C.1

DYNAMICAL CHIRAL SYMMETRY BREAKING IN FOUR-FERMI THEORIES

By

John W. Smith

B. Sc. (Physics) University of British Columbia

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

in

THE FACULTY OF GRADUATE STUDIES

PHYSICS

We accept this thesis as conforming

to the required standard

THE UNIVERSITY OF BRITISH COLUMBIA

October 1991

Abstract

Dynamical symmetry breaking of discrete chiral symmetry in four-fermi models is studied. A variational method is used to determine the effective potential. This potential is then examined to determine the critical coupling for which a phase transistion between massless and massive states occurs. Two trial ground states are used in the variational calculation and the results are the same in each case. The first is the ground state of a free massive fermion and the other is a generalized Bogoliobov-Valatin transformation of a free massless fermion ground state. In each case dynamical symmetry breaking occurs, if the coupling is fine-tuned. The results are shown to be valid for physical dimensions 1+1, 2+1 and 3+1 and compared with those of other variational methods and the 1/N expansion.

Table of Contents

List of Tables

List of Figures

Acknowledgement

I would like to thank my thesis supervisor, Dr. Gordon Semenoff, for suggesting this project and his guidance throughout the course of my research. In addition I wish to thank Gil Gat and Baruch Rosenstein for their helpful discussions regarding four-fermi theories. Finally, I would like to thank my wife, Ann Marie, for her continued support over the last two years.

Chapter 1

Introduction

In modern physics there exists many similarities between particle and condensed matter physics which can be classified under the general heading of quantum critical phenomenon [1]. The example of dynamical symmetry breaking, which is the subject of this thesis, is a prime example of this fact. A system which undergoes dynamical symmetry breaking has a second order phase transition which means that its correlation length diverges. In statistical mechanics there is a natural length scale such as the lattice spacing. In the example of spontaneous magnetization you need long range correlations in order for the spins at large distances to line up and break rotational symmetry. Similarly in field theory, a momentum cutoff is introduced which eliminates the divergences and corresponds to some small length scale (much less than the Compton wavelength and ranges of observed forces). Since the cutoff is not detected at experimentally accessible energies, it is necessary that the theory be independent of this length scale (renormalizable) and so physical field theories are those with long range correlations.

Nambu and Jona-Lasinio [2] are regarded as being the first to use the paradigm of dynamical symmetry breaking in particle physics. They used a four-fermi interaction with a chiral symmetry and a BCS or quasi-particle mechanism to generate the masses of the nucleons dynamically. The drawback of their model was that it is non-renormalizable, but one can still get some relevant results concerning the excitations of nucleons which are bound states of the theory. It was not until the problem of the renormalizability of the electro-weak theory (weak interactions are short range so the exchange particles are massive, but this is forbidden by renormalization) came to the forefront of particle physics that Weinberg and Salam [3], expanding on the work of Higgs and Goldstone [4], applied the idea of spontaneous symmetry breaking of the vacuum by a scalar field. In general, spontaneous symmetry breaking in the standard model is assumed to happen by some unknown mechanism. Just as Landau in the early 1950's was able to present a phenomenology of superfluid behavior which account for many of the observed facts without resorting to an explicit knowledge of the microscopic model; so too are particle physicists able to assume asymptotic freedom and chiral symmetry breaking in the quark and parton model to explain much of the behavior for particle physics, but the detailed dynamics are not known.

The color confinement problem in QCD has required a more detailed understanding of how quarks produce bound states. The earliest discussion by Wilson and Nambu [5] was drawn in analogy with the theory of superconductivity. Quark pair condensates give rise to a color electric 'Meisnner' effect which confines the interaction of the pair to a flux tube which when broken simply gives rise to a quark and antiquark pair and no isolated quarks. It is a current belief that confinement accompanies spontaneous breaking of the approximate chiral symmetry of the standard model and that the light mesons are pseudo-Goldstone bosons. In recent years the original analysis of chiral symmetry breaking suggested by Nambu has been applied to quarks [6, 7, 8, 9]. The benefit of this is that one can obtain quantitative results concerning the structure of the QCD vacuum, break chiral symmetry and obtain bound states within the same theory. All of these approaches rely [6, 7, 8, 9] on a BCS type wave functional and a variational method to arrive at the results.

One interesting model which displays some of the features of QCD was suggested by Gross and Neveu [10]. It of a scalar - scalar four-fermi interaction in two dimensions, which has a discrete chiral symmetry. It displays the properties of asymptotic freedom and dynamical symmetry breaking. They chose to use the $1/N$ expansion to analyze the model because it can be systematically improved unlike the variational method, gives good results in leading order for very large N and goes beyond perturbation theory because it sums an infinite number of diagrams. The results of applying the $1/N$ expansion to the problem of dynamical symmetry breaking for four-fermi interactions was reviewed by Rosenstein and Park [11].

Alternatively, some authors [12, 13, 14, 15] have applied a variational method to the Gross-Neveu model and have found that dynamical symmetry breaking occurs and the results are the same as the $1/N$ expansion when N is taken to be large. They find that the model has two phases associated with different methods of renormalization. These phases are distinguished by the fact that one makes the theory trivial and the other may produce a stable theory. The approaches [12, 22, 14, 15] are characterized by the fact that they assume the variational parameter is the expectation value of the mass operator. In [13] a restricted BCS anstaz is made and the parameter of rotation is used as the variational parameter.

In this thesis, the Gross-Neveu model in D dimensions will be explored in chapter 3, using a variational calculation. Two different types of ground state ansatzae will be used. The first trial ground state will be that of a massive free fermion. We compute the expectation value of the Hamiltonian of the Gross-Neveu model in this state and obtain a function of the mass, which characterizes the state, and the four-fermi coupling constant of the Gross-Neveu model. We then determine the physical 'equilibrium' value of the mass parameter by minimizing the expectation value. If the minimum occurs for zero mass, we say that the vacuum displays chiral symmetry. If the minimum is for non-zero mass we say that the chiral symmetry is spontaneously broken.

