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MULTIPLE SCATTERING THEORY WITH PROPER WAVE FUNCTION SYMMETRY APPLIED TO PION-DEUTERON SCATTERING AT THRESHOLD

by

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Abstract

The scattering amplitude for pions on deuterons is calculated in the threshold limit using Watson's multiple scattering theory. Care is taken to use wave functions with proper symmetry throughout and it is shown that the results are identical with those obtained using unsymmetrized wave functions in intermediate states. Terms up to second order in the multiple scattering series are calculated, gradually increasing the complexity of the assumptions until all quantitatively relevant features are taken into account. Specifically, we treat single scattering, double elastic scattering, double charge-exchange scattering, and second order binding correction terms. New quantitative results are obtained which account for non-zero binding energy of the deuteron and nucleon excitation in the propagators, Lorentz-invariant and inelastic scattering kinematic factors in the two-body scattering amplitudes, phase-shift fitted pion-nucleon scattering amplitudes up to P-waves, an S-wave Gartenhaus deuteron wave function, and relativistic effects in high-momentum intermediate states. In addition, a general method utilizing graphs analogous to Feynman diagrams is presented which easily reproduces each order contribution

of the multiple scattering series (for constant two-body T matrices) and allows one to sum the whole series in closed form. In particular, we find the sum of the whole series for π^--deuteron scattering at threshold, including all isospin-flipping terms, a result incorrectly obtained in previous literature. We also find the series sum for π^- scattering on an arbitrary nucleus of neutrons and protons, including charge-exchange scattering. (This result does not appear in the literature). From the series sum we then calculate the higher-order contribution with a Hulthen and then a Gartenhaus S-wave deuteron wave function, first neglecting charge-exchange and then including it. We find the higher-order contribution to be roughly twenty per cent of the first and second order terms combined (at threshold). Our best estimate of the pion-deuteron scattering amplitude at threshold (the pi-d scattering length) is $F_{\pi d} = -.0273$ fermis.

Because pion-deuteron scattering is a three-body problem and because of the similarities with multiple scattering theory, we have included a short discussion of the Faddeev equations. We give particular emphasis to wave function symmetry in the Faddeev approach.

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Acknowledgement

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In addition, the author gives his sincere appreciation to the United States government for its undying interest in his activities and for allowing him to complete his graduate life in Canada.

Introduction and Motivation

The most important part of any calculation is not the result but the reason for undertaking the task at all. The organization of the finished thesis does not reflect the chronological order of events usually, so the beginning of this thesis will be devoted to an explanation of how and why the body of the enclosed work began and developed.

It was suggested by D.S. Beder that low energy scattering of pions on deuterons should be re-investigated because early work by various people failed to account for the Pauli exclusion principle in intermediate states.[1] The low energy range was chosen because here the effects of the exclusion principle would be more dramatic (for a detailed explanation of this statement see chapter 2).

But why look at deuterons at all? The looming presence of the TRIUMF meson facility has influenced the course of more than one graduate research proposal and having a straight-forward calculation ready before the machine is in operation would give experimentalists a chance to make the real world fit the theory. Besides it looks good to

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[1] See for example references (2), (10), (11), (19), (22), (23), (24), and (26).

everyone if a camaraderie between various groups is manifest. In addition, maintaining an interest in strong interaction physics is not incompatible with investigating pion-deuteron scattering. Very little progress has been made in recent years on the pi-nucleon interaction and one might hope for some clues to the two-body problem from the three-body one. In support of this notion, Lovelace has said,[2]

> My opinion is that two-particle systems are now finished. By this I do not mean that we have done everything we hoped to do, but rather that we have done everything we are going to be able to do. I think the future of strong interactions now lies with many-particle systems.

More specifically, the pi-deuteron scattering amplitude depends on the off-shell pi-nucleon scattering amplitudes which are not well understood. Many recent strong interaction theories inter-relate the off-shell and on-shell scattering amplitudes and a knowledge of off-shell behavior would lead to better understanding of on-shell properties. Of course off-shell behavior cannot be determined experimentally from the pi-nucleon interaction alone; therefore, the pi-deuteron scattering amplitude is the simplest candidate to yield off-shell information on the pi-nucleon interaction. Finally, the methods used in treat-

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[2] See reference (20), p.437

pion-deuteron scattering might shed some light on the less-understood two and three-particle resonances of the hadrons.

With the previous statements in mind, the work was undertaken[3] using well-known techniques, adding more and more complications in an effort to find the quantitatively correct pion-deuteron scattering amplitude in the threshold limit (zero energy pions). But when the exclusion principle was applied by symmetrizing (which we take to mean making symmetric or anti-symmetric) intermediate state wave functions, divergences appeared, terms which looked out of place crept in, and chaos ensued. People[4] suggested throwing away the nasty terms or cancelling them by adding others with opposite sign, but no suggestion could be justified. Finally after normalizing the symmetrized propagators correctly, a consistent and justifiable treatment was discovered which eliminated the divergent terms. It was then found that the final results are independent of the choice between symmetrized and unsymmetrized intermediate state wave functions (provided initial and final states

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[3] See the last footnote of chapter 4, section F.

[4] Private communication with members of the University of British Columbia physics department, including Rubin Landau and Leonard Scherk.

are properly symmetrized always). Moyer and Koltun (reference (22)) point out that it is unnecessary to use symmetrized wave functions in intermediate states in the Lippmann-Schwinger equation, $T = V + VGT$, because V is already symmetric in all target particles. However, it is quite another matter to draw the same conclusion for the Watson multiple scattering series for T. We demonstrate in this thesis exactly how the symmetrization effects in the usual multiple scattering terms and the binding correction terms cancel in pairs to each order.

Even though it was found that symmetrization of intermediate states is unnecessary, the time spent worrying about the divergent terms was not wasted. It was believed that the sum of the whole multiple scattering series, divergences and all, would be finite; therefore, a method was devised which allows one to easily sum all the orders of scattering. After the divergence problems were eliminated, the method of summing the whole series still remained valid and so laborious methods of the past were simplified enormously. This in turn facilitated the correcting of previously incorrectly evaluated series sums in the literature.[5]

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[5]See the Appendix of reference (28).

1.Multiple Scattering Ideas from Born to Watson

Although Watson's multiple scattering theory did not appear until the early 1950's, the basic notions of multiple scattering had been formulated well before. In particular, the earliest interpretation of multiple scattering is found in the Born series iterative solution of the Schrödinger equation.[4] The relativistic analog of this is manifest in Feynman's diagrammatic approach to quantum electro-dynamics (QED). The Born, Feynman, and Watson approaches share the properties of starting with Green's function techniques to solve a differential equation and iterating the solution (in multiples of some relevant scattering parameter). A physical interpretation is then attached to each type of term in the expansion and a diagram can be drawn to represent each term, thus reducing messy algebraic manipulations to graph problems and associating an integral (which can be written by inspection) with each graph. Historically and in practice, graphs are not used in the Born or Watson methods because the number of terms retained is usually small and the expressions relatively simple compared to their relativistic

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[4] See any quantum mechanics text on the Born approximation. What we here call the Born series is perhaps more correctly termed Green's function theory.

counterparts in quantum electro-dynamics. We emphasize the association of graphs with Born and Watson terms here partly for comparison with QED but also with some foresight regarding summation of Watson's series.

In what follows we will derive the Born series by formally solving the Schrödinger equation to obtain an integral equation for the wave function; iteration of the integral equation produces a series expansion for the wave function. The scattering amplitude is defined and using the series expansion of the wave function we obtain the Born series for the scattering amplitude. Graphs are then associated with each term in the expansion of the scattering amplitude.

The Feynman approach is similar to the Born work except relativistic equations (Dirac and Maxwell) for the wave functions replace the Schrödinger equation and complications arise from the more difficult equations and from invoking particle statistics (Bose-Einstein and Fermi-Dirac) on the wave functions. The complexity of the solutions is greatly simplified by the use of Feynman diagrams which we introduce and use to comment upon a few relevant difficulties which will appear in the Watson work. We only treat the Feynman approach heuristically owing to the complexity of the problem.

In the Watson work, we lay the preliminary groundwork of scattering theory and then derive the Watson multiple scattering series in terms of the T matrix, closely related to the scattering amplitude. The Watson work is an extension of the Born work in that the scattering potential is broken up into a sum of two-body potentials between the incident particle and each constituent of the scatterer (nucleus), but then the total scattering amplitude is expressed in terms of two-body scattering amplitudes rather than in terms of the total potential. The similarities to the Born and Feynman subsections should be obvious by the end of the Watson derivations and we will avoid lengthy comparisons since they will be undertaken when we try to explain away symmetrization difficulties in Watson's theory.

1a. The Born Series

We consider the scattering of a particle of mass m and momentum $\hbar k$ by a potential $V(\vec{r})$ (where $\vec{r}$ is the position vector from the potential source to the particle). The Born series solution begins with the Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right]\psi(\vec{r}) = \frac{\hbar^2 k^2}{2m}\psi(\vec{r}) \qquad (1a\text{-}1)$$

re-arranged to

$$(\nabla^2 + k^2)\psi(\vec{r}) = \frac{2m}{\hbar^2} V(\vec{r})\psi(\vec{r}) \qquad (1a\text{-}2)$$

Our choice of Green's function $G(\vec{r},\vec{r}')$ satisfies[6]

$$(\nabla^2 + k^2) G(\vec{r},\vec{r}') = \frac{2m}{\hbar^2}\delta^3(\vec{r}-\vec{r}')$$
$$G(\vec{r},\vec{r}') = G(\vec{r}',\vec{r}) \qquad (1a\text{-}3)$$

This implies that

$$(\nabla'^2 + k^2) G(\vec{r},\vec{r}') = \frac{2m}{\hbar^2}\delta^3(\vec{r}-\vec{r}') \qquad (1a\text{-}4)$$

where ∇' implies derivatives with respect to $\vec{r}'$. Let $(\nabla^2 + k^2)^{-1}$ be the appropriate inverse operator of (∇^2+k^2). (see equation (1a-11)).

[6] See equations (1d-45) and (1a-11)..

Then multiplying (1a-3) on the left by $(\nabla^2 + k^2)^{-1}$

$$G(\vec{r},\vec{r}') = \frac{2m}{\hbar^2}(\nabla^2+k^2)^{-1}\delta^3(\vec{r}-\vec{r}') \qquad (1a\text{-}5)$$

Multiplying (1a-2) on the left similarly

$$\psi(\vec{r}) = \frac{2m}{\hbar^2}(\nabla^2+k^2)^{-1} V(\vec{r})\,\psi(\vec{r}) \qquad (1a\text{-}6)$$

Multiplying (1a-5) by $V(\vec{r}')\psi(\vec{r}')$ and integrating over $\vec{r}'$

$$C + \int d^3r' G(\vec{r},\vec{r}')\,V(\vec{r}')\psi(\vec{r}') = \frac{2m}{\hbar^2}(\nabla^2+k^2)^{-1} V(\vec{r})\,\psi(\vec{r})$$

and using (1a-6) we immediately obtain

$$\psi(\vec{r}) = C + \int G(\vec{r},\vec{r}')\,V(\vec{r}')\,\psi(\vec{r}')\,d^3r' \qquad (1a\text{-}7)$$

were C is a constant of integration (i.e., a function of $\vec{r}$ but not $\vec{r}'$). To determine C, note that as $\vec{r}\to\infty$, $G(\vec{r},\vec{r}') \to 0$ (see equation (1a-11)),and in this limit $\psi(\vec{r})$ must reduce to a plane wave (the initial state wave function) travelling along the $\vec{k}$ direction so that

$$C = \psi_i = (2\pi)^{-3/2} e^{i\vec{k}\cdot\vec{r}} \qquad (1a\text{-}8)$$

The complete solution of the problem is thus

$$\psi(\vec{r}) = \psi_i(\vec{r}) + \int G(\vec{r},\vec{r}')\,V(\vec{r}')\,\psi(\vec{r}')\,d^3r' \qquad (1a\text{-}9)$$

with ψ_i the initial state (given in (1a-8)).

Iterating (1a-9) for Ψ produces the Born series expansion for the wave function in the form

$$\Psi(\vec{r}) = \Psi_i(\vec{r}) + \int G(\vec{r},\vec{r}')V(\vec{r}')\Psi_i(\vec{r}')\,d^3r' + \int G(\vec{r},\vec{r}')V(\vec{r}')G(\vec{r}',\vec{r}'')V(\vec{r}'')\Psi_i(\vec{r}'')\,d^3r'\,d^3r'' + \ldots \tag{1a-10}$$

The solution of (1a-3) with appropriate boundary conditions is

$$G(\vec{r},\vec{r}') = \frac{-2m}{4\pi\hbar^2}\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \tag{1a-11}$$

For $\vec{r} >> \vec{r}'$ we can write

$$|\vec{r}-\vec{r}'| \longrightarrow r - \frac{\vec{r}\cdot\vec{r}'}{r} \tag{1a-12}$$

so that

$$G(\vec{r},\vec{r}') \longrightarrow \frac{-2m}{4\pi\hbar^2}\,\frac{e^{ikr}}{r}\,e^{-ik\left(\frac{\vec{r}\cdot\vec{r}'}{r}\right)} \tag{1a-13}$$

and therefore one obtains

$$\Psi(\vec{r}) \longrightarrow \Psi_i(\vec{r}) - \frac{2m}{4\pi\hbar^2}\,\frac{e^{ikr}}{r}\left[\int e^{-ik\left(\frac{\vec{r}\cdot\vec{r}'}{r}\right)}V(\vec{r}')\Psi_i(\vec{r}')\,d^3r' + \int e^{-ik\left(\frac{\vec{r}\cdot\vec{r}'}{r}\right)}V(\vec{r}')G(\vec{r}',\vec{r}'')V(\vec{r}'')\Psi_i(\vec{r}'')\,d^3r'\,d^3r'' + \ldots\right] \tag{1a-14}$$

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[5] See reference (15), p.303, eq.364b.

From (1a-14) we can read off the scattering amplitude, $f(\theta)$, defined as the coefficient of $(2\pi)^{-3/2}\,\frac{e^{ikr}}{r}$, where the scattering angle, θ, is defined as follows: Write a final state plane wave, ψ_f, in terms of the final momentum $\vec{k}_f$ (more correctly, wave vector) as

$$(2\pi)^{3/2}\,\psi_f(\vec{r}') = e^{i\vec{k}_f\cdot\vec{r}'} \qquad \text{(1a-15)}$$

where $\vec{k}_f = k_f\,\vec{r}/r$. For elastic scattering we write therefore

$$k_f = k_i = k \quad (i \text{ stands for initial})$$
$$\theta = \text{angle between } \vec{k}_i \text{ and } \vec{k}_f$$

Now rewrite $f(\theta)$ from (1a-14) by using (1a-15) to obtain

$$f(\theta) = -\frac{2m(2\pi)^3}{4\pi\hbar^2}\left[\int \psi_f^*(\vec{r}')\,V(\vec{r}')\,\psi_i(\vec{r}')\,d^3r' + \int \psi_f^*(\vec{r}')\,V(\vec{r}')\,G(\vec{r}',\vec{r}'')\,V(\vec{r}'')\,\psi_i(\vec{r}'')\,d^3r\,d^3r'' + \cdots\right] \qquad \text{(1a-16)}$$

where the θ-dependence on the right side of (1a-16) is implicit in that

$$\psi_i(\vec{r}) = (2\pi)^{-3/2}\,e^{i\vec{k}_i\cdot\vec{r}}$$
$$\psi_f(\vec{r}) = (2\pi)^{-3/2}\,e^{i\vec{k}_f\cdot\vec{r}}$$
$$\vec{k}_f\cdot\vec{k}_i = k^2\cos\theta$$

and $*$ means complex conjugate.

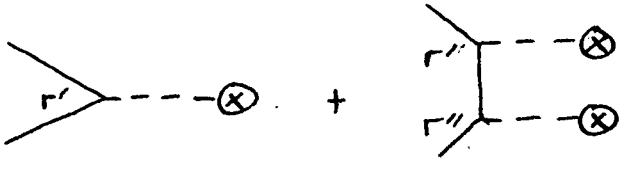
The equation (1a-16) is interpreted as follows: The complete scattering amplitude $f(\theta)$ is the amplitude to scatter once

$$\int \psi_f^*(\vec{r}') V(\vec{r}') \psi_i(\vec{r}') \, d^3r'$$

plus the amplitude to scatter, propagate, and scatter again,

$$\int \psi_f^*(\vec{r}') V(\vec{r}') G(\vec{r}', \vec{r}'') V(\vec{r}'') \psi_i(\vec{r}'') \, d^3r' d^3r''$$

etc. Diagrammatically we would represent the single, double, etc. scattering terms, respectively, as



(1a-17)

where solid lines represent the incoming, propagating, and scattered particle, and a dotted line with a ⊗ represents one interaction with the potential V. This summarizes the relevant features of the Born series.

1b. Quantum Electro-dynamics

No attempt will be made to derive quantum electo-dynamics; rather we will show the way Feynman's diagrammatic approach is applied to some simple scattering problems after some very intuitive discussion on the origin of the diagrams. The resolution of some difficulties in QED which are relevant to Watson's theory will also be covered.[6]

Consider the Dirac equation for an electron

$$(\not{p} - m)\,\psi(x) = 0 \qquad (1b\text{-}1)$$

where p is the electron four-momentum operator, m is the electron mass, x is the electron four-vector position coordinates, and $\psi(x)$ is a four component spinor wave function representing the field of the electron. The slash through p means

$$\not{p} = p_\mu \gamma_\mu = p_0 \gamma_0 - \vec{p} \cdot \vec{\gamma}$$

where the γ_μ are 4x4 Dirac matrices. Feynman writes in analogy to (1a-9) the integral equation for the wave

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[6] For a more detailed account of all that follows on QED see references (1) and (12).

$$\psi(x) = \psi_i(x) + e\int d^4y\, S_F(x-y)\, \not{A}(y)\, \psi(y) \qquad (1b\text{-}2)$$

where $\psi_i(x)$ is a solution of the free-field equation (1b-1) before scattering, e is the charge of the electron ($e < 0$), $S_F(x-y)$ is the propagator for Dirac particles (corresponding to $G(\vec{r},\vec{r}')$), and A is the electro-magnetic four-potential (corresponding to $V(\vec{r})$). Looking back at section 1a, we see that the equation for $G(\vec{r},\vec{r}')$,(1a-5), is heuristically obtained from (1a-2) by inverting the operator on the left of $\psi(\vec{r})$ in (1a-2) (with V=0); i.e.,

$$G(\vec{r},\vec{r}') \sim (\nabla^2 + k^2)^{-1} \times 1 \qquad (1b\text{-}3)$$

realizing that the delta function is the unit operator, 1, in the coordinate representation. In analogy, $S_F(p)$, the Fourier transorm of $S_F(x-y)$ is obtained from (1b-1) by inverting the operator to the left of $\psi(x)$, so that in momentum space

$$S_F(p) \sim \frac{1}{\not{p} - m} \qquad (1b\text{-}4)$$

Everything in (1b-2) is determined except A. The electro-magnetic four-potential satisfies Maxwell's equations

$$\partial_\mu \partial_\mu A_\nu(x) = j_\nu(x) \qquad (1b\text{-}5)$$

where $j_\nu(x)$ is a four-current (the source of the potentials). Again in analogy to (1a-5) and (1b-4) we obtain the photon propagator, D_F, from (1b-5) by inverting the operator to the left of A as

$$D_F(q^2) \sim \frac{1}{q^2} \qquad (1b\text{-}6)$$

where q is the four-momentum of the photon field (note that q's Fourier transform is $-i\partial_\mu$). Then in analogy to (1a-9) and (1b-2),

$$A_\mu(x) = \int d^4y\, D_F(x-y)\, j_\mu(y) \qquad (1b\text{-}7)$$

where $D_F(x-y)$ is the Fourier transform of $D_F(q^2)$. Put (1b-7) into (1b-2) to get

$$\psi(x) = \psi_i(x) + e \int d^4y d^4z\, S_F(x-y)\, D_F(y-z)\, \not{j}\, \psi(y) \qquad (1b\text{-}8)$$

When we obtained our expression for the scattering amplitude (1a-16), part of the left-most G (in (1a-9)) became the e^{ikr}/r of (1a-16) and the rest of this left-most G became ψ_f^* in (1a-16). In analogy the expression for

the scattering amplitude in the QED case removes the left-most S_F of (1b-8) to become (aside from kinematic factors)

$$f \sim \int d^4y\, d^4z\ \bar{\psi}_f(y)\, D_F(y-z) \not{j}(z)\, \psi(y) \qquad \text{(1b-9)}$$

where $\bar{\psi}_f(y) = \psi_f^\dagger(y)\, \gamma_0$

the $\dagger$ means hermitian adjoint, and γ_0 is one of the Dirac matrices. The iteration of (1b-8) for ψ into (1b-9) gives the analog of (1a-16); namely,

$$f \sim \int \Big[\bar{\psi}_f(y)\, D_F(y-z) \not{j}(z)\, \psi_i(y)$$

$$+\ \bar{\psi}_f(y)\, D_F(y-\omega) \not{j}(\omega)\, S_F(y-v)\, D_F(v-z) \not{j}(z) \psi_i(v)$$

$$+ \ldots \Big]\, d\tau \qquad \text{(1b-10)}$$

Just as the solution of (1a-16) is determined in principle once the potential source V is known, the solution of (1b-10) is determined once the current source j is known. Let's see how this works with an example.

Suppose we want to treat coulomb scattering of an electron by a point charge Q located at the origin.

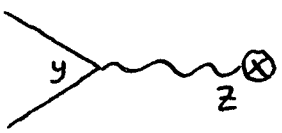
Then the appropriate four-current is

$$j_0(x) \sim Q\,\delta^3(\vec{x}), \quad j_1 = j_2 = j_3 = 0$$

By (1b-10), to first order the scattering amplitude is[6]

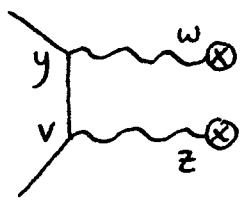
$$f \sim Q \int \bar{\psi}_f(y)\, D_F(y-z)\, \delta^3(\vec{z})\, \psi_i(y)\, dy^4 dz^4 \qquad (1b\text{-}11)$$

and corresponds to the diagram



(1b-12)

where the two solid lines represent the incoming and outgoing electron (ψ_i and ψ_f), the wavy line represents the propagating photon (D_F), and the ⊗ represents the source of the photon ($Q\,\delta^3(\vec{z})$). The ψ's both connect to the point labeled y, the photon propagates from z to y, and the source of the photons is at point z. The next higher term of (1b-10) is represented diagrammatically by



(1b-13)

where internal solid lines connecting two photon lines represent a propagating electron (S_F), and the coordinate

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[6] See reference (1), Vol.I, p.100. The f of (1b-11) as written will contain an energy-conserving δ function which should be removed.

labels at each vertex correspond to those used in (1b-10). The idea is to first draw and label the diagrams and then write the corresponding integrals from the diagrams by inspection rather than start with the more cumberson (1b-10).

Now we come to one of the most important aspects of QED that will carry over to multiple scattering theory. In everything done so far we have not symmetrized the terms as required by Bose-Einstein and Fermi-Dirac statistics (we could also symmetrize the Born work but choose to symmetrize only QED here to avoid duplication and because the problems which arise in QED are relevant to multiple scattering).

Consider electron-positron scattering. The relevant current is now the current of the positron as seen by the electron (or vice versa) and is given by[7]

$$j_\mu(x) = -e\,\bar{\psi}_f^{p}\,\gamma_\mu\,\psi_i^{p}(x) \qquad (1b\text{-}14)$$

where the superscripts on the ψ's mean positron states. The choice of j is motivated by the fact that j_o should be

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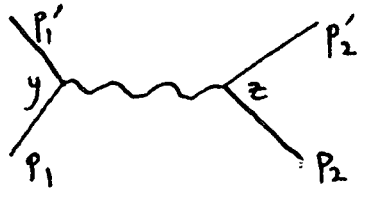
[7] See reference 1, p.135.

density which is something on the order of

$$-e\,\psi_f^{p\dagger}\psi_i^{p}$$

A second order graph looks like (second order in e, first order in e^2)

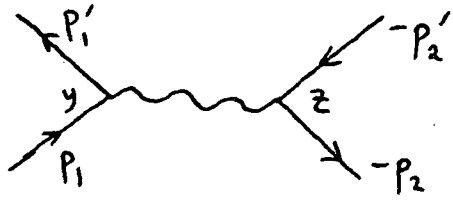


(1b-15)

(where subscript 1 refers to electron, 2 to positron, unprimed p's are initial momenta, primed p's are final momenta) and corresponds to a term

$$e^2\int \bar{\psi}_f(y)\gamma_\mu \psi_i(y) D_F(y-z)\bar{\psi}_f^{\rho}(z)\gamma_\mu \psi_i^{\rho}(z)\,d\tau \qquad (1b\text{-}16)$$

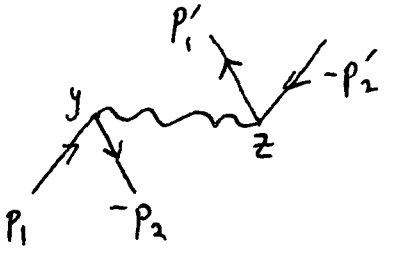
Feynman states that electrons going forward in time are equivalent to positrons running backward in time so that we would more correctly draw (1b-15) with arrows indicating the direction of time for an electron as



(1b-17)

in which we replace a positron with momentum p_2 going forward in time by an electron with momentum $-p_2$ going backward in time. Let's try to symmetrize the above diagram

by interchanging electron lines where possible. At y an electron entering from the past can proceed into the future as shown with momentum p_1' but it can also go back into the past with momentum $-p_2$; at z the electron coming in from the future with momentum $-p_2'$ can scatter into the past with momentum $-p_2$ as shown or it can scatter into the future with momentum p_1'. We would draw the "exchange diagram" resulting from the above observation as

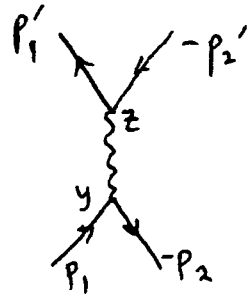


(1b-18)

in which lines p_1' and $-p_2$ of (1b-17) have been interchanged to obtain (1b-18). In other words, we can interchange electron lines as long as we conserve charge; i.e., something like

is not allowed because the electron converts to a positron (the direction of the arrow reverses as we follow the continuous world line of the electron on the left). It is more conventional to twist the photon line of (1b-18)

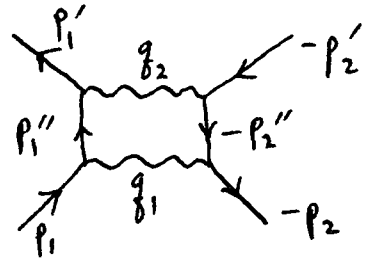
into



(1b-19)

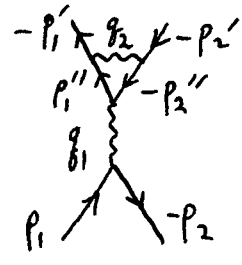
so that the exchange diagram of the second order electron-positron scattering corresponds to annihilation into a photon followed by decay into another electron-positron pair. Fermi-Dirac statistics require that (1b-19) adds to (1b-17) with a relative minus sign (two fermions were exchanged).

