THE CLOSED BOSONIC STRING ON A PP-WAVE BACKGROUND

by

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Abstract

We study the classical solutions and spectrum of the closed bosonic string on a pp-wave background. The classical solution is found in two distinct cases, and in one of those cases it is then quantized. Classical scalar, vector, and graviton fields are studied on the same background, and the spectrum of the fields is found for comparison. Finally, the quantized string is compared to the graviton in the appropriate limits and we conclude that the ζ function regularization of the normal ordering constant is completely successful at matching the classical graviton spectrum to linear order in α'. Additional evidence in favour of the ζ function regularization is found by considering the BRST central charges.

Contents

1 Introduction

Spaces on which exact string solutions can be found are of great interest. Because of this and the recent interest in the AdS/CFT correspondence, the pp-wave spacetime is an important example. While the superstring has been studied in detail, however, the bosonic string has been much less thoroughly examined. Most work on the superstring also uses an RR 5-form to source the curvature of the metric, while the bosonic string requires that we use the NS-NS antisymmetric 3-form H instead. Polchinski [1] provides a collection of results for a general curved spacetime. The action and β functions found there for the bosonic string in a general background provided much of the starting point for this work. Because of the generality, however, the details tend to apply only to Minkowski space. Other works which study the bosonic string on similar backgrounds include [2] and [3]. If conformal invariance is ignored and no source is taken for the pp-wave metric, then it is simple to show that the string oscillators become like massive particles. We are interested in the conformally invariant theory, however, because H will have an interaction term with the closed string. Of course it is also possible to include a non-constant dilaton Φ which could in principle replace the 3-form entirely. Because the non-zero Ricci curvature appears in the $_{++}$ component, however, the dilaton would have to grow quadratically in the time coordinate x^+, so I will not consider this possibility here.

An important area of research is understanding tractable interacting string theories. Because a conformal field theory formulation exists for the bosonic string on this spacetime, there is a set of tools to begin studying interactions in this case. The pp-wave spacetime is especially interesting because it is the BMN limit of Anti de Sitter space. Before an interacting theory can be studied, however, we must know how to quantize the free theory. The normal ordering constant is of particular interest, as are the frequencies of the oscillators in the mode expansion for the embedding functions. The classical wavefunctions for the free states are also useful, as they are incorporated into the vertex operators of the interacting theory.

A complete set of classical graviton wavefunctions for the space is not difficult to find. From that, we can determine a light cone momentum for the gravitons in the classical theory, and require that it matches string theory in the appropriate limit. This provides a constraint on the space of possible normal ordering constants for the quantized string. There is still enough freedom, however, to choose constants with very different physical results. In order to choose one, we can use the ζ function regularization. There is also a recent paper, [4], which uses a covariant BRST quantization to find the ordering constant for small energies compared to the string scale. Their result can be extended to confirm the ζ function regularization for the general case. When we match the string theory to the gravitational theory, we find that we can nearly match the classical graviton spectrum as long as the combination of constants $\left|\alpha' p^{+} \sqrt{\mu}\right| < 1$, but we only have perfect agreement for $\alpha' = 0$. While it is possible to alter the ordering constant to match the graviton spectrum for any energy, this would contradict the BRST and ζ function results. This leaves the question of when the two theories should match and why they do not in general. Classical gravity is in fact the $\alpha' = 0$ limit of the string theory, so the $O(\alpha')$ disagreement is quite natural. A proper examination of the first order contributions from α' to the gravitational calculation is not a small task, but quick estimates and dimensional considerations suggest exactly the right sort of term to match the string spectrum for $\left|\alpha' p^{+} \sqrt{\mu}\right| < 1$.

In section 2 I will begin with a brief look at the metric of the pp-wave. I will solve the unperturbed β function assuming a constant dilaton, $\nabla\Phi = 0$, in order to ensure consistency of the theory. In section 3 I will begin to study the excitations allowed here by working out equations of motion from the bosonic string action. I will first consider a simple case with an unusual dispersion relation, and then I will consider more general forms for H. In section 4 I quantize the theory for a specific choice of H and work out the details in a convenient choice of mode expansion. The level-matching condition and mass spectrum will be found for an arbitrary state. The momenta are all solved for in terms of the mode expansion for the embedding functions X. Finally, the normal ordering constant is found through the ζ function regularization, and then verified through the extension of the BRST calculation. Section 5 contains a discussion of classical gravity on the pp-wave background. I solve the perturbed β functions in order to find the energy spectrum of the various states arising in this theory, and I will find a common zero-mode part which describes the overall motion of the particle, as well as a part which is proportional to the antisymmetric field strength H in the direction of the polarization. In section 6 the two calculations are compared and I determine which states can be considered gravitons. I also examine the conditions under which the theories should match, and how well they match in that region as well as close to it. I then present further evidence in favour of the ζ function regularization by demonstrating what can happen if the ordering constant takes other values. Section 7 summarizes the results.

2 The pp-wave

First I will establish the basic properties of the background. I am always going to assume that the background dilaton and antisymmetric three-form are constants, so $\nabla_\mu \Phi = 0$ and $\partial_\mu H_{\nu\rho\sigma} = 0$. I am going to work with the metric given by

$$ds^2 = -2dx^+ dx^- - x^i \mu_{ij} x^j dx^+ dx^+ + d\vec{x} \cdot d\vec{x} \tag{2.1}$$

Here μ_{ij} is a matrix of free parameters. In the limit where $\mu_{ij} = 0$, this becomes Minkowski space. The matrix form of the metric here is

$$G_{\mu\nu} = \begin{pmatrix} -x^i \mu_{ij} x^j & -1 & & 0 \\ -1 & 0 & & \\ & & 1 & \\ & 0 & & \ddots \end{pmatrix} \qquad G^{\mu\nu} = \begin{pmatrix} 0 & -1 & & 0 \\ -1 & x^i \mu_{ij} x^j & & \\ & & 1 & \\ & 0 & & \ddots \end{pmatrix} \tag{2.2}$$

In order to solve the Einstein equation for the background, we are obviously going to need the Christoffel symbols and Ricci tensor. Starting with the standard formulas, we can work out the geometric properties of the space from the metric.

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} G^{\mu\sigma} \left(G_{\sigma\nu,\rho} + G_{\sigma\rho,\nu} - G_{\nu\rho,\sigma} \right) \tag{2.3}$$

There is only one component of $G_{\mu\nu}$ with a non-zero partial derivative, so the only contributions to the Christoffel symbol will come from $G_{++,i} = -\mu_{ij} x^j$. With this in mind, it is not difficult to see that there are only two kinds of non-zero Christoffel symbols. They are

$$\Gamma^i_{++} = \Gamma^-_{+i} = \frac{1}{2} \left(\mu_{ij} + \mu_{ji} \right) x^j \tag{2.4}$$

We do not need to know every component of the Riemann tensor, only the ones which will be contracted in order to form the Ricci tensor. The Ricci tensor is $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$, so we need $R^+{}_{\mu+\nu}$, $R^-{}_{\mu-\nu}$, and $R^i{}_{\mu i\nu}$. The Riemann tensor is given by

$$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu_{\nu\sigma,\rho} - \Gamma^\mu_{\nu\rho,\sigma} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho} \tag{2.5}$$

Since $\Gamma^{+}_{\mu\nu} = 0$, we can immediately see that $R^{+}{}_{\mu\nu\rho} = 0$. Also, we know that $\Gamma^{\mu}_{-\nu} = 0$ and $\Gamma^{\mu}_{\nu\rho,-} = 0$, so we know that $R^{\mu}{}_{\nu-\rho} = 0$. The only remaining component to calculate is

$$R^{i}{}_{\mu i\nu} = \Gamma^{i}_{\mu\nu,i} - \Gamma^{i}_{\mu i,\nu} + \Gamma^{i}_{i\rho}\Gamma^{\rho}_{\mu\nu} - \Gamma^{i}_{\nu\rho}\Gamma^{\rho}_{\nu i} = \delta^{+}_{\mu}\delta^{+}_{\nu}\,\mathrm{tr}(\mu) \tag{2.6}$$

From this, we can immediately see that the Ricci tensor is

$$R_{\mu\nu} = \begin{cases} \mathrm{tr}(\mu) & \mu = \nu = + \\ 0 & \text{otherwise} \end{cases} \tag{2.7}$$

The Einstein equation is equivalent to the first of the three β functions for the bosonic string. Setting these β functions to 0 at every order in α' gives us conformal invariance, but we will only be working to first order. To this order, the functions are

$$\beta^{G}_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_{\mu}\nabla_{\nu}\Phi - \frac{\alpha'}{4}H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} = 0 \tag{2.8a}$$

$$\beta^{B}_{\mu\nu} = -\frac{\alpha'}{2}\nabla^{\lambda}H_{\lambda\mu\nu} + \alpha'\nabla^{\lambda}\Phi H_{\lambda\mu\nu} = 0 \tag{2.8b}$$

$$\beta^{\Phi} = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla^{2}\Phi + \alpha'\nabla_{\lambda}\Phi\nabla^{\lambda}\Phi - \frac{\alpha'}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} = 0 \tag{2.8c}$$

With the assumption we are making that the dilaton is constant, this system simplifies, and can be solved to give

$$4R_{\mu\nu} = H_{\mu\rho\sigma}H_{\nu}{}^{\rho\sigma} \tag{2.9a}$$

$$= H_{\mu ij}H_{\nu ij} - 2H_{\mu i+}H_{\nu i-} - 2H_{\mu i-}H_{\nu i+} \quad + 2H_{\mu+-}H_{\nu-+} + 2x^{i}\mu_{ij}x^{j}H_{\mu k-}H_{\nu k-} \tag{2.9b}$$

$$R_{--} = 0 \quad\Rightarrow\quad 0 = H_{-ij}H_{-ij} \quad\Rightarrow\quad H_{-ij} = 0 \quad \forall i,j \tag{2.9c}$$

$$R_{+-} = 0 \quad\Rightarrow\quad 0 = H_{+-i}H_{+-i} \quad\Rightarrow\quad H_{+-i} = 0 \quad \forall i \tag{2.9d}$$

$$R_{kl} = 0 \quad\Rightarrow\quad 0 = H_{kij}H_{lij} \quad\Rightarrow\quad H_{ijk} = 0 \quad \forall i,j,k \tag{2.9e}$$

$$R_{+k} = 0 \quad\Rightarrow\quad 0 = H_{+ij}H_{kij} \tag{2.9f}$$

$$R_{++} = \mathrm{tr}(\mu) \quad\Rightarrow\quad 4\,\mathrm{tr}(\mu) = H_{+ij}H_{+ij} \tag{2.9g}$$

Since H is antisymmetric, this tell us that all components of $H_{\mu\nu\rho}$ are 0 except for

$$\sum_{i,j} H_{+ij}^2 = 4\,\mathrm{tr}(\mu) \tag{2.10}$$

The last β function, Eq. (2.8c), also tells us that $D = 26$. This will be assumed throughout. We now move on to study the closed bosonic string with the background I have just described. The constraint of Eq. (2.10) is not enough to completely determine H, so there will be a choice to make of exactly which background to study. As we will find, not all allowed backgrounds behave the same way.

3 Choice of $H_{\mu\nu\rho}$

There are many possible antisymmetric fields, H_{+ij} which will satisfy the condition that the β functions, Eq. (2.8), are 0. In this section I will explore what happens in some of the more tractable cases. There are still more complicated cases where μ_{ij} and H_{+ij} are arbitrary symmetric and antisymmetric matrices respectively, but this most general case will turn out to make predictions difficult. It is not considered in any detail here.

3.1 A Simple Case

I will start by considering the case where H_{+ij} only has a single non-zero component and its antisymmetric partner. To make the discussion even simpler, I will assume that the two indices on the non-zero H are in directions where μ have the same value, $\mu_i = \mu_j$. Here the μ matrix in the metric is

$$\mu_{ij} = \begin{pmatrix} \mu_1 & & & & & \\ & \mu_1 & & & 0 & \\ & & \mu_4 & & & \\ & & & \mu_5 & & \\ & 0 & & & \mu_6 & \\ & & & & & \ddots \end{pmatrix} \tag{3.1}$$

And in order to satisfy Eq. (2.10) with only the one component, the antisymmetric field H has to be

$$\begin{aligned} H_{+23} &= \sqrt{2\,\mathrm{tr}(\mu)} \\ H_{\mu\nu\rho} &= 0 \quad \text{otherwise} \end{aligned} \tag{3.2}$$

$H_{\mu\nu\rho}$ has been chosen, but we still have some freedom to choose the gauge field $B_{\mu\nu}$ which is equivalent. The equation relating the two is

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{3.3}$$

and I will choose a gauge where the field does not increase in x^+, which is like the time coordinate. One such gauge choice is

$$\begin{aligned} B_{+2} &= \sqrt{2\,\mathrm{tr}(\mu)}\,x^3 \\ B_{\mu\nu} &= 0 \quad \text{otherwise} \end{aligned} \tag{3.4}$$

We now wish to take a brief look at what happens to the bosonic string when it is placed in this background. In this section I will only go as far as finding a mode expansion and oscillator frequencies for the string. The action for the bosonic string in an arbitrary background is given in Polchinski.

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma g^{1/2} \left(g^{ab} G_{\mu\nu} + i\epsilon^{ab} B_{\mu\nu}\right) \partial_a X^\mu \partial_b X^\nu \tag{3.5}$$
$$= \frac{1}{4\pi\alpha'} \int d^2\sigma \left(2G_{\mu\nu}\partial_+ X^\mu \partial_- X^\nu + iB_{\mu\nu}(-i\partial_+ X^\mu \partial_- X^\nu + i\partial_- X^\mu \partial_+ X^\nu)\right)$$
$$= \frac{1}{2\pi\alpha'} \int d^2\sigma \left(G_{\mu\nu} + B_{\mu\nu}\right) \partial_+ X^\mu \partial_- X^\nu \tag{3.6}$$

Where I have simplified it by choosing worldsheet coordinates where $\sigma^\pm = \sigma^0 \pm \sigma^1$ and $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$. The coordinates σ^0 and σ^1 are often referred to as τ and σ in the literature, and the $\sigma^\pm$ coordinates are the light-cone coordinates on the worldsheet. In light-cone coordinates, the worldsheet metric is

$$g_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{3.7}$$

With these coordinates, we also have $\sqrt{g}\epsilon^{+-} = -i$. I will use these conventions for the worldsheet coordinates throughout this paper.

