TOWARDS A DESCRIPTION OF LOW–ENERGY HADRONIC PHYSICS USING THE SKYRME MODEL

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Abstract

We prove that the generalization of the Skyrme quartic lagrangian from flavor SU(2) to flavor SU(3) is not unique. Under the general assumptions of chiral symmetry, Lorentz invariance and restriction to two time–derivatives, there are two independent SU(3) forms. We apply within the framework of the Yabu and Ando approach, the new "alternate" lagrangian to a calculation of octet and decuplet baryon mass differences and find good agreement with experiment. We consider the alternate lagrangian to construct an eta–nucleon bound state model for the N(1535) resonance. Also, we have considered the problem of describing the S and P wave π–nucleon scattering within the framework of the Skyrme model. We go beyond the adiabatic approximation by considering the introduction of time–derivative interactions between pions and collective coordinates. A truncation scheme of unphysically open channels is introduced in K–matrix formalism and a unitary S–matrix is reconstructed. We compare our results with the Δ–isobar model and with phase shift analyses. Our calculation reproduces well the essential features of the P waves. For S waves, very attractive background contributions lead to poor agreement with the phase shift data.

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Dedication

A Fortunato et Maria pour leur soutien et leurs encouragements tout au long de cette aventure.

To Jack and Sally Nafe for their friendship.

> At Shelter Island the results of experiments [...] by Nafe, Nelson and Rabi[1] on the fine and hyperfine structure of hydrogen were presented. These precision experiments [...] indicated that deviations existed from the predictions of the Dirac equation for the spectrum of an electron in a Coulomb field.
>
> 1. J.E. Nafe, E.B. Nelson and I.I. Rabi, Phys. Rev. **71** (1947) 914.

from: S.S. Schweber, Rev. Mod. Phys. **58** (1986) 449.

Chapter 1

The SU(2) Skyrme Model

1.1 Foreword

When I began, in 1988, research in the field of the Skyrme model, my supervisor Dr. B.K. Jennings had words to this effect: "The Skyrme model is a mature field, but there should still be interesting things to do in this area!". Indeed, the ideas of the Skyrme model had been put forth by T.H.R. Skyrme in the late 1950's and early 1960's, and although they were literally forgotten by the particle and nuclear physics community for a good 20 years, there has been an important resurgence of activity following the works of the Syracuse [BNRS83] and Princeton [W83a] [W83b] [ANW83] groups in 1983. Since then, the Skyrme model has been a very hot topic of research, with well over 100 papers a year appearing in this field. As for the second part of Dr. Jennings' proposition, I let the reader who will persevere through this thesis make his (or her) own opinion about my work.

When Skyrme [S61] [S62] set out to construct the model that has come to be known by his name, he was mainly following theoretical guiding lines [S88]. The most important of which was that he strongly believed physical models should have classical mechanical analogues. In that sense, he quite disliked theories with fundamental fermionic degrees of freedom since half–integer spin fermions are quantum mechanical in nature. He felt that only the bosons, with integer spins, should be used as fundamental particles and therefore considered a theory where only spin 0 pi–meson fields entered.

Certainly, this was going against the grain of accepted theories of particle physics of the 1950's where the nucleons where introduced as fundamental fermion fields in the Dirac lagrangian, coupled covariantly to pions. We remind the reader that at the time of Skyrme's work, strangeness had just been discovered, and the corresponding particle zoo that would lead to Gell-Mann and Ne'eman's flavor SU(3) classification of particles was just emerging[1] [GN64]. But there was an important difficulty in the creation of a purely bosonic theory: although it is a simple matter to construct integer spin representations out of half–integer ones, viz. the familiar addition of angular momenta, for example $1/2 \oplus 1/2 = 1 \oplus 0$, it is quite unobvious how to proceed in the other case where integer representations are used to build half–integer ones. Skyrme, however, succeeded in such a scheme and in one of his papers [S58] on this subject, he was able to write: " The boson field has been fundamental, and the fermion field is brought in to describe some of the nonlinear effects ".

The above quotation reveals one of the basic theoretical ideas employed by Skyrme to create fermions out of bosons: nonlinearity. For some nonlinear differential equations, nonlinearity compensates dispersion in just such a way as to allow localized, extended and stable solutions to exist. They are called solitons[2]; they can move and scatter off each other without losing their identity. Since the discovery of solitons by J. Scott–Russell in 1834, –it has become textbook folklore [L81] to recount how he gave chase on horseback, for a couple of miles, to a soliton–like wave making its ways through the waters of winding English countryside channels– the concept has become of great importance to nonlinear science. The early, but exhaustive review of Scott *et al.* [SCM73] contains a discussion of several areas, from hydrodynamics to particle

[1] A discussion of flavor SU(3) symmetry can be found in chapter 2 of this thesis; certainly, the fact that the Skyrme model has anything interesting to say about the strange sector of baryons highlights Skyrme's unusual physical foresight.

[2] A precise definition of a soliton can be found in the textbook by Rajaraman [R82].

physics, where solitons have found useful applications.

The particular area that will interest us in this thesis pertains to the particle–nuclear physics interface. The Skyrme model purports to describe in a unified way the strong interactions of baryons and mesons [S61] [S62]. The mesons are taken as the fundamental punctual fields and the baryons are postulated to be the solitons of the theory. Indeed, the previous paragraph hinted at the particle–like character of the soliton, perhaps making such an identification plausible. In the next section, we will explicitly present, using the simplest version of the Skyrme model, how these ideas can be formulated in more precise mathematical and physical terms. A discussion of the spin–statistics of the "skyrmion"[3] will be given in subsection 2.2.3. At the risk of repeating ourselves, we emphasize that the crux of the Skyrme approach is that there need not be any fundamental fermions in the model.

It is believed by most physicists that Quantum Chromodynamics (QCD) is the fundamental theory of strong interactions. QCD is an $\mathrm{SU}(N_c = 3)_c$ color gauge field theory of Yang–Mills type; here N_c is the number of colors. The particle fields are quarks and gluons. Each quark flavor is labelled by the quantum numbers of one of the three states in the fundamental triplet representation of the exact $\mathrm{SU}(3)_c$ symmetry. The gluons are the force carriers of the theory, and are analogous to the familiar photon of electromagnetism. Unfortunately, however, QCD is intractable[4] in the long wavelength limit. Due to a diverging coupling constant, it is impossible to carry out perturbative calculations. In such a case, one must revert to model building, with the aim that the model reflect those characteristics of QCD believed to be important in the description of strong interaction observables. For example, in modelling static

[3]The name skyrmion was coined to the solitons of the Skyrme model by Pak and Tze in an early paper on chiral solitons [PT79]. This has become customary terminology.

[4]By discretizing QCD on a space–time lattice, approximate results can be obtained for the mass spectrum of hadrons as well as magnetic moments and charge radii of baryons. Their accuracy, however, is presently limited by available computer power. This is the method of Lattice Gauge Theory.

properties of baryons, a familiar approach consists in confining three massive quarks to a spherical bag[5] of radius R. Of course, this reflects the belief that QCD is a confining theory. The procedure gives rise to a few phenomenological parameters, in this example the bag radius R, that are chosen to reproduce experiment.

What resurrected the Skyrme model was the suggestion by Witten [W79], based on ideas put forth by 't Hooft [tH74a] [tH74b], that in the limit of large N_c QCD approximates to an effective theory of non–interacting mesons with baryons identified as the solitons of the theory. But this is just the type of model that Skyrme constructed! The other feature taken from QCD, that the Skyrme model embodies, is chiral symmetry.

Chiral symmetry can be presented by considering the quark (q) sector of the QCD lagrangian (we can omit in this case the gluons, as well as the color degrees of freedom)

$$L = \sum_{i=1}^{N_f} \bar{q}_i (i\gamma^\mu \partial_\mu - m_i) q_i, \tag{1.1}$$

where $\bar{q} = q^\dagger \gamma^0$, γ^μ are the Dirac matrices, $\partial_\mu \equiv \partial/\partial x_\mu$ and m_i is the mass of the ith quark in the sum over flavors. Throughout this thesis, the Minkowskian metric $g = diag(1, -1, -1, -1)$ is used and we will implicitly assume summation over repeated Lorentz indices $\mu = 0, 1, 2, 3$. Restricting the number of flavors N_f to 2, the case of up and down quarks, we see that in the limit $m_i \to 0$, the lagrangian 1.1 is invariant under the separate infinitesimal 2×2 flavor rotations

$$q \to q + i\frac{\vec{\tau} \cdot \vec{\epsilon}}{2} q, \tag{1.2}$$

$$q \to q + i\frac{\vec{\tau} \cdot \vec{\epsilon}'}{2} \gamma_5 q, \tag{1.3}$$

where $\vec{\epsilon}$, $\vec{\epsilon}'$, are infinitesimal parameters, $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, and $\vec{\tau}$ are the familiar Pauli matrices. Transformation 1.2 is called a vector (V) transformation and eq. 1.3 an axial

[5]For a review of quark bag models, including the refinement of introducing mesons coupled to the quarks, we refer to the paper of A.W. Thomas [T83].

vector (A) transformation. The terminology $R \equiv 1/2(V+A)$, for right transformations, and $L \equiv 1/2(V-A)$, for left transformations, will also be used. This $SU(2)_L \times SU(2)_R$ symmetry is called chiral symmetry. According to Pagels [P75], "chiral $SU(2)_L \times SU(2)_R$ is the most accurate hadron symmetry after isotopic invariance". So it is certainly a feature models of QCD should incorporate within their framework. But chiral symmetry is not obviously reflected in the particle spectrum: if it were, there should correspond to each hadron an opposite parity partner because the axial vector SU(2) has opposite parity to vector SU(2). The symmetry is then said to be hidden. It is spontaneously broken down to isospin SU(2) by the action of the group generators on the vacuum state: the axial charges do not annihilate the vacuum. For each of the broken generators of the group there arises a massless particle called a Goldstone boson [CL84]. In the case we are discussing, $SU(2)_L \times SU(2)_R \rightarrow SU(2)$, three Pauli matrices are lost implying the creation of three Goldstone bosons. In Nature, the Goldstone bosons of QCD are the pions. Indeed, this gives to the pions a rather fundamental role. A completely similar discussion holds for the case of a larger flavor space.

Armed with the knowledge that the Skyrme model is a possible physical realization of QCD in the long wavelength regime of strong interactions, we will henceforth put aside the discussion of the important issue of finding the precise link between the model and the theory. We will instead, in this thesis, investigate some of the phenomenological predictions the Skyrme model makes and compare them to the plentiful experimental data that exist for low–energy strong interaction physics. But before this, we introduce the Skyrme model.

1.2 A Basic Introduction to the Skyrme Model

The chiral group $SU(2)_L \times SU(2)_R$ possesses the same Lie algebra as the group SO(4) of 4–dimensional rotations. To construct a chirally invariant mesonic lagrangian, we can therefore consider the scalar products of 4–vectors of the form

$$(\sigma(t,\vec{x}),\ \vec{\pi}(t,\vec{x})). \tag{1.4}$$

σ is a scalar isoscalar sigma meson introduced as the chiral partner of the pseudoscalar isovector $\vec{\pi}$. Kinetics are introduced by taking derivatives of the fields. Restricting the model to low–energies, that is, to smallest number of derivatives, the unique 2–derivative chiral and Lorentz invariant lagrangian is

$$L^{(2)} = \frac{1}{2}\int d^3x\ \{\partial_\mu\sigma\partial^\mu\sigma\ +\ \partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi}\}. \tag{1.5}$$

The σ field is a rather fictitious particle. It therefore is appropriate to consider a constraint which forces the fields to lie on a 3–dimensional sphere of radius f_π:

$$\sigma^2 + \vec{\pi}\cdot\vec{\pi} = f_\pi^2. \tag{1.6}$$

This eliminates σ in favor of the the $\vec{\pi}$ triplet. The pion decay constant defines the energy scale of the model; experiment gives $f_\pi = 93.2$ MeV.

It is useful to reformulate the simple lagrangian eq. 1.5 and constraint eq. 1.6 in terms of the unitary matrix

$$\begin{aligned} U &= \frac{1}{f_\pi}(\sigma(t,\vec{x}) + i\vec{\tau}\cdot\vec{\pi}(t,\vec{x})) \\ &= e^{i\vec{\tau}\cdot\vec{\phi}(t,\vec{x})}. \end{aligned} \tag{1.7}$$

Here, $\vec{\tau}$ are the Pauli matrices. The three angles $\vec{\phi}$ are related to the chiral fields by

$$\sigma = f_\pi \cos|\vec{\phi}|, \tag{1.8}$$

$$\vec{\pi} = f_\pi \frac{\vec{\phi}}{|\vec{\phi}|} \sin|\vec{\phi}|, \tag{1.9}$$

where $|\vec{\phi}| = \sqrt{\vec{\phi}\cdot\vec{\phi}}$. The constraint eq. 1.6 then is just the unitarity condition for U

$$UU^\dagger = U^\dagger U = 1 \tag{1.10}$$

and the lagrangian eq. 1.5 takes the simple form

$$L^{(2)} = \frac{f_\pi^2}{4}\int d^3x\ tr\{\partial_\mu U \partial^\mu U^\dagger\}, \tag{1.11}$$

where tr denotes the trace of the matrix. We finally recast $L^{(2)}$ in the form that will subsequently be used in this thesis by introducing

$$\begin{aligned} \ell_\mu &= U^\dagger \partial_\mu U \\ &= i\sum_{a=1}^{3} \ell_{\mu a}(t,\vec{x})\tau_a \end{aligned} \tag{1.12}$$

for some real space–time dependent functions $\ell_{\mu a}(t,\vec{x})$. The second form in eq. 1.12 is understood by noticing that for a finite element U of the group, $U \approx 1 + i\vec{\tau}\cdot\vec{\epsilon}$ (ϵ infinitesimal), implying that to first order in ϵ, ℓ_μ must be proportional to $\vec{\tau}$. The cyclic property of the trace and $U^\dagger\partial_\mu U = -\partial_\mu U^\dagger U$ allow us to write

$$L^{(2)} = -\frac{f_\pi^2}{4}\int d^3x\ tr\{\ell_\mu \ell^\mu\}. \tag{1.13}$$

In the limit of small fields, $U \approx 1 + i\vec{\tau}\cdot\vec{\phi}$, $L^{(2)}$ reduces to the Klein–Gordon lagrangian and we identify the physical pion field with $f_\pi\vec{\phi}$. The simple lagrangian $L^{(2)}$ with constraint eq. 1.6 is the well–known nonlinear σ–model [GL60].

It is important to consider the properties of U under chiral symmetry transformations. A simple way to determine these is to require that the σ and $\vec{\pi}$ fields in eq. 1.7 transform in just such a way as to leave the meson–quark coupling

$$\sim \bar{q}(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5)q \tag{1.14}$$

invariant (see also eq. 1.1). If $R \in \mathrm{SU}(2)_R$ and $L \in \mathrm{SU}(2)_L$ are arbitrary unitary constant matrices, then global invariance under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ imposes that

$$U \to LUR^\dagger. \tag{1.15}$$

We further have, for a general SU(2) matrix A: purely axial rotations

$$U \to AUA \tag{1.16}$$

and purely vector, or isospin, rotations

$$U \to AUA^\dagger. \tag{1.17}$$

Under global chiral transformations,

$$\begin{aligned} \ell_\mu &\to RU^\dagger L(L^\dagger \partial_\mu U R^\dagger) \\ &= R\ell_\mu R^\dagger \end{aligned} \tag{1.18}$$

and $L^{(2)}$ is indeed a chiral invariant due to pairwise cancellation of $RR^\dagger$ and the cyclic property of the trace. The vacuum of the theory corresponds to U = a constant. Isospin rotations eq. 1.17 leave it invariant but axial rotations eq. 1.16 do not. This is symptomatic, as advertised in section 1.1, of spontaneous symmetry breaking. The pions are massless Goldstone bosons. But $m_\pi = 138$ MeV, a mass that is small (≈ 0) compared to other hadrons, but nevertheless finite. It is therefore customary to introduce a pion mass term which explicitly breaks chiral symmetry down to isospin symmetry. The choice, to agree with the familiar $1/2\ m_\pi^2 \vec{\pi} \cdot \vec{\pi}$ term in the small field limit, is taken to be

$$L^{(mass)} = \frac{m_\pi^2 f_\pi^2}{4} \int d^3x\ tr\{U + U^\dagger - 2\}. \tag{1.19}$$

We now turn to a discussion of the solitons of the mesonic theory. The field U in eq. 1.7 defines a mapping from usual space–time to the surface of a 4–dimensional sphere

S^3 in internal symmetry space. These mappings are topologically non–trivial in the sense that they are characterized by the number of times S^3 is covered. Now if U is to represent a soliton, it must be a finite energy configuration. This is achieved with $U \to$ constant, $U_{\pm\infty}$, as $|\vec{x}| \to \pm\infty$ (we consider only static fields; moving solutions can be obtained by performing a Lorentz boost). A continuous deformation of U that would interpolate between different vacua as $|\vec{x}| \to \pm\infty$ cannot occur because it would cost an infinite amount of energy. Hence, the soliton U is classified into different topological sectors according to the difference $U_{+\infty} - U_{-\infty}$.

In $1+1$ dimensions, these observations are quantified by defining a topological current

$$j_\mu = \frac{1}{\kappa}\epsilon_{\mu\nu}\partial^\nu\phi \tag{1.20}$$

which is trivially conserved: $\partial_\mu j^\mu = 0$. So the charge

$$Q = \int_{-\infty}^{+\infty} dx\ j_0 = \frac{1}{\kappa}(\phi_{+\infty} - \phi_{-\infty}) = n \tag{1.21}$$

equals an integer n called the winding number or topological index. Here κ is a normalisation factor and $\epsilon_{\mu\nu}$ is a 2–dimensional Levi–Civita tensor.

In $3+1$ dimensions, our usual space–time, Skyrme wrote down the form the topological current should take:

$$B^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\ tr(\ell_\nu\ell_\rho\ell_\sigma). \tag{1.22}$$

This is conserved by virtue of the cyclic property of the trace

$$\partial_\mu B^\mu = -\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\ tr(\ell_\mu\ell_\nu\ell_\rho\ell_\sigma) = 0. \tag{1.23}$$

We wish to emphasize that these topological currents are fundamentally different from

the perhaps more familiar Noether currents[6] which are associated with continuous symmetries of the lagrangian. The topological charge is

$$B = \int d^3x B^0. \tag{1.24}$$

The bold step that Skyrme took was to identify the topological charge eq. 1.24 with the physical baryon number. By trial and error, he arrived at a topologically non–trivial field configuration with one unit of baryon number:

$$U_0 = e^{i\vec{\tau}\cdot\hat{r}F(r)}, \tag{1.25}$$

where $\hat{r}$ is a unit radial vector and $F(r)$ is a function to be determined variationally. $F(r)$ is often referred to as the chiral angle. Most importantly, $F(r)$ satisfies the boundary conditions

$$\begin{aligned} F(0) &= \pi, \\ F(\infty) &= 0. \end{aligned} \tag{1.26}$$

The form eq. 1.25 for U has come to be known as the hedgehog due to the fact that the field $\vec{\phi} = \hat{r}F(r)$ points in a radial direction at all spatial points. The picture is that of a spiny sphere, just like a rolled–up hedgehog.

We are now beginning to have an outline of the Skyrme model, with the lagrangian $L^{(2)} + L^{(mass)}$ and the hedgehog baryon–like solution eq. 1.25. There remain, however,

[6]For the lagrangian $L = \int d^3x\mathcal{L}(\phi_i(t,\vec{x}), \partial_\mu\phi_i(t,\vec{x}))$, the Noether current is

$$j^\mu = \sum_{i=1}^{n} \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\delta\phi_i,$$

a result derived with the use of the Euler–Lagrange equation of motion

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\right) - \frac{\delta\mathcal{L}}{\delta\phi_i} = 0.$$

$\delta\phi_i$ is an infinitesimal transformation which leaves L invariant (see the textbook of Cheng and Li [CL84] for more details).

two questions that we now consider. The first pertains to the fact that with $L^{(2)}+L^{(mass)}$ alone, there exists no variational solution $F(r)$ satisfying the boundary conditions eq. 1.26. This can be understood by considering the scaling properties ($\vec{x} \to \lambda\vec{x}$) of the (static) hamiltonian

$$\begin{aligned} H^{(2)} + H^{(mass)} &= -(L^{(2)} + L^{(mass)}) \\ &\to \lambda H^{(2)} + \lambda^3 H^{(mass)}. \end{aligned} \tag{1.27}$$

The minimum energy of the system, namely zero, will occur for $\lambda = 0$ since $H^{(2)}$ and $H^{(mass)}$ are positive. Hence the soliton collapses to zero size.

The way to circumvent this difficulty was proposed by Skyrme. If a 4–derivative term is introduced in the lagrangian, the hamiltonian

$$H^{(2)} + H^{(4)} \to \lambda H^{(2)} + \frac{1}{\lambda} H^{(4)} \tag{1.28}$$

will clearly have a minimum for

$$\lambda = \sqrt{\frac{H^{(4)}}{H^{(2)}}}. \tag{1.29}$$

The 4–derivative term stabilizes the soliton to a finite spatial extension; the form chosen by Skyrme is

$$L^{(4a)} = \frac{1}{32e^2} \int d^3x \; tr([\ell_\mu, \ell_\nu]^2), \tag{1.30}$$

where e is a free parameter and the commutator $[\ell_\mu, \ell_\nu] = \ell_\mu \ell_\nu - \ell_\nu \ell_\mu$. This is the unique 4–derivative $SU(2)_L \times SU(2)_R$ chiral invariant lagrangian with only two time–derivatives[7].

[7]This result is proven in section 2.4, eq. 2.62. For $SU(3)_L \times SU(3)_R$, this "Skyrme term" is not unique, a point we will expound in chapter 2.

We have arrived at the basic version of the Skyrme model. To recapitulate:

$$\begin{aligned} L &= L^{(2)} + L^{(4a)} + L^{(mass)}, \\ L^{(2)} &= -\frac{f_\pi^2}{4}\int d^3x \quad tr\ell_\mu\ell^\mu \quad , \qquad \ell_\mu = U^\dagger\partial_\mu U \quad , \\ L^{(4a)} &= \frac{1}{32e^2}\int d^3x \quad tr[\ell_\mu,\ell_\nu][\ell^\mu,\ell^\nu] \quad , \qquad U \in SU(2) \quad , \\ L^{(mass)} &= \frac{f_\pi^2 m_\pi^2}{4}\int d^3x \quad tr(U+U^\dagger-2) \quad , \end{aligned} \tag{1.31}$$

and the field U is taken to be of the hedgehog form (eq. 1.25)

$$\begin{gathered} U = \cos F(r) + i\vec{\tau}\cdot\hat{r}\sin F(r), \\ F(0) = \pi; \quad F(\infty) = 0. \end{gathered} \tag{1.32}$$

The energy of the system corresponds to the mass of the soliton M^{cl}. Evaluating the trace in eq. 1.32 yields

$$\begin{aligned} &M^{cl} = H = -L = \\ &4\pi\int_0^\infty dr r^2 \ \{ \ \frac{f_\pi^2}{2}[F'^2 + 2\frac{\sin^2 F}{r^2}] + \frac{\sin^2 F}{2e^2r^2}\ [2F'^2 + \frac{\sin^2 F}{r^2}] + m_\pi^2 f_\pi^2[1-\cos F]\}, \end{aligned} \tag{1.33}$$

The profile $F(r)$ is determined by solving the Euler–Lagrange equation of motion (see footnote 6 in this chapter) subject to the appropriate boundary conditions. The resulting second order nonlinear differential equation of motion satisfied by $F(r)$ is

$$\begin{aligned} &(\frac{1}{4}x^2 + 2\sin^2 F)\ F'' + \frac{1}{2}xF' \\ &\quad + \sin(2F)F'^2 - \frac{1}{4}\sin(2F) - \frac{2}{x^2}\sin^3 F\cos F - \frac{1}{4}\tilde{m}_\pi^2 x^2 \sin F = 0, \end{aligned} \tag{1.34}$$

where we have introduced the dimensionless radial variable $x = e2f_\pi r$ and dimensionless mass $\tilde{m}_\pi = m_\pi/(e2f_\pi)$. F' and F'' denote the first and second derivatives of F with respect to the argument. This equation is solved numerically. We have shown in figure 1.1 the chiral angle and its first two derivatives for the choice of parameters

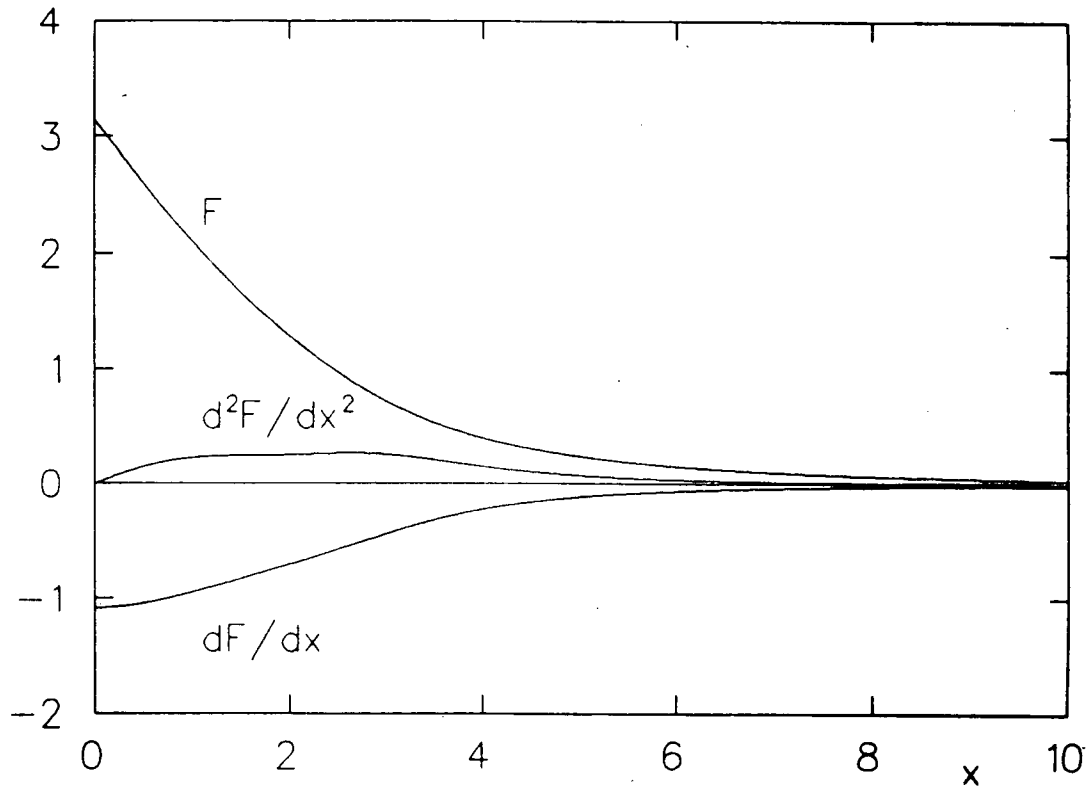

Figure 1.1: Chiral angle and its first two derivatives obtained by solving eq. 1.35 numerically for the choice of parameters $f_\pi = 93$ MeV, $e = 4$, $m_\pi = 138$ MeV. The abscissa is the dimensionless radius $x = e2f_\pi r$.

$f_\pi = 93$ MeV, $e = 4$ and $m_\pi = 138$ MeV (we will use this set of parameters quite frequently, especially in our π–N scattering work presented in chapter 3).

The second question to which we now turn deals with the quantum numbers of the soliton. As is clear from our discussion, we have only considered classical variables. Indeed, the hedgehog is a purely classical object[8]. So we cannot expect it to be an eigenstate of isospin and spin as physical particles are. The way to resolve this problem starts by noticing that the hedgehog remains invariant under the peculiar combination of infinitesimal isospin transformation $\vec{I} = [\vec{\tau}/2, \ ]$ (for $A = exp(\frac{i}{2}\vec{\tau}\cdot\vec{\epsilon})$ in eq. 1.17) and angular momentum transformation $\vec{s} = -i\vec{r}\times\vec{\nabla}$:

$$(\vec{I} + \vec{s})U_0 = 0. \tag{1.35}$$

We say that the hedgehog is a singlet under the "extended spherical" symmetry operator

$$\vec{K} = \vec{I} + \vec{s}. \tag{1.36}$$

$\vec{K}$ is also referred to as the "grand spin". Eq. 1.35 implies that spatial rotations are equivalent to isospin transformations. Another way to understand this is

$$\begin{aligned} AU_0A^\dagger &= e^{iA\vec{\tau}A^\dagger\cdot\hat{r}F(r)} \\ &= e^{i\vec{\tau}D^{(1)\dagger}\cdot\hat{r}F(r)} \end{aligned} \tag{1.37}$$

where

$$D^{(1)\dagger}_{ab} = \frac{1}{2}tr(\tau_a A\tau_b A^\dagger) \tag{1.38}$$

are elements of the familiar rotation matrices; in this case in the adjoint representation. For the SU(2) D–matrices, we use the convention of Holzwarth [H90]. From eq. 1.35, we conclude that for the hedgehog, $\vec{I}\cdot\vec{I} = \vec{s}\cdot\vec{s}$.

[8]In this sense, the Skyrme model differs markedly from the traditional particle physics picture of particles as quantum mechanical bound states of fundamental fields.

The previous discussion suggests that to adorn the soliton with good spin and isospin quantum numbers, we should quantize the variable A. This first requires that A be promoted to the role of a time–dependent variable $A(t)$: it parametrizes the collective rotations of the hedgehog. Substituting the new ansatz

$$U = A(t)U_0A(t)^\dagger \tag{1.39}$$

in the Skyrme lagrangian 1.32, we have:

$$L = -M^{cl} + \Theta_\pi \; tr(\dot{A}\dot{A}^\dagger), \tag{1.40}$$

where M^{cl} is given in eq. 1.34 and Θ_π is a model–dependent moment of inertia

$$\Theta_\pi = \int d^3x \; \frac{2\sin^2 F}{3}(f_\pi^2 + \frac{1}{e^2}(F'^2 + \frac{\sin^2 F}{r^2})). \tag{1.41}$$

As for the vector current –identified to the isospin I_a– and the spin s_a –identified from the appropriate components of the energy momentum tensor–, they take the simple forms

$$s_a = -i\Theta_\pi \; tr(\tau_a A^\dagger \dot{A}), \tag{1.42}$$

$$I_a = -i\Theta_\pi \; tr(\tau_a A\dot{A}^\dagger). \tag{1.43}$$

A suitable parametrization of the independent components of A is then introduced. In the approach to quantization due to Adkins *et al.* [ANW83],

$$A(t) = a_0(t) + i\vec{\tau}\cdot\vec{a}(t); \quad a_0^2 + \vec{a}\cdot\vec{a} = 1 \tag{1.44}$$

and the canonical quantization prescription is applied by identifying the conjugate momentum

$$\pi_i = \frac{\delta L}{\delta \dot{a}_i} \tag{1.45}$$

with

$$\pi_i = -i\frac{\partial}{\partial a_i}. \tag{1.46}$$

The hamiltonian, spin current eq. 1.42 and isospin current eq. 1.43 are now quantum mechanical operators acting on baryon wavefunctions which are polynomials in a_0, $\vec{a}$. In this thesis, we take the slightly different approach of reference [HS86b] in choosing as three independent collective coordinates the familiar Euler angles α, β, γ. In the problem at hand, the Euler angles define finite rotations between a frame fixed to the soliton and a frame fixed in space. This latter frame will be referred to as the "laboratory frame". The body–fixed frame, which is rotating in space–isospace (viz. eq. 1.35), will be referred to as the "intrinsic frame". A similar case of transformations between frames is familiar to nuclear physicists who consider the properties of deformed nuclei. The collective coordinates $A = A(\alpha, \beta, \gamma)$ are now explicit functions of the Euler angles. If we define the collective velocity as

$$\Omega_a = -i\ tr(\tau_a A^\dagger \dot{A}), \tag{1.47}$$

then the hamiltonian for the time–dependent model takes the following form:

$$\begin{aligned} H &= \sum_{a=1}^{3} \frac{\delta L}{\delta \Omega_a}\Omega_a - L, \\ &= M^{cl} + \frac{\Theta_\pi}{2}\sum_{a=1}^{3}\Omega_a^2. \end{aligned} \tag{1.48}$$

This becomes, upon using eq. 1.42 or eq. 1.43,

$$\begin{aligned} H &= M^{cl} + \frac{1}{2\Theta_\pi}\vec{s}\cdot\vec{s}, \\ &= M^{cl} + \frac{1}{2\Theta_\pi}\vec{I}\cdot\vec{I}, \end{aligned} \tag{1.49}$$

which is just the familiar hamiltonian for a spherical top. Its quantization is a standard

exercise in quantum mechanics, leading to the rotational spectrum

$$\begin{aligned} H &= M^{cl} + \frac{1}{2\Theta_\pi} s(s+1) \\ &= M^{cl} + \frac{1}{2\Theta_\pi} I(I+1). \end{aligned} \tag{1.50}$$

The properly normalized quantum mechanical wavefunction for the baryons is

$$\Psi^B_{(IM_I)(s=I,M_s)}(\alpha,\beta,\gamma) = (-)^{I+M_I}\sqrt{\frac{2I+1}{8\pi^2}}\, D^{(I)}_{M_s,\ -M_I}(\alpha,\beta,\gamma), \tag{1.51}$$

where M_s, M_I are respectively the z–components of the spin and isospin quantum numbers of the baryon state. We again follow the convention of the Siegen group for the labelling of this state [HEHW84].