The idea here is to find a free field approximation to the solution of the full field theory. The free fermions can be thought of as the asymptotic fermions of the theory. A weakness of this type of ansatz is that there is no mechanism for including any bound states such as Goldstone bosons or others such as the threshold modes (with mass twice the physical fermion mass) which appear in the Nambu-Jona Lasinio model. We speculate that a more general variational picture, motivated partially by the Lehman-Symanzik-Zimmerman [16] picture of scattering theory might work for this. However, we have not explored that avenue here. Our results should be regarded as simply addressing the question of what is the nature of the fermionic part of the spectrum. In the LSZ picture, this means that we take

$$\psi(x) \approx \psi_{in}(x) + \dots, \tag{1.1}$$

where $\psi(x)$ is the interacting field and $\psi_{in}(x)$ is the asymptotic free field and the variational method determines the spectrum of ψ_{in}. More elaborate versions would attempt to also determine nonlinear corrections to this formula where other asymptotic states are included.

If one could determine all orders in the above formula, one would have the exact solution of the theory - the so-called Hagg-Nishijima-Zimmermann [17] expansion of the interacting fields in terms of free asymptotic fields. If, as we speculate, the coefficients of the terms in this formula are computable by variational calculations, there is some hope of making the latter systematic and classifying orders by the degree of nonlinearity of the terms that we are computing. This expansion would be controlled by the magnitude of many-particle amplitudes, if they are small then corrections that they give to the formula are small. This is similiar to the virial expansions in statistical physics.

The main content of this thesis is a preliminary study of the viability of this idea where we examine just the linear term in the expansion. If we have a linear term with coefficient

$$\psi(x) \approx \int dy\, U(x-y)\psi_{in}(y) \; + \; \dots \tag{1.2}$$

can we determine $U(x-y)$ by a variational proceedure?

We may view this as broadening the previous variational ansatz somewhat by asking how general can one make a *practical* ansatz where there is a linear relation between the physical and interacting particle. This is the analog of asking, in a bosonic field theory, how general could one choose a Gaussian ansatz for the ground state wave-functional and still solve the variational problem explicitly. We consider a free massless hamiltonian, construct its ground state, and take as the variational ansatz an arbitrary translationally invariant Bogoliubov-Valatin transformation of that ground state. In this variational problem, the variational parameters are a matrix-valued function. We derive a compact expression for the energy functional and require that its first variation vanishes. We then find an explicit solution of the resulting variational equation.

It turns out that the solution to the general variational ansatz is identical to that found in our first method of using the mass of a free fermion as the variational parameter. Thus, that method is more general than it seems at first sight. To our knowledge, fermionic variational problems have not been discussed to this degree of generality in previous literature. Our central result suggests that they didn't have to be, the simpler starting point is equivalent to the most general.

A result of our calculations is a value for the critical coupling constant. We find that chiral symmetry is broken for sufficiently strong attractive four-fermi interaction and that the critical coupling is computable using our method. The result agrees with previous computations of the critical coupling in leading order of the large N expansions, of the same model.

As you will see our treatment of the problem is designed to be elegant and notationally simple so that its application to more complicated theories will be straightforward. Before attacking the problem of chiral symmetry breaking we shall elaborate on some of the ideas discussed in this chapter. In chapter 2 ideas such as how dynamical symmetry breaking arises through the formation of a non-symmetric ground state will be discussed and the nature of phase transitions will be outlined as well.

Chapter 2

Symmetries in Physics

Recognizing symmetries in physical systems reduces the number of parameters needed to describe a system's behavior. The conservation of momentum and energy are simply consequences of space and time invariance of a system. An example, which is relevant to field theory, is that local gauge symmetry in QED forbids the presence of a mass term in the electromagnetic field. The existence of such symmetries in quantum field theory often coincides with some unusual features, such as the fact that only massless QED is renormalizable. However, even if the equations of motion respect a symmetry, the energy spectrum may not. Just try and stand your pen on its end and you'll see that it's lowest energy state breaks rotational symmetry in space. In QCD, the approximate chiral flavour symmetry of the strong interactions implies that the quark masses are almost zero on the typical strong interaction scale. PCAC, current algebra and many low-energy theorems rely on the chirally symmetric theory [18], but hadrons with mass are known to exist. Since some results in QCD depend on the mass being zero and others non-zero it is crucial that one be able achieve a massive theory without putting mass in explicitly and dynamical symmetry breaking of chiral symmetry can do that. The important thing to recognize is that only by looking at the energy spectrum of a theory can it be determined if a symmetry is preserved. When the equations describing a theory are symmetric and the states of a theory aren't, this is known as spontaneous symmetry breaking. In this chapter the it will be made precise what is symmetry breaking and its implications.

2.1 Classical Symmetries and Conservation Laws

Using the Lagrangian formulation of mechanics one can show that symmetries and conservation laws are related. The connection was made precise by Noether's Theorem [18]. A dynamical field theory can be described by the action functional S in D dimensions with field $\psi_i(x)$

$$S = \int d^D x \ \mathcal{L}(\psi_i(x), \partial_\mu \psi_i(x)) \tag{2.3}$$

where $\mathcal{L}$ is the Lagrange density. A general transformation of the fields can be written

$$\psi \to \psi + \delta\psi, \tag{2.4}$$

and

$$\partial_\mu \psi \to \partial_\mu \psi + \delta \partial_\mu \psi, \tag{2.5}$$

resulting in the transformation

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}. \tag{2.6}$$

A transformation which leaves the equations of motion unchanged is

$$\delta\mathcal{L} = \partial_\mu K^\mu. \tag{2.7}$$

Another symmetry is defined as those symmetries which leave the Lagrangian density itself unchanged with a change of fields. If this is true then there is a conserved Noether current, J^μ, associated with each symmetry. To obtain J^μ, look at the expression for $\delta\mathcal{L}$ under the transformations given in 2.4

$$\delta\mathcal{L} = \sum_i [\frac{\delta\mathcal{L}}{\delta\psi_i}\delta\psi_i + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_i)}\delta(\partial_\mu\psi_i)]. \tag{2.8}$$

Using the Euler-Lagrange equations for the field,

$$\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_i)} - \frac{\delta\mathcal{L}}{\delta\psi_i} = 0. \tag{2.9}$$

Then substituting this into 2.8 and including 2.7 we obtain the total change in $\mathcal{L}$ (assuming $\delta(\partial_\mu \psi_i) = \partial_\mu(\delta\psi_i)$)

$$\delta\mathcal{L} = \sum_i \partial_\mu[\frac{\delta\mathcal{L}}{\delta(\partial_\mu \psi_i)}\delta\psi_i], \tag{2.10}$$

Thus since symmetry of the Lagrangian requires $\delta\mathcal{L} = 0$, there is a conserved current

$$\partial_\mu J^\mu = 0, \tag{2.11}$$

with a corresponding 'charge'

$$Q^a = -i\int d^3x J_0(x), \tag{2.12}$$

which satisfies the equation

$$\frac{dQ}{dt} = 0. \tag{2.13}$$

This is just a conservation law for the charge which results from the symmetries in the action 2.3.