I have used the symmetries of G and B to factor out $G_{\mu\nu} + B_{\mu\nu}$ in Eq. (3.6) because they are the background fields, and will both be known in any given situation. They can be combined to give a single matrix, and in this case that tensor is

$$G_{\mu\nu} + B_{\mu\nu} = \begin{pmatrix} -x^i \mu_{ij} x^j & -1 & \sqrt{2\,\mathrm{tr}(\mu)}x^3 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \\ -\sqrt{2\,\mathrm{tr}(\mu)}x^3 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{pmatrix} \tag{3.8}$$

We can see that when the action is expanded in terms of the components, one of the equations of motion will be

$$\partial_+ \partial_- X^+ = 0 \tag{3.9}$$

Rather than dealing with a fully covariant quantization of the string on this background, I will work in light cone gauge, where

$$X^+ = x^+ + \alpha' p^+ \sigma^0 = x^+ + \alpha' p^+ \frac{\sigma^+ + \sigma^-}{2} \tag{3.10}$$

This satisfies the equation of motion Eq. (3.9). Now the action can be further simplified by inserting this solution for X^+ and finding the action in terms of the remaining components.

$$\begin{aligned} S = \frac{1}{2\pi\alpha'} \int d^2\sigma \Bigg(- \left(\mu_1 (X^k X^k) + \mu_i (X^i X^i)\right) \left(\frac{\alpha' p^+}{2}\right)^2 - \frac{\alpha' p^+}{2} (\partial_+ + \partial -) X^- \\ - \sqrt{2\,\mathrm{tr}(\mu)} X^3 \frac{\alpha' p^+}{2} (\partial_+ - \partial_-) X^2 - \partial_+ \vec{X} \cdot \partial_- \vec{X} \Bigg) \end{aligned} \tag{3.11}$$

Here k is summed over the directions of the B field, 2 and 3, and i is summed over the remaining transverse coordinates, 4 through 25. At this point I find it useful to define the following constants

$$\mu_t = \frac{1}{2}\mathrm{tr}(\mu) = \mu_1 + \frac{1}{2}\sum_{i=4}^{25} \mu_i, \qquad C_a = \alpha' p^+ \sqrt{\mu_a} \tag{3.12}$$

Now we can take the action of Eq. (3.11) and vary it with respect to the transverse fields, X^i to get the equations of motion.

$$\frac{\delta \mathcal{L}}{\delta X^\mu} = \sum_{\pm} \partial_\pm \frac{\delta \mathcal{L}}{\delta \partial_\pm X^\mu} \tag{3.13}$$

$$-\frac{C_1^2}{2} X^2 = 2\partial_+ \partial_- X^2 - C_t \partial_1 X^3 \tag{3.14a}$$

$$-\frac{C_1^2}{2} X^3 = 2\partial_+ \partial_- X^3 + C_t \partial_1 X^2 \tag{3.14b}$$

$$-\frac{C_i^2}{2} X^i = 2\partial_+ \partial_- X^i \tag{3.14c}$$

Once we fourier transform the solution in the worldsheet coordinates, the solution has the form $e^{-i\omega\sigma^0 - in\sigma^1}$. With this ansatz the equations of motion become

$$(-\omega^2 + n^2 + C_1^2)X^2 = -2C_t in X^3 \tag{3.15a}$$

$$(-\omega^2 + n^2 + C_1^2)X^3 = 2C_t in X^2 \tag{3.15b}$$

$$-\omega^2 + n^2 + C_i^2 = 0 \tag{3.15c}$$

In the extra transverse directions, we just find a set of 22 massive oscillators with $\omega_n = \sqrt{n^2 + C_i^2}$. In the two directions with the antisymmetric tensor field, however, we have a system of two coupled equations. The two embedding functions must have the same frequency, so we can find the full four degrees of freedom from the fact that the coupled system produces four solutions for ω. The two equations together lead to

$$\left(-\omega^2 + n^2 + C_1^2\right)^2 = 4C_t^2 n^2 \tag{3.16}$$

$$\omega^2 = n^2 + C_1^2 \pm 2nC_t = (n \pm C_t)^2 - \frac{1}{2}\sum_{i=4}^{25} C_i^2 \tag{3.17}$$

If we define $\omega_n = \sqrt{(n + C_t)^2 - \frac{1}{2}\sum C_i^2}$, then we can write the four solutions to the equations of motion with the frequencies $\omega_{\pm n}$ and $-\omega_{\pm n}$. There are still more constraints to impose, the first coming from the equations of motion. It relates the modes in X^3 to the ones in X^2 so that there are only four degrees of freedom in total. The result is

$$X^2 = i\sqrt{\frac{\alpha'}{2}}\sum_n \Big(\alpha_n e^{-i(\omega_n\sigma^0 + n\sigma^1)} + \tilde{\alpha}_n e^{i(\omega_n\sigma^0 - n\sigma^1)} + \beta_n e^{-i(\omega_{-n}\sigma^0 + n\sigma^1)} + \tilde{\beta}_n e^{i(\omega_{-n}\sigma^0 - n\sigma^1)}\Big) \tag{3.18a}$$

$$X^3 = i\sqrt{\frac{\alpha'}{2}}\sum_n \Big(-i\alpha_n e^{-i(\omega_n\sigma^0 + n\sigma^1)} - i\tilde{\alpha}_n e^{i(\omega_n\sigma^0 - n\sigma^1)} + i\beta_n e^{-i(\omega_{-n}\sigma^0 + n\sigma^1)} + i\tilde{\beta}_n e^{i(\omega_{-n}\sigma^0 - n\sigma^1)}\Big) \tag{3.18b}$$

We can compare this to the well known mode expansion of the closed bosonic string on

Minkowski space.

$$X^i = x^i + \alpha' p^i \sigma^0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n=1}^{\infty} \left(\frac{\alpha_n^i}{n} e^{-in\sigma^+} + \frac{\tilde{\alpha}_n^i}{n} e^{-in\sigma^-} \right)$$

where the identification that $\alpha_n^{i\dagger} = \alpha_{-n}^i$ and $\tilde{\alpha}_n^{i\dagger} = \tilde{\alpha}_{-n}^i$ must be made. This means that on Minkowski space there are two real degrees of freedom summed over all n, or equivalently two complex degrees of freedom summed over positive n. Now we are worried that in our theory we have four independent complex constants summed over all n rather than just $n > 0$, the new mode expansion has four complex degrees of freedom summed over all n corresponding to the two dimensions, while in Minkowski space there were only half as many degrees of freedom per dimension. The solution is to remember that the embedding functions X^i must be real. When we set $X^2 = X^{2\dagger}$ we find

$$\tilde{\beta}_{-n} = \alpha_n^\dagger, \quad \beta_{-n} = \tilde{\alpha}_n^\dagger \tag{3.19}$$

This reduces the number of degrees of freedom by $\frac{1}{2}$ and we now have only two complex constants, α_n and $\tilde{\alpha}_n$, summed over all n, which is equivalent to four real degrees of freedom summed over all n. This is the same number as we would get from the same two embedding functions for a closed bosonic string in the two dimensions of Minkowski space.

The constants α, $\tilde{\alpha}$, β, and $\tilde{\beta}$ are promoted to creation and annihilation operators when the theory is quantized, but they are arbitrary at the classical level. In these directions, we see that ω_n will be complex as long as the remaining transverse C_i^2 are larger than the fractional part of C_t squared. It is not unreasonable to see complex ω, since we know that a strong enough electric field can tear an open string or accelerate a charged particle to infinity. The antisymmetric background field has an interaction with the closed string, and this simply suggests that it too has the ability to make the solutions diverge exponentially if it is large enough.

3.2 General Case

I now want to see what happens when the bosonic string interacts with more general pp-wave backgrounds than the one considered in section 3.1. The full generality is not tractable, but with only a small number of simplifying assumptions we can make predictions about ω.

Starting with the general case when μ_{ij} is an arbitrary matrix with positive entries, we can always find an orthonormal coordinate basis that will diagonalize μ. Only the sum $\sum_{i,j} H_{+ij}^2$ is determined, so there are many different possible fields that could source the metric. $H_{\mu\nu\rho}$, however, will always have a single $+$ index and since the sum of the squares of the components of H must be a constant, I will make the assumption that each H_{+ij} is a constant as well. We can always choose the gauge where $B_{+i} = \eta_{ij} X^j$ and see the trivial manner in which H_{+ij} and η_{ij} are related.

$$H_{+ij} = \partial_+ B_{ij} + \partial_i B_{j+} + \partial_j B_{+i} = \eta_{ij} - \eta_{ji} \tag{3.20}$$

Since H_{+ij} only depends on the anti-symmetrized form of η_{ij}, we clearly still have the gauge freedom to make it antisymmetric. Now for simplicity in the upcoming calculations, we can redefine η so that $H_{+ij} = \frac{\eta_{ij}}{\alpha' p^+}$ and now $B_{+i} = \frac{1}{2\alpha' p^+} \eta_{ij} X^j$. With this general form of the background, and still working in the light cone gauge, the action is

$$S = \frac{1}{2\pi\alpha'} \left(-\frac{\alpha' p^+}{2} (\partial_+ + \partial_-) X^- - \frac{\alpha' p^+}{4} \sum_i \mu_i X^{i2} + \sum_i \partial_+ X^i \partial_- X^i \right. \\ \left. - \frac{1}{4} \sum_{i,j} \eta_{ij} X^j (\partial_+ - \partial_-) X^i \right) \tag{3.21}$$

and this action leads to the equations of motion

$$\frac{1}{2} \left(-\partial_0^2 + \partial_1^2 - C_i^2 \right) X^i + \frac{1}{2} \sum_j \eta_{ij} \partial_1 X^j = 0 \tag{3.22}$$

Since the equations of motion are linear in X^i and the frequency ω only appears with X^i in the ith equation of motion, the oscillator frequencies ω^2 are the eigenvalues of the characteristic matrix

$$\begin{pmatrix} n^2 + C_2^2 & in\eta_{23} & in\eta_{24} & \\ -in\eta_{23} & n^2 + C_3^2 & in\eta_{34} & \cdots \\ -in\eta_{24} & -in\eta_{34} & n^2 + C_4^2 & \\ & \vdots & & \ddots \end{pmatrix} \tag{3.23}$$

In general, this is a 24×24 matrix, so it has extremely long eigenvalues, and there is very little that we can say about them.

If we make the simplifying assumption that the only non-zero components of the antisymmetric tensor are $\eta_{2j\ 2j+1}$ with j running over the range $j = 1, 2, \ldots, 12$ to get a block-diagonal form with 2×2 blocks, then the system becomes easily solvable. Each block now has two eigenvalues, given by

$$\omega^2 = \frac{1}{2}\left(C_{2j}^2 + C_{2j+1}^2\right) + n^2 \pm \frac{1}{2}\sqrt{\left(C_{2j}^2 - C_{2j+1}^2\right)^2 + 4n^2\eta_{2j\ 2j+1}^2} \tag{3.24}$$

If η_{ij} is large, then clearly this will have a negative value. To see how large η_{ij} can be before the theory becomes unstable, we set $\omega = 0$ in Eq. (3.24) and solve for η_{ij}.

$$\eta_{2j\ 2j+1} = \sqrt{\frac{\left(n^2 + C_{2j}^2\right)\left(n^2 + C_{2j+1}^2\right)}{n^2}} \tag{3.25}$$

For η_{ij} larger than this, we will find a complex mode. The right hand side has a minimum at $n = \sqrt{C_{2j}C_{2j+1}}$, so it will be easiest to get a complex spectrum for an integer n adjacent to this value. This is the n which gives the most negative value, but there could easily be more imaginary modes. If we complete the square on the right hand side of Eq. (3.24) we find a result which helps us study the behaviour when the conformal invariance of Eq. (2.10) is satisfied. To begin, we take the equivalent form of Eq. (3.24)

$$\omega^2 = \left(n \pm \sqrt{\frac{C_{2j}^2 + C_{2j+1}^2}{2}}\right)^2 \pm \left(\frac{1}{2}\sqrt{\left(C_{2j}^2 - C_{2j+1}^2\right)^2 + 4n^2\eta_{2j\ 2j+1}^2} - n\sqrt{2\left(C_{2j}^2 + C_{2j+1}^2\right)}\right) \tag{3.26}$$

If this is exactly a perfect square, then ω is linear in n just as on flat space, only now it will be shifted by the root mean squared of the two values of μ_i involved in this 2×2 block. In order for this to happen, η_{ij} must satisfy the condition that

$$\eta_{2j\ 2j+1}^2 = 2\left(C_{2j}^2 + C_{2j+1}^2\right) - \frac{1}{4n^2}\left(C_{2j}^2 - C_{2j+1}^2\right)^2 \tag{3.27}$$

So we see that the last term, which comes paired with η whenever the two μ_i are different, gives an n dependence to $\eta_{2j\ 2j+1}$. This means that there can be at most one mode which is a perfect square as long as the two C_i are not identical. Now to get back to the existence of complex modes, the perfect square term in Eq. (3.26) will be close to 0 when n is adjacent to the root mean squared. Either the n immediately before or after the RMS will cause this term to be $\leq \frac{1}{4}$. At this point, we can say

$$n \approx \sqrt{\frac{C_{2j}^2 + C_{2j+1}^2}{2}} \tag{3.28}$$

In order for a mode with this n to have a complex frequency, we would have to find

$$\eta_{2j\ 2j+1}^2 \gtrsim 2\left(C_{2j}^2 + C_{2j+1}^2\right) + \frac{1}{2}\left(\frac{2C_{2j}C_{2j+1}}{C_{2j}^2 + C_{2j+1}^2} - 1\right) \tag{3.29}$$

This formula has neglected the perfect square term in Eq. (3.26), so η will actually have to be slightly larger in order to overcome the small minimum of that term.

Suppose now that we take

$$\eta_{2j\ 2j+1} = \sqrt{2\left(C_{2j}^2 + C_{2j+1}^2\right)} \tag{3.30}$$

This is large enough to satisfy Eq. (3.29) because the last term there is always 0 or negative. If we now compute the sum that appears in the β function, we find

$$\sum_{i,j} \eta_{ij}^2 = 2 \sum_{j\ \text{even}} 2\left(C_{2j}^2 + C_{2j+1}^2\right) \tag{3.31a}$$

$$\sum_{i,j} H_{+ij}^2 = 4 \sum_i \mu_i \tag{3.31b}$$

If every component of η_{ij} has exactly the value proposed in Eq. (3.30), then it is a matter of comparing the minimum of the perfect square with the contribution from the term involving $C_{2j}^2 - C_{2j+1}^2$. If the latter is larger, then there will be at least one unstable mode. In order to satisfy conformal invariance, if any block-off-diagonal component of η_{ij} is smaller than this, it will give rise to at least one other component which is larger than this value. If any components of η_{ij} are larger, then there is an even stronger chance to find an unstable mode for that pair of coordinates. The increase in η_{ij} above Eq. (3.30) would have to be very small in order to preserve even a chance that the mode is not complex. Also recall that from the full formula we found in Eq. (3.24) led to the η necessary to get an unstable state having a minimum at $n = \sqrt{C_{2j}C_{2j+1}}$, while the mode which we study has $n = \sqrt{\frac{C_{2j}^2 + C_{2j+1}^2}{2}}$. The two are equal when $C_{2j} = C_{2j+1}$, but as the two separate, there will be more modes which have complex frequencies, and the one which is most easily made complex will have the

$n \approx \sqrt{C_{2j}C_{2j+1}}$. Another point to note is that if $\mu_{2j} \neq \mu_{2j+1}$, then the difference of the C_i can always be increased without bound by increasing p^+. A full spectrum of stable states will only exist for all values of the light cone momentum p^+ if the μ_i come in identical pairs.

