

Wavy Line Wilson Loops in the AdS/CFT Correspondence

by

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Abstract

Wilson loops which are close to straight, infinite lines are considered in the context of the AdS/CFT correspondence. The work concentrates on CFT side calculations. The one line self energy and the connected correlator of two lines are considered. Results are presented as functionals of the “waviness” function which describes the departure of the Wilson loop from a straight line. The calculations are performed to second order in waviness. The one wavy line expectation value is computed to second order in the 't Hooft coupling, while the two wavy line correlator is calculated to third (leading) order in the 't Hooft coupling. Comparisons of the CFT results to AdS side calculations show behaviour consistent with the AdS/CFT correspondence.

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Chapter 1

Introduction

1.1 Motivation

It is a long standing problem in physics to understand the strongly coupled kinematical regime of non-abelian gauge theories, the most important example of which is QCD, the theory of the strong nuclear force. At low energies, the coupling constant in these theories becomes too large for perturbation theory to be applicable. For example, in QCD, at low energies we have confinement, that is quarks are bound together in such a way that they may not be separated from one another. The quark-antiquark potential for example, is something that cannot be calculated using perturbative methods. Lattice QCD, a non-perturbative numerical technique, has had some success in solving strong coupling problems, however it is desirable to obtain full analytical control of the dynamics. Thus, researchers have looked to other models as means of obtaining analytical results.

There is a history of attempting to describe the strong coupling degrees of freedom by strings. The essential idea here is that when two quarks are bound together, the gluon field lines form a *flux tube* or string which binds the particles together, see figure 1.1.

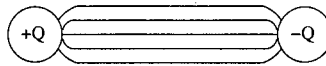


Figure 1.1: The field lines between two quarks in the low energy regime of QCD.

The flux tube dynamics would be given by the non-perturbative calculation of the Wilson loop:

$$W(C) = \text{Tr} \mathcal{P} \exp i \oint_C ds A(x(s)) \cdot \dot{x}(s) \quad (1.1)$$

but the non-perturbative calculation is not, in general, possible. Before the strong interaction was understood as a gauge theory, there was an attempt to describe the quark bound states as excitations of a quantum string. This was the beginning of modern string theory. The Regge relation between spin and mass of the hadrons, for example, was well modelled by the early string theories. The advent of QCD however, replaced string theory as the explanation of the strong force. Then in the early 1970's 't Hooft [2] showed that there existed a limit of non-abelian gauge theory where the rank of the gauge group N is sent to infinity, which could be interpreted as a kind of string theory. This did not solve the problem of non-perturbative calculations, but did provide a connection between gauge theories and

strings.

String theory was revived in the 1980's as a candidate for a quantum description of gravity and a grand unified theory. Much work has been done in this field and recently (1998) a new connection between strings and gauge theory has been discovered. Maldacena [3] conjectured a duality between supersymmetric Yang-Mills theory (SYM) in four dimensions and superstring theory on the ten dimensional space $AdS_5 \times S^5$. This duality has been expanded upon and verified by many researchers. The most exciting aspect of this conjecture is that the limit of strong coupling in the gauge theory is equivalent to a weakly coupled limit of the string theory which happens to be ten dimensional supergravity. This means that the non-perturbative regime of SYM may be calculated using perturbative supergravity.

The Maldacena conjecture, also known as the AdS/CFT correspondence, represents a major breakthrough in the work to understand strongly coupled gauge theory. As mentioned above, the potential between two strongly interacting particles should be described by the Wilson loop. The connection between the Wilson loop in SYM and the AdS/CFT correspondence has been investigated recently by many researchers. The results for a straight line Wilson loop have been previously obtained. However in order to probe the inner workings of the AdS/CFT correspondence, it is an interesting problem to understand Wilson lines which are close to straight. We call these the *wavy* lines, and we calculate results for their interactions with one another as functions of their *waviness*.

1.2 Large N Field Theory

The AdS/CFT correspondence involves supersymmetric Yang-Mills theory, as was previously mentioned. In fact the limit of interest for the correspondence is one in which the rank of the gauge group N of SYM is taken to infinity. This is called the *large N limit*. This section will briefly review the concept of the large N limit of non-abelian field theory. We will follow (closely at times) the treatment given in [4].

Consider $SU(N)$ gauge theory with fermionic matter in the fundamental representation. The standard form of the action is:

$$S = \int d^4x \left[-\frac{1}{4} \text{Tr} F_{\mu\nu}^2 + \bar{\psi}_i \gamma^\mu (i\partial_\mu \delta_{ij} - g A_\mu^a t_{ij}^a) \psi_j - m \bar{\psi}_i \delta_{ij} \psi_j \right] \quad (1.2)$$

where,

$$F_{\mu\nu} = \partial_\mu A_\nu^a t^a - \partial_\nu A_\mu^a t^a - ig [A_\mu^a t^a, A_\nu^b t^b] \quad (1.3)$$

and t^a where $a = 1, \dots, (N^2 - 1)$ are the generators (matrices) of the fundamental representation of $SU(N)$, thus $i, j = 1, \dots, N$. For a review of gauge theories, and the notation used above, see [1]. We begin by making the following redefinition of the fields:

$$A_\mu^a \rightarrow g A_\mu^a, \quad \psi_j \rightarrow g \psi_j \quad (1.4)$$

which allows us to write the action in the following form:

$$S = \frac{1}{g^2} \int d^4x \left[-\frac{1}{4} \text{Tr} F_{\mu\nu}^2 + \bar{\psi}_i \gamma^\mu (i\partial_\mu \delta_{ij} - A_\mu^a t_{ij}^a) \psi_j - m \bar{\psi}_i \delta_{ij} \psi_j \right] \quad (1.5)$$

where now $F_{\mu\nu}$ contains no factors of g . It was 't Hooft [2] who considered expressing the gauge degrees of freedom as:

$$A_\mu^a t_{ij}^a \rightarrow (A_\mu)_{ij} \quad (1.6)$$

in this language the propagators have the following index structure:

$$\langle (A_\mu(x))_{ij} (A_\nu(y))_{kl} \rangle \sim (\delta_{il} \delta_{kj} - \frac{1}{N} \delta_{ij} \delta_{kl}) \quad (1.7)$$

$$\langle \bar{\psi}_i(x) \psi_j(y) \rangle \sim \delta_{ij} \quad (1.8)$$

and therefore to leading order in the large N limit, the second term in the gauge field propagator may be ignored. In fact, this second term disappears entirely if one considers the gauge group $U(N)$ instead of $SU(N)$, and then everything that follows here is exactly (instead of approximately) true. In non-abelian gauge theory, the gauge field $A_\mu^a(x)$ always transforms in the adjoint representation of the gauge group. The interpretation of the picture which emerges here is that an adjoint field $\phi^a(x)$ may be represented as a direct product of fundamental and anti-fundamental fields, $\phi_i^a(x) \phi_j^a(x) = \phi_{ij}^a(x)$. In group theory language this is the statement:

$$\bar{N} \otimes N = \text{adjoint} \oplus \text{singlet} \quad (1.9)$$

but for large N the singlet contribution is suppressed, as per (1.7). Thus the adjoint gauge fields are in a sense quark/anti-quark composites, stressing the *flux tube* interpretation mentioned in section 1.1. 't Hooft developed a digram notation based on this fact, sometimes called the *fat graph* notation,

$$\langle (A_\mu(x))_{ij} (A_\nu(y))_{kl} \rangle \sim \begin{array}{c} j \xrightarrow{\hspace{1cm}} \bar{k} \\ \bar{i} \xleftarrow{\hspace{1cm}} l \end{array}$$

in this notation the three and four point vertices, and the quark-gluon vertex are given by the diagrams shown in figure 1.2. Gauge fields (adjoint) are represented by double lines, and quarks (fundamental) are represented by single lines.

We now introduce the quantity λ which is known as the 't Hooft coupling, $\lambda = g^2 N$. Glancing back at (1.5), it can be seen that the three and four point vertices come with a factor of $1/g^2 = N/\lambda$ whereas the gauge field propagator is proportional to $g^2 = \lambda/N$. Also, in a given diagram, when an arrowed line closes on itself (forms a loop) it supplies a factor of $\delta_{ii} = N$. Consider now the set of all vacuum diagrams involving gauge fields alone, an example of

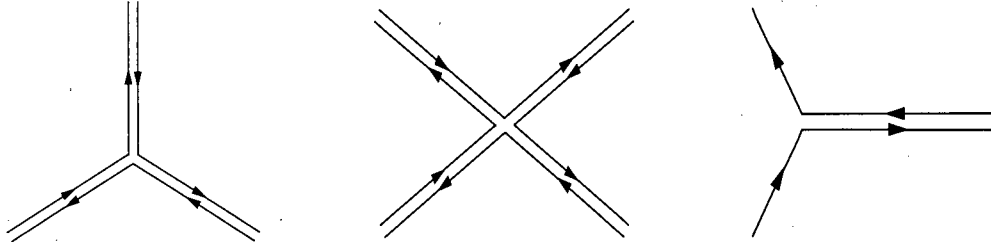


Figure 1.2: Vertices in the 't Hooft model.

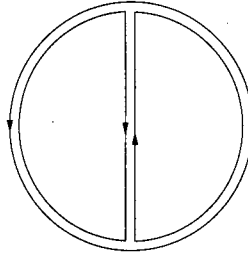


Figure 1.3: A vacuum diagram composed entirely of gauge fields.

which is shown in figure 1.3. A diagram with V vertices, E propagators, and F loops will therefore be proportional to:

$$N^{V-E+F} \lambda^{E-V} \quad (1.10)$$

We have chosen the letters V and E because if we collapse the double lines to single ones, then the diagram has V vertices, and E edges, as in figure 1.4. The letter F is chosen because each loop forms a *face* in the double-line diagram.

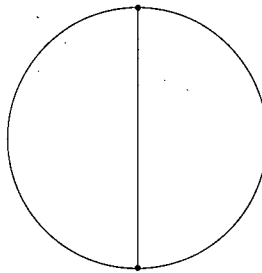


Figure 1.4: The vertices and edges of the diagram in figure 1.3.

The combination $\chi = V - E + F$ is recognized as the Euler character, which implies a connection between the diagrams and surfaces of a given genus g , since $\chi = 2 - 2g$. Thus we have that a given diagram is proportional to:

$$N^{2-2g} \lambda^{E-V} \quad (1.11)$$

and so diagrams corresponding to surfaces of higher genus are suppressed by successive powers of $1/N^2$. An example of a higher genus diagram is shown in figure 1.5.

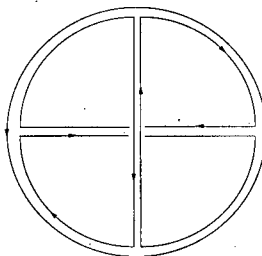


Figure 1.5: A diagram associated with a surface of genus $g = 1$.

Referring to figure 1.3, we see that $V = 2$, $E = 3$, and $F = 3$, and therefore the power of N associated with this diagram is $N^{2-3+3} = N^2$ or equivalently the genus g is 0. The genus 0 graphs are given a special name, they are called *planar* graphs. This is because they can be drawn on a plane. In contrast figure 1.5 shows a graph with $V = 4$, $E = 6$, and $F = 2$, and thus is proportional to N^0 or is genus 1 and obviously can not be drawn on a plane due to the crossing central propagators: a handle would need to be added to the plane in order to draw this graph.

The sum of all the vacuum diagrams would take the following form:

$$\sum_g N^{2-2g} \mathcal{F}_g(\lambda) \quad (1.12)$$

$$\mathcal{F}_g(\lambda) = \sum_n C_n^g \lambda^n \quad (1.13)$$

where the function $\mathcal{F}_g(\lambda)$ is the sum of all diagrams of genus g , which is naturally a power series in the 't Hooft coupling. For example $\mathcal{F}_0(\lambda)$ would be the sum of all planar diagrams. The point of interest here is that in the large N limit, we have a perturbative genus expansion. In string theory, the very same type of expansion arises in the calculation of amplitudes. It is this connection to string theory which makes the 't Hooft model so appealing: it connects QCD-type quantum field theory to string theory.

We have used the all-gauge vacuum diagrams to elucidate the salient features of large N field theory, however there are natural generalizations to the addition of the fermions and even Fadeev-Popov ghost particles. The genus expansion is retained in all of these cases. The typical calculation in a large N field theory involves taking λ to be finite and small, this is achieved by scaling the Yang-Mills coupling $g \rightarrow 0$ as N is scaled to infinity. Then, since

the planar diagrams dominate, regular perturbative (in λ) calculations can be performed ignoring all but the planar graphs. This is precisely the limit used in the calculations in this thesis.

1.3 String Theory

String theory is a vast and highly technical field of physics research, and no attempt will be made here to cover it completely. The purpose of this section is to introduce (without proof or derivation) the aspects of string theory which are important for understanding the AdS/CFT correspondence. There are two standard texts on string theory [5] and [6], while a good warm-up review paper is [7].

1.3.1 Preliminary Concepts

A concept which is usually introduced in a first course in General Relativity is the action associated with a particle in a given background space-time geometry. This action is the proper length of the particle's worldline, which when extremized gives the trajectory or equation of motion of the particle. The action is:

$$S = -m \int d\tau \sqrt{-\frac{dx^\mu(\tau)}{d\tau} G_{\mu\nu} \frac{dx^\nu(\tau)}{d\tau}} \quad (1.14)$$

where the embedding functions $x^\mu(\tau)$ describe the worldline of the particle, m is the particle's mass, and $G_{\mu\nu}$ is the metric of the background space-time. The functions which extremize this action are the geodesics of the space given by $G_{\mu\nu}$. The starting point of string theory is to generalize these concepts from a zero dimensional object (a particle) to a one dimensional object - a string.

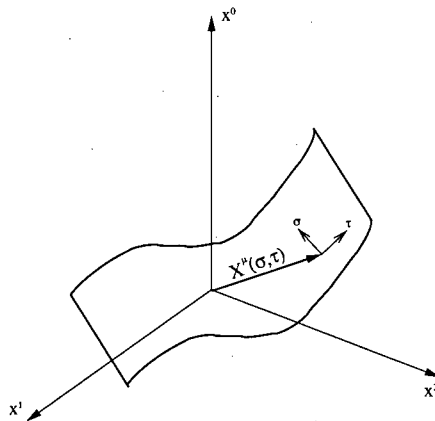


Figure 1.6: The string, sweeping out its worldsheet as it propagates.

The natural form for the string action is the proper area of the *worldsheet* swept out by the string as it propagates through space-time. The generalization of the embedding functions are $X^\mu(\tau, \sigma)$, where σ can be thought of as the coordinate that runs along the string at any given time, while τ can be thought of as the “worldsheet time”, see figure 1.6. The action, known as the Nambu action, is therefore:

$$S = -T \int d\tau d\sigma \left[-\det_{ab} \partial_a X^\mu G_{\mu\nu} \partial_b X^\nu \right]^{\frac{1}{2}} \quad (1.15)$$

where $a, b = 0, 1$ are worldsheet indices such that 0 refers to τ and 1 to σ . The quantity T is the string tension, a measure of the energy per unit length of the string. The program of string theory is to quantize the above action, and examine the associated Hilbert space of physical states. We discuss this next.

1.3.2 Quantization and Spectrum

In order to quantize string theory, an action other than, but equivalent to the Nambu action is used. It is called the Polyakov action, in a flat space-time background it is:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu \quad (1.16)$$

where h^{ab} is the worldsheet metric, which requires gauge fixing prior to quantization. The determinant of h^{ab} is denoted by h . Without going into any detail, the gauge choice is $h^{ab} = \eta^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and then the equation of motion derived from the Polyakov action is simply the wave equation:

$$(-\partial_\tau^2 + \partial_\sigma^2) X^\mu = 0 \quad (1.17)$$

The boundary conditions which are consistent with the vanishing of surface terms in the action allow for solutions describing *closed* strings, which are loops of string, and *open* strings which are bits of string that have ends. The ends of open strings can have either Neumann (free), or Dirichlet (fixed) boundary conditions.

Take for instance an open string with free ends, the solution of (1.17) with these boundary conditions is:

$$X^\mu(\tau, \sigma) = x^\mu + l^2 p^\mu \tau + il \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma \quad (1.18)$$

where we have introduced the characteristic string length $l = 1/\sqrt{\pi T}$. The program of quantization involves treating the α_n^μ 's as ladder operators obeying quantum commutation relations. A ground state with momentum p is defined via:

$$\alpha_n^\mu |0; p\rangle = 0 \quad \forall \quad n > 0 \quad (1.19)$$

and then excited states are formed by acting on the ground state with the raising operators α_{-n}^μ , for positive n . Without going into any detail, the states in string theory have different

masses, i.e. different values of $p^\mu p_\mu$. The interpretation is therefore that each string theory state corresponds to a particle of a given mass, the higher the excitation of the string, the larger the mass of the particle. Figure 1.7 gives a heuristic picture of this for the first three states of the open string.

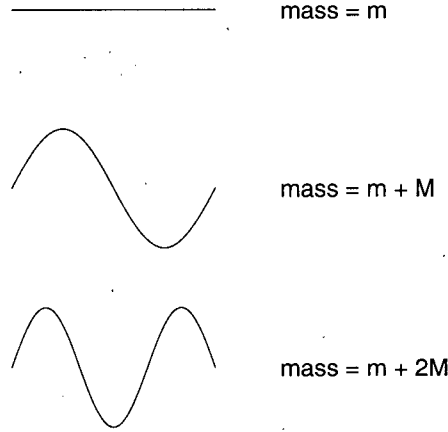


Figure 1.7: The tower of mass states for the open string.

The m in figure 1.7, refers to the ground state mass, i.e. the mass of the state $|0; p\rangle$. Subsequent states are equally spaced (in mass) above m by a mass spacing M . This mass spacing is proportional to the string tension T .

The treatment given here of the quantization of the string is vastly simplified, and many technical points have been left out. The purpose however, is to introduce the basic concepts at play.

1.3.3 The Superstring

In this section we introduce the supersymmetric generalization of string theory, which is called superstring theory. Supersymmetry is a symmetry which transforms fermionic and bosonic degrees of freedom into one another, the most important consequence of which is that every bosonic particle has a supersymmetric sister particle which is a fermion, and vice-versa. For a review of supersymmetry, see [8], the standard text on the subject is [9].

The superstring action, which is a generalization of the Polyakov action is:

$$S = -\frac{T}{2} \int d^2\sigma \{ \partial_a X^\mu \partial^a X_\mu - i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \} \quad (1.20)$$

where we have assumed a flat space-time background, and the worldsheet metric has been fixed as previously mentioned. The $\psi^\mu(\tau, \sigma)$ represent a D -plet (assuming that $\mu = 0, \dots, (D-1)$) of two component Majorana fermions transforming in the vector representation of the

Lorentz group $SO(D-1, 1)$, and ρ^a are the two dimensional Dirac matrices obeying $\{\rho^a, \rho^b\} = -2\eta^{ab}$. When this action is quantized, the Hilbert space of mass states includes bosons as well as an equal number of fermions.

One point which was left out of our discussion of quantization (section 1.3.2) was the critical dimension. In order to avoid states in the Hilbert space of string theory which have negative norm, i.e.,

$$\langle A|A \rangle < 0 \tag{1.21}$$

it is necessary to choose a specific value for D , the number of space-time dimensions. For the superstring, it is found that only the choice $D = 10$ ensures that these unphysical negative probability states disappear. We say that the critical dimension for superstring theory is ten.

The aspect of string theory which is most important to the understanding of the AdS/CFT correspondence is the spectrum of massless particles and the effective theory describing their dynamics and interactions. This is because the mass levels in string theory are separated by an energy scale given by the characteristic string length l which should physically be the Planck length, and so the mass spacing M is about 10^{19} GeV. Thus for earthly energy scales only the lowest mass states of string theory should come into play, these are the massless states. Amongst the massless states in superstring theory, is a closed string state which has spin equal to two. This is a graviton, superstring theory includes (amongst other things) a quantized description of general relativity. In fact if we consider only closed strings, the low energy effective theory of the massless states is 10 dimensional supergravity. Supergravity is a supersymmetric and quantized generalization of general relativity which was first realized by Freedman, van Nieuwenhuizen, and Ferrara in 1976, see [10].

The specific details of supergravity (or SUGRA) will not be important to us in this thesis. Instead we will move on to describe string interactions, and special dynamical objects which arise in string theory called D-branes.

1.3.4 String Interactions

We now discuss how strings interact with one another. The picture to keep in mind is shown in figure 1.8, on the left is a one-loop Feynman diagram. The Feynman diagram shows two point particles entering from the bottom, interacting and then emerging from the top. The figure on the right shows the same process, however the point particles have been replaced by closed strings.

The action of string theory contains an important symmetry which we have not yet alluded to. This is the conformal symmetry of the worldsheet. Consider the equation of motion (1.17) for a string with a worldsheet metric of Euclidean signature. This is obtained from the standard Minkowski signature metric via the rotation $\tau \rightarrow i\tau$. The effect is to flip the sign on the τ derivative, and so we are left with the 2-D Laplace equation, which is invariant under conformal transformations. Conformal transformations include straightforward

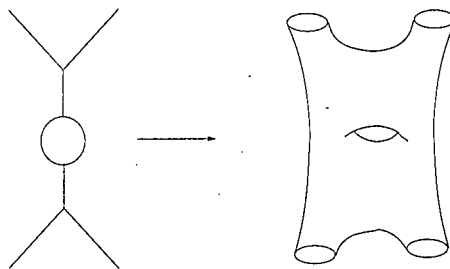


Figure 1.8: String interactions, as compared to those of point particles.

rescaling of the coordinates. This means that we can shrink the legs of the diagram in figure 1.8 down to zero size, as shown in figure 1.9.

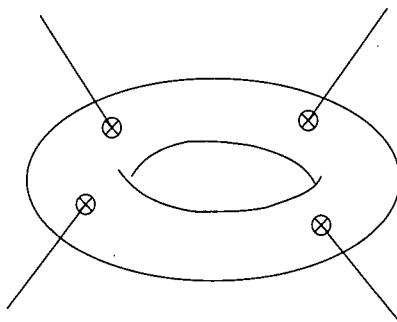


Figure 1.9: A conformally transformed picture of the interaction in figure 1.8.

The \otimes symbols on the torus in figure 1.9 reflect the fact that the external legs cannot be made to disappear entirely. An operator, specifying the manner in which the external particle attaches to the torus, must be inserted on to the worldsheet (the torus). This is called a *vertex operator*. A general string theory amplitude can thus be calculated in the following way. Take for example the four-point function, of which the process in the figures is the one-loop contribution to. For the first contribution, the torus is replaced by the sphere. We identify two of the four external states as the incoming or initial and outgoing or final states $|\phi_i\rangle$ and $|\phi_f\rangle$. We attach the initial and final states to the worldsheet via their vertex operators, and propagate the string between the vertex operators using a propagator P , see figure 1.10.

In calculating an amplitude we integrate over all possible positionings of the vertex operators on the surface, modded-out by the conformal transformations relating one positioning to another:

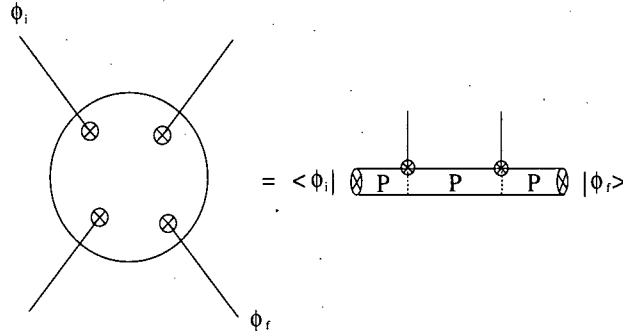


Figure 1.10: The leading contribution to the four-point function.

$$\mathcal{A} \sim \int \langle \phi_i | V P V P V P V | \phi_f \rangle \quad (1.22)$$

where we have represented the vertex operator by the symbol V . Just as in field theory where the higher loop contributions are suppressed by powers of the coupling constant, in string theory we find the following factor of the string coupling constant g_s associated with the g -loop contribution to the M -point function:

$$(g_s^2)^g g_s^{M-2} \quad (1.23)$$

And so we see that in string theory there is a genus expansion, that is we sum up the contributions of placing our vertex operators on surfaces of progressively higher genus g , and each contribution is suppressed by g_s^2 as compared to the previous one.

Our treatment of string interactions has been entirely schematic. This is because the details will not come into play in the thesis. What is important is that the reader have a diagrammatic picture of how strings interact with one another.

1.3.5 D-Branes

Recall from our discussion in section 1.3.2 that open strings may have two sorts of boundary conditions at their ends, free (Neumann) or fixed (Dirichlet). Now consider mixing these boundary conditions by applying Neumann to one subset of the (ten) space-time dimensions, and Dirichlet to the remaining dimensions. That is, set:

$$\begin{aligned} \partial_\sigma X^\mu|_{\sigma=0,\pi} &= 0, & \mu &= 0, 1, \dots, p \\ X^\mu|_{\sigma=0,\pi} &= C^\mu, & \mu &= p+1, \dots, 9 \end{aligned} \quad (1.24)$$

where we have placed the ends of the string at $\sigma = 0, \pi$, and the C^μ are constants. What we have here is a $p+1$ dimensional hypersurface along which the strings ends are attached and

free to propagate within, but are fixed with respect to motion perpendicular to the hypersurface, see figure 1.11. The hypersurface is called a *D-brane*, short for “Dirichlet Membrane,” since Dirichlet boundary conditions are enforced on the hypersurface.

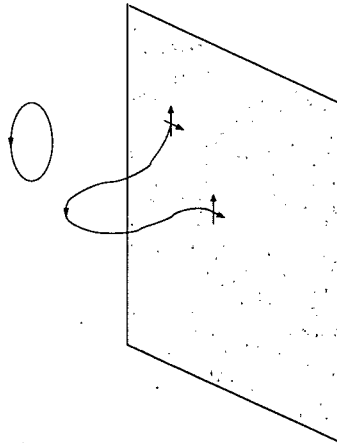


Figure 1.11: A D-brane with an attached open string, closed strings are not constrained by D-branes.

Closed strings are not constrained by D-branes, since they have no ends. However a D-brane may produce a closed string, via an open string on the D-brane bringing its ends together and detaching from the membrane, thus D-branes interact with closed strings. Let us now describe the spectrum of massless states associated with open bosonic strings attached to the D-brane as pictured in figure 1.11. There are two types of these states:

- $\alpha_{-1}^{\mu}|0;p\rangle$ where $\mu = 0, 1, \dots, p$. These states correspond to a gauge field A_{μ} living in the D-brane world volume.
- $\alpha_{-1}^m|0;p\rangle$ where $m = p + 1, \dots, 9$. These states correspond to $(9 - p)$ scalar fields ϕ_m whose vacuum expectation values describe the transverse position of the D-brane in the space-time. The quantum fluctuations of the scalar fields describe fluctuations in the shape of the D-brane.

For the superstring, the bosonic degrees of freedom listed above have fermionic superpartners. The second item above shows us that D-branes are dynamical objects in string theory. In fact D-branes can interact with one another as well as with string states in the “bulk” of space-time. One can even write down various low-energy effective actions describing the dynamics of D-branes in various configurations. The configuration which is of prime interest in the AdS/CFT correspondence is a set of N parallel and coincident D3-branes (i.e. $p = 3$), in a flat space-time background in the low energy limit of superstring theory. The dynamics of this system is dimensionally reduced $\mathcal{N} = 1$, ten dimensional supersymmetric Yang-Mills theory or SYM, which is exactly $\mathcal{N} = 4$, four dimensional SYM¹. The gauge group of the

¹The symbol \mathcal{N} refers to the number of supersymmetries which the theory is invariant under. See [8].

SYM is $SU(N)$, and the relation between the string coupling constant g_s and the Yang-Mills coupling constant g_{YM} is:

$$g_{YM}^2 = 4\pi g_s \quad (1.25)$$

The action of $\mathcal{N} = 4$, four dimensional SYM with gauge group $SU(N)$ is given in the appendix (A.1). This theory is known as a *conformal field theory* or CFT which means that it is invariant under conformal transformations of the space-time coordinates. Since conformal transformations include the scaling of distances, CFT's are scale invariant and so there is no concept of size or distance. The fact that $\mathcal{N} = 4$, four dimensional SYM is a CFT will play a central role in the AdS/CFT correspondence.

1.4 The AdS/CFT Correspondence

The Maldacena conjecture [3], or AdS/CFT correspondence, asserts that there is an exact duality between superstring theory on the background space-time $AdS_5 \times S^5$ (to be described shortly), and $\mathcal{N} = 4$, four (flat) dimensional SYM, (the CFT). In order to appreciate this remarkable connection between gauge theory and string theory, it is necessary to understand the properties of five dimensional Anti-de Sitter space AdS_5 .

1.4.1 Anti-de Sitter Space

Anti-de Sitter space is a solution of Einstein's equations of general relativity with constant, negative curvature. This means that there is a non-zero cosmological constant term in the action. It is also a space of so-called maximal symmetry, which means it has as many symmetries as a flat space of equal dimension. The simplest way to understand the symmetries of $(p+2)$ -dimensional anti-de Sitter space AdS_{p+2} , is to present the space as an embedding in a $(p+3)$ -dimensional space. We can represent AdS_{p+2} as the locus of points:

$$y^2 = y^I y_I = y_0^2 + y_{p+2}^2 - \sum_{i=1}^{p+1} y_i^2 = R^2 \quad (1.26)$$

in the $(p+3)$ -dimensional space, described by the metric:

$$ds^2 = -dy_0^2 - dy_{p+2}^2 + \sum_{i=1}^{p+1} dy_i^2 \quad (1.27)$$

The isometries of the above metric are described by the group $SO(2, p+1)$. This means that the length y^2 is invariant under this group of transformations. Since the embedding is just $y^2 = R^2$, it inherits the symmetry group of the target space, and so AdS_{p+2} has $SO(2, p+1)$ as its isometry group.

