Number of Closed Strings Emitted from a Decaying D-brane

by

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Abstract

We have calculated in the covariant gauge the total number of massless closed strings emitted from a decaying Dp-brane using fermionization technology on the time component of the boundary state. This verifies the computation carried out in the less well known temporal gauge despite differences in renormalization schemes.

In addition we have attempted to use the fermionic technique to calculate the total number of closed strings emitted. Our result is difficult to interpret due to the ambiguity involved with rotating it back to Minkowski space.

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Dedication

I would like to dedicate my thesis to my fiancee Carmen Romero as well as to thank her for her love and support over the past two years.

Chapter 1

Introduction

Perturbed slightly from the maximum of its potential an open string tachyon will undertake a rolling process towards the minimum of the potential, the closed string vacuum [5]. It has been shown [2, 7] that as the brane decays it radiates closed strings.

The main goal of this thesis was to find the total number of strings emitted by the Dp-brane. This calculation has been previously carried out level by level in [2] in the temporal gauge. We have been motivated to attempt this same calculation in the familiar covariant gauge because there is some doubt that the use of the temporal gauge in [2] is properly justified.

We find that our result for the total number of massless strings emitted agrees with [2]. This is despite the fact that the coupling constants g and $\bar{g}$, defined below, are given by a different renormalization procedure in the fermionic approach than in the bosonic approach. For the special case of the half S-brane we find the result to be coupling independent, allowing us to find explicit values for the number of particles emitted.

Our attempts at calculating the total number of closed strings emitted by an unstable Dp-brane in the covariant gauge have been unsuccessful. This is due to the fact that fermionization of the time component of the boundary state requires an analytic continuation to periodic Euclidean time. Thus any physical result requires the reinstatement of non-compact Euclidean time followed by a Wick rotation back to Minkowski time. As we will explain in chapter 5 there appears no clear way of rotating back to Minkowski space, preventing us from obtaining a meaningful result.

1.1 The Basics

The conformal field theory discussed in this paper has applications in many fields of physics [1], but we will be discussing it in the context of bosonic string theory. We will be dealing with three objects in bosonic string theory; the closed string, the open string and the D-brane. Oscillations of the strings are interpreted as bosons, for example the first excited state of the open and closed strings are interpreted as spin 1 and spin 2 objects, respectively.

A string is embedded in space-time via the parameterizations $X^\mu(\sigma, \tau)$ where τ is analogous to proper time in the case of the point particle. The string is an extended object which is why we require a second parameter σ. The standard convention is to set the length of string to 2π for the closed string and π for the open string. The closed string obeys the condition $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$. The endpoints of the open string are forced to lie on an object known as a D-brane [4]. The D-brane is sometimes called a Dp-brane in the literature to signify that it extends in p spatial dimensions.

As a string moves through space-time it traces out an object known as the world-sheet. Physics is parameterization invariant so we are free to use any parameterization of the world-sheet we wish. Two very useful parameterizations of the world-sheet are: the cylinder

$$w = \sigma + i\tau \quad \bar{w} = \sigma - i\tau \tag{1.1}$$

and the disk

$$z = \pm e^{iw} \quad \bar{z} = \pm e^{-iw} \tag{1.2}$$

where $+$ is used in the case of the closed string and $-$ is used in the case of the open string. These parameterizations are used frequently throughout this thesis.

1.2 Tachyon Action

The tachyon action is described by the usual non-decaying world-sheet action plus an exactly marginal boundary operator,

$$S = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\tau \int_0^\pi d\sigma \; \partial X^0 \cdot \partial X^0 - \int_{-\infty}^{\infty} d\tau S_{bdy} + ... \tag{1.3}$$

given by

$$S_{bdy} = \left(\frac{g}{2}e^{iX^0} + \frac{\bar{g}}{2}e^{-iX^0} + A\partial_\tau X^0\right)\Big|_{\sigma=0}. \tag{1.4}$$

It is also possible to include a boundary interaction at the $\sigma = \pi$ boundary with different couplings. When we do include an interaction at this boundary we take the couplings to be the same as on the $\sigma = 0$ boundary. We note that we have performed a wick rotation in (1.3), so that X^0 is time in Euclidean space, and the three dots indicate the usual action for the spatial directions and ghosts. Also in (1.3) we have set $\alpha' = 1$ and we will keep this convention throughout the rest of the thesis. We are assuming that the tachyon condensate is space independent, resulting in the usual equation of motion

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu(\sigma,\tau) = 0 \tag{1.5}$$

as well as a boundary condition given by

$$\left(-\partial_\sigma X^0 + i\frac{g}{2}e^{iX^0} - i\frac{\bar{g}}{2}e^{-iX^0}\right)\Big|_{\sigma=0} = 0 \tag{1.6}$$

at the end of the string.

Equation (1.3) defines a conformal field theory for all complex values of g and $\bar{g}$, however, since the Minkowski space tachyon field is real g and $\bar{g}$ must also be real. We are interested in two tachyon profiles, the half S-brane for which $\bar{g} = 0$ and as the full S-brane first discussed by Sen in [6] for which $g = \bar{g}$. The former corresponds to a tachyon decaying to a stable configuration and the latter to a tachyon coming up from the bottom of its potential and then falling back into it.

The topological term associated with A in (1.3) does not affect the equations of motion or the boundary condition. For a non-compact boson it can be interpreted as a Bloch wave-number. Further discussion on this can be found in [1]. In this discussion the authors conclude that any value of $A \in (0,1)$ allows us to treat X^0 as if it had the compactification $X^0 \to X^0 + 2\pi$. We will show that we need to treat X^0 as if it were compactified in order to use the fermionization technology developed in chapter 2. Since X^0 is not actually compactified we will wish at some point in our calculation to decompactify X^0 by integrating out the A dependence over all allowed values of $A \in (0,1)$. It is important to note that when we include an interaction at the $\sigma = \pi$ end of the strip and wish to decompactify we must integrate over $A = A_{\sigma=\pi} - A_{\sigma=0}$, is the difference between the parameters at each boundary.

1.3 Boundary State Formalism

The boundary state technique is useful for calculating open string partition functions

$$Z[\beta] = Tre^{-\beta H}, \tag{1.7}$$

where H is the Hamiltonian obtained from (1.3). We can rewrite this partition function as a closed string correlation function by gluing together the ends of the open string world-sheet. This gluing procedure is illustrated in Fig 1.1. The incoming part of the world-sheet at $\tau = 0$ is connected to the outgoing part of the world-sheet at $\tau = \pi$ creating a cylinder and making

τ periodic. As a result the definitions of τ and σ can be interchanged expressing the correlator as a time evolution operator sandwiched between the boundary states, which encode information about the boundary conditions

$$Z[\beta] = \langle B_1 | e^{\frac{2\pi^2}{\beta}\hat{H}} | B_2 \rangle. \tag{1.8}$$

The action (1.3) is now that of a free boson living on a Euclidean cylinder

$$S' = \frac{1}{4\pi}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \ \left((\partial_\tau X^0)^2 + (\partial_\sigma X^0)^2\right) + ... \tag{1.9}$$

The operator $\hat{H}$ is the Hamiltonian of this action. The Euclidean time parameter is $\alpha = \frac{2\pi^2}{\beta}$ because we have used conformal invariance to rescale the closed string world-sheet to match the convention $0 \leq \sigma \leq 2\pi$.

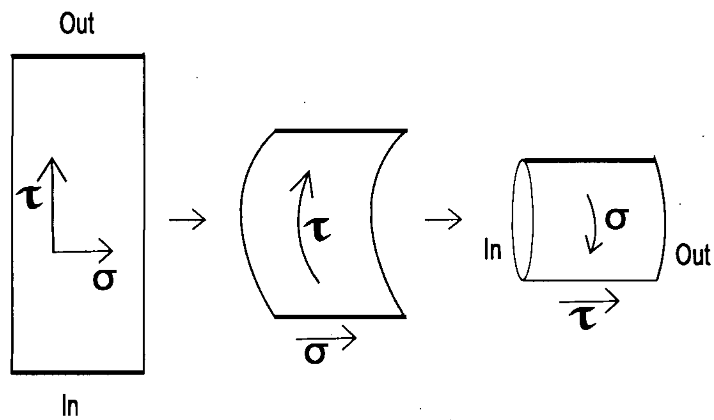

Figure 1.1: Gluing process used to convert an open string world-sheet into a closed string world-sheet

We can use the boundary state formalism to represent the interaction of closed strings with a decaying D-brane. The boundary state is a closed string state which is annihilated by the boundary condition that would be imposed on an open string embedding function. As described above this means interchanging the coordinates $\sigma \leftrightarrow \tau$ and re-scaling the closed string world-sheet to match the usual convention $0 \leq \sigma \leq 2\pi$. After this process the boundary condition (1.6) becomes the boundary state condition:

$$\left(-\frac{1}{2\pi}\partial_\tau X^0 + i\frac{g}{2}e^{iX^0} - i\frac{\bar{g}}{2}e^{-iX^0}\right)\Big|_{\tau=0} |B\rangle_{X^0} = 0. \tag{1.10}$$

The spatial component $|B\rangle_{\vec{X}}$ and the ghost component $|B\rangle_{bc}$ of the boundary state are given in [8]. The full boundary state of the tachyon is given by:

$$|B\rangle = \mathcal{N}_p\,|B\rangle_{X^0} \otimes |B\rangle_{\vec{X}} \otimes |B\rangle_{bc} \tag{1.11}$$

where the normalization constant

$$\mathcal{N}_p = \pi^{\frac{11}{2}}(2\pi)^{6-p} \tag{1.12}$$

and the states

$$|B\rangle_{\vec{X}} = \int \frac{d^{25-p}k_\perp}{(2\pi)^{25-p}} \exp\left(\sum_{s=1}^{25}\sum_{n=1}^{\infty}\frac{1}{n}\,(-1)^{d_s}\alpha^s_{-n}\tilde{\alpha}^s_{-n}\right)\left|k_\parallel = 0, k_\perp\right\rangle \tag{1.13}$$

$$|B\rangle_{bc} = \exp\left(-\sum_{n=1}^{\infty}(\tilde{b}_{-n}c_{-n} + b_{-n}\tilde{c}_{-n})\right)(c_0+\tilde{c}_0)c_1\tilde{c}_1\,|0\rangle \tag{1.14}$$

are the same as for the non-decaying Dp-brane. A few comments on notation: if the direction of X_s is along the Dp-brane, it has a Neumann boundary condition with $d_s = 1$ and momentum denoted by $k_\parallel$. Similarly a direction transverse to the Dp-brane has a Dirichlet boundary condition with $d_s = 0$ and momentum denoted by $k_\perp$. The $\left|k_\parallel = 0, k_\perp\right\rangle$ is the eigenstate of these momenta. The state $|0\rangle$ given in (1.14) is the SL(2,C) invariant ground state. In addition we remind the reader of the boson's commutation relations

$$[\alpha^\mu_m, \alpha^\nu_{-n}] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_{-n}] = m\eta^{\mu\nu}\delta_{m,n} \tag{1.15}$$

and the ghost's anticommutation relations

$$\{c_m, b_{-n}\} = \delta_{m,n}. \tag{1.16}$$

The boundary state in the time direction is

$$\begin{aligned} |B\rangle_{X^0} &= i\int\frac{dk^0}{2\pi}\left(\tilde{f}(k^0) + \tilde{h}(k^0)\alpha^0_{-1}\tilde{\alpha}^0_{-1} + ...\right)|0,0;k^0\rangle \\ &= i\int\frac{dk^0}{2\pi}\int_{\mathcal{C}} dx^0 e^{ik^0x^0}\left(f(x^0) + h(x^0)\alpha^0_{-1}\tilde{\alpha}^0_{-1} + ...\right)|0,0;k^0\rangle, \end{aligned} \tag{1.17}$$

where x^0 is the position zero mode of X^0 and ... indicates higher order oscillations. The contour $\mathcal{C}$ is the real axis. Other authors, [2] for example, define the above boundary state without an integration over x^0 and instead add a prescription to later integrate over an arbitrary contour. This gives rise to choices such as Hartle-Hawking contour discussed in [2].

In [1] $|B\rangle_{X^0}$ is expressed in a basis of x^0. We can also do this by integrating over k^0 in the second line of (1.17) to get

$$|B\rangle_{X^0} = i \int dx^0 \left(f(x^0)|x^0\rangle + h(x^0)\alpha_{-1}^0\tilde{\alpha}_{-1}^0|x^0\rangle + ...\right) \tag{1.18}$$

where the ... indicates higher order oscillator numbers and $|x^0\rangle$ is the position eigenstate. Using the method of fermionization, the function $f(x^0)$ was found in [1] to be:

$$f(x^0) = \frac{1}{1+\pi g e^{ix^0}} + \frac{1}{1+\pi \bar{g} e^{-ix^0}} - 1 \tag{1.19}$$

1.3.1 Renormalization scheme comparison

Dependence on g and $\bar{g}$ of $f(x^0)$ and $h(x^0)$ differ from those found in most other papers because of differences in renormalization prescriptions. The usual expression for $f(x^0)$ is given in [2]

$$f(x^0) = \frac{1}{1+\sin(\pi g')e^{ix^0}} + \frac{1}{1+\sin(\pi g')e^{-ix^0}} - 1 \tag{1.20}$$

for the case of the full-brane($\bar{g} = g$, $\bar{g}' = g'$) and

$$f(x^0) = \frac{1}{1+(\pi g')e^{ix^0}} \tag{1.21}$$

for the half-brane ($\bar{g} = 0$, $\bar{g}' = 0$). In [2] g' and $\bar{g}'$ are the renormalized versions of the couplings in (1.3). In [1] the relation between g' and $\bar{g}'$ was found to be given by:

$$\sin^2 \pi\sqrt{g'\bar{g}'} = \pi^2 g\bar{g}\ ,\ \frac{g'}{\bar{g}'} = \frac{g}{\bar{g}} \tag{1.22}$$

with the special case that $g' = g$ when $\bar{g} = 0$. Comparing (1.19) and (1.21) we find the fermionic approach agrees with the bosonic approach at least for the full S-brane and half S-brane.

In this thesis we extend the work done in [1] by calculating $h(x^0)$

$$h(x^0) = f(x^0) - 2(1 - \pi^2 g\bar{g}). \tag{1.23}$$

We can compare this to the values obtained in [2]:

$$h(x^0) = f(x^0) - (1 + \cos(2\pi g')) \tag{1.24}$$

for the full S-brane and

$$h(x^0) = f(x^0) - 2 \tag{1.25}$$

for the half S-brane. [1] Again these results agree at least for the cases of the full S-brane and half S-brane.