If all of the eigenvalues of the μ_{ij} matrix are the same, as is frequently assumed, then we can drop all indices from the μ_i and C_i constants. If this happens, then any orthonormal coordinate system for the transverse directions will give the same matrix $\mu_{ij} \propto \delta_{ij}$ and so we have the extra freedom to block-off-diagonalize η_{ij}. With η_{ij} in this form, we are still in the case of Eq. (3.26) and can simplify further.

$$\omega^2 = n^2 + C^2 \pm n\eta_{2j\ 2j+1} = (n \pm C)^2 \pm n\left(\eta_{2j\ 2j+1} - 2C\right) \tag{3.32}$$

This is the perfect square described above when $\eta_{2j\ 2j+1} = 2C$ for every even j, and this choice will also satisfy Eq. (2.10). This is now the only choice of H_{+ij} that will give a complete collection of stable oscillator states for the string.

4 String Spectrum

Here I will examine a choice of background in which there are no problems with complex frequencies. This time I will use

$$\mu = \begin{pmatrix} \mu_1 & & & & & & \\ & \mu_1 & & & & & \\ & & \mu_2 & & & & \\ & & & \mu_2 & & & \\ & & & & \mu_3 & & \\ & & & & & \mu_3 & \\ & & & & & & \ddots \end{pmatrix} \tag{4.1a}$$

$$H_{+(2j)(2j+1)} = 2\sqrt{\mu_j} \ \forall j = 1..\frac{D-2}{2} \tag{4.1b}$$

$$H_{\mu\nu\rho} = 0 \text{ otherwise} \tag{4.1c}$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{4.1d}$$

$$B_{+(2j)} = 2\sqrt{\mu_j} x^{2j+1} \tag{4.1e}$$

$$B_{\mu\nu} = 0 \text{ otherwise} \tag{4.1f}$$

This choice of fields will satisfy Eq. (2.10) and because the μ_j pairs are always equal, it will be free of instabilities. The goal is to find which states are physical, and the energy and momentum for all allowed states.

Throughout this section, I will use j rather than i when I wish to denote an index that runs over the pairs, rather than running over the whole $i = 2 \ldots 25$ range. j should therefore be understood to represent a positive integer from 1 to 12, just as it does above.

4.1 Equations of Motion

I will begin by following the same procedure as in section 3.1. When these values are substituted into the action and light cone gauge is imposed on X^+, the result is

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \left(-\frac{\alpha' p^+}{2} (\partial_+ + \partial_-) X^- + \partial_+ \vec{X} \cdot \partial_- \vec{X} \right. \\ \left. -\frac{1}{4} C_j^2 \left((X^{2j})^2 + (X^{2j+1})^2 \right) - C_j X^{2j+1} (\partial_+ - \partial_-) X^{2j} \right) \tag{4.2}$$

where $C_j = \alpha' p^+ \sqrt{\mu_j}$ as before. The equations of motion for the transverse embedding functions are again found.

$$-\frac{C_j^2}{2} X^{2j} = 2\partial_+\partial_- X^{2j} - \alpha' p^+ \sqrt{\mu_j}\partial_1 X^{2j+1} \tag{4.3a}$$

$$-\frac{C_j^2}{2} X^{2j+1} - \alpha' p^+ \sqrt{\mu_j}\partial_1 X^{2j} = 2\partial_+\partial_- X^{2j+1} \tag{4.3b}$$

This time, however, we find a perfect square as predicted by Eq. (3.26) when the pair of equations is solved for the oscillator frequency.

$$\omega_n^j = \pm n \pm C_j \tag{4.4}$$

This is a simple enough form to substitute into the mode expansion for X. With α, β, and their tildes all arbitrary at the classical level, the solutions to the equations of motion are

$$\begin{aligned} X^{2j} = i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \Big(& \alpha_n^j e^{-i(n\sigma^0 + C_j\sigma^0 + n\sigma^1)} + \tilde{\alpha}_n^j e^{-i(-n\sigma^0 - C_j\sigma^0 + n\sigma^1)} \\ & + \beta_n^j e^{-i(-n\sigma^0 + C_j\sigma^0 + n\sigma^1)} + \tilde{\beta}_n^j e^{-i(n\sigma^0 - C_j\sigma^0 + n\sigma^1)} \Big) \\ & + i\sqrt{\frac{\alpha'}{2}} \left(\alpha_0^{2j} e^{-iC_j\sigma^0} + \tilde{\alpha}_0^{2j} e^{iC_j\sigma^0} \right) \end{aligned} \tag{4.5a}$$

$$\begin{aligned} X^{2j+1} = i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \Big(& -i\alpha_n^j e^{-i(n\sigma^0 + C_j\sigma^0 + n\sigma^1)} - i\tilde{\alpha}_n^j e^{-i(-n\sigma^0 - C_j\sigma^0 + n\sigma^1)} \\ & + i\beta_n^j e^{-i(-n\sigma^0 + C_j\sigma^0 + n\sigma^1)} + i\tilde{\beta}_n^j e^{-i(n\sigma^0 - C_j\sigma^0 + n\sigma^1)} \Big) \\ & + i\sqrt{\frac{\alpha'}{2}} \left(\alpha_0^{2j+1} e^{-iC_j\sigma^0} + \tilde{\alpha}_0^{2j+1} e^{iC_j\sigma^0} \right) \end{aligned} \tag{4.5b}$$

These are the classical embedding functions for the coordinates, so we require that they are real valued functions. If we set $X^{2j\dagger} = X^{2j}$, we can solve for β and $\tilde{\beta}$ in terms of α and $\tilde{\alpha}$.

$$\begin{gathered} \tilde{\beta}_{-n}^j = -\alpha_n^{j\dagger},\ \beta_{-n}^j = -\tilde{\alpha}_n^{j\dagger},\ \text{if } n \neq 0 \\ \tilde{\alpha}_0^i = -\alpha_0^{i\dagger} \end{gathered} \tag{4.6}$$

This allows us to replace those constants in the mode expansion for X. At the same time, I wish to change the normalization and reorder the sum by taking $n \to -n$ in the β terms. With these alterations, a complete and real solution of the transverse equations of motion is

$$X^{2j} = i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{\sqrt{|n+C_j|}}\left(\alpha_n^j e^{-i(n\sigma^0+C_j\sigma^0+n\sigma^1)} - \alpha_n^{j\dagger} e^{-i(-n\sigma^0-C_j\sigma^0-n\sigma^1)}\right.$$
$$\left. -\tilde{\alpha}_n^j e^{-i(n\sigma^0+C_j\sigma^0-n\sigma^1)} + \tilde{\alpha}_n^{j\dagger} e^{-i(-n\sigma^0-C_j\sigma^0+n\sigma^1)}\right)$$
$$+ i\sqrt{\frac{\alpha'}{|C_j|}}\left(\alpha_0^{2j} e^{-iC_j\sigma^0} - \alpha_0^{2j\dagger} e^{iC_j\sigma^0}\right) \quad (4.7a)$$

$$X^{2j+1} = i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{\sqrt{|n+C_j|}}\left(-i\alpha_n^j e^{-i(n\sigma^0+C_j\sigma^0+n\sigma^1)} - i\alpha_n^{j\dagger} e^{-i(-n\sigma^0-C_j\sigma^0-n\sigma^1)}\right.$$
$$\left. -i\tilde{\alpha}_n^j e^{-i(n\sigma^0+C_j\sigma^0-n\sigma^1)} - i\tilde{\alpha}_n^{j\dagger} e^{-i(-n\sigma^0-C_j\sigma^0+n\sigma^1)}\right)$$
$$+ i\sqrt{\frac{\alpha'}{|C_j|}}\left(\alpha_0^{2j+1} e^{-iC_j\sigma^0} - \alpha_0^{2j+1\dagger} e^{iC_j\sigma^0}\right) \quad (4.7b)$$

Although I could choose any normalization, this one gives relatively simple results for most of the calculations to follow. Note that by resumming half of the terms, I have replaced all instances of $n - C$ with $n + C$. This makes the $\sqrt{|n+C|}$ in the normalization match the frequency for all of the terms. Also, a factor of $\sqrt{2}$ has been changed in the zero-mode term because there are only half as many possible states when $n = 0$.

An equivalent representation would be to take

$$Z^j = \frac{1}{2}\left(X^{2j} + iX^{2j+1}\right) \quad (4.8a)$$
$$= i\sqrt{\frac{\alpha'}{2}}\frac{1}{\sqrt{|n+C_j|}}\left(\alpha_n^j e^{-i(n\sigma^0+C_j\sigma^0+n\sigma^1)} + \alpha_n^{j\dagger} e^{-i(-n\sigma^0-C_j\sigma^0+n\sigma^1)}\right) \quad (4.8b)$$
$$+ \frac{i}{2}\sqrt{\frac{\alpha'}{|C_j|}}\left((\alpha_0^{2j} + i\alpha_0^{2j+1})e^{-iC_j\sigma^0} - (\alpha_0^{2j\dagger} + i\alpha_0^{2j+1\dagger})e^{iC_j\sigma^0}\right) \quad (4.8c)$$

Here Z is a complex scalar field, so both Z and $\bar{Z}$, and their derivatives will appear in the action. From this, momenta, commutators, constraints, and a spectrum of states can be found. Since this is equivalent, however, I will focus on the X coordinates for which we already have the action and a real solution. Clearly the zero mode quantities will also be simpler to calculate in terms of X coordinates.

4.2 Commutators

We wish to find the commutators of the oscillator modes. The simplest way to do this will be to express each of the constants, α, as a linear combination of the embedding functions for the positions and momenta. Obviously, in order to do this the mode expansions of the transverse momenta will be needed. The momenta in question are

$$P^i(\sigma^0,\sigma^1) = P_i(\sigma^0,\sigma^1) = \frac{\delta\mathcal{L}}{\delta\partial_0 X^i} = \frac{1}{4\pi\alpha'}\partial_0 X^i \tag{4.9}$$

which are found by carefully taking the σ^0 derivative of Eq. (4.7) to find

$$\begin{aligned}P_{2j}(\sigma^0,\sigma^1) = \frac{1}{4\pi\sqrt{2\alpha'}}\sum_{n\neq 0}\frac{n+C_j}{\sqrt{|n+C_j|}}&\left(\alpha_n^j e^{-i(n\sigma^0+C_j\sigma^0+n\sigma^1)} + \alpha_n^{j\dagger} e^{i(n\sigma^0+C_j\sigma^0+n\sigma^1)}\right.\\ &\left.- \tilde{\alpha}_n^j e^{-i(n\sigma^0+C_j\sigma^0-n\sigma^1)} - \tilde{\alpha}_n^j e^{i(n\sigma^0+C_j\sigma^0-n\sigma^1)}\right)\\ &+ \frac{1}{4\pi\sqrt{\alpha'}}\frac{C_j}{\sqrt{|C_j|}}\left(\alpha_0^{2j} e^{-iC_i\sigma^0} + \alpha_0^{2j\dagger} e^{iC_j\sigma^0}\right)\end{aligned} \tag{4.10a}$$

$$\begin{aligned}P_{2j+1}(\sigma^0,\sigma^1) = \frac{i}{4\pi\sqrt{2\alpha'}}\sum_{n\neq 0}\frac{n+C_j}{\sqrt{|n+C_j|}}&\left(-\alpha_n^j e^{-i(n\sigma^0+C_j\sigma^0+n\sigma^1)} + \alpha_n^{j\dagger} e^{i(n\sigma^0+C_j\sigma^0+n\sigma^1)}\right.\\ &\left.- \tilde{\alpha}_n^j e^{-i(n\sigma^0+C_j\sigma^0-n\sigma^1)} + \tilde{\alpha}_n^j e^{i(n\sigma^0+C_j\sigma^0-n\sigma^1)}\right)\\ &+ \frac{1}{4\pi\sqrt{\alpha'}}\frac{C_j}{\sqrt{|C_j|}}\left(\alpha_0^{2j+1} e^{-iC_i\sigma^0} + \alpha_0^{2j+1\dagger} e^{iC_j\sigma^0}\right)\end{aligned} \tag{4.10b}$$

With these classical solutions for the position and momentum of the string, the constants can be written as

$$\begin{aligned}\alpha_n^j = \frac{\sqrt{|n+C_j|}}{2\sqrt{2\alpha'}} e^{i(n+C_j)\sigma^0}\int_0^{2\pi} d\sigma^1 e^{in\sigma^1}&\left(\left(X^{2j}+iX^{2j+1}\right)\frac{1}{2\pi i}\right.\\ &\left.+\left(P^{2j}+iP^{2j+1}\right)\frac{2\alpha'}{n+C_j}\right)\end{aligned} \tag{4.11a}$$

$$\alpha_n^{j\dagger} = -\frac{\sqrt{|n+C_j|}}{2\sqrt{2\alpha'}} e^{-i(n+C_j)\sigma^0} \int_0^{2\pi} d\sigma^1 e^{-in\sigma^1} \left(\left(X^{2j} - iX^{2j+1} \right) \frac{1}{2\pi i} \right.$$

$$\left. - \left(P^{2j} - iP^{2j+1} \right) \frac{2\alpha'}{n+C_j} \right) \quad (4.11\text{b})$$

$$\tilde{\alpha}_n^{j} = -\frac{\sqrt{|n+C_j|}}{2\sqrt{2\alpha'}} e^{i(n+C_j)\sigma^0} \int_0^{2\pi} d\sigma^1 e^{-in\sigma^1} \left(\left(X^{2j} - iX^{2j+1} \right) \frac{1}{2\pi i} \right.$$

$$\left. + \left(P^{2j} - iP^{2j+1} \right) \frac{2\alpha'}{n+C_j} \right) \quad (4.11\text{c})$$

$$\tilde{\alpha}_n^{j\dagger} = \frac{\sqrt{|n+C_j|}}{2\sqrt{2\alpha'}} e^{-i(n+C_j)\sigma^0} \int_0^{2\pi} d\sigma^1 e^{in\sigma^1} \left(\left(X^{2j} + iX^{2j+1} \right) \frac{1}{2\pi i} \right.$$

$$\left. - \left(P^{2j} + iP^{2j+1} \right) \frac{2\alpha'}{n+C_j} \right) \quad (4.11\text{d})$$

where α_n^j is one of the constants shared by the even-odd pair X^{2j} and X^{2j+1}. When the positions and momenta are now promoted to operators, their equal time canonical commutation relations are

$$\left[X^i(\sigma^0,\sigma^1), X^j(\sigma^0,\sigma^{1\prime})\right] = \left[P^i(\sigma^0,\sigma^1), P^j(\sigma^0,\sigma^{1\prime})\right] = 0 \quad (4.12)$$

$$\left[X^i(\sigma^0,\sigma^1), P^j(\sigma^0,\sigma^{1\prime})\right] = i\delta^{ij}\delta(\sigma^1 - \sigma^{1\prime}) \quad (4.13)$$

where here both i and j clearly run over the entire range of transverse indices. Taking the commutators of the α operators will now give results which can be easily simplified. I will demonstrate one such process, as the rest will all be similar.

