Hamiltonian Treatment of the (2+1)-dimensional Yang-Mills Theory

by

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Abstract

The (2+1)-dimensional Yang-Mills theory is studied in the functional Schrödinger formalism — using the machinery laid out by Karabali and Nair. The low-lying spectrum of the theory is computed by analyzing correlators of the Leigh-Minic-Yelnikov ground-state wavefunctional in the Abelian limit. The contribution of the WZW measure is treated by a controlled approximation and the resulting spectrum is shown to reduce to that obtained by Leigh et al., at large momentum. The inclusion of fundamental Fermions is done from first-principles, and it is found that the requirement of gauge invariance spoils the commutativity between gauge and matter fields.

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Dedication

To my wife, Kelley.

Chapter 1

Background

1.1 Introduction

The study of non-Abelian gauge theories is complicated by the fact that by restricting oneself to states in the Hilbert space that are gauge invariant, or, alternatively, representatives of equivalence classes of states related by gauge transformations, one almost necessarily introduces interactions that complicate calculations. In perturbative treatments gauge invariance is achieved at the expense of the form of the bare propagator of the gauge-boson, or the introduction of ghosts, while in functional treatments Gauss' law must either be imposed as a constraint, or solved implicitly through a change of variables [1], as is also the case in lattice calculations. Manifestly gauge invariant calculations [2–5] may also be done in the Effective Renormalization Group formalism [6–8] which necessitates the introduction of heavy regularization machinery. Finally, there exist studies of the anisotropic version of weakly-coupled (2+1)-dimensional Yang-Mills theory in which one dimension is descretized, replacing the continuum theory with infinitely many (integrable) chiral sigma models [9, 10].

One promising solution to the constraint problem is the use of Wilson-line variables [1], for example in the work by Karabali and Nair on (2+1)-dimensional Yang-Mills theory [11]. The strength of their approach is that the field variables are encoded into a single variable along with a reality condition that allows it to be treated holomorphically, opening up the rich structure of complex analysis. Recent calculations by Leigh, Minic and Yelnikov based on this work have also yielded a candidate ground-state wave-functional as well as approximate analytical predictions for the $J^{PC} = 0^{++}$ and $J^{PC} = 0^{--}$ glueball masses [12] which are in good agreement with lattice calculations. However, the effect of the non-trivial configuration space measure, i.e., the WZW action, was omitted from calculations without sufficient justification.

In the following section we begin by reviewing the Hamiltonian quantization of Yang-Mills theory. We then discuss the use of Wilson-line variables to obtain manifestly gauge-invariant degrees of freedom. Subsequently, we review the formalism of Karabali and Nair, and outline the procedure used by Leigh et al. to obtain the vacuum wave-functional and glueball spectrum.

In part II we attempt a conservative approximation of the glueball spectrum that incorporates the WZW functional measure by expanding relevant operators about the Abelian limit. We then compare our results to that of [12] and analyze the robustness of the solution.

In part III we develop the techniques necessary for including fundamental matter using the functional Schrödinger picture. We show that in order for gauge-invariant Fermionic fields to be consistent with the overall theory, the matter and gauge fields fail to commute (even classically), and therefore that the transformation is non-canonical. Finally, the feasibility of this approach is discussed.

1.2 Yang-Mills theory

1.2.1 "Old" Canonical Quantization

We will begin with a review of the the "old"[1] canonical treatment of Yang-Mills theory. Specifically, we will study $SU(N)$ Yang-Mills theory in the Hamiltonian formalism, and denote the gauge potential by $A_i = -iA_i^a t^a$, $i = 1 \dots D$, where the $N \times N$ Hermitian matrices t^a generate the $\mathfrak{su}(N)$ Lie algebra $[t^a, t^b] = if^{abc}t^c$ and we choose $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$, while the gauge covariant derivative in the fundamental representation is $D_i = \partial_i + A_i$. The classical

[1]By "old" we mean that the modern BRST formalism is not used to implement first-class constraints

action of the theory is then

$$S_{YM} = -\frac{1}{2g_{YM}^2}\int d^3x \;\mathrm{tr}\, F^{\mu\nu}F_{\mu\nu}, \tag{1.1}$$

where $F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the field strength. The action can be written in a form that simplifies the process of obtaining the Hamiltonian, by by treating $F_{\mu\nu}$ as an auxiliary variable [13]:

$$S[A,F] \to -\frac{1}{g_{YM}^2}\int d^3x \;\mathrm{tr}\left[\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - F^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right)\right]. \tag{1.2}$$

"Integrating out" $F_{\mu\nu}$ shows that it equals $[D_\mu, D_\nu]$, which can be substituted into the action (1.2) to obtain (1.1). More precisely, that what we are doing is quantum mechanically sound can be seen by considering the ordinary (Euclidean) integral $\int dF dy e^{-\frac{1}{2}F^2 - iFY(y)} = \sqrt{2\pi}\int dy e^{-\frac{1}{2}Y(y)^2} = \sqrt{2\pi}\int dF dy e^{-\frac{1}{2}F^2 - iFY(y)}\delta(F + iY(y))$. Defining $E_i = F_{0i}$ and $B = \frac{1}{2}\epsilon_{ij}F_{ij}$ (forshadowing the restriction to 2 dimensions), the action becomes

$$\begin{aligned} S &= -\frac{1}{g_{YM}^2}\int d^3x \;\mathrm{tr}\left[B^2 + E_iE_i - 2E_i\left(\partial_0 A_i - \partial_i A_0 + [A_0, A_i]\right)\right], \\ &= -\frac{1}{g_{YM}^2}\int d^3x \;\mathrm{tr}\left[B^2 + E_iE_i - 2E_i\dot{A}_i - 2A_0\left(\partial_i E_i + [A_i, E_i]\right)\right], \end{aligned} \tag{1.3}$$

after integration by parts. Taking the trace gives

$$S = \frac{1}{g_{YM}^2}\int d^3x \left[-\frac{1}{2}\left(B^aB^a + E_i^aE_i^a\right) + E_i^a\dot{A}_i^a + A_0^a\left(D_iE_i\right)^a\right]. \tag{1.4}$$

This shows that E_i is the canonically conjugate momentum to A_i, and we shall take the equal-time Poisson bracket to be $\left[A_i^a(x), E_j^b(y)\right]_{PB} = \delta_{ij}\delta^{ab}\delta^{(2)}(x-y)$. Canonical quantization now proceeds by obtaining the canonical Hamiltonian and making the replacement $[\,,\,]_{PB} \to i\hbar\,[\,,\,]$. The Hamiltonian is then

$$H = T + V = \int d^2x \frac{g_{YM}^2}{2}E_i^aE_i^a + \frac{1}{2g_{YM}^2}B^aB^a - \frac{1}{g_{YM}^2}A_0^a\left(D_iE_i\right)^a. \tag{1.5}$$

In the Hamiltonian formalism the component A_0 is used up as a Lagrange multiplier in enforcing the Gauss' law constraint $\mathbf{D} \cdot \mathbf{E} = 0$, and is subsequently ignored. Thus at this point we have yet to fix a gauge, so that the theory is invariant under (time independent) gauge transformations.

1.2.2 Gauge-independent degrees of freedom

It is well known that from knowledge of all the holonomies of the gauge connection we can recover A_μ uniquely up to a gauge transformation [14]. Define the path-ordered phase $U(P_{x,y})$ as the phase accumulated by parallel transporting some field ψ along a smooth path from $y \to x$ due to the connection A_μ. Infinitesimally,

$$D_\mu \psi(x) \epsilon^\mu \equiv [\psi(x+\epsilon) - U(P_{x+\epsilon,x})\psi(x)] \tag{1.6}$$

Therefore $U(P_{x+\epsilon,x}) = 1 - \epsilon^\mu A_\mu + O(\epsilon^2)$, so that

$$U(P_{x,y}) \equiv \mathcal{P} e^{-\int_y^x dx^\mu A_\mu}. \tag{1.7}$$

where the integral is taken along a curve joining y and x, and $\mathcal{P}$ denotes the path-ordering of the exponential [13, 15]. $U(P_{x,y})$ depends on the path joining y to x, owing to the curvature of the connection. By definition, $U(P_{x,y})\psi(y)$ and $\psi(x)$ transform the same way under a gauge transformation, so that the path-ordered phase transforms bi-locally: $U(P_{x,y}) \to g(x)U(P_{x,y})g^{-1}(y)$. Additionally, the path-ordered phases give an $N \times N$ matrix representation of the holonomy group under multiplication, since

$$U(P_{x,y})U(Q_{y,z}) = U(P_{x,y} \circ Q_{y,z}). \tag{1.8}$$

Given an *open* path-ordered phase (i.e., Wilson line), we can recover A_μ as long dx^μ does not vanish along the curve at x. Let $S_{x,y}$ be a straight line segment joining y to x.

Then

$$A_\mu(x) = -\frac{\partial}{\partial x^\mu} U(S_{y+x,y})|_{y=x}. \tag{1.9}$$

To reconstruct the connection from holonomies around *closed* loops (i.e., Wilson loops), first consider the effect of moving around a square spanned by $\epsilon \mathbf{e}^1$ and $\epsilon \mathbf{e}^2$, labeled $C_x \equiv P_{x,x}$ (Figure 1.1)

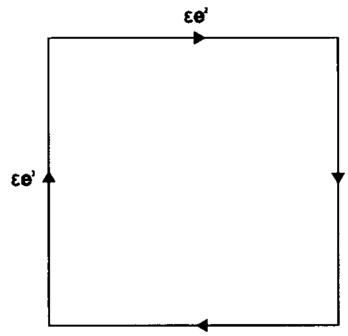

Figure 1.1: Closed contour $P_{x,x}$ supporting Wilson loop

$$\begin{aligned}
U(C_x) &= U(x, x+\epsilon\mathbf{e}^2) U(x+\epsilon\mathbf{e}^2, x+\epsilon\mathbf{e}^1+\epsilon\mathbf{e}^2) \\
&\times U(x+\epsilon\mathbf{e}^1+\epsilon\mathbf{e}^2, x+\epsilon\mathbf{e}^1) U(x+\epsilon\mathbf{e}^1, x), \\
&\approx \exp\left[\epsilon A_2(x) + \frac{\epsilon^2}{2} A_{2,2}(x)\right] \exp\left[\epsilon A_1(x) + \frac{\epsilon^2}{2} A_{1,1}(x) + \epsilon^2 A_{1,2}(x)\right] \\
&\times \exp\left[-\epsilon A_2(x) - \epsilon^2 A_{2,1}(x) - \frac{\epsilon^2}{2} A_{2,2}(x)\right] \exp\left[-\epsilon A_1(x) - \frac{\epsilon^2}{2} A_{1,1}(x)\right], \\
&= 1 + \epsilon^2 \left[A_{1,2}(x) - A_{2,1}(x) - A_1(x)A_2(x) + A_2(x)A_1(x)\right] + O(\epsilon^3).
\end{aligned}$$

Thus, knowledge of the holonomy around an infinitesimal loop gives the curvature of the connection:

$$U(C_x) = 1 - \epsilon^2 F_{12}(x) + O(\epsilon^3). \tag{1.10}$$

Knowing the curvature allows us to reconstruct the connection, up to gauge transformations. If A_μ is non-singular at the origin, then we can impose the coordinate gauge condition

$x^\mu A_\mu = 0$, from which $x^\mu F_{\mu\nu} = x^\mu \partial_\mu A_\nu - x^\mu \partial_\nu A_\mu = (1 + x^\mu \partial_\mu) A_\nu$, so that [16, 17]

$$\begin{aligned} \alpha x^\mu F_{\mu\nu}(\alpha x) &= (1 + \alpha x^\mu \partial_\mu) A_\nu(\alpha x) \\ &= \frac{d}{d\alpha} \left[\alpha A_\nu(\alpha x) \right], \end{aligned}$$

and

$$A_\mu(x) = \int_0^1 d\alpha\, \alpha x^\sigma F_{\sigma\mu}(\alpha x). \tag{1.11}$$

The phases $U(C_x)$ therefore encode all information about the field. It is conventional to define the object $W_C(x) \equiv \operatorname{tr} U(C_x)$, also called a Wilson loop, which is gauge invariant because of the cyclicity of the trace. It is then possible to reconstruct $U(C_x)$ from $W_C(x)$ modulo gauge transformations, although the construction is non-trivial [14]. For the time being, however, we will consider only open path-ordered phases. Note from equation (1.9) that we can also write

$$A_\mu = -\frac{\partial}{\partial x^\mu} U(S_{z+x,z} \circ S_{z,y} \circ S_{z,y}^{-1})|_{z=x}, \tag{1.12}$$

$$= -\frac{\partial}{\partial x^\mu} U(S_{z+x,y}) U^{-1}(S_{z,y})|_{z=x}. \tag{1.13}$$

The path $S_{x,y}$ is specific to the index being varied, so define $M_\mu(x) \equiv U(S_{(\mu)x,y})$ where $S_{(\mu)x,y}$ is a straight line segment directed along dx^μ. Then

$$A_\mu = -(\partial_\mu M_\mu) M_\mu^{-1}. \text{ (no summation)} \tag{1.14}$$

Note that we obtain a pure gauge if all the matrices M_μ are equal, corresponding to the vanishing curvature of the connection. Although the matrices M_μ transform bi-locally, the gauge transformations at the points y do not affect A_μ, as can be seen from equation (1.14). One may choose to set $y = \infty$, so that $g(y) = 1$; however, there is a symmetry that must be dealt with if the matrices $M_\mu \in G$ are to be considered as variables in their own right, *defined* by (1.14): A_μ is invariant under the replacement $M_\mu \to M_\mu V(x)$ provided that

$\partial_\mu V(x) = 0$.

1.3 Karabali-Nair formalism

1.3.1 2+1 Dimensions

In 2 spatial dimensions it is convenient to introduce the complex coordinates $z = x - iy$ and $\bar{z} = x + iy$. Correspondingly, $\partial \equiv \frac{\partial}{\partial z} = \frac{\partial x^i}{\partial z}\frac{\partial}{\partial x^i} = \frac{1}{2}(\partial_1 + i\partial_2)$ and $\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} = \frac{\partial x^i}{\partial \bar{z}}\frac{\partial}{\partial x^i} = \frac{1}{2}(\partial_1 - i\partial_2)$. In the Hamiltonian formalism, $A_0 = 0$, while the remaining two components of A_μ become $A \equiv A_z = \frac{1}{2}(A_1 + iA_2)$ and $\bar{A} \equiv A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2)$. In two dimensions, the magnetic field has only one independent component, given by $B = {}^*\left([D_i, D_j]\, dx^i \wedge dx^j\right)$, which becomes

$$
\begin{aligned}
B^a t^a &= \left(\partial_1 A_2^a - \partial_2 A_1^a + f^{abc} A_1^a A_2^b\right) t^a, \\
&= \left(i\partial_1 A_2 - i\partial_2 A_1 + [A_1, A_2]\right), \\
&= 2\frac{1}{4}\left(\partial_1 - i\partial_2\right)\left(A_1 + iA_2\right) - 2\frac{1}{4}\left(\partial_1 + i\partial_2\right)\left(A_1 - iA_2\right) + 2[\bar{A}, A], \\
&= 2\left(\bar{\partial} A - \partial \bar{A}\right) + 2[\bar{A}, A]
\end{aligned}
\tag{1.15}
$$

We can again define matrices M_z and $M_{\bar{z}}$ that satisfy

$$A = -(\partial M_z) M_z^{-1}, \tag{1.16}$$

$$\bar{A} = -(\bar{\partial} M_{\bar{z}}) M_{\bar{z}}^{-1}. \tag{1.17}$$

However, solving equation (1.16) iteratively, we obtain

$$M_z(\boldsymbol{r}) = 1 - \int_{-\infty}^{z} dz' A(z',\bar{z}) M(z',\bar{z}) \tag{1.18}$$

$$= 1 - \int_{-\infty}^{z} dz' A(z',\bar{z}) + \int_{-\infty}^{z} dz' A(z',\bar{z}) \int_{-\infty}^{z'} dz'' A(z'',\bar{z})... \tag{1.19}$$

$$= 1 - \int_{-\infty}^{z} dz' A(z',\bar{z}) + \frac{1}{2!}\mathcal{P} \int_{-\infty}^{z} dz' A(z',\bar{z}) \int_{-\infty}^{z} dz'' A(z'',\bar{z})... \tag{1.20}$$

$$= \mathcal{P} e^{-\int_{-\infty}^{z} dz' A(z',\bar{z})} \tag{1.21}$$

so that instead $M_z \in G^{\mathbb{C}}$, the complexification of G. The apparent over-abundance of degrees of freedom is resolved by the constraint that $A_{\bar{z}}$ be related to A_z by conjugation.