In order to obtain the metric of AdS_{p+2} , we find a solution to (1.26) and plug it into (1.27). One such solution is:

$$\begin{aligned}
y_0 &= \frac{1}{2u} [1 + u^2(R^2 + \vec{x}^2 - t^2)] \\
y^i &= Rux^i \quad i = 1, \dots, p \\
y^{p+1} &= \frac{1}{2u} [1 - u^2(R^2 - \vec{x}^2 + t^2)] \\
y_{p+2} &= Rut
\end{aligned} \tag{1.28}$$

where $\vec{x} = x^i$ is a p -vector. Plugging this into (1.27), we obtain the metric of AdS_{p+2} :

$$ds^2 = R^2 \left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2) \right) \tag{1.29}$$

or, setting $z = 1/u$,

$$ds^2 = R^2 \left(\frac{dz^2 + dx_i^2 - dt^2}{z^2} \right) \tag{1.30}$$

Anti-de Sitter space has a boundary which will be of great interest to us. Consider taking the limit $z \rightarrow 0$ in (1.30), essentially a blown-up Minkowski space is what is obtained. Although this is not quite true, the boundary of AdS_{p+2} is conformally equivalent to $(p+1)$ -dimensional Minkowski space M_{p+1} . In fact, the action of the isometry group of AdS_{p+2} (that is $SO(2, p+1)$) on the boundary, is simply the action of the conformal group on M_{p+1} . We say that the boundary of AdS is conformally flat.

It is often convenient to work with a metric of Euclidean signature, which may be obtained by rotating the time coordinate $t \rightarrow -it$. In this thesis we work with Euclidean metrics in all calculations. The Euclidean AdS_{p+2} metric is simply:

$$ds^2 = R^2 \left(\frac{dz^2 + dx_1^2 + \dots + dx_{p+1}^2}{z^2} \right) \tag{1.31}$$

and its isometry group is $SO(1, p+2)$. Its boundary is conformally equivalent to $(p+1)$ -dimensional Euclidean space E_{p+1} , and the action of $SO(1, p+2)$ on the boundary is the action of the conformal group on E_{p+1} .

What we have established here is that the boundary of AdS space is a "flat space" of one less dimension which has an enriched symmetry not found in standard Minkowski or Euclidean spaces, this is conformal symmetry. Although they will not be important in this thesis, the conformal transformations on Euclidean space are:

$$\begin{aligned}
&\text{Dilations: } x^\mu \rightarrow \lambda x^\mu \\
&\text{Special Conformal Transformations: } x^\mu \rightarrow \frac{x^\mu + \alpha^\mu x^2}{1 + 2\alpha \cdot x + \alpha^2 x^2}
\end{aligned} \tag{1.32}$$

where μ is a Euclidean index.

1.4.2 SUGRA Solitons and String Theory D-Branes

A soliton is a stable, classical solution to the equations of motion of a quantum field theory. It can be thought of as a background field around which an action can be expanded, and so the quantum excitations of the fields in question would be fluctuations about the soliton configuration. In gravity, the fields are the geometry of space-time, and so solitons in gravity theories are background space-times. We mentioned earlier that the low energy effective theory of closed strings was supergravity, a supersymmetric generalization of general relativity. We will now introduce a soliton in SUGRA which is central to the AdS/CFT correspondence. Since it will not illuminate the work of this thesis to give a proper derivation of the SUGRA soliton solution at hand, we will simply state the result. For an in-depth treatment, see [11], which also includes many general calculations concerning AdS/CFT left out of this thesis. The solution is:

$$\begin{aligned} ds^2 &= f^{-1/2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{1/2} (dr^2 + r^2 d\Omega_5^2) \\ f &= 1 + R^4/r^4 \end{aligned} \quad (1.33)$$

where $d\Omega_5^2$ is the metric on the five-sphere S^5 , and R is a constant. This space has two interesting limits, the first is to send $r \rightarrow \infty$, and thus $f \rightarrow 1$. We have just ten dimensional flat space, although five of the dimensions are expressed in polar coordinates. The other limit of interest is to take $r \rightarrow 0$, and thus $f \rightarrow R^4/r^4$, which yields:

$$ds^2 = \frac{r^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2 \quad (1.34)$$

Making the coordinate redefinition $x^\mu \rightarrow R^{-2}x^\mu$, and comparing to (1.29) with $u = r$, we see that we have the metric of $AdS_5 \times S^5$ where the radii of the five sphere and the AdS space are equal.

The soliton (1.33) is known as a 3-brane. This is because it can be viewed as a flat, three (spatial) dimensional space (or brane) sitting in a higher dimensional space which is isotropic in the dimensions transverse to the brane. Here the brane coordinates are (t, x_i) , while the transverse coordinates are r and the five angles on S_5 . The coordinate r is the “transverse distance” away from the brane, i.e. the brane is located at $r = 0$. Notice that the metric blows-up at $r = 0$, although this is an artifact of the coordinate system and no physical quantity is infinite at this location, it happens to signal the existence of a “horizon”. A horizon is the “surface” of a black-hole, the point of no return for massless particles propagating in the space-time. Thus here, $r = 0$ is called the “horizon” of the space. Also note that for space-time to be curved, there must be a source of stress-energy. Although we have not shown it here, the 3-brane is charged with respect to certain gauge fields which are part of the particle content of SUGRA. It is this charge which generates the required stress-energy, and is thus the source for the curvature. In other words the charged brane is the *source* for the soliton solution.

When one considers SUGRA in the background (1.33), i.e. we quantize the theory by expanding the fields about this solution, the following behaviour is found. For low energies, the

fields out at $r = \infty$ (i.e. those that propagate in essentially flat space), do not interact with the fields around $r = 0$ (i.e. those that propagate in $AdS_5 \times S^5$). Thus in the low energy limit, we have two decoupled SUGRA's, one in the near horizon which has $AdS_5 \times S^5$ as it's background, and one far away from the 3-brane which sees just flat, ten dimensional space.

During our discussion of D-branes in string theory (section 1.3.5), we mentioned that D-branes may interact with closed strings in the bulk. Let us consider the system of N coincident D3-branes we mentioned previously. We already asserted that the theory describing the (massless, read low energy) open strings living on the branes was $\mathcal{N} = 4$ SYM with gauge group $SU(N)$. However the full system (at all energies) should be described by an action encoding the dynamics of the closed strings in the bulk, and an action describing brane-bulk interactions in addition to the brane action. That is, the full action describing the dynamics of the system should have the form:

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}} \quad (1.35)$$

Recall from our discussion concerning the quantization of strings (section 1.3.2) that at low energies S_{bulk} is just SUGRA. Thus, it is not too surprising to find that string theory D-branes are charged with respect to the SUGRA gauge fields mentioned in the soliton discussion above. The string theory statement is that D-branes interact with closed strings in the bulk (via S_{int}) in such a way that D-branes appear charged with respect to certain closed string states which are these gauge fields of SUGRA. In fact, our configuration of D-branes is a source for the SUGRA soliton (1.33), as long as one is looking at the system far away from the D-branes and the energy is low. Here far away means $r \gg l$, where r is the distance from the D-branes and l is the characteristic string length scale introduced previously. Thus a dual description of the physics of the SUGRA soliton (1.33) exists using string theory with D-branes, in particular with the configuration of D-branes which gave us SYM as the low energy brane theory. Using this dual description one can draw a relation between the constant R in (1.33), and the constants describing the string theory D-branes:

$$R^4 = 4\pi g_s \alpha'^2 N \quad (1.36)$$

where $\alpha' = 1/(2\pi T) = l^2/2$, where T is the string tension, and l is the characteristic string length scale. There is an interesting way to take the low energy limit of this system of coincident D-branes. We explained earlier that in order to excite higher mass states in string theory an energy proportional to T was necessary. By sending l (or α') to zero, we're increasing that mass gap, which has the same effect as just lowering the energy of our string states. What happens to our system is that $S_{\text{int}} \rightarrow 0$ as $\alpha' \rightarrow 0$. What we're left with is two decoupled theories, SYM at $r = 0$, and SUGRA in flat, ten dimensional space in the bulk at $r \neq 0$. The SUGRA space is flat because the D-brane charge is decoupled from the SUGRA fields.

Notice that the SUGRA soliton solution also had two decoupled theories for low energy, and one of them was the same as one from the D-brane description, i.e. SUGRA in flat, ten dimensional space at large r . The Maldacena conjecture is that the other two theories at $r = 0$ are similarly equal, i.e. SUGRA on $AdS_5 \times S^5$ and $\mathcal{N} = 4$, $SU(N)$ SYM in four

dimensions. In fact, the conjecture goes beyond this, to say that *closed string theory* and not just SUGRA (its low energy limit) on $AdS_5 \times S^5$ is equivalent to the SYM.

We're interested in the region $r \rightarrow 0$, however on the string theory side we're also taking $\alpha' \rightarrow 0$. Maldacena introduced the near-horizon limit, where $U = r/\alpha'$ is kept fixed as r and therefore α' is sent to zero. Replacing r by U in (1.34) and in light of (1.36), we have that the near-horizon geometry of the SUGRA soliton is:

$$ds^2 = \alpha' \left[\frac{U^2}{\sqrt{4\pi g_s N}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{4\pi g_s N} \frac{dU^2}{U^2} + \sqrt{4\pi g_s N} d\Omega_5^2 \right] \quad (1.37)$$

which is still (of course) $AdS_5 \times S^5$, however we have “zoomed-in” on the region $r = 0$ and blown it up so that all values of U correspond to $r = 0$. Introducing new coordinates:

$$\begin{aligned} x^\mu &\rightarrow \frac{x^\mu}{\sqrt{4\pi g_s N}} \\ y^i &= \frac{\theta^i}{U} \quad \text{where } i = 1, \dots, 6 \quad \text{and} \quad \theta^i \theta^i = \theta^2 = 1 \end{aligned} \quad (1.38)$$

we have,

$$ds^2 = \alpha' \sqrt{4\pi g_s N} \frac{1}{y^2} (dx^\mu dx_\mu + dy^i dy^i) \quad (1.39)$$

where the θ^i are coordinates on the five-sphere. Comparing with (1.30), we see that the point $y^i = 0$ corresponds to the boundary of AdS_5 and a point on the five-sphere. This is the four dimensional flat space with conformal symmetry mentioned earlier. In the AdS/CFT correspondence, we are to imagine that a “hologram” of closed string theory on the space (1.37) is projected onto its boundary at $y^i = 0$, as a four dimensional field theory which adopts the symmetry of that space - i.e. conformal symmetry. Thus we are to imagine that our CFT: $\mathcal{N} = 4$, $SU(N)$ SYM, lives on the boundary of the near-horizon space $AdS_5 \times S^5$, and encodes all the information of the string theory.

1.4.3 The Correspondence

The most common application of the AdS/CFT correspondence involves taking the large N limit of the CFT. Recall from section 1.2, concerning large N , that the effective coupling constant is the 't Hooft coupling $\lambda = g_{YM}^2 N$, which is held fixed while $N \rightarrow \infty$. Now recall (1.25) and (1.36), we have:

$$\lambda = g_s N = \frac{R^4}{l^4} = \frac{R^4}{\alpha'^2} \quad (1.40)$$

If we consider a fixed value of R , we see that the 't Hooft coupling is related to α' via $\alpha'^2 \sim 1/\lambda$. Physical effects which are due to the stringy nature of string theory are controlled

by α' since the characteristic string length $l = \sqrt{2\alpha'}$. This means that stringy corrections on the AdS side of the correspondence are related to 't Hooft coupling corrections to the CFT side results. Now note that keeping R fixed another relation can be gleaned, namely $g_s \alpha'^2 \sim 1/N$. Recall from (1.23) that g_s , the string coupling constant, controls quantum corrections to the string theory. These corrections come from loop diagrams which are represented by string worldsheets with higher genera. Now recall (1.12), in large N field theory the non-planar (i.e. higher genus) diagrams give corrections controlled by $1/N^2$. Thus 't Hooft's connection between large N field theory and string theory is finally realized. The AdS/CFT correspondence tells us that higher genus string theory corrections are related to higher genus (non-planar) CFT corrections.

There are two limits of interest in the AdS/CFT correspondence. The first is $\lambda \ll 1$. Here we're in the perturbative regime of the CFT. However, since $N \rightarrow \infty$, $g_s \rightarrow 0$ which means only the simplest string interaction diagrams should be kept, these are the tree-level diagrams (i.e. no loops). Since in quantum field theories the tree level diagrams reproduce the classical results, this is also called the classical limit of the string theory. The radius of the AdS space (and five-sphere) here is much less than the characteristic string length l . This means that the space is highly curved and that the strings "see" the curvature very strongly. So here we have a duality between classical closed string theory and a perturbative CFT. It has been found that α' corrections to the string theory agree order by order with $1/\sqrt{\lambda}$ corrections to the CFT. It is not clear however, whether order g_s corrections to the string theory (quantum effects) would agree with order $1/N^2$ (non-planar) corrections to the CFT, even though the correspondence tells us that there must exist some relation.

The second limit of interest is $\lambda \gg 1$. Here we have the non-perturbative, strong coupling regime of the CFT. We still take $N \rightarrow \infty$, and $g_s \rightarrow 0$, however λ and so $g_s N$ is fixed to a large instead of a small value. The radius of $AdS_5 \times S^5$ here is very large, much larger than the string scale, and so this limit is like holding R fixed and taking $\alpha' \rightarrow 0$, which we know is the low energy limit where the string theory reduces to SUGRA. Since g_s is still taken to zero, the SUGRA is tree-level or classical, and on the background $AdS_5 \times S^5$ with large radius. This is a very exciting development indeed, for now non-perturbative CFT calculations which are impossible to do, can be done using classical supergravity. It is not clear whether or not the non-planar (i.e. $1/N^2$) corrections to the CFT would match up with SUGRA corrections of order g_s (i.e. loop diagrams or quantum effects). It is also not clear whether or not α' corrections to SUGRA (i.e. stringy effects) would match up with order $1/\sqrt{\lambda}$ corrections to the strong coupling CFT results, even though, again the correspondence does predict some relation.

It is these two limits, and mostly the second, that have been investigated by researchers. The pure form of the Maldacena conjecture asserts an exact equivalence between closed string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$, $SU(N)$ SYM in four dimensions for all values of g_s and N . This form of the conjecture has not been established yet. The prescription for the AdS/CFT correspondence in the strongly coupled CFT regime was suggested by Witten [12] and can be represented as follows:

$$\left\langle \exp \int d^4x \phi_0(x) \mathcal{O}(x) \right\rangle_{\text{CFT}} = \exp \left(-S_{\text{SUGRA}} [AdS_5 \times S^5]_{\phi|_B = \phi_0} \right) \quad (1.41)$$

On the left we have an expectation value in the CFT, on the right what is essentially the partition function of SUGRA on $AdS_5 \times S^5$, evaluated at the classical solutions for all its fields so that the action is minimized. The field ϕ represents a given field in the SUGRA, whose functional form on the boundary B is ϕ_0 . The operator $\mathcal{O}(x)$ is an operator of the CFT which is dual to the boundary field ϕ_0 . In practice, one would solve the classical equations of motion for the SUGRA field ϕ and impose the boundary condition $\phi|_B = \phi_0$ in order to compute the right hand side. The left hand side would then be computed via Feynman diagrams, assuming the operators $\mathcal{O}(x)$ were known. Different SUGRA fields couple to the CFT via different operators $\mathcal{O}(x)$.

We will not delve into the details here, but the various $\mathcal{O}(x)$ operators have been discovered, and many tests of (1.41) have been executed and found to be true, see [12–16]. In fact there is no evidence to date suggesting that the AdS/CFT correspondence is false. A rather different “corollary” of the Maldacena conjecture which has also been widely and successfully tested is a form of (1.41) where on the left hand side one has the Wilson loop. It is this form of AdS/CFT which is most interesting to the work of this thesis, and we shall describe it next.

1.5 Wilson Loops in AdS/CFT

In this section we’ll review the Wilson loop operator in gauge theory, and show the construction of such an operator in the AdS/CFT correspondence. We will see that the dual string theory description of this Wilson loop is a surface of minimal area in AdS_5 whose boundary is the loop. Finally we will review tests of this form of the AdS/CFT correspondence which have been successfully carried out in the literature.

1.5.1 The Wilson Loop

In non-abelian gauge theory the Wilson loop operator is defined as:

$$W(C) = \text{Tr} \mathcal{P} \exp i \oint_C ds A(x(s)) \cdot \dot{x}(s) \quad (1.42)$$

where $x_\mu(s)$ defines the trajectory of the loop C , $A_\mu(x(s))$ is the non-abelian gauge field, and the exponential is path ordered, which means that in the Taylor expansion fields with higher values of s are placed to the left. This is important because in non-abelian field theory the fields are represented using matrices which do not commute. The trace is over the fundamental² representation, which means that we are to interpret $A_\mu(x(s)) = A_\mu^a(x(s))T^a$ where T^a are the generators (matrices) of the fundamental representation of the gauge group of the theory. For example if the gauge group is $SU(N)$, then $a = 1, \dots, (N^2 - 1)$, and each T^a is an $N \times N$ matrix, forming a basis for the traceless, hermitian $N \times N$ matrices. The

²Other representations can be considered, but they won’t concern us here.

trace would then be over the $N \times N$ matrix indices.

An important application of the Wilson loop operator is to take the loop C to be a rectangle elongated in the time direction, with width in the spatial dimension R , see figure 1.12.

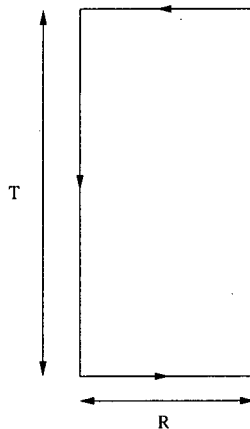


Figure 1.12: The Wilson loop which is related to the quark-antiquark potential.

In the limit $T \gg R$, it is a general result of quantum field theory that:

$$\langle W(C) \rangle = A(R) e^{-TV(R)} \quad (1.43)$$

where $V(R)$ is the potential energy between a quark and an antiquark (particles in the fundamental and anti-fundamental representation respectively) who are separated by the distance R . The factor $A(R)$ is dependent only upon R . In fact the two vertical edges of the rectangle can be viewed as the worldlines of these two particles: they are at constant positions separated by a distance R . Notice that this means that the quarks are idealized sources, whose trajectories are unaffected by their mutual interaction. This is analogous to assuming that the quarks are infinitely massive, and so immovable.

1.5.2 Construction of the Wilson Loop in AdS/CFT

It is a trivial matter to add massive particles in the (anti) fundamental representation (i.e. quarks) to a standard gauge theory, c.f. (1.2). However, if we are to construct a Wilson loop operator for the AdS/CFT correspondence which has the property (1.43), we must find a way to add such particles to the brane theory, i.e. the CFT. As it stands there are no such particles in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. Maldacena [17] suggested considering a system of $(N + 1)$ D-branes, where the extra D-brane is placed very far away from the other coincident N . The strings on the extra D-brane can be ignored, except for strings which stretch from the extra D-brane to the other N . In the low energy limit, these stretched strings behave as massive particles in the (anti) fundamental representation of the gauge group $SU(N)$. As is explained very nicely in [18], this situation is in

fact the Higgs mechanism breaking the gauge group of the $(N + 1)$ D-brane theory which is $SU(N + 1)$ down to $SU(N)$, and producing a massive “W-boson” in the (anti) fundamental representation.

Recall from section 1.3.5 on D-branes that in addition to the $(p + 1)$ -dimensional gauge field A_μ living on the D-brane, there are also $(9 - p)$ scalar fields Φ_i which describe the shape of the D-brane in the transverse coordinates. These fields can be seen in the action of $\mathcal{N} = 4$ SYM, (A.1), here p , of course is three. Our stretched string W-boson is of course charged and acts as a source for A_μ , but it is also coupled to the six scalars Φ_i . One can understand this in the following way: the stretched string pulls on and deforms the N D-branes and thus affects the Φ_i . The form of the Wilson loop, derived in the appendix of [18], is:

$$W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \oint_C d\tau (iA_\mu(x)\dot{x}_\mu + \Phi_i(x)|\dot{x}|\theta_i) \quad (1.44)$$

where θ_i are the coordinates on the five-sphere introduced in (1.38), and we are using four dimensional Euclidean (rather than Minkowski) space. The coupling constant here has been absorbed into the fields, to restore it, a factor of g should appear in front of the integral.

In order to suppress any corrections to (1.43) due to fluctuations of the W-boson itself, the particle’s mass must be taken to infinity, as per the discussion in section 1.5.1. Recall (1.37), this is the near horizon geometry of the SUGRA soliton. Because of the dual SUGRA and string theory description of the system, we can interpret this as $AdS_5 \times S^5$ space replacing the D-branes as the description of the system, when we are very close to the D-branes. It happens that the extra separated D-brane appears in this geometry at a fixed value of $U = U_0$ (and a point on the five-sphere), and that the mass of the W-boson (the stretched string) is proportional to U_0 . Thus U_0 should be taken to infinity, which means that the stretched strings end on the boundary of the AdS space, where the CFT lives. So the ends of the stretched strings describe the worldlines of the W-bosons, and these are the Wilson loops as per the discussion of figure 1.12 above.

1.5.3 Minimal Area Ansatz

We now describe the quantity on the AdS side corresponding to our Wilson loop. Looking at the right hand side of (1.41), we replace S_{SUGRA} with the action of string theory on $AdS_5 \times S^5$. This is justified since in the limit of interest the two theories are equivalent. The string theory action, using the metric (1.39) for the background space-time, is just a curved space generalization of (1.16):

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{-h} h^{ab} \frac{\partial_a X_\mu \partial_b X_\mu + \partial_a Y^i \partial_b Y^i}{Y^2} + \text{fermions} \quad (1.45)$$

where the fermionic part of the string theory, that is the superpartners of the bosonic fields, have been indicated by “+ fermions”. When this action is minimized in the large λ limit, it is found that the fermionic components drop away and the classical minimal action becomes:

$$S = \frac{\sqrt{\lambda}}{2\pi} \min \left[\int d^2\sigma \frac{1}{Y^2} \sqrt{\det(\partial_a X_\mu \partial_b X_\mu + \partial_a Y \partial_b Y)} \right] \quad (1.46)$$

which is the area of a surface of minimal area in AdS_5 , which sits at a point θ_i on the five-sphere. Maldacena [17] conjectured that with the following boundary conditions on the string embedding functions:

$$X^\mu|_B = x^\mu(\tau), \quad Y^i|_B = \theta^i Y, \quad Y|_B = 0. \quad (1.47)$$

that the left hand side of (1.41) should be replaced by the expectation value of the Wilson loop (1.44). The boundary conditions say that the surface of minimal area is open and has as its boundary the Wilson loop C . Thus, we have:

$$\langle W(C) \rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi} \mathcal{A}(C)} \quad (1.48)$$

where $\mathcal{A}(C)$ is the area of the minimal surface. In fact this is not quite correct. A look at (1.46) reveals that the area of the surface whose boundary is the curve C at $Y = 0$ is infinite. If we evaluate the area up to a small non-zero value of $Y = Y_0$, we find the divergence proportional to L/Y_0 , where L is the length of the curve C . Recall that the mass M of the W-boson was proportional to the radial position of the separated D-brane $U_0 = 1/Y_0$, thus the infinity in the area is a reflection of the infinite mass of the W-boson. It was understood in [18] that the proper relation is:

$$\langle W(C) \rangle = e^{-\frac{\sqrt{\lambda}}{2\pi} \mathcal{A}(C) + ML(C)} = e^{-\frac{\sqrt{\lambda}}{2\pi} \hat{\mathcal{A}}(C)} \quad (1.49)$$

which leaves both sides of the equation finite. $\hat{\mathcal{A}}(C)$ is now a “regularized” area, i.e. the infinite piece has been subtracted away. What we have now is a relation between a Wilson loop operator in the CFT, and a surface of minimal area in the space AdS_5 . Since the surface of minimal area came from the string action, we can interpret it as a macroscopic string worldsheet whose boundary is the Wilson loop. Figure 1.13 shows the rectangular Wilson loop on the boundary of AdS_5 with the associated string worldsheet. A slice at a given time reveals that according to the interpretation given to figure 1.12, we have a string connecting the two quarks. Alas we have a realization of the string description of the “flux tube” joining two strongly coupled quarks.

In fact the macroscopic string worldsheet is that of the W-boson(s). As we said in section 1.5.2, the ends of the stretched strings describe their worldlines which is the curve of the Wilson loop. If we turned off any interaction between two W-bosons, their associated stretched strings would be straight. With the interaction, the surface of minimal area is formed. This is shown in figure 1.14, at a given time.

1.5.4 Successful Tests Performed to Date

We will now review the various tests of (1.49) which have been performed in the literature. The most basic Wilson loop is the straight line, parameterized by $x_\mu = (s, 0, 0, 0)$. This happens to be a rather special object in terms of its supersymmetry properties, it is called

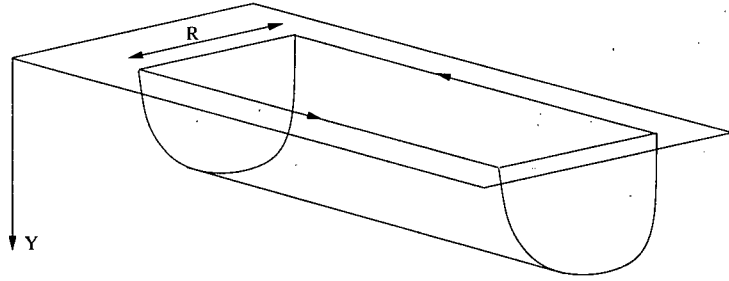


Figure 1.13: Rectangular Wilson loop at the boundary of AdS_5 , with a surface of minimal area whose boundary is the loop.

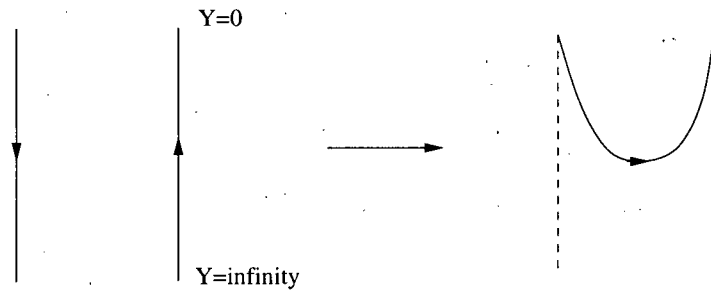


Figure 1.14: The stretched strings corresponding to W-bosons. On the left with their interaction turned off, on the right with their interaction restored.

a BPS object. This means that supersymmetry guarantees that quantum corrections due to bosons cancel those of fermions, and thus the exact answer can be calculated by considering only the leading term in the perturbation series. What is found is [23]:

$$\langle W(C) \rangle = 1, \quad C = \text{straight line} \quad (1.50)$$

independent of the coupling constant. It is easy to show that the regularized area is trivially zero for the straight line, and so the same result is found on the AdS side of the correspondence. In fact on the CFT side the explicit cancellation of the quantum corrections up to one loop order were shown in [21, 22].

The next shape of contour considered by researchers is the circle. On the AdS side, the surface of minimal area has been solved in [19] yielding the regularized area -2π . Thus we have:

$$\langle W(C) \rangle_{AdS} = e^{\sqrt{\lambda}}, \quad C = \text{circle} \quad (1.51)$$

independent of radius. In 2000 Semenoff et al. [21] summed all the ladder diagrams (diagrams with no internal vertices) for the circle in the CFT and found the result (good to all orders in λ):

$$\langle W(C) \rangle_{\text{ladders}} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}), \quad C = \text{circle} \quad (1.52)$$

where I_1 is the Bessel function. In the large λ limit this result agrees precisely with the AdS calculation. This implies cancellations between the various internal vertex diagrams. These cancellations were shown explicitly at the one loop level in [21]. In [23] the $1/N^2$ and $1/\sqrt{\lambda}$ corrections to the strong coupling result were considered and shown to agree with the corresponding g_s and α' corrections to the AdS side result.

For the rectangular loop (figure 1.12), the AdS side result was obtained in [17]:

$$\langle W(C) \rangle_{AdS} = \exp \left(\frac{4\pi^2}{\Gamma^4(1/4)} \sqrt{\lambda} \frac{T}{L} \right), \quad C = \text{rectangle} \quad (1.53)$$

where L is R from figure 1.12. Also in [21, 22], it was shown that the sum of all ladder diagrams for the rectangle yielded the following result:

$$\langle W(C) \rangle_{\text{ladders}} = \exp \left(\frac{1}{\pi} \sqrt{\lambda} \frac{T}{L} \right), \quad C = \text{rectangle} \quad (1.54)$$

which is not in exact agreement with the AdS result. This implies that the internal vertex diagrams do not cancel entirely, even though they do at the one loop level [21].

Researchers have also considered the correlator of two Wilson loops [19, 20]. However the shapes of loops have been confined to circles and rectangles. Other topics considered include finite temperature, monopoles, and g_s and α' corrections. Many references can be found in [4].

Chapter 2

Methods of Calculation

We now begin the main calculational portion of the thesis. We will calculate the expectation value of a single, and then two Wilson loops which are close to straight lines. The calculations will take place on the CFT side of the correspondence, in the regime where the 't Hooft coupling $\lambda = g_{YM}^2 N \ll 1$. All calculations are performed in Euclidean space, obtained from Minkowski space via the rotation $t \rightarrow it$. The wavy line is represented as follows:

$$x^\mu(s) = (s, \vec{\xi}(s)) \quad s \in [-\infty, +\infty] \quad (2.1)$$

where $\vec{\xi}(s)$ is called the “waviness function”, and is assumed to be small so that the curve is a small deviation from the straight line $x^\mu = (s, 0, 0, 0)$. What is meant by “small” here is that $|\partial_s \vec{\xi}(s)| \ll 1$, since we assume that $\vec{\xi}(s)$ is “well behaved” (i.e. has compact support) this translates into a bound on $|\vec{\xi}(s)|$ itself. Note that the Lorentz index μ can be up or down without effect, since the metric is Euclidean: $\text{diag}(+, +, +, +)$. The calculations will be performed to various orders in waviness, for example second order in waviness would be $\mathcal{O}(\xi^2)$. We will use two methods of calculation to evaluate the expectation values of the wavy lines, they are as follows:

- The *General Method* in which the Wilson loop operator is expanded perturbatively and evaluated, and then the information concerning the shape of the loop is entered at the last step.
- The *Operator Insertion Method* in which operators representing the waviness are sandwiched between regular straight Wilson line operators.