2.2 Gauge Symmetry

An example of a symmetry in elementary particles is the conservation of electric charge Q. This symmetry is associated with the arbitrary phase transformation of a complex field. Consider the Dirac field ψ and the transformation

$$\psi \rightarrow e^{ie\theta(x)}\psi. \tag{2.14}$$

The fact that ψ is complex leaves the QED Lagrangian

$$\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu - m\psi + e\bar{\psi}\gamma^\mu\psi A^\mu + F^{\mu\nu}F_{\mu\nu}, \tag{2.15}$$

invariant, if A_μ transforms as

$$A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\theta(x), \tag{2.16}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that a mass term $m\bar{\psi}\psi$ is invariant, but the vector field mass term $A_\mu A^\mu$ is not. This symmetry is called a local U(1) symmetry, because $e\theta(x)$ is a generator of the abelian group U(1). The conserved current associated with gauge symmetry is

$$J^\mu = F^{\mu\nu}\partial_\nu\theta + ie\theta\bar{\psi}\gamma^\mu\psi, \tag{2.17}$$

and the conserved charge operator which is associated with electric charge is the result of constant gauge or phase symmetry and is

$$Q^a = -i\int d^3x J_0(x) = \int d^3x\psi^\dagger\psi, \tag{2.18}$$

where the first term in 2.17 is eliminated using Gauss' law and partial integration (assuming θ vanishes at the boundary). These currents are conserved in quantum field theory regardless of whether symmetry breaking occurs [18].

QCD and electro-weak interactions are simliar to QED because they also use local gauge symmetric interactions, but transformations of the Dirac field now have the form

$$\psi \rightarrow e^{i\tau\cdot\theta(x)}\psi, \tag{2.19}$$

where τ are the generators of a non-abelian gauge group like SU(2) (the Pauli matrices being one representation). So in these non-abelian gauge theories there is more than one coupling and at least two fundamental fermions and this leads to the rich structure of non-abelian gauge theories. An interesting feature of gauge theories is that they can be rewritten in terms of an four-fermi interaction, this can be a shown using functional methods. For instance the Lagrangian 2.15 results in the coulomb interaction

$$V = \int d^{D-1}x\int d^{D-1}y\psi^\dagger(x)\psi(x)\frac{1}{|x-y|}\psi^\dagger(y)\psi(y). \tag{2.20}$$

In fact it is potentials of this form that are considered by those who look at dynamical symmetry breaking for QCD [6]. So considering four-fermi theories is very simliar to looking at gauge theories.

2.3 Chiral Symmetry

In this section the relationship between mass and chiral symmetry will be examined. Consider Dirac's action for free fermions in D dimensions

$$S = \int d^D x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x). \tag{2.21}$$

This action has a symmetry for $m = 0$ under the discrete chiral transformations

$$\psi(x) \rightarrow e^{i\frac{\pi}{2}\gamma_5}\psi(x) = \gamma_5\psi(x) \tag{2.22}$$

and

$$\bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{-i\frac{\pi}{2}\gamma_5} = -\bar{\psi}(x)\gamma_5, \tag{2.23}$$

where $\gamma_\mu\gamma_5 + \gamma_5\gamma_\mu = 0$ and $\gamma_5^2 = 1$. This symmetry persists for a suitable choice of interactions, such as minimal coupling to a gauge field A,

$$\partial_\mu \rightarrow \partial_\mu - ieA_\mu, \tag{2.24}$$

or a four-fermion interaction

$$g_4[(\bar{\psi}\psi(x))^2] + g_5[(\bar{\psi}i\gamma_5\psi(x))^2]. \tag{2.25}$$

The conserved current of this symmetry is given by

$$Q_5 = \int d^{D-1}x\psi^\dagger(x)\frac{\pi}{2}\gamma_5\gamma_\mu\psi(x). \tag{2.26}$$

The generator of the transformation

$$e^{i\frac{\pi}{2}\gamma_5}\psi(x) = e^{i\frac{\pi}{2}Q_5}\psi(x)e^{-i\frac{\pi}{2}Q_5}, \tag{2.27}$$

which can be shown by taylor expanding the generator and applying the anti-commutation relations between fields and Dirac matrices.

If these symmetries are respected by the vacuum, then

$$e^{-i\frac{\pi}{2}Q_5}|0\rangle = |0\rangle. \tag{2.28}$$

Taking the expectation value of the mass operator, in the ground state, by first rotating the vacuum and then the operators one finds a contradiction

$$\langle 0|e^{i\frac{\pi}{2}Q_5}\bar{\psi}\psi e^{-i\frac{\pi}{2}Q_5}|0\rangle = \langle 0|\bar{\psi}\psi|0\rangle = -\langle 0|\bar{\psi}\psi|0\rangle, \tag{2.29}$$

and therefore the expectation value of the mass operator must be identically zero. If the vacuum isn't chirally symmetric, then the expectation value behaves like

$$\sigma_c = \lim_{m\to 0}\langle 0|\bar{\psi}\psi|0\rangle \neq 0, \tag{2.30}$$

and chiral symmetry is spontaneously broken by the quantum field theory.

2.4 Phase Transistions

The definition of σ_c in 2.30 is analogous to that of the magnetization in a ferromagnet. The idea of spontaneous symmetry breaking originated in the theory of phase transistions. It was Landau who first pointed out the vital importance of symmetry in phase transistions. He stated that it is impossible to change symmetry gradually; the symmetry element is either there or it is not. Although this is not strictly true, all phase transistions need to break a symmetry and this is characterized by an order parameter similiar to 2.30 mentioned above. The theory of Gross and Neveu [10] will be introduced here and used as an example to illustrate how the order parameter arises. In addition the different types of phase transistions will be characterized.

