for $k = 1, 2, 4$. (b),(c),(d) denote the integrands of I_{2k}, I_{3k} and I_{4k}. The following symbols are used :

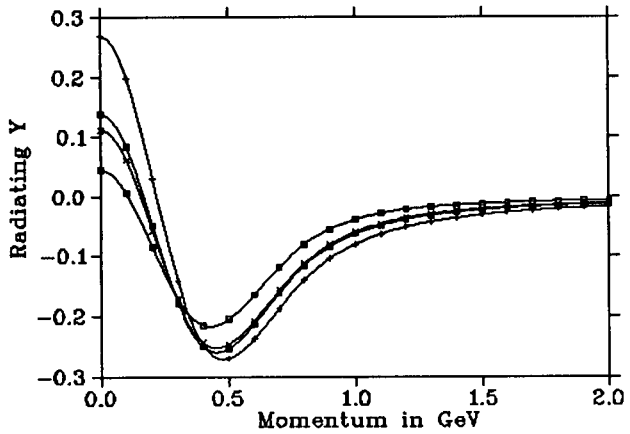
$$\begin{aligned} \square &= uds \text{ basis, } Y = \Lambda \\ \times &= SU(6) \text{ basis, } Y = \Lambda \\ \blacksquare &= uds \text{ basis, } Y = \Sigma^0 \\ + &= SU(6) \text{ basis, } Y = \Sigma^0 \end{aligned}$$

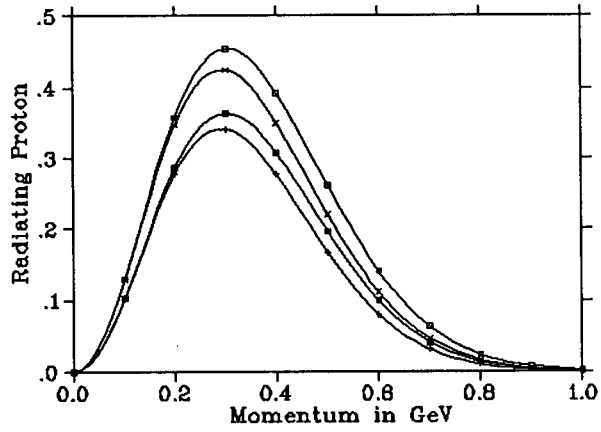
The I_{5k} must be integrated numerically over θ and momentum. Surface plots are given for the Λ uds basis case only. One axis is labelled by $\cos\theta + 1$ and so goes between

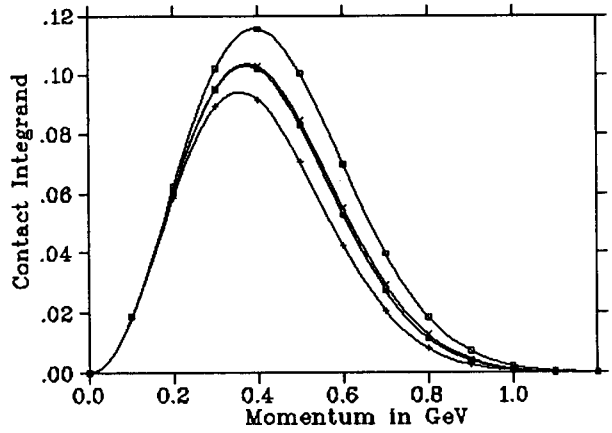
0 and 2. θ is the angle between the vector $\vec{p}$ ($\vec{u}$ for the case of the radiating Y diagram) and the z axis. Surface plots for the other cases have the same behaviour and so were omitted.

All integrands are multiplied by the constant

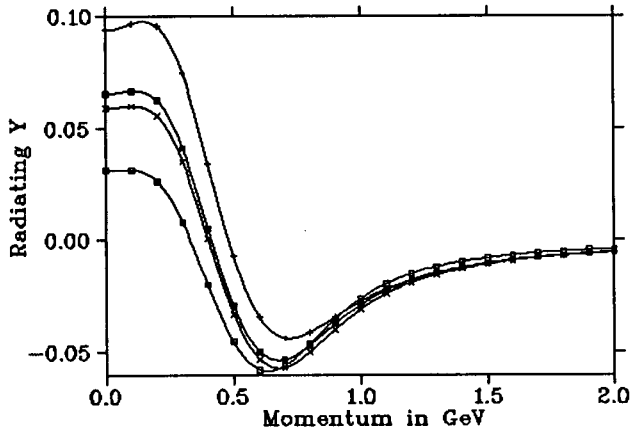
$$\frac{1}{2\pi A_p A_y}.$$

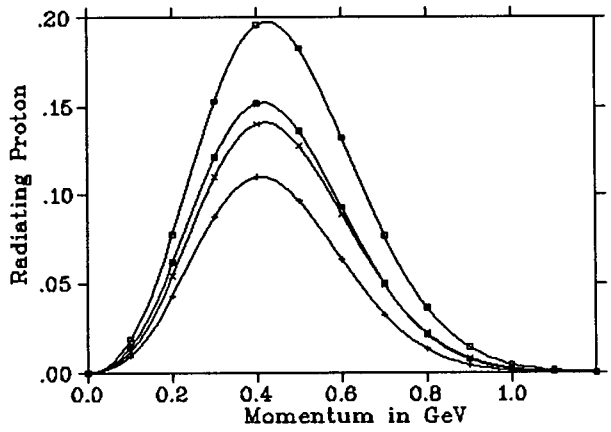

Radiating Y
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
0.0
0.5
1.0
1.5
2.0
Momentum in GeV

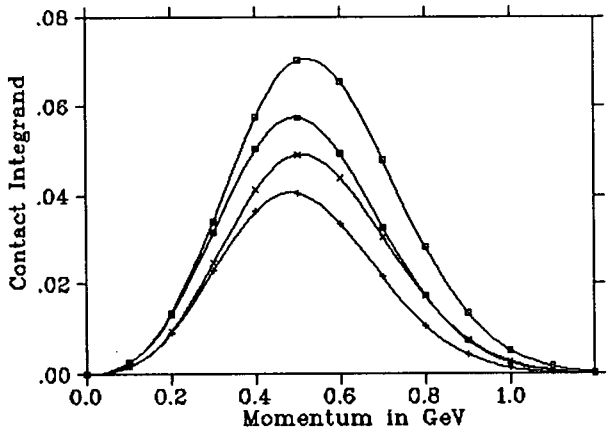

Radiating Proton
.5
.4
.3
.2
.1
.0
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV


Contact Integrand
.12
.10
.08
.06
.04
.02
.00
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV

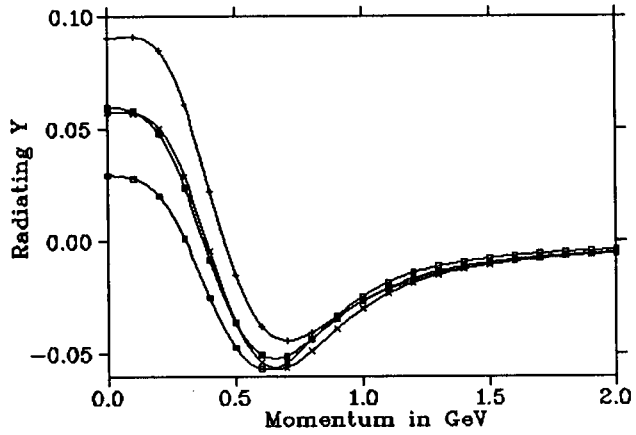
(a)

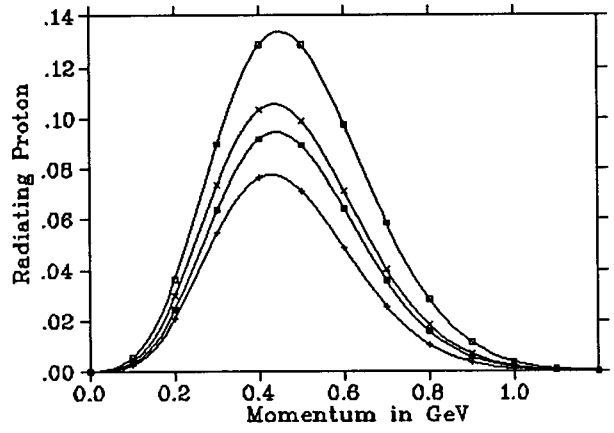

0.10
0.05
0.00
−0.05
Radiating Y
0.0
0.5
1.0
1.5
2.0
Momentum in GeV

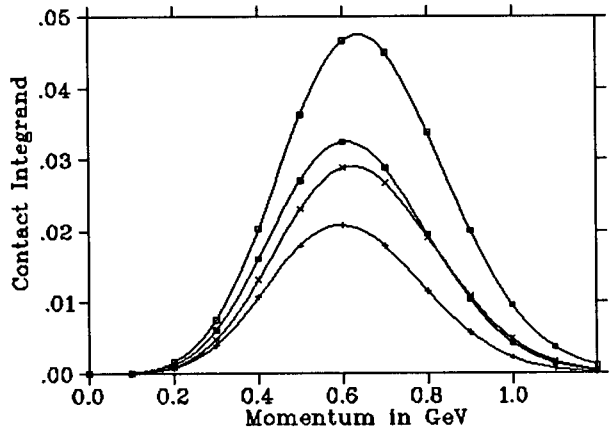

.20
.15
.10
.05
.00
Radiating Proton
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV


.08
.06
.04
.02
.00
Contact Integrand
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV

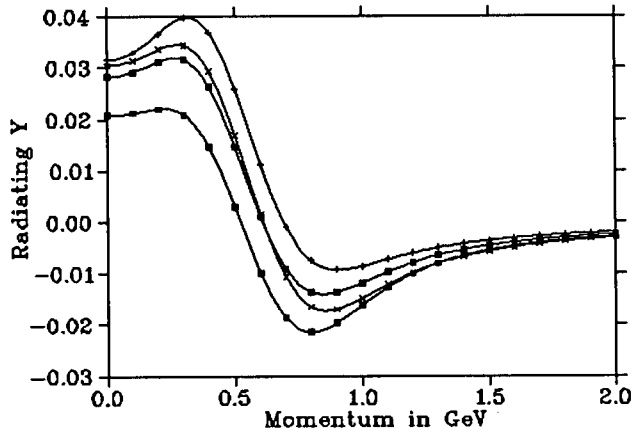
(b)

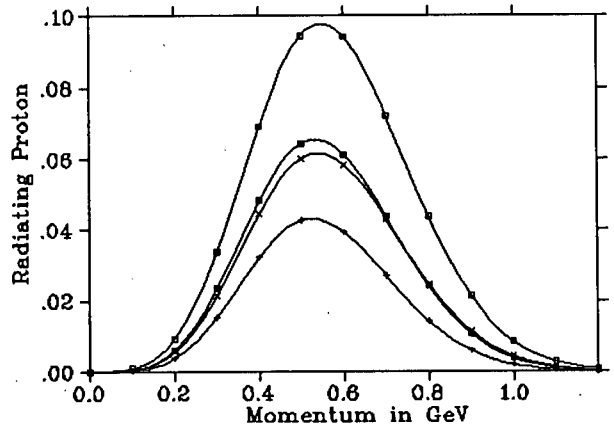

0.10
0.05
0.00
−0.05
Radiating Y
0.0
0.5
1.0
1.5
2.0
Momentum in GeV

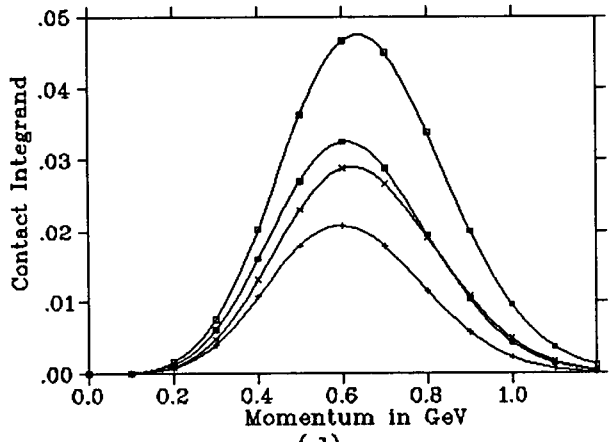

.14
.12
.10
.08
.06
.04
.02
.00
Radiating Proton
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV


.05
.04
.03
.02
.01
.00
Contact Integrand
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV

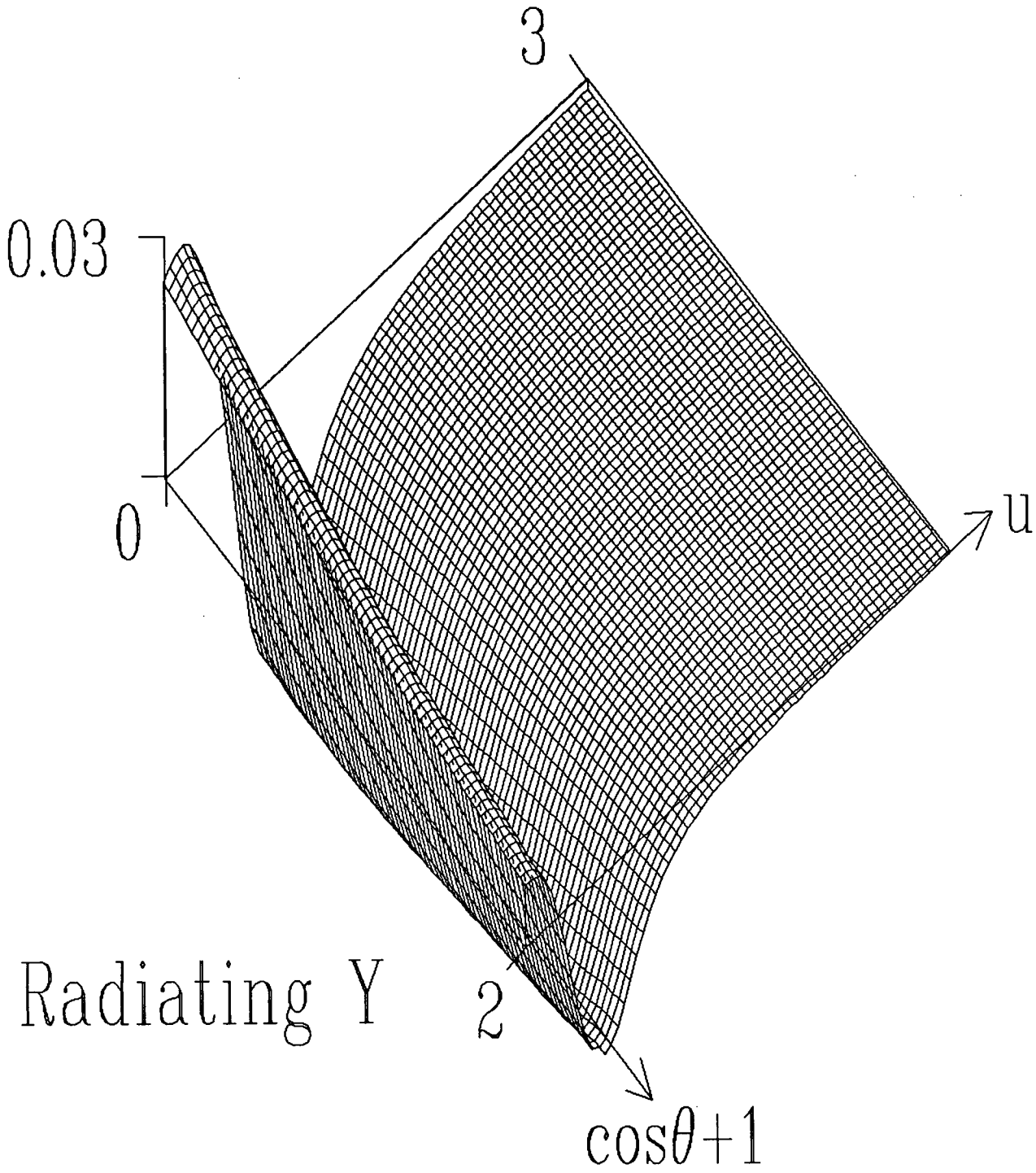
(c)

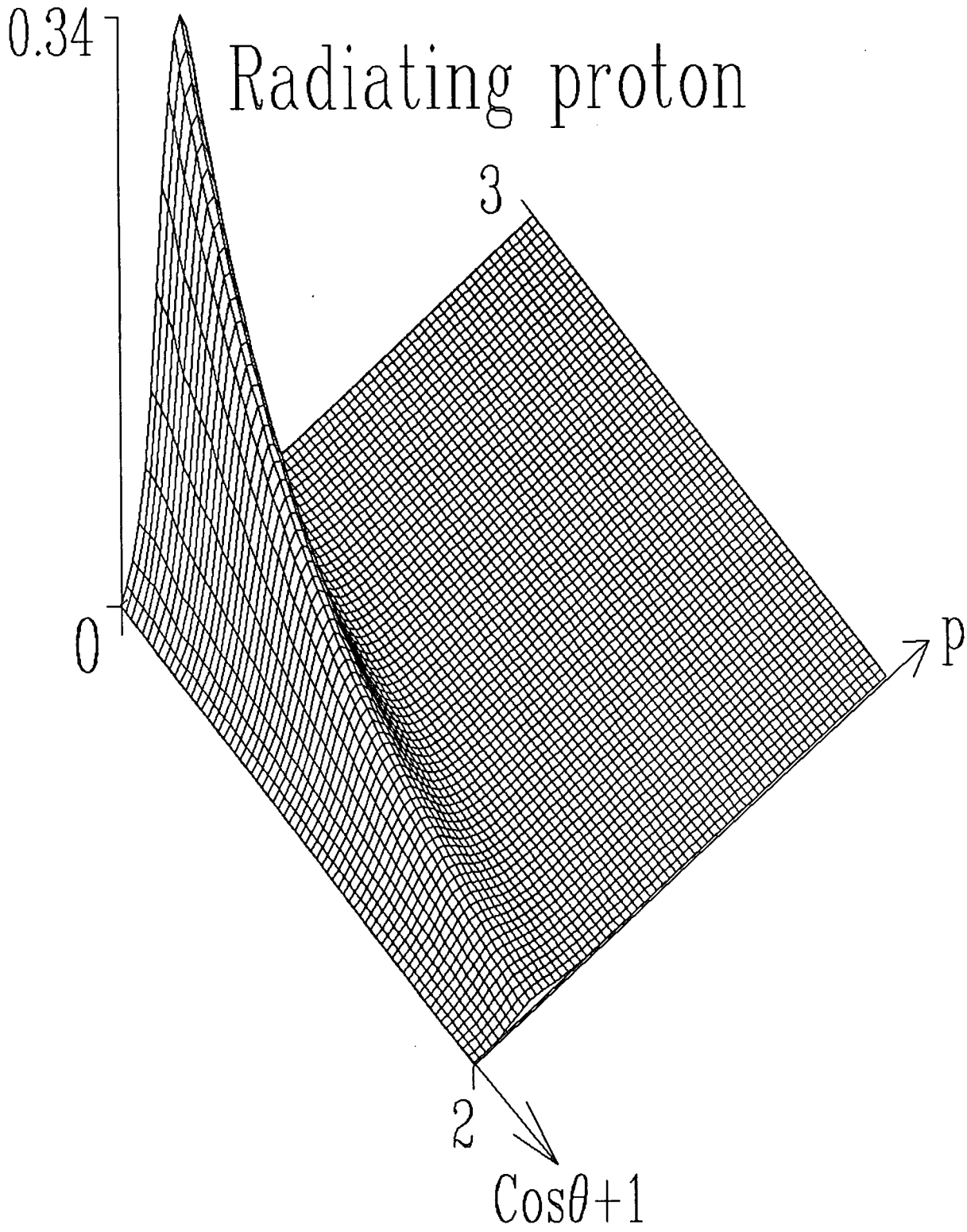

0.04
0.03
0.02
0.01
0.00
−0.01
−0.02
−0.03
Radiating Y
0.0
0.5
1.0
1.5
2.0
Momentum in GeV

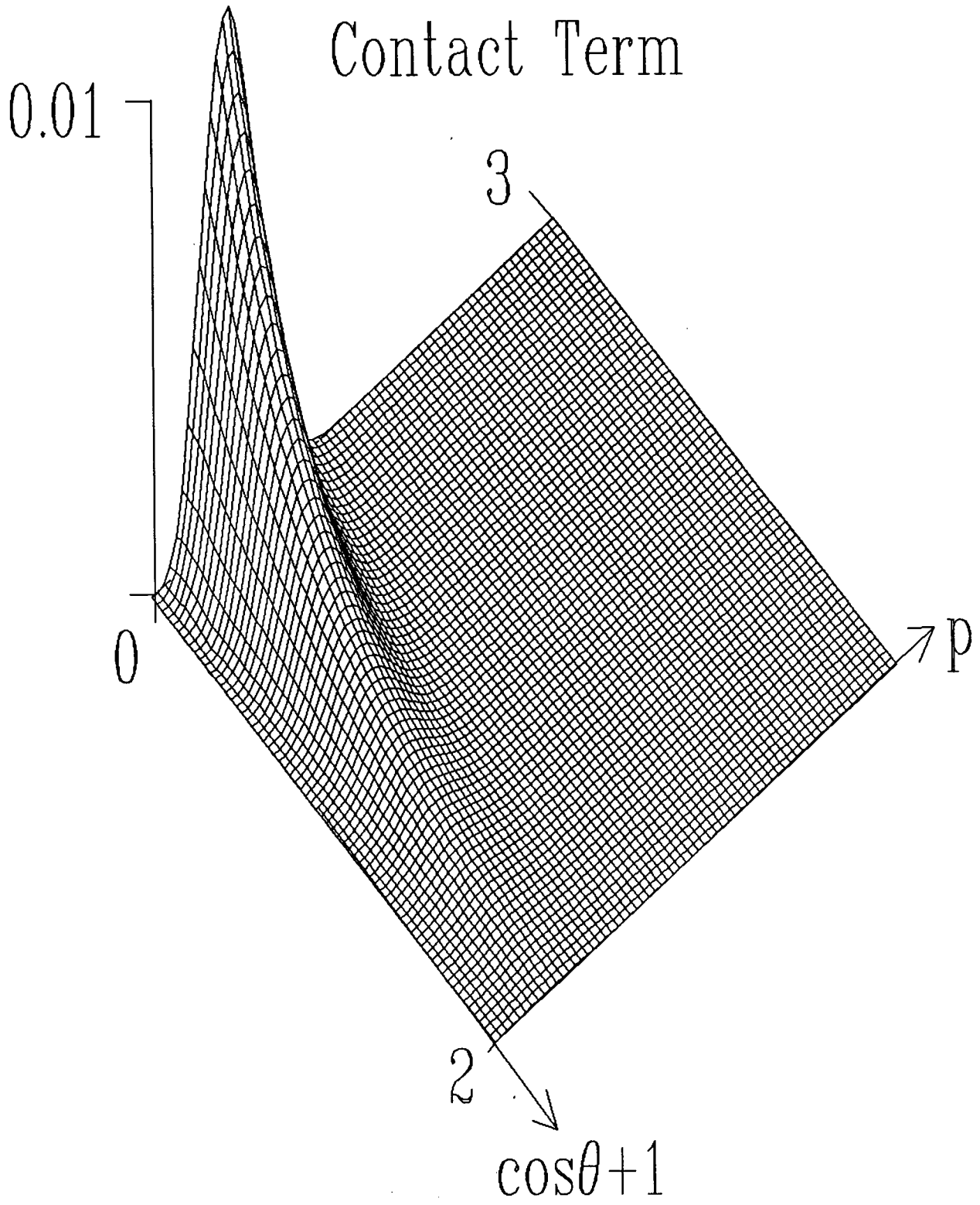

.10
.08
.06
.04
.02
.00
Radiating Proton
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV


.05
.04
.03
.02
.01
.00
Contact Integrand
0.0
0.2
0.4
0.6
0.8
1.0
Momentum in GeV

(d)


3
0.03
0
u
Radiating Y
2
cosθ+1


0.34
Radiating proton
3
0
p
2
Cosθ+1


Contact Term
0.01
0
3
2
p
cosθ+1

3.3 Spin Space

We wish to calculate the spin piece

$$\langle \chi_f^J | \vec{\sigma} \cdot \vec{\epsilon} | \chi_i^{J'} \rangle,$$

where χ_f^J is the spin wave function in the final state in which quarks 1 and 2 are combined to spin J and all three quarks are combined to spin S_f, with z component m_f; $\chi_i^{J'}$ is the spin wave function in the initial state in which quarks 1 and 2 are combined to spin J' and all three quarks are combined to spin S_i, with z component m_i.

Now

$$\begin{aligned} |\chi_i^{J'}\rangle &= \sum_{m'_\alpha, M'} \langle \tfrac{1}{2}, m'_1; \tfrac{1}{2}, m'_2 | J', M' \rangle \langle J', M'; \tfrac{1}{2}, m'_3 | S_i, m_i \rangle | \tfrac{1}{2}, m'_1 \rangle | \tfrac{1}{2}, m'_2 \rangle | \tfrac{1}{2}, m'_3 \rangle \\ \text{and } \langle \chi_f^J | &= \sum_{m_\alpha, M} \langle J, M | \tfrac{1}{2}, m_1; \tfrac{1}{2}, m_2 \rangle \langle S_f, m_f | J, M; \tfrac{1}{2}, m_3 \rangle \langle \tfrac{1}{2}, m_1 | \langle \tfrac{1}{2}, m_2 | \langle \tfrac{1}{2}, m_3 |. \end{aligned}$$

Where the m_α under the summation sign denotes m_1, m_2, m_3, similarly m'_α denotes m'_1, m'_2, m'_3. We have, since the interaction acts only on the third quark,

$$\begin{aligned} \langle \chi_f^J | \vec{\sigma} \cdot \vec{\epsilon} | \chi_i^{J'} \rangle &= \sum_{m_\alpha, M, m'_\alpha, M'} \langle J, M | \tfrac{1}{2}, m_1; \tfrac{1}{2}, m_2 \rangle \langle S_f, m_f | J, M; \tfrac{1}{2}, m_3 \rangle \\ &\times \langle \tfrac{1}{2}, m_1 | \tfrac{1}{2}, m'_1 \rangle \langle \tfrac{1}{2}, m_2 | \tfrac{1}{2}, m'_2 \rangle \langle \tfrac{1}{2}, m_3 | \vec{\sigma} \cdot \vec{\epsilon} | \tfrac{1}{2}, m'_3 \rangle \\ &\times \langle \tfrac{1}{2}, m'_1; \tfrac{1}{2}, m'_2 | J', M' \rangle \langle J', M'; \tfrac{1}{2}, m'_3 | S_i, m_i \rangle \\ &= \sum_{m_\alpha, M, m'_\alpha, M'} \langle J, M | \tfrac{1}{2}, m_1; \tfrac{1}{2}, m_2 \rangle \langle \tfrac{1}{2}, m'_1; \tfrac{1}{2}, m'_2 | J', M' \rangle \\ &\times \delta_{m_1, m'_1} \delta_{m_2, m'_2} \langle S_f, m_f | J, M; \tfrac{1}{2}, m_3 \rangle \\ &\times \langle J', M'; \tfrac{1}{2}, m'_3 | S_i, m_i \rangle \langle \tfrac{1}{2}, m_3 | \vec{\sigma} \cdot \vec{\epsilon} | \tfrac{1}{2}, m'_3 \rangle \\ &= \sum_{m_\alpha, M, m'_3, M'} \langle J, M | \tfrac{1}{2}, m_1; \tfrac{1}{2}, m_2 \rangle \langle \tfrac{1}{2}, m_1; \tfrac{1}{2}, m_2 | J', M' \rangle \\ &\times \langle S_f, m_f | J, M; \tfrac{1}{2}, m_3 \rangle \langle J', M'; \tfrac{1}{2}, m'_3 | S_i, m_i \rangle \langle \tfrac{1}{2}, m_3 | \vec{\sigma} \cdot \vec{\epsilon} | \tfrac{1}{2}, m'_3 \rangle. \end{aligned}$$