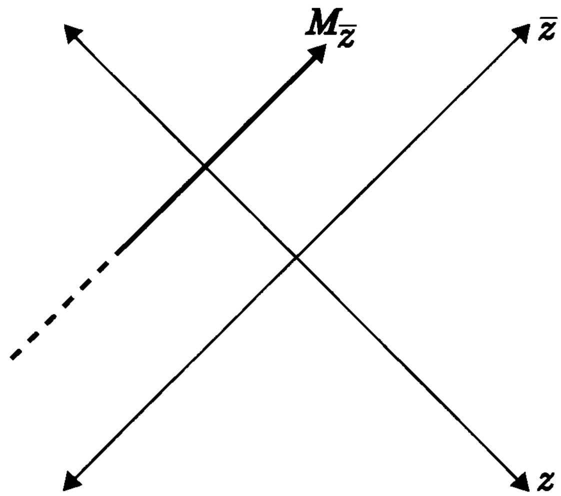

Figure 1.2: Complexified Wilson loop $M_{\bar{z}}$

The contour giving rise to $M_{\bar{z}}$ is shown in Figure 1.2, along with a schematic representation of the "lightcone coordinates" z and $\bar{z}$. For "small" gauge configurations the contour may be taken to extend to the point at infinity as indicated. If $G = SU(N)$ then the generators t^a are Hermitian, so that $\bar{A} = -A^\dagger = M_z^{\dagger-1}\bar{\partial}M_z^\dagger = -\bar{\partial}M_z^{\dagger-1}M_z^\dagger$. Therefore we have the constraint that $M_{\bar{z}} = M_z^{\dagger-1}$ and with this understanding we can drop the subscript on M_z. The matrix M is an element of $SL(N,\mathbb{C}) = G^{\mathbb{C}}$.

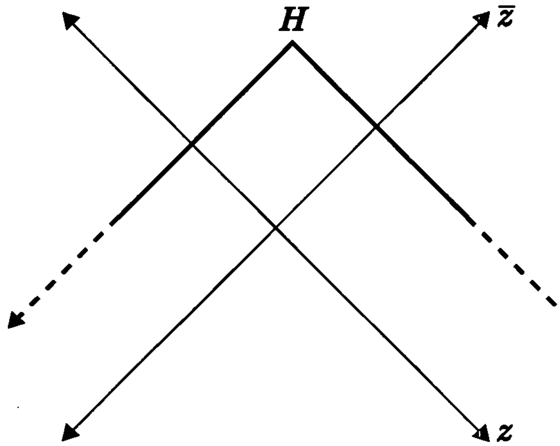

Figure 1.3: Complexified Wilson loop H

1.3.2 Gauge Symmetry

Under a gauge transformation, $M \to M^g = gM$, so that $H \equiv M^\dagger M$ is a gauge-invariant object. Any $SL(N,\mathbb{C})$ matrix M can be uniquely written[2] as $M = h\rho$, where

$$h \in SU(N),$$
$$\rho^\dagger = \rho,$$
$$\det \rho = 1.$$

The interpretation of H as a closed Wilson loop is shown in Figure 1.3. Since ρ is obtained from $H = M^\dagger M$, which is gauge invariant, all gauge information is contained within h. The matrix M is related to ρ by a constant gauge transformation. Therefore matrices M with the same ρ lie on the gauge orbit through ρ, and all gauge-independent information can be extracted from H. It is reasonable to expect that wave-functionals of physical states (which are gauge-invariant) can be written in terms of H.

[2]Specificically, $\rho = U^\dagger \sqrt{D} U$ where $D = U(M^\dagger M)U^\dagger$ is diagonal.

1.3.3 "Holomorphic" Symmetry

In addition to the gauge symmetry there is the invariance of A under the replacement $M \to M\bar{V}(\bar{z})$ and $M^\dagger \to V(z)M^\dagger$, discussed earlier. Here $V(z)$ and $\bar{V}(\bar{z})$ are independent holomorphic and anti-holomorphic function, respectively, and this symmetry is loosely termed *"holomorphic" invariance.* Under such a transformation, $H \to V(z)H\bar{V}(\bar{z})$, but physical operators and wave-functionals corresponding to physical states should be "holomorphically" invariant.

1.3.4 Inner product

In order to define an inner product between states we need to determine the measure $d\mu(\mathcal{C})$ on the configuration space $\mathcal{C}$, i.e., the space of gauge potentials $\mathcal{A}$ modulo the (small) gauge group $\mathcal{G}_*$. The metric on $\mathcal{A}$ is just the Euclidean one, i.e.,

$$ds^2_{\mathcal{A}} = \int dx\, \delta A^{a_i} \delta A^{a_i} = -8 \int \mathrm{tr}\,(\delta A_z \delta A_{\bar{z}}). \tag{1.22}$$

Recall the parameterization,

$$A = -\partial M M^{-1}, \tag{1.23}$$

$$\bar{A} = M^{\dagger -1} \bar{\partial} M^\dagger. \tag{1.24}$$

Now, using $\partial(MM^{-1}) = 0$,

$$\begin{aligned}
\delta A &= -(\partial \delta M)M^{-1} - (\partial M)\delta M^{-1}, \\
&= -\partial\left(\delta M M^{-1}\right) + \delta M \partial M^{-1} + \delta M M^{-1} \delta M M^{-1}, \\
&= -\partial\left(\delta M M^{-1}\right) - \delta M M^{-1}(\partial M)M^{-1} + (\partial M)M^{-1}\delta M M^{-1}, \\
&= -\partial\left(\delta M M^{-1}\right) + \left[A, \delta M M^{-1}\right], \\
&= -D\left(\delta M M^{-1}\right).
\end{aligned}$$

where $D = \partial + [A, \cdot]$ acts in the adjoint representation. Similarly,

$$\begin{aligned}
\delta \bar{A} &= \delta M^{\dagger -1} \bar{\partial} M^{\dagger} + M^{\dagger -1} \bar{\partial} \delta M^{\dagger}, \\
&= -M^{\dagger -1} \delta M^{\dagger} M^{\dagger -1} \bar{\partial} M^{\dagger} + \bar{\partial} \left(M^{\dagger -1} \delta M^{\dagger} \right) - \left(\bar{\partial} M^{\dagger -1} \right) \delta M^{\dagger}, \\
&= -M^{\dagger -1} \delta M^{\dagger} M^{\dagger -1} \bar{\partial} M^{\dagger} + \bar{\partial} \left(M^{\dagger -1} \delta M^{\dagger} \right) + M^{\dagger -1} \left(\bar{\partial} M^{\dagger} \right) M^{\dagger -1} \delta M^{\dagger}, \\
&= \bar{\partial} \left(M^{\dagger -1} \delta M^{\dagger} \right) - \left[\bar{A}, M^{\dagger -1} \delta M^{\dagger} \right], \\
&= \bar{D} \left(M^{\dagger -1} \delta M^{\dagger} \right),
\end{aligned}$$

so that

$$ds^2_{\mathcal{A}} = 8 \int \mathrm{tr} \left(D \left[\delta M M^{-1} \right] \bar{D} \left[M^{\dagger -1} \delta M^{\dagger} \right] \right). \tag{1.25}$$

On the other hand, the metric of $SL(N, \mathbb{C})$ is

$$ds^2_{SL(N,\mathbb{C})} = 8 \int \mathrm{tr} \left(\delta M M^{-1} \right) \left(M^{\dagger -1} \delta M^{\dagger} \right). \tag{1.26}$$

The Jacobian determinant is therefore $\det(D\bar{D})$, and

$$d\mu(\mathcal{C}) = \frac{d\mu(\mathcal{A})}{vol(\mathcal{G}_*)} = \det(D\bar{D}) \frac{d\mu(\mathcal{M})}{vol(\mathcal{G}_*)}, \tag{1.27}$$

Finally, from the discussion in section (1.3.2), the space of Hermitian matrices (H) with unit determinant is given by $\mathcal{H} = SL(N, \mathbb{C})/SU(N)$, so that

$$d\mu(\mathcal{H}) = \frac{d\mu(\mathcal{M})}{vol(\mathcal{G}_*)}, \tag{1.28}$$

and

$$d\mu(\mathcal{C}) = \det(D\bar{D}) d\mu(\mathcal{H}). \tag{1.29}$$

Up to complexification, $\det(D\bar{D}) = \det \not{D}$ where $\not{D} = \gamma^i D_i$ is the two-dimensional Dirac operator. This determinant was evaluated by Polyakov and Wiegmann [18] by investigating the chiral anomaly that occurs when coupling the gauge potential to Fermions in 2 dimensions. Instead, we may evaluate (and regularize) the determinant directly.

1.3.5 Evaluation of determinant

Define $e^{\Gamma} \equiv \det(D\bar{D})$ where D and $\bar{D}$ are in the adjoint representation. Considered as matrices, these have the form

$$D^{ab}_{\boldsymbol{x},\boldsymbol{x}'} \equiv D^{ab}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}'), \tag{1.30}$$

$$\bar{D}^{ab}_{\boldsymbol{x},\boldsymbol{x}'} \equiv \bar{D}^{ab}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}'), \tag{1.31}$$

where

$$D^{ab} = \delta^{ab}\partial + f^{acb}A^c. \tag{1.32}$$

Then

$$\Gamma = \ln\det(D\bar{D}) = \operatorname{tr}\ln(D\bar{D}). \tag{1.33}$$

One approach is to vary with respect to A^a and $\bar{A}^a$ separately, to obtain a functional differential equation for Γ. The solution to these equations gives Γ up to (divergent) $A^a, \bar{A}^a$-independent terms. Since $\delta\left(\ln\det M\right) = \operatorname{tr}\left(M^{-1}\delta M\right)$,

$$\frac{\delta\Gamma}{\delta A^a(\boldsymbol{x})} = \operatorname{tr}\left(D^{-1}\frac{\delta D}{\delta A^a(\boldsymbol{x})}\right), \tag{1.34}$$

$$= \int dr''\, dr'\, (D^{-1})^{bc}_{\boldsymbol{x}',\boldsymbol{x}''}\frac{\delta D^{cb}_{\boldsymbol{x}'',\boldsymbol{x}'}}{\delta A^a(\boldsymbol{x})}, \tag{1.35}$$

$$= \int dr''\, dr'\, (D^{-1})^{bc}_{\boldsymbol{x}',\boldsymbol{x}''} f^{abc}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}')\delta^{(2)}(\boldsymbol{x}''-\boldsymbol{x}'), \tag{1.36}$$

$$= f^{abc}(D^{-1})^{bc}_{\boldsymbol{x},\boldsymbol{x}'}|_{\boldsymbol{x}'\to\boldsymbol{x}}, \tag{1.37}$$

$$= -i\ \operatorname{tr}\left(D^{-1}_{\boldsymbol{x},\boldsymbol{x}'}T^a\right)|_{\boldsymbol{x}'\to\boldsymbol{x}}, \tag{1.38}$$

where $(T^a)^{bc} = -if^{abc}$ are the generators in the adjoint representation. We are therefore tasked with determining (and regularizing) $(D^{-1})^{bc}_{\boldsymbol{x},\boldsymbol{x}'}|_{\boldsymbol{x}'\to\boldsymbol{x}}$, i.e. solving

$$D^{ab}_{\boldsymbol{x},\boldsymbol{x}''}(D^{-1})^{bc}_{\boldsymbol{x}'',\boldsymbol{x}'} = \left(\delta^{ab}\partial_{\boldsymbol{x}} + A^{ab}(\boldsymbol{x})\right)\delta(\boldsymbol{x}-\boldsymbol{x}'')_{\boldsymbol{x},\boldsymbol{x}''}(D^{-1})^{bc}_{\boldsymbol{x}'',\boldsymbol{x}'}, \tag{1.39}$$

$$\stackrel{!}{=} \delta^{ac}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}'), \tag{1.40}$$

where $A^{ab} \equiv f^{acb}A^c$ (it is to be understood that repeated spatial variables are to be integrated over). Suppose $(D^{-1})^{bc}_{\boldsymbol{x},\boldsymbol{x}'} = M^{bd}_{\boldsymbol{x}}G_{\boldsymbol{x},\boldsymbol{x}'}(M^{-1})^{dc}_{\boldsymbol{x}'}$ for some yet to be determined matrix M. Then

$$\left(\delta^{ab}\partial_{\boldsymbol{x}} + A^{ab}(\boldsymbol{x})\right)M^{bd}_{\boldsymbol{x}}G_{\boldsymbol{x},\boldsymbol{x}'}(M^{-1})^{dc}_{\boldsymbol{x}'} \\ = \left((\partial M)^{ad}_{\boldsymbol{x}} + A^{ab}_{\boldsymbol{x}}M^{bd}_{\boldsymbol{x}}\right)G_{\boldsymbol{x},\boldsymbol{x}'}(M^{-1})^{dc}_{\boldsymbol{x}'} + \delta^{ac}\delta^{(2)}(\boldsymbol{x}-\boldsymbol{x}'), \tag{1.41}$$

which gives the requirement that

$$\partial M^{ab} + A^{ac}M^{cb} = 0. \tag{1.42}$$

Thus M^{ab} is the solution to $(\partial + A)M = 0$ in the adjoint representation. With our normalization convention for the Lie algebra, we obtain

$$M^{ab} = 2\,\mathrm{tr}\left(t^a M t^b M^{-1}\right), \tag{1.43}$$

$$(M^\dagger)^{ab} = 2\,\mathrm{tr}\left(t^a M^\dagger t^b M^{\dagger-1}\right). \tag{1.44}$$

Therefore $D^{-1}(\boldsymbol{x},\boldsymbol{x}') = M(\boldsymbol{x})G(\boldsymbol{x},\boldsymbol{x}')M^{-1}(\boldsymbol{x}')$, where it is to be understood that all objects are in the adjoint representation.