We emphasize that the rotational band eq. 1.50 contains only states which satisfy $|\vec{I}| = |\vec{s}|$. An important point, which is however left arbitrary in the SU(2) model [FR68], is the spin, or isospin of the ground state. One can actually choose $s = I = 0$, in which case the baryons are bosons and $\Psi^B(-A) = \Psi^B(A)$, or choose $s = I = 1/2$, in which case the baryons are fermions and $\Psi^B(-A) = -\Psi^B(A)$. It is, of course, the second choice[9] that is adopted, with the physical nucleon $(I,s) = (1/2, 1/2)$ identified with the ground state and the Δ isobar $(I,s) = (3/2, 3/2)$ identified with the first excited state of the rotational band. Higher rotational levels, such as the $(I,s) = (5/2, 5/2)$, are considered as exotics[10].

We have tried to present the Skyrme model in a brief but clear way. Our presentation contains those points that are important to a comprehension of the next two chapters of this thesis. Each of these chapters begins with further introductory comments, now specific to the work they present. For other details of the Skyrme model which are not discussed in this thesis, we refer the reader to the many good review articles that

[9] If one enlarges the flavor symmetry group from SU(2) to SU(3), then the soliton must be quantized as a fermion [W83b]. We will outline how this comes about in subsection 2.2.3 of the thesis.

[10] We will have more to say about the exotic $\frac{5}{2}^*$ state in chapter 3.

exist in the field. Our favorite one is the beginner's level, but oh so useful!, review of Adkins [A87]. In particular, one can find there a test of the Skyrme model in its predictions of the properties of SU(2) baryons, the nucleon and the Δ isobar. Without entering details, the verdict is that the Skyrme model reproduces charge radii, magnetic moments, etc. to an overall accuracy of 30%. This is quite reasonable in view of the fact that only f_π and e are adjusted to the data, in this case to reproduce the nucleon and Δ masses. An intermediate level presentation is that of Holzwarth and Schwesinger [HS86b] and an advanced level one can be found in Zahed and Brown [ZB86] or Schwesinger *et al.* [SWHH89]. These papers give a broad overview of the physics questions that have been addressed within the framework of the Skyrme model, including the many refinements to the model that have been considered.

1.3 What Is New in this Thesis...

This brief section may serve as a Statement of Originality for our work. In chapter 2 of this thesis, the reader will find our contributions to the SU(3) sector of the Skyrme model. We have observed that within the context of the Skyrme model, the lagrangian based on a gradient expansion of the chiral field that has been used by workers in the field is not unique. There exists a second possibility, introducing no new assumptions beyond those that were already used: chiral symmetry, Lorentz invariance and two time–derivatives. After proving our claim, we apply the "alternate" lagrangian to a calculation of the SU(3) octet and decuplet baryon mass spectra. We will find that some improvement is achieved when the alternate form is used instead of the usual Skyrme stabilization term. We also apply the alternate lagrangian to the following question: the role of the isoscalar eta meson in the Callan–Klebanov approach to the Skyrme model. The Callan–Klebanov model creates the hyperon spectrum of baryons by binding kaons

to SU(2) skyrmions. The motivation for this model rests on the assumption that SU(3) flavor symmetry is badly broken, i.e. $m_K >> m_\pi$. The same criterion applies for the mass of the eta, but this point seems to have received very little attention in the literature. Using the Skyrme lagrangian, the dynamics of the eta field are trivially described by the Klein–Gordon lagrangian. But for the alternate lagrangian, this is no longer the case. The eta couples in a non–trivial way to the hedgehog and there exists an S wave bound state for a range of values of a new constant x parametrizing the strength of the coupling. We suggest that in some qualitative ways, this bound state can be identified with a know nucleon resonance, N(1535), which until now has been difficult to describe in the context of the Skyrme model.

In chapter 3 we turn to our work on low–energy S and P wave π–N scattering in the framework of the SU(2) Skyrme model. We address the well–known difficulties that the model has had in describing these partial waves. Here, we quote Skyrme [S62], who had remarkably predicted the problems[11] that would later plague the Siegen and Stanford π–N scattering calculations:

> [The] P wave meson–particle interaction [is] repulsive on the average. There is no indication of the strong attraction observed in the pion–nucleon resonant state [Δ resonance], but this would hardly be expected in a static classical treatment where the rotational splitting of the particle states has been ignored.

In our calculation, we consider such "rotational splittings" by including in the S–matrix the plane wave Born approximation contributions coming from time–derivative interactions between the skyrmion and the meson fluctuations. We find it necessary to implement further approximations in order to carry out the complete calculation. In

[11] and partly suggested how to resolve them.

particular, we close unphysically open channels in the background amplitudes (those obtained from scattering a meson off a static hedgehog) and construct the scattering S–matrix from the K–matrix in a piecewise way. For the P waves, this method obtains good results: we now have the delta resonance appearing in the P_{33} channel and at the right energy value; we find strong attraction in the P_{11} channel and therefore account for the Roper resonance. Finally, we reproduce the splitting of the P_{13} and P_{31} channels. This calculation represents the first semi–quantitative calculation of P wave π–N phase shifts in the context of the Skyrme model, for energies up to approximately 300 MeV above threshold. Our results justify a posteriori some of the approximations we have implemented. In the case of the S waves, the situation is not as nice. Here, it seems that using the plane wave Born approximation to calculate contributions from time–derivative interactions is too drastic an approximation. Although we can reproduce the isospin splitting between the S_{11} and S_{31} adequately by considering only time–derivative interactions, we find that inclusion of the background contributions ruins the agreement.

The appropriate acknowledgement to previous work in the field will be found in chapters 2 and 3. In particular, acknowledgement of the colleagues with whom I have worked is done by providing the references to our joint papers.

Chapter 2

Investigation of Topics Concerning the SU(3) Skyrme Model

2.1 Introduction

A description of low– and medium– energy hadronic physics must go beyond the familiar nucleon, delta and pion to include particles endowed with an additional degree of freedom called strangeness[1] [G53][NN53]. Strangeness is observed to be conserved by the strong interactions and it is therefore a good quantum number to use in labelling particle states. Given the definition of the strangeness of a particle as

$$S = 2(Q - I_3) - B, \tag{2.1}$$

where Q is the charge, I_3 the third component of isospin and B the baryon number, it is useful to introduce the concept of hypercharge as

$$Y = B + S. \tag{2.2}$$

The hypercharge Y and isospin $\vec{I}$ are incorporated together in the symmetry group of 3–dimensional unitary unimodular transformations SU(3); this is the flavor symmetry group of strong interactions [G62][N61]. It leads to the often pictured octet and decuplet representations of elementary particles. The particles forming the lowest–lying baryon octet and decuplet can be seen in figure 2.1. The lowest–lying meson octet consists of the pions, the kaons and antikaons and the eta particles.

In the Skyrme model, the extension to the SU(3) flavor symmetry sector proceeds through the enlargement of the chiral group $\mathrm{SU_L(N_f)} \times \mathrm{SU_R(N_f)}$ by letting the number

[1] We consider the other flavors: bottom, charm, top, as belonging to the realm of high energy physics.

of flavors N_f go from 2 to 3. This larger group is spontaneously broken to the flavor group SU(3) by the choice of vacuum $U = 1$. The symmetry breaking generates 8 massless Goldstone bosons, namely the three pions, two kaons and two antikaons and an eta, each corresponding to a broken generator of the group. The nonlinear Skyrme lagrangian is now described in terms of 8 fundamental fields, the dynamics of which give rise to the soliton solutions on which is based the description of the baryon sector.

The enlarged chiral symmetry group imposes modifications on the Skyrme model. The simple hedgehog solution described in chapter 1 must be changed in order to now reside in SU(3) space [W83b]. As for the Skyrme lagrangian, there are two new ingredients that must be taken into account. The first is that to ensure invariance of the lagrangian under the correct parity transformation $\mathcal{P}U(\vec{x}) = U^\dagger(-\vec{x})$ which explicitly takes into account the pseudoscalar nature of the meson octet field U, a five derivative term, the Wess–Zumino term, must be introduced [W83a][WZ71]. This term has profound physical consequences in the theory, providing a criterion indicating whether the soliton must be quantized as a fermion or as a boson. It also leads to a constraint from which a spin 1/2 baryon octet and a spin 3/2 baryon decuplet arise, in agreement with experiment, as the lowest–lying baryon representations.

The second ingredient is that the extension of the 4 derivative Skyrme stabilization term is not unique [PSW91][PEGJ90]. This is the observation we describe in this chapter and is new to the field. Imposing the general assumption of invariance under chiral and Lorentz transformations and the requirement that there be at most two time derivatives in the lagrangian, we demonstrate that there are two, and only two, acceptable fourth order forms. The proof follows the presentation of the paper by Pari, Eisenberg, Gal and Jennings [PEGJ90]. There is, of course, the usual Skyrme term, but now supplemented by a new term to which we refer to as the "alternate"

lagrangian. This generalized lagrangian finds support from Chiral Perturbation Theory [GL84][GL85], a framework constructed out of a gradient expansion of chirally invariant terms, with symmetry breaking terms treated as perturbations. Indeed, both the Skyrme and alternate lagrangians can be extracted through a recombination of the parameters appearing in the lagrangian of Gasser and Leutwyler [GL85]. This important result, that there are two acceptable fourth order stabilization forms, seems, however, to have been overlooked in the context of soliton models.

At this stage, it is perhaps useful to do a brief digression to explain how the SU(3) Skyrme model evolved from providing quite disappointing results for the baryon mass spectrum to providing quite good results. All the problems and their resolution are rooted in the treatment of the flavor symmetry breaking terms (breaking chiral symmetry represents no problems in this context, it gives rise to classical contributions to octet and decuplet average masses). The initial calculations of baryon properties in SU(3) took the reasonable approach of assuming that flavor symmetry was a good symmetry, whose breaking could be treated to first order perturbatively [G83][P85][C85]. After all, one obtains the famous Gell-Mann–Okubo mass relation (which is satisfied to better than 1%)

$$2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma, \tag{2.3}$$

by evaluating in first order perturbation theory the effect of a flavor symmetry breaking term proportional to the eighth component (hypercharge) of an octet. Following this approach to treating flavor symmetry breaking in the Skyrme model, it was unfortunately found that the overall mass splitting amongst both the octet members (47 MeV as compared to the experimental $M_\Xi - M_\Lambda = 380$ MeV) and the decuplet members (36 MeV as compared to the experimental $M_\Omega - M_\Delta = 440$ MeV) was much smaller

than required [P85]. And these disappointing results[2] coming from a "best fit" to the data imposed an extremely small pion decay constant $f_\pi = 23$ MeV (experimental value equals 93 MeV). Not surprisingly, the Gell-Mann–Okubo relation is, however, well satisfied by the poor calculated values.

It was immediately realized that the resolution of the above mentioned difficulties proceeded through a better treatment of the flavor symmetry breaking. Obviously, the symmetry breaking is too small; since such terms do not come with free parameters (meson masses and decay constants that parametrize them are fixed at their experimental values) that can be dialed to fit the spectrum, new calculational techniques had to be considered. Chronologically, the method due to Callan *et al.* [CK85][CHK88]: the Bound State Approach to Strangeness in the Skyrme Model, came first. The main assumption in this framework is to take $m_K >> m_\pi$ (equivalently, $m_s >> m_u = m_d$ in quark language), which corresponds to the case of badly broken flavor symmetry. Solitons are constructed out of the nonlinear pion dynamics in the SU(2) (nonstrange) sector with small kaon vibrations introduced about the SU(2) background soliton. The hyperons are interpreted as kaon–soliton bound states. Of crucial importance here is the role that the Wess-Zumino term plays: attractive for antikaons and repulsive for kaons, leading to the correct hyperon spectroscopy.

The second method is due to Yabu and Ando [YA88]. Here, the flavor symmetry breaking terms are treated exactly. This is achieved by introducing eight Euler angles to parametrize the SU(3) collective coordinates and subsequently diagonalizing the hypercharge dependent part of the collective SU(3) hamiltonian. Equivalently, more recent treatments expand the baryonic wavefunction in terms of SU(3) symmetric wavefunctions, and the eigenvalue problem reduces to a matrix diagonalization [KLL89]. The

[2]For a complete set of references dealing with the SU(3) symmetric Skyrme model, see the paper of Masak [M89].

eigenvalues obtained for each baryon state are added to the isospin invariant baryon mass relation to represent the effects of the flavor symmetry breaking. The results of the exact diagonalization have been well reproduced by calculating to second order in the flavor perturbation, with third order effects found to be negligible [PSW89]. Importantly, cancellation of large second order contributions preserve the form of the Gell-Mann–Okubo relation.

In this chapter, we will discuss the results of two calculations centered around phenomenological applications of the alternate Skyrme lagrangian. So as to make no enemies[3], one will be within the framework of the Yabu and Ando approach and the other, the Callan–Klebanov approach. In collaboration with Schwesinger and Walliser [PSW91], we have investigated the consequences the alternate lagrangian has on the baryon mass splittings spectrum when it replaces the Skyrme term. The usual flavor symmetry breaking terms are treated exactly following Yabu and Ando; we find it crucial, however, to introduce additional terms, which lift the degeneracy $f_\pi = f_K = f_\eta$ usually assumed in such calculations, in order to get good results. Since the numerical work is due to the Siegen part of the collaboration, we present results after a brief outline of the ingredients that enter in the calculation. We will see that baryon octet and decuplet mass splittings are well reproduced, showing improvement over the usual Skyrme model.

In our second calculation, we study, along lines similar to those developed by Callan and Klebanov for the treatment of the strange degrees of freedom, the question of including the eta meson[4] alongside the kaons. To this end, we introduce the alternate lagrangian which provides a coupling between the skyrmion and the eta field. In the usual Skyrme lagrangian, no such coupling exists, perhaps explaining why the role of the

[3]Or alternatively, please no one.

[4]By eta meson, we mean the isoscalar member of the meson octet. This does not correspond to the physical meson.

eta received so little attention in previous calculations. Indeed, it is while considering this question that we were led to introduce the alternate lagrangian. We find an S wave eta–skyrmion bound state and quantize its rotational excitations. The lowest rotational band member is qualitatively identified as the N(1535) nucleon resonance. This work contains three new contributions: the observation that the new–found generality in the model should be applied to the Callan–Klebanov version of the Skyrme model; a first thorough study of the role of the eta meson in the context of this model; and the demonstration that there exists a bound state that in some qualitative ways can be identified with a known nucleon resonance. This work was presented in the paper "The Role of the Eta Meson in the Callan–Klebanov Approach to the Skyrme Model" by Pari [P91].

2.2 The SU(3) Skyrme Model

The obvious generalization that must be carried out in the extension of the Skyrme model from flavor SU(2) to flavor SU(3) is that of the hedgehog ansatz. Whereas before it was constructed in terms of pions, it must now reside in the SU(3) sector. Following Witten [W83b] and Balachandran *et al.* [BLRS85], the requirement that the SU(3) soliton $U_0^{[SU(3)]}$ satisfies the extended spherical symmetry is imposed:

$$-i\ (\vec{r}\times\vec{\nabla})\ U_0^{[\mathrm{SU}(3)]}(\vec{x}) + [\ \vec{G}, U_0^{[\mathrm{SU}(3)]}(\vec{x})\ ] = 0, \tag{2.4}$$

where $\vec{G}$ are three SU(3) generators spanning any SU(2) subgroup of SU(3) and satisfying the Lie algebra

$$[G_i, G_j] = i\sum_{i=1}^{3}\epsilon_{ijk}G_k. \tag{2.5}$$

When $U_0 \subset$ SU(2) and $\vec{G} \equiv \vec{\tau}/2$, the Pauli matrices, eq. 2.4 states, as we have seen in eq. 1.35, that the hedgehog is a singlet under $\vec{K}$ transformations. Constraint eq. 2.4,

along with the fact that we want a baryon number one solution, leads to the result that $U_0^{[\mathrm{SU}(3)]}$ is simply the embedding of the SU(2) hedgehog in SU(3):

$$\begin{aligned} U_0^{[\mathrm{SU}(3)]} &= e^{i\sum_{i=1}^{3}\lambda_i\hat{r}_iF(r)} \\ &= \begin{pmatrix} U_0^{[\mathrm{SU}(2)]}(\vec{x}) & 0 \\ 0 & 1 \end{pmatrix} \\ U_0^{[\mathrm{SU}(2)]} &= e^{i\sum_{i=1}^{3}\tau_i\hat{r}_iF(r)}. \end{aligned} \tag{2.6}$$

Here, λ_i, $i = 1, 2, 3$, are the first three of eight Gell-Mann matrices [GN64] denoting the fundamental representation of SU(3). They satisfy the commutation and anticommutation relations

$$[\lambda_\alpha, \lambda_\beta] = 2i\sum_{\gamma=1}^{8} f_{\alpha\beta\gamma}\lambda_\gamma \tag{2.7}$$

$$\{\lambda_\alpha, \lambda_\beta\} = 2\sum_{\gamma=1}^{8} d_{\alpha\beta\gamma}\lambda_\gamma + \frac{4}{3}\,\delta_{\alpha\beta} \tag{2.8}$$

with $1 \leq \alpha, \beta, \gamma \leq 8$. They are traceless 3×3 hermitian matrices normalized according to $\frac{1}{2}tr(\lambda_\alpha\lambda_\beta) = \delta_{\alpha\beta}$. The constants $f_{\alpha\beta\gamma}$, $d_{\alpha\beta\gamma}$ are respectively totally antisymmetric and symmetric structure constants for SU(3).

In the symmetric case, constant flavor transformations $A \in \mathrm{SU}(3)$

$$U_0 \to AU_0A^\dagger;\ AA^\dagger = A^\dagger A = 1,$$

generate eight zero modes. It is the quantization of these zero modes[5] that produces the "rotational" spectrum of baryon states. We put rotational in quotations since we mean SU(3) rotations. Accordingly, the eigenspectrum will be labelled by the isospin and the hypercharge quantum numbers in a given representation of the flavor group. Further comments concerning the quantum numbers, in particular the interrelation between

[5]For a definition of zero modes, see section 3.3.

spin and isospin of the baryons, which are no longer equal in SU(3) in contradistinction to the SU(2) case, must await the explicit introduction of the lagrangian (viz. the Wess–Zumino term). As in the SU(2) case, the constant matrix A is promoted to the role of collective coordinate by making it time–dependent. There are now 8 collective velocities Ω_α. A particularly useful form to represent these is

$$\Omega_\alpha = -i\ tr(\lambda_\alpha A^\dagger \frac{\partial A}{\partial t}); \quad \alpha = 1, ..., 8, \tag{2.9}$$

which expresses the fact that $A^\dagger \partial A/\partial t$ is spanned by the generators of the SU(3) Lie algebra. So the SU(3) Skyrme model ansatz is

$$U^{[\mathrm{SU(3)}]}(\vec{x}, t) = A(t)\ U_0^{[SU(3)]}(\vec{x})\ A^\dagger(t). \tag{2.10}$$

This form, with 8 collective coordinates, assumes that the deformation due to the kaonic components of the classical solution is not exceedingly large. This assumption is common to both the SU(3) symmetric approach and the Yabu and Ando approach, although the treatment of the explicit symmetry breaking terms is different. In the opposite case, where the assumption is made that on typical strong interaction scales the kaon mass is much larger than that of the pions, the rotator approach ceases to be valid. Modifications to eq. 2.10 were suggested by Callan and Klebanov: only the isospin symmetry is treated with collective coordinates, whereas strangeness degrees of freedom are carried by kaons vibrating about the SU(2) skyrmion:

$$U = \sqrt{U_0}\ U_K\ \sqrt{U_0}, \tag{2.11}$$
$$U_K = e^{\frac{i}{f_\pi}\sum_{a=4}^{7} \lambda_a K_a(\vec{x},t)},$$

with $K_a(\vec{x}, t)$ a space–time dependent profile determined by minimization of the energy. Alternatively, the form symmetric in kaon fluctuations [BDR89]

$$U = \sqrt{U_K}\ U_0\ \sqrt{U_K} \tag{2.12}$$

leads to the same physical results. For our work in the Callan–Klebanov approach, we choose the symmetric expression eq. 2.12 to carry out calculations.

The standard SU(3) Skyrme model lagrangian is taken to be:

$$\begin{aligned}
L &= L^{(2)} + L^{(4a)} + L^{(mass)} + L^{(WZ)} + L^{(SB)} \quad , \\
L^{(2)} &= -\frac{f_\pi^2}{4}\int d^3x \quad tr\ell_\mu\ell^\mu \quad , \qquad \ell_\mu = U^\dagger\partial_\mu U \quad , \\
L^{(4a)} &= \frac{1}{32e^2}\int d^3x \quad tr[\ell_\mu,\ell_\nu][\ell^\mu,\ell^\nu] \quad , \qquad U \in SU(3) \quad , \\
L^{(mass)} &= \frac{f_\pi^2 m_\pi^2}{4}\int d^3x \quad tr(U+U^\dagger-2) \quad , \\
S^{(WZ)} &= \frac{iN_c}{240\pi^2}\int_Q d\Sigma^{\mu\nu\lambda\rho\sigma}\ tr(\ell_\mu\ell_\nu\ell_\lambda\ell_\rho\ell_\sigma) \quad , \\
L^{(SB)} &= \frac{f_\pi^2}{6}(m_K^2 - m_\pi^2)\int d^3x \quad tr(1-\sqrt{3}\lambda_8)(U+U^\dagger-2).
\end{aligned} \tag{2.13}$$

The 2–derivative term $L^{(2)}$ and the 4–derivative term $L^{(4a)}$ just repeat the usual SU(2) expressions. Comments concerning the symmetry breaking $L^{(mass)}+L^{(SB)}$ are presented in a subsection below. Upon substitution of the rotating hedgehog ansatz eq. 2.10, we find, after evaluating the trace, the lagrangian

$$L = -M^{cl} + \frac{1}{2}\Theta_\pi\sum_{i=1}^{3}\Omega_i^2 + \frac{1}{2}\Theta_K\sum_{a=4}^{7}\Omega_a^2 + \frac{N_c}{2\sqrt{3}}\,\Omega_8 - \frac{1}{2}\Gamma(1-D_{88}) \tag{2.14}$$

with

$$A^\dagger\dot{A} = \frac{i}{2}\sum_{\alpha=1}^{8}\lambda_\alpha\Omega_\alpha, \tag{2.15}$$

and

$$D_{\alpha\beta} = \frac{1}{2}tr(\lambda_\alpha A\lambda_\beta A^\dagger). \tag{2.16}$$

Here, the soliton mass M^{cl}, the pionic and kaonic moments of inertia Θ_π and Θ_K, and the strength of SU(3) symmetry breaking Γ are model dependent integrals involving the chiral angle $F(r)$. Explicit expressions for these integrals will be given in section

2.4. In this expression, terms proportional to Ω_8^2 vanish. This is related to the fact that Ω_8 is a redundant variable: under right hypercharge global transformations θ_8^R, $A \to A\, exp(i\lambda_8\theta_8^R)$, $exp(i\lambda_8\theta_8^R)$ commutes with the static hedgehog. We also note that the lagrangian with rotating hedgehog solution is invariant under left SU(3) transformations ($\mathcal{L}$): $A \to \mathcal{L}A$, and under right isospin SU(2) transformations ($\mathcal{R}$) $A \to A\mathcal{R}$. This nicely fits in with our expectations that the theory should have flavor and spin symmetries SU(3) $\times$ SU(2). Finally, the linear contribution proportional to Ω_8 is due to the Wess–Zumino term. We return to this point later.

2.2.1 Symmetry Breaking Terms

The symmetry breaking terms considered here, $L^{(mass)}+L^{(SB)}$, perform both the explicit breaking of chiral symmetry down to SU(3) flavor symmetry through contributions proportional to $tr(U + U^\dagger - 2)$ and the breaking of flavor symmetry down to isospin multiplets[6] through terms proportional to $tr(\lambda_8(U + U^\dagger - 2))$. In this latter case, it is interesting to see how this comes about by considering

$$U = e^{\frac{i}{f_\pi}\sum_{\alpha=1}^{8}\lambda_\alpha\phi_\alpha} \tag{2.17}$$

residing in the meson sector of the theory. Expanding $L^{(mass)} + L^{(SB)}$ to second order in meson octet fields $\phi_\alpha(\vec{x})$, we find

$$L^{(mass)} + L^{(SB)} = \frac{1}{2}m_\pi^2\sum_{i=1}^{3}\phi_i^2 \;+\; \frac{1}{2}m_K^2\sum_{a=4}^{7}\phi_a^2 \;+\; \frac{1}{2}m_\eta^2\phi_8^2. \tag{2.18}$$

The pions, kaons, and eta now have different masses, fixed by the experimental values of m_π and m_K. The eta mass is obtained through the Gell-Mann–Okubo relation for meson masses

$$3m_\eta^2 = 4m_K^2 - m_\pi^2, \tag{2.19}$$

[6]We do not break isospin symmetry. This could be done through $tr(\lambda_3(U + U^\dagger - 2))$, but it is physically a small effect.

a relation satisfied to within 3% accuracy (also, compare this to eq. 2.3 for the baryon octet masses).

It is, however, the baryon sector that interests us. From eq. 2.14, we observe that there, the flavor symmetry breaking is a quantal effect, surviving only because of the presence of time–dependent coordinates:

$$D^8_{88} = \frac{1}{2}\, tr(\lambda_8 A \lambda_8 A^\dagger). \tag{2.20}$$

D^8 denotes the octet (or adjoint) representation of the SU(3) rotation matrices. The strength Γ of the symmetry breaking, an important quantity in view of our introductory discussion, is a model dependent quantity carrying no free parameter.

2.2.2 The Wess–Zumino Term

The importance of the Wess–Zumino term in the context of soliton models was discussed in two ground breaking papers by Witten [W83a][W83b]. Although this term carries little bearing on the original material we present in this chapter, its profound physical consequences on the realm of solitons cannot be overlooked.

The simplest way to see that such a term must enter in the gradient expansion, is to realize that $L = L^{(2)} + L^{(4a)}$ (symmetry breaking terms can be put aside in this discussion) possesses unphysical invariance laws. L is invariant under the naive parity transformation $\vec{x} \to -\vec{x}$, and also under $U \to U^\dagger$. This last symmetry transforms the meson field $\phi_\alpha \to -\phi_\alpha$, $\alpha = 1, ..., 8$, (viz. eq. 2.17) and imposes mesons to appear in either even or odd numbers in any strong interaction relation. For example, $K^+K^- \to \pi^+\pi^0\pi^-$ would not be permitted. The correct parity transformation, which takes into account the pseudoscalar nature of the meson octet field is $\mathcal{P}U(\vec{x}) \to U^\dagger(-\vec{x})$. The simplest Lorentz and chiral invariant term that will preserve $\mathcal{P}$, but not the two

previously discussed symmetries, appears in the equation of motion for U as:

$$\rightsquigarrow \epsilon^{\mu\nu\lambda\rho}\ \ell_\mu \ell_\nu \ell_\lambda \ell_\rho \tag{2.21}$$

where $\epsilon^{\mu\nu\lambda\rho}$ is the 4–dimensional Levi–Civita tensor. Finding an action[7] which yields this term is not an easy matter. The obvious candidate $\epsilon^{\mu\nu\lambda\rho}\ tr(\ell_\mu \ell_\nu \ell_\lambda \ell_\rho)$ vanishes due to a combination of the cyclic property of the trace and antisymmetry of the Levi–Civita symbol. There does, however, exist an action. It is the 5–dimensional expression due to Wess and Zumino [WZ71]:

$$S^{(WZ)} \ = \ \frac{iN_c}{240\pi^2}\int_Q d\Sigma^{\mu\nu\lambda\rho\sigma}\ tr(\ell_\mu \ell_\nu \ell_\lambda \ell_\rho \ell_\sigma). \tag{2.22}$$

Here, Q is a 5–dimensional volume having usual space–time ∂Q as a boundary. The measure $d\Sigma^{\mu\nu\lambda\rho\sigma}$ means $\epsilon^{\mu\nu\lambda\rho\sigma} d^5x$. The normalization $iN_c/(240\pi^2)$ factor is a nontrivial issue: Witten extracted it by comparing the prediction of the gauged (electromagnetism) Wess–Zumino term to QCD for the decay $\pi^0 \to \gamma\gamma$. The two frameworks agree if the arbitrary constant in the Wess–Zumino term is set equal to the number of colors[8] N_c.

We will describe how to simplify the unwieldy looking expression eq. 2.22 for a given choice of ansatz, say $U = AU_0A^\dagger$, or $U = MU_0M$, etc. For the latter ansatz, the unpublished result we obtain will allow us to conclude in section 2.5 that the Wess–Zumino term contributes no η–soliton or eta–kaon interaction to second order in the meson fields. So following the method of Kaymakçalan [BLRS85], the trick is to rewrite the 5–dimensional volume element in terms of a differential form with totally antisymmetric properties:

$$\epsilon^{\mu\nu\lambda\rho\sigma}\ d^5x = dx_\mu \wedge dx_\nu \wedge dx_\lambda \wedge dx_\rho \wedge dx_\sigma, \tag{2.23}$$

[7]The action is $\int dtL$, L is the lagrangian.

[8]Color is an additional degree of freedom, coming in $N_c = 3$ varieties, bestowed on quarks by theorists. The ensuing SU(3) color gauge theory is believed to be <u>the</u> theory of strong interactions.

expand derivatives, viz. expressions like $\partial_\alpha(AU_0A^\dagger)$, and finally regroup terms to explicitly make the result appear as a total derivative. Using Stokes theorem for differential forms[9], we can reduce the expression for the Wess–Zumino term to an integral over usual 4–dimensional space–time. For the ansatz $U = AU_0A^\dagger$, the result is [BLRS85]:

$$\begin{aligned} S^{(WZ)}(U) &= S^{(WZ)}(U_0) \\ &\quad -\frac{iN_c}{48\pi^2}\int_{\partial Q} d^4x\ tr\{u^3a - a^3u - \frac{1}{2}(ua)^2 \\ &\quad +U_0(u-a)^3U_0^\dagger a - a^3U_0(u-a)U_0^\dagger - \frac{1}{2}(aU_0(u-a)U_0^\dagger)^2\}. \end{aligned} \tag{2.24}$$

Here, we have used the shorthand notation $u = U_0^\dagger\partial U_0$, $a = A^\dagger\partial A$, and deleted explicit reference to contracted indices. Hence, $u^3a = \epsilon^{\mu\nu\lambda\rho}u_\mu u_\nu u_\lambda a_\rho$, and similarly for the other terms. The further simplification of this expression due to the vanishing of terms with powers of a greater than one (observe that only one time–derivative can appear due to the Levi–Civita symbol) leads to the contribution of the Wess–Zumino term in the Skyrme lagrangian eq. 2.14:

$$L^{(WZ)} = \frac{N_c}{2\sqrt{3}}\,\Omega_8. \tag{2.25}$$

For the other ansatz $U = MU_0M$ used in section 2.5, the general result we find is the action:

$$\begin{aligned} S^{(WZ)}(U) &= S^{(WZ)}(U_0) \\ &\quad +\frac{iN_c}{48\pi^2}\int_{\partial Q} d^4x\ tr\{m^3\bar{u} - \bar{m}^3u + u^3\bar{m} - \bar{u}^3m \\ &\quad +\frac{1}{2}\bar{m}u\bar{m}u - \frac{1}{2}m\bar{u}m\bar{u} - U_0\bar{m}^3U_0^\dagger m + U_0^\dagger m^3U_0m + mU_0\bar{m}u^2U_0^\dagger \\ &\quad +mU_0u^2\bar{m}U_0^\dagger + \bar{m}u\bar{m}U_0^\dagger mU_0 + m^2U_0u\bar{m}U_0^\dagger - mU_0\bar{m}^2uU_0^\dagger \\ &\quad -\frac{1}{2}mU_0\bar{m}U_0^\dagger mU_0\bar{m}U_0^\dagger + \frac{1}{2}mU_0\bar{m}U_0^\dagger mU_0\bar{m}uU^\dagger\}, \end{aligned} \tag{2.26}$$

[9]Stokes theorem is $\int_Q d\omega = \int_{\partial Q}\omega$, where ω is a differential form.

where

$$u = U_0^\dagger \partial U_0; \quad \bar{u} = U_0 \partial U_0^\dagger; \quad m = M^\dagger \partial M; \quad \bar{m} = M \partial M^\dagger.$$

We finally note that $S^{(WZ)}(U_0) = 0$ for the hedgehog ansatz because there are only three independent Pauli matrices contracted antisymmetrically with four Lorentz indices. This null result explains why no mention of the Wess–Zumino term was made in our discussion of the SU(2) Skyrme model.