From the unitarity properties of the Clebsch-Gordan coefficients (equation (3.5.4) of ref.[13])

$$\sum_{m_1,m_2} \langle J,M|j_1,m_1;j_2,m_2\rangle\langle j_1,m_1;j_2,m_2|J',M'\rangle = \delta_{M,M'}\delta_{J,J'}, \tag{3.273}$$

we get

$$\begin{aligned}\langle\chi_f^J|\vec{\sigma}\cdot\vec{\epsilon}|\chi_i^{J'}\rangle &= \sum_{m_3,M,m_3',M'} \delta_{M,M'}\delta_{J,J'}\langle S_f,m_f|J,M;\tfrac{1}{2},m_3\rangle\langle J',M';\tfrac{1}{2},m_3'|S_i,m_i\rangle \\ &\times \langle\tfrac{1}{2},m_3|\vec{\sigma}\cdot\vec{\epsilon}|\tfrac{1}{2},m_3'\rangle \\ &= \sum_{m_3,M,m_3'} \langle S_f,m_f|J,M;\tfrac{1}{2},m_3\rangle\langle J',M;\tfrac{1}{2},m_3'|S_i,m_i\rangle\langle\tfrac{1}{2},m_3|\vec{\sigma}\cdot\vec{\epsilon}|\tfrac{1}{2},m_3'\rangle\delta_{J,J'}.\end{aligned}$$

Now

$$\vec{\sigma}\cdot\vec{\epsilon} = \sum_R \epsilon_{-R}\sigma_R(-1)^R \tag{3.274}$$

in spherical tensor notation.

The Wigner-Eckart theorem (equation (5.4.1) of ref.[13]),

$$\langle j',m'|T(k,q)|j,m\rangle = (-1)^{j-m}\frac{\langle j',m';j,-m|k,q\rangle}{\sqrt{2k+1}}\langle j'\|\vec{T}(k)\|j\rangle, \tag{3.275}$$

(T is a tensor of rank k, component q), in our case becomes,

$$\langle\tfrac{1}{2},m_3|\sigma_R|\tfrac{1}{2},m_3'\rangle = (-1)^{\frac{1}{2}-m_3'}\frac{\langle\frac{1}{2},m_3;\frac{1}{2},-m_3'|1,R\rangle}{\sqrt{3}}\langle\tfrac{1}{2}\|\vec{\sigma}\|\tfrac{1}{2}\rangle.$$

Equation (3.275) defines the *reduced* or *double-bar* matrix element. The reduced matrix element is easily calculated from (3.275) with $j=\frac{1}{2}$, $m=\frac{1}{2}$, $j'=\frac{1}{2}$, $m'=\frac{1}{2}$, (and S denotes $|\vec{S}|$),

$$\begin{aligned}\langle\tfrac{1}{2},\tfrac{1}{2}|S|\tfrac{1}{2},\tfrac{1}{2}\rangle &= (-1)^{\frac{1}{2}-\frac{1}{2}}\frac{\langle\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}|1,0\rangle}{\sqrt{2(1)+1}}\langle\tfrac{1}{2}\|\vec{S}\|\tfrac{1}{2}\rangle, \\ \Rightarrow \langle\tfrac{1}{2}\|\vec{\sigma}\|\tfrac{1}{2}\rangle &= \sqrt{6} \\ \text{since (in natural units) } \vec{S} &\equiv \frac{1}{2}\vec{\sigma}.\end{aligned}$$

So,

$$\begin{aligned}\langle\chi_f^J|\vec{\sigma}\cdot\vec{\epsilon}|\chi_i^{J'}\rangle &= \sum_{m_3,M,m_3',R}\langle S_f,m_f|J,M;\tfrac{1}{2},m_3\rangle\langle J,M;\tfrac{1}{2},m_3'|S_i,m_i\rangle \\ &\times (-1)^{\frac{1}{2}-m_3'}\frac{\langle\frac{1}{2},m_3;\frac{1}{2},-m_3'|1,R\rangle}{\sqrt{3}}\sqrt{6}(-1)^R\epsilon_{-R}\delta_{J,J'} \\ &= \sum_{m_3,M,m_3',R}\frac{\sqrt{2S_f+1}}{(-1)^{J-\frac{1}{2}+m_f}}\frac{\sqrt{2S_i+1}}{(-1)^{J-\frac{1}{2}+m_i}}\frac{\sqrt{3}}{(-1)^{\frac{1}{2}-\frac{1}{2}+R}}\frac{(-1)^{\frac{1}{2}-m_3'}}{\sqrt{3}}\epsilon_{-R}(-1)^R \\ &\times \sqrt{6}\,\delta_{J,J'}\begin{pmatrix} J & \frac{1}{2} & S_f \\ M & m_3 & -m_f\end{pmatrix}\begin{pmatrix} J & \frac{1}{2} & S_i \\ M & m_3' & -m_i\end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m_3 & -m_3' & -R\end{pmatrix}.\end{aligned}$$

Where we have converted our Clebsch-Gordan coefficients into '3-j symbols'. The definition of the '3-j symbols' is given by (equation (3.7.3) ref.[13])

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} \equiv \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}}\langle j_1,m_1;j_2,m_2|j_3,-m_3\rangle. \tag{3.276}$$

Using the symmetry properties of these symbols [13] we obtain

$$\begin{pmatrix} J & \frac{1}{2} & S_f \\ M & m_3 & -m_f\end{pmatrix} = \begin{pmatrix} \frac{1}{2} & S_f & J \\ m_3 & -m_f & M\end{pmatrix},$$

$$\begin{pmatrix} J & \frac{1}{2} & S_i \\ M & m_3' & -m_i\end{pmatrix} = \begin{pmatrix} S_i & \frac{1}{2} & J \\ m_i & -m_3' & -M\end{pmatrix},$$

$$\text{and}\quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ m_3 & -m_3' & -R\end{pmatrix} = (-1)^2\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ -m_3 & m_3' & R\end{pmatrix}.$$

Now we have,

$$\begin{aligned}\langle\chi_f^J|\vec{\sigma}\cdot\vec{\epsilon}|\chi_i^{J'}\rangle &= \sum_{m_3,M,m_3',R}\sqrt{(2S_f+1)(2S_i+1)}(-1)^{-2J+\frac{3}{2}-m_f-m_i-m_3'}\epsilon_{-R}\sqrt{6}\,\delta_{J,J'} \\ &\times \begin{pmatrix} S_i & \frac{1}{2} & J \\ m_i & -m_3' & -M\end{pmatrix}\begin{pmatrix} \frac{1}{2} & S_f & J \\ m_3 & -m_f & M\end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ -m_3 & m_3' & R\end{pmatrix}.\end{aligned} \tag{3.277}$$

This can be expressed in terms of the '6-j symbol' defined by,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \equiv \frac{(-1)^{j_1+j_2+l_1+l_2}}{\sqrt{(2j_3+1)(2l_3+1)}} \sum_{m_1,m_2} \langle j_1, m_1; j_2, m_2 | j_3, m_1+m_2\rangle \times \langle j_3, m_1+m_2; l_1, M-m_1-m_2 | l_2, M\rangle \times \langle j_2, m_2; l_1, M-m_1-m_2 | l_3, M-m_1\rangle \langle j_1, m_1; l_3, M-m_1 | l_2, M\rangle. \tag{3.278}$$

Hence, as a consequence of the reality of our Clebsch-Gordan coefficients, the 6-j symbol is real. Comparing (3.277) with equation (6.2.8) of ref.[13]

$$\sum_{\mu_1,\mu_2,\mu_3} (-1)^{l_1+l_2+l_3+\mu_1+\mu_2+\mu_3} \begin{pmatrix} j_1 & l_2 & l_3 \\ \tilde{m}_1 & \mu_2 & -\mu_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -\mu_1 & \tilde{m}_2 & \mu_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ \mu_1 & -\mu_2 & \tilde{m}_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ \tilde{m}_1 & \tilde{m}_2 & \tilde{m}_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}, \tag{3.279}$$

and noting that

$$\begin{aligned} m_f &= M + m_3, \\ m_i &= M + m_3', \\ \Rightarrow (-1)^{-2J+\frac{3}{2}-m_f-m_i-m_3'} &= (-1)^{1+J-m_3-m_3'-M} \times \frac{(-1)^{-2J+\frac{3}{2}-m_i}}{(-1)^{J+1}}, \end{aligned}$$

we get (since J is an integer),

$$\langle \chi_f^J | \vec{\sigma}\cdot\vec{\epsilon} | \chi_i^{J'}\rangle = (-1)^{-J+\frac{1}{2}-m_i}\sqrt{6}\,\delta_{J,J'}\sqrt{(2S_f+1)(2S_i+1)} \begin{Bmatrix} S_i & S_f & 1 \\ \frac{1}{2} & \frac{1}{2} & J \end{Bmatrix} \times \sum_R \epsilon_{-R} \begin{pmatrix} S_i & S_f & 1 \\ m_i & -m_f & R \end{pmatrix}$$

For the square of the spin amplitude we have,

$$\langle \chi_f^{J'} | \vec{\sigma}\cdot\vec{\epsilon} | \chi_i^{J'}\rangle^* \langle \chi_f^J | \vec{\sigma}\cdot\vec{\epsilon} | \chi_i^J\rangle = (-1)^{-(J+J')+1-2m_i} 6(2S_f+1)(2S_i+1) \begin{Bmatrix} S_i & S_f & 1 \\ \frac{1}{2} & \frac{1}{2} & J' \end{Bmatrix}$$

$$\times \begin{Bmatrix} S_i & S_f & 1 \\ \frac{1}{2} & \frac{1}{2} & J \end{Bmatrix} \sum_{R'} \sum_{R} \epsilon^*_{-R'} \cdot \epsilon_{-R}$$

$$\times \begin{pmatrix} S_i & S_f & 1 \\ m_i & -m_f & R \end{pmatrix} \begin{pmatrix} S_i & S_f & 1 \\ m_i & -m_f & R' \end{pmatrix},$$

but $(-1)^{1-2m_i} = 1$ since the intrinsic spin of the proton S_i is half-integral[2]. Summing over polarization states λ and using the identity [26]

$$\sum_{\lambda} \epsilon_\mu(\lambda)\epsilon_\nu(\lambda) = -g_{\mu\nu} \tag{3.280}$$

which in the transverse gauge becomes,

$$\sum_{\lambda} \epsilon^*_i(\lambda)\epsilon_j(\lambda) = \delta_{ij} \tag{3.281}$$

we get

$$\begin{aligned} \langle \chi_f^{J'} | \vec{\sigma} \cdot \vec{\epsilon} | \chi_i^{J'} \rangle^* \langle \chi_f^{J} | \vec{\sigma} \cdot \vec{\epsilon} | \chi_i^{J} \rangle &= 6(2S_f+1)(2S_i+1) \begin{Bmatrix} S_i & S_f & 1 \\ \frac{1}{2} & \frac{1}{2} & J' \end{Bmatrix}^* \\ &\times \begin{Bmatrix} S_i & S_f & 1 \\ \frac{1}{2} & \frac{1}{2} & J \end{Bmatrix} \sum_R \begin{pmatrix} S_i & S_f & 1 \\ m_i & -m_f & R \end{pmatrix}^2 \end{aligned} \tag{3.282}$$

3.3.1 Evaluation of the 6-j symbols

Using relation (6.3.4) of ref.[13],

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ \frac{1}{2} & j_3 - \frac{1}{2} & j_2 - \frac{1}{2} \end{Bmatrix} = (-1)^{j_1+j_2+j_3} \sqrt{\frac{(j_1+j_2+j_3+1)(j_2+j_3-j_1)}{2j_2(2j_2+1)2j_3(2j_3+1)}} \tag{3.283}$$

$$\Rightarrow \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{Bmatrix}^2 = 1/4.$$

[2]We consider only S state mixings in our compositions so, since the total angular momentum $J = \frac{1}{2}$ for the Y or proton, S_i and S_f are both one-half.