Regularization

The propagator, $D^{-1}(\boldsymbol{x}, \boldsymbol{x}')$, transforms bilocally, i.e.,

$$D^{-1}(\boldsymbol{x}, \boldsymbol{x}') \to g(\boldsymbol{x})D^{-1}(\boldsymbol{x}, \boldsymbol{x}')g^{-1}(\boldsymbol{x}'). \tag{1.45}$$

It is, however, holomorphically invariant. We want to consider the coincident limit $\bar{x}' \to \bar{x}$ and write $D^{-1}(\boldsymbol{x}, \boldsymbol{x})$ as a power series in $\bar{\epsilon} \equiv \bar{x} - \bar{x}'$ (the limit $x' \to x$ can be taken without difficulty). The coefficient $D_n^{-1}(\boldsymbol{x})$ of $\bar{\epsilon}^n$ transforms as $D_n^{-1}(\boldsymbol{x}) \to g(\boldsymbol{x})D_n^{-1}(\boldsymbol{x})g^{-1}(\boldsymbol{x}')$ meaning that it is not gauge covariant. Therefore, introduce the gauge covariant regulator

$$D_{cov}^{-1}(\boldsymbol{x}, \epsilon) = D^{-1}(\boldsymbol{x}, \epsilon)\Omega(\boldsymbol{x}', \boldsymbol{x}), \tag{1.46}$$

where

$$\Omega(\boldsymbol{x}', \boldsymbol{x}) \equiv \mathcal{P}\exp\left(-\int_{x}^{x'} d\boldsymbol{x} \cdot \mathbf{A}\right), \tag{1.47}$$

parallel-transports between the points $\boldsymbol{x}$ and $\boldsymbol{x}'$ in a guage-covariant manner. Then, let

$$\begin{aligned}
D_{cov}^{-1}(\boldsymbol{x}, \bar{\epsilon}) &= M(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{x}')M^{-1}(\boldsymbol{x}')e^{-\bar{A}(\bar{x}'-\bar{x})} + O(\bar{\epsilon}^2), \\
&= M(\boldsymbol{x})G(\boldsymbol{x}, \boldsymbol{x}')\left[M^{-1}(\boldsymbol{x}) - M^{-1}(\boldsymbol{x})\bar{\partial}M(\boldsymbol{x})M^{-1}(\boldsymbol{x})(\bar{x}' - \bar{x})\right] \times \\
&\left[1 + \bar{A}(\bar{x} - \bar{x}')\right] + O(\bar{\epsilon}^2), \\
&= G(\boldsymbol{x}, \boldsymbol{x}')\left[1 + \bar{\partial}M(\boldsymbol{x})M^{-1}(\boldsymbol{x}')(\bar{x} - \bar{x}')\right]\left[1 + \bar{A}(\bar{x} - \bar{x}')\right] + O(\bar{\epsilon}^2), \\
&= \frac{1}{\pi\bar{\epsilon}} + \frac{1}{\pi}\left[\bar{A} + \bar{\partial}M(\mathbf{x})M^{-1}(\mathbf{x})\right] + O(\bar{\epsilon}^2),
\end{aligned}$$

so that

$$D_{reg}^{-1}(\boldsymbol{x}, \bar{\epsilon}) = \frac{1}{\pi}\left[\bar{A} + \bar{\partial}M(\boldsymbol{x})M^{-1}(\boldsymbol{x})\right]. \tag{1.48}$$

The regularization procedure used above is sufficient for determining D_{reg}^{-1}, but holomorphic invariance is not manifest. Therefore, following [11], we may instead regularize

$G(\boldsymbol{x}, \boldsymbol{x}')$ by replacing it with a new function $\mathcal{G}(\boldsymbol{x}, \boldsymbol{x}')$ that is holomorphically covariant. Let $K^{ab}(x, \bar{x}) = H^{ab}(\mathbf{x}) = M^{\dagger ac}(\mathbf{x}) M^{cb}(\mathbf{x}')$ be the adjoint representation of H. To be certain, $K^{ab}(x, \bar{y})$ equals $H^{ab}(\mathbf{x})$ only when $\bar{y} = \bar{x}$. K^{ab} transforms holomorphically as

$$K^{ab}(x, \bar{x}) \to V^{ac}(x) K^{cd}(x, \bar{x}) \bar{V}^{db}(\bar{x}), \tag{1.49}$$

or more generally,

$$K^{ab}(x, \bar{y}) \to V^{ac}(x) K^{cd}(x, \bar{y}) \bar{V}^{db}(\bar{y}). \tag{1.50}$$

The above requirements are satisfied by the choice

$$\mathcal{G}^{ab}(\mathbf{x}, \mathbf{y}) = \int_{\mathbf{w}} G(\mathbf{x}, \mathbf{w}) \sigma(\mathbf{w}, \mathbf{y}, \epsilon) \left[K^{-1}(y, \bar{w}) K(y, \bar{y})\right]^{ab}, \tag{1.51}$$

$$\bar{\mathcal{G}}^{ab}(\mathbf{x}, \mathbf{y}) = \int_{\mathbf{w}} \bar{G}(\mathbf{x}, \mathbf{w}) \sigma(\mathbf{w}, \mathbf{y}, \epsilon) \left[K(w, \bar{y}) K^{-1}(y, \bar{y})\right]^{ab}, \tag{1.52}$$

where $\sigma(\mathbf{x}, \mathbf{y}, \epsilon) = \frac{1}{\pi\epsilon} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{\epsilon}}$ tends to the Dirac delta function $\delta^{(2)}(\mathbf{x} - \mathbf{y})$ as $\epsilon \to 0$. Thus we damp out short-distance (UV) behaviour by keeping ϵ strictly positive. Let its antiderivative be

$$S(\mathbf{x}) = -\frac{1}{\pi \bar{x}} e^{-\frac{|\mathbf{x}|^2}{\epsilon}} + h(\bar{x}), \tag{1.53}$$

so that $\partial S(\mathbf{x}) = \sigma(\mathbf{x})$, and $h(\bar{x})$ is arbitrary. Also, denote $\left[K^{-1}(y, \bar{w}) K(y, \bar{y})\right]^{ab}$ by $f^{ab}(\bar{w})$. Then, on suppressing indices,

$$\begin{aligned}
\mathcal{G}(\mathbf{x}, \mathbf{y}) &= \int_{\mathbf{w}} G(\mathbf{x}, \mathbf{w}) \partial_w \left[S(\mathbf{w} - \mathbf{y})\right] f(\bar{w}), \\
&= \int_{\mathbf{w}} \partial_w \left[G(\mathbf{x}, \mathbf{w}) S(\mathbf{w} - \mathbf{y}) f(\bar{w})\right] - \int_{\mathbf{w}} \partial_w \left[G(\mathbf{x}, \mathbf{w})\right] S(\mathbf{w} - \mathbf{y}) f(\bar{w}), \\
&= -\int_{\mathbf{w}} \partial_w \left[G(\mathbf{x}, \mathbf{w}) \left(\frac{1}{\pi(\bar{w} - \bar{y})} e^{-\frac{|\mathbf{w}-\mathbf{y}|^2}{\epsilon}}\right) f(\bar{w})\right] - \frac{1}{\pi(\bar{x} - \bar{y})} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{\epsilon}} f(\bar{x}).
\end{aligned}$$

There is a subtle point that must be considered: $G(\mathbf{x}) = \lim_{\eta \to 0^+} \frac{x}{x\bar{x} + \eta^2}$. In performing the regularization we keep η finite (along with ϵ) and merely analyze the behaviour of the result as $\eta \to 0$. In the second term above, the derivative leads to a delta sequence, because if η is small then $\partial_w G(\mathbf{x},\mathbf{w})$ only has support near $\mathbf{w}\approx\mathbf{x}$, so we can approximate the rest of the integrand (which is well-behaved) with its value at this point. Similarly, in the steps to follow, we would expand in powers of η^2, in which case we would obtain terms that vanish as $\eta\to 0$.

The first term of $\mathcal{G}(\mathbf{x},\mathbf{w})$ is

$$\frac{1}{\pi^2}\int_{\mathbf{w}}\partial_w\left[\frac{1}{(\bar{x}-\bar{w})}\frac{1}{(\bar{w}-\bar{y})}e^{-\frac{|\mathbf{w}-\mathbf{y}|^2}{\epsilon}}f(\bar{w})\right] \tag{1.54}$$

$$=-\frac{1}{\pi^2}\int_{\mathbf{w}}\partial_w\left[\frac{1}{(\bar{x}-\bar{w}-\bar{y})}\frac{1}{\bar{w}}e^{-\frac{|\mathbf{w}|^2}{\epsilon}}f(\bar{w}+\bar{y})\right] \tag{1.55}$$

Now $dx\wedge dy=-\frac{i}{2}dw\wedge d\bar{w}$, so upon integration we obtain

$$\frac{i}{2\pi^2}\int dw\,d\bar{w}\,\partial_w\left[\frac{1}{(\bar{x}-\bar{w}-\bar{y})}\frac{1}{\bar{w}}e^{-\frac{w\bar{w}}{\epsilon}}f(\bar{w}+\bar{y})\right]$$
$$=-\frac{i}{2\pi^2\epsilon}\int dw\,d\bar{w}\,\frac{1}{(\bar{x}-\bar{w}-\bar{y})}e^{-\frac{w\bar{w}}{\epsilon}}f(\bar{w}+\bar{y}). \tag{1.56}$$

We may rotate the integration contour as follows: $w\to iw$, so that the integral becomes

$$\begin{aligned}\frac{1}{\pi\epsilon}\int d\bar{w}\,\frac{1}{(\bar{x}-\bar{w}-\bar{y})}\delta\left(\frac{\bar{w}}{\epsilon}\right)f(\bar{w}+\bar{y}) &= \frac{1}{\pi}\frac{1}{\bar{x}-\bar{y}}f(\bar{y}),\\ &= \frac{1}{\pi}\frac{1}{\bar{x}-\bar{y}},\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}^{ab}(\mathbf{x},\mathbf{y}) &= \frac{1}{\pi}G(\mathbf{x},\mathbf{y})\left[\delta^{ab}-e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{\epsilon}}\left(K^{-1}(y,\bar{x})K(y,\bar{y})\right)^{ab}\right],\\ &\approx \frac{1}{\pi}\frac{1}{\bar{x}-\bar{y}}\left[\delta^{ab}-e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{\epsilon}}\left(1+\bar{\partial}K^{-1}(y,\bar{y})K(y,\bar{y})(\bar{x}-\bar{y})+\ldots\right)^{ab}\right].\end{aligned}$$

Then, as $\mathbf{y} \to \mathbf{x}$, since $K^{ab} = (K^{-1})^{ba}$

$$\mathcal{G}^{ab}(\mathbf{x},\mathbf{x}) = \frac{1}{\pi}\left(K^{-1}\bar{\partial}K\right)^{ab}(\mathbf{x}), \tag{1.57}$$

$$\bar{\mathcal{G}}^{ab}(\mathbf{x},\mathbf{x}) = -\frac{1}{\pi}\left(\partial K K^{-1}\right)^{ab}(\mathbf{x}). \tag{1.58}$$

After regularization,

$$G(\mathbf{r},\mathbf{r})\delta^{ab} \to \mathcal{G}^{ab}(\mathbf{r},\mathbf{r}) = \frac{1}{\pi}\left(H^{-1}\bar{\partial}H\right)^{ab}, \tag{1.59}$$

so that,

$$\begin{aligned}\mathcal{G}^{ab}(\mathbf{r},\mathbf{r}) &= \frac{1}{\pi}\left[M^{-1}M^{\dagger-1}\left(\bar{\partial}M^{\dagger}M + M^{\dagger}\bar{\partial}M\right)\right]^{ab}, \\ &= \frac{1}{\pi}\left[M^{-1}\left(\bar{A} + \bar{\partial}MM^{-1}\right)M\right]^{ab}.\end{aligned}$$

So

$$(D^{-1})^{ab}_{reg}(\mathbf{r},\mathbf{r}) = \frac{1}{\pi}\left[\bar{A} + \bar{\partial}MM^{-1}\right]^{ab}, \tag{1.60}$$

$$(\bar{D}^{-1})^{ab}_{reg}(\mathbf{r},\mathbf{r}) = \frac{1}{\pi}\left[A - M^{\dagger-1}\partial M^{\dagger}\right]^{ab}. \tag{1.61}$$

Then,

$$\begin{aligned}\frac{\delta\Gamma}{\delta A^a(\mathbf{r})} &= -\frac{2i}{\pi}f^{abc}f^{dbc}\ \mathrm{tr}\left[t^d(\bar{A} + (\bar{\partial}M)M^{-1})\right], \\ &= -\frac{c_A i}{\pi}2\ \mathrm{tr}\left[t^a(\bar{A} + (\bar{\partial}M)M^{-1})\right], \end{aligned}\tag{1.62}$$

$$\frac{\delta\Gamma}{\delta\bar{A}^a(\mathbf{r})} = -\frac{c_A i}{\pi}2\ \mathrm{tr}\left[t^a(A + M^{\dagger-1}\partial M^{\dagger})\right], \tag{1.63}$$

where c_A is the quadratic Casimir in the adjoint representation, i.e., $(T^cT^c)_{ab} \equiv -f^{cad}f^{cdb} = c_A\delta^{ab}$, equal to N for $SU(N)$. The solution to these differential equations is the Wess-Zumino-Witten action [19, 20], i.e.,

$$\Gamma = 2c_A S_{WZW}(M^{\dagger}M), \tag{1.64}$$

where

$$S_{WZW}[H] = \frac{1}{2\pi}\int d^2x\ \mathrm{tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi}\int d^3x\, \varepsilon^{\mu\nu\alpha}\, \mathrm{tr}(H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\alpha H). \tag{1.65}$$

In the second line, the field H has been extended into the third dimension, where the boundary of the extended space is $\mathbb{C}_\infty$, upon which the physical fields live. For Hermitian models, this can be written as an integral over 2-dimensional space only [11, 21].

Summarizing, the change of variables from $(A, \bar{A}) \to H$ results in a Jacobian determinant so that the measure on the configuration space $\mathcal{C}$ (the space of gauge potentials modulo allowable gauge transformations) is [11]

$$d\mu(\mathcal{C}) = \det(D\bar{D})d\mu(H), \tag{1.66}$$

$$= \left[\frac{\det'(\partial\bar{\partial})}{\int d^2x}\right]^{\dim G} e^{2c_A S_{WZW}[H]}d\mu(H). \tag{1.67}$$

Inner products and expectation values are calculated using this measure, as

$$\langle\Psi_1|\mathcal{O}|\Psi_2\rangle = \int d\mu(H)e^{2c_A S_{WZW}[H]}\Psi_1^*[H]\mathcal{O}\Psi_2[H], \tag{1.68}$$

so that expectation values are essentially correlators of the Euclidean, Hermitian WZW theory. These, in turn, can be found (at least in principle) by solving the Knizhnik-Zamolodchikov equations for the unitary theory and performing an analytic continuation to the Hermitian case [11].