2.2.3 Some Points Concerning the Quantization of the SU(3) Skyrme Model

We now give a brief description of the quantization of the SU(3) collective degrees of freedom along lines suggested by Balachandran *et al.* [BLRS85]. It is useful to locally parametrize the matrix $A \equiv A(\xi(t))$ with eight time–dependent variables ξ_α, $\alpha = 1, ..., 8$. This just defines a general coordinate transformation from the Ω_α set we introduced in eq. 2.9 to the new set

$$\dot{\xi}_\alpha = -iN_{\alpha\beta} tr(A^\dagger \dot{A} \lambda_\beta) = N_{\alpha\beta}\Omega_\beta, \tag{2.27}$$

where $N_{\alpha\beta}$ are the elements of a general coordinate transformation matrix, the form of which we need not explicitly specify. Defining

$$R_\alpha = -\frac{\partial L}{\partial \dot{\xi}_\beta} N_{\beta\alpha} = -\pi_\beta N_{\beta\alpha} \tag{2.28}$$

through the conjugate momentum π_β, we obtain from the lagrangian eq. 2.14

$$R_\alpha = -2\Theta_\pi \Omega_i \delta_{i\alpha} - 2\Theta_K \Omega_a \delta_{a\alpha} + \frac{1}{\sqrt{3}} N_c \delta_{\alpha 8}, \tag{2.29}$$

where $1 \leq i \leq 3$; $4 \leq a \leq 7$. We observe that

$$R_8 = \frac{1}{\sqrt{3}} N_c = \sqrt{3} \tag{2.30}$$

is constant. In terms of the R_α's, we can write the hamiltonian

$$\begin{aligned} H &= \pi_\alpha \dot{\xi}_\alpha - L \\ &= \sum_{i=1}^{3} \frac{1}{8\Theta_\pi} R_i^2 + \sum_{a=4}^{7} \frac{1}{8\Theta_K} R_a^2 \\ &= \sum_{i=1}^{3} (\frac{1}{8\Theta_\pi} - \frac{1}{8\Theta_K}) R_i^2 + \sum_{\alpha=1}^{8} \frac{1}{8\Theta_K} R_\alpha^2 - \frac{3}{8\Theta_K} \\ &= \sum_{i=1}^{3} (\frac{1}{2\Theta_\pi} - \frac{1}{2\Theta_K}) C(\mathrm{SU}(2)) + \sum_{\alpha=1}^{8} \frac{1}{2\Theta_K} C(\mathrm{SU}(3)) - \frac{3}{8\Theta_K} \end{aligned} \tag{2.31}$$

In the last line, we have explicitly introduced the SU(2) and SU(3) quadratic Casimir operators[10] $C(\mathrm{SU}(2))$, $C(\mathrm{SU}(3))$. When operating on SU(2) symmetric wavefunctions, $C(\mathrm{SU}(2))\Psi = J(J+1)\Psi$, where J labels the SU(2) representation; whereas for SU(3) symmetric wavefunctions, $C(\mathrm{SU}(3))\Psi = \frac{1}{3}(p^2 + q^2 + 3p + 3q + pq)\Psi$, where (p, q) labels the SU(3) representation: e.g. $(1,1) \equiv$ octet, $(3,0) \equiv$ decuplet, etc.

Classically, the R_α's satisfy the relation

$$\{R_\alpha, R_\beta\}_{PB} = -2i f_{\alpha\beta\gamma} R_\gamma, \tag{2.32}$$

where $\{,\}_{PB}$ is the Poisson bracket defined by

$$\{R_\alpha, R_\beta\}_{PB} = \sum_\gamma (\frac{\partial R_\alpha}{\partial \xi_\gamma} \frac{\partial R_\beta}{\partial \pi_\gamma} - \frac{\partial R_\alpha}{\partial \pi_\gamma} \frac{\partial R_\beta}{\partial \xi_\gamma}). \tag{2.33}$$

Following the standard canonical quantization prescription, the Poisson bracket is replaced by the commutator

$$[R_\alpha, R_\beta] = -2i f_{\alpha\beta\gamma} R_\gamma, \tag{2.34}$$

and $C(\mathrm{SU}(2))$, $C(\mathrm{SU}(3))$, are treated as operators operating on collective wavefunctions.

[10] A Casimir operator is an operator that commutes with all the group generators.

In the case of unbroken SU(3) flavor symmetry, the baryon wavefunctions are matrix elements of SU(3) rotation operators. These are the SU(3) D–matrices

$$\Psi(A) \sim D^{(p,q)}_{(I,I_z,Y),(S,-S_z,Y_R)}(A), \tag{2.35}$$

to be compared to the analogous SU(2) D–matrices discussed in chapter 1. This result should not surprise since C(SU(3)) is the natural extension of C(SU(2)). The quantum numbers (p,q) label the SU(3) representation. The right indices S, $-S_z$, are identified, based on our comment (following eq. 2.14) that the lagrangian has SU(2) invariance under right transformations, with the spin quantum numbers. To find out which value they can take, we consider the defining relation for the realization of right transformations $\mathcal{R} = exp(-i\sum_{\alpha=1}^{8}\lambda_\alpha\theta_\alpha^R)$ in collective coordinate space

$$\Psi(A\mathcal{R}) = e^{-i\sum_{\alpha=1}^{8} R_\alpha\theta_\alpha^R}\Psi(A), \tag{2.36}$$

where R_α are just the operators in eq. 2.34. Let us consider a $\theta_3^R = 2\pi$ rotation $\mathcal{R} = exp(i2\pi\lambda_3/2)$ around the z–axis in the SU(2) spin subspace of right transformations. There, $\mathcal{R} = exp(i2\pi\lambda_3/2) = exp(i\pi\sqrt{3}\lambda_8)$ since λ_8 is proportional to the unit matrix. From eq. 2.36 and eq. 2.30

$$\begin{aligned}\Psi(Ae^{i2\pi\lambda_3/2}) &= e^{-i\pi\sqrt{3}R_8}\Psi(A)\\ &= e^{-i3\pi}\Psi(A) = -\Psi(A).\end{aligned} \tag{2.37}$$

This proves, owing to our use of the physical result $N_c = 3$, that for an odd number of color the soliton is quantized as a fermion. Furthermore, if we now specifically refer to the D–matrix as a baryon wavefunction and note that R_8 has eigenvalue $\sqrt{3}Y_R$, we arrive at

$$Y_R = 1. \tag{2.38}$$

This then forces the low–lying representations to be respectively a spin–1/2 octet and a spin–3/2 decuplet, just as is observed in nature. This brief discussion highlights the crucial importance the Wess–Zumino terms plays through the appearance of the constraint 2.30 in determining the spectrum of baryons.

We conclude this subsection by pointing out that the set of left–indices (I, I_z, Y) correspond to flavor quantum numbers, in accord with the fact that the lagrangian is invariant under left SU(3) transformations $A \to \mathcal{L}A$. We have been able to avoid explicitly introducing left operators L_α defined by

$$L_\alpha = -\frac{\partial L}{\partial \dot{\xi}_\beta}\bar{N}_{\beta\alpha};\ \dot{\xi}_\alpha = -i\bar{N}_{\alpha\beta}tr(A\dot{A}^\dagger\lambda_\beta), \tag{2.39}$$

(compare this to eq. 2.27 and eq. 2.28) because

$$\sum_{\alpha=1}^{8} L_\alpha^2 = \sum_{\alpha=1}^{8} R_\alpha^2 \tag{2.40}$$

and the Wess–Zumino term does not constrain L_8. Hence these operators serve in specifying the SU(3) flavor quantum numbers without explicitly coming in at the hamiltonian level.

2.3 The Alternate SU(3) Lagrangian

In this section, we demonstrate that in SU(3) there exists a <u>second</u> term of fourth order in the field derivative that may enter in stabilizing the skyrmion, over and above the unique structure in SU(2) proposed by Skyrme [S61]. This naturally raises the question as to whether these two terms are exhaustive, or if other forms may also be admissible. We show here that one may in general entertain four acceptable structures of fourth order in the field derivative, which reduce to three when the constraint that no term give rise to expressions containing four time derivatives of the field is imposed[11].

[11]This constraint eliminates an independent chiral term present in both SU(2) and SU(3) which can be put in the form of the so–called quartic symmetric term discussed in ref. [DGH84].

Further, in SU(3) these reduce to two, without recourse to particular parametrizations of the chiral field, such as the usual embedding of the hedgehog ansatz in SU(3), while in SU(2) only one form survives, whence the uniqueness noted originally by Skyrme [S61].

The Skyrme lagrangian density[12] has the general structure[13]

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)}, \tag{2.41}$$

where

$$\mathcal{L}^{(2)} = -\frac{f_\pi^2}{4} tr(\ell_\mu \ell^\mu) \tag{2.42}$$

is the usual nonlinear σ–model term, involving the field derivative bilinearly. Here,

$$\ell_\mu = U^\dagger \partial_\mu U \tag{2.43}$$

and U is the meson field matrix, with $U^\dagger U = UU^\dagger = 1$. As is well known, the term $\mathcal{L}^{(2)}$ alone cannot lead to stable soliton solutions, and so a term $\mathcal{L}^{(4)}$ (or $\mathcal{L}^{(6)}$, etc.) containing the fourth (or sixth, etc.) power of the field derivative is required. We here construct the stabilization terms out of four powers of ℓ_μ (as usual, rejecting higher derivatives of the field in the lagrangian density). Since the lagrangian density must be a Lorentz scalar, we require structures of the form $\ell_\mu \ell^\mu \ell_\nu \ell^\nu$, $\ell_\mu \ell_\nu \ell^\nu \ell^\mu$ or $\ell_\mu \ell_\nu \ell^\mu \ell^\nu$. We further must take traces of these strings of ℓs to construct quantities invariant under the chiral symmetry group $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ (recall that under global chiral transformations, $U \to AUB^\dagger$ where $A \in \mathrm{SU}(3)_L$ and $B \in \mathrm{SU}(3)_R$, and $\ell_\mu \to B\ell_\mu B^\dagger$). The quantities ℓ_μ are spanned by the traceless generators of the SU(3) Lie algebra,

$$\ell_\mu = i \sum_{\alpha=1}^{8} l_{\mu\alpha} \lambda_\alpha, \tag{2.44}$$

[12]Here, we choose to describe things in terms of lagrangian densities $\mathcal{L}$ related to the lagrangian by $L = \int d^3x \mathcal{L}$.

[13]Throughout this section we omit explicit reference to the symmetry breaking terms and the Wess–Zumino term, viz. eq. 2.14. They do not affect the points considered here.

where λ_α are the usual eight generators of SU(3) presented in eqs 2.7 and 2.8 and $l_{\mu\alpha} = l_{\mu\alpha}(\vec{x}, t)$ are general real space-time dependent functions. The ℓ_μ are themselves traceless, and so the traces in the lagrangian can only involve two or four factors of ℓ_μ. Thus we may have

$$tr(\ell_\mu \ell^\mu \ell_\nu \ell^\nu), \quad tr(\ell_\mu \ell^\mu) tr(\ell_\nu \ell^\nu), \tag{2.45}$$

$$tr(\ell_\mu \ell_\nu \ell^\mu \ell^\nu), \quad tr(\ell_\mu \ell_\nu) tr(\ell^\mu \ell^\nu), \tag{2.46}$$

$$tr(\ell_\mu \ell_\nu \ell^\nu \ell^\mu), \quad tr(\ell_\mu \ell_\nu) tr(\ell^\nu \ell^\mu). \tag{2.47}$$

We omit the terms in eq. 2.47 since they are redundant with two of the previous four terms by reason of the cyclic property of the trace. Note also that

$$\epsilon^{\mu\nu\lambda\rho} tr(\ell_\mu \ell_\nu \ell_\lambda \ell_\rho), \quad \epsilon^{\mu\nu\lambda\rho} tr(\ell_\mu \ell_\nu) tr(\ell_\lambda \ell_\rho), \tag{2.48}$$

vanish identically [W83a], also by virtue of the cyclic property of the trace. The appearance in eqs. 2.45 and 2.46 of traces of four 3×3 traceless matrices will make it useful below to exploit the well-known [GL85] relationship for four such matrices

$$\begin{aligned} tr(ABCD) \;+\; & tr(ABDC) + tr(ACBD) + tr(ACDB) \\ & + tr(ADBC) + tr(ADCB) \\ & = tr(AB)tr(CD) + tr(AC)tr(BD) + tr(AD)tr(BC). \end{aligned} \tag{2.49}$$

We can now write the general form of the fourth-order stabilizing term as

$$\begin{aligned} \mathcal{L}_4 \;=\; & a\; tr(\ell_\mu \ell_\nu \ell^\mu \ell^\nu) + b\; tr(\ell_\mu \ell^\mu \ell_\nu \ell^\nu) \\ & + c\; tr(\ell_\mu \ell_\nu) tr(\ell^\mu \ell^\nu) + d\; tr(\ell_\mu \ell^\mu) tr(\ell_\nu \ell^\nu), \end{aligned} \tag{2.50}$$

with a, b, c, d arbitrary parameters. We immediately impose the requirement that there be no terms involving the time-derivative of the field to the fourth power, $(\partial_0 U)^4$

or ℓ_0^4, in order that the usual quantization procedure may be applied[14] [ANW83]. The implications of this requirement are obvious when use is made of eq. 2.49 with $A = B = C = D = \ell_0$; this immediately leads to

$$tr(\ell_0^4) = \frac{1}{2}[tr(\ell_0^2)]^2. \tag{2.51}$$

Equation 2.51 holds equally well for ℓ_0 in SU(2). Using eq. 2.51 in eq. 2.50, we find the constraint

$$a + b + 2(c + d) = 0, \tag{2.52}$$

valid for both SU(2) and SU(3). (Incidentally, eq. 2.52 also insures that under the hedgehog ansatz there is no terms of fourth order in the radial derivative of the profile function $F(r)$, preventing terms which would destabilize the soliton.)

Having reduced the available independent fourth–order forms from four to three by reason of eq. 2.52, it is convenient to rewrite the stabilizing lagrangian density as

$$\mathcal{L}^{(4)} = \frac{x}{32e^2}\mathcal{L}^{(4a)} + \frac{1-x}{16e^2}\mathcal{L}^{(4b)} + y\mathcal{L}^{(4c)} \tag{2.53}$$

where

$$\mathcal{L}^{(4a)} = tr([\ell_\mu, \ell_\nu]^2) \tag{2.54}$$

is the original choice of Skyrme [S61], corresponding to $x = 1$, $y = 0$;

$$\mathcal{L}^{(4b)} = tr(\ell_\mu \ell_\nu)tr(\ell^\mu \ell^\nu) - tr(\ell_\mu \ell^\mu)tr(\ell_\nu \ell^\nu) \tag{2.55}$$

[14]Without such a requirement, the canonical momenta would be related nonlinearly to the time–derivative of the collective coordinates, rather than linearly, rendering the quantization of the theory difficult.

is the new form to which we refer to as the "alternate" lagrangian, where $x = y = 0$; and

$$\begin{aligned}\mathcal{L}^{(4c)} &= 2tr(\ell_\mu \ell_\nu \ell^\mu \ell^\nu) + 4tr(\ell_\mu \ell^\mu \ell_\nu \ell^\nu) \\ &\quad -2tr(\ell_\mu \ell_\nu)tr(\ell^\mu \ell^\nu) - tr(\ell_\mu \ell^\mu)tr(\ell_\nu \ell^\nu). \end{aligned} \tag{2.56}$$

Each of these three Lagrangian densities fulfills the condition of eq. 2.52 separately, with $a = -b = 1,\ c = d = 0;\ a = b = 0,\ c = -d = 1;\ a = 2,\ b = 4,\ c = -2,\ d = -1$ respectively. Using eq. 2.44 in eq. 2.56, we find

$$\mathcal{L}^{(4c)} \equiv 0. \tag{2.57}$$

This result follows from eq. 2.49, or it may be derived by using the following relation satisfied amongst SU(3) structure constants:

$$\sum_{e=1}^{8}\{2d_{abe}d_{cde} + d_{ace}d_{dbe} - f_{ace}f_{bde}\} = \frac{1}{3}(\delta_{ac}\delta_{bd} - \delta_{ab}\delta_{cd}) + \delta_{ad}\delta_{bc}, \tag{2.58}$$

where $1 \leq a, b, c, d \leq 8$.

Thus our main conclusion is that the most general SU(3) form subject to the constraint of two time derivatives is

$$\mathcal{L}^{(4)} = \frac{x}{32e^2}\mathcal{L}^{(4a)} + \frac{1-x}{16e^2}\mathcal{L}^{(4b)}, \tag{2.59}$$

involving only the two parameters x and e. The generalization which we point out for the SU(3) Skyrme model lagrangian was not noticed in this context prior to our work. Certainly it has bearing on all previous SU(3) calculations which were done with the usual Skyrme term $\mathcal{L}^{(4a)}$ alone. After this work was completed, it came to our attention

that Gasser and Leutwyler [GL85] have also observed, in the related context of Chiral Perturbation Theory, that SU(3) allows for only three independent combinations of fourth–order terms in what we have called ℓ_μ, before the imposition of the requirement that there be no ℓ_0^4 term.

2.4 A Calculation of the Octet and Decuplet Baryon Mass Spectra Using the Alternate Lagrangian Within the Yabu and Ando Approach

Let us begin by considering a few points of interest that arise when we consider $\mathcal{L}^{(4)}$ as given in eq. 2.59 under the restriction that U be the usual embedding of the SU(2) hedgehog in SU(3) described in eq. 2.10. For the static part

$$\vec{\ell} = U^{\dagger[SU(3)]}\vec{\nabla}U^{[SU(3)]} = A(t)\ U_0^{\dagger[SU(3)]}(\vec{x})\vec{\nabla}U_0^{[SU(3)]}(\vec{x})\ A^\dagger(t), \tag{2.60}$$

the pairwise cancellations of A, $A^\dagger$ in $L^{(4)}$ imply that we may restrict

$$\ell_m = i\sum_{i=1}^{3} l_{mi}\lambda_i, \tag{2.61}$$

with m now taking values over 1,2,3 only. Equation 2.59 becomes:

$$\begin{aligned}\mathcal{L}^{(4)}_{static} &= \frac{1}{16e^2}\sum_{i,j,k,l,m,n=1}^{3} l_{mi}l_{nj}l_{mk}l_{nl}(x\ [\ tr(\lambda_i\lambda_j\lambda_k\lambda_l) - tr(\lambda_i\lambda_j\lambda_l\lambda_k)] \\ &\quad +(1-x)\ [\ tr(\lambda_i\lambda_j)tr(\lambda_k\lambda_l) - tr(\lambda_i\lambda_k)tr(\lambda_j\lambda_l)]) \\ &= \frac{1}{4e^2}\sum_{i,j,m,n=1}^{3}(l_{mi}l_{ni}l_{mj}l_{nj} - l_{mi}l_{nj}l_{mi}l_{nj}).\end{aligned} \tag{2.62}$$

The purely static part of the lagrangian density hence reduces to the SU(2) form and is independent of x, being parametrized by e alone. Thus for the static part of the SU(3) lagrangian density, the usual Skyrme form ($x = 1$) or the simpler $L^{(4b)}$ construct ($x = 0$) apply equally well, and the resulting equation for the profile function with the hedgehog ansatz is unmodified from the usual form. The form eq. 2.59 therefore does not

introduce any problems with respect to the classical stability of the soliton [ANW83]. Since for the SU(2) case, eq. 2.61 holds even for the time component ($m = 0$), it is clear that there one may in general consider only $L^{(4a)}$, justifying Skyrme's choice as indeed unique within our assumptions, or one may take $L^{(4b)}$ as a simpler algebraic form [KY88]. The equivalence does not depend on the hedgehog form in SU(2).

When dynamic terms are considered, that is, when one has ℓ_0^2 terms, the usual hedgehog ansatz for SU(3), eq. 2.10, leads to the classical SU(3) lagrangian eq. 2.14. The classical mass arising from the static parts is

$$M^{cl} = 4\pi \int_0^\infty dr r^2 \; \{ \frac{f_\pi^2}{2}[F'^2 + 2\frac{\sin^2 F}{r^2}] + \frac{\sin^2 F}{2e^2 r^2} \; [2F'^2 + \frac{\sin^2 F}{r^2}]\}, \tag{2.63}$$

again unchanged from the result in ref. [YA88] and containing no dependence on x. As for the moments of inertia, we have

$$\Theta_\pi = \frac{2\pi}{3} \int_0^\infty dr r^2 \sin^2 F \{4f_\pi^2 + \frac{4}{e^2}[F'^2 + \frac{\sin^2 F}{r^2}]\}, \tag{2.64}$$

and

$$\Theta_K = \frac{\pi}{2} \int_0^\infty dr r^2 (1 - \cos F)\{4f_\pi^2 + \frac{(4-3x)}{e^2}[F'^2 + \frac{2\sin^2 F}{r^2}]\}. \tag{2.65}$$

Note that the nonstrange moment of inertia Θ_π is again dependent only on f_π and e and not on x, i.e, it is unchanged from its previous form [YA88], while Θ_K now depends on x as well, and is not fixed in value even after f_π and e are determined. We expect that $\Theta_K > 0$ remains as a constraint[15] here in order that the masses of baryons in the SU(3) irreducible representations of higher dimension – of possibly dubious legitimacy within the skyrmion framework – do not fall below the baryon octet (viz. eq. 2.31). This freedom to modify Θ_K even after f_π and e, and hence Θ_π, are fixed may imply sizable effects in SU(3) beyond those found in SU(2) calculations.

[15]In a general fit to the baryon mass spectrum using the lagrangian 2.59, this constraint restricts the range of values x can take [KPE91]. In this section, however, we consider only $x = 0$ and $x = 1$; both values satisfy $\Theta_K > 0$.

In this section, we want to investigate some of these phenomenological consequences by presenting results for the mass–splitting spectrum of octet baryons when the alternate lagrangian $L^{(4b)}$ replaces the usual Skyrme lagrangian $L^{(4a)}$. This corresponds to setting $x = 0$ in eq. 2.59. We compare these results to those obtained when $x = 1$; this is just the usual Skyrme model.

As emphasized at the beginning of this chapter, in the Introduction, attempts to calculate the baryon mass–splittings have been unsatisfactory due to the insufficient strength of the symmetry breaking terms. When calculating within the framework of Yabu and Ando, it is rather the product of the kaonic inertia Θ_K times the symmetry breaking Γ that parametrizes the magnitude of the baryon mass–splittings. Indeed, the Yabu and Ando scheme reduces to the problem of finding the eigenvalues ε^{SB} for each baryon by diagonalizing the equation

$$(C(SU(3)) + 2\Theta_K\Gamma(1 - D^8_{88}))\Psi^R_{(I,I_z,Y),(S,-S_z,1)} = \varepsilon^{SB}_{(I,I_z,Y),(S,-S_z,1)}\Psi^R_{(I,I_z,Y),(S,-S_z,1)}, \quad (2.66)$$

where $\Psi^R_{(I,I_z,Y),(S,-S_z,1)}$ is the baryon wavefunction for a given representation R of SU(3). This shows how the product $\Theta_K\Gamma$ enters crucially. To get good results, it must be increased beyond the value obtained for the usual Skyrme model.

By choosing the alternate lagrangian $L^{(4b)}$, we immediately gain, for reasonable values of the parameters f_π and e, a doubling of the kaonic inertia. A feel for this can be obtained by simply observing that now $4 - 3x = 4$ instead of 1 for $L^{(4a)}$. It is however found that this is not yet sufficient to produce good results (unless Θ_K takes unreasonable values [KPE91]). We must further increase Γ. An additional physically plausible term, which is also present in chiral perturbation theory [GL85], is of kinetic origin and lifts the degeneracy $f_\pi = f_K = f_\eta$. It takes the form

$$L^{(SB)} = - \frac{f_K^2 - f_\pi^2}{12} \int d^3x \quad tr(1 - \sqrt{3}\lambda_8)(U\ell_\mu\ell^\mu + \ell_\mu\ell^\mu U^+)$$

$$+ \quad \frac{f_K^2 m_K^2 - f_\pi^2 m_\pi^2}{6} \int d^3x \quad tr(1 - \sqrt{3}\lambda_8)(U + U^+ - 2). \tag{2.67}$$

We note that this term reduces to $L^{(SB)}$ of eq. 2.14 when $f_\pi = f_K$. In the meson sector, $L^{(\tilde{SB})}$ leads to a Gell-Mann–Okubo–like constraint amongst meson decay constants:

$$3f_\eta^2 = 4f_K^2 - f_\pi^2. \tag{2.68}$$

With experimental values $f_\pi = 93$ MeV and $f_K = 114$ MeV, we calculate $f_\eta = 120$ MeV. We have searched, but not found, an experimental measurement of this quantity. In the baryon sector, we get the following result for the hedgehog ansatz eq. 2.10:

$$\begin{aligned} L^{(mass)} + L^{(\tilde{SB})} &= -\frac{1}{2}\tilde{\Gamma}(1 - D_{88}^8) \\ &= -\frac{1}{2}\{\frac{2}{3}(f_K^2 - f_\pi^2)\int d^3x \ (F'^2 + \frac{2\sin^2 F(r)}{r^2})\cos F \\ &\quad + \frac{4}{3}(f_K^2 m_K^2 - f_\pi^2 m_\pi^2)\int d^3x(1 - \cos F(r))\}(1 - D_{88}^8), \end{aligned} \tag{2.69}$$

where time–derivative contributions from $L^{(\tilde{SB})}$ have been neglected, they are found to be small. The main increase is due to modification in the mass term, the second term in eq. 2.67. Indeed,

$$\frac{\tilde{\Gamma}}{\Gamma} \approx \frac{f_K^2 m_K^2 - f_\pi^2 m_\pi^2}{f_\pi^2 m_K^2 - f_\pi^2 m_\pi^2} = 1.55, \tag{2.70}$$

a substantial increase.

The combined increases in the symmetry breaker $\Theta_K\tilde{\Gamma}$ due to the alternate lagrangian and the dynamical symmetry breaking terms are sufficient to produce octet and decuplet baryon mass–splittings in agreement with experiment; without further ado, we present results. The masses of octet and decuplet baryons are shown in figure 2.1. For the parameters f_π, m_π, m_K, the experimental values were adopted. The strength $e = 4.1$ of the stabilization term is adjusted to the nucleon–delta split, and $f_K = 120$ MeV is pushed slightly above its experimental value of 114 MeV. With these

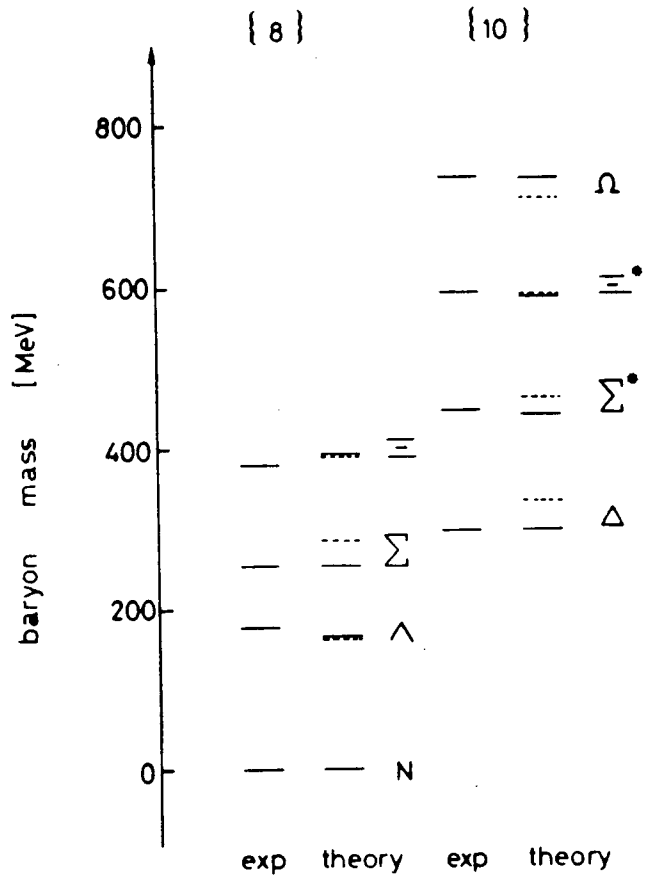

Figure 2.1: The mass splitting spectra in MeV of octet and decuplet baryons. The calculated nucleon mass is shifted in order to agree with the experimental nucleon mass. All other states are shifted by the same amount. Full line (theory) are results for $L^{(4b)}$; dashed line (theory) are results for $L^{(4a)}$.

numbers the baryon spectrum relative to the nucleon's mass is nicely reproduced. The absolute value for the nucleon mass is notoriously high, $M_N = 2330$ MeV, requiring that the whole spectrum be shifted downwards in energy. For comparison, the results for the usual Skyrme model $L^{(4a)}$ (with the smaller Θ_K) are also depicted in figure 2.1 (dashed lines, $M_N = 2586$ MeV , the symmetry breaking strength $\tilde{\Gamma}$ is kept fixed). The increase of the kaonic moment of inertia has two direct consequences: (i) the mass of the $\sum$ baryon is lowered and (ii) the splitting of the decuplet baryons is increased appreciably. If we stick to a collective hamiltonian of form eq. 2.31, a large Θ_K together with a strong symmetry breaking $\tilde{\Gamma}$ is the only possibility to get a reasonable fit to the baryon spectrum.

2.5 The Role of the Eta Meson in the Callan–Klebanov Approach to the Skyrme Model

Following the initial phenomenological failure of the SU(3) Skyrme model based on unbroken SU(3) flavor symmetry[16], the bound state approach to strangeness of Callan and Klebanov [CK85],[CHK88], has emerged as a viable approach for the extension of the Skyrme model beyond the two flavor sector[17]. In this model, the large SU(3) flavor symmetry breaking suggested by the pion–kaon mass difference is taken into account by constructing hyperons as bound states of SU(2) hedgehog solitons and kaons. More precisely, the strangeness degrees of freedom are incorporated as vibrational modes about the zero modes[18] corresponding to isospin rotations of the classical soliton. For the quantum numbers of the baryon states, it is found that the spin arises as a coupling between the kaon angular momentum and the spin of the skyrmion, and, remarkably,

[16]See, for example, the paper of Masak [M89] and references therein.
[17]For an alternative point of view, see section 2.4.
[18]Comments about zero modes are presented in section 3.3 of this thesis.

that the bound kaon carries no isospin, therefore imposing that the isospin of the system be solely associated with the collective rotations of the hedgehog. The strangeness quantum number of the state, on the other hand, is carried by the kaons alone, with a contribution of S $= -1$ for each bound kaon. This scheme obtains good agreement with the experimental values of the hyperon mass splittings [CHK88],[BDR89], and static observables such as the hyperon magnetic moments [NR89],[KM90].

An interesting question which has been considered only rather briefly [KM90] in the context of the bound state model of Callan and Klebanov is the role of the pseudoscalar eta meson ($m_\eta = 548.8$ MeV). Its comparable mass to that of the kaons ($m_{\frac{K+K}{2}} =$ 495.7 MeV) suggests that it also be treated as a heavy degree of freedom relative to the pions ($m_\pi = 138$ MeV), and correspondingly as a vibrational perturbation about the SU(2) soliton background, in a way completely analogous to that followed for kaons. To implement this, we extend the previous kaon–skyrmion ansätze to include an additional eta field. A natural choice [BDR89],[S84], which introduces the kaons ($\phi_4, ..., \phi_7$) and eta ($\phi_8 \equiv \eta$) symmetrically about the skyrmion U_0 is

$$U = \sqrt{U_{K+\eta}}\, U_0 \sqrt{U_{K+\eta}} \tag{2.71}$$

where

$$\sqrt{U_{K+\eta}} = \exp^{i \sum_{i=4}^{8} \frac{\lambda_i \phi_i}{F_\pi}}, \quad U_0 = \exp^{i \sum_{i=1}^{3} A \lambda_i A^\dagger \hat{r}_i F(r)}. \tag{2.72}$$

Here, $F_\pi = 2f_\pi = 186$ MeV is the pion decay constant, λ_i are the Gell–Mann matrices for SU(3), $\hat{r}_i$ are the three components of a radial unit vector and $F(r)$ is the chiral angle. Collective coordinates allowing the hedgehog to acquire definite spin and isospin are introduced via the time–dependent unitary matrix $A \in$ SU(2). With this ansatz, we can consider different lagrangians to investigate the issue of the existence of eta–skyrmion bound states.