In addition relation (6.3.3) of ref.[13] gives us,

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ \frac{1}{2} & j_3-\frac{1}{2} & j_2+\frac{1}{2} \end{array} \right\} = (-1)^{j_1+j_2+j_3}\sqrt{\frac{(j_1+j_2-j_3+1)(j_1+j_3-j_2)}{(2j_2+1)(2j_2+2)2j_3(2j_3+1)}} \quad (3.284)$$

$$\Rightarrow \left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{array} \right\}^2 = 1/36.$$

3.3.2 Evaluation of the 3-j symbols

The 3j-symbol can be evaluated from the relation ((3.7.10) ref.[13])

$$\begin{aligned} \left(\begin{array}{ccc} j_1 & j_2 & (j_1+j_2) \\ m_1 & m_2 & -(m_1+m_2) \end{array} \right) &= (-1)^{j_1-j_2+m_1+m_2} \\ &\times \sqrt{\frac{(2j_1)!(2j_2)!(j_1+j_2+m_1+m_2)!}{(2j_1+2j_2+1)!(j_1+m_1)!}} \\ &\times \sqrt{\frac{(j_1+j_2-(m_1+m_2))!}{(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!}} \quad (3.285) \\ \Rightarrow \left(\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ m_i & -m_f & R \end{array} \right)^2 &= [(-1)^{m_i-m_f}]^2\frac{(1+m_i-m_f)!(1+R)!}{6}. \end{aligned}$$

3.3.3 Spin summation and squaring the amplitude

Assigning equal a priori probabilities to each of the initial spin states and summing over the possible final spin states

$$|\bar{\mathcal{M}}|^2 = \frac{1}{2S_i+1}\sum_{m_f,m_i}|\mathcal{M}|^2 \quad (3.286)$$

For example with $J=0, J'=0$, $S_i=\frac{1}{2}$ and $S_f=\frac{1}{2}$

$$\begin{aligned} \frac{1}{2}\sum_{m_f,m_i}|\langle\chi_f^\rho|\vec{\sigma}\cdot\vec{\epsilon}|\chi_i^\rho\rangle|^2 &= 12\left\{ \begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{array} \right\}^2 \sum_{m_f=\pm\frac{1}{2},m_i=\pm\frac{1}{2}R=\pm1} \left(\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 1 \\ m_i & -m_f & R \end{array} \right)^2 \\ &= 2. \quad (3.287) \end{aligned}$$

With $J=1, J'=1$ $S_i=\frac{1}{2}$ and $S_f=\frac{1}{2}$

$$\frac{1}{2}\sum_{m_f,m_i}|\langle\chi_f^\lambda|\vec{\sigma}\cdot\vec{\epsilon}|\chi_i^\lambda\rangle|^2 = 12\left\{\begin{array}{ccc}\frac{1}{2}&\frac{1}{2}&1\\ \frac{1}{2}&\frac{1}{2}&1\end{array}\right\}^2\sum_{m_f=\pm\frac{1}{2},m_i=\pm\frac{1}{2}R=\pm1}\left(\begin{array}{ccc}\frac{1}{2}&\frac{1}{2}&1\\ m_i&-m_f&R\end{array}\right)^2$$
$$= \frac{2}{9}. \tag{3.288}$$

For the cross term:

$$\frac{1}{2}\sum_{m_f,m_i}\langle\chi_f^\rho|\vec{\sigma}\cdot\vec{\epsilon}|\chi_i^\rho\rangle^*\langle\chi_f^\lambda|\vec{\sigma}\cdot\vec{\epsilon}\chi_i^\lambda\rangle = 12\left\{\begin{array}{ccc}\frac{1}{2}&\frac{1}{2}&1\\ \frac{1}{2}&\frac{1}{2}&0\end{array}\right\}\left\{\begin{array}{ccc}\frac{1}{2}&\frac{1}{2}&1\\ \frac{1}{2}&\frac{1}{2}&1\end{array}\right\}$$
$$\times\sum_{m_f=\pm\frac{1}{2},m_i=\pm\frac{1}{2}R=\pm1}\left(\begin{array}{ccc}\frac{1}{2}&\frac{1}{2}&1\\ m_i&-m_f&R\end{array}\right)^2$$
$$= -\frac{2}{3}. \tag{3.289}$$

3.4 Full Amplitude

The full invariant amplitude is now obtained in terms of the expressions (3.241)–(3.245), with the assistance of the SMP algebraic manipulation package. The full wave functions were coded, along with the matrix relevant elements, into the file "WAVEFN.DEF". These definitions were then used (within the SMP environment) to simplify the amplitude as much as possible. A FORTRAN formula for the amplitude was generated. The relevant input data and commands, along with the output, is detailed in Appendix C.

3.5 Determination of the Strong Coupling Constant

We need to insert a value for g_{Kus} into the invariant amplitude. Unfortunately, the strong coupling constant for interactions at the quark level is not known. However it is possible to express the proton-kaon-Y vertex in terms of quark-kaon-quark vertex. We

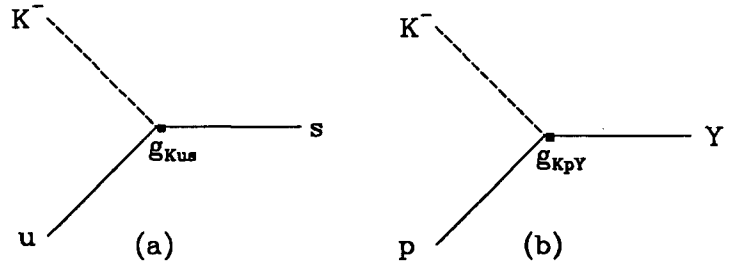


Figure 3.3: Strong vertices for relating g_{Kus} to g_{KpY}

follow the treatment in ref.[37], in which the process is assumed to be a single quark transition. We first consider the strong vertex at which a u quark absorbs a kaon and is transformed into an s quark as in Fig.3.3 (a). The amplitude for the process is

$$H_{Kus} = \sum_{i=1}^{3} \int \bar{\Phi}_Y(\vec{x}_1, \vec{x}_2, \vec{x}_3) i g_{Kus} \gamma_5 \psi_K(\vec{x}_i) V_+ \Phi_p(\vec{x}_1, \vec{x}_2, \vec{x}_3)\, d^3\vec{x}_1\, d^3\vec{x}_2\, d^3\vec{x}_3$$

in coordinate representation. The sum is over the three quarks in the baryons. Φ_Y and Φ_p include spin-flavour dependence and are given by equation (1.58). We now do a two component reduction and take the nonrelativistic limit (E_s, m_s are the energy and mass of the quark in the final state, the s quark; E_u, m_u the energy and mass of the initial state quark, the u quark). Neglecting terms of $O(p/m)^2$ and disregarding, for the purposes of this calculation, the difference in mass between the strange and up quark, we get,

$$\begin{aligned} H_{Kus} &= g_{Kus} \sum_i \int \Phi_Y^\dagger(\vec{x}_1, \vec{x}_2, \vec{x}_3)\left(-\frac{\vec{\sigma}^{(i)} \cdot \vec{p}_s}{E_s + m_s} + \frac{\vec{\sigma}^{(i)} \cdot \vec{p}_u}{E_u + m_u}\right)\psi_K(\vec{x}_i) \\ &\times V_+ \Phi_p(\vec{x}_1, \vec{x}_2, \vec{x}_3)\, d^3\vec{x}_1\, d^3\vec{x}_2\, d^3\vec{x}_3 \end{aligned}$$

$$= 3g_{Kus}\int d^3\vec{x}_1\, d^3\vec{x}_2\, d^3\vec{x}_3\, \Phi_Y^\dagger(\vec{x}_1,\vec{x}_2,\vec{x}_3)\left(\frac{\vec{\sigma}^{(3)}\cdot\vec{\nabla}\psi_K(\vec{x}_3)}{2m_u}\right)V_+^{(3)}\Phi_p(\vec{x}_1,\vec{x}_2,\vec{x}_3) \tag{3.290}$$

The complete permutational symmetry of the baryon wave functions allows us to write the interaction in terms of the third quark. The $\vec{\nabla}$ operator in (3.290) arises from taking $\vec{p}_K = \vec{p}_s - \vec{p}_u$ and so acts only on the kaon wave function.

Applying this procedure to the analogous baryon-meson-baryon vertex, Fig. 3.3 (b), we find in terms of the baryon centre of mass $\vec{X}$

$$H_{KpY} = g_{KpY}\int \Phi_Y^\dagger(\vec{X})\frac{(\vec{\sigma}\cdot\vec{\nabla}\psi_K(\vec{X}))}{2m_p}V_+\Phi_p(\vec{X})\, d^3\vec{X}.$$

Where $\langle Y|V_+|p\rangle \equiv 1$.

Neglecting the difference between the centre of mass and the position of the third quark and equating the amplitudes we find

$$\frac{g_{KpY}}{2m_p} = 3\frac{g_{Kus}}{2m_u}\langle Y\uparrow|V_+^{(3)}\sigma_z^{(3)}|P\uparrow\rangle.$$

Where P and Y are the 3-quark spin-flavour wave functions. The $\uparrow$ indicates we are taking the spin up state for the Y and proton. Therefore using table 3.11 and the spin matrix element results

$$\begin{aligned}
\langle\chi_+^\rho|\sigma_z^{(3)}|\chi_+^\rho\rangle &= 1,\\
\langle\chi_+^\rho|\sigma_z^{(3)}|\chi_+^\lambda\rangle &= 0,\\
\langle\chi_+^\lambda|\sigma_z^{(3)}|\chi_+^\rho\rangle &= 0,\\
\langle\chi_+^\lambda|\sigma_z^{(3)}|\chi_+^\lambda\rangle &= -1/3,
\end{aligned}$$

we get

$$g_{Kus}(Y=\Lambda) = \frac{g_{Kp\Lambda}}{3}\frac{m_u}{m_p}\sqrt{6} \tag{3.291}$$

$$\text{and } g_{Kus}(Y=\Sigma^0) = \frac{g_{Kp\Sigma^0}}{3}\frac{m_u}{m_p}9\sqrt{2}. \tag{3.292}$$

Using the values

$$g_{Kp\Lambda} = -13.2,$$
$$g_{Kp\Sigma^0} = 6.0,$$

obtained from ref.[31], we get

$$g_{Kus}(Y = \Lambda) = -4.83, \tag{3.293}$$
$$\text{and } g_{Kus}(Y = \Sigma^0) = 11.40. \tag{3.294}$$

Form factor corrections to these coupling constants will be small [38] due to the fact that the decay momenta ($|\vec{P}_Y|$ and $|\vec{k}|$) are equal.

3.6 Phase space

Recall that the amplitude for the process was given by

$$\mathcal{S}_{Yp} = Z(2\pi)^4\delta^4(p_K + P_p - (k + P_Y))\mathcal{M}$$

Squaring this amplitude, summing over final and averaging over initial spin states, and dividing by $\mathcal{V}T$ gives us the transition probability per unit time per unit volume

$$\frac{|\bar{\mathcal{S}}|^2}{\mathcal{V}T} = \frac{(2\pi)^4\delta^4(p_K + P_p - (k + P_Y))|\bar{\mathcal{M}}|^2|\psi_K(0)|^2 m_u^5 m_s}{4m_K k^0 \mathcal{V}^3 E_1 E_2 E_3 E_1' E_2' E_3'} \frac{(E_3 + m_s)(E_3' + m_u)}{4m_u m_s}. \tag{3.295}$$

However[3] we want the decay rate per decaying K^-p atom; therefore we divide by the number of decaying particles per unit volume. Since there are $\frac{1}{\mathcal{V}}$ K^-p atoms per unit volume we divide (3.295) by $\frac{1}{\mathcal{V}}$.

The decay rate is given by the transition probability per unit time per decaying particle, integrated over the number of final states in volume $\mathcal{V}$,

$$\Gamma = \int \frac{(2\pi)^4\delta^4(p_K + P_p - (k + P_Y))|\bar{\mathcal{M}}|^2|\psi_K(0)|^2 m_u^4 (E_3 + m_s)(E_3' + m_u)}{16 m_K k^0 \mathcal{V}^2 E_1 E_2 E_3 E_1' E_2' E_3'} \frac{\mathcal{V}\, d^3\vec{k}}{(2\pi)^3} \frac{\mathcal{V}\, d^3\vec{P}_Y}{(2\pi)^3}$$

[3]The bar in $\bar{\mathcal{S}}$ and $\bar{\mathcal{M}}$ indicates that the spin summation has been carried out.

$$= \frac{\delta(m_K + m_p - (k^0 + E_Y))|\bar{\mathcal{M}}(\vec{P}_Y = -\vec{k})|^2|\psi_K(0)|^2 m_u^4 (E_3 + m_s)(E_3' + m_u)}{64\pi^2 m_K k^0 E_1 E_2 E_3 E_1' E_2' E_3'} d^3\vec{k}. \tag{3.296}$$

Taking $E_1 = E_2 = E_1' = E_2' = m_u$, $E_3 = m_s$ and using conservation of energy for the third quark $E_3' = E_3 + m_K - k^0$, we find

$$\Gamma = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{k=0}^{\infty} \frac{\delta(m_K + m_p - (E_Y + k^0))|\bar{\mathcal{M}}(\vec{P}_Y = -\vec{k})|^2|\psi_K(0)|^2(m_s + m_K - k^0 + m_u)}{32\pi^2 m_K k^0 (m_s + m_K - k^0)} \times \sin\theta\, d\theta\, d\phi\, |\vec{k}|^2\, d|\vec{k}|.$$

Substituting

$$W \equiv E_Y + k^0 \;\Rightarrow\; \frac{dW}{dk^0} = \frac{E_Y + k^0}{E_Y} = \frac{W}{E_Y}$$

we get

$$\Gamma = \int_{W=m_Y}^{\infty} \frac{\delta(m_K + m_p - W)|\bar{\mathcal{M}}(\vec{P}_Y = -\vec{k})|^2|\psi_K(0)|^2(m_s + m_K - k^0 + m_u)k^0 E_Y\, dW}{8\pi m_K (m_s + m_K - k^0) W}$$
$$= \frac{|\psi_K(0)|^2 k^0 E_Y (m_s + m_K - k^0 + m_u)}{8\pi m_K (m_K + m_p)(m_s + m_K - k^0)}|\bar{\mathcal{M}}(\vec{P}_Y = -\vec{k})|^2. \tag{3.297}$$

Where $E_Y = \sqrt{m_Y^2 + (k^0)^2}$.

As mentioned previously the principal (§2.5.4) quantum number for the kaon wave function from which capture takes place is not precisely known. However, if we assume the kaon wave function at the origin is the same for all decay modes, it cancels in the branching ratio.