1.3.6 Hamiltonian

In order to quantize the theory in the H variables, the Yang-Mills Hamiltonian

$$H = T + V = \frac{g_{YM}^2}{2}\int E_i^a E_i^a + \frac{1}{2g_{YM}^2}\int B^a B^a, \tag{1.69}$$

must be written in terms of a single variable only. Therefore, we will use the fact that, when acting on physical (i.e. gauge invariant) states, the Gauss' law operator $I \equiv \mathbf{D} \cdot \mathbf{E} = 2(D\bar{E} + \bar{D}E)$ vanishes. We can solve for E, as

$$E^a(\mathbf{z}) = \int_w \left[\bar{D}^{-1}(\mathbf{z}, \mathbf{w})\right]^{ab} \left(\frac{1}{2}I - D\bar{E}\right)^b (\mathbf{w}), \tag{1.70}$$

where $\left[\bar{D}\bar{D}^{-1}(\mathbf{z}, \mathbf{w})\right]^{ab} = \delta^{ab}\delta(\mathbf{z} - \mathbf{w})$. The kinetic energy operator is then

$$\begin{aligned} T &= 2g_{YM}^2 \int_x E^a(\mathbf{x})\bar{E}^a(\mathbf{x}) \\ &= 2g_{YM}^2 \int_{x,y} \left[\bar{D}^{-1}(\mathbf{x}, \mathbf{y}) \left(\frac{1}{2}I - D\bar{E}\right)(\mathbf{y})\right]^a \bar{E}^a(\mathbf{x}). \end{aligned} \tag{1.71}$$

Since I vanishes on physical states, we move it toward the right, using

$$\begin{aligned} I^b(\mathbf{y})\bar{E}^a(\mathbf{x}) &= \bar{E}^a(\mathbf{x})I^b(\mathbf{y}) + \left[I^b(\mathbf{y}), \bar{E}^a(\mathbf{x})\right] \\ &= \bar{E}^a(\mathbf{x})I^b(\mathbf{y}) - i\delta(\mathbf{x} - \mathbf{y})f^{abc}\bar{E}^c(\mathbf{y}). \end{aligned} \tag{1.72}$$

Therefore,

$$\begin{aligned} T = -2g_{YM}^2 \int_{x,y} \left[\bar{D}^{-1}(\mathbf{x}, \mathbf{y})D\bar{E}(\mathbf{y})\right]^a \bar{E}^a(\mathbf{x}) + g_{YM}^2 \int_{x,y} \bar{D}^{ab-1}(\mathbf{x}, \mathbf{y})\bar{E}^a(\mathbf{x})I^b(\mathbf{y}) \\ - 2g_{YM}^2 \int_x \frac{1}{2} tr \left[\bar{D}^{-1}(\mathbf{x}, \mathbf{y})T^c\right]_{\mathbf{y}\to\mathbf{x}} \bar{E}^c(\mathbf{x}), \end{aligned} \tag{1.73}$$

since $T_{ab}^c = -if^{abc}$. Taking the coincident limit in the integrand leads to a divergence, so we must replace D^{-1} with its regularized counterpart. Then

$$-\frac{1}{2}\,\mathrm{tr}\left[\bar{D}^{-1}(\mathbf{x}, \mathbf{y})T^a\right]_{\mathbf{y}\to\mathbf{x}} \to \frac{ic_A}{2\pi}\left(A - M^{\dagger -1}\partial M^\dagger\right)^a, \tag{1.74}$$

where $c_A\delta_{ab} = f_{amn}f_{bmn}$, so that the kinetic energy becomes

$$T = -2g_{YM}^2 \int_{x,y} \left[\bar{D}^{-1}(\mathbf{x},\mathbf{y})D\bar{E}(\mathbf{y})\right]^a \bar{E}^a(\mathbf{x}) + g_{YM}^2 \int_{x,y} \bar{D}^{ab-1}(\mathbf{x},\mathbf{y})\bar{E}^a(\mathbf{x})I^b(\mathbf{y})$$
$$+ 2im \int_x \left(A - M^{\dagger -1}\partial M^\dagger\right)^a \bar{E}^a(\mathbf{x}), \quad (1.75)$$

where $m \equiv \frac{g_{YM}^2 c_A}{2\pi}$ is essentially the 't Hooft coupling. The potential energy is simply

$$V = \frac{1}{2g_{YM}^2} \int B^a B^a, \quad (1.76)$$

where $B^a t^a = 2\left(\bar{\partial}A - \partial\bar{A}\right) + 2[\bar{A}, A]$.

The Gauss' law operator is a generator for complex $SL(N,\mathbb{C})$ gauge transformations because, properly, gauge transformations involve g^{-1}, (which equals $g^\dagger$ if g is unitary), which gives rise to the commutator in the infinitesimal. Let $g = e^\theta$ where θ is complex and not antihermitian (as for $SU(N)$). Infinitesimally, $g \approx 1+\theta$, $g^{-1} \approx 1-\theta$, then

$$\begin{aligned} A_i^{(g)} &= gA_ig^{-1} - (\partial_i g)g^{-1}, \\ &\approx A_i + \theta A_i - A_i\theta - \partial_i\theta + O(\theta^2), \\ &= A_i - D_i\theta, \\ &= -i(A_i^a - i(D_i\theta)^a)t^a. \end{aligned} \quad (1.77)$$

Now,

$$\begin{aligned} \int dx\, [\theta^a(\mathbf{x})I^a(\mathbf{x}), A_j^c(\mathbf{y})] &= \int dx\, \left[\theta^a(\mathbf{x})(\delta^{ab}\partial_i + f^{adb}A(\mathbf{x})_i^d)E_i^b(\mathbf{x}), A_j^c(\mathbf{y})\right], \\ &= -\int dx\, \theta^a(\mathbf{x})(\delta^{ab}\partial_i + f^{adb}A(\mathbf{x})_i^d)i\delta^{bc}\delta_{ij}\delta(\mathbf{x}-\mathbf{y}), \\ &= -i(-\delta^{ac}\partial_j + f^{adc}A(\mathbf{y})_j^d)\theta^a(\mathbf{y}), \\ &= i(D_j\theta)^c(\mathbf{y}), \\ &= -\delta A_j^c(\mathbf{y}). \end{aligned} \quad (1.78)$$

Furthermore,

$$\Psi[A_i^a + \delta A_i^a] \approx \Psi[A_i^a] + \int dx\, \delta A_j^b(\mathbf{x}) \frac{\delta}{\delta A_j^b(\mathbf{x})} \Psi[A_i^a]$$
$$= \Psi[A_i^a] - i \int dx\, \delta A_j^b(\mathbf{x}) E_j^b(\mathbf{x}) \Psi[A_i^a],$$
$$= \Psi[A_i^a] + \int dx\, \theta^b(\mathbf{x}) I^b(\mathbf{x}) \Psi[A_i^a], \tag{1.79}$$

provided that surface terms vanish. For physical states, $I^b(\mathbf{x})\Psi_{phys}[A_i^a] = 0$, expressing the fact that the wave-functional is gauge invariant.

WZW current

We wish to gather everything in terms of the WZW current $J = \frac{c_A}{\pi}\partial H H^{-1}$. In the Hermitian WZW theory only correlators made up of integrable representations of the current algebra are well defined, so that all objects of interest can be written in terms of the WZW current $J = \frac{c_A}{\pi}\partial H H^{-1}$ [22]. First, note that

$$\partial H H^{-1} = \partial M^\dagger M^{\dagger -1} + M^\dagger \partial M M^{-1} M^{\dagger -1},$$
$$= \partial M^\dagger M^{\dagger -1} - M^\dagger A M^{\dagger -1}, \tag{1.80}$$

so that

$$A = M^{\dagger -1}(-\partial H H^{-1}) M^\dagger + M^{\dagger -1} \partial M^\dagger,$$
$$= M^{\dagger -1}(-\partial H H^{-1}) M^\dagger - \partial M^{\dagger -1} M^\dagger,$$
$$\bar{A} = \bar{\partial} M^{\dagger -1} M^\dagger. \tag{1.81}$$

Therefore $(A, \bar{A})$ can be thought of as a field-dependent complex $SL(N,\mathbb{C})$-valued gauge transformation of $(-\partial H H^{-1}, 0)$. In the A-representation, the wave-functional $\Psi(A, \bar{A})$ may alternatively be written in terms of $M^\dagger$ and J. Since the Gauss' law operator $\mathbf{D} \cdot \mathbf{E}$ is the generator of infinitesimal gauge transformations (and vanishes on physical states), we can perform a sequence of such transformations $M^\dagger \to M^\dagger g^\dagger$ until $M^\dagger = 1$, without affecting $\Psi(A, \bar{A})$. Hence, when acting on physical states, we may everywhere let $(A, \bar{A}) \to (-\partial H H^{-1}, 0)$. In terms of the generators,

$$A = -iA^a t^a \to -\frac{\pi}{c_A} J = -\frac{\pi}{c_A} J^a t^a, \tag{1.82}$$

so that $A^a \to -i\frac{\pi}{c_A} J^a$, and $\frac{\delta}{\delta A^a} \to i\frac{c_A}{\pi}\frac{\delta}{\delta J^a}$. In particular, we now have

$$\begin{aligned} D^{ab} &= \delta^{ab}\partial - i\frac{\pi}{c_A} f^{acb} J^c = \delta^{ab}\partial + i\frac{\pi}{c_A} f^{abc} J^c, \\ \bar{E}^a(\mathbf{x}) &= \frac{c_A}{2\pi}\frac{\delta}{\delta J^a(\mathbf{x})}. \end{aligned} \tag{1.83}$$

The first term in the kinetic energy becomes

$$m \int_x J^a(\mathbf{x}) \frac{\delta}{\delta J^a(\mathbf{x})},$$

while the second term is

$$-2g_{YM}^2 \int_{x,y} \left[\bar{D}^{-1}(\mathbf{x},\mathbf{y}) D \bar{E}(\mathbf{y})\right]^a \bar{E}^a(\mathbf{x}).$$

Using the substitution $\bar{A} \to 0$, $M^\dagger \to 1$ up to a holomorphic factor (recalling that $\bar{D}^{-1}$ is holomorphically invariant), so that $\bar{D}^{-1}(\mathbf{x},\mathbf{y}) \equiv M^{\dagger -1}(\mathbf{x}) \bar{G}(\mathbf{x},\mathbf{y}) M^\dagger(\mathbf{y}) \to \frac{1}{\pi}\frac{1}{x-y}$, the second term becomes

$$\begin{aligned} &-\frac{2}{\pi} g_{YM}^2 \int_{x,y} \frac{1}{x-y}\left[\delta^{ab}\partial_y + i\frac{\pi}{c_A} f^{abc} J^c(\mathbf{y})\right] \bar{E}^b(\mathbf{y}) \bar{E}^a(\mathbf{x}), \\ &= -\frac{m c_A}{\pi^2} \int_{x,y} \frac{1}{x-y}\left[\delta^{ab}\partial_y + i\frac{\pi}{c_A} f^{abc} J^c(\mathbf{y})\right] \frac{\delta}{\delta J^a(\mathbf{x})}\frac{\delta}{\delta J^b(\mathbf{y})}, \\ &= \frac{m c_A}{\pi^2} \int_{x,y} \frac{1}{y-x}\left[\delta^{ab}\overleftarrow{\partial_x} - i\frac{\pi}{c_A} f^{bac} J^c(\mathbf{x})\right] \frac{\delta}{\delta J^a(\mathbf{x})}\frac{\delta}{\delta J^b(\mathbf{y})}, \\ &= m \int_{x,y} \Omega^{ab}(\mathbf{x}-\mathbf{y}) \frac{\delta}{\delta J^a(\mathbf{x})}\frac{\delta}{\delta J^b(\mathbf{y})}, \end{aligned} \tag{1.84}$$

where $\Omega^{ab}(\mathbf{x}-\mathbf{y}) \equiv \frac{c_A}{\pi^2} D_x^{ab}\bar{G}(\mathbf{y}-\mathbf{x})$, so that

$$T = m\left[\int_x J^a(\mathbf{x})\frac{\delta}{\delta J^a(\mathbf{x})} + \int_{x,y}\Omega^{ab}(\mathbf{x}-\mathbf{y})\frac{\delta}{\delta J^a(\mathbf{x})}\frac{\delta}{\delta J^b(\mathbf{y})}\right]. \tag{1.85}$$

Under the substitution $B = 2\left(\bar{\partial}A - \partial\bar{A}\right) + 2[\bar{A}, A] \rightarrow \frac{2\pi}{c_A}\bar{\partial}J$, the potential energy becomes

$$V = \frac{\pi}{mc_A}\int_x \bar{\partial}J^a(\mathbf{x})\bar{\partial}J^a(\mathbf{x}). \tag{1.86}$$

The entire Hamiltonian is then

$$H = m\left[\int_x J^a(\mathbf{x})\frac{\delta}{\delta J^a(\mathbf{x})} + \int_{x,y}\Omega^{ab}(\mathbf{x}-\mathbf{y})\frac{\delta}{\delta J^a(\mathbf{x})}\frac{\delta}{\delta J^b(\mathbf{y})}\right] + \frac{\pi}{mc_A}\int_x \bar{\partial}J^a(\mathbf{x})\bar{\partial}J^a(\mathbf{x}). \tag{1.87}$$

1.3.7 Vacuum wavefunctional

A remarkable side-effect of the non-trivial measure on the configuration space is that the wave-functional $\Psi[J] = 1$ is both normalizable and an eigenstate of T, a solution to the Schrödinger equation can be obtained [22] by expanding around this infinite coupling limit. Writing $\Psi[J] = e^P$, we note that $H\Psi[J] = 0$ implies $e^{-P}He^P = V - [H, P] + ... = 0$, which leads to a recursion relation for P in powers of m. The resulting wave-functional resembles that of [23], which was obtained by a perturbative resummation in the strong-coupling regime, as well as of [24, 25], obtained my Monte Carlo methods, and for example [26], though the solution is local in J rather than in B. Using this wave-functional, Karabali and Nair were able to exhibit the mass gap of the theory as well as demonstrate area-law behaviour of the Wilson loop, giving evidence for confinement [22, 27]. However, while the solution can be resummed to second order in J, giving [22]

$$P = -\frac{1}{2g^2}\int_{x,y} B(\boldsymbol{x})\frac{1}{[m + (m^2 - \nabla^2)^{\frac{1}{2}}]}B(\boldsymbol{y}), \tag{1.88}$$

the solution cannot be invariant order-by-order in J, as this variable transforms as a connection. Therefore the inclusion of terms of higher order in J is necessary for gauge invariance (or holomorphy), while terms of higher order in m are needed for consistency with the Schrödinger equation. For computational purposes, some trade off is ultimately necessary.