We begin by describing the General Method.

2.1 General Method

In this method a straightforward expansion of the Wilson loop operator is used:

$$W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \left\{ 1 + g \oint_C ds \left(iA_\mu(x) \dot{x}_\mu + \Phi_i(x) |\dot{x}| \theta_i \right) + \frac{g^2}{2!} \oint ds_1 ds_2 \left(iA_\mu(y) \dot{y}_\mu + \Phi_i(y) |\dot{y}| \theta_i \right) \left(iA_\nu(z) \dot{z}_\nu + \Phi_j(z) |\dot{z}| \theta_j \right) + \dots \right\} \quad (2.2)$$

In order to demonstrate this method, consider calculating the $\mathcal{O}(g^2 N)$ term arising from the expectation value of $W(C)$. Any term with a single field will disappear since the gauge

group generators are traceless. Thus the expectation value of the above expression, to the order of interest is:

$$\langle W(C) \rangle = 1 + \left\langle \frac{1}{N} \text{Tr} \mathcal{P} \frac{g^2}{2!} \oint ds_1 ds_2 \left(iA_\mu(y) \dot{y}_\mu + \Phi_i(y) |\dot{y}| \theta_i \right) \left(iA_\nu(z) \dot{z}_\nu + \Phi_j(z) |\dot{z}| \theta_j \right) \right\rangle \quad (2.3)$$

Note that the path ordering implies that we should consider the two orderings of the $s_1 s_2$ integration domain, i.e. one where $s_1 > s_2$ and one where $s_2 > s_1$. However, due to the symmetry of the trace (of two fields), these contributions are equal, and add to give the whole domain. The next step is to perform the Wick contractions, using the formalism in appendix A. Note that $A_\mu = A_\mu^a T^a$, and $\Phi_i = \phi_i^a T^a$ where T^a are the generators of $SU(N)$ in the fundamental representation. The properties of the T^a appear in appendix A. We have:

$$= 1 + \frac{g^2}{2N} \oint ds_1 ds_2 \left\{ - \langle A_\mu^a(y(s_1)) A_\nu^b(z(s_2)) \rangle \frac{\delta^{ab}}{2} \dot{y}_\mu \dot{z}_\nu + \langle \phi_i^c(y(s_1)) \phi_j^d(z(s_2)) \rangle \frac{\delta^{cd}}{2} \theta_i \theta_j |\dot{z}| |\dot{y}| \right\} \quad (2.4)$$

where we have evaluated the trace. Noting that in the large N approximation $\delta^{ab} \delta^{ab} = N^2 - 1 \simeq N^2$, we have:

$$= 1 + \frac{g^2 N}{16\pi^2} \oint ds_1 ds_2 \frac{|\dot{y}(s_1)| |\dot{z}(s_2)| - \dot{y}(s_1) \cdot \dot{z}(s_2)}{|y(s_1) - z(s_2)|^2} \quad (2.5)$$

where we have also used $\theta_i \theta_i = 1$.

The above expression is true for any shape of curve, but we are interested in the wavy line. For example, let's look at the contribution to the $\mathcal{O}(g^2 N)$ term at leading order in waviness. We'll need:

$$|\dot{x}(s)| = \sqrt{1 + \dot{\xi}(s) \cdot \dot{\xi}(s)} \quad (2.6)$$

$$|x(s) - x(s')|^2 = (s - s')^2 + [\xi(s) - \xi(s')]^2 \quad (2.7)$$

$$\dot{x}(s) \cdot \dot{x}(s') = 1 + \dot{\xi}(s) \cdot \dot{\xi}(s') \quad (2.8)$$

and expanding we find,

$$\frac{\dot{x}(s) \cdot \dot{x}(s') - |\dot{x}(s)| |\dot{x}(s')|}{|x(s) - x(s')|^2} \simeq -\frac{1}{2} \left[\dot{\xi}(s) - \dot{\xi}(s') \right]^2 \frac{1}{(s - s')^2} \quad (2.9)$$

Thus at the end of the day, our result to $\mathcal{O}(g^2 N)$, $\mathcal{O}(\xi^2)$ is,

$$\langle W(C) \rangle = 1 + \frac{g^2 N}{16\pi^2} \oint_C ds_1 ds_2 \frac{[\dot{\xi}(s_1) - \dot{\xi}(s_2)]^2}{2(s_1 - s_2)^2} \quad (2.10)$$

This is how the general method calculations are performed.

2.2 Operator Insertion Method

2.2.1 Derivation of the Operators

This method requires that we find the operators which should be sandwiched between straight line Wilson loops to encode the waviness. In this section we find those operators. For the purpose of simplifying the notation, we absorb the coupling constant g into the scalar and gauge fields. To restore the coupling one can use the simple rule that for each power of a field, a power of the coupling should be added.

The Wilson loop operator is the following:

$$W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \int_C ds E(s) \quad (2.11)$$

Where the exponent $E(s)$ is as follows:

$$E = iA_\mu(x(s)) \dot{x}_\mu(s) + \Phi_j(x(s)) |\dot{x}(s)| \theta_j \quad (2.12)$$

First consider the variation of this exponent. To first order, we have:

$$\begin{aligned} \delta E &= i\delta x_\rho \dot{x}_\mu(s) \partial_\rho A_\mu(x(s)) + iA_\mu(x(s)) \delta \dot{x}_\mu(s) + \delta x_\rho |\dot{x}(s)| \partial_\rho \Phi_j(x(s)) \theta_j \\ &= i\delta x_\rho \dot{x}_\mu(s) \partial_\rho A_\mu(x(s)) - i\delta x_\rho(s) \dot{x}_\mu(s) \partial_\mu A_\rho(x(s)) \\ &\quad + i \frac{d}{ds} [A_\mu(x(s)) \delta x_\mu(s)] + \delta x_\rho |\dot{x}(s)| \partial_\rho \Phi_j(x(s)) \theta_j \end{aligned} \quad (2.13)$$

We can use this to get the first order (single-insertion) operator:

$$\begin{aligned} \delta W(C) &= \frac{1}{N} \text{Tr} \int ds \mathcal{P} e^{\int_s^\infty ds' E(s')} \\ &\quad \times \left[i\dot{x}_\mu \delta x_\rho (\partial_\rho A_\mu - \partial_\mu A_\rho) + \delta x_\rho |\dot{x}| \partial_\rho \Phi_j \theta_j + i \frac{d}{ds} [A_\mu(x(s)) \delta x_\mu(s)] \right] \\ &\quad \times \mathcal{P} e^{\int_{-\infty}^s ds'' E(s'')} \end{aligned} \quad (2.14)$$

Now let's consider the total derivative term separately:

$$\frac{1}{N} \text{Tr} \int ds \mathcal{P} e^{\int_s^\infty ds' E(s')} \left[i \frac{d}{ds} [A_\mu(x(s)) \delta x_\mu(s)] \right] \mathcal{P} e^{\int_{-\infty}^s ds'' E(s'')} \quad (2.15)$$

Using integration by parts, we have:

$$= \frac{1}{N} \text{Tr} \int ds \left[-\mathcal{P} e^{\int_s^\infty ds' E(s')} \{ (-E(s)) i\delta x_\mu A_\mu + i\delta x_\mu A_\mu E(s) \} \mathcal{P} e^{\int_{-\infty}^s ds'' E(s'')} \right] \quad (2.16)$$

$$= \frac{1}{N} \text{Tr} \int ds \left[-\mathcal{P}e^{\int_s^\infty ds' E(s')} \delta x_\mu \{ -[A_\rho, A_\mu] \dot{x}_\rho + i[\Phi_j, A_\mu] |\dot{x}| \} \mathcal{P}e^{\int_{-\infty}^s ds'' E(s'')} \right] \quad (2.17)$$

Therefore, we have:

$$\delta W(C) = \frac{1}{N} \text{Tr} \int ds \mathcal{P}e^{\int_s^\infty ds' E(s')} \delta x_\rho [i\dot{x}_\mu F_{\rho\mu} + |\dot{x}| D_\rho \Phi_j \theta_j] \mathcal{P}e^{\int_{-\infty}^s ds'' E(s'')} \quad (2.18)$$

where $D_\mu Z = \partial_\mu Z - i[A_\mu, Z]$, and $F_{\rho\mu} = \partial_\rho A_\mu - \partial_\mu A_\rho - i[A_\rho, A_\mu]$. This is the first order operator insertion, we'll call the operator \mathcal{O}_1 :

$$\mathcal{O}_1 = \delta x_\rho [i\dot{x}_\mu F_{\rho\mu} + |\dot{x}| D_\rho \Phi_j \theta_j] \quad (2.19)$$

The second order variation has two parts. One is to simply insert \mathcal{O}_1 in two places, as follows:

$$\frac{1}{N} \text{Tr} \int ds \int dt \mathcal{P}e^{\int_s^\infty ds_1 E(s_1)} \mathcal{O}_1(s) \mathcal{P}e^{\int_t^s ds_2 E(s_2)} \mathcal{O}_1(t) \mathcal{P}e^{\int_{-\infty}^t ds_3 E(s_3)} \quad (2.20)$$

The other piece of the second order variation is a 2^{nd} order operator inserted at a single point. To find this operator we'll take the strategy of varying $\delta W(C)$, that is (2.18) again, keeping only single point insertions. The variations of \mathcal{O}_1 itself will involve:

$$\delta(i\delta x_\rho \dot{x}_\mu F_{\rho\mu}) = i\delta x_\rho \delta x_\tau \dot{x}_\mu \partial_\tau F_{\rho\mu} + i\delta x_\rho \delta \dot{x}_\mu F_{\rho\mu} \quad (2.21)$$

Now, we should be careful about the interpretation of the term $\omega = \delta x_\rho \delta \dot{x}_\mu F_{\rho\mu}$ from (2.21). If we consider this as arising from a second partial derivative of $W(C)$, we should have:

$$\frac{\delta}{\delta x_\mu} \frac{\delta}{\delta x_\rho} \omega = -\frac{d}{ds} F_{\rho\mu} \quad (2.22)$$

but this vanishes on the basis of the symmetry of the partial derivatives, and the antisymmetry of $F_{\rho\mu}$. We will also require:

$$\delta(\delta x_\rho |\dot{x}| D_\rho \Phi_j) = \delta x_\rho \delta x_\tau |\dot{x}| \partial_\tau D_\rho \Phi_j \quad (2.23)$$

Note that the variation of $|\dot{x}|$ starts at 2^{nd} order in δx . We also have to consider variations of the exponentials which will yield single point insertions. These arise as a result of the total derivative terms in δE , see (2.13). Thus, keeping only single point terms:

$$\mathcal{P}e^{\int_s^\infty ds' \delta E(s')} \simeq 1 + i \int_s^\infty ds' \frac{d}{ds'} [A_\tau(s') \delta x_\tau(s')] = 1 - i A_\tau(s) \delta x_\tau(s) \quad (2.24)$$

Similarly:

$$\mathcal{P}e^{\int_{-\infty}^s ds'' \delta E(s'')} \simeq 1 + i A_\tau(s) \delta x_\tau(s) \quad (2.25)$$

Putting together all the pieces, we have:

$$\delta^2 W(C)_{\text{single}} = \frac{1}{N} \text{Tr} \int ds \mathcal{P} e^{\int_s^\infty ds' E(s')} \delta x_\rho \delta x_\tau \{i\dot{x}_\mu D_\tau F_{\rho\mu} + |\dot{x}| D_\tau D_\rho \Phi_j \theta_j\} \mathcal{P} e^{\int_{-\infty}^s ds'' E(s'')} \quad (2.26)$$

Thus our 2nd order operator is:

$$\mathcal{O}_2 = \delta x_\rho \delta x_\tau \{i\dot{x}_\mu D_\tau F_{\rho\mu} + |\dot{x}| D_\tau D_\rho \Phi_j \theta_j\} \quad (2.27)$$

We therefore have:

$$\begin{aligned} \delta^2 W(C) &= \frac{1}{N} \text{Tr} \int ds \int dt \mathcal{P} e^{\int_s^\infty ds_1 E(s_1)} \mathcal{O}_1(s) \mathcal{P} e^{\int_t^s ds_2 E(s_2)} \mathcal{O}_1(t) \mathcal{P} e^{\int_{-\infty}^t ds_3 E(s_3)} \\ &\quad + \frac{1}{N} \text{Tr} \int ds \mathcal{P} e^{\int_s^\infty ds_1 E(s_1)} \mathcal{O}_2(s) \mathcal{P} e^{\int_{-\infty}^s ds_2 E(s_2)} \end{aligned} \quad (2.28)$$

2.2.2 Calculational Method

Here we will give an example of how the operator insertion method works for real calculations. Consider that $x_\mu(s) = (s, 0, 0, 0)$, i.e. we have a straight line. Then we have:

$$\begin{aligned} \dot{x}_\mu &= (1, 0, 0, 0) \\ |\dot{x}| &= 1 \\ \delta x_\mu(s) &= (0, \xi_m(s)) \end{aligned} \quad (2.29)$$

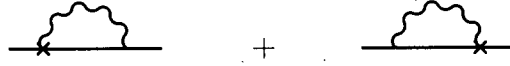
where $\xi_m(s)$ is our three-vector describing the “waviness” of our Wilson line. Consider calculating the first correction to the expectation value of a straight Wilson line, due to the line being wavy. To do this we simply calculate the expectation value of (2.18), which now reads:

$$\begin{aligned} \delta W(C) &= \frac{1}{N} \text{Tr} \int ds \mathcal{P} \exp \left\{ \int_s^\infty ds' [iA_0(x(s')) + \Phi_j(x(s'))\theta_j] \right\} \\ &\quad \times \xi_m(s) [iF_{m0}(x(s)) + D_m \Phi_k(x(s))\theta_k] \mathcal{P} \exp \left\{ \int_{-\infty}^s ds'' [iA_0(x(s'')) + \Phi_i(x(s''))\theta_i] \right\} \end{aligned} \quad (2.30)$$

This expression contains of course an infinite number of contributions corresponding to various powers of $g^2 N$, but let's look at the leading term which is $\mathcal{O}(g^2 N)$. Expanding (2.30) we have:

$$\begin{aligned} \delta W(C) &= \frac{1}{N} \text{Tr} \int ds \int_s^\infty ds' \xi_m(s) [iA_0(x(s')) + \Phi_j(x(s'))\theta_j] [iF_{m0}(x(s)) + D_m \Phi_k(x(s))\theta_k] \\ &\quad + \frac{1}{N} \text{Tr} \int ds \int_{-\infty}^s ds'' \xi_m(s) [iF_{m0}(x(s)) + D_m \Phi_k(x(s))\theta_k] [iA_0(x(s'')) + \Phi_i(x(s''))\theta_i] \\ &\quad + \dots \end{aligned} \quad (2.31)$$

We can represent the $\mathcal{O}(g^2 N)$ contribution from these two terms by Feynman diagrams,



where the X denotes the insertion of \mathcal{O}_1 and the wiggly line here denotes both gauge and scalar fields. Let's now evaluate the first of these diagrams, corresponding to the first term in our expansion:

$$\left\langle \frac{1}{N} \text{Tr} \int ds \int_s^\infty ds' \xi_m(s) \left\{ iA_0(x(s')) + \Phi_j(x(s'))\theta_j \right\} \left\{ i\nabla_m A_0(x(s)) - i\partial_0 A_m(x(s)) \right. \right. \\ \left. \left. + [A_m(x(s)), A_0(x(s))] + \nabla_m \Phi_k(x(s))\theta_k - i[A_m(x(s)), \Phi_k(x(s))\theta_k] \right\} \right\rangle \quad (2.32)$$

Keeping only single propagator terms, as the diagram suggests, we find the $\mathcal{O}(g^2 N)$ contribution:

$$\left\langle \frac{1}{N} \text{Tr} \int ds \int_s^\infty ds' \xi_m(s) \left\{ -A_0(x(s')) \nabla_m A_0(x(s)) + \Phi_j(x(s')) \nabla_m \Phi_k(x(s)) \theta_k \theta_j \right\} \right\rangle \quad (2.33)$$

evaluating the trace,

$$\frac{1}{N} \int ds \int_s^\infty ds' \xi_m(s) \left\{ -\langle A_0^a(x(s')) \nabla_m A_0^b(x(s)) \rangle \frac{\delta^{ab}}{2} + \langle \Phi_j^c(x(s')) \nabla_m \Phi_k^d(x(s)) \rangle \frac{\delta^{cd}}{2} \theta_k \theta_j \right\} \quad (2.34)$$

and so finally, and restoring the coupling constant:

$$\frac{g^2 N}{8\pi^2} \int ds \int_s^\infty ds' \xi_m(s) \frac{\partial}{\partial x_m(s)} \left\{ -\frac{1}{|x(s) - x(s')|^2} + \frac{1}{|x(s) - x(s')|^2} \right\} = 0 \quad (2.35)$$

This result is zero, as it should be considering the example provided of the General Method, where we saw that the result (2.10) is second order in waviness. However the purpose here is to demonstrate how the operator method calculations are performed.

2.3 An Efficient Calculation Scheme

We can develop a simple *10-D Calculus* for evaluating the large number of graphs which arise in both methods of calculation. We do this by making the following replacements:

$$\partial_\mu \rightarrow (\partial_\mu, 0, 0, 0, 0, 0, 0) \quad (2.36)$$

$$\dot{x}_\mu \rightarrow (\dot{x}_\mu, -i|\dot{x}|\theta_j) \quad (2.37)$$

$$A_\mu^a \rightarrow (A_\mu^a, \phi_j^a), \quad (2.38)$$

note that on the right hand side μ runs from 0,...,3 while on the left hand side μ runs from 0,...,9. The roman indices such as j run from 4,...,9. We define a 10-D gauge field propagator as follows:

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \frac{\delta_{\mu\nu} \delta^{ab}}{4\pi(x-y)^2} \quad (2.39)$$

This allows us to consider *only* the diagrams involving all gauge fields (no scalars), and then replace everything with their real values at the end, according to the above definitions. We can even replace the three and four-point vertices involving gauge and scalar fields with just the gauge field vertices. This works because our theory, $\mathcal{N} = 4$ SYM, comes from a dimensionally reduced pure gauge field (and supergauge field) theory in ten spacetime dimensions. This simplification will provide us with an enormous economy of calculation. For example, the Wilson loop operator becomes:

$$W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left\{ i \int_C ds A_\mu(x(s)) \dot{x}_\mu(s) \right\} \quad (2.40)$$

and $(\dot{x} \cdot \dot{y} - |\dot{x}||\dot{y}|)$ in the 4-D notation is $(\dot{x} \cdot \dot{y})$ in the 10-D notation. Also our operators (2.19) and (2.27) become:

$$\mathcal{O}_1 = i\delta x_\rho \dot{x}_\mu F_{\rho\mu}, \quad \mathcal{O}_2 = i\delta x_\rho \delta x_\tau \dot{x}_\mu D_\tau F_{\rho\mu} \quad (2.41)$$

Chapter 3

Expectation Value of a Single Wavy Line

In this chapter the expectation value of a single wavy line will be computed to $\mathcal{O}(g^4 N^2)$ and $\mathcal{O}(\xi^2)$. We begin by using the general method of calculation. Throughout, the efficient 10-D calculus will be used. The operator insertion method will be used as well, but only to $\mathcal{O}(g^2 N)$ and $\mathcal{O}(\xi^2)$.

3.1 General Method

Recall that the single wavy line is represented as follows:

$$x^\mu(s) = (s, \vec{\xi}(s)) \quad (3.1)$$

We will need (2.8) and (2.9) as well. At leading order in $g^2 N$, there is only one contribution (2.5), which we calculated previously. At the next order, $\mathcal{O}(g^4 N^2)$, there are three terms. The first is:

$$\text{Diagram} = \left\langle \frac{i^4 g^4}{4! N} \text{Tr } \mathcal{P} \oint ds_1 ds_2 ds_3 ds_4 A_\mu(x) A_\nu(y) A_\rho(z) A_\sigma(w) \dot{x}_\mu \dot{y}_\nu \dot{z}_\rho \dot{w}_\sigma \right\rangle = \Sigma_1$$

The path ordering implies $4!$ permutations of the ordering of the s_i , however they are all equivalent and so we can choose one and cancel the $4!$ in the denominator. The next step is to perform the Wick contractions, recall that we are interested only in planar diagrams and so the gauge field lines may not cross.

$$\frac{g^4}{4N} \int_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 \dot{x}_\mu \dot{y}_\nu \dot{z}_\rho \dot{w}_\sigma \langle A_\mu^a(x) A_\sigma^d(w) \rangle \langle A_\nu^b(y) A_\rho^c(z) \rangle \delta^{ad} \delta^{bc} \text{Tr} (T^a T^b T^c T^d) \quad (3.2)$$

We use the identity for the gauge group matrices:

$$T^a T^a = \frac{N}{2} \mathbf{1} \quad (3.3)$$

to evaluate the trace:

$$\begin{aligned}
& \delta^{ad} \delta^{bc} \text{Tr} (T^a T^b T^c T^d) \\
&= \delta^{ad} \text{Tr} (T^a T^b T^b T^d) \\
&= \frac{N}{2} \delta^{ad} \text{Tr} (T^a T^d) \\
&= \frac{N}{4} \delta^{aa} = \frac{N}{4} (N^2 - 1) \\
&\simeq \frac{N^3}{4}
\end{aligned} \tag{3.4}$$

The result is then,

$$\frac{g^4 N^2}{64\pi^2} \int_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 \frac{(\dot{x} \cdot \dot{w})}{|x - w|^2} \frac{(\dot{y} \cdot \dot{z})}{|y - z|^2} \tag{3.5}$$

and finally reverting to our 4-D notation, which implies the calculation of the following four diagrams in the 4-D language:



the result is:

$$\begin{aligned}
\Sigma_1 = \frac{g^4 N^2}{64\pi^2} \int_{s_1 > s_2 > s_3 > s_4} ds_1 ds_2 ds_3 ds_4 & \frac{(\dot{x}(s_1) \cdot \dot{x}(s_4)) - |\dot{x}(s_1)| |\dot{x}(s_4)|}{|x(s_1) - x(s_4)|^2} \\
& \times \frac{(\dot{x}(s_2) \cdot \dot{x}(s_3)) - |\dot{x}(s_2)| |\dot{x}(s_3)|}{|x(s_2) - x(s_3)|^2}
\end{aligned} \tag{3.6}$$

There's no need to evaluate this amplitude for the wavy line, because it begins at $\mathcal{O}(\xi^4)$ as can be seen from (2.9).

The next $\mathcal{O}(g^2 N^4)$ contribution comes from the one-loop corrected, single gluon exchange graph,

$$\text{Diagram} = \Sigma_2$$

where the loop contains all corrections as outlined in appendix B. The corrected propagator is given in (B.2), using (A.4) we can express this in configuration space:

$$-\delta^{ab} g^4 N \frac{\Gamma^2(\omega - 1)}{2^5 \pi^{2\omega} (2 - \omega)(2\omega - 3)} \frac{1}{[x^2]^{2\omega - 3}} \tag{3.7}$$

the resulting expression is then straightforward, given (2.5), it is:

$$\Sigma_2 = g^4 N^2 \frac{\Gamma^2(\omega - 1)}{2^7 \pi^{2\omega} (2 - \omega)(2\omega - 3)} \oint ds_1 ds_2 \frac{\dot{x}(s_1) \cdot \dot{x}(s_2) - |\dot{x}(s_1)| |\dot{x}(s_2)|}{[(x(s_1) - x(s_2))^2]^{2\omega - 3}} \quad (3.8)$$

This expression is divergent in the physical dimension ($\omega = 2$), however this divergence will cancel against our next (and last) contribution.

The last contribution at $\mathcal{O}(g^2 N^4)$ comes from the following Feynman diagram:

$$\begin{aligned} \text{Diagram} &= \left\langle \frac{i^3 g^3}{3! N} \text{Tr } \mathcal{P} \oint ds_1 ds_2 ds_3 A_\mu(x) A_\nu(y) A_\rho(z) \dot{x}_\mu \dot{y}_\nu \dot{z}_\rho \right. \\ &\quad \left. \times -g \int d^{2\omega} w f^{abc} \partial_\rho A_\lambda^a(w) A_\rho^b(w) A_\lambda^c(w) \right\rangle = \Sigma_3 \end{aligned}$$

This amplitude will involve the structure: $f^{abc} \text{Tr} [T^a T^b T^c]$ where the group matrices correspond to the three fields at x, y , and z respectively. This expression is completely antisymmetric in a, b, c since f^{abc} is. This implies that we should introduce the path ordering symbol, $\epsilon(s_1 s_2 s_3)$. We define it to be equal to 1 for $s_1 > s_2 > s_3$ and antisymmetric under any transposition of the s_i . Therefore,

$$\mathcal{P} \oint ds_1 ds_2 ds_3 = \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) \quad (3.9)$$

For each ordering, there are $3!$ equivalent ways of contracting the Wilson loop fields with the cubic vertex. Therefore we can choose one and eliminate the $3!$ from the denominator. In light of these arguments we have:

$$\begin{aligned} -\frac{i^3 g^4}{N} \text{Tr } \mathcal{P} \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) \int d^{2\omega} w f^{abc} \text{Tr} [T^d T^e T^f] \dot{x}_\mu \dot{y}_\nu \dot{z}_\rho \\ \times \langle A_\mu^d(x) \partial_\rho A_\lambda^a(w) \rangle \langle A_\nu^e(y) A_\rho^b(w) \rangle \langle A_\rho^f(z) A_\lambda^c(w) \rangle \end{aligned} \quad (3.10)$$

Now noting that:

$$\begin{aligned} f^{abc} \text{Tr} (T^a T^b T^c) &= f^{abc} \text{Tr} \left(\frac{1}{2} [T^a, T^b] T^c + \frac{1}{2} \{T^a, T^b\} T^c \right) \\ &= \frac{i}{2} f^{abc} \text{Tr} (f^{abd} T^d T^c) \\ &= \frac{i}{4} f^{abc} f^{abd} \delta^{dc} \simeq i \frac{N^3}{4} \end{aligned}$$

and that:

$$\langle A_\mu^d(x) \partial_\rho A_\lambda^a(w) \rangle = -\frac{\partial}{\partial x^\rho} \delta_{\mu\lambda} \delta^{ad} \Delta(x - w) \quad (3.11)$$

where:

$$\Delta(x) = \frac{\Gamma(\omega - 1)}{4\pi^\omega} \frac{1}{[x^2]^{\omega-1}} \quad (3.12)$$

and assembling everything we have:

$$\begin{aligned} & \frac{g^4 N^2}{4} \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) (\dot{x}(s_1) \cdot \dot{x}(s_3)) \left(\dot{x}(s_2) \cdot \frac{\partial}{\partial x(s_1)} \right) \\ & \times \int d^{2\omega} w \Delta(x(s_1) - w) \Delta(x(s_2) - w) \Delta(x(s_3) - w) \end{aligned} \quad (3.13)$$

replacing the 10-D values with their proper values gives:

$$\begin{aligned} \Sigma_3 = & \frac{g^4 N^2}{4} \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) (\dot{x}(s_1) \cdot \dot{x}(s_3) - |\dot{x}(s_1)| |\dot{x}(s_3)|) \left(\dot{x}(s_2) \cdot \frac{\partial}{\partial x(s_1)} \right) \\ & \times \int d^{2\omega} w \Delta(x(s_1) - w) \Delta(x(s_2) - w) \Delta(x(s_3) - w) \end{aligned} \quad (3.14)$$

We have now assembled the three contributions at $\mathcal{O}(g^4 N^2)$, and we can proceed to evaluate them for the wavy line. There are divergences in these terms which we will show will cancel and leave a finite result.