The Gross-Neveu Lagrangian density is given by

$$\mathcal{L} = \bar{\psi}^a(x)(i\gamma_\mu\cdot\partial^\mu)\psi^a(x) + \frac{\lambda}{2N}(\bar{\psi}^a(x)\psi^a(x))^2,\ a = 1..N. \tag{2.31}$$

The Lagrangian density is $O(2N)$ symmetric and invariant under the discrete γ^5 transformation

$$\psi^a \rightarrow \gamma^5 \psi^a. \tag{2.32}$$

Alternatively the Lagrangian 2.31 can be written

$$\mathcal{L}_\sigma = \bar{\psi}^a (i\gamma_\mu \cdot \partial^\mu)\psi^a - \frac{N\sigma^2}{2\lambda} + \sigma(\bar{\psi}^a \psi^a) \tag{2.33}$$

These two Lagrangian densities give equivalent theories because of the additional equation of motion 2.9 of the σ field,

$$\sigma = \frac{\lambda}{N} \bar{\psi}^a \psi^a. \tag{2.34}$$

In a system with many degrees of freedom the fundamental object is the partition function in statistical mechanics and the generating functional in the path integral approach to field theory, both of which depend on the action and have a similiar form. The generating functional for 2.33 is

$$Z(J) = \int d\psi d\bar{\psi} d\sigma exp[i \int (\mathcal{L}_\sigma + J\sigma)]. \tag{2.35}$$

and a derivation of it from an infinite dimensional quantum system can be found in [18]. This object is usually described as the vacuum to vacuum expectation value in the presence of an external source, where one takes the limit$J \rightarrow 0$ when one wants to look at physical quantities. The benefit of the large N approach is that fermion fields are quadratic and the large N expansion resums a certain class of diagrams to a certain order. Also the validity of the approximation is based on the largeness of N and not the size of λ.

It is commonly known [18] that

$$W = -ilnZ \tag{2.36}$$

is analogous to the free energy in statistical mechanics when complex time is used and for real time generates the connected Green's functions for a field theory

$$G^{(n)}(x_1, ..., x_n) = \frac{\delta lnW(J))}{\delta J(x_1)...\delta J(x_n)}|_{J=0}. \tag{2.37}$$

The classical σ field, σ_c, is defined by

$$\sigma_c(x) = \frac{\delta W}{\delta J(x)} = \langle 0|\sigma(x)|0\rangle. \tag{2.38}$$

The Legendre transformation of $W(J)$,

$$\Gamma(\sigma_c) = \int d^D x(\sigma_c(x)J(x)) - W(J), \tag{2.39}$$

is equivalent to the expectation value of the Hamiltonian of a system [19] in the state where the vacuum expectation value of σ is σ_c [10]. This fact is the starting point for the variational method which will be discussed in the next chapter. translational invariance of the vacuum requires tha σ_c be independent of space-time and so

$$\Gamma = \int d^D x V(\sigma_c). \tag{2.40}$$

This is just the thermodynamic potential in statistical mechanics and in field theory it is called the effective potential.

Recalling what was said about the partition function 2.35 it is now time to take the limit $J \rightarrow 0$. From the properties of the Legendre transform we get

$$J = \frac{\delta\Gamma}{\delta\sigma_c} = 0. \tag{2.41}$$

This is just the condition that the effective potential be extremized, along with the stability condition that the extremum be a minimum

$$\frac{\partial^2\Gamma}{\partial\sigma_c^2} > 0. \tag{2.42}$$

At a critical coupling (or temperature in statistical physics) the minimum of the potential will tend towards some non-zero value of σ_c and the system undergoes a phase transistion. Note that the condition 2.30 in the previous section is the same as the conclusion stated here; when the assignment $J \to m$ is made. Thus the statement for the existence of symmetry breaking in quantum field theory is as follows. When a critical coupling exists such that the effective potential has a minimum at some non-zero value of a composite field operator, the symmetry which that composite field breaks classically has been spontaneously broken.

There is one more aspect that is relevant to field theory that needs to be explained. There are two ways that the phase transistion between regions seperated by the critical coupling can occur and they are illustrated in figures 2.1 and 2.2. The first order phase transistion, seen in figure 2.1, is characterized by a discontinuous jump from $\sigma_c = 0$ to $\sigma_c \neq 0$ and an example of this type of transistion is when water boils. A second order phase transistion, seen in figure 2.2 is continuous and is similiar to that of a ferromagnet, whose magnetization rises from zero below some critical temperature. It is the second order phase transistion which is most interesting. The expression 2.35 takes into account the microscopic fluctuations in the vacuum, whereas particles exist on a much larger scale and so there needs to be structure on a large scale. By looking at the correlation length, which is proportional to $\Gamma''(\sigma_c)^{-1/2}$ [1] the large scale structure can be found. In the first order phase transistion the correlation length is finite and this is reflected by that fact that multiple phases can exist in a material (bubbles boil up in water). Whereas the second derivative of Γ vanishes at λ_c, the correlation length diverges. So now the microscopic fluctuations are irrelevant and the theory wll be massive, if one is near the critical coupling.

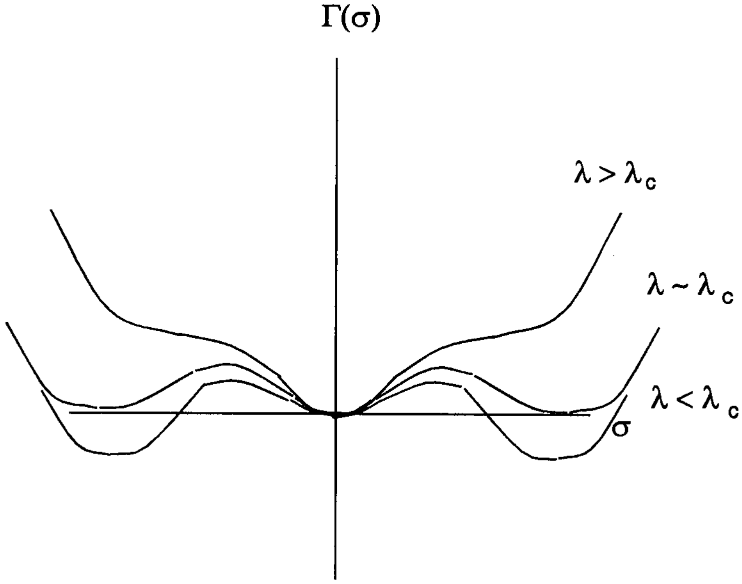

Figure 2.1: First order phase transistion.