1.4 Spectrum in the planar limit

Leigh et al. instead proposed the following *Ansatz* for the vacuum wave-functional [12]:

$$\Psi[J] = \exp\left(-\frac{\pi}{2c_A m^2}\int \bar{\partial}JK(L)\bar{\partial}J\right) + ..., \tag{1.89}$$

where the kernel K is a power series in the holomorphic-covariant Laplacian $\Delta \equiv \frac{\{D,\bar{\partial}\}}{2} \equiv m^2 L$. Here and onwards traces over adjoint indices are implicit. The general form of $\Psi[J]$ was chosen to reflect the notion, as argued in [28], that the variables $\bar{\partial}J$ somehow represent the correct degrees of freedom of the system (i.e., those in which the ground-state wave-functional is Gaussian). The use of the covariant Laplacian is then required by consistency with holomorphic invariance of the theory. Terms in the exponent of higher order in $\bar{\partial}J$ that can't be absorbed into the kernel are denoted by ellipsis; therefore, the *Ansatz* is not the most general one. Note that $\bar{\partial}JK(L)\bar{\partial}J = \bar{\partial}JK\left(\frac{\bar{\partial}\partial}{m^2}\right)\bar{\partial}J + O(J^3)$. Writing $\Psi[J] = e^P$, the action of kinetic energy term in (1.87) on $\Psi[J]$ becomes

$$T\Psi[J] = \left[TP + m\int_x\int_y \Omega^{ab}(\boldsymbol{x}-\boldsymbol{y})\frac{\delta P}{\delta J^a(\boldsymbol{x})}\frac{\delta P}{\delta J^b(\mathbf{y})}\right]\Psi[J]. \tag{1.90}$$

To second order in $\bar{\partial}J$, the second term in brackets yields

$$\frac{\pi}{c_A m}\int d^2x\, \bar{\partial}J\left[\frac{\partial\bar{\partial}}{m^2}K^2\left(\frac{\partial\bar{\partial}}{m^2}\right)\right]\bar{\partial}J + ... \tag{1.91}$$

The action of T on terms of the form $\mathcal{O}_n \equiv \int \bar{\partial}J(\Delta^n)\bar{\partial}J$ is less straight-forward. In reference [28] Leigh et al. argue that holomorphic invariance requires mixing between terms of different order in $\bar{\partial}J$ in such a way that $T\mathcal{O}_n = (2+n)m\mathcal{O}_n + ...$, with the result explicitly demonstrated for $\mathcal{O}_0$ and $\mathcal{O}_1$. Whether this result acquires corrections for higher values of n is not known. Formally, then, the Schrödinger equation becomes

$$H\Psi[J] = \left[\frac{\pi}{c_A m}\int \bar{\partial}J\left(-\frac{1}{2L}\frac{d}{dL}\left[L^2K(L)\right] + LK^2(L) + 1\right)\bar{\partial}J\right]\Psi = E\Psi[J]. \tag{1.92}$$

The eigenvalue equation

$$-\frac{1}{2L}\frac{d}{dL}\left[L^2K(L)\right] + LK^2(L) + 1 = 0 \tag{1.93}$$

is then solvable using the substitution $K = -\frac{U'}{2U}$ which casts it into Bessel form. The unique, normalizable solution with correct UV asymptotics was found by Leigh et al., to be [12]:

$$K(L) = \frac{1}{\sqrt{L}}\frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})}. \tag{1.94}$$

As mentioned earlier, the form of the *Ansatz* is somewhat tautological once the variables $\bar{\partial}J$ are assumed to be the correct physical ones. However, as many of the resulting steps, for example, the requirement of a holomorphic covariant Laplacian, ruin the precise Gaussian form of the wave-functional, the *Ansatz* needs to be motivated further. Note that $\bar{\partial}J = -\frac{2\pi}{c_A}M^{\dagger-1}BM^{\dagger}$, so that in many expressions $\bar{\partial}J$ simply reduces to the magnetic field. At large momentum (weak coupling) the field modes behave like free fields, and the well-known (Abelian) Maxwell-field ground-state wave-functional [29, 30] factors from the full wave-functional:

$$\Psi[A]_{UV} = e^{-\frac{1}{2g^2}\int_{x,y}B(x)\frac{1}{(-\nabla^2)^{1/2}}B(y)}. \tag{1.95}$$

On the other hand, many have argued [31] or found [22] that the in the low-momentum (strong coupling) limit the wave-functional becomes

$$\Psi[A]_{IR} = e^{-\frac{1}{2mg^2}\int_x B(x)^2}. \tag{1.96}$$

Thus the *Ansatz* may be seen as the minimal (but certainly not unique) way to interpolate between between these two limits while retaining the necessary holomorphic symmetry, and is analogous to the *Ansatz* proposed in [32]. As was pointed out by Greensite and Olejník in [33], the Abelian wave-functional equation (1.95) can be made to satisfy the non-Abelian version of Gauss' law exactly whilst solving the Schrödinger equation to zeroth order in g by replacing $\boldsymbol{\nabla}^2$ with its gauge-covariant form $\mathbf{D}^2$. There it was found that this substitution leads to a confining state. But $\{D, \bar{D}\} = \frac{1}{2}\mathbf{D}^2$ so that $\Delta = \frac{1}{2}\{D, \bar{\partial}\}$ is precisely the form of this operator where gauge covariance has been supplanted by holomorphic covariance. Therefore the *Ansatz* has many of the necessary properties, though these are certainly not sufficient.

Indeed, the solution found by Leigh et al., does not agree with the exact (series) solution obtained by Karabali and Nair [22] at higher orders in the 't Hooft coupling m. However, when solving the Schrödinger equation to second order in $\bar{\partial}J$, only terms that are required for consistency were kept. Furthermore, the wave-functional obtained by Leigh et. al., is only motivated in the strict large-N limit, and so does not contain all leading $\frac{1}{N}$ corrections. Thus, while the *Ansatz* leads to an approximate, closed form wave-functional, from which correlation functions can be computed, it sacrifices corrections only seen in an exact, order-by-order treatment like that in [22].

1.4.1 Mass Spectrum

In order to extract meaningful information from the ground state Ψ it is necessary to compute correlation functions as in equation (1.68). Attempting to solve the Knizhnik-Zamolodchikov equations directly, one encounters (already at the four-point level) non-trivial expressions involving hypergeometric functions [34], whereas the calculation wish to attempt contains an infinite series of such correlators (from expanding the wave-functional), with the next-leading-order being a six-point function. The need for approximation is therefore evident.

In [28] the operator $\mathrm{tr}\,\bar{\partial}J\bar{\partial}J$ was found to be even under parity and charge conjugation, and so creates 0^{++} states (see Appendix E). A good starting point then is the calculation of $\langle(\bar{\partial}J\bar{\partial}J)_x(\bar{\partial}J\bar{\partial}J)_y\rangle$. As we have already noted, such a computation is presently intractable. However, in [12, 28] it is argued that in the large-N limit, the variables $\bar{\partial}J$ represent the correct physical degrees of freedom so that integration over these variables can be done as in a free theory, in the sense that

$$\left\langle \bar{\partial}J_x^a \bar{\partial}J_y^b \right\rangle \sim \delta^{ab} K^{-1}(|\boldsymbol{x}-\boldsymbol{y}|). \tag{1.97}$$

$K^{-1}(|\boldsymbol{x}-\boldsymbol{y}|)^2$ is then identified with a particular two-point function probing 0^{++} glueball states, so that the mass spectrum of the vacuum can be read off from the analytic structure of $K^{-1}(k)^2$. Essentially, it is argued that in the $\bar{\partial}J$ configuration space the WZW measure can be neglected. The interpretation of this statement will be explored further in this paper.

The asymptotic form of $K^{-1}(|\boldsymbol{x}-\boldsymbol{y}|)^2$ was found to be [12]

$$K^{-1}(|\boldsymbol{x}-\boldsymbol{y}|)^2 \to \frac{1}{32\pi|\boldsymbol{x}-\boldsymbol{y}|}\sum_{n,m=1}^{\infty}(M_nM_m)^{3/2}e^{-(M_n+M_m)|\boldsymbol{x}-\boldsymbol{y}|}, \tag{1.98}$$

where $M_n \equiv \frac{j_{2,n}m}{2}, n = 1,2,3...$ and $j_{2,n}$ are the zeros of $J_2(z)$. Comparing to the two-point function of the free Boson, equation (D.1), we can see that the 0^{++} states have masses given by various combinations of M_n. This result is in good agreement with lattice calculations presented in [35, 36] and in the following chapter we will attempt to give justification for this agreement in light of the approximations that have been made.

Chapter 2

Controlled Approximation

2.1 Abelian Expansion

The statement that integration over the variables $\bar{\partial}J$ can be done explicitly should not be interpreted literally, as the change of variables from H to $\bar{\partial}J$ involves a further factor $\det(D\bar{\partial})^{-1}$ where D is now the holomorphic covariant derivative with J as connection. Although the derivatives D and $\bar{\partial}$ are related to the original expressed in terms of $(A, \bar{A})$ through conjugation by $M^{\dagger -1}$, the determinant also suffers from a multiplicative anomaly which is expressed by the Polyakov-Wiegmann identity that relates $S_{WZW}[gh]$ to $S_{WZW}[g]$ and $S_{WZW}[h]$ (see [18, 37]). Therefore we shall approach the problem by exploring it in a particular limit where the WZW action at least allows for tractable calculations.

We now attempt to calculate $\langle(\bar{\partial}J\bar{\partial}J)_x(\bar{\partial}J\bar{\partial}J)_y\rangle$ by performing the path integral in equation (1.68) in the Abelian limit. Following [27] we write $H = e^{\varphi}$ and expand terms of the form $H^{-1}f(\varphi)H$ in powers of φ. In the adjoint representation, $\varphi^{ab} = f^{abc}\varphi^c$, so that an expansion in φ is necessarily an expansion in the structure constants. In the limit of small φ (see e.g., [27, 38]), we can write the wave-functional and WZW factor in Gaussian form and calculate the four-point function as in a Euclidean field theory with action given by $2c_A S[H] + \ln \Psi^*[H]\Psi[H]$. In all cases we expand terms inside the exponential, but not the exponential itself, i.e., we are performing a selective resummation [27].

The measure factor becomes the exponential of the free complex scalar field action, $e^{2c_A S[H]} \approx e^{-\frac{c_A}{2\pi}\int d^2z\, \partial\varphi^a\bar{\partial}\varphi^a}$. The complete measure factor can be integrated, so that the volume of the configuration space $\mathcal{C}$ is finite, which leads to the existence of a mass-gap [27]. By approximating the WZW factor in this way we hope to make the calculation tractable and at the same time capture non-trivial effects. Already we notice that in the Abelian limit the zero mode causes the volume of $\mathcal{C}$ to diverge - it was already noted in [27] that the WZW factor cannot be obtained in the Abelian limit; therefore the approximation may break down at low momentum.

The correlation function $\langle(\bar{\partial}J\bar{\partial}J)_x(\bar{\partial}J\bar{\partial}J)_y\rangle$ gives, up to a disconnected diagram, the square of the two-point function:

$$\langle(\bar{\partial}J\bar{\partial}J)_x(\bar{\partial}J\bar{\partial}J)_y\rangle = 2\left\langle\bar{\partial}J_x\bar{\partial}J_y\right\rangle\left\langle\bar{\partial}J_x\bar{\partial}J_y\right\rangle. \tag{2.1}$$

To expand the wave-functional, note that [39]

$$J = \frac{c_A}{\pi}\int_0^1 ds\, e^{s\varphi^a t^a}(\partial\varphi^b t^b)e^{-s\varphi^c t^c}, \tag{2.2}$$

$$\approx \frac{c_A}{\pi}\partial\varphi + ..., \tag{2.3}$$

where we have dropped higher powers of φ, so that $\bar{\partial}J \approx \frac{c_A}{\pi}\frac{\nabla^2}{4}\varphi$. That we are truly in the Abelian limit can be seen by allowing terms inside equation (2.2) to commute, thereby obtaining equation (2.3) exactly.

Keeping in mind that $\mathbf{k}\cdot\mathbf{z} = k\bar{z} + \bar{k}z$, we can Fourier transform the fields as

$$\varphi^a(\mathbf{z}) = \frac{1}{2\pi}\int d^2k\,\varphi^a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{z}}, \tag{2.4}$$

so that the WZW term becomes

$$\begin{aligned}\int d^2z\,\partial\varphi^a\bar{\partial}\varphi^a &= -\frac{1}{(2\pi)^2}\int d^2z\int d^2k_1\int d^2k_2\,\varphi^a(\mathbf{k}_1)\bar{k}_1 e^{i\mathbf{k}_1\cdot\mathbf{z}}\varphi^a(\mathbf{k}_2)k_2 e^{i\mathbf{k}_2\cdot\mathbf{z}},\\ &= \int d^2k\,\varphi^a(\mathbf{k})\frac{\mathbf{k}^2}{4}\varphi^a(-\mathbf{k}),\end{aligned} \tag{2.5}$$

Also, $(L\varphi)(\mathbf{k}) \approx -\frac{\mathbf{k}^2}{4m^2}\varphi(\mathbf{k})$, and we can write the exponent of the wave-functional as

$$\int d^2z\,\bar{\partial}J^a K\left(\frac{\partial\bar{\partial}}{m^2}\right)\bar{\partial}J^a \approx \left(\frac{c_a}{\pi}\right)^2\int d^2k\,\varphi^a(\mathbf{k})\frac{\mathbf{k}^2}{4}K\left(-\frac{\mathbf{k}^2}{4m^2}\right)\frac{\mathbf{k}^2}{4}\varphi^a(-\mathbf{k}). \tag{2.6}$$

Finally,

$$e^{2c_A S[H]}\Psi^*[H]\Psi[H] \approx \exp\left\{-\frac{c_A}{2\pi}\int d^2k\,\varphi^a(\mathbf{k})\frac{2\mathbf{k}^2}{4}\left[\frac{2}{\mathbf{k}^2}+\frac{1}{m^2}K\left(-\frac{\mathbf{k}^2}{4m^2}\right)\right]\frac{\mathbf{k}^2}{4}\varphi^a(-\mathbf{k})\right\}. \tag{2.7}$$

To calculate $\langle\bar{\partial}J(x)\bar{\partial}J(y)\rangle$, it is sufficient to know $\langle\varphi^a(x)\varphi^b(y)\rangle$, as the former can be obtained from the latter by repeated differentiation. Therefore we have to analyze the field theory with effective kernel given by

$$\mathcal{K}\left(-\frac{\mathbf{k}^2}{4m^2}\right) \equiv \frac{1}{2}\frac{4m^2}{\mathbf{k}^2}+K\left(-\frac{\mathbf{k}^2}{4m^2}\right). \tag{2.8}$$

2.2 Analytic Structure

Let us begin by writing the effective kernel in terms of the formal parameter $y \equiv 4\sqrt{L}$:

$$\mathcal{K}(L) = -\frac{1}{2}\frac{1}{L}+\frac{1}{\sqrt{L}}\frac{J_2(4\sqrt{L})}{J_1(4\sqrt{L})} \tag{2.9}$$

$$= \frac{4}{y}\left[-\frac{2}{y}+\frac{J_2(y)}{J_1(y)}\right]. \tag{2.10}$$

As a first step towards finding the inverse of the kernel, consider the identity [40]

$$J_{\nu-1}(y)+J_{\nu+1}(y)=\frac{2\nu}{y}J_\nu(y), \tag{2.11}$$

from which we immediately see that

$$\mathcal{K}(L) = -\frac{4}{y}\frac{J_0(y)}{J_1(y)}. \tag{2.12}$$

A comparison between the behaviour of K and $\mathcal{K}$ is shown in Figure 2.1; (note that all quantities here are dimensionless). Another identity of interest is

$$\frac{J_1(y)}{J_0(y)} = 2y\sum_{s=1}^{\infty}\frac{1}{j_{0,s}^2-y^2}, \tag{2.13}$$

where $j_{\nu,s}$ denotes the s^{th} zero of $J_\nu(y)$. This result can be derived using common Bessel function identities along with the infinite product representation,

$$J_\nu(y) = \frac{\left(\frac{1}{2}y\right)^\nu}{\Gamma(\nu+1)} \prod_{s=1}^{\infty} \left(1 - \frac{y^2}{j_{\nu,s}^2}\right), \tag{2.14}$$

and the fact that $J_\nu(y)$ is a meromorphic function.