First some notational convenience:

$$\xi(i) = \vec{\xi}(s_i) \quad (3.15)$$

$$s_{ij} = (s_i - s_j) \quad (3.16)$$

$$\dot{\xi}(j) = \frac{\partial}{\partial s_j} \vec{\xi}(s_j) \quad (3.17)$$

$$(3.18)$$

The meat of the calculation comes from Σ_3 , which as we will see, will cancel out the divergent term Σ_2 and leave over a finite term which will have the same ξ dependence as (2.10). Now let us expand Σ_2 out to second order in waviness:

$$\Sigma_2 = -g^4 N^2 \frac{\Gamma^2(\omega - 1)}{2^7 \pi^{2\omega} (2 - \omega)(2\omega - 3)} \oint ds_1 ds_2 \frac{[\dot{\xi}(1) - \dot{\xi}(2)]^2}{2 (s_{12})^{2\omega-3}} \quad (3.19)$$

The calculation of Σ_3 begins by using Feynman parameters (4.41) in order to put it in the following form:

$$\begin{aligned} \Sigma_3 = & -\frac{g^4 N^2}{4} \frac{\Gamma(2\omega - 3)}{2^6 \pi^{2\omega}} \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) (|\dot{x}(s_1)| |\dot{x}(s_3)| - \dot{x}(s_1) \cdot \dot{x}(s_3)) \\ & \times \dot{x}(s_2) \cdot \frac{\partial}{\partial x(s_1)} \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma) \frac{1}{D^{2\omega-3}} \end{aligned} \quad (3.20)$$

where

$$D = \alpha\beta|x(s_1) - x(s_2)|^2 + \alpha\gamma|x(s_1) - x(s_3)|^2 + \gamma\beta|x(s_3) - x(s_2)|^2 \quad (3.21)$$

The expansion of the $|\dot{x}(s_1)||\dot{x}(s_3)| - \dot{x}(s_1) \cdot \dot{x}(s_3)$ factor starts at second order in waviness. This means we should expand the other factors to zeroth order. This means we take:

$$\dot{x}(s_2) \cdot \frac{\partial}{\partial x(s_1)} \approx \frac{\partial}{\partial s_1} \quad (3.22)$$

$$D \approx \alpha\beta s_{12}^2 + \alpha\gamma s_{13}^2 + \gamma\beta s_{23}^2 \quad (3.23)$$

Thus to second order in waviness Σ_3 takes the following form:

$$\begin{aligned} \Sigma_3 = & -\frac{g^4 N^2 \Gamma(2\omega - 3)}{4 \cdot 2^6 \pi^{2\omega}} \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) \frac{1}{2} [\dot{\xi}(1) - \dot{\xi}(3)]^2 \\ & \times \frac{\partial}{\partial s_1} \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma) \\ & \times \frac{1}{(\alpha\beta s_{12}^2 + \alpha\gamma s_{13}^2 + \gamma\beta s_{23}^2)^{2\omega-3}} \end{aligned} \quad (3.24)$$

We now integrate by parts. This generates a surface term which is zero, and two other terms. We use the derivative of the path ordering symbol:

$$\frac{\partial}{\partial s_1} \epsilon(s_1 s_2 s_3) = 2\delta(s_{12}) - 2\delta(s_{13}), \quad (3.25)$$

Thus (3.24) becomes:

$$\begin{aligned} \Sigma_3 = & -\frac{g^4 N^2 \Gamma(2\omega - 3)}{4 \cdot 2^7 \pi^{2\omega}} \oint ds_2 ds_3 \epsilon(s_1 s_2 s_3) [\dot{\xi}(1) - \dot{\xi}(3)]^2 \\ & \times \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma) \\ & \times \frac{1}{(\alpha\beta s_{12}^2 + \alpha\gamma s_{13}^2 + \gamma\beta s_{23}^2)^{2\omega-3}} \Big|_{s_1=-\infty}^{\infty} \\ & + \frac{g^4 N^2 \Gamma(2\omega - 3)}{4 \cdot 2^6 \pi^{2\omega}} \oint ds_1 ds_2 ds_3 \left[\{\delta(s_{12}) - \delta(s_{13})\} [\dot{\xi}(1) - \dot{\xi}(3)]^2 \right. \\ & \quad \left. + \epsilon(s_1 s_2 s_3) \frac{1}{2} \frac{\partial}{\partial s_1} (\dot{\xi}(1) - \dot{\xi}(3))^2 \right] \\ & \times \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma) \\ & \times \frac{1}{(\alpha\beta s_{12}^2 + \alpha\gamma s_{13}^2 + \gamma\beta s_{23}^2)^{2\omega-3}} \end{aligned} \quad (3.26)$$

$$\begin{aligned} & + \frac{g^4 N^2 \Gamma(2\omega - 3)}{4 \cdot 2^6 \pi^{2\omega}} \oint ds_1 ds_2 ds_3 \left[\{\delta(s_{12}) - \delta(s_{13})\} [\dot{\xi}(1) - \dot{\xi}(3)]^2 \right. \\ & \quad \left. + \epsilon(s_1 s_2 s_3) \frac{1}{2} \frac{\partial}{\partial s_1} (\dot{\xi}(1) - \dot{\xi}(3))^2 \right] \\ & \times \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma) \\ & \times \frac{1}{(\alpha\beta s_{12}^2 + \alpha\gamma s_{13}^2 + \gamma\beta s_{23}^2)^{2\omega-3}} \end{aligned} \quad (3.27)$$

The first term is a surface term and is zero since the integrand is well behaved. The second term, which contains $\{\delta(s_{12}) - \delta(s_{13})\}$, is zero for $s_1 = s_3$. For $s_1 = s_2$ however, we have:

$$\begin{aligned} & \frac{g^4 N^2 \Gamma(2\omega - 3)}{4} \frac{1}{2^6 \pi^{2\omega}} \oint ds_2 ds_3 \left[\dot{\xi}(2) - \dot{\xi}(3) \right]^2 \frac{1}{(s_{23}^2)^{2\omega-3}} \\ & \times \int_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma) \frac{(\alpha\beta\gamma)^{\omega-2}}{[(1-\gamma)\gamma]^{2\omega-3}} \end{aligned} \quad (3.28)$$

Now, it is shown in appendix D that:

$$\int_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma) \frac{(\alpha\beta\gamma)^{\omega-2}}{[(1-\gamma)\gamma]^{2\omega-3}} = \frac{\Gamma^2(\omega - 1)}{(2\omega - 3)} \frac{1}{(2 - \omega)\Gamma(2\omega - 3)} \quad (3.29)$$

And thus this term of Σ_3 is:

$$\frac{g^4 N^2 \Gamma^2(\omega - 1)}{2^7 \pi^{2\omega} (2 - \omega)(2\omega - 3)} \oint ds_2 ds_3 \left[\dot{\xi}(2) - \dot{\xi}(3) \right]^2 \frac{1}{2 (s_{23}^2)^{2\omega-3}} \quad (3.30)$$

which cancels Σ_2 , that is (3.19), identically.

We now have the remaining term in (3.27) to deal with. This term is:

$$\begin{aligned} & \frac{g^4 N^2 \Gamma(2\omega - 3)}{4} \frac{1}{2^6 \pi^{2\omega}} \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) \left\{ \frac{1}{2} \frac{\partial}{\partial s_1} \left(\dot{\xi}(1) - \dot{\xi}(3) \right)^2 \right\} \\ & \times \int_0^1 d\alpha d\beta d\gamma (\alpha\beta\gamma)^{\omega-2} \delta(1 - \alpha - \beta - \gamma) \\ & \times \frac{1}{(\alpha\beta s_{12}^2 + \alpha\gamma s_{13}^2 + \gamma\beta s_{23}^2)^{2\omega-3}} \end{aligned} \quad (3.31)$$

Since this is finite in the actual space-time dimension of 4, we can set ω to 2 and attempt to evaluate this integral. Because we don't know the explicit s -dependence of ξ , we can only integrate over s_2 . The path ordering symbol ϵ implies the following:

$$\begin{aligned} \oint ds_1 ds_2 ds_3 \epsilon(s_1 s_2 s_3) &= \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_3 \int_{s_3}^{s_1} ds_2 - \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_3 \int_{-\infty}^{s_3} ds_2 \\ &- \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_3 \int_{s_1}^{\infty} ds_2 - \int_{-\infty}^{\infty} ds_3 \int_{-\infty}^{s_3} ds_1 \int_{s_1}^{s_3} ds_2 \\ &+ \int_{-\infty}^{\infty} ds_3 \int_{-\infty}^{s_3} ds_1 \int_{-\infty}^{s_1} ds_2 + \int_{-\infty}^{\infty} ds_3 \int_{-\infty}^{s_3} ds_1 \int_{s_3}^{\infty} ds_2 \end{aligned} \quad (3.32)$$

We will also need the following integral:

$$\int dx \frac{1}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \arctan \frac{2ax + b}{\sqrt{4ac - b^2}} \quad 4ac > b^2 \quad (3.33)$$

In our case x is s_2 while:

$$\sqrt{4ac - b^2} = 2|s_{13}| \sqrt{\alpha\beta\gamma} \quad (3.34)$$

$$2as_1 + b = 2\beta\gamma s_{13} \quad (3.35)$$

$$2as_3 + b = -2\beta\alpha s_{13} \quad (3.36)$$

In light of these equations we can write (3.31) as:

$$\begin{aligned} & \frac{g^4 N^2}{2^8 \pi^4} \int_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma) \\ & \times \left[\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_3 \left\{ \left[\frac{1}{2} \frac{\partial}{\partial s_1} \left(\dot{\xi}(1) - \dot{\xi}(3) \right)^2 \right] \right. \right. \\ & \quad \times \frac{1}{|s_{13}|} \frac{1}{\sqrt{\alpha\beta\gamma}} \left[2 \arctan \left(\text{sign}(s_{13}) \sqrt{\frac{\beta\gamma}{\alpha}} \right) + 2 \arctan \left(\text{sign}(s_{13}) \sqrt{\frac{\beta\alpha}{\gamma}} \right) - \pi \right] \Big\} \\ & + \int_{-\infty}^{\infty} ds_3 \int_{-\infty}^{s_3} ds_1 \left\{ \left[\frac{1}{2} \frac{\partial}{\partial s_1} \left(\dot{\xi}(1) - \dot{\xi}(3) \right)^2 \right] \right. \\ & \quad \times \frac{1}{|s_{13}|} \frac{1}{\sqrt{\alpha\beta\gamma}} \left[2 \arctan \left(\text{sign}(s_{13}) \sqrt{\frac{\beta\gamma}{\alpha}} \right) + 2 \arctan \left(\text{sign}(s_{13}) \sqrt{\frac{\beta\alpha}{\gamma}} \right) + \pi \right] \Big\} \Bigg] \end{aligned} \quad (3.37)$$

Now note the following identities, which are proven in appendix D:

$$\int_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma) \frac{1}{\sqrt{\alpha\beta\gamma}} = 2\pi \quad (3.38)$$

$$\int_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma) \frac{1}{\sqrt{\alpha\beta\gamma}} \arctan \left(\sqrt{\frac{\beta\gamma}{\alpha}} \right) = \frac{\pi^2}{3} \quad (3.39)$$

In order to tame the monstrous (3.37) we introduce the following symbol:

$$\theta_{\pm} = \mp 4 \int_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma) \frac{1}{\sqrt{\alpha\beta\gamma}} \arctan \left(\text{sign}(s_{13}) \sqrt{\frac{\beta\gamma}{\alpha}} \right) \pm 2\pi^2 \quad (3.40)$$

Note that the arctan terms of two different arguments in (3.37) are equal due to the symmetry between α , β , and γ . Now also note that in (3.37) we can interchange the s_1 and s_3 variables in the second integration in order to combine it with the first. This flips the signs on the

arctan terms in the last line of (3.37), because $\text{sign}(s_{13}) \rightarrow \text{sign}(s_{31})$. We then have that (3.37) is equal to:

$$\frac{g^4 N^2}{2^8 \pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_3 \frac{1}{s_{13}} \left\{ \frac{\theta_+}{2} \frac{\partial}{\partial s_3} \left(\dot{\xi}(1) - \dot{\xi}(3) \right)^2 + \frac{\theta_-}{2} \frac{\partial}{\partial s_1} \left(\dot{\xi}(1) - \dot{\xi}(3) \right)^2 \right\} \quad (3.41)$$

Now we will perform integration by parts on both of the terms in (3.41). Many surface terms are generated in this process. However the non-trivial ones are all zero for smooth curves. These non-trivial surface terms are proportional to:

$$\int_{-\infty}^{\infty} ds_1 \frac{\left[\dot{\xi}(1) - \dot{\xi}(3) \right]^2}{s_{13}} \bigg|_{s_3=s_1} \quad (3.42)$$

which is zero for a smooth curve. The result of the integration by parts is then, for smooth curves:

$$\begin{aligned} & -\frac{g^4 N^2}{2^8 \pi^4} \int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{s_1} ds_3 \frac{\left[\dot{\xi}(1) - \dot{\xi}(3) \right]^2}{s_{13}^2} \left(\frac{\theta_+ - \theta_-}{2} \right) \\ & = -\frac{g^4 N^2}{2^8 \pi^4} \frac{1}{2} \oint ds_1 ds_3 \frac{\left[\dot{\xi}(1) - \dot{\xi}(3) \right]^2}{s_{13}^2} \frac{2\pi^2}{3} \end{aligned} \quad (3.43)$$

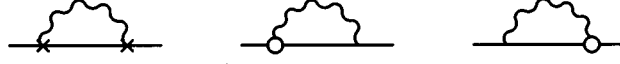
where on the left hand side we have used the fact that $s_1 > s_3$ to set $\text{sign}(s_{13}) = +1$ so that (3.39) may be used to evaluate the integral in (3.40). The resulting expression is symmetric in s_1 and s_3 and so the half domain integral can be replaced by 1/2 of the full domain integral. In conclusion, and using (2.10), we have:

$$\langle W(C) \rangle = 1 + \left\{ \frac{g^2 N}{2^4 \pi^2} - \frac{g^4 N^2}{2^7 \pi^2} \frac{1}{3} \right\} \oint ds_1 ds_3 \frac{\left[\dot{\xi}(1) - \dot{\xi}(3) \right]^2}{2s_{13}^2} \quad (3.44)$$

which is our final result.

3.2 Operator Insertion Method

As an explicit check of the operator insertion method, we will reproduce the $\mathcal{O}(g^2 N)$ term from (3.44) using this method. We showed earlier that the order ξ contribution from this method is zero, consistent with the general method calculation which starts at second order in ξ . Thus we begin here with the $\mathcal{O}(\xi^2)$ contribution. From (2.28), we know that this will involve the three diagrams:



where the open circle denotes the insertion of the second order operator (2.27), and the X denotes the insertion of (2.19) as before. We'll use the 10-D calculus for our calculations, and we'll begin with the first diagram, which we'll call Λ_1 , thus expanding the first line of (2.28) to order $g^2 N$:

$$\Lambda_1 = \frac{i^2}{N} \int ds \int ds' \theta(s-s') \delta x_\mu(s) \delta x_\nu(s') \dot{x}_\rho(s) \dot{x}_\tau(s') \left\langle \text{Tr}[F_{\mu\rho}(x(s)) F_{\nu\tau}(x(s'))] \right\rangle \quad (3.45)$$

where $\theta(s-s')$ is the Heaviside function. We should now specify what δx_μ is in the 10-D language, it turns out to be simply $(0, \xi_i, 0, \dots, 0)$, i.e. the extra 6 dimensions are just zero. Now, the commutator terms in the field strength tensors will not contribute at order $g^2 N$. This is because each field contains a single power of the coupling constant, and so the commutator terms would only contribute at $g^4 N^2$. In light of this we have:

$$\begin{aligned} \Lambda_1 = & -\frac{1}{N} \int ds \int ds' \theta(s-s') \delta x_\mu(s) \delta x_\nu(s') \dot{x}_\rho(s) \dot{x}_\tau(s') \frac{\delta^{ab}}{2} \\ & \times \left\{ \left\langle \partial_\mu A_\rho^a(x(s)) \partial_\nu A_\tau^b(x(s')) \right\rangle - \left\langle \partial_\mu A_\rho^a(x(s)) \partial_\tau A_\nu^b(x(s')) \right\rangle \right. \\ & \left. - \left\langle \partial_\rho A_\mu^a(x(s)) \partial_\nu A_\tau^b(x(s')) \right\rangle + \left\langle \partial_\rho A_\mu^a(x(s)) \partial_\tau A_\nu^b(x(s')) \right\rangle \right\} \end{aligned} \quad (3.46)$$

and therefore,

$$\begin{aligned} \Lambda_1 = & -\frac{g^2 N}{2} \int ds \int ds' \theta(s-s') \left[(\dot{x}(s) \cdot \dot{x}(s')) \left(\delta x(s) \cdot \frac{\partial}{\partial x(s)} \right) \left(\delta x(s') \cdot \frac{\partial}{\partial x(s')} \right) \right. \\ & + (\delta x(s) \cdot \delta x(s')) \left(\dot{x}(s) \cdot \frac{\partial}{\partial x(s)} \right) \left(\dot{x}(s') \cdot \frac{\partial}{\partial x(s')} \right) \\ & - (\delta x(s) \cdot \dot{x}(s')) \left(\dot{x}(s) \cdot \frac{\partial}{\partial x(s)} \right) \left(\delta x(s') \cdot \frac{\partial}{\partial x(s')} \right) \\ & \left. - (\delta x(s') \cdot \dot{x}(s)) \left(\dot{x}(s') \cdot \frac{\partial}{\partial x(s')} \right) \left(\delta x(s) \cdot \frac{\partial}{\partial x(s)} \right) \right] \Delta(x(s) - x(s')) \end{aligned} \quad (3.47)$$

Now we move back to our 4-D definitions, then apply (2.29). This gives:

$$\dot{x}(s) \cdot \dot{x}(s') \rightarrow \dot{x}(s) \cdot \dot{x}(s') - |\dot{x}(s)| |\dot{x}(s')| = 1 - 1 \cdot 1 = 0 \quad (3.48)$$

$$\delta x(s) \cdot \delta x(s') \rightarrow \xi(s) \cdot \xi(s') \quad (3.49)$$

$$\dot{x}(s) \cdot \delta x(s') = (1, 0, 0, 0, -i\theta_j) \cdot (0, \vec{\xi}, 0, \dots, 0) = 0 \quad (3.50)$$

$$\delta x(s) \cdot \frac{\partial}{\partial x(s)} \rightarrow \xi(s) \cdot \nabla_{x(s)} \quad (3.51)$$

$$\dot{x}(s) \cdot \frac{\partial}{\partial x(s)} \rightarrow \frac{\partial}{\partial x_0(s)} \quad (3.52)$$

and so we have:

$$\begin{aligned} \Lambda_1 &= -\frac{g^2 N}{2} \int ds \int ds' \theta(s-s') \xi(s) \cdot \xi(s') \frac{\partial}{\partial x_0(s)} \frac{\partial}{\partial x_0(s')} \frac{1}{4\pi^2 (x(s) - x(s'))^2} \\ &= -\frac{g^2 N}{16\pi^2} \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \frac{1}{(s-s')^2} \end{aligned} \quad (3.53)$$

Some comments are in order. Due to the symmetry of the integrand, we have replaced the half-domain integral with one half of the full domain integral. Also, integration by parts has been used twice, once on s and once on s' . Note that $\dot{\xi}(s) = \partial_s \xi(s)$.

Now we consider the two remaining diagrams involving the insertions of the second order operator. Not surprisingly these two diagrams are equal to one another in the sense that they add to give the full domain integral. For this reason we'll just look at the first one, which we'll call Λ_2 , its order $g^2 N$ contribution is:

$$\Lambda_2 = \left\langle \frac{i^2}{N} \text{Tr} \int ds \int ds' \theta(s-s') \left[A_\alpha(x(s)) \dot{x}_\alpha(s) \right] \left[\dot{x}_\mu(s') \delta x_\rho(s') \delta x_\tau(s') D_\tau F_{\rho\mu}(x(s')) \right] \right\rangle \quad (3.54)$$

As before, the various commutator terms in $D_\tau F_{\rho\mu}$ do not contribute at this order, therefore:

$$\begin{aligned} \Lambda_2 &= -\frac{1}{N} \int ds \int ds' \theta(s-s') \dot{x}_\mu(s') \dot{x}_\alpha(s) \delta x_\rho(s') \delta x_\tau(s') \\ &\quad \times \frac{\delta^{ab}}{2} \left[\left\langle A_\alpha^a(x(s)) \partial_\tau \partial_\rho A_\mu^b(x(s')) \right\rangle - \left\langle A_\alpha^a(x(s)) \partial_\tau \partial_\mu A_\rho^b(x(s')) \right\rangle \right] \end{aligned} \quad (3.55)$$

The first term is proportional to $\dot{x}(s) \cdot \dot{x}(s')$ which, as we saw in the evaluation of Λ_1 , is zero. The second term is proportional to $\dot{x}(s) \cdot \delta x(s')$, which is similarly zero. Thus,

$$\Lambda_2 = 0 \quad (3.56)$$

and therefore the last diagram is similarly zero. Thus the order $g^2 N$ correction to the straight line, as computed by the operator insertion method is:

$$-\frac{g^2 N}{16\pi^2} \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \frac{1}{(s-s')^2} \quad (3.57)$$

Now note the following,

$$\oint ds ds' \frac{\dot{\xi}^2(s)}{(s-s')^2} = \oint ds ds' \partial_{s'} \frac{\dot{\xi}^2(s)}{(s-s')} = 0 \quad (3.58)$$

because of the total derivative in s' . This means that:

$$-\frac{g^2 N}{16\pi^2} \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \frac{1}{(s-s')^2} = \frac{g^2 N}{16\pi^2} \oint ds ds' \frac{[\dot{\xi}(s) - \dot{\xi}(s')]^2}{2(s-s')^2} \quad (3.59)$$

which is in complete agreement with (3.44).

3.3 Strong Coupling Calculation

According to the AdS/CFT correspondence, the Wilson line expectation value is related to the regularized area of the minimal surface in AdS_5 whose boundary is the Wilson line. Representing the metric of AdS_5 as:

$$ds^2 = \frac{1}{y^2} (dx_\mu^2 + dy^2) \quad (3.60)$$

the boundary is at $y = 0$, which is conformally equivalent to 4-D Euclidean space, here the index μ runs from 0, ..., 3. The specific relation is:

$$\langle W(C) \rangle = e^{-\frac{\sqrt{\lambda}}{2\pi} \text{Area}} \quad (3.61)$$

where $\lambda = g^2 N$, and the area is regularized as per the discussion in section 1.5.3. We will calculate this minimal area now for the wavy line, following the work in [26]. The area of a two dimensional surface is given by:

$$S = \int d\tau d\sigma \sqrt{|\det h_{ab}|} \quad (3.62)$$

where,

$$h_{ab} = \partial_a X^\mu g_{\mu\nu} \partial_b X^\nu \quad (3.63)$$

is the induced metric on the 2-D surface embedded in the space given by $g_{\mu\nu}$, while $a, b = 0, 1$ corresponding to the surface coordinates τ and σ respectively. The embedding functions for the surface are given by the functions $X^\mu(\tau, \sigma)$. For the case at hand,

$$\partial_a X^\mu g_{\mu\nu} \partial_b X^\nu = (\partial_a x^\mu \partial_b x_\mu + \partial_a y \partial_b y) \frac{1}{y^2} \quad (3.64)$$

where on the right hand side the index μ runs from 0, ..., 3. Taking the determinant on the a, b indices, we have,

$$S = \int \frac{d\tau d\sigma}{y^2} \sqrt{(x'^2 + y'^2)(\dot{x}^2 + \dot{y}^2) - (x' \cdot \dot{x} + y' \dot{y})^2} \quad (3.65)$$

where z' denotes $\partial_\sigma z$ and \dot{z} denotes $\partial_\tau z$. Since the area is independent of the coordinate system chosen on the surface, we can choose the following description:

$$y(\tau, \sigma) = \tau \quad x_0(\tau, \sigma) = \sigma \quad (3.66)$$

This simplifies the expression for the area greatly,

$$S = \int \frac{d\tau d\sigma}{\tau^2} \sqrt{(1 + \Delta'^2)(1 + \dot{\Delta}^2) - (\Delta' \cdot \dot{\Delta})^2} \quad (3.67)$$

where,

$$\Delta_i(\tau, \sigma) = x_i(\tau, \sigma) \quad (3.68)$$

We mentioned earlier that the boundary ($\tau = 0$) of our surface is given by the Wilson loop. Thus, we have,

$$x_\mu(0, \sigma) = (\sigma, \Delta_i(0, \sigma)) = (\sigma, \xi_i(\sigma)) \quad (3.69)$$

where $\xi_i(\sigma)$ is our familiar waviness function, which is taken to be small in the sense given in the discussion at the start of chapter 2. Now, in order to regularize the area as per (1.49), the result when $\Delta_i = 0$ should be subtracted from the above. This is because the waviness contributes a very small amount to the already infinite curve length L , and because a straight line has zero regularized area, which means the divergent term is simply the straight line area itself. From now on S will denote this regularized area. Thus, expanding S to second order in Δ , we have:

$$S = \int \frac{d\tau d\sigma}{\tau^2} \left\{ \frac{1}{2} \dot{\Delta}^2 + \frac{1}{2} \Delta'^2 \right\} \quad (3.70)$$

Our purpose now is to minimize this area, thus we determine the equation of motion for $\Delta(\tau, \sigma)$, using the standard variational methods.

$$0 = \partial_a \left(\frac{\delta \mathcal{L}}{\delta (\partial_a \Delta)} \right) = \partial_0 \left(\frac{1}{\tau^2} \dot{\Delta} \right) + \partial_1 \left(\frac{1}{\tau^2} \Delta' \right) \quad (3.71)$$

Therefore,

$$\tau^2 \partial_\tau \left(\tau^{-2} \dot{\Delta} \right) + \Delta'' = 0 \quad (3.72)$$

or,

$$\Delta'' = \frac{2}{\tau} \dot{\Delta} - \ddot{\Delta} \quad (3.73)$$

Now we integrate (3.70) by parts. The first term is integrated by parts on τ :

$$S_1 = \frac{1}{2} \int d\sigma \left\{ \frac{1}{\tau^2} \dot{\Delta} \cdot \Delta \Big|_\epsilon^\infty - \int_\epsilon^\infty d\tau \Delta \cdot \left(-\frac{2}{\tau^3} \dot{\Delta} + \frac{1}{\tau^2} \ddot{\Delta} \right) \right\} \quad (3.74)$$

The second term is integrated by parts on σ :

$$S_2 = \frac{1}{2} \int_{\epsilon}^{\infty} d\tau \left\{ \frac{1}{\tau^2} \Delta' \cdot \Delta \Big|_{\sigma=-\infty}^{\infty} - \int d\sigma \Delta \cdot \Delta'' \right\} \quad (3.75)$$

Now, we use the equation of motion (3.73) on the second term in S_2 , which then cancels against the second term of S_1 . Also, the first term of S_2 vanishes since Δ is well behaved. Thus,

$$S = -\frac{1}{2\tau^2} \int d\sigma \dot{\Delta} \cdot \Delta \Big|_{\tau=\epsilon} \quad (3.76)$$

In order to continue with our calculation, we'll need to solve the equation of motion for $\Delta(\tau, \sigma)$, with the boundary condition $\Delta(0, \sigma) = \xi(\sigma)$. We begin by putting the equation of motion in momentum space (conjugate to σ):

$$\ddot{\Delta} - \frac{2}{\tau} \dot{\Delta} - p^2 \Delta = 0 \quad (3.77)$$

This is now a 2nd order ODE in τ alone. The boundary condition is now $\Delta(\tau=0, p) = \xi(p)$, where functions with the argument p are the fourier transforms of their σ counterparts. The solution of the ODE, with the desired boundary condition is:

$$\Delta(\tau, p) = e^{-|p|\tau} (1 + |p|\tau) \xi(p) \quad (3.78)$$

Placing this expression back in coordinate space, we have:

$$\Delta(t, s) = \int \frac{ds' \xi(s')}{\pi} \frac{2t^3}{((s-s')^2 + t^2)^2} \quad (3.79)$$

where we have changed notation from (τ, σ) to (t, s) . Now note that:

$$\lim_{a \rightarrow 0} \frac{a^3}{(x^2 + a^2)^2} = \frac{\pi}{2} \delta(x) \quad (3.80)$$

and so we can see that $\Delta(0, s) = \xi(s)$. We will also need to calculate:

$$\dot{\Delta}(t, s) = \int \frac{ds' \xi(s')}{\pi} \left\{ \frac{6t^2}{[(s-s')^2 + t^2]^2} - \frac{8t^4}{[(s-s')^2 + t^2]^3} \right\} \quad (3.81)$$

and therefore,

$$\lim_{t \rightarrow 0} \frac{\dot{\Delta}(t, s)}{t^2} = \int \frac{ds' \xi(s')}{\pi} \frac{6}{(s-s')^4} \quad (3.82)$$

Placing everything into (3.76), we arrive at the final result:

$$S = \frac{1}{2\pi} \int ds' ds \frac{\xi'(s) \cdot \xi'(s')}{(s-s')^2} = -\frac{1}{2\pi} \int ds' ds \frac{[\xi'(s) - \xi'(s')]^2}{2(s-s')^2} \quad (3.83)$$

where integration by parts has been used twice, once on s and once on s' , and as before $\xi'(s) = \partial_s \xi(s)$. The second equality follows from (3.58).

3.4 Discussion

Comparison of (3.44) and (3.83) shows the same dependence on ξ . Following (3.61), we have:

$$\ln \left(1 + \left[\frac{g^2 N}{2^4 \pi^2} - \frac{g^4 N^2}{2^7 \pi^2} \frac{1}{3} + \dots \right] I \right) = \frac{\sqrt{g^2 N}}{4\pi^2} I \quad (3.84)$$

where,

$$I = \oint ds ds' \frac{[\dot{\xi}(s) - \dot{\xi}(s')]^2}{2(s - s')^2} \quad (3.85)$$

The conjecture therefore is that if the CFT result at $\mathcal{O}(\xi^2)$ were known to all orders in $\lambda = g^2 N$, then the logarithm could be resummed, and the large λ limit could be taken to give the string theory result. For the straight line, recall that the result is $\langle W \rangle = 1$ and is protected in the sense that the straight line is a BPS object. So here we see how perturbing the line unbalances whatever cancellations conspire to give $\langle W \rangle = 1$. In particular we see that the dependence on the waviness seems independent of the coupling strength. Results in the AdS/CFT correspondence which are independent of λ have been discovered before, c.f. [24].

Chapter 4

Connected Correlator of Two Wavy Lines

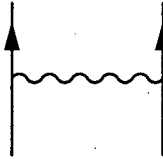
4.1 General Method

We wish to calculate the correlator between two Wilson loops which are stretched out in such a way as to resemble two (almost) straight lines. We will only be interested in the connected portion of the correlator, that is, only interactions between the Wilson lines will be considered. In other words the Wilson line's self-energies will be ignored. We will perform our calculation out to order $g^6 N^3$, and to $\mathcal{O}(\xi^2)$. Let us define our two wavy lines as follows:

$$C_1 : \quad x^\mu(s) = (s, \vec{\xi}(s)) \quad (4.1)$$

$$C_2 : \quad x'^\mu(s') = (s', \vec{\xi}'(s') + \vec{L}) \quad (4.2)$$

The prime does not denote a derivative here, it signifies that the quantity is associated with the second Wilson line. From now on derivatives will be represented by dots. We will explicate the calculations with Feynman diagrams, such as the following:



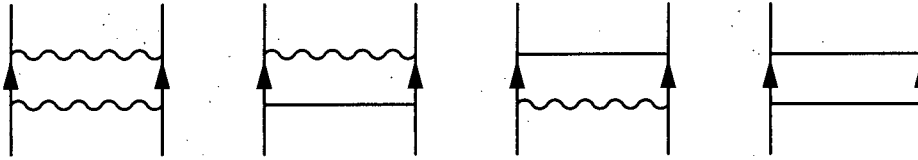
The arrowed line on the left is $x_\mu(s)$, that on the right is $x'_\mu(s')$. This diagram shows the exchange of a single gauge field between the lines. Note the arrows are in the same direction: we will be calculating the correlation of like-oriented Wilson lines. Because the gauge group generators are traceless, any diagram with a single line emanating from one of the Wilson lines will vanish. Thus our calculation starts with each Wilson line having two fields emanating from it.