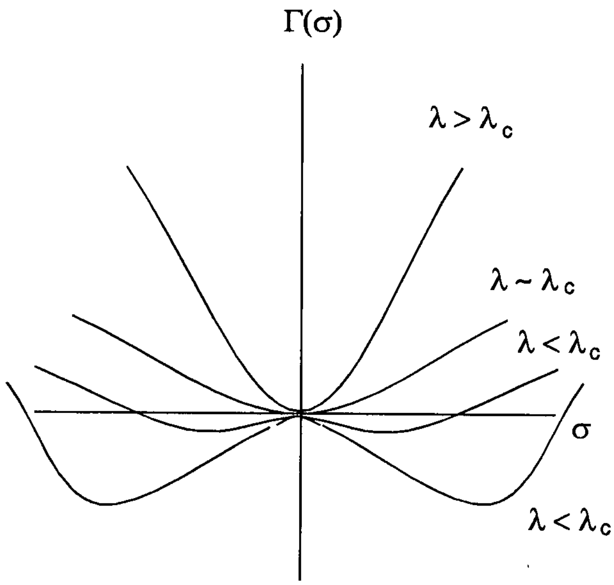

Figure 2.2: Second order phase transistion.

Chapter 3

The Variational Method of Dynamical Symmetry Breaking

So far we have shown that a theory which exhibits dynamical symmetry breaking has three characteristics. First of all the ground state doesn't respect the symmetry being broken. Second, the phase transistion is characterized by the expectation value of a composite operator. Finally, the phase transition should occur at some critical value of the coupling. It is the effective potential which will yield this valuable information and the most straightforward way of calculating the effective potential is to look at the ground state expectation value of the Hamiltonian. Since this can't be done exactly for a non-linear theory it is necessary to use an approximate method. A variational calculation of the ground state expectation value can be done, this is a non-perturbative method and it's validity does not generally depend on the coupling. But there is an added bonus because the variational method gives an ansatz for the form of the vacuum state. The symmetry breaking terms are placed into the theory as parameters of the ansatz and then the effective potential is minimized with respect to these parameters. The variational method has been tried by various authors [12, 14, 13] and the results concerning the existence of dynamical symmetry breaking confirm those of the $1/N$ expansion. Our motivation for doing this problem again is that the previous results use specific ansatz's for the ground state; whereas we will use the most general 'Gaussian' expression for a translationally invariant vacuum to minimize the potential. The advantage of this method over large N is that it is valid for any N and may sum up diagrams which aren't summed by large N. The drawbacks are that one is never sure how good the result is because it is not a formal expansion in some small parameter and it must be compared with results from other methods to test its validity.

In the first part of this chapter a simple ansatz for the ground state will be used to demonstrate the ideas we have discussed. In the second part the most general ansatz for the ground state will be examined.

3.1 The Massive Ground State

The simplest ansatz for the ground state wavefunction, which will exhibit chiral symmetry breaking, is to assume that the ground state is the solution of a massive free particle. If the ground state is taken to be that of a massive free particle hamiltonian

$$H_m = \int d^{D-1}x \psi^\dagger(x)(i\alpha \cdot \nabla + \beta m)\psi(y), \tag{3.43}$$

where $(\beta, \alpha) = (\gamma^0, \gamma^0\gamma^i)$ are the Dirac matrices, which obey the usual anti-commutation relations. The mass is treated as a variational parameter and we then compute the expectation value of the N particle interaction

$$H = \int d^{D-1}x[\psi^{a\dagger}(x)(i\alpha \cdot \nabla)\psi^a(y) + \frac{\lambda}{2\Lambda^{D-2}N}(\psi^{a\dagger}(x)\beta\psi^a(x))^2] \tag{3.44}$$

in the ground state to get $\langle H \rangle$ as a function of m. We then find the value of m for which the potential is a minimum.

The eigenvalue equation for a massive free Hamiltonian for a Dirac particle in momentum space is given by

$$h_m\psi_E = (\alpha \cdot p + \beta m)\psi_E, \tag{3.45}$$

where $|E| = \sqrt{p^2 + m^2}$. The fermion field operators which appear in the Lagrangian density 2.31 have the form

$$\psi(x) = \sum_{E>0} \psi_E a(E) e^{ipx} + \sum_{E<0} \psi_E b^\dagger(E) e^{-ipx}, \tag{3.46}$$

where a is the annihilation operator for particles and $b^\dagger$ is the creation operator for anitparticles, which satisfy the usual anticommutation relations. The vacuum state is defined by $a|0\rangle = 0$ and $b|0\rangle = 0$.

Using 3.45,3.44 and 3.46 the vacuum expectation value of the massless four-fermi Hamiltonian can be rewritten

$$\langle H\rangle = \langle 0| \int d^{D-1}x[(\psi^{a\dagger}(x)h_m(x,y)\psi^a(y) - m\psi^{a\dagger}(x)\beta\psi^a(x)) + \frac{\lambda}{2\Lambda^{D-2}N}(\psi^{a\dagger}(x)\beta\psi^a(x))^2]|0\rangle, \tag{3.47}$$

where the mass term has been added and subtracted from the hamiltonian for calculational convenience. The factor Λ^{D-2} has been introduced to make the coupling dimensionless, where Λ is the momentum cutoff of the theory. The fermionic operators should be normal ordered to eliminate any constant divergences and so each occurrence of the term $\psi^\dagger\psi$ is replaced by $\frac{1}{2}[\psi^\dagger, \psi]$. It is convenient to introduce a factor $\epsilon_m(x, y)$ which is the result of normal ordering the operators. Applying the creation annihilation operators to the vacuum can make the assignment

$$\epsilon_m(x,y) \equiv \langle 0|[\psi(x), \psi^\dagger(y)]|0\rangle = -\sum_{E>0} \psi_E\psi_E^\dagger e^{ip(x-y)} + \sum_{E<0} \psi_E\psi_E^\dagger e^{ip(x-y)}. \tag{3.48}$$