Now consider fourth order electron-positron scattering.[8] The most obvious graph is



(1b-20)

and exactly as in the second order case, interchange of lines $-p''_2$ and p_1 gives the graph



(1b-21)

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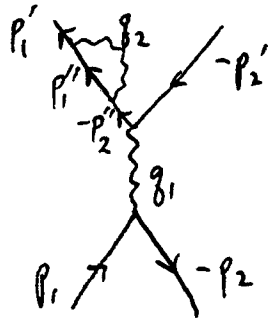
[8] Ibid., pp.148-151.

where we interchanged the location of the bottom of line $-p_2''$ with the top of line p_1 in obtaining (1b-21) from (1b-20). The fermion interchange requires that (1b-21) adds to (1b-20) with a relative minus sign. Interchange of the right side of line q_2 with the top of line q_1 in (1b-21) yields



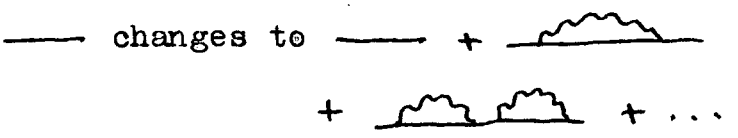
(1b-22)

which must add to (1b-21) with a relative plus sign (boson interchange). The integral corresponding to (1b-22) diverges, and in general all graphs with photon bubbles on electron lines like

(1b-23)

will diverge.

To eliminate these divergence problems one proceeds very crudely as follows: The electron propagator (drawn as a solid line and given by (1b-4))is modified by adding to it all bubbles; graphically we write



(1b-24)

The infinite series of divergent terms can be summed and the result changes the propagator to

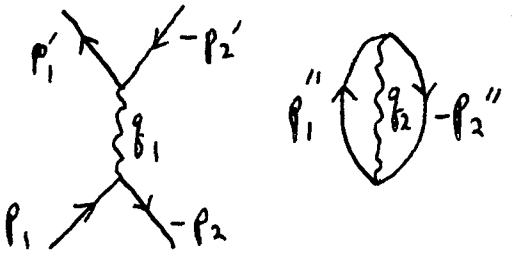
$$\frac{1}{\not{p} - m + \Delta} \tag{1b-25}$$

where Δ is the infinite contribution of one bubble. One then says that the mass m in (1b-4) is the "bare" mass of the electron and that the physically measured mass is

$$m - \Delta = m_{electron} \tag{1b-26}$$

So the theory becomes "renormalized" by changing m to m_e in all electron propagators. The same type of divergences will occur in multiple scattering theory upon symmetrizing.

One last exchange graph must be discussed. Appropriate interchange of photons and electrons in (1b-20) will lead to the graph



(1b-27)

whose corresponding integral also diverges. Graphs such

as these are called disconnected and are thrown out in QED. The justification for throwing them away is this---- the vacuum is constantly producing such bubbles by spontaneous emission and reabsorption and since everything is measured relative to the vacuum one ignores such graphs.

The ideas discussed for QED will appear again when we use Watson's theory.

1c.Preliminary Scattering Notions

In obtaining the Born series and QED results, we worked with the basic field equations and interaction potentials. In multiple scattering theory, especially when applied to strong interactions (nuclear physics), the basic interaction potentials and field equations are not known. Therefore the expansion for the total scattering amplitude of a projectile on a composite system of scatterers (nucleus) in powers of the two-body potentials between the projectile and scattering constituents is replaced by an expansion in powers of some more directly measurable quantity. Since the objective is to treat scattering of a particle by a collection of other particles, the most directly measurable quantities involved are the differential cross sections for the incoming particle scattering on each of the constituents of the scatterer. Any quantities closely related to the experimentally observed two-particle differential cross sections are likely candidates to use as a replacement for the two-body potentials. Historically one chooses to expand in powers of the so-called "T matrix". In the next several pages we develop the groundwork which defines the T matrix and

relates it to the potential V, the scattering amplitude f, and the differential cross section.

We imagine a two-body scattering experiment to start with a free particle approaching a scatterer at time $t=-\infty$, the free particle state unaffected at this time by the presence of the scatterer. As the particle moves closer to the scatterer, the scatterer starts to modify the free particle wave function and continues to do so until the particle gets sufficiently far away from the scatterer again, with the wave function becoming a new free particle state (or superposition of many free particle states) at $t=+\infty$. Define the Heisenberg S matrix by

$$\langle \chi_b | S | \chi_a \rangle = \lim_{t\to\infty} \int \chi_b^{\dagger}(\vec{x},t)\, \psi_a(\vec{x},t)\, d^3x \qquad (1c\text{-}1)$$

where χ_b is the free particle final state describing the asymptotic system after the scattering, and ψ_a describes the system at time t (similarly χ_a is the free particle state before the scattering event). By our now familiar Green's function work,

$$\psi_a(\vec{x},t) = \chi_a(\vec{x},t) + \int d^3x'\, G(\vec{x},\vec{x}')\, V(\vec{x}')\, \psi_a(\vec{x}',t) \qquad (1c\text{-}2)$$

which comes from (1a-9). Now put (1c-2) into (1c-1) getting

$$\langle \chi_b | S | \chi_a \rangle = \langle \chi_b | \chi_a \rangle \qquad (1c\text{-}3)$$
$$+ \lim_{t\to\infty} \int d^3x\, d^3x'\, \chi_b^*(\vec{x},t)\, G(\vec{x},\vec{x}')\, V(\vec{x}')\, \psi_a(\vec{x}',t)$$

Now sacrifice rigor for brevity.[9] The time-dependent part of the product of χ^*_b and ψ_a contributes a factor $e^{i(E_b-E_a)t}$ and the propagator G behaves like

$$\frac{1}{\nabla^2 + k^2} \propto \frac{1}{-E_b + E_a}$$

Here the k^2 part brought in the initial energy E_a and the ∇^2 part brought in the energy at $t=+\infty$ which is E_b (the minus sign in front of E_b comes from the fact that $E = p^2/2m \propto -\nabla^2/2m$). Piecing this together transforms the S matrix (1c-3) into

$$\langle \chi_b | S | \chi_a \rangle = \langle \chi_b | \chi_a \rangle \qquad (1c\text{-}4)$$
$$- \lim_{t\to\infty} \frac{e^{i(E_b-E_a)t}}{E_b - E_a} \int \chi_b^*(\vec{x})\, V(\vec{x})\, \psi_a(\vec{x})\, d^3x$$

where the wave functions in the integral have their time-dependences explicitly removed. Now use the identity

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[9] Those appalled by what follows can consult reference (15), p.178, eq.16 and preceeding pages.

$$\lim_{\substack{t\to\infty \\ \eta\to 0}} \frac{e^{i(E_b-E_a)t}}{E_b-E_a-i\eta} = 2\pi i\,\delta(E_b-E_a) \qquad (1c\text{-}5)$$

to write (1c-4) as

$$\langle \chi_b|S|\chi_a\rangle = \langle \chi_b|\chi_a\rangle - 2\pi i\,\delta(E_b-E_a)\,\langle \chi_b|V|\psi_a\rangle \qquad (1c\text{-}6)$$

which is exact (even though derived heuristically here). The T matrix is usually defined in terms of the S matrix by

$$\langle \chi_b|S|\chi_a\rangle = \langle \chi_b|\chi_a\rangle - 2\pi i\,\delta(E_b-E_a)\,\langle \chi_b|T|\chi_a\rangle \qquad (1c\text{-}7)$$

As (1c-7) is not a particularly useful form for our purposes, we led the reader up to (1c-6) so that we can make the correspondence[10]

$$\langle \chi_b|T|\chi_a\rangle = \langle \chi_b|V|\psi_a\rangle \qquad (1c\text{-}8)$$

an expression we will make direct use of in obtaining the Lippman-Schwinger equation, the cornerstone of Watson's theory.

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[10] Ibid., p.178, eq.16.

We have now defined the new expansion parameter T (which we will use instead of V) and have related T to V by (1c-8). Next we must show how T is directly related to the scattering amplitude f, or equivalently, how it is related to the differential cross section, the quantity determined experimentally. As a first step in obtaining an expression for the differential cross section in terms of T, we state that (by the definition of S) the transition probability, P_{ba}, to go from state χ_a to state χ_b is

$$P_{ba} = |\langle \chi_b | S | \chi_a \rangle|^2 \qquad (1c\text{-}9)$$

In most scattering experiments one doesn't measure scattered states χ_b which overlap initial states χ_a because the detectors would be swamped by the incoming beam. Thus we can say

$$\langle \chi_b | \chi_a \rangle = 0 \qquad (1c\text{-}10)$$

so that

$$P_{ba} = (2\pi)^2 [\delta(E_b - E_a)]^2 |\langle \chi_b | T | \chi_a \rangle|^2 \qquad (1c\text{-}11)$$

using (1c-7). The square of the δ function is tricky;

to treat it properly we paraphrase Bjorken and Drell.[11] If we consider transitions in a time interval from $-t_o/2$ to $+t_o/2$ the δ function would be smeared out and we would replace it according to

$$2\pi\delta(E_b - E_a) = \int_{-\infty}^{\infty} e^{i(E_b - E_a)t}\, dt \qquad (1c\text{-}12)$$

$$\longrightarrow \int_{-\frac{t_o}{2}}^{\frac{t_o}{2}} e^{i(E_b - E_a)t}\, dt = \frac{2\sin\left[\frac{t_o}{2}(E_b - E_a)\right]}{E_b - E_a}$$

We then have, squaring the above equation and integrating both sides over E_b

$$(2\pi)^2\,\delta(0) \longrightarrow 2\pi t_o \qquad (1c\text{-}13)$$

This allows us to write

$$(2\pi)^2\left[\delta(E_b - E_a)\right]^2 = (2\pi)^2\delta(0)\delta(E_b - E_a) \longrightarrow t_o\, 2\pi\delta(E_b - E_a) \qquad (1c\text{-}14)$$

The above is the desired result which converts the square of the δ function into a single energy-conserving δ function. Combining (1c-14) with (1c-11), the transition probability

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[11] See reference (1), p.101.

per unit time is found to be

$$\frac{P_{ba}}{t_o} = 2\pi\,\delta(E_b - E_a)\left|\langle \chi_b | T | \chi_a \rangle\right|^2 \qquad (1c\text{-}15)$$

The T matrix conserves momentum so it is customary to factor out the momentum conserving part by defining a new T through

$$\langle \chi_b | T | \chi_a \rangle = \delta^3(\vec{p}_b - \vec{p}_a)\, T_{ba} \qquad (1c\text{-}16)$$

where $\vec{p}_a$ and $\vec{p}_b$ are total momenta of projectile plus target for initial and final states, respectively. In a manner analogous to our treatment of the squared energy δ function, we smear out the squared momentum δ function over a finite volume of space, v_o, to write the transition probability per unit space-time as

$$\frac{P_{ba}}{v_o t_o} = (2\pi)^{-2}\,\delta^3(\vec{p}_b - \vec{p}_a)\,\delta(E_b - E_a)\left|T_{ba}\right|^2 \qquad (1c\text{-}17)$$

The transition rate per unit space-time into the interval $d^3p_1' d^3p_2'$ is then

$$dR = \frac{P_{ba}}{v_o t_o}\, d^3p_1' d^3p_2' \qquad (1c\text{-}18)$$

$$= (2\pi)^{-2}\,\delta^3(\vec{p}_b - \vec{p}_a)\,\delta(E_b - E_a)\left|T_{ba}\right|^2 d^3p_1' d^3p_2'$$

where subscript 1 refers to projectile and 2 to target momentum in the final state. The cross section is the transition rate per incident flux of particles summed over all final states, or[12]

$$\sigma = (2\pi)^6 \int (2\pi)^{-2} \delta^3(\vec{p}_b - \vec{p}_a)\, \delta(E_b - E_a) \frac{|T_{ba}|^2}{v_{rel}} d^3p_1' d^3p_2' \tag{1c-19}$$

where v_{rel} is the relative velocity of projectile and target. Now work in the center of momentum system of the projectile and target; i.e.,

$$E_1 + E_2 = E_a = E_b = E_1' + E_2'$$
$$\vec{p}_1 + \vec{p}_2 = \vec{p}_a = \vec{p}_b = \vec{p}_1' + \vec{p}_2' = 0$$
$$v_{rel} = |\vec{v}_1 - \vec{v}_2| \tag{1c-20}$$
$$v_1 = p_1/E_1$$
$$v_2 = p_2/E_2$$

Differentiating both sides of the first of the above equations and using the second of the above equations we have

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[12] The $(2\pi)^6$ comes from flux considerations; see e.g., reference (15), p.87.

$$dE_b = p_1'\left(\frac{1}{E_1'} + \frac{1}{E_2'}\right) dp_1' \qquad (1c\text{-}21)$$

where we also used the relativistic connection between all E's and corresponding p's

$$E = \sqrt{p^2+m^2}$$

Multiplying (1c-21) by p_1 and re-arranging gives

$$p_1'^2\, dp_1' = p_1'\left[\frac{1}{E_1'} + \frac{1}{E_2'}\right]^{-1} dE_b' \qquad (1c\text{-}22)$$

The expression (1c-19) for the cross section when differentiated with respect to solid angle and using (1c-22) becomes an expression for the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4 \left|T_{ba}\right|^2 p_1'}{v_{rel}\left(\frac{1}{E_1'} + \frac{1}{E_2'}\right)} \qquad (1c\text{-}23)$$

where p_1' is the projectile momentum in the final state, E_1' and E_2' are the projectile and target final (total) energies, respectively, v_{rel} is the relative velocity of the target and projectile given by

$$v_{rel} = p_1\left(\frac{1}{E_1} + \frac{1}{E_2}\right) \qquad (1c\text{-}24)$$

unprimed p's and E's refer to the initial state, and (1c-23) is the differential cross section in the center of momentum system of the projectile and scatterer.

By definition of the scattering amplitude, f, given by the statements following (1a-14) we can write[13]

$$\frac{d\sigma}{d\Omega} = |f|^2 \qquad (1c\text{-}25)$$

which establishes the correspondence between $|f|^2$ and $|T_{ba}|^2$; only the relative phase between f and T_{ba} is left undetermined. Taking the non-relativistic limit of (1c-23) and (1c-24) and letting the scatterer's mass become infinite (fixed scatterer) we obtain the limiting case

$$\frac{d\sigma}{d\Omega} \xrightarrow[\substack{p_1 \to 0 \\ m_2 \to \infty}]{} \left| (2\pi)^2 T_{ba} m_1 \right|^2 \qquad (1c\text{-}26)$$

which can be compared with results we will obtain using section 1a since there the non-relativistic limit and fixed scatterer were assumed. If we take equation (1a-16) and write it in its non-expanded form it becomes

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[13] See reference (8), p.57.

$$f = -\frac{2m(2\pi)^3}{4\pi}\int \chi_b^* V \psi_a \, d\tau \qquad (1c\text{-}27)$$

using our new notation for free and interacting states. But by (1c-8) and (1c-16),

$$T_{ba} = \int \chi_b^* V \psi_a \, d\tau \qquad (1c\text{-}28)$$

Comparing (1c-28) with (1c-27) we get the relative phase between f and T_{ba} from

$$T_{ba} = -\frac{4\pi}{2m}\frac{1}{(2\pi)^3} f \qquad (1c\text{-}29)$$

By taking the square root of (1c-23) we can define the relation between f and T_{ba} for elastic scattering (primed and unprimed p's and E's equivalent then)[14]

$$f = -\frac{(2\pi)^2 T_{ba}}{\left(\frac{1}{E_1} + \frac{1}{E_2}\right)} \qquad (1c\text{-}30)$$

So far this section has been rather long-winded but necessary to establish conventions; let us briefly review

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[14] See reference (15), p.223, eq.6.

what has been done. We wanted to expand the total scattering amplitude for a projectile on a composite system of scatterers in terms of some new quantity T closely related to the more directly measurable f, the scattering amplitude of the projectile on one constituent of the composite scatterer rather than expand in powers of the harder-to-determine two-body potentials between the projectile and scattering constituents. The first step was to define T (equations (1c-7) and (1c-1)) and find the correspondence between T and the scattering potential V (equation (1c-8)). Finally, it was necessary to relate T to f (and thus to the differential cross section also) to connect T to a real world experiment (equations (1c-30) and (1c-25)). Having accomplished all this, we now proceed to develop the Lippmann-Schwinger equation and Watson's multiple scattering theory.

1d. Watson's Multiple Scattering Series

Consider a scattering event with a total Hamiltonian H describing the system. Begin by breaking up H into the free Hamilitonian K and the potential V according to

$$H = K + V \tag{1d-1}$$

and define eigenstates of H and K by

$$\begin{aligned} H\psi &= E\psi \\ K\chi &= E\chi \end{aligned} \tag{1d-2}$$

Note that H, K, and V are operators in the above. Now find an expression for ψ in terms of χ. Substituting (1d-1) for K in the second equation of (1d-2)

$$(H - V)\chi = E\chi \tag{1d-3}$$

Re-write the first equation of (1d-2) and (1d-3) as

$$\begin{aligned} (E - H)\psi &= 0 \\ (E - H)\chi &= -V\chi \end{aligned}$$

Subtracting the second of the above from the first gives

$$(E - H)(\psi - \chi) = V\chi \qquad (1d\text{-}4)$$

Multiplying (1d-4) on the left by $(E-H)^{-1}$ and adding χ to both sides one gets

$$\psi = \left[1 + \left(\frac{1}{E-H}\right)V\right]\chi \qquad (1d\text{-}5)$$

To show the analogy of this treatment to section 1a, invert (1d-5) for χ in terms of ψ by first multiplying by V yielding

$$V\psi = \left[V + V\left(\frac{1}{E-H}\right)V\right]\chi \qquad (1d\text{-}6)$$

and now combine the two terms on the right into one by writing

$$V\psi = \left[1 + V\left(\frac{1}{E-H}\right)\right]V\chi$$

$$= \left[(E - H) + V\right]\left(\frac{1}{E-H}\right)V\chi$$

$$= (E-K)\left(\frac{1}{E-H}\right)V\chi$$

Inverting for χ we find

$$\chi = \left(\frac{1}{V}\right)(E - H)\left(\frac{1}{E-K}\right)V\psi \qquad (1d\text{-}7)$$

Now put (1d-7) back into (1d-5) and cancel operators with their inverses to obtain the Lippmann-Schwinger equation for the wave function

$$\Psi = \chi + \left(\frac{1}{E-K}\right) V \Psi \qquad (1d\text{-}8)$$

This is identical to (1a-9) of the Born work. In other words, judicious juggling of (1d-5) reproduces (1a-9).

Leaving this digression, we proceed to find an operator expression for the T matrix. Equation (1c-8) states that

$$\langle \chi_b | T | \chi_a \rangle = \langle \chi_b | V | \psi_a \rangle \qquad (1c\text{-}8)$$

and (1d-5) into (1c-8) gives

$$\langle \chi_b | T | \chi_a \rangle = \langle \chi_b | V \left[1 + \left(\frac{1}{E-H}\right) V\right] | \chi_a \rangle$$

which immediately implies

$$T = V + V \left(\frac{1}{E-H}\right) V \qquad (1d\text{-}9)$$

Invert the above to get V in terms of T by combining the two terms into one, writing

$$T = (E-K) \left(\frac{1}{E-H}\right) V$$

and invert to get

$$V = (E-H)\left(\frac{1}{E-K}\right)T$$

Putting the above into the rightmost V of (1d-9) leads us to

$$T = V + V\left(\frac{1}{E-K}\right)T \qquad (1d\text{-}10)$$

This is the T matrix form of the famous Lippmann-Schwinger equation which we will use ad nauseum throughout the rest of our multiple scattering work.[15]

We now proceed with the idea of doing nuclear physics. Consider the scattering of a single projectile on a compound system of n scatterers. The total potential V is then written as the sum of all two-body potentials between the projectile and each scatterer; i.e.,

$$V = \sum_{i=1}^{n} V_i \qquad (1d\text{-}11)$$

Let t_i represent the two-body T matrix for the projectile scattering on particle i. Then the two-body Lippmann-Schwinger equations are

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[15] Ibid., p.751, eq.252b. See also p.198, eq.85 for our equation (1d-8) which Goldberger and Watson refer to as the Lippmann-Schwinger equation.

$$t_i = v_i + v_i g_i t_i \qquad (1d\text{-}12)$$

where

$$g_i = \frac{1}{E_i - K_i} \qquad (1d\text{-}13)$$

in which E_i is the total energy of the incoming projectile plus target particle i in the initial state before the scattering, and K_i is the free energy operator with the property

$$K_i|\chi\rangle = \left[\left(m + \frac{p^2}{2m}\right) + \left(m_i + \frac{p_i^2}{2m_i}\right)\right]|\chi\rangle \qquad (1d\text{-}14)$$

where p and m are the projectile momentum and mass in state $|\chi\rangle$ and p_i and m_i are the target momentum and mass in state $|\chi\rangle$. The total T matrix for the complete scattering satisfies the appropriate Lippmann-Schwinger equation also; namely,

$$T = V + VGT \qquad (1d\text{-}15)$$

where V is given in (1d-11) and

$$G = \frac{1}{E - K} \qquad (1d\text{-}16)$$

in which E is the total energy of projectile plus all

target particles in the initial state before scattering and

$$K = \sum_{i=1}^{n} K_i \qquad (1d\text{-}17)$$

Invert (1d-12) by writing

$$t_i = v_i (1 + g_i t_i)$$

so that

$$v_i = t_i \left(\frac{1}{1+g_i t_i} \right)$$

and thus

$$V = \sum_{i=1}^{n} t_i \left(\frac{1}{1+g_i t_i} \right) \qquad (1d\text{-}18)$$

Iteration of (1d-15) gives

$$T = V + VGV + VGVGV + \ldots \qquad (1d\text{-}19)$$

and putting (1d-18) into (1d-19) we find

$$T = \sum_{i=1}^{n} t_i \left(\frac{1}{1+g_i t_i} \right) \qquad (1d\text{-}20)$$

$$+ \left[\sum_{i=1}^{n} t_i \left(\frac{1}{1+g_i t_i} \right) \right] G \left[\sum_{j=1}^{n} t_j \left(\frac{1}{1+g_j t_j} \right) \right] + \ldots$$

Now expand the terms in parentheses according to

$$\frac{1}{1+g_i t_i} = 1 - g_i t_i + g_i t_i g_i t_i - + \ldots \qquad (1d\text{-}21)$$

and re-group terms in powers of the t's to obtain

$$T = \sum_{i=1}^{n} t_i + \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} t_i G t_j - \sum_{i=1}^{n} t_i g_i t_i \right\} \ldots$$

$$+ \ldots$$

Re-arranging the sums in each order to separate out terms with the same t_i appearing next to itself (separated by propagators of course) we have[16]

$$T = \sum_{i=1}^{n} t_i + \left\{ \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} t_i G t_j + \sum_{i=1}^{n} t_i (G - g_i) t_i \right\}$$

$$+ \ldots \qquad (1d\text{-}22)$$

Terms which contain factors $(G-g_i)$ represent binding corrections which are worthy of consideration in themselves, but one customarily writes[17]

$$G \approx g_i \,, \quad i = 1, n \qquad (1d\text{-}23)$$

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[16] See reference (22).

[17] See the work of Koltun in reference (29), for example.

and ignores the contribution of these binding correction terms. We will return to this point when we discuss symmetrization. This leads us to Watson's formula for the total T matrix in terms of the two-body T matrices,

$$T = \sum_{i=1}^{n} t_i + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} t_i G t_j + \sum_{\substack{i=1 \\ i\neq j,\, j\neq k}}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} t_i G t_j G t_k + \cdots \qquad (1d\text{-}24)$$

and establishes the final result associated with the title of this section.

Now we have to state that everything leading to (1d-24) was done so crudely that we are compelled to make some verbal explanations. First, the way we write the propagators as in (1d-13) or (1d-16) is something like shovelling all the dirt under the rug. There are problems with the denominators when they become zero. Suffice to say that we should make the replacement[17]

$$G = \lim_{\eta\to 0} \frac{1}{E-K+i\eta} \qquad (1d\text{-}25)$$

$$= P\left\{\frac{1}{E-K}\right\} - i\pi\,\delta(E-K)$$

where P means take the principal value of the appropriate

[17] See reference (15), p.72, eq.52 and p.74, eq.63.

integral in which G appears. In summary, (1d-16) is sloppy and should be replaced by (1d-25), and similarly a replacement should be made for (1d-13). Second, the t's that we have written in (1d-24) are two-body T matrices and (1d-24) implies that the projectile scatters off particle i and all other n-1 scatterers sit as spectators; this is certainly an approximation (the so-called impulse approximation) and the more correct statement of Watson's result is that all the t's in (1d-24) are really τ's, where τ_i is the appropriate T matrix for scattering of the projectile on particle i in the presence of all the n-1 other scatterers.[18] Third, we have not dealt with the problem of symmetrization of the wave functions (nuclei are composed of many identical particles); though we will not go through this section again properly symmetrizing everything, let us just state that all states, initial, final, and intermediate, must be properly symmetrized. One then finds that symmetrized intermediate states give exactly the same results as unsymmetrized intermediate states (provided initial and final states are always symmetrized in both cases) when properly handled. We will return to this point in section 2b and related

[18] Ibid., p.754, eq.265.

appendices. Note that our statements do not necessarily apply when the projectile is the same as one of the nuclear constituents. The reader should pay particular attention to Goldberger and Watson, reference (15), pages 131 to 133 and the last sentence of page 750 which continues over to page 751. We quote it as follows:

> We naturally assume, however, that the target wave functions g are appropriately symmetrized in the coordinates of identical bound particles.