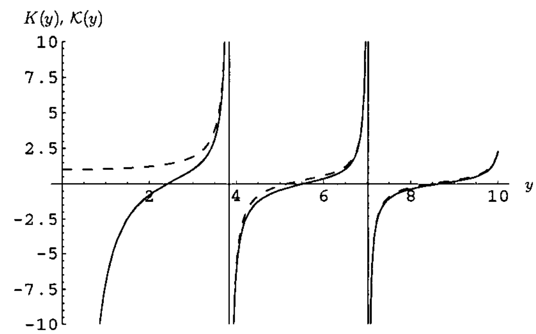

Figure 2.1: Comparison of $K(y)$ (dashed line) and $\mathcal{K}(y)$ (solid line)

Then

$$\mathcal{K}(L)^{-1} = \frac{1}{2}\sum_{s=1}^{\infty} \frac{y^2}{y^2 - j_{0,s}^2}. \tag{2.15}$$

We therefore take the inverse effective kernel to have the following analytical structure

$$\mathcal{K}^{-1}\left(-\frac{\mathbf{k}^2}{4m^2}\right) = \frac{1}{2}\sum_{s=1}^{\infty} \frac{y^2}{y^2 - j_{0,s}^2}, \tag{2.16}$$

$$= \frac{1}{2}\sum_{s=1}^{\infty} \left(1 - \frac{M_s^2}{\mathbf{k}^2 + M_s^2}\right), \tag{2.17}$$

where $M_s \equiv \frac{j_{0,s}m}{2}$. Following the treatment of [12], the real space kernel has the following asymptotic behaviour:

$$\mathcal{K}^{-1}(|\boldsymbol{x}-\boldsymbol{y}|) = -\frac{1}{2}\sum_{s=1}^{\infty}\frac{M_s^2}{2\pi}K_0\left(M_s|\boldsymbol{x}-\boldsymbol{y}|\right), \tag{2.18}$$

$$\rightarrow -\frac{1}{4}\sum_{s=1}^{\infty}M_s^{\frac{3}{2}}\frac{1}{\sqrt{2\pi|\boldsymbol{x}-\boldsymbol{y}|}}e^{-M_s|\boldsymbol{x}-\boldsymbol{y}|}, \tag{2.19}$$

while the four-point function is

$$\mathcal{K}^{-1}(|\boldsymbol{x}-\boldsymbol{y}|)^2 = \frac{1}{32\pi|\boldsymbol{x}-\boldsymbol{y}|}\sum_{r,s=1}^{\infty}(M_r M_s)^{\frac{3}{2}}e^{-(M_r+M_s)|\boldsymbol{x}-\boldsymbol{y}|}. \tag{2.20}$$

Comparing to the propagator of the free Boson evaluated at fixed time, equation (D.1), we see that a multitude of particles with masses $M_r + M_s$ have been identified. A comparison between our masses and those obtained in [12] is given in table 2.2. Large-N lattice results [35, 36, 41] given for comparison in [12] have also been reproduced. We have omitted the spurious pole due to $j_{0,1}$ as it has no obvious value to compare to. As noted in [12], the discrepancy with the 0^{++*} lattice result may suggest that this is in fact two states (corresponding to $M_1 + M_2$ and $M_2 + M_2$) which are not resolved by the lattice calculation, or it may indicate a low-momentum breakdown of our calculation.

State	Lattice[a]	Leigh, et., al.	Our prediction
0^{++}	4.065 ± 0.055	4.10	4.40
0^{++*}	6.18 ± 0.13	5.41	5.65
0^{++*}		6.72	6.90
0^{++**}	7.99 ± 0.22	7.99	8.15
0^{++***}	9.44 ± 0.38	9.27	9.40

[a] See [35, 36, 41]

Table 2.1: Comparison of 0++ glueball masses given in units of string tension $\sqrt{\sigma} \approx \sqrt{\frac{\pi}{2}}$

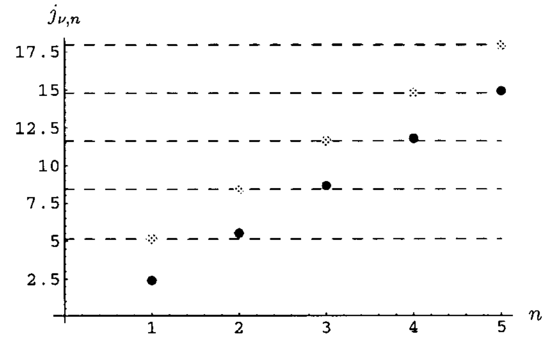

Figure 2.2: Zeros of J_2 (grey) and J_0 (black)

2.3 Discussion

The calculation presented here shows that in the Abelian limit the mass spectrum probed by $J^{PC} = 0^{++}$ operators comprises sums of pairs of zeros of $J_0(y)$, in contrast with the result of Leigh et al., which expresses these masses in terms of zeros of $J_2(y)$. Thus we find a correction due to the inclusion of the WZW action to lowest order in f^{abc}. Asymptotically $J_\nu(y) \to \sqrt{\frac{2}{\pi y}} \cos\left(y - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$ so that $J_2(y) \to -J_0(y)$ as $y \to \infty$. Therefore our results reproduce those in [12] at large momentum and give justification for approximations made therein.

That the results agree at large momentum suggests that the analytic structure of the wave-functional is rather robust against short-distance corrections. At low momentum (small y), however, an additional pole arises due to $j_{0,1}$ (see Figure 2.2), which is not seen in [12], and gives a constituent mass of $M_1 \approx 0.96\sqrt{\sigma}$. This correction can be seen in Figure 2.1. Combinations using M_1 do not seem to appear in lattice calculations presented in [35, 36]. It is possible that this signals a breakdown of the Abelian approximation at low momentum that can be corrected by continuing the expansion to higher order.

Chapter 3

Matter

3.1 Functional Schrödinger Picture

As the functional Schrödinger presentation of quantum field theory is not conventionally emphasized, it will be reviewed here briefly.

3.1.1 Bosonic case

With ordinary Quantum mechanics, we are motivated to view the field $\phi(\mathbf{x}, t)$ and it's conjugate momentum $\pi(\mathbf{x}, t)$ as operators acting on a Hilbert space. In the Schrödinger picture, these operators aren't explicitly time dependent and we simply write $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$. In canonical quantization we replace the fundamental Poisson bracket $[\phi(\mathbf{x}), \pi(\mathbf{y})]_{PB} = \delta(\mathbf{x}-\mathbf{y})$ with the commutator $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\hbar\delta(\mathbf{x}-\mathbf{y})$. We then must find operators that obey this relation. This is normally done via the introduction of creation and annihilation operators that act on a Fock space.

However, one can also consider operators that act on a space of wave functionals on the configuration space. The configuration space is the space of all (classical) fields $\phi(\mathbf{x})$, i.e., eigenvalues of the operator $\hat{\phi}(\mathbf{x})$ in the same sense that the configuration space of a quantum mechanical point particle is the space of all positions $\mathbf{x}$, i.e., eigenvalues of the position operator $\hat{\mathbf{x}}$. A wave functional is then a map $\Psi : \phi \rightarrow \Psi[\phi] \in \mathbb{C}$ from field configurations onto complex numbers, and represents the probability of the system to be in the configuration $\phi(\mathbf{x})$ [29]. It is just the functional analog of the wavefunction. In analogy with ordinary Quantum mechanics, we then make the following algebraic substitution:

$$\hat{\phi}(\mathbf{x}) \to \phi(\mathbf{x}), \tag{3.1}$$

$$\hat{\pi}(\mathbf{x}) \to -i\hbar \frac{\delta}{\delta\phi(\mathbf{x})}. \tag{3.2}$$

Note that from functional differentiation,

$$\frac{\delta\phi(\mathbf{x})}{\delta\phi(\mathbf{y})} = \delta(\mathbf{x}-\mathbf{y}), \tag{3.3}$$

so

$$\left[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] \Psi[\phi] = -i\hbar\phi(\mathbf{x})\frac{\delta}{\delta\phi(\mathbf{y})}\Psi[\phi] + i\hbar\frac{\delta}{\delta\phi(\mathbf{y})}\left(\phi(\mathbf{x})\Psi[\phi]\right), \tag{3.4}$$

$$= i\hbar\delta(\mathbf{x}-\mathbf{y})\Psi[\phi]. \tag{3.5}$$

The Hamiltonian for the free real field is then ($\hbar \to 1$)

$$H = \frac{1}{2}\int d^3x \, \left[\pi(\mathbf{x})^2 + [\nabla\phi(\mathbf{x})]^2 + m^2\phi(\mathbf{x})^2\right], \tag{3.6}$$

$$\to \frac{1}{2}\int d^3x \left[-\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + [\nabla\phi(\mathbf{x})]^2 + m^2\phi(\mathbf{x})^2\right], \tag{3.7}$$

$$= \frac{1}{2}\int d^3x \left[-\frac{\delta^2}{\delta\phi(\mathbf{x})^2} + \phi(\mathbf{x})(m^2 - \nabla^2)\phi(\mathbf{x})\right], \tag{3.8}$$

$$\tag{3.9}$$

The time-dependent Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi] = H\Psi[\phi]. \tag{3.10}$$

The (un-normalized) ground state of the real field is then

$$\Psi[\phi] = e^{-\frac{1}{2}\int d^3x\, \phi(\mathbf{x})\sqrt{m^2-\nabla^2}\phi(\mathbf{x}) - iE_0 t}. \tag{3.11}$$

Using the expansion,

$$\phi(\mathbf{x}) = \int d^3k \, \frac{1}{\sqrt{(2\pi)^3\omega_k}} \phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\,, \tag{3.12}$$

$$\phi(\mathbf{k}) = \int d^3x \, \sqrt{\frac{\omega_k}{(2\pi)^3}} \phi(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{3.13}$$

where $\omega_k^2 = \mathbf{k}^2 + m^2$, we have

$$\frac{\delta}{\delta\phi(\mathbf{x})} = \int d^3k \, \frac{\delta\phi(\mathbf{k})}{\delta\phi(\mathbf{x})} \frac{\delta}{\delta\phi(\mathbf{k})}, \tag{3.14}$$

$$= \int d^3k \, \sqrt{\frac{\omega_k}{(2\pi)^3}} e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{\delta}{\delta\phi(\mathbf{k})}, \tag{3.15}$$

$$\tag{3.16}$$

The kinetic part of the Hamiltonian becomes

$$\frac{1}{2}\int d^3x \, \frac{\delta^2}{\delta\phi(\mathbf{x})^2} = -\frac{1}{2}\int d^3x \int d^3k \, d^3k' \, \frac{\sqrt{\omega_k\omega_{k'}}}{(2\pi)^3} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \frac{\delta}{\delta\phi(\mathbf{k})} \frac{\delta}{\delta\phi(\mathbf{k}')}, \tag{3.17}$$

$$= -\frac{1}{2}\int d^3k \, \omega_k \frac{\delta}{\delta\phi(\mathbf{k})} \frac{\delta}{\delta\phi(-\mathbf{k})}\,. \tag{3.18}$$

The full Hamiltonian becomes

$$H = \frac{1}{2}\int d^3k \, \omega_k \left[-\frac{\delta}{\delta\phi(\mathbf{k})} \frac{\delta}{\delta\phi(-\mathbf{k})} + \phi(\mathbf{k})\phi(-\mathbf{k}) \right], \tag{3.19}$$

$$\Psi[\phi] = e^{-\frac{1}{2}\int d^3k \, \phi(\mathbf{k})\phi(-\mathbf{k}) - iE_0 t}. \tag{3.20}$$

Inserting into the Schrödinger equation gives

$$E_0 = \frac{1}{2}\int d^3k \, \omega_k \delta(0), \tag{3.21}$$

which agrees with the vacuum energy one finds using the traditional (operator) method.

3.1.2 Fermionic fields

In analogy with the Bosonic case, we will develop a functional description of the Schrödinger picture of Fermionic fields. We will first describe the Hermitian case since a complex field can be described by a pair of real fields.

Hermitian field

To describe a real Fermionic field, i.e., a Majorana field, we need a representation of the following equal-time anticommutation algebra:

$$\{\psi_a(\vec{x}), \psi_b(\vec{x}\,')\} = \delta_{ab}\delta(\vec{x} - \vec{x}\,'). \tag{3.22}$$

Since the field operator is it's own conjugate momentum, we cannot follow the Bosonic case where $\pi(\vec{x}) = -i\frac{\delta}{\delta\phi(\vec{x})}$. Instead, we construct them in a way first proposed by Floreanini and Jackiw (F-J) [42], namely

$$\psi_a(\vec{x}) = \alpha\theta_a(\vec{x}) + \beta\frac{\delta}{\delta\theta_a(\vec{x})}, \tag{3.23}$$

where $\theta_a(\vec{x})^* = \theta_a(\vec{x})$ is a real Grassmann number, i.e., $\{\theta_a(\vec{x}), \theta_b(\vec{x}\,')\} = 0$. Then

$$\psi_a(\vec{x})\psi_b(\vec{y}) = \left(\alpha\theta_a(\vec{x}) + \beta\frac{\delta}{\delta\theta_a(\vec{x})}\right)\left(\alpha\theta_b(\vec{y}) + \beta\frac{\delta}{\delta\theta_b(\vec{y})}\right), \tag{3.24}$$

$$\begin{aligned} = {} & \alpha^2\theta_a(\vec{x})\theta_b(\vec{y}) + \alpha\beta\delta_{ab}\delta(\vec{x} - \vec{y}) + \alpha\beta\theta_a(\vec{x})\frac{\delta}{\delta\theta_b(\vec{y})} \\ & -\alpha\beta\theta_b(\vec{y})\frac{\delta}{\delta\theta_a(\vec{x})} + \beta^2\frac{\delta}{\delta\theta_a(\vec{x})}\frac{\delta}{\delta\theta_b(\vec{y})}, \end{aligned} \tag{3.25}$$

so that

$$\{\psi_a(\vec{x}), \psi_b(\vec{x}\,')\} = 2\alpha\beta\delta_{ab}\delta(\vec{x} - \vec{x}\,'). \tag{3.26}$$

We can therefore choose complex numbers α and β such that $\alpha\beta = \frac{1}{2}$. For example,

$$\psi_a(\vec{x}) = \frac{1}{\sqrt{2}}\left(\theta_a(\vec{x}) + \frac{\delta}{\delta\theta_a(\vec{x})}\right), \tag{3.27}$$

and

$$\psi_a(\vec{x}) = \frac{1}{i\sqrt{2}} \left(\theta_a(\vec{x}) - \frac{\delta}{\delta\theta_a(\vec{x})} \right). \tag{3.28}$$

Non-Hermitian field

We can combine two Hermitian fields ψ_1 and ψ_2 (dropping spinor indices) into a single complex field $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$ which obeys $\{\psi^\dagger, \psi\} = \frac{1}{2}\{\psi_1, \psi_1\} + \frac{1}{2}\{\psi_2, \psi_2\} = \delta(\mathbf{x} - \mathbf{y})$. For example, we may use the form

$$\psi_i(\vec{x}) = \frac{1}{\sqrt{2}} \left(\theta_i(\vec{x}) + \frac{\delta}{\delta\theta_i(\vec{x})} \right), \tag{3.29}$$

so that

$$\psi(\vec{x}) = \frac{1}{\sqrt{2}} \left[\psi_1(\vec{x}) + i\psi_2(\vec{x}) \right], \tag{3.30}$$

$$= \frac{1}{\sqrt{2}} \left(\theta(\vec{x}) + \frac{\delta}{\delta\theta^*(\vec{x})} \right), \tag{3.31}$$

$$\psi^\dagger(\vec{x}) = \frac{1}{\sqrt{2}} \left[\psi_1(\vec{x}) - i\psi_2(\vec{x}) \right], \tag{3.32}$$

$$= \frac{1}{\sqrt{2}} \left(\theta^*(\vec{x}) + \frac{\delta}{\delta\theta(\vec{x})} \right), \tag{3.33}$$

where $\theta(\vec{x}) = \frac{1}{\sqrt{2}}[\theta_1(\vec{x}) + i\theta_2(\vec{x})]$,and $\theta^*(\vec{x}) = \frac{1}{\sqrt{2}}[\theta_1(\vec{x}) - i\theta_2(\vec{x})]$. This is the Floreanini - Jackiw representation, and is suited to the description of neutral spin-$\frac{1}{2}$ particles [43]. Then $\{\theta(\vec{x}), \theta^*(\vec{x}\,')\} = \{\theta(\vec{x}), \theta(\vec{x}\,')\} = 0$.