Using this in subsequent formulas will greatly simplify our notation. The expression 3.48 can be simplified. An operator can be defined in terms of its eigenvalues and eigenvectors, so that given

$$h\psi_E = E\psi_E, \tag{3.49}$$

the operator h can be written

$$h(x-y) = \sum_E E\psi_E\psi_E^\dagger e^{ip(x-y)}. \tag{3.50}$$

Similiarily an operator could be defined as

$$\epsilon(x-y) = -\sum_E \frac{E}{|E|}\psi_E\psi_E^\dagger e^{ip(x-y)}. \tag{3.51}$$

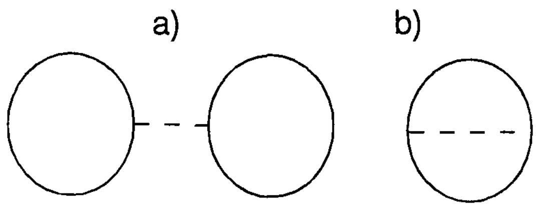

Figure 3.3: Feynman diagrams representing the four-fermi interaction terms: a) direct b) exchange.

which is the same as that in 3.48. The operator ϵ to be used in 3.47 can be thought of as the sign of the operator which generates the eigenvectors ψ_E and is effectively the operator

$$\epsilon_m = \int \frac{d^{D-1}p}{(2\pi)^{D-1}} \frac{h}{|h|} e^{ip(x-y)}. \tag{3.52}$$

Returning to evaluating 3.47 we get

$$\begin{aligned} \langle H \rangle = & -\frac{N}{2} TR[tr(h_m \epsilon_m)] + \frac{N}{2} TR[tr(\beta m \epsilon_m)] + \\ & \frac{\lambda}{8\Lambda^{D-2}} \{NTR[tr(\beta\epsilon_m)^2] - TR[tr((\beta\epsilon_m)^2)]\}. \end{aligned} \tag{3.53}$$

The small trace tr is over the Lorentz indices, the big trace TR is over the space variables and the particle flavour indices have been traced over already (we assumed there were N flavours). Notice the interaction term contains a positive direct term and and a negative exchange term corresponding to the Feynman diagrams in figure 3.3.

The explicit integral form of the ground state energy can now be found using 3.52 and 3.53. Using the fact that $|h_m| = \sqrt{p^2 + m^2}$, completing all traces and using the properties of the Dirac matrices, the expression for the ground state energy becomes

$$\frac{\langle H \rangle}{V} = D'N(-I_1 + m^2 I_0) + \frac{\lambda}{2\Lambda^{D-2}} D'(D'N - 1) m^2 I_0^2. \tag{3.54}$$

n	D=1+1	D=2+1	D=3+1
1	2	3	4
0	$ln(4\Lambda/m)$	1	2
-1	-2	-1	$ln(4\Lambda/m)$
-2	-4	-3	-2

Table 3.1: Table of the degrees of divergences of the integrals I_n in D dimensions.

The factor D' comes from completing the traces in D dimensions; it is 2 for 1+1 dimensions, 4 for 3+1 dimensions and is either 2 or 4 for 2+1 dimensions, because of the different ways of representing fermions in 2+1 dimensions [11] (we choose the four dimensional Dirac matrices representation). The intregal I_n is given by

$$I_n = \int_0^\infty \frac{d^{D-1}k}{2(2\pi)^{D-1}} E^{2n-1} \tag{3.55}$$

and has the property

$$\frac{\partial}{\partial m} I_n = (2n-1) m I_{n-1}. \tag{3.56}$$

Note that the integral I_0 in 3.54 is just the condensate σ_c 2.30 calculated using the normal ordered expectation value $tr(\beta\epsilon)$. The divergences of 3.55 for different dimensions and values of n can be seen in table 3.1.

Now the effective potential or ground state energy 3.54 can be used to determine the value of the critical coupling. Minimizing the ground state energy with respect to the free parameter m

$$\frac{\partial \langle H \rangle}{\partial m} = 0. \tag{3.57}$$

The resulting equation can be written

$$ND'(mI_0 - m^3 I_{-1}) + \frac{\lambda D'(D'N-1)}{\Lambda^{D-2}}(-m^3 I_0 I_{-1} + m I_0^2) = 0. \tag{3.58}$$

From 3.58 the critical coupling can be determined

$$\lambda_c = \frac{-N\Lambda^{D-2}}{(D'N-1)} \frac{1}{I_0(m=0)}. \tag{3.59}$$

Thus for $|\lambda| > |\lambda_c|$, the potential has a negative slope at the $m = 0$ and the minimum will be at $m \neq 0$. Thus chiral symmetry has been broken. Note also that the critical coupling is negative so that the potential is attractive and may result in bound states. Also at mass $m = 0$, by looking at the table for the integrals 3.1 it can be seen that the critical coupling 3.59 takes on a finite value in the limit $\Lambda \to \infty$. The expression 3.59 for the critical coupling is the same as that for the $1/N$ expansion [11]. The difference between the methods is that in the $1/N$ expansion, $\sigma_c = I_0$ is the variable used to minimize the effective potential.

To make sure the potential has a absolute minimum, the second derivitave of the potential 3.54 must be positive and finite around the critical coupling. The second derivative of the potential is

$$\frac{\partial^2\langle H\rangle}{V\partial m^2} = ND'(-4m^2I_{-1}+I_0-3m^4I_{-2})+\frac{\lambda D'(D'N-1)}{\Lambda^{D-2}}(m^4I_{-1}^2+3m^4I_0I_{-2}-5m^2I_0I_{-1}+I_0^2). \tag{3.60}$$

The positivity at critical coupling can be seen by inspection. If a theory is to be consistent, then it must be possible to renormalize λ to eliminate the divergences in the first and second derivatives and retain the critical behavior. This means tuning λ to be sufficiently close to λ_c and therefore λ_c is an ultraviolet fixed by point by definition [18]. Note that the divergence of I_1 in 3.54 is just the massive ground state energy, which will be subtracted from the excited states, and is irrelevant with regard to divergences, but contains some vacuum structure.