One final comment is necessary. Our derivation of Watson's multiple scattering series, (1d-24), is not the derivation of Watson but follows the simpler but less rigorous arguments of Moyer and Koltun (reference (22)). Watson finds coupled integral equations for the wave function Ψ which formally solve equation (1d-8). These coupled equations are as follows:[19]

$$\Psi = \chi + \sum_{i=1}^{n} G \tau_i \phi_i \qquad \text{(1d-26)}$$

$$\phi_i = \chi + \sum_{i \neq j = 1}^{n} G \tau_j \phi_j$$

Iteration of (1d-26) combined with (1c-8) reproduces (1d-24) except for the replacement of the t's by τ's (defined on the previous page).

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[19] Ibid., p.751, eqs.253a and 253b.

2.Simplified Low Order Pi-Deuteron Scattering Using Watson's Multiple Scattering Theory

Our ultimate goal is to obtain the pi-d scattering amplitude using Watson's multiple scattering theory and taking all quantitatively relevant effects into account. Naturally this is a large order to serve, so rather than present the meal in its entirety, we choose to offer it in several courses. To keep the arguments as simple as possible, we will concentrate on π^--d scattering in the threshold limit (the pion stikes the deuteron with zero momentum in the center of mass and lab systems). The incident projectile is chosen to be a pion (rather than a proton or neutron) to avoid extra symmetrization difficulties and because the two-body pi-nucleon interaction is well known (phenomenologically at least). Also pion beams will be readily available once the TRIUMF meson facility is in operation. We choose a π^- rather than a π^0 to avoid coulomb effects in intermediate states of the charge exchange process (to be covered in detail shortly) although this is a rather moot point since the advantage is lost in the elastic scattering contribution. A deuteron is chosen as the target nucleus because it is the simplest multi-particle aggregate, the deuteron wave

function is well-known, and we naively expect symmetrization to make a big difference here (see the third paragraph below). Finally, we consider zero energy scattering because in this limit we can throw away the delta function part of the propagators (to be explained when we take this step) and the expressions become simpler.[20]

The menu of the pi-d banquet begins with calculations of low order scattering terms neglecting symmetrization of intermediate state wave functions. Then we discuss how symmetrization of intermediate states is handled and why it has no effect on the results. We assume constant two-body T matrices throughout this section (i.e., the choice (2a-6)). This approximation is a highly unrealistic one but is introduced nevertheless because it greatly simplifies the calculations. In momentum space the constant two-body T matrix is just a constant multiplied by an overall momentum conserving delta function. We abandon the constant two-body T matrix assumption in chapter 4.

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[20] See equation (1d-25).

Note that the symmetrization effects we expect to see only occur in second and higher order scattering terms but not in first order (there is no sum over intermediate states in first order). Now usually when one does a calculation of a sum of terms one expects the first order to dominate, and each successive order to contribute much less than the previous one. However, by accident nature has chosen to make the π^--neutron scattering amplitude almost equal in magnitude and opposite in sign to the π^--proton scattering amplitude at threshold. The first order contribution in the deuteron is roughly the sum of these two scattering amplitudes so that the first order contribution is very small, and in fact about equal in magnitude to the second order contributions. For this reason the deuteron is an unusually good target for studying higher order (second and beyond) effects.

Two remarks on symmetrization in the deuteron are now in order. First, if one neglects the small D-wave admixture in the deuteron wave function, the state is then pure S-wave (the orbital angular momentum between the neutron and proton is zero). Now consider double charge-exchange (second order charge-exchange) scattering, to be referred

to from here on as DCE, in which a π^- enters and strikes the proton converting the proton to a neutron and itself to a π^0; thus one has an intermediate state consisting of two neutrons and a π^0; the π^0 proceeds to strike the original neutron converting it to a proton and the π^0 back to a π^-. Since the original proton and neutron were in a relative S-state, an incoming zero energy pion shouldn't excite the nucleons so we expect the intermediate state nucleons (two neutrons) to still be predominantly in an S-state. In addition, the deuteron has total angular momentum J=1 and since L=0 (S-state), we must conclude that the spin is S=1. So if we don't allow for any spin-flipping mechanism in the pi-nucleon interaction, the intermediate state of two neutrons must also have S=1. This means that the spin state of the two neutrons is symmetric (triplet) and the space part of the wave function is also symmetric (S-state) so that the total wave function for the intermediate state is symmetric, which is forbidden by the Pauli exclusion principle. Therefore we expect symmetrization of the wave functions to drastically reduce the contribution of the DCE term in the multiple scattering series.

We have made two contradictory statements so far.

In an earlier paragraph we stated that symmetrization of intermediate states has no effect whereas we also showed that we expect a large effect in the DCE term. In fact, there is a term coming from symmetrization which does reduce the DCE contribution, but there are other terms from the binding corrections which cancel the effects of symmetrization. This will all become clearer when we present the detailed calculations.

Secondly, we wish to comment on the validity of symmetrizing intermediate states in general. Some might doubt the whole symmetrization requirement on the grounds that intermediate states don't behave in the same way as external (initial and final) states; for example, energy is not conserved in intermediate collisions (particles are not on their mass shells). One is thus tempted to ask if symmetrization is required. If one believes in quantum field theory, the answer is an unequivocal affirmation of the symmetrization requirement. For details the reader is referred to Appendix 1.

2a. Low Order Multiple Scattering Terms for π^--d Scattering Neglecting Symmetrization

We now use Watson's theory and calculate the low order terms of (1d-24) with unsymmetrized intermediate state wave functions. Our aim is to start with the simplest cases and build up the work in stages until we can generalize our conclusions. The crudeness of the approximations we make here will be checked in chapter 4 when we do the problem in all its complexity. Suffice to say that additional complications to be introduced in chapter 4 would only obscure the relevant features and would not affect the general argument here.

The deuteron wave function is taken to be

$$\Psi_D(\vec{r},\vec{R}) = \Psi_D(\vec{r})\,\frac{e^{-i\,\vec{P}\cdot\vec{R}}}{(2\pi)^{3/2}} \qquad (2a\text{-}1)$$

where $\Psi_D(\vec{r})$ is a Hulthen function given in Appendix 2. We are neglecting the small D-state admixture. In the above, $\vec{P}$ and $\vec{R}$ are the momentum and position coordinates of the deuteron center of mass, $\vec{r}$ is the relative position coordinate of the two nucleons, $\vec{r}_1$ and $\vec{r}_2$ are the nucleon position coordinates in the lab system, and we write

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$
$$\vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) \qquad (2a\text{-}2)$$

We assume throughout this section that the mass of the neutron and proton are equal and also take the π^- and π^0 masses equal so that it is convenient to define the center of mass momentum $\vec{P}$ and relative momentum $\vec{k}$ in terms of the nucleon lab momenta $\vec{p}_1$ and $\vec{p}_2$ as

$$\begin{aligned} \vec{P} &= \vec{p}_1 + \vec{p}_2 \\ \vec{k} &= \tfrac{1}{2}(\vec{p}_2 - \vec{p}_1) \end{aligned} \qquad \text{(2a-3)}$$

It is also convenient to write the relative position vector part of (2a-1) in terms of its Fourier transform[21]

$$\psi_D(\vec{r}) = (2\pi)^{-3/2} \int e^{-i\vec{k}\cdot\vec{r}}\, \psi_D(\vec{k})\, d^3k \qquad \text{(2a-4)}$$

As a complete set of intermediate states we take the plane wave states

$$\begin{aligned} \chi_n &= \frac{e^{-i\vec{p}_{\pi_n}\cdot\vec{r}_\pi}}{(2\pi)^{3/2}} \frac{e^{-i\vec{P}_n\cdot\vec{R}}}{(2\pi)^{3/2}} \frac{e^{-i\vec{k}_n\cdot\vec{r}}}{(2\pi)^{3/2}} \\ &= \frac{e^{-i\vec{p}_{\pi_n}\cdot\vec{r}_\pi}}{(2\pi)^{3/2}} \frac{e^{-i\vec{p}_{1_n}\cdot\vec{r}_1}}{(2\pi)^{3/2}} \frac{e^{-i\vec{p}_{2_n}\cdot\vec{r}_2}}{(2\pi)^{3/2}} \end{aligned} \qquad \text{(2a-5)}$$

where the subscript n refers to the particular intermediate state and all the r's, p's, and k's are defined as before and in analogy $\vec{r}_\pi$ is the position vector of the pion in

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[21] See Appendix 2.

the lab and $\vec{p}_{\pi_n}$ is the pion lab momentum in the intermediate state n.

We choose a particularly simple form for the two-body T matrix operator in coordinate space

$$\hat{t}_1 = (2\pi)^3 t_o \delta^3(\vec{r}_\pi - \vec{r}_1) \qquad (2a\text{-}6)$$

$$\hat{t}_2 = (2\pi)^3 t_o \delta^3(\vec{r}_\pi - \vec{r}_2)$$

where t_o is a constant. This choice is motivated by noticing that

$$\int \frac{e^{i\vec{p}'_\pi \cdot \vec{r}_\pi}}{(2\pi)^{3/2}} \frac{e^{i\vec{p}'_1 \cdot \vec{r}_1}}{(2\pi)^{3/2}} \hat{t}_1 \frac{e^{-i\vec{p}_\pi \cdot \vec{r}_\pi}}{(2\pi)^{3/2}} \frac{e^{-i\vec{p}_1 \cdot \vec{r}_1}}{(2\pi)^{3/2}}$$

$$= t_o \delta^3(\vec{p}'_\pi + \vec{p}'_1 - \vec{p}_\pi - \vec{p}_1)$$

That is, t sandwiched between plane wave pion-nucleon states is a constant multiplied by an overall momentum-conserving delta function. The constant t_o will vary depending on whether we scatter off a proton elastically ($t_o = t_p$), off a neutron elastically ($t_o = t_n$), or charge-exchange scatter ($t_o = t_{ce}$).

Let's start by calculating the single scattering terms. From (1d-24), these are

$$\langle \psi_b | \hat{t}_1 | \psi_a \rangle + \langle \psi_b | \hat{t}_2 | \psi_a \rangle$$

where the subscript "b" stands for the final state, "a" for the initial state and

$$\psi_a = \frac{e^{-i\vec{p}_{\pi_a}\cdot\vec{r}_\pi}}{(2\pi)^{3/2}}\,\psi_D(\vec{r},\vec{R})$$

$$\psi_b = \frac{e^{-i\vec{p}_{\pi_b}\cdot\vec{r}_\pi}}{(2\pi)^{3/2}}\,\psi_D(\vec{r},\vec{R}) \qquad (2a\text{-}7)$$

where $\vec{p}_{\pi_a}$ and $\vec{p}_{\pi_b}$ are the initial and final pion momenta which we will set equal to zero in the end (threshold limit). If we let particle 1 be the proton and 2 the neutron the $\hat{t}_1$ term will give a factor t_p and the $\hat{t}_2$ term a factor t_n but neither will contribute any t_{ce} factors because a single charge-exchange cannot leave the final nucleons in the deuteron state (because there will be two neutrons). Let us just concentrate on the $\hat{t}_1$ term (the $\hat{t}_2$ term is treated analogously). Combining (2a-7), (2a-6), (2a-4), and (2a-3)

$$\langle\psi_b|t_1|\psi_a\rangle = \int \frac{e^{i\vec{p}_{\pi_b}\cdot\vec{r}_\pi}}{(2\pi)^{3/2}}\,\frac{e^{i\vec{P}_b\cdot\vec{R}}}{(2\pi)^{3/2}}\,\frac{e^{i\vec{k}_b\cdot\vec{r}}}{(2\pi)^{3/2}}$$

$$\psi_D^*(k_b)\,(2\pi)^3\,t_p\,\delta^3(\vec{r}_\pi-\vec{r}_1)\,\frac{e^{-i\vec{p}_{\pi_a}\cdot\vec{r}_\pi}}{(2\pi)^{3/2}}\,\frac{e^{-i\vec{P}_a\cdot\vec{R}}}{(2\pi)^{3/2}} \qquad (2a\text{-}8)$$

$$\frac{e^{-i\vec{k}_a\cdot\vec{r}}}{(2\pi)^{3/2}}\,\psi_D(\vec{k}_a)\,d^3k_a\,d^3k_b\,d^3r\,d^3R\,d^3r_\pi$$

where $\vec{P}_a$ and $\vec{P}_b$ are the initial and final deuteron center of mass momenta. Now use (2a-2) to convert the delta function part of the T matrix into one involving cm and relative coordinates to obtain

$$\delta^3(\vec{r}_\pi - \vec{r}_1) = \delta^3\left(\vec{r}_\pi + \frac{\vec{r}}{2} - \vec{R}\right) \qquad (2a\text{-}9)$$

The integral over R is easy with the above delta function; the remaining position vector integrals just give delta functions over the appropriate momenta; and we can also integrate over k_b to remove one delta function yielding

$$\langle \psi_b | \hat{t}_1 | \psi_a \rangle = t_p \delta^3(\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_a} - \vec{P}_a) \int \psi_D^*\left(\vec{k}_a + \frac{\vec{P}_a - \vec{P}_b}{2}\right) \psi_D(\vec{k}_a)\, d^3k_a \qquad (2a\text{-}10)$$

Now take the threshold limit setting $\vec{P}_a$ and $\vec{P}_b$ equal to zero and notice that the resulting integral over the ψ's just gives a factor of one (because the wave functions are unit normalized). The result is

$$\langle \psi_b | \hat{t}_1 | \psi_a \rangle = t_p \delta^3(\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_a} - \vec{P}_a) \qquad (2a\text{-}11)$$

where the delta function insures momentum conservation. To obtain the scattering amplitude one throws away the

delta function (see (1c-16) and (1c-30)). The t_2 term is handled exactly the same way and not surprisingly

$$\langle \Psi_b | \hat{t}_2 | \Psi_a \rangle = t_n \delta^3(\vec{P}_{\pi_b} + \vec{P}_b - \vec{P}_{\pi_a} - \vec{P}_a) \qquad (2a\text{-}12)$$

in the threshold limit. Since all future calculations in this section are treated along the same lines as the previous one, we put most of the details of succeeding calculations in the appendices.

Now consider second order scattering terms. According to (1d-24) these are expressed by

$$\langle \Psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \Psi_a \rangle + \langle \Psi_b | \hat{t}_2 \hat{G} \hat{t}_1 | \Psi_a \rangle$$

In this case we must insert the plane wave states of (2a-5) so that

$$\langle \Psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \Psi_a \rangle = \langle \Psi_b | \hat{t}_1 | \chi_n \rangle \langle \chi_n | \hat{G} | \chi_m \rangle \langle \chi_m | \hat{t}_2 | \Psi_a \rangle \qquad (2a\text{-}13)$$

and similarly for the $\hat{t}_2 \hat{G} \hat{t}_1$ term. The calculation of each of the above matrix elements is straightforward and we refer the reader to Appendix 4. Only two comments are necessary. For elastic scattering the $\hat{t}_1 \hat{G} \hat{t}_2$ term will contribute a term proportional to $t_n t_p$ and so will

the $\hat{t}_2\hat{G}\hat{t}_1$ term since they represent scattering off one nucleon and then the other. But for charge-exchange scattering, only the $\hat{t}_2\hat{G}\hat{t}_1$ term contributes because it represents a π^- scattering off the proton converting the proton to a neutron and itself to a π^o and then charge-exchanging back again off the original neutron; however, the $\hat{t}_1\hat{G}\hat{t}_2$ term cannot contribute to charge-exchange because it represents a π^- striking a neutron first and thus cannot convert the neutron to a proton and still conserve charge. The result is that we get a term proportional to $2t_n t_p$ from elastic scattering and one proportional to $-t_{ce}^2$. The minus sign comes from the fact that the neutron and proton are interchanged after the scattering by the charge-exchange process. (By choosing this minus sign we are effectively symmetrizing the initial and final states). It is also a simplifying assumption to neglect the kinetic energy of the nucleons in the propagator G as well as taking the binding energy of the deuteron equal to zero in the propagator. Summing (2a-13) over intermediate states one then finds for second order scattering[22]

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[22] See Appendix 4.

$$\langle \psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \psi_a \rangle + \langle \psi_b | \hat{t}_2 \hat{G} \hat{t}_1 | \psi_a \rangle$$

$$= \left[2 t_n t_p - t_{ce}^2 \right] \left\langle \frac{1}{r} \right\rangle \left[-(2\pi)^2 m_\pi \right] \delta^3 (\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_a} - \vec{P}_a) \qquad (2a\text{-}14)$$

where we define the expectation value of 1/r by

$$\left\langle \frac{1}{r} \right\rangle = \int \psi_D^*(r) \frac{1}{r} \psi_D(r) \, d^3 r \qquad (2a\text{-}15)$$

Application of (1c-30) to convert the T matrix to the scattering amplitude gives (up to second order terms)[23]

$$F_{\pi d} = f_n + f_p + \left(2 f_n f_p - f_{ce}^2 \right) \left\langle \frac{1}{r} \right\rangle$$

$$f_n = f_{\pi^- n \to \pi^- n} \qquad (2a\text{-}16)$$

$$f_p = f_{\pi^- p \to \pi^- p}$$

$$f_{ce} = f_{\pi^0 n \to \pi^- p} = f_{\pi^- p \to \pi^0 n}$$

the f's being the various two-body scattering amplitudes. In writing (2a-16) we have neglected terms of order m_π / m_n compared to unity. Equation (2a-16) is well-known.[24]

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[23] See Appendix 5.

[24] See references (2),(10),(19),(22),(26), especially (10), eq.A.3.

There is one curious result left to discuss which occurs in all orders of scattering past third. For example, consider elastic scattering from the p to the n, then back to the p, and finally back to the n once more (a fourth order contribution). The appropriate T matrix element is (from (1d-24))

$$\langle \psi_b | \hat{t}_2 \hat{G} \hat{t}_1 \hat{G} \hat{t}_2 \hat{G} \hat{t}_1 | \psi_a \rangle \qquad (2a\text{-}17)$$

and treating this just as we did the second order elastic terms of (2a-16) we find

$$\langle \psi_b | \hat{t}_2 \hat{G} \hat{t}_1 \hat{G} \hat{t}_2 \hat{G} \hat{t}_1 | \psi_a \rangle \propto t_n^2 t_p^2 \left\langle \frac{1}{r^3} \right\rangle \qquad (2a\text{-}18)$$

making the same approximations as before. We claim that in general the n^{th} order scattering term will be proportional to the expectation value of $(1/r)^{n-1}$.[25] But the expectation value of $(1/r)^n$ diverges for n greater than two if $\psi_D(r=0) \neq 0$. To see this in a specific case write

$$\left\langle \frac{1}{r^n} \right\rangle = \int \psi_D^*(\vec{r}) \frac{1}{r^n} \psi_D(\vec{r}) d^3r \qquad (2a\text{-}19)$$

and recall that

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[25] See section 3a.

$$\psi_D(\vec{r}) \propto \frac{e^{-\alpha r} - e^{-\beta r}}{r}$$

for our Hulthen choice of ψ . Therefore the integral in (2a-19) diverges for $n > 2$ so we conclude that every term in the multiple scattering series past third order will diverge. The divergence results from our bad (unphysical) choice of two-body T matrix (2a-6) but it is still possible to reconcile the difficulty and get a finite result for the complete pi-d scattering amplitude without abandoning (2a-6). For example, if one sums all the terms of the multiple scattering series for pi-d scattering the $(1/r)^n$ terms add in a geometric series and the sum of all terms can be written in closed form. The result neglecting charge-exchange is[26]

$$F_{\pi d} = \left\langle \frac{f_n + f_p + 2f_n f_p / r}{1 - (f_n f_p / r^2)} \right\rangle \qquad (2a\text{-}20)$$

where we neglected terms of order m_π / m_n.

To summarize, in pi-d scattering all terms of the multiple scattering series past third order diverge for

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[26] See reference (2), reference (19), and chapter 3.

our simple choice of constant two-body T matrices. Nevertheless, one can still obtain a finite result for the complete pi-d scattering amplitude by summing the multiple scattering series to all orders without abandoning the constant T matrix approximation. The scattering amplitude will always be finite (even when we include charge-exchange) although this has not been shown in general. We show how to always obtain finite results in chapter 3.

2b. Symmetrization of Intermediate States

We will now demonstrate how to incorporate symmetrized intermediate states into Watson's multiple scattering theory. We will also show that this procedure leads to no new results so that one is justified in neglecting symmetrization of intermediate states in general (at least when the projectile is not the same as one of the scatterers). First, however, we would like to show how direct insertion of symmetrized intermediate states into Watson's series (1d-24) leads to incorrect results. The point is that binding corrections which are usually ignored (e.g., ref.(19)) become fundamentally important when intermediate states are symmetrized.

To begin, the initial and final deuteron states must be completely antisymmetric in space, spin, and isospin variables. Since the deuteron wave function is spacially symmetric (S and D-waves) and the spin part is symmetric (spin one), the isospin part must be antisymmetric so we write the isospin part as

$$\frac{1}{\sqrt{2}}\left[|np\rangle - |pn\rangle\right]$$

where the notation $|np\rangle$ means particle 1 is a neutron and particle 2 a proton, etc. The three symmetric isospin

states are similarly written

$$|nn\rangle$$
$$|pp\rangle$$
$$\frac{1}{\sqrt{2}}|np+pn\rangle$$

and the above four isospin states form a complete set which we will sum over in intermediate states. Note that we will not write the spin part of the wave functions because we always work with two-body T matrices that don't flip spin (in this section) so it is unnecessary to pull the spin part through all the calculations.

The intermediate states must also be totally antisymmetric and since the spin part is symmetric (spin one does not change) we must have the space part and isospin part of opposite symmetry (one symmetric, the other antisymmetric). Let χ^S and χ^A be the space symmetric and space antisymmetric parts of the allowed intermediate state wave functions. Then symmetrizing (2a-5) we must have for plane wave intermediate states

$$\chi^{S}_{A\,n} = \frac{e^{-i\vec{p}_{\pi n}\cdot\vec{r}_{\pi}}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}}\left[\frac{e^{-i\vec{p}_{1n}\cdot\vec{r}_1}}{(2\pi)^{3/2}}\frac{e^{-i\vec{p}_{2n}\cdot\vec{r}_2}}{(2\pi)^{3/2}} \pm \frac{e^{-i\vec{p}_{1n}\cdot\vec{r}_2}}{(2\pi)^{3/2}}\frac{e^{-i\vec{p}_{2n}\cdot\vec{r}_1}}{(2\pi)^{3/2}}\right]$$

$$= \frac{e^{-i\vec{p}_{\pi n}\cdot\vec{r}_{\pi}}}{(2\pi)^{3/2}}\frac{e^{-i\vec{P}_n\cdot\vec{R}_n}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}}\left[e^{-i\vec{k}_n\cdot\vec{r}} \pm e^{i\vec{k}_n\cdot\vec{r}}\right] \qquad (2b\text{-}1)$$

The allowed intermediate states then are

$$|\chi^S\rangle \frac{1}{\sqrt{2}}\left[|np\rangle - |pn\rangle\right]$$
$$|\chi^A\rangle |nn\rangle \qquad (2b\text{-}2)$$
$$|\chi^A\rangle |pp\rangle$$
$$|\chi^A\rangle \frac{1}{\sqrt{2}}\left[|np\rangle + |pn\rangle\right]$$

We must now decide how to insert a complete set of states with proper normalization. When states are not symmetrized, one writes the unit operator as

$$\hat{1} = \sum_n |\chi_n\rangle\langle\chi_n|$$

so that multiplying both sides by $\langle\chi_m|$ gives

$$\langle\chi_m| = \sum_n \langle\chi_m|\chi_n\rangle\langle\chi_n| = \langle\chi_m$$

as it should since for unsymmetrized plane wave states,

$$\langle\chi_m|\chi_n\rangle = \delta_{mn}$$

But for symmetrized plane wave states normalized as above

$$\langle\chi_m^{Sym}|\chi_n^{Sym}\rangle = 2\delta_{mn}$$

so the unit operator must be written

$$\hat{1} = \frac{1}{2} \sum_n |\chi_n^{Sym}\rangle\langle\chi_n^{Sym}| \qquad (2b\text{-}3)$$

where the sum is over all states of the forms given in (2b-2).