We are using the large N limit of SYM theory, and so we should include only planar diagrams. A planar diagram has no lines (gauge fields and scalars) which cross one another in travelling from one Wilson line to the other. However, two lines which travel from the Wilson lines to an internal vertex may cross.

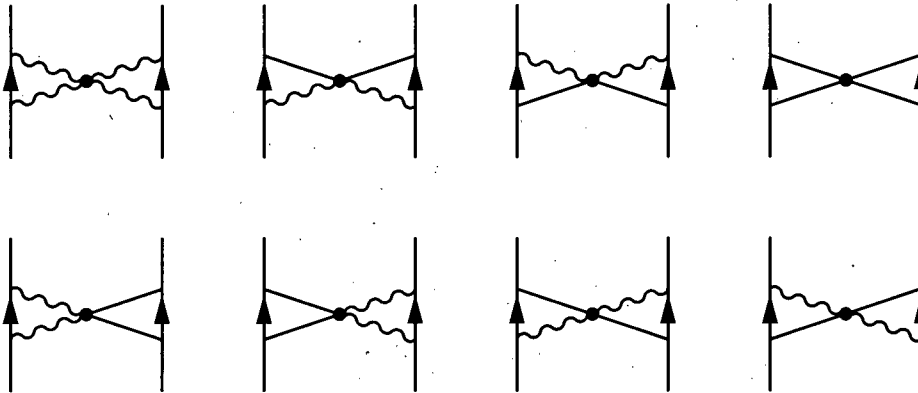
The expansion of (A.7) is straight forward:

$$\begin{aligned}
W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \Bigg\{ & 1 + g \oint_C ds \left(iA_\mu(x) \dot{x}_\mu + \Phi_i(x) |\dot{x}| \theta_i \right) \\
& + \frac{g^2}{2!} \oint ds_1 ds_2 \left(iA_\mu(y) \dot{y}_\mu + \Phi_i(y) |\dot{y}| \theta_i \right) \left(iA_\nu(z) \dot{z}_\nu + \Phi_j(z) |\dot{z}| \theta_j \right) + \dots \Bigg\},
\end{aligned}
\tag{4.3}$$

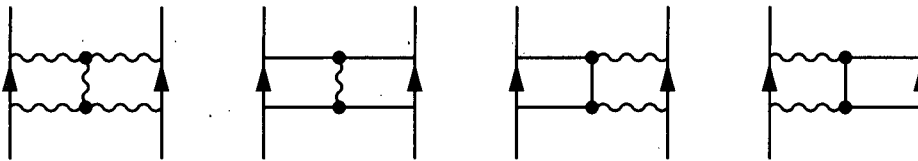
we can use it to enumerate all the diagrams which may contribute up to $\mathcal{O}(g^6 N^3)$. Note that the very first term of any loop-loop correlator is just $\text{Tr}(\mathbf{1})^2/N^2 = 1$. But we're after the next non-zero term. We already know that the "one-rung" ladder diagrams are zero. At the next order, $\mathcal{O}(g^4 N^2)$, we have:

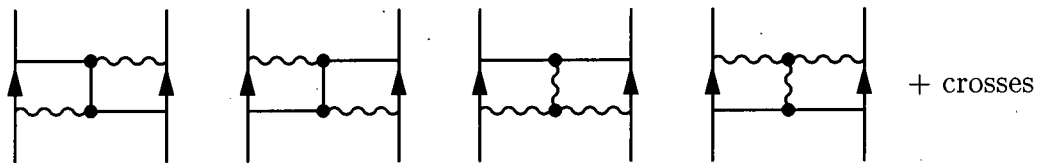


We will call these the "two rung ladders". The solid lines represent the exchange of the scalar fields Φ . At the next order, that is $\mathcal{O}(g^6 N^3)$ there are many more contributions:

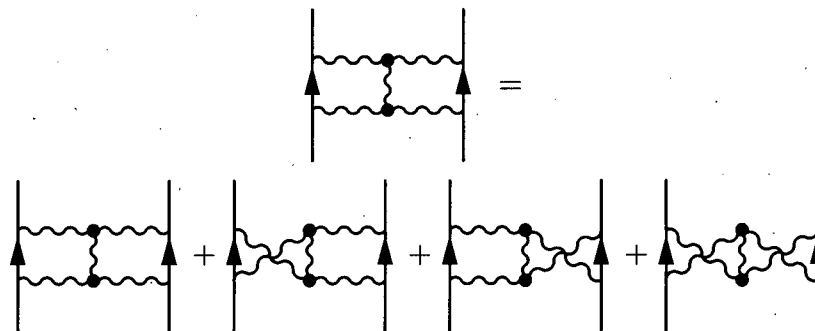


We'll call these the "X-diagrams". Note that they have an internal vertex (the four-point vertex) represented by a dot. After this we also have the following diagrams:

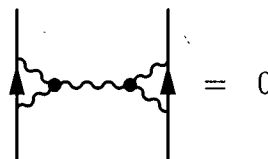




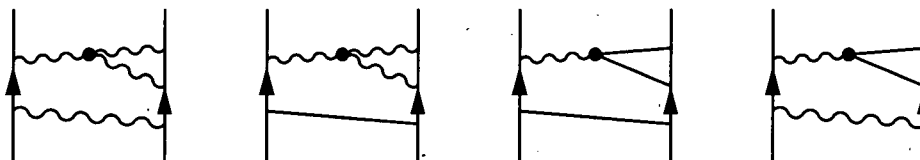
We'll call these the "H-diagrams," they contain two internal 3-point vertices. Each H-diagram actually contains four diagrams. These are itself plus its "crosses", for example:

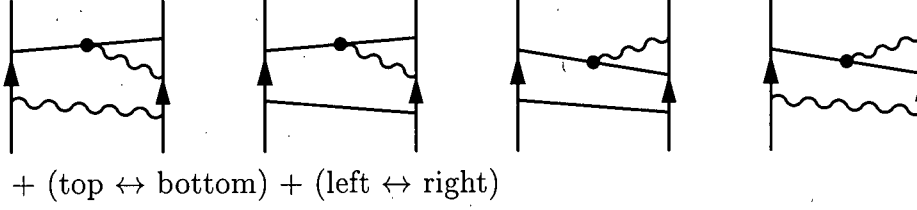


The effect of the crosses is simply to multiply the result of the main diagram by four. This is because the symmetry of the trace of two fields makes each cross diagram equal to one another and their parent. Note that the following type of diagram:

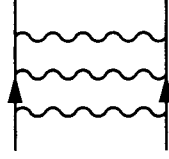


this is true for any exchange of gauge for scalar fields. The reason is that the trace of two fields, on say, the left line is symmetric, but the three-point vertex is completely antisymmetric, as it depends on the structure constants of the gauge group. The next class of diagram is:

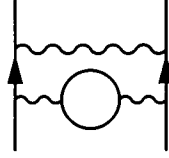




We'll call these the "IY Diagrams." The interchanges (" \leftrightarrow ") denote that there are also 16 additional IY diagrams which are reflections of the diagrams shown about vertical (for "left \leftrightarrow right") and horizontal (for "top \leftrightarrow bottom") axes. There are also the "3-rung ladders", these are the diagram below plus all interchanges of scalars for gauge fields:



Finally, the last class of diagrams at $\mathcal{O}(g^6 N^3)$ are the 1-loop corrected 2-rung ladders. These are graphs like:



plus all combinations where one of the two rungs is 1-loop corrected. The 1-loop corrected propagators for the scalar and gauge fields are computed in appendix B.

We will use our efficient 10-D calculus for all the calculations, and so only the "all gauge field diagrams" will be computed, as before.

4.1.1 The 2-Rung Ladder Diagrams

From (4.3) one can deduce that:

$$\begin{aligned}
 \begin{array}{c} y \\ \uparrow \\ z \end{array} & \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} y' \\ \uparrow \\ z' \end{array} = \left\langle \frac{g^4}{N^2} \frac{1}{2!} \text{Tr } \mathcal{P} \left[\oint ds_1 ds_2 A_\mu(y) A_\nu(z) \dot{y}_\mu \dot{z}_\nu \right] \right. \\
 & \left. \times \frac{1}{2!} \text{Tr } \mathcal{P} \left[\oint ds'_1 ds'_2 A_\sigma(y') A_\tau(z') \dot{y}'_\sigma \dot{z}'_\tau \right] \right\rangle
 \end{aligned}$$

Note that the path ordering implies that we should consider the two orderings of the $s_1 s_2$ (and $s'_1 s'_2$) integration domain, i.e. one where $s_1 > s_2$ and one where $s_2 > s_1$. However, due

to the symmetry of the trace (of two fields), these contributions are equal, and add to give the whole domain. The next step is to perform the Wick contractions:

$$= \frac{g^4}{4N^2} \oint ds_1 ds_2 ds'_1 ds'_2 \left\{ \langle A_\mu^a(y(s_1)) A_\sigma^c(y'(s'_1)) \rangle \frac{\delta^{ab}}{2} \langle A_\nu^b(z(s_2)) A_\tau^d(z'(s'_2)) \rangle \frac{\delta^{cd}}{2} \dot{y}_\mu \dot{z}_\nu \dot{y}'_\sigma \dot{z}'_\tau \right\} \quad (4.4)$$

$$= \frac{g^4}{256\pi^4 N^2} \oint ds_1 ds_2 ds'_1 ds'_2 \left\{ \dot{y}_\mu \dot{z}_\nu \dot{y}'_\sigma \dot{z}'_\tau \delta^{ab} \delta^{cd} \delta^{ac} \delta^{bd} \frac{\delta_{\mu\sigma}}{(y - y')^2} \frac{\delta_{\nu\tau}}{(z - z')^2} \right\} \quad (4.5)$$

$$= \frac{g^4}{256\pi^4 N^2} (N^2 - 1) \oint ds_1 ds_2 ds'_1 ds'_2 \frac{(\dot{y} \cdot \dot{y}') (\dot{z} \cdot \dot{z}')}{(y - y')^2 (z - z')^2} \quad (4.6)$$

In the large N limit, we can ignore the -1 in $N^2 - 1$, which gives the final result:

$$= \frac{1}{N^2} \frac{g^4 N^2}{256\pi^4} \oint ds_1 ds_2 ds'_1 ds'_2 \frac{(\dot{x}(s_1) \cdot \dot{x}'(s'_1)) (\dot{x}(s_2) \cdot \dot{x}'(s'_2))}{|x(s_1) - x'(s'_1)|^2 |x(s_2) - x'(s'_2)|^2} \quad (4.7)$$

The reason we multiply and divide by N^2 is because the loop-loop correlator is naturally suppressed by a factor of N^2 , however the perturbative expansion is in powers of $g^2 N$, and so we keep this notation to make things explicit. Now we are in a position to switch back to our 4-D space and insert our wavy lines from (4.2). From our calculus, we have:

$$\dot{x}(s) \cdot \dot{x}'(s') \rightarrow \dot{x}(s) \cdot \dot{x}'(s') - |\dot{x}(s)| |\dot{x}'(s')| \quad (4.8)$$

Now insert the wavy lines:

$$|\dot{x}(s)| = \sqrt{1 + \dot{\xi}(s) \cdot \dot{\xi}(s)} \quad (4.9)$$

$$|x(s) - x'(s')|^2 = (s - s')^2 + [\xi(s) - \xi'(s') - L]^2 \quad (4.10)$$

$$\dot{x}(s) \cdot \dot{x}'(s') = 1 + \dot{\xi}(s) \cdot \dot{\xi}'(s') \quad (4.11)$$

Note that ξ and L are spatial vectors. We expand the quantity in our result to 2^{nd} order in waviness:

$$\frac{\dot{x}(s) \cdot \dot{x}'(s') - |\dot{x}(s)| |\dot{x}'(s')|}{|x(s) - x'(s')|^2} \simeq -\frac{1}{2} [\dot{\xi}(s) - \dot{\xi}'(s')]^2 \frac{1}{(s - s')^2 + L^2} \quad (4.12)$$

By looking at (4.7) we see immediately that our result begins only at 4^{th} order in waviness. Because of (B.1, B.2) the 1-loop corrected 2-rung ladders are also zero at 2^{nd} order in waviness.

4.1.2 The 3-Rung Ladder Diagrams

After having seen that the 2-rung ladders vanish at 2^{nd} order in waviness, it is not surprising to find that the 3-rung ladders do as well. Rather than writing out the full expression for the 3-rung ladders, we can make a general statement about the n -rung ladders, which is that

they are proportional to $(\dot{x}(s_i) \cdot \dot{x}'(s'_i))^n$. Unless $n = 1$, in which case we get zero due to the tracelessness of the gauge group matrices, this expression is certainly zero to 2^{nd} order in waviness.

4.1.3 The X Diagrams

The next class of diagrams are the X-diagrams. These graphs have a single internal vertex. The vertex factor can be gleaned from the action (A.1). Again we use our calculus in order to streamline the calculation. The X diagram is:

$$\begin{aligned}
 \text{Diagram} &= \left\langle \frac{g^4}{N^2} \frac{1}{2!} \text{Tr } \mathcal{P} \left[\oint ds_1 ds_2 A_\mu(y) A_\nu(z) \dot{y}_\mu \dot{z}_\nu \right] \right. \\
 &\quad \times \frac{1}{2!} \text{Tr } \mathcal{P} \left[\oint ds'_1 ds'_2 A_\sigma(y') A_\tau(z') \dot{y}'_\sigma \dot{z}'_\tau \right] \\
 &\quad \left. \times \frac{-g^2}{4} \int d^2\omega w f^{abc} f^{ade} A_\rho^b(w) A_\zeta^c(w) A_\rho^d(w) A_\zeta^e(w) \right\rangle
 \end{aligned}$$

Evaluating the traces, we have:

$$\begin{aligned}
 &\frac{-g^6}{64N^2} \oint ds_1 \dots ds'_2 \int d^2\omega w \left\langle A_\mu^r(y) A_\nu^r(z) A_\sigma^t(y') A_\tau^t(z') A_\rho^b(w) A_\zeta^c(w) A_\rho^d(w) A_\zeta^e(w) \right. \\
 &\quad \left. \times f^{abc} f^{ade} \dot{y}_\mu \dot{z}_\nu \dot{y}'_\sigma \dot{z}'_\tau \right\rangle
 \end{aligned} \tag{4.13}$$

naively, we have $4!$ ways of contracting the fields here. However, if $b = c$ or $d = e$ we will get zero due to the antisymmetry of the structure constants f^{abc} . This reduces the number of contractions to 16. Let's take a closer look at (4.13). Because the Lorentz indices in the vertex fields are contracted in pairs, it means we will have only three possible factors emerging from the z, z', y, y' . These are $(\dot{y} \cdot \dot{z})(\dot{y}' \cdot \dot{z}')$, $(\dot{y} \cdot \dot{y}')(\dot{z} \cdot \dot{z}')$, and $(\dot{y} \cdot \dot{z}')(\dot{y}' \cdot \dot{z})$. The first combination accompanies the factor $f^{abc} f^{abc}$, while the last two accompany the factor $f^{abc} f^{acb}$ and thus pick-up a minus sign. There are four ways of Wick contracting which yield combinations two and three, while since the first combination is symmetric under $(y \leftrightarrow z, y' \leftrightarrow z')$ it receives double the number, eight. At the end of the day we have:

$$\begin{aligned}
 &\frac{-g^6 N^3}{64N^2} \oint ds_1 ds'_1 ds_2 ds'_2 \int d^2\omega w \frac{\Gamma^4(\omega - 1)}{2^8 \pi^{4\omega}} \\
 &\quad \times \left[\frac{1}{(w - y)^2} \frac{1}{(w - z)^2} \frac{1}{(w - y')^2} \frac{1}{(w - z')^2} \right]^{\omega-1} \\
 &\quad \times \{ 8(\dot{y} \cdot \dot{z})(\dot{y}' \cdot \dot{z}') - 4(\dot{y} \cdot \dot{y}')(\dot{z} \cdot \dot{z}') - 4(\dot{y} \cdot \dot{z}')(\dot{y}' \cdot \dot{z}) \}
 \end{aligned} \tag{4.14}$$

where we have used $f^{abc}f^{abc} = N(N^2 - 1) \simeq N^3$. Referring back to (4.8), and noting the expansion:

$$\dot{x}(s) \cdot \dot{x}'(s') - |\dot{x}(s)||\dot{x}'(s')| \simeq -\frac{1}{2} [\dot{\xi}(s) - \dot{\xi}'(s')]^2 \quad (4.15)$$

we can see immediately that the X-diagrams begin only at 4th order in waviness.

4.1.4 The IY Diagrams

Now we consider the other class of diagrams containing a single internal vertex, the IY diagrams. Here, we have expanded the Wilson loop out to three terms on one side, and our usual two on the other:

$$\begin{aligned} \text{Diagram} &= \left\langle \frac{g^5}{N^2} \frac{-1}{2!} \text{Tr } \mathcal{P} \left[\oint ds_1 ds_2 A_\mu(y) A_\nu(z) \dot{y}_\mu \dot{z}_\nu \right] \right. \\ &\quad \times \frac{-i}{3!} \text{Tr } \mathcal{P} \left[\oint ds'_1 ds'_2 ds'_3 A_\gamma(x') A_\sigma(y') A_\tau(z') \dot{x}'_\gamma \dot{y}'_\sigma \dot{z}'_\tau \right] \\ &\quad \left. \times -g \int d^2w w f^{abc} \partial_\rho A_\lambda^a(w) A_\rho^b(w) A_\lambda^c(w) \right\rangle \end{aligned}$$

After contracting z and z' there remain 3! ways of contracting the cubic vertex. Our previous comments concerning the path ordering conspiring to give the full domain integral still apply to the left hand side of this diagram. On the right hand side however, we do not have the same symmetry since we have a trace of 3 fields. In fact we will end up with the structure: $f^{abc} \text{Tr} [T^a T^b T^c]$ where the gauge group matrices correspond to the three fields at x' , y' , and z' respectively. This implies that we should introduce the path ordering symbol, $\epsilon(s'_1 s'_2 s'_3)$, as we did when evaluating the Σ_3 diagram for the single wavy line, see section 3.1. Recall that it is equal to 1 for $s'_1 > s'_2 > s'_3$ and antisymmetric under any transposition of s'_i . We'll work out one contraction explicitly, then the others will be obvious. We contract $A_\mu(y) \rightarrow A_\lambda^c(w)$, $A_\gamma(x') \rightarrow A_\lambda^a(w)$, and $A_\sigma(y') \rightarrow A_\rho^b(w)$:

$$\begin{aligned} &\frac{ig^6}{N^2} \frac{1}{4 \cdot 3!} \oint ds_1 ds_2 \oint ds'_1 ds'_2 ds'_3 \epsilon(s'_1 s'_2 s'_3) \dot{x}'_\gamma \dot{y}'_\sigma \dot{z}'_\tau \dot{y}_\mu \dot{z}_\nu \\ &\quad \times \delta^{vc} \delta^{sa} \delta^{tb} \delta_{\nu\tau} \delta_{\mu\lambda} \delta_{\gamma\lambda} \delta_{\sigma\rho} \frac{\partial}{\partial x'^\rho} f^{abc} \text{Tr} (T^s T^t T^v) \\ &\quad \times \int d^2w w \Delta(w - x') \Delta(w - y') \Delta(w - y) \Delta(z - z') \end{aligned} \quad (4.16)$$

We have used integration by parts to exchange the derivative by w for a derivative by x' , this generates a minus sign. The Δ symbol is shorthand for the propagator, and is defined in

(A.5), now we use the equation just below (3.10) to evaluate $f^{abc} \text{Tr} [T^a T^b T^c]$. Then (4.16) yields:

$$\begin{aligned}
 & -\frac{1}{N^2} \frac{g^6 N^3}{16 \cdot 3!} \oint ds_1 ds_2 \oint ds'_1 ds'_2 ds'_3 \epsilon(s'_1 s'_2 s'_3) \\
 & \quad \times (\dot{z} \cdot \dot{z}') (\dot{x}' \cdot \dot{y}) (\dot{y}' \cdot \partial_{x'}) \\
 & \quad \times \int d^{2\omega} w \Delta(w - x') \Delta(w - y') \Delta(w - y) \Delta(z - z')
 \end{aligned} \tag{4.17}$$

The other contractions proceed in the same manner, the final result being:

$$\begin{aligned}
 & -\frac{1}{N^2} \frac{g^6 N^3}{16 \cdot 3!} \oint ds_1 ds_2 \oint ds'_1 ds'_2 ds'_3 \epsilon(s'_1 s'_2 s'_3) \\
 & \quad \times (\dot{z} \cdot \dot{z}') \left\{ (\dot{x}' \cdot \dot{y}) \dot{y}' \cdot (\partial_{x'} - \partial_y) + (\dot{x}' \cdot \dot{y}') \dot{y} \cdot (\partial_{y'} - \partial_{x'}) \right. \\
 & \quad \left. + (\dot{y}' \cdot \dot{y}) \dot{x}' \cdot (\partial_y - \partial_{y'}) \right\} \times \int d^{2\omega} w \Delta(w - x') \Delta(w - y') \Delta(w - y) \Delta(z - z')
 \end{aligned} \tag{4.18}$$

The “top \leftrightarrow bottom” graphs mentioned in section 4.1 are covered by the path ordering, while the “left \leftrightarrow right” still need to be considered. However, we see now that all the IY diagrams are proportional to terms like $(\dot{z} \cdot \dot{z}')(\dot{x}' \cdot \dot{y})$, which are zero at $\mathcal{O}(\xi^2)$.

4.1.5 The H Diagrams

Every other class of diagrams has turned out to be zero at $\mathcal{O}(\xi^2)$. The H diagram however, will not be:

$$\begin{aligned}
 & \text{Diagram: } \left(\begin{array}{c} y \\ \uparrow \\ z \end{array} \right) \left(\begin{array}{c} w \\ \bullet \\ x \end{array} \right) \left(\begin{array}{c} y' \\ \uparrow \\ z' \end{array} \right) = \left\langle \frac{g^4}{N^2} \frac{1}{2!} \text{Tr } \mathcal{P} \left[\oint ds_1 ds_2 A_\mu(y) A_\nu(z) \dot{y}_\mu \dot{z}_\nu \right] \right. \\
 & \quad \times \frac{1}{2!} \text{Tr } \mathcal{P} \left[\oint ds'_1 ds'_2 A_\rho(y') A_\lambda(z') \dot{y}'_\rho \dot{z}'_\lambda \right] \\
 & \quad \times \frac{g^2}{2!} \int d^{2\omega} w \int d^{2\omega} x f^{abc} \partial_\sigma A_\tau^a(w) A_\sigma^b(w) A_\tau^c(w) \\
 & \quad \left. \times f^{def} \partial_\alpha A_\beta^d(x) A_\alpha^e(x) A_\beta^f(x) \right\rangle \times 4
 \end{aligned}$$

The factor of 4 at the end accounts for the the cross diagrams, as explained in section 4.1. There are 9 ways of choosing the internal propagator contraction, and then 4 ways of choosing the contractions of the Wilson lines with the remaining vertex fields. This makes

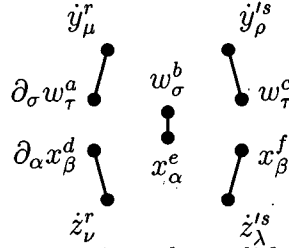
36 diagrams. However these 36 diagrams are closely related, and there's a relatively simple way of calculating everything. First we evaluate the traces in the above expression, and note that the path ordering just gives us the full integration domains, as for the X diagrams.

$$\left\langle \frac{g^6}{N^2} \frac{1}{8} \oint ds_1 ds_2 ds'_1 ds'_2 \int d^2 w w \int d^2 x \dot{y}_\mu \dot{z}_\nu \dot{y}'_\rho \dot{z}'_\lambda \right. \\ \left. \times f^{abc} f^{def} A_\mu^r(y) A_\nu^r(z) A_\rho^s(y') A_\lambda^s(z') \partial_\sigma A_\tau^a(w) A_\sigma^b(w) A_\tau^c(w) \partial_\alpha A_\beta^d(x) A_\alpha^e(x) A_\beta^f(x) \right\rangle \quad (4.19)$$

Now note that the final outcome of this calculation will look like this:

$$\frac{1}{N^2} \frac{g^6 N^3}{8} \oint ds_1 ds_2 ds'_1 ds'_2 \left\{ I \right\} \\ \times \int d^2 w w \int d^2 x \Delta(x-w) \Delta(y-w) \Delta(y'-w) \Delta(z-x) \Delta(z'-x) \quad (4.20)$$

where the N^3 has come from the $f^{abc} f^{abc}$ which occurs in each term. The I is what we'll have to calculate. We'll invent some notation $\dot{y}_\alpha^r \equiv A_\alpha^r(y) \dot{y}_\alpha$, and a diagram style:



Here we are representing Wick contractions by solid lines between the terms. Using the space-time and gauge index delta functions which arise from the contractions, it is easy to read off the diagram's contribution to I , it is:

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} (\dot{y} \cdot \dot{y}') (\dot{z} \cdot \dot{z}') \left(\frac{\partial}{\partial y} \cdot \frac{\partial}{\partial z} \right) \quad (4.21)$$

The "matrix" on the left is really just the labels of the two structure constants which are being contracted together. For example, the above "matrix" denotes $\delta^{ad} \delta^{be} \delta^{cf} f^{abc} f^{def}$. We're just interested in the sign here, since the N^3 portion is absorbed in our prefactors in (4.20). The order of the top row will always be abc , if the order of the bottom row is an odd number of exchanges away from def then we get a minus sign, otherwise plus. Note here that we have used the following property of the derivatives of the propagators:

$$\partial_w \Delta(w-y) = -\partial_y \Delta(w-y), \quad (4.22)$$

twice. Now that we have established a language of diagrams and notation, we can consider all the contributions to I . We will use dashed lines in our diagrams to denote other contractions which give contributions which are related to the solid-line contractions by exchanges

of Wilson line coordinates. We begin with our first diagram:

$$= \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} (\dot{y} \cdot \dot{y}') (\dot{z} \cdot \dot{z}') \left(\frac{\partial}{\partial y} \cdot \frac{\partial}{\partial z} \right) \\ - (z \leftrightarrow z') - (y \leftrightarrow y') \\ + (z \leftrightarrow z', y \leftrightarrow y')$$

The exchanges in brackets refer to the dashed lines in the diagram. They are the contributions from the three other ways (other than the solid lines) of contracting the Wilson lines to the internal vertices. For instance, when we use the dashed lines on the top of the diagram instead of the solid ones, we exchange y for y' . Similarly, using the dashed lines on the bottom, z and z' are exchanged. Lastly we can use the dashed lines on the bottom and the top, which of course combines the two previous exchanges. Because the exchange of y with y' or z with z' exchanges the gauge group indices, a sign is picked-up relative to the solid line contribution. Let's move on to the next diagram,

$$= \begin{pmatrix} a & b & c \\ d & f & e \end{pmatrix} (\dot{z} \cdot \dot{z}') \left(\dot{y} \cdot \frac{\partial}{\partial z} \right) \left(\dot{y}' \cdot \frac{\partial}{\partial y} \right) \\ - (z \leftrightarrow z') - (y \leftrightarrow y') \\ + (z \leftrightarrow z', y \leftrightarrow y') \\ \oplus (y \leftrightarrow z, y' \leftrightarrow z')$$

here we've added another bit of diagram notation, the dotted line. The dotted line represents a different choice of internal vertex contraction ($w \rightarrow x$) as compared to the solid line internal vertex contraction. The meaning is to consider the whole diagram again, this time with a new internal vertex contraction. Since the dotted line is "mirrored" with respect to the original internal vertex contraction, the effect is to simply exchange y and z , and y' and z' . In pictures, this is:

$$= \text{Diagram 1} + \text{Diagram 2}$$

So we have the four contributions from the original solid and dashed lines, plus four more related to the previous four by $(y \leftrightarrow z, y' \leftrightarrow z')$. This will be the last addition to our diagram notation. The next diagram is:

$$= \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} (\dot{z} \cdot \dot{y}) \left(\dot{z}' \cdot \frac{\partial}{\partial z} \right) \left(\dot{y}' \cdot \frac{\partial}{\partial y} \right) \\ - (z \leftrightarrow z') - (y \leftrightarrow y') \\ + (z \leftrightarrow z', y \leftrightarrow y')$$

There are three more diagrams to consider. They have derivatives acting on the internal vertex propagator $\Delta(w - x)$. The first one is:

$$= - \begin{pmatrix} a & b & c \\ e & d & f \end{pmatrix} (\dot{y}' \cdot \dot{y}) \left(\dot{z}' \cdot \frac{\partial}{\partial y} \right) \left(\dot{z} \cdot \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'} \right) \right) \\ - (z \leftrightarrow z') - (y \leftrightarrow y') \\ + (z \leftrightarrow z', y \leftrightarrow y') \\ \oplus (y \leftrightarrow z, y' \leftrightarrow z')$$