1.4 The boundary as a source of closed strings

In QFT the average number $\bar{N}$ of particles emitted from a classical source $j(x)$ is given by

$$\bar{N} = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{2E} \left|\tilde{j}(\vec{k})\right|^2 \tag{1.26}$$

where $\tilde{j}(\vec{k})$ is the fourier transform of $j(x)$ and $E^2 = \vec{k}^2 + m^2$. Since we are interested in calculating the total number of massless strings emitted we should take our source to be the one point function for creation of massless strings from the D-brane

$$\tilde{j}(\vec{k}) = \langle\psi_B|c_0^-|B\rangle. \tag{1.27}$$

Here $c_0^- = \frac{c_0 - \tilde{c}_0}{2}$ and $|\psi_B\rangle$ is the BRST invariant state at the massless level which we have derived in the appendix. As a result the total number of massless particles emitted should be given by

$$\bar{N}_{M^2=0} = \int \frac{d^{25}k}{(2\pi)^{25}} \frac{1}{2E} |\langle\Psi_B|c_0^-|B\rangle|^2. \tag{1.28}$$

As was discussed in [2] the analogous form of equation (1.26) for the total number of closed strings emitted from the D-brane can be obtained from the optical theorem. On the Lorentzian cylinder it takes the form

$$\bar{N} = Im\left(\langle B|\, \frac{b_0^+ c_0^-}{L_0 + \tilde{L}_0 - i\epsilon}\, |B\rangle\right) \tag{1.29}$$

where $b_0^+ = \frac{b^0 + \tilde{b}^0}{2}$, $c_0^- = \frac{c^0 - \tilde{c}^0}{2}$, L_0 and $\tilde{L}_0$ are the left and right moving Virasoro zero modes respectively. The infinitesimal ϵ is used for regularization purposes.

We can prove the validity of our expression for the total number of massless particles emitted (1.28) by deriving it from (1.29). We begin by inserting a complete set of closed string states

$$|\Psi_c\rangle\langle\Psi_c| = 1 \tag{1.30}$$

[1] We have used the fact that $\alpha_{-1}^0\tilde{\alpha}_{-1}^0 \mapsto -\alpha_{-1}^0\tilde{\alpha}_{-1}^0$ under the Wick rotation $X^0 \mapsto iX^0$.

after the c_0^- in equation (1.29). This conveniently splits (1.29) into a sum over mass levels

$$\begin{aligned}\bar{N} &= \bar{N}_{M^2=-4} + \bar{N}_{M^2=0} + \ldots \\ &= Im\left(\langle B|b_0^+ c_0^-|\Psi\rangle\langle\Psi|_{M^2=-4}(L_0+\tilde{L}_0)^{-1}|B\rangle\right) \\ &\quad + Im\left(\langle B|b_0^+ c_0^-|\Psi\rangle\langle\Psi|_{M^2=0}(L_0+\tilde{L}_0)^{-1}|B\rangle\right) + \ldots . \qquad (1.31)\end{aligned}$$

If we are interested in calculating the number of particles with $M^2 = n$ we need to know the explicit sum over states given by $|\Psi\rangle\langle\Psi|_{M^2=n}$.

In the covariant gauge, the most general from $|\Psi\rangle\langle\Psi|_{M^2=0}$ can take while ensuring a ghost number of 0 is

$$|\Psi\rangle\langle\Psi|_{M^2=0} = \int \frac{d^{26}k}{(2\pi)^{26}} e_{\mu\nu} e^*_{\rho\sigma} |0,0;k\rangle x_g \langle 0,0;k| . \qquad (1.32)$$

The ghost contribution x_g is given by

$$x_g = -i\left(|\uparrow\uparrow\rangle\langle\downarrow\downarrow| - |\downarrow\downarrow\rangle\langle\uparrow\uparrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| - |\downarrow\uparrow\rangle\langle\uparrow\downarrow|\right) . \qquad (1.33)$$

The states $|\downarrow\downarrow\rangle$,$|\uparrow\downarrow\rangle$,$|\downarrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are the four degenerate ground states of the ghost system. The state $|\downarrow\downarrow\rangle$ is defined to be annihilated by the ghosts b_0 and $\tilde{b}_0$. The ghosts c_0 and $\tilde{c}_0$ can be used to construct the other ground states $|\uparrow\downarrow\rangle = c_0|\downarrow\downarrow\rangle$, $|\downarrow\uparrow\rangle = \tilde{c}_0|\downarrow\downarrow\rangle$ and $|\uparrow\uparrow\rangle = \tilde{c}_0 c_0|\downarrow\downarrow\rangle$. The properties of Grassman variables and hermiticity demand that the non-vanishing inner product is defined by $\langle\downarrow\downarrow|\tilde{c}_0 c_0|\downarrow\downarrow\rangle = i$. Also worth noting is that the $SL(2,C)$ ground state given in equation (1.14) is related to $|\downarrow\downarrow\rangle$ by the conformal transformation from the disk to the cylinder, so that as $z = e^{-i\sigma^1+\sigma^2} \mapsto w = \sigma^1 + i\sigma^2$;

$$|0\rangle \to b_{-1}\tilde{b}_{-1}|\downarrow\downarrow\rangle . \qquad (1.34)$$

After inserting (1.32) into the expression for $\bar{N}_{M^2=0}$ and after some manipulation we obtain

$$\begin{aligned}\bar{N}_{M^2=0} &= Im\left(\int \frac{d^{26}k}{(2\pi)^{26}} \frac{1}{k^2} |\langle\Psi_B|c_0^-|B\rangle|^2\right) \\ &= \pi \int \frac{d^{26}k}{(2\pi)^{26}} \delta(k^{02} - E^2) |\langle\Psi_B|c_0^-|B\rangle|^2 \\ &= \int \frac{d^{25}k}{(2\pi)^{25}} \frac{1}{2E} |\langle\Psi_B|c_0^-|B\rangle|^2 \qquad (1.35)\end{aligned}$$

where $|\Psi_B\rangle$ is the on-shell BRST invariant state for a string with $M^2 = 0$ level, the form of which is derived in the appendix. This calculation has put the particles on-shell so that $E^2 = \vec{k}^2$. We notice that the last line of (1.35) now justifies our conjecture (1.27) because it agrees with (1.28).

At first glance the calculation of the total number of closed strings emitted seems fairly straightforward. It was given in (1.29) as

$$\bar{N} = Im\left(\langle B| \frac{b_0^+ c_0^-}{L_0 + \tilde{L}_0 - i\epsilon} |B\rangle \right). \tag{1.36}$$

We remind the reader that

$$\frac{1}{x - i\epsilon} = i\pi\delta(x) + y \tag{1.37}$$

where y is the principal value. Since y is real, this allows us to express (1.29) as

$$\bar{N} = Im\left(i\pi \langle B| \delta(L_0 + \tilde{L}_0 - i\epsilon) |B\rangle \right). \tag{1.38}$$

We can now rewrite the total number of particles emitted as a product of partition functions by expanding the delta function in (1.38), giving

$$\bar{N} = Im\left(i\pi \int_{-\infty}^{\infty} d\lambda \langle B| (b_0^+ c_0^-) q^{L_0 + \tilde{L}_0 - i\epsilon} |B\rangle \right) \tag{1.39}$$

where $q = e^{i\lambda}$. The spatial and ghost partition functions are well known [4]. At first glance fermionization seems promising for calculating the time partition function because it allows us to find an explicit expression for $|B\rangle_{X^0}$ and $L_0 + \tilde{L}_0$ after rotating to Euclidean space. There is, however, one drawback to this approach, finding a consistent way to rotate our final result back to Minkowski space. We will discuss this difficulty after we thoroughly develop the fermionic technology. We note that this problem is absent in the temporal gauge because no analytic continuation is required.

Chapter 2

Fermionization of X^0

Here we will briefly summarize the process of fermionization and extend the work done in [1] and [3]. The goal is to express the boundary state in terms of fermion variables. For simplicity we label $X^0 = X$ and ignore the spatial and ghost parts of the action. It was shown in [3] that fermionizing X^0 directly leads to the identification $X \to X + \sqrt{2}\pi$, which is inconsistent with the self-dual compactification scheme discussed in the introduction. A way to sidestep this is to first add an extra degree of freedom obeying a Dirichlet boundary condition. Introducing this new boson Y, the Euclidean action becomes:

$$S = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \ \left((\partial_\tau X)^2 + (\partial_\sigma X)^2 + (\partial_\tau Y)^2 + (\partial_\sigma Y)^2 \right) + \ldots \quad (2.1)$$

The boundary state condition now becomes the set of equations

$$\left(-\frac{1}{2\pi}\, \partial_\tau X + i\frac{g}{2} e^{iX} - i\frac{\bar{g}}{2} e^{-iX} \right) \Big|_{\tau=0} |BD\rangle = 0 \quad (2.2)$$

$$Y|_{\tau=0} |BD\rangle = 0 \quad (2.3)$$

for Y which is thus uncoupled and hence exactly solvable. When calculating quantities such as physical amplitudes we can simply factor out the contribution of the Y boson. We construct the fields

$$\phi_1 = \frac{1}{\sqrt{2}} (X + Y) \quad , \quad \phi_2 = \frac{1}{\sqrt{2}} (X - Y) \quad (2.4)$$

which are used to define the mapping to the fermions

$$\begin{aligned}
\psi_{1L}(z) &= \zeta_{1L} : e^{-\sqrt{2} i \phi_{1L}(z)} : \quad , \quad \psi^\dagger_{1L}(z) =: e^{\sqrt{2} i \phi_{1L}(z)} : \zeta^\dagger_{1L} & (2.5)\\
\psi_{2L}(z) &= \zeta_{2L} : e^{\sqrt{2} i \phi_{2L}(z)} : \quad , \quad \psi^\dagger_{2L}(z) =: e^{-\sqrt{2} i \phi_{2L}(z)} : \zeta^\dagger_{2L} & (2.6)\\
\psi_{1R}(\bar{z}) &= \zeta_{1R} : e^{\sqrt{2} i \phi_{1R}(\bar{z})} : \quad , \quad \psi^\dagger_{1R}(\bar{z}) =: e^{-\sqrt{2} i \phi_{1R}(\bar{z})} : \zeta^\dagger_{1R} & (2.7)\\
\psi_{2R}(\bar{z}) &= \zeta_{2R} : e^{-\sqrt{2} i \phi_{2R}(\bar{z})} : \quad , \quad \psi^\dagger_{2R}(\bar{z}) =: e^{\sqrt{2} i \phi_{2R}(\bar{z})} : \zeta^\dagger_{2R}. & (2.8)
\end{aligned}$$

z and $\bar{z}$ are the complex co-ordinates corresponding to the conformal transformations $z = e^{\tau+i\sigma}$, $\bar{z} = e^{\tau-i\sigma}$ and the $\zeta_{aL/R}$ are co-cycles used to make the fermions anti-commute with each other. The co-cycles are constructed from the zero modes of the bosons

$$\sqrt{2}\phi_a = \varphi_{aL} + \varphi_{aR} - i\pi_{aL}\ln z - i\pi_{aR}\ln\bar{z} + \dots \tag{2.9}$$

and are given in reference [3] as

$$\begin{aligned}\zeta_{1L} = \zeta_{1R} &= \exp\left(-i\frac{\pi}{2}\left(\pi_{1L} + \pi_{1R} + 2\pi_{2L} + 2\pi_{2R}\right)\right)\\ \zeta_{2L} = \zeta_{2R} &= \exp\left(-i\frac{\pi}{2}\left(\pi_{2L} + \pi_{2R}\right)\right).\end{aligned} \tag{2.10}$$

We now see the advantage of introducing the fields ϕ_1 and ϕ_2 and of making the mappings (2.5)-(2.8): it compactifies X and Y at the self-dual radius making them compatible with the compactification method discussed in the introduction.

The fermion bilinear operators obtained from operator product expansions of (2.5)-(2.8) are given by

$$:\psi^\dagger_{1L}(z)\psi_{1L}(z) := i\sqrt{2}\partial_\tau\phi_{1L}(z) \tag{2.11}$$
$$:\psi^\dagger_{2L}(z)\psi_{2L}(z) := -i\sqrt{2}\partial_\tau\phi_{2L}(z) \tag{2.12}$$
$$:\psi^\dagger_{1R}(\bar{z})\psi_{1R}(\bar{z}) := -i\sqrt{2}\partial_\tau\phi_{1R}(\bar{z}) \tag{2.13}$$
$$:\psi^\dagger_{2R}(\bar{z})\psi_{2R}(\bar{z}) := i\sqrt{2}\partial_\tau\phi_{2R}(\bar{z}). \tag{2.14}$$

The fermion action on the cylinder is

$$S = \frac{i}{2\pi}\int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \left(\psi^\dagger_{aL}(\partial_\tau + i\partial_\sigma)\psi_{aL} + \psi^\dagger_{aR}(\partial_\tau - i\partial_\sigma)\psi_{aR}\right). \tag{2.15}$$

When quantized the fermions have mode expansions:

$$\begin{aligned}\psi_{La}(z) &= \sum_n \psi_{a,n} z^{-n} \quad , \quad \psi_{Ra}(\bar{z}) = \sum_n \tilde{\psi}_{a,n}\bar{z}^{-n}\\ \psi^\dagger_{La}(z) &= \sum_n \psi^\dagger_{a,n} z^{-n} \quad , \quad \psi^\dagger_{Ra}(\bar{z}) = \sum_n \tilde{\psi}^\dagger_{a,n}\bar{z}^{-n}\end{aligned} \tag{2.16}$$

where n runs over $\mathbb{Z}$ represents wavefunctions which are periodic on cylinder and $\mathbb{Z} + 1/2$ represents wavefunctions which are antiperiodic. These correspond to Ramond(R) and Neveu-Schwarz (NS) sectors of the theory respectively. Quantization also gives the non-vanishing anticommutation relations:

$$\left\{\psi_{a,m}, \psi^{\dagger}_{b,-n}\right\} = \delta_{ab}\delta_{mn} \quad , \quad \left\{\tilde{\psi}_{a,m}, \tilde{\psi}^{\dagger}_{b,-n}\right\} = \delta_{ab}\delta_{mn} \tag{2.17}$$

Integrating equations (2.11)-(2.14) gives:

$$\pi_{1L} = \int_0^{2\pi} \frac{d\sigma}{2\pi} : \psi^{\dagger}_{1L}(z)\psi_{1L}(z) : \tag{2.18}$$

$$\pi_{2L} = -\int_0^{2\pi} \frac{d\sigma}{2\pi} : \psi^{\dagger}_{2L}(z)\psi_{2L}(z) : \tag{2.19}$$

$$\pi_{1R} = -\int_0^{2\pi} \frac{d\sigma}{2\pi} : \psi^{\dagger}_{1R}(z)\psi_{1R}(z) : \tag{2.20}$$

$$\pi_{2R} = \int_0^{2\pi} \frac{d\sigma}{2\pi} : \psi^{\dagger}_{2R}(z)\psi_{1L}(z) : \tag{2.21}$$

so that in the NS sector we have

$$\pi_{aL} = (-1)^{a-1} \sum_{n=\frac{1}{2}}^{\infty} \left(\psi^{\dagger}_{a,-n}\psi_{a,n} - \psi_{a,-n}\psi^{\dagger}_{a,n}\right) \tag{2.22}$$

$$\pi_{aR} = (-1)^{a} \sum_{n=\frac{1}{2}}^{\infty} \left(\tilde{\psi}^{\dagger}_{a,-n}\tilde{\psi}_{a,n} - \tilde{\psi}_{a,-n}\tilde{\psi}^{\dagger}_{a,n}\right) \tag{2.23}$$

and in the R sector we have

$$\pi_{aL} = (-1)^{a-1} \sum_{n=1}^{\infty} \left(\psi^{\dagger}_{a,-n}\psi_{a,n} - \psi_{a,-n}\psi^{\dagger}_{a,n} + \psi^{\dagger}_{a,0}\psi_{a,0} - \frac{1}{2}\right) \tag{2.24}$$

$$\pi_{aR} = (-1)^{a} \sum_{n=1}^{\infty} \left(\tilde{\psi}^{\dagger}_{a,-n}\tilde{\psi}_{a,n} - \tilde{\psi}_{a,-n}\tilde{\psi}^{\dagger}_{a,n} + \tilde{\psi}^{\dagger}_{a,0}\tilde{\psi}_{a,0} - \frac{1}{2}\right). \tag{2.25}$$

Equations (2.22)-(2.25) express the momentum of ϕ_1 and ϕ_2 in terms fermion number operators and imply that the momentum is discretized. Consequently X and Y are compactified as discussed in the introduction.