We are now prepared to calculate the DCE term with symmetrized wave functions. From the previous paragraphs we write

$$\langle\psi_b|\hat{t}_1^{ce}\hat{G}\hat{t}_2^{ce}|\psi_a\rangle = \frac{1}{4}\left\langle\psi_b\left(\frac{np-pn}{\sqrt{2}}\right)\middle|\hat{t}_1^{ce}\middle|\chi_n^A(nn)\right\rangle$$

$$\left\langle\chi_n^A(nn)\middle|G\middle|\chi_m^A(nn)\right\rangle\left\langle\chi_m^A(nn)\middle|\hat{t}_2^{ce}\middle|\psi_a\left(\frac{np-pn}{\sqrt{2}}\right)\right\rangle \qquad (2b\text{-}4)$$

In the above we have only written those intermediate states which give a non-zero contribution. That is, a π^- scattering off a deuteron and charge-exchanging to a π^o can only leave behind two neutrons in the intermediate state. First remove the isospin parts in (2b-4) as follows: Since t_2 operates on nucleon 2 we have

$$\left\langle nn\middle|\hat{t}_2^{ce}\middle|\frac{np-pn}{\sqrt{2}}\right\rangle = \left\langle nn\middle|\frac{nn}{\sqrt{2}}\right\rangle\hat{t}_2^{ce} = \frac{\hat{t}_2^{ce}}{\sqrt{2}} \qquad (2b\text{-}5)$$

and similarly

$$\left\langle\frac{np-pn}{\sqrt{2}}\middle|\hat{t}_1^{ce}\middle|nn\right\rangle = \left\langle\frac{np-pn}{\sqrt{2}}\middle|pn\right\rangle\hat{t}_1^{ce} = -\frac{\hat{t}_1^{ce}}{\sqrt{2}} \qquad (2b\text{-}6)$$

We can thus use (2b-5) and (2b-6) to reduce (2b-4) to

$$\langle \psi_b | \hat{t}_1^{ce} \hat{G} \hat{t}_2^{ce} | \psi_a \rangle = -\frac{1}{8} \langle \psi_b | \hat{t}_1^{ce} | \chi_n^A \rangle \cdot$$

$$\langle \chi_n^A | \hat{G} | \chi_m^A \rangle \langle \chi_m^A | \hat{t}_2^{ce} | \psi_a \rangle \qquad \text{(2b-7)}$$

The terms in (2b-7) can now be handled in a straightforward manner as was done in the non-symmetrized case. The result is[27]

$$\langle \psi_b | \hat{t}_1^{ce} \hat{G} \hat{t}_2^{ce} | \psi_a \rangle = -\frac{1}{4} \left[-(2\pi)^2 m_\pi \right] t_{ce}^2$$

$$\left[\left\langle \frac{1}{r} \right\rangle - \Delta \right] \delta^3 \left(\vec{p}_{n_b} + \vec{p}_b - \vec{p}_{n_a} - \vec{p}_a \right) \qquad \text{(2b-8)}$$

where $$\Delta = \frac{1}{2\pi^2} \int \frac{1}{p^2} d^3p \qquad \text{(2b-9)}$$

Because the deuteron wave function was symmetrized in isospin space, neutron and proton are not always particle 2 and particle 1, repsectively, as they were in the non-symmetrized case. Therefore the $\hat{t}_1 \hat{G} \hat{t}_2$ term is not zero this time but is equal to the $\hat{t}_2 \hat{G} \hat{t}_1$ term (by symmetry). So the complete DCE contribution to the T matrix is twice the right side of (2b-8). The corresponding DCE scattering

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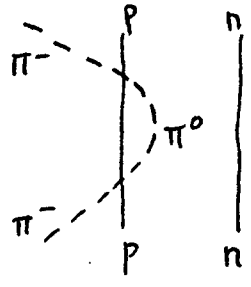
[27] See Appendix 6.

amplitude is

$$F_{\pi d}^{DCE} = -\frac{1}{2} f_{ce}^2 \left[\left\langle \frac{1}{r} \right\rangle - \Delta \right] \qquad (2b\text{-}10)$$

We see that the Δ decreases the DCE contribution as we expected. Unfortunately, however, Δ is inifinite and in fact it numerically is equivalent to the bubble graph



(2b-11)

which we expect to be infinite by analogy with the Feynman QED work. The infinity is really no problem since it results from an unphysical choice of two-body T matrix, (2a-6), so that a more realistic choice for the t's would make Δ finite. But even more perplexing is the fact that when we calculate the elastic scattering contribution in second order, symmetrizing intermediate states the same way, we get

$$F_{\pi d}^{\text{2nd ord. el.}} = \frac{1}{2} \left[2 f_n f_p \left\langle \frac{1}{r} \right\rangle + \left(f_n^2 + f_p^2 \right) \Delta \right] \qquad (2b\text{-}12)$$

This just cannot be correct for the following reason. The intermediate state for elastic scattering consists of a

neutron and a proton (not two neutrons as in DCE) so that the particles are not identical. This means that symmetrizing should not give different results from non-symmetrized calculations.

We can resolve the difficulties encountered from symmetrization by going back to (1d-22) instead of (1d-24). Then we see that the complete second order contribution is

$$\left[t_1 G t_2 + t_2 G t_1 + t_1 G t_1 + t_2 G t_2\right] - \left[t_1 g_1 t_1 + t_2 g_2 t_2\right] \qquad (2b\text{-}13)$$

When we neglected nucleon excitation in the propagators we assumed that

$$G = g_1 = g_2$$

but this is only true if the intermediate states are not symmetrized. The important point to remember is that the g's are two-body propagators (one pion and one nucleon) and the G's are three-body propagators (pion and both nucleons) so that symmetrized intermediate states are sandwiched between G and the t's, but unsymmetrized (pion and one nucleon) states must be sandwiched between the g's

and the t's.[28] When this is done properly we find the following results for DCE:

$$t_1^{ce} G t_2^{ce} + t_2^{ce} G t_1^{ce} \propto \tfrac{1}{2}\left[\Delta - \left\langle \tfrac{1}{r} \right\rangle\right]$$

$$t_1^{ce} G t_1^{ce} + t_2^{ce} G t_2^{ce} \propto \tfrac{1}{2}\left[\Delta - \tfrac{1}{r}\right]$$

$$t_1^{ce} g_1 t_1^{ce} + t_2^{ce} g_2 t_2^{ce} \propto \Delta$$

so that the complete second order contribution to the scattering amplitude with intermediate state symmetrization included is

$$F_{\pi d}^{DCE} = f_{ce}^2 \left[\tfrac{1}{2}\left(\Delta - \left\langle \tfrac{1}{r} \right\rangle\right) + \tfrac{1}{2}\left(\Delta - \left\langle \tfrac{1}{r} \right\rangle\right) - \Delta \right]$$

$$= - f_{ce}^2 \left\langle \tfrac{1}{r} \right\rangle$$

which is the same result we found without symmetrizing intermediate states (see (2a-16)). Similarly, the second order elastic result is the same as the unsymmetrized result when binding corrections are properly handled, and in general, symmetrization of intermediate states is unnecessary since the results are always the same as un-

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[28] The author is deeply indebted to D.S. Beder for pointing this out and thus resolving the symmetrization difficulties.

symmetrized ones (providing that the incoming particle is different from the target particles). However, initial and final states must always be symmetrized. In section 2a we did not explicitly symmetrize initial and final deuteron states but the results are correct because we accounted for symmetrization by choosing the minus sign in (2a-16) next to f_{ce}^2.

Moyer and Koltun, reference (22), mention that intermediate states need not be symmetrized, but their argument does not analyze the situation in detail as we have done. Theirs is the only paper to even mention the equivalence of symmetrized and unsymmetrized calculations, and on this account deserves considerable credit. Note that Moyer and Koltun discuss symmetrization based on the Lippmann-Schwinger equation whereas we work with the multiple scattering series.

In conclusion, we can ignore symmetrization of intermediate state wave functions in all subsequent work of this paper.

3. Summation of the Multiple Scattering Series To All Orders

It has been shown that the multiple scattering series diverges in each term past third order; here we want to sum the whole series and obtain finite results. To accomplish our goal, it is necessary to write the various scattering amplitudes for each order of scattering. In general this is not possible, when non-constant two-body T matrices are used, for example; i.e., the integrals to each order cannot be evaluated (except numerically). But if we keep the approximations we have made in the previous sections (constant t's, zero binding and no nucleon excitation in G) it is possible to evaluate the integrals and obtain closed analytic expressions for each order of scattering. It is also possible to write the scattering amplitude immediately for each order of scattering by looking at the appropriate graphs. We therefore will state the rules for obtaining the scattering amplitude to each order and then sum the terms to all orders. No attempt will be made to derive the rules that will be given because they are easily established by the techniques of Appendix 4 although it is tedious to do so.

However, we will make reference to previous calculations of second order terms to indicate the origin of our graph rules. We also postpone numerical evaluation of series sums until chapter 4.

The sum of the elastic multiple scattering series for pions on deuterons has been evaluated long ago (in 1953 by Brueckner, reference (2)) but the charge-exchange process was neglected. Not until 1972 was an attempt made by Kolybasov and Kudryavtsev (reference (19)) to sum the series including charge-exchange and π^0 elastic scattering. Unfortunately their result is incorrect, which they state in a note added in proof. But their later paper (reference (28)) which they claim corrects their previous error is also wrong. In the present section we show how to obtain the series sum including charge-exchange by a new method which "folds" the charge-exchange and π^0 elastic scattering contributions into the original series sum neglecting these processes. This technique greatly reduces the complexity of the problem and avoids the tedious labor of summing a great many extra graphs. In addition to our pi-deuteron series sum, we evaluate the series sum neglecting charge-exchange and π^0 elastic scattering for pions on an arbitrary nucleus of N neutrons and Z protons.

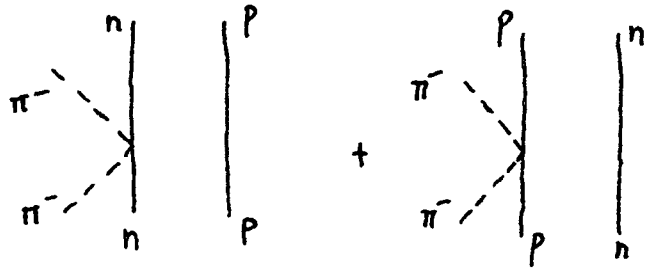
Using our new technique we then fold in the charge-exchange contribution so that we evaluate the series sum for an arbitrary nucleus including elastic π^- and charge-exchange scattering. We do not include elastic π^0 scattering in this more general case because the expressions then become gargantuan. Nevertheless, our technique does allow one to persist, if desired.

3a. Graph Rules and Order by Order Summation

We begin by looking back at (2a-16), the first order contribution to the scattering amplitude is

$$f_n + f_p \tag{3a-1}$$

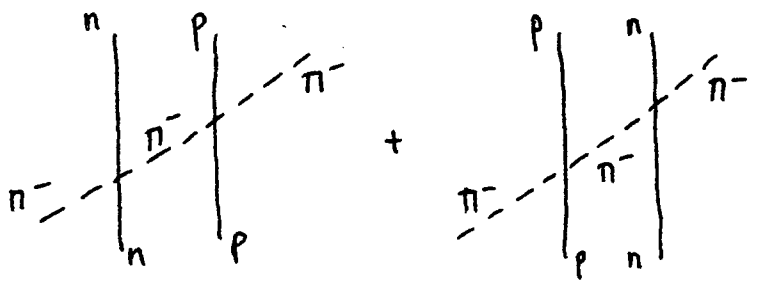
For simplicity, let graphs corresponding to (3a-1) be drawn as



(3a-2)

where it is understood that deuteron lines should be joined to the beginning and end of the two nucleon lines in (3a-2) and all that follows. If we ignore initial and final lines the relevant feature of (3a-2) is a pi-neutron vertex in the first graph and a pi-proton vertex in the second. Thus, the amplitude which we write by inspection from (3a-2) is just (3a-1), where f_n is written for the pi-neutron vertex and f_p for the pi-proton vertex.

Now look at second order elastic terms. The diagrams are



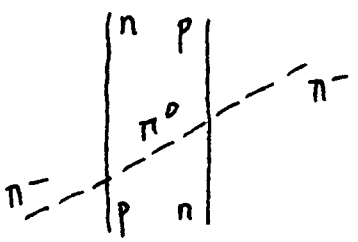
(3a-3)

and from (2a-16) the corresponding scattering amplitude is

$$\frac{2f_n f_p}{r} \qquad (3a\text{-}4)$$

where it is understood that (3a-4) is to be averaged over the deuteron wave function (take the expectation value). That is, instead of taking the expectation value of each term in the multiple scattering series and then adding all the terms, we first add the whole series and then take the expectation value (both methods must give the same results). Neglecting external lines in (3a-3), the first graph has a pi-neutron vertex, a pi-proton vertex, and an internal pion propagating; the two vertices contribute the factors f_n and f_p and the propagating pion gives the factor $1/r$. The second graph contributes the same so that (3a-4) results by inspection from (3a-3).

Now look at the DCE graph



(3a-5)

with the corresponding amplitude from (2a-16)

$$-f_{ce}^2/r \qquad (3a\text{-}6)$$

Each pion-nucleon vertex in (3a-5) contributes a factor f_{ce} because the pion changes its charge after each scattering, the propagating pion contributes a factor 1/r, and a minus sign is necessary because the proton and neutron have exchanged places in the fnal state with respect to the initial states.

We can now state all the graph rules. In analogy to QED rules, we choose to abbreviate our rules with the letters MSG (multiple scattering graphs). The rules are as follows:

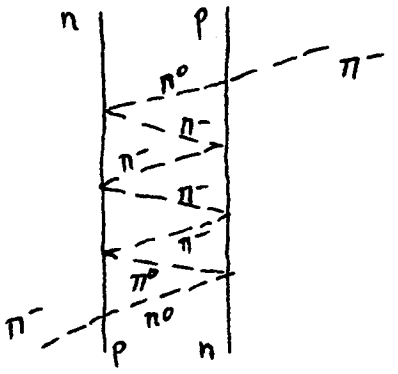
MSG Rules

1. Draw all possible graphs with no bubble diagrams allowed (no graphs like (2b-11), for example).
2. For each vertex associate a factor f_n, f_p, f_{ce}, or f_o corresponding to elastic π^--neutron, π^--proton, charge-exchange, or elastic π^o scattering, repectively.
3. For each internal propagating pion line associate a factor 1/r.
4. If two nucleons are interchanged in the final state, multiply by -1.

(3a-7)

Keep in mind that the above rules apply only under the assumptions of page 72 and in the threshold limit.

As a check to see if you can apply the MSG rules to a more complicated case, the amplitude associated with the diagram

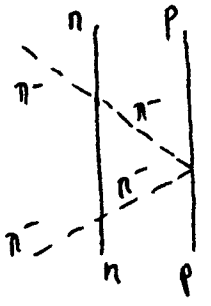


(3a-8)

is given by

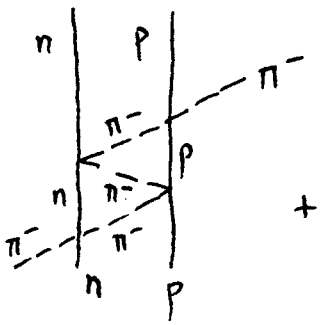
$$-\frac{f_{ce}^{4} f_n^{2} f_p f_o}{r^7} \qquad (3a\text{-}9)$$

Using the MSG rules we draw the graphs and write the corresponding amplitudes for the next few orders of elastic scattering (neglect the charge-exchange process for now). We obtain for the third order processes



$$+\ n \longleftrightarrow p \ = \ \frac{f_n f_p (f_n + f_p)}{r^2} \qquad (3a\text{-}10)$$

where $n \longleftrightarrow p$ means the same graph with the pion striking the proton first. To fourth order we get



$$+ \quad n \longleftrightarrow p = \frac{2 f_n^2 f_p^2}{r^3} \tag{3a-11}$$

The pattern is obvious so we write the sum of all odd order terms as

$$(f_n + f_p)\left[1 + \frac{f_n f_p}{r^2} + \left(\frac{f_n f_p}{r^2}\right)^2 + \cdots\right] \tag{3a-12}$$

and the sum of all even order terms as

$$\frac{2 f_n f_p}{r}\left[1 + \frac{f_n f_p}{r^2} + \left(\frac{f_n f_p}{r^2}\right)^2 + \cdots\right] \tag{3a-13}$$

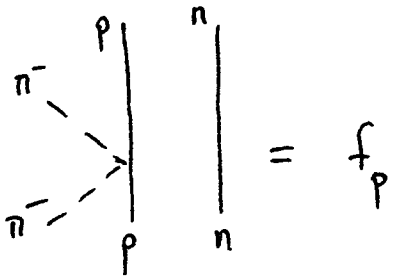
and summing (3a-12) and (3a-13) gives the complete pi-d scattering amplitude neglecting charge-exchange

$$F_{\pi d}(f_{ce}=0) = \left[f_n + f_p + \frac{2 f_n f_p}{r}\right]\left[1 + \frac{f_n f_p}{r^2} + \left(\frac{f_n f_p}{r^2}\right)^2 + \cdots\right]$$

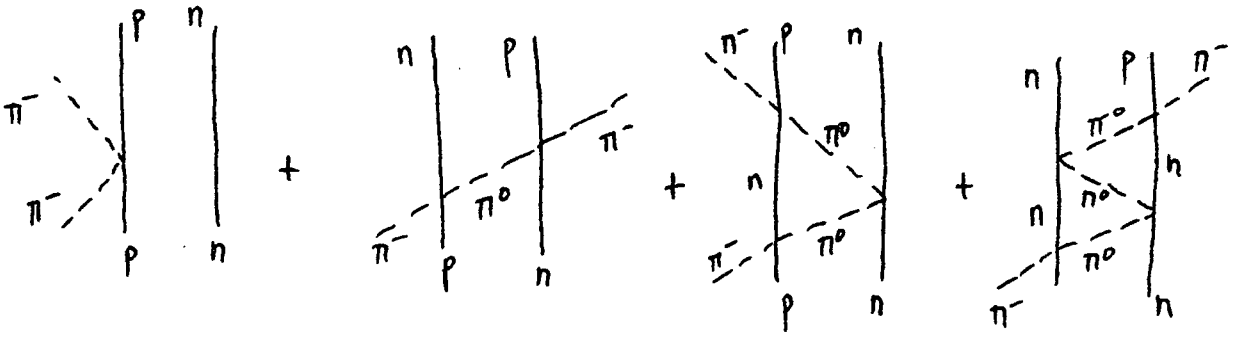
$$= \frac{f_n + f_p + \frac{2 f_n f_p}{r}}{1 - \frac{f_n f_p}{r^2}} \tag{3a-14}$$

where it is understood that the right side of (3a-14) is to be averaged over the deuteron wave function (take the expectation value). We have thus shown that the series of divergent terms can be summed to a finite result, the result (3a-14) having been obtained previously by Brueckner.

Now consider the charge-exchange contribution. We could in principle write the graphs to each order as before by including extra factors of f_{ce} and f_o. However, the combinations are so many and varied that it is difficult to see a pattern to each order so that one has trouble summing the series proceeding as before. Instead, suppose we replace every f_p in (3a-14) by some factor which takes into account all ways to scatter from a π^- on a p to a π^- off a p with any and all possible f_{ce} and f_o scatterings in between. That is, replace the vertex



by the sum of graphs



$$+ \ldots$$

$$= f_p - \frac{f_{ce}^2}{r} + \frac{f_{ce}^2 f_o}{r^2} - \frac{f_{ce}^2 f_o^2}{r^3} + - \ldots$$

$$= f_p - \frac{f_{ce}^2}{r}\left(1 + \frac{f_o}{r}\right)^{-1} \qquad (3a\text{-}15)$$

In other words, we have summed all paths which start with a π^- on a proton with a neutron as a spectator and finish with a π^- off a proton and a neutron as spectator, and no intermediate elastic scattering (of π^-) allowed. There is no similar replacement for the pi-neutron vertex because the pion would have to first elastically scatter over to the proton before it could charge-exchange, and such contributions are accounted for in (3a-15). Suffice to say, then, that the series sum including charge-exchange and elastic π^o scattering is obtained by replacing all f_p's in (3a-14) by the expression (3a-15). We then find that the sum of the multiple scattering series including

charge-exchange and elastic π^0 scattering is

$$F_{\pi D} = \frac{f_n + f_p + \frac{2f_n f_p}{r} - \frac{f_{ce}^2}{r}\left(1+\frac{f_o}{r}\right)^{-1}\left(1+\frac{2f_n}{r}\right)}{1 - \frac{f_n f_p}{r^2} + \frac{f_{ce}^2 f_n}{r^3}\left(1+\frac{f_o}{r}\right)^{-1}} \qquad (3a\text{-}16)$$

The dubious reader is urged to verify (3a-16) to any order by expanding the denominator. To the author's knowledge, (3a-16) is a new result not before obtained. We remark that the result (3a-16) is finite (as is seen by multiplying numerator and denominator by $r^3(1+f_o/r)$; writing the expectation value of the resulting quantity with an explicit deuteron wave function makes the finiteness manifest).

One curious feature of (3a-16) is the third term in the denominator which is not present in (3a-14). In practice f_n is negative so that the denominator of (3a-14) has no zeroes. But the denominator of (3a-16) does have zeroes for positive (but small) r so that $F_{\pi d}$ has a pole when charge-exchange is included. The quantitative ramifications of this fact will be discussed in the end of the next chapter.

In writing our MSG rules we neglected terms of order m_π/m_n. They are easily accounted for but we choose not to introduce them here because they do not add anything to the discussion. Furthermore, since we will see that the

choice of constant two-body T matrices is a poor one, there is no point in trying to improve the results for the series sum since the sum cannot be so easily evaluated for non-constant two-body t's.

Although our MSG rules were only written for pions on deuterons, it is a simple matter to extend them to include scattering of a projectile on any size system of scatterers, each different from the projectile. We show how to sum the series for scattering on an arbitrary nucleus in the next section.

3b. Series Sums for Arbitrary Nuclei

It is too cumbersome to try to sum the multiple scattering series for an arbitrary projectile striking an arbitrary nucleus. However, we would like to demonstrate the generality of our method more fully so that the reader should have no difficulty (in principle) applying it to other series sums.

Let us briefly review how we summed the multiple scattering series for pi-d scattering. First we summed the series of elastic scattering graphs neglecting charge-exchange. Let us refer to this sum as the skeleton graph sum. Next we included charge-exchange scattering by replacing the vertex f_p by the same vertex plus all ways to charge-exchange and finish with a π^- coming off a p. Last, we replaced the vertex product f_{ce}^2 by the same product plus all ways to charge-exchange off a proton, elastically scatter a π^0, and finish with a charge-exchange off a neutron producing a proton.

Now consider π^- scattering on a nucleus consisting of N neutrons, Z protons, where N+Z=A is the total number of nuclear particles. Let's first sum the skeleton graphs (π^--p and π^--n elastic scattering terms only). Assume

that the first scattering occurs on a neutron. There are N ways for this to happen so the amplitude for scattering on a neutron first is Nf_n. Then the pion can succesively stike N-1 other neutrons as many times as it likes before striking a proton. This brings in the factor $\left[1-(N-1)\frac{f_n}{r}\right]^{-1}$ where r is some average value of r_{ij}, the relative distance between target particles i and j (see ref.(22)).

Next there are Z protons to choose from so we get a factor Zf_p/r, and then the pion can strike Z-1 protons any number of times before striking a neutron again. This gives a factor

$$\left(Z f_p/r\right)\left[1-(Z-1)\frac{f_p}{r}\right]^{-1} \qquad (3b\text{-}1)$$

and in analogy, the next neutron scatterings bring in a factor

$$\left(N f_n/r\right)\left[1-(N-1)\frac{f_n}{r}\right]^{-1} \qquad (3b\text{-}2)$$

Then the factors (3b-1) and (3b-2) come in alternatively, on and on. Therefore the total amplitude assuming a neutron was struck first is

$$\frac{N f_n\left[1-(N-1)\frac{f_n}{r}\right]^{-1}\left\{1+Z\frac{f_p}{r}\left[1-(Z-1)\frac{f_p}{r}\right]^{-1}\right\}}{1-\left\{N\frac{f_n}{r}\left[1-(N-1)\frac{f_n}{r}\right]^{-1}\right\}\left\{Z\frac{f_p}{r}\left[1-(Z-1)\frac{f_p}{r}\right]^{-1}\right\}} \qquad (3b\text{-}3)$$

Similarly, the total amplitude assuming a proton was struck first is obtained from (3b-3) by the replacements $N \leftrightarrow Z$, $f_n \leftrightarrow f_p$, which gives

$$\frac{Zf_p\left[1-(Z-1)\frac{f_p}{r}\right]^{-1}\left\{1+\frac{Nf_n}{r}\left[1-(N-1)\frac{f_n}{r}\right]^{-1}\right\}}{1-\left\{\frac{Nf_n}{r}\left[1-(N-1)\frac{f_n}{r}\right]^{-1}\right\}\left\{\frac{Zf_p}{r}\left[1-(Z-1)\frac{f_p}{r}\right]^{-1}\right\}} \qquad (3b\text{-}4)$$

The complete pi-nucleus scattering amplitude at threshold neglecting charge-exchange is the sum of (3b-3) and (3b-4).

If we want to include f_{ce}, we can do so by the replacement in (3b-3) and (3b-4)

$$f_p \rightarrow f_p - N\frac{f_{ce}^2}{r} \qquad (3b\text{-}5)$$

since after charge-exchanging off one proton there are N neutrons to choose from for the next (and final) charge-exchange. We could go on to include f_o scattering but the technique should be obvious without doing so. The reader is warned, however, that including f_o makes the expressions for the series sums very complicated. In practice there is little need to include f_o scattering because[29]

$$f_o = \frac{1}{2}\left(f_n + f_p\right)$$

[29] See reference (18) for example.

and since $f_n \approx -f_p$ at threshold we have $f_o \approx 0$.

We feel that we have demonstrated our technique of summing the multiple scattering series in sufficient generality. We remark that the results (3b-3), (3b-4), and (3b-5) are new and do not exist in the literature to the author's knowledge. If one persists in doing nuclear physics, equations (3b-3), (3b-4), and (3b-5) may be of some use.

4.Realistic Pion-Deuteron Calculations

The preceeding two chapters analyzed pi-d multiple scattering for constant two-body T matrices so that general features would not be obscured by the full complexity of the problem. Now we wish to include those complications which will quantitatively change the results of previous work; i.e., we want to find the scattering amplitude in the threshold limit which we believe to be the experimentally observed amplitude (to date no reliable experimental value exists but the new meson facilities now in production should soon provide an answer to compare with our results). Our approach will be to add in one complication at a time, calculating the scattering amplitude anew at each successive step. Finally we discuss additional complications and compare our results to previous calculations in the literature.

4a.Pi-d Scattering Complications One Step at a Time

In our previous calculations with constant two-body T matrices, we saw that terms beyond third order diverged (see (2a-18), for example). However, if more realistic two-body T matrices are used, the integrals associated with each order scattering term will be reduced drastically at high intermediate state pion energies due to the energy-dependence of the t's. The integrals will then be finite and owing to the smallness of the pi-nucleon scattering amplitudes the magnitudes of each term past second order is small compared to the first and second order terms. Therefore our main work will not go beyond second order in the multiple scattering series. The results for constant two-body T matrices will also only be retained up to second order for comparison with improved calculations of this section but one should keep in mind that for constant two-body T matrices keeping terms only up to second order is not necessarily a good approximation to summing the whole series. In the following, we present numbers at each stage, but also present all stages together at the end in Table II for comparison.