The minus sign in front of the “matrix” comes from the fact that (4.22) was used once. We have also expressed the internal propagator derivative in a nicer form by using integration by parts:

$$\begin{aligned} & \partial_x \Delta(x - w) \Delta(y - w) \Delta(y' - w) \Delta(z - x) \Delta(z' - x) \\ &= -\Delta(x - w) \Delta(y - w) \Delta(y' - w) \partial_x \Delta(z - x) \Delta(z' - x) \\ & \quad - \Delta(x - w) \Delta(y - w) \Delta(y' - w) \Delta(z - x) \partial_x \Delta(z' - x) \\ &= (\partial_z + \partial_{z'}) \Delta(x - w) \Delta(y - w) \Delta(y' - w) \Delta(z - x) \Delta(z' - x) \end{aligned} \quad (4.23)$$

The second diagram with derivatives acting on the internal vertex propagator is the following:

$$= - \begin{pmatrix} a & b & c \\ e & f & d \end{pmatrix} (\dot{y} \cdot \dot{z}') \left(\dot{y}' \cdot \frac{\partial}{\partial y} \right) \left(\dot{z} \cdot \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'} \right) \right)$$

$$\begin{aligned}
& - (z \leftrightarrow z') - (y \leftrightarrow y') \\
& + (z \leftrightarrow z', y \leftrightarrow y') \\
& \oplus (y \leftrightarrow z, y' \leftrightarrow z')
\end{aligned}$$

and finally, the very last diagram is:

$$= \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} (\dot{y}' \cdot \dot{z}') \left(\dot{y} \cdot \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \right) \right) \left(\dot{z} \cdot \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial z'} \right) \right)$$

$$\begin{aligned}
& - (z \leftrightarrow z') - (y \leftrightarrow y') \\
& + (z \leftrightarrow z', y \leftrightarrow y')
\end{aligned}$$

where (4.23) has been used twice, once as before, and once on ∂_w . Now we are in a position to assemble all the contributions and give a final expression for I . We begin by writing out the contributions for each diagram in their full form. For the first diagram we have,

$$(\dot{y} \cdot \dot{y}') (\dot{z} \cdot \dot{z}') [\partial_y \cdot \partial_z - \partial_{y'} \cdot \partial_z - \partial_y \cdot \partial_{z'} + \partial_{y'} \cdot \partial_{z'}] \quad (4.24)$$

for the second diagram,

$$\begin{aligned}
& (\dot{z} \cdot \dot{z}') [- (\dot{y} \cdot \partial_z) (\dot{y}' \cdot \partial_y) + (\dot{y}' \cdot \partial_z) (\dot{y} \cdot \partial_{y'}) + (\dot{y} \cdot \partial_{z'}) (\dot{y}' \cdot \partial_y) - (\dot{y}' \cdot \partial_{z'}) (\dot{y} \cdot \partial_{y'})] \\
& + (\dot{y} \cdot \dot{y}') [- (\dot{z} \cdot \partial_y) (\dot{z}' \cdot \partial_z) + (\dot{z}' \cdot \partial_y) (\dot{z} \cdot \partial_{z'}) + (\dot{z} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_z) - (\dot{z}' \cdot \partial_{y'}) (\dot{z} \cdot \partial_{z'})]
\end{aligned} \quad (4.25)$$

for the third diagram,

$$\begin{aligned}
& (\dot{z} \cdot \dot{y}) (\dot{z}' \cdot \partial_z) (\dot{y}' \cdot \partial_y) - (\dot{z} \cdot \dot{y}') (\dot{z}' \cdot \partial_z) (\dot{y} \cdot \partial_{y'}) - (\dot{z}' \cdot \dot{y}) (\dot{z} \cdot \partial_{z'}) (\dot{y}' \cdot \partial_y) \\
& + (\dot{z}' \cdot \dot{y}') (\dot{z} \cdot \partial_{z'}) (\dot{y} \cdot \partial_{y'})
\end{aligned} \quad (4.26)$$

for the fourth diagram,

$$\begin{aligned}
& - (\dot{z} \cdot \dot{z}') \left[+ (\dot{y} \cdot \partial_z) (\dot{y}' \cdot \partial_y) - (\dot{y}' \cdot \partial_z) (\dot{y} \cdot \partial_{y'}) - (\dot{y} \cdot \partial_{z'}) (\dot{y}' \cdot \partial_y) + (\dot{y}' \cdot \partial_{z'}) (\dot{y} \cdot \partial_{y'}) \right. \\
& \quad \left. - (\dot{y}' \cdot \partial_z) (\dot{y} \cdot \partial_y) + (\dot{y}' \cdot \partial_{z'}) (\dot{y} \cdot \partial_y) + (\dot{y} \cdot \partial_z) (\dot{y}' \cdot \partial_{y'}) - (\dot{y} \cdot \partial_{z'}) (\dot{y}' \cdot \partial_{y'}) \right] \\
& - (\dot{y} \cdot \dot{y}') \left[+ (\dot{z} \cdot \partial_y) (\dot{z}' \cdot \partial_z) - (\dot{z}' \cdot \partial_y) (\dot{z} \cdot \partial_{z'}) - (\dot{z} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_z) + (\dot{z}' \cdot \partial_{y'}) (\dot{z} \cdot \partial_{z'}) \right. \\
& \quad \left. - (\dot{z}' \cdot \partial_y) (\dot{z} \cdot \partial_z) + (\dot{z}' \cdot \partial_{y'}) (\dot{z} \cdot \partial_z) + (\dot{z} \cdot \partial_y) (\dot{z}' \cdot \partial_{z'}) - (\dot{z} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_{z'}) \right]
\end{aligned} \quad (4.27)$$

and for the fifth diagram,

$$\begin{aligned}
& -(\dot{y} \cdot \dot{z}') \left[(\dot{y}' \cdot \partial_y) (\dot{z} \cdot \partial_z) + (\dot{y}' \cdot \partial_y) (\dot{z} \cdot \partial_{z'}) + (\dot{y}' \cdot \partial_{y'}) (\dot{z} \cdot \partial_{z'}) + (\dot{y}' \cdot \partial_y) (\dot{z} \cdot \partial_{z'}) \right] \\
& + (\dot{y}' \cdot \dot{z}') \left[(\dot{y} \cdot \partial_{y'}) (\dot{z} \cdot \partial_z) + (\dot{y} \cdot \partial_{y'}) (\dot{z} \cdot \partial_{z'}) + (\dot{y} \cdot \partial_y) (\dot{z} \cdot \partial_{z'}) + (\dot{y} \cdot \partial_{y'}) (\dot{z} \cdot \partial_{z'}) \right] \\
& + (\dot{y} \cdot \dot{z}) \left[(\dot{y}' \cdot \partial_y) (\dot{z}' \cdot \partial_{z'}) + (\dot{y}' \cdot \partial_y) (\dot{z}' \cdot \partial_z) + (\dot{y}' \cdot \partial_{y'}) (\dot{z}' \cdot \partial_z) + (\dot{y}' \cdot \partial_y) (\dot{z}' \cdot \partial_z) \right] \\
& - (\dot{y}' \cdot \dot{z}) \left[(\dot{y} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_{z'}) + (\dot{y} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_z) + (\dot{y} \cdot \partial_y) (\dot{z}' \cdot \partial_z) + (\dot{y} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_z) \right]
\end{aligned} \tag{4.28}$$

and finally, the sixth diagram,

$$\begin{aligned}
& -(\dot{y} \cdot \dot{z}') \left[(\dot{y}' \cdot \partial_{y'}) (\dot{z} \cdot \partial_z) + (\dot{y}' \cdot \partial_y) (\dot{z} \cdot \partial_z) + (\dot{y}' \cdot \partial_{y'}) (\dot{z} \cdot \partial_{z'}) + (\dot{y}' \cdot \partial_y) (\dot{z} \cdot \partial_{z'}) \right] \\
& + (\dot{y}' \cdot \dot{z}') \left[(\dot{y} \cdot \partial_y) (\dot{z} \cdot \partial_z) + (\dot{y} \cdot \partial_{y'}) (\dot{z} \cdot \partial_z) + (\dot{y} \cdot \partial_y) (\dot{z} \cdot \partial_{z'}) + (\dot{y} \cdot \partial_{y'}) (\dot{z} \cdot \partial_{z'}) \right] \\
& + (\dot{y} \cdot \dot{z}) \left[(\dot{y}' \cdot \partial_{y'}) (\dot{z}' \cdot \partial_{z'}) + (\dot{y}' \cdot \partial_y) (\dot{z}' \cdot \partial_{z'}) + (\dot{y}' \cdot \partial_{y'}) (\dot{z}' \cdot \partial_z) + (\dot{y}' \cdot \partial_y) (\dot{z}' \cdot \partial_z) \right] \\
& - (\dot{y}' \cdot \dot{z}) \left[(\dot{y} \cdot \partial_y) (\dot{z}' \cdot \partial_{z'}) + (\dot{y} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_{z'}) + (\dot{y} \cdot \partial_y) (\dot{z}' \cdot \partial_z) + (\dot{y} \cdot \partial_{y'}) (\dot{z}' \cdot \partial_z) \right]
\end{aligned} \tag{4.29}$$

Now we can eliminate most of the terms by considering the following. Terms of the form $(\dot{y}' \cdot \dot{z})(\dot{y} \cdot \partial_y)(\dot{z}' \cdot \partial_{z'})$ become $[\dot{x}'(s'_1) \cdot \dot{x}(s_2) - |\dot{x}'(s'_1)| |\dot{x}(s_2)|] \partial_{s_1} \partial_{s'_2}$. But this is a total derivative in s_1 (and s'_2) and so vanishes. Also note that terms like $(\dot{y}' \cdot \dot{y}')(\dot{z} \cdot \dot{z}')$ are zero at 2^{nd} order in waviness. Applying these two rules, and summing all contributions from the six diagrams, we arrive at:

$$\begin{aligned}
I = & (\dot{y} \cdot \dot{z}' - |\dot{y}| |\dot{z}'|) (-4) \partial_{s_1} \partial_{s'_2} + (\dot{y}' \cdot \dot{z} - |\dot{y}'| |\dot{z}|) (-4) \partial_{s'_1} \partial_{s_2} \\
& + (\dot{y} \cdot \dot{z} - |\dot{y}| |\dot{z}|) (4) \partial_{s_1} \partial_{s_2} + (\dot{y}' \cdot \dot{z}' - |\dot{y}'| |\dot{z}'|) (4) \partial_{s'_1} \partial_{s'_2}
\end{aligned} \tag{4.30}$$

Now recall (4.15) and note (3.58). Our final form for the connected loop-loop correlator thus becomes:

$$\begin{aligned}
& \langle W_1 W_2 \rangle_{\text{connected}} = 1 \\
& + \frac{1}{N^2} \frac{g^6 N^3}{2} \oint ds_1 ds_2 ds'_1 ds'_2 \left\{ -\dot{\xi}(s_2) \cdot \dot{\xi}'(s'_1) \partial_{s_2} \partial_{s'_1} - \dot{\xi}(s_1) \cdot \dot{\xi}'(s'_2) \partial_{s_1} \partial_{s'_2} \right. \\
& \quad \left. + \dot{\xi}(s_1) \cdot \dot{\xi}(s_2) \partial_{s_1} \partial_{s_2} + \dot{\xi}'(s'_1) \cdot \dot{\xi}'(s'_2) \partial_{s'_1} \partial_{s'_2} \right\} \\
& \times \int d^{2\omega} w \int d^{2\omega} z \Delta(z - w) \Delta(x(s_1) - w) \Delta(x'(s'_1) - w) \Delta(x(s_2) - z) \Delta(x'(s'_2) - z)
\end{aligned} \tag{4.31}$$

Finally we have a non-zero result. We see that the only contribution to the loop-loop correlator at 2^{nd} order in waviness and up to order $g^6 N^3$ is from the H diagrams.

4.2 Operator Insertion Method

We will reproduce the two loop connected correlator result (4.31), using the operator insertion method. Sticking to the 10-D calculus our operators are,

$$\mathcal{O}_1 = i\delta x_\rho \dot{x}_\mu F_{\rho\mu} = i\delta x_\rho \dot{x}_\mu (\partial_\rho A_\mu - \partial_\mu A_\rho - i[A_\rho, A_\mu]) \quad (4.32)$$

$$\begin{aligned} \mathcal{O}_2 &= i\delta x_\rho \delta x_\tau \dot{x}_\mu D_\tau F_{\rho\mu} \\ &= i\delta x_\rho \delta x_\tau \dot{x}_\mu \left(\partial_\tau \partial_\rho A_\mu - \partial_\tau \partial_\mu A_\rho - i[\partial_\tau A_\rho, A_\mu] \right. \\ &\quad \left. - i[A_\rho, \partial_\tau A_\mu] - i[A_\tau, \partial_\rho A_\mu] + i[A_\tau, \partial_\mu A_\rho] - [A_\tau, [A_\rho, A_\mu]] \right) \end{aligned} \quad (4.33)$$

We'll represent, as before, the insertion of \mathcal{O}_1 by an X in a Feynman diagram, and \mathcal{O}_2 by an open circle. Thus the following emissions from a Wilson line are possible:

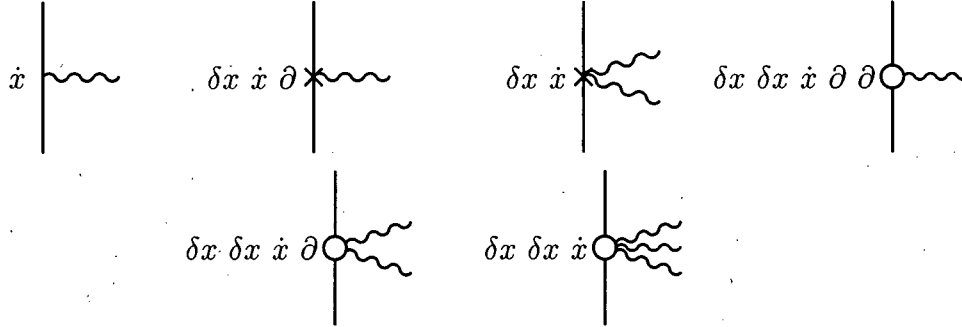
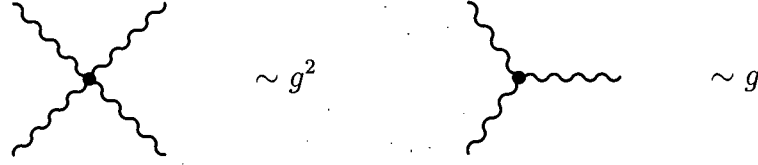


Figure 4.1: The possible emissions from a Wilson line, in the operator insertion method.

The factors shown to the left of the vertices are the objects carrying Lorentz indices which become contracted with one another in the evaluation of a diagram. These will play an important role in our calculation. There are some rules for narrowing down the diagrams which will contribute:

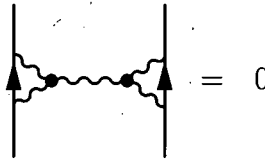
- Since we are limiting the calculation to order $g^6 N^3$, we can have a combined maximum of six gauge fields emanating from the two Wilson lines, since each gauge field contributes a single power of the coupling constant. For diagrams with internal vertices, there must be less than six. Recall that there are two fundamental vertices:



- We are calculating only to order ξ^2 in waviness. We will show in what follows that there are no contributions linear in ξ , and therefore we must have either two insertions of \mathcal{O}_1 or one insertion of \mathcal{O}_2 , in each diagram.
- Note from figure (4.1) that each regular emission from a Wilson line contains a factor of \dot{x} , while an \mathcal{O}_1 emission contains a factor of $(\delta x \dot{x} \partial)$ and a factor of $(\delta x \dot{x})$. Similarly an \mathcal{O}_2 emission contains a factor of $(\delta x \delta x \dot{x} \partial \partial)$, a factor of $(\delta x \delta x \dot{x} \partial)$, and a factor of $(\delta x \delta x \dot{x})$. In a given diagram all these objects must contract with one another, and with possible factors of ∂ coming from 3-point vertices, to give a scalar (no indices) result. And yet (3.48) - (3.52) tell us that $\dot{x} \cdot \dot{x} = 0$, and $\delta x \cdot \dot{x} = 0$.
- Because the gauge group matrices are traceless, \mathcal{O}_1 and \mathcal{O}_2 are traceless, and so there is a minimum of two insertions per Wilson line of one of the six emissions pictured in figure 4.1.

Now, there is simply no diagram consistent with the aforementioned rules but which contains a single insertion of \mathcal{O}_1 and which is not proportional to either $\dot{x} \cdot \dot{x}$ or $\delta x \cdot \dot{x}$. This is not too difficult to see: there will be at least four factors of \dot{x} , then (for say the single line emission from \mathcal{O}_1) the additional factors $\delta x \partial$. In order to avoid $\dot{x} \cdot \dot{x}$ and $\delta x \cdot \dot{x}$, one must contract δx with ∂ and would need an additional factor of ∂ for each factor of \dot{x} , which is at least four factors of ∂ . These could only be provided by four 3-point vertices, and this would result in an order $g^8 N^4$ contribution. Similarly, the double line emission from \mathcal{O}_1 does not yield a result consistent with the above rules either. So we are assured that there are no contributions linear in ξ .

Following the logic of the preceding paragraph, we can determine which diagrams will contribute at order ξ^2 . Again, we will have at least four factors of \dot{x} . Now consider two factors of $(\delta x \partial)$ arising from two insertions of \mathcal{O}_1 or one insertion of \mathcal{O}_2 . We would need two additional factors of ∂ in order to avoid $\dot{x} \cdot \dot{x}$ and $\delta x \cdot \dot{x}$. These can only come from two 3-point vertices. Recall from section 4.1 that the following type of diagrams are zero:



And so the only diagrams which may contribute are the H diagrams shown in figure 4.2.

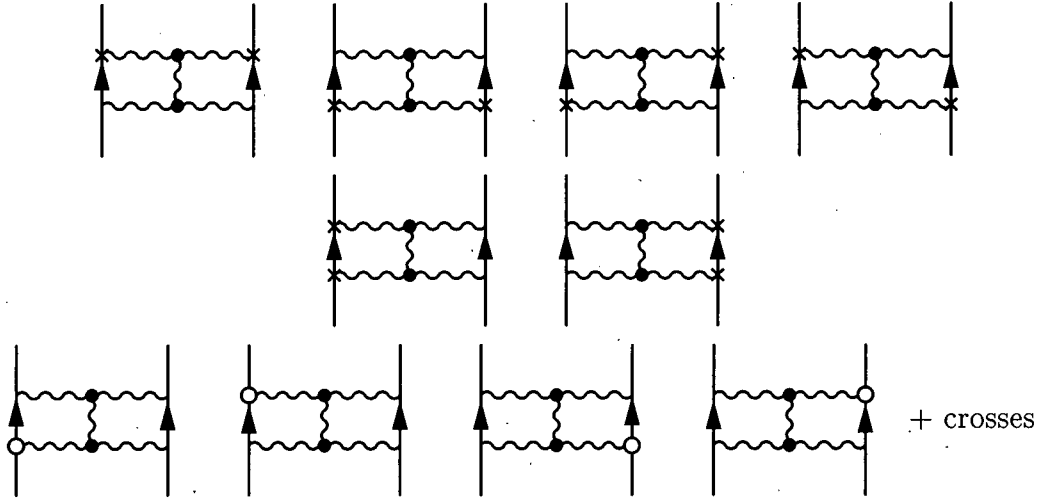
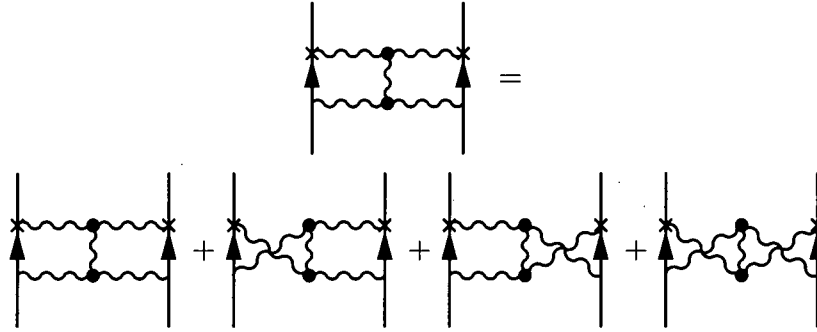


Figure 4.2: The diagrams which can possibly contribute to the two loop correlator.

The cross diagrams are not shown explicitly, however every H diagram implies four diagrams, for example:



As was mentioned during the general method calculation in section 4.1, these diagrams are planar. In the operator insertion method the effect of the cross diagrams is the following: let the positions of the gauge fields on the left Wilson line be s and s' , on the right Wilson line t and t' , then we have:

$$\int ds \int ds' \theta(s - s') \int dt \int dt' \theta(t - t') \rightarrow \oint ds ds' dt dt' \quad (4.34)$$

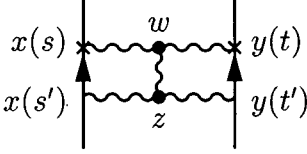
that is, adding the cross diagrams allows one to replace the half integration domains with their full domains.

In fact figure (4.2) is it, there are no other diagrams which contribute. For example, if we consider the other possible emissions from \mathcal{O}_1 and \mathcal{O}_2 , these have less factors of ∂ than the

ones considered above. This means we would need more 3-point vertices to compensate, and already without them, we would be over our total of six gauge fields emanating from the two lines. Furthermore, the H diagrams involving the single \mathcal{O}_2 insertion vanish due to the following: The emission is:

$$\delta x_\rho \delta x_\tau \dot{x}_\mu (\partial_\tau \partial_\rho A_\mu - \partial_\tau \partial_\mu A_\rho)$$

note that ρ and τ will never be contracted together, since they are not both contracted to gauge fields individually. This means that the H diagrams will end up containing either a factor of $\dot{x} \cdot \dot{x}$ or $\delta x \cdot \dot{x}$, and will vanish. So we are left with the six H diagrams involving two \mathcal{O}_1 insertions. We begin by evaluating the first diagram:



$$= \left\langle \frac{1}{N} \text{Tr} \int ds \int ds' i^2 \delta x_\rho(s) \dot{x}_\mu(s) \dot{x}_\sigma(s') F_{\rho\mu}(x(s)) A_\sigma(x(s')) \right. \\ \times \frac{1}{N} \text{Tr} \int dt \int dt' i^2 \delta y_\alpha(s) \dot{y}_\beta(t) \dot{y}_\lambda(t') F_{\alpha\beta}(y(t)) A_\lambda(y(t')) \\ \times \frac{g^2}{2!} \int d^2\omega w \int d^2\omega z f^{abc} \partial_\tau A_\gamma^a(w) A_\tau^b(w) A_\gamma^c(w) \\ \left. \times f^{def} \partial_\nu A_\phi^d(z) A_\nu^e(z) A_\phi^f(z) \right\rangle = \Lambda_1$$

where the cross diagrams are implicitly included according to (4.34). Evaluating the traces, writing out $F_{\mu\nu}$ explicitly, and restoring the coupling constant for each field emanating from a Wilson line, we have:

$$\Lambda_1 = \frac{g^6}{8N^2} \int ds ds' \int dt dt' \int d^2\omega w \int d^2\omega z f^{abc} f^{def} \\ \times \delta x_\rho(s) \dot{x}_\mu(s) \dot{x}_\sigma(s') \delta y_\alpha(s) \dot{y}_\beta(t) \dot{y}_\lambda(t') \\ \times \left\langle \left(\partial_\rho A_\mu^r(x(s)) - \partial_\mu A_\rho^r(x(s)) \right) A_\sigma^r(x(s')) \right. \\ \times \left(\partial_\alpha A_\beta^s(y(t)) - \partial_\beta A_\alpha^s(y(t)) \right) A_\lambda^s(y(t')) \\ \left. \times \partial_\tau A_\gamma^a(w) A_\tau^b(w) A_\gamma^c(w) \partial_\nu A_\phi^d(z) A_\nu^e(z) A_\phi^f(z) \right\rangle \quad (4.35)$$

Now note that we must have ρ and α contracted together, or else δx will not be contracted with δy and that will force either $(\delta x \cdot \dot{y})$ and/or $(\delta y \cdot \dot{x})$ and/or $(\dot{x} \cdot \dot{y})$ to appear in the final

expression, and we know these are zero. The only way for ρ and α to be contracted together is to contract $\partial_\mu A_\rho^r(x(s))$ and $\partial_\beta A_\alpha^s(y(t))$ with $\partial_\tau A_\gamma^a(w)$ and $A_\gamma^c(w)$. There are two ways to do this. This leaves three choices for the internal propagator ($w \rightarrow z$), however the choice $\langle A_\tau^b(w) A_\nu^e(z) \rangle$ results in σ and λ being contracted which gives the factor $(\dot{x} \cdot \dot{y})$ which is of course zero. Each remaining choice for the internal propagator leaves two ways in which $A_\sigma^r(x(s'))$ and $A_\lambda^s(y(t'))$ can be contracted with the remaining two fields in the 3-point vertex at z . Thus there are $2 \cdot 2 \cdot 2 = 8$ terms in the diagram. Note that the terms $\partial_\rho A_\mu^r(x(s))$ and $\partial_\alpha A_\beta^s(y(t))$ from the two factors of $F_{\mu\nu}$ will never contribute since they cannot possibly create a contraction between ρ and α , this will be true for all the diagrams we consider in this section.

We can use the diagram notation developed in section 4.1.5 to simplify our calculations. In fact the factors surrounding "I" in (4.20) are the same here. For example the first set of contributions for the H diagram at hand is:

$$\begin{aligned}
 & \begin{array}{c} \dot{x}(s) \cdot \partial x_\rho^r(s) \quad \dot{y}(t) \cdot \partial y_\alpha^t(t) \\ \partial_\tau w_\gamma^a \quad w_\tau^b \quad w_\gamma^c \\ \partial_\nu z_\phi^d \quad z_\nu^e \quad z_\phi^f \\ \dot{x}_\sigma^r(s') \quad \dot{y}_\lambda^t(t') \end{array} = - \begin{pmatrix} a & b & c \\ e & d & f \end{pmatrix} (\dot{x}(s) \cdot \partial_{x(s)}) (\dot{y}(t') \cdot \partial_{x(s)}) (\dot{y}(t) \cdot \partial_{y(t)}) \\
 & \times (\dot{x}(s') \cdot [\partial_{x(s')} + \partial_{y(t')}] (\delta x(s) \cdot \delta y(t)) \\
 & - [x(s') \leftrightarrow y(t')] - [x(s) \leftrightarrow y(t)] \\
 & + [x(s') \leftrightarrow y(t'), x(s) \leftrightarrow y(t)]
 \end{aligned}$$

where variants of (4.22) and (4.23) have been used. In light of the relations (3.48) - (3.52), and noting that the above contributions are proportional to $\xi(s) \cdot \xi'(t)$, one can see that we have total derivatives in s' and t' and so our first set of contributions are all zero. The second set of contributions come from the second choice for the internal propagator:

$$\begin{aligned}
 & \begin{array}{c} \dot{x}(s) \cdot \partial x_\rho^r(s) \quad \dot{y}(t) \cdot \partial y_\alpha^t(t) \\ \partial_\tau w_\gamma^a \quad w_\tau^b \quad w_\gamma^c \\ \partial_\nu z_\phi^d \quad z_\nu^e \quad z_\phi^f \\ \dot{x}_\sigma^r(s') \quad \dot{y}_\lambda^t(t') \end{array} = \begin{pmatrix} a & b & c \\ d & f & e \end{pmatrix} (\dot{x}(s) \cdot \partial_{x(s)}) (\dot{x}(s') \cdot \partial_{x(s)}) (\dot{y}(t') \cdot \partial_{x(s')}) \\
 & \times (\dot{y}(t) \cdot \partial_{y(t)}) (\delta x(s) \cdot \delta y(t)) \\
 & - [x(s') \leftrightarrow y(t')] - [x(s) \leftrightarrow y(t)] \\
 & + [x(s') \leftrightarrow y(t'), x(s) \leftrightarrow y(t)]
 \end{aligned}$$

but this is zero too, since it is a total derivative in either s' or t' . Thus all 8 terms are zero. So $\Lambda_1 = 0$. By symmetry, we can see that,

$$\Lambda_2 = \begin{array}{c} \text{Diagram: Two vertical lines with arrows pointing up. A wavy line connects the left line at a point marked with an 'x' to the right line at a point marked with a dot. Another wavy line connects the left line at a point marked with a dot to the right line at a point marked with an 'x'. The two wavy lines cross each other in the middle.} \end{array} = 0$$

as well.