$$\left[\alpha_n^j, \alpha_m^{j\dagger}\right] = -\frac{\sqrt{|n+C_j|\,|m+C_j|}}{8\pi i} e^{i(n-m)\sigma^0} \iint d\sigma^1 d\sigma^{1\prime} e^{in\sigma^1 - im\sigma^{1\prime}}$$

$$\cdot \left(\left(-\left[X^{2j}(\sigma^0,\sigma^1), P^{2j}(\sigma^0,\sigma^{1\prime})\right] - \left[X^{2j+1}(\sigma^0,\sigma^1), P^{2j+1}(\sigma^0,\sigma^{1\prime})\right] \right) \frac{1}{m+C_j} \right.$$

$$\left. + \left(\left[P^{2j}(\sigma^0,\sigma^1), X^{2j}(\sigma^0,\sigma^{1\prime})\right] + \left[P^{2j+1}(\sigma^0,\sigma^1), X^{2j+1}(\sigma^0,\sigma^{1\prime})\right] \right) \frac{1}{n+C_j} \right)$$

$$
\begin{aligned}
&= \frac{\sqrt{|n+C_j|\,|m+C_j|}}{8\pi} e^{i(n-m)\sigma^0} \iint d\sigma^1 d\sigma^{1\prime} e^{in\sigma^1 - im\sigma^{1\prime}} \\
&\quad \cdot \left(\frac{2\delta(\sigma^1 - \sigma^{1\prime})}{m+C_j} + \frac{2\delta(\sigma^{1\prime} - \sigma^1)}{n+C_j} \right) \\
&= \frac{\sqrt{|n+C_j|\,|m+C_j|}}{8\pi} \left(\frac{2}{m+C_j} + \frac{2}{n+C_j} \right) e^{i(n-m)\sigma^0} \int_0^{2\pi} d\sigma^1 e^{i(n-m)\sigma^1} \\
&= \frac{\sqrt{|n+C_j|\,|m+C_j|}}{4} \left(\frac{2}{m+C_j} + \frac{2}{n+C_j} \right) e^{i(n-m)\sigma^0} \delta_{nm} \\
&= \frac{|n+C_j|}{n+C_j} \delta_{nm}
\end{aligned}
$$

There are, of course, many more terms in the first line, but they all involve commutators like $[X, X]$, $[P, P]$, or $\left[X^{2j}, P^{2j+1}\right]$ and so they were not included. The results of all possible commutators for the oscillators are

$$
\left[\alpha_n^{j_1}, \alpha_m^{j_2\dagger}\right] = \left[\tilde{\alpha}_n^{j_1}, \tilde{\alpha}_m^{j_2\dagger}\right] = \frac{n+C_{j_1}}{|n+C_{j_1}|} \delta_{nm} \delta^{j_1 j_2} \tag{4.14}
$$

and all other commutators are 0. The $n = 0$ terms were separated in the mode expansion, however, so this has not yet been demonstrated for them. The zero-modes have a different number of degrees of freedom and a slightly different normalization, which is why they need to be treated separately. The zero mode oscillators are given by

$$
\alpha_0^{i(\dagger)} = \frac{1}{\sqrt{2}} e^{\pm i C_i \sigma^0} \int_0^{2\pi} d\sigma^1 \left(P^i \frac{2\sqrt{\alpha' |C_i|}}{C_i} \mp i X^i \frac{\sqrt{|C_i|}}{2\pi\sqrt{\alpha'}} \right) \tag{4.15}
$$

where the $+$ choice of sign refers to the $\alpha_0^\dagger$. The results when commutators are calculated will be identical to the other oscillators.

$$
\left[\alpha_0^i, \alpha_0^{j\dagger}\right] = \frac{C_i}{|C_i|} \delta^{ij} \tag{4.16}
$$

The results are simply ± 1 for all non-zero commutators because the normalization of Eq. (4.7) was carefully chosen to produce just this result.

4.3 Creation Operators

I will now come back to the choice of labeling the four constants in the mode expansion of X armed with the sign of the $[\alpha, \alpha^\dagger]$ commutator. We have seen that α and $\tilde{\alpha}$ commute and have different signs of the σ^0 coefficient relative to the coefficient of σ^1, so it is reasonable to consider them to be the left and right-moving vibrations of the string. This justifies that one of them has a tilde, conforming with standard notation for the closed string on Minkowski space. The remaining question is to distinguish the creation and annihilation operators. We look at the simple relationship

$$\langle 0| \alpha_n \alpha_n^\dagger |0\rangle - \langle 0| \alpha_n^\dagger \alpha_n |0\rangle = \langle 0| \left[\alpha_n, \alpha_n^\dagger\right] |0\rangle = \pm 1 \tag{4.17}$$

Suppose we first look at this with $n + C_i > 0$ and assume that $\alpha_n^\dagger$ is the annihilation operator. Then $\langle 0| \alpha_n^\dagger \alpha_n |0\rangle = -1$, but this is just $\|\alpha_n |0\rangle\| = -1$ which cannot be true for a physical state. I have chosen the definition of X so that for $n + C_i > 0$ both $\alpha_n^\dagger$ and $\tilde{\alpha}_n^\dagger$ are creation operators which give states with positive norm. The same method easily verifies that for $n + C_i < 0$ we will find that α_n and $\tilde{\alpha}_n$ are now the creation operators, and $\alpha_n^\dagger$ and $\tilde{\alpha}_n^\dagger$ will annihilate the vacuum.

4.4 Constraints

The next task is to determine the constraints. This is done by restoring the worldsheet metric, g^{ab}, in the action and determining the equations of motion under variations of it as if it were a free field. The original form of the action before the worldsheet coordinates were fixed was Eq. (3.5). We are only interested in the result of varying the diagonal fields g^{++} and g^{--} so for these purposes the second term can be ignored as it contains $\sqrt{g}\epsilon^{ab}$ rather than the full worldsheet metric. For the purposes of this section only, we can take the action to be

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma g^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \tag{4.18}$$

The equation of motion found by minimizing this action under a variation of g^{ab} is now

$$G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu = 0 \tag{4.19}$$

I have neglected $\sqrt{g}$ in Eq. (3.5) for simplicity. If included, it would lead to the addition of $-\frac{1}{2} g_{ab} g^{cd} G_{\mu\nu} \partial_c X^\mu \partial_d X^\nu$ to the equation of motion, but since the only indices which are not contracted are in g_{ab} which is off-diagonal, this term will not contribute to the diagonal part which we are interested in. The diagonal terms in the equation are

$$-\sum_i \mu_i X_i^2 \partial_\pm X^+ \partial_\pm X^+ - 2\partial_\pm X^+ \partial_\pm X^- + \partial_\pm \vec{X} \cdot \partial_\pm \vec{X} = 0 \tag{4.20}$$

According to the definition of the matrix μ, this equation is not correct, but it should be clear that in situations like this μ_i represents the entry which comes paired with ${x^i}^2$ in the line element ds^2. With a $_+$ index we call this expression L, and with a $_-$ it is $\tilde{L}$. If these are going to be zero with the mode expansion of Eq. (4.7), we can split the constraints according to the fourier transform, so that

$$L_n = \int_0^{2\pi} \frac{d\sigma^1}{2\pi} e^{in\sigma^1} L \tag{4.21}$$

$\tilde{L}_n$ is similarly defined. The only way that L and $\tilde{L}$ can be 0 is if $L_n = 0$ and $\tilde{L}_n = 0$ for all n. The difference $L_0 - \tilde{L}_0$ will give the level-matching condition, just as it does in Minkowski space.

$$\sum_j \sum_n \frac{n + C_j}{|n + C_j|} n \left(\alpha_n^j \alpha_n^{j\dagger} + \alpha_n^{j\dagger} \alpha_n^j - \tilde{\alpha}_n^j \tilde{\alpha}_n^{j\dagger} - \tilde{\alpha}_n^{j\dagger} \tilde{\alpha}_n^j \right) = 0 \tag{4.22}$$

The zero modes do not appear in the level-matching condition, just as the center of mass motion of the closed string did not appear in the condition on Minkowski space. For the purposes of identifying physical states, it can often be helpful to separate the sides of the absolute value. The condition then looks like

$$\sum_j \sum_{n>-C} n\alpha_n^{j\dagger} \alpha_n^j - \sum_j \sum_{n<-C} n\alpha_n^j \alpha_n^{j\dagger} = \sum_j \sum_{n>-C} n\tilde{\alpha}_n^{j\dagger} \tilde{\alpha}_n^j - \sum_j \sum_{n<-C} n\tilde{\alpha}_n^j \tilde{\alpha}_n^{j\dagger} \tag{4.23}$$

Unlike in Minkowski space, there can be negative contributions to either side of the level-matching condition by taking $C > 1$ and $-C < n < 0$.

The mass-shell condition is more complicated here because of the metric. What we are really aiming for is an understanding of the momenta, not simply the first term of the sum of the $_{++}$ and $_{--}$ constraints. We will, however, use that sum in order to determine $\partial_0 X^-$

which is a necessary part of the last remaining component of the momentum.

$$L+\tilde{L}=-\frac{(\alpha' p^+)^2}{2}\sum_i \mu_i X_i^2+\partial_+\vec{X}\cdot\partial_+\vec{X}+\partial_-\vec{X}\cdot\partial_-\vec{X}-\alpha' p^+\partial_0 X^-=0 \tag{4.24}$$

4.5 Momenta

In order to study the spectrum of the bosonic string, we will need to know all of the momenta. Taking the derivative of the action in the standard way gives the following general form.

$$\begin{aligned}P_\mu &= \frac{\delta\mathcal{L}}{\delta\partial_0 X^\mu}\\ &= \frac{1}{2\pi\alpha'}\left(G_{\mu\nu}\frac{1}{2}(\partial_+ +\partial_-)X^\nu+B_{\mu\nu}\frac{1}{2}(\partial_- -\partial_+)X^\nu\right)\\ &= \frac{1}{4\pi\alpha'}\left(G_{\mu\nu}\partial_0 X^\nu - B_{\mu\nu}\partial_1 X^\nu\right)\end{aligned} \tag{4.25}$$

The first two momenta will not require us to do any work. While calculating the commutators, we already found the transverse momenta Eq. (4.10). Although the dependence on the worldsheet coordinates was not always written out, what we really have is $P_i(\sigma^0,\sigma^1)$ and not the net momentum of the whole string, $P_i=\int d\sigma^1 P_i^2$. In general we can integrate out σ^1 in order to get the net momentum of the string. For the transverse momenta we can trivially see that all of the oscillators are removed except for the zero-modes, demonstrating that these terms do in fact describe the center of mass motion of the string. The first of the light cone momenta is just as simple to find.

$$\begin{aligned}P_-(\sigma^0,\sigma^1) &= -\frac{1}{4\pi\alpha'}\partial_0 X^+\\ &= \frac{-p^+}{4\pi}\end{aligned} \tag{4.26}$$

$$P_- = \frac{-p^+}{2} \tag{4.27}$$

The final component of the momentum, P_+, requires a longer computation. Eq. (4.25) gives us that

$$P_+=\frac{1}{4\pi\alpha'}\left(-\sum_i \mu_i X_i^2\partial_0 X^+ - \partial_0 X^- - 2\sum_j \sqrt{\mu_j}X^{2j+1}\partial_1 X^{2j}\right) \tag{4.28}$$

We have not calculated any quantities involving X^- before, but fortunately we find it appearing with a worldsheet time derivative so that we can substitute the "mass-shell" condition in order to express P_+ in terms of only the transverse oscillators. Eq. (4.24) tells us that

$$-\partial_0 X^- = \frac{\alpha' p^+}{2}\mu_i X_i^2 - \frac{1}{2\alpha' p^+}\left((\partial_0 X_i)^2 + (\partial_1 X_i)^2\right) \tag{4.29}$$

So after replacing $\alpha' p^+ = \frac{C_i}{\sqrt{\mu_i}}$, P_+ now becomes

$$P_+ = \frac{1}{4\pi\alpha'}\left(-\sum_i \frac{C_i}{2}\sqrt{\mu_i}X_i^2 - \sum_i \frac{1}{2C_i}\sqrt{\mu_i}\left((\partial_0 X_i)^2 + (\partial_1 X_i)^2\right) - 2\sum_j \sqrt{\mu_j}X_{2j+1}\partial_1 X_{2j}\right) \tag{4.30}$$

Now I will expand P_+ in terms of the mode expansion for X^i. After integrating out the σ^1 coordinate in order to find the conserved light cone momentum of the whole string, all terms which alter the state, such as $\alpha_n^\dagger \tilde{\alpha}_m^\dagger$, will be removed as they will either come with non-zero integers in the σ^1 exponential or cancel algebraically from the expression. What is left is the quadratic terms involving only one raising operator and its corresponding lowering operator. The first few steps involve some very long expressions, but once the integral is done, what is left is

$$\begin{aligned} P_+ = &-\sum_n \sum_j \frac{\sqrt{\mu_j}}{4C_j}\frac{n^2 + C_j^2 + (n+C_j)^2 + 2nC_j}{|n+C_j|}\left(\alpha_n^j \alpha_n^{j\dagger} + \alpha_n^{j\dagger}\alpha_n^j + \tilde{\alpha}_n^j \tilde{\alpha}_n^{j\dagger} + \tilde{\alpha}_n^{j\dagger}\tilde{\alpha}_n^j\right) \\ &-\sum_i \frac{\sqrt{\mu_i}}{2}\frac{C_i}{|C_i|}\left(\alpha_0^i \alpha_0^{i\dagger} + \alpha_0^{i\dagger}\alpha_0^i\right) \end{aligned} \tag{4.31a}$$

$$\begin{aligned} = &-\sum_n \sum_j \frac{\sqrt{\mu_j}}{2}\frac{|n+C_j|}{C_j}\left(\alpha_n^j \alpha_n^{j\dagger} + \alpha_n^{j\dagger}\alpha_n^j + \tilde{\alpha}_n^j \tilde{\alpha}_n^{j\dagger} + \tilde{\alpha}_n^{j\dagger}\tilde{\alpha}_n^j\right) \\ &-\sum_i \frac{\sqrt{\mu_i}}{2}\frac{C_i}{|C_i|}\left(\alpha_0^i \alpha_0^{i\dagger} + \alpha_0^{i\dagger}\alpha_0^i\right) \end{aligned} \tag{4.31b}$$