A] For constant two-body T matrices, zero deuteron binding energy, equal mass nucleons ($m_n = m_p \equiv m_N$), equal mass pions ($m_{\pi^-} = m_{\pi^0}$) in the propagator, the scattering amplitude is (to second order)

$$F_{\pi d} = F_{\pi d}^{1st} + F_{\pi d}^{2nd\ el.} + F_{\pi d}^{2nd\ ce.}$$

where

$$F_{\pi d}^{1st} = (f_n + f_p)\left(\frac{1}{m_\pi} + \frac{1}{m_D}\right)^{-1}\left(\frac{1}{m_\pi} + \frac{1}{m_N}\right)$$

$$F_{\pi d}^{2nd\ el.} = 2 f_n f_p \left[\left(\frac{1}{m_\pi} + \frac{1}{m_D}\right)^{-1}\left(\frac{1}{m_\pi} + \frac{1}{m_N}\right)^{2} m_\pi\right]\left\langle\frac{1}{r}\right\rangle$$

$$F_{\pi d}^{2nd\ ce.} = -f_{ce}^2 \left[\left(\frac{1}{m_\pi} + \frac{1}{m_D}\right)^{-1}\left(\frac{1}{m_\pi} + \frac{1}{m_N}\right)^{2} m_\pi\right]\left\langle\frac{1}{r}\right\rangle$$

(4a-1)

The above results come from (2a-16) and Appendix 5. For the f's we first take the scattering lengths (pi-nucleon scattering amplitudes in the threshold limit) from reference (7), the Samaranayake and Woolcock data,

$$f_n = -.143 \text{ fm}$$
$$f_p = .118 \text{ fm} \qquad (4a\text{-}2)$$
$$f_{ce} = -.185 \text{ fm}$$

For the masses we use

$$m_N = 939.0 \text{ Mev}$$
$$m_D = (939.6 + 938.3 - 2.2) \text{ Mev} = 1875.7 \text{ Mev}$$
$$m_\pi = 139.6 \text{ Mev} \qquad (4a\text{-}3)$$

and for $\langle 1/r \rangle$ using a Hulthen wave function (see Appendix 3) we find

$$\langle 1/r \rangle = .594 \text{ fm}^{-1} \qquad (4a\text{-}4)$$

and with an S-wave Gartenhaus wave function (see reference (21) and (4a-14))[29]

$$\langle 1/r \rangle = .446 \text{ fm}^{-1} \qquad (4a\text{-}5)$$

so that with the above values the numerical results for (4a-1) are the following:

Table I: Comparison of Hulthen and Gartenhaus Results for $F_{\pi d}$

	Ψ_{Hulthen}	$\Psi_{\text{Gartenhaus}}$
$F_{\pi d}^{1st}$	-.0268	-.0268
$F_{\pi d}^{2nd \text{ el.}}$	-.0246	-.0185
$F_{\pi d}^{2nd \text{ ce.}}$	-.0250	-.0188

with all amplitudes above and from here on in fermis.

[29] The Hulthen and Gartenhaus wave functions are simple and accurate enough for our purposes.

With the introduction of subsequent complications we do not repeat the calculations with the Gartenhaus wave function except for the last, most complicated case. However, we feel the above results give an indication of the further reduction to be expected in each subsequent complication.

B] The first complication introduced is to account for a pole in the propagator resulting from the unequal nucleon and pion masses and the non-zero binding energy of the deuteron. That is, as shown in Appendix 4, we take the propagator G to be

$$G = \frac{1}{E-K}$$

where

$$E = (m_n + m_p - B + m_{\pi^-})$$

$$K = 2m_n + m_{\pi^0} + \frac{p^2_{\pi_m}}{2m_{\pi^0}} \quad \text{(for ce)}$$

so that

$$G = \left[1.1 \text{ MeV} - \frac{p^2_{\pi_m}}{2m_{\pi^0}} \right]^{-1}$$

instead of

$$G = -\left[\frac{p^2_{\pi_m}}{2m_{\pi}} \right]^{-1}$$

Of course this complication has no effect on first order terms. When the appropriate integral with the new G was

computed taking the pole into account, the second order charge exchange term changed negligibly, becoming

$$F_{\pi d}^{2nd\ ce} = -.0248 \text{ fm} \qquad (4a\text{-}6)$$

and the second order elastic term didn't change at all to three significant figures. Thus the effect of unequal masses and non-zero binding is insignificant.

C] The next complication involves the nucleon excitation in the propagator (as shown in Appendix 4) so that now we have

$$E = m_n + m_p - B + m_{\pi^-}$$

$$K = 2m_n + m_{\pi^0} + \frac{p_m^2}{4m_N} + \frac{k_m^2}{m_N} + \frac{p_{\pi_m}^2}{2m_{\pi^0}}$$

for charge-exchange and similarly for elastic scattering. We then find a substantial effect with the results

$$F_{\pi d}^{2nd\ el.} = -.0166 \text{ fm}$$
$$F_{\pi d}^{2nd\ ce.} = -.0168 \text{ fm} \qquad (4a\text{-}7)$$

Of course the first order terms are again unaffected.

D] The next complication arises when we keep track of the various reference frames of each scattering event.

For example, if we look at a second order term of the multiple scattering series

$$\hat{T}_{12} = \hat{t}_1 \hat{G} \hat{t}_2 \qquad (4a\text{-}7)$$

and choose to evaluate $\hat{T}_{12}$ in the pi-d center of momentum frame, then $\hat{t}_1$ and $\hat{t}_2$ must also be evaluated in the pi-d cm frame. In our previous work we made the very crude approximation that the t's are constant in all reference frames, an approximation which violates Lorentz-invariance of the theory. Suppose now we say that each two-body t is constant in the pi-nucleon cm frame; i.e., $\hat{t}_i$ is constant in the cm frame of the pion and nucleon i. We want to convert $\hat{t}_i$ from the pi-nucleon i cm frame to the pi-d cm frame because the latter is what must be used in (4a-7) and in general for every two-body t of the multiple scattering series.

Consider the matrix element

$$\langle \chi_n | t_i | \chi_m \rangle$$

in which state χ_n contains a pion with total lab energy (not just kinetic energy) E_{π_n} and nucleon i has total lab energy E_{N_n} and similarly for state χ_m. Denote the corresponding total energies of the pion and nucleon i

in their cm frame by putting a (*) over each **E**. Then by reference (15), page 86, equation 112, the conversion of the two-body t's from the pi-nucleon cm frame to the lab frame is given by

$$\langle \chi_n | t_i | \chi_m \rangle_{lab} \sqrt{E_{\pi_n} E_{N_n} E_{\pi_m} E_{N_m}}$$

$$= \langle \chi_n | t_i | \chi_m \rangle_{\pi N_i \, cm} \sqrt{E^*_{\pi_n} E^*_{N_n} E^*_{\pi_m} E^*_{N_m}} \qquad (4a\text{-}8)$$

In our previous work we used

$$\langle \chi_n | t_i | \chi_m \rangle_{\pi N_i \, cm}$$

which is a constant, but we should have used

$$\langle \chi_n | t_i | \chi_m \rangle_{lab}$$

which is not a constant by (4a-8). (Note that the lab frame and the pi-d cm frame are identical in the threshold limit). It is a straightforward matter to find the various energies in (4a-8) from the known momenta of the particles. We just mention that for simplicity we choose to evaluate the E's non-relativistically so that we write

$$E_{\pi_n} = m_\pi + \frac{p^2_{\pi_n}}{2m_\pi}$$

and similarly for the other E's.

Let us refer to the application of (4a-8) as the inclusion of Lorentz factors. Since this effectively makes the two-body t's non-constant it is appropriate to include another complication along with the Lorentz factors. Looking back at (1c-30), E_1 is say the pion total energy and E_2 the struck nucleon total energy (in the pi-nucleon cm frame). But should E_1 and E_2 be the energies of the pion and nucleons before or after the scattering? Actually (1c-30) only holds for elastic scattering in which the energies of the two particles don't change after the scattering; that is not the case in multiple scattering theory so we have to generalize (1c-30) for inelastic scattering. The details are covered in Appendix 8 and the generalized result is

$$\delta^3\left(\vec{P}_{\pi_n}+\vec{P}_{N_n}-\vec{P}_{\pi_m}-\vec{P}_{N_m}\right)f = -(2\pi)^2\left[\frac{P^*_{\pi_n}}{P^*_{\pi_m}}\left(\frac{1}{E^*_{\pi_n}}+\frac{1}{E^*_{N_n}}\right)\left(\frac{1}{E^*_{\pi_m}}+\frac{1}{E^*_{N_m}}\right)\right]^{-\frac{1}{2}}\langle\chi_n|t|\chi_m\rangle_{\pi N\ cm} \qquad (4a\text{-}9)$$

so that f is effectively no longer constant but is multiplied by the square root of the term in brackets in (4a-9).

The application of (4a-8) and (4a-9) changes the previous results for the pi-d scattering amplitudes to

$$F_{\pi d}^{1st} = -.0267 \text{ fm}$$

$$F_{\pi d}^{2nd\ el.} = -.0103 \text{ fm} \qquad (4a\text{-}10)$$

$$F_{\pi d}^{2nd\ ce.} = -.0104 \text{ fm}$$

The results (4a-10) hold for unequal pion and nucleon masses, non-zero binding energy, and nucleon excitation in the propagator, and constant two-body scattering amplitudes but with Lorentz factors and inelastic scattering factors included.

E] Now let us drop the assumption of constant two-body scattering amplitudes and rather fit the two-body scattering amplitudes with partial wave phase shift data. As a first step we only include S-waves (the first term in the partial wave expansion of f). One usually parameterizes the two-body scattering amplitudes in terms of the pion momentum in the pi-nucleon cm frame, but for inelastic scattering the pi momentum is different after the scattering. Therefore to satisfy time reversal invariance the parameterization of the f's must be invariant under the interchange of initial and final pion momenta. If q_i and q_f are the initial and final pion momenta in the pi-

nucleon cm frame then we find that the appropriate quantity to use in the parameterization of the f's is[30]

$$q = \sqrt{q_i q_f} \tag{4a-11}$$

instead of q_i. With the parameterization for the S-wave phase shifts given in Appendix 10 we find for the pi-d scattering amplitudes

$$\begin{aligned} F_{\pi d}^{1st} &= -.00760 \text{ fm} \\ F_{\pi d}^{2nd\ el.} &= -.0115 \\ F_{\pi d}^{2nd\ ce.} &= -.0115 \text{ fm} \end{aligned} \tag{4a-12}$$

In obtaining the result (4a-12) we work with complex scattering amplitudes yet the results written in (4a-12) are real. The reason for this is that we only keep the real part of the pi-d scattering amplitude because in the threshold limit $F_{\pi d}$ must be real. This follows from the optical theorem which states that the total cross section is related to the forward scattering amplitude by[31]

$$\sigma = \frac{4\pi}{q} \operatorname{Im} F(\theta = 0)$$

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[30] See Appendix 9.

[31] See reference (27), p.74.

In the threshold limit we have $q \rightarrow 0$ so that ImF must also vanish in this limit to keep σ finite. Thus F is real in the threshold limit.[32]

F] The next complication we introduce is the inclusion of P-waves in the two-body scattering amplitudes (keeping the second term of the partial wave expansion). It is unnecessary to go beyond the P-wave terms because the low energy contribution from the higher partial waves is negligible as one can see from the data. In addition to including P-waves we also allow for a spin-flipping term in the pi-nucleon scattering amplitude (see Appendices 11 and 12 for our choice of P-wave parameterization and our treatment of the spin-flipping term). The spin-flip part of the pi-nucleon scattering amplitude complicates the calculations because it changes the symmetry of the intermediate states depending on the total spin of the two nucleons after each scattering; nevertheless, carefully accounting for the proper symmetrization gives the same results as not symmetrizing the intermediate states

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[32] This conclusion only follows if we neglect absorption effects like $\pi d \rightarrow NN \rightarrow \pi d$. Nevertheless our retention of only the real part of $F_{\pi d}$ calculated from complex $f_{\pi N}$'s is admittedly but an ansatz; the appropriate off-shell behavior of the $f_{\pi N}$'s which guarantees a real $F_{\pi d}$ merits further study.

(as in our previous work). The results for P-waves and spin-flip in the pi-nucleon scattering amplitudes are found to be

$$F_{\pi d}^{1st} = .000229 \text{ fm}$$

$$F_{\pi d}^{2nd\ el.} = -.0135 \text{ fm} \qquad (4a\text{-}13)$$

$$F_{\pi d}^{2nd\ ce.} = -.0140 \text{ fm}$$

G] It is well-known that the Hulthen wave function for the deuteron is not a good approximation for small r (relative position coordinate of the two nucleons). A better approximation is obtained with the Gartenhaus S-wave deuteron wave function (see reference(21)) of the form

$$\psi_D(r) \propto (1 - e^{-c_1 r})(1 - e^{-c_2 r})\left(\frac{e^{-\alpha r} - e^{-\beta r}}{r}\right) \qquad (4a\text{-}14)$$

where c_1, c_2, α, β are all constants. With the improved Gartenhaus wave function and all other previous complications combined we find

$$F_{\pi d}^{1st} = -.00394 \text{ fm}$$

$$F_{\pi d}^{2nd\ el.} = -.0118 \text{ fm} \qquad (4a\text{-}15)$$

$$F_{\pi d}^{2nd\ ce.} = -.0119 \text{ fm}$$

The results (4a-15) represent our best estimate of the first and second order terms of the multiple scattering series. We have collected the results of each successive approximation in Table II.

Before we consider other terms of the multiple scattering series, some remarks are in order concerning the single scattering contribution. Looking at the first column of Table II we see an erratic fluctuation of the single scattering terms beyond approximation 4. The implication is that we have little confidence in the final result (approximation 7) for the single scattering contribution. Let us examine the single scattering contribution in each approximation. In approximation 1, each single scattering term (f_n and f_p essentially) is one order of magnitude larger than either double scattering term, but the two single scattering contributions are of opposite sign so that when we add them together there is a partial cancellation; the total single scattering contribution is thus an order of magnitude smaller than either single scattering term. The inclusion of Lorentz and inelastic factors in approximation 4 has little effect in single scattering because there is no integration over intermediate states and these extra factors are only important

at higher momenta where they cut-off the two-body T matrices. When we go to approximation 5 there is a considerable decrease in the magnitude of the single scattering contribution. This results because at low momenta (but not zero momentum) the S-wave phase shifts are such that the cancellation between f_n and f_p is even greater than at zero momentum. For example, at zero momentum we have

$$f_n + f_p = (4/3)a_3 + (2/3)a_1 = -.025 \text{ fm}$$

where the a's are the pi-nucleon S-wave scattering lengths. However, at q=68 Mev/c (pion momentum in the pi-nucleon cm frame)

$$f_n + f_p \approx \frac{1}{q}\left[(4/3)S_{31} + (2/3)S_{11}\right] = -.00356 \text{ fm}$$

In the range $0 < q < 68$ Mev/c there is no experimental data for the S-wave phase shifts and we have no idea how to extrapolate the phase shifts in this range. It is precisely in this small q range that the major contribution to the single scattering terms occurs. If one assumes a q^{2l+1} phase-shift dependence for the l^{th} -wave phase shift for small q then it is impossible to fit the S-wave scattering lengths and the known phase shifts at q=68 Mev/c

simultaneously. In other words, $q^{2l+1} = q$ for S-waves ($l=0$) so the scattering amplitude is constant for small q; but $f_n + f_p$ is not constant in the range $0 < q < 68$ Mev/c (as we just showed) and therefore some other (and arbitrary) parameterization different from q^{2l+1} must be chosen for the S-wave phase shifts. Therefore the extrapolation of the S-wave phase shifts for $0 < q < 68$ Mev/c is arbitrary and we have little confidence in our single scattering calculation until the S-wave phase shifts are known better. The situation is not so critical for the P-waves because the q^{2l+1} dependence fits the P-wave scattering lengths and P-wave phase shifts at q=68 Mev/c simultaneously. Since the S and P-wave contributions do not interfere in single scattering, we can say with confidence that the P-wave single scattering contribution is (from approximations 5 and 6, column 1, Table II)

$$F_{\pi d}^{1st}\ (\text{P-wave}) = .000229 - (-.00760)$$
$$= .00783 \text{ fm}$$

The S-wave contribution remains an open question until better (lower energy) pi-nucleon data is available. Note that the second order scattering terms are not sensitive to our choice of the S-wave phase shifts in the range

$0 < q < 68$ Mev/c because there is no delicate cancellation of $f_n + f_p$ but rather the contribution looks like $f_n f_p$. Therefore we are confident of our second order scattering results.

Table II: First and Second Order Pi-d Scattering Amplitudes for Various Complications. (all F's in fermis)

Complications	$F_{\pi d}^{1st}$	$F_{\pi d}^{2nd\ el.}$	$F_{\pi d}^{2nd\ ce.}$	$F_{\pi d}^{1+2}$
1. constant two-body f's and t's, $m_n=m_p$, $m_{\pi^-}=m_{\pi^0}$, $B=0$ } in G	-.0268	-.0246	-.0250	-.0764
2. Same as 1 but $m_n\neq m_p$, $m_{\pi^-}\neq m_{\pi^0}$, $B\neq 0$ } in G	-.0268	-.0246	-.0248	-.0762
3. Same as 2 plus nucleon excitation in G.	-.0268	-.0166	-.0168	-.0602
4. Same as 3 but Lorentz and inelastic factors in t's.	-.0267	-.0103	-.0104	-.0474
5. Same as 4 but f's not constant and only S-waves in f's.	-.00760	-.0115	-.0115	-.0306
6. Same as 5 but also P-waves and spin-flipping in f's.	+.000229	-.0135	-.0140	-.0272
7. Same as 6 but Gartenhaus Ψ_D instead of Hulthen	-.00394	-.0118	-.0119	-.0276

4b.Binding Corrections

In addition to those terms we have already considered in the multiple scattering series (first, second order elastic, and second order charge-exchange) we must comment on those remaining; the first of these is the binding corrections.

Looking back at (1d-22), the second order binding correction terms are given by

$$T_{\pi d}^{2nd\ bind.} = t_1(G-g_1)t_1 + t_2(G-g_2)t_2 \qquad (4b-1)$$

The three-body propagator G is straightforward and is given by (A4-4). The two-body propagators g_1 and g_2 are a real problem to evaluate in our case, however. One defines the two-body propagator g_i (i=1,2) by (1d-13) with E_i the initial (before scattering) energy of the incident pion plus nucleon i. Unfortunately the Fermi motion of the nucleons in the deuteron allows a whole range of initial state energies for each nucleon. The choice for E_i in the propagator g_i is therefore completely arbitrary and for lack of any information we take $E_i=m_\pi+m_N$ (note that this is consistent with our choice of E in G when we set the binding energy equal to zero). We therefore write

$$G-g_1 \approx -\left[\frac{p^2}{2m_\pi} + \frac{\left(\frac{\vec{p}}{2}+\vec{k}\right)^2}{2m_N} + \frac{\left(\frac{\vec{p}}{2}-\vec{k}\right)^2}{2m_N}\right]^{-1}$$

$$+\left[\frac{p^2}{2m_\pi} + \frac{\left(\frac{\vec{p}}{2}-\vec{k}\right)^2}{2m_N}\right]^{-1} \qquad (4b\text{-}2)$$

and similarly for $G-g_2$, replacing the $\vec{p}/2-\vec{k}$ in the second term by a $\vec{p}/2+\vec{k}$. In G we have summed the kinetic energies of all three particles and in g_1 we summed the pion and nucleon 1 kinetic energies; $\vec{p}$ is the intermediate state pion momentum (in the lab frame) and $\vec{k}$ is the intermediate state relative momentum for the two nucleons (see (2a-3)). Note that while the individual terms $t_1 G t_1$ and $t_1 g_1 t_1$ are divergent for constant two-body T matrices, the difference $t_1(G-g_1)t_1$ is finite. With the choice (4b-2), Hulthen ψ_p, and constant two-body T matrices the contribution of second order binding terms to the pi-d scattering amplitude is

$$F_{\pi d}^{\text{2nd bind.}} = -.00818 \text{ fm} \qquad (4b\text{-}3)$$

accurate to two per cent. Including Lorentz and inelastic factors in the two-body T matrices gives a substantial reduction due to the cut-off at higher energies

(in the intermediate state) with the result

$$F_{\pi d}^{2nd\ bind.} = -.00353 \text{ fm} \qquad (4b\text{-}4)$$

again accurate to two per cent. Comparing (4b-4) with the last column of approximation 4, Table II we see that the second order binding correction is less than ten per cent of the first plus second order scattering terms. For this reason and because we are not sure how to choose g_i, it was considered unprofitable to introduce any further complications in calculating the binding corrections to second order. We are only interested in an order-of-magnitude estimate. Note that the binding corrections become more important for tighter-bound nuclei (see reference (22)).

4c. Higher Order Corrections

To investigate the contribution of the multiple scattering terms beyond second order it is necessary to invoke the constant two-body T matrix approximation. If more complicated t's are used (Lorentz factors, phase shifts, etc.) it becomes impossible to sum the series analytically and worse than that, the evaluation of third and higher order integrals becomes a formidable task. We therefore return to the results we obtained for the series sum using the MSG rules of chapter 3.

First consider Brueckner's well known result (3a-14) which neglects charge-exchange scattering. The sum of all terms up to and including second order is (neglecting binding corrections)

$$f_n + f_p + 2 f_n f_p \langle \tfrac{1}{r} \rangle_H = -.045\ f_m \qquad (4c\text{-}1)$$

where the subscript H means we used a Hulthen deuteron wave function. Evaluating the series sum (3a-14) for a Hulthen wave function we find

$$\left\langle \frac{f_n + f_p + \frac{2 f_n f_p}{r}}{1 - \frac{f_n f_p}{r^2}} \right\rangle_H = -.0443\ f_m \qquad (4c\text{-}2)$$

This means that the contribution of all terms beyond second order is -1.6% of the first plus second order terms. Therefore in the approximation of neglecting charge-exchange the contribution of third and higher order terms is quite small.

If we include charge-exchange, the sum of the first and second order terms is now

$$f_n + f_p + \left(2f_n f_p - f_{ce}^2\right)\left\langle \frac{1}{r} \right\rangle_H = -.065\, f_m \qquad (4c\text{-}3)$$

For the series sum with charge-exchange but neglecting f_o scattering, the appropriate expression to evaluate is (3a-16) with $f_o=0$ and we find

$$\left\langle \frac{f_n + f_p + \frac{2f_n f_p}{r} - \frac{f_{ce}^2}{r}\left(1 + \frac{2f_n}{r}\right)}{1 - \frac{f_n f_p}{r^2} + \frac{f_{ce}^2 f_n}{r^3}} \right\rangle_H = -.0573\, f_m \qquad (4c\text{-}4)$$

In this case the higher order correction is -11.8% of the first plus second order terms. Including f_o in (3a-16) gives us for the complete multiple scattering series

$$\left\langle \frac{f_n + f_p + \frac{2f_n f_p}{r} - \frac{f_{ce}^2}{r}\left(1 + \frac{f_o}{r}\right)^{-1}\left(1 + \frac{2f_n}{r}\right)}{1 - \frac{f_n f_p}{r^2} + \frac{f_{ce}^2 f_n}{r^3}\left(1 + \frac{f_o}{r}\right)^{-1}} \right\rangle_H = -.0496\, f_m \qquad (4c\text{-}5)$$

Now the higher order correction is -23.7% of the first plus second order terms. As our best estimate of the higher order contribution, we evaluated (3a-16) with the Gartenhaus wave function instead of the Hulthen to find

$$\left\langle \frac{f_n + f_p + \frac{2f_n f_p}{r} - \frac{f_{ce}^2}{r}\left(1+\frac{f_o}{r}\right)^{-1}\left(1+\frac{2f_n}{r}\right)}{1 - \frac{f_n f_p}{r^2} + \frac{f_{ce}^2 f_n}{r^3}\left(1+\frac{f_o}{r}\right)^{-1}} \right\rangle = -.0447\ fm \qquad (4c\text{-}6)$$

The sum of the first and second order terms in the Gartenhaus case is

$$f_n + f_p + \left(2f_n f_p - f_{ce}^2\right)\left\langle \frac{1}{r} \right\rangle_G = -.0554\ fm \qquad (4c\text{-}7)$$

so that comparing (4c-6) with (4c-7) we find the higher order correction is -19.3% of the first plus second order terms. This represents our best estimate of the contribution of the higher order terms of the multiple scattering series. We therefore conclude (and/or guess) that the higher order scattering terms contribute roughly -20% of the first plus second order terms (not including binding corrections) even for non-constant t's.

4d.Relativistic Corrections

All our results up to now have been non-relativistic. We must therefore estimate the corrections arising when the problem is treated relativistically. There is no need to modify the deuteron wave function since it is accurate enough for our work here. The Lorentz and inelastic factors in the two-body T matrices could be treated relativistically but the kinematics becomes much more complicated. In addition, the Lorentz factors are ratios of energies and so an over-estimate of the energy in the numerator (by treating it non-relativistically) is compensated by an over-estimate in the denominator. For simplicity and because we do not belive it to be the dominant relativistic contribution, we neglect relativistic modifcations in the Lorentz and inelastic factors. By far the greatest effect is in the propagator. Instead of using the non-relativistic energy

$$E = m + \frac{p^2}{2m}$$

we choose to replace this with the relativistic expression

$$E = \sqrt{p^2+m^2}$$

for all intermediate state energies in the propagator so

that (A4-4) is replaced by[33]

$$G = \frac{1}{E-K}$$

$$K = \left[\left|\frac{\vec{p}}{2}+\vec{k}\right|^2 + m_N^2\right]^{\frac{1}{2}} + \left[\left|\frac{\vec{p}}{2}-\vec{k}\right|^2 + m_N^2\right]^{\frac{1}{2}} + \left[p^2+m_\pi^2\right]^{\frac{1}{2}}$$

$$\vec{p} \equiv \vec{p}_{\pi_m}\,, \quad \vec{k} \equiv \vec{k}_m \qquad (4d\text{-}1)$$

and E is unchanged. For convenience and because it has little effect, we neglect the binding energy and take nucleon masses equal ($m_n=m_p=m_N$) and pion masses equal ($m_{\pi^0}=m_{\pi^-}$) in the propagator. Of course the relativistic corrections we are making have no effect on the single scattering terms.