Moving on to the next H diagram,

$$\begin{aligned} \begin{array}{c} \text{Diagram: Two vertical lines. Left line has points x(s) (marked with x) and x(s') (marked with dot). Right line has points y(t) (marked with dot) and y(t') (marked with x). A wavy line connects x(s) to y(t) with a dot labeled w. Another wavy line connects x(s') to y(t') with a dot labeled z. The two wavy lines cross each other in the middle.} \end{array} \sim \delta x_\rho(s) \dot{x}_\mu(s) \dot{x}_\sigma(s') \delta y_\alpha(s) \dot{y}_\beta(t) \dot{y}_\lambda(t') \\ \times \left\langle \left(\partial_\rho A_\mu^r(x(s)) - \partial_\mu A_\rho^r(x(s)) \right) A_\sigma^r(x(s')) \right. \\ \times A_\lambda^s(y(t)) \left(\partial_\alpha A_\beta^s(y(t')) - \partial_\beta A_\alpha^s(y(t')) \right) \\ \left. \times \partial_\tau A_\gamma^a(w) A_\tau^b(w) A_\gamma^c(w) \partial_\nu A_\phi^d(z) A_\nu^e(z) A_\phi^f(z) \right\rangle = \Lambda_3 \end{aligned}$$

Now we'll have a different rule to ensure the $\rho - \alpha$ contraction. We will have to have $\partial_\mu A_\rho^r(x(s))$ contract with either $\partial_\tau A_\gamma^a(w)$ or $A_\gamma^c(w)$, and $\partial_\beta A_\alpha^s(y(t'))$ contract with either $\partial_\nu A_\phi^d(z)$ or $A_\phi^f(z)$. Then the remaining two fields from the two 3-point vertices contract together. There are then $2 \cdot 2 = 4$ terms in this H diagram. The first term is:

$$\begin{aligned} \begin{array}{c} \text{Diagram: A central vertex with six lines. Top-left: line with dot, labeled \dot{x}(s) \cdot \partial x_\rho^r(s). Top-right: line with dot, labeled \dot{y}_\lambda^t(t). Middle-left: line with dot, labeled \partial_\tau w_\gamma^a. Middle-right: line with dot, labeled w_\gamma^c. Bottom-left: line with dot, labeled \partial_\nu z_\phi^d. Bottom-right: line with dot, labeled z_\phi^f. Bottom: line with dot, labeled \dot{x}_\sigma^r(s'). Bottom-right: line with dot, labeled \dot{y}(t') \cdot \partial y_\alpha^t(t').} \end{array} = - \begin{pmatrix} a & b & c \\ e & f & d \end{pmatrix} \left(\dot{x}(s) \cdot \partial_{x(s)} \right) \left(\dot{y}(t) \cdot \partial_{x(s)} \right) \left(\dot{y}(t') \cdot \partial_{y(t')} \right) \\ \times \left(\dot{x}(s') \cdot [\partial_{x(s')} + \partial_{y(t')}] \right) \left(\delta x(s) \cdot \delta y(t') \right) \end{aligned}$$

The second term is:

$$\begin{aligned}
 & \begin{array}{c} \dot{x}(s) \cdot \partial x_\rho^r(s) \\ \partial_\tau w_\gamma^a \\ \partial_\nu z_\phi^d \\ \dot{x}_\sigma^r(s') \end{array} \begin{array}{c} \dot{y}_\lambda^t(t) \\ w_\gamma^b \\ z_\nu^e \\ \dot{y}(t') \cdot \partial y_\alpha^t(t') \end{array} \\
 & = \begin{pmatrix} a & b & c \\ e & d & f \end{pmatrix} \left(\dot{x}(s) \cdot \partial_{x(s)} \right) \left(\dot{y}(t) \cdot \partial_{x(s)} \right) \left(\dot{y}(t') \cdot \partial_{y(t')} \right) \\
 & \quad \times \left(\dot{x}(s') \cdot \partial_{y(t')} \right) \left(\delta x(s) \cdot \delta y(t') \right)
 \end{aligned}$$

The third term is:

$$\begin{aligned}
 & \begin{array}{c} \dot{x}(s) \cdot \partial x_\rho^r(s) \\ \partial_\tau w_\gamma^a \\ \partial_\nu z_\phi^d \\ \dot{x}_\sigma^r(s') \end{array} \begin{array}{c} \dot{y}_\lambda^t(t) \\ w_\gamma^b \\ z_\nu^e \\ \dot{y}(t') \cdot \partial y_\alpha^t(t') \end{array} \\
 & = \begin{pmatrix} a & b & c \\ d & f & e \end{pmatrix} \left(\dot{x}(s) \cdot \partial_{x(s)} \right) \left(\dot{y}(t) \cdot [\partial_{x(s)} + \partial_{y(t)}] \right) \\
 & \quad \times \left(\dot{y}(t') \cdot \partial_{y(t')} \right) \left(\dot{x}(s') \cdot [\partial_{x(s')} + \partial_{y(t')}] \right) \left(\delta x(s) \cdot \delta y(t') \right)
 \end{aligned}$$

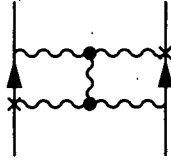
The fourth and final term is:

$$\begin{aligned}
 & \begin{array}{c} \dot{x}(s) \cdot \partial x_\rho^r(s) \\ \partial_\tau w_\gamma^a \\ \partial_\nu z_\phi^d \\ \dot{x}_\sigma^r(s') \end{array} \begin{array}{c} \dot{y}_\lambda^t(t) \\ w_\gamma^b \\ z_\nu^e \\ \dot{y}(t') \cdot \partial y_\alpha^t(t') \end{array} \\
 & = - \begin{pmatrix} a & b & c \\ f & d & e \end{pmatrix} \left(\dot{x}(s) \cdot \partial_{x(s)} \right) \left(\dot{y}(t) \cdot [\partial_{x(s)} + \partial_{y(t)}] \right) \\
 & \quad \times \left(\dot{y}(t') \cdot \partial_{y(t')} \right) \left(\dot{x}(s') \cdot \partial_{y(t')} \right) \left(\delta x(s) \cdot \delta y(t') \right)
 \end{aligned}$$

Assembling our result using (3.48) - (3.52), and noting that any derivatives with respect to $x(s')$ or $y(t)$ will be total derivatives, and cause the associated term to vanish, we find that:

$$\Lambda_3 = -4 \xi(s) \cdot \xi'(t') \partial_s^2 \partial_{t'}^2 \Delta(z-w) \Delta(x(s)-w) \Delta(y(t)-w) \Delta(x(s')-z) \Delta(y(t')-z) \quad (4.36)$$

where the factor of 4 is due to the fact that the four terms of the H diagram are equal to one another. By symmetry, one can see that:

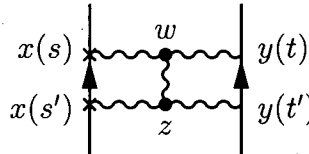
$$\Lambda_4 = \text{Diagram} = \Lambda_3 \quad \text{with prime} \leftrightarrow \text{unprime}$$


and therefore:

$$\Lambda_4 = -4 \xi(s') \cdot \xi'(t) \partial_s^2 \partial_t^2 \Delta(z-w) \Delta(x(s)-w) \Delta(y(t)-w) \Delta(x(s')-z) \Delta(y(t')-z) \quad (4.37)$$

since there is an inherent $w \leftrightarrow z$ symmetry.

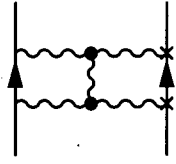
Next we should consider the following H diagram:

$$\begin{aligned} \text{Diagram} &\sim \delta x_\rho(s) \dot{x}_\mu(s) \dot{x}_\sigma(s') \delta y_\alpha(s) \dot{y}_\beta(t) \dot{y}_\lambda(t') \\ &\times \left\langle \left(\partial_\rho A_\mu^r(x(s)) - \partial_\mu A_\rho^r(x(s)) \right) \left(\partial_\alpha A_\beta^r(x(s')) - \partial_\beta A_\alpha^r(x(s')) \right) \right. \\ &\quad \times A_\sigma^s(y(t)) A_\lambda^s(y(t')) \\ &\quad \left. \times \partial_\tau A_\gamma^a(w) A_\tau^b(w) A_\gamma^c(w) \partial_\nu A_\phi^d(z) A_\nu^e(z) A_\phi^f(z) \right\rangle = \Lambda_5 \end{aligned}$$


To ensure the $\rho - \alpha$ contraction, the very same rule emerges as for Λ_3 , that is, $\partial_\mu A_\rho^r(x(s))$ must contract with either $\partial_\tau A_\gamma^a(w)$ or $A_\gamma^c(w)$, and $\partial_\beta A_\alpha^s(x(s'))$ with either $\partial_\nu A_\phi^d(z)$ or $A_\phi^f(z)$. Then the remaining two fields from the two 3-point vertices contract together. In light of this it is not surprising that:

$$\Lambda_5 = \Lambda_3 \quad \text{with } y(t') \leftrightarrow x(s') \text{ and } (-1)$$

where the (-1) comes because there is a switch in the structure constant coefficients relative to Λ_3 . By symmetry, it is evident that:



$$\Lambda_6 = \text{Diagram} = \Lambda_5 \quad \text{with } y(t) \leftrightarrow x(s) \text{ and } y(t') \leftrightarrow x(s')$$

Therefore, we have:

$$\Lambda_5 = 4 \xi(s) \cdot \xi(s') \partial_s^2 \partial_{s'}^2 \Delta(z-w) \Delta(x(s)-w) \Delta(y(t)-w) \Delta(x(s')-z) \Delta(y(t')-z) \quad (4.38)$$

$$\Lambda_6 = 4 \xi'(t) \cdot \xi'(t') \partial_t^2 \partial_{t'}^2 \Delta(z-w) \Delta(x(s)-w) \Delta(y(t)-w) \Delta(x(s')-z) \Delta(y(t')-z) \quad (4.39)$$

Referring back to (4.20) and assembling the results of Λ_3 - Λ_6 , we have our final result:

$$\begin{aligned} \delta \langle W_1 W_2 \rangle_{\text{connected}} = & \frac{1}{N^2} \frac{g^6 N^3}{2} \oint ds ds' dt dt' \left\{ -\dot{\xi}(s) \cdot \dot{\xi}'(t') \partial_s \partial_{t'} - \dot{\xi}(s') \cdot \dot{\xi}'(t) \partial_{s'} \partial_t \right. \\ & \left. + \dot{\xi}(s) \cdot \dot{\xi}(s') \partial_s \partial_{s'} + \dot{\xi}'(t) \cdot \dot{\xi}'(t') \partial_t \partial_{t'} \right\} \\ & \times \int d^{2\omega} w \int d^{2\omega} z \Delta(z-w) \Delta(x(s)-w) \Delta(y(t)-w) \Delta(x(s')-z) \Delta(y(t')-z) \end{aligned} \quad (4.40)$$

where integration by parts has been used to transfer two derivatives in each term to the waviness functions. This is the correction to the result for two straight, like oriented lines which is just unity, thus this result agrees exactly with (4.31).

4.3 Interaction Between a Straight and a Wavy Line at Large Separation

4.3.1 Perturbative Calculation

Ideally, one would like to be able to reduce the expression for the interaction of the two wavy lines, (4.40), down to an integral over only the Wilson loop parameters which the waviness functions depend on. This involves integrating out w and z and two of s , s' , t , and t' , depending on the term. It turns out that this is an intractable integration. The first step is to use the Feynman parameters formula:

$$\prod_i A_i^{-n_i} = \frac{\Gamma(\sum n_i)}{\prod_i \Gamma(n_i)} \int_0^1 dx_1 \cdots dx_k x_1^{n_1-1} \cdots x_k^{n_k-1} \frac{\delta(1 - \sum x_i)}{[\sum_i A_i x_i]^{\sum n_i}} \quad (4.41)$$

on the five propagators in order to integrate out w and z , this leaves an integral over the four Wilson loop parameters and four Feynman parameters (the delta function eliminates one). The integrations over the two Wilson loop parameters are possible, however the resulting four dimensional Feynman parameter integration is entirely intractable. The calculations leading to the intractable integral are included in appendix C.

A result which is tractable, and interesting, is the limit where the two Wilson lines are widely separated, and one of them is straight. What is meant by “widely separated” here is that the separation L is much larger than the domain of compact support of the waviness function $\xi(s)$. Recall from the discussion at the start of chapter 2 that $\xi(s)$ is also taken to be small in the sense that $|\dot{\xi}(s)| \ll 1$, thus there is an implicit sense in which L is much greater than $|\xi(s)|$ itself. In this limit we see that (4.40) becomes:

$$\begin{aligned} \langle W_1 W_2 \rangle &= \frac{1}{N^2} \frac{g^6 N^3}{2} \oint ds ds' dt dt' \dot{\xi}(s) \cdot \dot{\xi}(s') \partial_s \partial_{s'} \\ &\times \int d^{2\omega} w \int d^{2\omega} z \Delta(z - w) \Delta(x(s) - w) \Delta(y(t) - w) \Delta(x(s') - z) \Delta(y(t') - z) \end{aligned} \quad (4.42)$$

Now let us perform the integrations over t and t' . We'll stick to the physical dimension, i.e. $\omega = 2$. Also recall (A.5). The integration is:

$$I = \int dt \int dt' \frac{1}{|y(t) - w|^2} \frac{1}{|y(t') - z|^2} \quad \text{where } y(t) = (t, \vec{L}) \quad (4.43)$$

We use (4.41) on the two propagators to bring the integral into the form:

$$I = \int dt \int dt' \int_0^1 d\alpha \frac{1}{(\alpha(t - w_0)^2 + D)^2} \quad (4.44)$$

where,

$$D = \alpha(\vec{L} - \vec{w})^2 + (1 - \alpha)|x(t') - z|^2 \quad (4.45)$$

The integral over t is now easily performed using:

$$\int \frac{d^d x}{(2\pi)^d} \frac{1}{(x^2 + D)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{D} \right)^{n-d/2} \quad (4.46)$$

which yields,

$$I = \frac{2\pi}{\sqrt{4\pi}} \Gamma(3/2) \int dt' \int_0^1 d\alpha \frac{1}{\sqrt{\alpha}} \frac{1}{D^{3/2}} \quad (4.47)$$

The integration over t' is similarly integrated using the same general relation (4.46). The result is:

$$I = \pi \int_0^1 d\alpha \alpha^{-1/2} (1 - \alpha)^{-1/2} \frac{1}{\alpha(\vec{L} - \vec{w})^2 + (1 - \alpha)(\vec{L} - \vec{z})^2} \quad (4.48)$$

Then, taking the limit of large separation:

$$\lim_{L \rightarrow \infty} I = \frac{\pi}{L^2} \int_0^1 d\alpha \alpha^{-1/2} (1-\alpha)^{-1/2} = \frac{\pi^2}{L^2} \quad (4.49)$$

and so we find that our interaction will fall off as L^{-2} . Taking the limit in this way is justified given the remarks concerning the compact support of $\xi(s)$ at the beginning of the section. We should now venture to integrate out w and z in our original expression,

$$\begin{aligned} \langle W_1 W_2 \rangle &= \frac{\pi^2}{L^2} \left(\frac{1}{4\pi^\omega} \right)^5 \Gamma^5(\omega-1) \frac{1}{N^2} \frac{g^6 N^3}{2} \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \partial_s \partial_{s'} \\ &\times \int d^{2\omega} w \int d^{2\omega} z \left(\frac{1}{|w-z|^2} \right)^{\omega-1} \left(\frac{1}{|x(s)-w|^2} \right)^{\omega-1} \left(\frac{1}{|x(s')-z|^2} \right)^{\omega-1} \end{aligned} \quad (4.50)$$

where now,

$$x(s) = (s, 0, 0, 0) \quad (4.51)$$

In order to prove that our result will actually be finite, we stick to the general dimension 2ω for the integrations. Beginning with the integration over w , we use (4.41) on the first two propagators, then use (4.46) to integrate out w . The Feynman parameter α just drops out of this calculation, contributing a factor of 1. The result is:

$$\begin{aligned} \langle W_1 W_2 \rangle &= \frac{\pi^2}{L^2} \left(\frac{1}{4\pi^\omega} \right)^5 \frac{\Gamma^5(\omega-1) \Gamma(\omega-2)}{\Gamma^2(\omega-1)} \frac{(2\pi)^{2\omega}}{(4\pi)^\omega} \frac{1}{N^2} \frac{g^6 N^3}{2} \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \partial_s \partial_{s'} \\ &\times \int d^{2\omega} z \left(\frac{1}{|x(s)-z|^2} \right)^{\omega-2} \left(\frac{1}{|x(s')-z|^2} \right)^{\omega-1} \end{aligned} \quad (4.52)$$

In order to integrate out z the very same steps are made, using the same formulae for the integration. The result is:

$$\begin{aligned} \langle W_1 W_2 \rangle &= \frac{\pi^2}{2L^2} \left(\frac{1}{4\pi^\omega} \right)^5 \left(\frac{(2\pi)^{2\omega}}{(4\pi)^\omega} \right)^2 \Gamma^2(\omega-1) \Gamma(\omega-3) \\ &\times \frac{1}{N^2} \frac{g^6 N^3}{2} \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \partial_s \partial_{s'} (s-s')^{6-2\omega} \end{aligned} \quad (4.53)$$

where we have used the fact that $x_\mu(s) = (s, 0, 0, 0)$. Now consider the following factor in the expression above,

$$I = \Gamma(\omega-3) \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \partial_s \partial_{s'} (s-s')^{6-2\omega} \quad (4.54)$$

applying integration by parts to migrate the s and s' derivatives from the ξ 's over to the factor $(s-s')^{6-2\omega}$, we have:

$$I = \Gamma(\omega - 3) \int ds ds' \xi(s) \cdot \xi(s') (6 - 2\omega)(5 - 2\omega)(4 - 2\omega)(3 - 2\omega)(s - s')^{2-2\omega} \quad (4.55)$$

but,

$$\Gamma(\omega - 3) = \frac{\Gamma(\omega - 1)}{(\omega - 3)(\omega - 2)} \quad (4.56)$$

This means that the $4 - 2\omega = 2(2 - \omega)$ term in the numerator of I cancels the pole $\omega - 2$ in the denominator. We now have:

$$\begin{aligned} \langle W_1 W_2 \rangle &= \frac{2\pi^2}{L^2} \left(\frac{1}{4\pi^\omega} \right)^5 \left(\frac{(2\pi)^{2\omega}}{(4\pi)^\omega} \right)^2 \Gamma^3(\omega - 1) (5 - 2\omega)(3 - 2\omega) \\ &\times \frac{1}{N^2} \frac{g^6 N^3}{2} \oint ds ds' \frac{\xi(s) \cdot \xi(s')}{(s - s')^{2\omega-2}} \end{aligned} \quad (4.57)$$

Integration by parts may be used again on s and s' , in order to express our result in terms of $\dot{\xi}$. Since everything is finite, we can also set $\omega = 2$. The final result is:

$$\langle W_1 W_2 \rangle = -\frac{1}{N^2} \frac{1}{L^2} \frac{g^6 N^3}{2^{11} \pi^4} \oint ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \ln(\mu^2(s - s')^2) \quad (4.58)$$

where μ is an arbitrary constant with dimensions of 1/length, which does not affect the value of the integral. One might have expected a form similar to the single line result (3.44), and so the appearance of the logarithm here is very interesting. This interaction appears as an effective self-interaction in the sense that ξ at one point is correlated with ξ at another point on the same line, as was the case (of course) for the self interaction itself (3.44). However here, the effective self-interaction falls off less quickly (by two powers of $(s - s')$) than for the real self-interaction. Note that by using an argument analogous to (3.58), $\xi(s) \cdot \xi(s')$ above may be replaced by $-\frac{1}{2}[\dot{\xi}(s) - \dot{\xi}(s')]^2$.

4.3.2 Strong Coupling Calculation

Now we would like to calculate the interaction between a straight and a wavy line at large separation, i.e. the calculation carried out in section 4.3.1, on the AdS side of the correspondence. As was explained in [19], the interaction between two largely separated loops comes from the exchange of the lightest SUGRA particles between the two macroscopic worldsheets, see figure 4.3. The term “largely separated” is explained at the beginning of section 4.3.1.

We must determine what these lightest particles are, and then calculate how they couple to the worldsheets, i.e. determine the vertex operators. The result will have the schematic form:

$$\Lambda = \int_{\Omega_1} \int_{\Omega_2} \mathcal{V}_1 P \mathcal{V}_2 \quad (4.59)$$

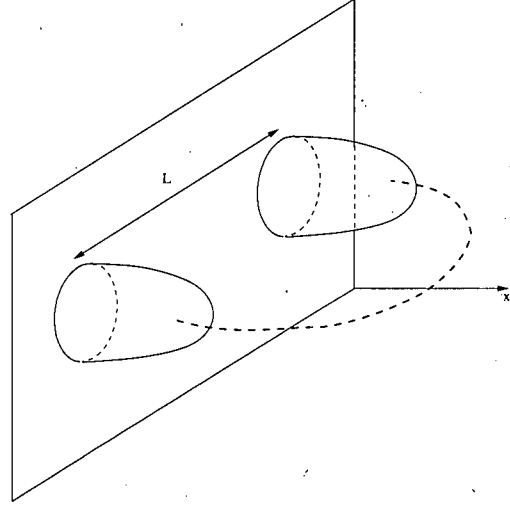


Figure 4.3: The interaction between two Wilson loops at strong coupling. The interaction is due to the exchange of SUGRA particles between the worldsheets.

where Ω_i refers to the worldsheet domains, \mathcal{V}_i refers to the vertex operator on that worldsheet, and P denotes the propagator for the SUGRA field which travels between the worldsheets.

The particle spectrum of SUGRA on the background $AdS_5 \times S^5$ was worked out in [25]. Since S^5 is a compact space, the fields are expanded in a series of spherical harmonics, times Kaluza-Klein mode fields (ϕ^k) on AdS_5 :

$$\phi(x^M) = \sum_k \phi^k(x^\mu) Y^k(\theta_i) \quad (4.60)$$

where $M = 0, \dots, 9$, x^μ are the coordinates on AdS_5 , θ_i are the five angles on S^5 , and Y^k are the 5-D spherical harmonic functions. Since we are considering largely separated loops, only the lightest fields will be important. As is explained in [19], in the strong coupling (therefore large radius) regime these lightest modes arise from perturbations of the AdS_5 metric. Following [25], the metric on AdS_5 is denoted $g_{\mu\nu}$, and it is perturbed as follows:

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} + h_{\mu\nu} \\ h_{\mu\nu} &= h'_{\mu\nu} - \frac{1}{3} g_{\mu\nu} h^\alpha_\alpha \\ h^\alpha_\alpha &= \pi^k(x) Y^k(\theta) \\ h'_{\mu\nu} &= H^k_{\mu\nu}(x) Y^k(\theta) \\ g^{\mu\nu} h_{\mu\nu} &= h^\mu_\mu \equiv -\frac{9}{15} \pi^k(x) Y^k(\theta) \end{aligned} \quad (4.61)$$

where h^α_α is the trace of the metric on S^5 , sum over k is implicit, and x are the coordinates on

AdS_5 . The $\pi^k(x)$ are Kaluza-Klein modes of a scalar field. It is shown in [25] and mentioned in [19], that this field mixes with another scalar field $b^k(x)$ which arises from fluctuations of a four-form gauge field which is present in the SUGRA. Thus there are two linear combinations of the π and b fields which are mass eigenstates:

$$\begin{aligned} t^k(x) &= \frac{1}{20(k+2)} [\pi^k(x) + 10k b^k(x)] & m^2 &= (k+4)(k+8) \quad k \geq 0 \\ s^k(x) &= \frac{1}{20(k+2)} [\pi^k(x) - 10(k+4) b^k(x)] & m^2 &= k(k-4) \quad k \geq 2 \end{aligned} \quad (4.62)$$

Solving for π^k we have:

$$\pi^k(x) = 10k s^k(x) + 10(k+4) t^k(x) \quad (4.63)$$

but since $s^k(x)$ is the lighter field, we ignore $t^k(x)$ entirely. Equation (2.44) from [25] then tells us that:

$$H_{(\mu\nu)}^k = \frac{4}{k+1} D_{(\mu} D_{\nu)} s^k(x) \quad (4.64)$$

where the brackets indicate the traceless, symmetric combination, and D_μ denotes the covariant derivative. Now, referring back to (4.61), we have:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{1}{3} g_{\mu\nu} \pi^k Y^k + H_{(\mu\nu)}^k Y^k + \frac{16}{75} g_{\mu\nu} \pi^k Y^k \quad (4.65)$$

where we have broken $H_{\mu\nu}^k$ into its traceless and trace pieces. Expressing this in terms of s^k , we have:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{10k}{3} g_{\mu\nu} s^k Y^k + \frac{4}{k+1} D_{(\mu} D_{\nu)} s^k Y^k + \frac{160k}{75} g_{\mu\nu} s^k Y^k \quad (4.66)$$

Let us now calculate $D_{(\mu} D_{\nu)} s^k(x)$. Note that,

$$\begin{aligned} D_{(\mu} D_{\nu)} \phi &= \left[\frac{1}{2} (D_\mu D_\nu + D_\nu D_\mu) - \text{Trace} \right] \phi \\ &= \left[\partial_\mu \partial_\nu - \Gamma_{\mu\nu}^\tau \partial_\tau - \text{Trace} \right] \phi \end{aligned} \quad (4.67)$$

where,

$$\Gamma_{\mu\nu}^\tau = \frac{1}{2} g^{\tau\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\sigma\nu} - \partial_\sigma g_{\mu\nu}) \quad (4.68)$$

The metric of AdS_5 is:

$$g_{\mu\nu} = \frac{1}{x_0^2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4.69)$$

since we are using a Euclidean signature. We find,

$$\Gamma_{\mu\nu}^\tau \partial_\tau = \frac{1}{x_0} (\delta_{\mu\nu} - 2\delta_{0\mu}\delta_{0\nu}) \partial_0 - \frac{1}{x_0} (\delta_{0\nu}\delta_{i\mu} + \delta_{i\nu}\delta_{0\mu}) \partial_i \quad (4.70)$$

where $i = 1, \dots, 4$. In light of this we have,

$$\begin{aligned} D_{(\mu} D_{\nu)} s^k &= \partial_\mu \partial_\nu s^k - \frac{1}{x_0} (\delta_{\mu\nu} - 2\delta_{0\mu}\delta_{0\nu}) \partial_0 s^k \\ &+ \frac{1}{x_0} (\delta_{0\nu}\delta_{i\mu} + \delta_{i\nu}\delta_{0\mu}) \partial_i s^k - \frac{1}{5} g_{\mu\nu} (x_0^2 (\partial_0^2 + \partial_i^2) s^k - 3x_0 \partial_0 s^k) \end{aligned} \quad (4.71)$$

Now we should determine the vertex operators. In order to do this we begin with the minimal action (1.46) which can be written:

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \sqrt{\det h} \quad h_{ab} = \partial_a X^\mu g_{\mu\nu} \partial_b X^\nu \quad (4.72)$$

We now consider the variation in h due to a variation in $g_{\mu\nu}$:

$$\delta\sqrt{\det h} = \text{Tr}(h^{-1}\delta h) \frac{\sqrt{\det h}}{2} \quad \delta h_{ab} = \partial_a X^\mu \delta g_{\mu\nu} \partial_b X^\nu \quad (4.73)$$

We have calculated our $\delta g_{\mu\nu}$ in (4.66). Note that we will be interested only in the Kaluza-Klein mode field $s^k(x)$ and not the spherical harmonic piece. This is because the two Wilson loops can be thought of as being at the same point on the five-sphere. Thus we have:

$$\begin{aligned} \delta h_{ab} &= \partial_a X^\mu g_{\mu\nu} \left(-\frac{90k}{75} - \frac{4}{5(k+1)} [x_0^2 (\partial_0^2 + \partial_i^2) - 3x_0 \partial_0] \right) s^k \partial_b X^\nu \\ &+ \partial_a X^\mu \frac{4}{k+1} \left(\partial_\mu \partial_\nu s^k - \frac{1}{x_0} (\delta_{\mu\nu} - 2\delta_{0\mu}\delta_{0\nu}) \partial_0 + \frac{1}{x_0} (\delta_{0\nu}\delta_{i\mu} + \delta_{i\nu}\delta_{0\mu}) \partial_i \right) s^k \partial_b X^\nu \end{aligned} \quad (4.74)$$

The piece of $\delta g_{\mu\nu}$ which is proportional to $g_{\mu\nu}$ gives rise to the following variation:

$$\delta g_{\mu\nu} = \mathcal{O}_{\mu\nu} = g_{\mu\nu} \mathcal{O}, \quad \Rightarrow \quad \delta\sqrt{\det h} = \mathcal{O}\sqrt{\det h} \quad (4.75)$$

whereas for a general variation $\delta g_{\mu\nu} = \mathcal{O}_{\mu\nu}$ we have:

$$\begin{aligned} \delta\sqrt{\det h} &= \frac{1}{2\sqrt{\det h}} \left\{ (\partial_1 X^\mu g_{\mu\nu} \partial_1 X^\nu) (\partial_0 X^\mu \mathcal{O}_{\mu\nu} \partial_0 X^\nu) \right. \\ &\quad - (\partial_0 X^\mu g_{\mu\nu} \partial_1 X^\nu) (\partial_1 X^\mu \mathcal{O}_{\mu\nu} \partial_0 X^\nu) \\ &\quad - (\partial_1 X^\mu g_{\mu\nu} \partial_0 X^\nu) (\partial_0 X^\mu \mathcal{O}_{\mu\nu} \partial_1 X^\nu) \\ &\quad \left. + (\partial_0 X^\mu g_{\mu\nu} \partial_0 X^\nu) (\partial_1 X^\mu \mathcal{O}_{\mu\nu} \partial_1 X^\nu) \right\} \end{aligned} \quad (4.76)$$

Note that worldsheet derivatives act only on embedding functions, whereas derivatives in the AdS_5 coordinates act only on the $s^k(x)$ field. Recall the embedding function for the wavy line:

$$X^\mu = (t, s, \vec{\Delta}(s, t)) \quad (4.77)$$

where t and s are the worldsheet coordinates σ^0 and σ^1 respectively, and the waviness vector $\vec{\Delta}$ is a three-component object. Since the embedding of the worldsheet coordinates is a direct mapping to X^0 and X^1 , we'll call the AdS_5 coordinates x_0 and x_1 by t and s respectively. We will require the following formulae:

$$\begin{aligned} \partial_0 X^\mu &= (1, 0, \dot{\Delta}) & \partial_1 X^\mu &= (0, 1, \Delta') \\ \partial_0 X^\mu g_{\mu\nu} \partial_0 X^\nu &= \frac{1}{t^2} (1 + \dot{\Delta}^2) & \partial_1 X^\mu g_{\mu\nu} \partial_1 X^\nu &= \frac{1}{t^2} (1 + \Delta'^2) \\ \partial_0 X^\mu g_{\mu\nu} \partial_1 X^\nu &= \partial_1 X^\mu g_{\mu\nu} \partial_0 X^\nu = \frac{1}{t^2} \dot{\Delta} \cdot \Delta' \\ \sqrt{\det h} &\simeq \frac{1}{t^2} \left(1 + \frac{1}{2} \dot{\Delta}^2 + \frac{1}{2} \Delta'^2 \right) \end{aligned} \quad (4.78)$$

where in the last line we have expanded to second order in waviness, which is the order we'll keep all the calculations to. We begin by calculating the trace piece from (4.74). We employ the notation $\partial_\mu = (\partial_t, \partial_s, \vec{\nabla})$:

$$\delta \sqrt{\det h_1} = \left(1 + \frac{1}{2} \dot{\Delta}^2 + \frac{1}{2} \Delta'^2 \right) \frac{1}{t^2} \left\{ -\frac{6k}{5} - \frac{4}{5(k+1)} [t^2(\partial_t^2 + \partial_s^2 + \nabla^2) - 3t\partial_t] \right\} s^k \quad (4.79)$$