Unit Conversion

The decay rate for the process

$$K^- p \rightarrow \text{ all modes}$$

is given by Burkhardt *et al.* [39] as

$$\Gamma_{\rm all} = 2W_p|\psi_K(0)|^2$$

with[4] $W_p = (0.560 \pm 0.135) GeV\, fm^3$

Since we have used natural units throughout and taken masses and the confinement parameter to be in GeV, our invariant amplitude is in units of $1/GeV$; as it must in order to yield the dimensions of a decay rate. In order to get $\Gamma/|\psi_K(0)|^2$ to have dimensions $GeV\, fm^3$ we multiply by

$$(\frac{\hbar c}{1000})^3 = (0.19732705359 GeV\, fm)^3.$$

The branching ratio, therefore, for the process $K^- p \to Y\gamma$ is given by

$$BR = \frac{k^0 E_Y (m_s + m_K - k^0 + m_u)}{8\pi m_K (m_K + m_p)(m_s + m_K - k^0)} \frac{|\bar{\mathcal{M}}(\vec{P}_Y = -\vec{k})|^2 (0.197 GeV\, fm)^3}{2W_p}. \quad (3.298)$$

3.7 Results and Discussion

Table 3.13 lists the branching ratios to $\Lambda\gamma$ and $\Sigma^0\gamma$ in the *uds* and $SU(6)$ basis with both PS and PV couplings. The confinement parameter and the quark constituent masses are taken from previous analyses [15],[24]. We take the same values as used in the previous NRQM calculation by Darewych *et. al* [32].

From table 3.13 it can be seen that only the Σ^0 *uds* (PS) prediction agrees with the current experimental range[1] of $(1.4 \pm 0.2)10^{-3}$. The experimental value for the $\Lambda\gamma$ branching ratio is $(0.86 \pm 0.07)10^{-3}$.

The fact that the *uds* calculation is closer to experiment than $SU(6)$ suggests that taking into account the strange quark mass difference is important. Setting $m_s = m_u$ in the *uds* amplitudes does *not* yield the same result as the $SU(6)$ amplitude. The two bases are not equivalent physical descriptions.

[4] It is interesting to note that if the experimental width of the 1s level[40] $\Gamma_{1s} = 620eV$ is taken, along with a hydrogenic 1s orbital we get $W_p = 0.565 GeV\, fm^3$ in good agreement with the result of ref.[39].

BR×10^{-3}	*uds*		*SU*(6)	
	PS	PV	PS	PV
$Y = \Lambda$	7.08	12.54	49.99	52.09
$Y = \Sigma^0$	1.36	3.71	3.50	3.17

Table 3.13: Branching ratios for $K^-p \to Y\gamma$ ($Y = \Lambda, \Sigma^0$). Results obtained from PS and PV couplings within the *uds* and *SU*(6) basis are also tabulated. $\alpha = 0.41$, $m_u = 0.42$, $m_s = 0.70\,GeV$ (parameter set 1).

BR×10^{-3}	*uds*		*SU*(6)	
	PS	PV	PS	PV
$Y = \Lambda$	6.39	8.82	68.92	73.44
$Y = \Sigma^0$	1.18	1.57	11.59	14.07

Table 3.14: Calculated branching ratios for $K^-p \to Y\gamma$ ($Y = \Lambda, \Sigma^0$). $\alpha = 0.41$, $m_u = 0.42$, $m_s = 0.70\,GeV$. Here only the largest component of the proton and Y wave functions is included.

BR×10^{-3}	*uds*		*SU*(6)	
	PS	PV	PS	PV
$Y = \Lambda$	10.52	18.29	69.24	70.70
$Y = \Sigma^0$	2.87	6.30	4.77	4.41

Table 3.15: Calculated branching ratios for $K^-p \to Y\gamma$ ($Y = \Lambda, \Sigma^0$). $\alpha = 0.32$, $m_u = 0.33$, $m_s = 0.55\,GeV$ (parameter set 2).

Λ/Λ_o (%)		Σ^0/Σ^0_o (%)		
PS	PV	PS	PV	
14	63	7	67	Proton diagram off
40	126	56	115	Y diagram off
—	9	—	2	Contact diagram off

Table 3.16: Contributions from the various diagrams to the invariant amplitude in the *uds* basis. Λ/Λ_o (%) denotes the percentage of the original $\Lambda\gamma$ branching ratio and Σ^0/Σ^0_o (%) the percentage of the original $\Sigma^0\gamma$ branching ratio, when one of the diagrams is 'switched off'. $\alpha = 0.41$, $m_u = 0.42$, $m_s = 0.70\,GeV$.

BR$\times 10^{-3}$	*uds*		*SU*(6)	
	PS	PV	PS	PV
$Y = \Lambda$	1.46	2.59	10.33	10.76
$Y = \Sigma^0$	1.36	3.71	3.50	3.17

Table 3.17: Branching ratios for $K^- p \rightarrow Y\gamma$ ($Y = \Lambda, \Sigma^0$). $\alpha = 0.41$, $m_u = 0.42$, $m_s = 0.70\,GeV$. Here $g_{Kus}(\Lambda)$ is set equal to $g_{Kus}(\Sigma^0)$.

Contributions from the Diagrams

In PS coupling graphs (1) and (2) (Fig. 2.2) add constructively. However graph (2), the proton radiation diagram, dominates. As can be seen from table 3.16, turning off the contribution from this graph reduces the amplitude to 14% for the lambda and 7.5% for the sigma of the original. The amplitude is nearly all imaginary; the real part coming entirely from the radiating Y diagram ($Y = \Sigma^0, \Lambda$). Turning off this real part changes the amplitude to only 1-2% (PS) and less than 1% (PV) of the original amplitude. This is contrary to the assumption made in ref.[32] where they neglect graphs (2),(3), and (4).

In PV coupling there is destructive interference between graphs (1) and (2) of Fig. 2.2 and the contact term graph (4) dominates. Table 3.16 shows that removing the contact term reduces the *uds* amplitude, to 9% for Λ and 2% for the Σ^0, of the original amplitude. The large contribution from the contact term is in agreement with that found by Workman and Fearing [31] where they perform an analogous calculation to ours within a pole model.

The branching ratios in the PS and PV coupling schemes were found to be roughly the same within the *SU*(6) basis but the PS/PV ratio is about 0.56 in the *uds* basis for both parameter sets. As mentioned in §2.10.2, the PS and PV results should be identical for an interaction taken over free quark states. Presumably, $PS/PV \neq 1$ is a result of the off-shell nature of the quarks in a baryon.

Turning to the ratio

$$\frac{K^- p \to \Sigma^0 \gamma}{K^- p \to \Lambda \gamma}, \tag{3.299}$$

which is independent of the uncertainty in W_p and so is perhaps more reliable than either of the individual branching ratios. Here most of the theoretical models do poorly; none predict the experimental result [1] of 1.71 ± 0.30 for the ratio in (3.299). We obtain values in the range 0.19 – 0.29 for PS and PV in both *uds* and $SU(6)$. Changing the constituent quark masses and the confinement parameter to $m_s = 0.55$, $m_u = 0.33$, and $\alpha = 0.32\,GeV$ we get 0.18 – 0.26 for the Σ^0/Λ ratio. This is in contrast to the other NRQM calculation, ref.[32], where they obtain 0.76 for the ratio (3.299). However our result agrees roughly with ref.[31] where they get 0.16 – 0.18 for (3.299), including only the Born diagrams. When the contribution from the $\Lambda(1405)$ resonance is added they find, with a variety of parameters, the $\Sigma^0\gamma$ branching ratio to be 'several times larger' [31] than the $\Lambda\gamma$ branching ratio. The cloudy bag model (CBM) [41] appears to do best here. They obtain 1.1 – 1.2 for the ratio (3.299), but is still below the experimentally observed result. It is interesting to note that when we use the same quark coupling constant for the Λ and Σ^0, which is what one would expect from a quark model, we get (see table 3.17) 0.93 (PS) and 1.43 (PV) for (3.299) in the *uds* basis, close to the experimental result.

It can be seen from tables 3.13 and 3.15 that, unfortunately, many of the results appear to be sensitive to the confinement parameter and the quark masses.

Comparing table 3.14 and 3.13 it can be seen that adding the excited components has only a small effect on the PS result but gives a large increase in the PV calculation.

Differences due to the kinematics (the photon momentum is larger in the $\Lambda\gamma$ reaction as compared to the $\Sigma^0\gamma$) and phase space factors contribute to the value of the Λ/Σ^0 ratio. However, our high value for the $\Lambda\gamma$ branching ratio may be a result of omitting

the $\Lambda(1405)$ resonance. Since the interaction in our calculations involves two spectator quarks and a freely propagating third quark it would describe broad resonances. However, there is insufficient binding between the three quarks in the intermediate state to generate a sharp resonance such as the $\Lambda(1405)$. Its contribution was found to be significant in refs.[32] and [31] (where they found it to interfere destructively with the other Born diagrams).

It would be interesting to see a future calculation including the $\Lambda(1405)$, along with the graphs we have estimated. However, further terms in the invariant amplitude would be needed to obtain a gauge invariant result for bound propagating quarks.

Our value of the ratio in (3.299) can be partially attributed to symmetry considerations: the Σ^0 has isospin one and therefore has (within the *uds* basis) a flavour wave function with M_λ symmetry. This must be combined with a symmetric (ground state) spatial wave function and a χ^λ spin wave function to yield an overall symmetric space-spin-flavour wave state. On the other hand, the Λ has isospin zero which gives rise to a χ^ρ spin wave function. These symmetry constraints on the quark spin wave functions lead to an an extra factor of 3 (from the 6-j symbols) in the $K^- p \to \Lambda\gamma$ amplitude and therefore the branching ratio will increase by a factor of 9. This causes the Λ to have a larger branching ratio despite $g_{Kus}(Y = \Lambda) < g_{Kus}(Y = \Sigma^0)$.

Thus our results appear to be qualitatively reasonable but not quantitatively rigorous, as was to be expected from the NRQM.