The renormalization prescription for variational methods, which has been developed by Stevenson and Barnes [20] and is used in previous variational approximations [12, 13, 15], is to equate the terms in the second derivative with a renormalized coupling constant λ_R which is positive and finite

$$\frac{\partial^2\langle H\rangle}{V\partial m^2}|_{m=\mu} = \lambda_R. \tag{3.61}$$

The renormalized coupling is the same in all dimensions. In 1+1 and 2+1 dimensions, all the divergences are in the condensate I_0, whereas in 3+1 dimensions the integral I_{-1} also has a logarithmic divergence, but by tuning the coupling to be near the critical coupling λ_c the theory can be renormalized.

For simplicity the renormalized coupling in 1+1 dimensions will be analyzed. Using 3.61 the coupling constant can be expressed as

$$\lambda = \frac{\lambda_R - NI_0(\mu)}{(2N-1)I_0^2(\mu)}, \tag{3.62}$$

where the non-divergent terms have been absorbed into λ_R. Looking at the expression 3.62 one can see that it has the form $\lambda - \lambda_c = \lambda_R/(2N-1)I_0(\mu)$ and so the new coupling has indeed been tuned to be near the critical value. In addition the coupling has a cutoff dependence $\lambda = 1/ln(\Lambda)$ in 1+1 dimensions,which is the criteria for asymptotic freedom. Alternatively $\lambda_c = const$ or $I_0 = const$ could have been chosen, this fixes the condensate to a finite value and results in a dependence of the mass on the cutoff of the form

$$m = 4\Lambda e^{-2\pi I_0}. \tag{3.63}$$

This is a fine tuning of the mass and means that for $\Lambda \to \infty$ the integral I_0 is finite. Thus the methods are essentially the same. Using either one of these methods makes the effective potential finite. So in the variational method, the fact that condensate is finite is a consequence of renormalization and not an additional restriction on renormalization.

3.2 The General Ground State

In this treatment of the variational problem for four-fermi theories the most general quadratic ground state wavefunctional will be used. The most general unitary transformation for non-relativistic fermions was discussed by Kaempffer [21], where it was shown that the superconducting wavefunction of BCS theory was but a special case. In this section we present the most general quadratic relativistic unitary transformation of a fermionic operator

$$\psi \to \mathcal{U}^{\dagger}\psi\mathcal{U}. \tag{3.64}$$

The matrix $\mathcal{U}(x, y)$ will be chosen to be translationally invariant (i.e. of the form $\mathcal{U}(x - y)$). The specific form of the transformation is

$$\mathcal{U} = exp(\int d^{D-1}x\psi^{\dagger}(x)ln(U(x-y))\psi(y)) \tag{3.65}$$

and the fact that the exponential is quadratic in the fields is why the transformation is referred to as being quadratic or gaussian. As seen in the case of the chiral symmetry, a taylor expansion of 3.65 will yield the transformations

$$\psi \to U\psi \tag{3.66}$$

and

$$\psi^{\dagger} \to \psi^{\dagger}U^{\dagger}. \tag{3.67}$$

Note that $U^{\dagger}U = 1$. These transformations can be thought of as acting on the wavefunctional or the ground state.

To procede we quantize the massless hamiltonian by solving its eigenvalue problem

$$\alpha \cdot p\psi_E = E\psi_E \tag{3.68}$$

and define the second quantized fermion field as in 3.46. Taking the general form of the variational ground state to be $\mathcal{U}|0\rangle$ the ground state energy can be calculated.

$$\langle H\rangle = -\frac{N}{2}TR[tr(U^{\dagger}h_0U\epsilon)] + \frac{\lambda}{8}(NTR[tr(U^{\dagger}\beta U\epsilon)^2] - TR[tr((U^{\dagger}\beta U\epsilon)^2)]), \tag{3.69}$$

where

$$\epsilon = \int d^{D-1}p\alpha \cdot \frac{p}{|p|}e^{ip(x-y)}. \tag{3.70}$$

This equation also contains a direct and exchange term much like the previous expression 3.53 for the ground state, but it is not clear at this point what affect the terms U might have. Taking the first functional variation of 3.69 to minimize the potential

$$\begin{aligned}\frac{N}{2}(-TR[tr(\delta U\epsilon U^\dagger h_0)] + TR[tr(\delta UU^\dagger h_0 U\epsilon U^\dagger)]) + \\ \frac{\lambda}{2}(NTR[tr(\beta U\epsilon U^\dagger)(-(\delta U\epsilon U^\dagger h_0) + \\ +tr(\delta UU^\dagger h_0 U\epsilon U^\dagger)]) - TR[tr(\delta U\epsilon U^\dagger\beta U\epsilon U^\dagger\beta)] + \\ TR[tr(\delta UU^\dagger\beta U\epsilon U^\dagger\beta U\epsilon U^\dagger)]) = 0, \qquad (3.71)\\ \qquad (3.72)\end{aligned}$$

where $\delta U^\dagger = -U^\dagger\delta UU^\dagger$ and the trace over Lorentz indices has been used to bring the term δU to the beginning of the trace.

To find a solution for 3.72 it is useful to use the expression

$$M = U\epsilon U^\dagger, \qquad (3.73)$$

where M is unitary, to simplify things. Since TR will just become a volume factor in this theory it can removed from 3.72 and so can a Lorentz trace and a momentum integral from ϵ. By applying the term U to the right of 3.72, the expression is simplified considerably

$$\begin{aligned}N[h_0, M] + \lambda(N[h_0, M]\int\frac{d^{D-1}p}{(2\pi)^{D-1}}tr[\beta M] - M\beta\int\frac{d^{D-1}p}{(2\pi)^{D-1}}M\beta + \\ \beta M\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\beta M) = 0, \qquad (3.74)\end{aligned}$$

where the integral in momentum space is given so the form of the equation to be solved is known. The general solution for M will just be a linear combination of the possible γ matrices [1]. It is simplest to solve 3.74 if we assume that $\lambda = 0$, then $M = \epsilon$ or in other