We begin by considering U_0 as a static background, first set $A = 1$ (intrinsic frame), and substitute the meson field matrix U of eq. 2.71 in the Skyrme model lagrangian eq. 2.14

$$L = L^{(2)} + L^{(4a)} + L^{(mass)} + L^{(SB)}. \tag{2.73}$$

As before, $L^{(2)}$ is the nonlinear σ–model lagrangian with two field derivatives, $L^{(4a)}$ is the four derivative term introduced by Skyrme to stabilize the classical soliton U_0 and $L^{(mass)}$ is a mass term which explicitly breaks the $SU(3)_L \times SU(3)_R$ down to diagonal SU(3). When we expand to second order in the meson fluctuations, we find the following results which possibly explain why the role of the eta field has hitherto not received much attention in the Callan–Klebanov model: (i) there is no eta–kaon coupling term. (ii) More importantly, the Skyrme stabilization term contributes no eta–hedgehog interaction whereas the nonlinear σ–model term only contributes the Klein–Gordon term of a free massless meson field[19]. (iii) Consideration of the mass term $L^{(mass)}$ leads to the following eta meson lagrangian

$$L_\eta^{(mass)} = \frac{1}{2} \int d^3x \{ \partial_\mu \eta \partial^\mu \eta + \gamma \, \eta^2 \}, \tag{2.74}$$

with γ defined by

$$\gamma = \frac{1}{3}(m_\pi^2(2 - \cos F(r)) - 4m_K^2). \tag{2.75}$$

Asymptotically, L_η represents the lagrangian of a free massive eta meson due to the fact that

$$F(r) \to 0 \Rightarrow \gamma \to -m_\eta^2, \tag{2.76}$$

[19]For a similar result obtained with a different ansatz, see the paper by M. Karliner and M.P. Mattis [KM86].

which follows after using the Gell-Mann–Okubo relation eq. 2.19 for meson masses. We find however that L_η does not support bound state solutions. (iv) Also, the Wess–Zumino action S_{WZ} (see eq. 2.14) must be included in the SU(3) Skyrme model [W83a],[W83b]; to second order in the fields, we have checked using the result given in eq. 2.26 that there are no $\eta - \eta$ or $\eta - K$ interactions.

It is clear that to provide a non–trivial potential for the eta meson we must go beyond the basic Skyrme model[20]. To this end, we consider the replacement of the Skyrme stabilization term $L^{(4a)}$ by the general form $L^{(4)}$ presented in eq. 2.59. Here, we note that $L^{(4)}$ differs from the Skyrme stabilization term $L^{(4a)}$ by providing a non–trivial potential between the eta and the skyrmion. $L^{(4)}$ does not, however, provide an eta–kaon coupling, allowing us to study the issue of eta–skyrmion bound states without making further reference to the strangeness degrees of freedom.

Replacing $L^{(4a)}$ by $L^{(4)}$ in the lagrangian of eq. 2.73, substituting the meson field U (with $A = 1$) of eq. 2.71 and expanding to second order in the meson fields, we find

$$L(U) = L(U_0) + L_\eta + \dots \tag{2.77}$$

where ... stands for second order terms involving kaon fields and higher order terms in the meson fluctuations. The lagrangian for the eta field now takes the form

$$L_\eta = \frac{1}{2} \int d^3x \{ \ \alpha \ \dot{\eta}^2 - \ \partial_i \eta \ \beta_{ij} \ \partial_j \eta + \gamma \ \eta^2 \}, \tag{2.78}$$

where

$$\alpha = 1 + \frac{4(1-x)}{e^2 F_\pi^2} \ (\frac{dF(r)}{dr}^2 + \frac{2 \sin^2 F(r)}{r^2}), \tag{2.79}$$

[20]The other four derivative chirally invariant term, the so–called quartic symmetric term discussed in [DGH84] is unacceptable in the present context. Not only does it lead to an eta–hedgehog potential plagued with "fall to the center" problem, but the position of the singularity depends on the strength parametrizing this term. In this context, we found the equation of motion for the eta profile to be intractable. G. Pari and G. Gat, unpublished.

$$\beta_{ij} = \delta_{ij} + \frac{4(1-x)}{e^2 F_\pi^2} \left(\frac{dF(r)}{dr}^2 (\delta_{ij} - \hat{r}_i \hat{r}_j) + \frac{\sin^2 F(r)}{r^2}(\delta_{ij} + \hat{r}_i \hat{r}_j)\right), \tag{2.80}$$

γ is defined in eq. 2.75, and $\dot{\eta}$ denotes the time derivative of the field. Using the canonical momentum

$$\pi^\eta = \frac{\delta L_\eta}{\delta \dot{\eta}} = \alpha\ \dot{\eta}, \tag{2.81}$$

we obtain, after the usual Legendre transformation, the hamiltonian

$$H_\eta = \frac{1}{2} \int d^3x \{\ \alpha\ \dot{\eta}^2 + \partial_i \eta\ \beta_{ij}\ \partial_j \eta -\ \gamma\ \eta^2\}. \tag{2.82}$$

It is useful to introduce at this stage a partial wave decomposition of the eta field

$$\eta(\vec{r}, t) \rightarrow Y_m^l(\Omega) \eta_l(r, t). \tag{2.83}$$

This allows us to write the hamiltonian for η_l as

$$H_{\eta_l} = \frac{1}{2} \int_0^\infty dr r^2\ \alpha \{\dot{\eta_l}^2 - \ddot{\eta}_l \eta_l\}, \tag{2.84}$$

where in the last equation, we have used the Euler-Lagrange equation for η_l obtained from the usual variation of the lagrangian:

$$\alpha \ddot{\eta}_l - \frac{1}{r^2} \frac{d}{dr}(r^2 \beta \frac{d\eta_l}{dr}) + (\frac{l(l+1)}{r^2}\ \delta - \gamma)\eta_l = 0 \tag{2.85}$$

$$\delta = 1 + \frac{4(1-x)}{e^2 F_\pi^2} \left(\frac{dF(r)}{dr}^2 + \frac{\sin^2 F(r)}{r^2}\right), \tag{2.86}$$

$$\beta = \hat{r}_i \beta_{ij} \hat{r}_j. \tag{2.87}$$

These equations are the same, except for the terms coming from $L^{(mass)}$, as the equations obtained when the eta field is introduced as a phase about the SU(2) hedgehog.

The time–translational invariance of the lagrangian allows us to extract the time dependence of η_l as

$$\eta_l(r,t) = \mathrm{e}^{-i\omega t}\eta_l(r). \tag{2.88}$$

With this form substituted in eq. 2.85, we have a classical eigenvalue equation which can be investigated numerically, given values of the parameters x, e and F_π, for bound states $\omega^2 < \mathrm{m}_\eta^2$. For our explorative purposes, we fix F_π, m_π and m_K at their respective experimental values of 186 MeV, 138 MeV and 495.7 MeV (with these values, we have $m_\eta = 566.8$ MeV). The parameter $e = 4.25$ is chosen to reproduce the nucleon–Δ split. We treat x as a free parameter (we will return to this point in the conclusion).

We find an $l = 0$ eta–hedgehog bound state. We have shown in figure 2.2 how the binding energy of this state depends on the parameter x. The binding energy monotonically decreases with increasing value of x, reaching zero when the strength of the alternate lagrangian $L^{(4b)}$ in $L^{(4)}$ is 14% of the usual Skyrme term. For $0.1 \geq 1 - x \geq 0$, the attractive potential leads to phase shifts that rapidly decrease to zero as $1 - x \to 0$, showing no elastic η–N resonance in the process. These phase shifts are plotted in figure 2.3 as a function of eta meson momentum for different values of x. We find no other S wave bound state. In the other partial waves, $l > 0$, the centrifugal term is repulsive enough to impede the formation of eta–hedgehog bound states.

The S wave bound state we have described does not have well defined spin or isospin quantum numbers. These can be generated in the standard way by quantizing collective coordinates associated with rotations of the hedgehog. Since the eta is an isoscalar field, the introduction of collective coordinates transforming the soliton from the intrinsic to the laboratory frame leaves it unaffected. Letting the SU(2) matrix $A \equiv A(t)$ in eq. 2.72 be time–dependent, we obtain (omitting the kaons as before) an

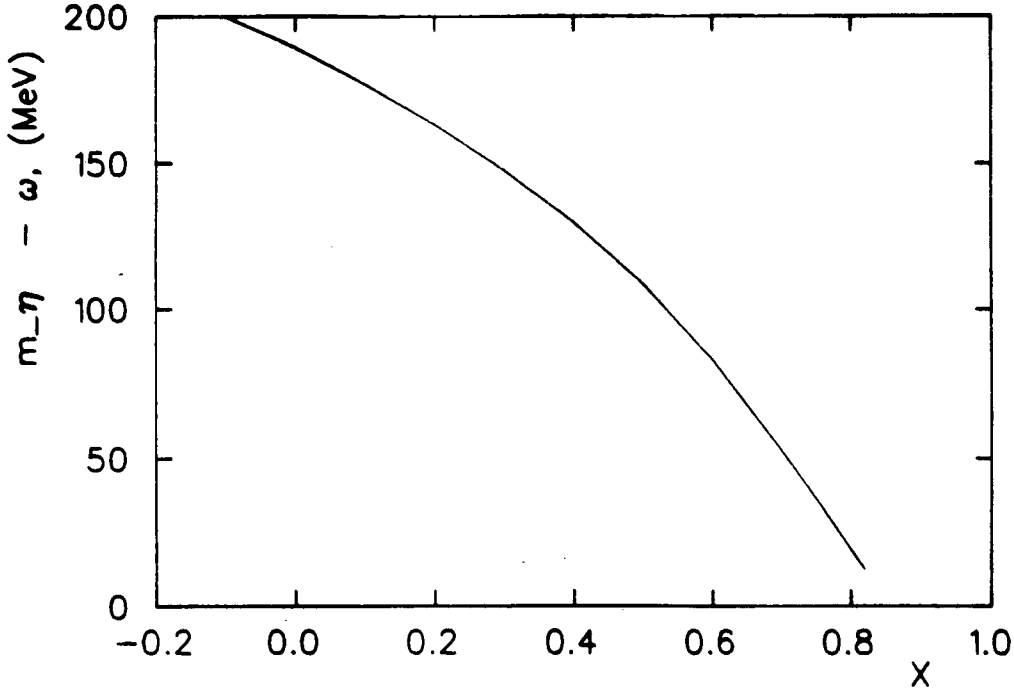

Figure 2.2: Binding energy (in MeV) of the eta meson to the hedgehog as a function of the parameter x appearing in $L^{(4)}$

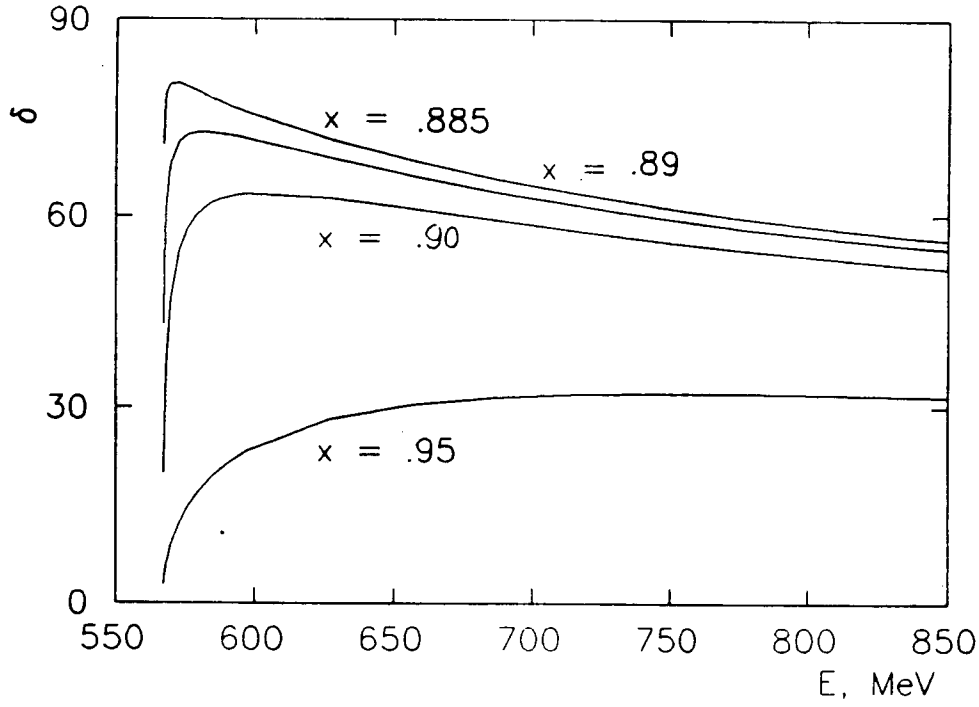

Figure 2.3: S wave eta–hedgehog phase shifts as a function of eta meson energy (MeV) for different values of x.

additional contribution to the lagrangian

$$\begin{aligned}\delta L^{time} &= 2\,\Theta_\pi\,\dot{K}_i\dot{K}_i + \frac{8(1-x)}{e^2F_\pi^2}\int d^3x\,\sin^2 F(r)\,(\,\epsilon_{ijk}\frac{\hat{r}_k}{r}\dot{K}_j\,\dot{\eta}\partial_i\eta \\ &+(\delta_{ij}-\hat{r}_i\hat{r}_j)\,\dot{K}_i\dot{K}_j\,\partial_k\eta\partial_k\eta\,),\end{aligned} \tag{2.89}$$

where $\dot{K}_i = -\frac{i}{2}tr(\lambda_i A^\dagger \dot{A})$, $i = 1,2,3$, are the collective velocities and Θ_π is the skyrmion moment of inertia eq. 1.41. We have expanded the interaction to second order in the collective velocities, rather than to first order only as is done in the strangeness case [CK85], because here the linear term vanishes identically for a purely radial S wave eta. From eq. 2.89, applied to S wave, the conjugate momentum to $\dot{K}_i$ is

$$\begin{aligned}\pi_i^K &= \frac{\delta L^{time}}{\delta \dot{K}_i} \\ &= 4\dot{K}_i\,(\,\Theta_\pi + \frac{32(1-x)\pi}{3e^2F_\pi^2}\int_0^\infty dr r^2\,\sin^2 F(r)\,(\frac{d\eta}{dr})^2\,),\end{aligned} \tag{2.90}$$

which implies the following form for the hamiltonian:

$$\delta H^{time} = \frac{1}{2}\,\frac{\vec{J}\cdot\vec{J}}{\Theta_\pi + \epsilon}, \tag{2.91}$$

where $\vec{J}$ is the angular momentum of the skyrmion and we use the shorthand ϵ for the integral in eq. 2.90. The effect of the S wave bound state on the mass relation is therefore just an effective shift of the skyrmion moment of inertia. The eta fields appearing in ϵ are to be interpreted as field operators. To this end, we introduce the Fourier decomposition

$$\hat{\eta}_l = \sum_n\,(\,\eta(r)_l^n e^{i\omega_n t}\hat{a}_n^\dagger + \eta(r)_l^n e^{-i\omega_n t}\hat{a}_n\,), \tag{2.92}$$

with n running over all bound states and $\eta(r)_l^n$ the given bound state profile obtained from solving eq. 2.85. The creation and annihilation operators $\hat{a}_n$, $\hat{a}_n^\dagger$ satisfy the usual

commutation rules for boson operators and we understand the integral ϵ as normal ordered.

We evaluate the rotational mass spectrum arising from δH^{time} in first order perturbation theory by using a product state for the eta–skyrmion system. Substituting eq. 2.92 and expanding the denominator in eq. 2.91, we find the mass relation

$$M = M_{cl} + \frac{1}{2\Theta_\pi}\, I(I+1) + \omega + (\kappa - \frac{\epsilon}{2\Theta_\pi^2})\, I(I+1). \tag{2.93}$$

Numerically $\epsilon = 6.1 \times 10^{-5}$MeV for $x = 0$ and decreases linearly with increasing x, for the bound state profile η_l normalized with respect to the nonlinear metric [CHK88] α

$$e^2 \int_0^\infty dr\; r^2 \alpha\; (\omega_n + \omega_m)\; \eta_l^n \eta_l^m = \delta_{mn}. \tag{2.94}$$

Using this normalization equation along with the decomposition eq. 2.92 in H_{η_l} of eq. 2.84, provides the result that ω in the mass formula is just the classical frequency. A parameter κ has been introduced in the mass relation to represent both the omitted higher order terms in the ϵ/Θ_π expansion and the fact that this series suffers from operator ordering ambiguities [CHK88].

The mass spectrum we have discussed is now compared to experiment [RPP88]: we first make the obvious observation that since the eta meson is a pseudoscalar particle, the bound state we have is a negative parity baryon state. The lowest mass rotational band member has the quantum numbers $J^p = \frac{1}{2}^-; I = \frac{1}{2}$ (recall that for the skyrmion, $I = S$). A possible candidate for this is the N(1535) negative parity S_{11} nucleon resonance which lies slightly above ηN threshold, and decays with a 45%–55% branching ratio to η+N suggesting a large η content of the state. A previous description of N(1535) in the context of the Skyrme model required the introduction an isoscalar scalar σ meson [SW89]; here, we obtain very naturally a state with the correct quantum numbers. From figure 2.2, we observe that for values of $1 - x$ of the order of 0.2, the

η–N bound state mass is close to η–N threshold. A small value of the free parameter κ could then be used to push the mass of the state above threshold, bringing it in agreement with that of N(1535). Such a scheme, to describe resonances within the context of the Callan–Klebanov model has previously been advocated in the strange sector for $\Lambda(1520)$ [BDR89]. A possible candidate for the $J^p = \frac{3}{2}^-; I = \frac{3}{2}$ rotational band member is the $\Delta(1700)$ π–N D–wave resonance. Here, since we predict for the resonances $M_{\Delta^*} - M_{\mathrm{N}^*} \sim M_\Delta - M_\mathrm{N}$, we do not account well for the observed splitting.

We have described, for the eta–baryon sector of the Callan–Klebanov model, the results obtained with the general lagrangian $L^{(4)}$ and made a comparison with experiment. Choosing small values for the parameter $1 - x$ is in agreement with the experimental information available (see ref. [BDR89]), albeit large uncertainties are associated with such data. On a general theoretical basis, $L^{(4)}$ should be used rather than the Skyrme term alone, suggesting that it becomes of interest to investigate how the strangeness sector of the Callan–Klebanov model, and the ensuing physical discussion of hyperons [DNR89], is modified by the new–found freedom.

Chapter 3

S and P Wave Pion–Nucleon Scattering in the SU(2) Skyrme Model

3.1 Introduction

Nonlinear models for mesonic fields which contain baryons as topological solitons allow for a natural description of meson–baryon scattering [HEHW84][WE84][EHH86] [MK85][KM86]. Small fluctuations in the underlying meson fields are introduced to describe the shape vibrations of the soliton. Due to the diffuse surface of the soliton which extends exponentially decreasing to infinity, the normal modes asymptotically have an oscillatory behaviour which allows an identification to scattering waves [EH82]. This situation is in contradistinction with the usual case of finding a set of discrete levels describing the normal modes of an object with a sharp surface.

The meson field lagrangian, with soliton–meson ansatz, is expanded in the intrinsic frame of the skyrmion to second order in the small meson fields. A multipole expansion of the meson field insures that the Euler–Lagrange equations of motion obtained from the usual variation of the lagrangian are a set of linear ordinary coupled differential equations [WE84][MK85]. These are then solved with respect to boundary conditions such that the solutions describe scattering waves. In the intrinsic frame, the scattering solutions are identified with phonon excitations of the soliton [HEHW84]. To recover the familiar description of the scattering event in terms of mesons scattering off a target baryon we must consider a recoupling scheme for the various angular momenta involved in the problem [HEHW84][MP85]. This implements a transformation from

the intrinsic frame to the laboratory frame where it is finally possible to compare the theoretical phase shifts and inelasticities with those extracted from experiment for the different channels defined by total isospin and total angular momentum (we restrict our observations to pion-nucleon scattering).

Extensive compilations of results obtained for these "standard" background, or adiabatic, calculations can be found in the literature. They represent principally the work of the Siegen group [HEHW84][WE84][EHH86] and the Stanford group [MK85][KM86]. The wealth of information predicted by the Skyrme model is impressive, given that with only two free parameters, e and f_π, it is possible not only to describe the spectrum of baryon resonances, as is done in naive quark models [IK79a][IK79b], but also the whole meson–baryon scattering amplitudes as a function of meson energy. If we restrict the discussion to SU(2), that is pions, nucleons and deltas, then good agreement with experiment is found in all channels except the lowest angular momentum partial waves. The pion–nucleon F waves are particularly well reproduced [HEHW84][MK85]. However, it is observed that the S and P waves[1] are in obvious disagreement with experiment [HEHW84][MK85]. This situation in many ways was originally a serious blow to the model since theoretical expectations had it that the Skyrme model should be particularly effective in the low–energy regime; that perhaps it represented a low–energy non–perturbative realization of quantum chromodynamics [W83a][W83b]. These early calculations, however, suffered from the severe "adiabatic" approximation which, in few words, corresponds to neglecting all interactions between the meson field and the time–dependent collective motions of the skyrmion.

Recently, it has been convincingly demonstrated how one should proceed to bring

[1] We follow the usual spectroscopy nomenclature to describe partial waves: a relative π–N angular momentum $l = 0,\ 1,\ 2,\ 3$, etc, corresponds to S, P, D, F, etc, waves. Where we will have to be more specific, we will introduce the total isospin T and total angular momentum J to describe the channel: S$(2T)(2J)$, P$(2T)(2J)$, etc.

the model in closer agreement with the experimental data for the low–energy P waves [KO89][V89][V91]. Of the ideas contained in these works, there are two observations we think are worth insisting upon. The first is that it is incorrect to neglect the coupling between the fluctuation field and the rotations of the soliton. Formally, this corresponds to retaining meson–baryon interactions involving time–derivatives of the collective coordinates. The other point we emphasize is that the zero mode solutions which appear in the background scattering calculations, reflecting the breaking of isospin symmetry in the lagrangian by the soliton solution, do not require any special kind of treatment. This point was not appreciated immediately and led to various suggestions[2] on how to treat the zero energy solutions in the background [SOY86][KVW88].

As observed by Verschelde [V89], the sum of the contributions from zero modes and linear time–derivative interaction reproduces the Δ–isobar model [EW88] for the P waves in πN elastic scattering. In its non–relativistic version, the physics of the Δ–isobar model is contained in the direct and crossed nucleon and Δ exchange Born graphs. Such a clear link between the Skyrme model and familiar phenomenology is an important result in view of the Skyrme model's above mentioned difficulties in reproducing even the basic qualitative features of π–N scattering.

For the S waves, the background calculations left degenerate the two isospin channels, in obvious disagreement with experiment [HEHW84][MK85]. It has, however, been shown by Uehara and Kondo [UK86] that the time–derivative interaction quadratic in the fluctuations restores the low–energy theorem for the scattering lengths. This shows that the soliton model in principle contains all the important ingredients necessary for

[2]Actually, the issue of identifying the Born terms in the Skyrme model is still rather controversial. This is attributed to the constrained nature of the degrees of freedom that are to be quantized for the soliton–pion interacting system. Also problematic in this context has been the issue of the order in N_c of the time–dependent terms responsible for the description of the delta resonance. These issues are interestingly addressed by Vershelde [V87][V88][VV89]. For a different point of view to the one we advocate in this thesis, we refer in particular to the articles of Uehara [U88][U89][U90] and ref. [HSU91] and to commentaries by Liang *et al.* [LLLS90] concerning ref. [KO89].

a satisfactory description of meson–baryon scattering.

In this chapter we describe how to realize such ideas in a full calculation of π–N phase shifts. The approach we follow finds its origins in ref. [H90], details of which we will review in section 3.4. We construct a scheme that allows us to go beyond the adiabatic approximation for meson–baryon scattering and address the problems plaguing the calculated background pion–nucleon S and P waves. The results obtained represent a marked improvement over background calculations and bring the Skyrme model in semi–quantitative agreement, especially for the P waves, with the phase shift analyses based on experiment. Our comparison is limited to pion energies up to 300 MeV above threshold. We also consider the results obtained for the potentials generated by the nonlinear σ–model, for a given soliton profile. In this case, however, we find unrealistic predictions and therefore will only present a limited discussion of this model. This latter approach has not appeared in the literature, most probably because it fails in reproducing observed features of the phase shifts.

The process will require that we judiciously implement approximations avoiding the overwhelming complexity of the complete calculation. This will first lead us to comment about the usual adiabatic calculations where we will find it necessary to truncate the scattering S–matrix in such a way that we can remove unphysical scattering of pions from spin 5/2, isospin 5/2, exotic baryon states[3] and close those channels which are the cause of large inelasticities setting in at unphysically low momenta in the pi–N channels. We will pay particular attention to preserving the unitarity of the S–matrix; for this reason we will carry out the truncation in the K–matrix. We then add to this background scattering the contributions arising from terms representing the coupling of time–derivatives of baryon collective coordinates to the meson fields. These interactions will be evaluated in first order Born approximation. To take into account the

[3]Recall that such states arise naturally in the Skyrme model rotational mass spectrum.

arbitrariness of the choice of representation for the fluctuations a specific interpolating field incorporating the metric associated with the fluctuations is substituted for by plane waves. We present the elastic phase shifts constructed from our background results and time–derivative interaction results for S and P waves. When discussing results, we will tend to first present the case of P wave π–N channels. This is the area in which we have invested most effort and for which we have found best agreement between the Skyrme model and phase shift analyses. The treatment of the π–N S waves then follows in a straightforward way.

The work presented in this chapter covers the material contained in the two papers "Low–Energy Pion–Nucleon P Wave Scattering in the Skyrme Model" by Holzwarth, Pari and Jennings [HPJ90] and "Low–Energy S Wave Pion–Nucleon Scattering in the Skyrme Model: a PWBA Analysis" by Holzwarth, Walliser and Pari [HWP90]. Our results represent the first successful calculation of low–energy P wave π–N phase shifts within the framework of the Skyrme model. We emphasize that this is due to our use of a better set of approximations over previous calculations [HEHW84][WE84][EHH86] [MK85]. We are not introducing any new parameters; indeed, our "favorite set": $f_\pi =$ 93 MeV, $e = 4$, $m_\pi = 138$ MeV, has only the dimensionless constant e taken as a free parameter and adjusted to the physical value of the nucleon–Δ mass split. Although our results for the S waves are not unsatisfactory, we will see that they lack in several features. As demonstrated very recently by Walliser [W91], the main problem resides in our use of the plane wave Born approximation. This approximation is known to be "risky" to use in nuclear physics. It should perhaps not be too surprising that Walliser's distorted wave Born approximation calculation, which takes into account the deformation of the pion profile due to solitonic effects, achieves better results.

3.2 Background Scattering

3.2.1 Introduction

We consider two particular soliton models: the nonlinear σ–model [GL60]

$$L_\sigma = \frac{f_\pi^2}{4}\int (tr(\partial_\mu U \partial^\mu U^\dagger) + m_\pi^2\, tr(U + U^\dagger - 2))d^3x, \tag{3.1}$$

and the Skyrme model [S58][S61]

$$L_S = L_\sigma + \frac{1}{32e^2}\int tr[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U]^2 d^3x, \tag{3.2}$$

with $U \in SU(2)$. Both models have serious drawbacks: the nonlinear σ–model does not stabilize the soliton against spherical collapse. But we can still use it for a given soliton profile to investigate its implications for the scattering of mesons off the chosen soliton background. As for the Skyrme model, it suffers at higher energies from an unphysical monotonic rise of phase shifts in all normal modes[4] [EHH86]. Fortunately, this does not seriously affect the scattering amplitudes in the low–energy domain we consider ($\leq$ 300 MeV above threshold, or equivalently, a momentum in units of the pion mass $k \leq 3$ m$_\pi$).

Starting with the B=1 static hedgehog

$$U_0 = e^{i\vec{\tau}\cdot\hat{r}\ F(r)}, \tag{3.3}$$

we introduce meson fields which will later be identified with scattering states by considering small fluctuations of the field U_0 (small in the sense that they are damped by $1/f_\pi$). This procedure might seem to contain a great deal of arbitrariness since one could easily think of a large number of ways to implement such a scheme:

$$\begin{aligned} U &= e^{i\vec{\tau}\cdot(\hat{r}\ F(r) + \frac{\vec{\phi}}{f_\pi})} = e^{i\vec{\tau}\cdot\frac{\vec{\xi}}{2f_\pi}}\ U_0\ e^{i\vec{\tau}\cdot\frac{\vec{\xi}}{2f_\pi}} \\ &= U_0\ e^{i\vec{\tau}\cdot\frac{\vec{\chi}}{f_\pi}} = \sqrt{U_0}\ e^{i\vec{\tau}\cdot\frac{\vec{\zeta}}{f_\pi}}\ \sqrt{U_0} \qquad etc., \end{aligned} \tag{3.4}$$

[4]Except, as can be seen in figure 3.1, in the E0 mode.

with Greek symbols labelling the small fluctuation field. Asymptotically, as the chiral profile $F(r) \to 0$, U resides in the meson sector of the theory and the model describes a theory of interacting mesons. For finite r, the mesons interact with the soliton and the potential is determined by the explicit form of chiral lagrangian we use. Of these meson–baryon ansätze, two have been particularly popular. The first form

$$U_D = e^{i\vec{\tau}\cdot(\hat{r}\ F(r)+\frac{\vec{\phi}}{f_\pi})} \tag{3.5}$$

is referred as the "direct expansion" form and was preferred by the initial workers doing meson–baryon scattering within the framework of the Skyrme model [WE84][EHH86] [MK85]. We will therefore extensively rely on the background scattering results obtained within this framework. In particular, we use for the Skyrme model the equations derived with this form of fluctuation by Mattis and Karliner [MK85]. For completeness we have included these differential equations in an appendix alongside those we have obtained for the nonlinear σ–model (see Appendix A). The other form we use in this thesis is the one proposed by Schnitzer [S84]:

$$U_S = U_\xi\ U_0\ U_\xi = e^{i\vec{\tau}\cdot\frac{\vec{\xi}}{2f_\pi}}\ U_0\ e^{i\vec{\tau}\cdot\frac{\vec{\xi}}{2f_\pi}}. \tag{3.6}$$

Here the fluctuations $\vec{\xi}$ are introduced symmetrically about the hedgehog through a chiral rotation. This form has the nice property of leading in a transparent way to all low–energy soft pion theorems; by expanding $L(U_S)$ to second order in $\vec{\xi}$, Schnitzer recovered the usual Weinberg lagrangian [W66]. U_S is also useful because it leads to less tedious calculations than the direct expansion form. We will use it when we consider the time–dependent interactions to be added in Born approximation to the background results in section 3.4.

Since we are free to choose any of the forms in eq. 3.4 the procedure described to introduce meson fields in the theory appears to contain a great deal of arbitrariness.

This is not the case, however, in the harmonic approximation[5] used to calculate background scattering: the resulting linear equations of motion for the mesons lead to the same phase shifts. Restricting ourselves to U_S and U_D, we see that the equivalence of the two forms leads, to first order, to the relations [SOY86]

$$\xi_L = \phi_L, \tag{3.7}$$

$$\vec{\xi}_T = \frac{\tan F}{F}\vec{\phi}_T, \tag{3.8}$$

where we have used the decomposition of the field in longitudinal and transverse components

$$\vec{\phi} = \phi_L \hat{r} + \vec{\phi}_T. \tag{3.9}$$

Because $\frac{\tan F}{F} \to 1$ as $r \to \infty$, the phase shifts extracted from the asymptotic part of the field are the same. Of course, we expect different results if we expand the model to 3rd order, but we do not touch this issue here.

The ansätze of eq. 3.4 refer to the intrinsic frame because they do not contain collective coordinates which rotate the hedgehog to states of good spin and isospin. It is in this frame that the background scattering phase shifts are most easily obtained. We therefore proceed, in the next paragraphs, to provide general details of such a calculation and show how to arrive at the physical laboratory frame π–N phase shifts.