As discussed in [3] the mapping to fermions described above is not one-to-one which requires us to investigate the compactification process. The goal of the next section will be to find the projection which makes the map one-to-one.

2.1 Compactification at the self-dual radius

As discussed above, when X and Y are compactified at the self dual radius they have momenta and winding numbers that take on discrete values

$$p_{X_L} + p_{X_R} = \sqrt{2}m_X \ , \ p_{X_L} - p_{X_R} = \sqrt{2}w_X \tag{2.26}$$

$$p_{Y_L} + p_{Y_R} = \sqrt{2}m_Y \ , \ p_{Y_L} - p_{Y_R} = \sqrt{2}w_Y \tag{2.27}$$

where $m_X, m_Y, w_X, w_Y \in \mathbb{Z}$. Writing write these conditions in terms in terms of the momenta of ϕ_1 and ϕ_2 gives the following system of equations

$$\pi_{1L} + \pi_{2L} + \pi_{1R} + \pi_{2R} = 2m_X \tag{2.28}$$

$$\pi_{1L} - \pi_{2L} + \pi_{1R} - \pi_{2R} = 2m_X \tag{2.29}$$

$$\pi_{1L} + \pi_{2L} - \pi_{1R} - \pi_{2R} = 2m_X \tag{2.30}$$

$$\pi_{1L} - \pi_{2L} - \pi_{1R} + \pi_{2R} = 2m_X. \tag{2.31}$$

In order for these equations to be satisfied the momenta $\pi_{1L}, \pi_{1R}, \pi_{2L}, \pi_{2R}$ must either all be integers or all be half-odd integers. As a result using (2.22)-(2.25) the fermions ψ_1 and ψ_2 must either both be Neveu-Schwarz(NS) or Ramond(R). The final step to obtain all of the bosonic states is to project out states that do not satisfy one of (2.28)-(2.31). For example we can take the projection operator to be

$$P = \frac{1}{2}[1 + (-1)^{\pi_{1L} - \pi_{2L} - \pi_{1R} + \pi_{2R}}]. \tag{2.32}$$

In the fermionic language this corresponds to all states in which the total fermion number is even.

2.2 The boundary state in fermion variables

We will write the double boundary state as $|BD\rangle$ to signify that it contains information about the boundary conditions of both the X and Y bosons. To find the fermionized version of $|BD\rangle$ we must first rewrite the boundary condition(2.2) in terms of the fermion variables. We find that the first term in (2.2) is given by

$$\partial_\tau X(0,\sigma) = \frac{1}{2i}\left[: \psi_L^\dagger(0,\sigma)\sigma^3\psi_L(0,\sigma) : - : \psi_R^\dagger(0,\sigma)\sigma^3\psi_R(0,\sigma) : \right] \tag{2.33}$$

where $\psi^\dagger_{L,R}$ and $\psi_{L,R}$ are

$$\psi^\dagger_{L,R} = \begin{pmatrix} \psi^\dagger_{1;L,R} & \psi^\dagger_{2;L,R} \end{pmatrix} \ , \ \psi_{L,R} = \begin{pmatrix} \psi_{1;L,R} \\ \psi_{2;L,R} \end{pmatrix}. \tag{2.34}$$

We use the Dirichlet boundary condition on Y to find that

$$\begin{aligned} e^{iX(0,\sigma)}|BD\rangle &= e^{i(X(0,\sigma)+Y(0,\sigma))}|BD\rangle \\ &= e^{\sqrt{2}i\phi_{1L}(0,\sigma)}e^{\sqrt{2}i\phi_{1R}(0,\sigma)}|BD\rangle \\ &= Z^2 :e^{\sqrt{2}i\phi_{1L}(0,\sigma)} :: e^{\sqrt{2}i\phi_{1R}(0,\sigma)} : |BD\rangle \\ &= Z^2\psi^\dagger_{1L}(0,\sigma)\zeta_{1L}\zeta^\dagger_{1R}\psi_{1R}(0,\sigma)|BD\rangle \\ &= \tfrac{1}{2}Z^2\psi^\dagger_L(0,\sigma)\left(1+\sigma^3\right)\psi_R(0,\sigma)|BD\rangle \end{aligned} \tag{2.35}$$

where the infinite constant Z^2 comes from normal ordering $e^{\sqrt{2}i\Phi_{1L}(0,\sigma)}$ and $e^{\sqrt{2}i\Phi_{1R}(0,\sigma)}$. Absorbing Z^2 into the coupling constant g is the only renormalization that is required. Note that the power of the cutoff makes g a marginal coupling. Through an analogous argument the last term in (2.2) is given by

$$\begin{aligned} e^{-iX(0,\sigma)}|BD\rangle &= e^{-i(X(0,\sigma)-Y(0,\sigma))}|BD\rangle \\ &= \tfrac{1}{2}Z^2\psi^\dagger_L(0,\sigma)\left(1-\sigma^3\right)\psi_R(0,\sigma)|BD\rangle \end{aligned} \tag{2.36}$$

so that the boundary condition written in fermion variables is given by

$$\begin{aligned} &[:\psi^\dagger_L\sigma^3\psi_L: - :\psi^\dagger_R\sigma^3\psi_R: \\ &+\pi g\psi^\dagger_L\left(1+\sigma^3\right)\psi_R - \pi\bar{g}\psi^\dagger_L\left(1-\sigma^3\right)\psi_R]|BD\rangle = 0. \end{aligned} \tag{2.37}$$

It was shown in [1] that the state $|BD\rangle$ which satisfies (2.37) and the level matching condition $\left(L_n - \tilde{L}_{-n}\right)|BD\rangle = 0$ is given by

$$|BD\rangle_{NS} = 2^{-\frac{1}{4}}\prod_{r=\frac{1}{2}}^{\infty}\exp\left[\psi^\dagger_{-r}U^{-1}i\sigma^1\tilde{\psi}_{-r} - \tilde{\psi}^\dagger_{-r}i\sigma^1U\psi_{-r}\right]|0\rangle \tag{2.38}$$

in the NS-NS sector and

$$\begin{aligned} |BD\rangle_R = 2^{-\frac{1}{4}}\prod_{n=1}^{\infty}&\exp\left[\psi^\dagger_{-n}U^{-1}i\sigma^1\tilde{\psi}_{-n} - \tilde{\psi}^\dagger_{-n}i\sigma^1U\psi_{-n}\right] \\ &\times\exp\left[\psi^\dagger_0U^{-1}i\sigma^1\tilde{\psi}_0\right]|-+-+\rangle \end{aligned} \tag{2.39}$$

in the R-R sector.[2] Since the Ramond sector contains eight zero modes, the ground state is 16-fold degenerate. Here we define $|----\rangle$ as

[2]Equations (2.38) and (2.39) differ from their counterparts in [1] by a factor of $2^{-1/4}$ because our definition of the normalization constant (1.12) already contains the factor of $2^{-1/4}$ from the X_0 boson state. The factor of $2^{-1/4}$ that remains in these equations comes from the Y boundary state.

the state which is annihilated by all positively moded operators as well as $\psi_{1,0}, \tilde{\psi}_{1,0}, \psi_{2,0}$ and $\tilde{\psi}_{2,0}$. The other 16 ground states are created by acting with various $\psi^\dagger_{a,0}$ and $\tilde{\psi}^\dagger_{a,0}$ on $|----\rangle$. The phase convention we adopt gives creation operators ordered as $\psi^\dagger_{1,0}, \tilde{\psi}^\dagger_{1,0}, \psi^\dagger_{2,0}, \tilde{\psi}^\dagger_{2,0}$ a plus sign. For example the ground state given in (2.39) is given by $|-+-+\rangle = \tilde{\psi}^\dagger_{1,0}\tilde{\psi}^\dagger_{2,0}\,|----\rangle$.

The matrix U has the property $UU^* = 1$ for real values and is given by

$$U = \begin{bmatrix} e^{-2\pi iA}\sqrt{1-\pi^2 g\bar{g}} & -i\pi g \\ -i\pi\bar{g} & e^{2\pi iA}\sqrt{1-\pi^2 g\bar{g}} \end{bmatrix}. \tag{2.40}$$

One may be surprised to discover that the parameter A defined in (1.4) reappears in this matrix despite the fact that is absent from the boundary state condition (1.10). The reason is that certain partition functions constructed from the boundary state depend on it. A thorough discussion of this can be found in [3].

Chapter 3

Calculation of $h(x^0)$

We see from equation (1.18) that

$$f(x^0,\hat{x}^0) = \langle x^0,\hat{x}^0|B\rangle_{X^0,\hat{X}^0} \tag{3.1}$$

and

$$h(x^0,\hat{x}^0) = \langle x^0,\hat{x}^0\,|\alpha_1\tilde{\alpha}_1|\,B\rangle_{X^0,\hat{X}^0} \tag{3.2}$$

where we have defined

$$x^0 = \frac{1}{\sqrt{2}}(x_L + x_R) \quad , \quad \hat{x}^0 = \frac{1}{\sqrt{2}}(x_L - x_R)\,. \tag{3.3}$$

The new variable $\hat{x}^0$ has been introduced because we have treated X^0 as if it were compactified. It is the variable whose conjugate momentum is the winding number. As a result we expect that once we decompactify $|B\rangle$, f and h should not depend on $\hat{x}^0$.

The position eigenstate itself can be expressed in terms of the momentum eigenstate via a fourier transform

$$\langle x^0,\hat{x}^0| = \sum_{p_{XL},p_{XR}} e^{i\frac{1}{\sqrt{2}}(p_{XL}+p_{XR})x^0 + i\frac{1}{\sqrt{2}}(p_{XL}-p_{XR})\hat{x}^0}\langle p_{XL},p_{XR}|. \tag{3.4}$$

so that the quantities we need are $C \equiv \langle p_{XL},p_{XR}|B\rangle$ and $D \equiv \langle p_{XL},p_{XR}|\alpha_1\tilde{\alpha}_1|B\rangle$.

At this point we will relate the inner product $\langle p_{XL},p_{XR}|B\rangle$ which we cannot calculate directly to the double boson boundary inner product $\langle p_{XL},p_{XR},p_{YL},p_{YR}|BD\rangle$ which we can calculate using the method of fermionization. Since the Y boson was introduced by hand we can set $p_{YL} = p_{YR} = 0$. As discussed in [1] the inner product of the zero momentum states of the Y boson with its Dirichlet state gives a factor of $2^{-\frac{1}{4}}$ so that

$$\langle p_{XL},p_{XR},p_{YL}=0,p_{YR}=0|BD\rangle = 2^{\frac{1}{4}}\langle p_{XL},p_{XR}|B\rangle. \tag{3.5}$$

Remembering that $p_{YL} = \frac{1}{\sqrt{2}}(\pi_{1L}-\pi_{2L})$ and $p_{YR} = \frac{1}{\sqrt{2}}(\pi_{1R}-\pi_{2R})$, allows us to unclutter our notation by defining $\pi_L \equiv \pi_{1L} = \pi_{2L} = \frac{1}{\sqrt{2}}p_{XL}$

and $\pi_R \equiv \pi_{1R} = \pi_{2R} = \frac{1}{\sqrt{2}} p_{XR}$. Similarly, the Y boson contribution to $\langle p_{XL}, p_{XR}, p_{YL} = 0, p_{YR} = 0 | \alpha_1 \tilde{\alpha}_1 | BD \rangle$ is $2^{-\frac{1}{4}}$ so that we can relate

$$\langle p_{XL}, p_{XR}, p_{YL} = 0, p_{YR} = 0 | \alpha_1 \tilde{\alpha}_1 | BD \rangle = 2^{\frac{1}{4}} \langle p_{XL}, p_{XR} | \alpha_1 \tilde{\alpha}_1 | B \rangle \quad (3.6)$$

where it is important to remember that the oscillators $\alpha_1, \tilde{\alpha}_1$ are associated with the X^0 boson.

3.1 NS sector

In section 2.1 we concluded that in the NS sector (π_L, π_R) are both integers.

The work done in [3] shows that the momentum eigenstate $|\pi_L, \pi_R\rangle$ corresponds to a fermion state that is filled up to a Fermi level.

$$\pi_L > 0, \pi_R > 0 \qquad \langle \pi_L, \pi_R | = \langle 0 | \prod_{r=\frac{1}{2}}^{\pi_L - \frac{1}{2}} (\psi^\dagger_{2r} \psi_{1r}) \prod_{r=\frac{1}{2}}^{\pi_R - \frac{1}{2}} (\tilde{\psi}_{2r} \tilde{\psi}^\dagger_{1r}) \quad (3.7)$$

$$\pi_L > 0, \pi_R < 0 \qquad \langle \pi_L, \pi_R | = \langle 0 | \prod_{r=\frac{1}{2}}^{\pi_L - \frac{1}{2}} (\psi^\dagger_{2r} \psi_{1r}) \prod_{r=\frac{1}{2}}^{-\pi_R - \frac{1}{2}} (\tilde{\psi}_{1r} \tilde{\psi}^\dagger_{2r}) \quad (3.8)$$

$$\pi_L < 0, \pi_R > 0 \qquad \langle \pi_L, \pi_R | = \langle 0 | \prod_{r=\frac{1}{2}}^{-\pi_L - \frac{1}{2}} (\psi^\dagger_{1r} \psi_{2r}) \prod_{r=\frac{1}{2}}^{\pi_R - \frac{1}{2}} (\tilde{\psi}_{2r} \tilde{\psi}^\dagger_{1r}) \quad (3.9)$$

$$\pi_L < 0, \pi_R < 0 \qquad \langle \pi_L, \pi_R | = \langle 0 | \prod_{r=\frac{1}{2}}^{-\pi_L - \frac{1}{2}} (\psi^\dagger_{1r} \psi_{2r}) \prod_{r=\frac{1}{2}}^{-\pi_R - \frac{1}{2}} (\tilde{\psi}_{1r} \tilde{\psi}^\dagger_{2r}) \quad (3.10)$$

We give these states explicitly in the dual form for the convenience of our later calculations.