As with the level-matching condition, it will be useful to separate the first term into two parts depending on the sign of $n + C_j$. With this separation, we will also be able to distinguish creation and annihilation operators, so we can normal order them by including the appropriate commutators.

$$P_+ = -\sum_{n+C>0}\sum_j \frac{\sqrt{\mu_j}}{C_j}(n+C_j)\left(\alpha_n^{j\dagger}\alpha_n^j + \tilde{}\right) + \sum_{n+C<0}\sum_j \frac{\sqrt{\mu_j}}{C_j}(n+C_j)\left(\alpha_n^j\alpha_n^{j\dagger} + \tilde{}\right)$$
$$-\frac{1}{2}\sum_j \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\sqrt{\mu_j}}{C_j}|n+C_j|\left(\left|\left[\alpha_n^j, \alpha_n^{j\dagger}\right]\right| + \tilde{}\right) - \frac{1}{2}\sum_i \sqrt{\mu_i}\frac{C_i}{|C_i|}\left|\left[\alpha_0^i, \alpha_0^{i\dagger}\right]\right|$$
$$-\sum_i \sqrt{\mu_i}\frac{C_i}{|C_i|} : \alpha_0^{i\dagger}\alpha_0^i : \quad (4.31c)$$

The commutators are known from Eq. (4.14) and Eq. (4.16), and can be used to further simplify this expression. The final commutator term matches the missing $n = 0$ term in the first commutator sum, once the tilde and non-tilde commutators have been added together and the factor of $\frac{1}{2}$ has been included to change the sum over even indices to a sum over all indices. It can then be included simply by summing over all integers in the first commutator sum. This gives

$$P_+ = -\sum_{n+C>0}\sum_j \frac{\sqrt{\mu_j}}{C_j}(n+C_j)\left(\alpha_n^{j\dagger}\alpha_n^j + \tilde{}\right) + \sum_{n+C<0}\sum_j \frac{\sqrt{\mu_j}}{C_j}(n+C_j)\left(\alpha_n^j\alpha_n^{j\dagger} + \tilde{}\right)$$
$$-\sum_j \sum_{n=-\infty}^{\infty} \frac{\sqrt{\mu_j}}{C_j}|n+C_j| - \sum_i \sqrt{\mu_i}\frac{C_i}{|C_i|} : \alpha_0^{i\dagger}\alpha_0^i : \quad (4.31d)$$

The $|n + C_j|$ which appears is just the absolute value of the oscillator frequency, and $\frac{\sqrt{\mu_j}}{C_j} = \frac{1}{\alpha' p^+}$. With this in mind, that term simply becomes an appropriately normalized ζ function. The standard method of determining the normal-ordering constant has appeared in a natural way. We define

$$A_i = \frac{1}{4}\sum_{n=-\infty}^{\infty} |\omega_n| \quad (4.32)$$

This is almost exactly how would define the constant in Minkowski space. The difference is that there we only had to sum over positive n, but we performed the sum for each transverse direction. If we wish to write expressions with A_i summed over every transverse direction, then we need to include an extra factor of $\frac{1}{2}$ in the definition, as I have done. This is in addition to the factor of $\frac{1}{2}$ that we include in order to split the single term into both A_i

and $\tilde{A}_i$, giving the overall factor of $\frac{1}{4}$. The final momentum is now

$$P_+ = -\sum_{n+C>0}\sum_j \frac{\sqrt{\mu_j}}{C_j}(n+C_j)\left(\alpha_n^{j\dagger}\alpha_n^j + \tilde{}\right) + \sum_{n+C<0}\sum_j \frac{\sqrt{\mu_j}}{C_j}(n+C_j)\left(\alpha_n^j\alpha_n^{j\dagger} + \tilde{}\right) \\ -\sum_i \frac{A_i+\tilde{A}_i}{\alpha' p^+} - \sum_i \sqrt{\mu_i}\frac{C_i}{|C_i|} : \alpha_0^{i\dagger}\alpha_0^i : \quad (4.33)$$

The normal ordering constant, A_i and $\tilde{A}_i$, is an infinite sum, so its value can be ambiguous. The most common way to calculate it is with a Hurwitz ζ function, which tells us that

$$\sum_{n=0}^{\infty}(n+c) = -\frac{1}{12}+\frac{c}{2}-\frac{c^2}{2} \quad (4.34)$$

With this we can split A_i into two pieces and then add them back together. I will break C into an integer part, $\lfloor C \rfloor$, and a fractional part, $\{C\}$, so that $C = \lfloor C \rfloor + \{C\}$. Now the normal ordering constant is

$$A_i = \tilde{A}_i = \frac{1}{4}\left(\sum_{n>-C_i}(n+C_i) - \sum_{n<C_i}(n+C_i)\right) \quad (4.35a)$$

$$= \frac{1}{4}\left(\sum_{n=0}^{\infty}(n+\{C_i\}) - \sum_{n=-\infty}^{-1}(n+\{C_i\})\right) \quad (4.35b)$$

$$= \frac{1}{4}\left(\sum_{n=0}^{\infty}(n+\{C_i\}) + \sum_{n=0}^{\infty}(n-\{C_i\})+\{C_i\}\right) \quad (4.35c)$$

$$= \frac{1}{4}\left(-\frac{1}{12}+\frac{\{C_i\}}{2}-\frac{\{C_i\}^2}{2}-\frac{1}{12}-\frac{\{C_i\}}{2}-\frac{\{C_i\}^2}{2}+\{C_i\}\right) \quad (4.35d)$$

$$= -\frac{1}{24}+\frac{\{C_i\}(1-\{C_i\})}{4} \quad (4.35e)$$

$$= -\frac{1}{24}+\frac{C_i}{4}-\frac{\lfloor C_i \rfloor}{4}-\frac{\{C_i\}^2}{4} \quad (4.35f)$$

Eq. (4.35e) is a compact form that makes the $C \to -C$ symmetry plain. As we make this change, the fractional parts $\{C\}$ and $1-\{C\}$ are interchanged, and A_i remains the same. We see immediately from this that as C_i approaches any integer A_i will become the constant on Minkowski space. Because we have not yet examined the string in the special case where $C_i \in \mathbb{Z}$, I will not make any claims about the spectrum at these points here, but the limit is clearly Minkowski space. For the specific case where $\mu_i = 0$, this background is Minkowski space so it is reassuring to see this constant in the limit as $C_i \to 0$. The alternate form, Eq. (4.35f), will be useful for matching the graviton in the next section. It also presents the $\alpha' \to 0$ limit in a simple form.

4.6 States

We will be most interested in the graviton on this space, so I will now work out P_+ on a state which has been excited twice in addition to an arbitrary number of zero-mode excitations. This will create all possible states with two indices, so the graviton must be among them. The states in question look like

$$\alpha^{j_1}_{N_1} \alpha^{j_2}_{N_2} \prod_i \left(\alpha^i_0\right)^{n_i} |p^+\rangle \tag{4.36}$$

I have neglected both $\tilde{}$ and † notations for the time being. Level-matching allows some combinations where neither or both operators have a tilde, and since it does not affect P_+, any tildes can be added as needed in order to construct physical states. Since for $n + C < 0$ the roles of creation and annihilation operators are switched, the raising operators in the state above should be taken to mean α_N if $N + C < 0$ and $\alpha^\dagger_N$ if $N + C > 0$. With the details now take care of, we act on Eq. (4.36) with the P_+ operator of Eq. (4.33) and find

$$P_+ = -\frac{\sqrt{\mu_{j_1}}}{C_{j_1}} |N_{j_1} + C_{j_1}| - \frac{\sqrt{\mu_{j_2}}}{C_{j_2}} |N_{j_2} + C_{j_2}| - \sum_i \frac{A_i + \tilde{A}_i}{\alpha' p^+} - \sum_i \sqrt{\mu_i} \frac{C_i}{|C_i|} n_i \tag{4.37}$$

From here on I will drop the $\frac{C}{|C|}$ in the last term and simply assume that $C_i > 0$. This is a reasonable assumption because of the definition of C in terms of basic quantities.

Since we want to compare P_+ to graviton states, it can help to add and subtract $\frac{2}{\alpha' p^+}$ and pull a factor of C_i out of the constant A_i and rewrite this as

$$P_+ = -\frac{\sqrt{\mu_{j_1}}}{C_{j_1}} \left(|N_{j_1} + C_{j_1}| - 1\right) - \frac{\sqrt{\mu_{j_2}}}{C_{j_2}} \left(|N_{j_2} + C_{j_2}| - 1\right) - \sum_i \sqrt{\mu_i} \left(n_i + \frac{1}{2}\right) - \frac{2 + A + \tilde{A}}{\alpha' p^+} \tag{4.38}$$

Now the $\left(n + \frac{1}{2}\right)$ has been separated off to match form of the solutions to the wave operator on the space, Eq. (5.8), and what is left over in the final term is a slightly different ordering constant.

$$A' = \tilde{A}' = -1 - \frac{1}{4}\sum_i \left(\{C_i\}^2 + \lfloor C_i \rfloor\right) \tag{4.39}$$

If we wish to see what happens to states with more than two excitations, as we will briefly later on, we will need to alter this formula. It is trivial to see that for every new excitation we will only need to add a term of $-\frac{|N+C|}{\alpha' p^+}$ with the appropriate N and C. The formula of Eq. (4.37) thus provides a clear template for an arbitrary state. Of course, in order for these states to be physical, they must also obey the level-matching condition, Eq. (4.23).

4.7 BRST

In [4] the bosonic string was quantized using the BRST formalism, which allowed the normal ordering constant and dimension of the space to be determined exactly. The calculation was only performed, however, for the special case where $0 < \alpha' p^+ \mu < 1$ and only one pair of μ_i was non-zero. In this section I will use the mode expansion and commutators of that paper and rederive the Virasoro anomaly without the assumption that $0 < \mathcal{C} < 1$. The result can then easily be extended to include more curved directions. The reason for the use of that paper's notation is to remind us that we are no longer working in the light-cone gauge, and the commutators and Virasoro operators can be properly found in the reference. The result will exactly match the ζ function regularization.

Here the relevant mode expansion of the complex field $Z = X^2 + iX^3$ is

$$Z(\sigma^+, \sigma^-) = e^{-i\mu\tilde{X}^+}\left(f(\sigma^+) + g(\sigma^-)\right) \tag{4.40a}$$

$$f(\sigma^+) = \sqrt{\alpha'} \sum_{N=-\infty}^{\infty} \frac{A_N}{\sqrt{|N - \mathcal{C}|}} e^{-i(N-\mathcal{C})\sigma^+} \tag{4.40b}$$

$$g(\sigma^-) = \sqrt{\alpha'} \sum_{N=-\infty}^{\infty} \frac{B_N}{\sqrt{|N + \mathcal{C}|}} e^{-i(N+\mathcal{C})\sigma^-} \tag{4.40c}$$

Recall Eq. (4.8a) where we found an expression for this same Z in the light cone gauge, except for a factor of 2. I have used $\mathcal{C} = \alpha' p^+ \mu$ to simplify the notation somewhat, as this is the dimensionless constant equivalent to the C that I have been using. Clearly the oscillator modes α_n^1 and $\tilde{\alpha}_n^1$ have been replaced by the modes A_N and B_N. The non-zero commutators for these modes are

$$\left[A_M, A_N^\dagger\right] = \text{sgn}(M-\mathcal{C})\delta_{MN} \tag{4.40d}$$

$$\left[B_M, B_N^\dagger\right] = \text{sgn}(M+\mathcal{C})\delta_{MN} \tag{4.40e}$$

This exactly matches our result. Neglecting the ghosts for the moment, the Virasoro constraints with this mode expansion are

$$\tilde{L}_n^X = \int_0^{2\pi} \frac{d\sigma}{2\pi\alpha'} e^{in\sigma} \left(-2 : \partial_+ X^+ \partial_+ X_0^- : + : \partial_+ \bar{f} \partial_+ f : + \mu \partial_+ X^+ J + : \partial_+ X^k \partial_+ X^k :\right) \tag{4.40f}$$

$$L_n^X = \int_0^{2\pi} \frac{d\sigma}{2\pi\alpha'} e^{-in\sigma} \left(-2 : \partial_- X^+ \partial_- X_0^- : + : \partial_- \bar{g} \partial_- g : - \mu \partial_+ X^+ J + : \partial_- X^k \partial_- X^k :\right) \tag{4.40g}$$

The only other result we will need from the paper is

$$J = \alpha' \sum_{n=-\infty}^{\infty} \left(\text{sgn}(N-\mathcal{C}) : A_N^\dagger A_N : + \text{sgn}(N+\mathcal{C}) : B_N^\dagger B_N :\right) \tag{4.40h}$$

is a part of the rather complicated mode expansion for X^- which is not important for our purposes. With the definitions from this paper now close at hand, we can begin to calculate the commutator of the Virasoro constraints.