Appendix A

FORTRAN PROGRAMS

The following program (KAONCAPTURE) calculates the branching ratios $K^- p \to Y\gamma$ using the methods outlined in chapter 2. The Gaussian integration routines are called from the routine SETWX. This routines returns the gaussian points and their weights. The boundary points of the intervals and the number of points in an interval can be changed. This was done many times with the same result so we are satisfied that the estimated integrals are reliable.

```
      PROGRAM Kaoncapture
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
     .,a1i,b1i,b2i,c1i,c2i,d1i,d2i,d3i,d4i,g1i
     .,VI(5),cl1u,cl2u,cl3u,cl4u,cs1u,cs2u,cs3u,cs4u
     .,cl1,cl2,cl3,cl4,cs1,cs2,cs3,cs4,cp1,cp2,cp3
     .,A(6),I2(6),I3,L,I1,
     .gt,gs,gl,mp,mtb,msb,mlb,mst,ms,mu,f,b,sixj
      DIMENSION Mps(2,2),Mpv(2,2)
      mp=0.93827231                ! proton rest mass
```

```
      mK=0.49367                    ! kaon rest mass
      mlb=1.11563                   ! Lambda baryon rest mass
      msb=1.19255                   ! Sigma baryon rest mass
      mu=0.42                       ! default u quark mass
      ms=0.7                        ! default s quark mass
      pi=3.141592653589793238
      e=dsqrt(4*pi/137.035989561) ! elementary charge
      convfac=(0.1973270539)**3   ! converts from GeV^-2 to GeV fm^3
      eu=2*e/3                      ! charge on u quark
      es=-e/3                       ! charge on s quark
      ed=es                         ! charge on d quark
      eK=-e                         ! charge on kaon
      alps=(0.41)**2                ! default confinement parameter
      Wp=0.56             ! total rate Kp->Y gamma=2Wp|psiK|^2 (in GeV fm^3)
C
      cp1=0.95            ! proton admixture coefficients
      cp2=0.25
      cp3=0.20
C
      cl1u=0.95           ! Lambda uds admixture coefficients
      cl2u=0.07
      cl3u=0.28
      cl4u=0.08
C
      cl1=0.97            ! Lambda SU(6) admixture coefficients
      cl2=0.18
```

```
      cl3=0.16
      cl4=-0.01
C
      cs1u=0.98          ! Sigma uds admixture coefficients
      cs2u=0.18
      cs3u=0.02
      cs4u=0.11
C
      cs1=0.97           ! Sigma SU(6) admixture coefficients
      cs2=0.17
      cs3=0.17
      cs4=-0.00
C
      gl=-13.2d0         ! strong coupling constant for Kp->sigma
      gs=6.0d0           ! strong coupling constant for Kp->lambda
C
      Call INTEGRATE
C
C                *** MAIN LOOP ***
C When :
C       J1=1 --> Yp= Lambda
C       J1=2 --> Yp= Sigma
C       J2=1 --> Basis= uds
C       J2=2 --> Basis= SU(6)
C
      Type *,'Enter 1 to activate proton diagram (0=off)'
```

```
Read (5,*) u1
Type *,'Enter 1 to activate imag part of Y diagram (0=off)'
Read (5,*) u2
Type *,'Enter 1 to activate real part of Y diagram (0=off)'
Read (5,*) u3
Type *,'Enter 1 to activate contact diagram (0=off)'
Read (5,*) u4
Type *,'Enter 1 for parameter defaults '
Read (5,*) L
if (l.eq.1) go to 2
Type *,'Enter alpha (in GeV): '
Read (5,*) alpha
alps=alpha**2
Type *,'Enter mu (in GeV): '
Read (5,*) mu
Type *,'Enter ms (in GeV): '
Read (5,*) ms
Type *,'Enter 1 for full wave functions '
Read (5,*) L
if (l.eq.1) go to 2
cp1=1
cp2=0
cp3=0
cl1u=1
cl2u=0
cl3u=0
```

```
      cl4u=0
      cl1=1
      cl2=0
      cl3=0
      cl4=-0
      cs1u=1
      cs2u=0
      cs3u=0
      cs4u=0
      cs1=1
      cs2=0
      cs3=0
      cs4=-0.00
2     DO 30 J1=1,2
      DO 15 J2=1,2
      q1=J1
      q2=J2
      CALL TRANS(q1,q2)
      mlt=3*mu*mst/(2*mu+mst)
      E3=mst
      alpst=alps*dsqrt(mlt/mu)
      kt=((mp+mK)**2-mtb**2)/(2*(mp+mK))      ! photon momentum
      ht=0.75*(1/alpst+1/alps)                ! h in text
      Cc=(27*b*f)/((2*pi)**3*2*dsqrt(2.0d0)) ! C in text
      Api=dexp(-(0.75*kt**2)/alps)            ! Ap in text
      Ayi=dexp(-(mlt*kt/mu)**2/(12*alpst))    ! Ay in text
```

```
      c1=(mlt/mu+3*alpst/alps)/(2*alpst)
      c2=dexp((c1-ht)*kt**2)*es*(mst-kt-E3)
      c3=(E3-mu-mK)*eu
      s(1)=c1*kt                              ! s2 in text
      s(2)=Abs((2*ht-c1)*kt)                  ! s1 in text
      p1=(mu**2-(E3-mK)**2)                   !lambda2 in text
      p2=(kt+E3)**2-mst**2                    !lambda1 in text
      rtp2=dsqrt(p2)                ! pole of integrand 2 (1 in text)
      qint=0
      a1i=(4*Pi/dsqrt(alps*alpst))**(1.5)
      b1i=4/dsqrt(3.0d0)*(2*Pi/dsqrt(alps*alpst))**(1.5)*(-1.5)
      b2i=4/dsqrt(3.0d0)*(2*Pi/dsqrt(alps*alpst))**(1.5)/alps
      c1i=4/dsqrt(3.0d0)*(2*Pi/dsqrt(alps*alpst))**(1.5)*(-1.5)
      c2i=4/dsqrt(3.0d0)*(2*Pi/dsqrt(alps*alpst))**(1.5)/alpst
      d1i=12*(Pi/dsqrt(alps*alpst))**(1.5)
      d2i=-8/alps*(Pi/dsqrt(alps*alpst))**(1.5)
      d3i=-8/alpst*(Pi/dsqrt(alps*alpst))**(1.5)
      d4i=16/(3*alps*alpst)*(Pi/dsqrt(alps*alpst))**(1.5)
      g1i=-16/dsqrt(alps*alpst)*(Pi/dsqrt(alps*alpst))**(1.5)
C
C     I0 integral
      ap=1
      ay=1
      bp=0
      by=0
      call setyi(1)    ! evaluate F1(p)
```

```
      call setyi(2)     ! evaluate G1(u)
      call calc(1)
C
C     Ip integral
      ap=3*kt/Dsqrt(6.0d0)
      bp=3/DSQRT(6.0d0)
      ay=1
      by=0
      call setyi(1)      ! evaluate F2(p)
      ap=0
      call setyi(2)      ! evaluate G2(u)
      call calc(2)
C
C     Iy integral
      ap=1
      bp=0
      ay=-mlt*kt/(mu*DSQRT(6.0d0))
      by=-3/DSQRT(6.0d0)
      call setyi(1)
      ay=(3-mlt/mu)*kt/DSQRT(6.0d0)
      call setyi(2)
      call calc(3)
      if (u.eq.1) go to 15
C
C     Ipy integral
      ap=3*kt/DSQRT(6.0d0)
```

```
      bp=3/DSQRT(6.0d0)
      ay=-mlt*kt/(mu*DSQRT(6.0d0))
      by=-bp
      call setyi(1)
      ap=0
      ay=(3-mlt/mu)*kt/DSQRT(6.0d0)
      call setyi(2)
      call calc(4)
C
C     Iq integral
      qint=1
      ap=3*kt/DSQRT(6.0d0)
      bp=3/DSQRT(6.0d0)
      ay=-mlt*kt/(mu*DSQRT(6.0d0))
      by=-bp
      call setyi(1)
      ap=0
      ay=(3-mlt/mu)*kt/DSQRT(6.0d0)
      call setyi(2)
      call calc(5)
C
17    DO 23 I1=1,5
      cxpart=c3*GI1(J1,J2,I1)*u1+c2*GI2(J1,J2,I1)*u2
      rpart=pole(I1)*pi/(2*rtp2)*c2*u3
      VI(I1)=DCMPLX(rpart,cxpart)
23    CONTINUE
```

```
      mgamp=DIMAG(VAMPL(q1,q2))**2+DREAL(VAMPL(q1,q2))**2
      Mps(J1,J2)=mgamp*(2*pi*gt*Cc*Ayi*Api)**2
      DO 24 I1=1,5
      cxpart=c3*GI1(J1,J2,I1)*u1-c2*GI2(J1,J2,I1)*u2+
     .GI3(J1,J2,I1)*eK/mK*u4
      rpart=pole(I1)*pi/(2*rtp2)*c2*u3
      VI(I1)=DCMPLX(rpart,cxpart)
24    CONTINUE
      mgamp=DIMAG(VAMPL(q1,q2))**2+DREAL(VAMPL(q1,q2))**2
      Mpv(J1,J2)=mgamp*(gt/(mst+mu)*mK*Cc*Ayi*Api*2*pi)**2
      Ey=dsqrt(mtb**2+kt**2)
      E3p=mst+mK-kt
      phase=Ey*kt*(E3p+mu)/(8*pi*(mK+mp)*mK*E3p)
300   FORMAT(' Y= Lambda, Basis=uds',$)
310   FORMAT(' Y= Sigma, Basis=uds ',$)
320   FORMAT(' Y=Lambda, Basis=SU(6):',$)
330   FORMAT(' Y=Sigma, Basis=SU(6):',$)
      open(unit=6,carriagecontrol='FORTRAN',status='OLD')
      if ((j1.eq.1) .and. (J2.eq.1)) type 300
      if ((j1.eq.2) .and. (J2.eq.1)) type 310
      if ((j1.eq.1) .and. (J2.eq.2)) type 320
      if ((j1.eq.2) .and. (j2.eq.2)) type 330
      TYPE *,'M^2 (PS)=',Mps(J1,J2)
      Mps(J1,J2)=phase*convfac*Mps(J1,J2)/(2*Wp)
      TYPE *,'Ratio (PS)=',Mps(J1,J2)
      TYPE *,'M^2 (PV)=',Mpv(J1,J2)
```

```
      Mpv(J1,J2)=phase*convfac*Mpv(J1,J2)/(2*Wp)
      TYPE *,'Ratio (PV)=',Mpv(J1,J2)
15    CONTINUE
30    CONTINUE
      STOP
      END

      SUBROUTINE setyi(I)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      a0(1,I)=ap**2
      a0(2,I)=2*ap*bp
      a0(3,I)=bp**2
      b0(1,I)=ay**2
      b0(2,I)=2*ay*by
      b0(3,I)=by**2
      if (qint.EQ.1) GO TO 34   ! if doing I5 do cos theta numerically.
      y(1,I)=-a0(1,I)*b0(1,I)/s(I)+a0(1,I)*b0(2,I)/s(I)**2
     .+a0(2,I)*b0(1,I)/s(I)**2-2*a0(2,I)*b0(2,I)/s(I)**3
      y(2,I)=a0(1,I)*b0(2,I)/s(I)+a0(2,I)*b0(1,I)/s(I)
     .-2*a0(2,I)*b0(2,I)/s(I)**2
      y(3,I)=-a0(1,I)*b0(3,I)/s(I)-a0(2,I)*b0(2,I)/s(I)
```

```
     .+a0(2,I)*b0(3,I)/s(I)**2-a0(3,I)*b0(1,I)/s(I)
     .+a0(3,I)*b0(2,I)/s(I)**2
      y(4,I)=a0(2,I)*b0(3,I)/s(I)+a0(3,I)*b0(2,I)/s(I)
      y(5,I)=-a0(3,I)*b0(3,I)/s(I)
34    RETURN
      END
C
      SUBROUTINE calc(I)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      pole(I)=FF(rtp2,2)
      spik=pole(I)
      CALL DOINT(I)
      RETURN
      END
C
      SUBROUTINE doint(I)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
```

```
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      GI1(J1,J2,I)=0      ! Integral for proton radiation diagram,
      GI2(J1,J2,I)=0      ! for Y radiation diagram,
      GI3(J1,J2,I)=0      ! and the contact graph.
      DO 10 I1=1,LIM
      GI1(J1,J2,I)=GI1(J1,J2,I)+WW(I1)*F1(XX(I1))
      GI2(J1,J2,I)=GI2(J1,J2,I)+WW(I1)*F2(XX(I1))
      GI3(J1,J2,I)=GI3(J1,J2,I)+WW(I1)*FF(XX(I1),1)
10    CONTINUE
11    RETURN
      END
C
      FUNCTION F1(X)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      F1=FF(X,1)/(X**2+p1)
      RETURN
      END
C
      FUNCTION F2(X)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
```

```
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      F2=(FF(X,2)-spik)/(X**2-p2)
      RETURN
      END
C
      FUNCTION FF(X,I)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      If (qint.EQ.0) GO TO 36
      FF=sing(X,I)
      GO TO 37
36    t1=0
      t2=0
      DO 35 I1=1,5
      t1=t1+y(I1,I)*X**I1
      t2=t2+y(I1,I)*(-X)**I1
35    CONTINUE
      t1p=DEXP(-ht*X**2)
      FF=0
```

```
      IF (t1p.EQ.0) GO TO 37
      FF=t1p*(DEXP(-s(I)*X)*t1+DEXP(s(I)*X)*t2)
37    RETURN
      END
C
      FUNCTION sing(X,I)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      t1=0
      DO 10 I1=1,LIM2
      t1=t1+WWT(I1)*q(XXT(I1),x,I)   ! Do numerical cos theta integration
10    CONTINUE
      sing=t1
      RETURN
      END
C
      FUNCTION q(u,X,I)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
```

```
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      t1=dexp(-ht*x*x)*X**2
      IF (t1.EQ.0) GO TO 38
      t2=u-1
      t1p=dexp(s(I)*X*t2)*dsqrt(a0(1,I)+a0(2,I)*t2*X+a0(3,I)*X**2)
     .*dsqrt(b0(1,I)+b0(2,I)*t2*X+b0(3,I)*X**2)
38    q=t1*t1p
      RETURN
      END
C
      FUNCTION VAMPL(Yp,bas)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
     .,a1i,b1i,b2i,c1i,c2i,d1i,d2i,d3i,d4i,g1i
     .,VI(5),cl1u,cl2u,cl3u,cl4u,cs1u,cs2u,cs3u,cs4u
     .,cl1,cl2,cl3,cl4,cs1,cs2,cs3,cs4,cp1,cp2,cp3
C
C  Main formulas for the amplitude as generated by SMP.
C  Includes spin summations, flavour and space matrix elements.
C
      IF (Yp.EQ.2) GO TO 94
C Yp=Lambda
```

```
      IF (bas.EQ.2) GO TO 92
C uds LAMBDA
      VAMPL = (1.D0/18.D0)*((VI(1)*(18*A1I*CL3U*CP2+18*B1I*
     $    CL1U*CP2+18*CL2U*CP2*D1I+18*2 ** 0.5D0*A1I
     $   *CL1U*CP1+9*2 ** 0.5D0*A1I*CL3U*CP3+(-9)*2
     $     ** 0.5D0*B1I*CL1U*CP3+18*2 ** 0.5D0*C1I*CL2U
     $   *CP1+(-9)*2 ** 0.5D0*CL2U*CP3*D1I)+VI(2)*(18*
     $    B2I*CL1U*CP2+18*CL2U*CP2*D2I+(-9)*2 **
     $    0.5D0*B2I*CL1U*CP3+(-9)*2 ** 0.5D0*CL2U*CP3*
     $    D2I)+VI(4)*(18*CL2U*CP2*D4I+(-9)*2 ** 0.5D0*
     $    CL2U*CP3*D4I)+VI(3)*(18*CL2U*CP2*D3I+18*2
     $    ** 0.5D0*C2I*CL2U*CP1+(-9)*2 ** 0.5D0*CL2U*
     $    CP3*D3I)+(-2)*CL4U*CP3*G1I*VI(5)) / 2 ** 0.5D0)
      GO TO 99
C SU(6) LAMBDA
92    T1 = VI(2)*((-36)*B2I*CL1*CP2+18*CL2*CP3*D2I+18*
     $    CL3*CP2*D2I+(-18)*CL4*CP2*D2I+18*2 ** 0.5D0*
     $    B2I*CL1*CP3+(-18)*2 ** 0.5D0*CL2*CP2*D2I+(-9)
     $   *2 ** 0.5D0*CL3*CP3*D2I)+VI(4)*(18*CL2*CP3*
     $    D4I+18*CL3*CP2*D4I+(-18)*CL4*CP2*D4I+(-18)
     $   *2 ** 0.5D0*CL2*CP2*D4I+(-9)*2 ** 0.5D0*CL3*
     $    CP3*D4I)
      T2 = VI(3)*((-36)*C2I*CL2*CP1+18*CL2*CP3*D3I+18*
     $    CL3*CP2*D3I+(-18)*CL4*CP2*D3I+18*2 ** 0.5D0*
     $    C2I*CL3*CP1+(-18)*2 ** 0.5D0*CL2*CP2*D3I+(-9)
     $   *2 ** 0.5D0*CL3*CP3*D3I)+2 ** 0.5D0*(9*A1I*CL4
```

```
     $    *CP3*VI(1)+(-18)*C1I*CL4*CP1*VI(1)+(-18)*C2I*
     $     CL4*CP1*VI(3)+9*CL4*CP3*D1I*VI(1)+9*CL4*CP3*
     $     D2I*VI(2)+9*CL4*CP3*D3I*VI(3)+9*CL4*CP3*D4I*
     $     VI(4)+2*CL4*CP3*G1I*VI(5))+2*2 ** 0.5D0*CL3*
     $     CP3*G1I*VI(5)
      VAMPL = ((-1.D0/36.D0))*((VI(1)*((-18)*A1I*CL2*CP3+(-18)*
     $     A1I*CL3*CP2+18*A1I*CL4*CP2+(-36)*B1I*CL1*
     $     CP2+(-36)*C1I*CL2*CP1+18*CL2*CP3*D1I+18*
     $     CL3*CP2*D1I+(-18)*CL4*CP2*D1I+(-36)*2 **
     $     0.5D0*A1I*CL1*CP1+(-18)*2 ** 0.5D0*A1I*CL2*
     $     CP2+(-9)*2 ** 0.5D0*A1I*CL3*CP3+18*2 ** 0.5D0*
     $     B1I*CL1*CP3+18*2 ** 0.5D0*C1I*CL3*CP1+(-18)*
     $     2 ** 0.5D0*CL2*CP2*D1I+(-9)*2 ** 0.5D0*CL3*CP3
     $    *D1I)+T1+T2) / 6 ** 0.5D0)
      GO TO 99
C Yp=sigma 0
94    IF (bas.EQ.2) GO TO 96
C SIGMA uds
      VAMPL = (1.D0/6.D0)*((VI(1)*(2*A1I*CP2*CS3U+2*B1I*CP2
     $    *CS1U+2*CP2*CS2U*D1I+2*2 ** 0.5D0*A1I*CP1
     $    *CS1U+(-1)*2 ** 0.5D0*A1I*CP3*CS3U+2 ** 0.5D0
     $    *B1I*CP3*CS1U+2*2 ** 0.5D0*C1I*CP1*CS2U+2
     $     ** 0.5D0*CP3*CS2U*D1I)+VI(2)*(2*B2I*CP2*CS1U
     $    +2*CP2*CS2U*D2I+2 ** 0.5D0*B2I*CP3*CS1U+2
     $     ** 0.5D0*CP3*CS2U*D2I)+VI(4)*(2*CP2*CS2U*D4I
     $    +2 ** 0.5D0*CP3*CS2U*D4I)+VI(3)*(2*CP2*CS2U*
```

```
     $    D3I+2*2 ** 0.5D0*C2I*CP1*CS2U+2 ** 0.5D0*CP3*
     $    CS2U*D3I)+(-2)*CP3*CS4U*G1I*VI(5)) / (2 ** 0.5D0
     $   *3 ** 0.5D0))
C      type *,cp1,cp2,cp3
C      type *,cs1u,cs2u,cs3u,cs4u
C      type *,VI(1)
C      type *,Vampl
      GO TO 99
C SIGMA SU(6)
96    T1 = VI(2)*((-4)*B2I*CP2*CS1+(-2)*CP2*CS3*D2I+2*
     $    CP2*CS4*D2I+(-2)*CP3*CS2*D2I+(-2)*2 ** 0.5D0
     $   *B2I*CP3*CS1+(-2)*2 ** 0.5D0*CP2*CS2*D2I+(-1
     $    )*2 ** 0.5D0*CP3*CS3*D2I+2 ** 0.5D0*CP3*CS4*
     $    D2I)
      T2 = VI(4)*((-2)*CP2*CS3*D4I+2*CP2*CS4*D4I+(-2)
     $   *CP3*CS2*D4I+(-2)*2 ** 0.5D0*CP2*CS2*D4I+(-1
     $    )*2 ** 0.5D0*CP3*CS3*D4I+2 ** 0.5D0*CP3*CS4*
     $    D4I)+VI(5)*(2*2 ** 0.5D0*CP3*CS3*G1I+2*2 **
     $    0.5D0*CP3*CS4*G1I)+VI(3)*((-4)*C2I*CP1*CS2+(
     $    -2)*CP2*CS3*D3I+2*CP2*CS4*D3I+(-2)*CP3*
     $    CS2*D3I+(-2)*2 ** 0.5D0*C2I*CP1*CS3+2*2 **
     $    0.5D0*C2I*CP1*CS4+(-2)*2 ** 0.5D0*CP2*CS2*D3I
     $   +(-1)*2 ** 0.5D0*CP3*CS3*D3I+2 ** 0.5D0*CP3*
     $    CS4*D3I)
      VAMPL = ((-1.D0/36.D0))*((VI(1)*(2*A1I*CP2*CS3+(-2)*A1I
     $   *CP2*CS4+2*A1I*CP3*CS2+(-4)*B1I*CP2*CS1+
```

```
     $     (-4)*C1I*CP1*CS2+(-2)*CP2*CS3*D1I+2*CP2*
     $     CS4*D1I+(-2)*CP3*CS2*D1I+(-4)*2 ** 0.5D0*A1I
     $    *CP1*CS1+(-2)*2 ** 0.5D0*A1I*CP2*CS2+(-1)*2
     $     ** 0.5D0*A1I*CP3*CS3+2 ** 0.5D0*A1I*CP3*CS4+(
     $     -2)*2 ** 0.5D0*B1I*CP3*CS1+(-2)*2 ** 0.5D0*C1I
     $    *CP1*CS3+2*2 ** 0.5D0*C1I*CP1*CS4+(-2)*2 **
     $     0.5D0*CP2*CS2*D1I+(-1)*2 ** 0.5D0*CP3*CS3*D1I
     $    +2 ** 0.5D0*CP3*CS4*D1I)+T1+T2) / 2 ** 0.5D0)
99     RETURN
       END
C
       SUBROUTINE TRANS(Yp,bas)
       IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
       IMPLICIT COMPLEX*16 V
       COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
      .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
      .s(2),y(5,2),J1,J2,
      .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
      .,a1i,b1i,b2i,c1i,c2i,d1i,d2i,d3i,d4i,g1i
      .,VI(5),cl1u,cl2u,cl3u,cl4u,cs1u,cs2u,cs3u,cs4u
      .,cl1,cl2,cl3,cl4,cs1,cs2,cs3,cs4,cp1,cp2,cp3
      .,A(6),I2(6),I3,L,I1,
      .gt,gs,gl,mp,mtb,msb,mlb,mst,ms,mu,f,b,sixj
       IF (Yp.EQ.2) GO TO 110
C Yp=Lambda
       mtb=mlb
```

```
      gt=gl/3*mu/mp*dsqrt(6.0d0)
      f= (1.D0/12.D0)*(2*3 ** 0.5D0*(2*CL1*CL1U+(-1)*
     $    CL2U*CL3+2 ** 0.5D0*CL2*CL2U+2 ** 0.5D0*CL2*
     $    CL3U)+2*3 ** 0.5D0*CL3*CL3U+2*6 ** 0.5D0*CL3
     $   *CL4U+2*6 ** 0.5D0*CL4*CL4U+12 ** 0.5D0*CL2U
     $   *CL4+(-1)*12 ** 0.5D0*CL3U*CL4)
      f=1
      GO TO 120
C Yp=sigma 0
110   mtb=msb
      gt=gs/3*mu/mp*9*dsqrt(2.0d0)
      f= (1.D0/12.D0)*(2*3 ** 0.5D0*(2*CS1*CS1U+CS2U*
     $    CS3+2 ** 0.5D0*CS2*CS2U+2 ** 0.5D0*CS2*CS3U)+
     $    (-2)*3 ** 0.5D0*CS3*CS3U+2*6 ** 0.5D0*CS3*
     $    CS4U+2*6 ** 0.5D0*CS4*CS4U+(-1)*12 ** 0.5D0*
     $    CS2U*CS4+12 ** 0.5D0*CS3U*CS4)
      f=1
120   IF (bas.EQ.2) GO TO 130
C uds basis
      mst=ms
      b=1
      f=1
      GO TO 140
C SU(6) basis
130   mst=mu
      b=3
```

```
140   RETURN
      END
C
      FUNCTION FAC(q)
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      IF (q.GE.0) GO TO 40
      TYPE *,'Argument of Factorial less than zero!'
      GO TO 55
40    FAC=1
      IF (q.LT.2) GO TO 55
      DO 45 J=2,q
      FAC=FAC*J
45    CONTINUE
55    RETURN
      END
C
C Subroutine to determine the points and weights for evaluating the integrals.
C
      SUBROUTINE INTEGRATE
      IMPLICIT REAL*8 (A-H,K,M-U,W-Z)
      IMPLICIT COMPLEX*16 V
      COMMON /WXCOM/ XX(100), WW(100), WXXVEC(100), LIM,
     .XXT(100), WWT(100), LIM2,GI1(2,2,5),GI2(2,2,5),GI3(2,2,5),
     .s(2),y(5,2),J1,J2,
     .a0(3,2),b0(3,2),pole(5),ht,p1,p2,rtp2,ap,bp,ay,by,spik,qint
```

```
     .,a1i,b1i,b2i,c1i,c2i,d1i,d2i,d3i,d4i,g1i
     .,VI(5),cl1u,cl2u,cl3u,cl4u,cs1u,cs2u,cs3u,cs4u
     .,cl1,cl2,cl3,cl4,cs1,cs2,cs3,cs4,cp1,cp2,cp3
     .,A(6),I2(6),I3,L,I1
      I1=4
      A(1)=.5          ! Intervals for numerical cos theta integration
      I2(1)=16
      A(2)=1.0
      I2(2)=16
      A(3)=1.5
      I2(3)=16
      A(4)=2.0
      I2(4)=16
      I3=0
42    CALL SETWX(I1,I2,A,I3)
      LIM2=LIM
      DO 48,I1=1,LIM2
      XXT(I1)=XX(I1)
      WWT(I1)=WW(I1)
48    CONTINUE
C
      I1=6
      A(1)=.25             ! Intervals for numerical momentum integration
      I2(1)=16
      A(2)=.5
      I2(2)=16
```

```
      A(3)=.75
      I2(3)=16
      A(4)=1.0
      I2(4)=16
      A(5)=1.25
      I2(5)=16
      A(6)=1.5
      I2(6)=12
      I3=8
43    CALL SETWX(I1,I2,A,I3)
      RETURN
      END
```