3.1.1 $\langle \pi_L \pi_R | BD \rangle_{NS}$

The boundary state in the NS sector was given in section 2.2

$$|BD\rangle_{NS} = 2^{-\frac{1}{4}} \prod_{r=\frac{1}{2}}^{\infty} \exp\left[\psi^\dagger_{-r} U^{-1} i\sigma^1 \tilde{\psi}_{-r} - \tilde{\psi}^\dagger_{-r} i\sigma^1 U \psi_{-r} \right] |0\rangle \quad (3.11)$$

and is repeated here for the convenience of the reader. Since the calculation of $f(x^0)$ was previously performed in [1] we will show only the technique used there by calculating $\langle \pi_L, \pi_R | B, D\rangle_{NS}$ for $\pi_L > 0, \pi_R > 0$

$$\begin{aligned}
\langle \pi_L, \pi_R | B, D\rangle_{NS} &= \prod_{r=\frac{1}{2}}^{\pi_L-\frac{1}{2}} -\langle 0| \left(\psi_r^\dagger \sigma^{12} \psi_r\right) \left(\tilde{\psi}_r^\dagger \sigma^{12\dagger} \tilde{\psi}_r\right) |BD\rangle_{NS} \\
&= \prod_{r=\frac{1}{2}}^{\pi_L-\frac{1}{2}} \langle 0| \left(\psi_r^\dagger \sigma^{12} \psi_r\right) \left(\psi_{-r}^\dagger (iU^{-1}\sigma^1)\sigma^{12\dagger}(-i\sigma^1 U)\psi_{-r}\right) |BD\rangle_{NS} \\
&= 2^{-1/4}\left(Tr(\sigma^{12}(iU^{-1}\sigma^1)\sigma^{12\dagger}(-i\sigma^1 U))\right)^{\pi_L} \\
&= 2^{-1/4}\,(-iU_{12})^{2\pi_L}\,\delta(\pi_L - \pi_R). \qquad (3.12)
\end{aligned}$$

In this equation the matrix σ_{12} is given by

$$\sigma_{12} = \frac{\sigma^1 - \sigma^2}{2} \qquad (3.13)$$

where σ^i, $i = 1, 2, 3$ are the Pauli spin matrices. Using equation (3.5) we find

$$\begin{aligned}
2^{\frac{1}{4}}\langle p_{XL}, p_{XR} | B\rangle &= \frac{(-iU_{12})^{\sqrt{2}p_{XL}}}{2^{1/4}}\sqrt{2}\delta(p_{XL} - p_{XR}) \\
C &= (-iU_{12})^{\sqrt{2}p_{XL}}\delta(p_{XL} - p_{XR}). \qquad (3.14)
\end{aligned}$$

In [1] this same technique was used to find

$$C = \begin{cases} [U_{22}]^{\sqrt{2}p_{XL}}\delta(p_{XL} + p_{XR}) & p_{XL} > 0, p_{XR} < 0 \\ [U_{11}]^{-\sqrt{2}p_{XL}}\delta(p_{XL} + p_{XR}) & p_{XL} < 0, p_{XR} > 0 \\ [-iU_{21}]^{-\sqrt{2}p_{XL}}\delta(p_{XL} - p_{XR}) & p_{XL} < 0, p_{XR} < 0 \\ 0 & |p_{XL}| \neq |p_{XR}| \end{cases} \qquad (3.15)$$

for the other momentum ranges. The only case left to consider is $p_{XL} = p_{XR} = 0$, one can easily check that this gives

$$\langle p_{XL} = 0 \;\; p_{XR} = 0 | B\rangle = 1 \qquad (3.16)$$

3.1.2 Calculation of $\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS}$

The next goal is to calculate $D = \langle p_{XL}\ p_{XR}|\alpha_1\tilde{\alpha}_1|B\rangle$. We must first find α_1^X and $\tilde{\alpha}_1^X$ in fermion variables. Notice that we have introduced the label X to emphasize that these are oscillators of the X boson.

Matching up like powers of z and $\bar{z}$ on both sides of the equations in (2.11)-(2.14) gives the oscillator modes of ϕ_1 and ϕ_2 in terms of the fermion modes

$$\alpha_{a,n} = (-1)^{a-1} \sum_{m=\mathbb{Z}+\frac{1}{2}} \psi^\dagger_{a,m}\psi_{a,n-m} \tag{3.17}$$

$$\tilde{\alpha}_{a,n} = (-1)^{a} \sum_{m=\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^\dagger_{a,m}\tilde{\psi}_{a,n-m} \tag{3.18}$$

where, as a reminder, a=1 gives oscillators associated with ϕ_1 and a=2 gives oscillators associated with ϕ_2. Matching up like powers of z and $\bar{z}$ in (2.4) we get

$$\alpha_1^X = \frac{1}{\sqrt{2}}(\alpha_{1,1} + \alpha_{2,1}) \ , \ \ \tilde{\alpha}_1^X = \frac{1}{\sqrt{2}}(\tilde{\alpha}_{1,1} + \tilde{\alpha}_{2,1}). \tag{3.19}$$

Using this with (3.17) and (3.18) we can write

$$\alpha_1^X\tilde{\alpha}_1^X = -\frac{1}{2}\left(\sum_{n=\mathbb{Z}+\frac{1}{2}} \psi^\dagger_n\sigma^3\psi_{1-n}\right)\left(\sum_{m=\mathbb{Z}+\frac{1}{2}} \tilde{\psi}^\dagger_m\sigma^3\tilde{\psi}_{1-m}\right). \tag{3.20}$$

The rest of the calculation is somewhat involved and given in the appendix, the results are

$$D = \begin{cases} [-iU_{12}]^{\sqrt{2}p_{XL}}\delta(p_{XL}-p_{XR}) & p_{XL}>0, p_{XR}>0 \\ -[U_{22}]^{\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}>0, p_{XR}<0 \\ -[U_{11}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}<0, p_{XR}>0 \\ [-iU_{21}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}-p_{XR}) & p_{XL}<0, p_{XR}<0 \\ 1-2U_{11}U_{22} & p_{XL}=p_{XR}=0 \\ 0 & |p_{XL}| \neq |p_{XR}| \end{cases} \tag{3.21}$$

3.2 Ramond Sector

In the Ramond sector π_L and π_R both take half-odd integer values. The calculations of $C = \langle p_{XL}\ p_{XR}|B\rangle$ and $D = \langle p_{XL}\ p_{XR}|\alpha_1\tilde{\alpha}_1|B\rangle$ in this sector are almost identical to that of the NS sector except that one must take the degenerate ground states into consideration.

In [1] $\langle p_{XL}\ p_{XR}|B\rangle$ was calculated to be the same as in the NS sector

$$C = \begin{cases} [-iU_{12}]^{\sqrt{2}p_{XL}}\delta(p_{XR}-p_{XR}) & p_{XL}>0, p_{XR}>0 \\ [U_{22}]^{\sqrt{2}p_{XL}}\delta(p_{XR}+p_{XL}) & p_{XL}>0, p_{XR}<0 \\ [\hat{U}_{11}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}<0, p_{XR}>0 \\ [-iU_{21}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}<0, p_{XR}<0 \\ 0 & |p_{XL}|\neq|p_{XR}| \end{cases} \tag{3.22}$$

The calculation of $\langle \pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_R$ is once again quite involved so we reserve it for the appendix and state the results here

$$D = \begin{cases} [-iU_{12}]^{\sqrt{2}p_{XL}}\delta(p_{XL}-p_{XR}) & p_{XL}>0, p_{XR}>0 \\ -[U_{22}]^{\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}>0, p_{XR}<0 \\ -[U_{11}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}<0, p_{XR}>0 \\ [-iU_{21}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}-p_{XR}) & p_{XL}<0, p_{XR}<0 \\ 0 & |p_{XL}|\neq|p_{XR}| \end{cases} \tag{3.23}$$

3.3 Summary and Final Calculation

In this section we summarize our results and use them to complete the calculation. In the preceding subsections we found that

$$C = \begin{cases} [-iU_{12}]^{\sqrt{2}p_{XL}}\delta(p_{XR}-p_{XR}) & p_{XL}>0, p_{XR}>0 \\ [U_{22}]^{\sqrt{2}p_{XL}}\delta(p_{XR}+p_{XL}) & p_{XL}>0, p_{XR}<0 \\ [U_{11}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}<0, p_{XR}>0 \\ [-iU_{21}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}<0, p_{XR}<0 \\ 1 & p_{XL}=p_{XR}=0 \\ 0 & |p_{XL}|\neq|p_{XR}| \end{cases} \tag{3.24}$$

where $\frac{1}{\sqrt{2}}p_{XL}$ and $\frac{1}{\sqrt{2}}p_{XR}$ are either both integers or both half-odd integers.

Now we can calculate the matrix element $f(x^0,\hat{x}^0) = \langle x^0,\hat{x}^0|B\rangle$ where

$\langle x^0, \hat{x}^0|$ was given by (3.4). We find that

$$\begin{aligned} \langle x^0, \hat{x}^0|B\rangle &= 1+\sum_{n=1}^{\infty}[(e^{i\hat{x}^0}U_{11})^n + (e^{-i\hat{x}^0}U_{22})^n \\ &\quad +(e^{ix^0}(-iU_{12}))^n + (e^{-ix^0}(-iU_{12}))^n] \\ &= \frac{1}{1-U_{11}e^{i\hat{x}_0}} + \frac{1}{1-U_{22}e^{-i\hat{x}_0}} \\ &\quad +\frac{1}{1+iU_{12}e^{ix_0}} + \frac{1}{1+iU_{21}e^{-ix_0}} - 3 \end{aligned} \tag{3.25}$$

The final step is to decompactify $f(x^0, \hat{x}^0)$ by integrating over all allowed values of A, giving

$$\begin{aligned} f(x^0) &= \int_0^1 dA\langle x^0, \hat{x}^0|B\rangle \\ &= \frac{1}{1+\pi g e^{ix^0}} + \frac{1}{1+\pi\bar{g}e^{-ix^0}} - 1. \end{aligned} \tag{3.26}$$

We see that the integration over A eliminates the coordinate $\hat{x}^0$ and gives the expected value of $f(x^0)$

The calculation of $h(x^0)$ is almost the same. The matrix element $D = \langle p_{XL}, p_{XR}|\alpha_1\tilde{\alpha}_1|B\rangle$ was calculated in the previous sections to be

$$D = \begin{cases} [-iU_{12}]^{\sqrt{2}p_{XL}}\delta(p_{XL}-p_{XR}) & p_{XL}>0, p_{XR}>0 \\ -[U_{22}]^{\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}>0, p_{XR}<0 \\ -[U_{11}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}+p_{XR}) & p_{XL}<0, p_{XR}>0 \\ [-iU_{21}]^{-\sqrt{2}p_{XL}}\delta(p_{XL}-p_{XR}) & p_{XL}<0, p_{XR}<0 \\ 1-2U_{11}U_{22} & p_{XL}=p_{XR}=0 \\ 0 & |p_{XL}|\neq|p_{XR}| \end{cases} \tag{3.27}$$

We can now calculate $h(x^0, \hat{x}^0) = \langle x^0, \hat{x}^0\,|\alpha_1\tilde{\alpha}_1|\,B\rangle_{X^0,\hat{X}^0}$

$$\begin{aligned} h(x^0, \hat{x}^0) &= 1+\sum_{n=1}^{\infty}[-(e^{i\hat{x}^0}U_{11})^n - (e^{-i\hat{x}^0}U_{22})^n \\ &\quad +(e^{ix^0}(-iU_{12}))^n + (e^{-ix^0}(-iU_{12}))^n] \\ &= \frac{-1}{1-U_{11}e^{i\hat{x}_0}} + \frac{-1}{1-U_{22}e^{-i\hat{x}_0}} \\ &\quad +\frac{1}{1+iU_{12}e^{ix_0}} + \frac{1}{1+iU_{21}e^{-ix_0}} + 1 - 2U_{11}U_{22}. \end{aligned} \tag{3.28}$$

Once again we can decompactify by integrating out A which gives

$$h(x^0) = f(x^0) - 2(1 - \pi^2 g\bar{g}). \tag{3.29}$$

Chapter 4

Number of Massless Strings Emitted

4.1 1-point functions

Following the discussion in section 1.4, the 1-point function for the production of a closed string field with $M^2 = 0$ from the boundary $|B\rangle$ is given by

$$\langle \Psi_B | c_0^- | B \rangle, \tag{4.1}$$

where $|\Psi_B\rangle$ is the on-shell BRST invariant state for the $M^2 = 0$ level

$$|\Psi_B\rangle = e_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0,0;k\rangle \otimes |\downarrow\downarrow\rangle \tag{4.2}$$

and we remind the reader that we are in Minkowski space.

The polarization tensor $e_{\mu\nu}$ has 24×24 components coming from restrictions $e_{\mu\nu}k^{\mu} = e_{\mu\nu}k^{\nu} = 0$, $e_{\mu\nu} \neq e_{\mu}k_{\nu}$ and $e_{\mu\nu} \neq k_{\mu}e_{\nu}$. A full review of how to obtain this state and its restrictions is given in the appendix.

The calculation of the 1-point function can be split into two inner products

$$\begin{aligned} \langle \Psi_B | \frac{c_0 - \tilde{c}_0}{2} | B \rangle &= \mathcal{N}_p \; e_{\mu\nu} \langle 0,0;k | \alpha_1^{\mu} \tilde{\alpha}_1^{\nu} | B \rangle_{X_0} | B \rangle_{\vec{X}} \\ &\quad \times \langle \downarrow\downarrow | \frac{c_0 - \tilde{c}_0}{2} | B \rangle_{bc}, \end{aligned} \tag{4.3}$$

where we can find $|B\rangle_{X^0}$ by analytically continuing (1.18) back to Minkowski space

$$|B\rangle_{X^0} = \int dt \left(f(t)|t\rangle - h(t)\alpha_{-1}^0 \tilde{\alpha}_{-1}^0 |t\rangle + ... \right) \tag{4.4}$$

where

$$f(t) = \frac{1}{1 + \pi g e^t} + \frac{1}{1 + \pi \bar{g} e^{-t}} - 1 \tag{4.5}$$

and

$$h(t) = f(t) - 2(1 - \pi^2 g \bar{g}). \tag{4.6}$$

The ghost part of (4.3) is evaluated to be

$$\begin{aligned}\langle\downarrow\downarrow|\frac{c_0-\tilde{c}_0}{2}|B\rangle_{bc} &= \frac{1}{2}\langle\downarrow\downarrow|(c_0-\tilde{c}_0)(c_0+\tilde{c}_0)c_1\tilde{c}_1 b_{-1}\tilde{b}_{-1}|\downarrow\downarrow\rangle \\ &= \langle\downarrow\downarrow|\tilde{c}_0 c_0|\downarrow\downarrow\rangle \\ &= i \end{aligned} \tag{4.7}$$

so that (4.3) simplifies to

$$\begin{aligned}\langle\Psi_B|\frac{c_0-\tilde{c}_0}{2}|B\rangle &= i\mathcal{N}_p\, e_{\mu\nu}\langle 0,0;k|\alpha_1^\mu\tilde{\alpha}_1^\nu|B\rangle_{X_0}|B\rangle_{\vec{X}} \\ &= i\mathcal{N}_p\,\langle 0,0;k|\left(e_{00}\alpha_1^0\tilde{\alpha}_1^0+\sum_{i=1}^{25}e_{ii}\alpha_1^i\tilde{\alpha}_1^i\right)|B\rangle_{X_0}|B\rangle_{\vec{X}} \\ &= -i\mathcal{N}_p\int_C dte^{iEt}(2\pi)^p\delta^p(k^i)F(t)\end{aligned} \tag{4.8}$$

where $F(t)$ is given by

$$F(t) = h(t)e_{00}+f(t)\left(\sum_{i=1}^{p}e_{ii}-\sum_{j=p+1}^{25}e_{jj}\right) \tag{4.9}$$

and $E=|\vec{k}_\perp|$ is the energy of the closed strings.