We will separate off the part of the Virasoro operator which corresponds to the Z and $\bar{Z}$ coordinates, as the rest will be unchanged by considering $\mathcal{C} > 1$. Expanding these in terms of A_N and B_N and performing the integral over σ leads to

$$\tilde{L}_n^f = \sum_N \frac{(N-\mathcal{C})(N+n-\mathcal{C})}{\sqrt{|N-\mathcal{C}||N+n-\mathcal{C}|}} : A_N^\dagger A_{N+n} : \tag{4.41a}$$

$$L_n^g = \sum_N \frac{(N+\mathcal{C})(N+n+\mathcal{C})}{\sqrt{|N+\mathcal{C}||N+n+\mathcal{C}|}} : B_N^\dagger B_{N+n} : \tag{4.41b}$$

The next step is to determine the commutator of normal ordered pairs so that we can find $\left[\tilde{L}_n^f, L_m^g\right]$. The identity

$$[ab, cd] = a[b,c]d + ac[b,d] + [a,c]db + c[a,d]b \tag{4.42}$$

will allow us to simplify the problem. Because a constant will commute with anything, the normal ordering is irrelevant in the commutator. This gives

$$\Big[: A_a^\dagger A_b :, : A_c^\dagger A_d :\Big] = \Big[A_a^\dagger A_b, A_c^\dagger A_d\Big] \tag{4.43a}$$

$$= A_a^\dagger A_d \delta_{bc} \operatorname{sgn}(b - \mathcal{C}) - A_c^\dagger A_b \delta_{ad} \operatorname{sgn}(a - \mathcal{C}) \tag{4.43b}$$

The other two terms involve commutators with either both creation or both annihilation operators, so they have been dropped. The first term is normal ordered unless $b = c$ and $a = d < \mathcal{C}$. The second term is the same except that a and d are interchanged with c and b, and if all four indices are equal the two will cancel. As a result, we can write this in terms of normal ordered pairs and add a commutator in the special cases where it is necessary. These cases can be identified by the Heaviside step function Θ. Now we get

$$\begin{aligned}\Big[: A_a^\dagger A_b :, : A_c^\dagger A_d :\Big] =&: A_a^\dagger A_d : \delta_{bc} \operatorname{sgn}(b - \mathcal{C}) - : A_c^\dagger A_b : \delta_{ad} \operatorname{sgn}(a - \mathcal{C}) \\ &- \delta_{ad}\delta_{bc} \operatorname{sgn}(a - \mathcal{C}) \operatorname{sgn}(b - \mathcal{C}) \left(\Theta(\mathcal{C} - a) - \Theta(\mathcal{C} - b)\right)\end{aligned} \tag{4.44}$$

We can now use this to find the Virasoro commutators. Clearly the part of $\tilde{L}$ which does not involve the A_N or B_n oscillators will commute with $\tilde{L}^f$, so the first thing to do is to verify that the J term in $\tilde{L}$ commutes with this as well. Using Eq. (4.40h) we find that

$$\Big[J, : A_N^\dagger A_{n+N} :\Big] = \alpha' \sum_M \Big(: A_M^\dagger A_{n+N} : \delta_{M,N} - : A_N^\dagger A_M : \delta_{M,n+N}\Big) = 0 \tag{4.45}$$

Now we can move on to the commutator of interest.

$$\begin{aligned}\Big[\tilde{L}_m^f, \tilde{L}_n^f\Big] = \sum_{N,M} & \frac{(N - \mathcal{C})(M - \mathcal{C})(N + n - \mathcal{C})(M + m - \mathcal{C})}{\sqrt{|N - \mathcal{C}||M - \mathcal{C}||N + n - \mathcal{C}||M + m - \mathcal{C}|}} \\ \cdot \Bigg(&: A_M^\dagger A_{N+n} : \delta_{M+m,N} \operatorname{sgn}(N - \mathcal{C}) - : A_N^\dagger A_{M+m} : \delta_{M,N+n} \operatorname{sgn}(M - \mathcal{C}) \\ &- \delta_{M,N+n}\delta_{0,n+m} \operatorname{sgn}(N - \mathcal{C}) \operatorname{sgn}(M - \mathcal{C}) \begin{cases} 1 & \text{if } M < \mathcal{C},\ N > \mathcal{C} \\ -1 & \text{if } M > \mathcal{C},\ N < \mathcal{C} \\ 0 & \text{otherwise} \end{cases} \Bigg)\end{aligned} \tag{4.46a}$$

$$
\begin{aligned}
&= \sum_N (N-\mathcal{C})\frac{(N-m-\mathcal{C})(N+n-\mathcal{C})}{\sqrt{|N-m-\mathcal{C}||N+n-\mathcal{C}|}} : A^\dagger_{N-m}A_{N+n} \\
&\quad - \sum_N (N+n-\mathcal{C})\frac{(N-\mathcal{C})(N+m+n-\mathcal{C})}{\sqrt{|N-\mathcal{C}||N+m+n-\mathcal{C}|}} : A^\dagger_N A_{N+m+n} : \\
&\quad - \delta_{n+m,0}\sum_N (N-\mathcal{C})(N+n-\mathcal{C})\begin{cases} 1 & n<\mathcal{C}-N<0 \\ -1 & n>\mathcal{C}-N>0 \\ 0 & \text{otherwise}\end{cases}
\end{aligned} \tag{4.46b}
$$

$$
\begin{aligned}
&= \sum_N ((N+m-\mathcal{C})-(N+n-\mathcal{C}))\frac{(N-\mathcal{C})(N+m+n-\mathcal{C})}{\sqrt{|N-\mathcal{C}||N+m+n-\mathcal{C}|}} : A^\dagger_N A_{N+m+n} : \\
&\quad + \delta_{n+m,0}\left(\sum_{\mathcal{C}-n<N<\mathcal{C}} (N-\mathcal{C})(N+n-\mathcal{C}) - \sum_{\mathcal{C}<N<\mathcal{C}+m} (N-\mathcal{C})(N-m-\mathcal{C})\right)
\end{aligned} \tag{4.46c}
$$

$$
\begin{aligned}
&= (m-n)\tilde{L}^f_{m+n} - \delta_{n+m,0}\left(\sum_{N=1}^{m}(N-\{\mathcal{C}\})(N-m-\{\mathcal{C}\})\right. \\
&\quad \left. - \sum_{N=1}^{-m}(N+\{\mathcal{C}\}-1)(N+m+\{\mathcal{C}\}-1)\right)
\end{aligned} \tag{4.46d}
$$

$$
\begin{aligned}
&= (m-n)\tilde{L}^f_{m+n} + \delta_{n+m,0}\left(\frac{1}{6}(m^3-m)+m\{\mathcal{C}\}(1-\{\mathcal{C}\})\right)\Theta(m) \\
&\quad + \delta_{n+m,0}\left(\frac{1}{6}(m^3-m)+m\{\mathcal{C}\}(1-\{\mathcal{C}\})\right)\Theta(-m)
\end{aligned} \tag{4.46e}
$$

$$
= (m-n)\tilde{L}^f_{m+n} + \delta_{n+m,0}\left(\frac{1}{6}(m^3-m)+m\{\mathcal{C}\}(1-\{\mathcal{C}\})\right) \tag{4.46f}
$$

The other commutator of interest is $[L^g_m, L^g_n]$ which will be identical since taking $\mathcal{C} \to -\mathcal{C}$ will just interchange $\{\mathcal{C}\}$ and $(1-\{\mathcal{C}\})$ in the result. The other commutators are just in flat space, and are not of interest here. The total result is then

$$
[L_m, L_n] = (m-n)L_{m+n} + \delta_{n+m,0}\left(\frac{D-26}{12}(m^3-m) + 2\left(a-1+\frac{1}{2}\{\mathcal{C}\}(1-\{\mathcal{C}\})\right)\right) \tag{4.47}
$$

Here the normal ordering constant a corresponds to $a=-A=-\tilde{A}$ in our notation. The commutator for $\tilde{L}_n$ is clearly the same. Requiring that the anomaly is 0 then corresponds to fixing the ordering constant. This tells us that the $D=26$ and that the normal ordering constant found by the ζ function regularization is correct. When more non-zero entries are included in μ_i the anomalies will simply add leading to a perfect match of Eq. (4.35e)

$$a = 1 - \frac{1}{4} \sum_i \{\mathcal{C}\} \left(1 - \{\mathcal{C}\}\right) \tag{4.48}$$

5 Gravitational Perturbations

In order to compare the string calculation to some known quantities, it helps to calculate classical wavefunctions with this metric. In this section I will solve the classical wave equations for massless scalar and vector fields. I will then solve the beta function for the wavefunction of a perturbation to the metric, the antisymmetric gauge field $B_{\mu\nu}$, and the dilaton. This will produce the graviton spectrum. Once the graviton spectrum is known in detail, it can be used to compare with the string solutions already found.

In order to match the string calculations, I will take the same form of the metric as I did in the previous section.

$$G_{\mu\nu} = \begin{pmatrix} -x^i \mu_{ij} x^j & -1 & & 0 \\ -1 & 0 & & \\ & & 1 & \\ & 0 & & \ddots \end{pmatrix} \tag{5.1}$$

We also make the same assumption about μ_{ij}, specifically that

$$\mu = \begin{pmatrix} \mu_1 & & & & & & \\ & \mu_1 & & & & & \\ & & \mu_2 & & & & \\ & & & \mu_2 & & & \\ & & & & \mu_3 & & \\ & & & & & \mu_3 & \\ & & & & & & \ddots \end{pmatrix} \tag{5.2}$$

The calculations of section 4 also require that

$$H_{+(2j)(2j+1)} = 2\sqrt{\mu_j} \tag{5.3}$$

This is not necessary in order to compute the spectrum of the gravitational perturbations, so I will leave H_{+ij} arbitrary for now. The assumption Eq. (5.3) will only be made at the end in order to compare with the string calculations. I will frequently raise and lower the transverse indices without worrying, since $G_{ij} = G^{ij} = \delta_{ij}$ for all transverse i and j. In this section there is no need to deal with more than one index pair at a time while treating them as a pair, so j will be just another index for all 24 transverse directions.

5.1 Scalar

If we look at the scalar wave equation in this background we get

$$\nabla^2 \varphi = \left(-2\partial_+\partial_- + \sum_{i=2}^{25} \mu_i x_i^2 \partial_-^2 + \vec{\partial} \cdot \vec{\partial} \right) \varphi = 0 \tag{5.4}$$

We define this operator to be $\Box$. In the $_{+/-}$ directions there are no complications and it is useful to perform a Fourier transform. In the transverse directions, however, the equation of motion has the form of an harmonic oscillator. I therefore assume a solution of the form

$$\varphi_{\vec{n}} = C_{\vec{n}} e^{-i(k_+ x^+ + k_- x^-)} \prod_i e^{-\frac{1}{2}a_i^2 x_i^2} \mathcal{H}_{n_i}(a_i x_i) \tag{5.5}$$

where $a_i = (\mu_i k_-^2)^{\frac{1}{4}}$, $C_{\vec{n}}$ is a constant, and $\mathcal{H}_n(x)$ are Hermite polynomials. In this basis, the transverse derivatives and x's act as linear combinations of raising and lowering operators on the components of $\vec{n}$ which index the Hermite polynomial. In particular,

$$x\mathcal{H}_n(x) = \frac{1}{2}\mathcal{H}_{n+1}(x) + n\mathcal{H}_{n-1}(x) \tag{5.6a}$$

$$\mathcal{H}_n'(x) = 2n\mathcal{H}_{n-1}(x) \tag{5.6b}$$

Of course taking a derivative of φ will take a derivative of the Hermite polynomial, and also bring down an x^i from the Gaussian, so it has both raising and lowering operator parts to it. Using these relationships, we can show that

$$\left(\sum_i \mu_i x_i^2 \partial_-^2 + \vec{\partial} \cdot \vec{\partial} \right) \varphi_{\vec{n}} = -\sum_i a_i^2 (2n_i + 1)\, \varphi_{\vec{n}} \tag{5.7}$$

Since this is an eigenfunction of the transverse part of the wave operator, $\Box$, it is obvious that $\varphi_{\vec{n}}$ is a solution of the scalar laplacian when its eigenvalue with respect to the light cone part of the wave operator is the negative of this. Solving, we find that the light cone momentum must satisfy

$$k_+ = \sum_i \sqrt{\mu_i} \left(n_i + \frac{1}{2} \right) \tag{5.8}$$

This provides a solution to the massless scalar wave equation.

5.2 Vector

Solving the wave equation for a massless vector field on this space does not advance our investigation of the graviton spectrum, but it is a simple case to demonstrate what a solution can look like on this space. The Maxwell equations for a massless vector field are $\nabla_\lambda \nabla^\lambda \xi^\mu - \nabla_\lambda \nabla^\mu \xi^\lambda = 0$. Since we are working in the light cone gauge, it helps to separate the $^+$ and $^-$ components. This gives

$$\Box \xi^+ = 0 \tag{5.9a}$$

$$\Box \xi^i = 2\mu_i x_i \partial_- \xi^+ + \partial_i \partial_\mu \xi^\mu \tag{5.9b}$$

$$\begin{aligned}(-\partial_+\partial_- + \vec{\partial}\cdot\vec{\partial})\xi^- = &\left(-\partial_+ + \sum_i \mu_i x_i^2 \partial_-\right)\left(\partial_+\xi^+ + \sum_j \partial_j \xi^j\right) \\ &+ 2\sum_i \mu_i x_i \left(\partial_-\xi^i - \partial_i \xi^+\right) - 2\,\mathrm{tr}(\mu)\xi^+\end{aligned} \tag{5.9c}$$

The last of these equations is not obviously in the form $\Box\xi^- = \ldots$, but we have not yet chosen a gauge for the field ξ^μ In Lorentz gauge, $\partial_\lambda \xi^\lambda = 0$, the equations simplify and take a more useful form.

$$\Box \xi^+ = 0 \tag{5.10a}$$

$$\Box \xi^i = 2\mu_i x_i \partial_- \xi^+ \tag{5.10b}$$

$$\Box \xi^- = 2\sum_i \mu_i x_i \left(\partial_-\xi^i - \partial_i \xi^+\right) - 2\,\mathrm{tr}(\mu)\xi^+ \tag{5.10c}$$

The vector wave equation has now been written as sourced scalar wave equation for the each of the components. As a result, the most natural basis for the space of solutions is the $\varphi_{\vec{n}}$ because they are the eigenvalues of $\Box$. With this ansatz, the component equations reduce to a set of simple algebraic equations for the coefficients $C_{\vec{n}}$ in the definition of $\varphi_{\vec{n}}$. For a single fixed choice of $\vec{n}$ we find that $C_{\vec{n}}$ remains arbitrary, but we now require non-zero contributions from $C_{\vec{n}'}$ where $\vec{n}'$ has had certain indices shifted. The solutions are

$$\xi^{+}_{\vec{n}} = \varphi^{+}_{\vec{n}} \tag{5.11a}$$

$$\xi^{i}_{\vec{n}} = \varphi^{i}_{\vec{n}} + \frac{i}{2}\left(\frac{\mu_i}{k_-^2}\right)^{\frac{1}{4}} \varphi^{+}_{n_i+1} - in_i \left(\frac{\mu_i}{k_-^2}\right)^{\frac{1}{4}} \varphi^{+}_{n_i-1} \tag{5.11b}$$

$$\xi^{-}_{\vec{n}} = \varphi^{-}_{\vec{n}} + \sum_i \left(\sqrt{\frac{\mu_i}{k_-^2}}\left(-\frac{1}{4}\varphi^{+}_{n_i+2} + n_i(n_i-1)\varphi^{+}_{n_i-2}\right)\right. \tag{5.11c}$$

$$\left. +i\frac{1}{2}\left(\frac{\mu_i}{k_-^2}\right)^{\frac{1}{4}} \varphi^{i}_{n_i+1} - in_i \left(\frac{\mu_i}{k_-^2}\right)^{\frac{1}{4}} \varphi^{i}_{n_i-1}\right) \tag{5.11d}$$

Where $\varphi^{\mu}_{\vec{n}}$ are the solutions to the homogeneous component equations $\Box\varphi^{\mu}_{\vec{n}} = 0$, and φ_{n_i+l} has had the ith index raised by l. To be precise, $\varphi_{n_i+l} = \varphi_{(n_2,n_3,\ldots,n_i+l,\ldots,n_{25})}$. This solution then has the same spectrum as the scalar field, Eq. (5.8).