With a Hulthen wave function, constant t's, no nucleon excitation (i.e., $K = (p^2+m^2)^{1/2}$) and no Lorentz or inelastic factors we find the results for our relativistic propagator are

$$F_{\pi d}^{\text{2nd el.}} = -.0354 \text{ fm}$$

$$F_{\eta d}^{\text{2nd ce.}} = -.0359 \text{ fm} \qquad (4d\text{-}2)$$

The increase over the first row's results of Table II

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[33] See Appendix 13 for justification of this replacement.

is obvious since $1/p^2$ dies out more rapidly than its relativistic counterpart $1/(p^2+m^2)^{1/2}$. When we include nucleon excitation (i.e., use the K of (4d-1)) we find

$$F_{\pi d}^{2nd\ el.} = -.0224 \text{ fm}$$
$$F_{\pi d}^{2nd\ ce.} = -.0227 \text{ fm} \qquad (4d\text{-}3)$$

Putting more in the denominator of the propagator makes the results smaller than (4d-2). When we now include the Lorentz and inelastic factors (but treat them non-relativistically) the results decrease to

$$F_{\pi d}^{2nd\ el.} = -.0120 \text{ fm}$$
$$F_{\pi d}^{2nd\ ce.} = -.0121 \text{ fm} \qquad (4d\text{-}4)$$

Comparing (4d-2) with the first row of Table II we see that the relativistic results are 1.44 times larger than the non-relativistic counterparts. Comparing (4d-4) with row 4 of Table II, the relativistic results with Lorentz factors are 1.16 times larger than the non-relativistic counterparts. In other words, the effect of introducing relativistic energies in the propagator is less significant when Lorentz and inelastic factors are included. This follows because the relativistic effects are larger at higher momenta and the Lorentz and inelastic factors

decrease the high-momentum contribution. If we include more detail (phase shifts, etc.) we expect the relativistic effects to be even less important. To test our beliefs, we calculate approximation 7 of Table II with our relativistic propagator (still neglecting binding energy in the propagator, however). That is, we run the most realistic case with our relativistic propagator and find

$$F_{\pi d}^{2nd\ el.} = -.0129 \text{ fm}$$
$$F_{\pi d}^{2nd\ ce.} = -.0131 \text{ fm} \qquad (4d\text{-}5)$$

These results are 1.10 times larger in magnitude than the non-relativistic counterparts (row 7, Table II). We therefore conclude that for the most realistic case calculated the relativistic second order correction is

$$\frac{-(-.0118-.0119)+(-.0129-.0131)}{(-.0118-.0119)} = 9.7\%$$

of the non-relativistic second order terms.

4e. Summary of Corrections and Best Estimate

We have shown in Table II, row 7, that the first and second order scattering terms of the multiple scattering series for pi-d threshold scattering are

$$F_{\pi d}^{1st} = -.00394 \text{ fm}$$

$$F_{\pi d}^{2nd\ el.} = -.0118 \text{ fm} \qquad (4e\text{-}1)$$

$$F_{\pi d}^{2nd\ ce.} = -.0119 \text{ fm}$$

Summing the above and taking 10% of the sum gives the second order binding correction

$$F_{\pi d}^{2nd\ bind.} = -.00276 \text{ fm} \qquad (4e\text{-}2)$$

Taking -20% of the sum of all terms in (4e-1) gives the higher order correction

$$F_{\pi d}^{high.\ ord.} = +.00552 \text{ fm} \qquad (4e\text{-}3)$$

Finally, taking 10% of the sum of the second order terms in (4e-1) estimates the relativistic correction to these

$$F_{\pi d}^{2nd\ rel.} = -.00237 \text{ fm} \qquad (4e\text{-}4)$$

Adding all three terms of (4e-1), (4e-2), (4e-3), and (4e-4) gives our best estimate of the pi-d scattering length

$$F_{\pi d}^{\text{best estimate}} = -.0273 \text{ fm} \qquad (4e\text{-}5)$$

4f. Review of Other Pi-d Literature and Discussion of Results

People have been estimating the pi-d scattering amplitude since 1950 and possibly earlier (see reference (11) and bibliography therein). In many cases the authors were interested in comparing their results with then-known pi-d scattering data (above threshold) so their results are not directly comparable with those here. We therefore will only briefly mention the approaches used in such papers. Other papers in which the pi-d scattering amplitude in the threshold limit is calculated will be discussed in more detail.

One of the earliest attempts to obtain the pi-d differential cross section was performed by Fernback, Green, and Watson (reference (11)). They did a very crude estimate by taking the product of the free particle scattering amplitudes with an overlap integral of the initial and final wave functions to get the differential cross section. Thus they neglected double scattering and all ramifications thereof. In an effort to find the contribution from higher order scattering terms, Brueckner (reference (2)) evaluated the scattering amplitude by solving the Schrödinger equation for the scattering of a fast particle by two heavy scatterers. He found the result (3a-14) although

his is more general because he allowed arbitrary momentum transfer. Of course his result is not very practical since it effectively assumes constant two-body T matrices for pi-nucleon scattering. In addition, a serious drawback of Brueckner's result is the neglect of charge-exchange scattering. It took thirteen years before Wilkin (reference (26)) pointed out the necessity of including charge-exchange. Wilkin included terms up to second order but assumed constant two-body T matrices and his results apply only to high energy scattering. Since Wilkin, others have evaluated the pi-d scattering amplitude (see references (24), (14), (3), (23)) at various energies using multiple scattering approaches taking more details into account, but the results are not comparable with ours since they were performed at higher energies. Dispersion relation approaches were tried by Fäldt (reference (10)) and Schiff and Tran Thanh Van (reference (25)) in an effort to treat the problem covariantly and also to tackle the difficulties of unitarity. Again their results do not apply at threshold so we cannot compare with ours.

After Brueckner, a paper which treats pion-nucleus scattering in the threshold limit is that of Moyer and Koltun (reference 22)). Unfortunately the lightest nucleus they treat is Helium so their numerical results are not

comparable with ours. We mention this paper because it treats the binding and higher order corrections in the threshold limit and finds their contribution to be non-negligible. However, their treatment of the binding corrections is substantially different from ours in that they write the difference $G-g_i$ of the binding correction as a single operator and solve a Lippmann-Schwinger type equation for it using separable Yamaguchi potentials. Their method is therefore closely related to Faddeev equation approaches (see next chapter) as far as binding corrections are concerned. Except for the binding corrections, all work of Moyer and Koltun assume constant two-body T matrices. Note that the sum of the multiple scattering series in their paper (equation A9) does not allow for charge-exchange scattering of a pion on a deuteron.

The most complete calculation of the pi-d scattering amplitude at threshold to date is by Kolybasov and Kudryavtsev (reference (19)). Because they used different pi-nucleon scattering lengths from ours their results are somewhat different but the relative effect of each complication introduced compares favorably with our results. For example, when they introduce nucleon excitation in the propagator they find a fifty to seventy per cent reduction in the magnitude of the double scattering term.

We find a sixty seven per cent reduction. When they include P-waves in the pi-nucleon interaction they find the P-wave contribution to be about thirty per cent of the S-wave in double scattering. We find it to be about twenty per cent. The reason for the discrepancy is mostly due to the neglect of Lorentz and inelastic factors in their calculations. In addition, they do not use a phase-shift parameterization for their pi-nucleon scattering amplitudes. Their results are obtained with a Hulthen instead of a Gartenhaus wave function and there is no treatment of binding corrections or relativistic effects. Finally, they calculate the series sum for constant two-body T matrices but their final expression is wrong (which they state in a note added in proof). There is little point in comparing their best estimate with ours since the result depends on the choice of S and P-wave scattering lengths and their choice differs from ours. Nevertheless, Kolybasov and Kudryavtsev find for their particular choice of pi-nucleon scattering lengths

$$F_{\pi d} = -.047 \text{ fm}$$

compared to our result

$$F_{\pi d} = -.0273 \text{ fm}$$

We expect their results to be higher than ours mainly because Lorentz and inelastic factors were neglected in their calculations and because they used a Hulthen instead of a Gartenhaus wave function for the deuteron.

We stated that our goal was to obtain the pi-d scattering amplitude at threshold accounting for all quantitatively relevant complications. We believe this goal has been achieved within our present knowledge of the pi-nucleon phase shifts. We have neglected the small D-state part of the deuteron wave function and real absorption effects ($\pi^- d \rightarrow 2n$). Our crude estimates indicate that these additional complications will introduce much less than a ten per cent correction. We are hoping therefore that this thesis puts the final nails in the coffin of the pi-d scattering length.[34]

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[34] Attributed to D.S. Beder while groping for the truth.

5.The Faddeev Equations and Symmetrization

Pion-deuteron scattering is a three-body problem. In 1960 the Russian mathematician Faddeev published an article showing how to solve the three-body problem once the two-body T matrices for all pairs of particles (three pairs) are known.[35] The Faddeev equations are closely related to Watson's multiple scattering series since both Faddeev and Watson start with the Lippmann-Schwinger equation. The beauty of Faddeev's approach is that he writes an integral equation for the complete three-body T matrix solely in terms of free two-body T matrices instead of Watson's bound two-body T matrices (the τ's). In addition, the Faddeev equations account for scattering of one nucleon on the other (for the case of pi-d scattering) and Watson's approach only treats scattering of the pion on each nucleon.

The purpose of this chapter is to quickly review the Faddeev equations with particular emphasis placed on intermediate state wave function symmetrization. One can consider this brief chapter to be a warning concerning symmetrization and divergent bubble graphs in the spirit of Appendix 6 and section 2b. No attempt will be made to solve the Faddeev equations.

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[35] See reference (9).

5a. Symmetrization of the Faddeev Equations

Rather than follow Faddeev's original approach, we choose to paraphrase the approach of Hetherington and Schick (reference (17)) because theirs demonstrates the analogy to Watson's multiple scattering series.

Let particle 2 be the pion and particles 1 and 3 the nucleons. Then we can write the Lippmann-Schwinger equation, (1d-10), for the complete three-particle T matrix as

$$T = V_1 + V_3 + (V_1 + V_3) G_2 T \qquad (5a\text{-}1)$$

where the three-particle Green's function is given by (see (1d-16))

$$G_2 = \frac{1}{E - K - V_2 + i\eta} \qquad (5a\text{-}2)$$

Here E is the total three-particle energy, K is the sum of all three energy operators (pion, nucleon 1, and nucleon 2) and V_i is the potential between particles j and k with $j \neq i$, $k \neq i$, $j \neq k$. One also defines the free three-particle Green's function

$$G = \frac{1}{E - K + i\eta} \qquad (5a\text{-}3)$$

By the easy-to-follow steps of Hetherington and Schick

one quickly obtains the Faddeev equations for the complete three-particle T matrix

$$T = \sum_{i,j\neq 2} T^{ij} \tag{5a-4}$$

where

$$T^{ij} = t_i \delta_{ij} + \sum_{k=1}^{3} t_i G^{ik} T^{kj} \tag{5a-5}$$

with t_i and G^{ik} given by

$$t_i = (1 - V_i G)^{-1} V_i \tag{5a-6}$$

$$G^{ik} = G(1 - \delta_{ik}) \tag{5a-7}$$

As Hetherington and Schick point out, iteration of (5a-5) gives the multiple scattering series

$$T^{ij} = t_i \delta_{ij} + t_i G^{ij} t_j + t_i G^{ik} t_k G^{kj} t_j + \ldots \tag{5a-8}$$

The point is that (5a-8) only applies if the intermediate state wave functions are not symmetrized. If one tries to symmetrize them, the same divergences associated with bubble graphs arise that we saw in Appendix 6. In addition, (5a-8) does not contain binding correction terms. The error is not in the Faddeev equations as Faddeev writes

them but rather in the way Hetherington and Schick write (5a-6). In (5a-6) they have associated the wrong expression with the two-body T matrix, t_i. The correct association is to write

$$t_i = (1 - v_i g_i)^{-1} v_i \qquad (5a\text{-}9)$$

where

$$g_i = [E_j + E_k - K_j - K_k + i\eta]^{-1} \qquad (5a\text{-}10)$$

$$j \neq k \neq i$$

E_j is the energy of particle j, etc., and it is understood that non-symmetrized two-particle states are summed over between g_i and v_i whereas in (5a-6) one sums over three-particle symmetrized states between G and v_i. From Hetherington and Schick, equation (10), one has

$$T^{ij} = \delta_{ij}(1 - v_i G)^{-1} v_i + \sum_{k=1}^{3} (1 - v_i G)^{-1} v_i G^{ik} T^{kj} \qquad (5a\text{-}11)$$

The correct Faddeev equations are obtained by writing (5a-11) in terms of the correct two-body t_i's, (5a-9), instead of the incorrect t_i's, (5a-6). One then finds with the help of

$$v_i = t_i (1 + g_i t_i)^{-1} \qquad (5a\text{-}12)$$

that the appropriate modification to the Faddeev equations as written by Hetherington and Schick is

$$T^{ij} = \tilde{t}_i \delta_{ij} + \sum_{k=1}^{3} \tilde{t}_i G^{ik} T^{kj} \qquad (5a\text{-}13)$$

where $$\tilde{t}_i = \left[1 - t_i (1+g_i t_i)^{-1} G\right] t_i (1+g_i t_i)^{-1} \qquad (5a\text{-}14)$$

Of course iteration of (5a-13) with $t_2=0$ reproduces the multiple scattering series with binding corrections, (1d-22). We urge the reader to remember that (5a-13) is valid for symmetrized intermediate states and (5a-5) is not. There seems to be no regard to this fact in many calculations based on the Faddeev approach.

In summary then, people who use equation (5a-5) with symmetrized intermediate states will not get correct numerical results. Worse than that, they will more than likely not even see their errors when they solve the integral equations numerically. This follows because it is customary to use separable Yamaguchi potentials in the two-body T matrices, t_i, which fall rapidly with increasing energy; this in turn suppresses the divergences associated with bubble terms.[35] Also, keeping only S-waves in the two-body interactions suppresses the divergences and makes them finite. Therefore, if Yamaguchi potentials, separable t_i's, and S-wave two-body interactions are the approximations

[35] See reference (17), for example.

employed in solving the Faddeev equations numerically, the contributions from bubble graphs (which should not be present) will not be overly large compared to other proper scattering terms and one will find incorrect results without realizing it.

The point we wish to emphasize is the following: Use equation (5a-5) but do not symmetrize the intermediate state wave functions. If you insist on symmetrizing intermediate states, (5a-13) must be used.

6. Conclusions

We have examined pi-deuteron scattering in the threshold limit, including all quantitatively relevant features of the problem, we believe. Along the tortuous path we investigated the effects of symmetrizing intermediate state wave functions in the multiple scattering series and found the results identical to those obtained with non-symmetrized intermediate states when properly treated. We also found that symmetrizing intermediate state wave functions in the Faddeev equations to solve the three-body problem leads to incorrect results unless one is very careful in interpreting the Faddeev equations. The moral here is, "Don't symmetrize intermediate states in the Faddeev equations if the projectile particle differs from the target particle". We have also demonstrated a technique in chapter 3 (using the MSG rules) that allows one to sum the multiple scattering series in closed form for any nucleus, taking into account all isospin-flipping mechanisms in the two-body scattering amplitudes. In particular, we applied this technique to π^--deuteron scattering to find the series sum including all isospin-flipping terms. We also evaluated the series sum for π^- scattering on an arbitrary nucleus of neutrons and protons including charge-exchange scattering terms.

The most important question remaining now is, "What are the pi-nucleon scattering amplitudes at very low energies, especially the S-wave scattering lengths?".[36] Our uncertainty in the pi-nucleon scattering lengths propagates an uncertainty in the pi-deuteron single scattering contribution and therefore an uncertainty in the pi-deuteron scattering length itself. We stress, as did Kolybasov and Kudryavstev (reference (19)) that the determination of the pi-nucleon scattering lengths is an important experimental task. Once these are known we can give a theoretically determined value for the pi-deuteron scattering length based on our choice for the off-shell pi-nucleon T matrix (a re-calculation of the single scattering contribution once the pi-nucleon scattering lengths are better known is simple and can then be added to our second and higher order results). If our theoretically determined pi-deuteron scattering length agrees with the experimental value then we can say with confidence that we understand the off-shell behavior of the pi-nucleon scattering amplitude. This would be of fundamental importance in understanding the complete pi-nucleon interaction and perhaps strong interactions in general.

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[36] This question is being asked on April 1, 1973.

It is well known (see reference (30)) that one can determine the scattering length for pions on light nuclei by observing the gamma ray emissions of pi-mesic atoms. One produces a pion bound in a high n quantum number shell of an atom and measures the frequency of the emitted gamma ray as the pion falls to the lowest orbit (K shell). The pion in the high n shell is not affected by the strong interaction of the nucleus because the pion in this shell is farther from the nucleus than the K shell orbit. The strong interaction does shift the energy of the pion in the K shell, however. If one knows the energy level of the high n shell from which the pion falls, the energy of the emitted gamma ray is the difference in energy between the (unshifted) high n shell and the (shifted) K shell to which the pion falls. The energy level shift (due to the strong interaction of the pion in the K shell with the nucleus) is proportional to the pi-nucleus scattering length (see reference (31)). Measurement of the level shift can be used to give quite accurate values of the pi-proton scattering length by applying the above procedure to pi-mesic hydrogen. One then finds a level shift of about 6 ev in magnitude (to be compared with the n=2 to n=1 transition for pi-mesic hydrogen which is 2.77 kev). The magnitude of the pi-proton scattering length is about

four times our estimate of the pi-deuteron scattering length so that one expects a level shift for pi-mesic deuterium on the order of 1 ev (magnitude). Therefore the experimentalist's task in determining the pi-deuteron scattering length via this method is quite difficult. Note also that this method will not allow one to find the pi-neutron scattering length.

Bibliography

(1)Bjorken, J.D., and Drell, S.D., Relativistic Quantum Mechanics (Vol.I) and Relativistic Quantum Fields (Vol.II), McGraw-Hill, New York(1964).

(2)Brueckner, K.A., Phys.Rev. 89, 834(1953).

(3)Carlson, C., SLAC Pre-print SLAC-PUB-706, Stanford University(1970).

(4)Davydov, A.S., Quantum Mechanics, Addison-Wesley, Inc., Reading, Mass.(1965).

(5)DeBenedetti, S., Nuclear Interactions, John Wiley & Sons, Inc., New York(1964).

(6)Dicke, R.H., and Wittke, J.P., Introduction to Quantum Mechanics, Addison-Wesley, Inc., Reading, Mass.(1960).

(7)Ebel, G., Pilkuhn, H., and Steiner, F., Nucl. Phys. B17, No.1(1970).

(8)Elton, L.R.B., Introductory Nuclear Theory, Second Ed., W.B. Saunders Co., Philadelphia(1966).

(9)Faddeev, L.D., Soviet Physics JETP, Vol.12, No.5, 1014(1961).

(10)Fäldt, G., Nucl. Phys. B10, 597(1969).

(11)Fernbach, S., Green, F.A., and Watson, K., Phys. Rev. 82, 980(1951).

(12)Feynman, R.P., Quantum Electro-dynamics, W.A. Benjamin, Inc., New York(1962).

(13)Frazer, W.R., Elementary Particles, Prentice-Hall, Inc., Englewood Cliffs, New Jersey(1966).

(14)Gibbs, W.R., Phys. Rev. 3C, No.3, 1127(1971).

(15)Goldberger, M., and Watson, K., Collision Theory, John Wiley & Sons, Inc., New York(1962).

(16)Herndon, D., Barbaro-Galtieri, A., and Rosenfeld, A., πN Partial-Wave Amplitudes, from Particle Data Group, UCRL-20030 πN(1970).

(17)Hetherington, J.H., and Schick, L.H., Phys. Rev. 137, No.4B, 939(1965).

(18)Jacob, J., and Chew, G., Strong-Interaction Physics, W.A. Benjamin, Inc., New York(1964).

(19)Kolybasov, V.M., and Kudryavtsev, A.E., Nucl. Phys. B41, 510(1972).

(20)Lovelace, C., in Strong Interactions and High Energy Physics, ed. by R.G. Moorhouse, Plenum Press, New York(1964).

(21)Moravcsik, M., Nucl. Phys. 7, 113(1958).

(22)Moyer, L.R., and Koltun, D.S., Phys. Rev. 182, 999 (1969).

(23)Pendleton, H., Phys. Rev. 131, 1833(1963).

(24)Rockmore, R.M., Phys. Rev. 105, 256(1957).

(25)Schiff, D., and Tran Thanh Van, J., Nucl. Phys. B3, 671(1967).

(26)Wilkin, C., Phys. Rev. Lett. 17, 561(1966).

(27)Williams, W.S.C., An Introduction to Elementary Particles, Second Ed., Academic Press, New York(1971).

(28)Kolybasov, V.M., and Kudryavtsev, A.E., Soviet Physics JETP 63, 35(1972).

(29)Koltun, D.S., in Advances in Nuclear Physics Vol.3, ed. by M. Barranger and E. Vogt, Plenum Press, New York(1970).

(30)Bailey, J., Phys. Lett. 33B, 369(1970).

(31)Deser, S., Goldberger, M.L., Baumann, K., and Thirring, W., Phys. Rev. 96, 774(1954).

Appendix 1: Field-theoretic Two-fermion Propagator

We wish to show here using field-theoretic arguments that the propagator for a two-fermion state only contains odd angular momentum states; this means that the Pauli exculsion principle applies in intermediate states (as well as initial and final states). We use the notation of reference (1) throughout.

The two-particle fermion propagator is

$$\langle 0 | T\{\psi(x_4)\bar{\psi}(x_3)\psi(x_2)\bar{\psi}_x(1)\}|0\rangle$$

where the ψ's are Dirac fields (i.e., operators) and the T operator is defined by

$$T[\psi(x_2)\bar{\psi}(x_1)] = \begin{cases} \psi(x_2)\bar{\psi}(x_1), & x_{2_0} > x_{1_0} \\ -\bar{\psi}(x_1)\psi(x_2), & x_{2_0} < x_{1_0} \end{cases} \qquad \text{(A1-1)}$$

and the o subscript on the x's means the time component. By Wick's theorem[37]

$$\langle 0 | T\{\psi(x_4)\bar{\psi}(x_3)\psi(x_2)\bar{\psi}(x_1)\}|0\rangle$$

$$= \underline{\psi(x_4)\bar{\psi}}(x_3)\underline{\psi(x_2)\bar{\psi}}(x_1) + \underline{\psi(x_4)\bar{\psi}(x_3)\psi(x_2)\bar{\psi}}(x_1) \qquad \text{(A1-2)}$$

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[37] See e.g. reference (1), Vol.II, p.181.

where the line connecting two ψ's means

$$\underbrace{\psi(x_2)\,\bar{\psi}(x_1)} = \langle 0 | T\{ \psi(x_2)\,\bar{\psi}(x_1)\} | 0 \rangle$$

$$\equiv i S_F(x_2 - x_1) \qquad \text{(A1-3)}$$

which is the single-particle fermion propagator.[38] The Pauli exclusion principle (or equivalently, Fermi-Dirac statistics) for the ψ's is contained in

$$\psi(x_2)\,\bar{\psi}(x_1) = -\,\bar{\psi}(x_1)\,\psi(x_2) \qquad \text{(A1-4)}$$

which applied to (A1-2) gives

$$\langle 0 | T\{ \psi(x_4)\,\bar{\psi}(x_3)\,\psi(x_2)\bar{\psi}(x_1)\} | 0 \rangle$$

$$= \underbrace{\psi(x_4)\,\bar{\psi}}(x_3)\,\underbrace{\psi(x_2)\,\bar{\psi}}(x_1)$$

$$- \underbrace{\psi(x_4)\,\bar{\psi}}(x_1)\,\underbrace{\psi(x_2)\,\bar{\psi}}(x_3) \qquad \text{(A1-5)}$$

For simplicity we neglect the extra bookkeeping of spin in what follows. The propagator is written is terms of its Fourier transform as[39]

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4} \left[\frac{e^{-ipx}}{p^2 - m^2 + i\epsilon}\right](\not{p} + m) \qquad \text{(A1-6)}$$

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[38] See equation (1b-4).

[39] See reference (1), p.95, eq.6.46.

and (A1-6) together with (A1-3) allows us to express (A1-2) in the form

$$\langle 0|T\{\psi(x_4)\bar{\psi}(x_3)\psi(x_2)\bar{\psi}(x_1)\}|0\rangle$$

$$= S_F(x_4-x_1)S_F(x_2-x_3) - S_F(x_4-x_3)S_F(x_2-x_1)$$

$$= \left[\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x_4-x_1)}}{p^2-m^2+i\epsilon}(\not{p}+m)\right]\left[\int\frac{d^4p'}{(2\pi)^4} \frac{e^{-ip'(x_2-x_3)}}{p'^2-m^2+i\epsilon}(\not{p}'+m)\right]$$

$$-\left[\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x_4-x_3)}}{p^2-m^2+i\epsilon}(\not{p}+m)\right]\left[\int\frac{d^4p'}{(2\pi)^4} \frac{e^{-ip'(x_2-x_1)}}{p'^2-m^2+i\epsilon}(\not{p}'+m)\right]$$

$$= \int \frac{d^4p\,d^4p'}{(2\pi)^8} \frac{1}{p^2-m^2+i\epsilon}\frac{1}{p'^2-m^2+i\epsilon}(\not{p}+m)(\not{p}'+m)$$

$$e^{-i\left(\frac{p+p'}{2}\right)(x_4+x_2-x_1-x_3)}\, e^{-i\left(\frac{p-p'}{2}\right)(x_4-x_2)}$$

$$\left[e^{-i\left(\frac{p-p'}{2}\right)(x_3-x_1)} - e^{+i\left(\frac{p-p'}{2}\right)(x_3-x_1)}\right]$$

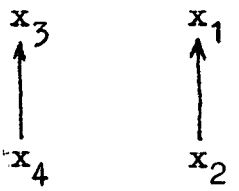
(A1-7)

The part of (A1-7) in brackets at the end is the desired result----only odd angular momentum waves propagate. Perhaps this is more easily appreciated if we go back to

the original form of the two-particle propagator

$$\langle 0 | T\{\Psi(x_4)\,\bar{\Psi}(x_3)\,\Psi(x_2)\bar{\Psi}(x_1)\}|0\rangle$$

and draw the diagrammatic correspondence to the above as



in which one of the fermions starts at location x_4 and propagates to location x_3 while the other goes from x_2 to x_1. Thus, x_3-x_1 is the relative coordinate between the two nucleons and therefore our interpretation of (A1-7) as a statement of the Pauli exclusion principle is confirmed.

Appendix 2: Deuteron Wave Function Conventions

The relative coordinate part of the deuteron wave function, $\psi_D(\vec{r})$, which we use in (2a-1) is chosen to be the Hulthen form[40]

$$\psi_D(\vec{r}) = N\left(\frac{e^{-\alpha r} - e^{-\beta r}}{r}\right) \qquad \text{(A2-1)}$$

where $$\alpha \equiv \frac{1}{R} = \frac{\sqrt{m_N B}}{\hbar} = (4.31\ \text{fm})^{-1}$$

$$\beta = 7\alpha \ , \quad B = 2.2\ \text{mev}$$

and N is a normalization constant which we now determine.