We now see that the 1 point function is in the form given in equation (1.35)

$$\langle\Psi_B|c_0^-|B\rangle(\vec{k}) = -i\mathcal{N}_p(2\pi)^p\delta^p(k^i)\left[I_h e_{00}+I_f\left(\sum_{i=1}^{p}e_{ii}-\sum_{j=p+1}^{25}e_{jj}\right)\right] \tag{4.10}$$

where the integrals I_f and I_h are given by

$$I_f = \int_{-\infty}^{\infty}dtf(t)e^{iEt} \tag{4.11}$$

$$\begin{aligned}I_h &= \int_{-\infty}^{\infty}dth(t)e^{iEt} \\ &= I_f-2(1-\pi^2 g\bar{g})\delta(E).\end{aligned} \tag{4.12}$$

We can never move to the rest frame of a particle with $M^2=0$ therefore the delta function on the right hand side of (4.12) always gives 0 allowing us to set $I_f=I_h$. This is another way of saying that $h(x^0)=f(x^0)$ up to terms that are non-singular in x^0

The integral I_f can be calculated

$$I_f = -i(e^{-iE\ln\pi g} - e^{iE\ln\pi\bar{g}})\frac{\pi}{\sinh(\pi E)}. \tag{4.13}$$

Using equation (1.26) with (4.10) as our $\tilde{j}(k)$ we can write down an expression for the number per volume of massless closed string modes emitted by the brane

$$\begin{aligned}
\bar{N} &= \sum_{\lambda}\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\frac{|\langle\Psi_B|\frac{c_0-\bar{c}_0}{2}|B\rangle|^2}{2E} \\
\frac{\bar{N}}{V_p} &= \mathcal{N}_p{}^2\int\frac{d^{25-p}k}{(2\pi)^{25-p}}\frac{|I_f|^2}{2E}\sum_{\lambda}\left|e_{00}+\sum_{i=1}^{p}e_{ii}-\sum_{j=p+1}^{25}e_{jj}\right|^2.
\end{aligned} \tag{4.14}$$

In the above equation the sum is over a basis of polarization tensors. This sum is well known [9] and we provide a brief review of how to evaluate it in the appendix. We can gain information by splitting the sum into three parts, a sum over the polarizations of the gravitonS_G, dilaton S_Φ and antisymmetric tensor S_B

$$\sum_{\lambda}\left|e_{00}+\sum_{i=1}^{p}e_{ii}-\sum_{j=p+1}^{25}e_{jj}\right|^2 = S_G+S_B+S_\Phi \tag{4.15}$$

with

$$S_G = \frac{1}{6}p(24-p),\ S_B = 0,\ S_\Phi = 24-\frac{1}{6}p(24-p). \tag{4.16}$$

We can use this result to find out how many gravitons and dilatons are emitted from a decaying Dp-brane[3]

$$\frac{\bar{N}_G}{V_p} = \mathcal{N}_p{}^2\frac{1}{6}p(24-p)I_p \tag{4.17}$$

$$\frac{\bar{N}_\Phi}{V_p} = \mathcal{N}_p{}^2\left(24-\frac{1}{6}p(24-p)\right)I_p \tag{4.18}$$

as well as the total number of particles emitted

$$\begin{aligned}
\frac{\bar{N}}{V_p} &= \frac{\bar{N}_G}{V_p}+\frac{\bar{N}_\Phi}{V_p} \\
&= 24\mathcal{N}_p{}^2 I_p
\end{aligned} \tag{4.19}$$

[3] $S_A = 0$ gives no antisymmetric tensors being emitted by the Dp-brane

where

$$
\begin{aligned}
I_p &= \int \frac{d^{25-p}k}{(2\pi)^{25-p}} |I_f|^2 \\
&= \int \frac{d^{25-p}k}{(2\pi)^{25-p}} \frac{\pi^2}{2E\sinh^2 \pi E} \left| e^{-iE\ln(\pi g)} - e^{iE\ln(-\pi \bar{g})} \right|^2. \qquad (4.20)
\end{aligned}
$$

In conclusion the total number of massless closed strings emitted is

$$
\begin{aligned}
\frac{\bar{N}}{V_p} &= 24\mathcal{N}_p{}^2 \int \frac{d^{25-p}k}{(2\pi)^{25-p}} |I_f|^2 \\
&= 24\mathcal{N}_p{}^2 \int \frac{d^{25-p}k}{(2\pi)^{25-p}} \frac{\pi^2}{2E\sinh^2 \pi E} \left| e^{-iE\ln(\pi g)} - e^{iE\ln(-\pi \bar{g})} \right|^2 (4.21)
\end{aligned}
$$

We see that for the case of the full brane we can proceed no further, our solution will depend on our choice of g. For the case of the half S-brane equation (4.11) is evaluated to be

$$
I_f = -i(e^{-iE\ln \pi g})\frac{\pi}{\sinh(\pi E)}. \qquad (4.22)
$$

Upon inserting this into the first line of (4.21) we see that

$$
\frac{\bar{N}}{V_p} = 24\mathcal{N}_p{}^2 \int \frac{d^{25-p}k}{(2\pi)^{25-p}} \frac{\pi^2}{2E\sinh^2 \pi E}. \qquad (4.23)
$$

is independent of g. In the next section we explicitly calculate the number of closed strings emitted by a decaying half S-brane.

4.2 Number of Massless Strings Emitted by Half Brane

Inserting (4.22) into the first line of (4.20) gives

$$
\begin{aligned}
I_p &= \int \frac{d^{25-p}k}{(2\pi)^{25-p}} \frac{\pi^2}{2E\sinh^2 \pi E} \\
&= \frac{\pi^2}{2} \frac{2\pi^{(25-p)/2}}{\Gamma((25-p)/2)(2\pi)^{25-p}} \int_0^\infty dx \frac{x^{23-p}}{\sinh^2(\pi x)} \qquad (4.24)
\end{aligned}
$$

which can be integrated numerically. We expect this integral to diverge when $p = 22, 23, 24, 25$ by inspection. Below we give a table showing the number of gravitons and dilations emitted per volume for various values of p.

p	I_p	N_G/V_p	N_Φ/V_p	N/V_p
0	9.47×10^{-18}	0	0.262	.262
1	3.14×10^{-17}	3.39×10^{-3}	.0179	.0213
2	1.07×10^{-16}	5.16×10^{-4}	1.28×10^{-3}	1.84×10^{-3}
3	3.72×10^{-16}	7.07×10^{-5}	9.09×10^{-5}	1.65×10^{-4}
4	1.32×10^{-15}	8.06×10^{-6}	6.48×10^{-6}	1.45×10^{-5}
5	4.85×10^{-15}	8.90×10^{-7}	4.61×10^{-7}	1.35×10^{-6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
11	2.23×10^{-11}	1.63×10^{-12}	1.14×10^{-14}	1.64×10^{-12}
12	1.04×10^{-10}	1.94×10^{-13}	0	1.94×10^{-13}
13	5.04×10^{-10}	2.37×10^{-14}	1.65×10^{-16}	2.38×10^{-14}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	2.46×10^{-4}	4.33×10^{-20}	3.47×10^{-20}	7.79×10^{-20}
21	$1/96\pi$	1.16×10^{-20}	1.49×10^{-20}	2.66×10^{-20}

Table 4.1: Number of particles emitted with $M^2 = 0$ for half S-brane.

Chapter 5

Total Number of Particles Emitted

We begin by reminding the reader the total number of particles emitted can be expressed as a product of partition functions

$$\bar{N} = Im\left(i\pi\int_{-\infty}^{\infty} d\lambda\langle B|\,(b_0^+c_0^-)q^{L_0+\tilde{L}_0-i\epsilon}\,|B\rangle\right). \tag{5.1}$$

We call these partition functions is because they were obtained from open string partition functions via the gluing procedure described in the introduction. The spatial and ghost partition functions are straight forward to complete giving

$$\begin{aligned} Z_{NN} &= \langle N|q^{L_0^i+\tilde{L}_0^i}|N\rangle \\ &= q^{-1/12}\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}2\pi\delta(0) \end{aligned} \tag{5.2}$$

in the Neumann directions, with $2\pi\delta(0)$ interpreted as a volume factor and

$$\begin{aligned} Z_{DD} &= \langle D|q^{L_0^j+\tilde{L}_0^j}|D\rangle \\ &= q^{-1/12}\prod_{n=1}^{\infty}\frac{1}{1-q^{2n}}\int\frac{dk}{2\pi}q^{\frac{k^2}{2}} \end{aligned} \tag{5.3}$$

in the Dirichlet directions, and

$$\begin{aligned} Z_{ghost} &= {}_{bc}\langle B|(b_0^+c_0^-)q^{L_0^g+\tilde{L}_0^g}|B\rangle_{bc} \\ &= iq^{\frac{2}{12}}\prod_{n=1}^{\infty}(1-q^{2n})^2 \end{aligned} \tag{5.4}$$

for the ghosts.

The only non-trivial part left to calculate is the partition function in the time direction. After analytic continuation to Euclidean space it seems as simple as computing

$$Z_{(BD|BD)} = \langle BD|q^{L_0^0+\tilde{L}_0^0}|BD\rangle_{NS} + \langle BD|q^{L_0^0+\tilde{L}_0^0}|BD\rangle_{R}. \tag{5.5}$$

and factoring out the contribution from the Y boson. We remind the reader that

$$|BD\rangle_{NS} = 2^{-\frac{1}{4}} \prod_{r=\frac{1}{2}}^{\infty} \exp\left[\psi^{\dagger}_{-r} U^{-1} i\sigma^1 \tilde{\psi}_{-r} - \tilde{\psi}^{\dagger}_{-r} i\sigma^1 U \psi_{-r}\right] |0\rangle \tag{5.6}$$

$$\begin{aligned} |BD\rangle_{R} = 2^{-\frac{1}{4}} \prod_{n=1}^{\infty} \exp\left[\psi^{\dagger}_{-n} U^{-1} i\sigma^1 \tilde{\psi}_{-n} - \tilde{\psi}^{\dagger}_{-n} i\sigma^1 U \psi_{-n}\right] \\ \times \exp\left[\psi^{\dagger}_{0} U^{-1} i\sigma^1 \tilde{\psi}_{0}\right] |-+-+\rangle \end{aligned} \tag{5.7}$$

and

$$(L_0^0 + \tilde{L}_0^0)_{NS} = \sum_{r=1/2}^{\infty} r \left(\psi^{\dagger}_{-r}\psi_r + \psi_{-r}\psi^{\dagger}_r + \tilde{\psi}^{\dagger}_{-r}\tilde{\psi}_r + \tilde{\psi}_{-r}\tilde{\psi}^{\dagger}_r\right) - \frac{1}{6} \tag{5.8}$$

$$(L_0^0 + \tilde{L}_0^0)_{R} = \sum_{r=1}^{\infty} r \left(\psi^{\dagger}_{-r}\psi_r + \psi_{-r}\psi^{\dagger}_r + \tilde{\psi}^{\dagger}_{-r}\tilde{\psi}_r + \tilde{\psi}_{-r}\tilde{\psi}^{\dagger}_r\right) + \frac{1}{3}. \tag{5.9}$$

The boundary state $\rangle BD|$ which satisfies the dual form of (1.10) is given by

$$\langle BD|_{NS} = 2^{-\frac{1}{4}} \langle 0| \prod_{r=\frac{1}{2}}^{\infty} \exp\left[\psi^{\dagger}_{r} \bar{U}^{-1} i\sigma^1 \tilde{\psi}_{r} - \tilde{\psi}^{\dagger}_{r} i\sigma^1 \bar{U} \psi_{r}\right] \tag{5.10}$$

$$\begin{aligned} \langle BD|_{R} = 2^{-\frac{1}{4}} \langle -+-+| \prod_{n=1}^{\infty} \exp\left[\psi^{\dagger}_{-n} \bar{U}^{-1} i\sigma^1 \tilde{\psi}_{n} - \tilde{\psi}^{\dagger}_{n} i\sigma^1 \bar{U} \psi_{n}\right] \\ \times \exp\left[-\tilde{\psi}^{\dagger}_{0} i\sigma^1 \bar{U} \psi_{0}\right] \end{aligned} \tag{5.11}$$

The matrices U and $\bar{U}$ are of the same form as (2.40) but differ only by the definition of the parameter A. We will label A_2 as the parameter living at the $\tau = 0$ boundary of the cylinder and A_1 as the parameter living at the $\tau = \frac{2\pi^2}{\beta}$ boundary. As discussed in the introduction we integrate over $A = A_1 - A_2$ to decompactify the boson. In [3] it was shown that (5.5) depends only on the eigenvalues of the matrix $(-i\sigma^1 U)(\bar{U} i\sigma^1)$. These eigenvalues are given by

$$\zeta = \cos(2\pi A)(1 - \pi^2 g\bar{g}) + \pi^2 g\bar{g} \pm i\sqrt{1 - (\cos(2\pi A)(1 - \pi^2 g\bar{g}) + \pi^2 g\bar{g})^2} \tag{5.12}$$

where $A = A_1 - A_2$.

Reference [1] also tells us that after evaluating (5.5) and factoring out the Z_{DD} contribution from the Y boson,

$$Z_{BB} = \frac{Z_{(BD|BD)}}{Z_{DD}} = \left(\sum_{n\epsilon\mathcal{Z}} \zeta^n q^{\frac{1}{2}n^2}\right) q^{-1/12} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}}. \tag{5.13}$$

We have attempted many different ways to bring our result back to Mikowski space and all have failed. There is only method which gives a clear indication as to why it fails. This method begins by checking that we get the expected result for the total number in the Neumann limit of the boundary state (5.2). Substituting our partition functions into (5.1) we get

$$\frac{\bar{N}}{V_{p+1}} = Im\left(-\pi \int_{-\infty}^{\infty} d\lambda \left(e^{-i\lambda/12} \prod_{n=1}^{\infty} \frac{1}{1-e^{2i\lambda n}}\right)^{24} \left(\int \frac{dk}{2\pi} e^{i\lambda\frac{k^2}{2}}\right)^{25-p}\right) \tag{5.14}$$

which can be split into two parts

$$\frac{\bar{N}}{V_{p+1}} = -\pi Im\left(\int_0^{\infty} d\lambda \frac{I^{25-p}(i\lambda)}{\eta^{24}(\frac{\lambda}{\pi})} + \int_0^{\infty} d\lambda \frac{I^{25-p}(-i\lambda)}{\eta^{24}(\frac{-\lambda}{\pi})}\right) \tag{5.15}$$

with

$$I(i\lambda) = \left(\int \frac{dk}{2\pi} e^{i\lambda\frac{k^2}{2}}\right). \tag{5.16}$$

We now observe that in the first integral λ is analytic in the upper half plane which allows us to rotate this contour to the positive imaginary axis. Similarly, in the second integral λ is analytic in the lower half plane which allows us to rotate this contour to the negative imaginary axis. After rotating these contours we see that these two integrals cancel one another giving

$$\bar{N} = 0 \tag{5.17}$$

exactly as expected for a static D-brane. It was thought that perhaps a similar rotation could be used for the unstable D-brane. One can see from (5.13),however, that it is impossible to consistently perform an analogous Wick rotation. The reason is that the n^2 in $q^{\frac{n^2}{2}}$ is the discretized version of the square of momentum in the time direction, so as we rotate back to Minkowski space $n^2 \rightarrow -n^2$. This makes $e^{-i\lambda\frac{n^2}{2}}$ non-analytic in the upper half plane, so that the first integral is neither analytic in the upper half plane nor in the lower half plane. The same argument can be applied to the second integral. This lack of analyticity prevents us from rotating these contours at all. The exception of course is $n = 0$ which is just the same as the Neumann case.