[1]There are 16 combinations in four dimensions $1, \gamma_\mu, \gamma_0\gamma_\mu, \gamma_5, \gamma_5\gamma_\mu, [\gamma_\mu, \gamma_\nu]$

words $\mathcal{U} = 1$ as expected. If $\lambda \neq 0$ then we make the assumption that $M = h_0 + \bar{M})$ and if we are near critical coupling then $\bar{M}$ will be small and the expression 3.74 will be

$$[h_0, \bar{M}] + \lambda\beta h_0 \int \{\beta, \bar{M}\} = 0. \qquad (3.75)$$

The solution for 3.75 can be found by inspection

$$\bar{M} = \beta m. \qquad (3.76)$$

Including a normalization factor because of the unitarity of M, $M^2 = 1$, we find

$$M = \frac{\alpha \cdot p + \beta m}{\sqrt{p^2 + m^2}}. \qquad (3.77)$$

This solution also satisifies the non-linear relation 3.74. Using the expression 3.77 in the ground state energy we find exactly the same expression as in the previous chapter 3.53. Thus even the most general translationally invariant ground state or Bogolioubov rotation, as it is sometimes called, yields a massive Hamiltonian as the optimized ground state when using the variational method. So dynamical symmetry breaking is a persistent feature in Gross-Neveu type models where the vacuum seems to like the presence of chirally nonsymmetric mass terms. Thus the critical behaviour will be the same as that found in the previous section.

Chapter 4

Conclusion

The study of dynamical symmetry breaking in four-fermi interactions touches upon many relevant topics in modern physics. In particle physics the existence of some form of spontaneous chiral symmetry breaking is necessary if the standard model is to remain a viable theory as researchers go to higher energy. Indeed many attempts to go beyond the standard model such as technicolor theories rely upon a good understanding of symmetry breaking. If the SSC, currently being planned, is unable to find the Higgs particles then dynamical symmetry will play an even larger role in particle physics. In statistical physics , where the idea of spontaneous symmetry breaking arose, it was successfully applied to one of the great triumphs of modern physics, the BCS theory of superconductivity. Chiral symmetry breaking has even been suggested as being pivotal in the origin of life [23], because the absence of chiral symmetry in nature has implications for the structure of the early universe and the structure of DNA.

Although symmetry breaking gives physical phenomenon an rich structure and its pervasive influence in physics reflects the unity of physics. It also emphasizes the poverty of methods which we use to deal with them. Take for instance the general proof by Kovner and Rosenstein [22] of the equivalence between the Gaussian variational method, Bogolioubov rotations and the truncation of Schwinger-Dyson equations for scalars. In this thesis we have seen the equivalence between two methods which at the outset seemed likeley to produce different results from each other.

To summarize the results of this thesis, the dynamical breaking of chiral symmetry in physics was explored using the variational method. It was found that dynamical symmetry breaking occurs for 1+1,2+1 and 3+1 dimensions. The value of the critical coupling was found to be the same as that of the $1/N$ expansion and the theory is renormalizable near the critical coupling. The coupling was found to be asymptotically free in 1+1 dimensions and has an ultraviolet fixed point. A general translationally invariant unitary transformation was applied to the vacuum to see what kind of terms might break the chiral symmetry of the 'massless' vacuum. It was found that one simply arrives at the ground state of a free massive particle. The reason for this rather surprising result may be that the method is automatically respecting Lorentz invariance, in that it seems to be finding a Lorentz invariant solution of the problem even when the general ansatz is not Lorentz invariant and there are very few ways that it could do this in the static, Hamiltonian picture that we use, besides creating mass.

The fact that the free massive fermion ansatz is sufficient to generate symmetry breaking is encouraging because it implies that other simple variational ansatzae may provide the necessary symmetry breaking terms in theories like QCD. Despite the fact that no new results were found concerning the phase transistion of the Gross-Neveu model in up to 3+1 dimensions, this thesis simplifies and generalizes the treatment of the problem for variational calculations considerably. It is hoped that the work done here may be applied to more difficult theories such as QCD, in order to at least confirm known results in a more elegant and quantitative way.

Bibliography

[1] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomenon* (1989) Academic Press.

[2] Y. Nambu and G. Jona-Lasino, Phys. Rev. 122 (1961) 345.

[3] S. Weinberg, Phys. Rev. Lett. 17 (1966) 616.

[4] P.W. Higgs, Phys. Rev. Lett. 12 (1964) 132.

[5] Y. Nambu, phys. Rev. D10 (1974) 4264.

[6] J.R. Finger and J.E. Mandula, Nucl. Phys. B199 (1982) 168;J. Finger, D. Horn and J.E. Mandula, Phys. Rev. D20 (1979) 3253.

[7] A. Amer, A. Le Yaouanc et al., Z. Phys. C17 (1983) 61.

[8] A. Le Yaouanc, L. Oliver et al., Phys. Rev D31 (1985) 137.

[9] A. Casher, Phys. Lett. B83 (1979) 395.

[10] D. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.

[11] B. Rosenstein, B.J. Warr and S.H. Park, Phys. Rep. 205 (1991) 59.

[12] J.I. Latorre and J. Soto, Phys. Rev. D34 (1986) 3111.

[13] I. Yotsuyanagi, Phys. rev. D39 (1989) 3034.

[14] S.K. Kim, K.S. Soh and J.H. yee, Phys. Rev. D41 (1990) 1345.

[15] S.K. Kim, J. Yang et al., Phys. Rev. D40 (1989) 2647.

[16] C. Itzykson and J. Zuber, *Quantum Field Theory* (1980) McGraw-Hill.

[17] K. Nishijima, *Fundamental Particles* (1963) W.A. Benjamin.

[18] T. Cheng and L. Li, *Gauge Theory of Elementary Particle Physics* (1984) Oxford University Press.

[19] R. Jackiw and A. Kerman, Phys. Lett. A71 (1979) 158.

[20] P.M. Stevenson, Phys. Rev. D30 (1984) 1712; T. Barnes and G.I. Ghandour, Phys. Rev. D22 (1971) 924.

[21] F.A. Kaempffer *Concepts in Quantum Mechanics* (1965) Academic Press.

[22] B. Rosenstein and A. Kovner, Phys. Rev. D 40, (1989) 523.

[23] A. Salam, The role of chirality in the origin of life, preprint IC preprint 277, (1990).