Using the direct expansion ansatz U_D, the results of the expansion to second order in the fluctuation for the lagrangian is

$$L = -M^{cl} + \frac{1}{2}\int d^3r\left(g_L\dot{\phi}_L^2 + \frac{\sin^2 F}{F^2} g_T \dot{\vec{\phi}}_T^{\,2}\right) - V_D(\phi), \tag{3.10}$$

[5]Interchangeably, we will use either "harmonic approximation" or "expansion to second order in the fluctuating field".

where we have found it convenient to adopt a form used by Walliser [W91]. If we use instead the Schnitzer ansatz U_S, then we find the lagrangian [SOY86]

$$L = -M^{cl} + \frac{1}{2}\int d^3r(\ g_L\dot{\xi}_L^2 + \cos^2 F g_T \dot{\vec{\xi}}_T^{\,2}) - V_S(\xi). \tag{3.11}$$

In these expressions, M^{cl} (eq. 1.34) is the static soliton mass contributing only to the baryon sector. The time derivatives $\dot{\phi}_L$, $\dot{\vec{\phi}}_T$, $\dot{\xi}_L$, $\dot{\vec{\xi}}_T$, of the longitudinal and transverse components come multiplied by the metrics

$$g_L = 1 + \frac{2}{e^2 f_\pi^2}\ \frac{\sin^2 F}{r^2}, \tag{3.12}$$

$$g_T = 1 + \frac{1}{f_\pi^2 e^2}(F'^2 + \frac{\sin^2 F}{r^2}). \tag{3.13}$$

These functions play an important role in the normalization of the fluctuations. The potentials $V_D(\phi)$, $V_S(\xi)$, are lengthy quadratic functions of the fields and their spatial derivatives for which we need not present explicit expressions.

It is important to note that in the expressions eqs 3.10, 3.11, there is no linear term in the fluctuation. This reflects the fact that the hedgehog is a local minimum with respect to vibrations $\vec{\phi}$ (or $\vec{\xi}$), i.e. the coefficient of the linear term is simply the defining differential equation for the soliton profile $F(r)$. But the vanishing of this contribution [OSY85][HS86a][U86] also seems to indicate a serious missing ingredient in the model: the lack of minimal Yukawa couplings (see eq. 1.14, for example), as in the familiar relativistic meson–baryon lagrangians, leading to baryon exchange diagrams. It turns out, however, that for the rotating hedgehog, a coupling involving the meson field linearly and the time–derivative of the zeroth component of the axial current survives [HHS87][S87a][V88][AZ88]. This is due to the fact that the rotating hedgehog is not a solution to the differential equation defining the static profile $F(r)$. The implications of this will become clear in section 3.4.

A Euler–Lagrange variation of the lagrangian eq. 3.10 leads to the linear equations of motion for the fluctuation $\vec{\phi}$. The equations separate into angular and radial parts if the field is expanded in standard vector spherical harmonics [WE84][MK85]:

$$\vec{\phi} = \sum_{KlM_K} \phi_{Kl}(r,t)\ \vec{Y}_{KlM_K}(\hat{r}), \tag{3.14}$$

$$\vec{Y}_{KlM_K}(\hat{r}) = \sum_{m\nu} \langle lm1\nu|KM_K\rangle\ \hat{e}_\nu\ Y_{lm}(\hat{r}), \tag{3.15}$$

where in these expressions, $\vec{Y}_{KlM_K}(\hat{r})$ are vector spherical harmonics constructed by adding vectorially the spherical isovector $\hat{e}_\nu$, $\nu = 0, \pm 1$, to spherical harmonics $Y_{lm}(\hat{r})$ with the usual recoupling Clebsch–Gordan coefficient. A further decoupling into positive and negative parity channels [WE84][MK85] occurs because of the definite parity of the spherical harmonics

$$\mathcal{P}Y_{lm}(\hat{r}) = (-)^l\ Y_{lm}(\hat{r}). \tag{3.16}$$

Applied to $\vec{\phi}$, we then have the magnetic modes with parity[6] $(-)^{K\pm 1}$

$$\vec{\phi}_M \rightsquigarrow \phi_{KK}(r,t)\ \vec{Y}_{KKM_K}(\hat{r}) \tag{3.17}$$

and the electric modes with parity $(-)^K$

$$\vec{\phi}_E \rightsquigarrow \phi_{KK-1}(r,t)\ \vec{Y}_{KK-1M_K}(\hat{r})\ +\ \phi_{KK+1}(r,t)\ \vec{Y}_{KK+1M_K}(\hat{r}). \tag{3.18}$$

The labelling $\vec{\phi}_E$, $\vec{\phi}_M$ follows the well-known nomenclature associated with the multipole decomposition of the electromagnetic $\vec{E}$ and $\vec{B}$ fields [J75]. We will repeatedly use this terminology, examples of which are the E0 (electric monopole) $K = 0$, $l = 1$; M1 (magnetic dipole) $K = 1$, $l = 1$; E1 (electric dipole) $K = 1$, $l = 0$; etc. Of these modes, the E0 has acquired the name of "breathing mode". This is due to the fact that

[6] We also include here the intrinsic pseudoscalar nature of $\vec{\phi}$.

it corresponds to a change $\hat{r}F(r) \rightarrow \hat{r}(F(r) + \delta F(r,t))$ which is simply a small radial vibration about the equilibrium profile $F(r)$. Finally, we specify the time dependence of $\phi_{Kl}(r,t)$ as

$$\phi_{Kl}(r,t) \rightarrow \phi_{Kl}(r)e^{-i\omega t}, \tag{3.19}$$

which leads us to search for the normal modes of the field. The complete equations for both the Skyrme and the nonlinear σ–model are presented in Appendix A. There is also presented a quick discussion on the technicalities involved in extracting the phase shifts from the differential equations.

The grand spin K arises from the unusual coupling of the field's isospin to its angular momentum:

$$\vec{K} = \vec{i} + \vec{l} \tag{3.20}$$

$$|\vec{K}| = l + 1, l, l - 1. \tag{3.21}$$

That such a decomposition is useful is due to the fact that in the intrinsic frame, the hedgehog is a singlet under $\vec{K}$–transformations.

$$\vec{K}U_0 = \frac{1}{2}[\vec{\tau}, U_0] - i(\vec{r} \times \vec{\nabla})U_0 = 0. \tag{3.22}$$

The physical interpretation that is given to $\vec{K}$ is that of "phonon spin" [HEHW84] carried by the fluctuation (see also section 1.2 for other comments concerning the grand spin). A precise identification requires a consideration of the Noether currents involved in the problem [W91].

In the intrinsic frame, we therefore have the physical picture of a phonon of spin $\vec{K}$ interacting with a baryon of spin $\vec{s}$. The total angular momentum $\vec{J}$ for the scattering event is then

$$\vec{J} = \vec{K} + \vec{s}. \tag{3.23}$$

This leads us to write, in the intrinsic frame a state of total angular momentum and total isospin as [HEHW84]

$$|(Ks)JM_J; sM_I\rangle_{int} = \sum_{M_K M_s} \langle KM_K sM_s|JM_J\rangle \ \ |KM_K\rangle_{int} \otimes |sM_s; sM_I\rangle_{int}. \tag{3.24}$$

The first ket on the right hand side of expression eq. 3.24 represents the phonon wavefunction whereas the second one is to be identified with the baryon's wavefunction in collective coordinate space. For the baryon wavefunction, we have explicitly made use of the result $|\vec{s}| = |\vec{I}|$, which explains why only the spin quantum number appears as a label alongside the magnetic quantum numbers.

In the laboratory frame, the fluctuations carry angular momentum $|\vec{l}|$ and isospin $|\vec{i}| = 1$. The baryon carries spin $|\vec{s}| = |\vec{I}|$. It is clear how to write states corresponding to channels of total isospin and angular momentum [HEHW84]:

$$|(ls)JM_J; (1s)TM_T\rangle_{lab} = \sum_{m_l M_s m_i M_I} \langle lm_l sM_s|JM_J\rangle\langle 1m_i sM_I|TM_T\rangle \ |lm_l; 1m_i\rangle_{lab} \otimes |sM_s; sM_I\rangle_{lab}, \tag{3.25}$$

where we have introduced T to denote the total channel isospin. The physical situation in the laboratory frame corresponds to the familiar one of a pion scattering off a target baryon, with $|lm_l; 1m_i\rangle_{lab}$ representing the pion and $|sM_s; sM_I\rangle_{lab}$ the baryon. It is there that we compare the model's predictions with experiment and we therefore now consider the geometric transformation relating the two frames.

The overlap is

$$_{lab}\langle (ls)JM_J; (1s)TM_T|(Ks')J'M'_J; T'M'_T\rangle_{int} = \delta_{TT'}\ \delta_{JJ'}\ \delta_{M_J M'_J}\ \delta_{M_T M'_T}\ \delta_{ll'}\ \delta_{ss'}\ C_{JTlsK}, \tag{3.26}$$

with recoupling coefficients

$$C_{JTlsK} = (-)^{l+s+J}((2K+1)(2s+1))^{1/2} \begin{Bmatrix} T & 1 & s \\ l & J & K \end{Bmatrix}. \tag{3.27}$$

The appearance of the 6–j symbol $\begin{Bmatrix} T & 1 & s \\ l & J & K \end{Bmatrix}$ reflects the fact that there are precisely 6 intertwined angular momenta in the problem. We may point out that a particularly useful identity in such simplifications is [E74]

$$\sum_{\mu_1\mu_2\mu_3} (-)^{\mu_1+\mu_2+\mu_3} \begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & \mu_2 & -\mu_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -\mu_1 & m_2 & \mu_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ \mu_1 & -\mu_2 & m_3 \end{pmatrix}$$

$$= (-)^{l_1+l_2+l_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}, \tag{3.28}$$

where we have introduced the familiar 3–j symbols related by a normalisation factor to the Clebsch-Gordan coefficients [E74]. To simplify the string of rotational D–matrices arising from the baryon wavefunctions and the transformation from intrinsic to laboratory fluctuation we have used the following integral over collective coordinates:

$$\frac{1}{8\pi^2} \int d\Omega \; D^{s}_{M_s - M_T} \; D^{1}_{\mu\nu} \; D^{s'\,*}_{M'_s - M'_T} =$$

$$(-)^{M'_s+M'_T} \begin{pmatrix} s & 1 & s' \\ M_s & \mu & -M'_s \end{pmatrix} \begin{pmatrix} s & 1 & s' \\ -M_T & \nu & M'_T \end{pmatrix}. \tag{3.29}$$

Here, we have used the baryon wavefunction eq. 1.51. We point out these details because they are very typical of the technical points that arise in Skyrme model calculations.

The result for the S–matrix is obtained by using the closure relation $\sum_{int} | >< | = 1$ twice and projecting onto laboratory frame state [HEHW84]

$$\mathrm{S}^{2T\,2J}_{ls;l's'} = \sum_K C_{JTlsK} \; \mathrm{S}^{K}_{ll'} \; C_{JTl's'K}. \tag{3.30}$$

We could have alternatively arrived at this important result by introducing time–independent collective coordinates about the hedgehog and carried out the calculation along group theoretical lines [MK85]. We emphasize that this static approximation is at the heart of the background scattering. If we use an analogy to rotational models of nuclear physics, the approximation corresponds to neglecting vibration–rotation coupling.

We have indicated the various angular momenta that label the laboratory and intrinsic S–matrices. The labeling of the laboratory frame S–matrix explicitly indicates conservation of total isospin and angular momentum. The incoming pion angular momentum l is not, however, necessarily equal to the outgoing pion momentum l' (similarly s is not necessarily equal to s'), reflecting, for example, the possibility of having $l = 3$ pions contributing to the inelastic process π–N $\rightarrow$ π–Δ in the $2J = 3$ channel. Parity precludes even l's from mixing with odd l's, and total angular momentum conservation restricts $\Delta l \leq 2$. We see that to calculate background scattering in the laboratory frame, it is sufficient to know only the few intrinsic frame channels which contribute through non–zero 6–j symbols in eq. 3.30 to the physical π–N channels.

3.2.2 Intrinsic Frame Background Scattering

We will now proceed to describe these intrinsic frame channels which contribute to the π–N S and P waves required in our calculation. For our descriptive purposes, we fix the parameters of the model f_π, m_π to their respective physical values of 93 MeV and 138 MeV and choose $e = 4$, a value which gives a reasonable Skyrme model nucleon–delta mass split as well as a reasonable value of the πNN coupling constant $f_{\pi\mathrm{NN}}$ (section 3.3). A further set of parameters: $f_\pi = 110$ MeV, $e = 4$ is used to investigate the sensitivity of the P wave Skyrme model results whereas we present for the S waves the results obtained with $f_\pi = 110$ MeV, $e = 5$.

It is perhaps useful at this stage to regroup all the parameter sets used in this chapter for calculations of S and P wave scattering and briefly outline the reason for our choices. We note that in the case of the nonlinear σ–model, only the set $f_\pi = 93$ MeV, $e = 4$, $m_\pi = 138$ MeV, is used for both S and P wave π–N scattering. For the Skyrme model, the various sets of parameters now follow, with f_π in MeV and m_π unchanged from its experimental value of 138 MeV.

Parameters sets used for π–N P wave scattering within the framework of the Skyrme model.

$f_\pi = 93$	$e = 4$	Gives best results for P waves; also good values for $f_{\pi\mathrm{NN}}$ and N–Δ mass split.
$f_\pi = 110$	$e = 4$	P_{11} channel is nicely reproduced; Used to check dependence of results on f_π.
$f_\pi = 54$	$e = 4.84$	Reproduces physical N and Δ masses; a curiosity, it fails to reproduce P waves.

Parameters sets used for π–N S wave scattering within the framework of the Skyrme model.

$f_\pi = 93$	$e = 4$	Consistent model should use same parameters for S and P waves.
$f_\pi = 110$	$e = 5$	Our best results for the S waves.

First, we consider the background results for the π–N P waves. The laboratory S–matrix eq. 3.30 contains the π–N P waves in the channels with isospin–spin labels $2T = 1, 3$; $2J = 1, 3$; pion orbital angular momentum $l = 1, 3$; target spin=isospin

$s = 1/2, 3/2, 5/2$. Of course, $s = 1/2$ corresponds to the nucleon, $s = 3/2$ to the Δ, but there is no sharp resonance in nature with $s = I = 5/2$ [MP85]. Unitarity of the S–matrix, however, requires the inclusion of this state. Other artifacts of the model, i.e. rotational band members with $s = 7/2, 9/2...$ are excluded by the triangular rules satisfied by the various angular momenta. We will return to the case of the exotic $s = 5/2$ state later in this section when we consider the truncation of the background amplitudes to a single channel.

For $2T = 2J = 1$, S^{11} couples only P waves in the π–N; ($s = 1/2$) and $\pi - \Delta$ ($s = 3/2$) channels, and the K–sum in eq. 3.30 involves only the $K = 0$ electric monopole (E0) and the $K = 1$ (M1) mode. In figure 3.1 we present the E0 phase shifts[7] δ^0_{11} which enter the r.h.s. of eq. 3.30 through $S^0_{11} = \exp(2i\delta^0_{11})$ (see also appendix A) for both models eqs 3.1 and 3.2. It may be seen that the nonlinear σ–model produces a very pronounced low–lying resonance near $k = 0.75\ m_\pi$ while the Skyrme model shows the well–known structure which hardly reaches 90^0 near $k = 3.5\ m_\pi$.

The other contribution the P_{11} channel receives is from the M1 mode. For this case, presented in figure 3.2, the phase shifts leave threshold with a negative scattering volume. For the Skyrme model, they start bending upwards at $k = 2.5\ m_\pi$, indicating the onset of the unphysical monotonic rise of the phase shifts[8] due to a peculiar momentum dependence introduced into the scattering potential by the Skyrme stabilization term [EHH86]. This effect leads us to restrict our considerations[9], as is obvious from the graphs, to low momenta $k \leq 3\ m_\pi$. For the nonlinear σ–model, the phase shifts asymptotically tend to zero at momenta beyond those considered here. The M1 mode

[7]Our notation for the intrinsic frame phase shift $\delta^K_{ll'}$ follows from eq. 3.30.

[8]This problem can be cured with the replacement of the Skyrme stabilization term by stabilization via vector mesons [SWHH89].

[9]Another reason, which will come later, is that having truncated the K–matrix eq. 3.31 to a single channel, we would not expect reasonable results above the π–Δ threshold $k_{th} = 2.93\ m_\pi$ because inelasticities start setting in.

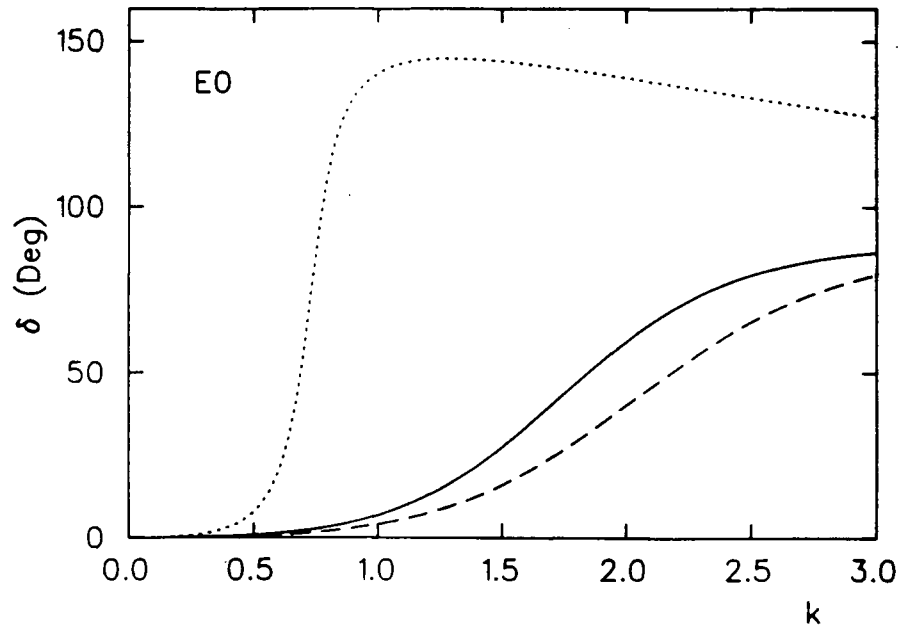

Figure 3.1: Phase shifts for the E0 breathing mode for Skyrme model with $e = 4$ and f_π equal 93 MeV (full line) and 110 MeV (dashed line), and for the nonlinear σ–model calculated with a Skyrme profile $F(r)$ for f_π=93 MeV and $e = 4$ (dotted line); $m_\pi = 138$ Mev. The abscissa is the absolute value of the pion three–momentum in units of m_π.

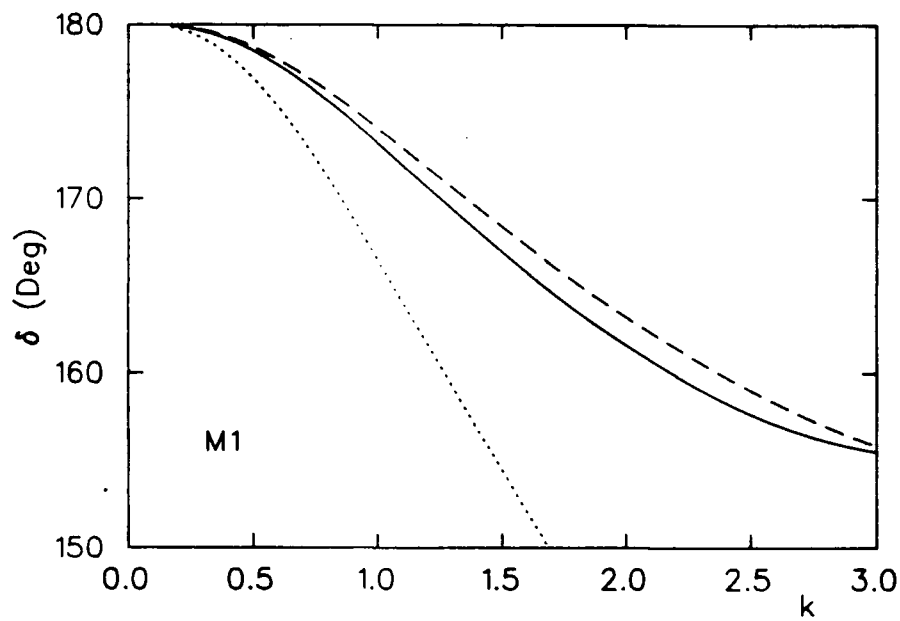

Figure 3.2: Phase shifts for the M1 magnetic dipole mode. Parameters, line types and axes as in 3.1.

contains the bound rotational zero mode at $k^2 = -m_\pi^2$ (this will be discussed at length in the section 3.3). This explains why the attractive potential felt by the fluctuation in this channel seemingly leads to a repulsive phase shift. In accordance with Levinson's theorem [GW64], the phase shifts are plotted starting at 180^0.

The $2T = 1, 2J = 3$ and $2T = 3, 2J = 1$ channels pick up only $K = 2$ contributions in addition to the M1 mode, and the π–N P waves couple also to the F waves in the $s = 3/2$ and $s = 5/2$ channels, for S^{13} and S^{31}, respectively. Only in S^{33} the π–N P waves couple to P and F waves in both, $s = 3/2$ and $s = 5/2$ channels, making S^{33} a 5×5 matrix, which picks up contributions from values $K = 0, 1, 2, 3$ on the r.h.s. of eq. 3.30. Figure 3.3 shows the K=2 phase shifts δ_{11}^2 for both models. Using a little foresight (see discussion preceding eq. 3.31), we present only the $l = l' = 1$ phase shifts: from these come the important contributions that survive when we truncate the laboratory S–matrix to the single π–N $\rightarrow$ π–N channel. For the Skyrme model, they remain close to zero; their influence on the low–energy behaviour of $S^{2T\,2J}$ is small. The nonlinear σ–model results, however, show rapidly increasing attraction for $k \geq 2$ m_π. The $K = 3$ contributions are unimportant in the low–energy region considered here; we accordingly consider the S–matrix for this channel as equal to one.

We now consider the background results necessary for the π–N S waves. For $2T = 2J = 1$, S^{11} couples π–N ($s = 1/2$) S waves $l = 0$ and $\pi - \Delta$ ($s = 3/2$) D waves ($l = 2$) together. The K–sum in eq. 3.30 is restricted to the single $K = 1$ electric dipole E1. Again, we present in figure 3.4 only the $l = l' = 0$ element due to our truncation procedure. Interestingly, δ_{00}^1 shows attractive behaviour for the Skyrme model and repulsive behavior for the nonlinear σ–model. This points towards a great dependence of the results on the precise form of the potential, even near threshold where we would perhaps not expect such sensitivity to the quartic Skyrme term (see the P wave backgrounds). For the $2T = 3$, $2J = 1$ channel, eq. 3.30 leads to the

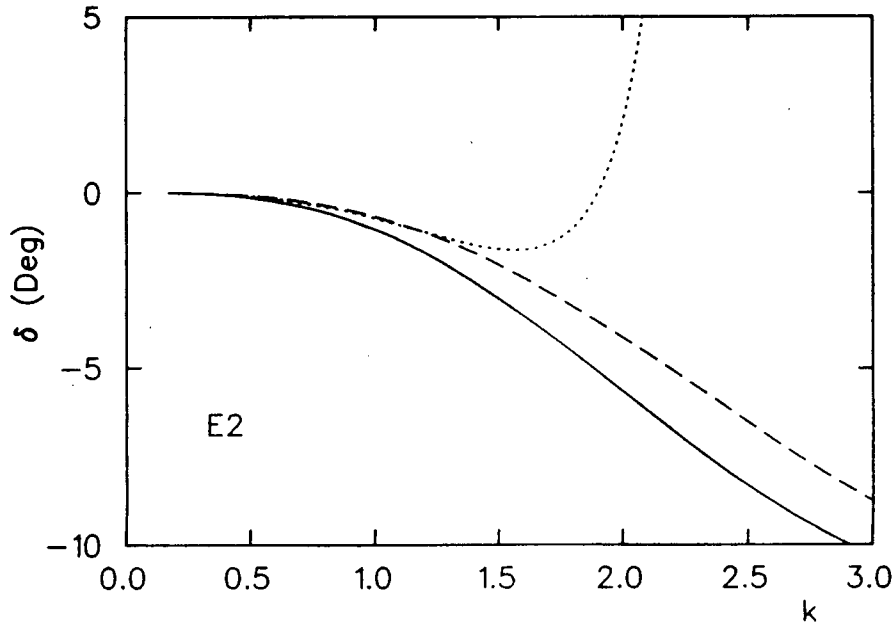

Figure 3.3: Phase shifts for the $l = l' = 1$ part of the electric quadrupole ($K = 2$) mode. Parameters and line types as in figure 3.1.

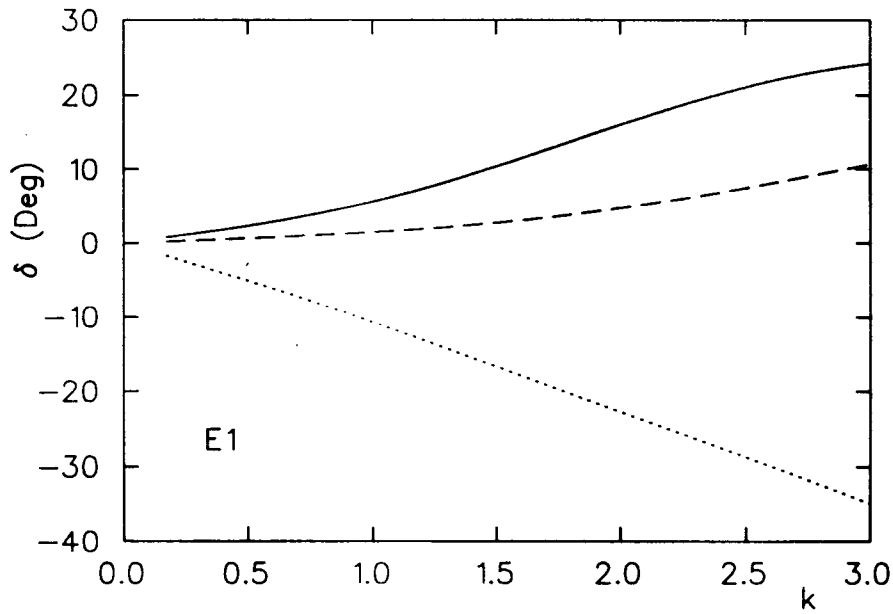

Figure 3.4: Phase shifts for the $l = l' = 0$ part of the electric dipole ($K = 1$) mode for the Skyrme model with $e = 4$, $f_\pi = 93$ MeV (full line) and $e = 5$, $f_\pi = 110$ MeV (dashed line); and for the nonlinear σ–model calculated with a Skyrme profile $F(r)$ for f_π=93 MeV and $e = 4$ (dotted line); $m_\pi = 138$ MeV. The abscissa is the absolute value of the pion three–momentum in units of m_π.

surprising result $S^{11} = S^{31}$ [HEHW84][MP85]; hence, we need not consider any further background contributions for the S waves. The identity is due to the fact that there are more independent scattering amplitudes in the laboratory frame than in the intrinsic frame: there results a "conspiracy" amongst the 6–j symbols, leading to the mentioned constraint. This is an unfortunate result, in violation of experiment and the Weinberg–Tomozawa rule for the description of low–energy S wave π–N scattering [W66][T66]. However, it is not catastrophic. The resolution[10] of this problem within the context of the Skyrme model has been known for some time and requires the introduction of time–derivative interactions [UK86] (see section 3.4).

3.2.3 Laboratory Frame Background Scattering and Truncation Procedure

With the intrinsic frame background results we obtain the corresponding physical scattering amplitudes by using eq. 3.30. We show in figures 3.5 and 3.6 (full line curves) the calculated phase shifts and inelasticities extracted from $S^{2T\,2J}_{1\;1/2;1\;1/2}$ corresponding to π–N P wave scattering in the Skyrme model. A quick comparison between the calculated phase shifts of figure 3.5 and the results of phase shift analyses (the data can be found in figure 3.10, for example) shows that the Skyrme model results achieve dismal agreement with the data. The P_{11} and P_{33} channels respectively show no sign of the Roper and Delta resonances; instead, there is repulsion at those values of momenta for which we would expect important attraction. The P_{13} and P_{31} channels are degenerate, but in reasonable agreement with experiment. As mentioned before, these initial background results were quite disappointing to workers in the field.

There is an important physical ingredient, however, which is not taken into account in the static background calculation. It is the opening of the π–N, π–Δ, π–$\frac{5}{2}^*$ at their

[10]In the context of the adiabatic approximation, the soft pion result for the S wave scattering lengths $a_1 = -2a_3$ is actually trivially satisfied by $a_1 = a_3 = 0$; there is no theoretical inconsistency.

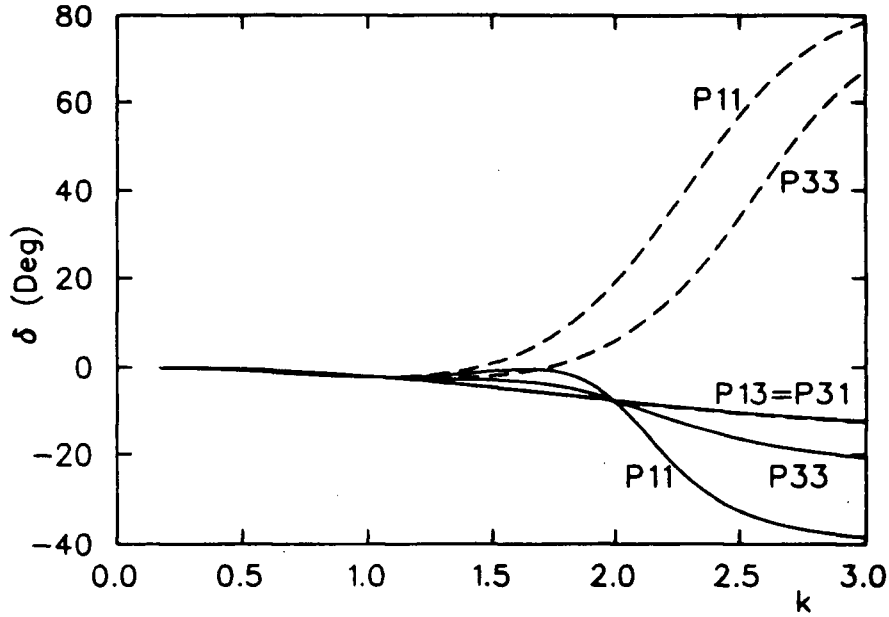

Figure 3.5: Untruncated (full line) and truncated (dashed line) background phase shifts δ for the four P wave channels as a function of absolute three–momentum k (units of m_π). Parameters are $f_\pi = 93$ MeV, $e = 4$ and $m_\pi = 138$ MeV. The four P wave truncated background phase shifts are indicated by dashed lines.

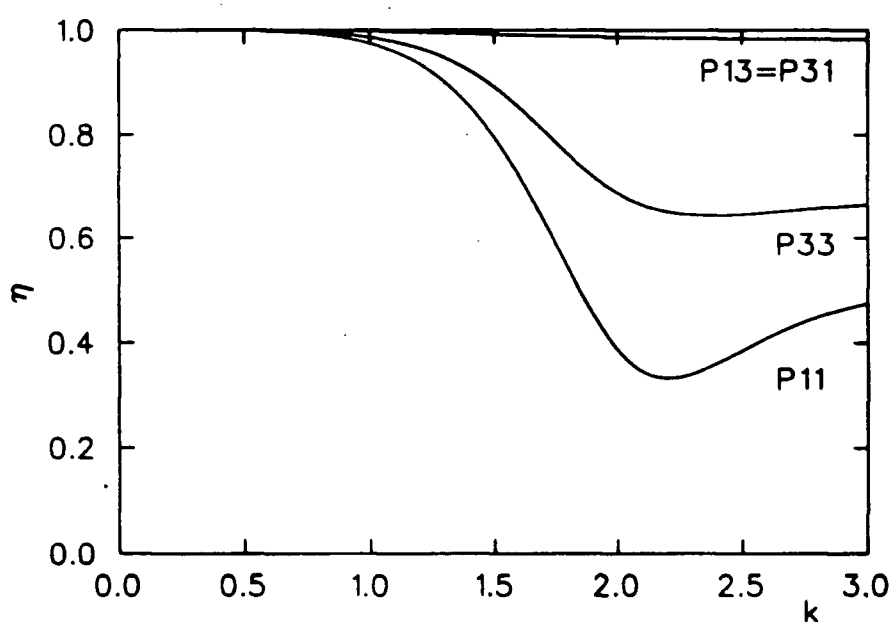

Figure 3.6: Background inelasticities η for the four P wave channels as a function of absolute three–momentum k (units of m_π). Parameters as in figure 3.5.

correct thresholds. Rather, in this approximation, N, Δ, $\frac{5}{2}^*$, are degenerate in mass and all channels open too soon at the π–N threshold. The consequence is that these unphysically open channels lead to large inelasticities where we expect elastic π–N scattering. This is shown in figure 3.6. In the soliton models, which does not include pion production because such terms would require an expansion of the lagrangian to third order in the fluctuations, the onset of inelasticities should come at the π–Δ threshold energy $\omega_{th} = M_\Delta - M_N + m_\pi = 3.1\ m_\pi$ (or $k_{th} = 2.93\ m_\pi$, (no recoil)). We would therefore like to cut off the coupling to the unphysically open $s = 3/2,\ 5/2$ channels and keep only the $l = l' = 1,\ s = 1/2$ channel. In order to retain unitarity of the S–matrix this can be done by solving coupled-channels equations directly in the laboratory frame without the unwanted channels [EHH86]. This procedure is, however, not straightforward. Instead, we suggest performing the truncation in the real symmetric K–matrix defined by

$$\mathrm{K}^{2T\,2J} = \frac{i}{k}\,\frac{1-\mathrm{S}^{2T\,2J}}{1+\mathrm{S}^{2T\,2J}}\,. \tag{3.31}$$

Truncating in the K–matrix can be done without affecting the reality and symmetry of the matrix; hence the reconstructed S–matrix obtained by inverting eq. 3.31 remains unitary. Such a truncation procedure certainly is a rather rough approximation to an exact calculation of the background K–matrix. However, it represents an acceptable method to implement the required closure of the unphysically open channels.