Chapter 6

Summary and Conclusion

We have calculated the total number of massless particles emitted by a decaying Dp-brane in the covariant gauge using the method of fermionization

$$\frac{\bar{N}}{V_p} = 24\mathcal{N}_p{}^2 \int \frac{d^{25-p}k}{(2\pi)^{25-p}} \frac{|I_f(E)|^2}{2E}. \tag{6.1}$$

For the full brane $|I_f(E)|^2$ can be found by taking $g = \bar{g}$ in (4.13). For the half brane $|I_f(E)|^2$ was given by (4.22). We have written $\frac{\bar{N}}{V_p}$ in this form to compare with the massless contribution to the total number obtained in [2]

$$\frac{\bar{N}}{V_p} = \mathcal{N}_p{}^2 \sum_s \frac{1}{2E_s} |I_f(E_s)|^2. \tag{6.2}$$

The sum s includes both the sum over level n as well as over the momenta transverse to the brane and they found that only left-right symmetric states contribute to their sum. Also, we are taking their result $|I_f(E_s)|^2$ as integrated along the real contour for both the full brane and half brane, so as to be able to make a comparison. In their expression $E_s = \sqrt{k_s^2 + M_s^2}$ so at the massless level $E_s = E$. So we see that their result agrees with ours, after picking up a factor of 24 from the 24 left-right symmetric states at this level.

We have also attempted to verify (6.2) using the fermionic technique but we have been unsuccessful due to the ambiguity in rotating our result back to Minkowski space.

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Appendix A

Calculation of $\langle \pi_L\ \pi_R | \alpha_1 \tilde{\alpha}_1 | BD \rangle$

A.1 $\langle \pi_L\ \pi_R | \alpha_1 \tilde{\alpha}_1 | BD \rangle_{NS}$

We begin by writing out some of the equations and results from section 3 for the convenience of the reader. The first of these is the boundary state (2.38) written in fermion variables in the NS sector

$$|BD\rangle_{NS} = 2^{-\frac{1}{4}} \prod_{r=\frac{1}{2}}^{\infty} \exp\left[\psi^{\dagger}_{-r} U^{-1} i\sigma^1 \tilde{\psi}_{-r} - \tilde{\psi}^{\dagger}_{-r} i\sigma^1 U \psi_{-r}\right] |0\rangle \tag{A.1}$$

and we will also require the dual form of the momentum states

$$\pi_L, \pi_R > 0 \qquad \langle \pi_L, \pi_R| = \langle 0| \prod_{r=\frac{1}{2}}^{\pi_L - \frac{1}{2}} (\psi^{\dagger}_{2r} \psi_{1r}) \prod_{r=\frac{1}{2}}^{\pi_R - \frac{1}{2}} (\tilde{\psi}_{2r} \tilde{\psi}^{\dagger}_{1r}) \tag{A.2}$$

$$\pi_L, -\pi_R > 0 \qquad \langle \pi_L, \pi_R| = \langle 0| \prod_{r=\frac{1}{2}}^{\pi_L - \frac{1}{2}} (\psi^{\dagger}_{2r} \psi_{1r}) \prod_{r=\frac{1}{2}}^{-\pi_R - \frac{1}{2}} (\tilde{\psi}_{1r} \tilde{\psi}^{\dagger}_{2r}) \tag{A.3}$$

$$-\pi_L, \pi_R > 0 \qquad \langle \pi_L, \pi_R| = \langle 0| \prod_{r=\frac{1}{2}}^{-\pi_L - \frac{1}{2}} (\psi^{\dagger}_{1r} \psi_{2r}) \prod_{r=\frac{1}{2}}^{\pi_R - \frac{1}{2}} (\tilde{\psi}_{2r} \tilde{\psi}^{\dagger}_{1r}) \tag{A.4}$$

$$\pi_L, \pi_R < 0 \qquad \langle \pi_L, \pi_R| = \langle 0| \prod_{r=\frac{1}{2}}^{-\pi_L - \frac{1}{2}} (\psi^{\dagger}_{1r} \psi_{2r}) \prod_{r=\frac{1}{2}}^{-\pi_R - \frac{1}{2}} (\tilde{\psi}_{1r} \tilde{\psi}^{\dagger}_{2r}) \tag{A.5}$$

Placed between these two states we have the $\alpha_1^X\tilde{\alpha}_1^X$ operator written in terms of fermion variables (3.20)

$$\alpha_1^X\tilde{\alpha}_1^X = -\frac{1}{2}\left(\sum_{n=\mathbb{Z}+\frac{1}{2}}\psi_n^\dagger\sigma^3\psi_{1-n}\right)\left(\sum_{m=\mathbb{Z}+\frac{1}{2}}\tilde{\psi}_m^\dagger\sigma^3\tilde{\psi}_{1-m}\right). \tag{A.6}$$

A.1.1 $\pi_L > 0, \pi_R > 0$

After acting with (A.6) to the left on the bra momentum state only two terms in each of the sums survive

$$\begin{aligned}
&\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS}\\
&= -\frac{1}{2}\delta(\pi_L-\pi_R)\\
&\times\langle\pi_L\ \pi_L|\left(\psi^\dagger_{-(\pi_L-1/2)}\sigma^3\psi_{\pi_L+1/2}+\psi^\dagger_{\pi_L+1/2}\sigma^3\psi_{-(\pi_L-\frac{1}{2})}\right)\\
&\times\left(\tilde{\psi}^\dagger_{\pi_L+1/2}\sigma^3\tilde{\psi}_{-(\pi_L-1/2)}+\tilde{\psi}^\dagger_{-(\pi_L-1/2)}\sigma^3\tilde{\psi}_{\pi_L+1/2}\right)|BD\rangle_{NS}.
\end{aligned}\tag{A.7}$$

Noticing that the right hand side of (A.7) can be broken down into the product of two inner products

$$\begin{aligned}
&\frac{2^{1/4}}{2}\delta(\pi_L-\pi_R)\langle\pi_L-1,\pi_L-1|BD\rangle_{NS}\\
&\times\langle 0|(\psi_r^\dagger\sigma^{12}\psi_r)(\tilde{\psi}_r^\dagger(\sigma^{12})^\dagger\tilde{\psi}_r)(\psi^\dagger_{-r}\sigma^3\psi_{r+1}+\psi^\dagger_{r+1}\sigma^3\psi_{-r})\\
&\times(\tilde{\psi}^\dagger_{r+1}\sigma^3\tilde{\psi}_{-r}+\tilde{\psi}^\dagger_{-r}\sigma^3\tilde{\psi}_{r+1})|BD\rangle_{NS}
\end{aligned}\tag{A.8}$$

where we have set $r=\pi_L-1/2$ in order to unclutter our equations.

The first inner product we have already calculated in (3.12)

$$\langle\pi_L-1,\pi_L-1|BD\rangle_{NS}=\tfrac{1}{2^{1/4}}\left(-iU_{12}\right)^{2(\pi_L-1)}. \tag{A.9}$$

To evaluate the second inner product we first act the very last term in brackets on $|BD\rangle_{NS}$ followed by acting $(\psi^\dagger_{-r}\sigma^3\psi_{r+1}+\psi^\dagger_{r+1}\sigma^3\psi_{-r})$ on everything to the right of it so as to eliminate the $r+1$ modes

$$\begin{aligned}
&\langle 0|(\psi_r^\dagger\sigma^{12}\psi_r)(\tilde{\psi}_r^\dagger(\sigma^{12})^\dagger\tilde{\psi}_r)(\psi^\dagger_{-r}\sigma^3\psi_{r+1}+\psi^\dagger_{r+1}\sigma^3\psi_{-r})\\
&\quad\times(-\psi^\dagger_{-(r+1)}(iU^{-1}\sigma^1)\sigma^3\tilde{\psi}_{-r}-\tilde{\psi}^\dagger_{-r}\sigma^3(i\sigma^1U)\psi_{-(r+1)})|BD\rangle_{NS}\\
&=-\langle 0|(\psi_r^\dagger\sigma^{12}\psi_r)(\tilde{\psi}_r^\dagger\sigma^{12\dagger}\tilde{\psi}_r)\\
&\quad\times(\psi^\dagger_{-r}\sigma^3(iU^{-1}\sigma^1)\sigma^3\tilde{\psi}_{-r}-\tilde{\psi}^\dagger_{-r}\sigma^3(i\sigma^1U)\sigma^3\psi_{-r})|BD\rangle_{NS}.
\end{aligned}\tag{A.10}$$

Next we anti-commute away the r modes in the final line of (A.10) to get

$$\begin{aligned}
& -\langle 0 | (\psi_r^\dagger \sigma^{12} \psi_r)(\tilde{\psi}_r^\dagger \sigma^{12\dagger} \tilde{\psi}_r) \\
& \quad \times (\psi_{-r}^\dagger \sigma^3 (iU^{-1}\sigma^1)\sigma^3 \tilde{\psi}_{-r} - \tilde{\psi}_{-r}^\dagger \sigma^3 (i\sigma^1 U)\sigma^3 \psi_{-r}) | BD \rangle_{NS} \\
= \ & \langle 0 | \psi_r^\dagger \sigma^{12} \left[\sigma^3 (iU^{-1}\sigma^1)\sigma^3 \right] \sigma^{12\dagger} \tilde{\psi}_r \\
& \quad - \tilde{\psi}_r^\dagger \sigma^{12\dagger} \left[\sigma^3 (i\sigma^1 U)\sigma^3 \right] \sigma^{12} \psi_r | BD \rangle_{NS} \\
= \ & -\langle 0 | \psi_r^\dagger \sigma^{12} (iU^{-1}\sigma^1) \sigma^{12\dagger} (i\sigma^1 U) \psi_{-r} \\
& \quad + \tilde{\psi}_r^\dagger \sigma^{12\dagger} (i\sigma^1 U) \sigma^{12} (iU^{-1}\sigma^1) \tilde{\psi}_{-r} | BD \rangle_{NS} \\
= \ & -\frac{1}{2^{1/4}} \left[Tr(\sigma^{12}(iU^{-1}\sigma^1)\sigma^{12\dagger}(i\sigma^1 U)) + Tr(\sigma^{12\dagger}(i\sigma^1 U)\sigma^{12}(iU^{-1}\sigma^1)) \right] \\
= \ & -\frac{2}{2^{1/4}} \left[Tr(\sigma^{12}(iU^{-1}\sigma^1)\sigma^{12\dagger}(i\sigma^1 U)) \right] \\
= \ & -\frac{2}{2^{1/4}} (iU^{-1}\sigma^1)_{11} (i\sigma^1 U)_{22} \\
= \ & \frac{2}{2^{1/4}} (-iU_{12})^2.
\end{aligned} \tag{A.11}$$

Substituting (A.9) and (A.11) into (A.8) we complete the desired calculation

$$\langle \pi_L\ \pi_R | \alpha_1 \tilde{\alpha}_1 | BD \rangle_{NS} = \frac{1}{2^{1/4}} (-iU_{12})^{2\pi_L} \delta(\pi_L - \pi_R) \tag{A.12}$$

A.1.2 $\pi_L > 0, \pi_R < 0$

Following the same lines of reasoning as in the previous subsection we can arrive at the equivalent of equation (A.8) by taking $\sigma^{12\dagger} \to \sigma^{12}$, so that

$$\begin{aligned}
& \langle \pi_L\ \pi_R | \alpha_1 \tilde{\alpha}_1 | BD \rangle_{NS} \\
= \ & \frac{2^{1/4}}{2} \delta(\pi_L + \pi_R) \langle \pi_L - 1, -(\pi_L - 1) | BD \rangle_{NS} \\
& \quad \times \langle 0 | (\psi_r^\dagger \sigma^{12} \psi_r)(\tilde{\psi}_r^\dagger \sigma^{12} \tilde{\psi}_r)(\psi_{-r}^\dagger \sigma^3 \psi_{r+1} + \psi_{r+1}^\dagger \sigma^3 \psi_{-r}) \\
& \quad \times (\tilde{\psi}_{r+1}^\dagger \sigma^3 \tilde{\psi}_{-r} + \tilde{\psi}_{-r}^\dagger \sigma^3 \tilde{\psi}_{r+1}) | BD \rangle_{NS}.
\end{aligned} \tag{A.13}$$

The first inner product was given in (3.15) by

$$\langle \pi_L - 1, -(\pi_L - 1) | BD \rangle_{NS} = \tfrac{1}{2^{1/4}} (U_{22})^{2(\pi_L - 1)} \tag{A.14}$$

and we can find the second inner product by taking $\sigma^{12\dagger} \to \sigma^{12}$ in the third last line of (A.11) and also remembering to pick up an extra minus sign between the second and third line, giving

$$\begin{aligned}
&\langle 0|(\psi_r^\dagger\sigma^{12}\psi_r)(\tilde{\psi}_r^\dagger\sigma^{12}\tilde{\psi}_r)(\psi_{-r}^\dagger\sigma^3\psi_{r+1}+\psi_{r+1}^\dagger\sigma^3\psi_{-r})\\
&\quad\times(\tilde{\psi}_{r+1}^\dagger\sigma^3\tilde{\psi}_{-r}+\tilde{\psi}_{-r}^\dagger\sigma^3\tilde{\psi}_{r+1})|BD\rangle_{NS}\\
=&\ \frac{2}{2^{1/4}}\left[Tr(\sigma^{12}(iU^{-1}\sigma^1)\sigma^{12}(i\sigma^1U))\right]\\
=&\ \frac{2}{2^{1/4}}(iU^{-1}\sigma^1)_{12}(i\sigma^1U)_{12}\\
=&\ -\frac{2}{2^{1/4}}(U_{22})^2. \qquad\qquad \text{(A.15)}
\end{aligned}$$

Substituting (A.14) and (A.15) into (A.13) we get the result

$$\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS}=-\tfrac{1}{2^{1/4}}\,(U_{22})^{2\pi_L}\,\delta(\pi_L+\pi_R). \qquad \text{(A.16)}$$

A.1.3 $\pi_L<0,\pi_R>0$

This time we take $\sigma^{12}\to\sigma^{12\dagger}$, so that (A.8) becomes

$$\begin{aligned}
&\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS}\\
=&\ \frac{2^{1/4}}{2}\delta(\pi_L+\pi_R)\langle\pi_L+1,-(\pi_L+1)|BD\rangle_{NS}\\
&\langle 0|(\psi_r^\dagger\sigma^{12\dagger}\psi_r)(\tilde{\psi}_r^\dagger\sigma^{12\dagger}\tilde{\psi}_r)(\psi_{-r}^\dagger\sigma^3\psi_{r+1}+\psi_{r+1}^\dagger\sigma^3\psi_{-r})\\
&\times(\tilde{\psi}_{r+1}^\dagger\sigma^3\tilde{\psi}_{-r}+\tilde{\psi}_{-r}^\dagger\sigma^3\tilde{\psi}_{r+1})|BD\rangle_{NS}.
\end{aligned} \qquad \text{(A.17)}$$