5.3 Graviton

To find the full graviton spectrum, we need to perturb the β function, including the Einstein equations, to linear order. The first order perturbations to the metric, antisymmetric gauge field B, and dilaton are $\gamma_{\mu\nu}$, $b_{\mu\nu}$, and ϕ respectively, and are all included despite any assumptions about the unperturbed fields. I will also sometimes use $h_{\mu\nu\rho} = \partial_\mu b_{\nu\rho} + \partial_\nu b_{\rho\mu} + \partial_\rho b_{\mu\nu}$ as a shorthand because it is the perturbation to the field strength, not the gauge field, which enters the equations. Finally, I will immediately choose the light cone gauge, $\gamma_{-\mu} = b_{-\mu} = 0$ in order to avoid extremely lengthy equations.

The background fields will eventually all be taken to be the same as when the real string spectrum was found. The unperturbed metric $G_{\mu\nu}$ is from Eq. (2.2), and the dilaton Φ is still a constant and does not enter any of the equations. For the moment I will allow a slightly more general form of the antisymmetric field $H_{\mu\nu\rho}$. From the Einstein equation we know that the only allowed non-zero components are H_{+ij}, so I will keep that assumption. I will also consider only the block-off-diagonal components where i is even and $j = i+1$. This will allow us to investigate the graviton behaviour when the string modes have imaginary frequencies, as well as the case in Eq. (4.1) which corresponds to stable string states.

The β functions expanded to first order in the perturbations yield the following equations.

$$\beta^G_{\mu\nu} = \delta R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - \frac{1}{4}H_{\mu\lambda\rho}h_\nu{}^{\lambda\rho} - \frac{1}{4}H_{\nu\lambda\rho}h_\mu{}^{\lambda\rho} - \frac{1}{2}\gamma^{\lambda\sigma}H_{\mu\lambda\rho}H_{\nu\sigma}{}^{\rho} = 0 \tag{5.12a}$$

$$\beta^B_{\mu\nu} = -\frac{1}{2}\nabla^\lambda h_{\lambda\mu\nu} - \frac{1}{2}\gamma^{\lambda\rho}\nabla_\lambda H_{\rho\mu\nu} - \frac{1}{2}G^{\lambda\rho}\delta\nabla_\lambda H_{\rho\mu\nu} + \nabla^\lambda\phi H_{\lambda\mu\nu} = 0 \tag{5.12b}$$

$$\beta^\Phi = -\frac{1}{2}\nabla^2\phi - \frac{1}{12}H^{\mu\nu\rho}h_{\mu\nu\rho} - \frac{1}{8}\gamma^{\lambda\rho}H^{\mu\nu}{}_{\rho}H_{\mu\nu\lambda} = 0 \tag{5.12c}$$

Where the perturbation to the Ricci tensor and covariant derivative are given by

$$\delta R_{\mu\nu} = -\frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu\nabla_\nu\gamma_{\rho\sigma} + \nabla_\rho\nabla_\sigma\gamma_{\mu\nu} - 2\nabla_\rho\nabla_{(\mu}\gamma_{\nu)\sigma}\right) \tag{5.13a}$$

$$\delta\nabla_\lambda H_{\mu\nu\rho} = -\left(\delta\Gamma^\sigma{}_{\lambda\mu}H_{\sigma\nu\rho} + \delta\Gamma^\sigma{}_{\lambda\nu}H_{\mu\sigma\rho} + \delta\Gamma^\sigma{}_{\lambda\rho}H_{\mu\nu\sigma}\right) \tag{5.13b}$$

$$\delta\Gamma^\sigma{}_{\lambda\mu} = \frac{1}{2}\gamma^{\sigma\rho}\left(\partial_\mu G_{\rho\lambda} + \partial_\lambda G_{\rho\mu} - \partial_\rho G_{\lambda\mu}\right) + \frac{1}{2}G^{\sigma\rho}\left(\partial_\mu\gamma_{\rho\lambda} + \partial_\lambda\gamma_{\rho\mu} - \partial_\rho\gamma_{\lambda\mu}\right) \tag{5.13c}$$

All of the covariant derivatives ∇_μ in Eq. (5.12) and Eq. (5.13a) are with respect to the unperturbed metric $G_{\mu\nu}$. Assuming that the boundary conditions are such that we can do a fourier transform in the $_{+/-}$ directions as we did for the scalar and vector wave equations, we can replace $\partial_\pm \to -ik_\pm$ any time we want. Since the two are equivalent, I will leave the derivatives alone until ready to read off the spectrum k_+.

In order to solve these β functions, it will be necessary to compute the form of the different components separately. Each component equation $\beta^G_{\mu\nu}$ has a $\Box\gamma_{\mu\nu}$ term in it, but if either index is $-$, then $\gamma_{\mu\nu} = 0$ and the equation does not involve the $\Box$ operator. Such equations are interpreted as constraints imposed by the light-cone gauge. The same is true for the equations of motion for b and ϕ. These constraints fix the dilaton ϕ and all of the components of γ and b with a $+$ index in terms of the purely transverse degrees of freedom γ_{ij} and b_{ij}.

$$\beta^G_{--} = 2\partial_-^2\phi - \frac{1}{2}\partial_-^2\gamma_i{}^i = 0 \tag{5.14a}$$

$$\beta^G_{-i} = -\frac{1}{2}\partial_-^2\gamma_{+i} + \frac{1}{2}\partial_-\left(\partial_j\gamma^{ij} - \partial_i(\gamma_j{}^j - 4\phi)\right) = 0 \tag{5.14b}$$

$$\beta^G_{+-} = -\frac{1}{2}\partial_-^2\gamma_{++} + \frac{1}{2}\partial_-\left(\partial_i\gamma_{+i} - \partial_+(\gamma_i{}^i - 4\phi) - \frac{1}{2}H_+{}^{ij}b_{ij}\right) = 0 \tag{5.14c}$$

$$\beta^B_{-i} = -\frac{1}{2}\partial_-^2 b_{+i} - \frac{1}{2}\partial_-\partial_j b_{ij} = 0 \tag{5.14d}$$

There is still one remaining constraint equation from β^B_{+-} but it is redundant. Next I look at the purely transverse β_{ij} equations. By imposing the constraints to remove all non-transverse degrees of freedom, they can be simplified to the following.

$$\beta^G_{ij} = -\Box\gamma_{ij} + \partial_- b_{ik} H_{+j}{}^k + \partial_- b_{jk} H_{+i}{}^k = 0 \tag{5.15a}$$

$$\beta^B_{ij} = -\Box b_{ij} + \partial_- \gamma_{ik} H_{+j}{}^k - \partial_- \gamma_{jk} H_{+i}{}^k = 0 \tag{5.15b}$$

The remaining β function components, β^G_{++}, β^G_{+i}, β^B_{+i}, and β^Φ are all identically zero once the constraints and transverse wave equations are solved. In order to verify this, it is necessary to remember that ∂_i and $\Box$ do not commute when further simplifying the equations. I will not demonstrate that calculation here, but it simply shows that the wave equations for the $_{+\mu}$ components are satisfied once the constraints determine those components in term of transverse degrees of freedom which also satisfy their own wave equations.

For the on-block components such as $i = 2$, $j = 3$, we find that the linear combinations β^G_{23}, $\beta^G_{22} - \beta^G_{33}$, and $\beta^G_{22} + \beta^G_{33} \pm 2i\beta^B_{23}$ yield a simple uncoupled system.

$$-\Box\gamma_{23} = 0 \tag{5.16a}$$

$$-\Box\left(\gamma_{22} - \gamma_{33}\right) = 0 \tag{5.16b}$$

$$-\left(\Box \pm iH_{+23}\partial_-\right)\left(\gamma_{22} + \gamma_{33} \pm 2ib_{23}\right) = 0 \tag{5.16c}$$

This immediately tells us that of the four degrees of freedom in the block, two have the plain scalar spectrum Eq. (5.8) and two have a shifted spectrum given by

$$k_+ = \sum_k \sqrt{\mu_k}\left(n_k + \frac{1}{2}\right) \pm H_{+23} \tag{5.17}$$

The off-block components are a similar but algebraically longer story. Here there are four indices to considered, two in each block, so rather than referring to them as i and j, I will work out the solution for the first two blocks, $(2,3)$ and $(4,5)$, and any other pair can be solved for in an identical manner. As for the previous calculation, there are linear combinations of β functions which diagonalize the equations. Here they are

$$\beta^G_{24} + \beta^G_{35} \pm i\left(\beta^B_{25} - \beta^B_{34}\right) = 0 \tag{5.18a}$$

$$\beta^G_{24} - \beta^G_{35} \pm i\left(\beta^B_{25} + \beta^B_{34}\right) = 0 \tag{5.18b}$$

The combinations lead to the uncoupled systems

$$-\left(\Box \pm \frac{i}{2}\left(H_{+23} + H_{+45}\right)\partial_-\right)\left(\gamma_{24} + \gamma_{35} \pm i\left(b_{25} - b_{34}\right)\right) = 0 \tag{5.19a}$$

$$-\left(\Box \pm \frac{i}{2}\left(H_{+45} - H_{+23}\right)\partial_-\right)\left(\gamma_{24} - \gamma_{35} \pm i\left(b_{25} + b_{34}\right)\right) = 0 \tag{5.19b}$$

I have only used half of the degrees of freedom in order to make these combinations. The other half will have an identical structure, only the roles of γ and b will be switched. At first glance this seems like twice as many degrees of freedom as we should have, but these are the same equations that we will find from the situation where the pairs of indices are reversed, so there should indeed be 8 tensor components for the whole set.

The result of all of this is that for a tensor with indices i_1 and i_2, $\gamma_{i_1 i_2}$ for example, there are four possible shifts to the spectrum.

$$k_+ = \sum_i \sqrt{\mu_i}\left(n_i + \frac{1}{2}\right) \pm \frac{1}{2}H_{i_1} \pm \frac{1}{2}H_{i_2} \tag{5.20}$$

where H_i is the component of H_{+ij} in the direction of i, where the indices are in increasing order. In the case of section 3 where H was entirely in the first two directions, this means that only the states with an index in the 2 or 3 direction will have any shift at all. The rest will have the scalar spectrum. Neither one will be complex, however. The classical gravitational theory has avoided the instability of the string states. When $H_{\mu\nu\rho}$ is defined as in section 4 then the result of Eq. (5.20) is that

$$k_+ = \pm\sqrt{\mu_{i_1}} \pm \sqrt{\mu_{i_2}} + \sum_i \sqrt{\mu_i}\left(n_i + \frac{1}{2}\right) \tag{5.21}$$

If both indices come from the same pair, then the two $\sqrt{\mu_i}$ are the same and two of the states will have no shift in the spectrum. Since this corresponds to the choice of H which gives a stable string theory, this is the result which we want to compare to the string calculations.

6 Comparison

With this general form now complete, we want to see if the predictions of string theory and classical gravity are consistent. We saw in Eq. (4.38) that the spectrum involves an absolute value function, so we will have to consider separate cases depending on how the value of C compares to the integers used to excite the state.

6.1 $|C| < 1$

The first of these cases is when C is small. Here we can take the state

$$\alpha_{\pm 1}^{j_1} \alpha_{\pm 1}^{j_2} \prod_i \left(\alpha_0^i\right)^{n_i} |p^+\rangle \tag{6.1}$$

Here I have again omitted the † which should appear on the α_{+1} terms because the sign of $N + C$ is positive, and used j to represent an index over the pairs rather than all transverse indices. For this state Eq. (4.38) reduces to

$$P_+ = \mp\sqrt{\mu_{j_1}} \mp \sqrt{\mu_{j_2}} - \sum_i \sqrt{\mu_i}\left(n_i + \frac{1}{2}\right) - \frac{2 + A' + \tilde{A}'}{\alpha' p^+} \tag{6.2}$$

This is a perfect match for the classical gravity result if we let $A' = \tilde{A}' = -1$. Since we are dealing with A' rather than the A of Eq. (4.35), this should correspond to Eq. (4.39). With the assumption that $|C| < 1$, this result is nearly a perfect match. The only discrepancy is the $\{C\}^2$ term. This term can be legitimately ignored in the $\alpha' \to 0$ limit, in which the string theory becomes an effective theory for the low energy states. The classical calculation certainly does not include any higher mass or higher rank particles, so $\alpha' = 0$ is indeed the limit in which the classical calculation was performed. Including a non-zero α' in the classical theory would require a more complete definition of the β function than the one considered here.

6.2 Non-zero α'

While the gravitational calculation is valid only in the $\alpha' = 0$ limit, the non-interacting string theory is valid for all α'. The next step in comparing the two theories is to look at the non-zero α' regime of the gravitational calculation. For this, we need to consider the $O(\alpha'^2)$ corrections to the β function. Rather than trying to solve the whole thing here, I will only demonstrate that the sort of term produced by the next order in the equations is exactly the sort of term needed to match the $\{C\}^2$ term on the string side.

The dimensionless expansion parameter in the β function is $\frac{\alpha'}{R_c^2}$ where R_c is the radius of curvature of the space. R_c is a purely geometric quantity, so without any calculations, we know that $R_c \sim \frac{1}{\sqrt{\mu}}$, as μ is the only dimensionful parameter in the metric. This is the only way that R_c can have dimensions of length. This means that the β function has an added term $O\left((\alpha'\mu)^2\right)$. When we take two derivatives, we can bring down a factor of k^2, so the change to the classical equations of motion will look something like $\alpha'\mu k^2$, which is the same as $Ck\sqrt{\mu}$. The whole thing now looks like

$$k_+k_- \sim k\sqrt{\mu} + Ck\sqrt{\mu} \tag{6.3}$$

The added term is of the correct form to contribute the $\frac{C^2}{\alpha' p^+}$ needed in order to match Eq. (4.39) in the $0 < C < 1$ regime. For larger C, the higher order contributions to the classical equations of motion certainly can't be ignored, as they will be the same order of magnitude as the terms considered. The perturbative approach we have taken to the β functions will fail when C is not small, and the classical calculation will be completely invalid.