Note that

$$\frac{\delta}{\delta\theta_1} = \frac{\delta\theta}{\delta\theta_1}\frac{\delta}{\delta\theta} + \frac{\delta\theta^*}{\delta\theta_1}\frac{\delta}{\delta\theta^*} = \frac{1}{\sqrt{2}} \left(\frac{\delta}{\delta\theta} + \frac{\delta}{\delta\theta^*} \right), \tag{3.34}$$

and

$$\frac{\delta}{\delta\theta_2} = \frac{i}{\sqrt{2}} \left(\frac{\delta}{\delta\theta} - \frac{\delta}{\delta\theta^*} \right), \tag{3.35}$$

hence

$$\frac{1}{2} \left(\frac{\delta}{\delta\theta_1} + i\frac{\delta}{\delta\theta_2} \right) = \frac{1}{\sqrt{2}}\frac{\delta}{\delta\theta^*}, \tag{3.36}$$

There is an alternative, inequivalent choice for ψ_1 and ψ_2 in terms of a single, real Grassmann variable $\theta_a(\vec{x})^* = \theta_a(\vec{x})$:

$$\psi_1(\vec{x}) = \frac{1}{\sqrt{2}}\left(\theta(\vec{x}) + \frac{\delta}{\delta\theta(\vec{x})}\right), \tag{3.37}$$

and

$$\psi_2(\vec{x}) = \frac{1}{i\sqrt{2}}\left(\theta(\vec{x}) - \frac{\delta}{\delta\theta(\vec{x})}\right), \tag{3.38}$$

Then

$$\psi(\vec{x}) = \theta(\vec{x})\,, \tag{3.39}$$

$$\psi^\dagger(\vec{x}) = \frac{\delta}{\delta\theta(\vec{x})}, \tag{3.40}$$

This representation was proposed by Duncan, Meyer-Ortmanns and Roskies (DMR) [44]. Both representations are suitable for charged particles spin-$\frac{1}{2}$ particles [43]. In this representation the complication between canonical conjugation and Hermitian conjugation is evident. $\psi^\dagger$ is therefore the Hermitian conjugate of ψ in the sense in which both are operators on a Hilbert space.

Vacuum wavefunctional

Consider a Fermionic system with finitely many degrees of freedom and Hamiltonian $H = h_{ij}\psi_i^\dagger\psi_j$. Let us work in the complex F-J representation. This has the form of a harmonic oscillator, so we take as *Ansatz* the Gaussian ground state $\Psi = e^{G_{ij}u_i^*u_j}$ where G is necessarily antisymmetric.

$$H\Psi = \frac{1}{2}\left(u_i^* + \frac{\partial}{\partial u_i}\right) h_{ij}\left(u_j + \frac{\partial}{\partial u_j^*}\right)\Psi, \tag{3.41}$$

$$= \frac{1}{2}\left(u_i^* + \frac{\partial}{\partial u_i}\right) h_{ij}\left(\delta_{jk} + G_{jk}\right) u_k\Psi, \tag{3.42}$$

$$= \frac{1}{2}\left(\delta_{il} + G_{il}\right) h_{ij}\left(\delta_{jk} + G_{jk}\right) u_l^* u_k\Psi + \frac{1}{2}\delta_{ik}h_{ij}\left(\delta_{jk} + G_{jk}\right)\Psi, \tag{3.43}$$

$$= \frac{1}{2}\left[(I - G)h(I + G)\right]_{lk} u_l^* u_k\Psi + \frac{1}{2}\operatorname{tr} h(I + G)\Psi. \tag{3.44}$$

For this state to be an eigenstate, the first term must vanish, so that $h-[G,h]-GhG$ must be symmetric or zero. Trivial solutions such as $G=\pm I$ are excluded. Suppose G and h are simultaneously diagonalized. Then $h(I-G^2)=0$ so that $G^2=I$. The eigenvalues of G are therefore ± 1. Since the exact form of G will not be needed here, the reader is invited to see [42] for a more detailed discussion.

3.2 Yang-Mills with Fermions

It is inviting to try to extend the formalism of Karabali and Nair to something more closely resembling QCD, namely a gauge field interacting with Fermions. We will attempt to take the first steps towards such a description, and utilize the ideas of chapter 1 as far as possible. Let us then assume that the state of the system can be represented by a wavefunctional $\Psi[A_i, u]$ where u is the Grassmann variable representation of the Fermion field in either the F-J or DMR representation. Physical wave-functionals are subject to the constraint that the Gauss' law operator (generator of gauge transformations),

$$\hat{G} = \int d^2x \left(\mathbf{D}\cdot\mathbf{E} - \psi^\dagger\psi \right), \tag{3.45}$$

vanishes. In the work by Karabali and Nair, it was required that $(\mathbf{D}\cdot\mathbf{E})\Psi_{phys}=0$ so that one may make the substitution $\Psi[A,\bar{A}] = \Psi[-\frac{\pi}{c_A}J, 0]$ throughout. When Fermions are introduced, this expression must be replaced with $\hat{G}\Psi = 0$, which presumably means that $\Psi[\mathbf{A},u] = \Psi[\mathbf{A}^g, u^g]$.

The field-dependent "gauge transformation" by $M^\dagger$, i.e., $\mathcal{O} \to U\mathcal{O}U^{-1}$ sends the pure Yang-Mills part of the Hamiltonian into the Karabali-Nair form. It also sends $\psi \to \chi = M^\dagger\psi$ and $\psi^\dagger \to \chi^\dagger \psi^\dagger M^{\dagger -1}$. These are both gauge invariant quantities. Thus under $(A,\bar{A}) \to (-\frac{\pi}{c_A}J, 0)$, the KKN Hamiltonian is obtained, plus a term proportional to j^0 (which was omitted in the treatment of Karabali and Nair), plus the gauged Dirac part.

$$H = H_{KKN} - \int d^2x\,(...)j^0 + \int d^2x\,\chi^\dagger\gamma^0\mathcal{D}\chi, \tag{3.46}$$

where $\mathcal{D} = \gamma\partial + \bar{\gamma}\bar{\partial} - \gamma\frac{\pi}{c_A}J(x)$.

The variables χ arise naturally from consistency with the field-dependent "gauge transformation". Evidently, these variables obey canonical anticommutation relations, so we may choose the Floreanini-Jackiw representation, as it is the easiest to implement.

$$\chi = \frac{1}{\sqrt{2}}\left(u + \frac{\delta}{\delta u^\dagger}\right),$$
$$\chi^\dagger = \frac{1}{\sqrt{2}}\left(u^\dagger + \frac{\delta}{\delta u}\right). \tag{3.47}$$

3.2.1 Holomorphic invariance

Under a "holomorphic transformation", $M \to M\bar{h}(\bar{z}), M^\dagger \to hM^\dagger, M^{\dagger-1} \to M^{\dagger-1}h^{-1}$, we find that

$$\chi \to h(z)\chi,$$
$$\chi^\dagger \to \chi^\dagger h^{-1}(z), \tag{3.48}$$

which is consistent with the fact that J transforms as a holomorphic connection. However, this would seem to require that

$$u \to h(z)u$$
$$u^\dagger \to uh^{-1}(z), \tag{3.49}$$

which is not consistent. On the other hand, we recall that we u and $u^\dagger$ are to be treated as independent variables, and so may transform independently. We may replace $u^\dagger$ by another variable $v^\dagger$ which transforms as $v^\dagger \to v^\dagger h^{-1}(z)$. Then the F-J representation becomes

$$\chi = \frac{1}{\sqrt{2}}\left(u + \frac{\delta}{\delta v^\dagger}\right),$$
$$\chi^\dagger = \frac{1}{\sqrt{2}}\left(v^\dagger + \frac{\delta}{\delta u}\right) \tag{3.50}$$

3.2.2 Non-canonical behaviour

We have thus far succeeded in finding a functional representation of the operators ψ and $\psi^\dagger$ that is both gauge and holomorphically invariant. The manner in which these variables arise is required for consistency with the functional manipulations done on the gauge theory sector. However, the price to be paid for this simplicity is that the gauge and matter field operators no longer commute (actually, the distinction between them is now a matter of viewpoint). Since M is a functional of A, it is no longer true that $[E,\chi] \sim [\frac{\delta}{\delta A},\chi] = 0$. Instead, we now have

$$\begin{aligned}
[E^a,\chi] &= \left[-\frac{i}{2}\frac{\delta}{\delta \bar{A}^a}, M^\dagger\right]\psi \\
&= -\frac{i}{2}\frac{\delta M^\dagger}{\delta \bar{A}^a}\psi.
\end{aligned} \tag{3.51}$$

Therefore, the "pure" Yang-Mills Hamiltonian H_{KKN} now acts on the fermionic fields in a rather non-trivial way. The additional terms coming from $[E^a,\chi]$ are also of the same order in g, and it is not clear how successfully these can be treated perturbatively if manifest gauge-invariance is to be maintained. However, recent work by Agarwal, Karabali and Nair [45] involves a similar treatment of a massive scalar field in the adjoint representation, definining $\chi = M^\dagger \phi M^{\dagger -1}$, which translates to $\chi^a = M^{a\dagger}\phi^a$ and is analogous to the Fermionic case presented here. There the contribution of $[E,\chi]$ to the Hamiltonian is included, giving the highly non-trivial form for the Yang-Mills field Hamiltonian coupled to adjoint scalars.

$$\begin{aligned}
H \;=\; & m\left[\int J_a(\vec{z})\frac{\delta}{\delta J_a(\vec{z})} + \int \Omega(\vec{w},\vec{z})_{ba}\frac{\delta}{\delta J_a(\vec{z})}\frac{\delta}{\delta J_b(\vec{w})}\right] \;+\; \frac{\pi}{mc_A}\int :\bar{\partial}J^a\bar{\partial}J^a: \\
& +im\int_{z,w}\Lambda_{cd}(\vec{w},\vec{z})f^{abc}\chi^a(\vec{w})\frac{\delta}{\delta\chi^b(\vec{w})}\frac{\delta}{\delta J^d(\vec{z})} \\
& -\frac{1}{2}\int\frac{\delta^2}{\delta\chi^a\delta\chi^a} + \int\left(\frac{2\pi}{c_A}\bar{\partial}\chi^a(\mathcal{D}\chi)^a + M^2\frac{\chi^a\chi^a}{2}\right),
\end{aligned} \tag{3.52}$$

where M is the mass of ϕ and

$$\Pi_{rs}(\vec{u},\vec{v}) = \int_x \bar{\mathcal{G}}_{ar}(\vec{x},\vec{u}) K_{ab}(\vec{x}) \mathcal{G}_{bs}(\vec{x},\vec{v}), \tag{3.53}$$

$$\Lambda_{ra}(\vec{w},\vec{z}) = -\left[\partial_z \Pi_{rs}(\vec{w},\vec{z})\right] \; K^{-1}_{sa}.(\vec{z}) \tag{3.54}$$

While the treatment is mostly qualitative, the authors analyze screening of the adjoint representations and give an estimate of the string-breaking energy which is within 8.8% of the latest lattice estimates.

Chapter 4

Conclusion

The research presented in this thesis looked at the problem of calculating correlation functions in (2+1)-dimensional Yang-Mills theory in the Karabali-Nair formalism, specifically in the vacuum proposed by Leigh et al. The effect of the WZW non-linear sigma model action that arises anomalously from the configuration space measure was incorporated in a controlled manner by expanding the path integral around the Abelian limit; that is, in powers of the structure constants.

In the Abelian limit it was found that the mass spectrum probed by $J^{PC} = 0^{++}$ operators comprises sums of pairs of zeros of $J_0(y)$, in contrast with the result of Leigh et al., which expresses these masses in terms of zeros of $J_2(y)$ (which approach those of $J_0(y)$ as $y \to \infty$). Thus, to lowest order in f^{abc}, we find a correction due to the inclusion of the WZW action. Therefore our results reproduce those in [12] at large momentum and give justification for approximations made therein. That the results agree at large momentum suggests that a robustness of the analytic structure of the wave-functional is against short-distance corrections. On the other hand, the calculation presented here does not agree with that of Leigh et al. [12] at low momentum.

This research also investigated the inclusion of Fermions in the fundamental representation in a way that preserves as much of the machinery developed by Karabali and Nair while still maintaining gauge invariance. It was found that Fermions could be described in the functional Schrödinger picture using either the Floreanini-Jackiw or Duncan-Meyer-Ortmans-Roskies representations in a way that preserves holomorphic invariance of the theory. On the other hand, the restriction of gauge invariance (which translates into holomorphic invariance) necessarily complicates the canonical commutation relations between the gauge and matter fields. Presently it is not known how to successfully deal with these complications, though other authors have been able to make some qualitative statements as well as preliminary strong-coupling numerical estimates [45].

4.1 Future work

If the theory is to be meaningfully expanded about the Abelian limit then the behaviour of the correlation function $\langle(\bar{\partial}J\bar{\partial}J)_x(\bar{\partial}J\bar{\partial}J)_y\rangle$ at low momentum needs to be understood. Whether the discrepancy in the mass spectrum in this regime signals a breakdown of the Abelian approximation and whether it survives at higher orders in f^{abc} (equivalently, φ) is a possible direction for future investigation. These issues are independent of the fact that the vacuum wavefunctional was motivated in the large N, and it should be possible to answer these questions within the constraints of the Ansatz.

The treatment of matter is indeed much more complicated than that of the pure Yang-Mills system, and many of the tools used thusfar are no longer available. On the other hand, it should be feasible to test the general approach by enlarging the theory to one that has a string theory or gravity dual. Thus far some relevant objects, e.g., Wilson loops, have been studied with some success[45].

Appendix A

The Adjoint Representations

Since this thesis contains many calculations in the adjoint representation, the relationship between the adjoint representations of the Lie Group and its associated Lie Algebra will be reviewed here.

The adjoint representation of the Lie group G with Lie algebra $\mathfrak{g}$ is given by the derivation of the conjugation action ghg^{-1} at the identity:

$$Ad(g)(X) \equiv gXg^{-1}. \tag{A.1}$$

The adjoint representation $ad(\mathfrak{g})$ of the Lie algebra is given by the derivative of the exponential map. Since $e^{tY}Xe^{-tY} = X + t[Y, X] + O(t^2)$,

$$ad(Y) \equiv [Y, \cdot]. \tag{A.2}$$

In other words, writing $g = e^{tY}$, we say that $ad(Y)$ is the derivative of $Ad(g) = 1 \cdot + t[Y, \cdot] + O(t^2)$ at the identity ($t = 0$).

A.1 Group representation

We can consider the adjoint action of the group G, on the algebra $\mathfrak{g}$, i.e., $X \to gXg^{-1}$, where $g \in G$ and $X \in T_e(G)$ is a generator of the group. Recall that the generators form a representation of the algebra $\mathfrak{g}$ and are derivatives near the identity, and thus form a vector space. Let $h(t)$ be a smooth curve through the identity $h(0) = 1$ such that $h'(t) = X$. Then

$$h'(0) \to gh'(0)g^{-1} \in T_e(G) \tag{A.3}$$

i.e. if $X = X^a t^a$ then $\tilde{X} = gXg^{-1} = \tilde{X}^a t^a$ also for some $\tilde{X}^a$. Assuming a compact Lie group for clarity, so that we can write

$$\begin{aligned}
\operatorname{tr}\left(t^a t^b\right) &= \lambda \delta^{ab}, \\
X^a &= \lambda^{-1} \operatorname{tr}\left(X t^a\right). \\
\tilde{X}^a &= \lambda^{-1} \operatorname{tr}\left(gXg^{-1}t^a\right), \\
&= \lambda^{-1} \operatorname{tr}\left(Xg^{-1}t^a g\right), \\
&= \lambda^{-1} \operatorname{tr}\left(X^b t^b g^{-1} t^a g\right), \\
&= \lambda^{-1} \operatorname{tr}\left(t^a g t^b g^{-1}\right) X^b, \\
&= g^{ab} X^b,
\end{aligned}$$

where $g^{ab} \equiv \lambda^{-1} \operatorname{tr}\left(t^a g t^b g^{-1}\right)$ is the adjoint representation of g. Furthermore, let $\tilde{\tilde{X}} = h\tilde{X}h^{-1} = (hg)X(hg)^{-1}$. Then

$$\begin{aligned}
\tilde{\tilde{X}}^a &= \lambda^{-1} \operatorname{tr}\left(t^a h g t^b g^{-1} h^{-1}\right) X^b, \\
&= \lambda^{-1} \operatorname{tr}\left(h^{-1} t^a h g t^b g^{-1}\right) X^b,
\end{aligned}$$

but also,

$$\begin{aligned}
\tilde{\tilde{X}}^a &= h^{ac} g^{cb} X^b, \\
&= \lambda^{-1} \operatorname{tr}\left(h^{-1} t^a h t^c\right) \lambda^{-1} \operatorname{tr}\left(t^c g t^b g^{-1}\right) X^b.
\end{aligned}$$

Therefore $\lambda^{-1} \operatorname{tr}\left(h^{-1}t^a h t^c\right) \lambda^{-1} \operatorname{tr}\left(t^c g t^b g^{-1}\right) = \lambda^{-1} \operatorname{tr}\left(h^{-1} t^a h g t^b g^{-1}\right)$. If $h \in G$, and $m = hg$, then $m^{ab} = h^{ac} g^{cb}$. Thus group multiplication is carried over naturally into the adjoint representation.