Once again we can evaluate the first inner product using (3.15)

$$\langle\pi_L+1,-(\pi_L+1)|BD\rangle_{NS}=\tfrac{1}{2^{1/4}}\,(U_{11})^{-2(\pi_L+1)} \qquad \text{(A.18)}$$

and taking $\sigma^{12}\to\sigma^{12\dagger}$ in the third last line of (A.11) we get

$$\begin{aligned}
&\langle 0|(\psi_r^\dagger\sigma^{12\dagger}\psi_r)(\tilde{\psi}_r^\dagger\sigma^{12\dagger}\tilde{\psi}_r)(\psi_{-r}^\dagger\sigma^3\psi_{r+1}+\psi_{r+1}^\dagger\sigma^3\psi_{-r})\\
&\times(\tilde{\psi}_{r+1}^\dagger\sigma^3\tilde{\psi}_{-r}+\tilde{\psi}_{-r}^\dagger\sigma^3\tilde{\psi}_{r+1})|BD\rangle_{NS}\\
=&\ \frac{2}{2^{1/4}}\left[Tr(\sigma^{12\dagger}(iU^{-1}\sigma^1)\sigma^{12\dagger}(i\sigma^1U))\right]\\
=&\ \frac{2}{2^{1/4}}(iU^{-1}\sigma^1)_{21}(i\sigma^1U)_{21}\\
=&\ -\frac{2}{2^{1/4}}(U_{11})^2. \qquad\qquad \text{(A.19)}
\end{aligned}$$

for the second inner product. Substituting (A.19) and (A.18) in (A.17) we find

$$\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS} = -\tfrac{1}{2^{1/4}}\left(U_{11}\right)^{-2\pi_L}\delta(\pi_L+\pi_R). \tag{A.20}$$

A.1.4 $\pi_L < 0, \pi_R < 0$

For this case we need to take $\sigma^{12} \to \sigma^{12\dagger}$ and $\sigma^{12\dagger} \to \sigma^{12}$ so that equation (A.8) becomes

$$\begin{aligned}
&\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS}\\
=\ &\frac{2^{1/4}}{2}\delta(\pi_L+\pi_R)\langle\pi_L+1,\pi_L+1|BD\rangle_{NS}\\
&\times\langle 0|(\psi_r^\dagger\sigma^{12\dagger}\psi_r)(\tilde{\psi}_r^\dagger\sigma^{12}\tilde{\psi}_r)(\psi_{-r}^\dagger\sigma^3\psi_{r+1}+\psi_{r+1}^\dagger\sigma^3\psi_{-r})\\
&\times(\tilde{\psi}_{r+1}^\dagger\sigma^3\tilde{\psi}_{-r}+\tilde{\psi}_{-r}^\dagger\sigma^3\tilde{\psi}_{r+1})|BD\rangle_{NS}.
\end{aligned} \tag{A.21}$$

Using (3.15) we can find the first inner product

$$\langle\pi_L+1,\pi_L+1|BD\rangle_{NS} = \tfrac{1}{2^{1/4}}\left(-iU_{21}\right)^{-2(\pi_L+1)} \tag{A.22}$$

and taking $\sigma^{12} \to \sigma^{12\dagger}$ and $\sigma^{12\dagger} \to \sigma^{12}$ in the third last line of (A.11), we get

$$\begin{aligned}
&\langle 0|(\psi_r^\dagger\sigma^{12\dagger}\psi_r)(\tilde{\psi}_r^\dagger\sigma^{12}\tilde{\psi}_r)(\psi_{-r}^\dagger\sigma^3\psi_{r+1}+\psi_{r+1}^\dagger\sigma^3\psi_{-r})\\
&\times(\tilde{\psi}_{r+1}^\dagger\sigma^3\tilde{\psi}_{-r}+\tilde{\psi}_{-r}^\dagger\sigma^3\tilde{\psi}_{r+1})|BD\rangle_{NS}\\
=\ &-\frac{2}{2^{1/4}}\left[Tr(\sigma^{12\dagger}(iU^{-1}\sigma^1)\sigma^{12}(i\sigma^1U))\right]\\
=\ &-\frac{2}{2^{1/4}}(iU^{-1}\sigma^1)_{22}(i\sigma^1U)_{11}\\
=\ &\frac{2}{2^{1/4}}(-iU_{21})^2.
\end{aligned} \tag{A.23}$$

for the second inner product. Substituting (A.21) and (A.22) into (A.23) we find

$$\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS} = \tfrac{1}{2^{1/4}}\left(-iU_{21}\right)^{-2\pi_L}\delta(\pi_L-\pi_R) \tag{A.24}$$

A.1.5 $\pi_L = 0, \pi_R = 0$

We see that by acting (3.20) to the left on the bra $\langle\pi_L = 0\ \pi_R = 0|$ only one term in each sum survives

$$\begin{aligned}
& \langle\pi_L = 0\ \pi_R = 0|\alpha_1\tilde{\alpha}_1|BD\rangle_{NS} \\
= & -\frac{1}{2}\langle 0|\left(\psi^\dagger_{1/2}\sigma^3\psi_{1/2}\right)\left(\tilde{\psi}^\dagger_{1/2}\sigma^3\tilde{\psi}_{1/2}\right)|BD\rangle_{NS} \\
= & -\frac{1}{2}\langle 0|\tilde{\psi}^\dagger_{1/2}\sigma^3[i\sigma^1 U]\sigma^3[iU^{-1}\sigma^1]\tilde{\psi}_{-1/2}|BD\rangle_{NS} \\
= & \frac{1}{2\cdot 2^{1/4}}Tr[\sigma^3(\sigma^1 U)\sigma^3(U^{-1}\sigma^1)] \\
= & -\frac{1}{2^{1/4}}(U_{12}U_{21} + U_{11}U_{22}) \\
= & \frac{1}{2^{1/4}}(1 - 2U_{11}U_{22}). \qquad (A.25)
\end{aligned}$$

A.2 $\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_R$

Again we will rewrite the fermion boundary state (2.39) in the Ramond sector

$$|BD\rangle_R = 2^{-\frac{1}{4}}\prod_{n=1}^{\infty}\exp\left[\psi^\dagger_{-n}U^{-1}i\sigma^1\tilde{\psi}_{-n} - \tilde{\psi}^\dagger_{-n}i\sigma^1 U\psi_{-n}\right] \times \exp\left[\psi^\dagger_0 U^{-1}i\sigma^1\tilde{\psi}_0\right]\ |-+-+\rangle \qquad (A.26)$$

as well as the momentum states from [1] for this sector

$$\pi_L, \pi_R > 0 \quad \langle\pi_L, \pi_R| = \langle 0|_1 \prod_{n=1}^{\pi_L - \frac{1}{2}}\left(\psi^\dagger_{2n}\psi_{1n}\right)\prod_{n=1}^{\pi_R-\frac{1}{2}}\left(\tilde{\psi}_{2n}\tilde{\psi}^\dagger_{1n}\right) \qquad (A.27)$$

$$\pi_L, -\pi_R > 0 \quad \langle\pi_L, \pi_R| = \langle 0|_2 \prod_{n=1}^{\pi_L - \frac{1}{2}}\left(\psi^\dagger_{2n}\psi_{1n}\right)\prod_{n=1}^{-\pi_R-\frac{1}{2}}\left(\tilde{\psi}_{1n}\tilde{\psi}^\dagger_{2n}\right) \qquad (A.28)$$

$$-\pi_L, \pi_R > 0 \quad \langle\pi_L, \pi_R| = \langle 0|_3 \prod_{n=1}^{-\pi_L - \frac{1}{2}}\left(\psi^\dagger_{1n}\psi_{2n}\right)\prod_{n=1}^{\pi_R-\frac{1}{2}}\left(\tilde{\psi}_{2n}\tilde{\psi}^\dagger_{1n}\right) \qquad (A.29)$$

$$\pi_L, \pi_R < 0 \quad \langle\pi_L, \pi_R| = \langle 0|_4 \prod_{n=1}^{-\pi_L - \frac{1}{2}}\left(\psi^\dagger_{1n}\psi_{2n}\right)\prod_{n=1}^{-\pi_R-\frac{1}{2}}\left(\tilde{\psi}_{1n}\tilde{\psi}^\dagger_{2n}\right) \qquad (A.30)$$

where the degenerate ground states are given by

$$|0\rangle_1 = |+--+\rangle \tag{A.31}$$
$$|0\rangle_2 = i|++--\rangle \tag{A.32}$$
$$|0\rangle_3 = -i|--++\rangle \tag{A.33}$$
$$|0\rangle_4 = |-++-\rangle. \tag{A.34}$$

In the Ramond sector the fermion modes are integers so that $\alpha_1^X\tilde{\alpha}_1^X$ is

$$\alpha_1^X\tilde{\alpha}_1^X = -\frac{1}{2}\left(\sum_{n=\mathbb{Z}}\psi_n^\dagger\sigma^3\psi_{1-n}\right)\left(\sum_{m=\mathbb{Z}}\tilde{\psi}_m^\dagger\sigma^3\tilde{\psi}_{1-m}\right). \tag{A.35}$$

A.2.1 $\pi_L > 0, \pi_R > 0$

Acting (A.35) to the left on the momentum state gives the same result as in the NS sector, only two terms in each sum survive,

$$\begin{aligned}
&\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_R \\
&= -\frac{1}{2}\delta(\pi_L-\pi_R) \\
&\langle\pi_L\ \pi_L|\left(\psi^\dagger_{-(\pi_L-1/2)}\sigma^3\psi_{\pi_L+1/2}+\psi^\dagger_{\pi_L+1/2}\sigma^3\psi_{-(\pi_L-\frac{1}{2})}\right) \\
&\times\left(\tilde{\psi}^\dagger_{\pi_L+1/2}\sigma^3\tilde{\psi}_{-(\pi_L-1/2)}+\tilde{\psi}^\dagger_{-(\pi_L-1/2)}\sigma^3\tilde{\psi}_{\pi_L+1/2}\right)|BD\rangle_R
\end{aligned} \tag{A.36}$$

and for $\pi_L > 1/2$ neither of these terms contains a zero mode. In this case the calculation is the same as in the NS sector, except that we still have to calculate the remaining zero mode parts

$$\begin{aligned}
&\langle\pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_R \\
= &\ \frac{1}{2^{1/4}}(-iU_{12})^{2\pi_L-1}\langle+--+|\exp[\psi_0^\dagger U^{-1}i\sigma^1\tilde{\psi}_0]|-+-+\rangle \\
= &\ \frac{1}{2^{1/4}}(-iU_{12})^{2\pi_L}\delta(\pi_R-\pi_L)
\end{aligned} \tag{A.37}$$

where we have used the fact that if c is a Grassman variable then $c^2 = 0$. A special case occurs when $\pi_L = 1/2$, equation (A.36) becomes

$$
\begin{aligned}
& \langle 1/2\ 1/2 | \alpha_1 \tilde{\alpha}_1 | BD \langle_R \\
= \quad & -\frac{1}{2} \langle + - - + | \left(\psi_0^\dagger \sigma^3 \psi_1 + \psi_1^\dagger \sigma^3 \psi_0 \right) \left(\tilde{\psi}_1^\dagger \sigma^3 \tilde{\psi}_0 + \tilde{\psi}_0^\dagger \sigma^3 \tilde{\psi}_1 \right) | BD \rangle_R \\
= \quad & -\frac{1}{2} \langle + - - + | \psi_0^\dagger \sigma^3 \psi_1 \tilde{\psi}_1^\dagger \sigma^3 \tilde{\psi}_0 + \psi_1^\dagger \sigma^3 \psi_0 \tilde{\psi}_0^\dagger \sigma^3 \tilde{\psi}_1 | BD \rangle_R \\
= \quad & -\frac{1}{2} \left(\langle - + - + | \psi_{1,1} \tilde{\psi}_{1,1}^\dagger + < + - + - | \tilde{\psi}_{2,1} \psi_{2,1}^\dagger \right) | BD \rangle_R \\
= \quad & \frac{-iU_{12}}{2 \cdot 2^{1/4}} (1 + < + - + - | \exp(\psi_0^\dagger U^{-1} i \sigma^1 \tilde{\psi}_0) | - + - + >) \\
= \quad & \frac{-iU_{12}}{2^{1/4}}
\end{aligned}
\tag{A.38}
$$

where in the second last line we have used the fact that

$$
(U^{-1} i \sigma^1)_{11} (U^{-1} i \sigma^1)_{22} - (U^{-1} i \sigma^1)_{12} (U^{-1} i \sigma^1)_{21} = 1. \tag{A.39}
$$

Thus we conclude that

$$
\langle \pi_L\ \pi_R | \alpha_1 \tilde{\alpha}_1 | BD \rangle_R = \frac{1}{2^{1/4}} (-iU_{12})^{2\pi_L} \delta(\pi_R - \pi_L) \tag{A.40}
$$

A.2.2 $\pi_L > 0, \pi_R < 0$

Following the same lines of reasoning as in the previous subsection we can immediately write down the solution for $\pi_L > 1/2$

$$
\begin{aligned}
& \langle \pi_L\ \pi_R | \alpha_1 \tilde{\alpha}_1 | BD \rangle_R \\
= \quad & -\frac{1}{2^{1/4}} i (U_{22})^{2\pi_L - 1} \langle + + - - | \exp[\psi_0^\dagger U^{-1} i \sigma^1 \tilde{\psi}_0] | - + - + \rangle \\
= \quad & -\frac{1}{2^{1/4}} (U_{22})^{2\pi_L} \delta(\pi_R + \pi_L)
\end{aligned}
\tag{A.41}
$$

so that the only thing left to calculate is

$$
\begin{aligned}
& \langle 1/2\ \ -1/2 | \alpha_1 \tilde{\alpha}_1 | BD \rangle_R \\
= & -\frac{i}{2} \langle + + - - | \left(\psi_0^\dagger \sigma^3 \psi_1 + \psi_1^\dagger \sigma^3 \psi_0 \right) \left(\tilde{\psi}_1^\dagger \sigma^3 \tilde{\psi}_0 + \tilde{\psi}_0^\dagger \sigma^3 \tilde{\psi}_1 \right) | BD \rangle_R \\
= & -\frac{i}{2} \langle + + - - | \psi_0^\dagger \sigma^3 \psi_1 \tilde{\psi}_1^\dagger \sigma^3 \tilde{\psi}_0 + \psi_1^\dagger \sigma^3 \psi_0 \tilde{\psi}_0^\dagger \sigma^3 \tilde{\psi}_1 | BD \rangle_R \\
= & -\frac{i}{2} \left(\langle - + - + | \psi_{1,1} \tilde{\psi}_{2,1}^\dagger + \langle + - + - | \psi_{2,1}^\dagger \tilde{\psi}_{1,1} \right) | BD \rangle_R \\
= & -\frac{U_{22}}{2 \cdot 2^{1/4}} (1 + \langle + - + - | \exp(\psi_0^\dagger U^{-1} i \sigma^1 \tilde{\psi}_0) | - + - + \rangle) \\
= & \frac{-U_{22}}{2^{1/4}}.
\end{aligned} \tag{A.42}
$$

A.2.3 $\pi_L < 0, \pi_R > 0$

Once again we can immediately write down the solution for $\pi_L < 1/2$

$$
\begin{aligned}
& \langle \pi_L\ \pi_R | \alpha_1 \tilde{\alpha}_1 | BD \rangle_R \\
= & \frac{1}{2^{1/4}} i (U_{11})^{-2\pi_L - 1} \langle - - + + | \exp[\psi_0^\dagger U^{-1} i \sigma^1 \tilde{\psi}_0] | - + - + \rangle \\
= & -\frac{1}{2^{1/4}} (U_{11})^{-2\pi_L} \delta(\pi_R + \pi_L).
\end{aligned} \tag{A.43}
$$