6.3 $C > 1$

When we try to make this comparison for $C > 1$, the two fail to match. This is not at all a surprise, as in the previous section we found that the form of the β function used on the classical side is only valid for very small C. Here we find that $\alpha_{-1}|p^+\rangle$ is no longer a physical state because $-1 + C > 0$. Attempting to use $\alpha_{-1}^\dagger |p^+\rangle$ will contribute an extra string scale term and the term to match the classical graviton spectrum will have the wrong sign. Instead of $-\frac{1}{\alpha' p^+} + \sqrt{\mu}$, we will get $\frac{1}{\alpha' p^+} - \sqrt{\mu}$. There are, of course, many other states which we can attempt to use, so it is helpful to see what contribution each individual excitation will make to the P_+ spectrum. I will use m to denote the largest integer smaller than C, so $m = \lfloor C \rfloor$. With n an arbitrary positive integer, we can work out the contributions to level-matching and P_+ from an excitation by α_N^j.

$$\begin{array}{ccc} N & \text{LM} & P_+ \\ & & \\ 1 & 1 & -\frac{1}{\alpha' p^+} - \sqrt{\mu_j} \\ -1 & -1 & \frac{1}{\alpha' p^+} - \sqrt{\mu_j} \\ -m & -m & -\sqrt{\mu_j} + \frac{m}{\alpha' p^+} \\ -m+n & -m+n & -\sqrt{\mu_j} - \frac{-m+n}{\alpha' p^+} \\ -m-n & m+n & \sqrt{\mu_j} - \frac{m+n}{\alpha' p^+} \end{array} \tag{6.4}$$

While we still have the $-\sqrt{\mu_j}$ contribution from the state $\alpha_1^{j\dagger}$, its partner, $\sqrt{\mu_j}$ is missing from the list unless we allow an extra string scale term.

If we choose to follow a single state, for example the graviton state $\alpha_1^\dagger \alpha_{-1} |p^+\rangle$ as it crosses the $C = 1$ threshold and picks up a † on the second oscillator, we see that P_+ reaches a maximum at $C = 1$ and then starts becoming negative and more massive again. As this happens, the $N = -1$ state will begin mimicking the behaviour of the $N = 1$ state from the $0 < C < 1$ range. The spectrum will be exactly the same over every integer of the domain for C, we simply have to take $N \to N - 1$ every time C crosses an integer.

6.3.1 Alternative ordering constants

If we did not know from two independent sources what the ordering constant A' should be, we might try using different values to see if classical gravity can survive for $C > 1$. Here I will demonstrate some things that can happen in this case. By a creative choice of A' and $\tilde{A}'$ we can match the graviton spectrum, but this will raise other issues besides the obvious violation of the BRST result from section 4.7.

I will define m and m' so that $m - 1 < C < m$ for the largest of all $\frac{D-2}{2}$ independent C_i and $A' = \tilde{A}' = -(m + m')$. Now the state

$$\alpha^{j_1}_{\pm(m+m')} \alpha^{j_2}_{\pm(m+m')} \prod_i \left(\alpha_0^i\right)^{n_i} |p^+\rangle \tag{6.5a}$$

is a physical state once the appropriate † notations are added, and it has the spectrum

$$P_+ = \mp\sqrt{\mu_{j_1}} \mp \sqrt{\mu_{j_2}} - 2\frac{m+m'}{\alpha' p^+} - \frac{A' + \tilde{A}'}{\alpha' p^+} - \sum_i \sqrt{\mu_i}\left(n_i + \frac{1}{2}\right) \tag{6.5b}$$

Here we see that we can match the graviton spectrum for A' and $\tilde{A}'$ any integer greater than the values of C_i for the polarizations we want to match. To make all polarizations valid, we only need to choose the largest of the C_i when we are defining m. There are infinitely many ways to do this for $m' \in \mathbb{Z}^+$ so we would like to know which one is correct. At first glance, $m' = 0$ seems the most likely as it becomes $A' = \tilde{A}' = 1$ when $|C| < 1$, matching the case of section 6.1. In fact, none of these solutions are very good. Take for example the state

$$\alpha^{j_3\dagger}_{m+m'}\tilde{\alpha}^{j_3}_{-m-m'}\alpha^{j_1\dagger}_{1}\alpha^{j_2\dagger}_{-1}\left|p^+\right\rangle \tag{6.6a}$$

which satisfies level matching and has

$$P_+ = -2\frac{m+m'}{\alpha' p^+} - \sqrt{\mu_{j_1}} - \sqrt{\mu_{j_2}} - \frac{A'+\tilde{A}'}{\alpha' p^+} - \sum_i \frac{\sqrt{\mu_i}}{2} \tag{6.6b}$$

$$= -\sqrt{\mu_{j_1}} - \sqrt{\mu_{j_2}} - \sum_i \frac{\sqrt{\mu_i}}{2} \tag{6.6c}$$

This is independent of the string scale α' and is even identical to one of the graviton states, but it cannot be a graviton because it has four stringy excitations and three independent indices. If we take this state in the case where all μ_i are equal, then this state can have four independent indices, yet it still behaves like a graviton in terms of its energy. In principle, it should be possible to produce states independent of the string scale with an arbitrarily large number of free indices. This is not a property we want to see in our spectrum, as it means there are infinitely many light particles. This provides even more evidence that we should not require the two theories to match for all C, and that the normal ordering constant found both by the ζ function and the BRST calculation is correct rather than a larger value. When we do not alter the constant, some of the graviton polarizations will pick up a string scale term in the mass in this region, but this is an acceptable result. It makes predictions about the physics at scales where the momentum is already string scale, so classical gravity cannot be expected to remain unchanged.

6.4 $C = 1$

For this case, we must go all the way back to the classical equations of motion Eq. (4.3). If we use the same solution as before, Eq. (4.7), the four terms with $n = -1$ are really only two terms, and variation of parameters is needed to restore the others. The terms look like $e^{\pm i\sigma^1}$ and $\sigma^0 e^{\pm i\sigma^1}$. For the sake of generality, I will consider the $n = -C$ terms where $C \in \mathbb{Z}$. The previous four $n = -C$ terms in the solution are replaced by

$$X^{2j} = X^{2j}_{n\neq -C} + i\sqrt{\frac{\alpha'}{2}}\left(\alpha^j_{-C}e^{-iC\sigma^1} - \alpha^{j\dagger}_{-C}e^{iC\sigma^1}\right) + i\sqrt{\frac{\alpha'}{2}}\sigma^0\left(\tilde{\alpha}^j_{-C}e^{-iC\sigma^1} - \tilde{\alpha}^{j\dagger}_{-C}e^{iC\sigma^1}\right) \tag{6.7a}$$

$$X^{2j+1} = X^{2j+1}_{n\neq -C} + i\sqrt{\frac{\alpha'}{2}}\left(i\alpha^j_{-C}e^{-iC\sigma^1} + i\alpha^{j\dagger}_{-C}e^{iC\sigma^1}\right) + i\sqrt{\frac{\alpha'}{2}}\sigma^0\left(i\tilde{\alpha}^j_{-C}e^{-iC\sigma^1} + i\tilde{\alpha}^{j\dagger}_{-C}e^{iC\sigma^1}\right) \tag{6.7b}$$

The $\frac{1}{\sqrt{n+C_j}}$ normalization has had to be removed because it becomes infinite. The numerical factor in the normalization isn't very important because it will be linear combinations of these operators which give the new creation and annihilation operators. Since $P_i \propto \partial_0 X^i$, the non-tilde oscillators do not contribute to P_i at all. The transverse momenta are

$$P^{2j} = P^{2j}_{n\neq -C} + \frac{i}{4\pi\alpha'}\sqrt{\frac{\alpha'}{2}}\left(\tilde{\alpha}^j_{-C}e^{-iC\sigma^1} - \tilde{\alpha}^{j\dagger}_{-C}e^{iC\sigma^1}\right) \tag{6.8a}$$

$$P^{2j+1} = P^{2j+1}_{n\neq -C} + \frac{i}{4\pi\alpha'}\sqrt{\frac{\alpha'}{2}}\left(i\tilde{\alpha}^j_{-C}e^{-iC\sigma^1} + i\tilde{\alpha}^{j\dagger}_{-C}e^{iC\sigma^1}\right) \tag{6.8b}$$

We can find the oscillator amplitudes in terms of X and P just as we did in section 4.2.

$$\tilde{\alpha}^j_{-n} = -i\sqrt{2\alpha'}\int_0^{2\pi} d\sigma^1 e^{in\sigma^1}\left(P^{2j} - iP^{2j+1}\right) \tag{6.9a}$$

$$\tilde{\alpha}^{j\dagger}_{-n} = i\sqrt{2\alpha'}\int_0^{2\pi} d\sigma^1 e^{-in\sigma^1}\left(P^{2j} + iP^{2j+1}\right) \tag{6.9b}$$

$$\alpha^j_{-n} = \frac{-i}{2\pi\sqrt{2\alpha'}}\int_0^{2\pi} d\sigma^1 e^{in\sigma^1}\left(X^{2j} - iX^{2j+1}\right) - \sigma^0\tilde{\alpha}^j_{-n} \tag{6.9c}$$

$$\alpha^{j\dagger}_{-n} = \frac{i}{2\pi\sqrt{2\alpha'}}\int_0^{2\pi} d\sigma^1 e^{-in\sigma^1}\left(X^{2j} + iX^{2j+1}\right) - \sigma^0\tilde{\alpha}^{j\dagger}_{-n} \tag{6.9d}$$

Since all P commute, we will trivially find $\left[\tilde{\alpha}_{-n}, \tilde{\alpha}^\dagger_{-n}\right] = 0$. The fact that the X commute will then give $\left[\alpha_{-n}, \alpha^\dagger_{-n}\right] = 0$. At this point it is clear that the creation-annihilation operator pairs will not be the same as the operator-conjugate pairs. With $\left[X^{i_1}, P^{i_2}\right] = i\delta^{i_1 i_2}\delta(\sigma^1 - \sigma^{1\prime})$, we find that the only non-zero commutators here are

$$\left[\alpha^{j_1}_{-n}, \tilde{\alpha}^{j_2\dagger}_{-n}\right] = \left[\alpha^{j_1\dagger}_{-n}, \tilde{\alpha}^{j_2}_{-n}\right] = 2i\delta^{j_1 j_2} \tag{6.10}$$

We can see from the procedure of section 4.3 that in order to get a state with a positive norm we want an operator $\mathcal{A}$ with the property that $\left[\mathcal{A}, \mathcal{A}^\dagger\right] = 1$. The correct linear combinations are

$$\mathcal{A}^j = \frac{\alpha^j_{-n} + i\tilde{\alpha}^j_{-n}}{2} \tag{6.11a}$$

$$\tilde{\mathcal{A}}^j = \frac{\alpha^{j\dagger}_{-n} + i\tilde{\alpha}^{j\dagger}_{-n}}{2} \tag{6.11b}$$

With these we find $\left[\mathcal{A}, \mathcal{A}^\dagger\right] = \left[\tilde{\mathcal{A}}, \tilde{\mathcal{A}}^\dagger\right] = 1$ and all other commutators are 0. This gives $\left\|\mathcal{A}^\dagger \left|0\right\rangle\right\| = \left\|\tilde{\mathcal{A}}^\dagger \left|0\right\rangle\right\| = 1$, and $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are now the annihilation operators.

The contribution that the new terms in the mode expansion for X make in the level-matching condition is

$$\frac{iC}{2}\left(\tilde{\alpha}\alpha^\dagger + \alpha^\dagger\tilde{\alpha} - \tilde{\alpha}^\dagger\alpha - \alpha\tilde{\alpha}^\dagger\right) = C\left(\mathcal{A}\mathcal{A}^\dagger + \mathcal{A}^\dagger\mathcal{A} - \tilde{\mathcal{A}}\tilde{\mathcal{A}}^\dagger - \tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}}\right) \tag{6.12}$$

and the contribution to P_+ has the form

$$-\frac{1}{2}\frac{\sqrt{\mu}}{C}\tilde{\alpha}^\dagger\tilde{\alpha} = -\frac{1}{2}\frac{\sqrt{\mu}}{C}\left(\mathcal{A}\mathcal{A}^\dagger + \tilde{\mathcal{A}}^\dagger\tilde{\mathcal{A}} - \mathcal{A}\tilde{\mathcal{A}} - \tilde{\mathcal{A}}^\dagger\mathcal{A}^\dagger\right) \tag{6.13}$$

The level matching appears very reasonable since it is the symmetrized number operator that appears in the level-matching condition for $C \notin \mathbb{Z}$. P_+, however, no longer has these physical states as eigenstates. This suggests that these states which have 0 energy at integer values of C are causing problems for the theory. This is certainly to be expected, and explains why little work has been done in this case.

7 Conclusions

The closed bosonic string on the pp-wave can be solved exactly in the light-cone gauge as long as interactions are neglected. There is no longer a center of mass motion of the form $\alpha' p^i \sigma^0$ in the transverse directions, as it is replaced by the zero mode terms $e^{\pm iC\sigma^0}$. These describe the center of mass orbits about the origin of the coordinates, as the metric takes the form of an harmonic well. In order to satisfy conformal invariance, a source is required for the curvature, and we have chosen an antisymmetric tensor field. This field also interacts with the strings, in particular acting on the worldsheet momentum excitations. The result of both backgrounds is typically a string for which at least some of the oscillations have imaginary frequencies, suggesting that the strings are stretched to infinity much in the same way as an open string in a strong electric field. In the special case when the two fields, H and the metric, are perfectly balanced in each direction so that $H_{+\ 2j\ 2j+1} = 2\sqrt{\mu_j}$ for j over all pairs of indices with equal μ in the metric, then the mode expansion becomes similar to that of Minkowski space. The difference is that the frequencies have all been shifted by $C = \alpha' p^+ \sqrt{\mu}$.

In this case I have quantized the theory, identified creation and annihilation operators, and found the spectrum of P_+ for an arbitrary state. The level-matching condition is now more complicated than on Minkowski space because there are states which contribute negative amounts, so that combinations like $\alpha_1^\dagger \alpha_{-1}^\dagger |p^+\rangle$ can be allowed. Which states contribute negative amounts depends on the sign of $n + C$, as does the identification of the creation operators with $\alpha^\dagger$ and $\tilde{\alpha}^\dagger$. The momentum component P_+ has an absolute value factor which ensures that it will be negative definite, except for a tachyonic part from the normal ordering constant, $A + \tilde{A}$. This constant's value can be predicted by the use of the ζ function regularization, and this is confirmed by the BRST quantization performed in [4]. The constant is similar to on Minkowski space, but the differences are important. The result involves the fractional part of C, suggesting that it could not be obtained using perturbation theory for α'.

Classical gravity on this background gives wavefunctions which have a simple basis in harmonic oscillator eigenstates. Working in this basis, we can find the energy of a scalar, vector, and graviton. All of the different graviton polarizations can potentially have different energies, but they can all be found with the correct basis of states and linear combinations of components. It turns out that the classical result exactly matches the $\alpha' \to 0$ limit of string theory, demonstrating that the low energy states of string theory again reproduce the gravitational theory just as they should. If we want to consider the first order corrections in α', the equations of motion for the classical calculation become much more difficult. Without performing any calculations, however, we can see that we would expect to find an extra factor of C in the first order corrections. This is exactly what is needed to match the string theory result Eq. (4.37) as long as $C < 1$. Beyond this limit, we cannot expect the low energy theory to match string theory, as many string scale massive states will become important. The fact that the classical graviton spectrum matches the low energy limit of the free string theory so well supports strings as a valid high energy theory.

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