Appendix B

Details of Integrals

To evaluate the integrals used in chapter 3 we will need the following results [42]

$$\int e^{-\frac{1}{2}\alpha^2\lambda^2+i\vec{\lambda}\cdot\vec{q}}d^3\vec{\lambda} = \left(\frac{2\pi}{\alpha^2}\right)^{\frac{3}{2}} e^{-\frac{q^2}{2\alpha^2}}, \tag{B.300}$$

$$\int_0^\infty \lambda\sin(\lambda q)e^{-\frac{1}{2}\alpha^2\lambda^2}d\lambda = \frac{q\sqrt{\pi}}{4}\left(\frac{2}{\alpha^2}\right)^{\frac{3}{2}} e^{-\frac{q^2}{2\alpha^2}}, \tag{B.301}$$

$$\int_0^\infty \lambda^3\sin(\lambda q)e^{-\frac{1}{2}\alpha^2\lambda^2}d\lambda = \frac{\sqrt{\pi}(3q\alpha^2-q^3)2^{7/2}}{16\alpha^7}e^{-\frac{q^2}{2\alpha^2}}, \tag{B.302}$$

$$\int_{-1}^1 ue^{i\lambda qu}du = \frac{2}{i\lambda q}\left(\cos(\lambda q)-\frac{1}{\lambda q}\sin(\lambda q)\right), \tag{B.303}$$

$$\int_0^\infty \lambda^2\cos(\lambda q)e^{-\frac{1}{2}\alpha^2\lambda^2}d\lambda = \frac{\sqrt{\pi}2^{5/2}(\alpha^2-q^2)}{8\alpha^5}e^{-\frac{q^2}{2\alpha^2}}. \tag{B.304}$$

Using (B.300) we get

$$\begin{aligned} a_Y = \int \psi_{000}e^{i\vec{\lambda}\cdot\vec{q}_Y}d^3\vec{\lambda} &= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{4\alpha_\lambda^3}{\sqrt{\pi}}}\int e^{-\frac{1}{2}\alpha_\lambda^2\lambda^2+i\vec{\lambda}\cdot\vec{q}_Y}d^3\vec{\lambda} \\ &= \left(\frac{2\sqrt{\pi}}{\alpha_\lambda}\right)^{\frac{3}{2}} e^{-\frac{q_Y^2}{2\alpha_\lambda^2}} \\ \text{Similarly } a_p &= \left(\frac{2\sqrt{\pi}}{\alpha}\right)^{\frac{3}{2}} e^{-\frac{q_p^2}{2\alpha^2}}. \end{aligned}$$

$$\begin{aligned} b_Y = \int \psi_{100}e^{i\vec{\lambda}\cdot\vec{q}_Y}d^3\vec{\lambda} &= \sqrt{\frac{2}{3}}\frac{1}{\sqrt{4\pi}}\sqrt{\frac{4\alpha_\lambda^3}{\sqrt{\pi}}}\int(\frac{3}{2}-\alpha_\lambda^2\lambda^2)e^{-\frac{1}{2}\alpha_\lambda^2\lambda^2+i\vec{\lambda}\cdot\vec{q}_Y}d^3\vec{\lambda} \\ &= \frac{4}{\sqrt{3}}\left(\frac{\sqrt{\pi}}{\alpha_\lambda}\right)^{\frac{3}{2}}((\frac{q_Y}{\alpha_\lambda})^2-\frac{3}{2})e^{-\frac{q_Y^2}{2\alpha_\lambda^2}} \\ \text{Similarly } b_p &= \frac{4}{\sqrt{3}}\left(\frac{\sqrt{\pi}}{\alpha}\right)^{\frac{3}{2}}((\frac{q_p}{\alpha})^2-\frac{3}{2})e^{-\frac{q_p^2}{2\alpha^2}}. \end{aligned}$$

$$\begin{aligned}
\text{Finally } c_Y &= \int \psi_{010} e^{i\vec{\lambda}\cdot\vec{q}_Y} d^3\vec{\lambda} \\
&= \sqrt{\frac{2}{3}}\sqrt{\frac{4\alpha_\lambda^3}{\sqrt{\pi}}} \int \alpha_\lambda \lambda e^{-\frac{1}{2}\alpha_\lambda^2\lambda^2 + i\lambda q_Y \cos\theta} \sqrt{\frac{3}{4\pi}} \cos\theta\, \lambda^2 \sin\theta\, d\theta\, d\phi\, d\lambda \\
&= \frac{4i}{\alpha_\lambda}\left(\frac{\sqrt{\pi}}{\alpha_\lambda}\right)^{\frac{3}{2}} q_Y e^{-\frac{q_Y^2}{2\alpha_\lambda^2}} \\
\text{Similarly } c_p &= \frac{4i}{\alpha}\left(\frac{\sqrt{\pi}}{\alpha}\right)^{\frac{3}{2}} q_p e^{-\frac{q_p^2}{2\alpha^2}}.
\end{aligned}$$