The truncated elastic π–N $\rightarrow$ π–N P wave phase shifts are presented in figure 3.5. In this figure, we emphasize the effects of the truncation procedure by plotting truncated and untruncated phase shift side by side. A comparison with phase shift analysis data (these can be found in figure 3.10, for example) shows that an important improvement is obtained through our simple truncation procedure. The Roper and Δ resonances now appear in the P_{11} and P_{33} channels respectively. Their order is however inverted

from what is observed experimentally, there not being enough attraction in the P_{33} channel. To bring the Δ pole in the K–matrix at its correct physical position, we will need to consider time–dependent interactions between the meson fluctuations and the collective coordinates (section 3.4). As for the Roper resonance, it occurs too soon, showing that there is too much attraction in P_{11} channel. This is due to the fact that the soliton is too soft against compressions. The $P_{13} \sim P_{31}$ channels are not affected by the truncation procedure. We also note that had we only removed the exotic π–$\frac{5}{2}^*$ channel and truncated the S–matrix to a 2×2 matrix, then the resulting situation, which we do not show, would have been intermediate between the untruncated case and the single channel case.

We apply a similar truncation procedure to the S waves S_{31} and S_{11}. The untruncated laboratory $S_{11} = S_{31}$ phase shifts are equal to the intrinsic δ^1_{00} due to cancelations amongst the 6–j symbols entering eq. 3.30. We therefore refer to figure 3.4 for these results. The implementation of the truncation procedure leads to the surprising result that the phase shifts remain the same after truncation. Only the inelasticities are affected, with the channel becoming of course elastic. This result for the S wave phase shifts requires an explanation. Let us parametrize the real and symmetric 2×2 K^{11}–matrix as

$$ikK = -\frac{1-S}{1+S} = i \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{3.32}$$

for a,b,c some real numbers and drop superscripts. The phase shift δ extracted from the (1,1) element of the S–matrix expressed in terms of a, b, c is

$$\tan(2\delta) = 2\,\frac{a + ac^2 - 2cb^2}{1 - a^2 - b^4 - a^2c^2 + c^2 + 2b^2a^2}\,. \tag{3.33}$$

Now if we truncate the K–matrix to a single element

$$ikK = ia, \tag{3.34}$$

we find the truncated phase shift

$$\tan(2\delta_t) = 2\,\frac{a}{1-a^2}\,. \tag{3.35}$$

The phase shifts δ and δ_t look very different. But for the S wave calculation, it turns out that "a" dominates, and "b" and "c" are quite small comparatively. Now since the contributions $-1 \leq b, c \leq 1$ are at least squared, $\delta \sim \delta_t$. A similar derivation explains why the truncated and untruncated P_{13} $(= \mathrm{P}_{31})$ channel remains the same.

3.3 Zero Modes

Formally, the topics discussed in this section should be comprised in the preceding section dealing with background scattering. The zero mode contributions to the S–matrix are part of the background. But their importance has been magnified by the recent observation of Kawarabayashi and Ohta [KO89] that the Born terms for baryon exchange coincide with the pole contributions of the zero energy solutions to the background scattering amplitudes. This has been followed by the demonstration due to Verschelde [V89][V91] that the rotational zero mode contributions supplemented with the linear time–derivative interaction lead to the successful Δ–isobar model of P wave π–N scattering. A similar result has been obtained by Holzwarth [H90] using a simpler framework. In section 3.4 we will add to the background calculation the contributions coming from the time–derivative interactions. The question as to whether this combination will actually reflect the observed features of the experimental P wave phase shifts therefore depends on the extent to which the zero modes dominate the low–energy background scattering amplitude for positive values of k^2 in a particular model and on the structure of the resulting form factors. In this section, we study these points which

have relevance to the P waves[11].

For the massless case, as considered by Verschelde [V89][V91], the dominant behaviour is evident because the coincidence of the rotational zero modes with the pion threshold leads to phase shifts depending linearly on pion momentum k, as compared to the cubic k^3–law for other P wave contributions. Since for the massive case we consider here this is no longer true one may wonder how well the scheme actually works. But before addressing these details, we present some simple concepts concerning the zero modes.

Zero modes [R82] are discrete states with energy $\omega = 0$. They arise in a theory when the classical ground state solution about which one is constructing the eigenspectrum breaks the original symmetry of the lagrangian. Take the lagrangian to possess an invariance, such as translational or rotational. Then it costs no energy to generate a new solution $\varphi' = \varphi + \delta\varphi$ where $\delta\varphi$ represents the infinitesimal change of the ground state solution φ under the considered global transformation. If group parameters α_i describing the transformation are introduced as time dependent collective coordinates $\alpha_i(t)$, the variation of φ along the collective motion defines the zero mode $\Psi_i^{(0)}$

$$\begin{aligned} \varphi' - \varphi &= \frac{\delta\varphi}{\delta\alpha_i}\delta\alpha_i \\ &= \Psi_i^{(0)}\delta\alpha_i. \end{aligned} \tag{3.36}$$

Here, $\delta\varphi/\delta\alpha_i$ is a general variation to be interpreted, for example, as a partial derivative in the case of translations $\vec{e}$

$$\begin{aligned} \varphi(\vec{x} - \vec{e}) &= \varphi(\vec{x}) + \frac{\partial\varphi}{\partial\vec{x}} \cdot \vec{e}, \\ \vec{\Psi}^{(0)} &= \frac{\partial\varphi}{\partial\vec{x}}, \end{aligned} \tag{3.37}$$

[11]Similar observations for the S waves, although using a different methodology, are due to the Siegen part of the collaboration in the paper of Holzwarth, Walliser and Pari [HWP90], and are therefore not reported in this thesis.

or as a commutator in the case of isospin symmetry generated by a rotation in isospace around an arbitrary axis $\vec{a}$

$$\begin{aligned} \varphi(\vec{x})' &= \varphi(\vec{x}) + [\frac{i}{2}\vec{\tau}\cdot\vec{a}, \varphi(\vec{x})], \\ \vec{\Psi}^{(0)} &= [\frac{i}{2}\vec{\tau}, \varphi(\vec{x})]. \end{aligned} \tag{3.38}$$

In usual 3+1 dimensions, we have translational modes corresponding to the 3 spatial directions and for the internal isospin symmetry SU(2) $\sim$ SO(3), we have zero modes corresponding to the 3 independent rotations.

We now consider the specific case of the Skyrme and nonlinear $\sigma-$ models and restrict ourselves in further discussion to the case of the isorotational zero mode which affects the π–N P waves. Taking φ to be the hedgehog ansatz U_0 of eq. 3.3, the zero modes corresponding to global isorotations around three orthogonal axes $\hat{a}_i$ are

$$\Psi_i^{(0)}(\vec{x}) = [U_0, \frac{i}{2}\hat{a}_i\cdot\vec{\tau}] = i(\hat{a}_i\times\hat{r})\cdot\vec{\tau}\sin F(r). \tag{3.39}$$

The spatial form of the zero modes shows that it has angular momentum $l = 1$. Under parity,

$$\mathcal{P}\Psi_i^{(0)}(\vec{x}) = \Psi_i^{(0)\dagger}(-\vec{x}) = \Psi_i^{(0)}(\vec{x}). \tag{3.40}$$

This therefore indicates that the zero modes we are considering appear in the M1 channel as a zero energy solution to the differential equation. A quick inspection of eq. A.3 reveals the zero energy solution[12]

$$\chi^{(0)} = F(r). \tag{3.41}$$

An important point to which we now turn concerns the normalization of the zero modes. The normalization condition can be found by considering the kinetic part of

[12]By rewriting the M1 mode differential equation in such a way as to have it in Schrödinger–like form, the radial radial dependence of the zero mode becomes $\chi^{(0)} = \sin F(r)$ as in eq. 3.39.

the lagrangian (whether it be 3.1 or 3.2) with soliton ansatz $U = \theta_0 + i\vec{\tau}\cdot\vec{\theta}$. Written in terms of the three independent real variables[13], we have

$$\mathcal{T} = \sum_{mn} \int \dot{\theta}_m \; \Gamma_{mn}(\vec{\theta},\vec{\nabla}\vec{\theta}\,) \; \dot{\theta}_n d^3x, \tag{3.42}$$

where $\Gamma_{mn}(\vec{\theta},\vec{\nabla}\vec{\theta})$ is a general function of the chiral angle and its spatial derivatives multiplying the time derivative of the baryonic variables. Now

$$\frac{\delta^2\mathcal{T}}{\delta\dot{\theta}_m(\vec{x})\delta\dot{\theta}_n(\vec{y})} = 2\;\Gamma_{mn}(\vec{y})\;\delta^3(\vec{x}-\vec{y}) \tag{3.43}$$

allows us to write the identity

$$\begin{aligned}
\mathcal{T} &= \frac{1}{2}\sum_{mn}\int\int \dot{\theta}_m(\vec{x})\;\frac{\delta^2\mathcal{T}}{\delta\dot{\theta}_m(\vec{x})\delta\dot{\theta}_n(\vec{y})}\;\dot{\theta}_n(\vec{y})d^3xd^3y \\
&= \frac{1}{2}\sum_{ijmn}\int\int \frac{\partial\theta_m(\vec{x})}{\partial\alpha_i}\;\frac{\delta^2\mathcal{T}}{\delta\dot{\theta}_m(\vec{x})\delta\dot{\theta}_n(\vec{y})}\;\frac{\partial\theta_n(\vec{y})}{\partial\alpha_j}d^3xd^3y\;\dot{\alpha}_i\dot{\alpha}_j \\
&= \frac{1}{2}\sum_i \Theta_i\dot{\alpha}_i^2 = \frac{\Theta}{2}\sum_i \dot{\alpha}_i^2.
\end{aligned} \tag{3.44}$$

In these expressions, the second line introduces explicitly the zero modes. The last line follows from the fact that the collective coordinates are chosen in such a way that the quadratic form is diagonal. Θ_i, $i = 1,2,3$, are inertia parameters and we have explicitly indicated the fact that all turn out to be equal in the SU(2) models we consider. We can then read off from the last two lines the normalization condition for the zero modes $\vec{\Psi}_i^{(0)} = \partial\vec{\theta}/\partial\alpha_i$

$$\frac{1}{\Theta}\int\int \Psi_{im}^{(0)\,*}(\vec{x})\;\mathcal{M}_{mn}(\vec{x},\vec{y})\;\Psi_{jn}^{(0)}(\vec{y})\;d^3xd^3y = \delta_{ij}. \tag{3.45}$$

This shows that the zero modes $\vec{\Psi}_i^{(0)}/\sqrt{\Theta}$ are normalized with respect to the metric

$$\mathcal{M}_{mn}(\vec{x},\vec{y}) = \frac{\delta^2\mathcal{T}}{\delta\dot{\theta}_m(\vec{x})\delta\dot{\theta}_n(\vec{y})}\;. \tag{3.46}$$

[13]Unitarity imposes $\theta_0 = \sqrt{1-\vec{\theta}\cdot\vec{\theta}}$.

We then have for the nonlinear σ–model lagrangian eq. 3.1

$$\mathcal{M}^{\sigma}_{mn}(\vec{x},\vec{y}) = f_\pi^2\ \delta^3(\vec{x}-\vec{y})\ \delta_{mn}\ \mathrm{M}_\sigma(r)$$

$$\mathrm{M}_\sigma(r) = 1 \tag{3.47}$$

and for the Skyrme model lagrangian eq. 3.2

$$\mathcal{M}^{S}_{mn}(\vec{x},\vec{y}) = f_\pi^2\ \delta^3(\vec{x}-\vec{y})\ \delta_{mn}\ \mathrm{M}_S(r)$$

$$\mathrm{M}_S(r) = 1 + \frac{1}{e^2 f_\pi^2}(F'^2 + \frac{sin^2 F}{r^2}). \tag{3.48}$$

For convenience, we can include the metric in the definition of the normalized zero modes $\vec{\Phi}_i^{(0)}$

$$\vec{\Phi}_i^{(0)} = \frac{f_\pi}{\sqrt{\Theta}} i(\hat{a}_i \times \hat{x}) \sin F \sqrt{\mathrm{M}(r)} \tag{3.49}$$

in order to have the usual form

$$\int \vec{\Phi}_i^{(0)*} \cdot \vec{\Phi}_j^{(0)}\, d^3x = \delta_{ij}. \tag{3.50}$$

We note that had $\vec{\Psi}^{(0)}$ described scattering states rather than zero energy bound states, we would have replaced δ_{mn} by $\delta^3(\vec{k}-\vec{k}')$ in the normalisation condition eq. 3.45.

We now go on to extract the zero mode contributions to the background scattering amplitude. By a redefinition of the wavefunction for the M1 mode through the absorption of the metrical factor $\sqrt{\mathrm{M}}$ in the wavefunction, we can rewrite the differential equation in the form of a non–relativistic Schrödinger equation for P wave potential scattering

$$(-\partial_r^2 - \frac{2}{r}\partial_r + \frac{2}{r^2} + V_1(r) - k^2)\Phi(r) = 0, \tag{3.51}$$

with $k^2 = \omega_{k^2}^2 - m_\pi^2$. The corresponding 1st order Born approximation to the partial scattering amplitude with appropriate boundary conditions is defined by [GW64]

$$f_1(k^2) = -\int (j_1(kr))^2 V_1(r) r^2 dr - \sum_n \frac{(\int j_1(kr) V_1(r) \Phi_n(r) r^2 dr)^2}{k^2 - k_n^2 + i\epsilon}, \tag{3.52}$$

where the n–sum picks up all normalized bound states Φ_n of eq. 3.51 and implies an integral over the scattering states. The radial part of the zero modes eq. 3.49 satisfies eq. 3.51 at a momentum $k_n^2 = -m_\pi^2$. The contribution $f_1^{(0)}(k^2)$ of the zero modes eq. 3.49 to the scattering amplitude eq. 3.52 therefore is

$$\begin{aligned} f_1^{(0)}(k^2) &= -\frac{((k^2 + m_\pi^2) \int j_1(kr) \sin F \sqrt{\mathrm{M}(r)} r^2 dr)^2}{(k^2 + m_\pi^2) \int \sin^2 F \ \mathrm{M}(r) r^2 dr} \\ &= -\frac{3}{2\pi\Theta} \frac{k^2}{m_\pi^2} \frac{f_{\pi\mathrm{NN}}^2(k^2)}{\omega_k^2}, \end{aligned} \tag{3.53}$$

with $\omega_k^2 = k^2 + m_\pi^2$ and the πNN–form factor defined as

$$f_{\pi\mathrm{NN}}(k^2) = \frac{m_\pi}{k} f_\pi \frac{4\pi}{3} \omega_k^2 \int \sin F(r) \sqrt{\mathrm{M}(r)} j_1(kr) r^2 dr. \tag{3.54}$$

We note that in deriving eq. 3.54, we used eq. 3.51 to replace the potential by derivative terms and simplified the result with the recurrence relation satisfied by spherical Bessel functions of order l:

$$r^2 j''_l(kr) + 2r j'_l(kr) + (r^2 - l(l+1)) j_l(kr) = 0. \tag{3.55}$$

Through $\mathrm{M}(r)$ the form factor eq. 3.54 depends explicitly on the form of the lagrangian. For $\mathrm{M}(r) = 1$ it coincides with the πNN form factor derived from the static pion cloud as suggested by Cohen [C86] and further discussed in reference [KMW87]. Its value at the pole $k^2 = -m_\pi^2$, the πNN coupling constant $f = f_{\pi\mathrm{NN}}(-m_\pi^2)$ (with experimental value of $f^2/4\pi = 0.08$), picks up only the asymptotic amplitude of $F(r)$, and the deviation of $\mathrm{M}(r)$ from 1 is irrelevant. However, as can be seen from figure 3.7 the actual form of $f_{\pi\mathrm{NN}}(k^2)$ for the same profile function $F(r)$ differs apprecia-

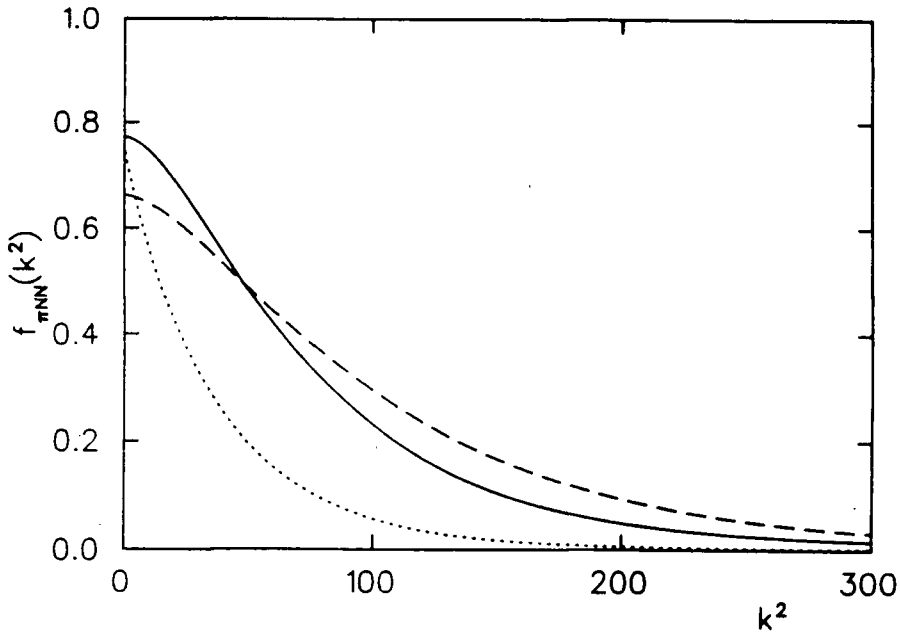

Figure 3.7: The pion–nucleon form factor $f_{\pi\mathrm{NN}}(k^2)$ as defined in eq. 3.54 for the Skyrme model and nonlinear σ–model. Parameters line types and axes as in figure 3.1.

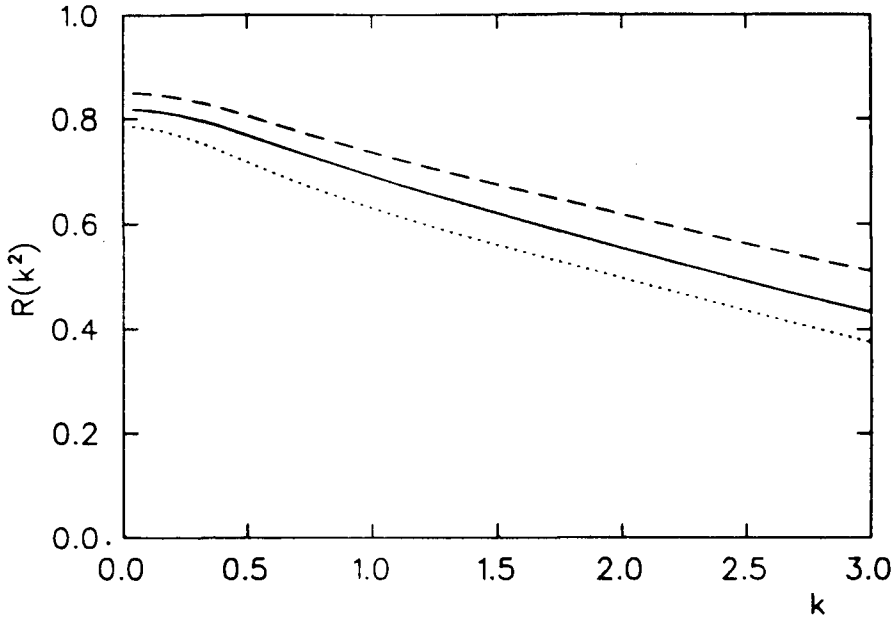

Figure 3.8: Ratios eq. 3.56 eq. between the zero mode contribution defined in eq. 3.53 and the real part of the total M1 amplitude (figure 3.2). Parameters, line types and axes as in figure 3.1.

bly for the lagrangians eqs 3.1, 3.2, which as previously noted have different metrics. From the origin of M(r) in eq. 3.49 it is evident that purely static considerations as in [C86][KMW87] cannot account for this specific feature of the dynamical form factor eq. 3.54. One may argue that deviations of M(r) from 1 do not indicate particularly attractive features of any model. Still, for consistency, they have to be included in eq. 3.54 and have a significant effect. The form factor we have derived here agrees with the one obtained by Saito [S87b].

This is an appropriate point to justify our choice of parameters used to calculate the background amplitudes contributing to P waves and in the presentation of the form factor. We choose for our comparison a chiral angle profile $F(r)$ created from the Skyrme lagrangian eq. 3.2 for a value of $e = 4$. It is evident from table 3.1 that with a profile derived from the Skyrme model (with f_π fixed at its experimental value of 93 MeV) it is not possible to simultaneously reproduce both, the πNN coupling constant $f = 1.0$ and the moment of inertia $\Theta = 0.70\ m_\pi^{-1}$ with good accuracy. But $e = 4$ seems

$e =$	3.0	3.5	4.0	4.5	Exp.
$\Theta_{NL\sigma}$	0.673	0.479	0.351	0.266	0.708
Θ_{SK}	1.772	1.183	0.829	0.606	
f	1.25	0.995	0.772	0.617	1.0

Table 3.1: Moments of inertia Θ in units of m_π^{-1} for nonlinear σ–model (NLσ) and Skyrme (SK) model and the pion-nucleon coupling constant $f = f_{\pi\mathrm{NN}}(k^2 = -m_\pi^2)$; calculated for a Skyrme profile $F(r)$ for $f_\pi = 93$ and different values of the parameter e in eq.(2).

a reasonable compromise for our present explorative purpose and it could probably be improved by changing the stabilization mechanism. In order to have a well–defined comparison we use the same profile function also as the background for the nonlinear σ–model eq. 3.1 although the resulting moment of inertia Θ_σ is too small by a factor of 2.

Both form factors intersect at $k^2 = -m_\pi^2$ with the value of $f_{\pi\mathrm{NN}} = 0.772$ which reflects the asymptotic normalization of the $e = 4$ soliton profile. However, while $\mathrm{M}_\sigma(r) = 1$ leads to the "hard" negative slope with rising k^2, the Skyrme metric $\mathrm{M}_S(r)$ creates a very soft form factor which stays nearly constant from $k^2 = -m_\pi^2$ to $k^2 = 0$ and leaves the threshold with an almost horizontal tangent. The second set of parameters which we have used to present Skyrme model results is $f_\pi = 110$ MeV, $e = 4$, $m_\pi = 138$ MeV. This choice was dictated by the study of the P_{11} channel in section 3.5. It allows for a slightly better description of the Roper resonance as well as P_{11} scattering volume.

Now we can consider the extent to which the zero energy mode eq. 3.49 dominates in the scattering amplitude eq. 3.52 for positive values of k^2. Since the zero mode contribution $f_1^{(0)}(k^2)$ is real, we choose to compare it to the real part of the scattering amplitude $f_1(k^2)$. Due to the close connection between real and imaginary parts of f_1 dictated by unitarity this implies that we can expect a dominant contribution of $f_1^{(0)}$ at most for small values of the phase shifts $\delta_{11}^1(k)$ (modulo π). Figure 3.8 shows the ratios

$$\begin{aligned} R(k^2) &= \frac{Re\{f_1(k^2)\}}{f_1^{(0)}} = \frac{Re\{\sin\delta_{11}^1\ e^{i\delta_{11}^1}\}}{k\ f_1^{(0)}} \\ &= \frac{1}{k}\sin\delta_{11}^1\cos\delta_{11}^1/f_1^{(0)}(k^2) \end{aligned} \tag{3.56}$$

for both lagrangians eqs 3.1, 3.2. The M1 mode phase shifts δ_{11}^1 that enter in eq. 3.56 were shown in figure 3.2. One striking feature of figure 3.8 is that we obtain basically parallel straight lines for the ratio[14] eq. 3.56. This means that in both models the k–dependence of the phase shifts is well represented by the k–dependence of the contributions $f_1^{(0)}$ deriving from the zero modes, although the ratios eq. 3.56 have fallen to about 50% near $k \sim 2.5\ m_\pi$. However, most important for the comparison with the

[14]This feature actually extends up to k–values where the M1 phase shifts for the Skyrme model cross the $\delta = 180^0$ line.

Δ–isobar model, at threshold the M1 scattering volume is still about 80% of the zero mode contribution for both models (note that for the attractive M1 potential the first term on the r.h.s. of eq. 3.52 is positive, i.e. reducing the negative value of eq. 3.53). Due to this dominance the difference in the low–energy M1 phase shifts in figure 3.2 just reflects the difference in the moments of inertia given in table 3.1.

3.4 Time–dependent Interactions

The introduction of time–dependent interactions [UK86][HHS87][S87a][VV89][JM89], those interactions which couple time–derivatives of the soliton's collective coordinates to the meson fluctuations, proceeds in a straightforward way from the background interactions. We let the hedgehog be time–dependent through the introduction of collective coordinates $\vec{\alpha}(t)$ (the Euler angles) parametrizing isorotations

$$U_0 = e^{i\vec{\tau}\cdot D^{\dagger}(\vec{\alpha})\cdot\hat{r}\ F(r)} = e^{i\vec{\tau}\cdot\hat{\gamma}\ F(r)}, \tag{3.57}$$

where $D(\vec{\alpha})$ are the familiar rotation matrices discussed in chapter 1. Here we introduce meson fluctuations according to the Schnitzer form eq. 3.6 and expand the soliton lagrangian to whichever order is appropriate. In contradistinction with our discussion of background scattering where everything was done in the intrinsic frame, only later to be transformed to the physical laboratory frame through a geometric recoupling, we directly consider time–derivative interactions in the laboratory frame. This is evident from our expansion about eq. 3.57, the rotating hedgehog.

We substitute U_S, with U_0 given by eq. 3.57, in the Skyrme lagrangian and find

$$L_{int} = -\frac{1}{2}\int d^3x\ tr\{\ (U_0^{\dagger}\dot{R}U_0 + \dot{L})(f_\pi^2\dot{H} + \frac{1}{4e^2}[\vec{H},[\dot{H},\vec{H}]])\ \}, \tag{3.58}$$

where $H_\mu = U_0^{\dagger}\partial_\mu U_0$, $R_\mu = U_\xi^{\dagger}\partial_\mu U_\xi$ and $L_\mu = \partial_\mu U_\xi U_\xi^{\dagger}$. Here we have neglected hard

pion terms [S84] of order $\sim (\partial U_\xi)^2(\partial U_0)^2$, $\sim (\partial U_\xi)^4$ and only retained terms with time–derivatives on the meson fluctuations: gradient terms $\vec{R}$, $\vec{L}$ do not have a significant effect near threshold since they are linear in momentum. The interaction lagrangian can be recast in a familiar form if $\dot{R} \equiv R_0$; $\dot{L} \equiv L_0$ are expanded to second order:

$$\dot{L} = \frac{i}{2f_\pi}\vec{\tau}\cdot(\dot{\vec{\xi}} - \frac{1}{2f_\pi}\vec{\xi}\times\dot{\vec{\xi}}) + \mathcal{O}(\xi^3), \tag{3.59}$$

$$\dot{R} = \frac{i}{2f_\pi}\vec{\tau}\cdot(\dot{\vec{\xi}} + \frac{1}{2f_\pi}\vec{\xi}\times\dot{\vec{\xi}}) + \mathcal{O}(\xi^3). \tag{3.60}$$

The result is

$$\begin{aligned} L_{int} &= L_{int}^{(1)} + L_{int}^{(2)} \\ &= \frac{1}{2f_\pi}\int \dot{\vec{\xi}}\cdot\vec{A}^{\,0}d^3x + \frac{1}{4f_\pi^2}\int(\dot{\vec{\xi}}\times\vec{\xi})\cdot\vec{V}^0 d^3x, \end{aligned} \tag{3.61}$$

where $\vec{A}^{\,0}$, $\vec{V}^0$ are the time components of the axial–vector and vector Noether currents associated with the rotating hedgehog:

$$\vec{A}^{\,0} = -\frac{i}{2}tr\{\vec{\tau}\cdot(f_\pi^2\dot{H} + \frac{1}{4e^2}[\vec{H},[\dot{H},\vec{H}]] - (H\to G))\}, \tag{3.62}$$

$$\vec{V}^0 = -\frac{i}{2}tr\{\vec{\tau}\cdot(f_\pi^2\dot{H} + \frac{1}{4e^2}[\vec{H},[\dot{H},\vec{H}]] + (H\to G))\}, \tag{3.63}$$

where $G_\mu = U_0\partial_\mu U_0^\dagger$. The form eq. 3.61 can be identified with the standard soft–pion interaction terms of Weinberg [W66] if the soliton currents are replaced by the standard relativistic bilinear forms

$$\vec{A}^{\,0} = \bar{\psi}\gamma^0\gamma_5\vec{\tau}\psi, \tag{3.64}$$

$$\vec{V}^0 = \bar{\psi}\gamma^0\vec{\tau}\psi, \tag{3.65}$$

where ψ represent nucleon spinors and γ^0, γ_5 Dirac matrices [BD64].

Eq. 3.61 highlights the physical character of the interactions; the simplified form of the time–derivative coupling is obtained by evaluating the trace of the integrand appearing in the expressions for the axial–vector and vector currents:

$$L_{int}^{(1)} = f_\pi \int (\dot{\hat{\gamma}} \cdot \dot{\vec{\xi}}) \ \sin F \cos F \ (1 + \frac{1}{e^2 f_\pi^2}(F'^2 + \frac{\sin^2 F}{r^2})) \ d^3x, \tag{3.66}$$

$$L_{int}^{(2)} = \frac{1}{2} \int (\dot{\hat{\gamma}} \times \hat{\gamma}) \cdot (\vec{\xi} \times \dot{\vec{\xi}}) \ \sin^2 F \ (1 + \frac{1}{e^2 f_\pi^2}(F'^2 + \frac{\sin^2 F}{r^2})) \ d^3x. \tag{3.67}$$

Perhaps the occurrence of eq. 3.66 will surprise since we explicitly mentioned following eq. 3.11 that no term linear in the fluctuation field remained in the lagrangian. But there is no paradox since we were then dealing with static background. In the time–dependent case, the rotating hedgehog eq. 3.57 is not a solution to the variational equations, and we therefore do not expect anymore a vanishing term linear in $\vec{\xi}$ [HHS87].

We emphasize that the interaction lagrangians eqs 3.66, 3.67, contain only classical variables. The question of promoting to quantum operators the collective coordinates $\vec{\alpha}$ and the meson fields $\vec{\xi}$ is a thorny issue. This is due to the fact that we are dealing with a constrained system. That is, the canonical momentum associated with the collective coordinates and the canonical momentum associated with the meson fluctuations are not independent of each other [HHS87][V87]. This is clear since the introduction of meson fluctuations and collective coordinates simultaneously really represents a duplication of rotational degrees of freedom.

The correct method to treat such constrained meson–soliton systems has been known for some time [T75] and employs the method of Dirac brackets in the quantization scheme [D64]. For the Skyrme model, Verschelde [V89][V91] has applied this intricate formalism with success to the case of both π–N P and S waves [VV90]. The important result he obtained, as mentioned before, was the equivalence of the phenomenological Δ–isobar model of π–N P waves with the sum of zero modes and linear

time–derivative interaction contributions.