Let's now calculate the special case $\pi_L = -1/2$

$$
\begin{aligned}
& \langle 1/2\ \ -1/2 | \alpha_1 \tilde{\alpha}_1 | BD \rangle_R \\
= & \frac{i}{2} \langle - - + + | \left(\psi_0^\dagger \sigma^3 \psi_1 + \psi_1^\dagger \sigma^3 \psi_0 \right) \left(\tilde{\psi}_1^\dagger \sigma^3 \tilde{\psi}_0 + \tilde{\psi}_0^\dagger \sigma^3 \tilde{\psi}_1 \right) | BD \rangle_R \\
= & \frac{i}{2} \langle - - + + | \psi_0^\dagger \sigma^3 \psi_1 \tilde{\psi}_1^\dagger \sigma^3 \tilde{\psi}_0 + \psi_1^\dagger \sigma^3 \psi_0 \tilde{\psi}_0^\dagger \sigma^3 \tilde{\psi}_1 | BD \rangle_R \\
= & -\frac{i}{2} \left(\langle - + - + | \psi_{2,1} \tilde{\psi}_{1,1}^\dagger + \langle + - + - | \psi_{1,1}^\dagger \tilde{\psi}_{2,1} \right) | BD \rangle_R \\
= & -\frac{U_{11}}{2 \cdot 2^{1/4}} (1 + \langle + - + - | \exp(\psi_0^\dagger U^{-1} i \sigma^1 \tilde{\psi}_0) | - + - + \rangle) \\
= & \frac{-U_{11}}{2^{1/4}}
\end{aligned} \tag{A.44}
$$

A.2.4 $\pi_L < 0, \pi_R < 0$

Once again we find

$$\begin{aligned}
&\langle \pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_R \\
=\ & -\frac{1}{2^{1/4}}(U_{21})^{-2\pi_L-1}\langle -++-|\exp[\psi_0^\dagger U^{-1} i\sigma^1\tilde{\psi}_0]|-+-+\rangle \\
=\ & \frac{1}{2^{1/4}}(U_{21})^{-2\pi_L}\delta(\pi_R-\pi_L)
\end{aligned} \tag{A.45}$$

for $\pi_L < -1/2$. Finally we consider the special case when $\pi_L = -1/2$

$$\begin{aligned}
&\langle -1/2\ -1/2|\alpha_1\tilde{\alpha}_1|BD\rangle_R \\
=\ & -\frac{1}{2}\langle -++-|\left(\psi_0^\dagger\sigma^3\psi_1+\psi_1^\dagger\sigma^3\psi_0\right)\left(\tilde{\psi}_1^\dagger\sigma^3\tilde{\psi}_0+\tilde{\psi}_0^\dagger\sigma^3\tilde{\psi}_1\right)|BD\rangle_R \\
=\ & -\frac{1}{2}\langle -++-|\psi_0^\dagger\sigma^3\psi_1\tilde{\psi}_1^\dagger\sigma^3\tilde{\psi}_0+\psi_1^\dagger\sigma^3\psi_0\tilde{\psi}_0^\dagger\sigma^3\tilde{\psi}_1|BD\rangle_R \\
=\ & -\frac{1}{2}\left(\langle -+-+|\psi_{2,1}\tilde{\psi}_{2,1}^\dagger + <+-+-|\tilde{\psi}_{1,1}\psi_{1,1}^\dagger\right)|BD\rangle_R \\
=\ & -\frac{iU_{21}}{2\cdot 2^{1/4}}(1+\langle +-+-|\exp(\psi_0^\dagger U^{-1}i\sigma^1\tilde{\psi}_0)|-+-+\rangle) \\
=\ & \frac{-iU_{21}}{2^{1/4}}
\end{aligned} \tag{A.46}$$

so that

$$\langle \pi_L\ \pi_R|\alpha_1\tilde{\alpha}_1|BD\rangle_R = \frac{1}{2^{1/4}}(-iU_{21})^{-2\pi_L}\delta(\pi_R-\pi_L) \tag{A.47}$$

Appendix B

BRST invariance of the $M^2 = 0$ state

We begin by writing down the most general closed string state with $M^2 = 0$ in $D = 26$ flat space-time dimensions

$$|\Psi_c\rangle = e_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1} + \beta_\mu\alpha^\mu_{-1}\tilde{b}_{-1} + \tilde{\beta}_\mu\tilde{\alpha}^\mu_{-1}b_{-1} + \gamma_\mu\alpha^\mu_{-1}\tilde{c}_{-1} + \tilde{\gamma}_\mu\tilde{\alpha}^\mu_{-1}c_{-1}$$
$$+B_{\tilde{c}}b_{-1}\tilde{c}_{-1} + \tilde{B}_c c_{-1}\tilde{b}_{-1} + Bb{-1}\tilde{b}_{-1} + Cc_{-1}\tilde{c}_{-1}|0,0;k\rangle|\downarrow\downarrow\rangle \quad \text{(B.1)}$$

and as was shown in [4] BRST invariance of the ground state demands $k^2 = M^2 = 0$ and that the other degenerate ground states are not allowed.

BRST invariance demands that the charges

$$Q_B = \sum_{n=-\infty}^{\infty}(c_n L^m_{-n}) + \sum_{m,n=-\infty}^{\infty}\frac{(m-n)}{2}\,\overset{\circ}{\circ}\, c_m c_n b_{-m-n}\,\overset{\circ}{\circ} \quad \text{(B.2)}$$

and

$$\tilde{Q}_B = \sum_{n=-\infty}^{\infty}(\tilde{c}_n \tilde{L}^m_{-n}) + \sum_{m,n=-\infty}^{\infty}\frac{(m-n)}{2}\,\overset{\circ}{\circ}\, \tilde{c}_m \tilde{c}_n \tilde{b}_{-m-n}\,\overset{\circ}{\circ} \quad \text{(B.3)}$$

annihilate any physical state of the closed string spectrum. The operators L^m_{-n}, $\tilde{L}^m_{-n}$ are the left and right moving matter Virasoro modes and $\overset{\circ}{\circ}\;\overset{\circ}{\circ}$ indicates creation-annihilation normal ordering. Acting the holomorphic BRST charge on $|\Psi_c\rangle$ gives

$$Q_B|\Psi_c\rangle = \left[\sqrt{\frac{\alpha'}{2}}(c_{-1}k_\mu\alpha^\mu_1 + c_1 k_\mu\alpha^\mu_{-1}) + \frac{\alpha' k^2}{4}c_0\right]|\Psi_c\rangle$$
$$= \sqrt{\frac{\alpha'}{2}}[k^\mu e_{\mu\nu}\tilde{\alpha}^\nu_{-1}c_{-1} + (k^\mu\beta_\mu)c_{-1}\tilde{b}_{-1} + (k_\mu\gamma^\mu)c_{-1}\tilde{c}_{-1}$$
$$+(\tilde{\beta}_\mu\tilde{\alpha}^\mu_{-1})(k_\nu\alpha^\nu_{-1}) + B_{\tilde{c}}k_\mu\alpha^\mu_{-1}\tilde{c}_{-1} + B\tilde{b}_{-1}(k_\mu\alpha^\mu_{-1})]|0,0;k\rangle|\downarrow\downarrow\rangle \quad \text{(B.4)}$$

which demands that $k^\mu e_{\mu\nu} = k^\mu\beta_\mu = k^\mu\gamma_\mu = B_{\tilde{c}} = \tilde{\beta}_\mu = B = 0$ for physical states. Acting the antiholomorphic BRST charge on $|\Psi_c\rangle$ gives

$$\tilde{Q}_B|\Psi_c\rangle = \left[\sqrt{\frac{\alpha'}{2}}(\tilde{c}_{-1}k_\mu\tilde{\alpha}_1^\mu + \tilde{c}_1 k_\mu\tilde{\alpha}_{-1}^\mu) + \frac{\alpha' k^2}{4}\tilde{c}_0\right]|\Psi_c\rangle$$
$$= \sqrt{\frac{\alpha'}{2}}(k^\nu e_{\mu\nu}\alpha_{-1}^\mu\tilde{c}_{-1} + (k^\mu\tilde{\beta}_\mu)\tilde{c}_{-1}b_{-1} - (k_\mu\tilde{\gamma}^\mu)c_{-1}\tilde{c}_{-1}$$
$$+(\beta_\mu\alpha_{-1}^\mu)(k_\nu\tilde{\alpha}_{-1}^\nu) + \tilde{B}_c k_\mu\tilde{\alpha}_{-1}^\mu c_{-1} + Bb_{-1}(k_\mu\tilde{\alpha}_{-1}^\mu))|0,0;k\rangle|\downarrow\downarrow\rangle \quad \text{(B.5)}$$

which demands that $k^\nu e_{\mu\nu} = k^\mu\tilde{\gamma} = \tilde{B}_c = \beta_\mu = 0$ for physical states.

Eliminating unphysical states (B.1) reduces to

$$|\Psi_c\rangle = e_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu + \gamma_\mu\alpha_{-1}^\mu\tilde{c}_{-1} + \tilde{\gamma}_\mu\tilde{\alpha}_{-1}^\mu c_{-1} + Cc_{-1}\tilde{c}_{-1}|0,0;k\rangle|\downarrow\downarrow\rangle, \quad \text{(B.6)}$$

with the restrictions $k^\mu e_{\mu\nu} = k^\nu e_{\mu\nu} = k^\mu\gamma_\mu = k^\mu\tilde{\gamma}_\mu = 0$ and $k^2 = 0$. Some of these remaining states, however, are BRST exact and have zero norm. By inspection eliminating the zero norm states sets $\gamma = \tilde{\gamma} = C = 0$. One can also check that polarizations of the form $e_{\mu\nu} = k_\mu e_\nu$ and $e_{\mu\nu} = e_\mu k_\nu$ have zero norm and can be eliminated.

In conclusion the 24x24 states that are BRST invariant and have positive norm are given by

$$|\Psi_B\rangle = e_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0,0;k\rangle|\downarrow\downarrow\rangle \quad \text{(B.7)}$$

with the restrictions $e_{\mu\nu}k^\mu = e_{\mu\nu}k^\nu = 0$, $e_{\mu\nu} \neq e_\mu k_\nu$ and $e_{\mu\nu} \neq k_\mu e_\nu$.

Appendix C

Polarization sums

We are interested in evaluating

$$\Pi_{\mu\nu\rho\sigma} = \sum_{\lambda} e^{\lambda}_{\mu\nu}(k) e^{\lambda}_{\rho\sigma}(k). \tag{C.1}$$

As discussed by Polchinski [4], the BRST invariant states (B.7) derived in the previous appendix transform as a 2-tensor under $SO(D-2)$. Since this is a reducible representation any tensor g_{ij} can be decomposed as

$$g_{ij} = \frac{1}{2}\left(g_{ij} + g_{ji} - \frac{2}{D-2}\delta_{ij} g_{kk}\right) + \frac{1}{2}\left(g_{ij} - g_{ji}\right) + \frac{1}{D-2}\delta_{ij} g_{kk}. \tag{C.2}$$

Under this decomposition reference [9] gives a method for computing (C.1) for the graviton. We can also use this method for the dilaton and antisymmetric tensor so that

$$\Pi_{\mu\nu\rho\sigma} = \begin{cases} \frac{1}{2}\left(\Pi_{\mu\rho}\Pi_{\nu\sigma} + \Pi_{\mu\sigma}\Pi_{\nu\rho} - \frac{2}{D-2}\Pi_{\mu\nu}\Pi_{\rho\sigma}\right) & \textit{Graviton} \\ \frac{1}{2}\left(\Pi_{\mu\rho}\Pi_{\nu\sigma} - \Pi_{\mu\sigma}\Pi_{\nu\rho}\right) & \textit{Antisymmetric Tensor} \\ \frac{1}{D-2}\Pi_{\mu\nu}\Pi_{\rho\sigma} & \textit{Dilaton} \end{cases} \tag{C.3}$$

where $\Pi_{\mu\nu}$ is the sum over polarizations of the gauge boson

$$\Pi_{\mu\nu} = \sum_{\lambda} e^{\lambda}_{\mu} e^{\lambda}_{\nu}. \tag{C.4}$$

Note that there is no actual gauge boson, this is just a mathematical trick.

Since (C.3) is rotationally invariant we choose a choice of coordinates such that $k^{\mu} = (k^0, 0, 0, ..., k^{25})$, $k^0 = k^{25}$ which gives linear polarizations $e^1_{\mu} = (0, 1, 0, ...)$; $e^2_{\mu} = (0, 0, 1, 0, ...)$; ...; $e^{24}_{\mu} = (0, 0, ..., 1, 0)$. This choice of coordinates gives non-zero $\Pi_{\mu\nu}$ as

$$\Pi_{ii} = 1 \; ; i = 1, .., 24 \tag{C.5}$$

with no sum on i.

Following equation (4.15) we see that we are interested in calculating the special case $\Pi_{\mu\mu\nu\nu}$ with no sum on μ or ν. Substituting (C.5) into (C.3) we see that the non-zero components of $\Pi_{\mu\mu\nu\nu}$ are

$$\Pi_{iijj} = \begin{cases} \delta_{ij} - \frac{1}{D-2} & \textit{Graviton} \\ 0 & \textit{Antisymmetric Tensor} \\ \frac{1}{D-2} & \textit{Dilaton} \end{cases} \tag{C.6}$$

with $i, j = 1 \ldots 24$ and again no summation of the i's and j's. As advertised these sums are independent of k^μ.

We can now simplify (4.15) to give

$$\begin{aligned} S &= S_G + S_B + S_\Phi = \sum_\lambda \left| e_{00} + \sum_{i=1}^{p} e_{ii} - \sum_{j=p+1}^{25} e_{jj} \right|^2 \\ &= \sum_\lambda \left| \sum_{i=1}^{p} e_{ii} - \sum_{j=p+1}^{24} e_{jj} \right|^2 \\ &= \sum_{i,j=1}^{p} \Pi_{iijj} + \sum_{i,j=p+1}^{24} \Pi_{iijj} - 2\sum_{i=1}^{p} \sum_{j=p+1}^{24} \Pi_{iijj}. \end{aligned} \tag{C.7}$$

and using equation (C.6) we can fully evaluate this sum for each of S_G,S_B and S_Φ independently

$$\begin{aligned} S_G &= \sum_{i,j=1}^{p} \left(\delta_{ij} - \frac{1}{24} \right) + \sum_{i,j=p+1}^{24} \left(\delta_{ij} - \frac{1}{24} \right) + \frac{1}{12} \sum_{i=1}^{p} \sum_{j=p+1}^{24} 1 \\ &= \frac{p}{6}(24 - p) \end{aligned} \tag{C.8}$$

$$S_B = 0 \tag{C.9}$$

$$\begin{aligned} S_\Phi &= \frac{1}{24} \left(\sum_{i,j=1}^{p} 1 + \sum_{i,j=p+1}^{24} 1 - 2 \sum_{i=1}^{p} \sum_{j=p+1}^{24} 1 \right) \\ &= 24 - \frac{p}{6}(24 - p). \end{aligned} \tag{C.10}$$

Adding these together gives

$$S = 24. \tag{C.11}$$