Angular integrals

When performing the angular integrals we will need the general formula ($u = \cos\theta$)

$$\int_{u=-1}^{1} e^{spu}(a_1 + a_2 pu + a_3 p^2)(b_1 + b_2 pu + b_3 p^2) p^2\, du = \sum_i \eta_i (p^i e^{-ps} + (-p)^i e^{ps}) \tag{B.305}$$

$$\text{where } \eta_1 \equiv \frac{-a_1 b_1}{s} + \frac{a_1 b_2}{s^2} + \frac{a_2 b_1}{s^2} - \frac{2a_2 b_2}{s^3}, \tag{B.306}$$

$$\eta_2 \equiv \frac{a_1 b_2}{s} + \frac{a_2 b_1}{s} - \frac{2a_2 b_2}{s^2}, \tag{B.307}$$

$$\eta_3 \equiv \frac{-a_1 b_3}{s} - \frac{a_2 b_2}{s} + \frac{a_2 b_3}{s^2} - \frac{a_3 b_1}{s} + \frac{a_3 b_2}{s^2}, \tag{B.308}$$

$$\eta_4 \equiv \frac{a_2 b_3}{s} + \frac{a_3 b_2}{s},\ \eta_5 \equiv -\frac{a_3 b_3}{s}. \tag{B.309}$$

All the angular integrals can be evaluated from this result by appropriate choice of the a_i and b_i.

Contour Integration

Since the integrand of the second integral in equation (3.264) is an even function of u,

$$G_J(\lambda_1) \int_0^\infty \frac{1}{u^2 - \lambda_1^2 - i\epsilon}\, du = \frac{G_J(\lambda_1)}{2} \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{u^2 - \lambda_1^2 - i\epsilon}\, du. \tag{B.310}$$

Neglecting terms of $O(\epsilon)^2$ and defining

$$z \equiv \lambda_1 + \frac{i\epsilon}{2\lambda_1},$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{u^2 - \lambda_1^2 - i\epsilon}\, du &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(u+z)(u-z)}\, du. \\ &= \lim_{R\to\infty} \left\{ \oint_C \frac{(u+z)^{-1}}{u-z}\, du - \oint_{C_R} \frac{(u+z)^{-1}}{u-z}\, du \right\} \quad \text{(B.311)}\end{aligned}$$

The contour C is to be closed on the top half of the Argand plane. Contour C_R represents the semi-circle part of C. Since C is an anti-clockwise contour it is postive. From the Cauchy residue theorem [43]

$$\oint_C \frac{f(x)}{(x-a)}\, dx = 2\pi i f(a), \quad \text{(B.312)}$$

we get

$$\lim_{R\to\infty} \lim_{\epsilon\to 0} \oint_C \frac{(u+z)^{-1}}{u-z}\, du = \lim_{\epsilon\to 0} \frac{2\pi i}{2\lambda_1 + \frac{i\epsilon}{\lambda_1}} = \frac{\pi i}{\lambda_1}.$$

We must now show that the integral over contour C_R yields zero. It follows from the triangle inequality [44] that,

$$|u+z||u-z| \geq ||u| - |z||^2.$$

On C_R, $|u| = R$ and we get

$$\left| \oint_{C_R} \frac{1}{(u+z)(u-z)}\, du \right| \leq \frac{\pi R}{|R - |z||^2}$$

where πR is the length C_R. The desired limit is now evident; that is,

$$\lim_{R\to\infty} \oint_{C_R} \frac{1}{(u+z)(u-z)}\, du = 0.$$

Appendix C

SMP Procedures

The following commands define the wave functions and rules for evaluating the matrix elements within the SMP environment. They are contained within the file "WAVEFN.DEF". The following translations may be useful:

psip$\equiv \Psi_p$	psil$\equiv \Psi^\dagger_\Lambda$
psis$\equiv \Psi^\dagger_\Sigma$	N82ss$\equiv N_8{}^2 S_S$ etc
L2S$\equiv \Lambda^2 S$ etc	cr$\equiv \chi^\rho$
cl$\equiv \chi^\lambda$	pl$\equiv \phi_\Lambda$
ps$\equiv \phi_\Sigma$	prl$\equiv \phi^\rho_\Lambda$
pll$\equiv \phi^\lambda_\Lambda$ etc	Ps000$\equiv \Phi^S_{000}$
Prl20f$\equiv \Phi^{\rho\lambda}_{200}$ etc.	

Where the f denotes final state; that is the spatial wave function for the Y.

For the matrix elements:

pl*prp denotes $\langle \phi_\Lambda | V_+ | \phi^\rho_p \rangle$

cr^2 denotes $\langle \chi^\rho | \vec{\sigma} \cdot \vec{\epsilon} | \chi^\rho \rangle$

Ps00f*Ps000 denotes $\langle \Phi^S_{000} | Q | \Phi^S_{000} \rangle$

```
psip:cp1*N82ss+cp2*N82ssp+cp3*N82sm  /* Proton wave function

psil:If[uds=1,cl1u*L2S+cl2u*L2Sll+cl3u*L2Srr+cl4u*L2Srl,\
cl1*L82ss+cl2*L82ssp+cl3*L82sm+cl4*L12sm]  /* Lambda wave function
```

```
psis:If[uds=1,cs1u*S2S+cs2u*S2Sll+cs3u*S2Srr+cs4u*S2Srl,\
cs1*S82ss+cs2*S82ssp+cs3*S82sm+cs4*S102sm]   /* Sigma wave function

N82ss:(cr*prp+cl*plp)Ps000/Sqrt[2]
N82ssp:(cr*prp+cl*plp)Ps200/Sqrt[2]
N82sm:(Pl200*(cr*prp-cl*plp)+Pr200*(cr*plp+cl*prp))/2

L82ss:(cr*prl+cl*pll)Ps00f/Sqrt[2]
L82ssp:(cr*prl+cl*pll)Ps20f/Sqrt[2]
L82sm:(Pl20f*(cr*prl-cl*pll)+Pr20f*(cr*pll+cl*prl))/2
L12sm:(Pl20f*cr-cl*Pr20f)pal/Sqrt[2]

S82ss:(cr*prs+cl*pls)Ps00f/Sqrt[2]
S82ssp:(cr*prs+cl*pls)Ps20f/Sqrt[2]
S82sm:(Pl20f*(cr*prs-cl*pls)+Pr20f*(cr*pls+cl*prs))/2
S102sm:(Pl20f*cl+cr*Pr20f)pss/Sqrt[2]

L2S:Ps00f*pl*cr
L2Sll:Pll20f*pl*cr
L2Srr:Prr20f*pl*cr
L2Srl:Prl20f*pl*cl

S2S:Ps00f*ps*cl
S2Sll:Pll20f*ps*cl
S2Srr:Prr20f*ps*cl
S2Srl:Prl20f*ps*cr
```

```
'Flavour Matrix element results
pl*prp:1
pl*plp:0

prl*prp:Sqrt[2/3]
pll*prp:0
pal*prp:-1/Sqrt[3]
prl*plp:0
pll*plp:0
pal*plp:0

ps*prp:0
ps*plp:-1/Sqrt[3]

prs*prp:0
prs*plp:0
pls*prp:0
pls*plp:-Sqrt[2]/3
pss*prp:0
pss*plp:-1/3

'Space Matrix elements
Ps00f*Ps000:a
Ps00f*Ps200:b/Sqrt[2]
Ps00f*Pl200:-b/Sqrt[2]
```

```
Ps00f*Pr200:0

Prl20f*Ps000:0

Prl20f*Ps200:0

Prl20f*Pl200:0

Prl20f*Pr200:g/3

Prr20f*Ps000:0

Prr20f*Ps200:a/Sqrt[2]

Prr20f*Pl200:a/Sqrt[2]

Prr20f*Pr200:0

Pll20f*Ps000:c

Pll20f*Ps200:d/Sqrt[2]

Pll20f*Pl200:-d/Sqrt[2]

Pll20f*Pr200:0

Ps20f*Ps000:c/Sqrt[2]

Ps20f*Ps200:(a+d)/2

Ps20f*Pl200:(a-d)/2

Ps20f*Pr200:0

Pl20f*Ps000:-c/Sqrt[2]

Pl20f*Ps200:(a-d)/2

Pl20f*Pl200:(a+d)/2

Pl20f*Pr200:0

Pr20f*Ps000:0

Pr20f*Ps200:0

Pr20f*Pl200:0

Pr20f*Pr200:g/3
```

```
'Spin Matrix elements
cr^2:sfr
cr*cl:0
cl*cr:0
cl^2:sfl

SMP 1.6.2
Mon Feb  4 16:14:28 1991
uds:1;              /* uds basis
<"wavefn.def";      /* Load definitions
psil psip;          /* Calculate <Lambda|V|proton> amplitude
Ex[%];              /* Expand last expression (using definitons)
Rat[%];             /* Rationalize last expression over a common denominator
Fac[%];             /* Factorize last expression
Cb[%,{a,b,c,d,g}] /* Combine coefficients in last expression

                                                  1/2
        a (6cl1u cp1 sfr + 3cl3u cp3 sfr + 3 2    cl3u cp2 sfr)

                                                 1/2
                  + b (-3cl1u cp3 sfr + 3 2    cl1u cp2 sfr)

                                                 1/2
                  + d (-3cl2u cp3 sfr + 3 2    cl2u cp2 sfr)

                                                1/2
```

```
                    + 6c cl2u cp1 sfr + 2    cl4u cp3 g sfl
#0[7]:   ---------------------------------------------------
                                  1/2
                                 6 2

psis psip;    /* Calculate <Sigma 0|V|proton> amplitude
Ex[%];
Rat[%];
Fac[%];
Cb[%,{a,b,c,d,g}]

                                                    1/2
         -(a (6cp1 cs1u sfl - 3cp3 cs3u sfl + 3 2    cp2 cs3u sfl)

                                       1/2
                  + b (3cp3 cs1u sfl + 3 2    cp2 cs1u sfl)

                                       1/2
                  + d (3cp3 cs2u sfl + 3 2    cp2 cs2u sfl)

                                    1/2
                  + 6c cp1 cs2u sfl + 2    cp3 cs4u g sfr)
#0[12]:  ---------------------------------------------------
                                  1/2  1/2
                                 6 2    3
```

```
uds:0;              /* Now do same thing in SU(6) basis
<"wavefn.def";
psis psip;          /* for the Sigma
Ex[%];
Rat[%];
Fac[%];
Cb[%,{a,b,c,d,g}]
```

```
          -(a (12cp1 cs1 sfl + 6cp2 cs2 sfl + 3cp3 cs3 sfl - 3cp3 cs4 sfl

                                    1/2                    1/2
                               - 3 2    cp2 cs3 sfl + 3 2    cp2 cs4 sfl

                                    1/2
                               - 3 2    cp3 cs2 sfl)

                                          1/2
                        + b (6cp3 cs1 sfl + 6 2    cp2 cs1 sfl)

                                                      1/2
                        + c (6cp1 cs3 sfl - 6cp1 cs4 sfl + 6 2    cp1 cs2 sfl)

                        + d (6cp2 cs2 sfl + 3cp3 cs3 sfl - 3cp3 cs4 sfl

                                          1/2                    1/2
                                     + 3 2    cp2 cs3 sfl - 3 2    cp2 cs4 sfl
```

```
                                                  1/2
                                              + 3 2    cp3 cs2 sfl)

                                    + g (2cp3 cs3 sfr + 2cp3 cs4 sfr))
#0[19]:  ------------------------------------------------------------------
                                                  1/2
                                                 36 2

psil psip;        /* and for the Lambda
Ex[%];
Rat[Fac[%]];
Cb[%,{a,b,c,d,g}]

          -(a (-24cl1 cp1 sfr - 12cl2 cp2 sfr - 6cl3 cp3 sfr + 6cl4 cp3 sfr

                                       1/2                    1/2
                                  - 6 2    cl2 cp3 sfr - 6 2    cl3 cp2 sfr

                                       1/2
                                  + 6 2    cl4 cp2 sfr)

                                                    1/2
                              + b (12cl1 cp3 sfr - 12 2    cl1 cp2 sfr)

                              + c (12cl3 cp1 sfr - 12cl4 cp1 sfr
```

```
                                        1/2
                                - 12 2    cl2 cp1 sfr)

                        + d (-12cl2 cp2 sfr - 6cl3 cp3 sfr + 6cl4 cp3 sfr

                                    1/2                     1/2
                                + 6 2    cl2 cp3 sfr + 6 2    cl3 cp2 sfr

                                    1/2
                                - 6 2    cl4 cp2 sfr)

                        + g (-4cl3 cp3 sfl - 4cl4 cp3 sfl))
#0[25]:  ------------------------------------------------------------------
                                       1/2
                                    24 6
```

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