Recently, Holzwarth [H90] has demonstrated how it is possible to arrive at the Skyrme model Δ–isobar model link without having to consider the constrained nature of the meson–soliton system. This alternative method, supplemented by a few ideas concerning the choice of interpolating plane–wave meson field, is the one we will adhere to in this work. We therefore outline this scheme by beginning our considerations with $L_{int}^{(1)}$ of eq. 3.66. This is the crucial term for the P waves: it is responsible for shifting the Δ pole to its correct physical position.

Following Holzwarth, we quantize the collective coordinates according to the usual prescription described in chapter 1:

$$\vec{\Omega} \rightarrow \vec{I}/2\Theta, \tag{3.68}$$

where $\vec{\Omega}$ is the angular velocity of the collective coordinates defined through

$$\dot{\hat{\gamma}} = (\vec{\Omega} \times \hat{\gamma}), \tag{3.69}$$

Θ is the hedgehog's moment of inertia and $\vec{I}$ are the generators of SU(2) isospin. Expression eq. 3.69 is then symmetrized with respect to the non–commuting $\vec{\Omega}$, $\hat{\gamma}$ so as to be hermitian:

$$L_{int}^{(1)} \rightarrow \frac{f_\pi}{2} \int \epsilon_{abc}(\Omega_b D_{cd}^\dagger + D_{cd}^\dagger \Omega_b)\hat{r}_d \dot{\xi}_a \ \sin F \cos F(1 + \frac{1}{e^2 f_\pi^2}(F'^2 + \frac{\sin^2 F}{r^2}))d^3x. \tag{3.70}$$

For the meson field we must take into account the nature of the representation we choose (see eq. 3.4). Although the explicit form of the interaction lagrangian eq. 3.70 may look quite different for different choices of interpolating fields, the final results will be the same if $L_{int}^{(1)}$ is evaluated for the scattering waves (i.e. exact solutions to the background equations of appendix A) obtained in the chosen representation. In our present context, however, we will approximate matrix elements of $L_{int}^{(1)}$ by the plane-wave Born approximation, to be added to the background K–matrix. The quality of

such an approach then will depend on the choice of the representation, i.e. on which interpolating field is substituted by plane waves. However, it turns out that the actual form of eq. 3.70 suggests how to reduce this ambiguity. This expression evidently involves only the transverse components[15] of $\dot{\vec{\xi}}$. These appear in the kinetic energy eq. 3.11 with a metrical factor $\cos^2 F\ \mathrm{M}_S(r)$. For a plane wave Born approximation it therefore seems appropriate to substitute plane waves for transverse normal modes $\cos F\sqrt{M_S}\,\vec{\xi}$ in order to account for the specifics of the Schnitzer representation.

We therefore let

$$\cos F\sqrt{M_S}\,\vec{\xi} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}}\{\vec{a}(\vec{k})e^{i\omega_k t - i\vec{k}\cdot\vec{x}} + \vec{a}^\dagger(\vec{k})e^{-i\omega_k t + i\vec{k}\cdot\vec{x}}\}, \tag{3.71}$$

where $\vec{a}^\dagger(\vec{k})$, $\vec{a}(\vec{k})$ are second quantized creation and annihilation operators satisfying the usual bosonic commutation relations and operating on 1–meson states (isospin index α', momentum index $\vec{k}'$) according to the rule

$$a_\alpha(\vec{k})\ |\alpha',\ \vec{k}'> = (2\pi)^{3/2}\sqrt{2\omega_{k'}}\ \delta_{\alpha\alpha'}\ \delta^3(\vec{k}' - \vec{k})\ |0> . \tag{3.72}$$

Here, $|0>$ is the meson vacuum state and a similar relation holds for the creation operator $\vec{a}^\dagger(\vec{k})$.

With the formalism as outlined above, we can consider the vertex for the emission and absorption of a pion with momentum $\vec{k}$ and isospin orientation $\hat{e}$ by taking matrix elements of the form $< s' \mid L^{(1)}_{int} \mid \vec{k}, \hat{e}; s >$, where s', s denote final and initial baryon states. For the baryonic part, we need the following results derived with the use of the Wigner–Eckart theorem:

$$< N'|D^\dagger_{aj}|N > = -\frac{1}{3} < N'|\tau_a\sigma_j|N >, \tag{3.73}$$

$$< \Delta|D^\dagger_{aj}|N > = -\frac{1}{\sqrt{2}} < \Delta|T^{(+)}_a S^{(+)}_j|N >, \tag{3.74}$$

[15] $\dot{\vec{\xi}} = \dot{\vec{\xi}}\cdot\hat{\gamma}\ \hat{\gamma} + \dot{\vec{\xi}}_T = \dot{\vec{\xi}}_L\hat{\gamma} + \dot{\vec{\xi}}_T$.

which state that the first index a of $D^\dagger$ transforms as an isospin τ_a whereas the second index transforms as a spin σ_j. $T_a^{(+)}$ and $S_j^{(+)}$ are respectively the a and j components of isospin and spin $3/2 \to 1/2$ transition operators defined in the usual way via their Clebsch–Gordan coefficients [EW88]:

$$< 3/2a_\Delta|T_a^{(+)}|1/2a_N >=< 3/2a_\Delta|1a1/2a_N >, \tag{3.75}$$

$$< 3/2j_\Delta|S_j^{(+)}|1/2j_N >=< 3/2j_\Delta|1j1/2j_N > . \tag{3.76}$$

Lowering operators $\vec{T}^{(-)}$, $\vec{S}^{(-)}$ are defined in a similar way.

With these results, we immediately observe that $L_{int}^{(1)}$ cannot connect nucleon states. Taking matrix elements of eq. 3.70, eq. 3.73 leads to

$$\epsilon_{abc}(\Omega_b\tau_c + \tau_c\Omega_b) = 0 \tag{3.77}$$

since for nucleon states, $\vec{T} \equiv \vec{\tau}$. $L_{int}^{(1)}$ can, however, contribute to elastic π–N scattering through an intermediate Δ state. As shown by Holzwarth [H90],

$$\epsilon_{abc}(\Omega_b T_c^{(+)} + T_c^{(+)}\Omega_b) = \frac{3i}{\Theta}\, T_a^{(+)}. \tag{3.78}$$

This allows us to write the general form for the vertex as

$$< s' \mid L_{int}^{(1)} \mid \vec{k}, \hat{e}; s >= -i\frac{3}{2\Theta\omega_k} f_{\pi ss'}(k^2) \;\; < s' \mid (\hat{e}\cdot\vec{T}^{(\pm)})\,(\vec{k}\cdot\vec{S}^{(\pm)}) \mid s >, \tag{3.79}$$

with selection rule $s' = s \pm 1$. The πNΔ form factor entering this expression differs from the πNN form factor eq. 3.54 extracted from the zero modes only by a geometrical factor which arises from the different baryonic matrix elements [H90]

$$f_{\pi\mathrm{N}\Delta}(k^2) = \frac{3}{\sqrt{2}} f_{\pi\mathrm{NN}}(k^2). \tag{3.80}$$

The fact that the k-dependence is the same relies on our replacement eq. 3.71 which leaves the metric $\sqrt{M_S}$ in the same way as it appears in eq. 3.54. So the form factor reflects the model lagrangian but not the representation chosen for the fluctuations.

Second order perturbation theory with the vertex eq. 3.79 leads to the elastic π–N K–matrix. For pionic P waves we can express the TS matrix elements conveniently in terms of projectors $P_{2I\,2J}$ for total isospin I and spin J and obtain the familiar result [EW88]

$$(T^{(-)}T^{(+)}S^{(-)}S^{(+)})_{direct} = \mathrm{P}_{33}, \tag{3.81}$$

$$(T^{(-)}T^{(+)}S^{(-)}S^{(+)})_{crossed} = (\mathrm{P}_{33} + 4(\mathrm{P}_{13} + \mathrm{P}_{31}) + 16\mathrm{P}_{11})/9. \tag{3.82}$$

The elastic π–N K–matrix is

$$K^{(1)}_{\pi N}(k^2) = C(k^2)\left(\frac{\mathrm{P}_{33}}{\omega_k - \Delta M} - \frac{1}{9}\frac{\mathrm{P}_{33} + 4(\mathrm{P}_{13} + \mathrm{P}_{31}) + 16\mathrm{P}_{11}}{\omega_k + \Delta M} \right) \tag{3.83}$$

Here ΔM is the mass difference between Δ and N which in the soliton model is determined by the moment of inertia Θ

$$\Delta M = 3/(2\Theta) \tag{3.84}$$

and

$$C(k^2) = -\frac{1}{3 \cdot 4\pi}\left(\frac{3}{2\Theta}\frac{k/m_\pi}{\omega_k} f_{\pi N\Delta}(k^2)\right)^2. \tag{3.85}$$

The π–N K–matrix clearly shows how the moment of inertia Θ determines the position of the Δ resonance through the Δ–pole in the P_{33} channel. This points towards the crucial importance of the linear time–derivative interaction in reproducing the general features of the P_{33} channel.

We now turn to the second order time–derivative coupling $L^{(2)}_{int}$. As we shall shortly see, this term is crucial for the low–energy behaviour of the S waves but has little influence on the P waves. We are interested here in that part which couples the fluctuations and the collective velocity in a form that provides after quantization a coupling

$$L^{(2)}_{int} \Rightarrow \vec{t} \cdot \vec{\tau}/2\Theta \tag{3.86}$$

between pion and target isospins $\vec{t}$ and $\vec{\tau}/2$. This causes the splitting between phase shifts in channels with total isospin $2I = 1$ and $2I = 3$ according to

$$\vec{t} \cdot \vec{\tau} = -2(\mathrm{P}_{11} + \mathrm{P}_{13}) + (\mathrm{P}_{31} + \mathrm{P}_{33}). \tag{3.87}$$

$L^{(2)}_{int}$ will therefore be the only source of splitting between the P_{13} and P_{31} channels, which remained degenerate in the background calculation presented in figure 3.5.

To evaluate matrix elements of $L^{(2)}_{int}$, eq. 3.67, between N and Δ states, we first note that

$$\dot{\hat{\gamma}} \times \hat{\gamma} = \hat{\gamma} \cdot \vec{\Omega}\, \hat{\gamma} - \vec{\Omega} \tag{3.88}$$

requires us to use the identity in spherical components ($m,\ n,\ k,\ k' = 0, \pm 1$)

$$\hat{\gamma}_m \hat{\gamma}^*_n = \frac{1}{3}\delta_{mn} + 2\sqrt{\frac{2\pi}{3}}(-)^n \sum_{k,k'} \begin{pmatrix} 1 & 1 & 2 \\ m & -n & k \end{pmatrix} D^{2*}_{kk'} Y_{2k'}(\hat{r}). \tag{3.89}$$

This identity couples π–N S and P waves to π–Δ D and F waves, respectively. We consider here only the NN matrix elements to which only the first term in eq. 3.89 contributes

$$< N'|L^{(2)}_{int}|N > \Rightarrow -\frac{1}{6\Theta} \int \vec{\tau} \cdot (\vec{\xi} \times \dot{\vec{\xi}}) \sin^2 F\ \mathrm{M}_S\, d^3x. \tag{3.90}$$

In plane-wave approximation, we obtain the $\vec{t} \cdot \vec{\tau}$ coupling from eq. 3.90 by using the adjoint representation for the pion isospin [EW88]

$$i(t_c)_{ab} = \epsilon_{abc} \tag{3.91}$$

and substituting plane waves for Schnitzer fluctuations. The contribution of $L^{(2)}_{int}$ to the K–matrix is

$$\mathrm{K}^{(2)}_{\pi N} = -\frac{\omega_k}{3\Theta} \int (j_1(kr))^2\ \sin^2 F\ \mathrm{M}_S\, r^2 dr\ \left(-2(\mathrm{P}_{11} + \mathrm{P}_{13}) + (\mathrm{P}_{31} + \mathrm{P}_{33})\right). \tag{3.92}$$

The integrand in the form factor for this second order contact interaction turns out to be the square of the expression eq. 3.54 which appears in the K–matrix eq. 3.83 due to the linear interaction term.

In the piecewise approach we have been following, it seems like a reasonable approach to add all contributions in K–matrix formalism. We therefore take the background K–matrix eq. 3.31, along with eq. 3.83 due to the linear interaction term and eq. 3.92 due to the quadratic interaction term to obtain

$$(\mathrm{K}_{l=1}^{2T\,2J})_{\mathrm{NN}} = (\mathrm{K}_{l=1}^{2T\,2J})_{\mathrm{NN}}^{BG} + P_{2I\,2J}(\mathrm{K}_{\pi\mathrm{N}}^{(1)} + \mathrm{K}_{\pi\mathrm{N}}^{(2)}). \tag{3.93}$$

We have truncated these K–matrices to $l = 1$ states in π–N channels; the results for the corresponding reconstructed S–matrix in the form of elastic phase shifts (since we have only a single channel) are presented in section 3.5.

We conclude our discussion of time–derivative interactions by a presentation of some points of importance related to the π–N S waves. We decompose the low momentum π–N S wave amplitude in terms of [EW88]

$$f_S = a_1 P_1 + a_3 P_3 = b_0 + b_1\ (\vec{t}\cdot\vec{\tau}), \tag{3.94}$$

where a_1, a_3 (units of m_π^{-1}) correspond to the total isospin 1/2, 3/2 scattering lengths, b_0, b_1 (units of m_π^{-1}) are isospin averaged and isospin dependent S wave parameters, and P_1, P_3 are isospin 1/2, 3/2 projectors. Phase shift analysis based on experiment give the result

$$b_0 = \frac{1}{3}(a_1 + 2a_3) = -0.010, \tag{3.95}$$

$$b_1 = \frac{1}{3}(a_3 - a_1) = -0.091. \tag{3.96}$$

The Weinberg lagrangian [W66], based on a nonlinear realization of chiral symmetry, reproduces these results accurately, predicting $b_0 = 0$ and $b_1 = -0.089$. It therefore

becomes interesting to see whether the Skyrme model, also based on chiral symmetry can achieve good S wave results within the scheme we are proposing.

The large relative size of b_1 as compared to b_0 shows that the low energy S wave scattering is essentially dominated by the coupling of the pion's isospin to the nucleon's isospin. But we have seen how we can precisely take such an effect into account through the introduction of the coupling of the soliton's vector current to the pions, eq. 3.67. This quadratic interaction produces after quantization just the $\vec{t}\cdot\vec{\tau}$ interaction we need. Implemented in K–matrix formalism, we have for the S waves:

$$\tilde{\mathrm{K}}^{(2)}_{\pi N} = -\frac{\omega_k}{3\Theta}\int (j_0(kr))^2 \sin^2 F \; M_S \, r^2 dr \; (-2P_1 + P_3)) \,. \tag{3.97}$$

Comparing this to the P wave expression eq. 3.92, we note the expected replacement of $j_1(kr)$ by the $l = 0$ $j_0(kr)$ spherical Bessel function.

For π–N S waves, the linear interaction eq. 3.66 does not contribute: for the S wave fluctuation $\vec{\xi} \to j_0(kr)\hat{e}$, where $\hat{e}$ is an arbitrary isospin direction, $L^{(1)}_{int}$ is odd under parity transformations. To describe S waves, it would therefore seem sufficient to add to the K–matrix resulting from the quadratic interactions the background result contributions to S waves. Unfortunately, as will be seen in section 3.6, the very attractive Skyrme background (or repulsive for the nonlinear σ–model; see figure 3.4) has the effect of displacing the phase shift curves away from the results of the phase shift analyses. In the hope of improving agreement, we were led to further consider the effect of the gradient terms. These terms originate in the Skyrme stabilization term, and involve a gradient operating on the fluctuations. The interaction has the form

$$L^{grad}_{int} = \frac{1}{8e^2}\int d^3x \; tr\{ \; [\dot{A},\vec{H}]\vec{A}\dot{H} + [\dot{H},\vec{A}]\vec{H}\dot{A} + [\dot{A},\vec{A}]\vec{H}\dot{H} + [\dot{H},\vec{H}]\vec{A}\dot{A} \; \}, \tag{3.98}$$

where $H_\mu = U_0^\dagger \partial_\mu U_0$ and we use the shorthand

$$A_\mu = U_0^\dagger R_\mu U_0 + L_\mu; \; R_\mu = U_\xi^\dagger \partial_\mu U_\xi, \; L_\mu = \partial_\mu U_\xi U_\xi^\dagger. \tag{3.99}$$

Evaluating the trace in eq. 3.98 leads to the general result

$$L_{int}^{grad} = \frac{1}{e^2 f_\pi^2} \int d^3x \; \{ \; 2(\dot{\hat{\gamma}} \cdot \dot{\vec{\xi}})(\; s\partial_i s \; \hat{\gamma} \cdot \partial_i \vec{\xi} \; + \; s^2 c^2 \; \partial_i \hat{\gamma} \cdot \partial_i \vec{\xi} \;)$$
$$-(\dot{\hat{\gamma}} \cdot \partial_i \vec{\xi})(\; s\partial_i s \; \hat{\gamma} \cdot \dot{\vec{\xi}} \; + \; s^2 c^2 \; \partial_i \hat{\gamma} \cdot \dot{\vec{\xi}} \;)$$
$$-(\dot{\hat{\gamma}} \cdot \partial_i \hat{\gamma} \;)(\; s^4 \; \hat{\gamma} \cdot \dot{\vec{\xi}} \, \hat{\gamma} \cdot \partial_i \vec{\xi} \; + \; s^2 c^2 \; \dot{\vec{\xi}} \cdot \partial_i \vec{\xi} \;) \; \}. \tag{3.100}$$

where we use the abbreviated forms $s \equiv \sin F$, $c \equiv \cos F$. Fortunately, this result coalesces to a reasonable form when we restrict $\vec{\xi}$ to S waves:

$$\vec{\xi} \rightarrow \vec{\xi}(r), \; \partial_i \vec{\xi} \rightarrow \frac{d\vec{\xi}}{dr} \cdot \hat{r}_i; \tag{3.101}$$

and more importantly, exhibits the $\vec{t} \cdot \tau$ form. We obtain

$$L_{int}^{grad} = \frac{4\pi}{e^2 f_\pi^2} \int \sin F \frac{d(\sin F)}{dr} \; \vec{\Omega} \cdot (\frac{d\vec{\xi}}{dr} \times \dot{\vec{\xi}} \;) \; r^2 dr. \tag{3.102}$$

The resulting K–matrix is

$$\tilde{\mathrm{K}}_{\pi N}^{(3)} = -\frac{\omega_k}{4e^2 f_\pi^2 \Theta} \int (j_0(kr))^2 \frac{d}{dr}(r^2 \frac{d}{dr} \sin^2 F) dr \; (-2P_1 + P_3), \tag{3.103}$$

where we have performed an integration by parts.

Hence, in a similar way as for eq. 3.93 for the P waves, we write the $l = 0$ π–N K–matrix by adding the contributions eqs 3.31, 3.97, 3.103:

$$(\mathrm{K}_{l=0}^{2T})_{\mathrm{NN}} = (\mathrm{K}_{l=0}^{2T})_{\mathrm{NN}}^{BG} + P_{2I}(\; \tilde{\mathrm{K}}_{\pi\mathrm{N}}^{(2)} \; + \; \tilde{\mathrm{K}}_{\pi\mathrm{N}}^{(3)} \;). \tag{3.104}$$

The results for the reconstructed S–matrix in the form of elastic phase shifts are presented in section 3.6.

3.5 Results for Pion–Nucleon P Wave Scattering

We have seen in table 3.1 that with the experimental value $f_\pi = 93$ (MeV) the Skyrme stabilization does not allow for a simultaneous fit of both the πNN–coupling constant

$f = 1.0$ and the N–Δ split $\Delta = (M_\Delta - M_N)/m_\pi = 2.12$. Both numbers, of course, are crucial for a satisfactory description of the position and width of the Δ–resonance. But it would require a value of f_π as low as about 40 (with $e \approx 5.2$) to reproduce them in the Skyrme model. We therefore also present results for the Adkins–Nappi [AN84] set ($f_\pi = 54$, $e = 4.84$) chosen to reproduce M_N and M_Δ. This set leads to $f = 0.877$ which seems an acceptable compromise. Unfortunately, the lower value of ef_π lowers the breathing (E0) mode even further and thereby increases its influence on the low–energy phase shifts. We also find increased attraction for the (M1) mode which counteracts the larger value of f such that the ratio eq. 3.56 for $k \to 0$ goes down to 0.74 (as compared to 0.82 for the set $f_\pi = 93$, $e = 4$). In order to decrease the influence of the breathing mode we also consider the value $f_\pi = 110$ MeV (again with $e = 4$).

The calculated coupling constants f, the nucleon–Delta separation energies Δ, and the scattering volumes[16] for the E0, M1, E2 modes are given in table 3.2. The comparison of a_1 with the zero mode contribution from eq. 3.53

$$a_1^{(0)} = -\frac{\Delta}{\pi} f_{\pi NN}^2(0) \tag{3.105}$$

shows that for all cases the zero mode dominates the M1 phase shift at threshold. However, it is also evident that the $K = 1$ mode does not necessarily dominate over the $K = 0$ breathing mode. This is quite clear for the nonlinear σ–model which, as expected, is thereby ruled out as a reasonable model for P waves; but also for the Skyrme model with $f_\pi = 54$, a_0 is almost comparable to a_1 which makes this an unsuitable parameter set despite the improved value of f. The K=2 mode counteracts

[16]We denote intrinsic frame scattering volumes by a_K, where K is the multipole index in the intrinsic frame, and $a_{2I\ 2J}$ in the laboratory frame. There should be no confusion, given the context, with the scattering lengths a_1, a_3 for the S waves. For a definition of the scattering lengths and volumes, see appendix A.

f_π (93)	e	f (1.0)	Δ (2.12)	$a_1^{(0)}$	a_1	$a_1/a_1^{(0)}$	a_0	a_2
93	4	.772	1.809	-.343	-.281	.82	.105	-.021
110	4	.663	2.024	-.283	-.240	.85	.066	-.013
54	4.84	.877	2.120	-.519	-.382	.74	.282	-.053
93	NLσ	.772	4.274	-.765	-.601	.79	.620	-.015

Table 3.2: Pion–Nucleon coupling constant f, Nucleon–Δ split Δ, and P wave scattering volumes a_K, ($K = 0,\ 1,\ 2$) in the intrinsic frame, for different parameter sets f_π (MeV) and e. For $K = 1$ the zero mode contribution is listed as $a_1^{(0)}$, together with the ratio $a_1/a_1^{(0)}$. The units for Δ and a_K are m_π and m_π^{-3}. Experimental values are given in brackets. The last row shows the results for the nonlinear σ–model with Skyrme profile $F(r)$ calculated for $f_\pi = 93$ and $e = 4$.

K=0 but it is too weak to restore the zero mode values for the background (BG) scattering volumes a^{BG} after transformation to the laboratory frame.

In the different TJ–channels these are obtained from eq. 3.30 as

$$a_{11}^{BG} = a_0/3 + 2a_1/3,$$

$$a_{13}^{BG} = a_{31}^{BG} = a_1/6 + 5a_2/6,$$

$$a_{33}^{BG} = a_0/6 + 5a_1/12 + 5a_2/12. \tag{3.106}$$

(These relations are not affected by the truncation of the K–matrix in eq. 3.93 to π–N P waves only.) The individual contributions from $K = 0,\ 1,\ 2$, to TJ–channels are listed in table 3.3, together with the contributions from the time–derivative interactions $a^{(1)}$ and $a^{(2)}$. They all add up to the theoretical values for the scattering volumes given in the last columns of table 3.3. We also show the numbers obtained by replacing the whole background scattering by the M1 zero modes [H90]

$$a_{zm}^{BG} = -\frac{\Delta}{\pi} f^2 \left(\frac{2}{3}\mathrm{P}_{11} + \frac{1}{6}(\mathrm{P}_{13} + \mathrm{P}_{31}) + \frac{5}{12}\mathrm{P}_{33}\right) m_\pi^{-3} \tag{3.107}$$

P_{11}: (Exp.: $a_{11} = -.081$)

f_π	e		$a_0/3$	$2a_1/3$	$a_{11}^{(1)}$	$a_{11}^{(2)}$	a_{11}
		zm:		-.450	.306		-.144
93	4	zm:		-.229	.147		-.082
			.035	-.187	.147	.007	+.003
110	4	zm:		-.189	.126		-.063
			.022	-.160	.126	.004	-.008
54	4.84	zm:		-.346	.235		-.111
			.094	-.255	.235	.031	+.105

P_{13} and P_{31}: (Exp.: $a_{13} = -.030$, $a_{31} = -.045$)

f_π	e		$a_1/6$	$5a_2/6$	$a_{13}^{(1)} = a_{31}^{(1)}$	$a_{13}^{(2)}, a_{31}^{(2)}$	a_{13}, a_{31}
		zm:	-.113		.077		-.036
93	4	zm:	-.057		.037		-.020
			-.047	-.017	.037	.007,-.004	-.020,-.031
110	4	zm:	-.047		.032		-.016
			-.040	-.011	.032	.004,-.002	-.015,-.021
54	4.84	zm:	-.087		.059		-.028
			-.064	-.044	.059	.031,-.015	-.018,-.064

P_{33}: (Exp.: $a_{33} = .214$)

f_π	e		$a_0/6$	$5a_1/12$	$5a_2/12$	$a_{33}^{(1)}$	$a_{33}^{(2)}$	a_{33}
		zm:		-.279		.497		.218
93	4	zm:		-.143		.297		.154
			.018	-.117	-.009	.297	-.004	.185
110	4	zm:		-.118		.218		.100
			.011	-.100	-.005	.218	-.002	.122
54	4.84	zm:		-.216		.383		.167
			.047	-.159	-.022	.383	-.015	.234

Table 3.3: Scattering volumes in the different TJ–channels (in m_π^{-3}). The columns show the contributions from background (a_K), linear ($a^{(1)}$) and second order ($a^{(2)}$) time–derivative coupling. "zm" denotes the zero mode contributions. The first "zm" row in each channel is calculated with the experimental values for f (=1.0) and Δ (=2.12), i.e. it corresponds to the static Δ–isobar model.

which, together with the contributions from the linear time–derivative coupling (from eq. 3.83)

$$a_{11}^{(1)} = 4a_{13}^{(1)} = 4a_{31}^{(1)} = \frac{2}{3\pi} f_{\pi NN}^2(0) \frac{\Delta^2}{1+\Delta} m_\pi^{-3}, \tag{3.108}$$

$$a_{33}^{(1)} = \frac{1}{12\pi} f_{\pi NN}^2(0) \Delta^2 \frac{4+5\Delta}{\Delta^2-1} m_\pi^{-3}. \tag{3.109}$$

constitute the Δ–isobar model for the parameter set considered (note that for the Skyrme model we have $f = f_{\pi\mathrm{NN}}(-m_\pi^2) \approx f_{\pi\mathrm{NN}}(0)$).

Comparison with the experimental values shows that replacing the background by the (zm) contributions eq. 3.107 only, and using the experimental values for $f(=1.0)$ and $\Delta(=2.12)$, accounts almost perfectly for the observed scattering volumes for P_{33} and P_{13}, P_{31}, with only P_{11} being too low. This is of course as expected because it just repeats the static Δ–isobar model.

Especially sensitive is the P_{11} channel: first, we notice that with the calculated values of f and Δ, taking only the (zm) contributions improves the result for a_{11} for all parameter sets considered. This good agreement is a bit unfortunate because all additional terms act in the same direction (bringing attraction for both the E0 and M1 modes). For $f_\pi = 93$ this increases the value of a_{11} to almost zero such that the tiny second order time–derivative coupling can push it just above zero. For $f_\pi = 110$ the attraction in the breathing mode is sufficiently quenched to allow the overall result for a_{11} to be negative, although its absolute value is still too small. For the $f_\pi = 54$ set the large a_0 together with the smaller ratio (0.74) of $a_1/a_1^{(0)}$ leads to a sizable positive result for a_{11}.

For P_{13}, P_{31} already the (zm) contributions with calculated f values are too small. Further reduction through attractive terms in the M1 mode is however compensated by the K=2 contributions, but it is still not sufficient to reestablish agreement with

the experimental values. The isospin splitting due to $L_{int}^{(2)}$ for $f_\pi = 93$ is almost of the correct magnitude, but for $f_\pi = 110$ it is too small by a factor of two.

Also for P_{33} the calculated values of f being too small directly causes the values of a_{33} to be too low. This is especially apparent for $f_\pi = 110$. Here, however, the quenching of the negative a_1 values helps to improve the final results. The positive contributions of the breathing mode are half compensated by $K = 2$ so that for the P_{33} scattering volumes the $K = 0,\ 2$, modes are of minor importance.

In summary, it is evident that the stabilization mechanism must lead to good results for f and Δ in order to guarantee the sensitive cancelations between the large (zm) and $a^{(1)}$ contributions. The quenching of the $M1$ mode then can be compensated by the $K = 0$ and $K = 2$ contributions, except for the P_{11} channel. Fortunately, for P_{11} the Δ–isobar model result for a_1 (=-0.144) is too low by almost a factor of two which leaves some room for the positive breathing mode contribution. But it still seems quite difficult to obtain a completely satisfactory result in this channel.

With the scattering volumes fixed, the behaviour of the phase shifts for positive values of k is largely determined by the following three features: The calculated moment of inertia Θ fixes the k value where the P_{33} phase shift crosses the 90^0 line; the product ef_π then determines the relative position of the Roper and Δ resonances; finally the unphysical rise of the M1 and E2 phase shifts in the Skyrme model begins to affect the results above $k \simeq 2$.

The P_{11} and P_{33} phase shifts are shown in figure 3.9 for $f_\pi = 93$ and f_π=110, together with the Karlsruhe data [HKKP79]. It should be noted that for both parameter sets the moments of inertia are larger than 0.708 m_π^{-1}, the value of Θ which reproduces the experimental N–Δ mass difference (see table 3.1). The fact that the data for P_{33} lie between both theoretical curves is due to recoil. Without recoil the data points would

cross the 90^0 line at $k = 1.87\ m_\pi$. This shows that without including recoil it makes no sense to fine–tune the parameter set to the data. More significant is the slope of the P_{33} phase shifts near 90^0. Both theoretical curves are too steep which reflects the values of the calculated πNN coupling constants f being too small. But still the overall width of the Δ resonance is reasonably reproduced, better than one might have guessed from the actual values of f given in table 3.2. As expected from the P_{33} scattering volumes in table 3.3 the $f_\pi = 93$ set is superior at low energies. In contrast, for higher k–values, the $f_\pi = 93$ results are more affected by the lower position of the breathing mode, the rise of the Skyrme background phase shifts, and the missing inelasticities above the $\Delta\pi$ threshold due to our truncation of the K–matrix.

The P_{11} phase shifts show that with f_π chosen at its physical value of $f_\pi = 93$ the skyrmion is too soft with regard to the E0 compression. But although the choice $f_\pi = 110$ seems to put the calculated P_{11} phase shifts into an overall agreement with the data points the problem is not really cured. This becomes evident from figure 3.10 and figures 3.11, 3.12. Figure 3.10 shows that $f_\pi = 93$ is really doing well, not only for P_{33} but also for P_{13} and P_{31}, and for the shape of P_{11}, with only the onset of the rise in $\delta_{P_{11}}$ being too low. While for $f_\pi = 110$ everything else getting worse (note that the isospin splitting between P_{13} and P_{31} is $\propto f_\pi^{-2}$), the quality for $\delta_{P_{11}}$ is not really much better, although it does cross the 0^0 line near $k = 0.4\ m_\pi$ (figure 3.12) (as compared to $k = 1.5\ m_\pi$ for the Karlsruhe data), reflecting the bound zero mode at low k–values. As a curiosity we include in figure 3.13 also the P_{11} and P_{33} phase shifts for the set ($f_\pi = 54$, $e = 4.84$) where the E0 resonance (at k=2.13m_π) appears just above the Δ resonance (at $k = 1.87\ m_\pi$) which forces the P_{33} phase shift to jump from 90^0 to 270^0 within a few MeV. This clearly rules out this choice as an acceptable set of parameters.