Now we calculate the term proportional to $\partial_\mu \partial_\nu s^k$, using (4.76). First we'll need:

$$\begin{aligned} \partial_0 X^\mu \partial_\mu \partial_\nu s^k \partial_0 X^\nu &= \left(\partial_t + \dot{\Delta} \cdot \nabla \right)^2 s^k = \left(\partial_t^2 + 2\dot{\Delta} \cdot \nabla \partial_t + (\dot{\Delta} \cdot \nabla)^2 \right) s^k \\ \partial_1 X^\mu \partial_\mu \partial_\nu s^k \partial_1 X^\nu &= \left(\partial_s + \Delta' \cdot \nabla \right)^2 s^k = \left(\partial_s^2 + 2\Delta' \cdot \nabla \partial_s + (\Delta' \cdot \nabla)^2 \right) s^k \\ \partial_0 X^\mu \partial_\mu \partial_\nu s^k \partial_1 X^\nu &= \left(\partial_t + \dot{\Delta} \cdot \nabla \right) \left(\partial_s + \Delta' \cdot \nabla \right) s^k \\ &= \left(\partial_t \partial_s + \dot{\Delta} \cdot \nabla \partial_s + \Delta' \cdot \nabla \partial_t + (\Delta' \cdot \nabla)(\dot{\Delta} \cdot \nabla) \right) s^k \end{aligned} \quad (4.80)$$

Using these relations, we find that to second order in waviness, the contribution from the $\partial_\mu \partial_\nu s^k$ piece in (4.74) is:

$$\begin{aligned} \delta \sqrt{\det h_2} &= \frac{2}{k+1} \left\{ \partial_t^2 + \partial_s^2 + \frac{1}{2} \dot{\Delta}^2 (-\partial_t^2 + \partial_s^2) + \frac{1}{2} \Delta'^2 (\partial_t^2 - \partial_s^2) \right. \\ &\quad \left. + 2(\dot{\Delta} \cdot \nabla) \partial_t + 2(\Delta' \cdot \nabla) \partial_s + (\Delta' \cdot \nabla)^2 + (\dot{\Delta} \cdot \nabla)^2 - 2(\Delta' \cdot \dot{\Delta}) \partial_t \partial_s \right\} s^k \end{aligned} \quad (4.81)$$

Now we calculate the contribution from the $\Gamma_{\mu\nu}^0$ piece from (4.74). We'll require the following:

$$\begin{aligned}
\partial_0 X^\mu \Gamma_{\mu\nu}^0 \partial_t \partial_0 X^\nu &= \frac{1}{t} (-1 + \dot{\Delta}^2) \partial_t \\
\partial_0 X^\mu \Gamma_{\mu\nu}^0 \partial_t \partial_1 X^\nu &= \frac{1}{t} \dot{\Delta} \cdot \Delta' \partial_t \\
\partial_1 X^\mu \Gamma_{\mu\nu}^0 \partial_t \partial_1 X^\nu &= \frac{1}{t} (1 + \Delta'^2) \partial_t
\end{aligned} \tag{4.82}$$

Using these relations, we find that to second order in waviness, the contribution is:

$$\delta \sqrt{\det h_3} = -\frac{4 \dot{\Delta}^2}{k+1} \frac{1}{t} \partial_t s^k \tag{4.83}$$

Finally we have the contribution from the $\Gamma_{\mu\nu}^i$ piece from (4.74). We'll require the following:

$$\begin{aligned}
\partial_0 X^\mu \Gamma_{\mu\nu}^i \partial_i \partial_0 X^\nu &= \frac{2}{t} \dot{\Delta} \cdot \nabla \\
\partial_1 X^\mu \Gamma_{\mu\nu}^i \partial_i \partial_1 X^\nu &= 0 \\
\partial_0 X^\mu \Gamma_{\mu\nu}^i \partial_i \partial_1 X^\nu &= \frac{1}{t} (\partial_s + \Delta' \cdot \nabla)
\end{aligned} \tag{4.84}$$

Using these relations, we find that to second order in waviness, the contribution is:

$$\delta \sqrt{\det h_4} = \frac{4}{k+1} \frac{1}{t} \left(\dot{\Delta} \cdot \nabla - (\dot{\Delta} \cdot \Delta') \partial_s \right) s^k \tag{4.85}$$

Now we can assemble our vertex operator \mathcal{V} , it is:

$$\mathcal{V} = \delta \sqrt{\det h_1} + \delta \sqrt{\det h_2} + \delta \sqrt{\det h_3} + \delta \sqrt{\det h_4} \tag{4.86}$$

where the field s^k itself is dropped from each term. Now, the propagator for s^k is given in [19], with normalization from [24]. It is expressed in terms of a hypergeometric function:

$$\begin{aligned}
P &= \frac{\alpha_0}{B_k} W^k {}_2F_1(k, k-3/2, 2k-3; -4W) \\
W &= \frac{x_0 x'_0}{(x_0 - x'_0)^2 + \sum_{i=1}^4 |x_i - x'_i|^2}
\end{aligned} \tag{4.87}$$

where,

$$\begin{aligned}
\alpha_0 &= \frac{k-1}{2\pi^2} & B_k &= \frac{2^{3-k} N^2 k(k-1)}{\pi^2 (k+1)^2} \\
{}_2F_1(a, b, c; z) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 dt t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a}
\end{aligned} \tag{4.88}$$

Here, we are interested in largely separated Wilson loops, this means that W will be small. Also, the value of k for the lightest mode of s^k is 2, since we are interested only in the lightest mode, we can set $k=2$. The expansion of the propagator in this regime is:

$$P = \frac{\alpha_0}{B_k} W^2 (1 - 4W + 18W^2 + \dots) \quad (4.89)$$

We are interested only in the leading behaviour in the separation L , which as we will see is $1/L^2$. Also, we are interested only in the result at second order in waviness. In this limit the propagator simplifies further:

$$P = \frac{9}{8N^2} \frac{t^2 t'^2}{[(s - s')^2 + (t - t')^2 + (\vec{x} - \vec{x}')^2 + L^2]^2} \quad (4.90)$$

Our interaction will be:

$$\Lambda = \frac{\lambda}{4\pi^2} \int_{-\infty}^{\infty} ds \int_0^{\infty} dt \int_{-\infty}^{\infty} ds' \int_0^{\infty} dt' \vec{\nabla}_{\Delta=0}(s, t) P \overleftarrow{\nabla}(s', t') \Big|_{\vec{x}=\vec{x}'=0} \quad (4.91)$$

where the arrows indicate the direction the derivatives in the vertex operators act. Note that the waviness Δ has been set to zero in the first vertex operator. We wish to keep only the terms from the above integrations which are proportional to $1/L^2$, and are $\mathcal{O}(\Delta^2)$. Note that:

$$\mathcal{V}_{\Delta=0} = -\frac{12}{5} \frac{1}{t^2} - \frac{4}{15} \vec{\nabla}^2 + \frac{4}{5} \frac{1}{t} \partial_t + \frac{2}{5} (\partial_t^2 + \partial_s^2) \quad (4.92)$$

Now consider acting on the propagator with this vertex operator. Notice first that if we restrict the action of the derivatives to the numerator of the propagator alone (i.e. t^2), the result is zero. The non-zero terms are:

$$\begin{aligned} \mathcal{V}_{\Delta=0} P = \frac{9}{8N^2} \Bigg\{ & -\frac{4}{15} \left(\frac{-12 t^2 t'^2}{D^3} + \frac{24 t'^2 t^2 (\vec{x} - \vec{x}')^2}{D^4} \right) \\ & + \frac{4}{5} \left(\frac{-4 t'^2 t (t - t')}{D^3} \right) \\ & + \frac{2}{5} \left(\frac{-4 t'^2 t^2}{D^3} + \frac{24 t'^2 t^2 (t - t')^2}{D^4} \right) \\ & + \frac{2}{5} \left(\frac{-4 t'^2 t^2}{D^3} + \frac{24 t'^2 t^2 (s - s')^2}{D^4} \right) \Bigg\} \end{aligned} \quad (4.93)$$

where $D = (s - s')^2 + (t - t')^2 + (\vec{x} - \vec{x}')^2 + L^2$. Now note the following integrals:

$$\begin{aligned} A &= \int_{-\infty}^{\infty} ds \int_0^{\infty} dt \frac{t^4}{[(s - s')^2 + (t - t')^2 + L^2]^4} = \frac{\pi}{L^2} \frac{1}{16} + \mathcal{O}(1/L^7) \\ B &= \int_{-\infty}^{\infty} ds \int_0^{\infty} dt \frac{t^2}{[(s - s')^2 + (t - t')^2 + L^2]^3} = \frac{\pi}{L^2} \frac{1}{8} + \mathcal{O}(1/L^5) \\ C &= \int_{-\infty}^{\infty} ds \int_0^{\infty} dt \frac{s^2 t^2}{[(s - s')^2 + (t - t')^2 + L^2]^4} = \frac{\pi}{L^2} \frac{1}{48} + \mathcal{O}(1/L^5) \end{aligned} \quad (4.94)$$

These are the forms which will give terms of order $1/L^2$. The extra $\mathcal{O}(1/L^5)$ and $\mathcal{O}(1/L^7)$ terms arise because the integrations over t are half-domain but the integrand is shifted by t' . The assumption here is that t' is finite, or equivalently, that $\Delta(s', t')$ has compact support. We discard any terms in (4.93) which are not of one of the above three forms. When we apply the second vertex operator, any derivatives which do not act on the numerator alone, will give rise to terms subleading in $1/L$. Thus we consider only those terms of $\mathcal{V}(s', t')$ which contain powers of $\partial_{t'}$ alone. We also consider only those terms which contain Δ , this is because the interaction between two straight lines is zero. These terms are:

$$\begin{aligned} \mathcal{V}(s', t') = & \left(\frac{1}{2} \dot{\Delta}^2 + \frac{1}{2} \Delta'^2 \right) \frac{1}{t'^2} \left\{ -\frac{12}{5} - \frac{4}{15} [t'^2 \partial_{t'}^2 - 3t' \partial_{t'}] \right\} \\ & + \frac{1}{3} \left(-\dot{\Delta}^2 \partial_{t'}^2 + \Delta'^2 \partial_{t'}^2 \right) - \frac{4}{3} \frac{1}{t'} \dot{\Delta}^2 \partial_{t'} \end{aligned} \quad (4.95)$$

Since the derivatives act only on the numerator, which is t'^2 for all relevant terms in (4.93), we can replace $\partial_{t'}$ with $2/t'$ and $\partial_{t'}^2$ with $2/t'^2$. This yields:

$$\mathcal{V}(s', t') = -\frac{4}{t'^2} \dot{\Delta}^2 \quad (4.96)$$

We are now in a position to evaluate (4.91), it is:

$$\begin{aligned} \Lambda = & -4 \frac{\lambda}{4\pi^2} \frac{9}{8N^2} \left[B \left(\frac{48}{15} - \frac{16}{5} - \frac{48}{15} \right) + \frac{48}{5} (A + C) \right] \int_{-\infty}^{\infty} ds' \int_0^{\infty} dt' \dot{\Delta}^2 \\ = & -\frac{9}{20} \frac{\lambda}{\pi} \frac{1}{N^2} \frac{1}{L^2} \int_{-\infty}^{\infty} ds' \int_0^{\infty} dt' \dot{\Delta}^2(s', t') \end{aligned} \quad (4.97)$$

Recall from (3.78) that the waviness function is given by:

$$\Delta(s, t) = \frac{1}{\sqrt{2\pi}} \int dk e^{-iks} e^{-|k|t} (1 + |k|t) \xi(k) \quad (4.98)$$

Therefore,

$$\begin{aligned} \dot{\Delta} &= -\frac{1}{\sqrt{2\pi}} \int dk e^{-iks} e^{-|k|t} k^2 t \xi(k) \\ \int_{-\infty}^{\infty} ds' \dot{\Delta}^2 &= \int dk k^4 t^2 e^{-2|k|t} \xi(k) \cdot \xi(-k) \\ \int_{-\infty}^{\infty} ds' \int_0^{\infty} dt' \dot{\Delta}^2 &= \frac{1}{4} \int dk \frac{k^2}{|k|} \xi(k) \cdot \xi(-k) \\ &= \frac{1}{8\pi} \int dk \int ds ds' \frac{k^2}{|k|} e^{ik(s-s')} \xi(s) \cdot \xi(s') \\ &= \frac{1}{8\pi} \int dk \int ds ds' \frac{1}{|k|} e^{ik(s-s')} \dot{\xi}(s) \cdot \dot{\xi}(s') \end{aligned} \quad (4.99)$$

We need the following fourier transform:

$$\int dk \frac{1}{|k|} e^{ikx} = -\ln(x^2 \mu^2) \quad (4.100)$$

which is the principal value of the integral, and μ is an arbitrary constant with dimensions 1/length. Our final result is therefore:

$$\Lambda = \frac{1}{L^2} \frac{1}{N^2} \frac{\lambda}{8\pi^2} \frac{9}{20} \int ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \ln(\mu^2 (s - s')^2) \quad (4.101)$$

Note that by using an argument analogous to (3.58), $\dot{\xi}(s) \cdot \dot{\xi}(s')$ above may be replaced by $-\frac{1}{2}[\dot{\xi}(s) - \dot{\xi}(s')]^2$.

4.3.3 Discussion

Comparing (4.58) and (4.101) we see immediately that the form of the dependence on the waviness is identical. We saw the very same type of behaviour in section 3.4, where we compared the results for the single wavy line on the two sides of the correspondence. Following the prescription for the AdS/CFT correspondence:

$$\ln(\text{CFT Result}) = -\text{AdS Result} \quad (4.102)$$

we have:

$$\ln \left(1 - \frac{\lambda^3}{2^{11}\pi^4} I + \dots \right) = -\frac{9}{5} \frac{\lambda}{2^5\pi^2} I \quad (4.103)$$

where $I = \frac{1}{N^2 L^2} \int ds ds' \dot{\xi}(s) \cdot \dot{\xi}(s') \ln(\mu^2 (s - s')^2)$

The assumption is that the terms on the CFT side higher order in λ (represented by the “...” above) would all be directly proportional to I , and would re-sum the logarithm to give the AdS result in the limit of large λ . It would be interesting to verify this assumption. Comparing the coefficients on the two sides of the correspondence, we have:

$$\left(\frac{\lambda}{2^5\pi^2} \right)^2 \frac{\lambda}{2} + \dots \rightarrow \frac{9}{5} \frac{\lambda}{2^5\pi^2} \quad (4.104)$$

Thus in some way the interaction is screened in the large λ limit, ending up as being linear in λ .

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Appendix A

Action, Propagators, and Feynman Rules

Appendix A and B have been (mostly) taken directly from [21]. The Euclidean space action of D=4, $\mathcal{N} = 4$ SYM which we are considering is as follows:

$$S = \int d^4x \frac{1}{2g^2} \left\{ \frac{1}{2} (F_{\mu\nu}^a)^2 + (\partial_\mu \phi_i^a + f^{abc} A_\mu^b \phi_i^c)^2 + \bar{\psi}^a i \gamma^\mu (\partial_\mu \psi^a + f^{abc} A_\mu^b \psi^c) \right. \\ \left. + i f^{abc} \bar{\psi}^a \Gamma^i \phi_i^b \psi^c - \sum_{i < j} f^{abc} f^{ade} \phi_i^b \phi_j^c \phi_i^d \phi_j^e + \partial_\mu \bar{c}^a (\partial_\mu c^a + f^{abc} A_\mu^b c^c) + \xi (\partial_\mu A_\mu^a)^2 \right\} \quad (\text{A.1})$$

where the ϕ_i^a are six scalar fields, A_μ^a is the 4-D gauge field, c^a are ghost fields,

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$$

and f^{abc} are the structure constants of the $SU(N)$ Lie algebra,

$$[T^a, T^b] = i f^{abc} T^c$$

and so $a, b, c = 1, \dots, (N^2 - 1)$. The T^a are the $(N^2 - 1)$ generators of the fundamental representation, and so each is an $N \times N$ matrix. They are normalized as

$$\text{Tr } T^a T^b = \frac{1}{2} \delta^{ab}$$

and obey the identity

$$\sum_{c,d} f^{acd} f^{bcd} = N \delta^{ab}$$

Now, ψ^a is a sixteen component spinor obeying the Majorana condition

$$\psi(x) = C \psi^*(x) \quad (\text{A.2})$$

where C is the charge conjugation matrix. $\Gamma^A = (\gamma^\mu, \Gamma^i)$, for $\mu = 1, \dots, 4$ and $i = 5, \dots, 10$ are ten real 16×16 Dirac matrices (in the 10-dimensional Majorana representation with the Weyl constraint) obeying

$$\text{Tr } (\Gamma^A \Gamma^B) = 16 \delta^{AB}$$

We have chosen the covariant gauge fixing condition,

$$\partial_\mu A_\mu^a = 0 \quad (\text{A.3})$$

and we work in Feynman gauge, where the gauge parameter is chosen as $\xi = 1$. The appropriate action for ghost fields, $c^a(x)$ has been included. In Feynman gauge, the vector field propagator is

$$\Delta_{\mu\nu}^{ab}(p) = \text{~~~~~} = g^2 \delta^{ab} \frac{\delta_{\mu\nu}}{p^2},$$

the scalar propagator is

$$D_{ij}^{ab}(p) = \text{————} = g^2 \delta^{ab} \frac{\delta_{ij}}{p^2},$$

the fermion propagator is

$$S^{ab}(p) = \text{-----} = g^2 \delta^{ab} \frac{-\gamma \cdot p}{p^2},$$

and the ghost propagator is

$$C^{ab}(p) = \text{.....} = g^2 \delta^{ab} \frac{1}{p^2}.$$

The vertices can be easily deduced from the non-quadratic terms in the action (A.1).

We use the position-space propagators in 2ω -dimensions (the physical dimension is given by $\omega = 2$). These can be deduced from the Fourier transform:

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \frac{e^{ip \cdot x}}{[p^2]^s} = \frac{\Gamma(\omega - s)}{4^s \pi^\omega \Gamma(s)} \frac{1}{[x^2]^{\omega-s}} \quad (\text{A.4})$$

By setting $s = 1$ we find the Green function in 2ω dimensions:

$$\Delta(x) = \frac{\Gamma(\omega - 1)}{4\pi^\omega} \frac{1}{[x^2]^{\omega-1}} \quad \text{which satisfies} \quad -\partial^2 \Delta(x) = \delta^{2\omega}(x) \quad (\text{A.5})$$

Therefore, for example, the position space propagators for the scalar and gauge fields are:

$$\begin{aligned} \langle A_\mu^a(x) A_\nu^b(y) \rangle &= g^2 \delta^{ab} \delta_{\mu\nu} \Delta(x - y) \\ \langle \phi_i^a(x) \phi_j^b(y) \rangle &= g^2 \delta^{ab} \delta_{ij} \Delta(x - y) \end{aligned} \quad (\text{A.6})$$

The Wilson loop is defined in the following manner:

$$W(C) = \frac{1}{N} \text{Tr} \mathcal{P} \exp g \oint_C d\tau (iA_\mu(x)\dot{x}_\mu + \Phi_i(x)|\dot{x}|\theta_i) \quad (\text{A.7})$$

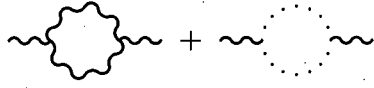
where $x_\mu(\tau)$ is a parameterization of the loop, and θ_i is a point on the five dimensional unit sphere ($\theta^2 = 1$). Also, $A_\mu = A_\mu^a T^a$ and $\Phi_i = \phi_i^a T^a$.

Appendix B

One Loop Self Energy of the Vector and Scalar Fields

Consider the one loop self-energies of the vector and scalar field. The dimension of space-time is $D = 2\omega$. The vector field obtains self-energy corrections from:

- N^2 colours of vector fields and ghost fields:



$$= \delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 2(3\omega-1) \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{2-2\omega}}$$

- $10 - 2\omega$ real scalar fields in the adjoint representation:



$$= -\delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot (10-2\omega) \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{2-2\omega}}$$

- four flavours of four-component Majorana fermions in the adjoint representation:



$$= -\delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 16(\omega-1) \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{2-2\omega}}$$

Note that these are the negative of the conventionally defined self-energies. Thus, to one loop order, the propagator for the unrenormalized gluon, in Feynman gauge is

$$\Delta_{\mu\nu}^{ab} = g^2 \delta^{ab} \frac{\delta_{\mu\nu}}{p^2} - g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 4(2\omega-1) \delta^{ab} \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^{6-2\omega}} \quad (\text{B.1})$$

Similarly, we can compute the one loop correction to the scalar propagator. It obtains corrections from:

- the scalar-vector intermediate state:



$$= \delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 4(2\omega-1) \frac{\delta_{ij}}{p^{6-2\omega}}$$

- and the fermion loop:

$$\text{---} \bigcirc \text{---} = -\delta^{ab} g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 8(2\omega-1) \frac{\delta_{ij}}{p^{6-2\omega}}$$

Thus, to one loop order, the (unrenormalized) scalar propagator is

$$D_{ij}^{ab} = g^2 \delta^{ab} \frac{\delta_{ij}}{p^2} - g^4 N \frac{\Gamma(2-\omega)\Gamma(\omega)\Gamma(\omega-1)}{(4\pi)^\omega \Gamma(2\omega)} \cdot 4(2\omega-1) \frac{\delta_{ij} \delta^{ab}}{p^{6-2\omega}} \quad (\text{B.2})$$

Note that, aside from vector indices, the scalar and vector propagators are identical. Also, note that the self-energy corrections have poles at $\omega = 2$ which arise from an ultraviolet divergence. If we were to compute correlators of local renormalized fields, it would be necessary to add a counterterm to the action in order to cancel these ultraviolet singularities. Here, for purpose of computing the Wilson loop, we leave them unrenormalized.

Appendix C

Integration of the Connected Correlator

It is desirable to integrate out the internal vertex positions w and z from (4.31), as well as the two Wilson loop parameters which are not the arguments of the waviness functions. We have succeeded in doing this, however have in the process introduced a four dimensional Feynman parameter integral which we find intractable. We will outline the process we have followed to execute these integrations and give our Feynman parameter integral result for completeness.

We began by using the Feynman parameter formula (4.41) on the five propagators of (4.31). We kept the dimension at 2ω until both w and z were integrated out, then ω was set to two. The general formula (4.46) was used to integrate out w and z , by completing the square of the denominator from the Feynman parameter form. These integrations were carried out by hand. Next each of the four terms in (4.31) was integrated over the two loop parameters which are not the arguments of ξ or ξ' . The two derivatives in each term were evaluated after one integration. These calculations were done using the software Maple. The result is:

$$\begin{aligned} \langle W_1 W_2 \rangle_{\text{connected}} = 1 \\ - \frac{1}{N^2} \frac{g^6 N^3}{2^{10} \pi^5} \oint ds dt \left[2 \dot{\xi}(s) \cdot \dot{\xi}'(t) \mathcal{F}_1(s, t) - \left(\dot{\xi}(s) \cdot \dot{\xi}(t) + \dot{\xi}'(s) \cdot \dot{\xi}'(t) \right) \mathcal{F}_2(s, t) \right] \end{aligned} \quad (\text{C.1})$$

where:

$$\begin{aligned} \mathcal{F}_i(s, t) = \int_0^1 da db dd dg dm \delta(1 - a - b - d - g - m) \\ \times \frac{amb}{\sqrt{gd(ab + am + mb)(a(g + m + b + d) + (g + m)(b + d))}} \\ \times \frac{F(s - t)^2 - G_i L^2}{[F(s - t)^2 + G_i L^2]^2} \end{aligned} \quad (\text{C.2})$$

where:

$$\begin{aligned} F &= amb[a(g + m + b + d) + (g + m)(b + d)] \\ G_1 &= (ab + am + mb)[(m + d)\{a(b + g) + bg\} + md(g + b)] \\ G_2 &= (ab + am + mb)[(m + b)\{a(d + g) + dg\} + mb(g + d)] \end{aligned} \quad (\text{C.3})$$

where L is the separation between the lines. It is not surprising that even after considerable effort this final integration was not able to be done. Even the asymptotic behaviour at large L was not successfully extracted from this expression.

Appendix D

Selected Integrals

We will show the evaluations for various integrals referenced in the body of the thesis. We begin with the following:

$$I_1 = \int_0^1 d\alpha d\beta d\gamma \delta(1 - \alpha - \beta - \gamma) \frac{1}{\sqrt{\alpha\beta\gamma}} \arctan \left(\sqrt{\frac{\beta\gamma}{\alpha}} \right) \quad (D.1)$$

We introduce the following representation of unity:

$$1 = \int_0^1 d\rho \delta(\rho - \alpha - \gamma) \quad \text{so that} \quad \rho = \alpha + \gamma \quad (D.2)$$

and then introduce scaled versions of α and γ :

$$\alpha' = \rho\alpha \quad \gamma' = \rho\gamma \quad (D.3)$$

this yields:

$$I_1 = \int_0^1 d\rho \int_0^1 \rho d\alpha' d\beta d\gamma' \delta(1 - \beta - \rho) \delta(\rho - \rho\alpha' - \rho\gamma') \frac{1}{\rho \sqrt{\alpha'\beta\gamma'}} \arctan \left(\sqrt{\frac{\beta\gamma'}{\alpha'}} \right) \quad (D.4)$$

Now a power of $1/\rho$ drops out of the delta function, and so ρ disappears from the integrand entirely and we can integrate over it, which yields:

$$I_1 = \int_0^1 d\beta \int_0^1 d\alpha d\gamma \delta(1 - \alpha - \gamma) \frac{1}{\sqrt{\alpha\beta\gamma}} \arctan \left(\sqrt{\frac{\beta\gamma}{\alpha}} \right) \quad (D.5)$$

The next step is to make the substitution $u^2 = \beta$ in order to integrate over β , this gives:

$$\begin{aligned} I_1 &= 2 \int_0^1 du \int_0^1 d\alpha d\gamma \delta(1 - \alpha - \gamma) \frac{1}{\sqrt{\alpha\gamma}} \arctan \left(u \sqrt{\frac{\gamma}{\alpha}} \right) \\ &= 2 \int_0^1 d\alpha d\gamma \delta(1 - \alpha - \gamma) \frac{1}{\sqrt{\alpha\gamma}} \left[u \arctan \left(u \sqrt{\frac{\gamma}{\alpha}} \right) - \frac{1}{2} \sqrt{\frac{\alpha}{\gamma}} \ln \left(1 + \frac{\gamma}{\alpha} u^2 \right) \right]_0^1 \end{aligned} \quad (D.6)$$

Evaluating the u limits, then proceeding to integrate over α , we arrive at:

$$I_1 = 2 \int_0^1 d\gamma \frac{1}{\sqrt{\gamma(1-\gamma)}} \arctan \left(\sqrt{\frac{\gamma}{1-\gamma}} \right) + \int_0^1 d\gamma \frac{1}{\gamma} \ln(1-\gamma) \quad (D.7)$$

The second integral is a well known form:

$$\int_0^1 dx \frac{1}{x} \ln(1-x) dx = -\frac{\pi^2}{6} \quad (\text{D.8})$$

however we'll need to work on the first. We use the following substitution:

$$\gamma = \sin^2 \theta \quad \theta : 0 \rightarrow \frac{\pi}{2} \quad (\text{D.9})$$

which yields, for the first integration of (D.7),

$$\int_0^{\pi/2} d\theta \, 4\theta = \frac{\pi^2}{2} \quad (\text{D.10})$$

And so we have that:

$$I_1 = \frac{\pi^2}{2} - \frac{\pi^2}{6} = \frac{\pi^2}{3} \quad (\text{D.11})$$

The second integral we consider is:

$$I_2 = \int_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma) \frac{1}{\sqrt{\alpha\beta\gamma}} \quad (\text{D.12})$$

We follow the same first steps as for I_1 , that is introducing ρ so that the integral over β may be evaluated, then α may be integrated by simply replacing it with $1-\gamma$ wherever it appears, this yields:

$$I_2 = 2 \int_0^1 d\gamma \gamma^{-1/2} (1-\gamma)^{-1/2} = 2 \frac{\Gamma^2(1/2)}{\Gamma(1)} = 2\pi \quad (\text{D.13})$$

The third and last integral we solve is the following:

$$I_3 = \int_0^1 d\alpha d\beta d\gamma \delta(1-\alpha-\beta-\gamma) \frac{(\alpha\beta\gamma)^{\omega-2}}{[(1-\gamma)\gamma]^{2\omega-3}} \quad (\text{D.14})$$

First we integrate over β so that β is replaced by $1-\alpha-\gamma$. This yields:

$$I_3 = \int_0^1 d\gamma \int_0^{1-\gamma} d\alpha [\alpha(1-\alpha-\gamma)]^{\omega-2} \gamma^{1-\omega} (1-\gamma)^{3-2\omega} \quad (\text{D.15})$$

Next we use the following change of variables:

$$\gamma = 1-y \quad \alpha = yx \quad (\text{D.16})$$

which yields:

$$\begin{aligned} I_3 &= \int_0^1 dy (1-y)^{1-\omega} \int_0^1 dx x^{\omega-2} (1-x)^{\omega-2} \\ &= \frac{1}{2-\omega} \frac{\Gamma^2(\omega-1)}{\Gamma(2\omega-2)} = \frac{1}{2-\omega} \frac{\Gamma^2(\omega-1)}{(2\omega-3)\Gamma(2\omega-3)} \end{aligned} \tag{D.17}$$