The normalization condition for N is given by

$$1 = N^2 \int \left[\frac{e^{-\alpha r} - e^{-\beta r}}{r}\right]^2 d^3r \qquad \text{(A2-2)}$$

The evaluation of the integral is elementary and we find

$$N^2 = \frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2} \qquad \text{(A2-3)}$$

The Fourier transform of $\psi_D(\vec{r})$ is found via simple

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[40] See reference (5), p.46.

integrations as follows:

$$\psi_D(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int e^{i\vec{k}\cdot\vec{r}}\, \psi_D(\vec{r})\, d^3r$$

$$= \frac{4\pi N}{(2\pi)^{3/2}} \left[\frac{1}{\alpha^2+k^2} - \frac{1}{\beta^2+k^2} \right] \qquad \text{(A2-4)}$$

and inverting the Fourier transform gives

$$\psi_D(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\vec{k}\cdot\vec{r}}\, \psi_D(\vec{k})\, d^3k \qquad \text{(A2-5)}$$

Appendix 3: Expectation Value of 1/r

Our objective is to calculate $\langle 1/r \rangle$ for the Hulthen wave function of Appendix 2 so that we want to find

$$\left\langle \frac{1}{r} \right\rangle = 4\pi N^2 \int_0^\infty \left(e^{-\alpha r} - e^{-\beta r}\right)^2 \frac{1}{r}\, dr \qquad \text{(A3-1)}$$

If we square the term in parentheses above we will have a sum of three separate integrals to evaluate, each of which diverges, although as we will see the sum of all three divergent integrals is finite. We therefore proceed by writing

$$I(\alpha, \beta) = \int_0^\infty \left(e^{-\alpha r} - e^{-\beta r}\right)^2 \frac{1}{r}\, dr$$

and differentiating with respect to α

$$\frac{\partial I}{\partial \alpha} = \int_0^\infty \left[-2e^{-2\alpha r} + 2e^{-(\alpha+\beta)r}\right] dr$$

$$= -\frac{1}{\alpha} + \frac{2}{\alpha+\beta} \qquad \text{(A3-2)}$$

Integrating (A3-2) to get $I(\alpha,\beta)$ back including the constant of integration (which we obtain by simple manipulations) we have

$$I(\alpha,\beta) = \ln\left[\frac{(\alpha+\beta)^2}{4\alpha\beta}\right] \qquad \text{(A3-3)}$$

and (A3-3) back into (A3-1) solves the problem with

$$\left\langle \frac{1}{r} \right\rangle = \frac{2\alpha\beta(\alpha+\beta)}{(\alpha-\beta)^2} \ln\left[\frac{(\alpha+\beta)^2}{4\alpha\beta}\right] \qquad \text{(A3-4)}$$

where we used (A2-3).

Using the values of α and β given in Appendix 2 we find the numerical value of $\langle 1/r \rangle$ to be

$$\left\langle \frac{1}{r} \right\rangle = .594 \ f_m^{-1} \qquad \text{(A3-5)}$$

Appendix 4: Second Order Unsymmetrized Calculations

Our goal here is to obtain (2a-14) starting from (2a-13). First write the separate matrix elements of (2a-13) with the aid of (2a-2) through (2a-7). For the first term of (2a-13) we can write

$$\langle \psi_b | \hat{t}_1 | \chi_n \rangle = [(2\pi)^{-3}]^2 \int e^{i\vec{k}_b \cdot \vec{r}} \, e^{i\vec{P}_b \cdot \vec{R}} \, e^{i\vec{p}_{\pi_b} \cdot \vec{r}_\pi}$$
$$\psi_D^*(\vec{k}_b) \, t_p \, \delta^3\left(\vec{r}_\pi + \frac{\vec{r}}{2} - \vec{R}\right) e^{-i\vec{k}_n \cdot \vec{r}} \, e^{-i\vec{P}_n \cdot \vec{R}} \, e^{-i\vec{p}_{\pi_n} \cdot \vec{r}_\pi}$$
$$d^3k_b \, d^3r_\pi \, d^3r \, d^3R \qquad \text{(A4-1)}$$

where we only consider elastic scattering (for charge-exchange replace t_p by t_{ce}). Integrating over $\vec{R}$ eliminates the delta function, integrating over $\vec{r}$ and $\vec{r}_\pi$ produces two more delta functions, and integrating over $\vec{k}_b$ eliminates one of these two new delta functions (just as we did in section 2a for single scattering) so that

$$\langle \psi_b | \hat{t}_1 | \chi_n \rangle = t_p \, \delta^3\left(\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_n} - \vec{P}_n\right) \psi_D^*\left(\frac{\vec{p}_n}{2} + \vec{k}_n\right) \qquad \text{(A4-2)}$$

where we set $\vec{p}_{\pi b} = \vec{P}_b = 0$ (threshold limit). Similarly for the third term of (2a-13) we find

$$\langle \chi_m | \hat{t}_2 | \psi_a \rangle = t_n \, \delta^3\left(\vec{p}_{\pi_m} + \vec{p}_m - \vec{p}_{\pi_a} - \vec{p}_a\right) \psi_D\left(-\frac{\vec{p}_m}{2} + \vec{k}_m\right) \qquad \text{(A4-3)}$$

with $\vec{p}_{\pi_a} = \vec{P}_a = 0$. The appropriate propagator term of (2a-13) is

$$\langle \chi_n | \hat{G} | \chi_m \rangle = [(2\pi)^{-3}]^3 \int e^{i\vec{p}_{\pi_n}\cdot\vec{r}_\pi} e^{i\vec{P}_n\cdot\vec{R}} e^{i\vec{k}_n\cdot\vec{r}}$$

$$\left[\frac{1}{E-K+i\eta}\right] e^{-i\vec{p}_{\pi_m}\cdot\vec{r}_\pi} e^{-i\vec{P}_m\cdot\vec{R}} e^{-i\vec{k}_m\cdot\vec{r}} d^3r_\pi d^3r d^3R$$

(A4-4)

where $E = m_D + m_{\pi^-} = (m_n + m_p - B) + m_{\pi^-}$

$$K = m_{\pi^-} + m_n + m_p + \frac{P_m^2}{2(m_n+m_p)} + \frac{k_m^2}{2\mu} + \frac{p_{\pi_m}^2}{2m_{\pi^-}}$$

$$\mu = \frac{m_n m_p}{m_n + m_p}$$

$B > 0$ is the binding energy of the deuteron, and it is understood that the integral is evaluated in the limit $\eta \to 0$. Now we make the approximation of neglecting the kinetic energy terms of the two nucleons in K, take $m_n = m_p$, and neglect the small binding energy of the deuteron. Then (A4-4) becomes:

$$\langle \chi_n | G | \chi_m \rangle = [(2\pi)^{-3}]^3 \int e^{i\vec{p}_{\pi_n}\cdot\vec{r}_\pi} e^{i\vec{P}_n\cdot\vec{R}} e^{i\vec{k}_n\cdot\vec{r}}$$

$$\left[-\frac{2m_\pi}{p_{\pi_m}^2}\right] e^{-i\vec{p}_{\pi_m}\cdot\vec{r}_\pi} e^{-i\vec{P}_m\cdot\vec{R}} e^{-i\vec{k}_m\cdot\vec{r}} d^3r_\pi d^3r d^3R$$

and integrating over all the coordinates immediately produces three delta functions according to

$$\langle \chi_n | \hat{G} | \chi_m \rangle = \left(-\frac{2m_\pi}{p_{\pi_m}^2} \right) \delta^3(\vec{P}_{\pi_n} - \vec{P}_{\pi_m})$$
$$\delta^3(\vec{P}_n - \vec{P}_m)\, \delta^3(\vec{k}_n - \vec{k}_m) \qquad \text{(A4-5)}$$

Referring back to (2a-13), we multiply (A4-2), (A4-3), and (A4-5) and integrate over the intermediate states to write

$$\langle \psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \psi_a \rangle = t_n t_p \int \delta^3(\vec{P}_{\pi_b} + \vec{P}_b - \vec{P}_{\pi_n} - \vec{P}_n)$$
$$\psi_D^*\left(\frac{\vec{P}_n}{2} + \vec{k}_n\right)\left[-\frac{2m_\pi}{p_{\pi_m}^2} \right] \delta^3(\vec{P}_{\pi_n} - \vec{P}_{\pi_m})\, \delta^3(\vec{P}_n - \vec{P}_m)$$
$$\delta^3(\vec{k}_n - \vec{k}_m)\, \delta^3(\vec{P}_{\pi_m} + \vec{P}_m - \vec{P}_{\pi_a} - \vec{P}_a)\, \psi_D\left(-\frac{\vec{P}_m}{2} + \vec{k}_m\right)$$
$$d^3p_{\pi_n} d^3k_n d^3P_n d^3p_{\pi_m} d^3k_m d^3P_m \qquad \text{(A4-6)}$$

Integration over the four momenta $\vec{p}_{\pi_m}$, $\vec{P}_m$, $\vec{k}_m$, and $\vec{P}_n$ eliminates four delta functions and one is left with

$$\langle \psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \psi_a \rangle = t_n t_p\, \delta^3(\vec{P}_{\pi_b} + \vec{P}_b - \vec{P}_{\pi_a} - \vec{P}_a)$$
$$\int \psi_D^*\left(-\frac{\vec{P}_{\pi_n}}{2} + \vec{k}_n\right) \psi_D\left(\frac{\vec{P}_{\pi_n}}{2} + \vec{k}_n\right)\left[-\frac{2m_\pi}{p_{\pi_n}^2} \right] d^3p_{\pi_n} d^3k_n$$

(A4-7)

It is customary to write the integral in coordinate space rather than momentum space so we proceed to write the integrand of (A4-7) in terms of its Fourier transform. First write the wave functions in terms of their transforms according to (A2-4) to convert (A4-7) to

$$\langle \psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \psi_a \rangle = t_n t_p \, \delta^3(\vec{p}_{\pi_b} + \vec{p}_b - \vec{p}_{\pi_a} - \vec{p}_a)$$
$$(2\pi)^{-3} \int d^3p \, d^3k \, d^3r \, d^3r' \, \psi_D^*(\vec{r}) \, e^{-i(-\frac{\vec{p}}{2} + \vec{k}) \cdot \vec{r}}$$
$$\psi(\vec{r}') \, e^{i(\frac{\vec{p}}{2} + \vec{k}) \cdot \vec{r}'} \left(-\frac{2m_\pi}{p^2} \right)$$

Integration over $\vec{k}$ gives a delta function and integration over $\vec{r}'$ eliminates this delta function so that we get

$$\langle \psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \psi_a \rangle = t_n t_p \, \delta^3(\vec{p}_{\pi_b} + \vec{p}_b - \vec{p}_{\pi_a} - \vec{p}_a)$$
$$\int d^3p \, d^3r \, \psi_D^*(\vec{r}) \psi_D(\vec{r}) \left(-\frac{2m_\pi}{p^2} \right) e^{i\vec{p} \cdot \vec{r}} \qquad \text{(A4-8)}$$

Now observe that

$$-\frac{2m_\pi}{p^2} = -\frac{m_\pi}{2\pi} \int e^{-i\vec{p} \cdot \vec{r}'} \frac{1}{r'} \, d^3r' \qquad \text{(A4-9)}$$

Putting (A4-9) back into (A4-8), integrating over $\vec{p}$ to produce a delta function, and integrating over $\vec{r}'$ then

gives the result

$$\langle \psi_b | \hat{t}_1 \hat{G} \hat{t}_2 | \psi_a \rangle = -(2\pi)^2 m_\pi t_n t_p \delta^3(\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_a} - \vec{P}_a)$$

$$\int \psi^*(\vec{r}) \frac{1}{r} \psi(\vec{r}) d^3r \qquad \text{(A4-10)}$$

Invoking the statements directly preceeding equation (2a-14) we obtain for all the double scattering terms

$$\langle \psi_b | \hat{t}_1 \hat{G} \hat{t}_2 + \hat{t}_2 \hat{G} \hat{t}_1 | \psi_a \rangle = -(2\pi)^2 m_\pi \delta^3(\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_a} - \vec{P}_a)$$

$$(2t_n t_p - t_{ce}^2) \int \psi^*(\vec{r}) \frac{1}{r} \psi(r) d^3r \qquad \text{(A4-11)}$$

which is just (2a-14).

Appendix 5: Conversion of the T matrix to the Scattering Amplitude for First and Second Order Terms

To convert the T matrix (2a-14) to the corresponding scattering amplitude, use (1c-30) and (1c-16) to write

$$F_{\pi d}^{2nd} = \frac{-(2\pi)^2}{\left(\frac{1}{m_\pi}+\frac{1}{m_D}\right)}\left[-(2\pi)^2 m_\pi \left(2t_n t_p - t_{ce}^2\right)\right]\left\langle\frac{1}{r}\right\rangle$$

and use of (1c-30) once more for the t's gives

$$F_{\pi d}^{2nd} = \left(2f_n f_p - f_{ce}^2\right)\left\langle\frac{1}{r}\right\rangle\left[\frac{m_\pi}{\left(\frac{1}{m_\pi}+\frac{1}{m_D}\right)}\right]$$
$$\left(\frac{1}{m_\pi}+\frac{1}{m_n}\right)\left(\frac{1}{m_\pi}+\frac{1}{m_p}\right) \qquad \text{(A5-1)}$$

and taking m_π/m_p as negligible compared with unity we reduce the above to

$$F_{\pi d}^{2nd} = \left(2f_n f_p - f_{ce}^2\right)\left\langle\frac{1}{r}\right\rangle \qquad \text{(A5-2)}$$

The single scattering terms (2a-11) and (2a-12) are converted to their corresponding scattering amplitudes using (1c-30) to immediately obtain

$$F_{\pi d}^{1st} = \left(f_n + f_p\right)\left[\frac{1}{\left(\frac{1}{m_\pi}+\frac{1}{m_D}\right)}\right]\left(\frac{1}{m_\pi}+\frac{1}{m_N}\right) \qquad \text{(A5-3)}$$

where in keeping with the approximations of section 2a we have set $m_p = m_n = m_N$. Again throwing away negligible terms, (A5-3) reduces to

$$F_{\pi d}^{1st} = f_n + f_p \qquad \text{(A5-4)}$$

and cominging (A5-2) with (A5-4) we get the desired pi-d scattering amplitude to second order

$$F_{\pi d}^{1+2} = f_n + f_p + \left(2 f_n f_p - f_{ce}^2\right)\left\langle \frac{1}{r} \right\rangle \qquad \text{(2a-16)}$$

Appendix 6: Second Order Symmetrized Calculations

We derive here the DCE contribution to the T matrix with symmetrized wave functions neglecting binding corrections. Start with (2b-7) and calculate each term separately. Using (2b-1) write

$$\langle \psi_b | \hat{t}_1^{ce} | \chi_n^A \rangle = \frac{1}{\sqrt{2}} \left[(2\pi)^{-3} \right]^2 \int e^{i \vec{p}_{\pi_b} \cdot \vec{r}_\pi} \, e^{i \vec{P}_b \cdot \vec{R}} \, e^{i \vec{k}_b \cdot \vec{r}}$$

$$\psi_D^*(\vec{k}_b) \, t_{ce} \, \delta^3 \left(\vec{r}_\eta + \frac{\vec{r}}{2} - \vec{R} \right) e^{-i \vec{p}_{\pi_n} \cdot \vec{r}_\pi} \, e^{-i \vec{P}_n \cdot \vec{R}}$$

$$\left[e^{-i \vec{k}_n \cdot \vec{r}} - e^{i \vec{k}_n \cdot \vec{r}} \right] d^3k_b \, d^3r_\pi \, d^3r \, d^3R$$

and using the results of Appendix 4 we get

$$\langle \psi_b | \hat{t}_1^{ce} | \chi_n^A \rangle = \frac{1}{\sqrt{2}} \, t_{ce} \, \delta^3 \left(\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_n} - \vec{P}_n \right)$$

$$\left[\psi_D^* \left(\frac{\vec{p}_n}{2} + \vec{k}_n \right) - \psi_D^* \left(\frac{\vec{p}_n}{2} - \vec{k}_n \right) \right] \qquad \text{(A6-1)}$$

Similarly,

$$\langle \chi_m^A | \hat{t}_2^{ce} | \psi_a \rangle = \frac{1}{\sqrt{2}} \, t_{ce} \, \delta^3 \left(\vec{p}_{\pi_m} + \vec{P}_m - \vec{p}_{\pi_a} - \vec{P}_a \right)$$

$$\left[\psi_D \left(\frac{\vec{p}_m}{2} - \vec{k}_m \right) - \psi_D \left(\frac{\vec{p}_m}{2} + \vec{k}_m \right) \right] \qquad \text{(A6-2)}$$

The symmetrized propagator is

$$\langle \chi^A_n | \hat{G} | \chi^A_m \rangle = \delta^3(\vec{p}_{\pi_n} - \vec{p}_{\pi_m})\, \delta^3(\vec{p}_n - \vec{p}_m)$$
$$\left[\delta^3(\vec{k}_n - \vec{k}_m) - \delta^3(\vec{k}_n + \vec{k}_m)\right]\left(-\frac{2m_\pi}{p^2_{\pi_m}}\right) \qquad \text{(A6-3)}$$

Combining (A6-1), (A6-2), and (A6-3) with (2b-7) yields

$$\langle \psi_b | \hat{t}^{ce}_1 \hat{G}\, \hat{t}^{ce}_2 | \psi_a \rangle = -\frac{t^2_{ce}}{16} \int \delta^3(\vec{p}_{\pi_b} + \vec{p}_b - \vec{p}_{\pi_n} - \vec{p}_n)$$
$$\left[\psi^*_D\left(\frac{\vec{p}_n}{2} + \vec{k}_n\right) - \psi^*_D\left(\frac{\vec{p}_n}{2} - \vec{k}_n\right)\right] \delta^3(\vec{p}_{\pi_n} - \vec{p}_{\pi_m})\, \delta^3(\vec{p}_n - \vec{p}_m)$$
$$\left(-\frac{2m_\pi}{p^2_{\pi_m}}\right)\left[\delta^3(\vec{k}_n - \vec{k}_m) - \delta^3(\vec{k}_n + \vec{k}_m)\right] \delta^3(\vec{p}_{\pi_m} + \vec{p}_m - \vec{p}_{\pi_a} - \vec{p}_a)$$
$$\left[\psi_D\left(\frac{\vec{p}_m}{2} - \vec{k}_m\right) - \psi_D\left(\frac{\vec{p}_m}{2} + \vec{k}_m\right)\right] d^3p_{\pi_n}\, d^3k_n\, d^3P_n$$
$$d^3p_{\pi_m}\, d^3k_m\, d^3P_m$$

and performing the integrals just as in Appendix 4 we find

$$\langle \psi_b | \hat{t}^{ce}_1 \hat{G}\, \hat{t}^{ce}_2 | \psi_a \rangle = -\frac{t^2_{ce}}{4}\, \delta^3(\vec{p}_{\pi_b} + \vec{p}_b - \vec{p}_{\pi_a} - \vec{p}_a)$$
$$\left[\int \psi^*_D\left(-\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right) \psi_D\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right)\left(\frac{-2m_\pi}{p^2_{\pi_n}}\right) d^3p_{\pi_n}\, d^3k_n\right.$$
$$\left. - \int \psi^*_D\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right) \psi_D\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right)\left(-\frac{2m_\pi}{p^2_{\pi_n}}\right) d^3p_{\pi_n} d^3k_n\right] \qquad \text{(A6-4)}$$

The first integral in brackets is just the one of (A4-7) so we can replace it as in Appendix 4. The second integral is reduced by substitution of variables first to write

$$-(2\pi)^2 m_\pi \Delta = \int \psi_D^*\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right)\psi_D\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right)\left(\frac{-2m_\pi}{p_{\pi_n}^2}\right) d^3p_{\pi_n} d^3k_n$$

$$= \int \psi_D^*(\vec{K})\,\psi_D(\vec{K})\left(\frac{-2m_\pi}{p_{\pi_n}^2}\right) d^3p\, d^3K$$

and since the wave functions are normalized,

$$\Delta = \frac{1}{2\pi^2}\int \frac{1}{p^2}\, d^3p \qquad \text{(2b-9)}$$

Thus, (A6-4) is reduced to

$$\langle \psi_b | \hat{t}_1^{ce} \hat{G} \hat{t}_2^{ce} | \psi_a \rangle = \left[-(2\pi)^2 m_\pi\right]\left(-\frac{t_{ce}^2}{4}\right)$$

$$\delta^3\left(\vec{p}_{\pi_b} + \vec{P}_b - \vec{p}_{\pi_a} - \vec{P}_a\right)\left[\left\langle \frac{1}{r}\right\rangle - \Delta\right] \qquad \text{(2b-8)}$$

Appendix 7: Pi-nucleon Scattering Lengths

We wish to show here how the results (4a-2) are obtained from the data of reference (7). One usually writes the pi-nucleon scattering amplitudes in terms of the isospin 3/2 and isospin 1/2 scattering amplitudes, $f_{3/2}$ and $f_{1/2}$ (e.g., reference (13), p.49)

$$f_n \equiv f_{\pi^- n \to \pi^- n} = f_{3/2}$$

$$f_p \equiv f_{\pi^- p \to \pi^- p} = (1/3)(2f_{1/2} + f_{3/2})$$

$$f_{ce} \equiv f_{\pi^- p \to \pi^0 n} = \frac{\sqrt{2}}{3}(f_{3/2} - f_{1/2}) \qquad \text{(A7-1)}$$

In the notation of reference (7), $f_{3/2} \equiv a_3$, $f_{1/2} \equiv a_1$, and according to their tables, ($m_\pi^{-1} = 1.414$ fm)

$$a_1 - a_3 = (.277)(1.414 \text{ fm}) = .392 \text{ fm}$$
$$a_1 + 2a_3 = (-.026)(1.414 \text{ fm}) = -.0367 \text{ fm}$$

implying

$$a_3 = -.143 \text{ fm}$$
$$a_1 = .249 \text{ fm} \qquad \text{(A7-2)}$$

which together with (A7-1) immediately gives (4a-2).

Appendix 8: Relation Between the T Matrix and the Scattering Amplitude for Inelastic Scattering

We wish to generalize equation (1c-30) to include inelastic scattering. Combining (1c-23) and (1c-24) we get for inelastic scattering

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 |T_{ba}|^2 \left[\frac{p_1'}{p_1} \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^{-1} \left(\frac{1}{E_1'} + \frac{1}{E_2'} \right)^{-1} \right]$$

where unprimed p's and E's denote before scattering and primed denote after scattering. Using (1c-25) and the phase established in section 1c we get the desired result

$$f = -(2\pi)^2 T_{ba} \left[\frac{p_1'}{p_1} \left(\frac{1}{E_1} + \frac{1}{E_2} \right)^{-1} \left(\frac{1}{E_1'} + \frac{1}{E_2'} \right)^{-1} \right]^{\frac{1}{2}}$$

which is just the result (4a-9) written in slightly different notation.

Appendix 9: Phase Shift Momentum-Dependence for Inelastic Scattering

It is customary to expand the two-body scattering amplitude f in partial waves by writing[41]

$$f \propto \sum_{\ell=0}^{\infty} \frac{e^{i\delta_\ell} \sin\delta_\ell}{q} P_\ell(\cos\theta) \qquad \text{(A9-1)}$$

where q is the momentum of either particle in the two-body cm frame, the δ_ℓ are the phase shifts, P_l are Legendre polynomials, and θ is the scattering angle in the cm frame. Unfortunately (A9-1) only applies for elastic scattering since we don't know whether to use q_i or q_f (initial and final) or what for inelastic scattering. Whatever we use for q in the inelastic case, it must be some combination of q_i and q_f that is invariant under the interchange of both these momenta so that time reversal invariance is satisfied. Our goal is to find the appropriate combination of q_i and q_f that replaces q in (A9-1) for inelastic scattering.

In the notation of reference (4), section 103, the equation 103.13 states that

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[41] See e.g., reference (27), p.69, eq.3.4.

$$f \propto \int \phi_b^*(\xi)\, e^{-i\vec{k}_b\cdot\vec{r}}\, V(\vec{r},\xi)\, \phi_a(\xi)\, e^{i\vec{k}_a\vec{r}}\, d\xi^3 dr \qquad \text{(A9-2)}$$

in the Born approximation (i.e., to first order in the Born series), where ϕ is the wave function of the target (subscripts a and b refer to initial and final), V is the interaction potential, $\vec{r}$ and $\vec{k}$ are the coordinate and momentum vectors of the projectile, and ξ are the target coordinates. If we assume V is separable so that

$$V(\vec{r},\xi) = V(\vec{r})\,\mathbb{V}(\xi) \qquad \text{(A9-3)}$$

equation (A9-2) becomes

$$f \propto \left[\int \phi_b^*(\xi)\,\mathbb{V}(\xi)\,\phi_a(\xi)\, d\xi\right] \int e^{i(\vec{k}_a-\vec{k}_b)\cdot\vec{r}}\, V(\vec{r})\, d^3r \qquad \text{(A9-4)}$$

Treat the term in brackets above as a proportionality factor independent of the k's and let

$$\vec{K} = \vec{k}_a - \vec{k}_b \qquad \text{(A9-5)}$$

$$\frac{\vec{K}\cdot\vec{r}}{Kr} = \cos\theta_r$$

which allows us to re-write (A9-4) as

$$f \propto \int_0^\infty \int_{-1}^{1} e^{iKr\cos\theta_r}\, V(\vec{r})\, d^3r \qquad \text{(A9-6)}$$

For simplicity and keeping in mind that we are dealing with strong interactions, take for V the Yukawa potential

$$V(\vec{r}) = V_o \frac{e^{-\alpha r}}{r} \qquad \text{(A9-7)}$$

where V_o and α are constants. Then we can integrate (A9-6) using (A9-7) to obtain

$$f \propto \frac{1}{\alpha^2 + k^2} = \alpha^2 + k_a^2 + k_b^2 - 2k_a k_b \cos\theta \qquad \text{(A9-8)}$$

where $$\cos\theta = \frac{\vec{k}_a \cdot \vec{k}_b}{k_a k_b}$$

For small k_a and k_b (low energy scattering) we can neglect the terms k_a^2 and k_b^2 compared to α^2 (but keep the $\cos\theta$ term to preserve the angle dependence). Then we can write (A9-8) by expanding the denominator as

$$f \propto \left[1 + \frac{2k_a k_b \cos\theta}{\alpha^2} + \left(\frac{2k_a k_b \cos\theta}{\alpha^2}\right)^2 + \cdots\right] \qquad \text{(A9-9)}$$

Looking back at (A9-1), the phase shifts are usually parameterized by writing[42]

$$\delta_\ell \propto q^{2\ell+1} \qquad \text{(A9-10)}$$

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[42] See reference (6), p.306, eq.16-98.

so that (A9-10) into (A9-1) gives for small

$$f \propto \sum_{\ell=0}^{\infty} \frac{\delta_\ell}{q} P_\ell(\cos\theta)$$

$$\propto \sum_{\ell=0}^{\infty} q^{2\ell} P_\ell(\cos\theta) \qquad \text{(A9-11)}$$

Keeping only the lowest power of q (small q) in each factor of $(\cos\theta)^\ell$ we can write (A9-11) as

$$f \propto \sum_{\ell=0}^{\infty} q^{2\ell} (\cos\theta)^\ell \qquad \text{(A9-12)}$$

and comparing each $(\cos\theta)^\ell$ term in (A9-12) and (A9-9) we make the association

$$q^2 \propto k_a k_b$$

or in our original notation

$$q^2 \propto q_i q_f \qquad \text{(A9-13)}$$

But when $q_i = q_f$ we must have $q = q_i$ so that the proportionality factor of (A9-13) is determined and we can write the final result for inelastic scattering

$$q_{\text{effective}} = \sqrt{q_i q_f} \qquad \text{(A9-14)}$$

Notice that (A9-14) is invariant under the interchange of q_i and q_f as it must be.