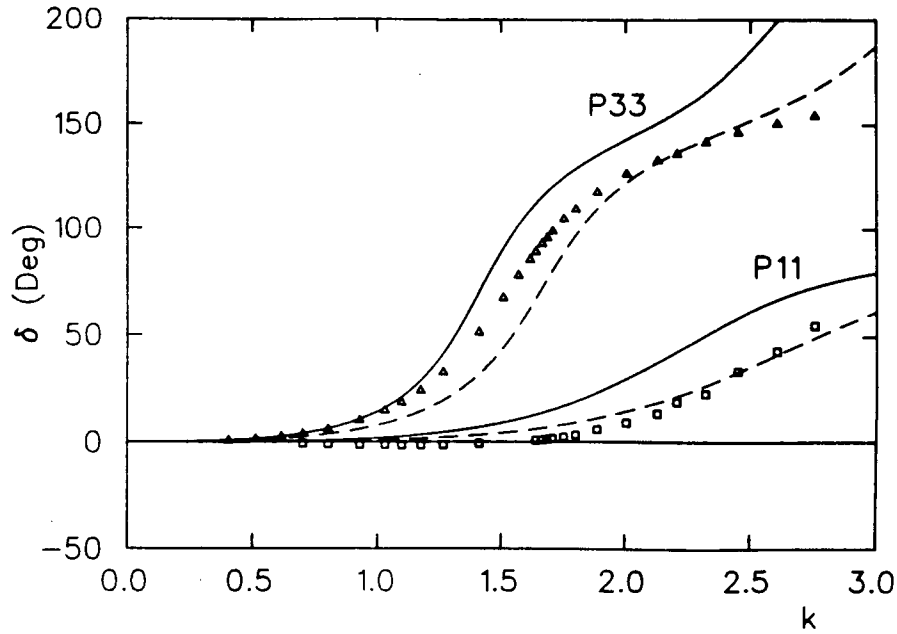

Figure 3.9: Elastic π–N phase shifts in the P_{11} and P_{33} channels calculated from the Skyrme model with e=4 and f_π = 93 MeV (full lines) and f_π = 110 MeV (dashed lines). The corresponding Karlsruhe data [HKKP79] are indicated by □ and △ with the center of mass momentum as abscissa in units of m_π.

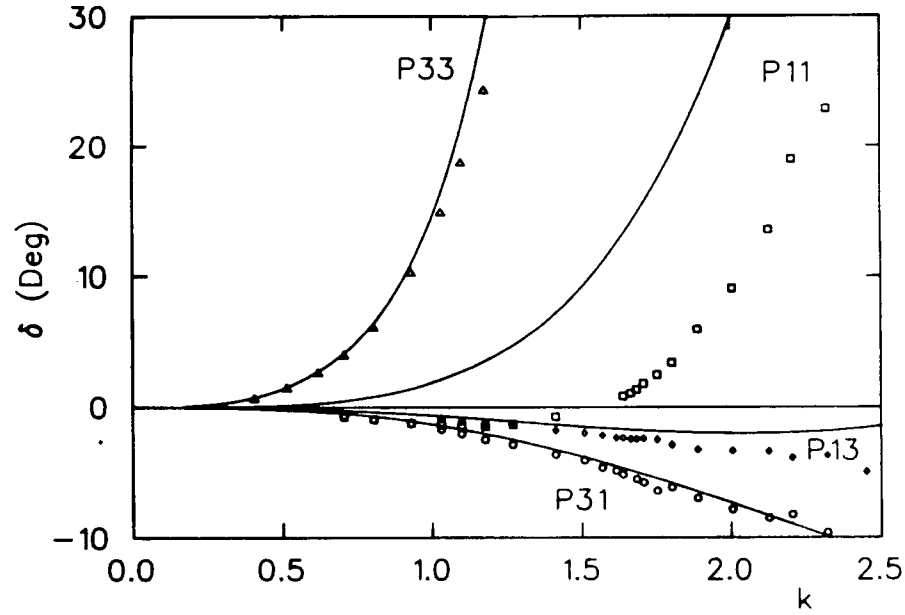

Figure 3.10: Elastic π–N phase shifts in the P_{11}, P_{33}, P_{31} and P_{13} channels calculated from the Skyrme model with $e = 4$ and $f_\pi = 93$ MeV together with the corresponding Karlsruhe data.

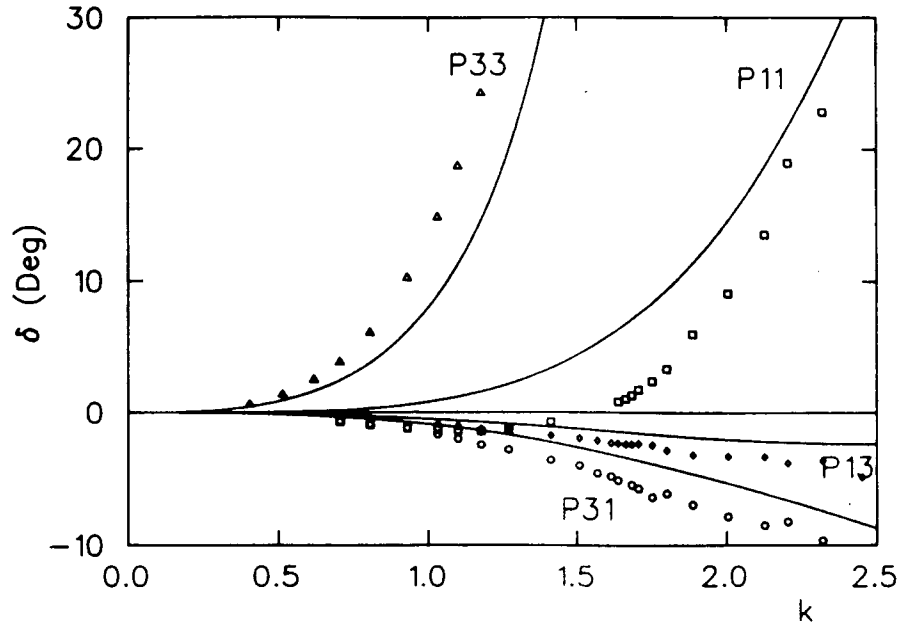

Figure 3.11: Elastic π–N phase shifts in the P_{11}, P_{33}, P_{31} and P_{13} channels calculated from the Skyrme model with $e = 4$ and $f_\pi = 110$ MeV together with the corresponding Karlsruhe data.

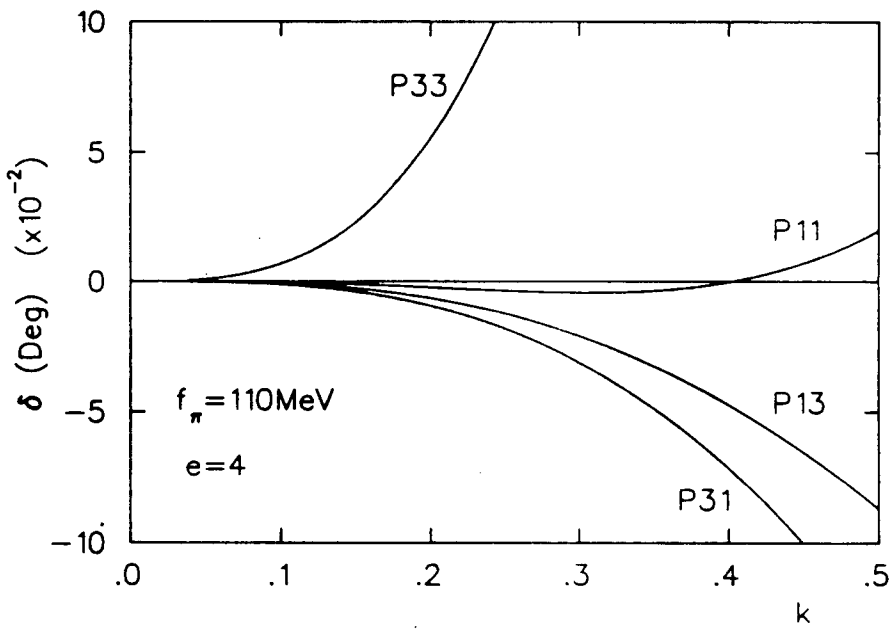

Figure 3.12: Enlarged portion of figure 3.11 which shows the small negative values of $\delta_{P_{11}}$ for small values of pion momentum k.

3.6 Results for Pion–Nucleon S wave Scattering

We have seen in the previous section that the π–N P wave scattering was best described by the set of parameters $f_\pi = 93$ MeV, $e = 4$. Given the aim of reproducing all partial waves with the same set of parameters, we begin by showing the S waves obtained in this case. The S_{11} and S_{31} phase shifts are plotted in figure 3.14 and compared to the Karlsruhe data [HKKP79]. The graphs reveal poor results for these parameters. In particular, the scattering length $a_3 \simeq 0$. Some improvement can be obtained if we allow the parameters to vary. Best results were found for $f_\pi = 110$ MeV, $e = 5$, which unfortunately produce a very small moment of inertia $\Theta = 0.335\ \mathrm{m}_\pi^{-1}$, as compared to the experimental value $\Theta = 0.708\ \mathrm{m}_\pi^{-1}$. The scattering lengths (units of m_π^{-1}) are however improved:

$$a_1 = 0.147 \qquad a_1^{expt} = 0.173$$

$$a_3 = -0.041 \qquad a_3^{expt} = -0.101$$

The most important feature that determines whether the calculated phase shifts will agree or not with the data is the background result that is added to the second order time–derivative interaction. Figure 3.15 shows that the second order time–derivative interaction (with gradient terms) reproduces perfectly the splitting of the S waves. Only a weakly repulsive background is needed to "tilt" the phase shifts to their correct position. Hence the importance of the interplay between the metrical factor in the time–derivative interactions and the background. For the Skyrme model the background is rather strongly attractive (see figure 3.4), leading to the poor results discussed above. As a curiosity, we show in figure 3.16 the results obtained for the nonlinear σ–model, which has a strongly repulsive background. As expected, the behaviour of the phase shifts is opposite that of the Skyrme model's.

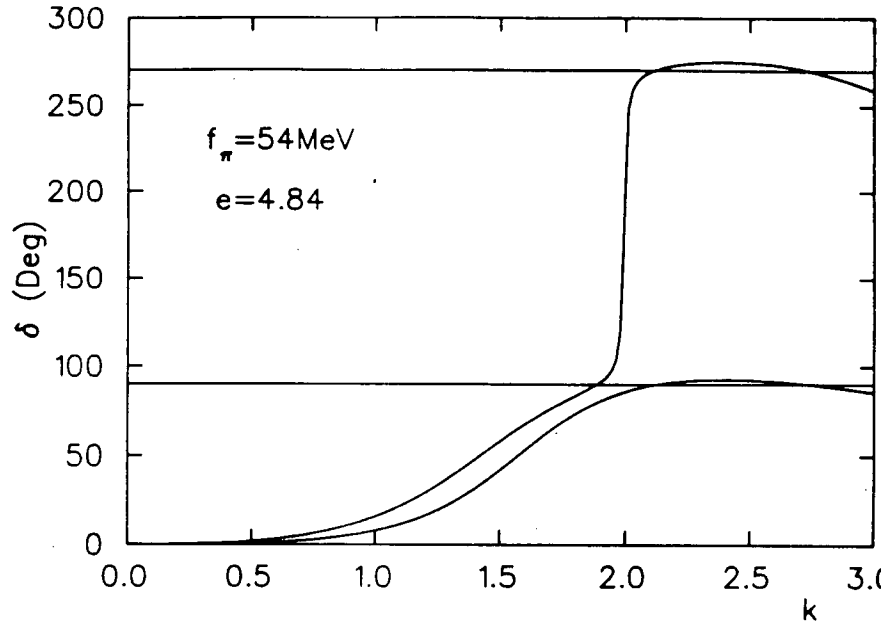

Figure 3.13: P_{11} and P_{33} phase shifts for the Skyrme model calculated with the parameter set of [AN84] $f_\pi = 54$ MeV and $e = 4.84$. The horizontal lines are drawn at 90^0 and 270^0.

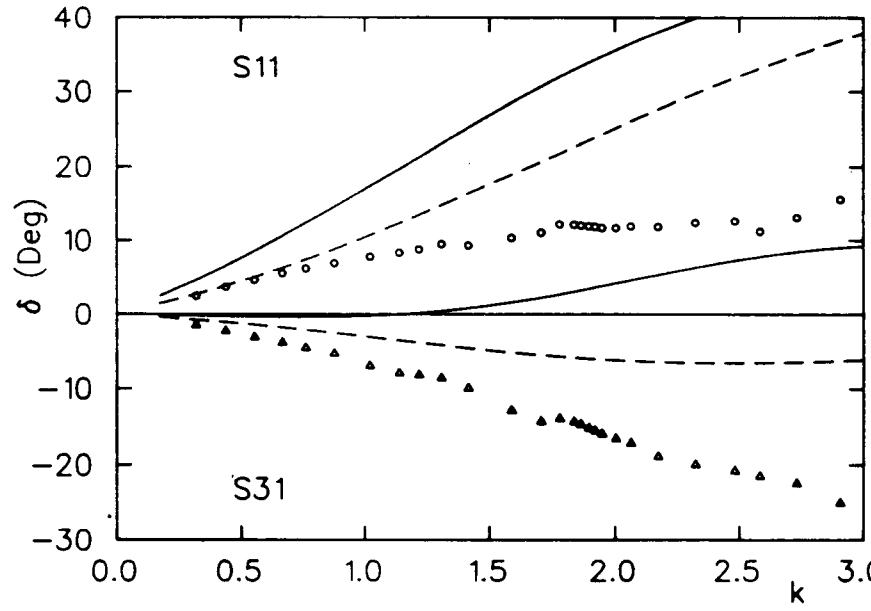

Figure 3.14: Elastic π–N phase shifts in the S_{11} and S_{33} channels calculated with the parameter sets $f_\pi = 93$ MeV and $e = 4$ (full line) and $f_\pi = 110$ MeV and $e = 5$ (dashed line) for the Skyrme model. The corresponding Karlsruhe data [HKKP79] are indicated by o and $\triangle$ with the center of mass momentum as abscissa in units of m_π.

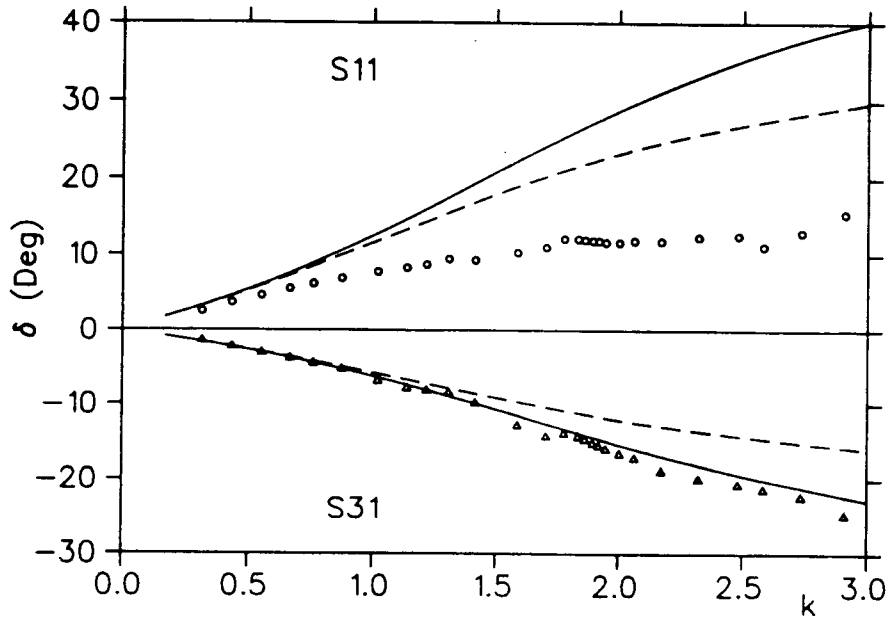

Figure 3.15: Phase shifts derived from the splitting interaction eq. 3.97 (full line). The dashed line further contains the gradient interactions eq. 3.103. In both cases, $f_\pi = 93$ MeV and $e = 4$, and we compare to the Karlsruhe data. Axes as in 3.14.

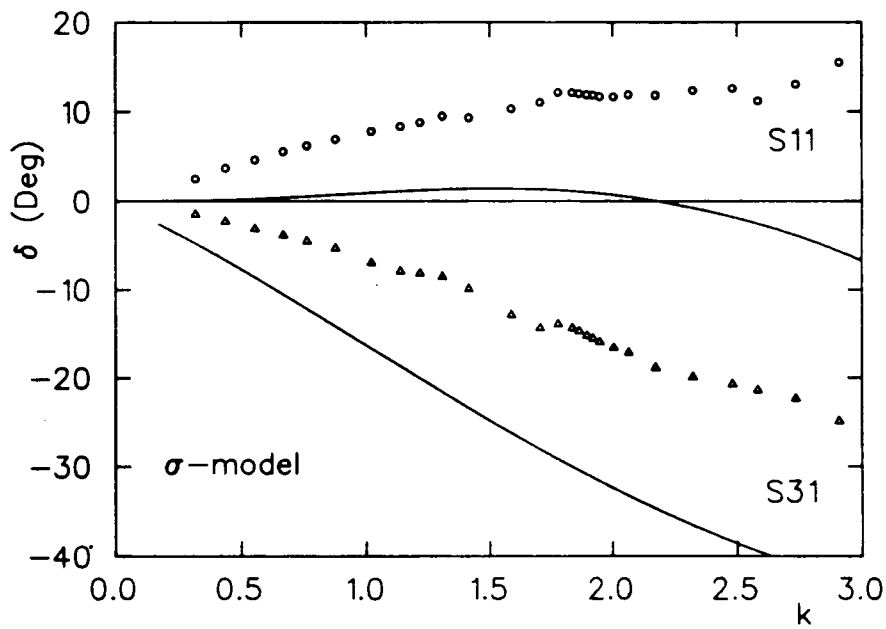

Figure 3.16: Phase shifts calculated for the nonlinear σ–model using a Skyrme model profile generated with the parameters $f_\pi = 93$ MeV and $e = 4$. Data and axes as in 3.14.

The problems we encountered have also plagued calculations by other workers in the field who have attempted to describe π–N S waves by going beyond the adiabatic approximation [JM89][JM90]. Recently, however, Walliser [W91] has obtained encouraging results for the S waves. In many ways, his approach is similar to ours, except for the removal of the plane wave Born approximation. His reformulation of the calculation solely in terms of direct fluctuations allowed him to use the distorted background waves in the evaluation of the time–derivative matrix elements rather than plane waves. This demonstrates that the PWBA is too drastic an approximation for the case of the S waves.

3.7 Conclusion

Verschelde's observation [V89][V91] that the zero mode contributions to the background scattering and the linear time–derivative interaction in soliton models add up to the Δ–isobar model for P wave pion–nucleon scattering has raised the question how the additional $K = 0,\ 1,\ 2$, contributions present in the background scattering would interfere with this result, especially for massive pions where the zero mode terms follow the same k^3–law for $k \to 0$ as all the other contributions. At the same time it is of interest to investigate the question how background, linear and quadratic time-derivative interactions combine to create the scattering phase shifts in the different TJ channels for k–values up to and beyond the resonance region.

We have investigated these points in two models: the Skyrme model and the non-linear σ–model. The extreme attraction produced by this latter model in the breathing mode allowed us to rule it out as a suitable framework to describe the P waves. We therefore restricted ourselves to the Skyrme model. The input for this model is three numbers: the pion decay constant $f_\pi = 93$ MeV, the pion mass $m_\pi = 138$ MeV and

one parameter e which is not fixed in the meson sector, but may be used to adjust the nucleon–Δ mass difference. A value of $e = 4$ reproduces the experimentally observed NΔ split reasonably well. With this input the Skyrme stabilization mechanism shows two major deficiencies: The pion–nucleon coupling constant f is too small by a factor of 0.77 (which is not too bad, anyway) and the soliton is too soft against compression which brings too much attraction into the P_{11} channel. Both of these problems are characteristic of the Skyrme stabilization term and cannot really be cured by changing f_π or e. Accepting these deficiencies we find it remarkable that with the physical value of f_π the essential features of the P wave phase shifts are reproduced by what is now a one-parameter model. The results are quite sensitive to the choice of f_π and clearly rule out a value adjusted to the nucleon mass, as previously noted in [MK85]. This is another indication that the quantum corrections to the classical soliton mass are not really understood and that the nucleon mass should not be used to fix parameters of the effective meson lagrangian.

Unfortunately the Skyrme stabilization term also leads to unphysically rising phase shifts just beyond the resonance region. This limits our discussion to values below $k \simeq 2.5\ m_\pi$. To go beyond this region would also have required a much more involved treatment of the background scattering than the one used here: we have truncated the adiabatic K–matrix to the elastic πN channels only, in order to exclude the coupling to the closed $\pi\Delta$ channels. Time–derivative interactions have been added in plane wave Born approximation to the K–matrix. Also this procedure probably could be improved by employing the true distorted (background) scattering waves instead.

It would certainly be desirable to have all the ingredients we have been considering, contained in one concise system of equations of motion, instead of the piecewise way we have put them together here. We have also used a very naive way of quantizing collective and fluctuational degrees of freedom, which ignores all the constraints arising

from redundancies in their definitions, and the complications which the time–derivative interactions may imply for a canonical formalism. However, the simple terms we have considered contain the essential physical features of the model and the results show that the idea of baryons being solitons in an effective meson theory again proves to be a very powerful tool: it produces a complete description of all essentials observed in low–energy elastic P wave pion–nucleon scattering. The Skyrme model achieves with one single parameter a quality of agreement which suggests that improvements in the stabilization mechanism and in the theoretical techniques will be able to provide quantitative agreement with the data.

Having established the formalism for the P waves, we applied this to the case of the π–N S waves. We found that the quadratic time–derivative interaction led to a perfect description of the isospin splitting between the S_{11} and S_{31}. The addition of background contributions, whether calculated from the Skyrme model or the nonlinear σ–model, destroyed the agreement. In this case, it appears that plane wave approximation is too drastic and that one should really use distorted waves [W91] to finally obtain quantitative results for the S waves.

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Appendix A

Background Scattering

In this appendix we present the ordinary differential equations that arise from the substitution in the Skyrme model and nonlinear σ–model (eqs 3.1 and 3.2) of the multipole expansion (eqs. 3.17, 3.18) for the fluctuational field. We will present the differential equations in a systematic way by enumerating the coefficients of the derivatives appearing in them. We use the concise notation: $s \equiv \sin F(r)$, $c \equiv \cos F(r)$, and a prime denotes the derivative of the function with respect to x, where x is the usual dimensionless Skyrme model radius defined by

$$x = e2f_\pi. \tag{A.1}$$

The dimensionless pion mass is

$$\tilde{m}_\pi = \frac{m_\pi}{e2f_\pi} \tag{A.2}$$

The parameter α is a label which is equal to one for the Skyrme model and zero for the nonlinear σ–model. We emphasize that the equations for the nonlinear σ–model are studied with a soliton profile obtained from the Skyrme model. As for the equations for the Skyrme model, they are taken (with minor modifications) from the papers of Walliser and Eckart [WE84] and Mattis and Karliner [MK85]. Further details can be found in these references.

The only magnetic mode we need to consider in this work is the the M1 mode. The differential equation for the M1 fluctuation χ (equivalent to ϕ_{11} in eq. 3.17) is

uncoupled and reads:

$$\chi'' = -\{\Phi_2\chi' + \Phi_3\chi\}/\Phi_1, \tag{A.3}$$

where

$$\begin{aligned}
\Phi_1 &= -s^2 - \alpha\, 4s^4/x^2 \\
\Phi_2 &= -2F'/F\Phi_1 - 2scF' - 2s^2/x - \alpha\, 16s^3cF'/x^2 \\
\Phi_3 &= (2F'^2/F^2 - F''/F)\Phi_1 - F'/F\Phi_2 - (s^2 + \alpha\, 4s^2(F'^2 + s^2/x^2))\omega^2
\end{aligned}$$

We extract the scattering phase shifts from this equation by considering the asymptotic $x \to \infty$ behaviour of the solution χ. Far from the origin, eq. A.3 has as solution linear combinations of spherical Bessel ($j_l(kx)$) and Neumann ($n_l(kx)$) functions[1] of order l:

$$\begin{aligned}
\chi &\rightsquigarrow A_l(k)j_l(kx) + B_l(k)n_l(kx) \\
&\rightsquigarrow \frac{1}{kx}(A_l(k)\sin(kx - \frac{l\pi}{2}) - B_l(k)\cos(kx - \frac{l\pi}{2})),
\end{aligned} \tag{A.4}$$

where $A_l(k)$, $B_l(k)$ are momentum dependent (in units of $e2f_\pi$) constants which are simply extracted from the numerical integration of the differential equation. In the second line of eq. A.4, we have introduced the asymptotic forms for $j_l(kx)$ and $n_l(kx)$. We need not worry about the normalization of the profile χ since it is the ratio B_l/A_l that interests us: if we introduce

$$\tan\delta_l(k) = -\frac{B_l(k)}{A_l(k)}, \tag{A.5}$$

straightforward trigonometry leads to

$$\chi \rightsquigarrow \frac{1}{kx}\sin(kx - \frac{l\pi}{2} + \delta_l) \tag{A.6}$$

[1]Eq. A.3 is for $l = 1$. We are providing a general discussion to account for other possibilities.

up to a non–important multiplicative constant. This last form is the form for a scattered wave (Schrödinger equation). Hence, δ_l as defined in eq. A.5 is identified to the phase shift. In the usual way, this parametrizes the S–matrix:

$$\mathrm{S} = e^{2i\delta_l}. \tag{A.7}$$

Of course, we must start the numerical integration of the differential equation by providing its small–x solution. For the Skyrme model, this is, up to a normalization constant and to leading order in x

$$\chi \rightsquigarrow 1 - \frac{|F'(0)|}{F(0)}x. \tag{A.8}$$

For the nonlinear σ–model, we prefer to rewrite eq. A.3 in terms of $\overline{\chi}_\sigma = (F/\sin F)\chi_\sigma$:

$$\overline{\chi}_\sigma'' = -\frac{2}{x}\overline{\chi}_\sigma' + (\frac{c}{s}(F'' + \frac{2F'}{x}) - F'^2 - \omega^2)\overline{\chi}_\sigma \tag{A.9}$$

with small–x solution

$$\overline{\chi}_\sigma \rightsquigarrow x, \tag{A.10}$$

where we consider only leading terms in x.

For the electric modes, the situation is slightly more complicated because we are now dealing with a system of coupled equations. Here, the grand spin value K is explicitly indicated because we need both the E0, E2 modes (P waves) and the E1 mode (S waves). Following [MK85], we make the following orthogonal transformation on the multipole expansion fields ϕ_{KK-1}, ϕ_{KK+1} of eq. 3.18:

$$\psi_K = \frac{\sqrt{K}}{\sqrt{2K+1}}\,(-\phi_{KK-1} + \frac{\sqrt{K+1}}{\sqrt{K}}\phi_{KK+1}), \tag{A.11}$$

$$\zeta_K = \frac{\sqrt{K}}{\sqrt{2K+1}}\,(\phi_{KK+1} + \frac{\sqrt{K+1}}{\sqrt{K}}\phi_{KK-1}). \tag{A.12}$$

This facilitates the numerical work by avoiding the simultaneous appearance of both ψ'' and ζ'' in the same equation. The coupled equations then read:

$$\begin{aligned} \psi''_K &= -\{\Gamma_2\psi'_K + \Gamma_3\psi_K + \Gamma_4\zeta'_K + \Gamma_5\zeta_K\}/\Gamma_1 \\ \zeta''_K &= -\{\Theta_2\zeta'_K + \Theta_3\zeta_K + \Theta_4\psi'_K + \Theta_5\psi_K\}/\Theta_1 \end{aligned} \tag{A.13}$$

where

$$\begin{aligned}
\Gamma_1 &= -2F^2x^2(x^2 + \alpha\ 8s^2) \\
\Gamma_2 &= -4F^2x^2(x + \alpha\ 8F'sc) \\
\Gamma_3 &= 2F^2\{\ (2c^2 - 2s^2 + K(K+1))(x^2 + \alpha\ 4s^2) \\
&\quad +\alpha\ (16s^2c^2 - 16x^2F''sc - 8x^2F'^2(c^2 - s^2)) \\
&\quad +\tilde{m}_\pi^2x^2c - x^2(x^2 + \alpha\ 8s^2)\omega^2\} \\
\Gamma_4 &= -\alpha\ 8\sqrt{K(K+1)}\ FF'x^2s^2 \\
\Gamma_5 &= \sqrt{K(K+1)}s^2\ \{16Fsc - 16x^2FF'' + 4x^2F(1 - 4F'^2)c/s + \alpha\ (16Fsc + 8x^2F'^2)\}
\end{aligned}$$

$$\begin{aligned}
\Theta_1 &= -2F^2x^2(x^2 + \alpha\ 4s^2) \\
\Theta_2 &= -4F^2x^3 + x^2F'\{\ (4x^2F - 4F^2x^2c/s + \alpha\ (-32F^2sc + 16Fs^2)\ \} \\
\Theta_3 &= K(K+1)F^2\ \{2x^2 + \alpha\ 8(xr^2F'^2 + s^2)\} \\
&\quad +2x^2\{-2F^2 - 2x^2F'^2 + 2Fx^2F'^2c/s \\
&\quad +F(2sc + 8s^3c/x^2 - 8scF'^2 - 8s^2F'' + \tilde{m}_\pi^2x^2s) \\
&\quad +\alpha\ (-8F^2F'^2 + 4s^2FF'' + 16scFF'^2 - 8s^2F'^2)\} \\
&\quad -2\omega^2F^2x^2(x^2 + \alpha\ 4(s^2 + x^2F'^2)) \\
\Theta_4 &= \alpha\ 8\sqrt{K(K+1)}\ xr^2F^3F' \\
\Theta_5 &= 8F^3\sqrt{K(K+1)}\ \{x^2c/(2s) + 2sc - 2x^2F'' - 2F'^2x^2c/s \\
&\quad \alpha\ (2sc + x^2F'' + 2F'^2x^2c/s)\ \}
\end{aligned}$$

For these equations, the two small–x solutions take the form to leading order in x:

$$\psi_a \rightsquigarrow x^{K+1} \tag{A.14}$$

$$\zeta_a \rightsquigarrow Q_a x^{K+1}\left(\frac{F(0)}{|F(0)'|x} - 1\right) \tag{A.15}$$

and

$$\psi_b \rightsquigarrow x^{K-1} \tag{A.16}$$

$$\zeta_b \rightsquigarrow Q_b x^{K-1}\left(\frac{F(0)}{|F(0)'|x} - 1\right). \tag{A.17}$$

Here, the constants Q_a and Q_b are

$$Q_a = \sqrt{\frac{K}{K+1}}\,\frac{1+\alpha\,(2F(0)'^2K + 10F(0)'^2)}{-1+\alpha\,(2F(0)'^2K - 4F(0)'^2)}, \tag{A.18}$$

$$Q_b = \sqrt{\frac{K+1}{K}}. \tag{A.19}$$

For this system, the extraction of phase shifts is analogous, albeit more complicated, to the 1–dimensional case presented above. We need to integrate the equations of motion twice with different linear combinations of the small–x solutions in order to generate two sets of asymptotic coefficients A_a, B_a, C_a, D_a, and A_b, B_b, C_b, D_b:

$$\psi_K^a \rightsquigarrow A_a(k)j_{K-1}(kx) + B_a(k)n_{K-1}(kx), \tag{A.20}$$

$$\zeta_K^a \rightsquigarrow C_a(k)j_{K+1}(kx) + D_a(k)n_{K+1}(kx), \tag{A.21}$$

and similarly for $a \to b$. These momentum–dependent coefficients define the scattering S–matrix according to [MK85]:

$$\mathrm{S} = -\begin{pmatrix} B_a + iA_a & D_a + iC_a \\ B_b + A_b & D_b + iC_b \end{pmatrix}^{-1} \begin{pmatrix} B_a - iA_a & D_a - iC_a \\ B_b - A_b & D_b - iC_b \end{pmatrix}. \tag{A.22}$$

Each element of this 2×2 S–matrix is parametrized according to

$$\mathrm{S}_{ij} = \eta_{ij} e^{2i\delta_{ij}}, \tag{A.23}$$

where δ_{ij} is the phase shift and η_{ij} denotes the inelasticity necessary to take into account the transfer of flux from one channel to the other.

We conclude this appendix by providing a few useful relations [EW88] that we will frequently use in this thesis: (i) the relation between the S–matrix and the scattering amplitude $f(k)$ is

$$f(k) = \frac{1}{2ik}(\mathrm{S} - 1); \tag{A.24}$$

(ii) the relation between the S–matrix and the K–matrix is

$$\mathrm{S} = \frac{1 + ik\mathrm{K}}{1 - ik\mathrm{K}}; \tag{A.25}$$

(iii) the scattering length ($l = 0$, S wave) and scattering volume ($l = 1$, P wave) a is defined by

$$a = \lim_{k \to 0} \frac{\mathrm{K}}{k^{-2l}}. \tag{A.26}$$

In all these expressions, k denotes the magnitude of the momentum.