Generally, for compact Lie groups, if $A = A^a t^a$ and $B = B^a t^a$, then $\lambda\,\mathrm{tr}(AB) = \lambda^2 A^a B^a = \mathrm{tr}(At^a)\,\mathrm{tr}(t^a B)$.

A.2 Algebra representation

To obtain the representation of the Lie algebra we differentiate at the identity. If $M = e^{\omega^a t^a} = 1 + \omega^a t^a + O(\omega^2)$, then

$$\begin{aligned}
M^{ab} &= \lambda^{-1}\ \mathrm{tr}(t^a M t^b M^{-1}), \\
&= \lambda^{-1}\ \mathrm{tr}(t^a(1+\omega^c t^c)t^b(1-\omega^d t^d)) + \ldots, \\
&= \delta^{ab} + \omega^c \lambda^{-1}\ \mathrm{tr}(t^a t^c t^b - t^a t^b t^c) + O(\omega^2), \\
&= \delta^{ab} + \omega^c \lambda^{-1}\ \mathrm{tr}(t^a[t^c, t^b]) + O(\omega^2), \\
&= \delta^{ab} + \omega^c (i f^{cbd}) \lambda^{-1}\ \mathrm{tr}(t^a t^d) + O(\omega^2), \\
&= \delta^{ab} + \omega^c (i f^{cba}) + O(\omega^2), \\
&= \delta^{ab} + \omega^c (-i f^{cab}) + O(\omega^2), \\
&= \delta^{ab} + \omega^c T^c_{ab} + O(\omega^2).
\end{aligned}$$

Thus $T^c_{ab} \equiv -if^{cab}$ furnish the adjoint representation of the Lie algebra. Given $[t^a, t^b] = if^{abc}t^c$, it follows from the Jacobi identity that the matrices $(T^a)_{bc} = -if^{abc}$ satisfy

$$[T^a, T^b] = if^{abc}T^c. \tag{A.4}$$

These matrices form the adjoint representation of the Lie algebra. As in the fundamental representation, we take the inner product to be given by the trace,

$$\mathrm{tr}\left(T^a T^b\right). \tag{A.5}$$

In the case of non-compact Lie groups this inner product can be diagonalized, so that we may take

$$\operatorname{tr}\left(T^aT^b\right) = \Lambda\delta^{ab} \tag{A.6}$$

for some positive Λ, and $\operatorname{tr}\left(T^aT^b\right)$ is known as the quadratic Casimir invariant of the adjoint representation. With a metric in place, we can recover the new structure constants

$$f^{abc} = -\frac{i}{\Lambda}\operatorname{tr}\left([T^a, T^b]T^c\right). \tag{A.7}$$

A.3 Casimir Invariant

Here we assume a compact Lie group, so that we can write $\operatorname{tr}\left(t^at^b\right) = d_r\delta^{ab}$ in some representation r. For any simple Lie algebra, the quantity

$$t^2 \equiv t^at^a\,) \tag{A.8}$$

commutes with all the other elements of the algebra:

$$\begin{aligned}\left[t^a, t^bt^b\right] &= (if^{abc}t^c)t^b + t^b(if^{abc}t^c),\\ &= if^{abc}\left\{t^c, t^b\right\},\\ &= 0,\end{aligned}$$

since f^{abc} is antisymmetric in its last two indices.

This object is an invariant of the algebra, known as the quadratic Casimir invariant. The (irreducible) matrix representation of t^2 is therefore proportional to the identity matrix (since t^2 commutes with all t^a, and with $\mathbf{1}$, it commutes with all $g \in G$, and by Schur's lemma, $t^2 \propto \mathbf{1}$):

$$t^at^a = c_2(r)\mathbf{1}, \tag{A.9}$$

Furthermore, $c_2(r)$ is sometimes labeled as c_r, including in this thesis to be consistent with literature. In the adjoint representation,

$$(T^c T^c)_{ab} = -f^{cad} f^{cdb} = c_A \delta^{ab}. \tag{A.10}$$

If t^a are suitably normalized, so that f^{abc} is completely antisymmetric, then this can be written as

$$f^{acd} f^{bcd} = \mathrm{Tr}\,(T^a T^b) = c_A \delta^{ab}. \tag{A.11}$$

Now, in a particular irreducible representation r,

$$\mathrm{tr}(t^a t^a) = c_r \dim(r), \tag{A.12}$$

but also

$$\mathrm{tr}(t^a t^a) = d_r \delta^{aa} = d_r \dim(G). \tag{A.13}$$

Therefore we obtain the formula

$$\frac{d_r}{c_r} = \frac{\dim r}{\dim G}. \tag{A.14}$$

Appendix B

SU(N)

B.1 Completeness

Suppose we write the inner product in the Lie algebra as

$$\left\langle t^a | t^b \right\rangle \equiv \mathrm{tr}\left(\frac{t^{\dagger a}}{\sqrt{\lambda}}\frac{t^b}{\sqrt{\lambda}}\right) = \delta^{ab}. \tag{B.1}$$

Viewed as N^2-dimensional vectors in complex Euclidean space, this is just the Euclidean norm. We therefore have a subspace spanned by n orthonormal vectors. If these were a complete set, then we would have the completeness relation

$$|t^a\rangle \langle t^a| = \mathbf{1}_{N^2\times N^2}, \tag{B.2}$$

(summation implied). However, since the generators are traceless, they are all orthogonal to $|1\rangle$ the N^2-dimensional vector corresponding to $\frac{1}{\sqrt{N}}\delta_{rs}$. For $\mathfrak{su}(N)$, $n = N^2 - 1$, so we simply project out the subspace $|1\rangle$:

$$|t^a\rangle \langle t^a| = \mathbf{1}_{N^2\times N^2} - |1\rangle \langle 1|. \tag{B.3}$$

In matrix form, this becomes (keeping in mind that the generators are also hermitian)

$$\frac{1}{\lambda} t^a_{pq} t^a_{rs} = \delta_{pr}\delta_{qs} - \frac{1}{N}\delta_{pq}\delta_{rs}. \tag{B.4}$$

B.2 Adjoint representation

Consider the tensor product of the antifundamental representation and fundamental representations of $SU(N)$, i.e., $\mathbf{N}\otimes\bar{\mathbf{N}}$. Now the trace (singlet) part is invariant, so transforms as the $\mathbf{1}$. The trace-free part stays trace-free under a transformation owing to the cyclic property of the trace. The trace-free part of X can be written as a linear combination of some basis of N^2-1 traceless matrices. There happen to be N^2-1 generators of $SU(N)$, which are also traceless. For convenience, assume X is traceless (since the trace-part transforms trivially), so that $X \in \mathfrak{su}(N)$:

$$X = X^a t^a. \tag{B.5}$$

Therefore the transformation law $X \to UXU^{-1}$ is just the adjoint representation of $SU(N)$, so

$$\mathbf{N}\otimes\bar{\mathbf{N}} = \mathbf{1}\oplus\mathbf{Adj}. \tag{B.6}$$

B.3 Quadratic Casimir

$SU(2)$ is a subgroup of $SU(N)$, since we can consider the subgroup of $N\times N$ matrices leaving all but the two indices fixed. Any complete representation of $SU(N)$ is therefore a (possibly reducible) representation of $SU(2)$. Three generators of $SU(N)$ generate a (possibly reducible) representation of $SU(2)$. Therefore $\mathrm{tr}(t^at^b) = d_{r,SU(N)}\delta^{ab} = d_{r,SU(2)}\delta^{ab}$ where the indices a and b range over values which are defined for both groups. This assists in obtaining the Casimirs [15].

The $\mathbf{N}$ of $SU(N)$ decomposes into $N-2$ singlets ($\mathbf{1}$) and one doublet ($\mathbf{2}$) $SU(2)$. From the point of view of $SU(2)$,

$$d_{\mathbf{N},SU(N)} = (N-2)d_{\mathbf{1},SU(2)} + d_{\mathbf{2},SU(2)} = \frac{1}{2}, \tag{B.7}$$

and

$$c_{\mathbf{N}} = d_{\mathbf{N}}\frac{\dim G}{\dim\mathbf{N}} = \frac{1}{2}\frac{N^2-1}{N}. \tag{B.8}$$

For $\mathbf{N} \otimes \bar{\mathbf{N}}$,

$$\begin{aligned}
\left[\mathbf{N} \otimes \bar{\mathbf{N}}\right]_{SU(N)} &= [\mathbf{2} \oplus (N-2)\mathbf{1}]_{SU(2)} \otimes [\mathbf{2} \oplus (N-2)\mathbf{1}]_{SU(2)}, \\
&= \left[\mathbf{2} \otimes \mathbf{2} + 2(N-2)\mathbf{2} \otimes \mathbf{1} + (N-2)^2\mathbf{1} \otimes \mathbf{1}\right]_{SU(2)}, \\
&= \left[\mathbf{3} + \mathbf{1} + 2(N-2)\mathbf{2} + (N-2)^2\mathbf{1}\right]_{SU(2)}, \mathbf{Adj}_{SU(N)}, \\
&= \left[\mathbf{3} + 2(N-2)\mathbf{2} + (N-2)^2\mathbf{1}\right]_{SU(2)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
d_{\mathbf{Adj},SU(N)} &= \left[R_{\mathbf{3}} + 2(N-2)R_{\mathbf{2}} + (N-2)^2 R_{\mathbf{1}}\right]_{SU(2)}, \\
&= 2 + 2(N-2)\frac{1}{2} + (N-2)^2 0, \\
&= N.
\end{aligned}$$

Consequently, since $\dim \mathbf{Adj} = \dim G$,

$$c_{\mathbf{N}} = \frac{1}{2}\frac{N^2-1}{N}, \tag{B.9}$$

$$c_A = N. \tag{B.10}$$

Appendix C

Bessel function identities

Since $J_\nu(z)$ is meromorphic, the following infinite product representation of $J_\nu(z)$ exists in terms of the roots $j_{\nu,s}$ [40]:

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,s}^2}\right) \tag{C.1}$$

C.1 Proof of $\frac{J_1(z)}{J_2(z)} = \frac{4}{z} + 2z \sum_{s=1}^{\infty} \frac{1}{z^2 - j_{2,s}^2}$

$$\begin{aligned}
\frac{d}{dz} J_2(z) &= -\frac{d}{dz} \frac{(\frac{z}{2})^2}{\Gamma(3)} \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{j_{2,s}^2}\right), \\
&= -\frac{2}{z} J_2(z) + \sum_{s=1}^{\infty} \frac{\frac{2z}{j_{2,s}^2}}{1 - \frac{z^2}{j_{2,s}^2}} J_2(z), \\
&= -\frac{2}{z} J_2(z) + 2z \sum_{s=1}^{\infty} \frac{1}{j_{2,s}^2 - z^2} J_2(z), \\
&= \frac{2}{z} J_2(z) - J_1(z) \frac{J_1(z)}{J_2(z)}, \\
&= \frac{4}{z} + 2z \sum_{s=1}^{\infty} \frac{1}{z^2 - j_{2,s}^2} \frac{J_1(z)}{J_2(z)}, \\
&= \frac{4}{z} + 2z \sum_{s=1}^{\infty} \frac{1}{z^2 - j_{2,s}^2}.
\end{aligned}$$

C.2 Proof of $\frac{J_1(z)}{J_0(z)} = 2z\sum_{s=1}^{\infty}\frac{1}{\gamma_{0,s}^2 - z^2}$

$$
\begin{aligned}
\frac{d}{dz}J_0(z) &= -\frac{d}{dz}\prod_{s=1}^{\infty}\left(1-\frac{z^2}{\gamma_{0,s}^2}\right),\\
&= \sum_{s=1}^{\infty}\frac{\frac{2z}{j_{0,s}^2}}{1-\frac{z^2}{\gamma_{0,s}^2}}J_0(z)\frac{J_1(z)}{J_0(z)},\\
&= 2z\sum_{s=1}^{\infty}\frac{1}{\gamma_{0,s}^2 - z^2}\frac{J_1(z)}{J_0(z)},\\
&= 2z\sum_{s=1}^{\infty}\frac{1}{\gamma_{0,s}^2 - z^2}.
\end{aligned}
$$

Appendix D

Free Boson in (2+1)-Dimensions

In order to make a connection between the correlators obtained in the Hamiltonian formalism, e.g., equation (2.20), with the (covariant) Källén-Lehmann spectral representation, we need to recognize the analytic structure of a 1-particle state when expressed non-covariantly. To do this, we analyze the two-point function of the free Boson for purely spatial separation:

$$\begin{aligned}
\Delta_F(\boldsymbol{x}-\boldsymbol{y}) &= \int \frac{d^2k}{(2\pi)^2}\frac{1}{\sqrt{k^2+m^2}}e^{i\mathbf{k}\cdot\mathbf{r}}, \\
&= \int_0^\infty \frac{k\,dk}{(2\pi)^2}\frac{2\pi J_0(kr)}{\sqrt{k^2+m^2}}, \\
&= \frac{1}{2\pi|\boldsymbol{x}-\boldsymbol{y}|}e^{-m|\boldsymbol{x}-\boldsymbol{y}|}. \qquad \text{(D.1)}
\end{aligned}$$

Appendix E

Quantum numbers

In 2 spatial dimensions, angular momentum is characterized by the quantum numbers associated with the representations of the $SO(2)$ subgroup of the $SO(2,1)$ Lorentz group. Vectors $(A, \bar{A})$ transform as $J = (-1, 1)$ whereas derivatives $(\partial, \bar{\partial})$ transform in the opposite way, as $J = (1, -1)$. In this thesis we are chiefly interested in the operator $\bar{\partial}J$ which has total $J_{\bar{\partial}J} = 0$.

Under parity, i.e. $(x^1, x^2) \to (x^1 - x^2)$, we have that $z \to \bar{z}$, leading to [28]:

$$\begin{aligned} A &\to \bar{A}, \\ M &\to M^{\dagger -1}, \\ H &\to H^{-1}, \\ \bar{\partial}J &\to -H^{-1}\bar{\partial}JH, \end{aligned}$$

so that the trace $\operatorname{tr}\bar{\partial}J\bar{\partial}J$ is even under parity. Finally, this operator is also even under charge conjugation, and so $J^{PC} = 0^{++}$.

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