Appendix 10: Pi-nucleon S-wave Phase Shift Parameters

If T denotes the total isospin quantum number of a pi-nucleon state (T=3/2 or 1/2) it is customary to expand the two-body pi-nucleon scattering amplitude for each value of T by writing[43]

$$f^{2T} = \frac{1}{q} \sum_{\ell=0}^{\infty} \left[(\ell+1) A^{2T}_{\ell+} + \ell A^{2T}_{\ell-} \right] P_\ell(\cos\theta) \qquad \text{(A10-1)}$$

where $$A^{2T}_{\ell\pm} = \frac{1}{2i} \left[\eta^{2T}_{\ell\pm} e^{2i\delta^{2T}_{\ell\pm}} - 1 \right] \qquad \text{(A10-2)}$$

and $\ell_\pm$ means $j=\ell\pm\frac{1}{2}$. (We are neglecting spin-flipping mechanisms for now). The η's are called absorption coefficients and the δ's are the phase shifts. It is also customary to re-label the δ's and η's to include the total angular momentum quantum number j as well as the orbital angular momentum quantum number ℓ and the total isospin quantum number T. One uses the spectroscopic notation S,P,D,F, etc. for ℓ =0,1,2,3, etc., respectively, and writes

$$\delta^1_{0-} = 0 \qquad \delta^1_{0+} \equiv S_{11} \qquad \eta^1_{0-} = 1 \qquad \eta^1_{0+} \equiv \eta_{11}$$

$$\delta^3_{0-} = 0 \qquad \delta^3_{1+} \equiv S_{31} \qquad \eta^3_{0-} = 1 \qquad \eta^3_{0+} \equiv \eta_{31}$$

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[43] See reference (18) for example.

Keeping S-waves means that we only keep the $\ell = 0$ term of (A10-1). The phase shifts and absorption coefficients we use are polynomial fits to the UCRL tables (reference (16)), and in particular we choose the data of Kirsopp. It was found necessary to consider momenta up to 1000 Mev/c to make our integrals convergent to one per cent accuracy. To be more specific, squaring the momentum-space deuteron wave function (A2-4) and integrating over k, it is necessary to integrate out to k=1000 Mev/c to get within one per cent of the integral evaluated out to infinite k. We list the parameterization employed as follows:

$0 \leq q < 100$ Mev/c: (all S's in degrees)

$S_{11} = (-5.95\times10^{-6}q^2 + 5.3\times10^{-4}q + 7.25\times10^{-2})q$

$S_{31} = (-1.66\times10^{-6}q^2 + (1.41\times10^{-4}q - 4.15\times10^{-2})q$

$\eta_{11} = \eta_{31} = 1.00$

$100 \leq q < 200$ Mev/c:

$S_{11} = 2.5\times10^{-2}q + 4.1$

$S_{31} = -7.7\times10^{-2}q + 3.3$

$\eta_{11} = \eta_{31} = 1.00$

$200 \leq q < 300$ Mev/c:

$S_{11} = 1.1\times10^{-2}q + 6.9$

$S_{31} = -9.3\times10^{-2}q + 6.5$, $\eta_{11} = \eta_{31} = 1.00$

$300 \leq q < 400$ Mev/c:

$S_{11} = 1.16 \times 10^{-1} q - 24.6$

$S_{31} = -7.1 \times 10^{-2} q - .1$

$\eta_{11} = -9.0 \times 10^{-4} q + 1.27$

$\eta_{31} = -7.0 \times 10^{-4} q + 1.21$

$400 \leq q < 500$ Mev/c:

$S_{11} = 2.93 \times 10^{-1} q - 95.0$

$S_{31} = -3.3 \times 10^{-2} q - 15.3$

$\eta_{11} = -5.2 \times 10^{-3} q + 2.99$

$\eta_{31} = -4.0 \times 10^{-3} q + 2.53$

$500 \leq q < 600$ Mev/c:

$S_{11} = 5.59 \times 10^{-1} q - 229$

$S_{31} = -3.22 \times 10^{-1} q + 129$

$\eta_{11} = 3.2 \times 10^{-3} q - 1.21$

$\eta_{31} = 1.3 \times 10^{-3} q - .12$

$600 \leq q < 700$ Mev/c:

$S_{11} = 2.57 \times 10^{-1} q - 48$

$S_{31} = -6.0 \times 10^{-3} q - 60.4$

$\eta_{11} = 3.0 \times 10^{-4} q + .53$

$\eta_{31} = 2.4 \times 10^{-3} q - .78$

$700 \leq q < 800$ Mev/c:

$S_{11} = 1.58 \times 10^{-1} q + 22.7$

$S_{31} = 9.4\times10^{-2}q - 131$

$\eta_{11} = 1.00\times10^{-4}q + .75$

$\eta_{31} = -4.0\times10^{-3}q + 3.7$

$800 \leq q < 1000$ Mev/c:

$S_{11} = -3.35\times10^{-1}q + 417$

$S_{31} = -2.48\times10^{-1}q + 143$

$\eta_{11} = -7.0\times10^{-4}q + 1.23$

$\eta_{31} = -3.5\times10^{-3}q + 3.3$

Appendix 11: Pi-nucleon P-wave Phase Shift Parameters

In analogy with Appendix 10, we list here the P-wave ($\ell = 1$) absorption coefficients and phase shifts. The change in notation appropriate here is

$$\delta^{1}_{1-} \equiv P_{11} \qquad \eta^{1}_{1-} \equiv \eta_{11}$$

$$\delta^{3}_{1-} \equiv P_{31} \qquad \eta^{3}_{1-} \equiv \eta_{31}$$

$$\delta^{1}_{1+} \equiv P_{13} \qquad \eta^{1}_{1+} \equiv \eta_{13}$$

$$\delta^{3}_{1+} \equiv P_{33} \qquad \eta^{3}_{1+} \equiv \eta_{33}$$

and we again fit the UCRL tables (Kirsopp data) with polynomials in q. The parameterization is as follows:

P_{11} (in degrees) = $(-1.83\times10^{-6})q^3$, $0 \leq q < 100$ Mev/c

1.00, $100 \leq q < 200$ Mev/c

$1.74\times10^{-6}q^3 - 6.85\times10^{-2}q$, $200 \leq q < 400$ Mev/c

$4.35q - 89$, $400 \leq q < 600$

172 , $600 \leq q < 1000$

η_{11} = 1.00 , $0 \leq q < 300$ Mev/c

$-7.0\times10^{-3}q + 3.1$, $300 \leq q < 400$ Mev/c

$1.0\times10^{-3}q - .1$, $400 \leq q < 600$ Mev/c

$-4.2\times10^{-3}q + 3.0$, $600 \leq q < 700$ Mev/c

$2.74\times10^{-3}q - 1.84$, $700 \leq q < 1000$ Mev/c

P_{13} = 0.0 , $0 \leq q < 200$ Mev/c

-3.0 , $200 \leq q < 300$ Mev/c

-5.0 , $300 \leq q < 600$ Mev/c

-10.0 , $600 \leq q < 1000$ Mev/c

η_{13} = 1.00 , $0 \leq q < 500$ Mev/c

$-1.69 \times 10^{-3} q + 1.85$, $500 \leq q < 1000$ Mev/c

$P_{31} = -7.90 \times 10^{-7} q^3$, $0 \leq q < 100$ Mev/c

$-1.14 \times 10^{-4} q^2 + 3.3 \times 10^{-3} q$, $100 \leq q < 500$ Mev/c

-26.8 , $500 \leq q < 700$ Mev/c

$-.148q + 77$, $700 \leq q < 1000$ Mev/c

$\eta_{31} = \eta_{33} = 1.00$, $0 \leq q < 400$ Mev/c

$-1.02 \times 10^{-3} q + 1.41$, $400 \leq q < 1000$ Mev/c

For P_{33} the appropriate momentum dependence is well-known from Chew-Low theory (see reference (27), page 233, eq.8.22) and is given by

$$\cot P_{33} = \frac{1.8 \times 10^5 \left(\sqrt{s} - 939\right)}{q^3} \left[1 - \frac{\sqrt{s} - 939}{297} \right]$$

where $\sqrt{s}$ is the total center of mass energy. We again have the problem of inelastic scattering so it is most convenient to parameterize $\sqrt{s}$ in terms of q. The choice

$$\sqrt{s} = 1.5\times10^{-3}q^2 + .32q + 1078$$

gives a reasonable fit to the data out to q=500 Mev/c and for higher momenta the P_{33} phase shift is fairly constant out to 1000 Mev/c so we take[44]

$$P_{33} = 172 \text{ degrees}, \quad 500 \leq q < 1000 \text{ Mev/c}$$

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[44]The inelastic scattering is handled by taking $q = q_{eff}$. given in (A9-14) in the $\sqrt{s}$ parameterization.

Appendix 12: Treatment of Spin-Flipping Mechanism

We show here how to treat the added complication when spin-flip terms are included in the pi-nucleon scattering amplitudes. Unfortunately we will go through considerable effort to find the spin-flip contribution (up to P-waves only).

The most general pi-nucleon scattering amplitude must have the form[45]

$$f^{2T} = \langle b | \frac{1}{q} \sum_{\ell=0}^{\infty} \left[(\ell+1) A_{\ell+}^{2T} + \ell A_{\ell-}^{2T} \right] P_\ell(\cos\theta)$$

$$+ i \vec{\sigma} \cdot (\hat{q}_b \times \hat{q}_a) \frac{1}{q} \sum_{\ell=0}^{\infty} \left(A_{\ell-}^{2T} - A_{\ell+}^{2T} \right) P'_\ell(\cos\theta) | a \rangle \qquad \text{(A12-1)}$$

where the prime on P_ℓ denotes derivative with respect to $\cos\theta$, $\hat{q}_b$ and $\hat{q}_a$ are unit vectors in the direction of the initial and final pion (in the pi-nucleon cm frame), $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli spin matrices written as a three-component vector, the A's are given by (A10-2), and $|a\rangle$ and $|b\rangle$ are the initial and final nucleon spin states. Compare (A12-1) with (A10-1) which neglects the spin-flip terms. If we only keep the terms up to P-waves in (A12-1) we have

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[45] See reference (18), p.28, just below eq.2-24.

$$f^{2T} = \langle b | \; A + B\vec{\sigma}\cdot(\hat{q}_b \times \hat{q}_a) \; | a \rangle \qquad \text{(A12-2)}$$

where $$A = \frac{1}{q}\left[A_{0+}^{2T} + \left(2A_{1+}^{2T} + A_{1-}^{2T}\right)\cos\theta \right] \qquad \text{(A12-3)}$$

$$B = \frac{1}{q}\left[A_{1-}^{2T} - A_{1+}^{2T} \right]$$

The spin state $|b\rangle$ must be the same as $|a\rangle$ because in the threshold limit no energy is available to flip spins. Therefore there can be no spin-flip contribution in the single scattering terms since initial and final spin states are the same. Now write a typical second order term

$$\langle b | \hat{t}_1 \hat{G} \hat{t}_2 | a \rangle = \langle b | \hat{t}_1 | n \rangle \langle n | \hat{G} | m \rangle \langle m | \hat{t}_2 | a \rangle$$

$$\propto \langle b | \hat{t}_1 | n \rangle \langle n | \hat{t}_2 | a \rangle$$

where in writing the proportionality we do so because we are only concerned with spin and angular variables, not momenta and energies (this will become clearer later). Then substituting (A12-3) in the above (with the appropriate conversion from f to t) gives

$$\langle b|\hat{t}_1 \hat{G} \hat{t}_2|a\rangle \propto \langle b|A_1 + B_1 \vec{\sigma}\cdot(\hat{q}_b \times \hat{q}_1)|n\rangle$$

$$\langle n|A_2 + B_2 \vec{\sigma}\cdot(\hat{q}_2 \times \hat{q}_a)|a\rangle$$

$$= \langle b|A_1|n\rangle\langle n|A_2|a\rangle + \langle b|A_1|n\rangle\langle n|B_2\vec{\sigma}\cdot(\hat{q}_2\times\hat{q}_a)|a\rangle$$

$$+ \langle b|B_1\vec{\sigma}\cdot(\hat{q}_b\times\hat{q}_1)|n\rangle\langle n|A_2|a\rangle$$

$$+ \langle b|B_1\vec{\sigma}\cdot(\hat{q}_b\times\hat{q}_1)|n\rangle\langle n|B_2\vec{\sigma}\cdot(\hat{q}_2\times\hat{q}_a)|a\rangle$$

(A12-4)

where A_1, A_2, B_1, B_2 are the same as (A12-3) but with the appropriate kinematic factors to convert the scattering amplitudes to the corresponding two-body T matrices, $\hat{q}_1$ and $\hat{q}_2$ are the pion directions in the cm of pion and nucleon 1 and 2, respectively, in state $|n\rangle$.

Now we evaluate the necessary kinematic factors of (A12-4) nonrelativistically. Let us first list all lab frame momenta and velocities (as determined by (2a-3)) as follows:

lab momentum of pion in state $|b\rangle$ = $\vec{p}_{\pi_b} = m_\pi \vec{v}_{\pi_b} = 0$

lab momentum of pion in state $|a\rangle$ = $\vec{p}_{\pi_a} = m_\pi \vec{v}_{\pi_a} = 0$

lab momentum of pion in state $|n\rangle$ = $\vec{p}_{\pi_n} = m_\pi \vec{v}_{\pi_n}$

(continued from previous page)

lab momentum of nucleon 1 in state $|n\rangle = \vec{p}_1 = -\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right)$

lab momentum of nucleon 2 in state $|n\rangle = \vec{p}_2 = -\left(\frac{\vec{p}_{\pi_n}}{2} - \vec{k}_n\right)$

(A12-5)

where we used $\vec{p}_n = -\vec{p}_{\pi_n}$ in (2a-3). Then the velocity of the cm of pion and nucleon 1 in state $|n\rangle$ is

$$\vec{V}_{CM_1} = \frac{\vec{p}_{\pi_n} + \vec{p}_1}{m_\pi + m_N} = \frac{\frac{\vec{p}_{\pi_n}}{2} - \vec{k}_n}{m_\pi + m_N} \qquad \text{(A12-6)}$$

and the velocity of the cm of pion and nucleon 2 in state $|n\rangle$ is

$$\vec{V}_{CM_2} = \frac{\vec{p}_{\pi_n} + \vec{p}_2}{m_\pi + m_N} = \frac{\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n}{m_\pi + m_N} \qquad \text{(A12-7)}$$

so that by definition and using (A12-5) through (A12-7), the q's in (A12-4) are

$$\vec{q}_{fb} = m_\pi\left(\vec{V}_{\pi_b} - V_{CM_1}\right) = -\left(\frac{m_\pi}{m_\pi + m_N}\right)\left(\frac{\vec{p}_{\pi_n}}{2} - \vec{k}_n\right) \qquad \text{(A12-8)}$$

$$\vec{q}_{fa} = m_\pi\left(\vec{V}_{\pi_a} - \vec{V}_{CM_2}\right) = -\left(\frac{m_\pi}{m_\pi + m_N}\right)\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right)$$

$$\vec{q}_{fi} = m_\pi\left(\vec{V}_{\pi_n} - \vec{V}_{CM_1}\right) = \vec{p}_{\pi_n} - \left(\frac{m_\pi}{m_\pi + m_N}\right)\left(\frac{\vec{p}_{\pi_n}}{2} - \vec{k}_n\right)$$

$$\vec{q}_{i2} = m_\pi\left(\vec{V}_{\pi_n} - \vec{V}_{CM_2}\right) = \vec{p}_{\pi_n} - \left(\frac{m_\pi}{m_\pi + m_N}\right)\left(\frac{\vec{p}_{\pi_n}}{2} + \vec{k}_n\right)$$

With (A12-8) we can perform the required cross products of (A12-4), but note that we want cross products of unit vectors so that all vectors in (A12-8) must be divided by their magnitudes before taking cross products, complicating the work. Thus, using (A12-8),

$$\hat{q}_b \times \hat{q}_1 = -\left(\frac{m_\pi}{m_\pi + m_N}\right)(\vec{p}_{\pi_n} \times \vec{k}_n)\,\frac{1}{q_b q_1}$$

$$\hat{q}_2 \times \hat{q}_a = -\left(\frac{m_\pi}{m_\pi + m_N}\right)(\vec{p}_{\pi_n} \times \vec{k}_n)\,\frac{1}{q_2 q_a} \qquad \text{(A12-9)}$$

To proceed, define $\cos\theta = \vec{p}_{\pi_n} \cdot \vec{k}_n / p_{\pi_n} k_n$ so that

$$|\vec{p}_{\pi_n} \times \vec{k}_n| = p_{\pi_n} k_n \sin\theta$$

If we pick an arbitrary direction of p_{π_n} and integrate over it this is equivalent to picking an arbitrary direction of $\vec{p}_{\pi_n} \times \vec{k}_n$ and integrating over it. So we write

$$\vec{p}_{\pi_n} \times \vec{k}_n = p_{\pi_n} k_n \sin\theta \left[\sin\alpha\cos\beta\,\hat{i} + \sin\alpha\sin\beta\,\hat{j} + \cos\alpha\,\hat{k}\right]$$

Integration over β gives zero for the $\hat{i}$ and $\hat{j}$ terms and integration over α gives zero for the $\hat{k}$ term. The magnitudes q_1, q_2, q_a, q_b depend on θ but not on α or β and the same is true for all other kinematic factors in (A12-4). Therefore integrating the second and third terms of (A12-4) (A_1B_2 and B_1A_2) over angular variables gives zero just as

above, using (A12-9). The first term of (A12-4) is the one we have already evaluated when we did the non-spin flip case. It remains to evaluate the last term of (A12-4) to get the spin-flip contribution.

Consider the expression

$$\vec{\sigma}\cdot(\vec{p}_{n_n}\times\vec{k}_n) = p_{n_n}k_n\sin\theta\left[\begin{pmatrix}0&1\\1&0\end{pmatrix}\sin\alpha\cos\beta\right.$$

$$\left.+\begin{pmatrix}0&-i\\i&0\end{pmatrix}\sin\alpha\sin\beta+\begin{pmatrix}1&0\\0&-1\end{pmatrix}\cos\alpha\right] \qquad \text{(A12-10)}$$

and consider also the integral

$$S=\int\left\langle\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\left\{\sigma\cdot(\vec{p}_{n_n}\times\vec{k}_n)\right\}_1|n\right\rangle$$

$$\left\langle n\left|\left\{\vec{\sigma}\cdot(\vec{p}_{n_n}\times\vec{k}_n)\right\}_2\right|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle d(\cos\alpha)\,d\beta \qquad \text{(A12-11)}$$

where the subscripts 1 and 2 mean the operator only operates on the first or second part of the two-nucleon spin states, and the notation

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

means both nucleons have their spins up (along the axis $\beta=0$),

$$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

means nucleon one is spin down and nucleon 2 spin up, etc.

In (A12-11) we have chosen

$$|a\rangle = |b\rangle = \left|\binom{1}{0}\binom{1}{0}\right\rangle$$

since the spins of the nucleons are parallel (spin 1) in the deuteron. The only possible intermediate spin state $|n\rangle$ is $\left|\binom{1}{0}\binom{1}{0}\right\rangle$ since each spin operator in (A12-11) only operates once on each nucleon. Then putting (A12-10) into (A12-11) we have

$$S = \left(p_{\pi_n} k_n \sin\theta\right)^2 \int (\cos\alpha)^2 \, d(\cos\alpha)\, d\beta$$

$$= \frac{4\pi}{3}\left(p_{\pi_n} k_n \sin\theta\right)^2 \qquad \text{(A12-12)}$$

The contribution of the spin-flip part of (A12-4) is thus

$$\frac{\frac{4\pi}{3}\left(p_{\pi_n} k_n \sin\theta\right)^2 B_1 B_2 \left(\frac{m_\pi}{m_\pi + m_N}\right)^2}{q_a q_b q_1 q_2}$$

whereas the non-spin-flip contribution is $4\pi A_1 A_2$ (the 4π comes from integrating over α and β). Numerical integration of these two contributions over θ, $\vec{p}_{\pi_n}$, and $\vec{k}_n$ shows the spin-flip part to be negligible.

Appendix 13: Relativistic Propagator

We would like to justify the relativistic replacement (4d-1) in the propagator for the non-relativistic (A4-4). For convenience we will consider a two-particle propagator of spin-zero particles. If p_1 and p_2 are the four-momenta of each particle then we want to show that the non-relativistic propagator

$$G_{\text{non-rel.}} = \left[E_i - \left(m + \frac{|\vec{p}_1|^2}{2m}\right) - \left(m + \frac{|\vec{p}_2|^2}{2m}\right) \right]^{-1}$$

is more correctly replaced by

$$G_{\text{rel.}} = \left[E_i - \left(|\vec{p}_1|^2 + m^2\right)^{\frac{1}{2}} - \left(|\vec{p}_2|^2 + m^2\right)^{\frac{1}{2}} \right]^{-1}$$

where E_i is the initial total energy of both particles and for convenience we take the mass of the particles equal.

From field theory the propagator for a single scalar spin-zero particle is

$$\frac{1}{p^2 - m^2 + i\epsilon} = \frac{1}{p^2 - m^2} - i\pi\,\delta(p^2 - m^2) \qquad \text{(A13-1)}$$

For two spin-zero particles the propagator is just the product of the single particle propagators

$$G(p_1, p_2) = \left[\left(\frac{1}{p_1^2 - m^2}\right) - i\pi\,\delta(p_1^2 - m^2)\right]\left[\left(\frac{1}{p_2^2 - m^2}\right) - i\pi\,\delta(p_2^2 - m^2)\right] \qquad \text{(A13-2)}$$

At high energies the delta-function part of the propagator dominates so we write

$$G(p_1, p_2) \approx -\pi^2 \delta(p_1^2 - m^2)\, \delta(p_2^2 - m^2) \qquad \text{(A13-3)}$$

Define the total energy by

$$S' = (p_1 + p_2)^2 \equiv (E_1 + E_2)^2 - |\vec{p}_1 + \vec{p}_2|^2$$

where $E_1 = (|\vec{p}_1|^2 + m^2)^{\frac{1}{2}}$, $E_2 = (|\vec{p}_2|^2 + m^2)^{\frac{1}{2}}$

and similarly for the initial state

$$S_i = E_i^2$$

In terms of S' we have

$$p_1^2 - m^2 = S' - 2E_2^2 - 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2$$

$$p_2^2 - m^2 = S' - 2E_1^2 - 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2 \qquad \text{(A13-4)}$$

so that (A13-4) allows us to re-write the two-particle propagator (A13-3) in terms of S'

$$G(S', p_2) \approx -\pi^2 \delta(S' - 2E_2^2 - 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2)$$
$$\delta(S' - 2E_1^2 - 2E_1E_2 - 2\vec{p}_1 \cdot \vec{p}_2) \qquad \text{(A13-5)}$$

Now use the Cauchy integral formula (write a dispersion relation)

$$G(S_i, p_2) = \frac{1}{2\pi i} \oint_C \frac{G(S', p_2)\, dS'}{S' - S_i} \qquad \text{(A13-6)}$$

where the contour C is appropriately chosen within the limits of Cauchy's theorem. Performing the integral over the delta functions in (A13-5) we obtain

$$G(S_i, p_2) \propto \frac{\delta(2E_1^2 - 2E_2^2)}{2E_1^2 + 2E_1E_2 + 2\vec{p}_1\cdot\vec{p}_2 - S_i} \qquad \text{(A13-7)}$$

We can neglect the $\vec{p}_1\cdot\vec{p}_2$ term in the denominator since its average over angles gives zero. We are then left with

$$G(S_i, p_2) \propto \frac{1}{(E_1+E_2)^2 - S_i} = \frac{1}{(E_1+E_2-\sqrt{S_i})} \quad \frac{1}{E_1+E_2+\sqrt{S_i}}$$

and the right-most factor is just for normalization so that we obtain our desired result

$$G(S_i, p_2) \propto G_{rel.} = \frac{1}{E_i - (E_1+E_2)} \qquad \text{(A13-8)}$$

The careful reader will note that the result (A13-8) only holds for high energies since (A13-6) can only be used in conjunction with (A13-5) for large values of S_i; that is

the integrand of (A13-6) is peaked for S' near S_i so if S_i is small the major contribution to (A13-6) may not come from large S', in which case (A13-5) may not be appropriate. We therefore take (A13-8) as a very heuristic, hand-waving result but believe it to be close to reality, nevertheless. One could test our belief by using the full propagator (A13-2) in (A13-6) but we leave that task to the reader.

One can proceed analogously for three particles instead of two and treat two of them as spin ½ Dirac particles with similar results. The general conclusion is that non-relativistic energies in propagators get replaced by their relativistic analogs.